Operator Theory:
Advances and Applications, Vol. 231, 117–172

Multiplication Properties in Gelfand–Shilov Pseudo-differential Calculus

Joachim Toft

Abstract. We consider modulation space and spaces of Schatten–von Neumann symbols where corresponding pseudo-differential operators map one Hilbert space to another. We prove Hölder–Young and Young type results for such spaces under dilated convolutions and multiplications. We also prove continuity properties for such spaces under the twisted convolution, and the Weyl product. These results lead to continuity properties for twisted convolutions on Lebesgue spaces, e.g., $L^p_{(\omega)}$ is a twisted convolution algebra when $1 \le p \le 2$ and appropriate weight ω.

Mathematics Subject Classification (2010). Primary 35S05, 47B10, 46F99, 47B37, 44A35; Secondary 42B35, 47B35.

Keywords. Twisted, convolution, Weyl product, ultra-distributions, Beurling, Schatten–von Neumann, modulation, Toeplitz.

0. Introduction

The aim of the paper is to extend the results in [51] on various types of products in pseudo-differential calculus to include convenient Banach spaces of Gelfand–Shilov functions and distributions. The family of Banach spaces consists of (weighted) Lebesgue spaces, modulation spaces and spaces of Schatten–von Neumann symbols in the pseudo-differential calculus. The products concern the usual multiplication and convolution, twisted convolution and the Weyl product. Especially we establish continuity properties for Lebesgue and modulation spaces under the twisted convolution and the Weyl product, and prove Young type results for Schatten–von Neumann symbols under the ordinary multiplication and convolution.

We recall that the composition of two Weyl operators corresponds to the Weyl product of the two operator symbols on the symbol level, and the twisted convolution appears when Weyl product is conjugated by symplectic Fourier transform. (See Section 1 for the details.) Convolution and multiplication products appear when investigating Toeplitz operators (also known as localization operators) in the framework of pseudo-differential calculus. More precisely, each Toeplitz operator is equal to a pseudo-differential operator, where the symbol of the pseudo-differential operator is a convolution between the Toeplitz symbol and a rank one symbol, which is an ordinary multiplication on the Fourier transform side. We remark that Toeplitz operators might be convenient to use when approximating certain pseudo-differential operators (see, e.g., [7, 44]), and in computation of kinetic energy in mechanics (cf. [31]).

The most of the questions here were carefully investigated in [51] in the case when the involved spaces are defined by weights of polynomial type (see, e.g., [28] for notations concerning the usual function and distribution spaces, and Section 1 for other notations). In particular, all function and distribution spaces in [51] stay between $\mathscr{S}$ and $\mathscr{S}'$. In the present paper we use the framework in [51], and extend the results in [51] such that we permit general moderate weights. This implies that the function and distribution spaces can be arbitrary close to Gelfand–Shilov spaces of the form $\mathcal{S}_s^s$ and Σ_s^s when $s \ge 1$, and their duals.

In several questions we may use similar arguments as in [51], while new types of difficulties appear in other questions, when passing from the distribution theory for Schwartz functions in [51], to corresponding theory for Gelfand–Shilov functions.

In order to be more specific, let $\mathscr{H}_1$ and $\mathscr{H}_2$ be modulation spaces which are Hilbert spaces (see [52]). Also let $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$, $p \in [1, \infty]$, be the set of Schatten–von Neumann operators of order p from $\mathscr{H}_1$ to $\mathscr{H}_2$, and let $s_p^w(\mathscr{H}_1, \mathscr{H}_2)$ be the set of all distributions $a \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ such that the corresponding Weyl operators $\operatorname{Op}^w(a)$ belong to $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$.

In general it is not complicated to establish continuity properties for spaces of the form $s_p^w = s_p^w(\mathscr{H}_1, \mathscr{H}_2)$ under the Weyl product and twisted convolution, because such questions can easily be reformulated into questions of compositions for Schatten–von Neumann operators on the operator level. It is more complicated to find continuity relations for dilated multiplications and convolutions on the s_p^w spaces, because such products take complicated forms on the operator level. In this situation we use certain Fourier techniques, similar to those in [44, Section 3] and [51], to get convenient integral formulas. By making appropriate estimates on these formulas in combination with duality and interpolation, we establish Young type results for s_p^w spaces under such products.

For Lebesgue and general modulation spaces, the situation is different. In fact, in contrast to spaces of Schatten symbols, it is complicated to find certain results under the Weyl product and the twisted convolution, while finding Hölder–Young results under convolutions and multiplications are straightforward. For example, continuity properties for modulation spaces under the Weyl product have been investigated in, e.g., [23, 27, 30, 40, 51]. In Section 2 we extend these properties by enlarging the family of weights in the definition of modulation and Lebesgue spaces. In particular we prove that $L^2_{(\omega)}$ is an algebra under the twisted convolution, when $\omega(X) = e^{c|X|}$ and $c \ge 0$.

For further considerations we recall some definitions. Let $t \in \mathbf{R}$ be fixed and let $a \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then the *pseudo-differential operator* $\operatorname{Op}_t(a)$ with *symbol* a is the continuous operator on $\mathcal{S}_{1/2}(\mathbf{R}^d)$, defined by the formula

$$(\operatorname{Op}_t(a)f)(x) = (2\pi)^{-d} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi. \tag{0.1}$$

The definition of $\operatorname{Op}_t(a)$ extends to each $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$, and then $\operatorname{Op}_t(a)$ is continuous from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$. (Cf., e.g., [28] or Section 1.) If $t = 1/2$, then $\operatorname{Op}_t(a)$ is equal to the Weyl operator $\operatorname{Op}^w(a)$ for a. If instead $t = 0$, then the standard (Kohn–Nirenberg) representation $a(x, D)$ is obtained.

The modulation spaces were introduced by Feichtinger in [13], and developed further and generalized in [14, 16–18, 22]. We are especially interested in the modulation spaces $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ which are the sets of Gelfand–Shilov distributions on $\mathbf{R}^d$ whose short-time Fourier transform (STFT) belong to the weighted and mixed Lebesgue spaces $L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$ and $L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$ respectively. Here, and $p, q \in [1, \infty]$, and we refer to (1.26) and (1.27) below for the definition of the latter space norms. In contrast to [51], the weight function ω here is allowed to belong to $\mathscr{P}_E(\mathbf{R}^{2d})$, the set of all moderated functions on the phase (or time-frequency shift) space $\mathbf{R}^{2d}$. We remark that the family $\mathscr{P}_E$ contain all polynomial type weights. It follows that ω, p and q to some extent quantify the degrees of asymptotic decay and singularity of the distributions in $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$. (We refer to [15] for a modern description of modulation spaces.)

In the Weyl calculus of pseudo-differential operators, operator composition corresponds on the symbol level to the *Weyl product* #, which on the symplectic Fourier transform side corresponds to the twisted convolution $*_\sigma$. Sometimes, the Weyl product is called the twisted product. A problem in this field is to find conditions on the weight functions ω_j and $p_j, q_j \in [1, \infty]$, for the mappings

$$(a_1, a_2) \mapsto a_1 \# a_2 \quad \text{and} \quad (a_1, a_2) \mapsto a_1 *_\sigma a_2$$

on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ to be uniquely extendable to continuous mappings from

$$\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d}) \quad \text{to} \quad \mathcal{M}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d}),$$

and from

$$\mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d}) \quad \text{to} \quad \mathcal{W}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d}).$$

Here the modulation spaces $\mathcal{M}^{p,q}_{(\omega)}$ and $\mathcal{W}^{p,q}_{(\omega)}$ are obtained by replacing the usual STFT with the symplectic STFT in the definition of modulation space norms. One part of such questions might be to find appropriate conditions on ω_j and $p_j, q_j \in [1, \infty]$ such that

$$\|a_1 *_\sigma a_2\|_{\mathcal{W}^{p_0,q_0}_{(\omega_0)}} \lesssim \|a_1\|_{\mathcal{W}^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{\mathcal{W}^{p_2,q_2}_{(\omega_2)}}, \tag{0.2}$$

when $a_j \in \mathcal{S}_{1/2}$, $j = 0, 1, 2$. Here and in what follows we let $A \lesssim B$ indicate $A \le cB$, for a suitable constant $c > 0$, and we write $A \asymp B$ when $A \lesssim B$ and $B \lesssim A$. Important contributions in this context can be found in [23, 27, 30, 40, 43, 52], where Theorem $0.3'$ in [27] and Theorem 6.4 in [52] seem to be the most general results so far (see also Theorem 2.2).

The result for twisted convolution on modulation spaces which corresponds to Theorem $0.3'$ in [27] and Theorem 6.4 in [52] is given by Theorem 0.1 below. Here the assumptions on the involved weight functions and Lebesgue exponents on the modulation spaces are

$$\omega_0(X,Y) \lesssim \omega_1(X-Y+Z,Z)\omega_2(Y-Z,X+Z), \quad X,Y,Z \in \mathbf{R}^{2d}, \tag{0.3}$$

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0} = 1-\Big(\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0}\Big) \tag{0.4}$$

and

$$0 \le \frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0} \le \frac{1}{p_j},\frac{1}{q_j} \le \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0}, \quad j=0,1,2. \tag{0.5}$$

Theorem 0.1. *Let* $\omega_0,\omega_1,\omega_2 \in \mathscr{P}_E(\mathbf{R}^{4d})$ *satisfy* (0.3), *and that* $p_j,q_j \in [1,\infty]$ *for* $j=0,1,2$, *satisfy* (0.4) *and* (0.5). *Then the map* $(a_1,a_2)\mapsto a_1 *_\sigma a_2$ *on* $\mathcal{S}(\mathbf{R}^{2d})$ *extends uniquely to a continuous map from* $\mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $\mathcal{W}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, *and for some constant* $C>0$, *the bound* (0.2) *holds for every* $a_1 \in \mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ *and* $a_2 \in \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

In Section 2 we also consider the case when $p_j=q_j=2$, and the involved weights $\omega_j(X,Y)$ are independent of the Y-variable, i.e., $\omega_j(X,Y)=\omega_j(X)$. In this case, $\mathcal{W}^{2,2}_{(\omega_j)}$ agrees with $L^2_{(\omega_j)}$, and the condition (0.3) is reduced to

$$\omega_0(X_1+X_2) \lesssim \omega_1(X_1)\omega_2(X_2) \tag{0.6}$$

Hence, Theorem 0.1 shows that the map $(a_1,a_2)\mapsto a_1 *_\sigma a_2$ extends to a continuous mapping from $L^2_{(\omega_1)} \times L^2_{(\omega_2)}$ to $L^2_{(\omega_0)}$, and that

$$\|a_1 *_\sigma a_2\|_{L^2_{(\omega_0)}} \lesssim \|a_1\|_{L^2_{(\omega_1)}}\|a_2\|_{L^2_{(\omega_2)}}, \tag{0.7}$$

holds when $a_1 \in L^2_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^2_{(\omega_2)}(\mathbf{R}^{2d})$. Here and in what follows, $p' \in [1,\infty]$ denotes the conjugate exponent to $p \in [1,\infty]$, i.e., p and p' should satisfy $1/p+1/p'=1$. The latter property is extended in Section 2 to involve mixed weighted norm spaces of Lebesgue type. As a special case we obtain the following generalization of (0.7).

Theorem 0.2. *Let* $\omega_j \in \mathscr{P}_E(\mathbf{R}^{2d})$, *and let* $p_j \in [1,\infty]$ *for* $j=0,1,2$ *satisfy* (0.6) *and*

$$\max_{j=0,1,2}\Big(\frac{1}{p_j},\frac{1}{p_j'}\Big) \le \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0} \le 1.$$

Then the map $(a_1,a_2)\mapsto a_1 *_\sigma a_2$ *extends uniquely to a continuous mapping from* $L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $L^{p_0}_{(\omega_0)}(\mathbf{R}^{2d})$, *and*

$$\|a_1 *_\sigma a_2\|_{L^{p_0}_{(\omega_0)}} \lesssim \|a_1\|_{L^{p_1}_{(\omega_1)}}\|a_2\|_{L^{p_2}_{(\omega_2)}} \tag{0.8}$$

when $a_1 \in L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d})$ *and* $a_2 \in L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

Theorem 0.2 and its extensions are used in the end of Section 2 to extend the class of possible window functions in the definition of modulation space norms.

In Section 5 we establish Young type results for dilated multiplications and convolutions for the spaces $s_p^w(\omega_1,\omega_2) \equiv s_p^w(\mathscr{H}_1,\mathscr{H}_2)$, when $\mathscr{H}_j$ for $j=1,2$ is modulation space $M^{2,2}_{(\omega_j)}(\mathbf{R}^d) = M^2_{(\omega_j)}(\mathbf{R}^d)$ with appropriate weights ω_j. The involved Schatten exponents should satisfy the Young condition

$$\frac{1}{p_1}+\frac{1}{p_2} = 1+\frac{1}{r}, \qquad 1\le p_1,p_2,r\le \infty, \tag{0.9}$$

and the involved dilation factors should satisfy

$$(-1)^{j_1}t_1^{-2} + (-1)^{j_2}t_2^{-2} = 1 \tag{0.10}$$

or

$$(-1)^{j_1}t_1^{2} + (-1)^{j_2}t_2^{2} = 1, \tag{0.11}$$

when $j_1,j_2\in\{0,1\}$. The conditions for the involved weight functions are

$$\begin{aligned} \vartheta(X_1+X_2) &\lesssim \vartheta_{j_1,1}(t_1X_1)\vartheta_{j_2,2}(t_2X_2),\\ \omega(X_1+X_2) &\lesssim \omega_{j_1,1}(t_1X_1)\omega_{j_2,2}(t_2X_2), \end{aligned} \tag{0.12}$$

where

$$\omega_{0,k}(X) = \vartheta_{1,k}(X) = \omega_k(X), \quad \vartheta_{0,k}(X) = \omega_{1,k}(X) = \vartheta_k(X). \tag{0.13}$$

With these conditions we prove

$$\|a_{1,t_1} * a_{2,t_2}\|_{s_r^w(1/\omega,\vartheta)} \le C^d \|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)}\|a_2\|_{s_{p_2}^w(1/\omega_2,\vartheta_2)}, \tag{0.14}$$

$$\|a_{1,t_1} a_{2,t_2}\|_{s_r^w(1/\omega,\vartheta)} \le C^d \|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)}\|a_2\|_{s_{p_2}^w(1/\omega_2,\vartheta_2)}, \tag{0.15}$$

for admissible a_1 and a_2. Here and in what follows we set $a_{j,t} = a_j(t\,\cdot\,)$. More precisely, in Section 5 we prove the following two theorems, as well as multi-linear extensions of these results (cf. Theorems 0.3′ and 0.4′), which generalize Theorem 3.3, Theorem 3.3′ and Corollary 3.5 in [44] and corresponding results in [51]. In fact, these results in [44] follow by letting $\mathscr{H}_1=\mathscr{H}_2=L^2$ in Theorems 0.3′ and 0.4′.

Theorem 0.3. *Let $p_1,p_2,r\in[1,\infty]$ satisfy* (0.9)*, and let $t_1,t_2\in\mathbf{R}\backslash 0$ satisfy* (0.10)*, for some choices of $j_1,j_2\in\{0,1\}$. Also let $\omega,\omega_j,\vartheta,\vartheta_j\in\mathscr{P}_E(\mathbf{R}^{2d})$ for $j=1,2$ satisfy* (0.12) *and* (0.13)*. Then the map $(a_1,a_2)\mapsto a_{1,t_1}*a_{2,t_2}$ on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$, extends uniquely to a continuous mapping from*

$$s_{p_1}^w(1/\omega_1,\vartheta_1)\times s_{p_2}^w(1/\omega_2,\vartheta_2)$$

to $s_r^w(1/\omega,\vartheta)$. Furthermore, (0.14) *holds for some constant*

$$C = C_0^2|t_1|^{-2/p_1}|t_2|^{-2/p_2},$$

where C_0 is independent of $a_1\in s_{p_1}^w(1/\omega_1,\vartheta_1)$, $a_2\in s_{p_2}^w(1/\omega_2,\vartheta_2)$, t_1,t_2 and d.

Moreover, $\operatorname{Op}^w(a_{1,t_1}*a_{2,t_2})\ge 0$ *when* $\operatorname{Op}^w(a_j)\ge 0$ *for each* $1\le j\le 2$.

Theorem 0.4. *Let* $p_1, p_2, r \in [1, \infty]$ *satisfy* (0.9)*, and let* $t_1, t_2 \in \mathbf{R}\backslash 0$ *satisfy* (0.11)*, for some choices of* $j_1, j_2 \in \{0, 1\}$*. Also let* $\omega, \omega_j, \vartheta, \vartheta_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ *for* $j = 1, 2$ *satisfy* (0.12) *and* (0.13)*. Then the map* $(a_1, a_2) \mapsto a_{1,t_1} a_{2,t_2}$ *on* $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$*, extends uniquely to a continuous mapping from*

$$s^w_{p_1}(1/\omega_1, \vartheta_1) \times s^w_{p_2}(1/\omega_2, \vartheta_2)$$

to $s^w_r(1/\omega, \vartheta)$*. Furthermore,* (0.15) *holds for some constant*

$$C = C_0^2 |t_1|^{-2/p_1'} |t_2|^{-2/p_2'},$$

where C_0 *is independent of* $a_1 \in s^w_{p_1}(1/\omega_1, \vartheta_1)$*,* $a_2 \in s^w_{p_2}(1/\omega_2, \vartheta_2)$*,* t_1, t_2 *and* d*.*

Some preparations to the dilated convolution and multiplication results in Section 5 are given in Sections 3 and 4. In Section 3 we introduce the notion of Gelfand–Shilov and Beurling tempered (quasi-)Banach and Hilbert spaces, and prove certain properties. Especially we establish embedding properties between such spaces, modulation spaces and Gelfand–Shilov spaces. These embeddings are also used in [54], when establishing Schatten–von Neumann results for operators with Gelfand–Shilov kernels. Furthermore we investigate certain relations for bases in the Hilbert space case.

In Section 4 we consider dual properties for $s^w_p(\mathscr{H}_1, \mathscr{H}_2)$. Here $\mathscr{H}_1$ and $\mathscr{H}_2$ belong to a broad class of Hilbert spaces containing any $M^{2,2}_{(\omega)}$ space. More precisely, assume that $p < \infty$. Then we prove that the dual for $s^w_p(\mathscr{H}_1, \mathscr{H}_2)$ can be identified with $s^w_p(\mathscr{H}_1', \mathscr{H}_2')$ for appropriate Hilbert spaces $\mathscr{H}_1'$ and $\mathscr{H}_2'$ through a unique extension of the L^2 form on $\mathcal{S}_{1/2}$. (Cf. Theorem 4.8.) In the last part of Section 4 we show some properties on bases and Hilbert–Schmidt operators. We use these results to establish estimates for generalized gamma functions evaluated in integer points (cf. Example 3.6).

In the last section we apply the results in Section 5 to prove that the class of trace-class symbols is invariant under compositions with odd entire functions. Here we also show how Theorem 0.3 can be used to define Toeplitz operators with symbols in dilated s^w_p spaces, and that such operators fulfill certain Schatten–von Neumann properties.

1. Preliminaries

In this section we introduce some notations and discuss basic results. We start by recalling some facts concerning Gelfand–Shilov spaces. Thereafter we recall some properties about pseudo-differential operators. Especially we discuss the Weyl product and twisted convolution. Finally we recall some facts about modulation spaces. The proofs are in general omitted, since the results can be found in the literature.

We start by considering Gelfand–Shilov spaces. Let $0 < h, s \in \mathbf{R}$ be fixed. Then $\mathcal{S}_{s,h}(\mathbf{R}^d)$ consists of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha|+|\beta|}\alpha!^s\,\beta!^s}$$

is finite. Here the supremum should be taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$.

Obviously $\mathcal{S}_{s,h} \hookrightarrow \mathscr{S}$ is a Banach space which increases with h and s. Here and in what follows we use the notation $A \hookrightarrow B$ when the topological spaces A and B satisfy $A \subseteq B$ with continuous embeddings. Furthermore, if $s > 1/2$, or $s = 1/2$ and h is sufficiently large, then $\mathcal{S}_{s,h}$ contains all finite linear combinations of Hermite functions. Since such linear combinations are dense in $\mathscr{S}$, it follows that the dual $(\mathcal{S}_{s,h})'(\mathbf{R}^d)$ of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is a Banach space which contains $\mathscr{S}'(\mathbf{R}^d)$.

The *Gelfand–Shilov spaces* $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ are the inductive and projective limits respectively of $\mathcal{S}_{s,h}(\mathbf{R}^d)$. This implies that

$$\mathcal{S}_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d), \tag{1.1}$$

and that the topology for $\mathcal{S}_s(\mathbf{R}^d)$ is the strongest possible one such that the inclusion map from $\mathcal{S}_{s,h}(\mathbf{R}^d)$ to $\mathcal{S}_s(\mathbf{R}^d)$ is continuous, for every choice of $h > 0$. The space $\Sigma_s(\mathbf{R}^d)$ is a Fréchet space with semi norms $\|\cdot\|_{\mathcal{S}_{s,h}}$, $h > 0$. Moreover, $\Sigma_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s > 1/2$, and $\mathcal{S}_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s \geq 1/2$. From now on we assume that $s > 1/2$ when considering $\Sigma_s(\mathbf{R}^d)$, and $s \geq 1/2$ when considering $\mathcal{S}_s(\mathbf{R}^d)$

The *Gelfand–Shilov distribution spaces* $\mathcal{S}_s'(\mathbf{R}^d)$ and $\Sigma_s'(\mathbf{R}^d)$ are the projective and inductive limit respectively of $\mathcal{S}_s'(\mathbf{R}^d)$. This means that

$$\mathcal{S}_s'(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}'(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s'(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}'(\mathbf{R}^d). \tag{1.1$'$}$$

We remark that in [20, 29, 34] it is proved that $\mathcal{S}_s'(\mathbf{R}^d)$ is the dual of $\mathcal{S}_s(\mathbf{R}^d)$, and $\Sigma_s'(\mathbf{R}^d)$ is the dual of $\Sigma_s(\mathbf{R}^d)$ (also in topological sense).

For each $\varepsilon > 0$ and $s > 1/2$ we have

$$\begin{aligned} &\mathcal{S}_{1/2}(\mathbf{R}^d) \hookrightarrow \Sigma_s(\mathbf{R}^d) \hookrightarrow \mathcal{S}_s(\mathbf{R}^d) \hookrightarrow \Sigma_{s+\varepsilon}(\mathbf{R}^d) \\ \text{and} \quad &\Sigma_{s+\varepsilon}'(\mathbf{R}^d) \hookrightarrow \mathcal{S}_s'(\mathbf{R}^d) \hookrightarrow \Sigma_s'(\mathbf{R}^d) \hookrightarrow \mathcal{S}_{1/2}'(\mathbf{R}^d). \end{aligned} \tag{1.2}$$

The Gelfand–Shilov spaces are invariant under several basic transformations. For example they are invariant under translations, dilations, tensor products and under (partial) Fourier transformations.

From now on we let $\mathscr{F}$ be the Fourier transform which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x, \xi\rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the usual scalar product on $\mathbf{R}^d$. The map $\mathscr{F}$ extends uniquely to homeomorphisms on $\mathscr{S}'(\mathbf{R}^d)$, $\mathcal{S}_s'(\mathbf{R}^d)$ and $\Sigma_s'(\mathbf{R}^d)$, and restricts to homeomorphisms on $\mathscr{S}(\mathbf{R}^d)$, $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$.

It follows from the following lemma that elements in Gelfand–Shilov spaces can be characterized by estimates of the form

$$|f(x)| \lesssim e^{-\varepsilon|x|^{1/s}} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\varepsilon|\xi|^{1/s}}. \tag{1.3}$$

The proof is omitted, since the result can be found in, e.g., [4, 20].

Lemma 1.1. *Let $f \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$. Then the following is true:*

(1) *if $s \ge 1/2$, then $f \in \mathcal{S}_s(\mathbf{R}^d)$, if and only if (1.3) holds for some $\varepsilon > 0$;*
(2) *if $s > 1/2$, then $f \in \Sigma_s(\mathbf{R}^d)$, if and only if (1.3) holds for each $\varepsilon > 0$.*

Gelfand–Shilov spaces can also easily be characterized by Hermite functions. We recall that the Hermite function h_α with respect to the multi-index $\alpha \in \mathbf{N}^d$ is defined by

$$h_\alpha(x) = \pi^{-d/4}(-1)^{|\alpha|}(2^{|\alpha|}\alpha!)^{-1/2}e^{|x|^2/2}(\partial^\alpha e^{-|x|^2}).$$

The set $\{h_\alpha\}_{\alpha\in\mathbf{N}^d}$ is an orthonormal basis for $L^2(\mathbf{R}^d)$. In particular,

$$f = \sum_\alpha c_\alpha h_\alpha, \quad c_\alpha = (f, h_\alpha)_{L^2(\mathbf{R}^d)}, \tag{1.4}$$

and

$$\|f\|_{L^2} = \|\{c_\alpha\}_\alpha\|_{l^2} < \infty,$$

when $f \in L^2(\mathbf{R}^d)$. Here and in what follows, $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbf{R}^d)}$ denotes any continuous extension of the L^2 form on $\mathcal{S}_{1/2}(\mathbf{R}^d)$.

The Hermite expansions can also be used to characterize distributions and their test function spaces. More precisely, let $p \in [1, \infty]$ be fixed. Then it is well known that f here belongs to $\mathscr{S}(\mathbf{R}^d)$, if and only if

$$\|\{c_\alpha \langle\alpha\rangle^t\}_\alpha\|_{l^p} < \infty \tag{1.5}$$

for every $t \in \mathbf{R}$. Furthermore, for every $f \in \mathscr{S}'(\mathbf{R}^d)$, the expansion (1.4) still holds, where the sum converges in $\mathscr{S}'$, and (1.5) holds for some choice of $t \in \mathbf{R}$, which depends on f.

The following proposition, which can be found in, e.g., [21], shows that similar conclusion for Gelfand–Shilov spaces hold, after the estimate (1.5) is replaced by

$$\|\{c_\alpha e^{t|\alpha|^{1/2s}}\}_\alpha\|_{l^p} < \infty. \tag{1.6}$$

(Cf. formula (2.12) in [21].)

Proposition 1.2. *Let $p \in [1, \infty]$, $f \in \mathcal{S}_{1/2}'\mathbf{R}^d)$, $s_1 \ge 1/2$, $s_2 > 1/2$ and let c_α be as in (1.4). Then the following is true:*

(1) *$f \in \mathcal{S}_{s_1}'(\mathbf{R}^d)$, if and only if (1.6) holds with $s = s_1$ for every $t < 0$. Furthermore, (1.4) holds where the sum converges in $\mathcal{S}_{s_1}'$;*
(2) *$f \in \Sigma_{s_2}'(\mathbf{R}^d)$, if and only if (1.6) holds with $s = s_2$ for some $t < 0$. Furthermore, (1.4) holds where the sum converges in Σ_{s_2}';*

(3) $f \in \mathcal{S}_{s_1}(\mathbf{R}^d)$, *if and only if* (1.6) *holds with* $s = s_1$ *for some* $t > 0$. *Furthermore,* (1.4) *holds where the sum converges in* $\mathcal{S}_{s_1}$*;*

(4) $f \in \Sigma_{s_2}(\mathbf{R}^d)$, *if and only if* (1.6) *holds with* $s = s_2$ *for every* $t > 0$. *Furthermore,* (1.4) *holds where the sum converges in* Σ_{s_2}.

