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Pseudo-Differential Operators, Generalized Functions and Asymptotics

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Preface

At the Eighth Congress of the International Society for Analysis, its Applications and Computations (ISAAC) held at the Peoples' Friendship University of Russia in Moscow on August 22–27, 2011, a new initiative on selecting contributions from two special sessions, one on Pseudo-Differential Operators and the other on Generalized Functions and Asymptotics, for one volume was taken to heart by many participants. This resonates well with the grandeur of ISAAC of considering Analysis, Applications and Computations on an international scale as a unified discipline. This can be achieved, notwithstanding the diversity of the disciplines, by building synergies among clusters consisting of several closely related disciplines. To that end, volumes on pseudo-differential operators and applications in mathematical sciences have been published since the ISAAC Congress held at York University in 2003. The present volume entitled "Pseudo-Differential Operators, Generalized Functions and Asymptotics" is another project with this vision in mind.

This volume contains three categories of papers, originated from the Eighth ISAAC Congress or solicited by invitations, corresponding to each of the three areas in the title. The category of papers on pseudo-differential operators contains such topics as elliptic operators associated to diffeomorphisms of smooth manifolds, analysis on singular manifolds with edges, heat kernels and Green functions of sub-Laplacians on the Heisenberg group and Lie groups with more complexities than but closely related to the Heisenberg group, L^p-boundedness of pseudo-differential operators on the torus, pseudo-differential operators and Gelfand–Shilov spaces, and pseudo-differential operators in the context of time-frequency analysis.

The second group of papers is on generalized functions. Various classes of distributions and algebras of generalized functions are used for various linear partial differential equations and some of these have nonregular coefficients. Moreover, nonlinear problems with nonregular initial values or boundary conditions are treated in this framework. Featured in this volume are also papers on stochastic and Malliavin-type differential equations in which generalized functions are instrumental. This second group of papers are related to the third collection of papers via the setting of Colombeau-type spaces and algebras in which microlocal analysis is developed by means of various techniques of asymptotics.

This volume contains interesting topics in pseudo-differential operators, generalized functions and asymptotics that are essential in modern mathematical sciences and engineering. It is a volume that put different but related areas of analysis on an equal footing. It is through working with colleagues with a diversity of related expertise and through regular meetings and publishing that we can deepen our understanding of a vast area of mathematics that has been known as analysis.

Operator Theory:
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Elliptic Theory for Operators Associated with Diffeomorphisms of Smooth Manifolds

Anton Savin and Boris Sternin

Abstract. In this paper we give a survey of elliptic theory for operators associated with diffeomorphisms of smooth manifolds. Such operators appear naturally in analysis, geometry and mathematical physics. We survey classical results as well as results obtained recently. The paper consists of an introduction and three sections. In the introduction we give a general overview of the area of research. For the reader's convenience here we tried to keep special terminology to a minimum. In the remaining sections we give detailed formulations of the most important results mentioned in the introduction.

Mathematics Subject Classification (2010). Primary 58J20; Secondary 58J28, 58J32, 19K56, 46L80, 58J22.

Keywords. Elliptic operator, index, index formula, cyclic cohomology, diffeomorphism, G-operator.

Introduction

The aim of this paper is to give a survey of index theory for elliptic operators associated with diffeomorphisms of smooth manifolds. Recall that the construction of index theory includes the following main stages:

1) (finiteness theorem) Here one gives conditions, called ellipticity conditions, under which the operators under consideration are Fredholm in relevant function spaces;
2) (index theorem) Here one presents and proves an index formula, that is, an expression for the index of an elliptic operator in terms of topological invariants of the symbol of the operator and the manifold, on which the operator is defined.

The first index theorem on high-dimensional manifolds was the celebrated Atiyah–Singer theorem [11] on the index of elliptic pseudodifferential operators (ψDO) on a closed smooth manifold. This theorem appeared as an answer to a question posed by Gelfand [26]. Note that the statement and the proof of the index formula relied on most up to date methods of analysis and topology and stimulated interactions between them.

After that index theorems were obtained for many other classes of operators. In this paper we consider the class of operators associated with diffeomorphisms of closed smooth manifolds. One of advantages of this theory is that, besides the mentioned interaction of analysis and topology, here an important role is played by the theory of dynamical systems.

1. Elliptic operators for a discrete group of diffeomorphisms (analytical aspects). The theory of elliptic operators associated with diffeomorphisms and the corresponding theory of boundary value problems with nonlocal boundary conditions go back to the paper by T. Carleman [17], where he considered the problem of finding a holomorphic function in a bounded domain Ω, which satisfies a nonlocal boundary condition, which relates the values of the function at a point $x \in \partial\Omega$ of the boundary and at the point $g(x) \in \partial\Omega$, where $g : \partial\Omega \to \partial\Omega$ is a smooth mapping of period two: $g^2 = Id$. A reduction of this boundary value problem to the boundary does not give usual integral equation as it was the case with the local boundary condition. Rather, it gives an integro-functional equation, which we call *equation associated with diffeomorphism g*. This paper motivated the study of a more general class of operators on closed smooth manifolds. Let us give the general definition of such operators.

On a closed smooth manifold M we consider operators of the form

$$D = \sum_{g\in G} D_g T_g : C^\infty(M) \to C^\infty(M), \tag{0.1}$$

where:

- G is a discrete group of diffeomorphisms of M;
- $(T_g u)(x) = u(g^{-1}(x))$ is the shift operator corresponding to the diffeomorphism g;
- $\{D_g\}$ is a collection of pseudodifferential operators of order $\le m$;
- $C^\infty(M)$ is the space of smooth functions on M. Of course, one can also consider operators acting in sections of vector bundles.

Operators (0.1) will be called *G-pseudodifferential operators* (G-ψDO) or simply G-operators.[1] Such operators were intensively studied (see the fundamental works of Antonevich [4, 5], and also the papers [2, 7] and the references cited there). In particular, an extremely important notion of *symbol of a G-operator* was introduced there. More precisely, two definitions of the symbol of a G-operator were given. First, the symbol was defined as a function on the cotangent bundle T^*M of the manifold taking values in operators acting on the space $l^2(G)$ of square integrable functions on the group. Second, the symbol was defined as an element of the crossed product [69] of the algebra of continuous functions on the cosphere

[1] In the literature such operators are also called functional-differential, nonlocal, noncommutative operators and operators with shifts.

bundle S^*M of the manifold and the group G. Further, we introduce the ellipticity condition in this situation, which is the requirement of invertibility of the symbol of the operator. It was proved that the two ellipticity conditions (they correspond to the two definitions of the symbol) are equivalent under quite general assumptions. Ellipticity implies Fredholm property of the operator in Sobolev spaces H^s.

Let us note here one essential difference between the theory of elliptic G-ψDO and a similar theory of ψDO. Namely, examples show (see [6, 1, 45]) that the ellipticity (and the Fredholm property of operator (0.1) in the Sobolev spaces H^s) essentially depends on the smoothness exponent s. Thus, there arise natural questions on the description of the possible values of s, for which a given G-operator is elliptic and the question about the dependence of the index on s. The answers to these questions are well known in the situation of an isometric action of the group, that is, if the diffeomorphisms preserve a Riemannian metric on the manifold. In this case the symbol and the index do not depend on s. First steps in the study of these questions for nonisometric actions were done in the papers [58, 52], where it was shown for the simplest nonisometric diffeomorphism of dilation of spheres that the set of s, for which a G-operator is elliptic, is always an interval and the index (inside this interval) does not depend on s.

2. Index of elliptic operators for a discrete group of diffeomorphisms. Let us now turn attention to the problem of computing the index of elliptic G-operators. The first formula for the index of G-operators was obtained in the paper [3] for a *finite group* G of diffeomorphisms.[2] In this case the index of a G-operator was expressed in terms of Lefschetz numbers of an auxiliary elliptic ψDO on M. Since the Lefschetz numbers are expressed by a formula [12] similar to the Atiyah–Singer index formula the index problem for a finite group is thus solved.

The index problem for infinite groups turned out to be much more difficult and required application of new methods related with noncommutative geometry of Connes [20, 22]. The first advance was done in the celebrated work of Connes [19]. There an index formula was obtained for operators of the form

$$D = \sum_{\alpha\beta} a_{\alpha\beta} x^\alpha (d/dx)^\beta \tag{0.2}$$

acting on the real line, where the coefficients $a_{\alpha\beta}$ are Laurent polynomials in the operators

$$(Uf)(x) = e^{ix} f(x), \quad (Vf)(x) = f(x-\theta),$$

and θ is some fixed number. The index theorem of Connes for such differential-difference operators is naturally formulated in terms of noncommutative geometry. Operators (0.2), which are also called operators on the noncommutative torus[3], were used in a mathematical formulation of the quantum Hall effect [22]. It became

[2] These results were rediscovered in [39].

[3] This name is motivated by the fact that the algebra generated by U and V is a noncommutative deformation of the algebra of functions on the torus $\mathbb{T}^2$.

clear after the cited papers of Connes that noncommutative geometry is not only useful, but also natural in the index problem for G-operators, and since then noncommutative geometry is used in all the papers on the index of G-operators, we are aware of. For instance, methods of noncommutative geometry were applied to solve the index problem for deformations of algebras of functions on toric manifolds in [34, 23, 24] and other papers.

Further progress in the solution of the index problem for G-operators was made in the monograph [38]. Namely, an index formula for operators (0.1) was obtained in the situation, when the action is isometric. Let us note here that this index formula for isometric actions contains all the above-mentioned formulas as special cases.

In the situation of a general (that is, *nonisometric*) action there were no index formulas until recently. There were only partial results. Namely, the index problem for $\mathbb{Z}$-operators (that is, operators for the group of integers) was reduced to a similar problem for an elliptic ψDO (see [4, 51, 56]). The first index formula in the nonisometric case was obtained in the paper [58] for operators associated with dilation diffeomorphism of spheres. The index formula for elliptic operators associated with the group $\mathbb{Z}$ was obtained in [49]. Finally, an index formula for an arbitrary torsion free group acting on the circle was stated in [44].

Let us mention several interesting examples of elliptic G-operators. Suppose that G preserves some geometric structure on the manifold (for instance, Riemannian metric, complex structure, spin structure,...). Then we can consider an elliptic operator associated with that structure and twist this operator using a G-projection (that is an operator of the form (0.1), which is a projection: $P^2 = P$) or an invertible G-operator. This construction produces an elliptic G-operator. For instance, if G acts isometrically, then one can take classical geometric operators (Euler, signature, Dolbeault, Dirac operators). The indices of the corresponding twisted G-operators were computed in [54]. If G acts by conformal diffeomorphisms of a Riemannian surface, then one can take the $\overline{\partial}$ operator. Indices of the corresponding twisted operators were computed in [43, 42]. In the papers [25, 37] there are index formulas for the twisted Dirac operator for group actions preserving the conformal structure on the manifold.

3. Operators associated with compact Lie groups. Let now G be a *compact Lie group* acting on M. Consider the class of operators of the form

$$D = \int_G D_g T_g dg : C^\infty(M) \longrightarrow C^\infty(M) \tag{0.3}$$

(cf. (0.1)), where dg is the Haar measure. Such operators relate values of functions on submanifolds of M of positive dimension. They were considered in [55, 66, 50]. In these papers a G-operator of the form (0.3) was represented as a pseudodifferential operator acting in sections of infinite-dimensional bundles [35], whose fiber is the space of functions on G. This method goes back to the papers of Babbage [13] and for a finite group gives a finite system of equations [4]. Moreover, the obtained operator, which we denote by $\mathcal{D}$, is G-invariant, and its restriction $\mathcal{D}^G$ to the subspace of G-invariant functions is isomorphic to the original operator D. Now if $\widehat{\mathcal{D}} = 1 + \mathcal{D}$ is transversally elliptic[4] with respect to the action of G, then this implies the Fredholm property, that is, the index of the operator $\widehat{D} = 1 + D$ is finite. The index formula and the corresponding topological invariants of the symbol of elliptic G-operators were computed in the papers cited above.

4. Other classes of *G*-operators. Operators associated with diffeomorphisms are not exhausted by operators of the form (0.1). In this section we consider other classes of operators appearing in the literature. Boundary value problems similar to Carleman's problem, with the boundary condition relating the values of the unknown function at different points on the boundary were considered (see the monograph by Antonevich, Belousov and Lebedev [1] and the references cited there). Finiteness theorems were proved and index theorems were obtained for the case of finite group actions (see also [48]). On the other hand, nonlocal boundary value problems, in which the boundary condition relates the values of a function on the boundary of the domain and on submanifolds, which lie inside the domain, were considered in [15, 62, 63, 64]. We also mention that G-operators on manifolds with singularities were considered in [1]. The symbol was defined and a finiteness theorem was proved.

An important extension of the notion of (Fredholm) index was obtained in [36]. Namely, given a C^*-algebra A (the algebra of scalars) one considers operators F acting on the spaces, which are A-modules. The index of a Fredholm operator in this setting, also called Mishchenko–Fomenko index

$$\operatorname{ind}_A F \in K_0(A) \tag{0.4}$$

is an element of the K-group of A. Also, in the cited paper a definition of pseudodifferential operators over C^*-algebras was given and an index theorem was proved. Note, however, that it is sometimes useful in applications to have not only the index (0.4) but some *numerical* invariants. Such invariants can be constructed using the approach of noncommutative geometry by pairing the index (0.4) with cyclic cocycles over A. In the papers [38, 53, 57] G-operators over C^*-algebras were defined for isometric actions and the finiteness theorem and the index formula were obtained.

5. Methods used in the theory of *G*-operators. Let us know write a few words about the methods used in obtaining these index formulas. The first approach, which appears naturally, is to try to adapt the known methods of obtaining index formulas for ψDOs in our more general setting of G-operators. This approach was successfully applied, for instance, in the book [38]. Note, however, that using this approach we obtain the proof of the index formula, which is quite nontrivial and relies on serious mathematical results, notions and constructions from noncommutative geometry and algebraic topology.

[4]This notion was introduced by Atiyah and Singer [8, 61] and actively studied since then (see especially [31, 32, 33] and the references cited there).

The second approach uses the idea of uniformization [55, 49] (see also [59, 47, 46]) to reduce the index problem for a G-operator to a similar problem for a *pseudodifferential* operator on a manifold of a higher dimension. The index of the latter operator can be found using the celebrated Atiyah–Singer formula. The attractiveness of this approach is based on the fact that this approach is quite elementary and does not require application of complicated mathematical apparatus, which was mentioned above. This method of *pseudodifferential uniformization* enabled to give simple and elegant index formulas.

Let us now describe the contents of the remaining sections of the paper. In Section 1 we recall the definitions of symbol and the finiteness theorem for G-operators associated with actions of discrete groups. Section 2 is devoted to index formulas for actions of discrete groups. We start with the index formula for isometric actions and then give an index formula for nonisometric actions. Finally, Section 3 is devoted to G-operators associated with compact Lie group actions. We show how pseudodifferential uniformization can be used to obtain a finiteness theorem for such operators.

1. Elliptic operators associated with actions of discrete groups

1.1. Main definitions

Let M be a closed smooth manifold and G a discrete group acting on M by diffeomorphisms. We consider the class of operators of the form

$$D = \sum_{g\in G} D_g T_g : C^\infty(M) \longrightarrow C^\infty(M), \tag{1.1}$$

where $\{D_g\}_{g\in G}$ is a collection of pseudodifferential operators of order $\leq m$ acting on M. We suppose that only finitely many D_g's are nonzero. Finally, $\{T_g\}$ stands for the representation of G by the shift operators

$$(T_g u)(x) = u(g^{-1}(x)).$$

Here and below an element $g \in G$ takes a point $x \in M$ to the point denoted by $g(x) \in M$.

Main problems:

1. Give *ellipticity conditions*, under which the operator
$$D : H^s(M) \longrightarrow H^{s-m}(M), \quad m = \operatorname{ord} D, \tag{1.2}$$
is Fredholm in the Sobolev spaces.
2. Compute the index of operator (1.2).

The first of these problems is treated in this section, while the second problem is treated in the subsequent section.

Below operators of the form (1.1) are called *G-pseudodifferential operators* or *G-operators* for short.

1.2. Symbols of operators

Definition of symbol. The action of G on M induces a representation of this group by automorphisms of the algebra $C(S^*M)$ of continuous functions on the cosphere bundle $S^*M = T_0^*M/\mathbb{R}_+$. Namely, an element $g \in G$ acts as a shift operator along the trajectory of the mapping $\partial g : S^*M \to S^*M$, which is the extension of g to the cotangent bundle and is defined as $\partial g = ({}^t(dg))^{-1}$, where $dg : TM \to TM$ is the differential. Consider the C^*-crossed product $C(S^*M) \rtimes G$ (e.g., see [16, 41, 68, 69]) of the algebra $C(S^*M)$ by the action of G. Recall that $C(S^*M) \rtimes G$ is the algebra, obtained as a completion of the algebra of compactly-supported functions on G with values in $C(S^*M)$ and the product of two elements is defined as:

$$ab(g) = \sum_{kl=g} a(k) k^{-1^*}(b(l)), \quad k, l \in G.$$

The completion is taken with respect to a certain norm.[5] Here for $k \in G$ by $k^{-1^*} : C(S^*M) \to C(S^*M)$ we denote the above-mentioned automorphism of $C(S^*M)$.

To define the symbol for G-operators, it is useful to replace the shift operator $T_g : H^s(M) \longrightarrow H^s(M)$ by a unitary operator. We fix a smooth positive density μ and a Riemannian metric on M and treat $H^s(M)$ as a Hilbert space with the norm

$$\|u\|^2_{H^s} = \int_M |(1+\Delta)^{s/2} u|^2 \mu,$$

where Δ is the nonnegative Laplacian. A direct computation shows that the operator

$$T_{g,s} = (1+\Delta)^{-s/2} \mu^{-1/2} T_g \mu^{+1/2} (1+\Delta)^{s/2} : H^s(M) \longrightarrow H^s(M)$$

is unitary. Here

$$\mu^{1/2} : L^2(M) \to L^2(M, \Lambda^{1/2})$$

is the isomorphism of L^2 spaces of scalar functions and half-densities on M defined by multiplication by the square root of μ. Note that the operator $T_{g,s}$ can be decomposed as $T_{g,s} = A_{g,s} T_g$, where $A_{g,s}$ is an invertible elliptic ψDO of order zero.

This implies that the class of operators (1.1) does not change if in (1.1) we replace T_g by $T_{g,s}$. Now we can give the definition of the symbol.

Definition 1. The *symbol* of operator

$$D = \sum_{g \in G} D_g T_{g,s} : H^s(M) \longrightarrow H^{s-m}(M), \tag{1.3}$$

where $\{D_g\}$ are pseudodifferential operators of order $\leq m$ on M, is an element

$$\sigma_s(D) \in C(S^*M) \rtimes G, \tag{1.4}$$

defined by the equality $\sigma_s(D)(g) = \sigma(D_g)$ for all $g \in G$.

[5] Below we consider the so-called maximal crossed product.

The symbol (1.4) is not completely convenient for applications, since it depends on the choice of Δ and μ. Here we give another definition of the symbol which is free from this drawback.

Trajectory symbol. So, let us try to define the symbol of the operator (1.2) using the method of frozen coefficients. Note that the operator is essentially nonlocal. More precisely, the corresponding equation $Du = f$ relates values of the unknown function u on the *orbit* $Gx_0 \subset M$, rather than at a single point $x_0 \in M$. For this reason, unlike the classical situation, we need to freeze the coefficients of the operator on the entire orbit of x_0. Freezing the coefficients of the operator (1.2) on the orbit of x_0 and applying Fourier transform $x \mapsto \xi$, we can define the symbol as a function on the cotangent bundle $T^*_0M = T^*M \setminus 0$ with zero section deleted. This function ranges in operators acting on the space of functions on the orbit. A direct computation gives the following expression for the symbol (see [4, 52]):

$$\sigma(D)(x_0,\xi) = \sum_{h\in G} \sigma(D_h)(g^{-1}(x_0), \partial g^{-1}(\xi))\mathcal{T}_h : l^2(G, \mu_{x_0,\xi,s}) \longrightarrow l^2(G, \mu_{x_0,\xi,s-m}). \tag{1.5}$$

Here we identify the orbit Gx_0 with the group G using the mapping $g(x_0) \mapsto g^{-1}$ and use the following notation:

- $(\mathcal{T}_h w)(g) = w(gh)$ is the right shift operator on the group;
- the expression $\sigma(D_h)(g^{-1}(x_0), \partial g^{-1}(\xi))$ acts as an operator of multiplication of functions on the group;
- the space $l^2(G, \mu_{x,\xi,s})$ consists of functions $\{w(g)\}$, $g \in G$, which are square summable with respect to the density $\mu_{x,\xi,s}$, which in local coordinates is defined by the expression [52]

$$\mu_{x,\xi,s}(g) = \left|\det \frac{\partial g^{-1}}{\partial x}\right| \cdot \left|{}^t\left(\frac{\partial g^{-1}}{\partial x}\right)^{-1}(\xi)\right|^{2s} \tag{1.6}$$

  More precisely, here we suppose that the manifold is covered by a finite number of charts and the diffeomorphism g^{-1} is written (in some pair of charts) as $x \mapsto g^{-1}(x)$. The density is unique (up to equivalence of densities).

Definition 2. The operator (1.5) is the *trajectory symbol* of operator (1.2) at $(x_0, \xi) \in T^*_0M$.

Note that in general, the dependence of the trajectory symbol on x, ξ is quite complicated. For instance, the symbol may be discontinuous. This is related with the fact that the structure of the orbits can be quite complicated.

Let us describe the relation between the symbols defined in Definitions 2 and 1. Given $(x, \xi) \in S^*M$, we define the representation (restriction to trajectory)

$$\begin{array}{rcl} \pi_{x,\xi} : C(S^*M) \rtimes G & \longrightarrow & \mathcal{B}l^2(G) \\ f & \longmapsto & \sum_h f(g^{-1}(x), \partial g^{-1}(\xi), h)\mathcal{T}_h \end{array}$$

of the crossed product in the algebra of bounded operators acting on the standard space $l^2(G)$ (cf. (1.5)). One can show that the diagram

$$\begin{array}{ccc} l^2(G,\mu_{x,\xi,s}) & \xrightarrow{\sigma(D)(x,\xi)} & l^2(G,\mu_{x,\xi,s-m}) \\ \downarrow \simeq & & \downarrow \simeq \\ l^2(G) & \xrightarrow[\pi_{x,\xi}(\sigma_s(D))]{} & l^2(G), \end{array} \tag{1.7}$$

commutes, where the vertical mappings are isomorphisms defined by multiplication by the square root of the densities. In other words, this commutative diagram shows that the restriction of the symbol $\sigma_s(D)$ to a trajectory gives the trajectory symbol $\sigma(D)$.

1.3. Ellipticity and finiteness theorem

The two definitions of the symbol give two notions of ellipticity.

Definition 3. Operator (1.2) is *elliptic*, if its trajectory symbol (1.5) is invertible on T_0^*M.

Definition 4. Operator (1.2) is called *elliptic*, if its symbol (1.4) is invertible as an element of the algebra $C(S^*M) \rtimes G$.

It turns out that these definitions of ellipticity are equivalent, at least for a quite large class of groups. More precisely, the commutative diagram (1.7) shows that ellipticity in the sense of Definition 4 implies ellipticity in the sense of Definition 3; the inverse assertion is more complicated and was proved in [2] for actions of amenable groups (recall that a discrete group G is amenable, if there is a G-invariant mean on $l^\infty(G)$; for more details see, e.g., [40]). We suppose that below all groups are amenable and we identify these two notions of ellipticity.

The following finiteness theorem is proved by standard techniques (see [2, 1]).

Theorem 1. *If operator* (1.2) *is elliptic then it is Fredholm.*

Remark 1. It is shown in the cited monographs [2, 1] that under quite general assumptions (namely, the action of G on M is assumed to be topologically free, that is, for any finite set $\{g_1,\ldots,g_n\} \subset G \setminus \{e\}$ the union $M^{g_1} \cup \cdots \cup M^{g_n}$ of the fixed point sets has an empty interior), the ellipticity condition is necessary for the Fredholm property. If the action is not topologically free, then one could give a finer ellipticity condition. We do not consider these conditions here and refer the reader to the monograph [2].

1.4. Examples

Let us illustrate the notion of ellipticity for G-operators on several explicit examples.

1. Operators for the irrational rotations of the circle. Consider the group $\mathbb{Z}$ of rotations of the circle $\mathbb{S}^1$ by multiples of a fixed angle θ not commensurable to π:

$$g(x) = x + g\theta, \quad x \in \mathbb{S}^1, \quad g \in \mathbb{Z}, \quad \theta \notin \pi\mathbb{Q}.$$

A direct computation shows that in this case the densities $\mu_{x,\xi,s}$ (see (1.6)) are equivalent to the standard density $\mu(g) = 1$ on the lattice $\mathbb{Z}$. Hence, in this case the symbol of the operator $D = \sum_{g\in\mathbb{Z}} D_g T_g$ is equal to

$$\sigma(D)(x,\xi) = \sum_h \sigma(D_h)(x - g\theta, \xi)\mathcal{T}^h : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad \text{where } \mathcal{T}u(g) = u(g-1).$$

Let us make two remarks. First, in this example, as in the classical theory of ψDOs, the symbol does not depend on s and therefore an operator is elliptic or not elliptic for all s simultaneously. The same property holds in the general case if the action is isometric. Second, in this case to check the ellipticity condition, it suffices to check that the symbol is invertible only for one pair of points $(x_0, \pm 1)$. Indeed, since $S^*\mathbb{S}^1 = \mathbb{S}^1 \cup \mathbb{S}^1$, the crossed product $C(S^*\mathbb{S}^1) \rtimes \mathbb{Z}$ is a direct sum of two simple algebras[6] of irrational rotations $C(\mathbb{S}^1) \rtimes \mathbb{Z}$. Hence, the mapping

$$\pi_{x_0,1} \oplus \pi_{x_0,-1} : C(S^*\mathbb{S}^1) \rtimes \mathbb{Z} \longrightarrow \mathcal{B}l^2(\mathbb{Z}) \oplus \mathcal{B}l^2(\mathbb{Z})$$

is a monomorphism. Therefore, the symbol $\sigma(D)$ is invertible if and only the trajectory symbols at the points $(x_0, \pm 1)$ are invertible.

2. Operators for dilations of the sphere [58]. On the sphere $\mathbb{S}^m$ we fix the North and the South poles. The complements of the poles are identified with $\mathbb{R}^m$ with the coordinates x and x', correspondingly. Let us choose the following transition function $x'(x) = x|x|^{-2}$. Consider the action of $\mathbb{Z}$ on $\mathbb{S}^m$, which in the x-coordinates is generated by the dilations

$$g(x) = \alpha^g x, \quad g \in \mathbb{Z}, x \in \mathbb{R}^m,$$

where α $(0 < \alpha < 1)$ is fixed. This expression defines a smooth action on the sphere. Let us compute the densities $\mu_{x,\xi,s}$.

Proposition 1. *Depending on whether x is a pole of the sphere or not, the density $\mu_{x,\xi,s}$ in (1.6) is equal to:*

$$\mu_{x,\xi,s}(g) = \begin{cases} \alpha^{|g|(m-2s)}, & \text{if } x \neq 0, x \neq \infty, \\ \alpha^{g(m-2s)}, & \text{if } x = 0, \\ \alpha^{-g(m-2s)}, & \text{if } x = \infty. \end{cases}$$

Proof. Indeed, given $g \leq 0$ the points $g^{-1}(x)$ remain in a bounded domain of the chart $\mathbb{S}^m \setminus \infty$. Thus, we can apply the formula (1.6), in which we use the x-coordinate in the domain and the range of the diffeomorphism g. We get $\partial g^{-1}/\partial x = \alpha^{-g} I$. Hence

$$\mu_{x,\xi,s}(g) = \left| {}^t\left(\frac{\partial g^{-1}}{\partial x}\right)^{-1}(\xi)\right|^{2s} = \alpha^{-gm} \cdot |\xi/\alpha^{-g}|^{2s} = \alpha^{-g(m-2s)}|\xi|^{2s}.$$

[6] That is, algebras without nontrivial ideals.

This gives the desired expression for the measure if $g \le 0$. Now if $g \to +\infty$, then the points $g^{-1}(x) = \alpha^{-g}x$ tend to infinity and we can apply formula (1.6), where we use the pair of coordinates x and x'. A computation similar to the previous one gives the desired expression for the measure at the poles of the sphere: $x = 0$ and $x = \infty$. □

Consider the operator

$$D = \sum_k D_k T^k : H^s(\mathbb{S}^m) \longrightarrow H^s(\mathbb{S}^m), \quad Tu(x) = u(\alpha^{-1}x). \tag{1.8}$$

According to the obtained expressions for the densities, this operator has the symbol $\sigma(D)(x,\xi)$ at each point $(x,\xi) \in T^*_0\mathbb{S}^m$. For example, consider the point $x = 0$. It follows from Proposition 1 that we obtain an expression for the symbol at this point

$$\sigma(D)(0,\xi) = \sum_k \sigma(D_k)(0,\xi)\mathcal{T}^k : l^2(\mathbb{Z},\mu_s) \longrightarrow l^2(\mathbb{Z},\mu_s), \quad \mu_s(n) = \alpha^{-n(m-2s)}.$$

Fourier transform $\{u(g)\} \mapsto \sum_g u(g)w^{-g}$ takes the latter operator to the operator of multiplication

$$\sigma_S(D)(\xi,w) = \sum_k \sigma(D_k)\,(0,\xi)\,w^k : L^2(\mathbb{S}^1) \longrightarrow L^2(\mathbb{S}^1),\ \xi \in \mathbb{S}^{m-1},\ |w| = \alpha^{-m/2+s}$$

by a smooth function on the circle $\mathbb{S}^1$ of radius $\alpha^{-m/2+s}$. This shows that in this example the ellipticity condition explicitly depends on the smoothness exponent s. It was proved in [58] that the set of values of s for which the operator (1.8) is elliptic is an open interval (possibly (semi)infinite or empty).

2. Index formulas for actions of discrete groups

In the previous section we defined the symbol of a G-operator as an element of the corresponding crossed product. If an operator D elliptic (its symbol is invertible) then D has Fredholm property and its index $\operatorname{ind} D$ is defined. To solve the index problem means to express the index in terms of the symbol of the operator and the topological characteristics of the G-manifold.

2.1. Isometric actions

The index problem for G-operators was solved in 2008 for isometric actions in [38]. Here we discuss the index formula from the cited monograph. This formula is proved under the following assumption.

Assumption 1.

1. G is a discrete group of *polynomial growth* (see [28]), i.e., the number of elements of the group, whose length is $\le N$ in the word metric on the group, grows at most as a polynomial in N as $N \to \infty$.
2. M is a Riemannian manifold and the action of G on M is isometric.

Smooth crossed product. Let D be an elliptic operator. Then its symbol is invertible and defines an element

$$[\sigma(D)] \in K_1(C(S^*M) \rtimes G)$$

of the odd K-group of the crossed product $C(S^*M) \rtimes G$ (e.g., see [16]). Note a significant difference between the elliptic theory of G-operators and the classical Atiyah–Singer theory: the algebra of symbols is not commutative and therefore we use K-theory of algebras instead of topological K-theory. Further, to give an index formula, we will use tools from noncommutative differential geometry. Note that noncommutative differential geometry does not apply in general to C^*-algebras. The point here is that in a C^*-algebra there is a notion of continuity, but there is no differentiability. Fortunately, in the situation at hand, one can prove that we only deal with differentiable elements. Let us formulate this statement precisely.

Proposition 2 (see [60]). *If the symbol $\sigma(D)$ is invertible, then the inverse $\sigma(D)^{-1}$ lies in the subalgebra*

$$C^\infty(S^*M) \rtimes G \subset C(S^*M) \rtimes G, \tag{2.1}$$

of $C^\infty(S^*M)$-valued functions on G, which (together with all their derivatives) tend to zero as $|g| \to \infty$ faster than an arbitrary power of $|g|$.

The subalgebra (2.1) is called the *smooth crossed product*.

So, we have

$$[\sigma(D)] \in K_1(C^\infty(S^*M) \rtimes G). \tag{2.2}$$

To write an index formula for D, we first define a topological invariant of the symbol. This invariant is called the Chern character of the element (2.2). Then we define a topological invariant of the manifold.

Equivariant Chern character. Following [38], let us define the Chern character as the homomorphism of groups

$$\operatorname{ch} : K_1(C^\infty(X) \rtimes G) \longrightarrow \bigoplus_{\langle g\rangle \subset G} H^{\mathrm{odd}}(X^g), \tag{2.3}$$

where we put for brevity $X = S^*M$, the sum runs over conjugacy classes of G, and X^g denotes the fixed-point set of g. Since g is an isometry by assumption, the fixed-point set is a smooth submanifold (e.g., see [18]).

We define the Chern character using the abstract approach of noncommutative geometry. To this end, it suffices to define a pair (Ω, τ), where:

1. $\Omega = \Omega_0 \oplus \Omega_1 \oplus \Omega_2 \oplus \cdots$ is a differential graded algebra, which contains the crossed product $C^\infty(X) \rtimes G$ as a subalgebra of Ω_0;

2. $\tau : \Omega \longrightarrow \bigoplus_{\langle g\rangle\subset G} \Lambda(X^g)$ is a homomorphism of differential complexes such that

$$\tau(\omega_2\omega_1) = (-1)^{\deg\omega_1 \deg\omega_2}\tau(\omega_1\omega_2), \qquad \text{for all } \omega_1, \omega_2 \in \Omega. \tag{2.4}$$

The algebra Ω is called the *algebra of noncommutative differential forms*, and the functional τ is called the *differential graded trace*.

If such a pair is given, then the Chern character associated with the pair (Ω, τ) is defined as

$$\operatorname{ch}(a) = \operatorname{tr}\tau\left[\sum_{n\geq 0} \frac{n!}{(2\pi i)^{n+1}(2n+1)!}(a^{-1}da)^{2n+1}\right], \quad [a] \in K_1(C^\infty(X) \rtimes G), \tag{2.5}$$

where tr is the trace of a matrix. A standard computation shows that the form in (2.5) is closed and its class in de Rham cohomology is determined by $[a]$ and defines the homomorphism (2.3). It remains to define the pair (Ω, τ):

1. We set $\Omega = \Lambda(X) \rtimes G$, where the differential on the smooth crossed product of the algebra $\Lambda(X)$ of differential forms on X and the group G is equal to

$$(d\omega)(g) = d(\omega(g)), \qquad \omega \in \Lambda(X) \rtimes G.$$

2. To define a differential graded trace $\tau = \{\tau_g\}$, we fix some $g \in G$ and introduce necessary notation. Let $\overline{G}$ be the closure of G in the compact Lie group of isometries of X. This closure is a compact Lie group. Let $C_g \subset \overline{G}$ be the centralizer[7] of g. The centralizer is a closed Lie subgroup in $\overline{G}$. Denote the elements of the centralizer by h, and the induced smooth Haar measure on the centralizer by dh.

Let $\langle g\rangle \subset G$ be the conjugacy class of g, i.e., the set of elements equal to zgz^{-1} for some $z \in G$. Further, for each $g' \in \langle g\rangle$ we fix some element $z = z(g, g')$, which conjugates g and $g' = zgz^{-1}$. Any such element defines a diffeomorphism $z : X^g \to X^{g'}$.

Let us define the trace as

$$\tau_g(\omega) = \sum_{g'\in\langle g\rangle} \int_{C_g} h^*\big(z^*\omega(g')\big)\Big|_{X^g} dh, \quad \text{where } \omega \in \Lambda(X) \rtimes G. \tag{2.6}$$

One can show that this expression does not depend on the choice of elements z and is indeed a differential graded trace.

Remark 2. For a finite group the Chern character (2.5) coincides with the one constructed in [65], [14].

[7] Recall that the centralizer of g is the subgroup of elements commuting with g.

Equivariant Todd class. Given $g \in G$, the normal bundle of the fixed-point submanifold $M^g \subset M$ is denoted by N^g. The differential dg defines an orthogonal endomorphism of N^g and the corresponding bundle of exterior forms

$$\Omega(N^g_{\mathbb{C}}) = \Omega^{ev}(N^g_{\mathbb{C}}) \oplus \Omega^{\mathrm{odd}}(N^g_{\mathbb{C}}).$$

Here $E_{\mathbb{C}}$ stands for the complexification of a real vector bundle E. Consider the expression (see [12])

$$\operatorname{ch}\Omega^{ev}(N^g_{\mathbb{C}})(g) - \operatorname{ch}\Omega^{\mathrm{odd}}(N^g_{\mathbb{C}})(g) \in H^{\mathrm{ev}}(M^g). \tag{2.7}$$

The zero-degree component of this expression is nonzero [10]. Hence the class (2.7) is invertible and the following expression is well defined

$$\operatorname{Td}_g(T^*_{\mathbb{C}}M) = \frac{\operatorname{Td}(T^*_{\mathbb{C}}M^g)}{\operatorname{ch}\Omega^{\mathrm{ev}}(N^g_{\mathbb{C}})(g) - \operatorname{ch}\Omega^{\mathrm{odd}}(N^g_{\mathbb{C}})(g)} \in H^*(M^g), \tag{2.8}$$

where Td on the right-hand side in the equality is the Todd class of a complex vector bundle, and the expression is well defined, since the forms have even degrees.

Index theorem.

Theorem 2 (see [38]). *Let D be an elliptic G-operator on a closed manifold M. Then*

$$\operatorname{ind} D = \sum_{\langle g\rangle \subset G} \left\langle \operatorname{ch}_g[\sigma(D)] \operatorname{Td}_g(T^*_{\mathbb{C}}M), [S^*M^g]\right\rangle, \tag{2.9}$$

where $\langle g\rangle$ runs over the set of conjugacy classes of G; $[S^*M^g] \in H_{\mathrm{odd}}(S^*M^g)$ is the fundamental class of S^*M^g; the Todd class is lifted from M^g to S^*M^g using the natural projection; the brackets $\langle,\rangle$ denote the pairing of cohomology and homology. The series in (2.9) *is absolutely convergent.*

In some situations the sum in (2.9) can be reduced to one summand equal to the contribution of the unit element.

