

Elliptic Theory for Operators Associated with Diffeomorphisms of Smooth Manifolds

Anton Savin and Boris Sternin

Abstract. In this paper we give a survey of elliptic theory for operators associated with diffeomorphisms of smooth manifolds. Such operators appear naturally in analysis, geometry and mathematical physics. We survey classical results as well as results obtained recently. The paper consists of an introduction and three sections. In the introduction we give a general overview of the area of research. For the reader's convenience here we tried to keep special terminology to a minimum. In the remaining sections we give detailed formulations of the most important results mentioned in the introduction.

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Introduction

The aim of this paper is to give a survey of index theory for elliptic operators associated with diffeomorphisms of smooth manifolds. Recall that the construction of index theory includes the following main stages:

- 1) (finiteness theorem) Here one gives conditions, called ellipticity conditions, under which the operators under consideration are Fredholm in relevant function spaces;
- 2) (index theorem) Here one presents and proves an index formula, that is, an expression for the index of an elliptic operator in terms of topological invariants of the symbol of the operator and the manifold, on which the operator is defined.

The first index theorem on high-dimensional manifolds was the celebrated Atiyah–Singer theorem [11] on the index of elliptic pseudodifferential operators (ψ DO) on a closed smooth manifold. This theorem appeared as an answer to a

question posed by Gelfand [26]. Note that the statement and the proof of the index formula relied on most up to date methods of analysis and topology and stimulated interactions between them.

After that index theorems were obtained for many other classes of operators. In this paper we consider the class of operators associated with diffeomorphisms of closed smooth manifolds. One of advantages of this theory is that, besides the mentioned interaction of analysis and topology, here an important role is played by the theory of dynamical systems.

1. Elliptic operators for a discrete group of diffeomorphisms (analytical aspects).

The theory of elliptic operators associated with diffeomorphisms and the corresponding theory of boundary value problems with nonlocal boundary conditions go back to the paper by T. Carleman [17], where he considered the problem of finding a holomorphic function in a bounded domain Ω , which satisfies a nonlocal boundary condition, which relates the values of the function at a point $x \in \partial\Omega$ of the boundary and at the point $g(x) \in \partial\Omega$, where $g : \partial\Omega \rightarrow \partial\Omega$ is a smooth mapping of period two: $g^2 = Id$. A reduction of this boundary value problem to the boundary does not give usual integral equation as it was the case with the local boundary condition. Rather, it gives an integro-functional equation, which we call *equation associated with diffeomorphism g* . This paper motivated the study of a more general class of operators on closed smooth manifolds. Let us give the general definition of such operators.

On a closed smooth manifold M we consider operators of the form

$$D = \sum_{g \in G} D_g T_g : C^\infty(M) \rightarrow C^\infty(M), \quad (0.1)$$

where:

- G is a discrete group of diffeomorphisms of M ;
- $(T_g u)(x) = u(g^{-1}(x))$ is the shift operator corresponding to the diffeomorphism g ;
- $\{D_g\}$ is a collection of pseudodifferential operators of order $\leq m$;
- $C^\infty(M)$ is the space of smooth functions on M . Of course, one can also consider operators acting in sections of vector bundles.

Operators (0.1) will be called *G -pseudodifferential operators (G - ψ DO)* or simply *G -operators*.¹ Such operators were intensively studied (see the fundamental works of Antonevich [4, 5], and also the papers [2, 7] and the references cited there). In particular, an extremely important notion of *symbol of a G -operator* was introduced there. More precisely, two definitions of the symbol of a G -operator were given. First, the symbol was defined as a function on the cotangent bundle T^*M of the manifold taking values in operators acting on the space $l^2(G)$ of square integrable functions on the group. Second, the symbol was defined as an element of the crossed product [69] of the algebra of continuous functions on the cosphere

¹In the literature such operators are also called functional-differential, nonlocal, noncommutative operators and operators with shifts.

bundle S^*M of the manifold and the group G . Further, we introduce the ellipticity condition in this situation, which is the requirement of invertibility of the symbol of the operator. It was proved that the two ellipticity conditions (they correspond to the two definitions of the symbol) are equivalent under quite general assumptions. Ellipticity implies Fredholm property of the operator in Sobolev spaces H^s .

Let us note here one essential difference between the theory of elliptic G - ψ DO and a similar theory of ψ DO. Namely, examples show (see [6, 1, 45]) that the ellipticity (and the Fredholm property of operator (0.1) in the Sobolev spaces H^s) essentially depends on the smoothness exponent s . Thus, there arise natural questions on the description of the possible values of s , for which a given G -operator is elliptic and the question about the dependence of the index on s . The answers to these questions are well known in the situation of an isometric action of the group, that is, if the diffeomorphisms preserve a Riemannian metric on the manifold. In this case the symbol and the index do not depend on s . First steps in the study of these questions for nonisometric actions were done in the papers [58, 52], where it was shown for the simplest nonisometric diffeomorphism of dilation of spheres that the set of s , for which a G -operator is elliptic, is always an interval and the index (inside this interval) does not depend on s .

2. Index of elliptic operators for a discrete group of diffeomorphisms. Let us now turn attention to the problem of computing the index of elliptic G -operators. The first formula for the index of G -operators was obtained in the paper [3] for a *finite group* G of diffeomorphisms.² In this case the index of a G -operator was expressed in terms of Lefschetz numbers of an auxiliary elliptic ψ DO on M . Since the Lefschetz numbers are expressed by a formula [12] similar to the Atiyah–Singer index formula the index problem for a finite group is thus solved.

The index problem for infinite groups turned out to be much more difficult and required application of new methods related with noncommutative geometry of Connes [20, 22]. The first advance was done in the celebrated work of Connes [19]. There an index formula was obtained for operators of the form

$$D = \sum_{\alpha\beta} a_{\alpha\beta} x^\alpha (d/dx)^\beta \quad (0.2)$$

acting on the real line, where the coefficients $a_{\alpha\beta}$ are Laurent polynomials in the operators

$$(Uf)(x) = e^{ix} f(x), \quad (Vf)(x) = f(x - \theta),$$

and θ is some fixed number. The index theorem of Connes for such differential-difference operators is naturally formulated in terms of noncommutative geometry. Operators (0.2), which are also called operators on the noncommutative torus³, were used in a mathematical formulation of the quantum Hall effect [22]. It became

²These results were rediscovered in [39].

³This name is motivated by the fact that the algebra generated by U and V is a noncommutative deformation of the algebra of functions on the torus \mathbb{T}^2 .

clear after the cited papers of Connes that noncommutative geometry is not only useful, but also natural in the index problem for G -operators, and since then noncommutative geometry is used in all the papers on the index of G -operators, we are aware of. For instance, methods of noncommutative geometry were applied to solve the index problem for deformations of algebras of functions on toric manifolds in [34, 23, 24] and other papers.

Further progress in the solution of the index problem for G -operators was made in the monograph [38]. Namely, an index formula for operators (0.1) was obtained in the situation, when the action is isometric. Let us note here that this index formula for isometric actions contains all the above-mentioned formulas as special cases.

In the situation of a general (that is, *nonisometric*) action there were no index formulas until recently. There were only partial results. Namely, the index problem for \mathbb{Z} -operators (that is, operators for the group of integers) was reduced to a similar problem for an elliptic ψ DO (see [4, 51, 56]). The first index formula in the nonisometric case was obtained in the paper [58] for operators associated with dilation diffeomorphism of spheres. The index formula for elliptic operators associated with the group \mathbb{Z} was obtained in [49]. Finally, an index formula for an arbitrary torsion free group acting on the circle was stated in [44].

Let us mention several interesting examples of elliptic G -operators. Suppose that G preserves some geometric structure on the manifold (for instance, Riemannian metric, complex structure, spin structure, . . .). Then we can consider an elliptic operator associated with that structure and twist this operator using a G -projection (that is an operator of the form (0.1), which is a projection: $P^2 = P$) or an invertible G -operator. This construction produces an elliptic G -operator. For instance, if G acts isometrically, then one can take classical geometric operators (Euler, signature, Dolbeault, Dirac operators). The indices of the corresponding twisted G -operators were computed in [54]. If G acts by conformal diffeomorphisms of a Riemannian surface, then one can take the $\bar{\partial}$ operator. Indices of the corresponding twisted operators were computed in [43, 42]. In the papers [25, 37] there are index formulas for the twisted Dirac operator for group actions preserving the conformal structure on the manifold.