Next we recall some properties in pseudo-differential calculus. Let $s \geq 1/2$, $a \in \mathcal{S}_s(\mathbf{R}^{2d})$, and $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ in (0.1) is a linear and continuous operator on $\mathcal{S}_s(\mathbf{R}^d)$. For general $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_t(a)$ is defined as the continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$ with distribution kernel given by

$$K_{a,t}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)((1-t)x+ty, x-y). \tag{1.7}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathcal{S}_s'(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense, since the mappings

$$\mathscr{F}_2 \quad \text{and} \quad F(x,y) \mapsto F((1-t)x+ty, y-x) \tag{1.8}$$

are homeomorphisms on $\mathcal{S}_s'(\mathbf{R}^{2d})$. In particular, the map $a \mapsto K_{a,t}$ is a homeomorphism on $\mathcal{S}_s'(\mathbf{R}^{2d})$.

For any $K \in \mathcal{S}_s'(\mathbf{R}^{d_1+d_2})$, we let T_K be the linear and continuous mapping from $\mathcal{S}_s(\mathbf{R}^{d_1})$ to $\mathcal{S}_s'(\mathbf{R}^{d_2})$, defined by the formula

$$(T_K f, g)_{L^2(\mathbf{R}^{d_2})} = (K, g \otimes \overline{f})_{L^2(\mathbf{R}^{d_1+d_2})}. \tag{1.9}$$

It is well known that if $t \in \mathbf{R}$, then it follows from Schwartz kernel theorem that $K \mapsto T_K$ and $a \mapsto \operatorname{Op}_t(a)$ are bijective mappings from $\mathscr{S}'(\mathbf{R}^{2d})$ to the set of linear and continuous mappings from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ (cf., e.g., [28]).

In this context we remark that the maps $K \mapsto T_K$ and $a \mapsto \operatorname{Op}_t(a)$ are uniquely extendable to bijective mappings from $\mathcal{S}_s'(\mathbf{R}^{2d})$ to the set of linear and continuous mappings from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$. In fact, the asserted bijectivity for the map $K \mapsto T_K$ follows from the kernel theorem [32, Theorem 2.2], by Lozanov–Crvenković, Perišić and Taskovic. This kernel theorem corresponds to Schwartz kernel theorem in the usual distribution theory. The other assertion follows from the fact that the map $a \mapsto K_{a,t}$ is a homeomorphism on $\mathcal{S}_s'$.

In particular, for each $a_1 \in \mathcal{S}_s'(\mathbf{R}^{2d})$ and $t_1, t_2 \in \mathbf{R}$, there is a unique $a_2 \in \mathcal{S}_s'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_{t_1}(a_1) = \operatorname{Op}_{t_2}(a_2)$. The relation between a_1 and a_2 is given by

$$\operatorname{Op}_{t_1}(a_1) = \operatorname{Op}_{t_2}(a_2) \qquad \Longleftrightarrow \qquad a_2(x,\xi) = e^{i(t_2-t_1)\langle D_x, D_\xi\rangle} a_1(x,\xi). \tag{1.10}$$

(Cf. [28].) Note here that the right-hand side makes sense, since it is equivalent to $\widehat{a}_2 = e^{i(t_2-t_1)\langle x,\xi\rangle}\widehat{a}_1$, and that the map $a \mapsto e^{it\langle x,\xi\rangle}a$ is continuous on $\mathcal{S}_s'$.

Let $t \in \mathbf{R}$ and $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$ be fixed. Then a is called a rank-one element with respect to t, if the corresponding pseudo-differential operator is of rank-one, i.e.,

$$\operatorname{Op}_t(a)f = (f, f_2)f_1, \qquad f \in \mathcal{S}_s(\mathbf{R}^d), \tag{1.11}$$

for some $f_1, f_2 \in \mathcal{S}_s'(\mathbf{R}^d)$. By straightforward computations it follows that (1.11) is fulfilled, if and only if $a = (2\pi)^{d/2} W_{f_1,f_2}^t$, where W_{f_1,f_2}^t is the t-Wigner distribution, defined by the formula

$$W^t_{f_2,f_1}(x,\xi) \equiv \mathscr{F}(f_2(x+t\,\cdot\,)\overline{f_1(x-(1-t)\,\cdot\,)})(\xi), \tag{1.12}$$

which takes the form

$$W^t_{f_2,f_1}(x,\xi) = (2\pi)^{-d/2}\int f_2(x+ty)\overline{f_1(x-(1-t)y)}e^{-i\langle y,\xi\rangle}\,dy,$$

when $f_1, f_2 \in \mathcal{S}_s(\mathbf{R}^d)$. By combining these facts with (1.10), it follows that

$$W^{t_2}_{f_2,f_1} = e^{i(t_2-t_1)\langle D_x,D_\xi\rangle}W^{t_1}_{f_2,f_1}, \tag{1.13}$$

for each $f_1, f_2 \in \mathcal{S}_s'(\mathbf{R}^d)$ and $t_1, t_2 \in \mathbf{R}$. Since the Weyl case is particularly important, we set $W^t_{f_2,f_1} = W_{f_2,f_1}$ when $t = 1/2$, i.e., W_{f_2,f_1} is the usual (cross-)Wigner distribution of f_1 and f_2.

For future references we note the link

$$\begin{gathered}(\operatorname{Op}_t(a)f,g)_{L^2(\mathbf{R}^d)} = (2\pi)^{-d/2}(a,W^t_{g,f})_{L^2(\mathbf{R}^{2d})},\\ a\in\mathcal{S}_s'(\mathbf{R}^{2d}) \quad\text{and}\quad f,g\in\mathcal{S}_s(\mathbf{R}^d)\end{gathered} \tag{1.14}$$

between pseudo-differential operators and Wigner distributions, which follows by straightforward computations (see also, e.g., [10, 11]).

Next we discuss the Weyl product, twisted convolution and related objects. Let $s \ge 1/2$ and let $a, b \in \mathcal{S}_s'(\mathbf{R}^{2d})$ be appropriate. Then the Weyl product $a\#b$ between a and b is the function or distribution which fulfills $\operatorname{Op}^w(a\#b) = \operatorname{Op}^w(a)\circ\operatorname{Op}^w(b)$, provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$. More general, if $t \in \mathbf{R}$, then the product $\#_t$ is defined by the formula

$$\operatorname{Op}_t(a\#_t b) = \operatorname{Op}_t(a)\circ\operatorname{Op}_t(b), \tag{1.15}$$

provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$.

The Weyl product can also, in a convenient way, be expressed in terms of the symplectic Fourier transform and twisted convolution. More precisely, let $s \ge 1/2$. Then the *symplectic Fourier transform* for $a \in \mathcal{S}_s(\mathbf{R}^{2d})$ is defined by the formula

$$(\mathscr{F}_\sigma a)(X) = \pi^{-d}\int a(Y)e^{2i\sigma(X,Y)}\,dY,$$

where σ is the symplectic form, given by

$$\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle, \qquad X=(x,\xi)\in\mathbf{R}^{2d},\ Y=(y,\eta)\in\mathbf{R}^{2d}.$$

We note that $\mathscr{F}_\sigma = T\circ(\mathscr{F}\otimes(\mathscr{F}^{-1}))$, when $(Ta)(x,\xi) = 2^d a(2\xi,2x)$. In particular, $\mathscr{F}_\sigma$ is continuous on $\mathcal{S}_s(\mathbf{R}^{2d})$, and extends uniquely to a homeomorphism on $\mathcal{S}_s'(\mathbf{R}^{2d})$, and to a unitary map on $L^2(\mathbf{R}^{2d})$, since similar facts hold for $\mathscr{F}$. Furthermore, $\mathscr{F}_\sigma^2$ is the identity operator.

Let $s \ge 1/2$ and $a, b \in \mathcal{S}_s(\mathbf{R}^{2d})$. Then the *twisted convolution* of a and b is defined by the formula

$$(a *_\sigma b)(X) = (2/\pi)^{d/2}\int a(X-Y)b(Y)e^{2i\sigma(X,Y)}\,dY. \tag{1.16}$$

The definition of $*_\sigma$ extends in different ways. For example, it extends to a continuous multiplication on $L^p(\mathbf{R}^{2d})$ when $p \in [1,2]$, and to a continuous map from $\mathcal{S}_s'(\mathbf{R}^{2d}) \times \mathcal{S}_s(\mathbf{R}^{2d})$ to $\mathcal{S}_s'(\mathbf{R}^{2d})$. If $a, b \in \mathcal{S}_s'(\mathbf{R}^{2d})$, then $a\#b$ makes sense if and only if $a *_\sigma \widehat{b}$ makes sense, and then

$$a\#b = (2\pi)^{-d/2} a *_\sigma (\mathscr{F}_\sigma b). \tag{1.17}$$

We also remark that for the twisted convolution we have

$$\mathscr{F}_\sigma(a *_\sigma b) = (\mathscr{F}_\sigma a) *_\sigma b = \check{a} *_\sigma (\mathscr{F}_\sigma b), \tag{1.18}$$

where $\check{a}(X) = a(-X)$ (cf. [42, 44, 45]). A combination of (1.17) and (1.18) gives

$$\mathscr{F}_\sigma(a\#b) = (2\pi)^{-d/2} (\mathscr{F}_\sigma a) *_\sigma (\mathscr{F}_\sigma b). \tag{1.19}$$

In the Weyl calculus it is in several situations convenient to use the operator A on $\mathcal{S}_s'(\mathbf{R}^{2d})$, defined by the formula

$$Aa(x,y) = (\mathscr{F}_2^{-1} a)((y-x)/2, -(x+y)), \quad a \in \mathcal{S}_s'(\mathbf{R}^{2d}). \tag{1.20}$$

Here and in what follows we identify operators with their distribution kernels. We note that $Aa(x,y)$ agrees with $(2\pi)^{d/2} K_a^w(-x,y)$, where K_a^w is the distribution kernel to the Weyl operator $\operatorname{Op}^w(a)$. If $a \in \mathcal{S}_s(\mathbf{R}^{2d})$, then Aa is given by

$$Aa(x,y) = (2\pi)^{-d/2} \int a((y-x)/2, \xi) e^{-i\langle x+y, \xi\rangle}\, dy.$$

In particular, the map $a \mapsto Aa$ is bijective from $\mathcal{S}_s'(\mathbf{R}^{2d})$ to the set of linear and continuous operators from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$, since similar facts are true for the Weyl quantization.

The operator A is important when using the twisted convolution, because for each $a, b \in \mathcal{S}_s(\mathbf{R}^{2d})$ we have

$$A(a *_\sigma b) = Aa \circ Ab. \tag{1.21}$$

(See [19, 42, 44, 45].)

In the following lemma we list some facts about the operator A. The result is a consequence of Fourier's inversion formula, and the verifications are left for the reader.

Lemma 1.3. *Let $s \ge 1/2$, A be as above, $a, a_1, a_2, b \in \mathcal{S}_s'(\mathbf{R}^{2d})$, where at least two of a_1, a_2, b should belong to $\mathcal{S}_s'(\mathbf{R}^{2d})$, and set $U = Aa$. Then the following is true:*

(1) $\check{U} = A\check{a}$, *if* $\check{a}(X) = a(-X)$;

(2) $J_{\mathscr{F}} U = A\mathscr{F}_\sigma a$, *where* $J_{\mathscr{F}} U(x,y) = U(-x,y)$;

(3) $A(\mathscr{F}_\sigma a) = (2\pi)^{d/2} \operatorname{Op}^w(a)$ *and* $(\operatorname{Op}^w(a) f, g) = (2\pi)^{-d/2} (Aa, \check{g} \otimes \overline{f})$ *when* $f, g \in \mathcal{S}_{1/2}(\mathbf{R}^d)$;

(4) *the Hilbert space adjoint of Aa equals $A\widetilde{a}$, where $\widetilde{a}(X) = \overline{a(-X)}$. Furthermore,*

$$(a_1 *_\sigma a_2, b) = (a_1, b *_\sigma \widetilde{a}_2) = (a_2, \widetilde{a}_1 *_\sigma b),$$

$$(a_1 *_\sigma a_2) *_\sigma b = a_1 *_\sigma (a_2 *_\sigma b).$$

A linear and continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^n)$ is called positive semi-definite (of order $s \ge 1/2$) when $(Tf, f)_{L^2} \ge 0$ for every $f \in \mathcal{S}_s(\mathbf{R}^d)$. We write $T \ge 0$ when T is positive semi-definite or order s. A distribution $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$ is called σ-positive (of order s) if Aa is a positive semi-definite operator. The set of all σ-positive distributions on $\mathbf{R}^{2d}$ is denoted by $\mathcal{S}_{s,+}'(\mathbf{R}^{2d})$. Since $\mathcal{S}_s$ increases with s and that $\mathcal{S}_{1/2}$ is dense in $\mathcal{S}_s$, it follows that

$$\mathcal{S}_{t,+}'(\mathbf{R}^{2d}) = \mathcal{S}_{s,+}'(\mathbf{R}^{2d}) \bigcap \mathcal{S}_t'(\mathbf{R}^{2d}), \qquad t \ge s.$$

The following result is an immediate consequence of Lemma 1.3.

Proposition 1.4. *Let $s \ge 1/2$ and $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$. Then*

$$a \in \mathcal{S}_{s,+}'(\mathbf{R}^{2d}) \quad \iff \quad Aa \ge 0 \text{ as operator} \quad \iff \quad \operatorname{Op}^w(\mathscr{F}_\sigma a) \ge 0.$$

We refer to [44, 45] for more facts about σ-positive functions and distributions in the framework of tempered distributions.

In the end of Section 5 we also discuss continuity for Toeplitz operators. Let $s \ge 1/2$, $a \in \mathcal{S}_s(\mathbf{R}^{2d})$ and $h_1, h_2 \in \mathcal{S}_s(\mathbf{R}^d)$. Then the Toeplitz operator $\operatorname{Tp}_{h_1,h_2}(a)$, with symbol a, and window functions h_1 and h_2, is defined by the formula

$$(\operatorname{Tp}_{h_1,h_2}(a)f_1, f_2) = (a(2\,\cdot\,)W_{f_1,h_1}, W_{f_2,h_2}) \tag{1.22}$$

when $f_1, f_2 \in \mathcal{S}_s(\mathbf{R}^d)$. The definition of $\operatorname{Tp}_{h_1,h_2}(a)$ extends in several ways (cf., e.g., [6, 26, 42, 44, 46, 49, 50, 52]).

In several of these extensions as well as in Section 5, we interpret Toeplitz operators as pseudo-differential operators, using the fact that

$$\begin{aligned} \operatorname{Tp}_{h_1,h_2}(a) &= \operatorname{Op}_t(a * u) \quad \text{when} \\ u(X) &= (2\pi)^{-d/2} W_{h_2,h_1}^t(-X), \end{aligned} \tag{1.23}$$

h_1, h_2 are suitable window functions on $\mathbf{R}^d$ and a is an appropriate distribution on $\mathbf{R}^{2d}$. The relation (1.23) is well known when $t = 0$ or $t = 1/2$ (cf., e.g., [6, 8, 38, 42, 44, 46–48, 50]). For general t, (1.23) is an immediate consequence of the case $t = 1/2$, (1.13), and the fact that

$$e^{it\langle D_x, D_\xi\rangle}(a * u) = a * (e^{it\langle D_x, D_\xi\rangle}u),$$

which follows by integration by parts.

Next we discuss basic properties for modulation spaces, and start by recalling the conditions for the involved weight functions. Let $0 < \omega, v \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$. Then ω is called *moderate* or *v-moderate* if

$$\omega(x + y) \lesssim \omega(x)v(y), \quad x, y \in \mathbf{R}^d. \tag{1.24}$$

Here the function v is called *submultiplicative*, if (1.24) holds when $\omega = v$. We note that if (1.24) holds, then

$$v(-x)^{-1} \lesssim \omega(x) \lesssim v(x).$$

Furthermore, for such ω it follows that (1.24) is true when

$$v(x) = Ce^{c|x|},$$

for some positive constants c and C. In particular, if ω is moderate on $\mathbf{R}^d$, then

$$e^{-c|x|} \lesssim \omega(x) \lesssim e^{c|x|},$$

for some constant $c > 0$.

The set of all moderate functions on $\mathbf{R}^d$ is denoted by $\mathscr{P}_E(\mathbf{R}^d)$. Furthermore, if v in (1.24) can be chosen as a polynomial, then ω is called of polynomial type, or polynomially moderate. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. If $\omega(x,\xi) \in \mathscr{P}_E(\mathbf{R}^{2d})$ is constant with respect to the x-variable (ξ-variable), then we write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}_E(\mathbf{R}^{2d})$ or in $\mathscr{P}_E(\mathbf{R}^d)$ depending on the situation.

Let $\phi \in \mathcal{S}_s'(\mathbf{R}^d)$ be fixed. Then the *short-time Fourier transform* $V_\phi f$ of $f \in \mathcal{S}_s'(\mathbf{R}^d)$ with respect to the *window function* ϕ is the Gelfand–Shilov distribution on $\mathbf{R}^{2d}$, defined by

$$V_\phi f(x,\xi) \equiv (\mathscr{F}_2(U(f \otimes \phi)))(x,\xi) = \mathscr{F}(f\,\overline{\phi(\,\cdot\, - x)})(\xi),$$

where $(UF)(x,y) = F(y, y-x)$. If $f, \phi \in \mathcal{S}_s(\mathbf{R}^d)$, then it follows that

$$V_\phi f(x,\xi) = (2\pi)^{-d/2} \int f(y)\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle}\, dy.$$

We recall that the short-time Fourier transform is closely related to the Wigner distribution, because

$$W_{f,\phi} f(x,\xi) = 2^d e^{2i\langle x,\xi\rangle} V_{\check{\phi}} f(2x, 2\xi), \tag{1.25}$$

which follows by elementary manipulations. In particular, Toeplitz operators can be expressed by the formula

$$(\mathrm{Tp}_{h_1,h_2}(a)f_1, f_2) = (aV_{\check{h}_1} f_1, V_{\check{h}_2} f_2). \tag{1.22$'$}$$

Let $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$, $p, q \in [1,\infty]$ and $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)\backslash 0$ be fixed. Then the mixed Lebesgue space $L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{1,(\omega)}} < \infty$, and $L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{2,(\omega)}} < \infty$. Here

$$\|F\|_{L^{p,q}_{1,(\omega)}} = \Big(\int \Big(\int |F(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q}, \tag{1.26}$$

and

$$\|F\|_{L^{p,q}_{2,(\omega)}} = \Big(\int \Big(\int |F(x,\xi)\omega(x,\xi)|^q \, d\xi \Big)^{p/q} dx \Big)^{1/p}, \tag{1.27}$$

with obvious modifications when $p = \infty$ or $q = \infty$. We note that these norms might attain $+\infty$.

The *modulation spaces* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are the Banach spaces which consist of all $f\in \mathcal S_{1/2}'(\mathbf{R}^d)$ such that $\|f\|_{M^{p,q}_{(\omega)}}<\infty$ and $\|f\|_{W^{p,q}_{(\omega)}}<\infty$ respectively. Here

$$\|f\|_{M^{p,q}_{(\omega)}}\equiv \|V_\phi f\|_{L^{p,q}_{1,(\omega)}},\quad \text{and}\quad \|f\|_{W^{p,q}_{(\omega)}}\equiv \|V_\phi f\|_{L^{p,q}_{2,(\omega)}}. \tag{1.28}$$

We remark that the definitions of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are independent of the choice of $\phi\in \mathcal S_{1/2}(\mathbf{R}^d)\setminus 0$ and different ϕ gives rise to equivalent norms. (See Proposition 1.5 below.) From the fact that

$$V_{\widehat\phi}\widehat f(\xi,-x)=e^{i\langle x,\xi\rangle}V_{\check\phi}f(x,\xi),\qquad \check\phi(x)=\phi(-x), \tag{1.29}$$

it follows that

$$f\in W^{q,p}_{(\omega)}(\mathbf{R}^d)\quad\Longleftrightarrow\quad \widehat f\in M^{p,q}_{(\omega_0)}(\mathbf{R}^d),\qquad \omega_0(\xi,-x)=\omega(x,\xi).$$

For convenience we set $M^p_{(\omega)}=M^{p,p}_{(\omega)}$, which agrees with $W^p_{(\omega)}=W^{p,p}_{(\omega)}$. Furthermore we set $M^{p,q}=M^{p,q}_{(\omega)}$ and $W^{p,q}=W^{p,q}_{(\omega)}$ when $\omega\equiv 1$.

The proof of the following proposition is omitted, since the results can be found in [5, 12, 13, 16–18, 22, 46–49, 52]. Here we recall that $p,p'\in[1,\infty]$ satisfy $1/p+1/p'=1$.

Proposition 1.5. *Let $p,q,p_j,q_j\in[1,\infty]$ for $j=1,2$, and $\omega,\omega_1,\omega_2,v\in\mathscr P_E(\mathbf{R}^{2d})$ be such that $v=\check v$, ω is v-moderate and $\omega_2\lesssim\omega_1$. Then the following is true:*

1. *$f\in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if (1.28) holds for any $\phi\in M^1_{(v)}(\mathbf{R}^d)\setminus 0$. Moreover, $M^{p,q}_{(\omega)}$ is a Banach space under the norm in (1.28) and different choices of ϕ give rise to equivalent norms;*
2. *if $p_1\le p_2$ and $q_1\le q_2$ then*
$$\Sigma_1(\mathbf{R}^d)\hookrightarrow M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d)\hookrightarrow M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d)\hookrightarrow \Sigma_1'(\mathbf{R}^d).$$
3. *the L^2 product $(\,\cdot\,,\,\cdot\,)_{L^2}$ on $\mathcal S_{1/2}$ extends uniquely to a continuous map from $M^{p,q}_{(\omega)}(\mathbf{R}^n)\times M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$ to $\mathbf{C}$. On the other hand, if $\|a\|=\sup|(a,b)|$, where the supremum is taken over all $b\in\mathcal S_{1/2}(\mathbf{R}^d)$ such that $\|b\|_{M^{p',q'}_{(1/\omega)}}\le 1$, then $\|\cdot\|$ and $\|\cdot\|_{M^{p,q}_{(\omega)}}$ are equivalent norms;*
4. *if $p,q<\infty$, then $\mathcal S_{1/2}(\mathbf{R}^d)$ is dense in $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and the dual space of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$, through the L^2-form $(\,\cdot\,,\,\cdot\,)_{L^2}$. Moreover, $\mathcal S_{1/2}(\mathbf{R}^d)$ is weakly dense in $M^{p',q'}_{(\omega)}(\mathbf{R}^d)$ with respect to the L^2-form.*

Similar facts hold if the $M^{p,q}_{(\omega)}$ spaces are replaced by $W^{p,q}_{(\omega)}$ spaces.

Proposition 1.5 (1) allows us be rather vague concerning the choice of $\phi\in M^1_{(v)}\setminus 0$ in (1.28). For example, if $C>0$ is a constant and $\mathscr A$ is a subset of $\mathcal S_{1/2}'$, then $\|a\|_{M^{p,q}_{(\omega)}}\le C$ for every $a\in\mathscr A$, means that the inequality holds for some choice of $\phi\in M^1_{(v)}\setminus 0$ and every $a\in\mathscr A$. Evidently, a similar inequality is true for any other choice of $\phi \in M^1_{(v)} \setminus 0$, with a suitable constant, larger than C if necessary.

Remark 1.6. By Theorem 3.9 in [52] and Proposition 1.5 (2) it follows that

$$\bigcap_{\omega \in \mathscr{P}_E} M^{p,q}_{(\omega)}(\mathbf{R}^d) = \Sigma_1(\mathbf{R}^d), \qquad \bigcup_{\omega \in \mathscr{P}_E} M^{p,q}_{(\omega)}(\mathbf{R}^d) = \Sigma_1'(\mathbf{R}^d)$$

More generally, let $s \ge 1$, and let $\mathcal{P}$ be the set of all $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ such that

$$\omega(x+y,\xi+\eta) \lesssim \omega(x,\xi) e^{c(|y|^{1/s}+|\eta|^{1/s})},$$

for some $c > 0$. Then

$$\bigcap_{\omega \in \mathcal{P}} M^{p,q}_{(\omega)}(\mathbf{R}^d) = \Sigma_s(\mathbf{R}^d), \qquad \bigcup_{\omega \in \mathcal{P}} M^{p,q}_{(1/\omega)}(\mathbf{R}^d) = \Sigma_s'(\mathbf{R}^d)$$

$$\bigcup_{\omega \in \mathcal{P}} M^{p,q}_{(\omega)}(\mathbf{R}^d) = \mathcal{S}_s(\mathbf{R}^d) \quad \text{and} \quad \bigcap_{\omega \in \mathcal{P}} M^{p,q}_{(1/\omega)}(\mathbf{R}^d) = \mathcal{S}_s'(\mathbf{R}^d),$$

and that

$$\Sigma_s(\mathbf{R}^d) \hookrightarrow M^{p,q}_{(\omega)}(\mathbf{R}^d) \hookrightarrow \mathcal{S}_s(\mathbf{R}^d) \quad \text{and} \quad \mathcal{S}_s'(\mathbf{R}^d) \hookrightarrow M^{p,q}_{(1/\omega)}(\mathbf{R}^d) \hookrightarrow \Sigma_s'(\mathbf{R}^d).$$

(Cf. Proposition 4.5 in [9], Proposition 4. in [25], Corollary 5.2 in [35] or Theorem 4.1 in [41]. See also [52, Theorem 3.9] for an extension of these inclusions to broader classes of Gelfand–Shilov and modulation spaces.)

We refer to Example 3.4 below and to [51, Remark 1.4] for other examples on interesting modulation spaces.

We finish the section by giving some remarks on the symplectic short-time Fourier transform. The symplectic short-time Fourier transform of $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ with respect to the window function $\Phi \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ is defined by

$$\mathcal{V}_\Phi a(X,Y) = \mathscr{F}_\sigma\big(a\,\Phi(\,\cdot\,-X)\big)(Y), \quad X,Y \in \mathbf{R}^{2d}.$$

Let $\omega \in \mathscr{P}_E(\mathbf{R}^{4d})$. Then $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ and $\mathcal{W}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ denote the modulation spaces, where the symplectic short-time Fourier transform is used instead of the usual short-time Fourier transform in the definitions of the norms. It follows that any property valid for $M^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ or $W^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ carry over to $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ and $\mathcal{W}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ respectively. For example, for the symplectic short-time Fourier transform we have

$$\mathcal{V}_{\mathscr{F}_\sigma \Phi}(\mathscr{F}_\sigma a)(X,Y) = e^{2i\sigma(Y,X)} \mathcal{V}_\Phi a(Y,X), \tag{1.30}$$

(cf. (1.29)) which implies that

$$\mathscr{F}_\sigma \mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d}) = \mathcal{W}^{q,p}_{(\omega_0)}(\mathbf{R}^{2d}), \qquad \omega_0(X,Y) = \omega(Y,X). \tag{1.31}$$

2. Twisted convolution on modulation spaces and Lebesgue spaces

In this section we discuss algebraic properties of the twisted convolution when acting on modulation spaces of the form $\mathcal{W}^{p,q}_{(\omega)}$. The most general result corresponds to Theorem 0.3′ in [27], which concerns continuity for the Weyl product on modulation spaces of the form $\mathcal{M}^{p,q}_{(\omega)}$. Thereafter we use this result to establish continuity properties for the twisted convolution when acting on weighted Lebesgue spaces. We will mainly follow the analysis in Section 2 in [51], and the proofs are similar.