Corollary 1 (see [38, 54]). *Suppose that either the action of G on M is free or G is torsion free. Then one has*

$$\operatorname{ind} D = \langle \operatorname{ch}_e[\sigma(D)] \operatorname{Td}(T^*_{\mathbb{C}}M), [S^*M]\rangle. \tag{2.10}$$

Let us note that the index formula (2.9) contains many other index formulas as special cases (see [38, 54] for details). Here we give two situations, in which the index formula can be applied.

Example 1. Index of twisted Toeplitz operators. Let M be an odd-dimensional oriented manifold. We suppose that M is endowed with a G-invariant spin-structure (i.e., the action of G on M lifts to an action on the spin bundle $S(M)$). Let $\mathcal{D}$ be the Dirac operator [12]

$$\mathcal{D} : S(M) \longrightarrow S(M),$$

acting on spinors. This operator is elliptic and self-adjoint.

Denote by $\Pi_+ : S(M) \longrightarrow S(M)$ the positive spectral projection of this operator.

We define the Toeplitz operator

$$\Pi_+U : \Pi_+(S(M)) \otimes \mathbb{C}^n \longrightarrow \Pi_+(S(M)) \otimes \mathbb{C}^n, \qquad (2.11)$$

where U is an invertible n by n matrix with elements in $C^\infty(M) \rtimes G$. Then the operator (2.11) is Fredholm (its almost-inverse is equal to Π_+U^{-1}). Let us suppose for simplicity that either G is torsion free, or the action is free. In this case, the formula (2.10) gives the following expression for the index.

Theorem 3. *The index of operator* (2.11) *is equal to*

$$\operatorname{ind}(\Pi_+U) = \int_M A(TM)\,\mathrm{ch}_e(U), \qquad (2.12)$$

where $A(TM)$ is the A-class of the tangent bundle, which in the Borel–Hirzebruch formalism is defined by the function

$$\frac{x/2}{\operatorname{sh} x/2}.$$

Examples 2. Operators on noncommutative torus. Let us fix $0 < \theta \le 1$. A. Connes in [22] considered differential operators of the form

$$D = \sum_{\alpha+\beta\le m} a_{\alpha\beta} x^\alpha \left(-i\frac{d}{dx}\right)^\beta : S(\mathbb{R}) \longrightarrow S(\mathbb{R}), \qquad (2.13)$$

in the Schwartz space $S(\mathbb{R})$ on the real line. Here the coefficients $a_{\alpha\beta}$ are Laurent polynomials in operators U, V

$$(Uf)(x) = f(x+1), \qquad (Vf)(x) = e^{-2\pi i x/\theta} f(x) \qquad (2.14)$$

of shift by one and product by exponential.

Let us show that the operators of the form (2.13) reduce to G-operators on a closed manifold. To this end, we consider the real line as the total space of the standard covering

$$\mathbb{R} \longrightarrow \mathbb{S}^1,$$

whose base is the circle of length θ. Then the Schwartz space becomes isomorphic to the space of smooth sections of a (nontrivial) bundle on the base $\mathbb{S}^1$, whose fiber is the Schwartz space $S(\mathbb{Z})$ of rapidly decaying sequences (that is, functions on the fiber). Then we apply Fourier transform

$$\mathcal{F} : S(\mathbb{Z}) \longrightarrow C^\infty(\mathbb{S}^1)$$

in each fiber and obtain a space, which is the space of smooth sections of a complex line bundle over the torus $\mathbb{T}^2$. These transformations define the isomorphism

$$S(\mathbb{R}) \simeq C^\infty(\mathbb{T}^2, \gamma) \qquad (2.15)$$

of the Schwartz space on the real line and the space

$$C^\infty(\mathbb{T}^2, \gamma) = \{g \in C^\infty(\mathbb{R} \times \mathbb{S}^1) \mid g(\varphi+\theta, \psi) = g(\varphi, \psi)e^{-2\pi i \psi}\}$$

of smooth sections of a complex line bundle γ on the torus. Here on $\mathbb{T}^2$ we consider the coordinates $0 \le \varphi \le \theta$, $0 \le \psi \le 1$. This isomorphism is defined by the formula

$$f(x) \longmapsto \sum_{n \in \mathbb{Z}} f(\varphi + \theta n) e^{2\pi i n \psi}.$$

(This formula was found earlier by S. Novikov.) Using the isomorphism (2.15), it is easy to obtain the correspondences between the operators:

operators on the line	operators on the torus
$-i\frac{d}{dx}$	$-i\frac{\partial}{\partial\varphi}$
x	$-i\frac{\theta}{2\pi}\frac{\partial}{\partial\psi} + \psi$
$e^{-2\pi i x/\theta}$	$e^{-2\pi i \varphi/\theta}$
$f(x) \to f(x+1)$	$g(\varphi,\psi) \mapsto g(\varphi+1,\psi)$

This table implies that on the torus we obtain G-operators, which can be studied using the finiteness theorem and the index formula formulated above. We refer the reader to [38] for details.

2.2. General actions

In this subsection we survey index formulas for elliptic operators associated with general actions of discrete groups (see recent papers [49] and [44]). Let D be an elliptic operator of the form (1.1). We will assume for simplicity that the inverse symbol $\sigma(D)^{-1}$ lies in the algebraic crossed product

$$C^\infty(S^*M) \rtimes_{alg} G \subset C(S^*M) \rtimes G,$$

which consists of compactly supported functions on the group. Such a symbol defines an element

$$[\sigma(D)] \in K_1(C^\infty(S^*M) \rtimes_{alg} G) \tag{2.16}$$

in K-theory. We would like to define the topological index as a numerical invariant associated with $[\sigma(D)]$. There is a standard procedure in noncommutative geometry of constructing such invariants. Namely, one takes the pairing of (2.16) with an element in *cyclic cohomology* of the same algebra. Let us recall this construction.

Cyclic cohomology. Pairing with *K*-theory. Let A be an algebra with unit. Recall (see [22]) that the *cyclic cohomology* $HC^*(A)$ of A is the cohomology of the bicomplex

$$\begin{array}{ccccccc}
 & & & & A^* & & \cdots \\
 & & & & \uparrow B & & \\
 & & A^* & \xrightarrow{b} & A^{*2} & & \cdots \\
 & & \uparrow B & & \uparrow B & & \\
A^* & \xrightarrow{b} & A^{*2} & \xrightarrow{b} & A^{*3} & & \cdots \\
\vdots & & \vdots & & \vdots & &
\end{array} \tag{2.17}$$

where b, B are some differentials and for simplicity we denote the space of multi-linear functionals on A^k by A^{*k}. In particular, an element $\varphi \in HC^n(A)$ of cyclic cohomology is represented by a finite collection of multi-linear functionals

$$\{\varphi_j(a_0,\dots,a_j)\}, \quad j = n, n-2, n-4, \dots,$$

such that $B\varphi_j + b\varphi_{j-2} = 0$.

To make the paper self-contained, we recall the formulas for the differentials in the bicomplex:

$$\begin{aligned}(b\varphi)(a_0,a_1,\dots,a_{j+1}) = \sum_{n=0}^{j} \varphi(a_0,a_1,\dots,a_n a_{n+1},\dots,a_{j+1}) \\ + (-1)^{j+1}\varphi(a_{j+1}a_0,a_1,\dots,a_j).\end{aligned} \tag{2.18}$$

and $B = Ns(Id-\lambda)$, where $\lambda = (-1)^n$(cyclic left shift),

$$s: A^{*(n+1)} \longrightarrow A^{*n}, \quad (s\varphi)(a_0,\dots,a_{n-1}) = \varphi(1,a_0,\dots,a_{n-1}),$$

and $N : A^{*n} \longrightarrow A^{*n}$, $N = Id + \lambda + \lambda^2 + \dots + \lambda^{n-1}$ is the symmetrization mapping.

The desired numerical invariants are defined using the pairing

$$\langle , \rangle : K_1(A) \times HC^{\mathrm{odd}}(A) \longrightarrow \mathbb{C} \tag{2.19}$$

of K-theory and cyclic cohomology. The value of this pairing on the classes $[a]$ and $[\varphi]$ is equal to

$$\langle a, \varphi\rangle = \frac{1}{\sqrt{2\pi i}} \sum_{k\geq 0} (-1)^k k! \varphi_{2k+1}(a^{-1},a,\dots,a^{-1},a).$$

Now to define the topological index of the element (2.16), it remains to choose a cocycle over the algebra. It turns out that the desired cocycle can be defined as a special *equivariant characteristic class* in cyclic cohomology.

Equivariant characteristic classes. Suppose that a discrete group G acts smoothly on a closed smooth manifold X. We shall also assume that X is oriented and the action is orientation-preserving. Let $E \in \mathrm{Vect}_G(X)$ be a finite-dimensional complex G-bundle on X. Connes defined (e.g., see [22]) equivariant characteristic classes of E with values in cyclic cohomology $HC^*(C^\infty(X) \rtimes_{alg} G)$ of the crossed product. However, the formulas for these classes were quite complicated and we do not give them here. A simple explicit formula was obtained in [27] for the most important characteristic class, namely, for the equivariant Chern character

$$\mathrm{ch}_G(E) \in HC^*(C^\infty(X) \rtimes_{alg} G). \tag{2.20}$$

More precisely, it was shown in the cited paper that the class $\mathrm{ch}_G(E)$ is represented by the collection of functionals $\{\mathrm{ch}^k_G(E)\}$ defined as

$$\begin{aligned}&\mathrm{ch}^k_G(E;a_0,a_1,\dots,a_k) \\ &\quad = \frac{(-1)^{(n-k)/2}}{((n+k)/2)!} \sum_{i_0+i_1+\dots+i_k=(n-k)/2} \int_X \mathrm{tr}_E\left[\left(a_0\theta^{i_0}\nabla(a_1)\theta^{i_1}\nabla(a_2)\dots\nabla(a_k)\theta^{i_k}\right)_e\right]\end{aligned} \tag{2.21}$$

(cf. Jaffe–Lesniewski–Osterwalder formula [29]). Here

$$\dim X = n, \quad k = n, n-2, n-4, \ldots,$$

∇_E is a connection in E and $\theta = \nabla_E^2$ is its curvature form, for a noncommutative form ω by ω_e we denote the coefficient of $T_e = 1$, while the operator

$$\nabla : C^\infty(X) \rtimes_{\rm alg} G \to \Lambda^1(X, \operatorname{End} E) \rtimes_{\rm alg} G$$

is defined as

$$\nabla\Bigl(\sum_g a_g T_g\Bigr) = \sum_g \bigl[da_g - a_g(\nabla_E - (g^{-1})^* \nabla_E)\bigr] T_g.$$

It is proved in the cited paper that the collection of functionals $\{\operatorname{ch}_G^k(E)\}$ defines a cocycle over $C^\infty(X) \rtimes_{\rm alg} G$, and the class of this cocycle in cyclic cohomology does not depend on the choice of ∇_E and coincides with the equivariant Chern character defined by Connes [22].

Explicit formulas for other characteristic classes can be obtained using standard topological techniques (operations in K-theory, see [9]). For the index theorem, we need the equivariant Todd class.

Proposition 3 ([49]). *The equivariant Todd class*

$$\operatorname{Td}_G(E) \in HC^*(C^\infty(X) \rtimes_{\rm alg} G) \tag{2.22}$$

of a complex G-bundle E on a smooth manifold X is equal to

$$\operatorname{Td}_G(E) = \operatorname{ch}_G(\Phi(E)),$$

here Φ is the multiplicative operation in K-theory, which corresponds to the function $\varphi(t) = t^{-1}(1+t)\ln(1+t)$.

Note that Φ can be expressed explicitly in terms of Grothendieck operations. For instance, if $\dim X \le 5$ then (see [49])

$$\begin{aligned}\Phi(E) &= 1 + \frac{E-n}{2} + \frac{-2(E^2 - 2nE + n^2) + 7(E + \Lambda^2 E - nE + n(n-1)/2)}{12}\\ &= \frac{3n^2 - 19n + 24}{24} + \frac{(-3n+13)}{12} E - \frac{1}{6} E \otimes E + \frac{7}{12} \Lambda^2 E \end{aligned}\tag{2.23}$$

where $n = \dim E$.

Index theorem.

Theorem 4 ([49]). *Let D be an elliptic operator associated with the action of group $\mathbb{Z}$. Then we have the index formula*

$$\operatorname{ind} D = (2\pi i)^{-n} \langle [\sigma(D)], \operatorname{Td}_{\mathbb{Z}}(\pi^* T^*_{\mathbb{C}} M)\rangle, \qquad \dim M = n, \tag{2.24}$$

where $\pi : S^*M \longrightarrow M$ is the natural projection and the brackets $\langle , \rangle$ denote the pairing of K-theory and cyclic cohomology (*see* (2.19)).

Remark 3. An index formula for operators on the circle associated with an action of an arbitrary torsion free group is announced in [44]. The index formula in this case has the same form as (2.24).

Examples. 1. Suppose that the (usual) Todd class $\mathrm{Td}(T^*_{\mathbb{C}}M)$ is trivial and the diffeomorphisms of the $\mathbb{Z}$-action are isotopic to the identity. Then one can show that the equivariant Todd class is equal to the transverse fundamental cycle (see [21]) of S^*M and the index formula (2.24) is written as:

$$\operatorname{ind} D = \frac{(n-1)!}{(2\pi i)^n (2n-1)!} \int_{S^*M} (\sigma^{-1} d\sigma)_e^{2n-1}, \qquad \sigma = \sigma(D). \tag{2.25}$$

2. Suppose that the group acts isometrically. Then formula (2.24) reduces to (2.10) This is obvious if we choose an invariant metric and connection on the cotangent bundle.

3. Elliptic operators for compact Lie groups

3.1. Main definitions

Let a compact Lie group G act smoothly on a closed smooth manifold M. An element $g \in G$ takes a point $x \in M$ to the point denoted by $g(x)$. We fix a G-invariant metric on M and the Haar measure on G.

Consider the representation $g \mapsto T_g$ of G in the space $L^2(M)$ by shift operators

$$T_g u(x) = u(g^{-1}(x)).$$

Definition 5. A *G-pseudodifferential operator* (G-ψDO) is an operator

$$D : L^2(M) \longrightarrow L^2(M)$$

of the form

$$D = 1 + \int\limits_G D_g T_g dg, \tag{3.1}$$

where D_g, $g \in G$ is a smooth family of pseudodifferential operators of order zero on M.

Consider the equation

$$u + \int_G D_g T_g u dg = f, \quad u, f \in L^2(M). \tag{3.2}$$

Note that if G is discrete, then we obtain the class of equations (1.1).

Example 1. Integro-differential equations on the torus. On the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with coordinates x_1, x_2, consider the integro-differential equation

$$\Delta u(x_1, x_2) + \alpha \frac{\partial^2}{\partial x_1^2} \int_{\mathbb{S}^1} u(x_1, y) dy = f(x_1, x_2),$$

where Δ stands for the nonnegative Laplace operator, and α is a constant. Let us write this equation as

$$\Delta u + \alpha \frac{\partial^2}{\partial x_1^2} \int_{\mathbb{S}^1} T_g dg u = f, \tag{3.3}$$

where T_g denotes the shift operator $T_g u(x_1, x_2) = u(x_1, x_2 - g)$, induced by the action of the circle $G = \mathbb{S}^1$ by shifts in x_2. Note that if we multiply the equation (3.3) on the left by the almost inverse operator Δ^{-1}, we obtain an equation of the type (3.2).

Example 2. Integro-differential equations on the plane. Consider the integro-differential equation

$$\Delta u(x,y) + \left(\alpha\frac{\partial^2}{\partial x^2} + \beta\frac{\partial^2}{\partial x\partial y} + \gamma\frac{\partial^2}{\partial y^2}\right)\int_{\mathbb{S}^1} u(x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi)d\varphi = f(x,y)$$

on the plane $\mathbb{R}^2_{x,y}$, where Δ is the Laplace operator, and α, β, γ are constants. This equation can be written as

$$\Delta u + \left(\alpha\frac{\partial^2}{\partial x^2} + \beta\frac{\partial^2}{\partial x\partial y} + \gamma\frac{\partial^2}{\partial y^2}\right)\int_{\mathbb{S}^1} T_\varphi d\varphi u = f, \tag{3.4}$$

where the shift operator T_φ is induced by the action of the circle $G = \mathbb{S}^1$ by rotations

$$(x,y) \longmapsto (x\cos\varphi + y\sin\varphi, -x\sin\varphi + y\cos\varphi)$$

around the origin. If we multiply the equation (3.4) on the left by the almost inverse operator Δ^{-1}, we obtain an equation similar to (3.2).

3.2. Pseudodifferential uniformization

Here we formulate an approach, called *pseudodifferential uniformization*, which enables one to reduce a G-pseudodifferential operator

$$D = 1 + \int_G D_g T_g dg : L^2(M) \longrightarrow L^2(M). \tag{3.5}$$

to a pseudodifferential operator and then apply the methods of the theory of pseudodifferential operators.

1. Reduction to a ψDO. This reduction is constructed as follows.

- The operator D is represented as an operator on the quotient M/G (the space of orbits).
- If the action of G on M has no fixed points, then M/G is a smooth manifold; moreover, D can be treated as a ψDO on M/G with operator-valued symbol in the sense of Luke [35] (explanation: this follows from the fact that the operator T_g acts only along the fibers of the infinite-dimensional bundle over M/G, but not along the base).
- If the fixed point set is nonempty, then M/G has singularities; to construct a ψDO in this case, we do the following.
- We lift D from M to the product $M \times G$ endowed with the diagonal action of G:

$$(x,h) \longmapsto (g(x), gh). \tag{3.6}$$

FIGURE 1. The orbits of the diagonal and vertical actions on the product $M \times G$.

- The action (3.6) is fixed point free. Hence, the obtained G-pseudodifferential operator on $M \times G$, which we denote by $\widetilde{D}$, can be represented (see above) as a ψDO on the smooth orbit space $(M \times G)/G \simeq M$.

  These steps give the commutative diagram

$$\begin{array}{ccc} L^2(M) & \xrightarrow{\quad D \quad} & L^2(M) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ L^2(M \times G) & \xrightarrow{\quad \widetilde{D} \quad} & L^2(M \times G) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ L^2(M, L^2(G)) & \xrightarrow{\ \mathcal{D}\ } & L^2(M, L^2(G)), \end{array} \tag{3.7}$$

where π^* is the induced mapping for the projection $\pi : M \times G \to M$, while $\mathcal{D}$ is a pseudodifferential operator on M.

Remark 4. The fact that $\mathcal{D}$ is a ψDO is clear for geometric reasons. Indeed, $\widetilde{D}$ has shifts along the orbits of the diagonal action of G (see Figure 1, left). Clearly, these orbits can be transformed into vertical orbits (see Figure 1, right) by a change of variables on $M \times G$. The shift operator along the vertical orbits is a ψDO on M.

2. Restriction of ψDO to the subspace of invariant sections. The mapping π^* in (3.7) is a monomorphism. Its range is the space of G-invariant sections. Hence, (3.7) gives a commutative diagram of the form

$$\begin{array}{ccc} L^2(M) & \xrightarrow{\quad D \quad} & L^2(M) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ L^2(M, L^2(G))^G & \xrightarrow{\ \mathcal{D}^G\ } & L^2(M, L^2(G))^G, \end{array}$$

where $\mathcal{D}^G$ stands for the restriction of $\mathcal{D}$ to the subspace of invariant sections, which we denote by $L^2(M, L^2(G))^G$.

3. Transverse ellipticity. It remains to give conditions, which imply that the restriction

$$L^2(M, L^2(G))^G \xrightarrow{\mathcal{D}^G} L^2(M, L^2(G))^G$$

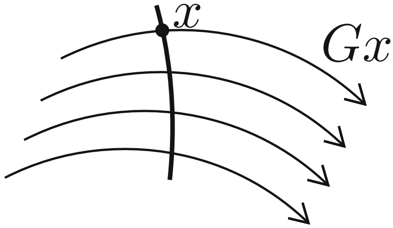

FIGURE 2. Transversal to the orbit.

of $\mathcal{D}$ to the subspace of G-invariant sections is Fredholm. Let us note that invariant sections are constant along the orbits of the group action. Hence, it suffices to impose the condition, which guarantees the Fredholm property, only along the transverse directions to the orbit (see Figure 2).

Definition 6 ([8, 61]). A pseudodifferential operator $\mathcal{D}$ is *transversally elliptic*, if its symbol $\sigma(\mathcal{D})(x,\xi)$ is invertible for all $(x,\xi)\in T^*_G M\setminus 0$, where

$$T^*_G M=\{(x,\xi)\in T^*M \mid \text{ covector } \xi \text{ is orthogonal to the orbit } Gx\}.$$

stands for the *transverse cotangent bundle*.

Theorem 5 ([55, 66]). *A transversally elliptic operator $\mathcal{D}$ restricts to a Fredholm operator on the subspace of G-invariant sections*

$$\mathcal{D}^G: L^2(M,L^2(G))^G \longrightarrow L^2(M,L^2(G))^G.$$

Let us summarize the above discussion.

1. To a G-pseudodifferential operator D we assigned a pseudodifferential operator $\mathcal{D}$ such that there is an isomorphism
$$D\simeq \mathcal{D}^G, \tag{3.8}$$
where $\mathcal{D}^G$ is the restriction of $\mathcal{D}$ to the subspace of invariant sections.
2. If $\mathcal{D}$ is transversally elliptic, then its restriction $\mathcal{D}^G$ is Fredholm. Hence, by virtue of the isomorphism (3.8) the original operator D is also Fredholm.

Since we have isomorphism (3.8), we obtain:

$$\operatorname{ind} D=\operatorname{ind}\mathcal{D}^G.$$

Using this equation and the theory of transversally elliptic pseudodifferential operators [8, 30, 67]), an index theorem for the G-operator D was obtained in [55, 50]. We only mention here that the main ingredients of the index formula are: 1) the definition of the symbol of D as an element of the crossed product of the algebra of functions on the transverse cotangent bundle by the group G; 2) a Chern character mapping on the K-theory of this algebra ranging in the basic cohomology of fixed point sets of the group action.

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The Singular Functions of Branching Edge Asymptotics

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Abstract. We investigate the structure of branching asymptotics appearing in solutions to elliptic edge problems. The exponents in powers of the half-axis variable, logarithmic terms, and coefficients depend on the variables on the edge and may be branching.

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Keywords. Singular functions of asymptotics, discrete, continuous, variable branching edge asymptotics, weighted edge Sobolev spaces.

Introduction

The solutions to elliptic problems on a manifold with edge are expected to have asymptotics of the form

$$u(r,x,y) \sim \sum_{j=0}^{J} \sum_{k=0}^{m_j} c_{jk}(x,y) r^{-p_j} \log^k r \tag{0.1}$$

as $r \to 0$, with exponents $p_j \in \mathbb{C}$, and $m_j \in \mathbb{N}$ $(= \{0,1,2,\dots\})$. Here (r,x,y) are the variables in an open stretched wedge $\mathbb{R}_+ \times X \times \Omega$ for a closed smooth manifold X of dimension n and an open set $\Omega \subseteq \mathbb{R}^q$. If the respective operator is a differential operator of the form

$$A = r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r,y)(-r\partial_r)^j (rD_y)^\alpha \tag{0.2}$$

for coefficients $a_{j\alpha}(r,y) \in C^\infty\big(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X)\big)$ (with $\mathrm{Diff}^{\nu}(X)$ being the space of differential operators of order ν on X) then the asymptotic data

$$\mathcal{P} := \{(p_j, m_j)\}_{0 \le j \le J} \subset \mathbb{C} \times \mathbb{N}, \tag{0.3}$$

$J = J(\mathcal{P}) \in \mathbb{N} \cup \{\infty\}$ are known to be determined by the leading conormal symbol

$$\sigma_{\mathrm{c}}(A)(y,z) := \sum_{j=0}^{\mu} a_{j0}(0,y)z^j, \tag{0.4}$$

regarded as a family of differential operators

$$\sigma_{\mathrm{c}}(A)(y,z) : H^s(X) \to H^{s-\mu}(X) \tag{0.5}$$

of order μ, smooth in $y \in \Omega$ and holomorphic in z. In the elliptic case it is known that the operators (0.5) are parameter-dependent elliptic of order μ where the parameter is $\operatorname{Im} z$ with z varying on a so-called weight line

$$\Gamma_\beta := \{z \in \mathbb{C} : \operatorname{Re} z = \beta\} \tag{0.6}$$

for every real β.

It is well known that for any fixed $y \in \Omega$ the operators (0.5) are bijective for all z off some discrete set $D(y) \subset \mathbb{C}$, where $D(y) \cap \{c < \operatorname{Re} z < c'\}$ is finite for every $c < c'$, cf. Bleher [1]. Those non-bijectivity points are just responsible for the exponents $-p_j$ in (0.1). More precisely, $\sigma_{\mathrm{c}}^{-1}(A)(y,z)$ is an $L_{\mathrm{cl}}^{-\mu}(X)$-valued meromorphic function with poles at the points p_j of (finite) multiplicities $m_j + 1$ and finite rank Laurent coefficients in $L^{-\infty}(X)$ at the powers $(z - p_j)^{-(k+1)}, 0 \leq k \leq m_j$. Here $L_{\mathrm{cl}}^{\nu}(X), \nu \in \mathbb{R}$, means the space of all classical pseudo-differential operators on X of order ν, and $L^{-\infty}(X) := \bigcap_{\nu \in \mathbb{R}} L_{\mathrm{cl}}^{\nu}(X)$ is the space of smoothing operators.

If $\sigma_{\mathrm{c}}(A)$ is independent of y we have constant discrete edge asymptotics of solutions, cf. the terminology below. Even in this case it is interesting to observe the nature of coefficients c_{jk} in (0.1) depending on the considered Sobolev smoothness $s \in \mathbb{R}$ of the solutions. The Sobolev smoothness of the coefficients c_{jk} in y also depends on $\operatorname{Re} p_j$. Clearly in general the leading conormal symbol $\sigma_{\mathrm{c}}(A)$ depends on y and then also the set $D(y)$. In this case the inverse $\sigma_{\mathrm{c}}^{-1}(A)$ is a y-dependent family of meromorphic operator functions with poles $p_j(y)$ varying in the complex plane and possible branchings where the multiplicities $m_j(y) + 1$ may have jumps, including the above-mentioned Laurent coefficients. These effects have been studied in a number of papers, cf. [12], [13] and [16]. In particular, also the Sobolev smoothness in y of the coefficients $c_{jk}(x,y)$ is branching. The program is going on, and in the present article we study some features of the functional analytic structure of singular functions in the variable branching case which are not yet analyzed so far.

The characterization of asymptotics of solutions to singular PDE-problems is a central issue of solvability theory of elliptic equations on a singular configuration. One of the classical papers in this connection is [9] of Kondratyev on boundary value problems on manifolds with conical singularities. Since then there appeared numerous investigations in this field, also on boundary problems for operators without the transmission property, or mixed and transmission problems, see, in particular, Eskin's book [3]. The present investigation is dominated by the pseudo-differential approach to generate asymptotics via parametrices and elliptic regularity, see, in particular, the monographs [11], [2], [5], and the references there. Note that a similar philosophy applies also for corner singularities where asymptotics appear in iterated form, cf. [15], or, the recent investigations, [4], [17].

This paper is organized as follows.

First in Section 1 we outline some necessary tools on constant discrete edge asymptotics in the frame of weighted edge spaces and corresponding subspaces. We then pass to a more detailed investigation of the singular functions and show some essential simplification compared with other expositions, say, [2] or [14], namely, that the cut-off functions may be chosen independently of the edge covariable η, modulo edge-flat remainders. We do that including the so-called continuous asymptotics.

In Section 2 we consider variable branching edge asymptotics, formulated in terms of smooth functions with values in analytic functionals that are pointwise discrete and of finite order. Basics and tools can be found in [11], [8]; the notion itself has been first established in [12], [13] and further studied in detail in [16]. Here we show a refinement of a result of [14] on the representation of singular functions with variable continuous asymptotics by analytic functionals without explicit dependence on the edge variable y. In particular, the preparations from Section 1 on η-independent cut-off functions allow us to find the claimed new representation in a unique way. We finally apply this result to the case of variable branching asymptotics and obtain the surprising effect that the pointwise discrete behaviour in y may be shifted into a new functional that gives rise to a localization of Sobolev smoothness of "coefficients of asymptotics", both in variable branching as well as in continuous asymptotics.

1. The constant discrete edge asymptotics

1.1. Edge spaces and specific operator-valued symbols

Let us first recall what we understand by abstract edge spaces modelled on a space with group action.

First if this space is a Hilbert space H such a group action is a family $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : H \to H$ with $\kappa_\lambda\kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \mathbb{R}_+$, and $\lambda \to \kappa_\lambda h$ represents a function in $C(\mathbb{R}_+, H)$ for every $h \in H$. As is known we have an estimate

$$\|\kappa_\lambda\|_{\mathcal{L}(H)} \leq c\big(\max\{\lambda, \lambda^{-1}\}\big)^M \tag{1.1}$$

for all $\lambda \in \mathbb{R}_+$, for some constants $c > 0, M > 0$, depending on κ (a proof may be found in [6]). We also need the case of a Fréchet space E written as a projective limit $\varprojlim_{j\in\mathbb{N}} E^j$ of Hilbert spaces, with continuous embeddings $E^j \hookrightarrow E^0$ for all j, where E^0 is endowed with a group action κ and $\kappa|_{E^j}$ defines a group action in E^j for every j. The constants c and M in (1.1) then may depend on j. Now

$\mathcal{W}^s(\mathbb{R}^q, H)$, $s \in \mathbb{R}$, is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} = \left\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \hat{u}(\eta)\|^2_H \, d\eta \right\}^{1/2} \tag{1.2}$$

with $\hat{u}(\eta) = (F_{y\to\eta}u)(\eta)$ being the Fourier transform, $\langle\eta\rangle = (1 + |\eta|^2)^{1/2}$. For a Fréchet space $E = \varprojlim_{j\in\mathbb{N}} E^j$ we have $\mathcal{W}^s(\mathbb{R}^q, E^j)$, $j \in \mathbb{N}$, and we set

$$\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j).$$

Recall that we obtain an equivalent norm to (1.2) when we replace $\langle\eta\rangle$ by a function $\eta \to [\eta]$, strictly positive, smooth, with $[\eta] = |\eta|$ for $|\eta| > C$ for some $C > 0$.

In the general discussion we often consider the Hilbert space case; the generalization to Fréchet spaces will be obvious. Observe that $\mathcal{W}^s(\mathbb{R}^q, H) \subset \mathcal{S}'(\mathbb{R}^q, H)$. For an open set $\Omega \subseteq \mathbb{R}^q$ by $\mathcal{W}^s_{\mathrm{loc}}(\Omega, H)$ we denote the space of all $u \in \mathcal{D}'(\Omega, H)$ such that $\varphi u \in \mathcal{W}^s(\mathbb{R}^q, H)$ for every $\varphi \in C^\infty_0(\Omega)$. Moreover, $\mathcal{W}^s_{\mathrm{comp}}(\Omega, H)$ denotes the subspace of all elements of $\mathcal{W}^s(\mathbb{R}^q, H)$ that have compact support in Ω. Clearly the spaces $\mathcal{W}^s(\mathbb{R}^q, H)$ depend on the choice of κ. If necessary we write $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ in order to indicate the specific group action κ. The case $\kappa = \mathrm{id}$ for all $\lambda \in \mathbb{R}_+$ is always admitted. Then we have

$$\mathcal{W}^s(\mathbb{R}^q, H)_{\mathrm{id}} = H^s(\mathbb{R}^q, H)$$

which is the standard Sobolev space of H-valued distributions. Observe that

$$\bigcap\nolimits_{s\in\mathbb{R}} \mathcal{W}^s(\mathbb{R}^q, H)_\kappa =: \mathcal{W}^\infty(\mathbb{R}^q, H)_\kappa = \mathcal{W}^\infty(\mathbb{R}^q, H)_{\mathrm{id}}, \tag{1.3}$$

i.e., the dependence on κ disappears when $s = \infty$. This is a consequence of (1.1). From the definition we have an isomorphism

$$\mathbb{K} := F^{-1}\kappa_{[\eta]}F : \mathcal{W}^s(\mathbb{R}^q, H)_{\mathrm{id}} \to \mathcal{W}^s(\mathbb{R}^q, H)_\kappa \tag{1.4}$$

for every $s \in \mathbb{R}$, in particular,

$$\mathbb{K} : \mathcal{W}^\infty(\mathbb{R}^q, H)_{\mathrm{id}} \to \mathcal{W}^\infty(\mathbb{R}^q, H)_{\mathrm{id}}.$$

We employ analogues of the spaces $\mathcal{W}^s(\mathbb{R}^q, H)$ for certain Hilbert spaces H based on the Mellin transform.

The analysis on a singular manifold refers to a large extent to the Mellin transform

$$Mu(z) = \int_0^\infty r^{z-1} u(r) \, dr$$

first for $u \in C^\infty_0(\mathbb{R}_+)$ and then extended to various distribution spaces, also vector-valued ones. For $u \in C^\infty_0(\mathbb{R}_+)$ we obtain an entire function in the complex z-plane. Function/distribution spaces on Γ_β always refer to $\rho = \operatorname{Im} z$ for $z \in \Gamma_\beta$, e.g., the Schwartz space $\mathcal{S}(\Gamma_\beta)$ or $L^2(\Gamma_\beta)$ with respect to the Lebesgue measure on $\mathbb{R}_\rho$. Recall that the Mellin transform induces a continuous operator $M : C^\infty_0(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C})$ with $\mathcal{A}(\mathbb{C})$ being the space of entire functions in z. In particular, for $u \in C^\infty_0(\mathbb{R}_+)$ we can form the weighted Mellin transform $M_\gamma : C^\infty_0(\mathbb{R}_+) \to \mathcal{S}(\Gamma_{1/2-\gamma})$

of weight $\gamma \in \mathbb{R}$, defined as $M_\gamma u := Mu|_{\Gamma_{1/2-\gamma}}$. As is well known, M_γ extends to an isomorphism $M_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2-\gamma})$, and then

$$(M_\gamma^{-1} g)(r) = \int_{\Gamma_{1/2-\gamma}} r^{-z} g(z)\, đz$$

for $đz = (2\pi i)^{-1} dz$. Analogously as standard Sobolev spaces based on L^2-norms and the Fourier transform we can form weighted Mellin–Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} = \left\{ \int_{\Gamma_{(n+1)/2-\gamma}} \int_{\mathbb{R}^n} \langle z, \xi \rangle^{2s} \left| (M_{\gamma-n/2, r\to z} F_{x\to\xi} u)(z,\xi) \right|^2 đz d\xi \right\}^{1/2},$$

with $F = F_{x\to\xi}$ being the Fourier transform in $\mathbb{R}^n$. Moreover, if X is a smooth closed manifold of dimension n we have analogous spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ for

$$X^\wedge := \mathbb{R}_+ \times X$$

based on the local spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ and defined with the help of charts and a partition of unity on X. Note that (in our notation) the meaning of γ depends on the dimension n. In the case $s = \infty$ we have a canonical identification

$$\mathcal{H}^{\infty,\gamma}(X^\wedge) = \mathcal{H}^{\infty,\gamma-n/2}(\mathbb{R}_+) \hat{\otimes}_\pi C^\infty(X) \cong C^\infty\big(X, \mathcal{H}^{\infty,\gamma-n/2}(\mathbb{R}_+)\big); \qquad (1.5)$$

here $\hat{\otimes}_\pi$ means the projective tensor product between the respective spaces.

