A Practice-Based Approach to Diagrams

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Abstract In this article, I propose an operational framework for diagrams. According to this framework, diagrams *do not work* like sentences, because we do not apply a set of explicit and linguistic rules in order to use them. Rather, we become able to manipulate diagrams in meaningful ways once we are familiar with some specific practice, and therefore we engage ourselves in a form of reasoning that is stable because it is shared. This reasoning constitutes at the same time discovery and justification for this discovery. I will make three claims, based on the consideration of diagrams in the practice of logic and mathematics. First, I will claim that diagrams are tools, following some of Peirce's suggestions. Secondly, I will give reasons to drop a sharp distinction between vision and language and consider by contrast how the two are integrated in a specific manipulation practice, by means of a kind of manipulative imagination. Thirdly, I will defend the idea that an inherent feature of diagrams, given by their nature as images, is their ambiguity: when diagrams are 'tamed' by the reference to some system of explicit rules that fix their meaning and make their message univocal, they end up in being less powerful.

Keywords Diagrammatic reasoning · Practice-based philosophy of mathematics · Peirce's diagrams · Manipulative imagination · Productive ambiguity

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1 Introduction: Diagrammatic Reasoning in Logic and Mathematics

The claim that diagrammatic reasoning has a role in the work of a logician or a mathematician, both in education and research, is not controversial. Even a scholar who defends a conception of logic or mathematics as mere games of symbols would not deny that objects such as circles or figures are helpful as heuristic tools in directing reasoning. A common saying is that in problem solving, if we draw the 'right' diagram, then we are half of the way to finding its solution. Diagrams seem to guide thought: their heuristic power is evident, both in the classroom—where concepts and theorems are often taught by making reference to objects such as sketches on the blackboard—and in research—when logicians and mathematicians interact and communicate by holding a pen in their hands, drawing, changing and deleting columns of formulas or shapes.

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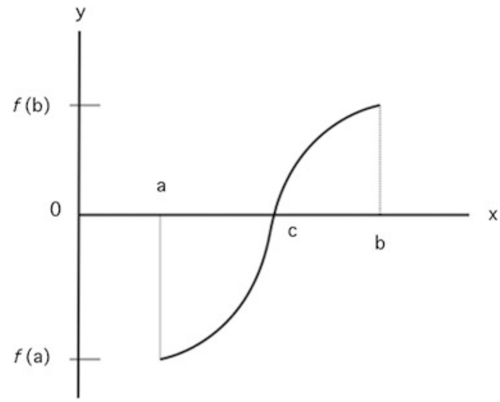


Fig. 1 A figure displaying the *Intermediate Zero Theorem*

Consider for example the case of the *intermediate zero theorem*.[1] According to this theorem, if a function f is continuous on the interval $[a, b]$ and f changes sign from negative to positive (or vice versa), then there is a c between a and b such that $f(c) = 0$. The figure displaying such a function is shown in Fig. 1.

By looking at the figure, we are inclined to believe that the theorem holds, because we can clearly 'see' it. The line of the argument would be the following: the horizontal axis divides the space on the right side of the vertical axis into two regions; in the region above the horizontal axis, the sign of the function is positive, while in the region below the horizontal axis, the sign of the function is negative. The function f would then be described as performing the action of *moving* from $f(a)$ to $f(b)$: if the function changes sign from negative to positive (or vice versa), then it *has to go* from below the horizontal axis to above it (or vice versa). The figure is straightforward: we have the impression that our reasoning corresponds to what we 'see'.

Nevertheless, a certain conception of logic and mathematics, by subscribing to the old distinction between a context of justification and a context of discovery, has relegated our natural disposition to use figures to display the content of our statements to the psychological context of discovery. This conception has discarded figures and diagrams as not being rigorous. I will call this view the *suspicious* view, both because it denies that diagrammatic reasoning plays a role in the context of justification, and because I want to put this approach into question. I mention here that I deliberately use the term 'suspicious' ambiguously; as I will show later in the article, ambiguity will become important in connection to diagrams.

According to the suspicious view, diagrams, despite their being good heuristic tools in discovery and explanation, are not sufficient when it comes to proof. The view does not directly deny the cognitive and computational advantages in using diagrams, but rather puts forward an image of logic and mathematics that is strongly dominated by proof. Proofs, in turn, are conceived as particular syntactic objects, namely derivations and thus verbal/symbolic entities. If formal proofs are at the core of mathematics, then diagrams cannot be part of them, unless they are 'tamed' by the definition of some syntactical rules that control their use. In such a framework, justification is defined by proof and rigor, and they alone lead to truth: mathematical knowledge happens only when we are *truly*

[1]The theorem is mentioned, among others, in [4].

justified in our belief, namely, when we have a formal proof, and not when we are merely justified in believing that a statement is true, as in the case of Fig. 1 for the intermediate zero theorem. As Fallis explains,

> even though there are many ways in which a mathematician might be justified in believing that a mathematical statement is true, there is only one way in which a mathematician feels that she is truly justified. Specifically, the mathematician has to know a proof of the mathematical statement ([10], p. 46).