In these investigations we need the following lemma, which is strongly related to Lemma 4.4 in [43] and Lemma 2.1 in [27]. The latter results were fundamental in the proofs of [43, Theorem 4.1] and for the Weyl product results in [27].

Lemma 2.1. *Let $s > 1/2$, $a_1 \in \mathcal{S}_s'(\mathbf{R}^{2d})$, $a_2 \in \mathcal{S}_s(\mathbf{R}^{2d})$, $\Phi_1, \Phi_2 \in \Sigma_s(\mathbf{R}^{2d})$ and $X, Y \in \mathbf{R}^{2d}$. Then the following is true:*

(1) *if $\Phi = \pi^d \Phi_1 \# \Phi_2$, then $\Phi \in \Sigma_s(\mathbf{R}^{2d})$, and the map*

$$Z \mapsto e^{2i\sigma(Z,Y)}(\mathcal{V}_{\Phi_1} a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\Phi_2} a_2)(X+Z,Y-Z)$$

belongs to $L^1(\mathbf{R}^{2d})$, and

$$\begin{aligned} &\mathcal{V}_\Phi(a_1 \# a_2)(X,Y) \\ &\quad = \int e^{2i\sigma(Z,Y)}(\mathcal{V}_{\Phi_1} a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\Phi_2} a_2)(X+Z,Y-Z)\,dZ; \end{aligned}$$

(2) *if $\Phi = 2^{-d}\Phi_1 *_\sigma \Phi_2$, then $\Phi \in \Sigma_s(\mathbf{R}^{2d})$, and the map*

$$Z \mapsto e^{2i\sigma(X,Z-Y)}(\mathcal{V}_{\Phi_1} a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\Phi_2} a_2)(Y-Z,X+Z)$$

belongs to $L^1(\mathbf{R}^{2d})$, and

$$\begin{aligned} &\mathcal{V}_\Phi(a_1 *_\sigma a_2)(X,Y) \\ &\quad = \int e^{2i\sigma(X,Z-Y)}(\mathcal{V}_{\Phi_1} a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\Phi_2} a_2)(Y-Z,X+Z)\,dZ. \end{aligned}$$

Proof. The L^1-continuity for the mapping in (1) and (2) follow immediately from Theorems 6.4 and 6.5 in [52]. The integral formula for $\mathcal{V}_\Phi(a_1 \# a_2)$ in (1) then follows by similar arguments as for the proof of [43, Lemma 4.4], based on repeated applications of Fourier's inversion formula. The details are left for the reader. This gives (1).

The integral formula $\mathcal{V}_\Phi(a_1 *_\sigma a_2)$ in (2) now follows from (1), (1.17), (1.18) and (1.30). The proof is complete. □

For completeness we also write down the following extension of Theorem 0.3′ in [27]. Here the involved weight functions should satisfy

$$\omega_0(X,Y) \lesssim \omega_1(X-Y+Z,Z)\omega_2(X+Z,Y-Z), \quad X,Y,Z \in \mathbf{R}^{2d}, \tag{2.1}$$

and the exponent $p_j, q_j \in [1,\infty]$ satisfy (0.4) and

$$0 \le \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0} \le \frac{1}{p_j}, \frac{1}{q_j} \le \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0}, \quad j = 0,1,2. \tag{2.2}$$

Theorem 2.2. *Let $\omega_j \in \mathscr{P}_E(\mathbf{R}^{4d})$ and $p_j, q_j \in [1,\infty]$, $j = 0,1,2$, satisfy* (0.4), (2.1) *and* (2.2). *Then the map $(a_1, a_2) \mapsto a_1 \# a_2$ on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ extends uniquely to a continuous map from $\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $\mathcal{M}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, and the bound*

$$\|a_1 \# a_2\|_{\mathcal{M}^{p_0,q_0}_{(\omega_0)}} \lesssim \|a_1\|_{\mathcal{M}^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{\mathcal{M}^{p_2,q_2}_{(\omega_2)}}, \tag{2.3}$$

holds for every $a_1 \in \mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

The proof of Theorem 2.2 is similar to the proof of [27, Theorem 0.3′], after Proposition 1.9 and Lemma 2.1 in [27] have been replaced by Theorem 4.19 in [52] and Lemma 2.1. The details are left for the reader.

We note that Theorem 0.1 is an immediate consequence of (1.31), (1.19) and Theorem 2.2. Another way to prove Theorem 0.1 is to use similar arguments as in the proof of Theorem 2.2, based on (2) instead of (1) in Lemma 2.1.

We are now able to state and prove mapping results for the twisted convolution on weighted Lebesgue spaces. We start with the extension of Theorem 0.2 from the introduction.

Theorem 0.2′. *Let $k \in \{1,2\}$, $\omega_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ and let $p_j, q_j \in [1,\infty]$ for $j = 0,1,2$ satisfy* (0.6) *and*

$$\max_{j=0,1,2}\left(\frac{1}{p_j}, \frac{1}{p_j'}, \frac{1}{q_j}, \frac{1}{q_j'}\right) \le \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0}, \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0} \le 1,$$

Then the map $(a_1, a_2) \mapsto a_1 *_\sigma a_2$ extends uniquely to a continuous mapping from $L^{p_1,q_1}_{k,(\omega_1)}(\mathbf{R}^{2d}) \times L^{p_2,q_2}_{k,(\omega_2)}(\mathbf{R}^{2d})$ to $L^{p_0,q_0}_{k,(\omega_0)}(\mathbf{R}^{2d})$, and

$$\|a_1 *_\sigma a_2\|_{L^{p_0,q_0}_{k,(\omega_0)}} \lesssim \|a_1\|_{L^{p_1,q_1}_{k,(\omega_1)}} \|a_2\|_{L^{p_2,q_2}_{k,(\omega_2)}} \tag{0.8$'$}$$

when $a_1 \in L^{p_1,q_1}_{k,(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^{p_2,q_2}_{k,(\omega_2)}(\mathbf{R}^{2d})$.

Remark 2.3. The condition of the Lebesgue exponents for Theorems 0.2 and 0.2′ in [51] should be the same as in Theorems 0.2 and 0.2′ respectively. In this context, the results here extend corresponding results in [51].

For the proof we need the following lemma.

Lemma 2.4. *Let $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ be such that $\omega(x,\xi) = \omega(x)$. Then*

$$M^2_{(\omega)}(\mathbf{R}^d) = W^2_{(\omega)}(\mathbf{R}^d) = L^2_{(\omega)}(\mathbf{R}^d).$$

Proof. It is obvious that $M^2_{(\omega)} = W^2_{(\omega)}$. We have to prove $M^2_{(\omega)} = L^2_{(\omega)}$. Let $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d) \setminus 0$, and let $v \in \mathscr{P}_E(\mathbf{R}^d)$ be such that ω is v-moderate. Then (1.24) and Parseval's formula give

$$\begin{aligned}\|f\|^2_{M^2_{(\omega)}} &\asymp \iint |f(y)\phi(y-x))\omega(x)|^2\,dxdy\\ &\lesssim \iint |f(y)\omega(y)|^2|\phi(y-x)v(y-x)|^2\,dxdy \asymp \|f\|^2_{L^2_{(\omega)}}.\end{aligned}$$

Since $\mathcal{S}_{1/2}$ is dense in $L^2_{(\omega)}$ and in $M^2_{(\omega)}$ it follows that $L^2_{(\omega)} \hookrightarrow M^2_{(\omega)}$.

In order to prove the opposite inclusion we note that $\phi_1 v \lesssim \phi_2$, when $\phi, \phi_1 \in \mathscr{P}_E$ are the Gauss functions $\phi_1(x) = e^{-|x|^2}$ and $\phi_2(x) = e^{-|x|^2/2}$. Hence (1.24) and Parseval's formula give

$$\begin{aligned}\|f\|^2_{L^2_{(\omega)}} &\lesssim \iint |f(y)\phi_1(y-x)\omega(y)|^2\,dxdy\\ &\lesssim \iint |f(y)\phi_1(y-x)v(y-x)\omega(x)|^2\,dxdy\\ &\lesssim \iint |f(y)\phi_2(y-x)\omega(x)|^2\,dxdy\\ &= \iint |\mathscr{F}(f\phi_2(\,\cdot\,-x))(\xi)\omega(x)|^2\,dxd\xi \asymp \|f\|^2_{M^2_{(\omega)}}.\end{aligned}$$

Hence $M^2_{(\omega)} \hookrightarrow L^2_{(\omega)}$.

The result now follows by combining these embeddings, and the proof is complete. □

Proof of Theorem 0.2′. By duality we may assume that

$$\max\Big(\frac{1}{p_j}, \frac{1}{p_j'}, \frac{1}{q_j}, \frac{1}{q_j'}\Big)$$

is attained when $j = 0$. Since $\mathcal{W}^2_{(\omega)} = \mathcal{M}^2_{(\omega)} = L^2_{(\omega)}$ when $\omega(X,Y) = \omega(X)$, in view of Lemma 2.4, the result follows from Theorem 0.1 in the case $p_0 = p_1 = p_2 = 2$.

Next we consider the case when the Young conditions

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0} = 1 \quad\text{and}\quad \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0} = 1 \tag{2.4}$$

are fulfilled.

First we consider the case when $p_2, q_2 < \infty$, and we let $a_1 \in L^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and that $a_2 \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then

$$\|a_1 *_\sigma a_2\|_{L^{p_0,q_0}_{(\omega_0)}} \le (2/\pi)^{d/2}\|\,|a_1| * |a_2|\,\|_{L^{p_0,q_0}_{(\omega_0)}} \lesssim \|a_1\|_{L^{p_1,q_1}_{(\omega_1)}}\|a_2\|_{L^{p_2,q_2}_{(\omega_2)}}, \tag{2.5}$$

by Young's inequality and (0.6). The result now follows in this case from the fact that $\mathcal{S}_{1/2}$ is dense in $L^{p_2,q_2}_{(\omega_2)}$, when $p_2, q_2 < \infty$.

In the same way, the case $p_1, q_1 < \infty$ follows. It remain to consider when $p_1, q_1 < \infty$ and $p_2, q_2 < \infty$ are violated. By (2.4) we get $p_1 = q_2 = \infty$ and

$p_2 = q_1 = 1$, or $p_1 = q_2 = 1$ and $p_2 = q_1 = \infty$, and it follows that $Y \mapsto a_1(X-Y)a_2(Y)e^{2i\sigma(X,Y)} \in L^1_{(\omega_0)}$ when $a_1 \in L^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$, and that (2.5) holds. This proves the result when (2.4) is fulfilled.

Next we let p_j and q_j be general. Then we may assume that $p_1, q_1 < \infty$ or $p_2, q_2 < \infty$, since otherwise, the Young condition (2.4) must hold, which has already been considered.

Therefore, by reasons of symmetry we may assume that $p_1, q_1 < \infty$, and we let $\mathcal{L}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ be the completion of $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then $\mathcal{L}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ possess the (complex) interpolation property

$$(\mathcal{L}^{p_1,q_1}_{(\omega)}(\mathbf{R}^{2d}), (\mathcal{L}^{p_2,q_2}_{(\omega)}(\mathbf{R}^{2d}))_{[\theta]} = \mathcal{L}^{p,q}_{(\omega)}(\mathbf{R}^{2d}),$$
$$\text{when} \qquad \frac{1-\theta}{p_1} + \frac{\theta}{p_2} = \frac{1}{p}, \quad \frac{1-\theta}{q_1} + \frac{\theta}{q_2} = \frac{1}{q}$$

and $p_1, q_1 < \infty$. (Cf. Chapter 5 in [2].) Hence, by multi-linear interpolation between the case $p_0 = p_1 = p_2 = 2$ and the case (2.4) it follows that $\mathcal{L}^{p_1,q_1}_{(\omega_1)} *_\sigma \mathcal{L}^{p_2,q_2}_{(\omega_2)} \hookrightarrow \mathcal{L}^{p_0,q_0}_{(\omega_0)}$, and that $(0.8)'$ holds when $a_1, a_2 \in \mathcal{S}_{1/2}$.

The result now follows for general $a_1 \in L^{p_1,q_1}_{(\omega_1)}$ and $a_2 \in L^{p_2,q_2}_{(\omega)}$ by density arguments, where a_2 is first approximated by elements in $\mathcal{S}_{1/2}$ weakly, and thereafter a_1 is approximated by elements in $\mathcal{S}_{1/2}$ in the norm convergence. The proof is complete. □

Corollary 2.5. *Let $\omega_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ for $j = 0, 1, 2$ and $p, q \in [1, \infty]$ satisfy (0.6), and $q \le \min(p, p')$. Then the map $(a_1, a_2) \mapsto a_1 *_\sigma a_2$ extends uniquely to a continuous mapping from $L^p_{(\omega_1)}(\mathbf{R}^{2d}) \times L^q_{(\omega_2)}(\mathbf{R}^{2d})$ or $L^q_{(\omega_1)}(\mathbf{R}^{2d}) \times L^p_{(\omega_2)}(\mathbf{R}^{2d})$ to $L^p_{(\omega_0)}(\mathbf{R}^{2d})$.*

In particular, if $p \in [1, 2]$ and in addition ω_0 is submultitplicative, then $(L^p_{(\omega_0)}(\mathbf{R}^{2d}), *_\sigma)$ is an algebra.

We finish the section by using Theorem $0.2'$ to prove that the class of permitted windows in the modulation space norms can be extended. More precisely we have the following.

Theorem 2.6. *Let $p, p_0, q, q_0 \in [1, \infty]$ and $\omega, v \in \mathscr{P}_E(\mathbf{R}^{2d})$ be such that $p_0, q_0 \le \min(p, p', q, q')$, $\check{v} = v$ and ω is v-moderate. Also let $f \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$. Then the following is true:*

1. *if $\phi \in M^{p_0,q_0}_{(v)}(\mathbf{R}^d) \setminus 0$, then $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if $V_\phi f \in L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$. Furthermore, $\|f\| \equiv \|V_\phi f\|_{L^{p,q}_{1,(\omega)}}$ defines a norm for $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and different choices of ϕ give rise to equivalent norms;*
2. *if $\phi \in W^{p_0,q_0}_{(v)}(\mathbf{R}^d) \setminus 0$, then $f \in W^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if $V_\phi f \in L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$. Furthermore, $\|f\| \equiv \|V_\phi f\|_{L^{p,q}_{2,(\omega)}}$ defines a norm for $W^{p,q}_{(\omega)}(\mathbf{R}^d)$, and different choices of ϕ give rise to equivalent norms.*

For the proof we note that (1.25) gives

$$\|W_{f,\check{\phi}}\|_{L^{p,q}_{k,(\omega_0)}} \asymp \|V_\phi f\|_{L^{p,q}_{k,(\omega)}}, \quad \text{when} \quad \omega_0(x,\xi)=\omega(2x,2\xi) \tag{2.6}$$

for $k=1,2$.

Finally, by Fourier's inversion formula it follows that if $f_1,g_2\in \mathcal S_{1/2}'(\mathbf R^d)$ and $f_2,g_1\in L^2(\mathbf R^d)$, then

$$W_{f_1,g_1} *_\sigma W_{f_2,g_2} = (\check f_2,g_1)_{L^2}W_{f_1,g_2}. \tag{2.7}$$

Proof of Theorem 2.6. We may assume that $p_0=q_0=\min(p,p',q,q')$. Assume that $\phi,\psi\in M^{p_0,q_0}_{(v)}(\mathbf R^d)\hookrightarrow L^2(\mathbf R^d)$, where the inclusion follows from the fact that $p_0,q_0\le 2$ and $v\ge c$ for some constant $c>0$. Since $\|V_\phi\psi\|_{L^{p_0,q_0}_{k,(v)}} = \|V_\psi\phi\|_{L^{p_0,q_0}_{k,(v)}}$ when $\check v=v$, the result follows if we prove that

$$\|V_\phi f\|_{L^{p,q}_{k,(\omega)}} \lesssim (\|\psi\|_{L^2})^{-2}\|V_\psi f\|_{L^{p,q}_{k,(\omega)}}\|V_\phi\psi\|_{L^{p_0,q_0}_{k,(v)}}, \tag{2.8}$$

for some constant C which is independent of $f\in\mathcal S_{1/2}'(\mathbf R^d)$ and $\phi,\psi\in M^{p_0,q_0}_{(v)}(\mathbf R^d)$. For reasons of homogeneity, it is then no restriction to assume that $\|\psi\|_{L^2}=1$.

If $p_1=p$, $p_2=p_0$, $q_1=q$, $q_2=q_0$, $\omega_0=\omega(2\,\cdot\,)$ and $v_0=v(2\,\cdot\,)$, then Theorem 0.2′ and (2.7) give

$$\begin{aligned}\|V_\phi f\|_{L^{p,q}_{k,(\omega)}} &\asymp \|W_{f,\check\phi}\|_{L^{p,q}_{k,(\omega_0)}} \asymp \|W_{f,\check\psi} *_\sigma W_{\psi,\check\phi}\|_{L^{p,q}_{k,(\omega_0)}}\\ &\lesssim \|W_{f,\check\psi}\|_{L^{p,q}_{k,(\omega_0)}}\|W_{\psi,\check\phi}\|_{L^{p_0,q_0}_{k,(v_0)}} \asymp \|V_\psi f\|_{L^{p,q}_{k,(\omega)}}\|V_\phi\psi\|_{L^{p_0,q_0}_{k,(v)}},\end{aligned}$$

and (2.8) follows. The proof is complete. □

3. Gelfand–Shilov tempered vector spaces

In this section we introduce the notion of Gelfand–Shilov and Beurling tempered (quasi-)Banach and Hilbert spaces, and establish embedding properties for such spaces. These results are applied in the next sections when discussing Schatten–von Neumann operators within the theory of pseudo-differential operators. The results are also applied in [54] where decomposition and Schatten–von Neumann properties for linear operators with Gelfand–Shilov kernels are established. We remark that some parts of the approach here are somewhat similar to the first part of Section 4 in [51], where related questions on tempered Hilbert spaces (with respect to Schwartz tempered distributions) are considered.

We start by introducing some notations on quasi-Banach spaces. A quasi-norm $\|\cdot\|_{\mathscr B}$ on a vector space $\mathscr B$ (over $\mathbf C$) is a non-negative and real-valued function $\|\cdot\|_{\mathscr B}$ on $\mathscr B$ which is non-degenerate in the sense

$$\|f\|_{\mathscr B}=0 \quad\Longleftrightarrow\quad f=0, \qquad f\in\mathscr B,$$

and fulfills

$$\begin{aligned} &\|\alpha f\|_{\mathscr B} = |\alpha|\cdot\|f\|_{\mathscr B}, && f\in\mathscr B,\ \alpha\in\mathbf C\\ \text{and}\quad &\|f+g\|_{\mathscr B}\le D(\|f\|_{\mathscr B}+\|g\|_{\mathscr B}), && f,g\in\mathscr B,\end{aligned} \tag{3.1}$$

for some constant $D \ge 1$ which is independent of $f, g \in \mathscr{B}$. Then $\mathscr{B}$ is a topological vector space when the topology for $\mathscr{B}$ is defined by $\| \cdot \|_{\mathscr{B}}$, and $\mathscr{B}$ is called a quasi-Banach space if $\mathscr{B}$ is complete under this topology.

Let $\mathscr{B}$ be a quasi-Banach space such that

$$\mathcal{S}_{1/2}(\mathbf{R}^d) \hookrightarrow \mathscr{B} \hookrightarrow \mathcal{S}_{1/2}'(\mathbf{R}^d), \tag{3.2}$$

and that $\mathcal{S}_{1/2}(\mathbf{R}^d)$ is dense in $\mathscr{B}$. We let $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$ be the sets of all $f \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$ such that $\check{f} \in \mathscr{B}$ and $\overline{f} \in \mathscr{B}$ respectively. Then $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$ are quasi-Banach spaces under the quasi-norms

$$\|f\|_{\check{\mathscr{B}}} \equiv \|\check{f}\|_{\mathscr{B}} \quad \text{and} \quad \|f\|_{\mathscr{B}^\tau} \equiv \|\overline{f}\|_{\mathscr{B}}$$

respectively. Furthermore, $\mathcal{S}_{1/2}(\mathbf{R}^d)$ is dense in $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$, and (3.2) holds after $\mathscr{B}$ have been replaced by $\check{\mathscr{B}}$ or $\mathscr{B}^\tau$. Moreover, if $\mathscr{B}$ is a Banach (Hilbert) space, then $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$ are Banach (Hilbert) spaces.

The L^2-*dual* $\mathscr{B}'$ of $\mathscr{B}$ is the set of all $\varphi \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$ such that

$$\|\varphi\|_{\mathscr{B}'} \equiv \sup |(\varphi, f)_{L^2(\mathbf{R}^d)}|$$

is finite. Here the supremum is taken over all $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$ such that $\|f\|_{\mathscr{B}} \le 1$. Let $\varphi \in \mathscr{B}'$. Since $\mathcal{S}_{1/2}$ is dense in $\mathscr{B}$, it follows from the definitions that the map $f \mapsto (\varphi, f)_{L^2}$ from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathbf{C}$ extends uniquely to a continuous mapping from $\mathscr{B}$ to $\mathbf{C}$.

From now on we assume that the (quasi-)Banach spaces $\mathscr{B}, \mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2, \dots$, and the Hilbert spaces $\mathscr{H}, \mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_2, \dots$ are "Gelfand–Shilov tempered" or "Beurling tempered" in the following sense.

Definition 3.1. Let $\mathscr{B}$ be a quasi-Banach space such that (3.2) is fulfilled.

(1) $\mathscr{B}$ is called *Beurling tempered*, or *B-tempered* (*of order* $s > 1/2$) on $\mathbf{R}^d$, if $\mathscr{B}, \mathscr{B}' \hookrightarrow \Sigma_s'(\mathbf{R}^d)$, and $\Sigma_s(\mathbf{R}^d)$ is dense in $\mathscr{B}$ and $\mathscr{B}'$;

(2) $\mathscr{B}$ is called *Gelfand–Shilov tempered*, or *GS-tempered* (*of order* $s \ge 1/2$) on $\mathbf{R}^d$, if $\mathscr{B}, \mathscr{B}' \hookrightarrow \mathcal{S}_s'(\mathbf{R}^d)$, and $\mathcal{S}_s(\mathbf{R}^d)$ is dense in both $\mathscr{B}$ and $\mathscr{B}'$;

(3) $\mathscr{B}$ is called *tempered* on $\mathbf{R}^d$, if $\mathscr{B}, \mathscr{B}' \hookrightarrow \mathscr{S}'(\mathbf{R}^d)$, and $\mathscr{S}(\mathbf{R}^d)$ is dense in $\mathscr{B}$ and $\mathscr{B}'$.

Remark 3.2. Let $\mathscr{B}$ be a quasi-Banach space such that (3.2) holds. Then it follows from (1.2) and the fact that $\mathcal{S}_{1/2}$ is dense in $\mathcal{S}_s$, Σ_s and $\mathscr{S}$, when $s > 1/2$, that the following is true:

(1) if $s > 1/2$ and $\mathscr{B}$ is GS-tempered of order s, then $\mathscr{B}$ is B-tempered of order s;

(2) if $s \ge 1/2$, $\varepsilon > 0$ and $\mathscr{B}$ is B-tempered of order $s+\varepsilon$, then $\mathscr{B}$ is GS-tempered of order s;

(3) if $s > 1/2$ and $\mathscr{B}$ is tempered, then $\mathscr{B}$ is GS- and B-tempered of order s.

We also note that Definition 3.1 (3) in the Hilbert space case might not be the same as [51, Definition 4.1]. In fact, $\mathscr{H}$ is a tempered Hilbert in the sense of Definition 3.1 (3), is the same as *both* $\mathscr{H}$ and $\mathscr{H}'$ are tempered in the sense of [51, Definition 4.1].

For future references we remark that $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$ are GS-tempered (B-tempered) quasi-Banach spaces, when $\mathscr{B}$ is a GS-tempered (B-tempered) quasi-Banach space, and that similar facts hold when $\mathscr{B}$ is a Banach or Hilbert space.

In the following analogy of [51, Lemma 4.2] we establish basic properties in the Hilbert space case.

Lemma 3.3. *Let $s \ge 1/2$ ($s > 1/2$), and let $\mathscr{H}$ be a GS-tempered (B-tempered) Hilbert space of order s on $\mathbf{R}^d$, with L^2-dual $\mathscr{H}'$. Then the following is true:*

(1) *$\mathscr{H}'$ is a GS-tempered (B-tempered) Hilbert space of order s, which can be identified with the dual space of $\mathscr{H}$ through the L^2-form;*

(2) *there is a unique map $T_{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}'$ such that*

$$(f,g)_{\mathscr{H}} = (T_{\mathscr{H}} f, g)_{L^2(\mathbf{R}^d)}, \quad f,g \in \mathscr{H}; \tag{3.3}$$

(3) *if $T_{\mathscr{H}}$ is the map in (2), $\{e_j\}_{j\in I}$ is an orthonormal basis in $\mathscr{H}$ and $\varepsilon_j = T_{\mathscr{H}} e_j$, then $T_{\mathscr{H}}$ is isometric, $\{\varepsilon_j\}_{j\in I}$ is an orthonormal basis in $\mathscr{H}'$ and*

$$(\varepsilon_j, e_k)_{L^2(\mathbf{R}^d)} = \delta_{j,k}.$$

Proof. We only prove the result when $\mathscr{H}$ is GS-tempered. The case when $\mathscr{H}$ is B-tempered follows by similar arguments and is left for the reader.

First assume that $f \in \mathscr{H}$, $g \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, and let $T_{\mathscr{H}} f \in \mathscr{H}'$ be defined by (3.3). Then $T_{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}'$ is isometric. Furthermore, since $\mathcal{S}_{1/2}$ is dense in $\mathscr{H}$, and the dual space of $\mathscr{H}$ can be identified with itself, under the scalar product of $\mathscr{H}$, the asserted duality properties of $\mathscr{H}'$ follow.