In this exposition a cut-off function ω on the half-axis is any $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ that is equal to 1 close to 0. It will be essential also to employ the spaces

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u + (1-\omega) v : u \in \mathcal{H}^{s,\gamma}(X^\wedge), v \in H^s_{\mathrm{cone}}(X^\wedge)\}. \qquad (1.6)$$

Here $H^s_{\mathrm{cone}}(X^\wedge)$ is defined as follows. Choose any diffeomorphism $\chi_1 : U \to V$ from a coordinate neighbourhood U on X to an open set $V \subset S^n$ (the unit sphere in $\mathbb{R}^{n+1}_{\tilde{x}}$), and let $\chi : \mathbb{R}_+ \times U \to \Gamma := \{\tilde{x} \in \mathbb{R}^{n+1} \setminus \{0\} : \tilde{x}/|\tilde{x}| \in V\}$ be defined by $\chi(r,x) := r\chi_1(x), r \in \mathbb{R}_+$. Then $H^s_{\mathrm{cone}}(X^\wedge)$ is the set of all $v \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+\times X}$ such that for any $\varphi \in C_0^\infty(U)$ we have $((1-\omega)\varphi v) \circ \chi^{-1} \in H^s(\mathbb{R}^{n+1})$, for every coordinate neighbourhood U on X. Concerning more details on those spaces, cf. [14] or [15]. In particular, $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge)$ are Hilbert spaces in suitable scalar products, and we have $\mathcal{H}^{0,0}(X^\wedge) = \mathcal{K}^{0,0}(X^\wedge) = r^{-n/2} L^2(\mathbb{R}_+ \times X)$ with L^2 referring to $drdx$ and dx associated with a fixed Riemannian metric on X, $n = \dim X$. Analogously as (1.5) we also have

$$\mathcal{K}^{\infty,\gamma}(X^\wedge) = \mathcal{K}^{\infty,\gamma-n/2}(\mathbb{R}_+) \hat{\otimes}_\pi C^\infty(X) = C^\infty\big(X, \mathcal{K}^{\infty,\gamma-n/2}(\mathbb{R}_+)\big). \qquad (1.7)$$

Here $\mathcal{K}^{\infty,\gamma-n/2}(\mathbb{R}_+)$ is endowed with its natural Fréchet topology. In order to formulate asymptotics of elements in $\mathcal{K}^{s,\gamma}(X^\wedge)$ we first fix so-called weight data (γ, Θ) for $\gamma \in \mathbb{R}$ and $\Theta = (\vartheta, 0], -\infty \le \vartheta < 0$. Define the Fréchet space

$$\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) = \varprojlim_{k\in\mathbb{N}} \mathcal{K}^{s,\gamma-\vartheta-(1+k)^{-1}}(X^\wedge)$$

of elements of flatness Θ relative to γ. For purposes below we also introduce the spaces $\mathcal{K}^{s,\gamma;e}(X^\wedge) := \langle r\rangle^{-e}\mathcal{K}^{s,\gamma}(X^\wedge)$, $\mathcal{K}^{s,\gamma;e}_\Theta(X^\wedge) := \langle r\rangle^{-e}\mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$ for any $s,\gamma,e\in\mathbb{R}$. In order to define subspaces with asymptotics we consider a sequence

$$\mathcal{P} = \{(p_j, m_j)\}_{j=0,1,\dots,J} \subset \mathbb{C}\times\mathbb{N} \tag{1.8}$$

for a $J = J(\mathcal{P}) \in \mathbb{N}\cup\{\infty\}$ such that $(n+1)/2-\gamma+\vartheta < \operatorname{Re} p_j < (n+1)/2-\gamma$ for all $0\le j\le J$, $J(\mathcal{P})<\infty$ for $\vartheta>-\infty$. In the case $\vartheta=-\infty$ and $J=\infty$ we assume $\operatorname{Re} p_j \to -\infty$ as $j\to\infty$. Such a $\mathcal{P}$ will be called a discrete asymptotic type associated with (γ,Θ). We set $\pi_{\mathbb{C}}\mathcal{P} := \{p_j\}_{j=0,1,\dots,J}$. Observe that for any $p\in\mathbb{C}, \operatorname{Re} p < (n+1)/2-\gamma$, and $c\in C^\infty(X)$ we have $\omega(r)c(x)r^{-p}\log^k r \in \mathcal{K}^{\infty,\gamma}(X^\wedge)$ for $k\in\mathbb{N}$ and any cut-off function ω. Given a discrete asymptotic type $\mathcal{P}$ for finite Θ we form the space

$$\mathcal{E}_\mathcal{P} := \{\omega(r)\sum_{j=0}^{J}\sum_{k=0}^{m_j} c_{jk} r^{-p_j}\log^k r : c_{jk}\in C^\infty(X)\}, \tag{1.9}$$

for some fixed cut-off function ω. This space is Fréchet in a natural way (in fact, isomorphic to a corresponding direct sum of finitely many copies of $C^\infty(X)$), and we have $\mathcal{K}^{s,\gamma}_\Theta(X^\wedge)\cap\mathcal{E}_\mathcal{P} = \{0\}$. Then the direct sum

$$\mathcal{K}^{s,\gamma}_\mathcal{P}(X^\wedge) := \mathcal{K}^{s,\gamma}_\Theta(X^\wedge) + \mathcal{E}_\mathcal{P} \tag{1.10}$$

is again a Fréchet space. The spaces (1.10) are examples of subspaces of $\mathcal{K}^{s,\gamma}(X^\wedge)$ with discrete asymptotics of type $\mathcal{P}$. The definition can be easily extended to asymptotic types $\mathcal{P} = \{(p_j,m_j)\}_{j=0,1,\dots,J}$ associated with $(\gamma,(-\infty,0])$ and $J\in\mathbb{N}\cup\{\infty\}$. In this case we form $\mathcal{P}_k := \{(p,m)\in\mathcal{P} : \operatorname{Re} p > (n+1)/2-\gamma-(k+1)\}, k\in\mathbb{N}$; then $\mathcal{P}_k$ is finite and associated with $\big(\gamma,(-(k+1),0]\big)$. Thus we have the spaces $\mathcal{K}^{s,\gamma}_{\mathcal{P}_k}(X^\wedge)$ and we set

$$\mathcal{K}^{s,\gamma}_\mathcal{P}(X^\wedge) := \varprojlim_{k\in\mathbb{N}} \mathcal{K}^{s,\gamma}_{\mathcal{P}_k}(X^\wedge).$$

Another technical tool that we employ later on are operator-valued symbols based on twisted symbolic estimates. Let H and $\tilde{H}$ be Hilbert spaces with group actions κ and $\tilde\kappa$, respectively.

By $S^\mu(\Omega\times\mathbb{R}^q; H,\tilde{H})$ for an open set $\Omega\subseteq\mathbb{R}^p$ we denote the set of all $a(y,\eta)\in C^\infty\big(\Omega\times\mathbb{R}^q,\mathcal{L}(H,\tilde{H})\big)$ such that

$$\|\tilde\kappa^{-1}_{[\eta]}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{[\eta]}\|_{\mathcal{L}(H,\tilde{H})} \le c[\eta]^{\mu-|\beta|} \tag{1.11}$$

for all $(y,\eta)\in K\times\mathbb{R}^q, K\Subset\Omega$, and $\alpha\in\mathbb{N}^p,\beta\in\mathbb{N}^q$, for constants $c=c(\alpha,\beta,K)>0$. Such a are called (operator-valued) symbols of order μ. For instance, if $a(y,\eta)$ is homogeneous of order μ for large $|\eta|$ then it is such a symbol. By $S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;H,\tilde{H})$ we denote the subspace of classical symbols, i.e., the set of those $a(y,\eta)\in S^\mu(\Omega\times\mathbb{R}^q;H,\tilde{H})$ with an asymptotic expansion into symbols that are homogeneous of order $\mu-j, j\in\mathbb{N}$, for large $|\eta|$. Let $S^{(\mu)}\big(\Omega\times(\mathbb{R}^q\setminus\{0\});H,\tilde{H}\big)$ be the space of those $a_{(\mu)}(y,\eta)\in C^\infty\big(\Omega\times(\mathbb{R}^q\setminus\{0\}),\mathcal{L}(H,\tilde{H})\big)$ such that $a_{(\mu)}(y,\lambda\eta) = \lambda^\mu\tilde\kappa_\lambda a_{(\mu)}(y,\eta)\kappa^{-1}_\lambda$

for all $\lambda \in \mathbb{R}_+$. Every $a(y,\eta) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \tilde{H})$ has a principal symbol of order μ, i.e., the unique $a_{(\mu)}(y,\eta) \in S^{(\mu)}\big(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \tilde{H}\big)$ such that

$$a(y,\eta) - \chi(\eta) a_{(\mu)}(y,\eta) \in S^{\mu-1}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \tilde{H})$$

for any fixed excision function χ.

If a consideration is valid in the classical as well as the general case we write as subscript (cl). If necessary we also write $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H})_{\kappa,\tilde{\kappa}}$ for the respective spaces of symbols. The spaces of symbols with constant coefficients will be denoted by $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \tilde{H})$. The spaces $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H})$ are Fréchet in a natural way, $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \tilde{H})$ are closed subspaces, and we have

$$S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \tilde{H}) = C^{\infty}\big(\Omega, S^{\mu}_{(\mathrm{cl})}\big(\mathbb{R}^q; H, \tilde{H}\big)\big).$$

In the case $p = 2q$ and $\Omega \times \Omega$ for $\Omega \subseteq \mathbb{R}^q$ instead of $\Omega \subseteq \mathbb{R}^p$ we also write (y, y') rather than y.

For every $a(y,y',\eta) \in S^{\mu}(\Omega \times \Omega \times \mathbb{R}^q; H, \tilde{H})$ the operator $\mathrm{Op}(a) : C_0^{\infty}(\Omega, H) \to C^{\infty}(\Omega, \tilde{H})$, defined by

$$\mathrm{Op}_y(a)u(y) := \iint e^{i(y-y')\eta} a(y,y',\eta) u(y')\, dy' đ\eta, \tag{1.12}$$

extends to a continuous map

$$\mathrm{Op}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, H) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{H}) \tag{1.13}$$

for any $s \in \mathbb{R}$. The continuity (1.13) has been established in [11, page 283] for all spaces $H, \tilde{H}$ that are of interest here. The case of general $H, \tilde{H}$ with group action was given in [18]. In the special case of $a(\eta) \in S^{\mu}(\mathbb{R}^q; H, \tilde{H})$ the operator $\mathrm{Op}(a)$ induces a continuous operator

$$\mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{H}) \tag{1.14}$$

for any $s \in \mathbb{R}$. Here

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{H}))} \le \sup_{\eta \in \mathbb{R}^q} [\eta]^{-\mu} \|\tilde{\kappa}^{-1}_{[\eta]} a(\eta) \kappa_{[\eta]}\|_{\mathcal{L}(H,\tilde{H})}. \tag{1.15}$$

Remark 1.1. Observe that (1.14) already holds for $a(\eta) \in C^{\infty}\big(\mathbb{R}^q, \mathcal{L}(H, \tilde{H})\big)$ when the 0th symbolic estimate (1.11) holds, namely,

$$\|\tilde{\kappa}^{-1}_{[\eta]} a(\eta) \kappa_{[\eta]}\|_{\mathcal{L}(H,\tilde{H})} \le c[\eta]^{\mu}$$

for all $\eta \in \mathbb{R}^q$, for some $c > 0$.

We will employ below a slight modification of such a construction. Let us start, in particular, with the case $H = \mathbb{C}$ with the trivial group action. Symbols in $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; \mathbb{C}, \tilde{H})$ are also referred to as potential symbols. Consider, for instance, the case of symbols $a(\eta)$ with constant coefficients, i.e., without y-dependence. Such symbols are realized as multiplications of $c \in \mathbb{C}$ by η-dependent families $f(\eta)$ of elements $\in \tilde{H}$. The symbolic estimates have the form

$$\|\kappa^{-1}_{[\eta]} D^{\beta}_{\eta} f(\eta)\|_{\mathcal{L}(\mathbb{C},\tilde{H})} = \|\kappa^{-1}_{[\eta]} D^{\beta}_{\eta} f(\eta)\|_{\tilde{H}} \le C[\eta]^{\mu-|\beta|}. \tag{1.16}$$

In our applications we have the situation that for a Fréchet space $E = \varprojlim_{j\in\mathbb{N}} E^j$ for Hilbert spaces E^j and the trivial group action id on all E^j we encounter E and tensor products $\tilde{H}\hat{\otimes}_\pi E$ rather than $\mathbb{C}$ and $\tilde{H}$. In our case E will be nuclear, and then we have

$$\tilde{H}\hat{\otimes}_\pi E = \varprojlim_{k\in\mathbb{N}} \tilde{H}\otimes_H E^j$$

with $\otimes_H$ being the Hilbert tensor product. Such things are well known, but details may be found, e.g., in [10, page 38]. From $f(\eta) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; \mathbb{C}, \tilde{H})$ we pass to the operator function $f(\eta) \otimes \mathrm{id}_E$. This can be interpreted as a symbol

$$f \otimes \mathrm{id}_E \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{H}\hat{\otimes}_\pi E) = \varprojlim_{j\in\mathbb{N}} S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E_j, \tilde{H}\otimes_H E^j).$$

In fact, instead of (1.16) we have the symbolic estimates

$$\begin{aligned}\|(\kappa^{-1}_{[\eta]} \otimes \mathrm{id}_{E^j}) D^\beta_\eta \big(f(\eta) \otimes \mathrm{id}_{E^j}\big)\|_{\mathcal{L}(E^j, \tilde{H}\otimes_H E^j)} &= \|\kappa^{-1}_{[\eta]} D^\beta_\eta f(\eta)\|_{\tilde{H}} \|\mathrm{id}_{E^j}\|_{\mathcal{L}(E^j,E^j)} \\ &= \|\kappa^{-1}_{[\eta]} D^\beta_\eta f(\eta)\|_{\tilde{H}} \le C[\eta]^{\mu-|\beta|}\end{aligned}$$

for every j. Similarly as (1.14) we obtain continuous operators

$$\mathrm{Op}_y(f \otimes \mathrm{id}_{E^j}) : H^s(\mathbb{R}^q, E^j) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{H}\otimes_H E^j). \tag{1.17}$$

The space on the right refers to the group action $\kappa_\lambda \otimes \mathrm{id}_E$, such that

$$\|u\|_{\mathcal{W}^t(\mathbb{R}^q, \tilde{H}\otimes_H E^j)} = \left\{ \int [\eta]^{2t} \left\| (\kappa^{-1}_{[\eta]} \otimes \mathrm{id}_{E^j}) \hat{u}(\eta) \right\|^2_{\tilde{H}\otimes_H E^j} d\eta \right\}^{1/2}$$

for every j. We have

$$\mathcal{W}^t(\mathbb{R}^q, \tilde{H}\hat{\otimes}_\pi E) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^t(\mathbb{R}^q, \tilde{H}\otimes_H E^j),$$

$t \in \mathbb{R}$, and it follows altogether

$$\mathrm{Op}_y(f \otimes \mathrm{id}_E) : H^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{H}\hat{\otimes}_\pi E). \tag{1.18}$$

1.2. Characterization of singular functions

In order to formulate the singular functions of discrete edge asymptotics we endow the Fréchet spaces $\mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)$ with the group action

$$(\kappa_\lambda u)(r, x) := \lambda^{(n+1)/2} u(\lambda r, x), \tag{1.19}$$

$\lambda \in \mathbb{R}_+$. The larger spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ are endowed with this group action as well, and we may consider κ_λ also over the spaces $\mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$ of functions of flatness Θ relative to γ. This allows us to define the spaces

$$\mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)\big) \supset \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)\big) \supset \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)\big).$$

The space $\mathcal{E}_{\mathcal{P}}$ of singular functions of cone asymptotics, defined for any fixed cut-off function ω, is not invariant under κ. Nevertheless, according to (1.10) it is desirable also to decompose $\mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)\big)$ into a flat part, namely, $\mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)\big)$

and a subspace generated by the singular functions. Here we proceed as follows. We first look at the case $\kappa = \mathrm{id}$ and observe that from (1.10) we have a direct sum

$$\mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge)\big)_{\mathrm{id}} = \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_{\Theta}^{s,\gamma}(X^\wedge)\big)_{\mathrm{id}} + \mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})_{\mathrm{id}}.$$

Clearly $\mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})_{\mathrm{id}}$ is a subspace of $\mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge)\big)_{\mathrm{id}}$. According to (1.4) we have an isomorphism

$$\mathbb{K} : \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge)\big)_{\mathrm{id}} \to \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge)\big)_{\kappa}.$$

Thus, applying (1.4) to the subspace $\mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})_{\mathrm{id}}$ we obtain a subspace of $\mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge)\big)_{\kappa}$ and a direct decomposition

$$\mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge)\big)_{\kappa} = \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_{\Theta}^{s,\gamma}(X^\wedge)\big)_{\kappa} + \mathbb{K}\mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})_{\mathrm{id}}. \tag{1.20}$$

By virtue of the definition of the operator $\mathbb{K}$ we have

$$\begin{aligned}\mathbb{K}\mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})_{\mathrm{id}} = \operatorname{span}\{&F^{-1}_{y\to\eta}[\eta]^{(n+1)/2}\omega(r[\eta])\hat{c}_{jk}(x,\eta)(r[\eta])^{-p_j}\log^k(r[\eta])\\ &: 0 \le k \le m_j, j = 0,1,\ldots,J, \hat{c}_{jk}(x,\eta) \in \hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\}\end{aligned} \tag{1.21}$$

for $\hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big) := F_{y\to\eta}H^s\big(\mathbb{R}^q_y, C^\infty(X)\big)$. This follows from the fact that

$$\begin{aligned}\mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})_{\mathrm{id}} = H^s(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}}) = \operatorname{span}\{&\omega(r)c_{jk}(x,y)r^{-p_j}\log^k r\\ &: 0 \le k \le m_j, j = 0,1,\ldots,J, c_{jk}(x,y) \in H^s\big(\mathbb{R}^q_y, C^\infty(X)\big)\}.\end{aligned} \tag{1.22}$$

The explicit form (1.21) gives us a first impression on the nature of singular terms of the edge asymptotics for a constant (in y) asymptotic type $\mathcal{P}$ and finite Θ.

Let us briefly comment the case $s = \infty$ where the $\mathcal{W}^s$-spaces do not depend on κ, cf. the relation (1.3). In that case we may choose the singular functions in the form (1.22) for $s = \infty$, i.e., the r-powers, logarithmic terms and the cut-off function ω do not contain η. In other words we have the direct decomposition

$$\begin{aligned}\mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{\infty,\gamma}(X^\wedge)\big)_{\mathrm{id}} &= \mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}_{\Theta}^{\infty,\gamma}(X^\wedge)\big)_{\mathrm{id}} + \mathcal{W}^\infty(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})_{\mathrm{id}}\\ &= H^\infty\big(\mathbb{R}^q, \mathcal{K}_{\Theta}^{\infty,\gamma}(X^\wedge)\big) + H^\infty(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}}).\end{aligned} \tag{1.23}$$

On the other hand we have

$$\mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{\infty,\gamma}(X^\wedge)\big)_{\kappa} = \mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}_{\Theta}^{\infty,\gamma}(X^\wedge)\big)_{\kappa} + \mathbb{K}\mathcal{W}^\infty(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})_{\mathrm{id}}, \tag{1.24}$$

cf. the relation (1.20) for $s = \infty$. By virtue of (1.3) the only formal difference between (1.23) and (1.24) for $s = \infty$ lies in the difference between $H^\infty(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})$ and $\mathbb{K}\mathcal{W}^\infty(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})_{\mathrm{id}}$.

Proposition 1.2. *Let $\mathcal{P} = \{(p_j, m_j)\}_{j=0,1,\ldots,J}$ be a discrete asymptotic type associated with the weight data (γ, Θ) for finite $\Theta = (\vartheta, 0]$. Then there is a direct decomposition*

$$\mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{\infty,\gamma}(X^\wedge)\big)_{\kappa} = \mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}_{\Theta}^{\infty,\gamma}(X^\wedge)\big)_{\kappa} + H^\infty(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}})$$

where

$$\begin{aligned}H^\infty(\mathbb{R}^q, \mathcal{E}_{\mathcal{P}}) = \operatorname{span}\big\{&\omega(r)c_{jk}(x,y)r^{-p_j}\log^k(r) : 0 \le k \le m_j,\\ &j = 0,1,\ldots,J, c_{jk} \in H^\infty\big(\mathbb{R}^q_y, C^\infty(X)\big)\big\}.\end{aligned} \tag{1.25}$$

Proof. We write down once again (1.21) for $s=\infty$, namely,

$$\begin{aligned}&\mathbb{K}\mathcal{W}^{\infty}(\mathbb{R}^q,\mathcal{E}_{\mathcal{P}})_{\mathrm{id}}=\operatorname{span}\big\{F^{-1}_{y\to\eta}[\eta]^{(n+1)/2}\omega(r[\eta])\hat{c}_{jk}(x,\eta)(r[\eta])^{-p_j}\\&\log^k(r[\eta]),0\le k\le m_j,j=0,1,\dots,J,\hat{c}_{jk}(x,\eta)\in\hat{H}^{\infty}\big(\mathbb{R}^q_\eta,C^{\infty}(X)\big)\big\}.\end{aligned}\tag{1.26}$$

First it is clear that $[\eta]^{-p_j}$ gives rise to a modification of the coefficients in $\hat{H}^{\infty}\big(\mathbb{R}^q_\eta,C^{\infty}(X)\big)$, since $[\eta]^M\hat{H}^{\infty}\big(\mathbb{R}^q_\eta,C^{\infty}(X)\big)=\hat{H}^{\infty}\big(\mathbb{R}^q_\eta,C^{\infty}(X)\big)$ for any real M. Moreover, writing $\log(r[\eta])=\log r+\log[\eta]$ we can dissolve $\log^k(r[\eta])$ as a sum of products between powers of $\log r$ and $\log[\eta]$. Also the $\log[\eta]$-terms are absorbed by $\hat{H}^{\infty}(\mathbb{R}^q_\eta,C^{\infty}(X))$, and hence we get rid of $[\eta]$ in (1.26), except for the cut-off function $\omega(r[\eta])$. In order to remove $[\eta]$ from the cut-off function we apply Taylor's-formula. Choose another cut-off function $\tilde{\omega}\succ\omega$ where $\tilde{\varphi}\succ\varphi$ or $\varphi\prec\tilde{\varphi}$ means that $\tilde{\varphi}$ is equal to 1 on $\operatorname{supp}\varphi$ such that $\tilde{\omega}(r)\big(\omega(r[\eta])-\omega(r)\big)=\omega(r[\eta])-\omega(r)$ for all r and η. Then

$$\begin{aligned}\omega(r[\eta])-\omega(r)=\tilde{\omega}(r)\bigg\{&\frac{(r[\eta])^{N+1}}{N!}\int_0^1(1-t)^N\omega^{(N+1)}(r[\eta]t)\,dt\\&-\frac{r^{N+1}}{N!}\int_0^1(1-t)^N\omega^{(N+1)}(rt)\,dt\bigg\}.\end{aligned}\tag{1.27}$$

If we verify that this function belongs to $\mathcal{W}^{\infty}\big(\mathbb{R}^q,\mathcal{K}^{\infty,\gamma}_{\Theta}(X^\wedge)\big)$ for sufficiently large N we may replace in the formula (1.26) $\omega(r[\eta])$ by $\omega(r)$, i.e., after the comments before on how to remove $[\eta]$ from $(r[\eta])^{-p_j}$ or $\log^k(r[\eta])$ we see altogether, that the singular functions of edge asymptotics for $s=\infty$ are of the form (1.25). The fact that a function $\psi\in C_0^{\infty}(\overline{\mathbb{R}}_+)$ of sufficiently high flatness at $r=0$, i.e., $r^{-N}\psi(r)\in C_0^{\infty}(\overline{\mathbb{R}}_+)$ for large and fixed N, belongs to $\mathcal{W}^{\infty}\big(\mathbb{R}^q,\mathcal{K}^{\infty,\gamma}_{\Theta}(X^\wedge)\big)$, follows from the fact that $\psi(r)$ may be regarded as an operator-valued symbol

$$\psi\in S^{\mu}(\mathbb{R}^q;\mathbb{C},\tilde{H}^j)$$

for $\tilde{H}^j:=\mathcal{K}^{s,\gamma-n/2-\vartheta-(1+j)^{-1}}(\mathbb{R}_+)$ and some $\mu=\mu(s)\in\mathbb{R}$, for all $j\in\mathbb{N}$.

In fact, it is clear that $\psi\in\tilde{H}^j$ for a fixed sufficiently large $N\in\mathbb{N}$. Moreover, $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ defined by (1.19) acts on $\tilde{H}^j$ for every j. Thus, by virtue of (1.1) we have $\|\kappa_\lambda\|_{\mathcal{L}(\tilde{H}^j)}\le c\max\{\lambda,\lambda^{-1}\}^M$ for constants $c,M>0$ depending on the space $\tilde{H}^j$, in fact, on s. The symbolic estimates (1.16) for ψ rather than $f(\eta)$, here independent of η, reduce to the estimate to $\beta=0$, and we have

$$\|\kappa^{-1}_{[\eta]}\psi\|_{\mathcal{L}(\mathbb{C},\tilde{H}^j)}\le\|\kappa^{-1}_{[\eta]}\|_{\mathcal{L}(\tilde{H}^j)}\|\psi\|_{\mathcal{L}(\mathbb{C},\tilde{H}^j)}\le c[\eta]^{\mu}\|\psi\|_{\tilde{H}^j}$$

for some μ and constants $c=c(j)>0$. Then, writing $E:=C^{\infty}(X)=\varprojlim_{j\in\mathbb{N}}E^j$, where we may take $E^j:=H^j(X)$, we obtain $\psi\otimes\mathrm{id}_{E^j}\in S^{\mu}(\mathbb{R}^q;E^j,\tilde{H}^j\otimes_H E^j)$. This gives us the continuity

$$\mathrm{Op}_y(\psi\otimes\mathrm{id}_{E^j}):H^{\tilde{s}}(\mathbb{R}^q,E^j)\to\mathcal{W}^{\tilde{s}-\mu}(\mathbb{R}^q,\tilde{H}^j\otimes_H E^j)$$

for every $\tilde{s} \in \mathbb{R}$, cf. (1.17) which entails

$$\mathrm{Op}_y(\psi \otimes \mathrm{id}_{E^j}) : H^\infty(\mathbb{R}^q, E^j) \to \mathcal{W}^\infty(\mathbb{R}^q, \tilde{H}^j \otimes_H E^j)$$

and

$$\mathrm{Op}_y\big(\psi \otimes \mathrm{id}_{C^\infty(X)}\big) : H^\infty\big(\mathbb{R}^q, C^\infty(X)\big) \to \mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma-n/2}(\mathbb{R}_+)\big)\hat{\otimes}_\pi C^\infty(X) = \mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge)\big).$$

Here we employed the relation $\mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge) = \mathcal{K}_\Theta^{\infty,\gamma-n/2}(\mathbb{R}_+)\hat{\otimes}_\pi C^\infty(X)$. For the second summand in (1.27) we argue as follows. The function $g(r) := \int_0^1 (1-t)^N \omega^{(N+1)}(rt)\,dt$ on $\overline{\mathbb{R}}_+$ belongs to $C^\infty(\overline{\mathbb{R}}_+)$ and is bounded on $\overline{\mathbb{R}}_+$ including all its r-derivatives. The same is true of $f(\eta) = g(r[\eta])$ as a function in $r \in \overline{\mathbb{R}}_+$. The notation $f(\eta)$ indicates that f is regarded as an operator-valued symbol. The operator of multiplication by $g(r)$ induces continuous operators $g : \tilde{H}^j \to \tilde{H}^j$ for all j. Thus $\mathrm{Op}_y(f) : \mathcal{W}^s(\mathbb{R}^q, \tilde{H}^j) \to \mathcal{W}^s(\mathbb{R}^q, \tilde{H}^j)$ is continuous for every $s \in \mathbb{R}$, cf. Remark 1.1. Setting $h(\eta) = f(\eta)\tilde{\omega}(r)(r[\eta])^{N+1}/N!$ and $\psi(r) = \tilde{\omega}(r)r^{N+1}/N!$ we have

$$\mathrm{Op}_y(h \otimes \mathrm{id}_{E^j}) = [\eta]^{N+1}\mathrm{Op}_y(f \otimes \mathrm{id}_{E^j})\mathrm{Op}_y(\psi \otimes \mathrm{id}_{E^j}).$$

From the first step of the proof we know that

$$\psi \otimes \mathrm{id}_{E^j} \in S^{\mu+N+1}(\mathbb{R}^q; E^j, \tilde{H}^j \otimes_H \tilde{H}^j).$$

It follows altogether

$$\mathrm{Op}_y(h) \otimes \mathrm{id}_{E^j} : H^{\tilde{s}}(\mathbb{R}^q, E^j) \to \mathcal{W}^{\tilde{s}-(\mu+N+1)}(\mathbb{R}^q, \tilde{H}^j \otimes_H E^j)$$

for every $\tilde{s} \in \mathbb{R}$, and finally

$$\mathrm{Op}_y(h) \otimes \mathrm{id}_E : H^\infty\big(\mathbb{R}^q, C^\infty(X)\big) \to \mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge)\big). \qquad \square$$

The case of variable discrete asymptotics will be prepared here by a number of specific observations. We saw that the space (1.21) is the image of

$$H^s(\mathbb{R}^q)\hat{\otimes}_\pi C^\infty(X)$$

under the action of a pseudo-differential operator

$$\mathrm{Op}_y(k)\hat{\otimes}_\pi \mathrm{id}_{C^\infty(X)} : H^s(\mathbb{R}^q)\hat{\otimes}_\pi C^\infty(X) \to \mathbb{K}\mathcal{W}^s(\mathbb{R}^q, \mathcal{E}_\mathcal{P})_{\mathrm{id}}$$

for symbols $k(\eta) \in S^0_{\mathrm{cl}}\big(\mathbb{R}^q; \mathbb{C}, \mathcal{K}^{\infty,\gamma-n/2}(\mathbb{R}_+)\big)$, $k(\eta) : c \to k(\eta)c, c \in \mathbb{C}$, where $k(\eta) := \sum_{j=0}^J \sum_{k=0}^{m_j} c_{jk}[\eta]^{(n+1)/2}\omega(r[\eta])(r[\eta])^{-p_j}\log^k(r[\eta])$ for arbitrary constants $c_{jk} \in \mathbb{C}$, $0 \leq j \leq m_j$, $j = 0, 1, \dots, J$.

Let us form the compact set $K := \pi_\mathbb{C}\mathcal{P} = \{p_j\}_{j=0,1,\dots,J}$ and choose any counter clockwise oriented (say, smooth) curve C surrounding K such that the winding number with respect to any $z \in K$ is equal to 1. The function

$$M_{r\to z}\left(\omega(r)\sum_{j=0}^J\sum_{k=0}^{m_j} c_{jk}(x)r^{-p_j}\log^k r\right)(z) =: f(z)$$

with M being the weighted Mellin transform for the weight $\gamma - n/2$ is meromorphic with poles at the points p_j of multiplicity $m_j + 1$ and Laurent coefficients $(-1)^k k! c_{jk}(x)$. This comes from the identity

$$M_{r\to z}\big(\omega(r)r^{-p}\log^k r\big)(z) = \frac{(-1)^k k!}{(z-p)^{k+1}}$$

for any $p \in \mathbb{C}, k \in \mathbb{N}$, modulo an entire function. For any compact set $K \subset \mathbb{C}$ by $\mathcal{A}'(K)$ we denote the space of analytic functionals carried by K, see [7, Vol. 1] or [8, Section 2.3]. The space $\mathcal{A}'(K)$ is a nuclear Fréchet space. Given another Fréchet space E we set $\mathcal{A}'(K,E) := \mathcal{A}'(K)\hat{\otimes}_\pi E$. Now

$$\mathcal{A}(\mathbb{C}) \ni h \to \langle \zeta_{f,z}, h\rangle := \int_C f(z)h(z)đz \tag{1.28}$$

is an analytic functional with carrier K, more precisely, $\zeta \in \mathcal{A}'\big(K, C^\infty(X)\big)$. It is of finite order in the sense of a linear combination of finite order derivatives of the Dirac measures at the points p_j. Inserting $h(z) := r^{-z}$ we just obtain

$$\langle \zeta_{f,z}, r^{-z}\rangle = \sum_{j=0}^{J}\sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j}\log^k r, \tag{1.29}$$

i.e., the singular functions are again reproduced as a linear superposition of r^{-z} with the density ζ.

The above-mentioned singular functions (1.21) of constant discrete edge asymptotics of type $\mathcal{P}$ may be written in the form

$$F^{-1}_{\eta\to y}\big\{[\eta]^{(n+1)/2}\omega(r[\eta])\langle\hat{\zeta}(\eta)_z, (r[\eta])^{-z}\rangle\big\}$$

where $\hat{\zeta}(\eta) \in \mathcal{A}'\big(K, \hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\big)$ is applied to $(r[\eta])^{-z}$; subscript z indicates the pairing with respect to z. The form of $\hat{\zeta}(\eta)$ is subordinate to $\mathcal{P}$ in the sense that $\langle\hat{\zeta}(\eta)_z, r^{-z}\rangle$ is a $\hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big)$-valued meromorphic function with poles at the points $p_j \in \pi_\mathbb{C}\mathcal{P}$ of multiplicity m_j+1. To have a notation, if E is a Fréchet space then a $\zeta \in \mathcal{A}'(K,E)$ is said to be subordinate to $\mathcal{P}$ if $\langle\zeta, r^{-z}\rangle$ is meromorphic with such poles and multiplicities, determined by $\mathcal{P}$. Let $\mathcal{A}'_\mathcal{P}(K,E)$ denote the subspace of all $\zeta \in \mathcal{A}'(K,E)$ of that kind.

2. Branching edge asymptotics

2.1. Wedge spaces with branching edge asymptotics

The role of the present section is to deepen and complete material from [16] on wedge space with variable branching edge asymptotics. To this end we first recall the notion of variable discrete asymptotic types.

Let $\mathcal{U}(\Omega)$ for an open set $\Omega \subseteq \mathbb{R}^q$ denote the system of all open subsets $U \subset \Omega$ with compact closure $\overline{U} \subset \Omega$.

Definition 2.1. A variable discrete asymptotic type $\mathcal{P}$ over an open set $\Omega \subseteq \mathbb{R}^q$ associated with weight data $(\gamma, \Theta), \Theta = (\vartheta, 0], -\infty < \vartheta < 0$, is a system of sequences of pairs

$$\mathcal{P}(y) = \{(p_j(y), m_j(y))\}_{j=0,1,\dots,J(y)} \tag{2.1}$$

for $J(y) \in \mathbb{N}$, $y \in \Omega$, such that $\pi_{\mathbb{C}}\mathcal{P} := \{p_j(y)\}_{j=0,1,\dots,J(y)} \subset \{(n+1)/2 - \gamma + \vartheta < \operatorname{Re} z < (n+1)/2 - \gamma\}$ for all $y \in \Omega$, and for every $b = (c, U), (n+1)/2 - \gamma + \vartheta < c < (n+1)/2 - \gamma$, $U \in \mathcal{U}(\Omega)$, there are sets $\{U_i\}_{0 \le i \le N}$, $\{K_i\}_{0 \le i \le N}$, for some $N = N(b)$, where $U_i \in \mathcal{U}(\Omega), 0 \le i \le N$, form an open covering of $\overline{U}$, moreover,

$$K_i \Subset \mathbb{C}, K_i \subset \{c - \varepsilon_i < \operatorname{Re} z < (n+1)/2 - \gamma\} \quad \text{for some} \quad \varepsilon_i > 0, \tag{2.2}$$

such that

$$\pi_{\mathbb{C}}\mathcal{P} \cap \{c - \varepsilon_i < \operatorname{Re} z < (n+1)/2 - \gamma\} \subset K_i \quad \text{for all} \quad y \in U_i \tag{2.3}$$

and

$$\sup_{y \in U_i} \sum_j \big(1 + m_j(y)\big) < \infty$$

where the sum is taken over those $0 \le j \le J(y)$ such that $p_j(y) \in K_i, i = 0, 1, \dots, N$.

We will say that a variable discrete asymptotic type $\mathcal{P}$ satisfies the shadow condition if $(p(y), m(y)) \in \mathcal{P}(y)$ implies $(p(y) - l, m(y)) \in \mathcal{P}(y)$ for every $l \in \mathbb{N}$, such that $\operatorname{Re} p(y) - l > (n+1)/2 - \gamma + \vartheta$, for all $y \in \Omega$. Observe that such a condition is natural when we ask the spaces of functions u with asymptotics (0.1) to be closed under multiplication by functions $\varphi \in C^\infty(\overline{\mathbb{R}}_+)$, and then the Taylor asymptotics of φ at $r = 0$ contributes to the asymptotics of φu. For any open $\tilde{\Omega} \subseteq \Omega$ we define the restriction $\mathcal{P}|_{\tilde{\Omega}} := \{\big(p(y), m(y)\big) \in \mathcal{P} : y \in \tilde{\Omega}\}$. We also define restrictions to $A \subseteq \mathbb{C}$ by setting $\mathrm{r}_A\mathcal{P} := \{\big(p(y), m(y)\big) \in \mathcal{P} : p(y) \in A\}$.

In future if $K \subset \mathbb{C}$ is a compact set and we are talking about a curve $C \subset \mathbb{C}\backslash K$ counter clockwise surrounding K we tacitly assume that the winding number is 1 with respect to every $z \in K$. It is well known, that for every K such a C always exists in an ε-neighbourhood of K for any $\varepsilon > 0$.

Parallel to variable discrete asymptotic types $\mathcal{P}$ we consider families of analytic functionals that are y-wise discrete and of finite order. Typical families of that kind are generated by functions $f(y, z) \in C^\infty\big(\Omega, \mathcal{A}(\mathbb{C} \setminus K)\big)$ that extend across K for every $y \in \Omega$ to a meromorphic function in z, with finitely many poles $p_0(y), p_1(y), \dots, p_J(y) \in K$ where $p_j(y)$ is of multiplicity $m_j(y) + 1$. The corresponding system $\mathcal{P}(y)$ of the form (2.1) is a variable discrete asymptotic type in the sense of Definition 2.1.

More generally, if we have a family of meromorphic functions $f(y, z)$, parametrized by $y \in \Omega$ we will say that f is subordinate to (2.1) if for every $y \in \Omega$ the system of poles is contained in $\pi_{\mathbb{C}}\mathcal{P}(y)$ and the multiplicities are $\le m_j(y) + 1$. With such an $f(y, z)$ we can associate a family of analytic functionals as follows. We fix $b = (c, U)$ as in Definition 2.1 and choose a pair (U_i, K_i) and a smooth curve $C_i \subset \{c-\varepsilon_i < \operatorname{Re} z < (n+1)/2-\gamma\}$ counter clockwise surrounding K_i, and then we form $\delta_i(y) \in \mathcal{A}'(K_i)$ by

$$\langle \delta_i(y)_z, h\rangle := \int_C f(y,z)h(z)\,đz,$$

$h \in \mathcal{A}(\mathbb{C})$. The family f is called smooth in $y \in \Omega$ if $\delta_i(y) \in C^\infty\big(U_i, \mathcal{A}'(K_i)\big)$ for all $i = 0, 1, \dots, N$, and if this is also the case for all $U \in \mathcal{U}(\Omega)$.

In the following constructions it will be convenient to fix for any given $U \in \mathcal{U}$ a system of $\varphi_i \in C_0^\infty(U_i)$, $i = 0, 1, \dots, N$, such that $\sum_{i=0}^N \varphi_i = 1$ for all $y \in \overline{U}$. This yields a family

$$\delta_U(y) := \sum_{i=0}^N \varphi_i(y)\delta_i(y) \in C^\infty\big(U, \mathcal{A}'(K)\big) \tag{2.4}$$

for $K := \bigcup_{i=0}^N K_i$ which has the property that $M_{r\to z}\big(\omega(r)\langle \delta_U(y)_w, r^{-w}\rangle\big)$ is a family of meromorphic functions over U equal to $f(y,z)|_U$ modulo a function in $C^\infty\big(U, \mathcal{A}\big(c-\varepsilon < \operatorname{Re} z < (n+1)/2-\gamma\big)\big)$, $\varepsilon = \min\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N\}$.

Let us summarize these observations in the analogous case of E-valued meromorphic functions and E-valued analytic functionals as follows.

Given a Fréchet space E and a family of E-valued functions $f(y,z)$ parametrized by $y \in \Omega$ and meromorphic in $(n+1)/2-\gamma+\vartheta < \operatorname{Re} z < (n+1)/2-\gamma$, we say that f is subordinate to $\mathcal{P}$ if every pole of $f(y,\cdot)$ belongs to a pair $(p(y), m(y)) \in \mathcal{P}$ where the multiplicity is less or equal $m(y)+1$.