3. Operators associated with compact Lie groups. Let now G be a *compact Lie group* acting on M . Consider the class of operators of the form

$$D = \int_G D_g T_g dg : C^\infty(M) \longrightarrow C^\infty(M) \quad (0.3)$$

(cf. (0.1)), where dg is the Haar measure. Such operators relate values of functions on submanifolds of M of positive dimension. They were considered in [55, 66, 50]. In these papers a G -operator of the form (0.3) was represented as a pseudodifferential operator acting in sections of infinite-dimensional bundles [35], whose fiber is the space of functions on G . This method goes back to the papers of Babbage [13] and for a finite group gives a finite system of equations [4]. Moreover, the obtained

operator, which we denote by \mathcal{D} , is G -invariant, and its restriction \mathcal{D}^G to the subspace of G -invariant functions is isomorphic to the original operator D . Now if $\widehat{\mathcal{D}} = 1 + \mathcal{D}$ is transversally elliptic⁴ with respect to the action of G , then this implies the Fredholm property, that is, the index of the operator $\widehat{D} = 1 + D$ is finite. The index formula and the corresponding topological invariants of the symbol of elliptic G -operators were computed in the papers cited above.

4. Other classes of G -operators. Operators associated with diffeomorphisms are not exhausted by operators of the form (0.1). In this section we consider other classes of operators appearing in the literature. Boundary value problems similar to Carleman's problem, with the boundary condition relating the values of the unknown function at different points on the boundary were considered (see the monograph by Antonevich, Belousov and Lebedev [1] and the references cited there). Finiteness theorems were proved and index theorems were obtained for the case of finite group actions (see also [48]). On the other hand, nonlocal boundary value problems, in which the boundary condition relates the values of a function on the boundary of the domain and on submanifolds, which lie inside the domain, were considered in [15, 62, 63, 64]. We also mention that G -operators on manifolds with singularities were considered in [1]. The symbol was defined and a finiteness theorem was proved.

An important extension of the notion of (Fredholm) index was obtained in [36]. Namely, given a C^* -algebra A (the algebra of scalars) one considers operators F acting on the spaces, which are A -modules. The index of a Fredholm operator in this setting, also called Mishchenko–Fomenko index

$$\mathrm{ind}_A F \in K_0(A) \tag{0.4}$$

is an element of the K -group of A . Also, in the cited paper a definition of pseudodifferential operators over C^* -algebras was given and an index theorem was proved. Note, however, that it is sometimes useful in applications to have not only the index (0.4) but some *numerical* invariants. Such invariants can be constructed using the approach of noncommutative geometry by pairing the index (0.4) with cyclic cocycles over A . In the papers [38, 53, 57] G -operators over C^* -algebras were defined for isometric actions and the finiteness theorem and the index formula were obtained.

5. Methods used in the theory of G -operators. Let us now write a few words about the methods used in obtaining these index formulas. The first approach, which appears naturally, is to try to adapt the known methods of obtaining index formulas for ψ DOs in our more general setting of G -operators. This approach was successfully applied, for instance, in the book [38]. Note, however, that using this approach we obtain the proof of the index formula, which is quite nontrivial and relies on serious mathematical results, notions and constructions from noncommutative geometry and algebraic topology.

⁴This notion was introduced by Atiyah and Singer [8, 61] and actively studied since then (see especially [31, 32, 33] and the references cited there).

The second approach uses the idea of uniformization [55, 49] (see also [59, 47, 46]) to reduce the index problem for a G -operator to a similar problem for a *pseudodifferential* operator on a manifold of a higher dimension. The index of the latter operator can be found using the celebrated Atiyah–Singer formula. The attractiveness of this approach is based on the fact that this approach is quite elementary and does not require application of complicated mathematical apparatus, which was mentioned above. This method of *pseudodifferential uniformization* enabled to give simple and elegant index formulas.

Let us now describe the contents of the remaining sections of the paper. In Section 1 we recall the definitions of symbol and the finiteness theorem for G -operators associated with actions of discrete groups. Section 2 is devoted to index formulas for actions of discrete groups. We start with the index formula for isometric actions and then give an index formula for nonisometric actions. Finally, Section 3 is devoted to G -operators associated with compact Lie group actions. We show how pseudodifferential uniformization can be used to obtain a finiteness theorem for such operators.

1. Elliptic operators associated with actions of discrete groups

1.1. Main definitions

Let M be a closed smooth manifold and G a discrete group acting on M by diffeomorphisms. We consider the class of operators of the form

$$D = \sum_{g \in G} D_g T_g : C^\infty(M) \longrightarrow C^\infty(M), \quad (1.1)$$

where $\{D_g\}_{g \in G}$ is a collection of pseudodifferential operators of order $\leq m$ acting on M . We suppose that only finitely many D_g 's are nonzero. Finally, $\{T_g\}$ stands for the representation of G by the shift operators

$$(T_g u)(x) = u(g^{-1}(x)).$$

Here and below an element $g \in G$ takes a point $x \in M$ to the point denoted by $g(x) \in M$.

Main problems:

1. Give *ellipticity conditions*, under which the operator

$$D : H^s(M) \longrightarrow H^{s-m}(M), \quad m = \text{ord } D, \quad (1.2)$$

is Fredholm in the Sobolev spaces.

2. Compute the index of operator (1.2).

The first of these problems is treated in this section, while the second problem is treated in the subsequent section.

Below operators of the form (1.1) are called *G -pseudodifferential operators* or *G -operators* for short.

1.2. Symbols of operators

Definition of symbol. The action of G on M induces a representation of this group by automorphisms of the algebra $C(S^*M)$ of continuous functions on the cosphere bundle $S^*M = T_0^*M/\mathbb{R}_+$. Namely, an element $g \in G$ acts as a shift operator along the trajectory of the mapping $\partial g : S^*M \rightarrow S^*M$, which is the extension of g to the cotangent bundle and is defined as $\partial g = ({}^t(dg))^{-1}$, where $dg : TM \rightarrow TM$ is the differential. Consider the C^* -crossed product $C(S^*M) \rtimes G$ (e.g., see [16, 41, 68, 69]) of the algebra $C(S^*M)$ by the action of G . Recall that $C(S^*M) \rtimes G$ is the algebra, obtained as a completion of the algebra of compactly-supported functions on G with values in $C(S^*M)$ and the product of two elements is defined as:

$$ab(g) = \sum_{kl=g} a(k)k^{-1*}(b(l)), \quad k, l \in G.$$

The completion is taken with respect to a certain norm.⁵ Here for $k \in G$ by $k^{-1*} : C(S^*M) \rightarrow C(S^*M)$ we denote the above-mentioned automorphism of $C(S^*M)$.

To define the symbol for G -operators, it is useful to replace the shift operator $T_g : H^s(M) \rightarrow H^s(M)$ by a unitary operator. We fix a smooth positive density μ and a Riemannian metric on M and treat $H^s(M)$ as a Hilbert space with the norm

$$\|u\|_{H^s}^2 = \int_M |(1 + \Delta)^{s/2}u|^2 \mu,$$

where Δ is the nonnegative Laplacian. A direct computation shows that the operator

$$T_{g,s} = (1 + \Delta)^{-s/2} \mu^{-1/2} T_g \mu^{1/2} (1 + \Delta)^{s/2} : H^s(M) \rightarrow H^s(M)$$

is unitary. Here

$$\mu^{1/2} : L^2(M) \rightarrow L^2(M, \Lambda^{1/2})$$

is the isomorphism of L^2 spaces of scalar functions and half-densities on M defined by multiplication by the square root of μ . Note that the operator $T_{g,s}$ can be decomposed as $T_{g,s} = A_{g,s} T_g$, where $A_{g,s}$ is an invertible elliptic ψ DO of order zero.

This implies that the class of operators (1.1) does not change if in (1.1) we replace T_g by $T_{g,s}$. Now we can give the definition of the symbol.

Definition 1. The *symbol* of operator

$$D = \sum_{g \in G} D_g T_{g,s} : H^s(M) \rightarrow H^{s-m}(M), \quad (1.3)$$

where $\{D_g\}$ are pseudodifferential operators of order $\leq m$ on M , is an element

$$\sigma_s(D) \in C(S^*M) \rtimes G, \quad (1.4)$$

defined by the equality $\sigma_s(D)(g) = \sigma(D_g)$ for all $g \in G$.

⁵Below we consider the so-called maximal crossed product.

The symbol (1.4) is not completely convenient for applications, since it depends on the choice of Δ and μ . Here we give another definition of the symbol which is free from this drawback.