Let $\{e_j\}_{j\in I}$ be an arbitrary orthonormal basis in $\mathscr{H}$, and let $\varepsilon_j = T_{\mathscr{H}} e_j$. Then it follows that $\|\varepsilon_j\|_{\mathscr{H}'} = 1$ and

$$(\varepsilon_j, e_k)_{L^2} = (e_j, e_k)_{\mathscr{H}} = \delta_{j,k}.$$

Furthermore, if

$$f = \sum \alpha_j e_j, \quad \varphi = \sum \alpha_j \varepsilon_j, \quad g = \sum \beta_j e_j \quad \text{and} \quad \gamma = \sum \beta_j \varepsilon_j$$

are finite sums, and we set $(\varphi,\gamma)_{\mathscr{H}'} \equiv (f,g)_{\mathscr{H}}$, then it follows that $(\,\cdot\,,\,\cdot\,)_{\mathscr{H}'}$ defines a scalar product on such finite sums in $\mathscr{H}'$, and that $\|\varphi\|^2_{\mathscr{H}'} = (\varphi,\varphi)_{\mathscr{H}'}$. By continuity extensions it now follows that $(\varphi,\gamma)_{\mathscr{H}'}$ extends uniquely to each $\varphi,\gamma \in \mathscr{H}'$, and that the identity $\|\varphi\|^2_{\mathscr{H}'} = (\varphi,\varphi)_{\mathscr{H}'}$ holds. This proves the result. □

From now on the basis $\{\varepsilon_j\}_{j\in I}$ in Lemma 3.3 is called the *dual basis* of $\{e_j\}_{j\in I}$.

Example 3.4. Let $\mathscr{H}_1 = H^2_s(\mathbf{R}^d)$ and $\mathscr{H}_2 = M^2_{(\omega_0)}(\mathbf{R}^d)$ where $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$. Then $\mathscr{H}_1$ is a tempered Hilbert space with dual $\mathscr{H}_1' = H^2_{-s}(\mathbf{R}^d)$. The space

$\mathscr{H}_2$ is B-tempered (GS-tempered) of order s, when $s \le 1$ ($s < 1$), and $\mathscr{H}_2' = M^2_{(1/\omega_0)}(\mathbf{R}^d)$.

We note that if $\omega_s(x,\xi) = \langle \xi \rangle^s$, then $M^2_{(\omega_s,0)} = H^2_s$, and we refer to [51, Example 4.3] for more examples on tempered Hilbert spaces.

In several situations, an orthonormal basis $\{e_j\}$ in a GS- or B-tempered Hilbert space $\mathscr{H}$ might be orthogonal in $L^2(\mathbf{R}^d)$. The following proposition shows that this is sufficient for $\{e_j\}$ being orthogonal in the dual $\mathscr{H}'$ of $\mathscr{H}$.

Proposition 3.5. *Let $\mathscr{H}$ be GS- or B-tempered Hilbert space on $\mathbf{R}^d$, $\{e_j\}_{j=1}^\infty$ and $\{e_{0,j}\}_{j=1}^\infty$ be orthonormal bases for $\mathscr{H}$ and $L^2(\mathbf{R}^d)$ respectively, and let $\{\varepsilon_j\}_{j=1}^\infty \in \mathscr{H}'$ be the dual basis of $\{e_j\}_{j=1}^\infty$. Then the following is true:*

1. *if $e_j = c_j e_{0,j}$ for every $j \ge 1$ and some $\{c_j\}_{j=1}^\infty \subseteq \mathbf{C}$, then $\varepsilon_j = (\overline{c_j})^{-1} e_{0,j}$;*
2. *if $\varepsilon_j = c_j e_j$ for every $j \ge 1$ and some $\{c_j\}_{j=1}^\infty \subseteq \mathbf{C}$, then $c_j > 0$, and $\{c_j^{1/2} e_j\}_{j=1}^\infty$ is an orthonormal basis for $L^2(\mathbf{R}^d)$;*
3. *if $e_j = c_j e_{0,j}$ and $\varepsilon_j = d_j e_{0,j}$ for every $j \ge 1$ and some $\{c_j\}_{j=1}^\infty \subseteq \mathbf{C}$ and $\{d_j\}_{j=1}^\infty \subseteq \mathbf{C}$, then*

$$c_j \overline{d_j} = \|e_{0,j}\|_{\mathscr{H}} \cdot \|e_{0,j}\|_{\mathscr{H}'} = \|e_j\|_{L^2} \|\varepsilon_j\|_{L^2} = 1.$$

Proof. (1) We have

$$\delta_{j,k} = (e_j, e_k)_{\mathscr{H}} = (e_j, \varepsilon_k)_{L^2} = c_j (e_{0,j}, \varepsilon_k)_{L^2},$$

giving that

$$\delta_{j,k} = c_k (e_{0,j}, \varepsilon_k)_{L^2} = (e_{0,j}, \overline{c_k} \varepsilon_k)_{L^2}. \tag{3.4}$$

Since $\{e_{0,j}\}_{j=1}^\infty$ is an orthonormal basis for L^2, it follows from (3.4) that $\overline{c_k} \varepsilon_k = e_{0,k}$, and (1) follows.

(2) We have

$$1 = (e_j, e_j)_{\mathscr{H}} = (e_j, \varepsilon_j)_{L^2} = \overline{c_j} (e_j, e_j)_{L^2} = \overline{c_j} \|e_j\|^2_{L^2},$$

giving that $c_j > 0$. Furthermore, if $f_j = c_j^{1/2} e_j$, then

$$(f_j, f_k)_{L^2} = (c_j/c_k)^{1/2} (e_j, \varepsilon_k)_{L^2} = (c_j/c_k)^{1/2} \delta_{j,k} = \delta_{j,k}.$$

Hence $\{f_j\}_{j=1}^\infty$ is an orthonormal basis of L^2.

(3) By applying appropriate norms on the identities $e_j = c_j e_{0,j}$ and, $\varepsilon_j = d_j e_{0,j}$ and using the fact that e_j, $e_{0,j}$ and ε_j are unit vectors in $\mathscr{H}$, L^2 and $\mathscr{H}'$ respectively, we get

$$1 = (e_j, \varepsilon_j)_{L^2} = c_j \overline{d_j} (e_{0,j}, e_{0,j})_{L^2} = c_j \overline{d_j},$$

and

$$|c_j| = \|e_j\|_{L^2} = 1/\|e_{0,j}\|_{\mathscr{H}} \quad \text{and} \quad |d_j| = \|\varepsilon_j\|_{L^2} = 1/\|e_{0,j}\|_{\mathscr{H}'}.$$

The assertion follows by combining these equalities. The proof is complete. □

Example 3.6. Let $\omega \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$ be positive. Then ω is called weakly sub-Gaussian type weight, if the following conditions are fulfilled:

- $e^{-\varepsilon|x|^2} \lesssim \omega(x) \lesssim e^{\varepsilon|x|^2}$, for every choice of $\varepsilon > 0$;
- for some fixed $R \ge 2$ and some constant $c > 0$, the relation

$$\omega(x+y)\omega(x-y) \asymp \omega(x)^2$$

holds when $Rc \le |x| \le c/|y|$.

The set of all weakly sub-Gaussian weights on $\mathbf{R}^d$ is denoted by $\mathscr{P}_Q^0(\mathbf{R}^d)$, and is a family of weights which contains $\mathscr{P}_E(\mathbf{R}^d)$. (Cf. Definition 1.1 in [52].)

Now consider the modulation spaces $M^2_{(\omega)}(\mathbf{R}^d)$, where $\omega \in \mathscr{P}_Q^0(\mathbf{R}^{2d})$ satisfies

$$\omega(x,\xi) = \omega_0(r) = \omega_0(r_1,\dots,r_d), \quad r_j = |(x_j,\xi_j)|, \quad j = 1,\dots,d$$

(i.e., $\omega(x,\xi)$ is rotation invariant under each coordinate pair (x_j,ξ_j)). Note that the window function $\phi(x)$ in the definition of modulation space norms with weights in $\mathscr{P}_Q^0$ is fixed and equal to the Gaussian $\pi^{-d/4}e^{-|x|^2/2}$. Then there is a constant $C > 0$ such that

$$C^{-1} \le \|h_\alpha\|_{M^2_{(\omega)}} \|h_\alpha\|_{M^2_{(1/\omega)}} \le C, \tag{3.5}$$

for every Hermite function h_α on $\mathbf{R}^d$.

In fact, if $\mathscr{H} = M^2_{(\omega)}$, then it follows from Theorem 4.17 in [52] that $\mathscr{H}' = M^2_{(1/\omega)}$ and that

$$\|f\|_{\mathscr{H}'} \asymp \|f\|_{M^2_{(1/\omega)}}, \quad f \in \mathscr{H}'.$$

The statement is now a consequence of Proposition 3.5, and the facts that $\{h_\alpha\}_{\alpha\in\mathbf{N}^d}$ and $\{h_\alpha/\|h_\alpha\|_{M^2_{(\omega)}}\}_{\alpha\in\mathbf{N}^d}$ are orthonormal bases for L^2 and $M^2_{(\omega)}$, respectively.

The relation (3.5) in combination with results in [52] can be used to establish estimates for generalized gamma functions in integer points, in a similar way as formula (30) in [24]. More precisely, let $\mathfrak{V}$ be the Bargmann transform, and let $A^2_{(\omega)}(\mathbf{C}^d)$ be the weighted Bargmann–Fock space of all entire functions F on $\mathbf{C}^d$ such that

$$\|F\|_{A^2_{(\omega)}} \equiv \pi^{-d/2}\left(\int_{\mathbf{C}^d} |F(z)\omega(2^{1/2}\overline{z})|^2 e^{-|z|^2}\, d\lambda(z)\right)^{1/2} < \infty.$$

(Cf., e.g., [1, 52].) Here we identify $x + i\xi$ in $\mathbf{C}^d$ with (x,ξ) in $\mathbf{R}^{2d}$, and $d\lambda(z)$ is the Lebesgue measure on $\mathbf{C}^d$. Then $(\mathfrak{V}h_\alpha)(z) = z^\alpha/(\alpha!)^{1/2}$, and the map $f \mapsto \mathfrak{V}f$ is isometric and bijective from $M^2_{(\omega)}$ to $A^2_{(\omega)}$, in view of Theorem 3.4 in [52]. Consequently, (3.5) is equivalent to

$$I_{(\omega_0)} \cdot I_{(1/\omega_0)} \asymp (\alpha!)^2, \tag{3.6}$$

where

$$I_{(\omega_0)} \equiv \pi^d \|z^\alpha\|^2_{A^2_{(\omega)}} = \int_{\mathbf{C}^d} |z^{2\alpha}|\omega_0(|z_1|,\dots,|z_d|)^2 e^{-|z|^2}\, d\lambda(z).$$

By writing $z_j = r_j e^{i\theta_j}$ in terms of polar coordinates for every $j = 1, \dots, d$, and taking r_j^2 and θ_j as new variables of integrations, we get

$$I_{(\omega_0)} \asymp \int_{[0,\infty)^d} t^\alpha \vartheta(t) e^{-\|t\|}\, dt,$$

where $\vartheta(t_1, \dots, t_d) = \omega_0(t_1^{1/2}, \dots, t_d^{1/2})^2$ and $\|t\| = t_1 + \cdots + t_d$, $t \in [0,\infty)^d$. Hence (3.6) is equivalent to

$$\int_{[0,\infty)^d} t^\alpha \vartheta(t) e^{-\|t\|}\, dt \cdot \int_{[0,\infty)^d} t^\alpha \vartheta(t)^{-1} e^{-\|t\|}\, dt \asymp (\alpha!)^2. \tag{3.7}$$

In particular, the formula (30) in the remark after Theorem 3.7 in [24] hold for the broad class $\mathscr{P}_Q^0$ of weights on $\mathbf{R}^d$.

The following result concerns continuous embeddings of the form

$$M^2_{(\omega_t)}(\mathbf{R}^d) \hookrightarrow \mathscr{B}, \mathscr{B}' \hookrightarrow M^2_{(1/\omega_t)}(\mathbf{R}^d), \tag{3.8}$$

when $\mathscr{B}$ is a quasi-Banach space. Here $M^2_{(\omega_t)}$ belongs to the extended family of modulation spaces in [52] and the weights ω_t are given by

$$\omega_t(x,\xi) = e^{t(|x|^{1/s} + |\xi|^{1/s})}, \tag{3.9}$$

when $s \ge 1/2$ and $t \in \mathbf{R}$.

Proposition 3.7. *Let $s > 1/2$, and let ω_t be given by* (3.9)*. Then the following is true:*

(1) *if $\mathscr{B}$ is a GS-tempered quasi-Banach space on $\mathbf{R}^d$ of order s, then* (3.8) *hold for every $t > 0$;*

(2) *if $\mathscr{B}$ is a B-tempered quasi-Banach space on $\mathbf{R}^d$ of order s, then* (3.8) *hold for some $t > 0$.*

We first prove Proposition 3.7 in the case that $\mathscr{B} = \mathscr{H}$ is a Hilbert space. Thereafter, the general result will follow from this case and Proposition 3.8 below.

Proof of Proposition 3.7 *when $\mathscr{B} = \mathscr{H}$ is a Hilbert space.* By Remark 1.6 it follows that

$$\Sigma_s \hookrightarrow M^2_{(\omega_t)} \hookrightarrow \mathcal{S}_s, \quad \mathcal{S}_s' \hookrightarrow M^2_{(1/\omega_t)} \hookrightarrow \Sigma_s' \tag{3.10}$$

when $t > 0$.

If $\mathscr{H}$ is GS-tempered, then it follows from these embeddings that $M^2_{(\omega_t)} \hookrightarrow \mathscr{H}, \mathscr{H}'$ holds for every $t > 0$. Furthermore, by Theorem 4.17 in [52] it follows that $\mathcal{S}_{1/2}$ is dense in these Hilbert spaces, and that the dual of $M^2_{(\omega_t)}$ is given by $M^2_{(1/\omega_t)}$.

Now if $\mathscr{H}$ is GS-tempered, then (3.10) gives

$$M^2_{(\omega_t)} \hookrightarrow \mathcal{S}_s \hookrightarrow \mathscr{H}, \mathscr{H}' \hookrightarrow \mathcal{S}_s' \hookrightarrow M^2_{(1/\omega_t)}, \quad t > 0,$$

and (1) follows.

In order to prove (2) we note that Theorem 3.9 and its proof in [52] implies that the topology for Σ_s is given by the semi-norms

$$f \mapsto \|f\|_{(t)} \equiv \|f\|_{M^2_{(\omega_t)}}, \quad t > 0.$$

Hence

$$\|f\|_{\mathscr H} \lesssim \|f\|_{M^2_{(\omega_t)}} \quad \text{and} \quad \|\varphi\|_{\mathscr H'} \lesssim \|\varphi\|_{M^2_{(\omega_t)}}, \quad f,\varphi \in M^2_{(\omega_t)}$$

hold for some choice of $t = t_0 > 0$, since $\Sigma_s \hookrightarrow \mathscr H$ and $\Sigma_s \hookrightarrow \mathscr H'$. This gives $M^2_{(\omega_{t_0})} \hookrightarrow \mathscr H$ and $M^2_{(\omega_{t_0})} \hookrightarrow \mathscr H'$. The assertion (2) now follows from these embeddings and duality. The proof is complete. □

With reference to the Hilbert spaces which occur in Example 3.6 we say that a Hilbert space $\mathscr H$ is of *Hermite type*, if $\{h_\alpha/\|h_\alpha\|_{\mathscr H}\}_\alpha$ is an orthonormal basis for $\mathscr H$,

$$(S_\pi f)(x) \equiv f(x_{\pi(1)},\dots,x_{\pi(d)}) \in \mathscr H \quad \text{when} \quad f \in \mathscr H$$

for every permutation π on $\{1,\dots,d\}$, and that $\|S_\pi f\|_{\mathscr H} = \|f\|_{\mathscr H}$ for every $f \in \mathscr H$ an permutation π.

Proposition 3.8. *Let $\mathscr B_1, \mathscr B_2$ be quasi-Banach spaces which are continuously embedded in $\mathcal S_{1/2}'(\mathbf R^d)$. Then the following is true:*

(1) *if $s \ge 1/2$, $\mathcal S_s(\mathbf R^d) \hookrightarrow \mathscr B_1$ and $\mathscr B_2 \hookrightarrow \mathcal S_s'(\mathbf R^d)$, then there are GS-tempered Hilbert spaces $\mathscr H_1$ and $\mathscr H_2$ of order s and of Hermite type such that*

$$\mathcal S_s(\mathbf R^d) \hookrightarrow \mathscr H_1 \hookrightarrow \mathscr B_1 \quad \text{and} \quad \mathscr B_2 \hookrightarrow \mathscr H_2 \hookrightarrow \mathcal S_s'(\mathbf R^d)$$

hold. Furthermore, $\mathscr H_1$ and $\mathscr H_2$ can be chosen such that $\mathscr H_1 \hookrightarrow \mathcal S_{s/\gamma}(\mathbf R^d)$ and $\mathcal S_{s/\gamma}'(\mathbf R^d) \hookrightarrow \mathscr H_2$ for every $\gamma \in (0,1)$;

(2) *if $s > 1/2$, $\Sigma_s(\mathbf R^d) \hookrightarrow \mathscr B_1$ and $\mathscr B_2 \hookrightarrow \Sigma_s'(\mathbf R^d)$, then there are B-tempered Hilbert spaces $\mathscr H_1$ and $\mathscr H_2$ of order s and of Hermite type such that*

$$\Sigma_s(\mathbf R^d) \hookrightarrow \mathscr H_1 \hookrightarrow \mathscr B_1 \quad \text{and} \quad \mathscr B_2 \hookrightarrow \mathscr H_2 \hookrightarrow \Sigma_s'(\mathbf R^d)$$

hold. Furthermore, $\mathscr H_1$ and $\mathscr H_2$ can be chosen such that $\mathscr H_1 \hookrightarrow \Sigma_{s/\gamma}(\mathbf R^d)$ and $\Sigma_{s/\gamma}'(\mathbf R^d) \hookrightarrow \mathscr H_2$ for every $\gamma \in (0,1)$;

(3) *if $\mathscr S(\mathbf R^d) \hookrightarrow \mathscr B_1$ and $\mathscr B_2 \hookrightarrow \mathscr S'(\mathbf R^d)$, then there are tempered Hilbert spaces $\mathscr H_1$ and $\mathscr H_2$ of Hermite type such that*

$$\mathscr S(\mathbf R^d) \hookrightarrow \mathscr H_1 \hookrightarrow \mathscr B_1 \quad \text{and} \quad \mathscr B_2 \hookrightarrow \mathscr H_2 \hookrightarrow \mathscr S'(\mathbf R^d)$$

hold.

Proof. We only prove (1). The other assertions follow by similar arguments and are left for the reader.

In order to prove (1) it is no restriction to assume that $\mathcal S_s$ is dense $\mathscr B_1$ by replacing $\mathscr B_1$ with the completion of $\mathcal S_s$ under the quasi-norm $\|\cdot\|_{\mathscr B_1}$. Let $f \in \mathscr B_1$. Since $\mathscr B_1 \hookrightarrow \mathcal S_{1/2}'$, it follows that

$$f = \sum_\alpha c_\alpha h_\alpha,$$

where h_α is the Hermite function of order α and its coefficients

$$c_\alpha = c_\alpha(f) = (f, h_\alpha)_{L^2}$$

satisfies

$$\sum_\alpha |c_\alpha|^2 e^{-c|\alpha|} < \infty,$$

for every $c > 0$.

The fact that $\mathcal{S}_s$ is continuously embedded in $\mathscr{B}_1$ implies that for every integer $j > 0$ we have

$$\|f\|^2_{\mathscr{B}_1} \le C_j D^{-2j} \sum_\alpha |c_\alpha|^2 e^{|\alpha|^{1/2s}/j},$$

where the constant $D \ge 1$ is the same as in (3.1), and the constant $C_j \ge 1$ is independent of f (cf. formula (2.12) in [21]).

For every integer $j \ge 1$, let

$$N_j = \sup\{\, |\alpha|\,;\, C_j j^2 e^{j^j - 1} e^{-|\alpha|^{1/2s}/j} \ge 1 \,\},$$

and define inductively

$$R_1 = N_1 \quad \text{and} \quad R_{j+1} = \max(R_j + 1, N_{j+1}), \quad j \ge 1.$$

Furthermore we set

$$I_0 = \{\, \alpha\,;\, |\alpha| \le R_1 \,\} \quad \text{and} \quad I_j = \{\, \alpha\,;\, R_j < |\alpha| \le R_{j+1} \,\},$$

and we let $m(\alpha) = \sup_{\alpha \in I_0} C_1 e^{|\alpha|^{1/2s}}$ when $\alpha \in I_0$, and $m(\alpha) = e^{j^j - 1} e^{2|\alpha|^{1/2s}/j}$ when $\alpha \in I_j$, $j \ge 1$. We note that R_j is finite and increases to ∞ as j tends to ∞.

Let $\mathscr{H}_1$ be the Hilbert space which consists of all $f \in \mathcal{S}_s'$ such that

$$\|f\|_{\mathscr{H}_1} \equiv \Big(\sum_\alpha |c_\alpha(f)|^2 m(\alpha) \Big)^{1/2}$$

is finite. We shall prove that $\mathscr{H}_1$ satisfies the requested properties. Since

$$\lim_{|\alpha| \to \infty} m(\alpha) e^{-c|\alpha|^{1/2s}} = 0$$

when $c > 0$, it follows that $\mathcal{S}_s$ is continuously embedded in $\mathscr{H}_1$. Furthermore, the fact that $m(\alpha) = m(\beta)$ when $|\alpha| = |\beta|$ implies that $f \mapsto S_\pi f$ is a unitary map on $\mathscr{H}_1$, for every permutation π on $\{1, \dots, d\}$.

It remains to prove that $\mathscr{H}_1$ is continuously embedded in $\mathscr{B}_1$ and in $\mathcal{S}_{s/\gamma}$ when $0 < \gamma < 1$. Let $f \in \mathcal{S}_s$, and let

$$f_j = \sum_{\alpha \in I_j} c_\alpha(f) h_\alpha, \quad j \ge 0.$$

Then

$$f = \sum_{j \ge 0} f_j, \quad c_\alpha(f_j) = \begin{cases} c_\alpha(f), & \alpha \in I_j \\ 0, & \alpha \notin I_j \end{cases} \quad \text{and} \quad \|f\|^2_{\mathscr{H}_1} = \sum_{j \ge 0} \|f_j\|^2_{\mathscr{H}_1}.$$

This gives

$$\begin{aligned}\|f\|_{\mathscr{B}_1} &\lesssim \sum_j D^j \|f_j\|_{\mathscr{B}_1} \le \Big(\sum_{\alpha\in I_0} |c_\alpha|^2 m(\alpha)\Big)^{1/2} + \sum_{j\ge 1} \Big(C_j \sum_{\alpha\in I_j} |c_\alpha|^2 e^{|\alpha|^{1/2s}/j}\Big)^{1/2} \\ &\le \Big(\sum_{\alpha\in I_0} |c_\alpha|^2 m(\alpha)\Big)^{1/2} + \sum_{j\ge 1} \frac{1}{j} \Big(\sum_{\alpha\in I_j} |c_\alpha|^2 e^{2|\alpha|^{1/2s}/j}\Big)^{1/2} \\ &\le \|f_0\|_{\mathscr{H}_1} + \sum_{j\ge 1}\frac{1}{j}\|f_j\|_{\mathscr{H}_1}.\end{aligned}$$

Hence, by Cauchy–Schwartz inequality we get

$$\|f\|_{\mathscr{B}_1} \le \|f_0\|_{\mathscr{H}_1} + \Big(\sum_{j\ge 1} \frac{1}{j^2}\Big)^{1/2} \Big(\sum_{j\ge 1} \|f_j\|_{\mathscr{H}_1}^2\Big)^{1/2} \lesssim \Big(\sum_{j\ge 0} \|f_j\|_{\mathscr{H}_1}^2\Big)^{1/2} = \|f\|_{\mathscr{H}_1},$$

which proves that $\mathscr{H}_1 \hookrightarrow \mathscr{B}_1$.

The inclusion $\mathscr{H}_1 \hookrightarrow \mathcal{S}_{s/\gamma}$ when $\gamma > 1$ follows if we prove that

$$e^{c|\alpha|^{\gamma/2s}} \lesssim m(\alpha), \tag{3.11}$$

for every $c > 0$. We claim that there is a constant C_0 which is independent of $j \ge 1$ and α such that

$$e^{c|\alpha|^{\gamma/2s}} \le C_0 e^{j^j-1} e^{2|\alpha|^{1/2s}/j}. \tag{3.12}$$

In fact, by applying the logarithm, (3.12) follows if we prove that for $r = 1/2s \le 1$ and constants $m_1, m_2 > 0$, the function

$$h(u,v) = m_1 u^u + u^{-1} v^r - m_2 v^{\gamma r}$$

is bounded from below, when $u, v \ge 1$.

In order to prove this we let $0 < \gamma_1, \gamma_2 < 1$ be chosen such that $\gamma_1 > \gamma$ and $\gamma_1 + \gamma_2 = 1$. Then the inequality on arithmetic and geometric mean-values gives that $h_0(u,v) \lesssim h(u,v)$, where

$$h_0(u,v) = u^{\gamma_2 u - \gamma_1} v^{\gamma_1 r} - m_1' v^{\gamma r} = v^{\gamma r}(u^{\gamma_2 u-\gamma_1} v^{(\gamma_1-\gamma)r} - m_1'),$$

for some $m_1' > 0$. Since $\gamma_1 > \gamma$, it follows that $h_0(u,v)$ tends to infinity when $u+v \to \infty$ and $u, v \ge 1$. The fact that h_0 is continuous then implies that $h_0(u,v)$ and thereby $h(u,v)$ is bounded from below when $u, v \ge 1$, which proves that (3.12) holds.

This gives

$$e^{c|\alpha|^{\gamma/2s}} \le C_0 e^{j^j-1} e^{2|\alpha|^{1/2s}/j} = C_0 m(\alpha), \quad \alpha \in I_j,$$

and (3.11) follows, which proves the first part of (1).

It remains to prove that $\mathscr{H}_2$ exists with the asserted properties. The fact that $\mathscr{B}_2$ is continuously embedded in $\mathcal{S}_s'$ implies that for every $j \ge 1$, there is a constant $C_j \ge 1$ such that

$$\sum_\alpha |c_\alpha|^2 C_j^{-1} e^{-|\alpha|^{1/2s}/j} \le \|f\|_{\mathscr{B}_2}^2.$$

Let

$$m(\alpha) = \sum_{j\ge 1} j^{-2} e^{-j^j} C_j^{-1} e^{-|\alpha|^{1/2s}/j},$$

and let $\mathscr{H}_2$ be the set of all $f \in \mathcal{S}_s'$ such that

$$\|f\|_{\mathscr{H}_2} \equiv \Big(\sum_\alpha |c_\alpha|^2 m(\alpha) \Big)^{1/2}$$

is finite.

By the definition it follows that $\|f\|_{\mathscr{H}_2} \lesssim \|f\|_{\mathscr{B}_2}$ when $f \in \mathscr{B}_2$, giving that $\mathscr{B}_2$ is continuously embedded in $\mathscr{H}_2$. Furthermore, if $c > 0$, then it follows that

$$\lim_{|\alpha|\to\infty} \frac{e^{-c|\alpha|^{1/2s}}}{m(\alpha)} = 0,$$

which implies that $\mathscr{H}_2$ is a Hilbert space.