Let $U \in \mathcal{U}, K \subseteq \mathbb{C}$, then $C^\infty\big(U, \mathcal{A}(\mathbb{C}\setminus K, E)\big)^\bullet$ will denote the subspace of all $f(y,z) \in C^\infty\big(U, \mathcal{A}(\mathbb{C}\setminus K, E)\big)$ that extend for every $y \in U$ to a meromorphic function across K, again denoted by $f(y,z)$, where poles and multiplicities minus 1 form a $\mathcal{P}$ as in Definition 2.1. If we specify $\mathcal{P}$ we also denote the space of such functions by $C^\infty\big(\Omega, \mathcal{A}_{\mathcal{P}}(\mathbb{C}, E)\big)$.

If $f(y,z)$ is any family of meromorphic functions parametrized by $y \in \Omega$ such that the pattern of poles together with multiplicities minus 1 is a $\mathcal{P}$ as in Definition 2.1 we may define smoothness in y as follows. First we fix any $y_0 \in \Omega$ and a $b = (c, U)$ and sets $K_i, U_i, i = 0, 1, \dots N$, as in Definition 2.1. Choose compact smooth curves $C_i \subset \{c-\varepsilon_i < \operatorname{Re} z < (n+1)/2-\gamma\}$ counter clockwise surrounding K_i and define $\delta_i(y) \in \mathcal{A}'(K_i, E)$ by $\langle \delta_i(y)_z, h\rangle := \int_{C_i} f(y,z)h(z)đz$, $h \in \mathcal{A}(\mathbb{C}), y \in U_i$. Then f is called smooth if $\delta_i \in C^\infty\big(U_i, \mathcal{A}'(K_i, E)\big)$ for $i = 0, 1, \dots, N$.

Remark 2.2. Consider the above-mentioned $f(y,z)$. Setting $f_i(y,z) := M_{r\to z}\omega(r)$ $\langle \delta_i(y)_z, r^{-z}\rangle$ with M being the weighted Mellin transform for any weight β such that $\Gamma_{1/2-\beta} \cap K_i = \emptyset$ we obtain an element in $C^\infty\big(U_i, \mathcal{A}(\mathbb{C}\setminus K_i, E)\big)$ subordinate to $\mathcal{P}|_{U_i}$. Clearly, in this case we have $f_i(y,z) \in C^\infty\big(U_i, \mathcal{A}(\mathbb{C}\setminus K_i, E)\big)$. Moreover, if $\{\varphi_i\}_{i=0,1,\dots,N}$ is a system $\varphi_j \in C_0^\infty(U_j)$ such that $\sum_{j=0}^N \varphi_j \equiv 1$ over $\overline{U} \subset \bigcup_{i=0}^N U_i$, then $f_b(y,z) := \sum_{i=0}^N \varphi_i(y) f_i(y,z)$ satisfies the relation $f|_U = f_b \mod C^\infty\big(U, \mathcal{A}(c-\varepsilon < \operatorname{Re} z < (n+1)/2-\gamma, E)\big)$ for $\varepsilon := \min\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N\}$.

Let us now recall from [16] the definition of weighted edge distributions of variable discrete asymptotic type $\mathcal{P}$, cf. Definition 2.1.

Definition 2.3. Let $\Omega \subseteq \mathbb{R}^q$ be open and let $\mathcal{P}$ be a variable discrete asymptotic type, cf. Definition 2.1 associated with the weight data (γ, Θ), $\Theta = (\vartheta, 0]$ finite. Then $\mathcal{W}^s_{\mathrm{loc}}\big(\Omega, \mathcal{K}^{s;\gamma}_{\mathcal{P}}(X^\wedge)\big)$ for $s \in \mathbb{R}$ is defined to be the set of all $u \in \mathcal{W}^s_{\mathrm{loc}}\big(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge)\big)$ such that for every $b := (c, U)$ for any $(n+1)/2 - \gamma + \vartheta < c < (n+1)/2 - \gamma$ and $U \in \mathcal{U}(\Omega)$ there exists a compact set $K_b \subset \{(n+1)/2 - \gamma + \vartheta < \operatorname{Re} z < (n+1)/2 - \gamma\}$ and a function $\hat{f}_b(y, z, \eta) \in C^\infty\big(U, \mathcal{A}(\mathbb{C} \setminus K_b, E^s)\big)^\bullet$ for

$$E^s := \hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big) \tag{2.5}$$

subordinate to $\mathcal{P}|_U$ and a corresponding $\hat{\delta}_b(y, \eta) \in C^\infty\big(U, \mathcal{A}'(K_b, E^s)\big)^\bullet$,

$$\langle \hat{\delta}_b(y, \eta)_z, h \rangle = \int_{C_b} \hat{f}_b(y, z, \eta) h(z)\, đz, \;\; h \in \mathcal{A}(\mathbb{C}), \tag{2.6}$$

with C_b counter clockwise surrounding K_b, such that

$$u(r, x, y) - F^{-1}_{\eta \to y}\{[\eta]^{(n+1)/2} \omega(r[\eta]) \langle \hat{\delta}_b(y, \eta)_z, (r[\eta])^{-z} \rangle\} \in \mathcal{W}^s_{\mathrm{loc}}\big(U, \mathcal{K}^{s,\gamma+\beta}(X^\wedge)\big) \tag{2.7}$$

for $\beta := \beta_0 + \varepsilon$ for any $0 < \varepsilon < \varepsilon(b)$, $\beta_0 := (n+1)/2 - \gamma - c$. Moreover, we set

$$\mathcal{W}^s_{\mathrm{comp}}\big(\Omega, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)\big) := \mathcal{W}^s_{\mathrm{loc}}\big(\Omega, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)\big) \cap \mathcal{W}^s_{\mathrm{comp}}\big(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge)\big).$$

For convenience, as a consequence of Definition 2.3, we characterize the space $\mathcal{W}^s_{\mathrm{loc}}\big(\Omega, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)\big)$ as the set of all $u \in \mathcal{W}^s_{\mathrm{loc}}(\Omega, \mathcal{K}^{s,\gamma}\big(X^\wedge)\big)$ such that for every $b = (c, U)$ the function $u|_U$ belongs to the space

$$\mathcal{W}^s_{\mathrm{loc}}\big(U, \mathcal{K}^{s,\gamma+\beta}(X^\wedge)\big) + \mathcal{W}^s_{b,\mathcal{P}}(U) \tag{2.8}$$

where $\mathcal{W}^s_{b,\mathcal{P}}(U) := \{F^{-1}_{\eta \to y}(\kappa_{[\eta]} \omega(r) \langle \hat{\delta}_b(y, \eta)_z, r^{-z} \rangle)\}$, $\hat{\delta}_b(y, \eta)$ as in (2.6) for an $\hat{f}_b(y, z, \eta)$ subordinate to $\mathcal{P}_b := \mathrm{r}_{K_b}(\mathcal{P}|_U)$.

Definition 2.3 expresses asymptotics of type $\mathcal{P}$ in terms of pairs U_i, K_i as in Definition 2.1, i.e., localizations in $y \in \Omega$ and $z \in \mathbb{C}$. Therefore, for simplicity we focus on an open set $U \in \mathcal{U}(\Omega)$ and a compact K in the complex plane, $K \subset \{c - \varepsilon < \operatorname{Re} z < (n+1)/2 - \gamma\}$ for some $\varepsilon > 0$, such that $\pi_{\mathbb{C}} \mathcal{P} \subset K$. This allows us to drop subscript b, i.e., we may write $K = K_b$, $\delta = \delta_b$,

$$\hat{\delta}(y, \eta) \in C^\infty\big(U, \mathcal{A}'(K, E^s)\big)^\bullet. \tag{2.9}$$

It is instructive to compare the notion of y-wise discrete asymptotics with continuous asymptotics where $\hat{\delta}(y, \eta) \in C^\infty\big(U, \mathcal{A}'(K, E^s)\big)$.

Formally, the singular functions of continuous asymptotics are as before, namely, of the form

$$F^{-1}_{\eta \to y}\{[\eta]^{(n+1)/2} \omega(r[\eta]) \langle \hat{\delta}(y, \eta)_z, (r[\eta])^{-z} \rangle\}.$$

In contrast to the latter explicit y-dependence of the analytic functionals there is also the case of constant continuous asymptotics carried by the compact set K. In this case we can proceed in an analogous manner as in the constant discrete case, outlined in Subsection 1.2. When we fix the position of K as above, i.e., $K \subset \{(n+1)/2 - \gamma + \vartheta < \operatorname{Re} z < (n+1)/2 - \gamma\}$, then we have

$$\omega(r)\langle \zeta_z, r^{-z} \rangle \subset \mathcal{K}^{\infty,\gamma}(X^\wedge)$$

for every $\zeta \in \mathcal{A}'\big(K, C^\infty(X)\big)$, and

$$\mathcal{E}_K := \{\omega(r)\langle \zeta_z, r^{-z} \rangle : \zeta \in \mathcal{A}'\big(K, C^\infty(X)\big)\} \tag{2.10}$$

is a continuous analogue of $\mathcal{E}_{\mathcal{P}}$ in (1.9). Again we have $\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) \bigcap \mathcal{E}_K = \{0\}$ for any $s \in \mathbb{R}$, and analogously as (1.10) we set

$$\mathcal{K}^{s,\gamma}_{\mathcal{C}}(X^\wedge) := \mathcal{K}^{s,\gamma}_\Theta(X^\wedge) + \mathcal{E}_K. \tag{2.11}$$

The notation $\mathcal{C}$ means that with K we associate a corresponding continuous asymptotic type. The space $\mathcal{E}_K$ is nuclear Fréchet in a natural way via an isomorphism

$$\mathcal{E}_K \cong \mathcal{A}'\big(K, C^\infty(X)\big). \tag{2.12}$$

Thus (2.11) is Fréchet in the topology of the direct sum. The group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ defined by (1.19) is also defined on $\mathcal{K}^{s,\gamma}_{\mathcal{C}}(X^\wedge)$ which allows us to define

$$\mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{C}}(X^\wedge)\big) := \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)\big) + \mathbb{K}H^s(\mathbb{R}^q, \mathcal{E}_K).$$

From (2.12) it follows that

$$H^s(\mathbb{R}^q_y, \mathcal{E}_K) = \{\omega(r)\langle \zeta(y)_z, r^{-z} \rangle : \zeta \in \mathcal{A}'\big(K, H^s\big(\mathbb{R}^q_y, C^\infty(X)\big)\big)\}. \tag{2.13}$$

Then

$$\begin{gathered}\mathbb{K}H^s(\mathbb{R}^q_y, \mathcal{E}_K) = \{F^{-1}_{\eta \to y} \kappa_{[\eta]}[\omega(r) F_{y' \to \eta} \langle \zeta(y')_z, r^{-z} \rangle] : \\ \zeta(y') \in \mathcal{A}'\big(K, H^s\big(\mathbb{R}^q_{y'}, C^\infty(X)\big)\big)\}.\end{gathered} \tag{2.14}$$

Let us now make some general remarks about managing analytic functionals. If E is a Fréchet space and $\mathcal{A}'(K, E)$ the space of E-valued analytic functionals carried by the compact set $K \subset \mathbb{C}$ we have

$$\mathcal{A}'(K, E) = \mathcal{A}'(K^{\mathrm{c}}, E) \tag{2.15}$$

where K^{c} means the complement of the unbounded connected component of $\mathbb{C} \setminus K$, cf. [8, Section 2.3]. Recall that the classical Cousin theorem also admits decompositions of the carrier, more precisely, if K_1, K_2 are compact sets in $\mathbb{C}$, then setting $K_1 + K_2 := (K_1 \cup K_2)^{\mathrm{c}}$ we have a non-direct sum of Fréchet spaces

$$\mathcal{A}'(K, E) = \mathcal{A}'(K_1, E) + \mathcal{A}'(K_2, E), \tag{2.16}$$

for any Fréchet space E, cf. also [11].

In the discussion so far we assumed that $K \bigcap \Gamma_{(n+1)/2-\gamma} = \emptyset$. However, in the edge calculus with continuous asymptotics also requires the case $K \bigcap \Gamma_{(n+1)/2-\gamma} \neq \emptyset$. Without loss of generality we may assume $K = K^{\mathrm{c}}$. Then (2.11) is not direct and only $\{z \in K : \operatorname{Re} z > (n+1)/2 - \gamma + \vartheta\}$ contributes to $\mathcal{C}$. Writing K as a sum $K = K_1 + K_2$ for $K_1 = \{z \in K : \operatorname{Re} z \leq (n+1)/2 - \gamma + \vartheta\}$, $K_2 = \{z \in K : \operatorname{Re} z \geq (n+1)/2 - \gamma + \vartheta\}$ we have a decomposition (2.16). Therefore, every $\zeta \in \mathcal{A}'(K, E)$

may be written as $\zeta = \zeta_1 + \zeta_2$ for suitable $\zeta_i \in \mathcal{A}'(K_i, E), i = 1, 2$. This leads to a decomposition of the space (2.14) as

$$\mathbb{K}H^s(\mathbb{R}^q, \mathcal{E}_K) = \mathbb{K}H^s(\mathbb{R}^q, \mathcal{E}_{K_1}) + \mathbb{K}H^s(\mathbb{R}^q, \mathcal{E}_{K_2}).$$

Clearly we have $\mathbb{K}H^s(\mathbb{R}^q, \mathcal{E}_{K_1}) \subset \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge)\big)$, but also K_2 gives rise to a flat contribution, namely, from $K_0 := K_2 \bigcap \Gamma_{(n+1)/2-\gamma+\vartheta}$. The notions and results that we are formulating here on continuous asymptotics have a natural modification for the case of arbitrary K. If necessary, we have to admit flat contributions.

Proposition 2.4. *For a compact set $K \subset \{(n+1)/2-\gamma+\vartheta < \operatorname{Re} z < (n+1)/2-\gamma\}$ we have*

$$\begin{aligned}\mathbb{K}H^s(\mathbb{R}^q_y, \mathcal{E}_K) = &\big\{\omega(r) F^{-1}_{\eta\to y}\kappa_{[\eta]}[F_{y'\to\eta}\langle\zeta(y')_z, r^{-z}\rangle] :\\ &\qquad \zeta(y') \in \mathcal{A}'\big(K, H^s\big(\mathbb{R}^q_{y'}, C^\infty(X)\big)\big)\big\}\end{aligned}$$

$\operatorname{mod} \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge)\big)$.

Proof. Let us first drop $C^\infty(X^\wedge)$; this can be tensor-multiplied to the result in the final step, cf. the considerations in connection with (1.17). For ζ we then have

$$\zeta \in \mathcal{A}'\big(K, H^s(\mathbb{R}^q_{y'})\big) = \mathcal{A}'(K)\hat{\otimes}_\pi H^s(\mathbb{R}^q_{y'}).$$

We employ the fact that ζ can be written as a convergent sum $\zeta = \sum_{j=0}^\infty \lambda_j\zeta_j v_j$ for $\lambda_j \in \mathbb{C}, \sum_{j=0}^\infty |\lambda_j| < \infty, \zeta_j \in \mathcal{A}'(K), v_j \in H^s(\mathbb{R}^q)$, tending to 0 in the respective spaces, as $j \to \infty$. Then, we form

$$\begin{aligned}k_j(\eta) &: c \to \omega(r[\eta])[\eta]^{(n+1)/2}\langle\zeta_{j,z}, (r[\eta])^{-z}\rangle c,\\ l_j(\eta) &: c \to \omega(r)[\eta]^{(n+1)/2}\langle\zeta_{j,z}, (r[\eta])^{-z}\rangle c,\end{aligned}$$

$c \in \mathbb{C}$ and write

$$d_j(\eta) := l_j(\eta) - k_j(\eta) = [\eta]^{(n+1)/2}\omega(r)\big(1-\omega(r[\eta])\big)\langle\zeta_{j,z}, (r[\eta])^{-z}\rangle.$$

We will show that

$$d_j(\eta) \in S^0_{\mathrm{cl}}\big(\mathbb{R}^q; \mathbb{C}, \mathcal{K}^{\infty,\beta}(\mathbb{R}_+)\big) \tag{2.17}$$

for every $\beta \in \mathbb{R}$ and that $d_j(\eta) \to 0$ in that symbol spaces as $j \to \infty$. This will give us

$$\operatorname{Op}_y(d_j) : H^s(\mathbb{R}^q) \to \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{\infty,\beta}(\mathbb{R}_+)\big).$$

For fixed $v \in H^s(\mathbb{R}^q)$ we can interpret $\operatorname{Op}_y(d_j)v = \operatorname{Op}_y(l_j)v - \operatorname{Op}_y(k_j)v$ as

$$\begin{aligned}&F^{-1}_{\eta\to y}\big[[\eta]^{(n+1)/2}\omega(r)\langle\zeta_{j,z}\hat{v}(\eta), (r[\eta])^{-z}\rangle\big]\\ &\quad - F^{-1}_{\eta\to y}\big[[\eta]^{(n+1)/2}\omega(r[\eta])\langle\zeta_{j,z}\hat{v}(\eta), (r[\eta])^{-z}\rangle\big],\end{aligned}$$

i.e., the difference between the respective singular functions for $\omega(r)$ and $\omega(r[\eta])$, respectively.

Let us now turn to (2.17) and set for the moment

$$d(\eta) = [\eta]^{(n+1)/2}\omega(r)\big(1-\omega(r[\eta])\big)\langle\zeta_z, (r[\eta])^{-z}\rangle,$$

i.e., we first drop subscript j. In order to show that $d(\eta) \in S^0_{\mathrm{cl}}\big(\mathbb{R}^q; \mathbb{C}, \mathcal{K}^{\infty,\beta}(\mathbb{R}_+)\big)$ we check the symbolic estimates

$$\|\kappa^{-1}_{[\eta]} D^\delta_\eta d(\eta)\|_{\mathcal{L}(\mathbb{C},\mathcal{K}^{s,\beta}(\mathbb{R}_+))} = \|\kappa^{-1}_{[\eta]} D^\delta_\eta d(\eta)\|_{\mathcal{K}^{s,\beta}(\mathbb{R}_+)} \le c[\eta]^{-|\delta|}, \tag{2.18}$$

$\delta \in \mathbb{N}^q$, cf. the relation (1.16). It suffices to do that for every $s \in \mathbb{N}$, and we first consider the case $s = 0$ and $\beta = 0$. Let $k^\beta(r) \in C^\infty_0(\mathbb{R}_+)$ be any function that is strictly positive and $k^\beta(r) = r^\beta$ for $0 < r < c_0$, $k^\beta(r) = 1$ for $r > c_1$, for some $0 < c_0 < c_1$. Then $\mathcal{K}^{s,\beta}(X^\wedge) = k^\beta(r)\mathcal{K}^{s,0}(X^\wedge)$. In particular, by virtue of $\mathcal{K}^{0,0}(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ we have $\mathcal{K}^{0,\beta}(\mathbb{R}_+) = k^\beta(r)L^2(\mathbb{R}_+)$ and

$$\|f\|_{\mathcal{K}^{0,\beta}(\mathbb{R}_+)} = \|k^{-\beta} f\|_{L^2(\mathbb{R}_+)}.$$

In connection with (2.18) we have to consider

$$\|\kappa^{-1}_{[\eta]} d(\eta)\|_{\mathcal{K}^{0,\beta}(\mathbb{R}_+)} = \|k^{-\beta}(r)\omega(r[\eta]^{-1})\big(1-\omega(r)\big)\langle\zeta_z, r^{-z}\rangle\|_{L^2(\mathbb{R}_+)}.$$

From the carrier of ζ we know that $\omega(r[\eta]^{-1})\langle\zeta_z, r^{-z}\rangle \in \mathcal{K}^{\infty,\gamma-n/2}(\mathbb{R}_+)$ for all ζ; together with the factor $k^{-\beta}(r)\big(1-\omega(r)\big)$ we get $k^{-\beta}(r)\big(1-\omega(r)\big)\omega(r[\eta]^{-1})\langle\zeta_z, r^{-z}\rangle \in L^2(\mathbb{R}_+)$. It follows that $\|\kappa^{-1}_{[\eta]} d(\eta)\|_{\mathcal{K}^{0,\beta}(\mathbb{R}_+)} \le c$ for all $\eta \in \mathbb{R}^q$. For the η-derivatives we obtain (2.18) in general. Let us check, for instance, the case $\delta = (1, 0, \dots, 0)$, i.e., $D^\delta_\eta = -i\partial_{\eta_1}$. In this case we have

$$\partial_{\eta_1} d(\eta) = (\partial_{\eta_1}[\eta]^{(n+1)/2})\omega(r)\big(1-\omega(r[\eta])\big)\langle\zeta_z, (r[\eta])^{-z}\rangle - r[\eta]^{(n+1)/2}(\partial_{\eta_1}[\eta])$$
$$\omega(r)\omega'(r[\eta])\langle\zeta_z, (r[\eta])^{-z}\rangle - [\eta]^{(n+1)/2}\omega(r)\big(1-\omega(r[\eta])\big)\langle\zeta_z, z[\eta]^{-1}(\partial_{\eta_1}[\eta])(r[\eta])^{-z}\rangle.$$

This gives us the desired estimate with $[\eta]^{-1}$ on the right. The general case may easily be treated in a similar manner. Now an elementary consideration shows that the constants $c = c(\zeta)$ in the symbolic estimates (2.18) tend to 0 as $\zeta \to 0$ in $\mathcal{A}'(K)$. Moreover, we can easily treat the case $\mathcal{K}^{s,\beta}(\mathbb{R}_+)$ rather than $\mathcal{K}^{0,\beta}(\mathbb{R}_+)$, $s \in \mathbb{N}$. This implies the asserted estimates for all $s \in \mathbb{R}$. In other words, as claimed above, $d_j(\eta) = l_j(\eta) - k_j(\eta)$ tends to 0 in $S^0_{\mathrm{cl}}\big(\mathbb{R}^q; \mathbb{C}, \mathcal{K}^{\infty,\beta}(\mathbb{R}_+)\big)$ as $j \to \infty$.

Now we characterize the difference between the singular terms defined with $\omega(r)$ and $\omega(r[\eta])$, respectively. It is equal to

$$\begin{aligned}
&F^{-1}_{\eta\to y}\omega(r)\big[\kappa_{[\eta]}F_{y'\to\eta}\langle\zeta(y')_z, r^{-z}\rangle\big] - F^{-1}_{\eta\to y}\omega(r[\eta])\big[\kappa_{[\eta]}F_{y'\to\eta}\langle\zeta(y')_z, r^{-z}\rangle\big]\\
=&F^{-1}_{\eta\to y}\omega(r)\big(1-\omega(r[\eta]\big)\big[\kappa_{[\eta]}F_{y'\to\eta}\langle\zeta(y')_z, r^{-z}\rangle\big]\\
=&F^{-1}_{\eta\to y}\omega(r)\big(1-\omega(r[\eta])\big)\kappa_{[\eta]}F_{y'\to\eta}\Big\langle\sum_{j=0}^\infty \lambda_j\zeta_{j,z}v_j(y'), r^{-z}\Big\rangle\\
=&\sum_{j=0}^\infty \lambda_j F^{-1}_{\eta\to y}\omega(r)\big(1-\omega(r[\eta])\big)\langle\zeta_{j,z}, (r[\eta])^{-z}\rangle\hat{v}_j(\eta) = \sum_{j=0}^\infty \lambda_j \mathrm{Op}_y(d_j)v_j.
\end{aligned}$$

This sum converges in $\mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{\infty,\beta}(\mathbb{R}_+)\big)$.

In fact, for every $t \geq 0$ we have

$$\begin{aligned}\Big\|\sum_{j=0}^{\infty}\lambda_j \mathrm{Op}_y(d_j)v_j\Big\|_{\mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{t,\beta}(\mathbb{R}_+))} &\leq \sum_{j=0}^{\infty}|\lambda_j|\,\|\mathrm{Op}_y(d_j)v_j\|_{\mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{t,\beta}(\mathbb{R}_+))} \\ &\leq \sum_{j=0}^{\infty}|\lambda_j|\,\|\mathrm{Op}_y(d_j)\|_{\mathcal{L}(H^s(\mathbb{R}^q),\mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{t,\beta}(\mathbb{R}_+)))}\|v_j\|_{H^s(\mathbb{R}^q)}.\end{aligned} \tag{2.19}$$

By virtue of (1.15) we have

$$\|\mathrm{Op}_y(d_j)\|_{\mathcal{L}(H^s(\mathbb{R}^q),\mathcal{W}^s(\mathbb{R}^q,\mathcal{K}^{t,\beta}(\mathbb{R}_+)))} \to 0$$

as $j \to \infty$. Then $v_j \to 0$ in $H^s(\mathbb{R}^q)$ as $j \to \infty$ shows the convergence of the right-hand side of (2.19) for every $t \geq 0$, and hence it follows that

$$\sum_{j=0}^{\infty}\lambda_j \mathrm{Op}_y(d_j)v_j \in \mathcal{W}^s\big(\mathbb{R}^q,\mathcal{K}^{\infty,\beta}(\mathbb{R}_+)\big).$$

So far we considered the case without $C^\infty(X)$. However, as illustrated at the beginning, a tensor product argument gives us the result in general. □

Let us finally discuss to what extent the singular functions of variable branching or continuous edge asymptotics depend on the specific choice of the function $\eta \to [\eta]$. The other "non-classical" ingredient, namely, the cut-off function ω has been considered before. After Proposition 2.4 it is clear that changing ω only causes a flat remainder. If we replace $[\eta]$ by an $[\eta]_1$ of analogous properties we obtain smoothing remainders with asymptotics. More precisely we have the following behaviour.

Remark 2.5. For any $\zeta \in \mathcal{A}'\big(K, H^s\big(\mathbb{R}^q_{y'}, C^\infty(X)\big)\big)$, $K \subset \{\mathrm{Re}\, z < (n+1)/2 - \gamma\}$, the difference

$$\omega(r)F^{-1}_{\eta\to y}\kappa_{[\eta]}\langle\hat{\zeta}_z, r^{-z}\rangle - \omega(r)F^{-1}_{\eta\to y}\kappa_{[\eta]_1}\langle\hat{\zeta}_z, r^{-z}\rangle \tag{2.20}$$

belongs to $\in \mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}^{\infty,\gamma}_{\mathcal{C}}(X^\wedge)\big)$, cf. the notation (2.11).

In fact, (2.20) has compact support in $\eta \in \mathbb{R}^q$. We have

$$\begin{aligned}&[\eta]^{(n+1)/2}\Big\langle\hat{\zeta}_z, (r[\eta])^{-z}\Big\rangle - [\eta]_1^{(n+1)/2}\Big\langle\hat{\zeta}_z, (r[\eta]_1)^{-z}\Big\rangle \\ &= [\eta]^{(n+1)/2}\frac{[\eta]^{(n+1)/2} - [\eta]_1^{(n+1)/2}}{[\eta]^{(n+1)/2}} \\ &\quad\times\Big\langle\hat{\zeta}_z, (r[\eta])^{-z}\Big\rangle + [\eta]^{(n+1)/2}\left(\frac{[\eta]_1}{[\eta]}\right)^{(n+1)/2}\left\langle\hat{\zeta}_z, (r[\eta])^{-z}\frac{[\eta]^{-z} - [\eta]_1{}^{-z}}{[\eta]^{-z}}\right\rangle.\end{aligned} \tag{2.21}$$

For the first summand we employ that

$$\frac{[\eta]^{(n+1)/2} - [\eta]_1^{(n+1)/2}}{[\eta]^{(n+1)/2}}\hat{\zeta} =: \hat{\nu} \in \mathcal{A}'\Big(K, \hat{H}^\infty\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\Big)$$

since $[\eta] = [\eta]_1$, for large $|\eta|$. Moreover, we have

$$\left(\frac{[\eta]_1}{[\eta]}\right)^{(n+1)/2} \frac{[\eta]^{-z} - [\eta]_1{}^{-z}}{[\eta]^{-z}} \hat{\zeta} =: \hat{\sigma} \in \mathcal{A}'\left(K, \hat{H}^\infty\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\right).$$

Thus (2.21) is equal to $[\eta]^{(n+1)/2}\langle(\hat{\nu}+\hat{\sigma})_z, (r[\eta])^{-z}\rangle$ and hence (2.20) is equal to

$$F^{-1}_{\eta\to y}[\eta]^{(n+1)/2}\langle(\hat{\nu}+\hat{\sigma})_z, (r[\eta])^{-z}\rangle$$

which belongs to $\mathcal{W}^\infty\big(\mathbb{R}^q, \mathcal{K}^{\infty,\gamma}_{\mathcal{C}}(X^\wedge)\big)$.

2.2. The Sobolev regularity of coefficients in branching edge asymptotics

Our next objective is to consider singular functions of continuous edge asymptotics, described in terms of smooth functions on $y \in \Omega$ with compact support with values in $\mathcal{A}'\big(K, \hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\big)$. We show that those functions may be represented by functionals without dependence on y. A similar result has been formulated in [14, Proposition 3.1.35], but here we give an alternative proof, and we obtain more information. For convenience we start with Schwartz functions in $y \in \mathbb{R}^q$ which covers the case of functions with compact support in $y \in \Omega$. In addition we always write $\omega(r)$ rather than $\omega(r[\eta])$ which is admitted for similar reasons as in Proposition 2.4, modulo flat remainders.

Theorem 2.6. *Let* $\hat{\zeta}(y,\eta) \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{A}'\big(K, \hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\big)\big)$, $K \subset \{(n+1)/2-\gamma+\vartheta < \operatorname{Re} z < (n+1)/2-\gamma\}$ *compact, and form*

$$f(r,y) := F^{-1}_{\eta\to y}\{[\eta]^{(n+1)/2}\omega(r)\langle\hat{\zeta}(y,\eta)_z, (r[\eta])^{-z}\rangle\} \tag{2.22}$$

(*the dependence on* $x \in X$ *is dropped in the notation*). *Then there is a unique* $\hat{\chi} \in \mathcal{A}'\big(K, \hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\big)$ *such that*

$$f(r,y) := F^{-1}_{\eta\to y}\{[\eta]^{(n+1)/2}\omega(r)\langle\hat{\chi}(\eta)_z, (r[\eta])^{-z}\rangle\}, \tag{2.23}$$

and the correspondence $\hat{\zeta} \to \hat{\chi}$ *defines an operator*

$$B : \mathcal{S}\big(\mathbb{R}^q, \mathcal{A}'\big(K, \hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\big)\big) \to \mathcal{A}'\big(K, \hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\big). \tag{2.24}$$

Proof. We employ some background on the pseudo-differential calculus with operator-valued symbols of the kind $S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^q; H, \tilde{H})$ with twisted symbolic estimates (1.11). In our case we set $\Omega = \mathbb{R}^q$ and look at the subspace $\mathcal{S}\big(\mathbb{R}^q_y, S^\mu_{(\mathrm{cl})}(\mathbb{R}^q_\eta; H, \tilde{H})\big)$. Given an $a_{\mathrm{L}}(y,\eta)$ in that space we have (by notation) the situation of a left symbol in the calculus of pseudo-differential operators $\mathrm{Op}_y(a_{\mathrm{L}})$, cf. the expression (1.12) where the respective amplitude function is a double symbol. It will be necessary to generate right symbols $a_{\mathrm{R}}(y',\eta)$ such that

$$\mathrm{Op}_y(a_{\mathrm{L}}) = \mathrm{Op}_y(a_{\mathrm{R}}). \tag{2.25}$$

A modification of the Kumano-go's global (in $\mathbb{R}^q$) pseudo-differential calculus is that $a_{\mathrm{L}} \to a_{\mathrm{R}}$ with (2.25) defines continuous operator

$$\mathcal{S}\big(\mathbb{R}^q_y, S^\mu_{(\mathrm{cl})}(\mathbb{R}^q_\eta; H, \tilde{H})\big) \to \mathcal{S}\big(\mathbb{R}^q_{y'}, S^\mu_{(\mathrm{cl})}(\mathbb{R}^q_\eta; H, \tilde{H})\big).$$

Using an expansion for a_{R} with remainder we have, in particular,

$$a_{\mathrm{R}}(y',\eta) = a_{\mathrm{L}}(y',\eta) + r_{\mathrm{R}}(y',\eta) \tag{2.26}$$

for

$$r_{\mathrm{R}}(y',\eta) = -\sum_{|\alpha|=1}\int_0^1\iint e^{-ix\xi}(D_y^\alpha\partial_\eta^\alpha a)(y'+x,\eta-t\xi)\,dx đ\xi dt. \tag{2.27}$$

Here $\partial_\eta^\alpha = \partial_{\eta_1}^{\alpha_1}\dots\partial_{\eta_q}^{\alpha^q}$ and $D_y^\alpha = (-i)^{|\alpha|}\partial_y^\alpha$ for $\alpha = (\alpha_1,\dots,\alpha_q), |\alpha| = \alpha_1+\dots+\alpha_q$. The map $a_{\mathrm{L}}(y,\eta)\to r_{\mathrm{R}}(y',\eta)$ defines a continuous operator

$$\mathcal{S}\big(\mathbb{R}^q_y, S^\mu_{(\mathrm{cl})}(\mathbb{R}^q_\eta;H,\tilde{H})\big)\to\mathcal{S}\big(\mathbb{R}^q_{y'}, S^{\mu-1}_{(\mathrm{cl})}(\mathbb{R}^q_\eta;H,\tilde{H})\big). \tag{2.28}$$

In our concrete situation similarly as before we first look at the case without $C^\infty(X)$; then a tensor product consideration gives us the result in general. We express $\hat{\zeta}(y,\eta)\in\mathcal{S}\big(\mathbb{R}^q,\mathcal{A}'\big(K,\hat{H}^s(\mathbb{R}^q_\eta)\big)\big)$ as an expansion

$$\hat{\zeta}(y,\eta) = \sum_{j=0}^\infty \lambda_j\zeta_j\varphi_j(y)\hat{v}_j(\eta)$$

for $\lambda_j\in\mathbb{C},\sum_{j=0}^\infty|\lambda_j|<\infty,\varphi_j\in\mathcal{S}(\mathbb{R}^q_y), v_j\in H^s(\mathbb{R}^q_{y'})$, tending to zero in the respective spaces. This allows us to write the function (2.22) in the form

$$f(r,y) = \sum_{j=0}^\infty\lambda_j F^{-1}_{\eta\to y}\{[\eta]^{(n+1)/2}\omega(r)\varphi_j(y)\langle\zeta_{j,z},(r[\eta])^{-z}\rangle\hat{v}_j(\eta)\} = \sum_{j=0}^\infty\lambda_j\mathrm{Op}_y(k_j)v_j$$

where $k_j(y,\eta)\in\mathcal{S}\big(\mathbb{R}^q_y,S^\mu(\mathbb{R}^q_\eta;\mathbb{C},\tilde{H}_l)\big)$ is defined by

$$k_j(y,\eta): c\to[\eta]^{(n+1)/2}\omega(r)\varphi_j(y)\langle\zeta_{j,z},(r[\eta])^{-z}\rangle c,$$

and $\tilde{H}_l, l\in\mathbb{N}$, is a scale of Hilbert spaces with κ-action such that

$$\mathcal{K}^{\infty,\gamma-n/2}_{\mathcal{C}}(\mathbb{R}_+) = \varprojlim_{l\in\mathbb{N}}\tilde{H}_l,$$

cf. equation (2.11). In other words we apply the above general relations on symbols to the case $H:=\mathbb{C}$ with the trivial group action and $\tilde{H}=\tilde{H}_l$ endowed with κ, for every fixed l. Writing for the moment $k_j(y,\eta) = k_{j,\mathrm{L}}(y,\eta)$ we obtain a right symbol $k_{j,\mathrm{R}}(y',\eta)$ which is of the form

$$k_{j,\mathrm{R}}(y',\eta)c = [\eta]^{(n+1)/2}\omega(r)\varphi_j(y')\langle\zeta_{j,z},(r[\eta])^{-z}\rangle c + r_{j,\mathrm{R}}(y',\eta)c,$$

where $r_{j,\mathrm{R}}$ is obtained from (2.27) for $a_{\mathrm{L}} = k_{j,\mathrm{L}}$. Let us consider for the moment the case $q=1$, and then write $y=y_1,\eta=\eta_1$. The general case is completely analogous. Later on in the function and symbol spaces we tacitly return again to $\mathbb{R}^q$ rather than $\mathbb{R}^1$. Then the remainder expression takes the form

$$r_{j,\mathrm{R}}(y',\eta) = -\int_0^1\iint e^{-ix\xi}r^{-(n+1)/2}\omega(r)(D_y\varphi_j)(y'+x)$$
$$\langle\zeta_{j,z},\big(\partial_\eta((r[\eta])^{-z+(n+1)/2})\big)|_{\eta-t\xi}\rangle\,dx đ\xi dt.$$

We now apply an element of Kumano-go's calculus for scalar symbols and observe that

$$d_j(z,y',\eta)=\int_0^1\iint e^{-ix\xi}(D_y\varphi_j)(y'+x)\big(\partial_\eta([\eta]^{-z+(n+1)/2})\big)|_{\eta-t\xi}\,dx\,d\!\!\bar{}\,\xi dt$$

belongs to $\mathcal{S}\big(\mathbb{R}_{y'},S^{-\mathrm{Re}\,z+(n+1)/2-1}(\mathbb{R}_\eta)\big)$ for every fixed z. In addition $d_j(z,y',\eta)$ is an entire function in z. This gives us

$$\begin{aligned} r_{j,\mathrm{R}}(y',\eta)&=-r^{-(n+1)/2}\omega(r)\langle\zeta_{j,z},r^{-z+(n+1)/2}d_j(z,y',\eta)\rangle\\ &=-r^{-(n+1)/2+1}\omega(r)\langle\hat{\delta}_{j,z}(y',\eta),(r[\eta])^{-z-1+(n+1)/2}\rangle \end{aligned}\tag{2.29}$$

for $\hat{\delta}_j(y',\eta):=\zeta_j d_j(z,y',\eta)/[\eta]^{-z-1+(n+1)/2}$. We now employ the fact that the pseudo-differential action with a right symbol $b(y',\eta)$, say, in the scalar case $b(y',\eta)\in\mathcal{S}\big(\mathbb{R}^q_{y'},S^\nu(\mathbb{R}^q_\eta)\big)$ for some ν, operating on $v\in H^s(\mathbb{R}^q)$ has the form

$$\mathrm{Op}_y(b)v=\int e^{iy\eta}\left\{\int e^{-iy'\eta}b(y',\eta)v(y')\,dy'\right\}d\!\!\bar{}\,\eta.$$