Polya, who famously devoted most of his work to the investigation of problem solving processes, claims that what is lost in rigor is made up in understanding [22]. Insight, understanding, and explanation, which are part and parcel of the work of the logician or of the mathematician, are not all necessarily included in a rigorous proof.

If mathematics is dominated in such a way by formal 'non-visual' proofs that preserve truth, all so called 'visual' proofs are discarded as non-rigorous. Diagrams are not reliable: they can mislead us and do not provide evidence. According to the standard and logocentric definition, proofs are

> syntactic objects consisting only of sentences arranged in a finite and inspectable way, and therefore diagrams can only be heuristic tools to prompt certain trains of inference ([2], p. 3).[2]

Against this assumption, Barwise and Etchemendy tried to expand formal logic by freeing it from having one mode of representation only, i.e. language, and by pushing it closer to reasoning, which is in their view a *heterogeneous* enterprise. As Shin sums up,

> all of us engage in and make use of valid reasoning, and in the process of reasoning human beings obtain information through many different kinds of media, including diagrams, maps, smells, sounds, as well as written or spoken statements ([26], p. 92).

However, Barwise and Etchemendy, as well as Shin later on, did not renounce the idea that proofs are at the core of logic and mathematics and that proofs are derivations. Therefore, their strategy was to force and 'tame' diagrammatic reasoning by defining syntactic rules to carry out diagrammatic manipulations: in their view, only 'tamed' diagrams, namely diagrams which we have control over, can become 'rigorous' elements of formal proofs.

By contrast, let us suppose that an alternative picture of logic and mathematics is possible. This picture does not have at its core formal proofs, but rather the practices that are shared by a community of scholars who in their ordinary work consider informal proofs, such as proofs based on induction or visual tools, as sufficient for being justified in believing that some statement holds. As Brown suggests, we should assume

> a somewhat more humble attitude toward our understanding of verbal/symbolic reasoning. First-order logic may be well understood, but what passes for acceptable proof in mathematics includes much more than that ([4], p. 164).

Moreover, as discussed by Mancosu, there are connections in mathematics that count as *necessary* and *truth preserving*, and others that are not held as such but nevertheless can count as *reasons* [17]. Rigor might preserve truth, but insight, understanding, and explanation hint at reasons. Furthermore, there are proofs that explain—which might be referred to as *causal* proofs—and proofs that convince but do not explain—which can

[2]Barwise and Etchemendy quote this passage from an article by Tennant (see [28]). They believe it expresses the 'dogma' of logocentricity that they want to challenge.

be referred to as *non-causal* proofs [3]. If we look at the practice of mathematics, we realize that verification is proof, but verification might not provide reasons: mathematicians are not satisfied with proving conjectures, since what they want is *reasons* for these conjectures [23].

Views such as the suspicious view, focusing on rigorous formal proofs, move away from the consideration of actual mathematical practice, both in contemporary mathematics as well as in the history of mathematics. According to Corfield, they apply what he calls the 'foundational filter'; because of this filter the only interesting questions in philosophy of mathematics are about the possible reduction of mathematics to some foundational system [7]. Corfield believes that the foundational filter is an "unhappy idea", because not only does it fail to detect the pulse of contemporary mathematics but it also "screens off the past to us as not-yet-achieved" ([7], p. 8). Mathematics is a human activity and, consequently, it is situated in time. The questions about mathematical practice "are to be addressed by an understanding of mathematical knowledge as historically situated rather than timeless" ([7], p. 15). The suggestion is to get free of the appeal to a timeless logic or mathematics that has inspired the assumptions behind the suspicious view. An interdisciplinary investigation may be helpful

> in the process demonstrating that philosophers, historians and sociologists working on pre-1900 mathematics are contributing to our understanding of mathematical thought, rather than acting as chroniclers of proto-rigorous mathematics ([7], p. 8).

The suspicious view discards as 'psychological', and therefore philosophically uninteresting, not only diagrammatic reasoning but also many other activities of the community of working logicians and mathematicians.

In this article, I will not give any normative criteria for the definition of what a proof should be. More modestly, I will simply claim that the search for the reasons why diagrams are apparently so effective in explanation and discovery is a philosophical issue. I will propose an *operational* framework, within which I will show that diagrams *do not work* like sentences: in fact, we do not necessarily apply a set of explicit and linguistic rules in order to use them. Rather, once we are familiar with some specific practice, we manipulate diagrams in meaningful ways, engaging ourselves in a form of reasoning that is stable because it is shared by the community and thus constitutes at the same time discovery and justification for that discovery. If this kind of operational framework works for diagrams, then a further issue will be to ask whether the same operational framework can be applied to other kinds of activities, and thus be generalized to a practice-based approach to logic and mathematics in general. This is a matter for further research.

In the following sections, I will make three claims based on the consideration of diagrams in the practice of logic and mathematics. First, I will claim that diagrams are tools and I will define what I intend by 'diagram' and by 'tool', following some of Peirce's suggestions. Secondly, I will give reasons to drop the opposition between vision and language, and consider by contrast how the two are integrated in a specific manipulation practice by means of a kind of *manipulative* imagination. Thirdly, I will defend the idea that an inherent feature of diagrams, given by their nature as images, is their ambiguity. Moreover, ambiguity promotes a wider variety of interpretation and understanding: when diagrams are 'tamed' by way of referencing to some system of explicit rules that fix their meaning and make their message univocal, they end up in being less powerful.