Trajectory symbol. So, let us try to define the symbol of the operator (1.2) using the method of frozen coefficients. Note that the operator is essentially nonlocal. More precisely, the corresponding equation $Du = f$ relates values of the unknown function u on the orbit $Gx_0 \subset M$, rather than at a single point $x_0 \in M$. For this reason, unlike the classical situation, we need to freeze the coefficients of the operator on the entire orbit of x_0 . Freezing the coefficients of the operator (1.2) on the orbit of x_0 and applying Fourier transform $x \mapsto \xi$, we can define the symbol as a function on the cotangent bundle $T_0^*M = T^*M \setminus 0$ with zero section deleted. This function ranges in operators acting on the space of functions on the orbit. A direct computation gives the following expression for the symbol (see [4, 52]):

$$\sigma(D)(x_0, \xi) = \sum_{h \in G} \sigma(D_h)(g^{-1}(x_0), \partial g^{-1}(\xi)) \mathcal{T}_h : l^2(G, \mu_{x_0, \xi, s}) \longrightarrow l^2(G, \mu_{x_0, \xi, s-m}). \quad (1.5)$$

Here we identify the orbit Gx_0 with the group G using the mapping $g(x_0) \mapsto g^{-1}$ and use the following notation:

- $(\mathcal{T}_h w)(g) = w(gh)$ is the right shift operator on the group;
- the expression $\sigma(D_h)(g^{-1}(x_0), \partial g^{-1}(\xi))$ acts as an operator of multiplication of functions on the group;
- the space $l^2(G, \mu_{x, \xi, s})$ consists of functions $\{w(g)\}$, $g \in G$, which are square summable with respect to the density $\mu_{x, \xi, s}$, which in local coordinates is defined by the expression [52]

$$\mu_{x, \xi, s}(g) = \left| \det \frac{\partial g^{-1}}{\partial x} \right| \cdot \left| \left(\frac{\partial g^{-1}}{\partial x} \right)^{-1}(\xi) \right|^{2s} \quad (1.6)$$

More precisely, here we suppose that the manifold is covered by a finite number of charts and the diffeomorphism g^{-1} is written (in some pair of charts) as $x \mapsto g^{-1}(x)$. The density is unique (up to equivalence of densities).

Definition 2. The operator (1.5) is the *trajectory symbol* of operator (1.2) at $(x_0, \xi) \in T_0^*M$.

Note that in general, the dependence of the trajectory symbol on x, ξ is quite complicated. For instance, the symbol may be discontinuous. This is related with the fact that the structure of the orbits can be quite complicated.

Let us describe the relation between the symbols defined in Definitions 2 and 1. Given $(x, \xi) \in S^*M$, we define the representation (restriction to trajectory)

$$\begin{aligned} \pi_{x, \xi} : C(S^*M) \times G &\longrightarrow \mathcal{B}l^2(G) \\ f &\longmapsto \sum_h f(g^{-1}(x), \partial g^{-1}(\xi), h) \mathcal{T}_h \end{aligned}$$

of the crossed product in the algebra of bounded operators acting on the standard space $l^2(G)$ (cf. (1.5)). One can show that the diagram

$$\begin{array}{ccc}
 l^2(G, \mu_{x,\xi,s}) & \xrightarrow{\sigma(D)(x,\xi)} & l^2(G, \mu_{x,\xi,s-m}) \\
 \downarrow \simeq & & \downarrow \simeq \\
 l^2(G) & \xrightarrow{\pi_{x,\xi}(\sigma_s(D))} & l^2(G),
 \end{array} \tag{1.7}$$

commutes, where the vertical mappings are isomorphisms defined by multiplication by the square root of the densities. In other words, this commutative diagram shows that the restriction of the symbol $\sigma_s(D)$ to a trajectory gives the trajectory symbol $\sigma(D)$.

1.3. Ellipticity and finiteness theorem

The two definitions of the symbol give two notions of ellipticity.

Definition 3. Operator (1.2) is *elliptic*, if its trajectory symbol (1.5) is invertible on T_0^*M .

Definition 4. Operator (1.2) is called *elliptic*, if its symbol (1.4) is invertible as an element of the algebra $C(S^*M) \rtimes G$.

It turns out that these definitions of ellipticity are equivalent, at least for a quite large class of groups. More precisely, the commutative diagram (1.7) shows that ellipticity in the sense of Definition 4 implies ellipticity in the sense of Definition 3; the inverse assertion is more complicated and was proved in [2] for actions of amenable groups (recall that a discrete group G is amenable, if there is a G -invariant mean on $l^\infty(G)$; for more details see, e.g., [40]). We suppose that below all groups are amenable and we identify these two notions of ellipticity.

The following finiteness theorem is proved by standard techniques (see [2, 1]).

Theorem 1. *If operator (1.2) is elliptic then it is Fredholm.*

Remark 1. It is shown in the cited monographs [2, 1] that under quite general assumptions (namely, the action of G on M is assumed to be topologically free, that is, for any finite set $\{g_1, \dots, g_n\} \subset G \setminus \{e\}$ the union $M^{g_1} \cup \dots \cup M^{g_n}$ of the fixed point sets has an empty interior), the ellipticity condition is necessary for the Fredholm property. If the action is not topologically free, then one could give a finer ellipticity condition. We do not consider these conditions here and refer the reader to the monograph [2].

1.4. Examples

Let us illustrate the notion of ellipticity for G -operators on several explicit examples.

1. Operators for the irrational rotations of the circle. Consider the group \mathbb{Z} of rotations of the circle \mathbb{S}^1 by multiples of a fixed angle θ not commensurable to π :

$$g(x) = x + g\theta, \quad x \in \mathbb{S}^1, \quad g \in \mathbb{Z}, \quad \theta \notin \pi\mathbb{Q}.$$

A direct computation shows that in this case the densities $\mu_{x,\xi,s}$ (see (1.6)) are equivalent to the standard density $\mu(g) = 1$ on the lattice \mathbb{Z} . Hence, in this case the symbol of the operator $D = \sum_{g \in \mathbb{Z}} D_g T_g$ is equal to

$$\sigma(D)(x, \xi) = \sum_h \sigma(D_h)(x - g\theta, \xi) \mathcal{T}^h : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z}), \quad \text{where } \mathcal{T}u(g) = u(g-1).$$

Let us make two remarks. First, in this example, as in the classical theory of ψ DOs, the symbol does not depend on s and therefore an operator is elliptic or not elliptic for all s simultaneously. The same property holds in the general case if the action is isometric. Second, in this case to check the ellipticity condition, it suffices to check that the symbol is invertible only for one pair of points $(x_0, \pm 1)$. Indeed, since $S^*\mathbb{S}^1 = \mathbb{S}^1 \cup \mathbb{S}^1$, the crossed product $C(S^*\mathbb{S}^1) \rtimes \mathbb{Z}$ is a direct sum of two simple algebras⁶ of irrational rotations $C(\mathbb{S}^1) \rtimes \mathbb{Z}$. Hence, the mapping

$$\pi_{x_0,1} \oplus \pi_{x_0,-1} : C(S^*\mathbb{S}^1) \rtimes \mathbb{Z} \longrightarrow \mathcal{B}l^2(\mathbb{Z}) \oplus \mathcal{B}l^2(\mathbb{Z})$$

is a monomorphism. Therefore, the symbol $\sigma(D)$ is invertible if and only the trajectory symbols at the points $(x_0, \pm 1)$ are invertible.

2. Operators for dilations of the sphere [58]. On the sphere \mathbb{S}^m we fix the North and the South poles. The complements of the poles are identified with \mathbb{R}^m with the coordinates x and x' , correspondingly. Let us choose the following transition function $x'(x) = x|x|^{-2}$. Consider the action of \mathbb{Z} on \mathbb{S}^m , which in the x -coordinates is generated by the dilations

$$g(x) = \alpha^g x, \quad g \in \mathbb{Z}, x \in \mathbb{R}^m,$$

where α ($0 < \alpha < 1$) is fixed. This expression defines a smooth action on the sphere. Let us compute the densities $\mu_{x,\xi,s}$.

Proposition 1. *Depending on whether x is a pole of the sphere or not, the density $\mu_{x,\xi,s}$ in (1.6) is equal to:*

$$\mu_{x,\xi,s}(g) = \begin{cases} \alpha^{|g|(m-2s)}, & \text{if } x \neq 0, x \neq \infty, \\ \alpha^{g(m-2s)}, & \text{if } x = 0, \\ \alpha^{-g(m-2s)}, & \text{if } x = \infty. \end{cases}$$

Proof. Indeed, given $g \leq 0$ the points $g^{-1}(x)$ remain in a bounded domain of the chart $\mathbb{S}^m \setminus \infty$. Thus, we can apply the formula (1.6), in which we use the x -coordinate in the domain and the range of the diffeomorphism g . We get $\partial g^{-1}/\partial x = \alpha^{-g}I$. Hence

$$\mu_{x,\xi,s}(g) = \left| \left(\frac{\partial g^{-1}}{\partial x} \right)^{-1}(\xi) \right|^{2s} = \alpha^{-gm} \cdot |\xi/\alpha^{-g}|^{2s} = \alpha^{-g(m-2s)} |\xi|^{2s}.$$

⁶That is, algebras without nontrivial ideals.