In order to analyze the expression we may apply a tensor product expansion

$$b(y',\eta)=\sum_{l=0}^\infty\gamma_l\psi_l(y')b_l(\eta)$$

with $\sum_{l=0}^\infty|\gamma_l|<\infty,\psi_l\in\mathcal{S}(\mathbb{R}^q),b_l\in S^\nu(\mathbb{R}^q)$, tending to zero in the considered spaces when $l\to\infty$. Then

$$\begin{aligned}\mathrm{Op}_y(b)v&=\int e^{iy\eta}\left\{\int e^{-iy'\eta}\sum_{l=0}^\infty\gamma_l\psi_l(y')b_l(\eta)v(y')\,dy'\right\}d\!\!\bar{}\,\eta\\ &=\int e^{iy\eta}\sum_{l=0}^\infty\gamma_l b_l(\eta)\widehat{\psi_l v}(\eta)\,dy'd\!\!\bar{}\,\eta.\end{aligned}$$

We have $\psi_l v\in H^s(\mathbb{R}^q_{y'}),\psi_l v\to 0$ in $H^s(\mathbb{R}^q_{y'})$, and we obtain altogether a sum

$$\mathrm{Op}_y(b)v=\sum_{l=0}^\infty\gamma_l\mathrm{Op}_y(b_l)(\psi_l v),$$

convergent in $H^{s-\nu}(\mathbb{R}^q)$. This consideration may be modified for the present case. Let us write (2.29) as

$$r_{j,\mathrm{R}}(y',\eta)=-[\eta]^{(n+1)/2-1}\omega(r)\langle\hat{\delta}_{j,z}(y',\eta),(r[\eta])^{-z}\rangle.\tag{2.30}$$

We have

$$\hat{\delta}_j(y',\eta)=\sum_{l=0}^\infty\gamma_l\psi_l(y')b_{jl}(\eta),$$

for $\hat{\delta}_j(y',\eta)\in\mathcal{A}'\big(K,\mathcal{S}\big(\mathbb{R}^q_{y'},S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta)\big)\big)$, where $\hat{b}_{jl}(\eta)\in\mathcal{A}'\big(K,S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta)\big)$. We employ the fact that the pairing $S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta)\times\hat{H}^s(\mathbb{R}^q_\eta)\to\hat{H}^s(\mathbb{R}^q_\eta)$ gives rise to a bilinear map

$$\big(\mathrm{id}_{\mathcal{A}'(K)}\otimes S^0_{\mathrm{cl}}(\mathbb{R}^q_\eta)\big)\times\hat{H}^s(\mathbb{R}^q_\eta)\to\mathcal{A}'(K)\hat{\otimes}_\pi\hat{H}^s(\mathbb{R}^q_\eta).$$

It follows that

$$r_{j,\mathrm{R}}(y',\eta) = -[\eta]^{(n+1)/2-1}\omega(r)\left\langle \sum_{l=0}^{\infty}\gamma_l\psi_l(y')b_{jl}(\eta), (r[\eta])^{-z}\right\rangle$$

and

$$\mathrm{Op}_y(r_{j,\mathrm{R}})v_j(y) = F^{-1}_{\eta\to y}\left\{-[\eta]^{(n+1)/2-1}\omega(r)\sum_{l=0}^{\infty}\gamma_l\langle b_{jl,z}(\eta),(r[\eta])^{-z}\rangle\widehat{\psi_l v_j}(\eta)\right\}.$$

For

$$\hat{\chi}_{j,\mathrm{rest}}(\eta) := \sum_{l=0}^{\infty}\gamma_l b_{jl}(\eta)\widehat{\psi_l v_j}(\eta) \in \mathcal{A}'\big(K,\hat{H}^s(\mathbb{R}^q)\big) \tag{2.31}$$

it follows that

$$\mathrm{Op}_y(r_{j,\mathrm{R}})v_j(y) = F^{-1}_{\eta\to y}\{-[\eta]^{(n+1)/2-1}\omega(r)\langle\hat{\chi}_{j,z}(\eta),(r[\eta])^{-z}\rangle\}.$$

Returning to (2.26) from (2.30) we obtain

$$r_{\mathrm{R}}(y',\eta) = -[\eta]^{(n+1)/2-1}\omega(r)\sum_{j=0}^{\infty}\lambda_j\big\langle\hat{\delta}_{j,z}(y',\eta),(r[\eta])^{-z}\big\rangle$$

and

$$F^{-1}_{\eta\to y}\big(F_{y'\to\eta}r_{j,\mathrm{R}})(y',\eta)\big) = -F^{-1}_{\eta\to y}\left\{[\eta]^{(n+1)/2-1}\omega(r)\sum_{j=0}^{\infty}\lambda_j\big\langle\hat{\chi}_{j,z}(\eta),(r[\eta])^{-z}\big\rangle\right\}.$$

By notation we have $k_{\mathrm{L}}(y,\eta) = \sum_{j=0}^{\infty}\lambda_j k_{j,\mathrm{L}}(y,\eta)$ where $k_{j,\mathrm{L}}(y,\eta)\to 0$ in $\mathcal{S}\big(\mathbb{R}^q_y, S^0(\mathbb{R}^q;\mathbb{C},\tilde{H}_l)\big)$ and then $k_{j,\mathrm{R}}(y',\eta)\to 0$ in $\mathcal{S}\big(\mathbb{R}^q_y, S^0(\mathbb{R}^q;\mathbb{C},\tilde{H}_l)\big)$ and $r_{j,\mathrm{R}}(y',\eta)\to 0$ in $\mathcal{S}\big(\mathbb{R}^q_y, S^{-1}(\mathbb{R}^q;\mathbb{C},\tilde{H}_l)\big)$ as $j\to\infty$.

This implies that $k_{\mathrm{R}}(y',\eta) = \sum_{j=0}^{\infty}\lambda_j k_{j,\mathrm{R}}(y',\eta)$. We obtain that $\hat{\chi}_{j,\mathrm{rest}}(\eta)\to 0$ in $\mathcal{A}'\big(K,\hat{H}^s(\mathbb{R}^q)\big)$ as $j\to\infty$, cf. (2.31), hence it follows an element

$$\hat{\chi}_{\mathrm{rest}}(\eta) := \sum_{j=0}^{\infty}\lambda_j\hat{\chi}_{j,\mathrm{rest}}(\eta)\in\mathcal{A}'\big(K,\hat{H}^s(\mathbb{R}^q)\big).$$

In a similar (simpler) manner we can treat the term $a_{\mathrm{L}}(y',\eta)$, cf. (2.26), which gives us a $\hat{\chi}_{\mathrm{main}}\in\mathcal{A}'\big(K,\hat{H}^s(\mathbb{R}^q_\eta)\big)$, and it follows altogether

$$\begin{aligned} f(r,y) = {} & F^{-1}_{\eta\to y}\{[\eta]^{(n+1)/2}\omega(r)\langle\hat{\chi}_{\mathrm{main}}(\eta)_z,(r[\eta])^{-z}\rangle\} \\ & - F^{-1}_{\eta\to y}\{[\eta]^{(n+1)/2-1}\omega(r)\langle\hat{\chi}_{\mathrm{rest}}(\eta)_z,(r[\eta])^{-z}\rangle\}. \end{aligned}$$

Note that $[\eta]^{-1}\hat{\chi}_{\mathrm{rest}}\in\mathcal{A}'\big(K,\hat{H}^{s+1}(\mathbb{R}^q_\eta)\big)\hookrightarrow\mathcal{A}'\big(K,\hat{H}^s(\mathbb{R}^q_\eta)\big)$. Analogous considerations apply for the $C^\infty(X)$-valued case. We thus obtain the claimed representation (2.23) where

$$\hat{\chi}(\eta) := \hat{\chi}_{\mathrm{main}}(\eta) - [\eta]^{-1}\hat{\chi}_{\mathrm{rest}}(\eta)\in\mathcal{A}'\big(K,H^s\big(\mathbb{R}^q,C^\infty(X)\big)\big).$$

Let us now prove the uniqueness of $\hat{\chi}$ in the formula (2.23). Without loss of generality we assume $K = K^{\mathrm{c}}$, cf. the relation (2.15). We have an isomorphism

$$\mathcal{A}'(K,E) \cong \{\omega(r)\langle \chi_z, r^{-z}\rangle : \chi \in \mathcal{A}'(K,E)\}$$

where on the right-hand side we talk about functions in $C^\infty(\mathbb{R}_+, E)$, and ω is a fixed cut-off function. Clearly we know much more about such functions; they belong to $\mathcal{K}^{\infty,\gamma}(\mathbb{R}_+, E)$ where $\gamma \in \mathbb{R}$ is any real such that $K \subset \{\mathrm{Re}\, z < 1/2 - \gamma\}$. The notation $\mathcal{K}^{\infty,\gamma}(\mathbb{R}_+, E)$ is an E-valued generalization of the above-mentioned $\mathcal{K}^{\infty,\gamma}(\mathbb{R}_+)$. Up to a translation in the complex plane we may assume $\gamma = 0$. Then the Mellin transform

$$M_{r\to w}\big(\omega(r)\langle \chi_z, r^{-z}\rangle\big) =: m(w)$$

gives us an element in $L^2(\Gamma_{1/2}, E)$ which is holomorphic in $\mathbb{C} \setminus K^{\mathrm{c}}$, and we can recover χ by forming

$$\chi : h \to \int_C m(w)h(w)\, đw, \quad h \in \mathcal{A}(\mathbb{C})$$

for any C counter clockwise surrounding K.

The multiplication of a $\chi \in \mathcal{A}'(K,E)$ by $g \in \mathcal{A}(\mathbb{C})$, defined by $\langle \chi, h\rangle := \langle \chi, gh\rangle$ gives us again an element in $\mathcal{A}'(K,E)$. Now looking at the expression (2.23) it suffices to recover

$$\hat{\vartheta}(\eta) := [\eta]^{(n+1)/2}\hat{\chi}(\eta) \in \mathcal{A}'\big(K, \hat{H}^{s-(n+1)/2}\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\big)$$

from

$$F_{y\to\eta}(f)(r,\eta) = \omega(r)\langle \hat{\vartheta}(\eta), (r[\eta])^{-z}\rangle = \omega(r)\langle [\eta]^{-z}\hat{\vartheta}(\eta), r^{-z}\rangle$$

the Mellin transform of which belongs to $\mathcal{A}\big(\mathbb{C}\setminus K, \hat{H}^{s-(n+1)/2}\big(\mathbb{R}^q_\eta, C^\infty(X)\big)\big)$ where

$$[\eta]^{-w}\hat{\vartheta}(\eta) : h \to \int_C M_{r\to w}\big(\omega(r)\langle [\eta]^{-z}\hat{\vartheta}(\eta), r^{-z}\rangle\big)h(w)\, đw.$$

Thus we find $[\eta]^{-w}\hat{\vartheta}(\eta)$ and hence $\hat{\vartheta}(\eta)$ itself by composing the result with the entire function $[\eta]^w$. In other words $\hat{\chi}$ in the formula (2.23) is unique. □

Let us now discuss the Sobolev regularity of coefficients in the singular functions of edge asymptotics. In order to illustrate what we mean we first look at constant discrete asymptotics of type $\mathcal{P}$. According to Proposition 2.4 the singular functions are finite linear combinations of expressions

$$\omega(r)F^{-1}_{\eta\to y}\{[\eta]^{(n+1)/2}(r[\eta])^{-p}\log^k(r[\eta])\hat{v}_{p,k}(\eta,x)\},$$

for $\hat{v}_{p,k}(\eta,x) \in \hat{H}^s\big(\mathbb{R}^q, C^\infty(X)\big), p \in \pi_{\mathbb{C}}\mathcal{P}$, and some $k \in \mathbb{N}$, cf. the formulas (1.19), (1.22) and (1.29). The η-dependence lies in

$$[\eta]^{(n+1)/2-p}\log^l[\eta]\hat{v}_{p,k}(\eta,x) =: \hat{w}_{p,k}(\eta,x) \tag{2.32}$$

for some $0 \le l \le k$, i.e.,

$$w_{p,k}(y,x) \in H^{s+\mathrm{Re}\,p-\varepsilon-(n+1)/2}\big(\mathbb{R}^q_y, C^\infty(X)\big), \tag{2.33}$$

for any $\varepsilon > 0$. The case of constant continuous asymptotics can be interpreted in terms of Sobolev regularity as well. Here in the representation as in Proposition 2.4 the analytic functional ζ is independent of y'. The meaning of the singular functions is a superposition of such functions with discrete asymptotics with exponents r^{-z} for $z \in K$, and ζ is just the "density" of the superposition. Then, taking into account what we obtained in the constant discrete case the Sobolev regularity which is determined by the occurring $[\eta]$-powers together with the $\hat{H}^s\big(\mathbb{R}^q_\eta, C^\infty(X)\big)$-valued character of $\hat{\zeta}$ is nothing else than

$$\inf_{z\in K}\big(s + \operatorname{Re} z - \varepsilon - (n+1)/2\big) \tag{2.34}$$

for any $\varepsilon > 0$.

Let us now draw some conclusions of Theorem 2.6 on a way to approximate the singular functions of branching edge asymptotics by singular functions of continuous asymptotics belonging to a decomposition of the considered compact set $K = \bigcup_{i=0}^N K_i$, where the K_i are as in (2.2). The decomposition (2.4) may also be applied to the E^s-valued case, cf. (2.5), i.e., we can write (2.9) in the form

$$\hat{\delta}(y,\eta) = \sum_{i=0}^N \varphi_i(y)\hat{\delta}_i(y,\eta) \tag{2.35}$$

for summands $\hat{\delta}_i(y,\eta) \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{A}'(K_i,E^s)\big)^\bullet$ (the Schwartz function is taken for convenience; it does not affect the results). The space $\mathcal{S}\big(\mathbb{R}^q, \mathcal{A}'(K_i,E^s)\big)^\bullet$ is closed in $\mathcal{S}\big(\mathbb{R}^q, \mathcal{A}'(K_i,E^s)\big)$. Let B_i denote the analogue of the operator B in the Theorem 2.6 now referring to K_i, i.e., $B_i : \mathcal{S}\big(\mathbb{R}^q, \mathcal{A}'(K_i,E^s)\big) \to \mathcal{A}'(K_i,E^s)$. Then, applying B_i to $\hat{\delta}_i(y,\eta) \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{A}'(K_i,E^s)\big)^\bullet$ we obtain an element

$$\hat{\chi}(y,\eta) := \sum_{i=0}^N \varphi_i(y)B_i\hat{\chi}_i(y,\eta) \tag{2.36}$$

which is now a kind of approximation of the branching pointwise discrete functional $\hat{\delta}(y,\eta)$ by $\hat{\chi}(y,\eta)$ which turns the asymptotics to a continuous behaviour over K_i where y varies over U_i. Since by Theorem 2.6 the singular functions associated with $\hat{\delta}(y,\eta)$ and $\hat{\chi}(y,\eta)$ remain the same, we obtain the following Sobolev regularity approximation of the coefficients in the singular functions of branching edge asymptotics.

Corollary 2.7. *Consider the branching discrete functional* $\hat{\delta}(y,\eta)$ *and the associated singular functions*

$$F^{-1}_{\eta\to y}\{[\eta]^{(n+1)/2}\omega(r)\langle\hat{\delta}(y,\eta),(r[\eta])^{-z}\rangle\}.$$

Then according to (2.35) *we may replace* $\hat{\delta}(y,\eta)$ *by the finite sum* (2.36)*, and from* (2.34) *we obtain the Sobolev regularity in the edge variables* $y \in U_i$*, namely,*

$$\inf_{z\in K_i}\big(s + \operatorname{Re} z - \varepsilon - (n+1)/2\big)$$

for any $\varepsilon > 0, i = 0, 1, \ldots, N$. *In other words the Sobolev regularity may be localized over* U_i *for the corresponding* K_i, *and, of course, the diameters both of* U_i *and* K_i *may be chosen as small as we want when we choose* N *sufficiently large.*

In other words, if we apply Theorem 2.6 to a $\hat{\delta}(y, \eta) \in \mathcal{S}\big(\mathbb{R}^q, \mathcal{A}'(K, E^s)\big)^\bullet$ with variable in y and in general branching patterns of y-wise discrete asymptotics, then "intuitively" the Sobolev regularity at a point $y \in \mathbb{R}^q$ has the form (2.33), now for $p = p(y)$. Clearly the Sobolev smoothness in correct form refers to an open set in the y-variables. But Corollary 2.7 tells us how to collapse such open sets to a single point, and then the Sobolev smoothness itself appears variable and branching under varying y.

Note that also the general continuous asymptotics carried by a compact set K can be interpreted in terms of decompositions into "small" parts K_i when we write $K = \sum_{i=0}^{N} K_i$. This allows us to read off the "content" of Sobolev regularity of singular functions as in Proposition 2.4 from the summands coming from K_i, and then we have similar relations as in Corollary 2.7.

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The Heat Kernel and Green Function of the Sub-Laplacian on the Heisenberg Group

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Abstract. We give a construction of the heat kernel and Green function of a hypoelliptic operator on the one-dimensional Heisenberg group $\mathbb{H}$, the sub-Laplacian $\mathcal{L}$. The explicit formulas are developed using Fourier–Wigner transforms, pseudo-differential operators of the Weyl type, i.e., Weyl transforms, and spectral analysis. These formulas are obtained by first finding the formulas for the heat kernels and Green functions of a family of twisted Laplacians L_τ for all non-zero real numbers τ. In the case when τ=1, L_1 is just the usual twisted Laplacian.

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1. Introduction

In this survey paper, we use the identification

$$\mathbb{R}^2 \ni (x, y) \leftrightarrow z = x + iy \in \mathbb{C}.$$

We consider the set $\mathbb{H}$ given by

$$\mathbb{H} = \mathbb{C} \times \mathbb{R}.$$

Then $\mathbb{H}$ becomes a non-commutative Lie group when equipped with the multiplication $\cdot$ given by

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{4}[z, w] \right), \quad (z, t), (w, s) \in \mathbb{H},$$

where $[z, w]$ is the symplectic form of z and w defined by

$$[z, w] = 2 \operatorname{Im}(z\overline{w}).$$

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Lemma 1.1. *The left Haar measure and right Haar measure on $\mathbb{H}$ are equal to the Lebesgue measure $dz\,dt$.*

Proof. Let f be a measurable function on $\mathbb{H}$. Then

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\mathbb{C}} f((w,s)\cdot(z,t))\,dz\,dt\\ &\quad=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(u+x, v+y, s+t+\frac{1}{2}(vx-uy)\right)\,dx\,dy\,dt.\end{aligned}$$

Let $x' = u + x$, $y' = v + y$, and $t' = s + t + \frac{1}{2}(vx - uy)$. Then the above integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y', t')\,dx'\,dy'\,dt'.$$

This shows that the Lebesgue measure is left-invariant. Similarly, we can show that it is also right-invariant. ☐

Since the left and right Haar measures are equal, it follows that $\mathbb{H}$ is unimodular. Having defined the Heisenberg group, we move on to introduce the heat kernel on $\mathbb{H}$. Consider the partial differential equation

$$\frac{\partial u}{\partial \rho}(z,t,\rho) = (\mathcal{L}u)(z,t,\rho)$$

with initial condition

$$u(z,t,0) = f(z,t),$$

where $(z,t)\in\mathbb{H}$, $\rho > 0$, f is a suitable function on $\mathbb{H}$, and $\mathcal{L}$ is the sub-Laplacian on $\mathbb{H}$ to be defined in Section 2. Its solution can be expressed formally as

$$u(z,t,\rho) = \left(e^{-\rho\mathcal{L}}f\right)(z,t), \quad (z,t)\in\mathbb{H},\ \rho>0.$$

The heat kernel K_ρ of $\mathcal{L}$ is the kernel of the integral operator $e^{-\rho\mathcal{L}}$, which satisfies

$$e^{-\rho\mathcal{L}}f = f *_{\mathbb{H}} K_\rho,$$

where the convolution $f *_{\mathbb{H}} K_\rho$ of f and K_ρ on $\mathbb{H}$ is given by

$$(f *_{\mathbb{H}} K_\rho)(z,t) = \int_{-\infty}^{\infty}\int_{\mathbb{C}} f((z,t)\cdot(-w,-s))K_\rho(w,s)\,dw\,ds, \quad (z,t)\in\mathbb{H},$$

for all suitable functions f on $\mathbb{H}$, provided that the integral exists. On the other hand, the Green function $\mathcal{G}$ of $\mathcal{L}$ is the kernel of the integral operator representing $\mathcal{L}^{-1}$, and

$$\mathcal{L}^{-1}f = f *_{\mathbb{H}} \mathcal{G}$$

for all suitable functions f on $\mathbb{H}$.

The formula for the heat kernel can be traced back to the independent works of Gaveau [5] and Hulanicki [13]. More recent derivation of the heat kernel can be found in Klingler [15]. The works of Hulanicki and Klingler are based on probability theory. The explicit expression for the Green function dates back to the work of Folland [3], in which the formula is given in terms of the distance on the Heisenberg group based on the analogy with the Green function for the Euclidean Laplacian, which is also given by a distance at least when the dimension of the Euclidean space is greater than 2. A geometric approach for finding the heat kernel and Green function of the sub-Laplacian on the Heisenberg group can be found in [1]. In [18], the formulas for the heat kernel and Green function for the twisted Laplacian L are derived by means of pseudo-differential operators of the Weyl type, i.e., Weyl transforms and the Fourier–Wigner transforms of Hermite functions, which form an orthonormal basis for $L^2(\mathbb{C})$. These facts can be found in the book [18] by Wong.

In this paper, the aim is to find the formulas for the heat kernel and Green function of the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}$. In order to do so, we first introduce a family of twisted Laplacians L_τ on $\mathbb{R}^2$, for $\tau \in \mathbb{R}$ such that $\tau \neq 0$. Using the more general formulas for the heat kernels and Green functions for these parametrized twisted Laplacians, we are able to compute the heat kernel and Green function for the sub-Laplacian on $\mathbb{H}$.

In Section 2, we first define the sub-Laplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{H}$ and then introduce a family of twisted Laplacians L_τ for $\tau \in \mathbb{R} \setminus \{0\}$ by taking the inverse Fourier transform of the sub-Laplacian with respect to t. The hypoellipticity of the sub-Laplacian and the ellipticity of the twisted Laplacians are briefly discussed. In Section 3, we give a result on the relationship between convolutions on the Heisenberg group and twisted convolutions. This can be used to relate the heat kernel of the sub-Laplacian to the heat kernels of the twisted Laplacians. In Section 4, we define the Fourier–Wigner transforms of Hermite functions. In Section 5, we define the Weyl transforms, give the formula for the product of two Weyl transforms and prove that the Fourier–Wigner transforms of Hermite functions given in Section 4 form an orthonormal basis for $L^2(\mathbb{R}^2)$. Weyl transforms are used again in Section 6 to prove that the twisted convolution of two Fourier–Wigner transforms of Hermite functions is again a Fourier–Wigner transform of Hermite functions. This fact and the spectral analysis of L_τ for $\tau \in \mathbb{R} \setminus \{0\}$ are then used in the same section to construct the heat kernels of the twisted Laplacians L_τ and hence the heat kernel of $\mathcal{L}$. The Green functions of the twisted Laplacians L_τ and the Green function of the sub-Laplacian $\mathcal{L}$ are given in Section 7.

The formulas for the heat kernel and Green function of the sub-Laplacian on the Heisenberg group are well known in the mathematical literature. To date several methods are available and the reconstructions of these formulas have become somewhat of an industry. This is due not only to the beauty and contents of the formulas, but also the need to construct similar formulas for other hypoelliptic operators. To wit, the techniques in this paper have the potential of constructing the heat kernel and Green function of the product of $L_\tau L_{-\tau}$ for a nonzero real number τ, which is a fourth-order operator whose eigenvalues have finite multiplicities. And the heat kernels and Green functions of these operators can enable us to find those for a fourth-order hypoelliptic operator on $\mathbb{H}$, in fact, a fourth-order sub-Laplacian. This is a project in progress, and is the motivation for this survey paper.

The results in this survey paper are valid for the n-dimensional Heisenberg group and we have chosen to work on the one-dimensional Heisenberg group $\mathbb{H}$ just for the simplicity and transparency of the notation. The inner products and the norms in $L^2(\mathbb{R})$, $L^2(\mathbb{R}^2)$, $L^2(\mathbb{C})$ and $L^2(\mathbb{R}^n)$ and so on are all denoted by $(\ ,\ )$ and $\|\ \|$ respectively and the space in which the inner product and norm is taken should be clear from the context.

2. The sub-Laplacian on $\mathbb{H}$ and twisted Laplacians

A vector field V on $\mathbb{H}$ is given by

$$V(x,y,t) = a(x,y,t)\frac{\partial}{\partial x} + b(x,y,t)\frac{\partial}{\partial y} + c(x,y,t)\frac{\partial}{\partial t},$$

where a, b, c are C^∞ functions in x, y, t.

A vector field V is left-invariant if it commutes with left translations, i.e.,

$$VL_{(w,s)} = L_{(w,s)}V,$$

for all $(w,s) \in \mathbb{H}$, where for all C^∞ functions f on $\mathbb{H}$,

$$(L_{(w,s)}f)(z,t) = f((w,s)\cdot(z,t)), \quad (z,t) \in \mathbb{H}.$$

Now, let $\gamma_1, \gamma_2, \gamma_3 : \mathbb{R} \to \mathbb{H}$ be curves in $\mathbb{H}$ given by

$$\begin{aligned}
\gamma_1(r) &= (r,0,0), \quad r \in \mathbb{R},\\
\gamma_2(r) &= (0,r,0), \quad r \in \mathbb{R},\\
\gamma_3(r) &= (0,0,r), \quad r \in \mathbb{R},
\end{aligned}$$

for all r in $\mathbb{R}$, and we define the vector fields X, Y, and T as follows. Let $f \in C^\infty(\mathbb{H})$. Then the function Xf is defined by

$$\begin{aligned}
(Xf)(x,y,t) &= \frac{\partial}{\partial r} f((x,y,t)\cdot\gamma_1(r))\Big|_{r=0}\\
&= \frac{\partial}{\partial r} f\left(x+r, y, t+\frac{1}{2}ry\right)\Big|_{r=0}\\
&= \frac{\partial f}{\partial x}(x,y,t) + \frac{1}{2}y\frac{\partial f}{\partial t}(x,y,t),
\end{aligned}$$

the function Yf is defined by

$$\begin{aligned}
(Yf)(x,y,t) &= \frac{\partial}{\partial r} f((x,y,t)\cdot\gamma_2(r))\Big|_{r=0}\\
&= \frac{\partial}{\partial r} f\left(x, y+r, t-\frac{1}{2}rx\right)\Big|_{r=0}\\
&= \frac{\partial f}{\partial y}(x,y,t) - \frac{1}{2}x\frac{\partial f}{\partial t}(x,y,t),
\end{aligned}$$

and the function Tf is defined by

$$\begin{aligned}(Tf)(x,y,t) &= \frac{\partial}{\partial r} f((x,y,t)\cdot\gamma_3(r))\Big|_{r=0}\\ &= \frac{\partial}{\partial r} f\left(x,y,t+r\right)\Big|_{r=0}\\ &= \frac{\partial f}{\partial t}(x,y,t)\end{aligned}$$

for all $(x,y,t)\in\mathbb{H}$. To summarize, we have

$$X=\frac{\partial}{\partial x}+\frac{1}{2}y\frac{\partial}{\partial t},\quad Y=\frac{\partial}{\partial y}-\frac{1}{2}x\frac{\partial}{\partial t}\quad\text{and}\quad T=\frac{\partial}{\partial t}.$$

It can be checked easily that

$$[X,Y]=-T,$$

and all other first-order commutators of X, Y and T are zero. Moreover, it can be checked easily that X, Y and T are left-invariant vector fields on $\mathbb{H}$. Indeed, direct calculations give for all (z,t) and (w,s) in $\mathbb{H}$,

$$\begin{aligned}(X(L_{(w,s)}f))(z,t) &= (Xf)((w,s)\cdot(z,t))\\ &= \left(\partial_1 f+\frac{1}{2}y\partial_3 f\right)\left(x+u,\, y+v,\, s+t+\frac{1}{2}(yu-xv)\right)\\ &= \frac{\partial f}{\partial x}\left(x+u,\, y+v,\, s+t+\frac{1}{2}(yu-xv)\right)\\ &\quad+\frac{1}{2}(v+y)\frac{\partial f}{\partial t}\left(x+u,\, y+v,\, s+t+\frac{1}{2}(yu-xv)\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}(L_{(w,s)}(Xf))(z,t) &= (Xf)((w,s)\cdot(z,t))\\ &= \frac{\partial f}{\partial x}\left(x+u,\, y+v,\, s+t+\frac{1}{2}(yu-xv)\right)\\ &\quad+\frac{1}{2}(v+y)\frac{\partial f}{\partial t}\left(x+u,\, y+v,\, s+t+\frac{1}{2}(yu-xv)\right).\end{aligned}$$

Thus, $X\left(L_{(w,s)}f\right)=L_{(w,s)}\left(Xf\right)$ for all $(w,s)\in\mathbb{H}$. Similarly, we have

$$YL_{(w,s)}=L_{(w,s)}Y\quad\text{and}\quad TL_{(w,s)}=L_{(w,s)}T$$

for all (w,s) in $\mathbb{H}$.

In addition, we can show that X, Y and T are linearly independent. Indeed, we set $(aX+bY+cT)f=0$ for all C^∞ function f on $\mathbb{R}^3$, where a, b and c are real numbers. We need to show that

$$a=b=c=0.$$

Let f be the function on $\mathbb{R}^3$ defined by

$$f(x,y,z)=x,\quad (x,y,z)\in\mathbb{R}^3.$$

Then

$$a\frac{\partial}{\partial x}f(x,y,t)+b\frac{\partial}{\partial y}f(x,y,t)+c\frac{\partial}{\partial t}f(x,y,t)=0\Rightarrow a=0.$$

Similarly, by setting $f(x,y,t)=y$ and $f(x,y,t)=t$, we get $b=0$ and $c=0$, as desired.

Now, let $\mathfrak{h}$ be the Lie algebra of all left-invariant vector fields on $\mathbb{H}$. Then X, Y and T form a basis for $\mathfrak{h}$. In fact, $\mathfrak{h}$ is the tangent space of $\mathbb{H}$ at the origin, which is a three-dimensional vector space.

Let $\mathcal{L}=-(X^2+Y^2)$. Then we call $\mathcal{L}$ the sub-Laplacian on the Heisenberg group $\mathbb{H}$ and it can be expressed as

$$\mathcal{L}=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{1}{4}\left(x^2+y^2\right)\frac{\partial^2}{\partial t^2}+\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}.$$

Now, by replacing $\frac{\partial}{\partial t}$ by $-i\tau$, where $\tau\in\mathbb{R}$ such that $\tau\neq 0$, we obtain a family of twisted Laplacians L_τ with respect to τ on $\mathbb{R}^2$, which can be written as

$$L_\tau=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{1}{4}(x^2+y^2)\tau^2-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\tau.$$

In the case when $\tau=1$, we get back our ordinary twisted Laplacian L_1. In fact, the twisted Laplacian L_τ is a perturbation of the Hermite operator by $-iN\tau$, where N is the rotation given by

$$N=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}.$$

Twisted Laplacians and their variants have been studied extensively in [2, 4, 6, 7, 8, 9, 14, 16].

The connections between $\mathcal{L}$ and L_τ can be explained by the following theorem. Before introducing the theorem, we give the definition of f^τ. Let $f\in\mathcal{S}(\mathbb{H})$. Then we define f^τ by

$$f^\tau(z)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{it\tau}f(z,t)\,dt,\quad z\in\mathbb{C},$$

which is the inverse Fourier transform of f with respect to t, at each z, evaluated at τ.

Theorem 2.1. *Let $g\in\mathcal{S}(\mathbb{H})$. Then for any $\tau\in\mathbb{R}$ such that $\tau\neq 0$,*

$$\left(\mathcal{L}g\right)^\tau(z)=\left(L_\tau g^\tau\right)(z),\quad z\in\mathbb{C}.$$

Proof. We only need to look at $(\frac{\partial g}{\partial t})^\tau$, and compute

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{it\tau}\frac{\partial g}{\partial t}(z,t)\,dt$$

for all z in $\mathbb{C}$. Integrating by parts, we get for all z in $\mathbb{C}$,

$$\begin{aligned}(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{it\tau}\frac{\partial g}{dt}(z,t)&=(2\pi)^{-1/2}\left(-\int_{-\infty}^{\infty}i\tau e^{it\tau}g(z,t)dt\right)\\&=-i\tau g^\tau(z).\end{aligned}$$

□

The theorem above justifies our replacement of $\frac{\partial}{\partial t}$ in the sub-Laplacian by $-i\tau$ to get the twisted Laplacian at τ. Now, we give some properties of the operators $\mathcal{L}$ and L_τ.

Theorem 2.2. *Let $\tau \in \mathbb{R}$ be such that $\tau \neq 0$. Then L_τ is elliptic on $\mathbb{R}^2$.*

Theorem 2.3. *$\mathcal{L}$ is nowhere elliptic on $\mathbb{R}^3$.*

Before we give the proofs of the above two theorems, we give a discussion on the ellipticity of operators. Let

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

where $a_\alpha \in C^\infty(\mathbb{R}^n)$, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ with $D_j = -i\frac{\partial}{\partial x_j}$, and the symbol of $P(x, D)$ is given by

$$P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha, \quad x, \xi \in \mathbb{R}^n.$$

Definition 2.4. Let $P_m(x, \xi)$ be the principal symbol of an operator of order m, i.e.,

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha, \quad x, \xi \in \mathbb{R}^n.$$

Let $x_0 \in \mathbb{R}^n$. Then $P(x, D)$ is elliptic at the point x_0 if

$$P_m(x_0, D) = 0,\, \xi \in \mathbb{R}^n \Rightarrow \xi = 0.$$

If $P(x, D)$ is elliptic at every point in $\mathbb{R}^n$, it is said to be elliptic everywhere on $\mathbb{R}^n$.

An operator $P(x, D)$ is hypoelliptic on $\mathbb{R}^n$ if

$$u \in \mathcal{D}'(\mathbb{R}^n),\, P(x, D)u \in C^\infty(\mathbb{R}^n) \Rightarrow u \in C^\infty(\mathbb{R}^n),$$

where $\mathcal{D}'(\mathbb{R}^n)$ represents the space of distributions of Laurent Schwartz.

We now give the proofs of the above two theorems.

Proof of Theorem 2.2. Since

$$L_\tau = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{1}{4}\left(x^2 + y^2\right)\tau^2 - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\tau,$$

the principal part of L_τ is $-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, whose symbol is $|\xi|^2 \in \mathbb{R}^2$, which vanishes only at $\xi = 0$, for all $x, y \in \mathbb{R}$. Thus, by the previous definition, L_τ is elliptic on $\mathbb{R}^2$ for all τ such that $\tau \neq 0$. □

Proof of Theorem 2.3. Replacing $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial t}$ by $i\xi$, $i\sigma$ and $i\lambda$ respectively, the symbol of $\mathcal{L}$ is given by

$$P_2(x,y,t,\xi,\sigma,\lambda) = -\left(i\xi - \frac{1}{2}yi\lambda\right)^2 - \left(i\sigma + \frac{1}{2}xi\lambda\right)^2$$
$$= \left(\xi - \frac{1}{2}y\lambda\right)^2 + \left(\sigma + \frac{1}{2}x\lambda\right)^2.$$

Observe that

$$P_2(x,y,t,\xi,\sigma,\lambda) = 0 \quad \Leftrightarrow \quad \xi = \frac{1}{2}y\lambda \quad \text{and} \quad \sigma = -\frac{1}{2}x\lambda.$$

Case 1: $x = y = 0$. Then

$$P_2(x,y,t,\xi,\sigma,\lambda) = 0 \Rightarrow \xi = \sigma = 0.$$

However, λ can take any real value.

Case 2: x or y is different from 0. Then P_2 vanishes along a line in the $\xi\lambda$-plane or $\sigma\lambda$-plane. Hence by our definition, $\mathcal{L}$ is not elliptic anywhere in $\mathbb{R}^3$. □

To end this section, we apply a simplified version of Hörmander's theorem to show the hypoellipticity of the sub-Laplacian on $\mathbb{H}$.

Theorem 2.5 (Hörmander). *Suppose X_j, for $j = 1, 2, \dots, N$, are vector fields and their commutators up to a certain order span its Lie algebra. Then $\sum_{j=1}^{N} X_j^2$ is hypoelliptic.*

We have shown that X, Y and $[X, Y]$ span the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$. Thus, by Hörmander's theorem [11], $\mathcal{L}$ is hypoelliptic on $\mathbb{R}^3$.

3. Convolutions on the Heisenberg group and twisted convolutions

Let f and g be measurable functions on $\mathbb{H}$. Then we define the convolution $f *_{\mathbb{H}} g$ of f and g on $\mathbb{H}$ by

$$(f *_{\mathbb{H}} g)(z,t) = \int_{-\infty}^{\infty} \int_{\mathbb{C}} f((z,t)\cdot(-w,-s))g(w,s)\,dw\,ds, \quad (z,t) \in \mathbb{H},$$

provided that the integral exists. For a parameter $\lambda \in \mathbb{R}$, the twisted convolution $f *_\lambda g$ of f and g is given by

$$(f *_\lambda g)(z,t) = \int_{\mathbb{C}} f(z-w)g(w)e^{i\lambda[z,w]}\,dw, \quad z \in \mathbb{C}.$$

The theorem below gives the connection between convolutions on $\mathbb{H}$ and twisted convolutions. It can be found in [20].