It remains to prove that $\mathcal{S}_{s/\gamma}' \hookrightarrow \mathscr{H}_2$ when $0 < \gamma < 1$, which follows if we prove that

$$m(\alpha) \lesssim e^{-c|\alpha|^{\gamma/2s}}, \tag{3.13}$$

for every $c > 0$. By the same arguments as in the last part of the proof we have

$$e^{-j^j} e^{-|\alpha|^{1/2s}/j} \le C_0 e^{-c|\alpha|^{\gamma/2s}},$$

where C_0 neither depends on j nor on α. This gives

$$m(\alpha) = \sum_{j\ge 1} j^{-2} e^{-j^j} e^{-|\alpha|^{1/2s}/j} \lesssim e^{-c|\alpha|^{\gamma/2s}} \sum_{j\ge 1} \frac{1}{j^2} \asymp e^{-c|\alpha|^{\gamma/2s}},$$

and (3.13) follows. The proof is complete. □

The end of the proof of Proposition 3.7. We only prove (1). The assertion (2) follows by similar arguments and is left for the reader.

Let $\mathscr{B}$ be a GS-tempered quasi-Banach space. By Proposition 3.8 there are GS-tempered Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $\mathscr{H}_1 \hookrightarrow \mathscr{B} \hookrightarrow \mathscr{H}_2$, and by the first part of the proof it follows that $M^2_{(\omega_t)} \hookrightarrow \mathscr{H}_j \hookrightarrow M^2_{(1/\omega_t)}$ for every $t > 0$, $j = 1, 2$. The result now follows by combining these inclusions. The proof is complete. □

4. Schatten–von Neumann classes and pseudo-differential operators

In this section we discuss Schatten–von Neumann classes of pseudo-differential operators from a Hilbert space $\mathscr{H}_1$ to another Hilbert space $\mathscr{H}_2$, or more generally, from a (quasi-)Banach space $\mathscr{B}_1$ to another (quasi-)Banach space $\mathscr{B}_2$. Schatten–von Neumann classes were introduced by R. Schatten in [36] in the case when $\mathscr{H}_1 = \mathscr{H}_2$ are Hilbert spaces. (See also [39].) The theory was thereafter extended in [3, 33, 37, 53] to the case when $\mathscr{H}_1$ is not necessarily equal to $\mathscr{H}_2$, and in [3, 33, 39], the theory was extended in such way that it includes linear operators from a Banach space $\mathscr{B}_1$ to another Banach space $\mathscr{B}_2$. Furthermore, the definitions and some of the results in [3, 33, 39] can easily be modified to permit $\mathscr{B}_1$ and $\mathscr{B}_2$ to be arbitrary quasi-Banach spaces.

We will mainly follow the organization in the second part of Section 4 in [51], and we remark that there are several similarities between the proofs in this section, and the proofs of analogous results in Section 4 in [51].

We start by recalling the definition of Schatten–von Neumann operators in the (quasi-)Banach space case. We remark however that this general setting is not needed for the main results in the present and next sections (e.g., Theorem 4.8 below). For the reader who is not interested in this general approach may therefore assume that the operators act on Hilbert spaces.

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be (quasi-)Banach spaces, and let T be a linear map from $\mathscr{B}_1$ to $\mathscr{B}_2$. For every integer $j \ge 1$, the *singular number* of T of order j is given by

$$\sigma_j(T) = \sigma_j(\mathscr{B}_1, \mathscr{B}_2, T) \equiv \inf \|T - T_0\|_{\mathscr{B}_1 \to \mathscr{B}_2},$$

where the infimum is taken over all linear operators T_0 from $\mathscr{B}_1$ to $\mathscr{B}_2$ with rank at most $j-1$. Therefore, $\sigma_1(T)$ equals $\|T\|_{\mathscr{B}_1 \to \mathscr{B}_2}$, and $\sigma_j(T)$ is non-negative which decreases with j.

For any $p \in (0, \infty]$ we set

$$\|T\|_{\mathscr{I}_p} = \|T\|_{\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)} \equiv \|\{\sigma_j(\mathscr{B}_1, \mathscr{B}_2, T)\}_{j=1}^{\infty}\|_{l^p}$$

(which might attain $+\infty$). The operator T is called a *Schatten–von Neumann operator* of order p from $\mathscr{B}_1$ to $\mathscr{B}_2$, if $\|T\|_{\mathscr{I}_p}$ is finite, i.e., $\{\sigma_j(\mathscr{B}_1, \mathscr{B}_2, T)\}_{j=1}^{\infty}$ should belong to l^p. The set of all Schatten–von Neumann operators of order p from $\mathscr{B}_1$ to $\mathscr{B}_2$ is denoted by $\mathscr{I}_p = \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$. We note that $\mathscr{I}_\infty(\mathscr{B}_1, \mathscr{B}_2)$ agrees with $\mathcal{B}(\mathscr{B}_1, \mathscr{B}_2)$, the set of linear and bounded operators from $\mathscr{B}_1$ to $\mathscr{B}_2$, and if $p < \infty$, then $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ is contained in $\mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$, the set of linear and compact operators from $\mathscr{B}_1$ to $\mathscr{B}_2$. The spaces $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ for $p \in (0, \infty]$ and $\mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$ are quasi-Banach spaces which are Banach spaces when $\mathscr{B}_1$, $\mathscr{B}_2$ are Banach spaces and $p \ge 1$. Furthermore, $\mathscr{I}_2(\mathscr{B}_1, \mathscr{B}_2)$ is a Hilbert space when $\mathscr{B}_1$ and $\mathscr{B}_2$ are Hilbert spaces. If $\mathscr{B}_1 = \mathscr{B}_2$, then the shorter notation $\mathscr{I}_p(\mathscr{B}_1)$ is used instead of $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$, and similarly for $\mathcal{B}(\mathscr{B}_1, \mathscr{B}_2)$ and $\mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$.

Now let $\mathscr{B}_3$ be an other Banach space (or quasi-Banach space) and let T_k and $T_{0,k}$ be linear and continuous operators from $\mathscr{B}_k$ to $\mathscr{B}_{k+1}$ such that the rank of $T_{0,k}$ is at most j_k for $k = 1, 2$. If T_0 is defined by

$$T_0 = T_{0,2} \circ T_1 + T_2 \circ T_{0,1} - T_{0,2} \circ T_{0,1} = T_{0,2} \circ T_1 + (T_2 - T_{0,2}) \circ T_{0,1},$$

then

$$T_2 \circ T_1 - T_0 = (T_2 - T_{0,2}) \circ (T_1 - T_{0,1}),$$

and it follows that the rank of T_0 is at most $j_1 + j_2$. Hence

$$\sigma_{j_1+j_2+1}(T_2 \circ T_1) \le \|T_2 \circ T_1 - T_0\| \le \|T_1 - T_{0,1}\| \cdot \|T_2 - T_{0,2}\|,$$

and by taking the infimum on the right-hand side of all possible $T_{0,1}$ and $T_{0,2}$ we get

$$\sigma_{j_1+j_2+1}(T_2 \circ T_1) \le \sigma_{j_1+1}(T_1)\sigma_{j_2+1}(T_2), \qquad j_1, j_2 \ge 0. \tag{4.1}$$

If $\mathscr{B}_j = \mathscr{H}_j$, $j = 1, 2, 3$, are Hilbert spaces, then (4.1) can be improved into

$$\sigma_{j+1}(T_2 \circ T_1) \le \sigma_{j+1}(T_1)\sigma_{j+1}(T_2), \quad j \ge 0. \tag{4.1$'$}$$

(Cf. [33, 39].)

In [33, 39], (4.1) is used to prove that if $p_1, p_2, r \in (0, \infty]$ satisfy the Hölder condition $1/p_1 + 1/p_2 = 1/r$, and $T_k \in \mathscr{I}_{p_k}(\mathscr{B}_k, \mathscr{B}_{k+1})$, then $T_2 \circ T_1 \in \mathscr{I}_r(\mathscr{B}_1, \mathscr{B}_3)$, and

$$\|T_2 \circ T_1\|_{\mathscr{I}_r(\mathscr{B}_1,\mathscr{B}_3)} \le C_r \|T_1\|_{\mathscr{I}_{p_1}(\mathscr{B}_1,\mathscr{B}_2)} \|T_2\|_{\mathscr{I}_{p_2}(\mathscr{B}_2,\mathscr{B}_3)}. \tag{4.2}$$

Here $C_r = 1$ when $\mathscr{B}_j$, $j = 1, 2, 3$ are Hilbert spaces, and $C_r = 2^{1/r}$ otherwise. In order to be self-contained we here give a proof of (4.2).

Let $T = T_2 \circ T_1$. Since $\sigma_{2j+2}(T) \le \sigma_{2j+1}(T)$, it follows by letting $j_1 = j_2 = j$ in (4.1) that

$$\begin{aligned}
\|T\|_{\mathscr{I}_r} &= \Big(\sum_{k \ge 0} \sigma_{k+1}(T)^r \Big)^{1/r} \le 2^{1/r} \Big(\sum_{j \ge 0} \sigma_{2j+1}(T)^r \Big)^{1/r} \\
&\le 2^{1/r} \Big(\sum_{j \ge 0} \sigma_{j+1}(T_1)^r \sigma_{j+1}(T_2)^r \Big)^{1/r} \\
&\le 2^{1/r} \Big(\sum_{j \ge 0} \sigma_{j+1}(T_1)^{p_1} \Big)^{1/p_1} \Big(\sum_{j \ge 0} \sigma_{j+1}(T_2)^{p_2} \Big)^{1/p_2} = 2^{1/r} \|T_1\|_{\mathscr{I}_{p_1}} \|T_2\|_{\mathscr{I}_{p_2}}.
\end{aligned}$$

This gives (4.2).

If $\mathscr{B}_j$, $j = 1, 2, 3$ are Hilbert spaces, then the same arguments, using (4.1)$'$ instead of (4.1), shows that (4.2) holds for $C_r = 1$. (Cf. [33] or Chapters 2 and 3 in [39].)

If $\mathscr{B}_1$ and $\mathscr{B}_2$ are Banach spaces, then we note that ranks and norms for the operators are invariant when passing from the operators to their adjoints. This implies that T belongs to $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$, if and only if the adjoint T^* of T belongs to $\mathscr{I}_p(\mathscr{B}_2', \mathscr{B}_1')$, and

$$\|T\|_{\mathscr{I}_p(\mathscr{B}_1,\mathscr{B}_2)} = \|T^*\|_{\mathscr{I}_p(\mathscr{B}_2',\mathscr{B}_1')}.$$

We recall that if $\mathscr{B}_1 = \mathscr{H}_1$ and $\mathscr{B}_2 = \mathscr{H}_2$ are Hilbert spaces, then there is an alternative way to compute the $\mathscr{I}_p$ norms. More precisely, let $\mathrm{ON}(\mathscr{H}_k)$, $k = 1, 2$, denote the family of orthonormal sequences in $\mathscr{H}_k$. If $T : \mathscr{H}_1 \to \mathscr{H}_2$ is linear, and $p \in (0, \infty]$, then it follows that

$$\|T\|_{\mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)} = \sup \Big(\sum |(Tf_j, g_j)_{\mathscr{H}_2}|^p \Big)^{1/p}$$

(with obvious modifications when $p = \infty$), where the supremum is taken over all $\{f_j\} \in \mathrm{ON}(\mathscr{H}_1)$ and $\{g_j\} \in \mathrm{ON}(\mathscr{H}_2)$.

Let $\{e_j\}$ be an orthonormal basis in $\mathscr{H}_1$, and let $S \in \mathscr{I}_1(\mathscr{H}_1)$. Then the trace of S is defined as

$$\operatorname{tr}_{\mathscr{H}_1} S = \sum (Se_j, e_j)_{\mathscr{H}_1},$$

and we recall that this is independent of the choice of the orthonormal basis $\{e_j\}$. For each pairs of operators $T_1, T_2 \in \mathscr{I}_\infty(\mathscr{H}_1, \mathscr{H}_2)$ such that $T_2^* \circ T_1 \in \mathscr{I}_1(\mathscr{H}_1)$, the sesqui-linear form

$$(T_1, T_2) = (T_1, T_2)_{\mathscr{H}_1, \mathscr{H}_2} \equiv \operatorname{tr}_{\mathscr{H}_1}(T_2^* \circ T_1)$$

of T_1 and T_2 is well defined. We refer to [3, 33, 39, 53] for more facts about Schatten–von Neumann classes.

Next we define symbol classes whose corresponding pseudo-differential operators belongs to $\mathscr{I}_p$ for some $p \in (0, \infty]$. Therefore, let $\mathscr{B}_1, \mathscr{B}_2 \hookrightarrow \mathcal{S}_{1/2}'(\mathbf{R}^d)$ be GS- or B-tempered (quasi-)Banach spaces, $t \in \mathbf{R}$ be fixed and let $p \in (0,, \infty]$. Then we let $s_p^A(\mathscr{B}_1, \mathscr{B}_2)$ and $s_{t,p}(\mathscr{B}_1, \mathscr{B}_2)$ be the sets of all $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ such that $Aa \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ and $\operatorname{Op}_t(a) \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ respectively. We also let $s_\sharp^A(\mathscr{B}_1, \mathscr{B}_2)$ and $s_{t,\sharp}(\mathscr{B}_1, \mathscr{B}_2)$ be the sets of all $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ such that $Aa \in \mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$ and $\operatorname{Op}_t(a) \in \mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$ respectively. These spaces are equipped by the (quasi-)norms

$$\|a\|_{s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)} \equiv \|\operatorname{Op}_t(a)\|_{\mathscr{I}_p(\mathscr{B}_1,\mathscr{B}_2)}, \qquad \|a\|_{s_p^A(\mathscr{B}_1,\mathscr{B}_2)} \equiv \|Aa\|_{\mathscr{I}_p(\mathscr{B}_1,\mathscr{B}_2)},$$
$$\|a\|_{s_{t,\sharp}(\mathscr{B}_1,\mathscr{B}_2)} \equiv \|a\|_{s_{t,\infty}(\mathscr{B}_1,\mathscr{B}_2)}, \qquad \|a\|_{s_\sharp^A(\mathscr{B}_1,\mathscr{B}_2)} \equiv \|a\|_{s_\infty^A(\mathscr{B}_1,\mathscr{B}_2)}.$$

Since the mappings $a \mapsto Aa$ and $a \mapsto \operatorname{Op}_t(a)$ are bijections from $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ to the set of linear and continuous operators from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$, it follows that $a \mapsto Aa$ and $a \mapsto \operatorname{Op}_t(a)$ restrict to isometric bijections from $s_p^A(\mathscr{B}_1, \mathscr{B}_2)$ and $s_{t,p}(\mathscr{B}_1, \mathscr{B}_2)$ respectively to $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$. Consequently, the properties for $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ carry over to $s_p^A(\mathscr{B}_1, \mathscr{B}_2)$ and $s_{t,p}(\mathscr{B}_1, \mathscr{B}_2)$. In particular, if $\mathscr{B}_1 = \mathscr{H}_1$ and $\mathscr{B}_2 = \mathscr{H}_2$ are Hilbert spaces, then the elements in $s_1^A(\mathscr{H}_1, \mathscr{H}_2)$ of finite rank (i.e., elements of the form $a \in s_1^A(\mathscr{H}_1, \mathscr{H}_2)$ such that Aa is a finite rank operator), are dense in $s_\sharp^A(\mathscr{H}_1, \mathscr{H}_2)$ and in $s_p^A(\mathscr{H}_1, \mathscr{H}_2)$ when $p < \infty$. Similar facts hold for $s_{t,\sharp}(\mathscr{H}_1, \mathscr{H}_2)$ and $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$. Since the Weyl quantization is particularly important in our considerations we also set

$$s_p^w = s_{t,p} \quad \text{and} \quad s_\sharp^w = s_{t,\sharp}, \quad \text{when} \quad t = 1/2.$$

If $\omega_1, \omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$, then we use the shorter notation $s_p^A(\omega_1, \omega_2)$ instead of $s_p^A(M^2_{(\omega_1)}, M^2_{(\omega_2)})$. Furthermore we set $s_p^A(\omega_1, \omega_2) = s_p^A(\mathbf{R}^{2d})$ when $\omega_1 = \omega_2 = 1$. In the same way the notations for $s_\sharp^A$, $s_{t,p}$, s_p^w, $s_{t,\sharp}$ and $s_\sharp^w$ are simplified.

Remark 4.1. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces. Then, except for the Hilbert–Schmidt case ($p = 2$), it is in general a hard task to find simple characterizations for $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$. Important questions therefore concern of finding convenient embedding properties between Schatten–von Neumann classes and well-known function and distribution spaces. We refer to Remark 4.5 in [51], for examples on such embeddings.

Remark 4.2. Let $t, t_1, t_2 \in \mathbf{R}$, $p \in [1, \infty]$, $\mathscr{B}_1, \mathscr{B}_2$ be GS- or B-tempered quasi-Banach spaces on $\mathbf{R}^d$ and let $a, b \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$. Then it follows by Fourier's inversion formula that the map $e^{it\langle D_x, D_\xi \rangle}$ is a homeomorphism on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ which extends uniquely to a homeomorphism on $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$. Furthermore, by (1.10) it follows that $e^{i(t_2-t_1)\langle D_x,D_\xi\rangle}$ restricts to an isometric bijection from $s_{t_1,p}(\mathscr{B}_1,\mathscr{B}_2)$ to $s_{t_2,p}(\mathscr{B}_1,\mathscr{B}_2)$.

The following proposition links $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$, $s_p^A(\mathscr{B}_1,\mathscr{B}_2)$ and other similar spaces to each others. Here $a^\tau(x,\xi)=\overline{a(x,-\xi)}$ is the "torsion" of $a\in\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$.

Proposition 4.3. *Let $t\in\mathbf{R}$, $\mathscr{B}_1,\mathscr{B}_2$ be GS- or B-tempered quasi-Banach spaces in $\mathbf{R}^d$, $a\in\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$, and let $p\in(0,\infty]$. Then $s_p^w(\mathscr{B}_1,\check{\mathscr{B}}_2)=s_p^A(\mathscr{B}_1,\mathscr{B}_2)$, and the following conditions are equivalent:*

1. $a\in s_p^w(\mathscr{B}_1,\mathscr{B}_2)$;
2. $\mathscr{F}_\sigma a\in s_p^w(\mathscr{B}_1,\check{\mathscr{B}}_2)$;
3. $\overline{a}\in s_p^w(\mathscr{B}_2',\mathscr{B}_1')$;
4. $a^\tau\in s_p^A(\mathscr{B}_1^\tau,\mathscr{B}_2^\tau)$;
5. $\check{a}\in s_p^w(\check{\mathscr{B}}_1,\check{\mathscr{B}}_2)$;
6. $\widetilde{a}\in s_p^w(\check{\mathscr{B}}_2',\check{\mathscr{B}}_1')$;
7. $e^{i(t-1/2)\langle D_\xi,D_x\rangle}a\in s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$.

Proof. Let $a_1=\mathscr{F}_\sigma a$, $a_2=\overline{a}$, $a_3=a^\tau$, $a_4=\check{a}$ and $a_5=\widetilde{a}$. Then the equivalences follow immediately from Remark 4.2 and the equalities

$$
\begin{aligned}
(\operatorname{Op}^w(a)f,g)&=(\operatorname{Op}^w(a_1)f,\check{g})=(f,\operatorname{Op}^w(a_2)g)\\
&=\overline{(\operatorname{Op}^w(a_3)(x,D)\overline{f},\overline{g})}=(\operatorname{Op}^w(a_4)\check{f},\check{g})=(\check{f},\operatorname{Op}^w(a_5)\check{g}),
\end{aligned}
$$

when $a\in\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ and $f,g\in\mathcal{S}_{1/2}(\mathbf{R}^d)$. (Cf. [51, Proposition 4.7].) The proof is complete. □

In Remarks 4.4 and 4.5 below we list some properties for $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$ and $s_p^A(\mathscr{H}_1,\mathscr{H}_2)$. These properties follow from well-known Schatten–von Neumann results in [3, 39, 53], in combination with (1.15), (1.21), (4.2), and the fact that the mappings $a\mapsto\operatorname{Op}_t(a)$ and $a\mapsto Aa$ are isometric bijections from $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$ and $s_p^A(\mathscr{B}_1,\mathscr{B}_2)$ respectively to $\mathscr{I}_p(\mathscr{B}_1,\mathscr{B}_2)$.

Remark 4.4. Let $p,p_j,q,r\in(0,\infty]$, $t\in\mathbf{R}$, $\mathscr{B}_j$ be GS- or B-tempered quasi-Banach spaces on $\mathbf{R}^d$, and let $\mathscr{H}_j$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$, $j=1,\dots,4$. Also let $C_r=1$ when $\mathscr{B}_1,\dots,\mathscr{B}_4$ are Hilbert spaces, and $C_r=2^{1/r}$ otherwise. Then the following is true:

1. the sets $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$ and $s_{t,\sharp}(\mathscr{B}_1,\mathscr{B}_2)$, are quasi-Banach space which increases with the parameter p. If in addition $r\le p<\infty$ and $p_1\le p_2$, then $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)\hookrightarrow s_{t,\sharp}(\mathscr{B}_1,\mathscr{B}_2)$, $s_{t,r}(\mathscr{B}_1,\mathscr{B}_2)$ is dense in $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$ and in $s_{t,\sharp}(\mathscr{B}_1,\mathscr{B}_2)$, and

$$
\|a\|_{s_{t,p_2}(\mathscr{B}_1,\mathscr{B}_2)}\le\|a\|_{s_{t,p_1}(\mathscr{B}_1,\mathscr{B}_2)},\quad a\in s_{t,\infty}(\mathscr{B}_1,\mathscr{B}_2). \tag{4.3}
$$

Moreover, if in addition $p \ge 1$ and $\mathscr{B}_j$, $j = 1, 2$, are Banach spaces, then $s_{t,p}(\mathscr{B}_1, \mathscr{B}_2)$ and $s_{t,\sharp}(\mathscr{B}_1, \mathscr{B}_2)$ are Banach spaces;

(2) if $\mathscr{B}_j = \mathscr{H}_j$, $j = 1, 2$, then equality is attained in (4.3), if and only if a is a rank one element, and then $\|a\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)} = (2\pi)^{-d/2}\|f_1\|_{\mathscr{H}_1}\|f_2\|_{\mathscr{H}_2}$, when a is given by (1.12);

(3) if $1/p_1 + 1/p_2 = 1/r$, $a_1 \in s_{t,p_1}(\mathscr{B}_1, \mathscr{B}_2)$ and $a_2 \in s_{t,p_2}(\mathscr{B}_2, \mathscr{B}_3)$, then $a_2 \#_t a_1 \in s_{t,r}(\mathscr{B}_1, \mathscr{B}_3)$, and

$$\|a_2 \#_t a_1\|_{s_{t,r}(\mathscr{B}_1,\mathscr{B}_3)} \le C_r \|a_1\|_{s_{t,p_1}(\mathscr{B}_1,\mathscr{B}_2)} \|a_2\|_{s_{t,p_2}(\mathscr{B}_2,\mathscr{B}_3)}. \tag{4.4}$$

On the other hand, for any $a \in s_{t,r}(\mathscr{H}_1, \mathscr{H}_3)$, there are elements $a_1 \in s_{t,p_1}(\mathscr{H}_1, \mathscr{H}_2)$ and $a_2 \in s_{t,p_2}(\mathscr{H}_2, \mathscr{H}_3)$ such that $a = a_2 \#_t a_1$ and equality holds in (4.4);

(4) if $\mathscr{B}_1 \hookrightarrow \mathscr{B}_2$ with dense embeddings and $\mathscr{B}_3 \hookrightarrow \mathscr{B}_4$, then $s_{t,p}(\mathscr{B}_2, \mathscr{B}_3) \hookrightarrow s_{t,p}(\mathscr{B}_1, \mathscr{B}_4)$.

Similar facts hold when the $s_{t,p}$ spaces and the product $\#_t$ are replaced by s_p^A spaces and $*_\sigma$, respectively.

In the next remark we make some further conclusions on dual forms for $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ and $s_p^A(\mathscr{H}_1, \mathscr{H}_2)$ when $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces. Here the forms $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}$ and $(\,\cdot\,,\,\cdot\,)_{s_2^A(\mathscr{H}_1,\mathscr{H}_2)}$ are defined by the formulas

$$(a,b)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)} = (\operatorname{Op}_t(a), \operatorname{Op}_t(b))_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)}, \quad a, b \in s_{t,2}(\mathscr{H}_1, \mathscr{H}_2)$$

and

$$(a,b)_{s_2^A(\mathscr{H}_1,\mathscr{H}_2)} = (Aa, Ab)_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)}, \qquad a, b \in s_2^A(\mathscr{H}_1, \mathscr{H}_2).$$

We also recall that $p' \in [1,\infty]$ is the conjugate exponent for $p \in [1,\infty]$, i.e., $1/p + 1/p' = 1$. Finally, the set l_0^∞ consists of all sequences in l^∞ which tends to zero at infinity, and l_0^1 consists of all sequences $\{\lambda_j\}_{j\in I}$ such that $\lambda_j = 0$ except for finite numbers of $j \in I$.

Remark 4.5. Let $p, p_j \in [1,\infty]$ for $1 \le j \le 2$, $t \in \mathbf{R}$, and let $\mathscr{H}_1, \mathscr{H}_2$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$. Then the following is true:

(1) the form $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}$ on $s_{t,1}(\mathscr{H}_1, \mathscr{H}_2)$ extends uniquely to a sesquilinear and continuous form on $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2) \times s_{t,p'}(\mathscr{H}_1, \mathscr{H}_2)$, and for every $a_1 \in s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ and $a_2 \in s_{t,p'}(\mathscr{H}_1, \mathscr{H}_2)$, it holds

$$(a_1, a_2)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)} = \overline{(a_2, a_1)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}},$$
$$|(a_1, a_2)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}| \le \|a_1\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)} \|a_2\|_{s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)} \quad \text{and}$$
$$\|a_1\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)} = \sup |(a_1, b)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}|,$$

where the supremum is taken over all $b \in s_{t,p'}(\mathscr{H}_1, \mathscr{H}_2)$ such that

$$\|b\|_{s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)} \le 1.$$

If in addition $p < \infty$, then the dual space of $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ can be identified with $s_{t,p'}(\mathscr{H}_1, \mathscr{H}_2)$ through this form;

(2) if $a \in s_{t,\sharp}(\mathscr{H}_1, \mathscr{H}_2)$, then

$$\operatorname{Op}_t(a)f = \sum_{j=1}^{\infty} \lambda_j (f, f_j)_{\mathscr{H}_1} g_j, \tag{4.5}$$

holds for some $\{f_j\}_{j=1}^{\infty} \in \mathrm{ON}(\mathscr{H}_1)$, $\{g_j\}_{j=1}^{\infty} \in \mathrm{ON}(\mathscr{H}_2)$ and non-negative decreasing sequence $\lambda = \{\lambda_j\}_{j=1}^{\infty} \in l_0^{\infty}$, where the operator on the right-hand side of (4.5) convergences with respect to the operator norm. Moreover, $a \in s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$, if and only if $\lambda \in l^p$, and then

$$\|a\|_{s_{t,p}} = \|\lambda\|_{l^p}$$

and the operator on the right-hand side of (4.5) converges with respect to the norm $\| \cdot \|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)}$;

(3) if $0 \le \theta \le 1$ is such that $1/p = (1-\theta)/p_1 + \theta/p_2$, then the (complex) interpolation formula

$$(s_{t,p_1}(\mathscr{H}_1, \mathscr{H}_2), s_{t,p_2}(\mathscr{H}_1, \mathscr{H}_2))_{[\theta]} = s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$$

holds with equality in norms.