2 Diagrams in the Practice

2.1 Diagrams Are Not Pictures of Abstract Objects

I propose an account of reasoning *in* and *from* diagrams based on the conception of diagrams as tools used within a specific practice. First, I will claim that if diagrams are considered as tools, then old problems that lie behind the suspicious view lose their strength. Secondly, I will clarify what I intend by 'diagram' and by 'tool'.

Traditionally, two problems have been put forward to conclude that figures are not sufficient for providing justification of mathematical statements. I will define these problems (i) the *generality* problem and (ii) the *appropriateness* problem. The problems state that:

(i) it is not possible to get to a general conclusion by looking at some particular diagram, for the reason that *that* particular diagram has specific properties and specific features that do not directly depend upon the statement that is to be proven: to check for the truth of the statement, a single diagram should be capable of representing *all* the possible specific ways in which the situation described by the statement can be true, which is impossible;
(ii) diagrams are most of the time inappropriate for proving the statement in question because they are never precise enough: these imprecisions can occur in our reasoning and thus bring us to false conclusions and misinterpretations.

For example, in the case of the intermediate zero theorem, (i) we cannot be sure that the function f would *in any case* behave as depicted in Fig. 1 and (ii) we cannot be sure that we have properly drawn the figure.

Nevertheless, the assumption behind the claim that diagrams are never sufficiently general and never sufficiently appropriate is that diagrams are depictions—though partial and imprecise—of abstract objects. I want to contend this claim and propose that diagrams are not pictures of abstract objects but tools for reasoning about abstract relationships. Before doing that, I will present Brown's view of diagrams as tools and explain why it is different from the view I propose.

Brown's strategy for refuting problems (i) and (ii) is to subscribe to a Platonist view of abstract objects. According to him, figures are not and cannot be representations in the sense of being pictures. His argument is that if they were such representations, then there would be a kind of structural similarity captured by the concept of *isomorphism* between them and the abstract objects they depict. Nonetheless, despite the fact that in a wide variety of cases a good diagram is isomorphic to the situation it represents, this is not always the case.

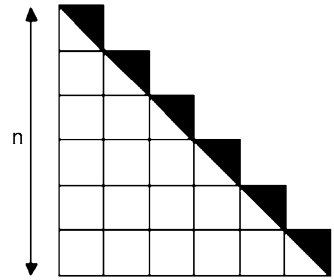
Take for example the following number theory result on the sum of natural numbers up to n:

$$1 + 2 + 3 + \cdots = \frac{n^2}{2} + \frac{n}{2}$$

Now consider one of the possible figures that can be used to represent this result (Fig. 2).

The area of the figure in Fig. 2 is composed by 1 square plus 2 squares plus 3 squares plus … n squares. Thanks to the arrangement of the squares, the area is *also* equal to $n \times n$ squares divided by 2 (the area of the white isosceles triangle with base and height

Fig. 2 A figure displaying the sum of n natural numbers



equal to n) plus half n (the area of the black triangles). Therefore, the figure shows that the equation holds: the same diagram that represents the right side of the equation also represents its left side.

However, Brown claims, the figure in Fig. 2 is strictly speaking just a 'picture', an illustration of the $n = 6$ case, and as a consequence

> we can claim that there is an isomorphism to some number structure with that cardinality. It is certainly not, however, isomorphic to all the numbers. True, it is homomorphic to the whole number structure. But note that a homomorphism to a larger structure is (at least in the case at hand) an isomorphism to a part ([4], p. 173).

Therefore, the diagram in Fig. 2 tells us something about the part of the structure it is isomorphic to, but nothing about the rest of the same structure. Brown's proposal then is to deny that figures such as the one in Fig. 2 are pictures at all, and assume that, in his words, "*some 'pictures' are not really pictures, but rather they are windows in Plato's heaven*": figures are tools in the same sense in which telescopes are tools for the unaided eye. For this reason, the problems (i) and (ii) do not apply, since diagrams are not depicting any possible situation and to some extent they are always inappropriate. In a nutshell, in Brown's view, it is possible to be a realist about abstract objects without being a realist about pictures [5].

Brown's solution is thus based on realism about abstract objects. My objection to his view is that it explains what is already mysterious—our common way of referring to figures in our reasoning—by means of even more mysterious entities: abstract objects in a Platonic heaven. Moreover, as Folina claims,

> telescopes are not *themselves* justificatory: it is not the telescope which is cited as the primary justification for an astronomical claim. Similarly for windows. . . . Rather, it is merely a tool which enables us to 'see' the evidence, or the abstract 'picture' ([11], p. 426).