Theorem 3.1. *Let $f, g \in L^1(\mathbb{H})$. Then*

$$(f *_{\mathbb{H}} g)^\tau = (2\pi)^{1/2} f^\tau *_{\tau/4} g^\tau.$$

Proof. We have

$$\begin{aligned}(2\pi)^{1/2}LHS &= \int_{-\infty}^{\infty} e^{it\tau}(f *_{\mathbb{H}} g)(z,t)\,dt\\ &= \int_{-\infty}^{\infty} e^{it\tau}\left(\int_{-\infty}^{\infty}\int_{\mathbb{C}} f((z,t)\cdot(-w,-s))g(w,s)\,dw\,ds\right)dt\\ &= \int_{-\infty}^{\infty} e^{it\tau}\left(\int_{-\infty}^{\infty}\int_{\mathbb{C}} f\left(z-w, t-s-\frac{1}{4}[z,w]\right)g(w,s)\,dw\,ds\right)dt.\end{aligned}$$

Let $t' = t - \frac{1}{4}[z,w]$. Then

$$LHS = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{it'\tau + i\tau\frac{1}{4}[z,w]}\left(\int_{-\infty}^{\infty}\int_{\mathbb{C}} f(z-w, t'-s)g(w,s)\,dw\,ds\right)dt'.$$

On the other hand,

$$\begin{aligned}RHS &= (2\pi)^{1/2}\int_{\mathbb{C}} f^\tau(z-w)g^\tau(w)e^{i\frac{\tau}{4}[z,w]}\,dw\\ &= (2\pi)^{1/2}(2\pi)^{-1/2}\int_{\mathbb{C}}\left(\int_{-\infty}^{\infty} f(z-w,\cdot-s)g(w,s)ds\right)^{\vee}(\tau)e^{\frac{i\tau}{4}[z,w]}\,dw\\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{it\tau + i\frac{\tau}{4}[z,w]}\int_{-\infty}^{\infty}\int_{\mathbb{C}} f(z-w,t-s)g(w,s)\,dw\,ds\,dt.\end{aligned}$$

Thus, $LHS = RHS$. □

By Theorems 2.1 and 3.1, we can relate the heat kernel of the sub-Laplacian to the heat kernels of the twisted Laplacians as given in the following theorem.

Theorem 3.2. *Let K_ρ be as given in Section 1. Then for suitable functions f on $\mathbb{H}$,*

$$e^{-\rho L_\tau} f^\tau = (e^{-\rho\mathcal{L}} f)^\tau = (f *_{\mathbb{H}} K_\rho)^\tau = (2\pi)^{1/2} f^\tau *_{\tau/4} K_\rho^\tau.$$

4. Fourier–Wigner transforms of Hermite functions

Let f and g be functions in the Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$. Then

$$V_\tau(f,g)(q,p) = |\tau|^{1/2}(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{i\tau qy} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy$$

for all $q, p \in \mathbb{R}$.

For $\tau \in \mathbb{R}$ such that $\tau \neq 0$, and for $k = 0, 1, 2, \ldots$, we define e_k^τ by

$$e_k^\tau(x) = |\tau|^{1/4} e_k(\sqrt{|\tau|}x), \quad x \in \mathbb{R},$$

where e_k is the Hermite function of degree k given by

$$e_k(x) = \frac{1}{(2^k k!\sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x),$$

for all x in $\mathbb{R}$, and H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k (e^{-x^2})$$

for all x in $\mathbb{R}$.

Theorem 4.1. *The set* $\{e_{j,k} : j,\, k = 0, 1, 2, \dots\}$ *is an orthonormal basis for* $L^2(\mathbb{R}^2)$.

The proof of this theorem will be given in the next section where the Weyl transforms are introduced.

For $j, k = 0, 1, 2, \dots$, we define $e^{\tau}_{j,k}$ by

$$e^{\tau}_{j,k} = V_{\tau}(e^{\tau}_j, e^{\tau}_k).$$

We now establish the connection of $\{e^{\tau}_{j,k} : j, k = 0, 1, 2, \dots\}$ with $\{e_{j,k} : j, k = 0, 1, 2, \dots\}$, which is studied in [18].

Theorem 4.2. *For* $\tau \in \mathbb{R}$ *such that* $\tau \neq 0$, *and for* $j, k = 0, 1, 2, \dots,$

$$e^{\tau}_{j,k}(q, p) = |\tau|^{1/2} e_{j,k} \left(\frac{\tau}{\sqrt{|\tau|}} q, \sqrt{|\tau|} p\right), \quad q, p \in \mathbb{R}.$$

Proof. By the Fourier–Wigner transform and a change of variables,

$$\begin{aligned}
e^{\tau}_{j,k}(q, p) &= V_{\tau}\left(e^{\tau}_j, e^{\tau}_k\right)(q, p) \\
&= |\tau|^{1/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{iqy\tau} e^{\tau}_j \left(y + \frac{p}{2}\right) \overline{e^{\tau}_k \left(y - \frac{p}{2}\right)}\, dy \\
&= |\tau| (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{iqy\tau} e_j \left(\sqrt{|\tau|}\left(y + \frac{p}{2}\right)\right) \overline{e_k \left(\sqrt{|\tau|}\left(y - \frac{p}{2}\right)\right)}\, dy \\
&= |\tau|^{1/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{\frac{iqy\tau}{\sqrt{|\tau|}}} e_j \left(y + \frac{p}{2}\sqrt{|\tau|}\right) \overline{e_k \left(y - \frac{p}{2}\sqrt{|\tau|}\right)}\, dy \\
&= |\tau|^{1/2} e_{j,k} \left(q \frac{\tau}{\sqrt{|\tau|}}, \sqrt{|\tau|} p\right)
\end{aligned}$$

for all $\tau \in \mathbb{R}$ with $\tau \neq 0$, and $q, p \in \mathbb{R}$. □

Theorem 4.3. $\{e^{\tau}_{j,k} : j, k = 0, 1, 2, \dots\}$ *forms an orthonormal basis for* $L^2(\mathbb{R}^2)$.

Proof. We first show orthogonality. For all nonnegative integers α, β, μ and ν,

$$(e^{\tau}_{\alpha,\beta}, e^{\tau}_{\mu,\nu}) = |\tau|^{1/2} |\tau|^{1/2} \int_{\mathbb{C}} e_{\alpha,\beta} \left(q \frac{\tau}{\sqrt{|\tau|}}, p\sqrt{|\tau|}\right) \overline{e_{\mu,\nu} \left(q \frac{\tau}{\sqrt{|\tau|}}, p\sqrt{|\tau|}\right)}\, dq\, dp.$$

Let $q' = q\frac{\tau}{\sqrt{|\tau|}}$ and $p' = p\sqrt{|\tau|}$. Then

$$(e^{\tau}_{\alpha,\beta}, e^{\tau}_{\mu,\nu}) = \int_{\mathbb{C}} e_{\alpha,\beta}(q', p') \overline{e_{\mu,\nu}(q', p')}\, dq'\, dp' = (e_{\alpha,\beta}, e_{\mu,\nu}).$$

Since $\{e_{j,k} : j,k = 0,1,2,\dots\}$ is an orthonormal set in $L^2(\mathbb{R}^2)$, so does $\{e^\tau_{j,k} : j,k = 0,1,2,\dots\}$. Secondly, we show that $\{e^\tau_{j,k} : j,k = 0,1,2,\dots\}$ spans the whole space. Let $f \in L^2(\mathbb{C})$ be such that $(f, e^\tau_{j,k}) = 0$ for all $j,k = 0,1,2,\dots$. We need to show $f(q,p) = 0$ for almost all $q,p \in \mathbb{R}$. But

$$\int_{\mathbb{C}} f(q,p)\sqrt{|\tau|}\overline{e_{j,k}\left(q\frac{\tau}{\sqrt{|\tau|}}, p\sqrt{|\tau|}\right)}\,dq\,dp = 0$$

for all $j,k = 0,1,2,\dots$. By letting $q' = q\frac{\tau}{\sqrt{|\tau|}}$, and $p' = p\sqrt{|\tau|}$, the above integral becomes

$$\int_{\mathbb{C}} f(Cq', Dp')\overline{e_{j,k}(q',p')}\,dq'\,dp' = 0$$

for all $j,k = 0,1,2,\dots$, where

$$C = \frac{\sqrt{|\tau|}}{\tau}, D = \frac{1}{\sqrt{|\tau|}}.$$

The preceding equation on the vanishing of the integral holds only when $f(q,p) = 0$ for all most all q, $p \in \mathbb{R}$. So, $\{e^\tau_{j,k} : j,k = 0,1,2,\dots\}$ is indeed an orthonormal basis for $L^2(\mathbb{C})$. □

5. Wigner transforms and Weyl transforms

Now we have another look at the Fourier–Wigner transform. Let q, $p \in \mathbb{R}^n$, and let f be a measurable function on $\mathbb{R}^n$. We define $\rho(q,p)f$ on $\mathbb{R}^n$ by

$$(\rho(q,p)f)(x) = e^{iqx + \frac{1}{2}iqp} f(x+p), \quad x \in \mathbb{R}^n.$$

Then $\rho(q,p) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary operator. Let f and g be in $\mathcal{S}(\mathbb{R}^n)$. Then we define the Fourier–Wigner transform $V(f,g)$ of f and g by

$$V(f,g)(q,p) = (2\pi)^{-n/2}(\rho(q,p)f, g), \quad q,\, p \in \mathbb{R}^n,$$

where $(\ ,\ )$ is the inner product in $L^2(\mathbb{R}^n)$.

An equivalent definition of the Fourier–Wigner transform is given in the following theorem, which can be found in [18].

Theorem 5.1. *Let f, g, q and p be the same as above. Then*

$$V(f,g)(q,p) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{iq\cdot y} f\left(y + \frac{p}{2}\right)\overline{g\left(y - \frac{p}{2}\right)}dy.$$

Proof. We have

$$\begin{aligned} V(f,g)(q,p) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{iq\cdot x + \frac{1}{2}q\cdot p} f(x+p)\overline{g(x)}dx \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{iq\cdot y} f\left(x + \frac{p}{2}\right)\overline{g\left(y - \frac{p}{2}\right)}dy, \end{aligned}$$

where a change of variable $x = y - \frac{p}{2}$ is used. □

Let f and g be in $\mathcal{S}(\mathbb{R}^n)$. Then the Wigner transform $W(f,g)$ of f and g is given by

$$W(f,g) = V(f,g)^\wedge.$$

Lemma 5.2. *For all f and g in $L^2(\mathbb{R}^n)$,*

$$W(f,g)(x,\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dp, \quad x,\xi\in\mathbb{R}^n.$$

Proof. By our definition of the Wigner transform, we have for all x and ξ in $\mathbb{R}^n$,

$$\begin{aligned}
W(f,g)(x,\xi) &= (2\pi)^{-3n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot q - i\xi\cdot y}\left(\int_{\mathbb{R}^n} e^{iq\cdot y} f\left(y+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dy\right)dq\,dp\\
&= (2\pi)^{-3n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-iq\cdot(x-y)}dq\right)e^{i\xi\cdot p} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy\,dp\\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\xi\cdot p}\left(\int_{\mathbb{R}^n}\delta(x-y) f\left(y+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dy\right)dp\\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dp,
\end{aligned}$$

where we have used the fact that

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{iq\cdot(y-x)}\,dq = \delta(y-x) = \delta(x-y),$$

and

$$\int_{\mathbb{R}^n}\delta(x-y)f(y)\,dy = f(x),$$

where δ is the Dirac delta. □

An important theorem involving Wigner transforms is the Moyal identity.

Theorem 5.3 (Moyal Identity). *For all functions f_1, f_2, g_1 and g_2 in $L^2(\mathbb{R}^n)$,*

$$(W(f_1,g_1), W(f_2,g_2)) = (f_1,f_2)\overline{(g_1,g_2)}.$$

Proof. By Plancherel's theorem, we have

$$\begin{aligned}
&(W(f_1,g_1),\, W(f_2,g_2)) = (V(f_1,g_1)^\wedge,\, V(f_2,g_2)^\wedge)\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1\left(x+\frac{p}{2}\right)\overline{g_1\left(x-\frac{p}{2}\right)f_2\left(x+\frac{p}{2}\right)}\,g_2\left(x-\frac{p}{2}\right)dx\,dp\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1(u)\overline{g_1(v)f_2(u)}g_2(v)\,du\,dv\\
&= (f_1,f_2)\overline{(g_1,g_2)},
\end{aligned}$$

where we have used the change of variables $u = x+\frac{p}{2}$ and $v = x-\frac{p}{2}$. □

Definition 5.4. Let $\sigma\in L^2(\mathbb{R}^n\times\mathbb{R}^n)$. Then for all functions f in $L^2(\mathbb{R}^n)$, the Weyl transform $W_\sigma f$ of f with symbol σ is given by

$$(W_\sigma f, g) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sigma(x,\xi)W(f,g)(x,\xi)dx\,d\xi, \quad g\in L^2(\mathbb{R}^n).$$

In fact, by the adjoint formula, we have

$$\begin{aligned}(W_\sigma f, g) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(q,p)V(f,g)^\wedge(q,p)\,dq\,dp\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{\sigma}(q,p)V(f,g)(q,p)\,dq\,dp\\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{\sigma}(q,p)(\rho(q,p)f,g)\,dq\,dp,\end{aligned}$$

and hence

$$W_\sigma f = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{\sigma}(q,p)\rho(q,p)f\,dq\,dp.$$

We also need a simple fact on the Wigner transform.

Theorem 5.5. *Let f and g be in $L^2(\mathbb{R}^n)$. Then*

$$W(g,f) = \overline{W(f,g)}.$$

Now, we introduce a theorem that plays a crucial role in the construction of the heat kernels of the twisted Laplacians in the next section. It is a result of Grossmann, Loupias and Stein in [10].

Theorem 5.6. *Let σ and τ be in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then we have*

$$W_\sigma W_\tau = W_\omega, \quad \textit{where} \quad \hat{\omega} = (2\pi)^{-n}\hat{\sigma} *_{1/4} \hat{\tau}.$$

Proof. Let $z = (q,p) \in \mathbb{C}$. Then by the previous theorem, we have

$$(W_\sigma f, g) = (2\pi)^{-n}\int_{\mathbb{R}^n} \overline{g(x)}\left(\int_{\mathbb{C}^n} \hat{\sigma}(z)(\rho(z)f)(x)\,dz\right)dx.$$

Then for all x in $\mathbb{R}^n$,

$$(W_\sigma(W_\tau f))(x) = (2\pi)^{-n}\int_{\mathbb{C}^n} \hat{\sigma}(z)(\rho(z)W_\tau f)(x)\,dz.$$

Since

$$\begin{aligned}(\rho(z)W_\tau f)(x) &= e^{iq\cdot x + \frac{1}{2}iq\cdot p}(2\pi)^{-n}\int_{\mathbb{C}} \hat{\tau}(w)(\rho(w)f)(x+p)\,dw\\ &= (2\pi)^{-n}\int_{\mathbb{C}^n} \hat{\tau}(w)(\rho(z)\rho(w)f)(x)\,dw\\ &= (2\pi)^{-n}\int_{\mathbb{C}^n} \hat{\tau}(w)(\rho(z+w)e^{\frac{1}{4}i[z,w]}f)(x)\,dw,\end{aligned}$$

where we have used

$$\rho(z)\rho(w) = \rho(z+w)e^{\frac{1}{4}i[z,w]}.$$

We have for all x in $\mathbb{R}^n$,

$$\begin{aligned}(W_\sigma(W_\tau f))(x) &= (2\pi)^{-2n}\int_{\mathbb{C}^n} \hat{\sigma}(z)\left(\int_{\mathbb{C}^n} \hat{\tau}(w)(\rho(z+w)e^{\frac{1}{4}i[z,w]}f)(x)\,dw\right)dz\\ &= (2\pi)^{-2n}\int_{\mathbb{C}^n} \rho(\zeta)f(x)\int_{\mathbb{C}^n} \hat{\sigma}(\zeta - w)\hat{\tau}(w)e^{\frac{1}{4}i[\zeta-w,w]}dw\,dz,\end{aligned}$$

where we have introduced a change of variable $z = \zeta - w$. Since $[w, w] = 0$, we let

$$\hat{\omega}(\zeta) = (2\pi)^{-n} \int_{\mathbb{C}^n} \hat{\sigma}(\zeta - w)\hat{\tau}(w)e^{\frac{1}{4}i[\zeta - w, w]}\, dw, \quad \zeta \in \mathbb{C},$$

to obtain $\hat{\omega} = (2\pi)^{-n}\hat{\sigma} *_{1/4} \hat{\tau}$. □

Now, we give the proof of Theorem 4.1 mentioned in the last section.

Proof of Theorem 4.1. By the Moyal identity for the Fourier–Wigner transform and the Plancherel theorem, we get for all nonnegative integers j_1, j_2, k_1 and k_2,

$$\begin{aligned}(e_{j_1,k_1}, e_{j_2,k_2}) &= (V(e_{j_1}, e_{k_1}), V(e_{j_2}, e_{k_2})) \\ &= (e_{j_1}, e_{j_2})(e_{k_1}, e_{k_2}) = 0\end{aligned}$$

unless $j_1 = j_2$ and $k_1 = k_2$; and if $j_1 = j_2$ and $k_1 = k_2$, we have

$$(e_{j_1,k_1}, e_{j_2,k_2}) = 1.$$

So, the set $\{e_{j,k} : j,\ k = 0, 1, 2, \dots\}$ is an orthonormal set in $L^2(\mathbb{R}^2)$. Secondly, we show that if $f \in L^2(\mathbb{R}^2)$ is such that $(f, e_{j,k}) = 0$, for $j,\ k = 0, 1, 2, \dots$, then $f = 0$ almost everywhere on $\mathbb{R}^2$. We let $g \in L^2(\mathbb{R}^2)$ be such that $\hat{g} = f$. Then, by the previous step, we have

$$\begin{aligned}(W_g e_j, e_k) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,\xi)W(e_j, e_k)(x,\xi)\, dx\, d\xi \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(q,p)V(e_j, e_k)(q,p)\, dq\, dp \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(q,p)e_{j,k}(q,p)\, dq\, dp = 0\end{aligned}$$

for $j,\ k = 0, 1, 2, \dots$. Then

$$W_g e_j = 0, \quad j = 0, 1, 2, \dots.$$

Now, let $h \in L^2(\mathbb{R})$ and ε be any positive number. Then we can find a finite linear combination of the e_j's such that

$$\left|\left|\sum a_{j_\lambda} e_{j_\lambda} - h\right|\right| < \epsilon.$$

So,

$$\|W_g h\| \le \left\|W_g\left(h - \sum a_{j_\lambda} e_{j_\lambda}\right)\right\| + \left\|W_g\left(\sum a_{j_\lambda} e_{j_\lambda}\right)\right\| \le \varepsilon ||W_g||_*,$$

where $\|W_g\|_*$ is the norm of W_g. Since ε is arbitrary, it follows that

$$W_g h = 0, \quad h \in L^2(\mathbb{R}).$$

But then for all h in $L^2(\mathbb{R})$,

$$\begin{aligned}(W_g h)(x) &= (2\pi)^{-1} \int_{\mathbb{C}} \hat{g}(q,p)(\rho(q,p)h)(x)\, dq\, dp \\ &= (2\pi)^{-1} \int_{\mathbb{C}} \hat{g}(q,p)e^{iqx + \frac{1}{2}iqp}h(x+p)\, dq\, dp = 0, \quad x \in \mathbb{R}.\end{aligned}$$

Let $p' = x + p$. Then

$$(W_g h)(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} h(p') \left(\int_{-\infty}^{\infty} \hat{g}(q, p' - x) e^{iqx + \frac{1}{2}iq(p'-x)} dq \right) dp' = 0, \;\; x \in \mathbb{R}.$$

Therefore for almost all x and p' in $\mathbb{R}$,

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{g}(q, p' - x) e^{iqx + \frac{1}{2}iq(p'-x)} dq = 0.$$

So, by the Fourier inversion formula, we have for almost all x and p' in $\mathbb{R}$,

$$(\mathcal{F}_2 g) \left(\frac{1}{2}p' + \frac{1}{2}x, p' - x \right) = 0,$$

where $\mathcal{F}_2$ denotes the Fourier transform with respect to the second variable. So, $g = 0$ and the proof is complete. □

6. Heat kernels of twisted Laplacians and the sub-Laplacian on $\mathbb{H}$

As our aim is to first compute the heat kernel of the twisted Laplacian L_τ, we need Theorem 5.6 that enables us to do so.

Theorem 6.1. *For all nonnegative integers α, β, μ and ν,*

$$e_{\alpha,\beta} *_{1/4} e_{\mu,\nu} = (2\pi)^{1/2} \delta_{\beta,\mu} e_{\alpha,\nu},$$

where

$$\delta_{\beta,\mu} = \begin{cases} 1, & \beta = \mu. \\ 0, & \beta \neq \mu. \end{cases}$$

Proof. Let φ, $\psi \in \mathcal{S}$. Then by the definition of the Weyl transform and the Moyal identity,

$$\begin{aligned}
(W_{\widehat{e_{\alpha,\beta}}} \varphi, \psi) &= (2\pi)^{-1/2} \int_{\mathbb{C}} \widehat{e_{\alpha,\beta}}(z) W(\varphi, \psi)(z) dz \\
&= (2\pi)^{-1/2} \int_{\mathbb{C}} \overline{W(e_\beta, e_\alpha)(z)} W(\varphi, \psi)(z) dz \\
&= (2\pi)^{-1/2} (W(\varphi, \psi), W(e_\beta, e_\alpha)) \\
&= (2\pi)^{-1/2} (\varphi, e_\beta) \overline{(\psi, e_\alpha)} \\
&= (2\pi)^{-1/2} (\varphi, e_\beta)(e_\alpha, \psi).
\end{aligned}$$

Hence for all φ in $\mathcal{S}$,

$$W_{\widehat{e_{\alpha,\beta}}} \varphi = (2\pi)^{-1/2} (\varphi, e_\beta) e_\alpha$$

and therefore

$$\begin{aligned}
W_{\widehat{e_{\alpha,\beta}}} W_{\widehat{e_{\mu,\nu}}} \varphi &= (2\pi)^{-1/2} (W_{\widehat{e_{\mu,\nu}}} \varphi, e_\beta) e_\alpha \\
&= (2\pi)^{-1} (\varphi, e_\nu)(e_\mu, e_\beta) e_\alpha \\
&= (2\pi)^{-1/2} W_{\delta_{\mu,\beta} \widehat{e_{\alpha,\nu}}} \varphi.
\end{aligned}$$

By the Weyl calculus in the Theorem 5.6, we have

$$W_{\widehat{e_{\alpha,\beta}}} W_{\widehat{e_{\mu,\nu}}} = W_\omega,$$

where

$$\hat{\omega} = (2\pi)^{-1} \widetilde{e_{\alpha,\beta}} *_{1/4} \widetilde{e_{\mu,\nu}}.$$

Since

$$\omega = (2\pi)^{-1/2} \delta_{\mu,\beta} \widehat{e_{\alpha,\nu}},$$

we have

$$\hat{\omega} = (2\pi)^{-1/2} \delta_{\mu,\beta} \widetilde{e_{\alpha,\nu}}.$$

So,

$$\delta_{\mu,\beta} e_{\alpha,\nu} = (2\pi)^{-1/2} e_{\alpha,\beta} *_{-1/4} e_{\mu,\nu}. \qquad \square$$

The preceding theorem gives us the following more general theorem.

Theorem 6.2. *For $\tau \in \mathbb{R}$ be such that $\tau \neq 0$. Then for all nonnegative integers α, β, μ and ν,*

$$e^\tau_{\alpha,\beta} *_{\tau/4} e^\tau_{\mu,\nu} = (2\pi)^{1/2} |\tau|^{-1/2} \delta_{\beta,\mu} e^\tau_{\alpha,\nu},$$

where $\delta_{\beta,\mu}$ is the Kronecker delta.

Proof. We have

$$\begin{aligned}(e^\tau_{\alpha,\beta} *_{\tau/4} e^\tau_{\mu,\nu})(z) &= \int_{\mathbb{C}} e^\tau_{\alpha,\beta}(z-w) e^\tau_{\mu,\nu}(w) e^{i\frac{\tau}{4}[z,w]}\, dw \\ &= |\tau| \int_{\mathbb{R}^2} e_{\alpha,\beta}\left(\frac{\tau}{\sqrt{|\tau|}}(q-x), \sqrt{|\tau|}(p-\xi)\right) \\ &\quad \times e_{\mu,\nu}\left(\frac{\tau}{\sqrt{|\tau|}}(q-x), \sqrt{|\tau|}(p-\xi)\right) e^{i\frac{\tau}{4}[z,w]}\, dx\, d\xi.\end{aligned}$$

Let $q' = \frac{\tau}{\sqrt{|\tau|}} x$ and $p' = \sqrt{|\tau|}\xi$. Then for all q and p in $\mathbb{R}$,

$$\begin{aligned}(e^\tau_{\alpha,\beta} *_{\tau/4} e^\tau_{\mu,\nu})(q,p) &= \int_{\mathbb{R}^2} e_{\alpha,\beta}\left(\frac{\tau}{\sqrt{|\tau|}} q - q', \sqrt{|\tau|} p - p'\right) \\ &\quad \times e_{\mu,\nu}(q',p') e^{\frac{i\tau}{2}\left(\frac{\sqrt{|\tau|}}{\tau} q' p - \frac{1}{\sqrt{|\tau|}} p' q\right)}\, dq'\, dp' \\ &= (e_{\alpha,\beta} *_{1/4} e_{\mu,\nu})\left(q\frac{\tau}{\sqrt{|\tau|}}, p\sqrt{|\tau|}\right) \\ &= (2\pi)^{1/2} |\tau|^{1/2} \delta_{\beta,\mu} e_{\alpha,\nu}\left(q\frac{\tau}{\sqrt{|\tau|}}, p\sqrt{|\tau|}\right). \qquad \square\end{aligned}$$

Theorem 6.3. *Let $\tau \in \mathbb{R}$ such that $\tau \neq 0$. Then for $j, k = 0, 1, 2, \ldots,$*

$$L_\tau e^\tau_{j,k} = (2k+1)|\tau| e^\tau_{j,k}.$$

In order to prove the theorem, we need a lemma.

Lemma 6.4. *Let* $x = q\frac{\tau}{\sqrt{|\tau|}}$ *and* $y = p\sqrt{|\tau|}$. *Then*

$$L_\tau^{(q,p)} = |\tau| L^{(x,y)}.$$

Proof. We have

$$\frac{\partial}{\partial q} = \frac{\partial}{\partial x}\frac{\tau}{\sqrt{|\tau|}}, \qquad \frac{\partial}{\partial p} = \frac{\partial}{\partial y}\sqrt{|\tau|}.$$

So,

$$\frac{\partial^2}{\partial q^2} = \frac{\partial^2}{\partial x^2}|\tau|, \qquad \frac{\partial^2}{\partial p^2} = \frac{\partial^2}{\partial y^2}|\tau|.$$

Therefore

$$\begin{aligned} L_\tau^{(q,p)} &= -\Delta + \frac{1}{4}(q^2+p^2)\tau^2 - i\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right)\tau \\ &= |\tau|\Delta + |\tau|\frac{1}{4}(x^2+y^2) - i|\tau|\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \\ &= |\tau|L^{(x,y)}. \end{aligned}$$

□

Proof of Theorem 6.3. The case when $\tau = 1$ is proved in [18]. Now, by Theorem 4.2 and Lemma 6.4, we get

$$\begin{aligned} (L_\tau e_{j,k}^\tau)(q,p) &= L_\tau |\tau|^{1/2} e_{j,k}\left(q\frac{\tau}{\sqrt{|\tau|}}, p\sqrt{|\tau|}\right) \\ &= |\tau| L |\tau|^{1/2} e_{j,k}(x,y) \\ &= |\tau||\tau|^{1/2} e_{j,k}(x,y) \\ &= (2k+1)|\tau||\tau|^{1/2} e_{j,k}\left(q\frac{\tau}{\sqrt{|\tau|}}, \sqrt{|\tau|}\right) \\ &= (2k+1)|\tau| e_{j,k}^\tau(q,p) \end{aligned}$$

for all q and p in $\mathbb{R}$. □

By Theorem 6.3 and the spectral theorem, for all functions f in $L^2(\mathbb{C})$, we have

$$e^{-\rho L_\tau} f = \sum_{k=0}^\infty \sum_{j=0}^\infty e^{-|\tau|(2k+1)\rho}(f, e_{j,k}^\tau) e_{j,k}^\tau, \quad \rho > 0,$$

and for $\rho > 0$, we have

$$e^{-\rho L_\tau} f = \sum_{k=0}^\infty e^{-(2k+1)|\tau|\rho} \sum_{j=0}^\infty (f, e_{j,k}^\tau) e_{j,k}^\tau.$$

To simplify the problem, we first compute the term $\sum_{j=0}^{\infty}(f, e^{\tau}_{j,k})e^{\tau}_{j,k}$. Since for $k = 0, 1, 2, \dots$

$$\begin{aligned} f *_{\tau/4} e^{\tau}_{k,k} &= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(f, e^{\tau}_{j,l})e^{\tau}_{j,l} *_{\tau/4} e^{\tau}_{k,k} \\ &= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(f, e^{\tau}_{j,l})(2\pi)^{1/2}|\tau|^{-1/2}\delta_{l,k}e^{\tau}_{j,k} \\ &= (2\pi)^{1/2}|\tau|^{-1/2}\sum_{j=0}^{\infty}(f, e^{\tau}_{j,k})e^{\tau}_{j,k}. \end{aligned}$$

We have for $k = 0, 1, 2, \dots,$

$$\sum_{j=0}^{\infty}(f, e^{\tau}_{j,k})e^{\tau}_{j,k} = (2\pi)^{-1/2}|\tau|^{1/2} f *_{\tau/4} e^{\tau}_{k,k}.$$

Therefore

$$e^{-\rho L_\tau} f = (2\pi)^{-1/2}|\tau|^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1)|\tau|\rho}e^{\tau}_{k,k} *_{-\tau/4} f, \quad \rho > 0.$$

Now, in order to find an explicit formula for the heat kernel of the twisted Laplacians, we need Mehler's formula, which can be found in the book [18].

Theorem 6.5 (Mehler's formula). *For all x and $y \in \mathbb{R}$ and all $w \in \mathbb{C}$ with $|w| < 1$,*

$$\sum_{k=0}^{\infty}\frac{h_k(x)h_k(y)}{2^k k!}w^k = (1 - w^2)^{-1/2}e^{-\frac{1}{2}\frac{1+w^2}{1-w^2}(x^2+y^2)+xy\frac{2w}{1-w^2}},$$

where the series is uniformly and absolutely convergent on the open disk $\{w \in \mathbb{C} : |w| < 1\}$.

Finally, by applying Theorem 4.2 and Mehler's formula, we are able to get the formula for our $e^{-\rho L_\tau} f$. Indeed, for all $z = (q, p) \in \mathbb{C}$ and $\rho > 0$,

$$\begin{aligned} (e^{-\rho L_\tau} f)(q, p) &= (2\pi)^{-1/2}|\tau|^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1)|\tau|\rho}e^{\tau}_{k,k}(q, p) \\ &= (2\pi)^{-1/2}|\tau|^{1/2}e^{-|\tau|\rho}\sum_{k=0}^{\infty} e^{-2k|\tau|\rho}e_{k,k}\left(q\frac{\tau}{\sqrt{|\tau|}}, p\sqrt{|\tau|}\right) \\ &= (2\pi)^{-1/2}|\tau|^{1/2}e^{-|\tau|\rho}\frac{1}{1 - e^{-2|\tau|\rho}}e^{-|\tau||z|^2\frac{1}{4}\frac{1+e^{-2|\tau|\rho}}{1-e^{-2|\tau|\rho}}} \\ &= \frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)}e^{-\frac{1}{4}|\tau||z|^2\coth(\tau\rho)}. \end{aligned}$$

Hence the heat kernel κ^{τ}_{ρ}, for $\rho > 0$, of L_τ is given by

$$\kappa^{\tau}_{\rho}(z, w) = \frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)}e^{-\frac{1}{4}|\tau||z-w|^2\coth(\tau\rho)}e^{i\frac{\tau}{4}[z,w]}, \quad z, w \in \mathbb{C}.$$

By Theorem 3.2, the heat kernel of $\mathcal{L}$ which we denote by K_ρ for $\rho > 0$ satisfies

$$K_\rho^\tau = (2\pi)^{-1/2} k_\rho^\tau, \quad \tau \in \mathbb{R} \setminus \{0\}.$$

where

$$k_\rho^\tau(z) = \frac{1}{4\pi} \frac{\tau}{\sinh(\tau\rho)} e^{-\frac{1}{4}|\tau||z-w|^2 \coth(\tau\rho)}, \quad z \in \mathbb{C}.$$

By taking the Fourier transform, we have the following theorem that gives the formula for the heat kernel K_ρ, $\rho > 0$, of the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}$.

Theorem 6.6. *For $\rho > 0$, and $\tau \in \mathbb{R}$ such that $\tau \neq 0$,*

$$K_\rho(z,t) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} e^{-it\tau} \frac{\tau}{\sinh(\tau\rho)} e^{-\frac{1}{4}|\tau||z|^2 \coth(\tau\rho)} d\tau, \ (z,t) \in \mathbb{H}.$$

7. The Green functions for the twisted Laplacians and the sub-Laplacian on $\mathbb{H}$

The Green function G_τ of L_τ is the kernel of the integral operator representing L_τ^{-1}, which can be obtained by integrating the heat kernel of the twisted Laplacian L_τ from 0 to ∞ with respect to time ρ. Then we get

$$\begin{aligned} G_\tau(z,w) &= \frac{1}{4\pi} \left(\int_0^\infty \frac{\tau}{\sinh(\tau\rho)} e^{-\frac{1}{4}|\tau||z-w|^2 \coth(\tau\rho)} \right) d\rho \, e^{-\frac{i}{4}\tau[z,w]} \\ &= \frac{1}{4\pi} \left(\int_0^\infty \frac{1}{(v^2-1)^{1/2}} e^{-\frac{1}{4}|\tau||z-w|^2 v^2} \, dv \right) e^{-\frac{i}{4}\tau[z,w]} \\ &= \frac{1}{4} K_0 \left(\frac{1}{4}|\tau||z-w|^2 \right) e^{-\frac{i}{4}\tau[z,w]}, \end{aligned}$$

where the change of variable $v = \coth(\rho\tau)$ is used, and K_0 is the modified Bessel function of order 0 given by

$$K_0(x) = \int_0^\infty e^{-x \cosh\delta} \, d\delta, \quad x > 0.$$

Similarly, the Green function $\mathcal{G}$ of the sub-Laplacian $\mathcal{L}$ is the kernel of the integral operator representing $\mathcal{L}^{-1}$. And it can be computed by integrating the heat kernel of $\mathcal{L}$ with respect to time ρ from 0 to ∞. More explicitly,

$$\begin{aligned} \mathcal{G}(z,w) &= \frac{1}{8\pi^2} \int_0^\infty \int_{-\infty}^\infty e^{-it\tau} \frac{\tau}{\sinh(\tau\rho)} e^{-\frac{1}{4}|z|^2 \coth(\tau\rho)} \, d\tau \, d\rho \\ &= \frac{1}{8\pi^2} \int_{-\infty}^\infty e^{-it\tau} \int_0^\infty \frac{\tau}{\sinh(\tau\rho)} e^{-\frac{1}{4}|\tau||z|^2 \coth(\tau\rho)} \, d\rho \, d\tau \\ &= \frac{1}{8\pi^2} \int_{-\infty}^\infty \int_0^\infty \frac{1}{(v^2-1)^{1/2}} e^{-\frac{1}{4}|\tau||z-w|^2 v^2} \, dv \, d\tau \\ &= \frac{1}{8\pi^2} \int_{-\infty}^\infty e^{-it\tau} K_0 \left(\frac{1}{4}|\tau||z|^2 \right) d\tau = \frac{1}{8\pi^2} \int_0^\infty \int_{-\infty}^\infty e^{it\tau} e^{-\frac{1}{4}|\tau||z|^2 \cosh\delta} \, d\tau \, d\delta. \end{aligned}$$

For all z in $\mathbb{C}$ and t in $\mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-it\tau} e^{-\frac{1}{4}|\tau|\,|z|^2 \cosh\delta}\, d\tau = \int_{-\infty}^{0} e^{-it\tau} e^{\frac{1}{4}\tau|z|^2\cosh\delta} d\tau + \int_{0}^{\infty} e^{-it\tau} e^{-\frac{1}{4}\tau|z|^2\cosh\delta}\, d\tau$$
$$= \frac{e^{\tau\left(\frac{1}{4}|z|^2\cosh\delta - it\right)}}{\frac{1}{4}|z|^2\cosh\delta - it}\Bigg|_{-\infty}^{0} + \frac{e^{-\tau\left(\frac{1}{4}|z|^2\cosh\delta + it\right)}}{\frac{1}{4}|z|^2\cosh\delta + it}\Bigg|_{0}^{\infty}$$
$$= \frac{\frac{1}{2}|z|^2\cosh\delta}{(|z|^4/16)\cosh^2\delta + t^2}.$$

So, for all z in $\mathbb{C}$ and t in $\mathbb{R}$,

$$\int_0^{\infty}\int_{-\infty}^{\infty} e^{-it\tau} e^{-\frac{1}{4}|\tau|\,|z|^2\cosh\delta} d\tau\, d\delta = \int_0^{\infty} \frac{\frac{1}{2}|z|^2\cosh\delta}{(|z|^4/16)\cosh^2\delta + t^2} d\delta$$
$$= \frac{|z|^2}{2}\int_0^{\infty} \frac{\cosh\delta}{(|z|^4/16)\cosh^2\delta + t^2} d\delta$$
$$= \frac{8}{|z|^2}\int_0^{\infty} \frac{\cosh\delta}{\cosh^2\delta + (16t^2/|z|^4)}\, d\delta$$
$$= \frac{8}{|z|^2}\int_0^{\infty} \frac{1}{\rho^2 + 1 + (16t^2/|z|^4)}\, d\rho$$
$$= \frac{4\pi}{|z|^2}\frac{1}{\sqrt{1 + (16t^2/|z|^4)}}$$
$$= \frac{4\pi}{\sqrt{|z|^4 + 16t^2}}.$$

Hence

$$\mathcal{G}(z,t) = \frac{1}{8\pi^2}|z|^2 \frac{1}{\sqrt{1 + (16t^2/|z|^4)}} = \frac{1}{2\pi}\frac{1}{\sqrt{|z|^4 + 16t^2}}$$

for all (z,t) in $\mathbb{H}$.