Similar facts hold when the $s_{t,p}$ spaces are replaced by s_p^A spaces.

In the sequel we assume that $\mathscr{B}_j = \mathscr{H}_j$, $j \ge 0$, are Hilbert spaces. A problem with the form $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}$ in Remark 4.5 is the somewhat complicated structure. In the following we show that there is a canonical way to replace this form with $(\,\cdot\,,\,\cdot\,)_{L^2}$. We start with the following result concerning polar decomposition of compact operators.

Proposition 4.6. *Let $t \in \mathbf{R}$, $p \in [1,\infty]$, $\mathscr{H}_1$ and $\mathscr{H}_2$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$ and let $a \in s_{t,\sharp}(\mathscr{H}_1, \mathscr{H}_2)$ ($a \in s_\sharp^A(\mathscr{H}_1, \mathscr{H}_2)$). Then*

$$a \equiv \sum_{j\in I} \lambda_j W^t_{g_j,\varphi_j} \qquad \Big(a \equiv \sum_{j\in I} \lambda_j W_{\check{g}_j,\varphi_j} \Big) \tag{4.6}$$

(*with norm convergence*) *for some orthonormal sequences $\{\varphi_j\}_{j\in I}$ in $\mathscr{H}_1'$ and $\{g_j\}_{j\in I}$ in $\mathscr{H}_2$, and a sequence $\{\lambda_j\}_{j\in I} \in l_0^{\infty}$ of non-negative real numbers which decreases to zero at infinity. Furthermore, $a \in s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ ($a \in s_p^A(\mathscr{H}_1, \mathscr{H}_2)$), if and only if $\{\lambda_j\}_{j\in I} \in l^p$, and*

$$\|a\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)} = (2\pi)^{-d/2} \|\{\lambda_j\}_{j\in I}\|_{l^p} \quad (\|a\|_{s_p^A(\mathscr{H}_1,\mathscr{H}_2)} = \|\{\lambda_j\}_{j\in I}\|_{l^p}).$$

Proof. By Remark 4.5 (2) it follows that if $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, then

$$\operatorname{Op}_t(a)f(x) = \sum_{j\in I} \lambda_j (f, f_j)_{\mathscr{H}_1} g_j \tag{4.7}$$

for some orthonormal sequences $\{f_j\}$ in $\mathscr{H}_1$ and $\{g_j\}$ in $\mathscr{H}_2$, and a sequence $\{\lambda_j\} \in l_0^{\infty}$ of non-negative real numbers which decreases to zero at infinity. Now let $\{\varphi_j\}_{j\in I}$ be an orthonormal sequence in $\mathscr{H}_1'$ such that $(\varphi_j, f_k)_{L^2} = \delta_{j,k}$. Then $(f, f_j)_{\mathscr{H}_1} = (f, \varphi_j)_{L^2}$, and the result follows from (4.7), and the fact that

$$\operatorname{Op}_t(W^t_{g_j,\varphi_j})f = (2\pi)^{-d/2}(f, \varphi_j)_{L^2} g_j = (2\pi)^{-d/2}(f, f_j)_{\mathscr{H}_1} g_j.$$

The proof is complete. □

We have now the following.

Proposition 4.7. *Let $p \in [1,\infty)$, and let $\mathscr{H}_1$ and $\mathscr{H}_2$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$. Then the following is true:*

(1) *$\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is dense in $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$, $s^A_p(\mathscr{H}_1, \mathscr{H}_2)$, $s_{t,\sharp}(\mathscr{H}_1, \mathscr{H}_2)$ and $s^A_\sharp(\mathscr{H}_1, \mathscr{H}_2)$;*

(2) *$\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is dense in $s_{t,\infty}(\mathscr{H}_1, \mathscr{H}_2)$ and $s^A_\infty(\mathscr{H}_1, \mathscr{H}_2)$ with respect to the weak* topology.*

Proof. By Proposition 4.6 it follows that any element in $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$, $s^A_p(\mathscr{H}_1, \mathscr{H}_2)$, $s_{t\sharp}(\mathscr{H}_1, \mathscr{H}_2)$ or in $s^A_\sharp(\mathscr{H}_1, \mathscr{H}_2)$ can be approximated in norm by finite sums of the forms in (4.6). The assertion (1) now follows from the facts that any φ_j and g_j can be approximated in norms by elements in $\mathcal{S}_{1/2}(\mathbf{R}^d)$, and that the map $(\varphi, g) \mapsto W^t_{g,\varphi}$ is continuous from $\mathcal{S}_{1/2}(\mathbf{R}^d) \times \mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$.

The assertion (2) now follows from (1) and the fact that $s_{t,1}$ is weakly dense in $s_{t,\infty}$, since $\mathscr{I}_1$ is weakly dense in $\mathscr{I}_\infty$. □

Next we prove that the duals for $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ and $s^A_p(\mathscr{H}_1, \mathscr{H}_2)$ can be identified with $s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')$ and $s^A_{p'}(\mathscr{H}_1', \mathscr{H}_2')$ respectively through the form $(\,\cdot\,,\,\cdot\,)_{L^2}$.

Theorem 4.8. *Let $t \in \mathbf{R}$, $p \in [1,\infty)$ and let $\mathscr{H}_1, \mathscr{H}_2$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$. Then the L^2 form on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ extends uniquely to a duality between $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ and $s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')$, and the dual of $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ can be identified with $s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')$ through this form. Moreover, if $\ell \in s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)^*$ and $a \in s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')$ are such that $\overline{\ell(b)} = (a,b)_{L^2}$ when $b \in s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$, then*

$$\|\ell\| = \|a\|_{s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')}.$$

The same is true if the $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ spaces are replaced by $s^A_p(\mathscr{H}_1, \mathscr{H}_2)$ spaces.

Proof. We only prove the assertion in the case $t = 1/2$. The general case follows by similar arguments and is left for the reader. Let $\ell \in s^w_p(\mathscr{H}_1, \mathscr{H}_2)^*$. Since the map $b \mapsto \operatorname{Op}^w(b)$ is an isometric bijection from $s^w_p(\mathscr{H}_1, \mathscr{H}_2)$ to $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$, it follows from Remark 4.5 (1) that for some $S \in \mathscr{I}_{p'}(\mathscr{H}_1, \mathscr{H}_2)$ and each orthonormal basis $\{f_j\} \in \operatorname{ON}(\mathscr{H}_1)$ we have

$$\begin{aligned} &\ell(b) = \operatorname{tr}_{\mathscr{H}_1}(S^* \circ \operatorname{Op}^w(b)) = \sum (\operatorname{Op}^w(b) f_j, S f_j)_{\mathscr{H}_2} \quad \text{and} \\ &\|\ell\| = \|S\|_{\mathscr{I}_{p'}(\mathscr{H}_1, \mathscr{H}_2)}, \end{aligned} \tag{4.8}$$

when $b \in s^w_p(\mathscr{H}_1, \mathscr{H}_2)$.

Now let $b \in s_p^w(\mathscr{H}_1, \mathscr{H}_2)$ be an arbitrary finite rank element. Then

$$b = \sum \lambda_j W_{g_j,\varphi_j} \quad \text{and} \quad \|b\|_{s_p^w(\mathscr{H}_1,\mathscr{H}_2)} = (2\pi)^{-d/2}\|\{\lambda_j\}\|_{l^p},$$

for some orthonormal bases $\{\varphi_j\} \in \mathrm{ON}(\mathscr{H}_1')$ and $\{g_j\} \in \mathrm{ON}(\mathscr{H}_2)$, and some sequence $\{\lambda_j\} \in l_0^1$. We also let $\{f_j\} \in \mathrm{ON}(\mathscr{H}_1)$ be the dual basis of $\{\varphi_j\}$ and a the Weyl symbol of the operator $T_{\mathscr{H}_2} \circ S \circ T_{\mathscr{H}_1'}$. Then $a \in s_{p'}^w(\mathscr{H}_1, \mathscr{H}_2)$ and $\|a\|_{s_{p'}^w(\mathscr{H}_1,\mathscr{H}_2)} = \|\ell\|$. By straightforward computations we also get

$$\begin{aligned}\ell(b) &= \mathrm{tr}_{\mathscr{H}_1}(S^* \circ \mathrm{Op}^w(b)) = \sum (\mathrm{Op}^w(b)f_j, Sf_j)_{\mathscr{H}_2}\\ &= (2\pi)^{-d/2}\sum \lambda_j (g_j, Sf_j)_{\mathscr{H}_2} = (2\pi)^{-d/2}\sum \lambda_j (g_j, \mathrm{Op}^w(a)\varphi_j)_{L^2(\mathbf{R}^d)}\\ &= (2\pi)^{-d}\sum \lambda_j (W_{g_j,\varphi_j}, a)_{L^2(\mathbf{R}^{2d})} = (2\pi)^{-d}(b,a)_{L^2(\mathbf{R}^{2d})}.\end{aligned}$$

Hence $\ell(b) = (2\pi)^{-d}(b,a)_{L^2(\mathbf{R}^{2d})}$. The result now follows from these identities and the fact that the set of finite rank elements are dense in $s_p^w(\mathscr{H}_1, \mathscr{H}_2)$. The proof is complete. □

An interesting question is wether Theorem 4.8 still holds after the Hilbert spaces $\mathscr{H}_k$ and $\mathscr{H}_k'$ have been replaced by appropriate Banach spaces.

We finish the section by a side result on bases and Hilbert–Schmidt operators on GS- or B-tempered Hilbert spaces.

Proposition 4.9. *Let $\mathscr{H}_j$ be GS- or B-tempered Hilbert space on $\mathbf{R}^{d_j}$ for $j = 1, 2$, and let T be a linear and continuous map from $\mathscr{H}_1$ to $\mathscr{H}_2$. Also let $\mathscr{H} = \mathscr{H}_2 \otimes (\mathscr{H}_1')^\tau$ (Hilbert tensor product). If K_T is the kernel of T, then $T \in \mathscr{I}_2(\mathscr{H}_1, \mathscr{H}_2)$, if and only if $K_T \in \mathscr{H}$, and*

$$\|T\|_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)} = \|K_T\|_{\mathscr{H}}. \tag{4.9}$$

Proof. First assume that $T \in \mathscr{I}_2(\mathscr{H}_1, \mathscr{H}_2)$, and let $\{e_{j,k}\}_{k=1}^\infty$ be an orthonormal basis for $\mathscr{H}_j$ and set $\varepsilon_{j,k} = T_{\mathscr{H}_j} e_{j,k}$, $j = 1, 2$. Then $\{\varepsilon_{j,k}\}_{k=1}^\infty$ is an orthonormal basis for $\mathscr{H}_j'$,

$$Te_{1,k} = \sum_l \lambda_{k,l} e_{2,l},$$

for some $\{\lambda_{k,l}\}_{k,l=1}^\infty$ and

$$\|T\|^2_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)} = (T,T)_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)} = \mathrm{tr}_{\mathscr{H}_1}(T^* \circ T),$$

giving that

$$\|T\|^2_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)} = \sum_k \|Te_{1,k}\|^2_{\mathscr{H}_2} = \sum_{k,l} |\lambda_{k,l}|^2. \tag{4.10}$$

Now let $N_1, N_2 > 0$ be integers and set

$$H_{N_1,N_2}(x,y) = \sum_{k\le N_1}\sum_{l\le N_2} \lambda_{k,l}(e_{2,l} \otimes \overline{\varepsilon_{1,k}})(x,y) = \sum_{k\le N_1}\sum_{l\le N_2} \lambda_{k,l} e_{2,l}(x)\overline{\varepsilon_{1,k}(y)} \in \mathscr{H}.$$

We shall prove that H_{N_1,N_2} has a limit H in $\mathscr{H}$ as $N_1, N_2 \to \infty$, and that $H = K_T$.

Since $\{\overline{\varepsilon_{1,k}}\}_{k=1}^{\infty}$ is an orthonormal basis for $(\mathscr{H}_1')^\tau$, we get

$$\|H_{N_1,N_2}\|_{\mathscr{H}}^2 = \sum_{k\le N_1}\sum_{l\le N_2} |\lambda_{k,l}|^2.$$

Hence (4.10) and the fact that $T \in \mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)$ imply that the limits

$$H_N = \lim_{N_2\to\infty} H_{N,N_2} \quad \text{and} \quad H = \lim_{N\to\infty} H_N$$

exist in $\mathscr{H}$, and that

$$\|H\|_{\mathscr{H}}^2 = \sum_{k,l} |\lambda_{k,l}|^2 = \|T\|_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)}^2. \tag{4.11}$$

In order to prove that $H = K_T$ we let

$$f = \sum_k c_k e_{1,k} \in \mathscr{H}_1 \quad \text{and} \quad g = \sum_l d_l \varepsilon_{2,l} \in \mathscr{H}_2'$$

be arbitrary, and we set

$$f_N = \sum_{k\le N} c_k e_{1,k} \quad \text{and} \quad g_N = \sum_{l\le N} d_l \varepsilon_{2,l}.$$

Then $\|f - f_N\|_{\mathscr{H}_1} \to 0$ and $\|g - g_N\|_{\mathscr{H}_2'} \to 0$ as $N \to \infty$. Furthermore,

$$(Tf_{N_1}, g_{N_2})_{L^2(\mathbf{R}^{d_2})} = \sum_{k\le N_1}\sum_{l\le N_2} \lambda_{k,l} c_k \overline{d_l} = (H, g_{N_2}\otimes \overline{f_{N_1}})_{L^2(\mathbf{R}^{d_2+d_1})}.$$

By letting $N_1, N_2 \to \infty$ we get

$$(Tf, g)_{L^2(\mathbf{R}^{d_2})} = (H, g\otimes \overline{f})_{L^2(\mathbf{R}^{d_2+d_1})}.$$

Hence $H = K_T$, and (4.9) follows.

If instead $K_T \in \mathscr{H}$, then it follows by similar arguments as in the first part of the proof that (4.9) and the first equality in (4.11) hold with $H = K_T$. Hence, the second inequality in (4.11) shows that $T \in \mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)$. The proof is complete. □

5. Young inequalities for weighted Schatten–von Neumann classes

In this section we establish Young type results for dilated convolutions and multiplications on $s_p^w(\mathscr{H}_1,\mathscr{H}_2)$, when $\mathscr{H}_1$ and $\mathscr{H}_2$ are appropriate modulation spaces of Hilbert type. Especially we prove multi-linear versions of Theorems 0.3 and 0.4. We will mainly follow the analysis in Section 5 in [51], and the proofs are similar. However, in order to be self-contained we here present proofs which are slightly condensed, where, at the same time, some misprints have been corrected.

We need some preparations for stating the results. If we have N convolutions, then the corresponding conditions compared to (0.9) is

$$p_1{}^{-1} + \cdots + p_N{}^{-1} = N - 1 + r^{-1}, \qquad 1 \le p_1,\dots,p_N, r \le \infty. \tag{0.9$'$}$$

In the same way, (0.10) should be replaced by

$$(-1)^{j_1}t_1^{-2}+\cdots+(-1)^{j_N}t_N^{-2}=1, \tag{0.10$'$}$$

and (0.11) by

$$(-1)^{j_1}t_1^{2}+\cdots+(-1)^{j_N}t_N^{2}=1. \tag{0.11$'$}$$

The condition (0.12) of the involved weight functions is modified into

$$\begin{aligned}\vartheta(X_1+\cdots+X_N) &\lesssim \vartheta_{j_1,1}(t_1X_1)\cdots\vartheta_{j_N,N}(t_NX_N),\\ \omega(X_1+\cdots+X_N) &\lesssim \omega_{j_1,1}(t_1X_1)\cdots\omega_{j_N,N}(t_NX_N),\end{aligned} \tag{0.12$'$}$$

where

$$\omega_{0,k}(X)=\vartheta_{1,k}(X)=\omega_k(X),\quad \vartheta_{0,k}(X)=\omega_{1,k}(X)=\vartheta_k(X). \tag{0.13$'$}$$

With these conditions we shall essentially prove estimates of the form

$$\|a_{1,t_1}*\cdots*a_{N,t_N}\|_{s_r^w(1/\omega,\vartheta)}\le C^d\|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s_{p_N}^w(1/\omega_N,\vartheta_N)}, \tag{0.14$'$}$$

and

$$\|a_{1,t_1}\cdots a_{N,t_N}\|_{s_r^w(1/\omega,\vartheta)}\le C^d\|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s_{p_N}^w(1/\omega_N,\vartheta_N)}. \tag{0.15$'$}$$

Here and in what follows we let a_s and b_t be given by $a_s=a(t\,\cdot\,)$ and $b_t=b(t\,\cdot\,)$ when $a,b\in\mathcal S_{1/2}'$ and $t\in\mathbf R\setminus 0$, and $a_{j,t}$ be given by $a_{j,t}=a_j(t\,\cdot\,)$ when $a_j\in\mathcal S_{1/2}'$, $j\in\mathbf N$, and $t\in\mathbf R\setminus 0$.

Theorem 0.3′. *Let $p_1,\dots,p_N,r\in[1,\infty]$ satisfy (0.9)′, and let $t_1,\dots,t_N\in\mathbf R\setminus 0$ satisfy (0.10)′, for some choices of $j_1,\dots,j_N\in\{0,1\}$. Also let $\omega,\omega_j,\vartheta,\vartheta_j\in\mathscr P_E(\mathbf R^{2d})$ for $j=1,\dots,N$ satisfy (0.12)′ and (0.13)′.*

Then the map $(a_1,\dots,a_N)\mapsto a_{1,t_1}*\cdots*a_{N,t_N}$ on $\mathcal S_{1/2}(\mathbf R^{2d})$, extends uniquely to a continuous mapping from

$$s_{p_1}^w(1/\omega_1,\vartheta_1)\times\cdots\times s_{p_N}^w(1/\omega_N,\vartheta_N)$$

to $s_r^w(1/\omega,\vartheta)$. Furthermore, (0.14)′ holds for some constant

$$C=C_0^N|t_1|^{-2/p_1}\cdots|t_N|^{-2/p_N},$$

where C_0 is independent of $a_1\in s_{p_1}^w(1/\omega_1,\vartheta_1)$, …,$a_N\in s_{p_N}^w(1/\omega_N,\vartheta_N)$, $t_1,\dots,t_N$ and d.

Moreover, $\operatorname{Op}^w(a_{1,t_1}*\cdots*a_{N,t_N})\ge 0$ when $\operatorname{Op}^w(a_j)\ge 0$ for each $1\le j\le N$.

Theorem 0.4′. *Let $p_1,\dots,p_N,r\in[1,\infty]$ satisfy (0.9)′, and let $t_1,\dots,t_N\in\mathbf R\setminus 0$ satisfy (0.11)′, for some choices of $j_1,\dots,j_N\in\{0,1\}$. Also let $\omega,\omega_j,\vartheta,\vartheta_j\in\mathscr P_E(\mathbf R^{2d})$ for $j=1,\dots,N$ satisfy (0.12)′ and (0.13)′.*

Then the map $(a_1,\dots,a_N)\mapsto a_{1,t_1}\cdots a_{N,t_N}$ on $\mathcal S_{1/2}(\mathbf R^{2d})$, extends uniquely to a continuous mapping from

$$s_{p_1}^w(1/\omega_1,\vartheta_1)\times\cdots\times s_{p_N}^w(1/\omega_N,\vartheta_N)$$

to $s_r^w(1/\omega,\vartheta)$. Furthermore, (0.15)′ holds for some constant

$$C=C_0^N|t_1|^{-2/p_1'}\cdots|t_N|^{-2/p_N'},$$

where C_0 *is independent of* $a_1 \in s^w_{p_1}(1/\omega_1, \vartheta_1)$, …,$a_N \in s^w_{p_N}(1/\omega_N, \vartheta_N)$, $t_1, \dots, t_N$ *and* d.

We need some preparations for the proofs. First we observe that the roles of multiplications and convolutions are essentially interchanged on the symplectic Fourier transform side, because

$$\mathscr{F}_\sigma(a_1 * \cdots * a_N) = \pi^{dN}(\mathscr{F}_\sigma a_1)\cdots(\mathscr{F}_\sigma a_N), \tag{5.1}$$

holds when $a_1, \dots, a_N \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Hence it follows immediately from Lemma 1.3 and Proposition 4.3 that Theorems 0.3′ and 0.4′ are equivalent to the following two propositions. Here the condition (0.13)′ should be replaced by

$$\omega_{0,k}(X) = \vartheta_{1,k}(-X) = \omega_k(X), \quad \vartheta_{0,k}(X) = \omega_{1,k}(-X) = \vartheta_k(X). \tag{5.2}$$

We also recall that $a \in \mathcal{S}'_{s,+}(\mathbf{R}^{2d})$, $s \ge 1/2$, if and only if the operator Aa is positive semi-definite (cf. Proposition 1.4).

Proposition 5.1. *Let* $p_1, \dots, p_N, r \in [1, \infty]$ *satisfy* (0.9)′, *and let* $t_1, \dots, t_N \in \mathbf{R} \setminus 0$ *satisfy* (0.10)′, *for some choices of* $j_1, \dots, j_N \in \{0, 1\}$. *Also let* $\omega, \omega_j, \vartheta, \vartheta_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ *for* $j = 1, \dots, N$ *satisfy* (0.12)′ *and* (5.2). *Then the continuity assertions in Theorem* 0.3′ *hold after the* s^w_p *spaces have been replaced by* s^A_p *spaces.*

Proposition 5.2. *Let* $p_1, \dots, p_N, r \in [1, \infty]$ *satisfy* (0.9)′, *and let* $t_1, \dots, t_N \in \mathbf{R} \setminus 0$ *satisfy* (0.11), *for some choices of* $j_1, \dots, j_N \in \{0, 1\}$. *Also let* $\omega, \omega_j, \vartheta, \vartheta_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ *for* $j = 1, \dots, N$ *satisfy* (0.12)′ *and* (5.2). *Then the continuity assertions in Theorem* 0.4′ *hold after the* s^w_p *spaces have been replaced by* s^A_p *spaces.*

Moreover, if $s > 1/2$ *and* $a_j \in \mathcal{S}'_{s,+}(\mathbf{R}^{2d}) \cap s^A_{p_j}(1/\omega_j, \vartheta_j)$ *for every* $j = 1, \dots, N$, *then* $a_{1,t_1} \cdots a_{N,t_N} \in \mathcal{S}'_{s,+}(\mathbf{R}^{2d}) \cap s^A_r(1/\omega, \vartheta)$.

When proving Propositions 5.1 and 5.2 we need some technical lemmas, and start with the following classification of Hilbert modulation spaces.

Lemma 5.3. *Let* $\omega \in \mathscr{P}_E(\mathbf{R}^{4d})$ *be such that* $\omega(x, y, \xi, \eta) = \omega(x, \xi)$, $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d) \setminus 0$ *and let* $F \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$. *Then* $F \in M^2_{(\omega)}$, *if and only if*

$$\|F\| \equiv \Big(\iiint |V_\phi(F(\,\cdot\,, y))(x, \xi)\omega(x, \xi)|^2 \, dx dy d\xi \Big)^{1/2} < \infty. \tag{5.3}$$

Furthermore, $F \mapsto \|F\|$ *in* (5.3) *defines a norm which is equivalent to any* $M^2_{(\omega)}$ *norm.*

Proof. We may assume that $\|\phi\|_{L^2} = 1$. Let $\Phi = \phi \otimes \phi$, and let $\mathscr{F}_1 F$ denotes the partial Fourier transform of $F(x, y)$ with respect to the x variable. By Parseval's formula we get

$$
\begin{aligned}
\|F\|^2_{M^2_{(\omega)}} &= \iiiint |(V_\Phi F)(x,y,\xi,\eta)\omega(x,\xi)|^2\,dxdyd\xi d\eta \\
&= \iint \Big(\iint |(\mathscr F\big(F\,\overline{\Phi(\,\cdot\,-(x,y))}\big)(\xi,\eta)\omega(x,\xi)|^2\,dyd\eta\Big)\,dxd\xi \\
&= \iint \Big(\iint |(\mathscr F_1\big(F(\,\cdot\,,z)\,\overline{\phi(\,\cdot\,-x)}\big)(\xi)\phi(z-y)\omega(x,\xi)|^2\,dydz\Big)\,dxd\xi \\
&= \iint \Big(\int |(\mathscr F_1\big(F(\,\cdot\,,z)\,\overline{\phi(\,\cdot\,-x)}\big)(\xi)\omega(x,\xi)|^2\,dz\Big)\,dxd\xi = \|F\|,
\end{aligned}
$$

where the right-hand side is the same as $\|F\|$ in (5.3). The proof is complete. □

We omit the proof of the next lemma, since the result follows immediately from [44, Lemma 3.2], and the fact that $\mathcal S_{1/2} \hookrightarrow \mathscr S$.

Lemma 5.4. *Let $s,t\in \mathbf R$ be such that $(-1)^js^{-2}+(-1)^kt^{-2}=1$, for some choice of $j,k\in\{0,1\}$, and that $a,b\in \mathcal S_{1/2}(\mathbf R^{2d})$. Also let $T_{j,z}$ for $j\in\{0,1\}$ and $z\in\mathbf R^d$ be the operator on $\mathcal S_{1/2}(\mathbf R^{2d})$, defined by the formula*

$$
(T_{0,z}U)(x,y)=(T_{1,z}U)(y,x)=U(x-z,y+z),\quad U\in \mathcal S_{1/2}(\mathbf R^{2d}).
$$

Then

$$
A(a(s\,\cdot\,)*b(t\,\cdot\,)) = (2\pi)^{d/2}|st|^{-d}\int (T_{j,sz}(Aa))(s^{-1}\,\cdot\,)(T_{k,-tz}(Ab))(t^{-1}\,\cdot\,)\,dz. \tag{5.4}
$$

We note that for the involved spaces in Theorems 0.3′ and 0.4′, and Propositions 5.1 and 5.2 we have

$$
s_p^A(1/\omega,\vartheta)\hookrightarrow s_p^A(\mathbf R^{2d})\hookrightarrow s_p^A(\omega,1/\vartheta),\quad \text{when}\quad \omega,\vartheta\ge c, \tag{5.5}
$$

for some constant $c>0$, and similarly when s_p^A is replaced by s_p^w. This is an immediate consequence of Remark 4.4 (4) and the embeddings $M^{2,2}_{(\omega)}\hookrightarrow M^{2,2}=L^2\hookrightarrow M^{2,2}_{(1/\omega)}$ which are valid when ω is bounded from below. In particular, if $C_B(\mathbf R^d)$ denotes the set of all continuous functions on $\mathbf R^d$, vanishing at infinity, then

$$
\begin{aligned}
s_1^A(1/\omega,\vartheta)\hookrightarrow s_1^A(\mathbf R^{2d})\hookrightarrow C_B(\mathbf R^{2d})\cap \mathscr F C_B(\mathbf R^{2d})\cap L^2(\mathbf R^{2d}),& \\
\text{when}\quad \omega,\vartheta\ge c,&
\end{aligned} \tag{5.6}
$$

and similarly when s_1^A is replaced by s_1^w. Here the latter embedding follows from Propositions 1.5 and 1.9 in [45].