This gives the desired expression for the measure if $g \leq 0$. Now if $g \rightarrow +\infty$, then the points $g^{-1}(x) = \alpha^{-g}x$ tend to infinity and we can apply formula (1.6), where we use the pair of coordinates x and x' . A computation similar to the previous one gives the desired expression for the measure at the poles of the sphere: $x = 0$ and $x = \infty$. \square

Consider the operator

$$D = \sum_k D_k T^k : H^s(\mathbb{S}^m) \longrightarrow H^s(\mathbb{S}^m), \quad Tu(x) = u(\alpha^{-1}x). \quad (1.8)$$

According to the obtained expressions for the densities, this operator has the symbol $\sigma(D)(x, \xi)$ at each point $(x, \xi) \in T_0^*\mathbb{S}^m$. For example, consider the point $x = 0$. It follows from Proposition 1 that we obtain an expression for the symbol at this point

$$\sigma(D)(0, \xi) = \sum_k \sigma(D_k)(0, \xi) \mathcal{T}^k : l^2(\mathbb{Z}, \mu_s) \longrightarrow l^2(\mathbb{Z}, \mu_s), \quad \mu_s(n) = \alpha^{-n(m-2s)}.$$

Fourier transform $\{u(g)\} \mapsto \sum_g u(g)w^{-g}$ takes the latter operator to the operator of multiplication

$$\sigma_S(D)(\xi, w) = \sum_k \sigma(D_k)(0, \xi) w^k : L^2(\mathbb{S}^1) \longrightarrow L^2(\mathbb{S}^1), \quad \xi \in \mathbb{S}^{m-1}, \quad |w| = \alpha^{-m/2+s}$$

by a smooth function on the circle \mathbb{S}^1 of radius $\alpha^{-m/2+s}$. This shows that in this example the ellipticity condition explicitly depends on the smoothness exponent s . It was proved in [58] that the set of values of s for which the operator (1.8) is elliptic is an open interval (possibly (semi)infinite or empty).

2. Index formulas for actions of discrete groups

In the previous section we defined the symbol of a G -operator as an element of the corresponding crossed product. If an operator D elliptic (its symbol is invertible) then D has Fredholm property and its index $\text{ind } D$ is defined. To solve the index problem means to express the index in terms of the symbol of the operator and the topological characteristics of the G -manifold.

2.1. Isometric actions

The index problem for G -operators was solved in 2008 for isometric actions in [38]. Here we discuss the index formula from the cited monograph. This formula is proved under the following assumption.

Assumption 1.

1. G is a discrete group of *polynomial growth* (see [28]), i.e., the number of elements of the group, whose length is $\leq N$ in the word metric on the group, grows at most as a polynomial in N as $N \rightarrow \infty$.
2. M is a Riemannian manifold and the action of G on M is isometric.

Smooth crossed product. Let D be an elliptic operator. Then its symbol is invertible and defines an element

$$[\sigma(D)] \in K_1(C(S^*M) \rtimes G)$$

of the odd K -group of the crossed product $C(S^*M) \rtimes G$ (e.g., see [16]). Note a significant difference between the elliptic theory of G -operators and the classical Atiyah–Singer theory: the algebra of symbols is not commutative and therefore we use K -theory of algebras instead of topological K -theory. Further, to give an index formula, we will use tools from noncommutative differential geometry. Note that noncommutative differential geometry does not apply in general to C^* -algebras. The point here is that in a C^* -algebra there is a notion of continuity, but there is no differentiability. Fortunately, in the situation at hand, one can prove that we only deal with differentiable elements. Let us formulate this statement precisely.

Proposition 2 (see [60]). *If the symbol $\sigma(D)$ is invertible, then the inverse $\sigma(D)^{-1}$ lies in the subalgebra*

$$C^\infty(S^*M) \rtimes G \subset C(S^*M) \rtimes G, \quad (2.1)$$

of $C^\infty(S^*M)$ -valued functions on G , which (together with all their derivatives) tend to zero as $|g| \rightarrow \infty$ faster than an arbitrary power of $|g|$.

The subalgebra (2.1) is called the *smooth crossed product*. So, we have

$$[\sigma(D)] \in K_1(C^\infty(S^*M) \rtimes G). \quad (2.2)$$

To write an index formula for D , we first define a topological invariant of the symbol. This invariant is called the Chern character of the element (2.2). Then we define a topological invariant of the manifold.

Equivariant Chern character. Following [38], let us define the Chern character as the homomorphism of groups

$$\text{ch} : K_1(C^\infty(X) \rtimes G) \longrightarrow \bigoplus_{\langle g \rangle \subset G} H^{\text{odd}}(X^g), \quad (2.3)$$

where we put for brevity $X = S^*M$, the sum runs over conjugacy classes of G , and X^g denotes the fixed-point set of g . Since g is an isometry by assumption, the fixed-point set is a smooth submanifold (e.g., see [18]).

We define the Chern character using the abstract approach of noncommutative geometry. To this end, it suffices to define a pair (Ω, τ) , where:

1. $\Omega = \Omega_0 \oplus \Omega_1 \oplus \Omega_2 \oplus \cdots$ is a differential graded algebra, which contains the crossed product $C^\infty(X) \rtimes G$ as a subalgebra of Ω_0 ;

2. $\tau : \Omega \rightarrow \bigoplus_{\langle g \rangle \subset G} \Lambda(X^g)$ is a homomorphism of differential complexes such that

$$\tau(\omega_2 \omega_1) = (-1)^{\deg \omega_1 \deg \omega_2} \tau(\omega_1 \omega_2), \quad \text{for all } \omega_1, \omega_2 \in \Omega. \quad (2.4)$$

The algebra Ω is called the *algebra of noncommutative differential forms*, and the functional τ is called the *differential graded trace*.

If such a pair is given, then the Chern character associated with the pair (Ω, τ) is defined as

$$\text{ch}(a) = \text{tr } \tau \left[\sum_{n \geq 0} \frac{n!}{(2\pi i)^{n+1} (2n+1)!} (a^{-1} da)^{2n+1} \right], \quad [a] \in K_1(C^\infty(X) \rtimes G), \quad (2.5)$$

where tr is the trace of a matrix. A standard computation shows that the form in (2.5) is closed and its class in de Rham cohomology is determined by $[a]$ and defines the homomorphism (2.3). It remains to define the pair (Ω, τ) :

1. We set $\Omega = \Lambda(X) \rtimes G$, where the differential on the smooth crossed product of the algebra $\Lambda(X)$ of differential forms on X and the group G is equal to

$$(d\omega)(g) = d(\omega(g)), \quad \omega \in \Lambda(X) \rtimes G.$$

2. To define a differential graded trace $\tau = \{\tau_g\}$, we fix some $g \in G$ and introduce necessary notation. Let \overline{G} be the closure of G in the compact Lie group of isometries of X . This closure is a compact Lie group. Let $C_g \subset \overline{G}$ be the centralizer⁷ of g . The centralizer is a closed Lie subgroup in \overline{G} . Denote the elements of the centralizer by h , and the induced smooth Haar measure on the centralizer by dh .

Let $\langle g \rangle \subset G$ be the conjugacy class of g , i.e., the set of elements equal to zgz^{-1} for some $z \in G$. Further, for each $g' \in \langle g \rangle$ we fix some element $z = z(g, g')$, which conjugates g and $g' = zgz^{-1}$. Any such element defines a diffeomorphism $z : X^g \rightarrow X^{g'}$.

Let us define the trace as

$$\tau_g(\omega) = \sum_{g' \in \langle g \rangle} \int_{C_g} h^*(z^* \omega(g')) \Big|_{X^g} dh, \quad \text{where } \omega \in \Lambda(X) \rtimes G. \quad (2.6)$$

One can show that this expression does not depend on the choice of elements z and is indeed a differential graded trace.

Remark 2. For a finite group the Chern character (2.5) coincides with the one constructed in [65], [14].

⁷Recall that the centralizer of g is the subgroup of elements commuting with g .

Equivariant Todd class. Given $g \in G$, the normal bundle of the fixed-point submanifold $M^g \subset M$ is denoted by N^g . The differential dg defines an orthogonal endomorphism of N^g and the corresponding bundle of exterior forms

$$\Omega(N_{\mathbb{C}}^g) = \Omega^{ev}(N_{\mathbb{C}}^g) \oplus \Omega^{\text{odd}}(N_{\mathbb{C}}^g).$$

Here $E_{\mathbb{C}}$ stands for the complexification of a real vector bundle E . Consider the expression (see [12])

$$\text{ch } \Omega^{ev}(N_{\mathbb{C}}^g)(g) - \text{ch } \Omega^{\text{odd}}(N_{\mathbb{C}}^g)(g) \in H^{ev}(M^g). \quad (2.7)$$

The zero-degree component of this expression is nonzero [10]. Hence the class (2.7) is invertible and the following expression is well defined

$$\text{Td}_g(T_{\mathbb{C}}^*M) = \frac{\text{Td}(T_{\mathbb{C}}^*M^g)}{\text{ch } \Omega^{ev}(N_{\mathbb{C}}^g)(g) - \text{ch } \Omega^{\text{odd}}(N_{\mathbb{C}}^g)(g)} \in H^*(M^g), \quad (2.8)$$

where Td on the right-hand side in the equality is the Todd class of a complex vector bundle, and the expression is well defined, since the forms have even degrees.