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Metaplectic Equivalence of the Hierarchical Twisted Laplacian

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Abstract. We use a metaplectic operator to prove that the hierarchical twisted Laplacian L_m is unitarily equivalent to the tensor product of the one-dimensional Hermite operator and the identity operator on $L^2(\mathbb{R}^{m+1})$, and we use this unitary equivalence to show that L_m is globally hypoelliptic in the Schwartz space and in the Gelfand–Shilov spaces.

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Keywords. Hierarchical Wigner transform, hierarchical twisted Laplacian, Hermite functions, Hermite operators, metaplectic operators, global hypoellipticity, Schwartz space, Gelfand–Shilov spaces.

1. Introduction

For all $x \in \mathbb{R}$ and all $v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m$, let

$$z = x + is(v),$$

where $s(v) = v_1 + v_2 + \cdots + v_m$. Then we let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ be linear partial differential operators on $\mathbb{R}^{m+1}$ defined by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i \sum_{l=1}^{m} \frac{\partial}{\partial v_l} + \frac{i}{2}\left(1 - \frac{1}{m}\right) s(v),$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i \sum_{l=1}^{m} \frac{\partial}{\partial v_l} + \frac{i}{2}\left(1 - \frac{1}{m}\right) s(v).$$

Then the hierarchical twisted Laplacian L_m is defined on $\mathbb{R}^{m+1}$ by

$$L_m = -\frac{1}{2}(Z\bar{Z} + \bar{Z}Z),$$

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where

$$Z = \frac{\partial}{\partial z} + \frac{1}{2}\bar{z} \quad \text{and} \quad \bar{Z} = \frac{\partial}{\partial \bar{z}} - \frac{1}{2}z.$$

By an easy calculation, we see that L_m is the linear partial differential operator on $\mathbb{R}^{m+1}$ given by

$$L_m = -\left(\frac{\partial^2}{\partial x^2} + \sum_{j,l=1}^{m} \frac{\partial^2}{\partial v_j \partial v_l}\right) + \frac{1}{4}\left(x^2 + \frac{s(v)^2}{m^2}\right) - i\left(\frac{s(v)}{m}\frac{\partial}{\partial x} - x\sum_{j=1}^{m}\frac{\partial}{\partial v_j}\right).$$

In the case when $m = 1$, L_1 is a linear partial differential operator on $\mathbb{R}^2$ and has the form

$$L_1 = -\Delta + \frac{1}{4}(x^2 + v_1^2) - i\left(v_1\frac{\partial}{\partial x} - x\frac{\partial}{\partial v_1}\right),$$

which is the ordinary twisted Laplacian and we denote it by L. It is a perturbation of the Hermite operator by a rotation operator.

The twisted Laplacian L comes up as the quantum-mechanical Hamiltonian of the motion of an electron in the infinite two-dimensional plane under the influence of a constant magnetic field perpendicular to the plane. The eigenvalues of the system are known as Landau levels and the corresponding eigenfunctions are the Wigner transforms of Hermite functions. The twisted Laplacian has been studied extensively in, e.g., [4, 5, 6, 7, 11, 12, 13, 14, 16, 17, 18].

The twisted Laplacian L can in fact be obtained from the sub-Laplacian on the one-dimensional Heisenberg group $\mathbb{C} \times \mathbb{R}$ by taking the Fourier transform with respect to the center, and is hence a linear partial differential operator on $\mathbb{R}^2$. Higher-dimensional twisted Laplacians on $\mathbb{R}^{2n}$ can be obtained similarly using the n-dimensional Heisenberg group $\mathbb{C}^n \times \mathbb{R}$. Therefore the twisted Laplacian is defined on even-dimensional Euclidean spaces. The hierarchical twisted Laplacian L_m on $\mathbb{R}^{m+1}$ can be seen as a twisted Laplacian on $\mathbb{R}^n$, where the dimension n can now be even or odd with $n > 1$.

The ordinary twisted Laplacian is well known to be elliptic, but not globally elliptic in the sense of Shubin defined in Section 25 of [10]. By explicit formulas of the heat kernel and the Green function of L, it is shown in [18] by Wong that L is globally hypoelliptic in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ and in [4] by Dasgupta and Wong that L is globally hypoelliptic in Gelfand–Shilov spaces. In [7], the global hypoellipticity of the twisted Laplacian is recaptured using the fact that L is unitarily equivalent to a tensor product of the ordinary Hermite operator and the identity operator.

The hierarchical twisted Laplacian L_m can be written in the form

$$L_m = \left(D_x - \frac{1}{2m}s(v)\right)^2 + \left(\sum_{j=1}^{m} D_{v_j} + \frac{1}{2}x\right)^2. \tag{1.1}$$

Its symbol is given by

$$\sigma_m(x, v; \xi, \eta) = \left(\xi - \frac{1}{2m}s(v)\right)^2 + \left(\sum_{j=1}^{m} \eta_j + \frac{1}{2}x\right)^2,$$

for all $x, \xi \in \mathbb{R}$ and $v, \eta \in \mathbb{R}^m$. Thus, L_m is not globally elliptic, in the sense that we cannot find positive constants C and R such that

$$|\sigma_m(x, v; \xi, \eta)| \geq C \left(1 + x^2 + \xi^2 + \sum_{j=1}^{m} v_j^2 + \sum_{j=1}^{m} \eta^2\right)$$

whenever

$$x^2 + \xi^2 + \sum_{j=1}^{m} v_j^2 + \sum_{j=1}^{m} \eta_j^2 \geq R.$$

In fact, it is not even elliptic.

The aim of this paper is to establish a unitary equivalence between the hierarchical twisted Laplacian L_m and the tensor product of the one-dimensional Hermite operator with the identity operator on $\mathbb{R}^m$. An immediate application of this unitary equivalence to the global hypoellipticity of the hierarchical twisted Laplacian is given.

In Section 2, we recall some of the definitions and results of the hierarchical twisted Laplacian from [9] that we need in this paper. In Section 3, we prove that the hierarchical twisted Laplacian L_m is unitarily equivalent to the tensor product of the one-dimensional Hermite operator with the identity on $\mathbb{R}^m$. Then in Section 4, we prove the global hypoellipticity of L_m using the global hypoellipticity of the Hermite operator.

2. Hierarchical Wigner transforms

Let $f \in \mathcal{S}(\mathbb{R}^m)$. Then for all $v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m$ and $w \in \mathbb{R}$, we define the function $\rho(w, v)f$ on $\mathbb{R}$ by

$$(\rho(w, v)f)(x) = e^{iw\cdot x + \frac{1}{2m}iw\cdot|v|} f(x \oplus v), \quad x \in \mathbb{R},$$

where

$$x \oplus v = (x + v_1, x + v_2, \ldots, x + v_m).$$

Now, we can define the hierarchical Fourier–Wigner transform $V(f, g)$ of two functions $f \in \mathcal{S}(\mathbb{R}^m)$ and $g \in \mathcal{S}(\mathbb{R})$ by

$$V(f, g)(w, v) = (2\pi)^{-m/2}(\rho(w, v)f, g)_{L^2(\mathbb{R})}$$

for all $v = (v_1, v_2, \ldots, v_m)$ in $\mathbb{R}^m$ and w in $\mathbb{R}$.

The following integral representation of hierarchical Fourier–Wigner transforms can be derived from the corresponding integral representation of multilinear Fourier–Wigner transforms in [3] using a density argument with tensor products.

Proposition 2.1. *Let $f \in \mathcal{S}(\mathbb{R}^m)$ and let $g \in \mathcal{S}(\mathbb{R})$. Then*

$$V(f,g)(w,v) = (2\pi)^{-m/2} \int_{\mathbb{R}} e^{iy\cdot w} f\left(\left(y - \frac{1}{2m}s(v)\right) \oplus v\right) \overline{g\left(y - \frac{1}{2m}s(v)\right)}\, dy$$

for all w in $\mathbb{R}$ and all $v = (v_1, v_2, \ldots, v_m)$ in $\mathbb{R}^m$.

The hierarchical Wigner transform $W(f,g)$ of f in $\mathcal{S}(\mathbb{R}^m)$ and g in $\mathcal{S}(\mathbb{R})$ is defined by

$$W(f,g) = V(f,g)^\wedge,$$

where $V(f,g)^\wedge$, also denoted by $\mathcal{F}V(f,g)$, is the Fourier transform of $V(f,g)$.

For $k = 0, 1, 2, \ldots$, the Hermite function e_k of order k is the function on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x)$$

for all x in $\mathbb{R}$, where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k (e^{-x^2})$$

for all x in $\mathbb{R}$.

For all nonnegative integers $j_1, j_2, \ldots, j_m$ and k, we define the function $e_{j_1,\ldots,j_m,k}$ on $\mathbb{R}^{m+1}$ by

$$e_{j_1,\ldots,j_m,k} = V(e_{j_1} \otimes \cdots \otimes e_{j_m}, e_k).$$

So,

$$e_{j_1,\ldots,j_m,k}(\xi, v) = (2\pi)^{-m/2} \int_{\mathbb{R}} e^{iy\xi} \prod_{l=1}^{m} e_{j_l}\left(y + v_l - \frac{1}{2m}s(v)\right) e_k\left(y - \frac{1}{2m}s(v)\right) dy$$

for all $\xi \in \mathbb{R}$ and $v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m$. It is obvious that when $m = 1$, we get back the classical Fourier–Wigner transform of Hermite functions.

Proposition 2.2. $\{e_{j_1,\ldots,j_m,k} : \; j_1, \ldots, j_m, k = 0, 1, 2, \ldots\}$ *is an orthonormal basis for $L^2(\mathbb{R}^{m+1})$.*

For $m = 1$, the results hitherto described can be found in the book [15] by Wong.

The following theorem gives the complete spectrum of the hierarchical twisted Laplacian. For a proof, see [9].

Theorem 2.3. *For all nonnegative integers $j_1, j_2, \ldots, j_m$ and k,*

$$L_m e_{j_1,\ldots,j_m,k} = (2k+1) e_{j_1,\ldots,j_m,k}.$$

3. A unitary equivalence

We define the linear operator $T : L^2(\mathbb{R}^{m+1}) \to L^2(\mathbb{R}^{m+1})$ by

$$(Tf)(y,v) = f\left(y - \frac{1}{2m}s(v), y + v - \frac{1}{2m}s(v), \dots, y + v_m - \frac{1}{2m}s(v)\right)$$

for all $f \in L^2(\mathbb{R}^{m+1})$, $v = (v_1, v_2, \dots, v_m) \in \mathbb{R}^m$ and $y \in \mathbb{R}$. We call T the twisting operator on $L^2(\mathbb{R}^{m+1})$. The following proposition is Proposition 3.2 in [9].

Proposition 3.1. $T : L^2(\mathbb{R}^{m+1}) \to L^2(\mathbb{R}^{m+1})$ *is a unitary operator and*

$$(T^{-1}f)(y,z) = f\left(\frac{1}{2m}s(z) + \frac{1}{2}y, z_1 - y, \dots, z_m - y\right)$$

for all $f \in L^2(\mathbb{R}^{m+1})$, $z = (z_1, z_2, \dots, z_m) \in \mathbb{R}^m$ *and* $y \in \mathbb{R}$.

Let $F \in L^2(\mathbb{R}^{m+1})$. Then we define JF on $\mathbb{R}^{m+1}$ by

$$(JF)(w,v) = (2\pi)^{-m/2} \int_{\mathbb{R}} e^{iyw} (TF)(y,v)\, dy \tag{3.1}$$

for all $w \in \mathbb{R}$ and $v \in \mathbb{R}^m$.

Theorem 3.2. $J : L^2(\mathbb{R}^{m+1}) \to L^2(\mathbb{R}^{m+1})$ *is a bijection and*

$$\|J\|_* = (2\pi)^{(-m+1)/2},$$

where $\| \ \|_*$ *in the norm in the* C^**-algebra of all bounded linear operators on* $L^2(\mathbb{R}^{m+1})$*. Furthermore, let* $G \in L^2(\mathbb{R}^{m+1})$*. Then*

$$\left(J^{-1}G\right)(y,z) = (2\pi)^{(m-1)/2} \int_{\mathbb{R}} e^{-it\left(\frac{1}{2m}s(z)+\frac{1}{2}y\right)} G(t, z_1 - y, \dots, z_m - y)\, dt$$

for all $y \in \mathbb{R}$ *and* $z \in \mathbb{R}^m$.

Proof. Let $F \in L^2(\mathbb{R}^{m+1})$. Then by Proposition 3.1, $TF \in L^2(\mathbb{R}^{m+1})$. It is easy to see that

$$(JF)(w,v) = (2\pi)^{(-m+1)/2} \left(\mathcal{F}_1^{-1}TF\right)(w,v)$$

for almost all $w \in \mathbb{R}$ and $v \in \mathbb{R}^m$, where $\mathcal{F}_1^{-1}TF$ is the inverse Fourier transform of TF with respect to the first variable. Therefore $JF \in L^2(\mathbb{R}^{m+1})$. Since $\mathcal{F}_1$ and T are unitary operators, it follows that

$$\|J\|_* = (2\pi)^{(-m+1)/2} \quad \text{and} \quad J^{-1} = (2\pi)^{(m-1)/2} T^{-1}\mathcal{F}_1.$$

Therefore for all $G \in L^2(\mathbb{R}^{m+1})$, we get by Proposition 3.1

$$(J^{-1}F)(y,z) = (2\pi)^{(m-1)/2} \int_{\mathbb{R}} e^{-it(\frac{1}{2m}s(z)+\frac{1}{2}y)} G(t, z_1 - y, \dots, z_m - y)\, dt. \qquad \square$$

The one-dimensional Hermite operator H on $L^2(\mathbb{R})$ is given by

$$H = -\frac{d^2}{dx^2} + x^2, \quad x \in \mathbb{R}.$$

Theorem 3.3. *Let* $H_1 = H \otimes I$*, where* I *is the identity operator on* $\mathbb{R}^m$*. Then*

$$JH_1J^{-1} = L_m.$$

Proof. Since $\{e_{j_1,j_2,\,\ldots,\,j_m,k} : j_1, j_2,\ \ldots,\ j_m, k\}$ forms an orthonormal basis for $L^2(\mathbb{R}^{m+1})$, it follows from Theorem 2.3 that it is enough to show that

$$(JH_1J^{-1})e_{j_1,\ldots,j_m,k} = (2k+1)e_{j_1,\ldots,j_m,k}.$$

By the definition of J, we get for all nonnegative integers $j_1, j_2, \ldots, j_m, k$,

$$J(e_k \otimes e_{j_1} \otimes e_{j_2} \otimes \cdots e_{j_m}) = e_{j_1,\ldots,j_m,k}. \tag{3.2}$$

On the other hand, for $k = 0, 1, 2, \ldots,$

$$He_k = (2k+1)e_k.$$

Therefore

$$H_1(e_k \otimes e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_m}) = (2k+1)e_k \otimes e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_m}. \tag{3.3}$$

Hence by (3.2) and (3.3), we get for all nonnegative integers $j_1, j_2, \ldots, j_m, k$,

$$JH_1J^{-1}e_{j_1,\ldots,j_m,k} = (2k+1)e_{j_1,\ldots,j_m,k}.$$ □

Remark 3.4. A more general class of operators *identifying* L_m with H_1 can be found by means of linear symplectic transformations and related metaplectic operators on pages 157–159 in the book [8] by Hörmander. To be more specific, in $\mathbb{R}^{2(m+1)}$, regarded as a symplectic vector space with variables $(x, v; \xi, \eta)$, we can find a linear symplectic mapping X changing the coordinates $(x, v; \xi, \eta)$ into $(y, z; \gamma, \zeta)$ such that

$$y = \xi - \frac{1}{2m}s(v), \quad \gamma = s(\eta) + \frac{1}{2}x,$$

$$z_j = v_j + y = v_j + \xi - \frac{1}{2m}s(v), \quad j = 1, 2, \ldots, m,$$

and

$$\zeta_j = \eta_j - \frac{1}{2m}x, \quad j = 1, 2, \ldots, m.$$

That this is possible is due to the fact that

$$\{y, \gamma\} = 1.$$

In fact, X can be so defined as to preserve the symplectic form and we can apply Theorem 18.5.9 in [8] to the effect that there is a unitary operator J, uniquely determined up to a constant factor with modulus 1, such that

$$J^{-1}\left(D_x - \frac{1}{2m}s(v)\right)J = y \quad \text{and} \quad J^{-1}\left(\sum_{j=1}^{m} D_{v_j} + \frac{1}{2}x\right)J = D_y.$$

Hence by (1.1),

$$J^{-1}L_mJ = D_y^2 + y^2 = H_1.$$

The unitary operator J in Theorems 3.2 and 3.4, which is distinguished by its connections with the Wigner theory, is a particular case of this general class of metaplectic operators. The detailed definition of the mapping X in this particular case can be written down easily from the definition of J in (3.1), which is the composition of a linear change of coordinates and a Fourier conjugation.

Remark 3.5. In fact, by conjugation with a suitable metaplectic operator, we can reduce to a simple canonical form any operator with quadratic symbol and hence compute the spectrum. See, for example, [1] for applications to the hypoellipticity of pseudo-differential operators with double characteristics.

4. Global hypoellipticity

In order to prove the global hypoellipticity of the hierarchical twisted Laplacian, we use the well-known fact about the global hypoellipticity of the Hermite operator which can be found in Shubin's book [10]. See also [7].

Theorem 4.1. *L_m is globally hypoelliptic in the sense that*

$$u \in \mathcal{S}'(\mathbb{R}^{m+1}),\ L_m u \in \mathcal{S}(\mathbb{R}^{m+1}) \Rightarrow u \in \mathcal{S}(\mathbb{R}^{m+1}).$$

Proof. Let $u \in \mathcal{S}'(\mathbb{R}^{m+1})$ be such that $L_m u \in \mathcal{S}(\mathbb{R}^{m+1})$. We need to show that $u \in \mathcal{S}(\mathbb{R}^{m+1})$. It is easy to see that $J : \mathcal{S}(\mathbb{R}^{m+1}) \to \mathcal{S}(\mathbb{R}^{m+1})$ is a bijection. Therefore $J^{-1}L_m u \in \mathcal{S}(\mathbb{R}^{m+1})$. But by Theorem 3.3,

$$H_1 J^{-1} u = J^{-1} L_m u.$$

Hence $H_1 J^{-1} u \in \mathcal{S}(\mathbb{R}^{m+1})$. Since H_1 is globally hypoelliptic, it follows that $J^{-1}u \in \mathcal{S}(\mathbb{R}^{m+1})$. Again using the fact that J^{-1} maps $\mathcal{S}(\mathbb{R}^{m+1})$ onto $\mathcal{S}(\mathbb{R}^{m+1})$, we get $u \in \mathcal{S}(\mathbb{R}^{m+1})$. □

Remark 4.2. Using the global hypoellipticity in [2] of the Hermite operator on the Gelfand–Shilov spaces $\mathcal{S}^\mu_\nu(\mathbb{R}^{m+1})$, where μ and ν are nonnegative real numbers such that $\mu \geq \frac{1}{2}$ and $\nu \geq \frac{1}{2}$, the same proof of Theorem 4.1 can be used to prove the global hypoellipticity of L_m on Gelfand–Shilov spaces $S^\mu_\mu(\mathbb{R}^{m+1})$ in the sense that

$$u \in \mathcal{S}'(\mathbb{R}^{m+1}),\ L_m u \in \mathcal{S}^\mu_\mu(\mathbb{R}^{m+1}) \Rightarrow u \in S^\mu_\mu(\mathbb{R}^{m+1})$$

because $J : \mathcal{S}^\mu_\mu(\mathbb{R}^{m+1} \to \mathcal{S}^\mu_\mu(\mathbb{R}^{m+1})$, $\mu \geq \frac{1}{2}$, is a bijection.

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The Heat Kernel and Green Function of a Sub-Laplacian on the Hierarchical Heisenberg Group

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Abstract. We give the hierarchical Heisenberg group underpinning the hierarchical twisted Laplacian discovered recently. This hierarchical twisted Laplacian is obtained by taking the inverse Fourier transform of a sub-Laplacian with respect to a subcenter of the hierarchical Heisenberg group. Using parametrized versions of Wigner transforms and Weyl transforms, we give formulas for the heat kernels and Green functions of the parametrized hierarchical twisted Laplacians. Taking the Fourier transform of the parametrized heat kernels so obtained, we give explicit formulas for the heat kernel and Green function of the hierarchical sub-Laplacian on the hierarchical Heisenberg group.

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Keywords. Hierarchical Heisenberg group, hierarchical sub-Laplacian, subcenter, Hermite functions, heat kernels, Green functions, global hypoellipticity.

1. Introduction

For all x in $\mathbb{R}$ and all $v = (v_1, v_2, \ldots, v_m)$ in $\mathbb{R}^m$, we let

$$z = x + is(v),$$

where $s(v) = \sum_{j=1}^m v_j$. Then we let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ be linear partial differential operators on $\mathbb{R}^{m+1}$ defined by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i \sum_{l=1}^m \frac{\partial}{\partial v_l} + \frac{i}{2}\left(1 - \frac{1}{m}\right) s(v),$$

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and
$$\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} + i\sum_{l=1}^{m} \frac{\partial}{\partial v_l} + \frac{i}{2}\left(1 - \frac{1}{m}\right) s(v).$$
The hierarchical twisted Laplacian L_m is defined on $\mathbb{R}^{m+1}$ by
$$L_m = -\frac{1}{2}(Z\overline{Z} + \overline{Z}Z),$$
where
$$Z = \frac{\partial}{\partial z} + \frac{1}{2}\overline{z} \quad \text{and} \quad \overline{Z} = \frac{\partial}{\partial \overline{z}} - \frac{1}{2}z.$$
In explicit detail,
$$L_m = -\left(\frac{\partial^2}{\partial x^2} + \sum_{j,l=1}^{m} \frac{\partial^2}{\partial v_j \partial v_l}\right) + \frac{1}{4}\left(x^2 + \frac{s(v)^2}{m^2}\right) - i\left(\frac{s(v)}{m}\frac{\partial}{\partial x} - x\sum_{j=1}^{m} \frac{\partial}{\partial v_j}\right).$$
If $m = 1$, then L_1 is the ordinary twisted Laplacian given by
$$L_1 = -\Delta + \frac{1}{4}(x^2 + v_1^2) - i\left(v_1\frac{\partial}{\partial x} - x\frac{\partial}{\partial v_1}\right),$$
which comes up as the quantum-mechanical Hamiltonian of the motion of an electron in the infinite two-dimensional plane under the influence of a constant magnetic field perpendicular to the plane. The twisted Laplacian L_1 and its various extensions have been studied in the works [2, 3, 4, 5, 7, 8, 11, 16, 17].

The twisted Laplacian L_1 can in fact be obtained from the sub-Laplacian on the one-dimensional Heisenberg group $\mathbb{C} \times \mathbb{R}$ by taking the inverse Fourier transform with respect to the center, and is therefore an elliptic partial differential operator on $\mathbb{R}^2$. Higher-dimensional twisted Laplacians on $\mathbb{R}^{2n}$ can be obtained similarly using the n-dimensional Heisenberg group $\mathbb{C}^n \times \mathbb{R}$. Therefore the twisted Laplacian is defined on even-dimensional Euclidean spaces. The hierarchical twisted Laplacian L_m on $\mathbb{R}^{m+1}$ can be seen as a twisted Laplacian on $\mathbb{R}^n$, where the dimension n can now be arbitrary but not equal to 1. This is a *raison d' être* of the adjective hierarchical in this paper and related papers. Another *raison d' être* is given at the end of Section 2. The heat semigroup generated by the hierarchical twisted Laplacian has been computed in [13]. A unitary equivalence between the hierarchical twisted Laplacian L_m and the tensor product of the one-dimensional Hermite operator with the identity operator on $\mathbb{R}^m$ can be established using a metaplectic operator as in [12]. It is also shown in [12] that the alluded unitary equivalence can be used to prove the global hypoellipticity of the hierarchical twisted Laplacian L_m to the effect that
$$u \in \mathcal{S}'(\mathbb{R}^{m+1}),\, L_m u \in \mathcal{S}(\mathbb{R}^{m+1}) \Rightarrow u \in \mathcal{S}(\mathbb{R}^{m+1}).$$

In this paper we first give the Lie group that underpins the hierarchical twisted Laplacian L_m first studied in [13] as an answer to the questions asked for the group underlying the hierarchical twisted Laplacian at the talk given by the first author at the International Conference on Generalized Functions held at the University of Vienna in 2009. The resulting Lie group is naturally dubbed the hierarchical Heisenberg group and is denoted by $\mathbb{H}^n_m$. On $\mathbb{H}^n_m$ is then constructed a sub-Laplacian $\mathcal{L}_m$ that we call the hierarchical sub-Laplacian. From the sub-Laplacian we generate a parametrized family of hierarchical twisted Laplacians L^λ_m for

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$$

with $s(\lambda) \neq 0$. These parametrized hierarchical twisted Laplacians include L_m as a special case. It is then noted that L^λ_m is in general not elliptic on $\mathbb{R}^{m+1}$ unless $m = 1$. The heat kernels and Green functions of the parametrized twisted Laplacians are then constructed using parametrized versions of Fourier–Wigner transforms, Wigner transforms and Weyl transforms as in [13]. Using the Green functions so constructed, we prove the hypoellipticity of the parametrized hierarchical twisted Laplacians. We then give formulas for the heat kernel and the Green function of the hierarchical sub-Laplacian $\mathcal{L}_m$ on the hierarchical Heisenberg group $\mathbb{H}^n_m$.

In Section 2 we introduce the hierarchical Heisenberg group $\mathbb{H}^n_m$ and then construct on it the hierarchical sub-Laplacian $\mathcal{L}_m$. The hierarchical twisted Laplacians parametrized by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ in $\mathbb{R}^m$ are obtained by taking the inverse Fourier transform of $\mathbb{H}^n_m$ with respect to a subcenter of $\mathbb{H}^n_m$. In Section 3 we give the Fourier–Wigner transforms, the Wigner transforms and the Weyl transforms, which are parametrized by λ in $\mathbb{R}^m$. These, together with the λ-version of the Hermite functions in [13], are then used to diagonalize the parametrized twisted Laplacians in Section 4. In Section 5 twisted convolutions based on the parameter λ are introduced and they are then used to give explicit formulas for the heat kernels of the parametrized hierarchical twisted Laplacians in Section 6. We give in Section 7 the Green functions of the parametrized hierarchical twisted Laplacians using the heat kernels obtained in Section 6. Since the methods are very similar to those developed in [17], we are content with simply writing down the results. The heat kernels and Green functions in, respectively, Sections 6 and 7 are then transferred back to the heat kernel and Green function of the hierarchical sub-Laplacian on the hierarchical Heisenberg group in, respectively, Sections 8 and 9.

We end this section with a note that for a function f in $L^1(\mathbb{R}^N)$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx, \quad \xi \in \mathbb{R}^N.$$

2. The hierarchical Heisenberg group

We consider the set $\mathbb{H}^n_m$ given by

$$\mathbb{H}^n_m = \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^m.$$

Then $\mathbb{H}^n_m$ is the set of points (x, v, t) in $\mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^n$, where

$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \qquad v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^{nm}$$

and

$$t = (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m.$$

Of particular note here is that

$$v_j \in \mathbb{R}^n, \quad j = 1, 2, \ldots, m,$$

and we define $s(v)$ by

$$s(v) = \sum_{j=1}^{m} v_j.$$

If we define the multiplication $\cdot$ on $\mathbb{H}_m^n$ by

$$(x, v, t) \cdot (y, w, s) = \left(x + y, v + w, (t + s) \oplus \left(\frac{1}{2} s(v) \cdot y - \frac{1}{2} s(w) \cdot x \right) \right)$$

for all (x, v, t) and (y, w, s) in $\mathbb{H}_m^n$, where $\oplus : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is the mapping defined by

$$t \oplus \alpha = (t_1 + \alpha, t_2 + \alpha, \ldots, t_m + \alpha)$$

for all $t = (t_1, t_2, \ldots, t_m)$ in $\mathbb{R}^m$ and all α in $\mathbb{R}$, then it can be easily checked that $\mathbb{H}_m^n$ becomes a group with respect to the group law $\cdot$ in which the identity element is $(0, 0, 0)$ and the inverse $(x, v, t)^{-1}$ of every element (x, v, t) in $\mathbb{H}_m^n$ is $(-x, -v, -t)$. We also need $t \ominus \alpha$, which is defined by

$$t \ominus \alpha = (t_1 - \alpha, t_2 - \alpha, \ldots, t_m - \alpha).$$

Note that if we let $m = 1$, then we get back the ordinary Heisenberg group $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. For simplicity, we work on $\mathbb{H}_m^1$ only, i.e., the case when $n = 1$.

We can easily check that the center $Z(\mathbb{H}_m^1)$ of the hierarchical Heisenberg group $\mathbb{H}_m^1$ is given by

$$Z(\mathbb{H}_m^1) = \{(0, w, t) \in \mathbb{H}_m^1 : s(w) = 0\}.$$

Thus, the subgroup $\{(0, 0, t) \in \mathbb{H}_m^1 : t \in \mathbb{R}^m\}$ of the hierarchical Heisenberg group $\mathbb{H}_m^1$ is a *subcenter* of $\mathbb{H}_m^1$.

Let $\mathfrak{h}_m^1$ be the Lie algebra of all left-invariant vector fields on $\mathbb{H}_m^1$. Let $\gamma_1 : \mathbb{R} \to \mathbb{H}_m^1$ be the curve in $\mathbb{H}_m^1$ given by

$$\gamma_1(r) = (r, 0, 0), \quad r \in \mathbb{R}.$$

For $j = 1, 2, \ldots, m$, let $\gamma_{2j} : \mathbb{R} \to \mathbb{H}_m^1$ and $\gamma_{3j} : \mathbb{R} \to \mathbb{H}_m^1$ be curves in $\mathbb{H}_m^1$ given by

$$\gamma_{2j}(r) = (0, re_j, 0) \quad \text{and} \quad \gamma_{3j}(r) = (0, 0, re_j)$$

for all r in $\mathbb{R}$, where e_j is the standard unit vector in $\mathbb{R}^m$ along the j^{th} coordinate axis. Then we define the left-invariant vector fields X, Y_j and T_j, $j = 1, 2, \ldots, m$, on $\mathbb{H}_m^1$ as follows. Let $f \in C^\infty(\mathbb{H}_m^1)$. Then the function Xf is defined by

$$\begin{aligned}
(Xf)(x, v, t) &= \frac{\partial}{\partial r} f((x, v, t) \cdot \gamma_1(r)) \Big|_{r=0} \\
&= \frac{\partial}{\partial r} f \left(x + r, v, t \oplus \frac{1}{2} s(v) r \right) \Big|_{r=0} \\
&= \frac{\partial f}{\partial x}(x, v, t) + \frac{1}{2} s(v) \sum_{k=1}^{m} \frac{\partial f}{\partial t_k}(x, v, t)
\end{aligned}$$

for all $(x,v,t) \in \mathbb{H}_m^1$. For $j = 1, 2, \ldots, m$, we define $Y_j f$ and $T_j f$ by

$$\begin{aligned}(Y_j f)(x,v,t) &= \frac{\partial}{\partial r} f((x,v,t)\cdot\gamma_{2j}(r))\Big|_{r=0} \\ &= \frac{\partial}{\partial r} f\left(x, v + re_j, t \ominus \left(\frac{1}{2}rx\right)\right)\Big|_{r=0} \\ &= \frac{\partial f}{\partial v_j}(x,v,t) - \frac{1}{2}x\sum_{k=1}^{m}\frac{\partial f}{\partial t_k}(x,v,t)\end{aligned}$$

and

$$\begin{aligned}(T_j f)(x,v,t) &= \frac{\partial}{\partial r} f((x,v,t)\cdot\gamma_{3j}(r))\Big|_{r=0} \\ &= \frac{\partial}{\partial r} f(x,v,t+re_j)\Big|_{r=0} \\ &= \frac{\partial f}{\partial t_j}(x,v,t)\end{aligned}$$

for all (x,v,t) in $\mathbb{H}_m^1$. It can be checked by easy computations that

$$[X, Y_j] = T_j, \quad j = 1, 2, \ldots, m,$$

and all other first-order commutators are equal to zero.

Now, if we let Y and T be vector fields on $\mathbb{H}_m^1$ such that

$$Y = \sum_{j=1}^{m} Y_j \quad \text{and} \quad T = \sum_{j=1}^{m} T_j,$$

then we get

$$Y = \sum_{j=1}^{m}\frac{\partial}{\partial v_j} + \frac{m}{2}x\sum_{k=1}^{m}\frac{\partial}{\partial t_k} \quad \text{and} \quad T = \sum_{k=1}^{m}\frac{\partial}{\partial t_k}.$$

It can be checked easily that

$$[X, Y] = -mT.$$

Let $\mathcal{L}_m = -(X^2 + Y^2)$. Then we call $\mathcal{L}_m$ the hierarchical sub-Laplacian on the Heisenberg group $\mathbb{H}_m^1$ and it can be expressed as

$$\begin{aligned}\mathcal{L}_m = &-\left(\frac{\partial^2}{\partial x^2} + \sum_{j,k=1}^{m}\frac{\partial^2}{\partial v_j \partial v_k}\right) - \frac{1}{4}\left(m^2x^2 + s(v)^2\right)\sum_{j,k=1}^{m}\frac{\partial^2}{\partial t_j \partial t_k} \\ &+ \left(s(v)\frac{\partial}{\partial x} - mx\sum_{j=1}^{m}\frac{\partial}{\partial v_j}\right)\left(\sum_{k=1}^{m}\frac{\partial}{\partial t_k}\right).\end{aligned}$$

Since the vector fields X and Y have analytic coefficients and commutators of arbitrary length generated by them do not span the Lie algebra $\mathfrak{h}_m^1$, it follows from a theorem of Hörmander [9] and related results [6, 18] that $\mathcal{L}_m$ is not hypoelliptic on $\mathbb{R}^{2m+1}$ unless $m = 1$. By taking the inverse Fourier transform of the hierarchical sub-Laplacian with respect to $t = (t_1, t_2, \dots, t_m)$, we get the parametrized hierarchical twisted Laplacians L_m^λ, $\lambda \in \mathbb{R}^m$, given by

$$L_m^\lambda = -\left(\frac{\partial^2}{\partial x^2} + \sum_{j,k=1}^{m} \frac{\partial^2}{\partial v_j \partial v_k}\right) + \frac{1}{4}\left(m^2x^2 + s(v)^2\right) s(\lambda)^2 + is(\lambda)\left(s(v)\frac{\partial}{\partial x} - mx\sum_{j=1}^{m}\frac{\partial}{\partial v_j}\right),$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m$. If we let

$$\lambda_j = -\frac{1}{m^2}, \quad j = 1, 2, \dots, m,$$

then

$$s(\lambda) = -\frac{1}{m}.$$

and

$$L_m^1 = -\left(\frac{\partial^2}{\partial x^2} + \sum_{j,k=1}^{m} \frac{\partial^2}{\partial v_j \partial v_k}\right) + \frac{1}{4}\left(x^2 + \frac{s(v)^2}{m^2}\right) + i\left(\frac{s(v)}{m}\frac{\partial}{\partial x} - x\sum_{j=1}^{m}\frac{\partial}{\partial v_j}\right),$$

which is the ordinary hierarchical twisted Laplacian described earlier.

It can be seen easily that L_m^λ is not elliptic unless $m = 1$. Thus, $\mathcal{L}_m$ is not hypoelliptic unless $m = 1$; and L_m^λ is not elliptic unless $m = 1$. This is another *raison d' être* of the hierarchy.

3. Hierarchical Wigner transforms and hierarchical Weyl transforms

The most basic Wigner transforms and Weyl transforms in the books [14, 15] need to be modified for the analysis of the hierarchical twisted Laplacian. To this end, let $f \in \mathcal{S}(\mathbb{R}^m)$. Then for all $v = (v_1, v_2, \dots, v_m) \in \mathbb{R}^m$ and $w \in \mathbb{R}$, we define the function $\rho(w, v)f$ on $\mathbb{R}$ by

$$(\rho(w, v)f)(x) = e^{iwx + \frac{1}{2m}iws(v)} f(x \oplus v), \quad x \in \mathbb{R}.$$

Now, we can define the hierarchical Fourier–Wigner transform $V(f, g)$ of f in $\mathcal{S}(\mathbb{R}^m)$ and g in $\mathcal{S}(\mathbb{R})$ by

$$V(f, g)(w, v) = (2\pi)^{-m/2}(\rho(w, v)f, g)_{L^2(\mathbb{R})}$$

for all $v = (v_1, v_2, \dots, v_m)$ in $\mathbb{R}^m$ and w in $\mathbb{R}$. The Wigner transform $W(f, g)$ of f in $\mathcal{S}(\mathbb{R}^m)$ and g in $\mathcal{S}(\mathbb{R})$ is defined by

$$W(f, g) = V(f, g)^\wedge,$$

where $V(f, g)^\wedge$, also denoted by $\mathcal{F}V(f, g)$, is the Fourier transform of $V(f, g)$.

Let $\sigma \in \mathcal{S}(\mathbb{R}^{m+1})$. Then for all f in $\mathcal{S}(\mathbb{R}^m)$, we define the hierarchical Weyl transform $W_\sigma f$ of f corresponding to the symbol σ to be the function on $\mathbb{R}$ by

$$(W_\sigma f, g)_{L^2(\mathbb{R})} = (2\pi)^{-m/2} \int_{\mathbb{R}^m} \int_{\mathbb{R}} \sigma(x,\xi) W(f,g)(x,\xi)\, dx\, d\xi$$

for all g in $\mathcal{S}(\mathbb{R})$.