According to Folina, Brown endorses Platonism in order to claim that figures can legitimately prove the truth of mathematical statements, but this is a non-starter. In fact, we can well accept the assumption that diagrams cannot be a part of formal proofs; the point is rather to claim that formal proofs are only a proper subset of a variety of justifications in mathematics. As in the criticism of Barwise and Etchemendy's program, the real challenge is not to make diagrams legitimate components of formal proofs, but rather to give an account of how they belong to other kinds of justifications in the variety. In this respect, Brown's view is not helpful.

I will side with Brown and agree with him that diagrams are not pictures of abstract objects but instruments, and therefore problems (i) and (ii) do not apply. Nevertheless, I will not claim that they are telescopes pointing at a Platonic heaven, because we are not sure of the existence of such a heaven. In the next section, I will discuss what I intend by saying that diagrams are tools.

2.2 Diagrams as Tools

For the purpose of this article, I use the label ‘diagram’ as a very general term to refer to two-dimensional displays.[3] I move within a Peircean perspective, and consider diagrams as instruments for thought along the following lines.

First, in order to be effective, diagrams must always be *interpreted* within a certain *context of use*. Secondly, the reference to them contributes to the very definition of this context and gives structure to the problem to solve. Thirdly, diagrams belong to the genus of ‘representation’ as “that character of a thing by virtue of which, for the predication of a certain mental effect, it may stand in place of another thing” ([13], vol. 1, par. 564; written in 1893). Nevertheless, this should not be taken literally, for diagrams do not ‘directly’ depict some abstract object whose existence is presupposed; rather, they embody a selection of *relevant* relations. Finally, diagrams are given *with an intention*, as all tools are, cognitive and epistemic tools being among them: they are conceived so as to achieve some particular aim, and the intention behind their creation must be acknowledged in order to appropriately interpret and use them.

Therefore, some of the physical features of a diagram refer to abstract elements that are not directly present in the diagram. By manipulating these physical features, the user—the ‘interpreter’—learns or genuinely discovers something new about the relations the diagram embodies. As Peirce rightly pointed out, this *dynamic* aspect of diagrams triggers “a state of activity” in the interpreter that leads to experimentation.[4] Diagrammatic reasoning would then bring logic and mathematics closer to the natural sciences: logicians and mathematicians experiment with the very same representations that constitute their instruments. Peirce goes even further by saying “all necessary reasoning without exception is diagrammatic” ([13], vol. 5, par. 162; written in 1903). Once more, I will not take any stance in the debate on what counts as necessary reasoning. My more modest suggestion is that in order to claim that diagrams are stable enough to provide justification, we have to consider the practice shared by the community of actors who experiment on them. Diagrams are representations used with the intention of embodying relations; moreover, they promote inference because they can be interpreted and manipulated in various ways according to the shared practice.

An important advantage of this operational approach is that it discards the opposition between visual reasoning and linguistic knowledge. In fact, the dichotomy of vision vs. language, which has led to the antithesis between visuocentric and logocentric views, is pernicious.

[3] I am not denying here the possibility that there are three-dimensional diagrams. I only want to exclude this possibility for the moment, because I am inclined to think that it implies additional considerations.

[4] “It is not, however, the statical Diagram-icon that directly shows this; but the Diagram-icon having being constructed with an Intention [. . .]. Now, let us see how the Diagram entrains its consequence. The Diagram sufficiently partakes of the percussivity of a Percept to determine, as its Dynamic, or Middle, Interpretant, a state [of] activity in the Interpreter, mingled with curiosity. As usual, this mixture leads to Experimentation.” In [21].

In the next sections, I will first discuss the risks of siding with or against vision or language without considering their continuous interaction; secondly, I will discuss two features of diagrammatic reasoning that emerge in this operational framework: the role of action and manipulative imagination, and the importance of ambiguity and multi-dimensionality of meaning.

3 Beyond Visuocentric and Logocentric Views

3.1 Not Only a Question of Visual Properties

Let us consider a first risk in endorsing the dichotomy of vision vs. language. Suppose that we want to refute logocentrism and defend a view diametrically opposed to it, a kind of 'visuocentrism' or 'optocentrism'. According to this approach, it suffices to look at a figure to get to its content and to the message it conveys; the meaning of a diagram, compared to the meaning of a sentence, would be easier to grasp because it is 'directly seen' and therefore extracted 'for free'.

Nevertheless, this view is ill posed, because it is not only visual perception that is at stake in diagrammatic reasoning. In fact, the diagram user—its interpreter—is not interested in many of the visual properties of the diagram; she only attends to a selection of them. For this reason, talking of visual thinking is misleading; the user rather considers possible spatial configurations and new constructions in the diagram.

Let me discuss one attempt to enhance the visual features of a diagram with the aim of making its meaning easier to grasp. Consider Byrne's edition of Euclid's *Elements* published in 1847 [6]. This edition covers Euclid's first 6 books and presents them in form of colored pictures, using as little text as possible, most of the times in the form of labels: Byrne's attempt was to make the *Elements* 'more visual' by adding colors to them. Nevertheless, this attempt is unsuccessful, if the aim is to make the Elements easier to understand. In fact, colors are introduced as a new code that must be learned in its symbolic use. As a consequence, the figures, instead of becoming more straightforward, end up becoming more complex: the student must learn how to interpret colors, and therefore she has to become familiar with a completely new language, the color code language, thus increasing the cognitive load of the task instead of reducing it (see Fig. 3). Furthermore, she is deprived of the instructions given in the text to construct the diagrams, and is therefore driven away from learning the practice of Euclidean geometry.