Index theorem.

Theorem 2 (see [38]). *Let D be an elliptic G -operator on a closed manifold M . Then*

$$\text{ind } D = \sum_{\langle g \rangle \subset G} \langle \text{ch}_g[\sigma(D)] \text{Td}_g(T_{\mathbb{C}}^*M), [S^*M^g] \rangle, \quad (2.9)$$

where $\langle g \rangle$ runs over the set of conjugacy classes of G ; $[S^*M^g] \in H_{\text{odd}}(S^*M^g)$ is the fundamental class of S^*M^g ; the Todd class is lifted from M^g to S^*M^g using the natural projection; the brackets \langle, \rangle denote the pairing of cohomology and homology. The series in (2.9) is absolutely convergent.

In some situations the sum in (2.9) can be reduced to one summand equal to the contribution of the unit element.

Corollary 1 (see [38, 54]). *Suppose that either the action of G on M is free or G is torsion free. Then one has*

$$\text{ind } D = \langle \text{ch}_e[\sigma(D)] \text{Td}(T_{\mathbb{C}}^*M), [S^*M] \rangle. \quad (2.10)$$

Let us note that the index formula (2.9) contains many other index formulas as special cases (see [38, 54] for details). Here we give two situations, in which the index formula can be applied.

Example 1. Index of twisted Toeplitz operators. Let M be an odd-dimensional oriented manifold. We suppose that M is endowed with a G -invariant spin-structure (i.e., the action of G on M lifts to an action on the spin bundle $S(M)$). Let \mathcal{D} be the Dirac operator [12]

$$\mathcal{D} : S(M) \longrightarrow S(M),$$

acting on spinors. This operator is elliptic and self-adjoint.

Denote by $\Pi_+ : S(M) \longrightarrow S(M)$ the positive spectral projection of this operator.

We define the Toeplitz operator

$$\Pi_+ U : \Pi_+(S(M)) \otimes \mathbb{C}^n \longrightarrow \Pi_+(S(M)) \otimes \mathbb{C}^n, \tag{2.11}$$

where U is an invertible n by n matrix with elements in $C^\infty(M) \rtimes G$. Then the operator (2.11) is Fredholm (its almost-inverse is equal to $\Pi_+ U^{-1}$). Let us suppose for simplicity that either G is torsion free, or the action is free. In this case, the formula (2.10) gives the following expression for the index.

Theorem 3. *The index of operator (2.11) is equal to*

$$\text{ind}(\Pi_+ U) = \int_M A(TM) \text{ch}_e(U), \tag{2.12}$$

where $A(TM)$ is the A -class of the tangent bundle, which in the Borel–Hirzebruch formalism is defined by the function

$$\frac{x/2}{\text{sh } x/2}.$$

Examples 2. Operators on noncommutative torus. Let us fix $0 < \theta \leq 1$. A. Connes in [22] considered differential operators of the form

$$D = \sum_{\alpha+\beta \leq m} a_{\alpha\beta} x^\alpha \left(-i \frac{d}{dx}\right)^\beta : S(\mathbb{R}) \longrightarrow S(\mathbb{R}), \tag{2.13}$$

in the Schwartz space $S(\mathbb{R})$ on the real line. Here the coefficients $a_{\alpha\beta}$ are Laurent polynomials in operators U, V

$$(Uf)(x) = f(x + 1), \quad (Vf)(x) = e^{-2\pi i x/\theta} f(x) \tag{2.14}$$

of shift by one and product by exponential.

Let us show that the operators of the form (2.13) reduce to G -operators on a closed manifold. To this end, we consider the real line as the total space of the standard covering

$$\mathbb{R} \longrightarrow \mathbb{S}^1,$$

whose base is the circle of length θ . Then the Schwartz space becomes isomorphic to the space of smooth sections of a (nontrivial) bundle on the base \mathbb{S}^1 , whose fiber is the Schwartz space $S(\mathbb{Z})$ of rapidly decaying sequences (that is, functions on the fiber). Then we apply Fourier transform

$$\mathcal{F} : S(\mathbb{Z}) \longrightarrow C^\infty(\mathbb{S}^1)$$

in each fiber and obtain a space, which is the space of smooth sections of a complex line bundle over the torus \mathbb{T}^2 . These transformations define the isomorphism

$$S(\mathbb{R}) \simeq C^\infty(\mathbb{T}^2, \gamma) \tag{2.15}$$

of the Schwartz space on the real line and the space

$$C^\infty(\mathbb{T}^2, \gamma) = \{g \in C^\infty(\mathbb{R} \times \mathbb{S}^1) \mid g(\varphi + \theta, \psi) = g(\varphi, \psi)e^{-2\pi i \psi}\}$$

of smooth sections of a complex line bundle γ on the torus. Here on \mathbb{T}^2 we consider the coordinates $0 \leq \varphi \leq \theta, 0 \leq \psi \leq 1$. This isomorphism is defined by the formula

$$f(x) \mapsto \sum_{n \in \mathbb{Z}} f(\varphi + \theta n) e^{2\pi i n \psi}.$$

(This formula was found earlier by S. Novikov.) Using the isomorphism (2.15), it is easy to obtain the correspondences between the operators:

operators on the line	operators on the torus
$-i \frac{d}{dx}$	$-i \frac{\partial}{\partial \varphi}$
x	$-i \frac{\theta}{2\pi} \frac{\partial}{\partial \psi} + \psi$
$e^{-2\pi i x / \theta}$	$e^{-2\pi i \varphi / \theta}$
$f(x) \rightarrow f(x + 1)$	$g(\varphi, \psi) \mapsto g(\varphi + 1, \psi)$

This table implies that on the torus we obtain G -operators, which can be studied using the finiteness theorem and the index formula formulated above. We refer the reader to [38] for details.

2.2. General actions

In this subsection we survey index formulas for elliptic operators associated with general actions of discrete groups (see recent papers [49] and [44]). Let D be an elliptic operator of the form (1.1). We will assume for simplicity that the inverse symbol $\sigma(D)^{-1}$ lies in the algebraic crossed product

$$C^\infty(S^*M) \rtimes_{alg} G \subset C(S^*M) \rtimes G,$$

which consists of compactly supported functions on the group. Such a symbol defines an element

$$[\sigma(D)] \in K_1(C^\infty(S^*M) \rtimes_{alg} G) \tag{2.16}$$

in K -theory. We would like to define the topological index as a numerical invariant associated with $[\sigma(D)]$. There is a standard procedure in noncommutative geometry of constructing such invariants. Namely, one takes the pairing of (2.16) with an element in *cyclic cohomology* of the same algebra. Let us recall this construction.

Cyclic cohomology. Pairing with K -theory. Let A be an algebra with unit. Recall (see [22]) that the *cyclic cohomology* $HC^*(A)$ of A is the cohomology of the bicomplex

$$\begin{array}{ccccccc}
 & & & & A^* & \cdots & \\
 & & & & \uparrow B & & \\
 & & & A^* & \xrightarrow{b} & A^{*2} & \cdots \\
 & & & \uparrow B & & \uparrow B & \\
 A^* & \xrightarrow{b} & A^{*2} & \xrightarrow{b} & A^{*3} & \cdots & \\
 \vdots & & \vdots & & \vdots & &
 \end{array} \tag{2.17}$$

where b, B are some differentials and for simplicity we denote the space of multi-linear functionals on A^k by A^{*k} . In particular, an element $\varphi \in HC^n(A)$ of cyclic cohomology is represented by a finite collection of multi-linear functionals

$$\{\varphi_j(a_0, \dots, a_j)\}, \quad j = n, n-2, n-4, \dots,$$

such that $B\varphi_j + b\varphi_{j-2} = 0$.

To make the paper self-contained, we recall the formulas for the differentials in the bicomplex:

$$(b\varphi)(a_0, a_1, \dots, a_{j+1}) = \sum_{n=0}^j \varphi(a_0, a_1, \dots, a_n a_{n+1}, \dots, a_{j+1}) + (-1)^{j+1} \varphi(a_{j+1} a_0, a_1, \dots, a_j). \quad (2.18)$$

and $B = Ns(Id - \lambda)$, where $\lambda = (-1)^n$ (cyclic left shift),

$$s : A^{*(n+1)} \longrightarrow A^{*n}, \quad (s\varphi)(a_0, \dots, a_{n-1}) = \varphi(1, a_0, \dots, a_{n-1}),$$

and $N : A^{*n} \longrightarrow A^{*n}$, $N = Id + \lambda + \lambda^2 + \dots + \lambda^{n-1}$ is the symmetrization mapping.