Proof of Proposition 5.1 *in the case $N=2$.* We only consider the case $j_1=1$ and $j_2=0$, i.e., $t^{-2}-s^{-2}=1$ when $t_1=s$ and $t_2=t$. The other cases follow by similar arguments and are left for the reader. We start to prove the theorem in the case $p_1=p_2=r=1$. By Propositions 4.6, 4.7 and a simple argument of approximations, it follows that we may assume that $a_1 = u$ and $a_2 = v$ are rank one elements in $\mathcal{S}_{1/2}$ and satisfy

$$\|u\|_{s_1^A(1/\omega_1,\vartheta_1)} \le C \quad \text{and} \quad \|v\|_{s_1^A(1/\omega_2,\vartheta_2)} \le C,$$

for some constant C. Then $Au = f_1 \otimes \overline{f_2}$, $Av = g_1 \otimes \overline{g_2}$ and

$$\|f_1\|_{M^2_{(\vartheta_1)}} \|f_2\|_{M^2_{(\omega_1)}} \lesssim \|u\|_{s_1^A(1/\omega_1,\vartheta_1)}, \quad \|g_1\|_{M^2_{(\vartheta_2)}} \|g_2\|_{M^2_{(\omega_2)}} \lesssim \|v\|_{s_1^A(1/\omega_2,\vartheta_2)},$$

for some vectors $f_1, f_2, g_1, g_2 \in \mathcal{S}_{1/2}$ such that

$$\|f_1\|_{M^2_{(\vartheta_1)}} \le C, \quad \|f_2\|_{M^2_{(\omega_1)}} \le C, \quad \|g_1\|_{M^2_{(\vartheta_2)}} \le C, \quad \|g_2\|_{M^2_{(\omega_2)}} \le C,$$

for some constant $C > 0$.

Set

$$F(x,z) = \overline{f_2(x/s + sz)}g_1(x/t + tz), \quad G(y,z) = \overline{f_1(y/s - sz)}g_2(y/t - tz).$$

It follows from (5.4) that

$$A(u_s * v_t)(x,y) = (2\pi)^{d/2}|st|^{-d} \int F(x,z)\overline{G(y,z)}\, dz.$$

This implies that

$$\begin{aligned} \|u_s * v_t\|_{s_1^A(1/\omega,\vartheta)} &\lesssim |st|^{-d} \int \|F(\,\cdot\,,z)\|_{M^2_{(\vartheta)}} \|G(\,\cdot\,,z)\|_{M^2_{(\omega)}}\, dz \\ &\lesssim |st|^{-d} I_1 \cdot I_2, \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} I_1 &= \Big(\iiint |V_\phi(F(\,\cdot\,,z))(x,\xi)\vartheta(x,\xi)|^2\, dxdzd\xi \Big)^{1/2} \\ I_2 &= \Big(\iiint |V_\phi(G(\,\cdot\,,z))(x,\xi)\omega(x,\xi)|^2\, dxdzd\xi \Big)^{1/2}, \end{aligned} \tag{5.8}$$

for some $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d) \setminus 0$. Hence, $I_1 \lesssim \|F\|_{M^2_{(\vartheta_0)}}$ and $I_2 \lesssim \|G\|_{M^2_{(\omega_0)}}$ by Lemma 5.3, when $\omega_0(x,y,\xi,\eta) = \omega(x,\xi)$ and $\vartheta_0(x,y,\xi,\eta) = \vartheta(x,\xi)$.

We need to estimate $\|F\|_{M^2_{(\vartheta_0)}}$ and $\|G\|_{M^2_{(\omega_0)}}$. In order to estimate $\|F\|_{M^2_{(\vartheta_0)}}$ we choose the window function $\Phi \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ as

$$\Phi(x,z) = \phi(x/s + sz)\phi(x/t + tz),$$

for some real-valued $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)$. By taking $(x_1/s + sz_1, x_1/t + tz_1)$ as new variables when evaluating $V_\Phi F$, and using $t^{-2} - s^{-2} = 1$, it follows by straightforward computations that

$$\begin{aligned} V_\Phi F(x,z,\xi,\zeta) &= (2\pi)^{-d} \iint F(x_1,z_1)\Phi(x_1 - x, z_1 - z)e^{-i\langle x_1,\xi\rangle - i\langle z_1,\zeta\rangle}\, dx_1dz_1 \\ &= |st|^{-d}\overline{V_\phi f_2(s^{-1}x + sz, s^{-1}\xi - (st^2)^{-1}\zeta)}V_\phi g_1(t^{-1}x + tz, t^{-1}\xi - (s^2t)^{-1}\zeta). \end{aligned}$$

Furthermore, by (0.12), (5.2) and the fact that $t^{-2} - s^{-2} = 1$, we obtain

$$\vartheta(x,\xi) = \vartheta\big((t^{-2}x + z) - (s^{-2}x + z), (t^{-2}\xi - (st)^{-2}\zeta) - (s^{-2}\xi - (st)^{-2}\zeta)\big)$$
$$\lesssim \omega_1(s^{-1}x + sz, s^{-1}\xi - (st^2)^{-1}\zeta)\vartheta_2(t^{-1}x + tz, t^{-1}\xi - (s^2t)^{-1}\zeta)$$

A combination of these relations now gives

$$|V_\Phi F(x,z,\xi,\zeta)\vartheta(x,\xi)| \lesssim |st|^{-d} J_1 \cdot J_2, \tag{5.9}$$

where

$$J_1 = |V_\phi f_2(s^{-1}x + sz, s^{-1}\xi - (st^2)^{-1}\zeta)\omega_1(s^{-1}x + sz, s^{-1}\xi - (st^2)^{-1}\zeta)|$$

and

$$J_2 = |V_\phi g_1(t^{-1}x + tz, t^{-1}\xi - (s^2t)^{-1}\zeta)\vartheta_2(t^{-1}x + tz, t^{-1}\xi - (s^2t)^{-1}\zeta)|.$$

By applying the L^2 norm and taking

$$s^{-1}x + sz, \quad t^{-1}x + tz, \quad s^{-1}\xi - (st^2)^{-1}\zeta, \quad t^{-1}\xi - (s^2t)^{-1}\zeta$$

as new variables of integration we get

$$\|F\|_{M^2_{(\vartheta)}} \lesssim |st|^{-2d} \|f_2\|_{M^2_{(\omega_1)}} \|g_1\|_{M^2_{(\vartheta_2)}}. \tag{5.10}$$

By similar computations it also follows that

$$\|G\|_{M^2_{(\omega)}} \lesssim |st|^{-2d} \|f_1\|_{M^2_{(\vartheta_1)}} \|g_2\|_{M^2_{(\omega_2)}}. \tag{5.11}$$

Hence, a combination of Proposition 4.6, (5.7), (5.8), (5.10) and (5.11) gives

$$\|u_s * v_t\|_{s_1^A(1/\omega,\vartheta)} \lesssim |st|^{-d} \|f_1\|_{M^2_{(\vartheta_1)}} \|f_2\|_{M^2_{(\omega_1)}} \|g_1\|_{M^2_{(\vartheta_2)}} \|g_2\|_{M^2_{(\omega_2)}}$$
$$\lesssim |st|^{-d} \|u\|_{s_1^A(1/\omega_1,\vartheta_1)} \|v\|_{s_1^A(1/\omega_2,\vartheta_2)}.$$

This proves the result in the case $p_1 = p_2 = r = 1$.

Next we consider the case $p_1 = \infty$, which implies that $p_2 = 1$ and $r = \infty$. Let $a \in s_\infty^A(1/\omega_1, \vartheta_1)$ and let $b, c \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then

$$(a_s * b_t, c) = |s|^{-4d}(a, \widetilde{b}_{t_0} * c_{s_0}),$$

where $\widetilde{b}(X) = \overline{b(-X)}$, $s_0 = 1/s$ and $t_0 = t/s$. We claim that

$$\|\widetilde{b}_{t_0} * c_{s_0}\|_{s_1^A(\omega_1, 1/\vartheta_1)} \lesssim |s^2/t|^{2d} \|b\|_{s_1^A(1/\omega_2,\vartheta_2)} \|c\|_{s_1^A(\omega, 1/\vartheta)} \tag{5.12}$$

Admitting this for a while, it follows by duality, using Theorem 4.8 that

$$\|a_s * b_t\|_{s_\infty^A(1/\omega,\vartheta)} \lesssim |s^2/t|^{2d} s^{-4d} \|a\|_{s_\infty^A(1/\omega_1,\vartheta_1)} \|b\|_{s_1^A(1/\omega_2,\vartheta_2)},$$

which gives (0.14). The result now follows in the case $p_1 = r = \infty$ and $p_2 = 1$ from the fact that $\mathcal{S}_{1/2}$ is dense in $s_1^A(1/\omega_2, \vartheta_2)$. In the same way the result follows in the case $p_2 = r = \infty$ and $p_1 = 1$.

For general $p_1, p_2, r \in [1, \infty]$ the result follows by multi-linear interpolation, using Theorem 4.4.1 in [2] and Remark 4.5 (3).

It remains to prove (5.12) when $b, c \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. The condition (0.10) is invariant under the transformation $(t,s) \mapsto (t_0, s_0) = (t/s, 1/s)$. Let

$$\widetilde{\omega} = 1/\omega_1, \quad \widetilde{\vartheta} = 1/\vartheta_1, \qquad \widetilde{\omega}_1 = 1/\omega,$$
$$\widetilde{\vartheta}_1 = 1/\vartheta, \quad \widetilde{\omega}_2 = \vartheta_2 \qquad \text{and} \quad \widetilde{\vartheta}_2 = \omega_2.$$

If $X_1 = -(X+Y)/s$ and $X_2 = Y/s$, then it follows that

$$\omega(X_1+X_2) \lesssim \vartheta_1(-sX_1)\omega_2(tX_2), \quad \vartheta(X_1+X_2) \lesssim \omega_1(-sX_1)\vartheta_2(tX_2),$$

is equivalent to

$$\widetilde{\omega}(X+Y) \lesssim \widetilde{\vartheta}_1(-s_0X)\widetilde{\omega}_2(t_0Y), \qquad \widetilde{\vartheta}(X+Y) \lesssim \widetilde{\omega}_1(-s_0X)\widetilde{\vartheta}_2(t_0Y).$$

Hence, the first part of the proof gives

$$\begin{aligned}
\|\widetilde{b}_{t_0} * c_{s_0}\|_{s_1^A(\omega_1, 1/\vartheta_1)} &= \|\widetilde{b}_{t_0} * c_{s_0}\|_{s_1^A(1/\widetilde{\omega}, \widetilde{\vartheta})} \\
&\lesssim |s_0t_0|^{-2d}\|\widetilde{b}\|_{s_1^A(1/\widetilde{\omega}_2, \widetilde{\vartheta}_2)}\|\widetilde{c}\|_{s_1^A(1/\widetilde{\omega}_1, \widetilde{\vartheta}_1)} \\
&= |s_0t_0|^{-2d}\|\widetilde{b}\|_{s_1^A(1/\vartheta_2, \omega_2)}\|\widetilde{c}\|_{s_1^A(\omega, 1/\vartheta)} \\
&= |s_0t_0|^{-2d}\|b\|_{s_1^A(1/\omega_2, \vartheta_2)}\|\widetilde{c}\|_{s_1^A(\omega, 1/\vartheta)},
\end{aligned}$$

and (5.12) follows. The proof in the case $N = 2$ is complete. □

We need the following lemma for the proof of Proposition 5.1 in the general case.

Lemma 5.5. *Let $\rho, t_1, \dots, t_N \in \mathbf{R} \setminus 0$ fulfill $(0.10)'$ and $\rho^{-2} + (-1)^{j_N} t_N^{-2} = 1$. For $t_j' = t_j/\rho$ set*

$$\omega_0(X) = \inf \omega_{j_1,1}(t_1'X_1)\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1}) \quad \textit{and}$$
$$\vartheta_0(X) = \inf \vartheta_{j_1,1}(t_1'X_1)\cdots\vartheta_{j_{N-1},N-1}(t_{N-1}'X_{N-1}),$$

where the infima are taken over all $X_1, \dots, X_{N-1}$ such that $X = X_1 + \dots + X_{N-1}$. Then the following is true:

(1) $\omega_0, \vartheta_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$;

(2) *for each $X_1, \dots X_{N-1} \in \mathbf{R}^{2d}$ it holds*

$$\omega_0(X_1 + \dots + X_{N-1}) \le \omega_{j_1,1}(t_1'X_1)\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1}), \quad \textit{and}$$
$$\vartheta_0(X_1 + \dots + X_{N-1}) \le \vartheta_{j_1,1}(t_1'X_1)\cdots\vartheta_{j_{N-1},N-1}(t_{N-1}'X_{N-1});$$

(3) *for each $X, Y \in \mathbf{R}^{2d}$ it holds*

$$\omega(X+Y) \lesssim \omega_0(\rho X)\omega_N(t_NY) \quad \textit{and} \quad \vartheta(X+Y) \lesssim \vartheta_0(\rho X)\vartheta_N(t_NY).$$

Proof. The assertion (2) follows immediately from the definitions of ω_0 and ϑ_0, and (3) is an immediate consequence of $(0.12)'$.

In order to prove (1) we assume that $X = X_1 + \cdots + X_{N-1}$. Since $\omega_{j_1,1} \in \mathscr{P}_E(\mathbf{R}^{2d})$, it follows that

$$\begin{aligned}\omega_0(X+Y) &\le \omega_{j_1,1}(t_1'(X_1+Y)) \cdots \omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1})\\ &\le \omega_{j_1,1}(t_1'X_1) \cdots \omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1})v(Y),\end{aligned}$$

for some $v \in \mathscr{P}_E(\mathbf{R}^{2d})$. By taking the infimum over all representations $X = X_1 + \cdots + X_N$, the latter inequality becomes $\omega_0(X+Y) \le \omega_0(X)v(Y)$. This implies that $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$, and in the same way it follows that $\vartheta_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$. The proof is complete. □

Proof of Proposition 5.1 *for general* N. We may assume that $N > 2$ and that the proposition is already proved for lower values on N. The condition on t_j is that $c_1t_1^{-2} + \cdots + c_Nt_N^{-2} = 1$, where $c_j \in \{\pm 1\}$. For symmetry reasons we may assume that $c_1t_1^{-2} + \cdots + c_{N-1}t_{N-1}^{-2} = \rho^{-2}$, where $\rho > 0$. Let $t_j' = t_j/\rho$, ω_0 and ϑ_0 be the same as in Lemma 5.5, and let $r_1 \in [1,\infty]$ be such that $1/r_1 + 1/p_N = 1 + 1/r$. Then $c_1(t_1')^{-2} + \cdots + c_{N-1}(t_{N-1}')^{-2} = 1$, $r_1 \ge 1$ since $p_N \le r$, and

$$1/p_1 + \cdots + 1/p_{N-1} = N - 2 + 1/r_1.$$

By the induction hypothesis and Lemma 5.5 (2) it follows that

$$b = a_{1,t_1'} * \cdots * a_{N-1,t_{N-1}'} = \rho^{d(2N-4)}(a_{1,t_1} * \cdots * a_{N-1,t_{N-1}})(\cdot/\rho)$$

makes sense as an element in $s_{r_1}^A(1/\omega_0,\vartheta_0)$, and

$$\|b\|_{s_{r_1}^A(1/\omega_0,\vartheta_0)} \lesssim \prod_{j=1}^{N-1} |t_j'|^{-2d/p_j} \|a\|_{s_{p_j}^A(1/\omega_j,\vartheta_j)}.$$

Since $1/r_1 + 1/p_N = 1 + 1/r$, it follows from Lemma 5.5 (3) that $b_\rho * a_{N,t_N}$ makes sense as an element in $s_r^A(1/\omega,\vartheta)$, and

$$\begin{aligned}\|(a_{1,t_1} * \cdots * a_{N-1,t_{N-1}}) * a_{N,t_N}\|_{s_r^A(1/\omega,\vartheta)} &= \rho^{-d(2N-4)}\|b_\rho * a_{N,t_N}\|_{s_r^A(1/\omega,\vartheta)}\\ &\le C_1\|a_1\|_{s_{p_1}^A(1/\omega_1,\vartheta_1)} \cdots \|a_N\|_{s_{p_N}^A(1/\omega_N,\vartheta_N)},\end{aligned}$$

where

$$C_1 \asymp \rho^{d(4-2N-2/r_1)}|t_N|^{-2d/p_N} \prod_{j=1}^{N-1} |t_j'|^{-2d/p_j} = \prod_{j=1}^{N} |t_j|^{-2d/p_j}.$$

This proves the extension assertions. The uniqueness as well as the symmetry assertions follow from the facts that $\mathcal{S}_{1/2}$ is dense in s_p^A when $p < \infty$ and dense in s_∞^A with respect to the weak* topology, and that at most one p_j is equal to infinity due to the Young condition. The proof is complete. □

Proof of Proposition 5.2. The continuity assertions follow by combining Proposition 4.3, Proposition 5.1 and (5.1).

When verifying the positivity statement we may argue by induction as in the proof of Proposition 5.1. This together with Proposition 1.4 and some simple arguments of approximation shows that it suffices to prove that $a_s b_t$ is positive semi-definite when $\pm s^2 \pm t^2 = 1$, $st \neq 0$, and $a, b \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ are σ-positive rank-one element.

For any $U \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ we set

$$U_{0,z}(x,y) = U_{1,z}(-y,-x) = U(x+z, y+z).$$

Then Lemmas 1.3 and 5.4 give

$$A(a_s b_t)(x,y) = (2/\pi)^{d/2}|st|^{-d} \int (Aa)_{j,z/s}(sx,sy)(Ab)_{k,-z/t}(tx,ty)\,dz,$$

for some choice of $j,k \in \{0,1\}$. Since $a, b \in C_+$ are rank-one elements, it follows that the integrand is of the form $\phi_z(x) \otimes \overline{\phi_z(y)}$ in all these cases. Consequently, $A(a_s b_t)$ is a positive semi-definite operator. □

Remark 5.6. We note that the arguments and conclusions in Remark 5.7 in [51] holds after $\mathscr{P}$ has been replaced by $\mathscr{P}_E$.

6. Some consequences

In this section we explain some consequences of the results in previous section. We omit the proofs since they are the same as corresponding results in Section 6 in [51], after the weight class $\mathscr{P}$ has been replaced by $\mathscr{P}_E$. It follows for example from Proposition 5.2, that $s_1^A(1/v, v)$ is stable under composition with odd entire analytic functions, when v is submultiplicative,

Thereafter we explain how the definition of Toeplitz operators can be extended to include appropriate dilations of s_p^w as permitted Toeplitz symbols.

We start by considering compositions of elements in $s_1^A(1/v, v)$ with analytic functions. In these considerations we restrict ourself to the case when $v = \check{v} \in \mathscr{P}_E(\mathbf{R}^{2d})$ is submultiplicative. We note that each element in $s_1^A(1/v, v)$ is a continuous function which turns to zero at infinity, since (5.6) shows that $s_1^A(1/v, v) \hookrightarrow C_B(\mathbf{R}^{2d})$.

A part of these investigations concerns σ-positive functions and distributions, and it is convenient to let $C_+(\mathbf{R}^{2d})$ denote the set of all continuous functions on $\mathbf{R}^{2d}$, which are σ-positive (cf. [44]).

It follows that any product of odd numbers of elements in $s_1^A(1/v, v)$ are again in $s_1^A(1/v, v)$. In fact, assume that $a_1, \dots, a_N \in s_1^A(1/v, v)$, $|\alpha|$ is odd, and that $t_j = 1$. Then it follows from Theorem 5.2 that $a_1^{\alpha_1} \cdots a_N^{\alpha_N} \in s_1^A(1/v, v)$, and

$$\|a_1^{\alpha_1} \cdots a_N^{\alpha_N}\|_{s_1^A(1/v,v)} \le C_0^{d|\alpha|} \prod \|a_j\|^{\alpha_j}_{s_1^A(1/v,v)}, \tag{6.1}$$

for some constant C_0 which is independent of α and d.

Furthermore, if in addition $a_1, \dots, a_N$ are σ-positive, then the same is true for $a_1^{\alpha_1} \cdots a_N^{\alpha_N}$. The following result is an immediate consequence of these observations.

Proposition 6.1. *Let* $a_1, \dots, a_N \in s_1^A(1/v, v)$*, where* $v = \check{v} \in \mathscr{P}_E(\mathbf{R}^{2d})$ *is submultiplicative,* C_0 *is the same as in* (6.1)*, and let* $R_1, \dots, R_N > 0$*. Also let* f, g *be odd analytic functions from the polydisc*

$$\{\, z \in \mathbf{C}^N \,;\, |z_j| < C_0 R_j \,\}$$

to $\mathbf{C}$*, with expansions*

$$f(z) = \sum_\alpha c_\alpha z^\alpha \quad \text{and} \quad g(z) = \sum_\alpha |c_\alpha| z^\alpha .$$

Then $f(a) = f(a_1, \dots, a_N)$ *is well defined and belongs to* $s_1^A(1/v, v)$*, and*

$$\|f(a)\|_{s_1^A(1/v,v)} \le g(C_0\|a_1\|_{s_1^A(1/v,v)}, \dots, C_0\|a_N\|_{s_1^A(1/v,v)}).$$

If in addition $a_1, \dots, a_N \in C_+(\mathbf{R}^{2d})$*, then* $g(a) \in C_+(\mathbf{R}^{2d})$*.*

For rank one elements we also have the following generalization of [44, Proposition 4.10].

Proposition 6.2. *Let* $v, v_1 \in \mathscr{P}_E(\mathbf{R}^{2d})$ *are even, submultiplicative and fulfill* $v_1 = v(\,\cdot\,/\sqrt{2}\,)$*. Also let* $u \in s_\infty^w(1/\omega, \omega)$ *be an element of rank one, and let* $a(X) = |u(X/\sqrt{2})|^2$*. Then* $a \in s_1^w(1/v_1, v_1)$*, and* $\operatorname{Op}^w(a) \ge 0$*.*

We finish the section by applying our results on Toeplitz operators (cf. (1.22)). The following result, parallel to Theorems 3.1 and 3.5 in [53], generalizes [46, Proposition 4.5].

Theorem 6.3. *Let* $p \in [1, \infty]$ *and* $\omega, \omega_0, \vartheta, \vartheta_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ *for* $j = 0, 1, 2$ *be such that*

$$\omega(X_1 - X_2) \lesssim \omega_0(\sqrt{2}\, X_1)\vartheta_2(X_2), \quad \vartheta(X_1 - X_2) \lesssim \vartheta_0(\sqrt{2}\, X_1)\vartheta_1(X_2).$$

Then the definition of $\operatorname{Tp}_{h_1,h_2}(a)$ *extends uniquely to each* $a \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ *and* $h_j \in M^2_{(\vartheta_j)}$ *for* $j = 1, 2$ *such that* $b = a(\sqrt{2}\,\cdot\,) \in s_p^w(1/\omega_0, \vartheta_0)$*, and*

$$\|\operatorname{Tp}_{h_1,h_2}(a)\|_{\mathscr{I}_p(M^2_{(1/\omega)}, M^2_{(\vartheta)})} \lesssim \|a(\sqrt{2}\,\cdot\,)\|_{s_p^w(1/\omega_0,\vartheta_0)} \|h_1\|_{M^2_{(\vartheta_1)}} \|h_2\|_{M^2_{(\vartheta_2)}}.$$

Furthermore, if $h_1 = h_2$ *and* $\operatorname{Op}^w(b) \ge 0$*, then* $\operatorname{Tp}_{h_1,h_2}(a) \ge 0$*.*

Proof. Since $W_{h_2,h_1} \in s_1^w(1/\vartheta_1, \vartheta_2)$, the result is an immediate consequence of (1.23) and Theorem 0.3. □

Appendix

In this appendix we prove basic results for pseudo-differential operators with symbols in modulation spaces, where the corresponding weights belong to $\mathscr{P}_E$. The arguments are in general similar as corresponding results in [47, 50].

The continuity results that we are focused on are especially Theorems A.1–A.3. Here Theorem A.1 is the extension of Feichtinger–Gröchenig's kernel theorem for modulation spaces with weights in $\mathscr{P}_E$. This result corresponds to Schwartz kernel theorem in distribution theory. The second result (Theorem A.2) concerns pseudo-differential operators with symbols in modulation spaces, which act on modulation spaces. Theorem A.3 gives necessary and sufficient conditions on symbols such that corresponding pseudo-differential operators are Schatten–von Neumann operators of certain degrees. Finally in Propositions A.4 and A.5 we establish preparatory results on Wigner distributions and pseudo-differential calculus in the context of modulation space theory.

Before stating the results we recall same facts on distribution kernels to linear operators in the background of Gelfand–Shilov spaces. Let $s \geq 1/2$ and let $K \in \mathcal{S}_s'(\mathbf{R}^{d_1+d_2})$. Then K gives rise to a linear and continuous operator $T = T_K$ from $\mathcal{S}_s(\mathbf{R}^{d_1})$ to $\mathcal{S}_s'(\mathbf{R}^{d_2})$, defined by the formula

$$Tf(x) = \langle K(x,\,\cdot\,), f\rangle, \tag{A.1}$$

which should be interpreted as (1.9) when $f \in \mathcal{S}_s(\mathbf{R}^{d_1})$ and $g \in \mathcal{S}_s(\mathbf{R}^{d_2})$.