To make Euclid diagrams easier to understand, attention should be focused on constructions and manipulations rather than on possibly new visual features. This example shows that diagrams are not simply 'seen' but must be 'read', i.e., interpreted. Their appropriate interpretation leads to the definition of their constructions and manipulations. Therefore, the assumption of a sharp distinction between vision and language along with the claim that vision matters does not give a good description of what happens in diagrammatic reasoning.

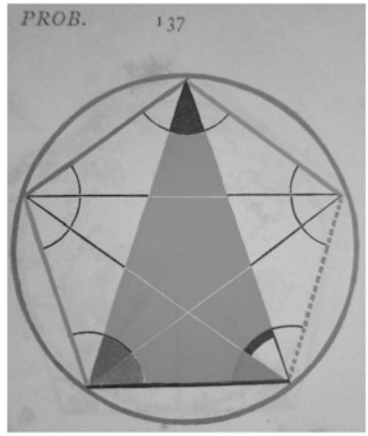


Fig. 3 A picture taken from Byrne's edition of Euclid's *Elements*

3.2 Not Only a Question of Expressivity

In the previous section, I claimed that diagrams are advantageous not simply because they are more 'visual' than sentences. In the same way, if the opposition between vision vs. language is assumed, not even a focus on the possible verbal/symbolic translations of the message conveyed by the diagram will yield a good description of what happens in diagrammatic reasoning.

Consider a case study from logic, which shows the contrast between the urge for rigor and the cognitive benefit of the diagrammatic representation.[5] In the 18th century, the mathematician Euler introduced his circles (D1) for the study of syllogisms. As Shin and Lemon explain, the representation in Euler's diagrams is governed by the convention according to which every object x in the domain is assigned a unique location in the plane, say $l(x)$, such that $l(x)$ is in the region R if and only if x is a member of the set the region represents [27]. Note that, despite its apparent naturalness, this move is already conventional since the choice of circles is arbitrary. Other logicians introduced systems that used points for objects and lines for sets [8, 15]. On the other hand, this convention exploits better than others the perceptual configuration that Lakoff and Nuñez have defined as the *Containment-schema*: circles and in general closed figures are more effective than lines in being interpreted as 'containing' the members of a set in the spatial region they identify [14].

Despite their straightforwardness, Euler circles have expressive limitations and retain some crucial ambiguities, for example when representing existential statements, or the empty set, or congruency among sets. In order to solve some of these ambiguities or limits, in 1881 Venn introduced his own system of diagrams (D2) based on 'primary diagrams'. Primary diagrams do not carry any particular information in themselves; in order to be meaningful, they need to be complemented by labels and shadings. Despite their greater generality, Venn diagrams present new expressive limitations, and for this reason Pierce, in the 20th century, modified them by introducing three new symbols (0, x, –), thus providing a new system (D3) by which existential statements, disjunctive information, probabilities, and relations could be represented (see Fig. 4).

[5]For a detailed discussion of this case-study, see [27].

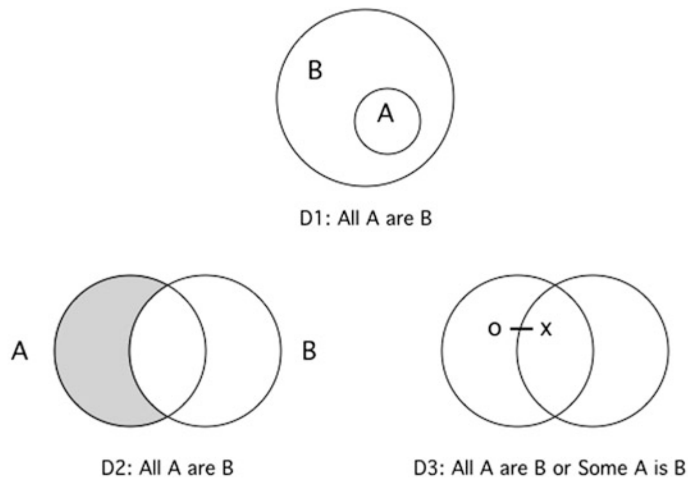


Fig. 4 Three examples of circles for syllogism: Euler (D1), Venn (D2) and Peirce (D3)

In going from D1 to D3, the new systems become thus more and more expressive. In the 90s, Shin continued on this path and proposed a formal system that improved Peirce's diagrams by making them even more expressively powerful [25].