The desired numerical invariants are defined using the pairing

$$\langle \cdot, \cdot \rangle : K_1(A) \times HC^{\text{odd}}(A) \longrightarrow \mathbb{C} \quad (2.19)$$

of K -theory and cyclic cohomology. The value of this pairing on the classes $[a]$ and $[\varphi]$ is equal to

$$\langle a, \varphi \rangle = \frac{1}{\sqrt{2\pi i}} \sum_{k \geq 0} (-1)^k k! \varphi_{2k+1}(a^{-1}, a, \dots, a^{-1}, a).$$

Now to define the topological index of the element (2.16), it remains to choose a cocycle over the algebra. It turns out that the desired cocycle can be defined as a special *equivariant characteristic class* in cyclic cohomology.

Equivariant characteristic classes. Suppose that a discrete group G acts smoothly on a closed smooth manifold X . We shall also assume that X is oriented and the action is orientation-preserving. Let $E \in \text{Vect}_G(X)$ be a finite-dimensional complex G -bundle on X . Connes defined (e.g., see [22]) equivariant characteristic classes of E with values in cyclic cohomology $HC^*(C^\infty(X) \rtimes_{\text{alg}} G)$ of the crossed product. However, the formulas for these classes were quite complicated and we do not give them here. A simple explicit formula was obtained in [27] for the most important characteristic class, namely, for the equivariant Chern character

$$\text{ch}_G(E) \in HC^*(C^\infty(X) \rtimes_{\text{alg}} G). \quad (2.20)$$

More precisely, it was shown in the cited paper that the class $\text{ch}_G(E)$ is represented by the collection of functionals $\{\text{ch}_G^k(E)\}$ defined as

$$\begin{aligned} & \text{ch}_G^k(E; a_0, a_1, \dots, a_k) \\ &= \frac{(-1)^{(n-k)/2}}{((n+k)/2)!} \sum_{i_0+i_1+\dots+i_k=(n-k)/2} \int_X \text{tr}_E [(a_0 \theta^{i_0} \nabla(a_1) \theta^{i_1} \nabla(a_2) \dots \nabla(a_k) \theta^{i_k})_e] \end{aligned} \quad (2.21)$$

(cf. Jaffe–Lesniewski–Osterwalder formula [29]). Here

$$\dim X = n, \quad k = n, n-2, n-4, \dots,$$

∇_E is a connection in E and $\theta = \nabla_E^2$ is its curvature form, for a noncommutative form ω by ω_e we denote the coefficient of $T_e = 1$, while the operator

$$\nabla : C^\infty(X) \rtimes_{\text{alg}} G \rightarrow \Lambda^1(X, \text{End } E) \rtimes_{\text{alg}} G$$

is defined as

$$\nabla \left(\sum_g a_g T_g \right) = \sum_g [da_g - a_g(\nabla_E - (g^{-1})^* \nabla_E)] T_g.$$

It is proved in the cited paper that the collection of functionals $\{\text{ch}_G^k(E)\}$ defines a cocycle over $C^\infty(X) \rtimes_{\text{alg}} G$, and the class of this cocycle in cyclic cohomology does not depend on the choice of ∇_E and coincides with the equivariant Chern character defined by Connes [22].

Explicit formulas for other characteristic classes can be obtained using standard topological techniques (operations in K -theory, see [9]). For the index theorem, we need the equivariant Todd class.

Proposition 3 ([49]). *The equivariant Todd class*

$$\text{Td}_G(E) \in HC^*(C^\infty(X) \rtimes_{\text{alg}} G) \quad (2.22)$$

of a complex G -bundle E on a smooth manifold X is equal to

$$\text{Td}_G(E) = \text{ch}_G(\Phi(E)),$$

here Φ is the multiplicative operation in K -theory, which corresponds to the function $\varphi(t) = t^{-1}(1+t)\ln(1+t)$.

Note that Φ can be expressed explicitly in terms of Grothendieck operations. For instance, if $\dim X \leq 5$ then (see [49])

$$\begin{aligned} \Phi(E) &= 1 + \frac{E-n}{2} + \frac{-2(E^2 - 2nE + n^2) + 7(E + \Lambda^2 E - nE + n(n-1)/2)}{12} \\ &= \frac{3n^2 - 19n + 24}{24} + \frac{(-3n + 13)}{12} E - \frac{1}{6} E \otimes E + \frac{7}{12} \Lambda^2 E \end{aligned} \quad (2.23)$$

where $n = \dim E$.

Index theorem.

Theorem 4 ([49]). *Let D be an elliptic operator associated with the action of group \mathbb{Z} . Then we have the index formula*

$$\text{ind } D = (2\pi i)^{-n} \langle [\sigma(D)], \text{Td}_{\mathbb{Z}}(\pi^* T_{\mathbb{C}}^* M) \rangle, \quad \dim M = n, \quad (2.24)$$

where $\pi : S^*M \rightarrow M$ is the natural projection and the brackets $\langle \cdot, \cdot \rangle$ denote the pairing of K -theory and cyclic cohomology (see (2.19)).

Remark 3. An index formula for operators on the circle associated with an action of an arbitrary torsion free group is announced in [44]. The index formula in this case has the same form as (2.24).

Examples. 1. Suppose that the (usual) Todd class $\text{Td}(T_{\mathbb{C}}^*M)$ is trivial and the diffeomorphisms of the \mathbb{Z} -action are isotopic to the identity. Then one can show that the equivariant Todd class is equal to the transverse fundamental cycle (see [21]) of S^*M and the index formula (2.24) is written as:

$$\text{ind } D = \frac{(n-1)!}{(2\pi i)^n (2n-1)!} \int_{S^*M} (\sigma^{-1} d\sigma)_e^{2n-1}, \quad \sigma = \sigma(D). \quad (2.25)$$

2. Suppose that the group acts isometrically. Then formula (2.24) reduces to (2.10) This is obvious if we choose an invariant metric and connection on the cotangent bundle.

3. Elliptic operators for compact Lie groups

3.1. Main definitions

Let a compact Lie group G act smoothly on a closed smooth manifold M . An element $g \in G$ takes a point $x \in M$ to the point denoted by $g(x)$. We fix a G -invariant metric on M and the Haar measure on G .

Consider the representation $g \mapsto T_g$ of G in the space $L^2(M)$ by shift operators

$$T_g u(x) = u(g^{-1}(x)).$$

Definition 5. A G -pseudodifferential operator (G - ψ DO) is an operator

$$D : L^2(M) \longrightarrow L^2(M)$$

of the form

$$D = 1 + \int_G D_g T_g dg, \quad (3.1)$$

where $D_g, g \in G$ is a smooth family of pseudodifferential operators of order zero on M .

Consider the equation

$$u + \int_G D_g T_g u dg = f, \quad u, f \in L^2(M). \quad (3.2)$$

Note that if G is discrete, then we obtain the class of equations (1.1).

Example 1. Integro-differential equations on the torus. On the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with coordinates x_1, x_2 , consider the integro-differential equation

$$\Delta u(x_1, x_2) + \alpha \frac{\partial^2}{\partial x_1^2} \int_{\mathbb{S}^1} u(x_1, y) dy = f(x_1, x_2),$$

where Δ stands for the nonnegative Laplace operator, and α is a constant. Let us write this equation as

$$\Delta u + \alpha \frac{\partial^2}{\partial x_1^2} \int_{\mathbb{S}^1} T_g dg u = f, \quad (3.3)$$

where T_g denotes the shift operator $T_g u(x_1, x_2) = u(x_1, x_2 - g)$, induced by the action of the circle $G = \mathbb{S}^1$ by shifts in x_2 . Note that if we multiply the equation (3.3) on the left by the almost inverse operator Δ^{-1} , we obtain an equation of the type (3.2).

Example 2. Integro-differential equations on the plane. Consider the integro-differential equation

$$\Delta u(x, y) + \left(\alpha \frac{\partial^2}{\partial x^2} + \beta \frac{\partial^2}{\partial x \partial y} + \gamma \frac{\partial^2}{\partial y^2} \right) \int_{\mathbb{S}^1} u(x \cos \varphi - y \sin \varphi, x \sin \varphi + y \cos \varphi) d\varphi = f(x, y)$$

on the plane $\mathbb{R}_{x,y}^2$, where Δ is the Laplace operator, and α, β, γ are constants. This equation can be written as

$$\Delta u + \left(\alpha \frac{\partial^2}{\partial x^2} + \beta \frac{\partial^2}{\partial x \partial y} + \gamma \frac{\partial^2}{\partial y^2} \right) \int_{\mathbb{S}^1} T_\varphi d\varphi u = f, \quad (3.4)$$

where the shift operator T_φ is induced by the action of the circle $G = \mathbb{S}^1$ by rotations

$$(x, y) \mapsto (x \cos \varphi + y \sin \varphi, -x \sin \varphi + y \cos \varphi)$$

around the origin. If we multiply the equation (3.4) on the left by the almost inverse operator Δ^{-1} , we obtain an equation similar to (3.2).