Before establishing the corresponding result for modulation with weights in $\mathscr{P}_E$, we present appropriate conditions on the involved weights and Lebesgue exponent. The involved weights are related to each others by the formulas

$$\frac{\omega_2(x,\xi)}{\omega_1(y,\eta)} \lesssim \omega(x,y,\xi,-\eta), \quad x,\xi \in \mathbf{R}^{2d_2},\ y,\eta \in \mathbf{R}^{2d_1} \tag{A.2}$$

or

$$\frac{\omega_2(x,\xi)}{\omega_1(y,\eta)} \asymp \omega(x,y,\xi,-\eta), \quad x,\xi \in \mathbf{R}^{2d_2},\ y,\eta \in \mathbf{R}^{2d_1}, \tag{A.2$'$}$$

and

$$\begin{aligned}\omega(x,y,\xi,\eta) \asymp\ & \omega_0((1-t)x+ty, t\xi-(1-t)\eta, \xi+\eta, y-x),\\ & x,y,\xi,\eta \in \mathbf{R}^d,\end{aligned} \tag{A.3}$$

or equivalently,

$$\begin{aligned}\omega_0(x,\xi,\eta,y) \asymp\ & \omega(x-ty, x+(1-t)y, \xi+(1-t)\eta, -\xi+t\eta),\\ & x,y,\xi,\eta \in \mathbf{R}^d.\end{aligned} \tag{A.3$'$}$$

We note that (A.2) and (A.3) imply

$$\frac{\omega_2(x,\xi)}{\omega_1(y,\eta)} \lesssim \omega_0((1-t)x+ty, t\xi+(1-t)\eta, \xi-\eta, y-x), \tag{A.4}$$

and that (A.2)′ and (A.3) imply

$$\frac{\omega_2(x,\xi)}{\omega_1(y,\eta)} \asymp \omega_0((1-t)x+ty, t\xi+(1-t)\eta, \xi-\eta, y-x), \tag{A.4$'$}$$

The Lebesgue exponents of the modulation spaces should satisfy conditions of the form

$$1/p_1 - 1/p_2 = 1/q_1 - 1/q_2 = 1 - 1/p - 1/q, \quad q \le p_2, q_2 \le p, \tag{A.5}$$

or

$$p_1 \le p \le p_2, \quad q_1 \le \min(p,p') \quad \text{and} \quad q_2 \ge \max(p,p'). \tag{A.6}$$

Theorem A.1. *Let $t \in \mathbf{R}$, $\omega_j \in \mathscr{P}_E(\mathbf{R}^{2d_j})$ for $j=1,2$ and $\omega \in \mathscr{P}_E(\mathbf{R}^{2d_2+2d_1})$ be such that* (A.2)′ *holds. Also let T is a linear and continuous map from $\mathcal{S}_{1/2}(\mathbf{R}^{d_1})$ to $\mathcal{S}_{1/2}'(\mathbf{R}^{d_2})$. Then the following conditions are equivalent:*

(1) *T extends to a continuous mapping from $M^1_{(\omega_1)}(\mathbf{R}^{d_1})$ to $M^\infty_{(\omega_2)}(\mathbf{R}^{d_2})$;*

(2) *there is a unique $K \in M^\infty_{(\omega)}(\mathbf{R}^{d_2+d_1})$ such that* (A.1) *holds for every $f \in \mathcal{S}_{1/2}(\mathbf{R}^{d_1})$;*

(3) *if in addition $d_1 = d_2 = d$ and* (A.3) *holds, then there is a unique $a \in M^\infty_{(\omega_0)}(\mathbf{R}^{2d})$ such that $Tf = \operatorname{Op}_t(a)f$ when $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$.*

Furthermore, if (1)–(2) *are fulfilled, then $\|T\|_{M^1_{(\omega_1)} \to M^\infty_{(\omega_2)}} \asymp \|K\|_{M^\infty_{(\omega)}}$, and if in addition $d_1 = d_2$ and $T = \operatorname{Op}_t(a)$ in* (3)*, then $\|K\|_{M^\infty_{(\omega)}} \asymp \|a\|_{M^\infty_{(\omega_0)}}$.*

Theorem A.2. *Let $t \in \mathbf{R}$ and $p,q,p_j,q_j \in [1,\infty]$ for $j=1,2$, satisfy* (A.5)*. Also let $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ and $\omega_1,\omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$ satisfy* (A.4)*. If $a \in M^{p,q}_{(\omega)}(\mathbf{R}^{2d})$, then $\operatorname{Op}_t(a)$ from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$ extends uniquely to a continuous mapping from $M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d)$ to $M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d)$, and*

$$\|\operatorname{Op}_t(a)\|_{M^{p_1,q_1}_{(\omega_1)} \to M^{p_2,q_2}_{(\omega_2)}} \lesssim \|a\|_{M^{p,q}_{(\omega_0)}}. \tag{A.7}$$

Moreover, if in addition a belongs to the closure of $\mathcal{S}_{1/2}$ under the $M^{p,q}_{(\omega_0)}$ norm, then $\operatorname{Op}_t(a) : M^{p_1,q_1}_{(\omega_1)} \to M^{p_2,q_2}_{(\omega_2)}$ is compact.

Theorem A.3. *Let $t \in \mathbf{R}$ and $p,q,p_j,q_j \in [1,\infty]$ for $j=1,2$, satisfy* (A.6)*. Also let $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ and $\omega_1,\omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$ satisfy* (A.4)′*. Then*

$$M^{p_1,q_1}_{(\omega_0)}(\mathbf{R}^{2d}) \hookrightarrow s_{t,p}(\omega_1,\omega_2) \hookrightarrow M^{p_2,q_2}_{(\omega_0)}(\mathbf{R}^{2d}).$$

For the proofs we also need the following extensions of Propositions 4.1 and 4.8 in [49].

Proposition A.4. *Let $t \in \mathbf{R}$, and let $p_j,q_j,p,q \in [1,\infty]$ be such that $p \le p_j, q_j \le q$, for $j=1,2$, and*

$$1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p + 1/q. \tag{A.8}$$

Also let $\omega_1,\omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$ and $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ be such that

$$\omega_0((1-t)x+ty, t\xi+(1-t)\eta, \xi-\eta, y-x) \lesssim \omega_1(x,\xi)\omega_2(y,\eta). \tag{A.9}$$

Then the map $(f_1,f_2)\mapsto W^t_{f_1,f_2}$ *from* $\mathcal S'_{1/2}(\mathbf R^d)\times \mathcal S'_{1/2}(\mathbf R^d)$ *to* $\mathcal S'_{1/2}(\mathbf R^{2d})$ *restricts to a continuous mapping from* $M^{p_1,q_1}_{(\omega_1)}(\mathbf R^d)\times M^{p_2,q_2}_{(\omega_2)}(\mathbf R^d)$ *to* $M^{p,q}_{(\omega_0)}(\mathbf R^{2d})$*, and*

$$\|W^t_{f_1,f_2}\|_{M^{p,q}_{(\omega_0)}} \lesssim \|f_1\|_{M^{p_1,q_1}_{(\omega_1)}}\|f_2\|_{M^{p_2,q_2}_{(\omega_2)}} \tag{A.10}$$

when $f_1,f_2\in \mathcal S'_{1/2}(\mathbf R^d)$.

Proposition A.5. *Let* $p\in[1,\infty]$, $\omega_j\in \mathscr P_E(\mathbf R^{2d_j})$, $j=1,2$, $\omega\in \mathscr P_E(\mathbf R^{2d_2+2d_1})$, *and let* T *be a linear and continuous operator from* $\mathcal S_{1/2}(\mathbf R^{d_1})$ *to* $\mathcal S'_{1/2}(\mathbf R^{d_2})$ *with distribution kernel* $K\in \mathcal S'_{1/2}(\mathbf R^{d_2+d_1})$. *Then the following is true:*

(1) *if* $d_1=d_2=d$ *and* $\omega_0\in \mathscr P_E(\mathbf R^{2d}\oplus \mathbf R^{2d})$ *satisfy* (A.3)$'$, $a\in \mathcal S'_{1/2}(\mathbf R^{2d})$ *and* $K=K_{a,t}$ *is given by* (1.7), *then* $K\in M^p_{(\omega)}(\mathbf R^{2d})$, *if and only if* $a\in M^p_{(\omega_0)}(\mathbf R^{2d})$, *and*
$$\|K\|_{M^p_{(\omega)}}\asymp \|a\|_{M^p_{(\omega_0)}};$$

(2) *if* (A.2)$'$ *holds, then* $T\in \mathscr I_2(M^2_{(\omega_1)},M^2_{(\omega_2)})$, *if and only if* $K\in M^2_{(\omega)}(\mathbf R^{d_2+d_1})$, *and then*
$$\|T\|_{\mathscr I_2}\asymp \|K\|_{M^2_{(\omega)}}. \tag{A.11}$$

For the proofs we note that (A.9) is the same as

$$\omega_0(x,\xi,\eta,y)\lesssim \omega_1(x-ty,\xi+(1-t)\eta)\omega_2(x+(1-t)y,\xi-t\eta). \tag*{(A.9)$'$}$$

Proof of Proposition A.4. We only prove the result when $p,q<\infty$. The straightforward modifications to the cases when $p=\infty$ or $q=\infty$ are left for the reader. Let $\phi_1,\phi_2\in \Sigma_1(\mathbf R^d)\setminus 0$, and let $\Phi=W^t_{\phi_1,\phi_2}$. Then Fourier's inversion formula gives

$$\begin{aligned}(V_\Phi(W^t_{f_1,f_2}))&(x,\xi,\eta,y)\\ &= e^{-i\langle y,\xi\rangle}F_1(x-ty,\xi+(1-t)\eta)\overline{F_2(x+(1-t)y,\xi-t\eta)},\end{aligned}$$

where $F_j=V_{\phi_j}f_j$. By applying the $L^{p,q}_{(\omega)}$-norm on the latter equality, and using (A.9)$'$, it follows from Minkowski's inequality that

$$\|W^t_{f_1,f_2}\|_{M^{p,q}_{(\omega_0)}}\lesssim \big(\|G_1*G_2\|_{L^r}\big)^{1/p}\le \Big(\int H(\eta)\,d\eta\Big)^{1/q},$$

where $G_j=|F_j\omega_j|^p$, $r=q/p\ge 1$ and

$$H(\eta)=\Big(\int\Big(\int\Big(\int G_1(y-x,\eta-\xi)G_2(x,\xi)\,dx\Big)^r dy\Big)^{1/r}d\xi\Big)^r.$$

Now let $r_j,s_j\in[1,\infty]$ for $j=1,2$ be chosen such that

$$1/r_1+1/r_2=1/s_1+1/s_2=1+1/r.$$

Then Young's inequality gives

$$H(\eta)\le \Big(\int \|G_1(\,\cdot\,,\eta-\xi)\|_{L^{r_1}}\|G_2(\,\cdot\,,\xi)\|_{L^{r_2}}\,d\xi\Big)^r$$

Hence an other application of Young's inequality gives

$$\|W^t_{f_1,f_2}\|_{M^{p,q}_{(\omega_0)}} \lesssim \Big(\int H(\eta)\, d\eta \Big)^{1/q} \lesssim \big(\|G_1\|_{L^{r_1,s_1}} \|G_2\|_{L^{r_2,s_2}} \big)^{1/p}$$

By letting $p_j = pr_j$ and $q_j = qs_j$, the last inequality gives (A.10). The proof is complete. □

Proof of Proposition A.5. (1) Let $\Phi, \Psi \in \mathcal{S}_{1/2}(\mathbf{R}^{2d}) \setminus 0$ be such that

$$\Phi(x,y) = (\mathscr{F}_2\Psi)((1-t)x + ty, x-y).$$

Then it follows by straightforward applications of Fourier's inversion formula that

$$|(V_\Phi K_{a,t})(x,y,\xi,\eta)| \asymp |(V_\Psi a)((1-t)x + ty, t\xi - (1-t)\eta, \xi + \eta, y - x)|.$$

The assertion now follows by applying the $L^p_{(\omega)}$ norm on the last equality.

Next we prove (2). Let $\{f_j\} \in \mathrm{ON}(M^2_{\omega_1})$ and $\{h_k\} \in \mathrm{ON}(M^2_{\omega_2})$. Then

$$\|T\|^2_{\mathscr{I}_2} = \sum_{j,k} |(Tf_j, h_k)_{M^2_{(\omega_2)}})|^2 = \sum_{j,k} |(K, h_k \otimes \overline{f_j})_{M^2_{(\omega_2)} \otimes L^2}|^2 \tag{A.12}$$

Next we consider the operator $T'_\vartheta = I_{M^2_{(\omega_2)}} \otimes \mathcal{R}_{1/\vartheta}$, where $\vartheta(x,\xi) = \omega_1(x,-\xi)$, which acts from $M^2_{(\omega_2)} \otimes M^2_{(1/\vartheta)}$ to $M^2_{(\omega_2)} \otimes M^2_{(\vartheta)}$ (Hilbert tensor products). Then (A.12) gives

$$\begin{aligned}\|T\|^2_{\mathscr{I}_2} &= \sum_{j,k} |(T'_{\omega_0} K, h_k \otimes \overline{f_j})_{M^2_{(\omega_2)} \otimes M^2_{(\omega_1)}}|^2 \\ &= \|T'_{\omega_0} K\|^2_{M^2_{(\omega_2)} \otimes M^2_{(\omega_0)}} = \|K\|^2_{M^2_{(\omega_2)} \otimes M^2_{(1/\omega_0)}} = \|K\|^2_{M^2_{(\omega)}},\end{aligned}$$

and the result follows. The proof is complete. □

Proof of Theorem A.1. Let T be extendable to a continuous map from $M^1_{(\omega_1)}(\mathbf{R}^{d_1})$ to $M^\infty_{(\omega_2)}(\mathbf{R}^{d_2})$. It follows from [32, Theorem 2.2] and Remark 1.6 that (A.1) holds for some $K \in \mathcal{S}'_{1/2}(\mathbf{R}^{d_2+d_1})$. We shall prove that K belongs to $M^\infty_{(\omega)}$.

From the assumptions and Proposition 1.5 (3) it follows that

$$|(K, g \otimes \overline{f})_{L^2}| \lesssim \|f\|_{M^1_{(\omega_1)}} \|g\|_{M^1_{(1/\omega_2)}}, \tag{A.13}$$

when $f \in \mathcal{S}_{1/2}(\mathbf{R}^{d_1})$ and $g \in \mathcal{S}_{1/2}(\mathbf{R}^{d_2})$. By letting $\Phi = \overline{g} \otimes f$ be fixed, and replacing f and g with

$$f_{y,\eta} = e^{-i\langle \cdot, \eta\rangle} f(\cdot - y) \quad \text{and} \quad g_{x,\xi} = e^{i\langle \cdot, \xi\rangle} f(\cdot - x),$$

(A.13) takes the form

$$|(V_\Phi K)(x,y,\xi,\eta)| \lesssim \|f_{y,\eta}\|_{M^1_{(\omega_1)}} \|g_{x,\xi}\|_{M^1_{(1/\omega_2)}}. \tag{A.13$'$}$$

If $v \in \mathscr{P}_E$ is chosen such that ω_1 is v-moderate, and $\phi_1 \in \mathcal{S}_{1/2} \setminus 0$, then

$$\|f_{y,\eta}\|_{M^1_{(\omega_1)}} \asymp \iint |(V_{\phi_1} f)(z-y, \zeta+\eta)\omega_1(z,\zeta)|\, dz d\zeta$$
$$\lesssim \omega_1(y,-\eta)\|f\|_{M^1_{(v)}} \asymp \omega_1(y,-\eta).$$

In the same way we get

$$\|g_{x,\xi}\|_{M^1_{(1/\omega_2)}} \lesssim \omega_2(x,\xi)^{-1}.$$

If these estimates are inserted into (A.13)$'$, we obtain

$$|(V_\Phi K)(x,y,\xi,\eta)\omega(x,y,\xi,\eta)| \lesssim 1,$$

By taking the supremum of the left-hand side it follows that $\|K\|_{M^\infty_{(\omega)}} < \infty$. Hence $K \in M^\infty_{(\omega)}$, and we have proved that (1) implies (2).

By straightforward computations it also follows that (2) gives (1). The details are left for the reader.

The equivalence between (2) and (3) follows immediately from Proposition A.5. The proof is complete. □

Proof of Theorem A.2. The conditions on p_j and q_j implies that

$$p' \le p_1, q_1, p_2', q_2' \le q', \quad 1/p_1 + 1/p_2' = 1/q_1 + 1/q_2' = 1/p' + 1/q'.$$

Hence Proposition A.4, and (A.4) show that

$$\|W^t_{g,f}\|_{\widetilde{M}^{p',q'}_{(1/\omega)}} \lesssim \|f\|_{M^{p_1,q_1}_{(\omega_1)}} \|g\|_{M^{p_2',q_2'}_{(1/\omega_2)}}$$

when $f \in M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d)$ and $g \in M^{p_2',q_2'}_{(1/\omega_2)}(\mathbf{R}^d)$.

The continuity is now an immediate consequence of (1.14) and Proposition 1.5 (4), except for the case $p = q' = \infty$, which we need to consider separately.

Therefore assume that $p = \infty$, and $q = 1$, and let $a \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then $p_1 = p_2$ and $q_1 = q_2$, and it follows from Proposition A.4 and the first part of the proof that $W^t_{g,f} \in M^{1,\infty}_{(1/\omega_0)}$, and that (A.7) holds. In particular,

$$|(\mathrm{Op}_t(a)f, g))| \lesssim \|f\|_{M^{p_1,q_1}_{(\omega_1)}} \|g\|_{M^{p_1',q_1'}_{(1/\omega_2)}},$$

and the result follows when $a \in \mathcal{S}_{1/2}$. The result now follows for general $a \in M^{\infty,1}_{(\omega_0)}$, by taking a sequence $\{a_j\}_{j\ge 1}$ in $\mathcal{S}_{1/2}$, which converges narrowly to a. (For narrow convergence see Theorems 4.15 and 4.19, and Proposition 4.16 in [52]).

It remains to prove that if a belongs to the closure of $\mathcal{S}_{1/2}$ under $M^{p,q}_{(\omega_0)}$ norm, then $\mathrm{Op}_t(a) : M^{p_1,q_1}_{(\omega_1)} \to M^{p_2,q_2}_{(\omega_2)}$ is compact. As a consequence of Theorem A.3, it follows that $\mathrm{Op}_t(a_0)$ is compact when $a_0 \in \mathcal{S}_{1/2}$, since $\mathcal{S}_{1/2} \hookrightarrow M^1_{(\omega_0)}$ when $\omega_0 \in \mathscr{P}_E$, and that every trace-class operator is compact. The compactness of $\mathrm{Op}_t(a)$ now follows by approximating a with elements in $\mathcal{S}_{1/2}$. The proof is complete. □

Proof of Theorem A.3. The first embedding in

$$M^{\infty,1}_{(\omega_0)} \hookrightarrow s_{t,\infty}(\omega_1,\omega_2) \hookrightarrow M^{\infty}_{(\omega_0)}$$

follows from Theorem A.2, and the second one from Proposition 1.5 (2) and Theorem A.1.

By Propositions 1.5 (3) and 4.7, Theorem 4.8 and duality, the latter inclusions give

$$M^{1}_{(\omega_0)} \hookrightarrow s_{t,1}(\omega_1,\omega_2) \hookrightarrow M^{1,\infty}_{(\omega_0)},$$

and we have proved the result when $p = 1$ and when $p = \infty$. Furthermore, by Proposition A.5 we have $M^2_{(\omega_0)} = s_{t,2}(\omega_1,\omega_2)$, and the result also holds in the case $p = 2$. The result now follows for general p from these cases and interpolation. (See, e.g., Proposition 5.8 in [52].) The proof is complete. □

Acknowledgment

I am very grateful to Stevan Pilipović and the unknown referee for kind supports and important advices, leading to several improvements of the original paper.

References

[1] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, *Comm. Pure Appl. Math.* **14** (1961), 187–214.

[2] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin Heidelberg New York, 1976.

[3] M.S. Birman and M.Z. Solomyak, Estimates for the singular numbers of integral operators (Russian), *Uspekhi Mat. Nauk.* **32** (1977), 17–84.

[4] J. Chung, S.-Y. Chung, and D. Kim, Characterizations of the Gelfand–Shilov spaces via Fourier transforms, *Proc. Amer. Math. Soc.* **124** (1996), 2101–2108.

[5] E. Cordero, Gelfand–Shilov window classes for weighted modulation spaces, *Integral Transforms Spec. Funct.* **18** (2007), 829–837.

[6] E. Cordero and K. Gröchenig, Time-frequency analysis of localization operators, *J. Funct. Anal.* **205**(1) (2003), 107–131.

[7] E. Cordero and K. Gröchenig, Symbolic calculus and Fredholm property for localization operators, *J. Fourier Anal. Appl.* **12** (2006), 371–392.

[8] E. Cordero and K. Okoudjou, Multilinear localization operators, *J. Math. Anal. Appl.* **325** (2007), 1103–1116.

[9] E. Cordero, S. Pilipović, L. Rodino, and N. Teofanov, Quasianalytic Gelfand–Shilov spaces with applications to localization operators, *Rocky Mountain J. Math.* **40** (2010), 1123–1147.

[10] M. de Gosson, Symplectic covariance properties for Shubin and Born-Jordan pseudodifferential operators, *Trans. Amer. Math. Soc.*, to appear.

[11] M. de Gosson and F. Luef, Preferred quantization rules: Born–Jordan vs. Weyl; applications to phase space quantization, *J. Pseudo-Differ. Oper. Appl.* **2** (2011), 115–139.

[12] H.G. Feichtinger, Banach convolution algebras of Wiener's type, in *Proc. Functions, Series, Operators in Budapest*, Colloquia Math. Soc. J. Bolyai, North-Holland Publ. Co., Amsterdam, Oxford, New York, 1980.

[13] H.G. Feichtinger, Modulation spaces on locally compact abelian groups, in *Wavelets and their Applications*, editors: M. Krishna, R. Radha, and S. Thangavelu, Allied Publishers Private Limited, 2003, 99–140.

[14] H.G. Feichtinger, Atomic characterizations of modulation spaces through Gabor-type representations, in *Proc. Conf. on Constructive Function Theory*, *Rocky Mountain J. Math.* **19** (1989), 113–126.

[15] H.G. Feichtinger Modulation spaces: looking back and ahead, *Sampl. Theory Signal Image Process* **5** (2006), 109–140.

[16] H.G. Feichtinger and K.H. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions I, *J. Funct. Anal.* **86** (1989), 307–340.

[17] H.G. Feichtinger and K.H. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions II*, *Monatsh. Math.* **108** (1989), 129–148.

[18] H.G. Feichtinger and K.H. Gröchenig, Gabor frames and time-frequency analysis of distributions, J. Funct. Anal. **146**(2) (1997), 464–495.

[19] G.B. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press, 1989.

[20] I.M. Gelfand and G.E. Shilov *Generalized Functions,* I–III, Academic Press, New York, London, 1968.

[21] T. Gramchev, S. Pilipović, and L. Rodino, Classes of degenerate elliptic operators in Gelfand–Shilov spaces, in *New Developments in Pseudo-Differential Operators*, editors: L. Rodino and M.W. Wong, Operator Theory: Advances and Applications **189**, Birkhäuser Verlag, Basel 2009, 15–31.

[22] K.H. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[23] K.H. Gröchenig, Composition and spectral invariance of pseudodifferential operators on modulation spaces, *J. Anal. Math.* **98** (2006), 65–82.

[24] K. Gröchenig and J. Toft, The range of localization operators and lifting theorems for modulation and Bargmann–Fock spaces, *Trans. Amer. Math. Soc.*, to appear.

[25] K. Gröchenig and G. Zimmermann, Spaces of test functions via the STFT, *J. Funct. Spaces Appl.* **2** (2004), 25–53.

[26] Z. He and M.W. Wong, Localization operators associated to square integrable group representations, Panamer. Math. J. **6**(1) (1996), 93–104.

[27] A. Holst, J. Toft, and P. Wahlberg, Weyl product algebras and modulation spaces, J. Funct Anal. **251** (2007), 463–491.

[28] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. I, III, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983, 1985.

[29] H. Komatsu, Ultradistributions. I, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **20** (1973), 25–205.

[30] D. Labate, Pseudodifferential operators on modulation spaces, *J. Math. Anal. Appl.* **262** (2001), 242–255.

[31] E.H. Lieb, Integral bounds for radar ambiguity functions and Wigner distributions, *J. Math. Phys.* **31** (1990), 594–599.

[32] Z. Lozanov-Crvenković, D. Perišić, and M. Tasković, Gelfand–Shilov spaces structural and kernel theorems, preprint, arXiv:0706.2268v2.

[33] A. Pietsch, *Operator Ideals*, Mathematische Monographien, Vol. 16, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[34] S. Pilipović, Tempered ultradistributions, *Boll. Un. Mat. Ital. B* **2**(7) (1988), 235–251

[35] S. Pilipović and N. Teofanov, Wilson bases and ultramodulation spaces, *Math. Nachr.* **242** (2002), 179–196.

[36] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer, Berlin, 1960.

[37] B.W. Schulze and N.N. Tarkhanov Pseudodifferential operators with operator-valued symbols, *Israel Math. Conf. Proc.* **16**, 2003.

[38] M.A. Shubin *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, Berlin, 1987.

[39] B. Simon *Trace Ideals and their Applications*, London Math. Soc. Lecture Note Series, Cambridge University Press, 1979.

[40] J. Sjöstrand, An algebra of pseudodifferential operators, *Math. Res. Letters* **1** (1994), 185–192.

[41] N. Teofanov, Modulation spaces, Gelfand–Shilov spaces and pseudodifferential operators, *Sampl. Theory Signal Image Process* **5** (2006), 225–242.

[42] J. Toft, *Continuity and Positivity Problems in Pseudo-Differential Calculus*, Thesis, University of Lund, 1996.

[43] J. Toft, Subalgebras to a Wiener type algebra of pseudo-differential operators, *Ann. Inst. Fourier* **51**(5) (2001), 1347–1383.

[44] J. Toft, Continuity properties for non-commutative convolution algebras with applications in pseudo-differential calculus, *Bull. Sci. Math.* **126**(2) (2002), 115–142.

[45] J. Toft Positivity properties for non-commutative convolution algebras with applications in pseudo-differential calculus, *Bull. Sci. Math.* **127**(2) (2003), 101–132.

[46] J. Toft, Continuity properties for modulation spaces with applications to pseudo-differential calculus, I, *J. Funct. Anal.* **207**(2) (2004), 399–429.

[47] J. Toft, Continuity properties for modulation spaces with applications to pseudo-differential calculus, II, *Ann. Global Anal. Geom.* **26** (2004), 73–106.

[48] J. Toft, Convolution and embeddings for weighted modulation spaces, in *Advances in Pseudo-Differential Operators*, editors: R. Ashino, P. Boggiatto, and M.W. Wong, Operator Theory: Advances and Applications **155**, Birkhäuser Verlag, Basel 2004, 165–186.

[49] J. Toft, Continuity and Schatten properties for pseudo-differential operators on modulation spaces, in *Modern Trends in Pseudo-Differential Operators*, editors: J. Toft, M. W. Wong, and H. Zhu, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 2007, 173–206.

[50] J. Toft, *Continuity and Schatten properties for Toeplitz operators on modulation spaces* in: J. Toft, M.W. Wong, H. Zhu (eds.) *Modern Trends in Pseudo-Differential*

Operators, Operator Theory: Advances and Applications **172**, Birkhäuser Verlag, Basel, 2007, 313–328.

[51] J. Toft, Multiplication properties in pseudo-differential calculus with small regularity on the symbols, *J. Pseudo-Differ. Oper. Appl.* **1** (2010), 101–138.

[52] J. Toft, The Bargmann transform on modulation and Gelfand–Shilov spaces, with applications to Toeplitz and pseudo-differential operators, *J. Pseudo-Differ. Oper. Appl.* **3** (2012), 145–227.

[53] J. Toft and P. Boggiatto, Schatten classes for Toeplitz operators with Hilbert space windows on modulation spaces, *Adv. Math.* **217** (2008), 305–333.

[54] J. Toft, A. Khrennikov, B. Nilsson, and S. Nordebo, Decompositions of Gelfand–Shilov kernels into kernels of similar class, *J. Math. Anal. Appl.* **396** (2012), 315–322.

Joachim Toft
Department of Computer Science,
Physics and Mathematics
Linnæus University
SE-35195 Växjö, Sweden
e-mail: joachim.toft@lnu.se