Nevertheless, as Shin and Lemon discuss, Peirce's (as well as Shin's) introduction of new conventions increased the expressive power of the single diagrams at the expense of the visual clarity that Euler's original system enjoys. The new conventions are more arbitrary and the new representations more confusing. This is true, despite the fact that Peirce's choice of the symbol '0' for the empty set is not wholly arbitrary. According to the authors, when Peirce's revision was completed, most of Euler's original ideas about visualization were lost, except for the choice of a geometrical object, the circle, still used to represent (possibly empty) sets. I would slightly modify their claim and say that what was retained in D3 was the mere *appearance* of the circles of the original diagrammatic system, but not their *function* of exploiting the Containment-schema. As Euler himself claimed, "we may employ, then, spaces formed at pleasure to represent every general notion, and mark the subject of a statement by a space containing A, and the attribute by another which contains B."[6] These are the 'spaces' that mattered for him and his system, and not the particular figure they were of. Circles could have been changed into squares, for example, without changing their function of 'containment' and, therefore, without changing the expressive power of the system. By the introduction of primary diagrams, this function of 'containment' loses its centrality, and new elements such as shadows and labels are introduced in order for the system to be meaningful. The strategy of augmenting the expressivity of diagrams by fixing their meaning through the introduction of these arbitrary conventions provides an interesting 'diagrammatic' extension of sentential logic, but at the same time deprives spatial tools of their effectiveness and straightforwardness, and leaves behind our perceptual and imaginative capacities.

Consider now the fascinating collection of 'visual proofs' that Nelsen gives in his two books, entitled *Proofs Without Words* (*I*, *II*) [18, 19]. On closer inspection, these proofs are

[6] Letter 103, *Of Syllogism, and their different Forms, when the first Proposition is Universal*. See [9].

not simply 'without words', since the use of a diagram is not only a matter of vision. As in the case of the intermediate zero theorem, we have to know the mathematical statement in question in order to find it 'in' or 'represented by' the diagram in Fig. 1. Without knowing the statement, reasoning with a diagram would be equivalent to a riddle such as 'From this figure, find the statement in it' or 'Which statement is represented by the figure?'. We always need some linguistic explanation or justification for our use of some specific diagram.

More generally, in practice, it is very difficult to distinguish purely graphical systems from purely sentential ones [24]. In most cases, diagrams and text are so strongly interconnected that they cannot be considered in isolation one from the other; furthermore, thanks to this dialectic, appropriate interpretations and operational procedures are defined.

Let me point out to two examples in which the dialog between images and text have been challenged by the invention of printing. First, consider Diderot and D'Alambert's *Encyclopédie*, published in France between 1751 and 1772. The images and text in it were conceived as interconnected and continuously referring one another; nevertheless, because of the limitations of the printing techniques available at the time, it was impossible to show on the same page the labeled images and the corresponding text. In many cases, the text was very far from the images it referred to. For this reason, the reader had to move from the images to the text without having both together; this was very awkward and contributed to the sharp distinction between images and text, which does not reflect the way the two were conceived at the beginning as being strongly interconnected.

Something similar happened for the first printed editions of Euclid's *Elements*, where it was even more crucial to show images and text on the same page [1]. In Euclid's *Elements*, we never find a simple two-step process from the verbal to the visual moment that is a mere illustration of the former. By contrast, the text and its accompanying figure are engaged each time in a fruitful and rational intercourse: the text gives the instructions to outline the figure that stands by its side, and to draw the conclusion from it. The verbal instructions do not assume the complete figure at the start, but rather walks the readers through its construction, and therefore the readers continuously go back and forth between verbal and visual. In following these instructions, they *actively imagine* drawing lines.

Instead of accepting the unuseful opposition between visual perception and linguistic knowledge, I propose to focus on this kind of 'manipulative' imagination at play when reasoning with diagrams. This is further discussed in the next two sections.

4 An Operational Framework for Diagrams

4.1 Looking at How We 'Act' upon the Diagrams

Diagrams are always considered from within a specific practice and context. Imagine the diagram of a circle. It has been used in logic, for example, by Euler, in Euclidean geometry, and in Cartesian geometry. Was it 'the same' circle in all these cases? Did it play the same role? Of course it did not. In fact, there is nothing like a specific set of rules that could be fixed once for all for all circles or for some specific kind of diagram in general. My proposal is that in diagrammatic reasoning what counts is not the appearance of a diagram and a list of explicit rules that can be applied to it, but rather a set of *procedures*:

when one learns to use a certain diagrammatic system for performing some inferences, she learns a *manipulation practice*. The diagram becomes the mathematician's worksite, where operations, plans, and experiments are made in order to find solutions and reasons for these solutions. While syntactic rules are piecemeal, procedures are holistic.

From this perspective, when drawing a diagram, the user never reproduces it *mechanically*. To go back to the intermediate zero theorem, in order to draw the diagram in Fig. 1 not only does she need to 'see' it, but also to appropriately interpret it, understand what it means in *that* context, and finally learn its construction. This operational aspect of working with diagrams has been neglected because of the suspicious view, thus encouraging a straightening of the opposition between vision and linguistic processes; this opposition has left out the consideration of the continuous interaction between the two formats in reasoning.