3.2. Pseudodifferential uniformization

Here we formulate an approach, called *pseudodifferential uniformization*, which enables one to reduce a G -pseudodifferential operator

$$D = 1 + \int_G D_g T_g dg : L^2(M) \longrightarrow L^2(M). \quad (3.5)$$

to a pseudodifferential operator and then apply the methods of the theory of pseudodifferential operators.

1. Reduction to a ψ DO. This reduction is constructed as follows.

- The operator D is represented as an operator on the quotient M/G (the space of orbits).
- If the action of G on M has no fixed points, then M/G is a smooth manifold; moreover, D can be treated as a ψ DO on M/G with operator-valued symbol in the sense of Luke [35] (explanation: this follows from the fact that the operator T_g acts only along the fibers of the infinite-dimensional bundle over M/G , but not along the base).
- If the fixed point set is nonempty, then M/G has singularities; to construct a ψ DO in this case, we do the following.
- We lift D from M to the product $M \times G$ endowed with the diagonal action of G :

$$(x, h) \mapsto (g(x), gh). \quad (3.6)$$



FIGURE 1. The orbits of the diagonal and vertical actions on the product $M \times G$.

- The action (3.6) is fixed point free. Hence, the obtained G -pseudodifferential operator on $M \times G$, which we denote by \tilde{D} , can be represented (see above) as a ψ DO on the smooth orbit space $(M \times G)/G \simeq M$.

These steps give the commutative diagram

$$\begin{array}{ccc}
 L^2(M) & \xrightarrow{D} & L^2(M) \\
 \pi^* \downarrow & & \downarrow \pi^* \\
 L^2(M \times G) & \xrightarrow{\tilde{D}} & L^2(M \times G) \\
 \downarrow \simeq & & \downarrow \simeq \\
 L^2(M, L^2(G)) & \xrightarrow{\mathcal{D}} & L^2(M, L^2(G)),
 \end{array} \tag{3.7}$$

where π^* is the induced mapping for the projection $\pi : M \times G \rightarrow M$, while \mathcal{D} is a pseudodifferential operator on M .

Remark 4. The fact that \mathcal{D} is a ψ DO is clear for geometric reasons. Indeed, \tilde{D} has shifts along the orbits of the diagonal action of G (see Figure 1, left). Clearly, these orbits can be transformed into vertical orbits (see Figure 1, right) by a change of variables on $M \times G$. The shift operator along the vertical orbits is a ψ DO on M .

2. Restriction of ψ DO to the subspace of invariant sections. The mapping π^* in (3.7) is a monomorphism. Its range is the space of G -invariant sections. Hence, (3.7) gives a commutative diagram of the form

$$\begin{array}{ccc}
 L^2(M) & \xrightarrow{D} & L^2(M) \\
 \downarrow \simeq & & \downarrow \simeq \\
 L^2(M, L^2(G))^G & \xrightarrow{\mathcal{D}^G} & L^2(M, L^2(G))^G,
 \end{array}$$

where \mathcal{D}^G stands for the restriction of \mathcal{D} to the subspace of invariant sections, which we denote by $L^2(M, L^2(G))^G$.

3. Transverse ellipticity. It remains to give conditions, which imply that the restriction

$$L^2(M, L^2(G))^G \xrightarrow{\mathcal{D}^G} L^2(M, L^2(G))^G$$

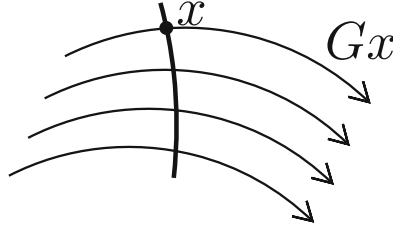


FIGURE 2. Transversal to the orbit.

of \mathcal{D} to the subspace of G -invariant sections is Fredholm. Let us note that invariant sections are constant along the orbits of the group action. Hence, it suffices to impose the condition, which guarantees the Fredholm property, only along the transverse directions to the orbit (see Figure 2).

Definition 6 ([8, 61]). A pseudodifferential operator \mathcal{D} is *transversally elliptic*, if its symbol $\sigma(\mathcal{D})(x, \xi)$ is invertible for all $(x, \xi) \in T_G^*M \setminus 0$, where

$$T_G^*M = \{(x, \xi) \in T^*M \mid \text{covector } \xi \text{ is orthogonal to the orbit } Gx\}.$$

stands for the *transverse cotangent bundle*.

Theorem 5 ([55, 66]). *A transversally elliptic operator \mathcal{D} restricts to a Fredholm operator on the subspace of G -invariant sections*

$$\mathcal{D}^G : L^2(M, L^2(G))^G \longrightarrow L^2(M, L^2(G))^G.$$

Let us summarize the above discussion.

1. To a G -pseudodifferential operator D we assigned a pseudodifferential operator \mathcal{D} such that there is an isomorphism

$$D \simeq \mathcal{D}^G, \tag{3.8}$$

where \mathcal{D}^G is the restriction of \mathcal{D} to the subspace of invariant sections.

2. If \mathcal{D} is transversally elliptic, then its restriction \mathcal{D}^G is Fredholm. Hence, by virtue of the isomorphism (3.8) the original operator D is also Fredholm.

Since we have isomorphism (3.8), we obtain:

$$\text{ind } D = \text{ind } \mathcal{D}^G.$$

Using this equation and the theory of transversally elliptic pseudodifferential operators [8, 30, 67]), an index theorem for the G -operator D was obtained in [55, 50]. We only mention here that the main ingredients of the index formula are: 1) the definition of the symbol of D as an element of the crossed product of the algebra of functions on the transverse cotangent bundle by the group G ; 2) a Chern character mapping on the K -theory of this algebra ranging in the basic cohomology of fixed point sets of the group action.

References

- [1] A. Antonevich, M. Belousov, and A. Lebedev, Functional differential equations II C^* -applications, Parts 1, 2, Number 94, 95 in *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman, Harlow, 1998.
- [2] A. Antonevich and A. Lebedev, Functional-Differential Equations I C^* -Theory, Number 70 in *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman, Harlow, 1994.
- [3] A.B. Antonevich. Elliptic pseudodifferential operators with a finite group of shifts, *Math. USSR-Izv.*, **7** (1973), 661–674.
- [4] A.B. Antonevich, *Linear Functional Equations, Operator Approach*, Universitetskoe, Minsk, 1988.
- [5] A.B. Antonevich, Strongly nonlocal boundary value problems for elliptic equations, *Izv. Akad. Nauk SSSR Ser. Mat.*, **53**(1) (1989), 3–24.
- [6] A.B. Antonevich and V.V. Brenner, On the symbol of a pseudodifferential operator with locally independent shifts, *Dokl. Akad. Nauk BSSR*, **24**(10) (1980), 884–887.
- [7] A.B. Antonevich and A.V. Lebedev, Functional equations and functional operator equations A C^* -algebraic approach, in *Proceedings of the St. Petersburg Mathematical Society, Vol. VI, Amer. Math. Soc. Transl. Ser. 2* **199**, 2000, 25–116.
- [8] M.F. Atiyah. *Elliptic Operators and Compact Groups*, Lecture Notes in Mathematics **401**, Springer-Verlag, Berlin, 1974.
- [9] M.F. Atiyah. *K-Theory*, Second Edition, The Advanced Book Program, Addison–Wesley, Inc., 1989.
- [10] M.F. Atiyah and G.B. Segal, The index of elliptic operators II, *Ann. Math.* **87** (1968), 531–545.
- [11] M.F. Atiyah and I.M. Singer, The index of elliptic operators on compact manifolds, *Bull. Amer. Math. Soc.* **69** (1963), 422–433.
- [12] M.F. Atiyah and I.M. Singer, The index of elliptic operators III, *Ann. Math.* **87** (1968), 546–604.
- [13] Ch. Babbage. An assay towards the calculus of functions, part II, *Philos. Trans. of the Royal Society* **106** (1816), 179–256.
- [14] P. Baum and A. Connes, Chern character for discrete groups, in *A fête of topology*, Academic Press, Boston, MA, 1988, 163–232.
- [15] A.V. Bitsadze and A.A. Samarskii, On some simple generalizations of linear elliptic boundary problems, *Sov. Math., Dokl.* **10** (1969), 398–400.
- [16] B. Blackadar, *K-Theory for Operator Algebras*, Second Edition, Mathematical Sciences Research Institute Publications **5**, Cambridge University Press, 1998.
- [17] T. Carleman, Sur la théorie des équations intégrales et ses applications, *Verh. Internat. Math.-Kongr. Zurich*, **1**, 1932, 138–151.
- [18] P.E. Conner and E.E. Floyd, *Differentiable Periodic Maps*, Academic Press, New York, 1964.
- [19] A. Connes, C^* -algèbres et géométrie différentielle, *C. R. Acad. Sci. Paris Sér. A-B* **290**(13) (1980), A599–A604.
- [20] A. Connes, Noncommutative differential geometry, *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 257–360.

- [21] A. Connes, Cyclic cohomology and the transverse fundamental class of a foliation, in *Geometric Methods in Operator Algebras*, Pitman Res. Notes in Math. **123**, Longman, Harlow, 1986.
- [22] A. Connes, *Noncommutative Geometry*, Academic Press Inc., San Diego, CA, 1994.
- [23] A. Connes and M. Dubois-Violette, Noncommutative finite-dimensional manifolds I, spherical manifolds and related examples, *Comm. Math. Phys.* **230**(3) (2002), 539–579.
- [24] A. Connes and G. Landi, Noncommutative manifolds, the instanton algebra and isospectral deformations, *Comm. Math. Phys.* **221**(1) (2001), 141–159.
- [25] A. Connes and H. Moscovici, Type III and spectral triples, in *Traces in Number Theory, Geometry and Quantum Fields*, Aspects Math., E38, Friedr. Vieweg, Wiesbaden, 2008, 57–71.
- [26] I.M. Gelfand, On elliptic equations, *Russian Math. Surveys* **15**(3) (1960), 113–127.
- [27] A. Gorokhovsky, Characters of cycles, equivariant characteristic classes and Fredholm modules, *Comm. Math. Phys.* **208**(1) (1999), 1–23.
- [28] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.*, **53** (1981), 53–73.
- [29] A. Jaffe, A. Lesniewski, and K. Osterwalder, Quantum K -theory I. The Chern character, *Comm. Math. Phys.* **118**(1) (1988), 1–14.
- [30] T. Kawasaki, The index of elliptic operators over V -manifolds, *Nagoya Math. J.* **84** (1981), 135–157.
- [31] Yu.A. Kordyukov, Transversally elliptic operators on G -manifolds of bounded geometry, *Russian J. Math. Phys.* **2**(2) (1994), 175–198.
- [32] Yu.A. Kordyukov, Transversally elliptic operators on G -manifolds of bounded geometry II, *Russian J. Math. Phys.* **3**(1) (1995), 41–64.
- [33] Yu.A. Kordyukov, Index theory and non-commutative geometry on foliated manifolds, *Russ. Math. Surv.* **64**(2) (2009), 273–391.
- [34] G. Landi and W. van Suijlekom, Principal fibrations from noncommutative spheres, *Comm. Math. Phys.* **260**(1) (2005), 203–225.
- [35] G. Luke, Pseudodifferential operators on Hilbert bundles, *J. Diff. Equations* **12** (1972), 566–589.
- [36] A.S. Mishchenko and A.T. Fomenko, The index of elliptic operators over C^* -algebras, *Izv. Akad. Nauk SSSR Ser. Mat.* **43**(4) (1979), 831–859, 967.
- [37] H. Moscovici, Local index formula and twisted spectral triples, in *Quanta of Maths*, Clay Math. Proc. **11**, Amer. Math. Soc., Providence, RI, 2010, 465–500.
- [38] V.E. Nazaïkinskii, A.Yu. Savin, and B.Yu. Sternin, *Elliptic Theory and Noncommutative Geometry, Operator Theory: Advances and Applications* **183**, Birkhäuser Verlag, Basel, 2008.
- [39] E. Park, Index theory of Toeplitz operators associated to transformation group C^* -algebras, *Pacific J. Math.* **223**(1) (2006), 159–165.
- [40] A.L.T. Paterson, *Amenability*, Mathematical Surveys and Monographs **29**, American Mathematical Society, Providence, RI, 1988.
- [41] G.K. Pedersen, *C^* -Algebras and Their Automorphism Groups*, London Mathematical Society Monographs **14**, Academic Press, London–New York, 1979.

- [42] D. Perrot. A Riemann–Roch theorem for one-dimensional complex groupoids. *Comm. Math. Phys.*, 218(2):373–391, 2001.
- [43] D. Perrot, Localization over complex-analytic groupoids and conformal renormalization, *J. Noncommut. Geom.* **3**(2) (2009), 289–325.
- [44] D. Perrot, On the Radul cocycle, *Oberwolfach reports* (2011), 53–55. DOI: 10.4171/OWR/2011/45.
- [45] L.E. Rossovskii, Boundary value problems for elliptic functional-differential equations with dilatation and contraction of the arguments, *Trans. Moscow Math. Soc.* (2001), 185–212.
- [46] A. Savin, E. Schrohe, and B. Sternin, On the index formula for an isometric diffeomorphism, arXiv:1112.5515, 2011.
- [47] A. Savin, E. Schrohe, and B. Sternin, Uniformization and an index theorem for elliptic operators associated with diffeomorphisms of a manifold, arXiv:1111.1525, 2011.
- [48] A. Savin and B. Sternin, Index defects in the theory of nonlocal boundary value problems and the η -invariant, *Sbornik: Mathematics* **195**(9) (2004), 1321–1358.
- [49] A. Savin and B. Sternin, Index of elliptic operators for a diffeomorphism. Journal of Noncommutative Geometry, V. 7, 2013. Preliminary Version: arxiv:1106.4195, 2011.
- [50] A.Yu. Savin, On the index of nonlocal elliptic operators for compact Lie groups, *Cent. Eur. J. Math.* **9**(4) (2011), 833–850.
- [51] A.Yu. Savin, On the index of nonlocal operators associated with a nonisometric diffeomorphism, *Mathematical Notes* **90**(5) (2011), 701–714.
- [52] A.Yu. Savin, On the symbol of nonlocal operators in Sobolev spaces, *Differential Equations*, **47**(6) (2011), 897–900.
- [53] A.Yu. Savin and B.Yu. Sternin, Index of nonlocal elliptic operators over C^* -algebras, *Dokl. Math.* **79**(3) (2009), 369–372.
- [54] A.Yu. Savin and B.Yu. Sternin, Noncommutative elliptic theory. Examples, *Proceedings of the Steklov Institute of Mathematics* **271** (2010), 193–211.
- [55] A.Yu. Savin and B.Yu. Sternin, Nonlocal elliptic operators for compact Lie groups, *Dokl. Math.* **81**(2) (2010), 258–261.
- [56] A.Yu. Savin. On the index of elliptic operators associated with a diffeomorphism of a manifold, *Doklady Mathematics* **82**(3) (2010), 884–886.
- [57] A.Yu. Savin and B.Yu. Sternin. On the index of noncommutative elliptic operators over C^* -algebras, *Sbornik: Mathematics* **201**(3) (2010), 377–417.
- [58] A.Yu. Savin and B.Yu. Sternin, Nonlocal elliptic operators for the group of dilations, *Sbornik: Mathematics* **202**(10) (2011), 1505–1536.
- [59] A.Yu. Savin, B.Yu. Sternin, and E. Schrohe. Index problem for elliptic operators associated with a diffeomorphism of a manifold and uniformization, *Dokl. Math.* **84**(3) (2011), 846–849.
- [60] L.B. Schweitzer, Spectral invariance of dense subalgebras of operator algebras, *Internat. J. Math.* **4**(2) (1993), 289–317.
- [61] I.M. Singer, Recent applications of index theory for elliptic operators, in *Partial Differential Equations*, Proc. Sympos. Pure Math., Vol. XXIII, Amer. Math. Soc., Providence, R.I., 1973, 11–31.

- [62] A.L. Skubachevskii, *Elliptic Functional Differential Equations and Applications*, Birkhäuser, Basel-Boston-Berlin, 1997.
- [63] A.L. Skubachevskii, Nonclassical boundary-value problems I, *Journal of Mathematical Sciences* **155**(2) (2008), 199–334.
- [64] A.L. Skubachevskii, Nonclassical boundary-value problems II, *Journal of Mathematical Sciences* **166**(4) (2010), 377–561.
- [65] J. Slominska, On the equivariant Chern homomorphism, *Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys.*, **24** (1976), 909–913.
- [66] B.Yu. Sternin, On a class of nonlocal elliptic operators for compact Lie groups. Uniformization and finiteness theorem, *Cent. Eur. J. Math.* **9**(4) (2011), 814–832.
- [67] M. Vergne, Equivariant index formulas for orbifolds, *Duke Math. J.* **82**(3) (1996), 637–652.
- [68] D.P. Williams, *Crossed Products of C^* -Algebras*, *Mathematical Surveys and Monographs* **134**, American Mathematical Society, Providence, RI, 2007.
- [69] G. Zeller-Meier, Produits croisés d'une C^* -algèbre par un groupe d'automorphismes, *J. Math. Pures Appl.* **47**(9) (1968), 101–239.

Anton Savin and Boris Sternin
Peoples' Friendship University of Russia
Miklukho-Maklaya str. 6
RU-117198 Moscow, Russia

and

Leibniz University of Hannover, Institut für Analysis
Welfengarten 1
D-30167 Hannover, Germany
e-mail: antonsavin@mail.ru
sternin@mail.ru