In the framework I propose, it is the operational aspect of working with diagrams, namely what is *done* with them, that must be taken into account and not their visual features nor the sentential information they can possibly carry. Diagram-users share something like the *experience* of seeing in a diagram what they *have to*, focusing their vision on a selection of relevant features, which will bring them, as rational agents, to understand and *reproduce* the relevant features of diagrams in a non mechanical way and without 'damages'. The antidote to the suspicious view consists thus neither in assuming a visuocentric view nor in fixing a set of syntactic rules, but in considering how diagrams are manipulated, in continuous interaction with language and within a specific practice, in order to infer some new conclusion.

My view moves from a purely syntactic approach to a semantic and, indeed, pragmatic approach to problem solving. As Grosholz claims in regard to the epistemology of mathematics, there are reasons for pursuing this approach and even considering the use of language in terms of its representational role in an historical context [12]. Diagrams and figures are inherently ambiguous: the operations on them are what fixes their meaning.

For example, consider someone who is inside the practice of Euclidean geometry. Without learning *in advance* any explicit rule, she perfectly knows that once she has constructed the figure, she is allowed to rotate or translate it, but she cannot for example stretch it differently from what can happen in other practices. This practice-based framework for diagrammatic reasoning presupposes the centrality of a form of *manipulative imagination* we refer to in our manipulation practices in logic and mathematics. This imagination is particularly effective because we are already familiar with it, since it derives from our perceptual experience and reproduces procedures that are to some extent similar to the manipulation of concrete objects. Yet, not all possible manipulations are allowed by the practice: among all the possible moves, only some of them are accepted.

Diagrams are cognitively advantageous because their use activates this manipulative form of imagination, as in the case of the Containment-schema for Euler circles, which is of course also informed by the context. The manipulations that are actively imagined on the diagram are controlled by interpretation and by the shared practice: diagrams do not offer a single message, but can be interpreted differently and, based on the interpretation, different actions can be performed on them in order to discover new relations. For this reason, one aspect of diagrams that becomes important is their inherent ambiguity: they can be 'read'—or interpreted—in different ways, and the practice of their manipulation and the procedures applied to them fix their meaning. Ambiguity is thus not a disadvantage in principle, but one of the strengths of diagrammatic reasoning: their multi-dimensionality

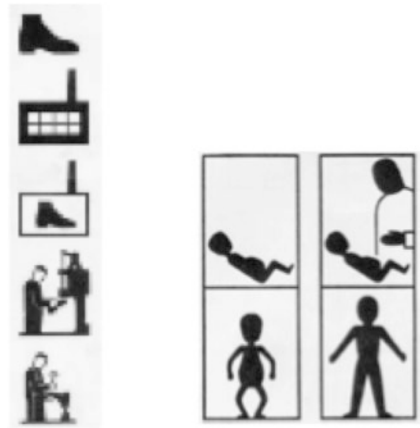


Fig. 5 Examples of pictures from Neurath's *Isotype* (taken from [20])

of meaning and non-unique interpretations can promote inference. In the next section, I will focus on this issue.

4.2 Diagrams Are (Hopefully) Ambiguous

I will first present an example to show that diagrams are ambiguous as are all images. Secondly, I will discuss how this ambiguity can be of help in mathematics by promoting inference.

Diagrams neither directly speak to the eyes nor convey a single message since, analogously to other images, they are inherently ambiguous. I want to mention here a particularly straightforward case that shows the ambiguous nature of images.

In the 1930s, the Austrian philosopher Neurath introduced *Isotype* (International System of Typographic Picture Education) to the aim of offering a tool that, in his view, could have solved the problems in communication caused by different levels of education among people, thus allowing free discussions of common problems and the dissemination of simple but important facts [20]. Isotype was meant to be a new way of conveying information that is at the same time easy to teach and learn, and is comprehensive and exact. It included a special dictionary and a special visual grammar, which, according to Neurath, created a new visual world analogous to the word world. As Neurath explains,

> the first step in Isotype is the development of easily understood and easily remembered symbols. The next step is to combine these symbolic elements.

In his view, icons such as the ones depicted in Fig. 5 are very easily recognized, no matter what the level of education is, and at the same time they express very complex situations.

Nevertheless, there are several problems in trying to provide a purely iconic system. First, a purely iconic system cannot easily convey changes of situations or convey that which contradicts some previous iconic representation. Let us suppose for example that at some point someone in our community decides that taking one's child for medical check-ups is not necessary for a child's health (in opposition to what the icons in Fig. 5 prescribe) and she shows this in Isotype by drawing the figure in Fig. 6.

Fig. 6 An example of a new (and ineffective) picture in *Isotype*, reworked from Fig. 5

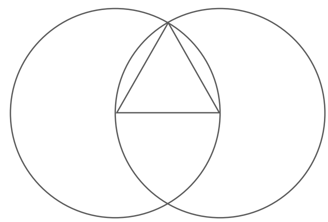


Fig. 7 The construction of an equilateral triangle starting from a given finite straight line

This time, the icons are evidently not as effective as in the previous case (i.e. as in Fig. 5). In fact, it is impossible to display *by them* the absence of the doctor or of anyone or anything else at all. This point expresses the difficulty in general of representing negation without using a specific symbol for it. Note that this impossibility is not too far from what we have seen in Euler circles' incapacity of representing empty sets.

Neurath's Isotype is based on the bad presupposition that there is in fact a distinction between a visual grammar on the one hand and a linguistic grammar on the other. By contrast, icons are not directly visual: each of them needs to be interpreted, to some extent by convention. To understand Isotype, the reader needs some background knowledge and, in the end, education. The same criticism can be formulated for any form of 'pictionary' that claims to be an universal dictionary.[7] Here we have a case similar to the 'colored' Euclidean geometry considered earlier: making it 'more visual' does not mean making it more straightforward. On the contrary, arbitrary conventions are introduced.

The above example shows that diagrams, analogously to other images, are inherently ambiguous since they do not convey one singular message that can be translated into language, but need interpretation in order to be understood. Nevertheless, in most cases, this is not a disadvantage but a strength.

Take the example of Euclid again, in Macbeth's reconstruction. Consider Book I, Proposition 1:

> *On a given finite straight line to construct an equilateral triangle.*

As Macbeth explains, the problem posed by this proposition is a *construction* problem. To get to the solution, it is necessary to reason *in* the diagram depicted in Fig. 7 [16].

[7] Still today, there are attempts to take this path, such as the pictionary 'Point it: Traveler's Language Kit', by Dieter Graf.

We start from the given finite line. Then, thanks to the construction of the two circles, we obtain a new figure. In order to get to the solution of the problem, the interpreter has to reason at different stages. At one stage, she must regard the lines in the diagram as the *radii* of the two circles: from this configuration, she concludes that all three lines are equal in length. At a second stage, she must regard the *same* lines as the *sides* of a triangle: from this new configuration, she concludes that the constructed figure is the desired equilateral triangle. Therefore, the diagram is in some way ambiguous, because it can be regarded as attending to different configurations. The interesting aspect is that all these possible configurations are compatible, and their compatibility is given by the fact that all configurations belong to the very same diagram.

As Macbeth explains, "one and the same lines are *now* regarded as parts of a circle and *later* as parts of a triangle" ([16], *italics mine*). In this case, there are three discernible levels of articulation: (i) primitive parts (points, lines, angles, and areas); (ii) geometrical figures (circles, triangles, and squares); and finally, (iii) the whole diagram. To find the solution of the problem stated by Euclid, one has to consider all these three levels together, precisely by exploiting the multidimensionality of meaning of the diagram. The generality of the diagram is given by the fact that it is constructed following the instructions in the text and is manipulated by imagination; the diagram is reliable because its use is considered inside a specific practice, Euclidean geometry, and its interpretation is intertwined with the rest of the shared system of knowledge and procedures that pertain to Euclidean geometry and serves as a guarantee for the correctness of the reasoning. The diagram is thus not a picture, but an instrument that promotes inference thanks to its possible manipulations. Moreover, the diagram does not need to be properly drawn, as long as the user is aware of the prescriptions contained in the instructions for its construction and is aware of its intended meaning.

We have seen in Fig. 2 something similar to the case described by Macbeth. In order to use the diagram for the sum of natural numbers up to n, we have to regard it *now* as a collection of squares going from 1 to n, corresponding to numbers from 1 to n, and *later* as a collage of triangles, the isosceles triangle with n as short sides, and the n half squares. The two configurations belonging to the *whole* diagram brought us to our conclusion. Moreover, consider Fig. 1 and the case of the intermediate zero theorem. In our linguistic description, we normally say that the function *goes* from below the horizontal axis to above it; we do that because we metaphorically reproduce its construction in our imagination.

My hypothesis is that, inside a specific practice, the space of the diagram perceived combines with the actions actually performed or imagined on it, in continuous interaction with linguistic knowledge. All these elements together contribute to mathematical meaning-making: manipulative imagination is at work to provide evidence in favor of some particular train of thought.

5 Conclusions

In this article, I tried to give arguments in favor of an operational framework for diagrammatic reasoning based on the practice of logic and mathematics. First, I presented what I defined as the suspicious view, according to which diagrams are not reliable enough to count as evidence for a conclusion. I claimed that this view is heavily based on a conception of proofs as syntactic objects and derivations, and I defended the idea that justification in practice is much more than proof only. Moreover, taking a Peircean perspective, I claimed that diagrams are a very special kind of representation, which is dynamic and needs an interpreter. According to the framework I propose, in order to give an account of diagrammatic reasoning it is necessary to focus on the practice shared by the community and on the actions performed on the diagram while considering two main aspects: (i) manipulative imagination; (ii) the role of ambiguity in triggering this imagination.

The dichotomy between visual thinking on the one hand and linguistic processes on the other has obscured the fact that what counts in diagrammatic reasoning is the manipulation practice, based on holistic procedures and not on the definition of explicit linguistic rules. According to this practice-based framework, it is the practice that fixes the meaning of the diagrams on each occasion, otherwise they are as ambiguous as other images are, and it is a kind of manipulative imagination that operates on them. Each practice is defined by procedures of manipulations and interconnected facts. All these elements taken together define in turn the system of knowledge shared by the community; this system encompasses diagrams, statements, particular notations and actions prescribed or allowed on them.

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