The following theorem gives a sufficient condition on the symbol σ to guarantee that a hierarchical Weyl transform is a Hilbert–Schmidt operator.

Theorem 3.1. *Let $\sigma \in L^2(\mathbb{R}^{m+1})$. Then $W_\sigma : L^2(\mathbb{R}^m) \to L^2(\mathbb{R})$ is a Hilbert–Schmidt operator and*

$$\|W_\sigma\|_{HS} = (2\pi)^{-m/2} \|\sigma\|_{L^2(\mathbb{R}^{m+1})},$$

where $\|W_\sigma\|_{HS}$ denotes the Hilbert–Schmidt norm of W_σ.

The results hitherto recapitulated can be found in [1] and [13].

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$ with $s(\lambda) \neq 0$, we define the linear partial differential operators Z_λ and $\overline{Z_\lambda}$ by

$$Z_\lambda = \frac{\partial}{\partial x} - i \sum_{l=1}^{m} \frac{\partial}{\partial v_l} - \frac{i}{2} s(\lambda)\, s(v) + \frac{m}{2} s(\lambda) x$$

and

$$\overline{Z_\lambda} = \frac{\partial}{\partial x} + i \sum_{l=1}^{m} \frac{\partial}{\partial v_l} - \frac{i}{2} s(\lambda)\, s(v) - \frac{m}{2} s(\lambda) x.$$

Then it can be checked easily that

$$L_m^\lambda = -\frac{1}{2}(Z_\lambda \overline{Z_\lambda} + \overline{Z_\lambda} Z_\lambda).$$

Now, we define $\rho^\lambda(w,v)$ for all w in $\mathbb{R}$ and v in $\mathbb{R}^m$ by

$$\rho^\lambda(w,v) = |s(\lambda)|^{1/2} \rho(s(\lambda)w, v).$$

For all f in $\mathcal{S}(\mathbb{R}^m)$ and g in $\mathcal{S}(\mathbb{R})$, the λ-hierarchical Fourier–Wigner transform $V^\lambda(f,g)$ of f and g is defined by

$$V^\lambda(f,g)(w,v) = (2\pi)^{-m/2} (\rho^\lambda(w,v) f, g)_{L^2(\mathbb{R})}.$$

In fact,

$$\begin{aligned} V^\lambda(f,g)(w,v) &= \sqrt{|s(\lambda)|}\, V(f,g)(s(\lambda)w, v) \\ &= \sqrt{|s(\lambda)|}(2\pi)^{-m/2} \int_{\mathbb{R}} e^{is(\lambda)yw} f\left(\left(y - \frac{1}{2m}s(v)\right) \oplus v\right) \\ &\quad \times \overline{g\left(y - \frac{1}{2m}s(v)\right)}\, dy. \end{aligned} \tag{3.1}$$

The λ-Wigner transform $W^\lambda(f,g)$ of f in $\mathcal{S}(\mathbb{R}^m)$ and g in $\mathcal{S}(\mathbb{R})$ is defined by

$$W^\lambda(f,g) = V^\lambda(f,g)^\wedge.$$

Then for $s(\lambda) \neq 0$,

$$W^\lambda(f,g)(x,\xi) = \sqrt{|s(\lambda)|}W(f,g)\left(\frac{x}{s(\lambda)},\xi\right), \quad x \in \mathbb{R},\, \xi \in \mathbb{R}^m. \tag{3.2}$$

The following theorem gives the Moyal identity for λ-hierarchical Wigner transforms.

Theorem 3.2. *For all f_1 and f_2 in $L^2(\mathbb{R}^m)$, and all g_1 and g_2 in $L^2(\mathbb{R})$,*

$$(W^\lambda(f_1,g_1),W^\lambda(f_2,g_2))_{L^2(\mathbb{R}^{m+1})} = (f_1,f_2)_{L^2(\mathbb{R}^m)}\overline{(g_1,g_2)_{L^2(\mathbb{R})}}.$$

Remark 3.3. Similarly, we have the Moyal identity for the λ-hierarchical Fourier–Wigner transform.

Using the λ-hierarchical Wigner transform, the λ-hierarchical Weyl transform $W_\sigma^\lambda f$ of a function f in $\mathcal{S}(\mathbb{R}^m)$ corresponding to the symbol σ in $\mathcal{S}(\mathbb{R}^{m+1})$ is defined to be the function on $\mathbb{R}$ by

$$(W_\sigma^\lambda f,g)_{L^2(\mathbb{R})} = (2\pi)^{-m/2}\int_{\mathbb{R}^m}\int_{\mathbb{R}} \sigma(x,\xi)W^\lambda(f,g)(x,\xi)\,dx\,d\xi$$

for all functions g in $\mathcal{S}(\mathbb{R})$. Using (3.2) and a change of variables, we get the following relationship between the ordinary hierarchical Weyl transform and the λ-hierarchical Weyl transform.

Proposition 3.4. *Let $\sigma \in \mathcal{S}(\mathbb{R}^{m+1})$. Then*

$$W_\sigma^\lambda = \sqrt{|s(\lambda)|}W_{\sigma_\lambda},$$

where σ_λ is given by

$$\sigma_\lambda(x,\xi) = \sigma(s(\lambda)x,\xi), \quad x \in \mathbb{R},\, \xi \in \mathbb{R}^m.$$

Using Proposition 3.4, we have the following result, which is an analog of Theorem 3.1 in [13], on the L^2-boundedness of λ-Weyl transforms with symbols in $L^2(\mathbb{R}^{m+1})$.

Proposition 3.5. *Let $\sigma \in L^2(\mathbb{R}^{m+1})$. Then $W_\sigma^\lambda : L^2(\mathbb{R}^m) \to L^2(\mathbb{R})$ is a bounded linear operator and*

$$\|W_\sigma^\lambda\|_{B(L^2(\mathbb{R}^m),L^2(\mathbb{R}))} \leq (2\pi)^{-m/2}\|\sigma\|_{L^2(\mathbb{R}^{m+1})},$$

where $\| \, \|_{B(L^2(\mathbb{R}^m),L^2(\mathbb{R}))}$ is the norm in the Banach algebra of all bounded linear operators from $L^2(\mathbb{R}^m)$ into $L^2(\mathbb{R})$.

Using Theorem 3.1 and Proposition 3.4, we have the following result on the Hilbert–Schmidt property of λ-hierarchical Weyl transforms.

Theorem 3.6. *Let $\sigma \in L^2(\mathbb{R}^{m+1})$. Then $W_\sigma^\lambda : L^2(\mathbb{R}^m) \to L^2(\mathbb{R})$ is a Hilbert–Schmidt operator and*

$$\|W_\sigma^\lambda\|_{HS} = (2\pi)^{-m/2}\|\sigma\|_{L^2(\mathbb{R}^{m+1})}.$$

4. Spectral analysis of λ-hierarchical twisted Laplacians

Let $\lambda \in \mathbb{R}^m$ be such that $s(\lambda) \neq 0$. Then for $k = 0, 1, 2, \ldots$, we define e_k^λ to be the function on $\mathbb{R}$ by

$$e_k^\lambda(x) = |s(\lambda)|^{1/4} e_k\left(\sqrt{|s(\lambda)|}x\right), \quad x \in \mathbb{R},$$

where e_k is the Hermite function of order k on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x)$$

for all x in $\mathbb{R}$ and H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k (e^{-x^2})$$

for all x in $\mathbb{R}$. Now, for all nonnegative integers $j_1, j_2, \ldots, j_m$ and k, we define the function $e^\lambda_{j_1,\ldots,j_m,k}$ on $\mathbb{R}^{m+1}$ by

$$e^\lambda_{j_1,\ldots,j_m,k} = V^\lambda \left(\otimes_{l=1}^m e^\lambda_{j_l}, e^\lambda_k\right).$$

It should be noted that $e^\lambda_{j_1,\ldots,j_m,k}$ is the *parametrized* version of the function $e_{j_1,\ldots,j_m,k}$ defined in [13] by

$$e_{j_1,\ldots,j_m,k} = V\left(\otimes_{l=1}^m e_{j_l}, e_k\right).$$

The following lemma is the precise *manifesto* of the connection.

Lemma 4.1. *For all λ in $\mathbb{R}^m$ with $s(\lambda) \neq 0$, and $j_1, j_2, \ldots, j_m, k = 0, 1, 2, \ldots$,*

$$e^\lambda_{j_1,\ldots,j_m,k}(w, v) = |s(\lambda)|^{(m+1)/4} e_{j_1,\ldots,j_m,k}\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}} w, \sqrt{|s(\lambda)|} v\right)$$

for all w in $\mathbb{R}$ and v in $\mathbb{R}^m$.

Proof. By (3.1) and a change of variables,

$$\begin{aligned} e^\lambda_{j_1,\ldots,j_m,k}(w, v) &= |s(\lambda)|^{1/2} V\left(\otimes_{l=1}^m e^\lambda_{j_l}, e^\lambda_k\right)(s(\lambda)w, v) \\ &= |s(\lambda)|^{1/2} (2\pi)^{-m/2} \int_{\mathbb{R}} e^{is(\lambda)yw} (\otimes_{l=1}^m e^\lambda_{j_l}) \left(\left(y - \frac{1}{2m}s(v)\right) \oplus v\right) \\ &\quad \times \overline{e^\lambda_k\left(y - \frac{1}{2m}s(v)\right)} dy \\ &= |s(\lambda)|^{(m+3)/4} (2\pi)^{-m/2} \int_{\mathbb{R}} e^{is(\lambda)yw} \\ &\quad \times (\otimes_{l=1}^m e_{j_l}) \left(\sqrt{|s(\lambda)|}\left\{\left(y - \frac{1}{2m}(v)\right) \oplus v\right\}\right) \\ &\quad \times \overline{e_k\left(\sqrt{|s(\lambda)|}\left\{y - \frac{1}{2m}s(v)\right\}\right)} dy \end{aligned}$$

$$= |s(\lambda)|^{(m+1)/4}(2\pi)^{-m/2}\int_{\mathbb{R}} e^{is(\lambda)yw/\sqrt{|s(\lambda)|}}$$

$$\times (\otimes_{l=1}^m e_{j_l})\left(\left(y-\frac{1}{2m}\sqrt{|s(\lambda)|}s(v)\right)\oplus\sqrt{|s(\lambda)|}v\right)$$

$$\times \overline{e_k\left(y-\frac{1}{2m}\sqrt{|s(\lambda)|}s(v)\right)}dy$$

$$= |s(\lambda)|^{(m+1)/4}e_{j_1,\ldots,j_m,k}\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}w,\sqrt{|s(\lambda)|}v\right)$$

for all λ in $\mathbb{R}^m$ with $s(\lambda)\neq 0$, and $j_1, j_2, \ldots, j_m, k = 0, 1, 2, \ldots$. □

Corollary 4.2. $\{e^\lambda_{j_1,\ldots,j_m,k} : j_1,\ldots,j_m,k = 0,1,2,\ldots\}$ *forms an orthonormal basis for* $L^2(\mathbb{R}^{m+1})$.

Corollary 4.2 follows from Lemma 4.1 and Proposition 4.1 in [13] to the effect that $\{e_{j_1,\ldots,j_m,k} : j_1,\ldots,j_k,k = 0,1,2,\ldots\}$ forms an orthonormal basis for $L^2(\mathbb{R}^{m+1})$. It can also be proved as in [17] using the Hilbert–Schmidt property of λ-hierarchical Weyl transforms in Theorem 3.6 and the Moyal identity for λ-hierarchical Fourier–Wigner transforms in Remark 3.3.

The following proposition shows that Z_λ and $\overline{Z_\lambda}$ are, respectively, annihilation and creation operators.

Proposition 4.3. *For all nonnegative integers* $j_1,\ldots,j_m$, *and positive integers* k,

$$Z_\lambda e^\lambda_{j_1,\ldots,j_m,k} = i\sqrt{|s(\lambda)|}(2k)^{1/2}e_{j_1,\ldots,j_m,k-1},$$

and for all nonnegative integers $j_1,\ldots,j_m,k$,

$$\overline{Z_\lambda}e_{j_1,\ldots,j_m,k} = i\sqrt{|s(\lambda)|}(2k+2)^{1/2}e_{j_1,\ldots,j_m,k+1}.$$

The proof of Proposition 4.3 follows from Lemma 4.1 and Proposition 4.2 in [13]. Using Proposition 4.3, we can easily obtain the following spectral decomposition of the λ-hierarchical twisted Laplacian.

Theorem 4.4. *For all nonnegative integers* $j_1,\ldots,j_m$ *and* k,

$$L^\lambda_m e^\lambda_{j_1,\ldots,j_m,k} = (2k+1)|s(\lambda)|e^\lambda_{j_1,\ldots,j_m,k}.$$

5. Twisted convolutions

For $z = (q,p)\in\mathbb{R}^2$ and $\zeta = (w,v)\in\mathbb{R}^{m+1}$, where $w\in\mathbb{R}$ and

$$v = (v_1, v_2, \ldots, v_m)\in\mathbb{R}^m,$$

we define $\zeta \boxplus z$ and $\zeta \boxminus z$ by

$$\zeta\boxplus z = (w+q, v_1+p,\ldots,v_m+p) \quad\text{and}\quad \zeta\boxminus z = (w-q, v_1-p,\ldots,v_m-p).$$

Now, we define the twisted convolution $\tau *_\lambda \sigma$ of τ in $L^2(\mathbb{R}^2)$ and σ in $L^2(\mathbb{R}^{m+1})$ by

$$(\tau *_\lambda \sigma)(\zeta) = \int_{\mathbb{C}} \sigma(\zeta \boxminus z)\tau(z)e^{-is(\lambda)[\zeta,z]}dz, \quad \zeta \in \mathbb{R}^{m+1},$$

where $[\zeta, z]$ is the symplectic form of ζ and z given by

$$[\zeta, z] = \frac{1}{2}wp - \frac{1}{2m}qs(v).$$

We also need the convolution $\tau *_- \sigma$ defined by

$$(\tau *_- \sigma)(\zeta) = \int_{\mathbb{C}} \sigma(\zeta \boxminus z)\tau(z)e^{-i[\zeta,z]}dz, \quad \zeta \in \mathbb{R}^{m+1}.$$

Theorem 5.1. *Let $\lambda \in \mathbb{R}^m$ be such that $s(\lambda) \neq 0$. Then for all nonnegative integers α, β, l, and $j_1, \ldots, j_m$,*

$$e^\lambda_{\alpha,\beta} *_\lambda e^\lambda_{j_1,\ldots,j_m,l} = (2\pi)^{1/2}|s(\lambda)|^{-1/2}\delta_{\alpha,l}e^\lambda_{j_1,\ldots,j_m,\beta}.$$

Proof. It is easy to see that

$$s(\lambda)[(w,v),(q,p)] = \left[\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}w, \sqrt{|s(\lambda)|}v\right), \left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}, \sqrt{|s(\lambda)|}p\right)\right]$$

for all $z = (q,p) \in \mathbb{R}^2$ and $\zeta = (w,v) \in \mathbb{R}^{m+1}$, where $w \in \mathbb{R}$ and

$$v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m.$$

So, by Lemma 4.1,

$$\begin{aligned}
&(e^\lambda_{\alpha,\beta} *_\lambda e^\lambda_{j_1,\ldots,j_m,l})(w,v)\\
&= \int_{\mathbb{C}} e^\lambda_{j_1,\ldots,j_m,l}(\zeta \boxminus z)e^\lambda_{\alpha,\beta}(z)e^{-is(\lambda)[\zeta,z]}dz\\
&= |s(\lambda)|^{(m+3)/4}\\
&\quad\times \int_{\mathbb{C}} e_{j_1,\ldots,j_m,l}\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}(w-q), \sqrt{|s(\lambda)|}(v_1 - p, \ldots, v_m - p)\right)\\
&\quad\times e_{\alpha,\beta}\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}q, \sqrt{|s(\lambda)|}p\right) e^{-i\left[\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}w, \sqrt{|s(\lambda)|}v\right), \left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}q, \sqrt{|s(\lambda)|}p\right)\right]}dz\\
&= |s(\lambda)|^{(m-1)/4}\\
&\quad\times \int_{\mathbb{C}} e_{j_1,\ldots,j_m,l}\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}w - q, \sqrt{|s(\lambda)|}v_1 - p, \ldots, \sqrt{|s(\lambda)|}v_m - p\right)\\
&\quad\times e_{\alpha,\beta}(q,p)e^{-i\left[\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}w, \sqrt{|s(\lambda)|}v\right), (q,p)\right]}dz
\end{aligned}$$

$$= |s(\lambda)|^{(m-1)/4}(e_{\alpha,\beta} *_- e_{j_1,\ldots,j_m,l})\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}w, \sqrt{|s(\lambda)|}v\right)$$

$$= (2\pi)^{1/2}|s(\lambda)|^{(m-1)/4}\delta_{\alpha,l}e_{j_1,\ldots,j_m,\beta}\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}}w, \sqrt{|s(\lambda)|}v\right) \tag{5.1}$$

for all w in $\mathbb{R}$ and

$$v = (v_1, v_2, \ldots, v_m)$$

in $\mathbb{R}^m$. Let us note that (5.1) is derived using Theorem 5.5 in [13]. Using Lemma 4.1 again, we get

$$e^\lambda_{\alpha,\beta} *_\lambda e_{j_1,\ldots,j_m,l}(w, v) = (2\pi)^{1/2}|s(\lambda)|^{-1/2}\delta_{\alpha,l}e^\lambda_{j_1,\ldots,j_m,\beta}(w, v)$$

for all w in $\mathbb{R}$ and

$$v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m,$$

as asserted. □

6. Heat kernels for λ-hierarchical twisted Laplacians

Theorem 6.1. *Let $\lambda \in \mathbb{R}^m$ be such that $s(\lambda) \neq 0$. Then for all f in $L^2(\mathbb{R}^{m+1})$ and $\rho > 0$,*

$$e^{-\rho L^\lambda_m} f = k^\lambda_\rho *_\lambda f,$$

where

$$k^\lambda_\rho(z) = \frac{1}{4\pi}\frac{|s(\lambda)|}{\sinh(|s(\lambda)|\rho)}e^{-\frac{1}{4}|s(\lambda)|\,|z|^2\coth(|s(\lambda)|\rho)}$$

for all z in $\mathbb{C}$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^{m+1})$. Then for $\rho > 0$, we get by means of Corollary 4.2 and Theorem 4.4,

$$e^{-\rho L^\lambda_m} f = \sum_{k=0}^\infty \sum_{j_1,\ldots,j_m=0}^\infty e^{-|s(\lambda)|(2k+1)\rho}(f, e^\lambda_{j_1,\ldots,j_m,k})_{L^2(\mathbb{R}^{m+1})}e^\lambda_{j_1,\ldots,j_m,k}.$$

By Theorem 5.1,

$$\begin{aligned}
e^\lambda_{k,k} *_\lambda f &= e^\lambda_{k,k} *_\lambda \sum_{l=0}^\infty \sum_{j_1,\ldots,j_m=0}^\infty (f, e^\lambda_{j_1,\ldots,j_m,l})_{L^2(\mathbb{R}^{m+1})}e^\lambda_{j_1,\ldots,j_m,l}\\
&= \sum_{l=0}^\infty \sum_{j_1,\ldots,j_m=0}^\infty (f, e^\lambda_{j_1,\ldots,j_m,l})_{L^2(\mathbb{R}^{m+1})}e^\lambda_{k,k} *_\lambda e^\lambda_{j_1,\ldots,j_m,l}\\
&= \sum_{l=0}^\infty \sum_{j_1,\ldots,j_m=0}^\infty (f, e^\lambda_{j_1,\ldots,j_m,l})_{L^2(\mathbb{R}^{m+1})}|s(\lambda)|^{-1/2}(2\pi)^{1/2}\delta_{k,l}e^\lambda_{j_1,\ldots,j_m,k}\\
&= |s(\lambda)|^{-1/2}(2\pi)^{1/2} \sum_{j_1,\ldots,j_m=0}^\infty (f, e^\lambda_{j_1,\ldots,j_m,k})_{L^2(\mathbb{R}^{m+1})}e^\lambda_{j_1,\ldots,j_m,k}.
\end{aligned}$$

Thus,

$$e^{-\rho L_m^\lambda} f = |s(\lambda)|^{1/2}(2\pi)^{-1/2} \sum_{k=0}^{\infty} e^{-|s(\lambda)|(2k+1)\rho} e_{k,k}^\lambda *_\lambda f.$$

Now, for $0 < r < 1$ and $z \in \mathbb{C}$,

$$\sum_{k=0}^{\infty} e_{k,k}(z) r^k = (2\pi)^{-1/2} \frac{1}{1-r} e^{-|z|^2 \frac{1}{4} \frac{1+r}{1-r}}.$$

By Lemma 4.1, for all λ with $s(\lambda) \neq 0$,

$$e_{k,k}^\lambda(q,p) = |s(\lambda)|^{1/2} e_{k,k}\left(\frac{s(\lambda)}{\sqrt{|s(\lambda)|}} q, \sqrt{|s(\lambda)|} p\right) \tag{6.1}$$

for all $z = q + ip$ in $\mathbb{C}$. Thus, by Lemma 4.1 and (6.1), we get

$$\sum_{k=0}^{\infty} e^{-|s(\lambda)|(2k+1)\rho} e_{k,k}^\lambda(z) = (2\pi)^{-1/2} \frac{|s(\lambda)|^{-1/2}}{2\sinh(|s(\lambda)|\rho)} e^{-\frac{1}{4}|s(\lambda)|\,|z|^2 \coth(|s(\lambda)|\rho)}$$

for all z in $\mathbb{C}$ and $\rho > 0$. Therefore

$$e^{-\rho L_m^\lambda} f = k_\rho^\lambda *_\lambda f, \quad \rho > 0, \tag{6.2}$$

as claimed. □

7. Green functions for λ-hierarchical twisted Laplacians

For $\lambda \in \mathbb{R}^m$ with $s(\lambda) \neq 0$, we let G_m^λ be the function on $\mathbb{R}^2$ such that

$$(L_m^\lambda)^{-1} f = G_m^\lambda *_\lambda f$$

for all f in $L^2(\mathbb{R}^{m+1})$.

Using the heat kernels obtained in the preceding section, we can integrate with respect to time ρ from 0 to ∞ as in [17] to find the Green functions of the λ-hierarchical twisted Laplacians. The result is encapsulated in the following theorem.

Theorem 7.1. *Let $\lambda \in \mathbb{R}^m$ be such that $s(\lambda) \neq 0$. Then*

$$G^\lambda(z) = \frac{1}{4\pi} K_0\left(\frac{1}{4}|s(\lambda)|\,|z|^2\right)$$

for all z in $\mathbb{C}$, where K_0 is the modified Bessel function of order 0 given by

$$K_0(x) = \int_0^\infty e^{-x\cosh t} dt, \quad x > 0.$$

8. The heat kernel for the sub-Laplacian

In order to get a heat kernel for the hierarchical sub-Laplacian $\mathcal{L}_m$ on the hierarchical Heisenberg group $\mathbb{H}^1_m$, we start with the initial value problem for the heat equation for $\mathcal{L}_m$ given by

$$\begin{cases} \dfrac{\partial u}{\partial \rho}(\zeta, t, \rho) = -(\mathcal{L}_m u)(\zeta, t, \rho), \\ u(\zeta, t, 0) = f(\zeta, t), \end{cases} \tag{8.1}$$

for all (ζ, t) in $\mathbb{H}^1_m$ and $\rho > 0$. Taking the inverse Fourier transform of (8.1) with respect to t, we get the initial value problem for the heat equation for the λ-hierarchical twisted Laplacian, i.e.,

$$\begin{cases} \dfrac{\partial u^\lambda}{\partial \rho}(\zeta, \rho) = -(L^\lambda_m u^\lambda)(\zeta, \rho), \\ u^\lambda(\zeta, 0) = f^\lambda(\zeta), \end{cases}$$

for all ζ in $\mathbb{R}^{m+1}$, $\rho > 0$, and all λ in $\mathbb{R}^m$ with $s(\lambda) \neq 0$. By Theorem 6.1,

$$u^\lambda(\zeta, \rho) = \int_{\mathbb{C}} f^\lambda(\zeta \boxminus z) k^\lambda_\rho(z) e^{-is(\lambda)[\zeta, z]} dz, \quad \zeta \in \mathbb{R}^{m+1}, \tag{8.2}$$

where $\lambda \in \mathbb{R}^m$ with $s(\lambda) \neq 0$. So, by taking the Fourier transform of the solution u^λ of (8.2) with respect to λ, we get

$$\begin{aligned} u(\zeta, t, \rho) &= \int_{\mathbb{C}} \left\{ (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i\lambda\cdot t} f^\lambda(\zeta \boxminus z) k^\lambda_\rho(z) e^{-is(\lambda)[\zeta, z]} d\lambda \right\} dz \\ &= (2\pi)^{-m/2} \int_{\mathbb{C}} \left(f(\zeta \boxminus z, \cdot) * K_\rho(\zeta, z, \cdot) \right)(t) dz, \end{aligned}$$

where

$$(\mathcal{F}_2^{-1} K_\rho)(\zeta, z, \lambda) = k^\lambda_\rho(z) e^{-is(\lambda)[\zeta, z]}, \quad z \in \mathbb{C},$$

and $(\mathcal{F}_2^{-1} K_\rho)(\zeta, z, \lambda)$ is the inverse Fourier transform of $K_\rho(\zeta, z, t)$ with respect to t in $\mathbb{R}^m$. Thus,

$$u(\zeta, t, \rho) = (2\pi)^{-m/2} \int_{\mathbb{C}} \int_{\mathbb{R}^m} f(\zeta \boxminus z, t - s) K_\rho(\zeta, z, s)\, ds\, dz, \tag{8.3}$$

where

$$K_\rho(\zeta, z, s) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-is\cdot\lambda} k^\lambda_\rho(z) e^{-is(\lambda)[\zeta, z]} d\lambda$$

for all ζ in $\mathbb{R}^{m+1}$, z in $\mathbb{C}$ and s in $\mathbb{R}^m$.

We call the function K_ρ, $\rho > 0$, the *heat kernel* of the sub-Laplacian $\mathcal{L}_m$. In glorious detail,

$$K_\rho(\zeta, z, s) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-is\cdot\lambda} \frac{1}{4\pi} \frac{|s(\lambda)|}{\sinh(|s(\lambda)|\rho)} e^{-\frac{1}{4}|s(\lambda)||z|^2 \coth(|s(\lambda)|\rho)} e^{-is(\lambda)[\zeta, z]} d\lambda. \tag{8.4}$$

9. The Green function for the sub-Laplacian

The Green function $\mathcal{G}_m$ of $\mathcal{L}_m$ is the kernel of the integral operator representing $\mathcal{L}_m^{-1}$, i.e.,

$$(\mathcal{L}_m^{-1} f)(\zeta, t) = (2\pi)^{-m/2} \int_{\mathbb{C}} \int_{\mathbb{R}^m} f(\zeta \boxminus z, t - s) \mathcal{G}_m(\zeta, z, s)\, ds\, dz$$

for all (ζ, t) in $\mathbb{H}_m^1$ and all suitable functions on $\mathbb{H}_m^1$, and can be obtained by integrating the heat kernel of $\mathcal{L}_m$ with respect to time ρ from 0 to ∞. Before formulating the result, we give the following lemma.

Lemma 9.1. *For all z in $\mathbb{C}$ and t in $\mathbb{R}$,*

$$\int_0^\infty \int_{-\infty}^\infty e^{-it\tau} e^{-\frac{1}{4}|\tau|\,|z|^2 \cosh \delta} d\tau\, d\delta = \frac{4\pi}{\sqrt{|z|^4 + 16t^2}}.$$

Proof. For all z in $\mathbb{C}$ and t in $\mathbb{R}$,

$$\int_{-\infty}^\infty e^{-it\tau} e^{-\frac{1}{4}|\tau|\,|z|^2 \cosh \delta} d\tau = \int_{-\infty}^0 e^{-it\tau} e^{\frac{1}{4}\tau |z|^2 \cosh \delta} d\tau + \int_0^\infty e^{-it\tau} e^{-\frac{1}{4}\tau |z|^2 \cosh \delta} d\tau$$
$$= \left. \frac{e^{\tau\left(\frac{1}{4}|z|^2 \cosh \delta - it\right)}}{\frac{1}{4}|z|^2 \cosh \delta - it} \right|_{-\infty}^0 + \left. \frac{e^{-\tau\left(\frac{1}{4}|z|^2 \cosh \delta + it\right)}}{\frac{1}{4}|z|^2 \cosh \delta + it} \right|_0^\infty = \frac{\frac{1}{2}|z|^2 \cosh \delta}{(|z|^4/16)\cosh^2 \delta + t^2}.$$

So, for all z in $\mathbb{C}$ and t in $\mathbb{R}$,

$$\int_0^\infty \int_{-\infty}^\infty e^{-it\tau} e^{-\frac{1}{4}|\tau|\,|z|^2 \cosh \delta} d\tau\, d\delta = \int_0^\infty \frac{\frac{1}{2}|z|^2 \cosh \delta}{(|z|^4/16)\cosh^2 \delta + t^2} d\delta$$
$$= \frac{|z|^2}{2} \int_0^\infty \frac{\cosh \delta}{(|z|^4/16)\cosh^2 \delta + t^2} d\delta = \frac{8}{|z|^2} \int_0^\infty \frac{\cosh \delta}{\cosh^2 \delta + (16t^2/|z|^4)} d\delta$$
$$= \frac{8}{|z|^2} \int_0^\infty \frac{1}{\rho^2 + 1 + (16t^2/|z|^4)} d\rho = \frac{4\pi}{|z|^2} \frac{1}{\sqrt{1 + (16t^2/|z|^4)}} = \frac{4\pi}{\sqrt{|z|^4 + 16t^2}}. \qquad \square$$

Theorem 9.2. *For all ζ in $\mathbb{R}^{m+1}$, z in $\mathbb{C}$ and $s = (s_1, s_2, \ldots, s_m)$ in $\mathbb{R}^m$,*

$$\mathcal{G}_m(\zeta, z, s) = 2(2\pi)^{-(m-2)/2} \delta(s_1 - s_2, \ldots, s_{m-1} - s_m) \frac{1}{\sqrt{|z|^4 + 16(s_m + [\zeta, z])^2}},$$

where δ is the Dirac delta on $\mathbb{R}^{m-1}$.

Proof. First, we note that

$$\int_0^\infty \frac{1}{4\pi} \frac{|s(\lambda)|}{\sinh(|s(\lambda)|\rho)} e^{-\frac{1}{4}|s(\lambda)|\,|z|^2 \coth(|s(\lambda)|\rho)} d\rho = \int_0^\infty \frac{1}{(v^2-1)^{1/2}} e^{-\frac{1}{4}|s(\lambda)|\,|z|^2 v} dv$$
$$= K_0\left(\frac{1}{4}|s(\lambda)|\,|z|^2\right) = \int_0^\infty e^{-\frac{1}{4}|s(\lambda)|\,|z|^2 \cosh \delta} d\delta. \qquad (9.1)$$

So, by Lemma 9.1, (8.4) and (9.1), we get for all ζ in $\mathbb{R}^{m+1}$, z in $\mathbb{C}$ and s in $\mathbb{R}^m$,

$$\int_0^\infty K_\rho(\zeta, z, s)\, d\rho = \int_0^\infty (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i\lambda\cdot(s \oplus [\zeta, z])} e^{-\frac{1}{4}|s(\lambda)|\,|z|^2 \cosh \delta} d\lambda\, d\delta.$$

Let

$$\begin{aligned} \lambda_1+\lambda_2+\cdots+\lambda_m &= \tau, \\ &\vdots \\ \lambda_1+\lambda_2 &= \mu_2, \\ \lambda_1 &= \mu_1. \end{aligned}$$

Then

$$\begin{aligned} &\int_0^\infty (2\pi)^{-m/2}\int_{\mathbb{R}^m} e^{-i\lambda\cdot(s\oplus[\zeta,z])}e^{-\frac{1}{4}|s(\lambda)|\,|z|^2\cosh\delta}d\lambda\,d\delta \\ &= \int_0^\infty (2\pi)^{-m/2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^{m-1}} \\ &\quad\times e^{-i(\mu_1(s_1+[\zeta,z]+(\mu_2-\mu_1)(s_2+[\zeta,z])+\cdots+(\tau-\mu_{m-1}(s_{m-1}+[\zeta,z]))} \\ &\quad\times d\mu_1\cdots d\mu_{m-1}d\tau e^{-i\tau s_m}e^{-\frac{1}{4}|\tau|\,|z|^2\cosh\delta}d\delta \\ &= \int_0^\infty (2\pi)^{(m-2)/2}\delta(s_1-s_2,\ldots,s_{m-1}-s_m) \\ &\quad\times\int_{-\infty}^\infty e^{-i\tau(s_m+[\zeta,z])}e^{-\frac{1}{4}|\tau|\,|z|^2\cosh\delta}d\tau\,d\delta \\ &= 2(2\pi)^{-(m-2)/2}\delta(s_1-s_2,\ldots,s_{m-1}-s_m)\frac{1}{\sqrt{|z|^4+16(s_m+[\zeta,z])^2}}, \end{aligned}$$

as required. □

So, by (8.3) and Theorem 9.2, for all suitable functions f on $\mathbb{H}^1_m$,

$$(\mathcal{L}_m^{-1}f)(\zeta,t) = 2(2\pi)^{-(m-2)/2}\int_{\mathbb{C}}\int_{\mathbb{R}^m}\frac{f(\zeta\boxminus z,t-s)\delta(s_1-s_2,\ldots,s_{m-1}-s_m)}{\sqrt{|z|^4+16(s_m+[\zeta,z])^2}}ds\,dz.$$

Now, let

$$\begin{aligned} \sigma_1 &= s_1-s_2, \\ \sigma_2 &= s_2-s_3, \\ &\vdots \\ \sigma_m &= s_{m-1}-s_m. \end{aligned}$$

Then

$$\begin{aligned} (\mathcal{L}_m^{-1}f)(\zeta,t) &= 2(2\pi)^{-(m-2)/2}\int_{\mathbb{C}}\int_{-\infty}^\infty \\ &\quad\times\int_{\mathbb{R}^{m-1}} f\left(\zeta\boxminus z,t_1-\sum_{j=1}^m\sigma_j-s_m,t_2-\sum_{j=2}^m\sigma_j-s_m,\ldots,t_{m-1}-s_m\right) \\ &\quad\times\delta(\sigma_1,\sigma_2,\ldots,\sigma_{m-1})d\sigma_1\cdots d\sigma_{m-1} \\ &\quad\times\frac{1}{\sqrt{|z|^4+16(s_m+[\zeta,z])^2}}ds_m\,dz \end{aligned}$$

$$
\begin{aligned}
&= 2(2\pi)^{-(m-2)/2} \int_{\mathbb{C}} \int_{-\infty}^{\infty} \frac{f(\zeta \boxminus z, t_1 - s_m, \ldots, t_m - s_m)}{\sqrt{|z|^4 + (s_m + [\zeta, z])^2}} ds_m\, dz \\
&= 2(2\pi)^{-(m-2)/2} \int_{\mathbb{C}} \int_{-\infty}^{\infty} f(\zeta \boxminus z, t \ominus s_m) \frac{1}{\sqrt{|z|^4 + (s_m + [\zeta, z])^2}} ds_m\, dz \\
&= 2(2\pi)^{-(m-2)/2} \int_{\mathbb{C}} \int_{-\infty}^{\infty} f(\zeta \boxminus z, t \ominus u \oplus [\zeta, z]) \frac{1}{\sqrt{|z|^4 + 16u^2}} du\, dz
\end{aligned}
$$

for all (ζ, t) in $\mathbb{H}^1_m$.

Thus, the function d on $\mathbb{C} \times \mathbb{R}$ given by

$$
d(z, u) = \frac{C_m}{\sqrt{|z|^4 + u^2}}, \quad (z, u) \in \mathbb{C} \times \mathbb{R},
$$

where

$$
C_m = 2(2\pi)^{-(m-2)/2}
$$

can be thought of as the Green function of $\mathcal{L}_m$.

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L^p-bounds for Pseudo-differential Operators on the Torus

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Abstract. We establish L^p bounds for a class of periodic pseudo-differential operators corresponding to symbols with limited regularity on the torus $\mathbb{T}^n$. The analysis is carried out using global representation of the symbols on $\mathbb{T}^n \times \mathbb{Z}^n$.

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1. Introduction

In this paper we study L^p-boundedness for periodic pseudo-differential operators or pseudo-differential operators on the torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. The group structure of $\mathbb{T}^n$ enables us to obtain globally defined symbols on $\mathbb{T}^n \times \mathbb{Z}^n$ and the corresponding $S^m_{\rho,\delta}$ Hörmander classes. The idea of the formulation of pseudo-differential operators on the circle $\mathbb{S}^1$ using Fourier series yielding global symbols was first suggested by Mikhail Semenovich Agranovich (cf. [1]). As has been pointed out in [17], despite of the intense research on periodic integral operators, the theory of periodic pseudo-differential operators has been difficult to find. Here, we consider periodic pseudo-differential operators in the framework of the pseudo-differential calculus on the torus recently developed in the works of M. Ruzhansky, V. Turunen and G. Vainikko (cf. [16], [17], [21]). The $S^m_{\rho,\delta}$ Hörmander classes can be defined on manifolds using charts, in the case of the torus the equivalence of this local and the global definition has been proved by W. McLean [14]. A different approach to obtain that equivalence based on extension and periodisation techniques was developed by M. Ruzhansky and V. Turunen [17].

One of the most interesting topics in the theory of pseudodifferential operators is to investigate the behavior of pseudodifferential operators of Hörmander's

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class $S^m_{\rho,\delta}$ in L^p. In a classical paper (cf. [10]) Charles Fefferman establishes L^p-bounds for $S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ classes, those estimates are obtained via real and complex interpolation from a L^∞-BMO bound. Extensions of the Fefferman's L^p estimates have been obtained in [2], [13] on $\mathbb{R}^n$; the references [7], [8] consider a non-homogeneous setting. L^p-bounds on the torus have been investigated for instance in [15], [16] and [17], and multipliers on compact Lie groups can be found in [18]. L^p bounds on the circle which can be routinely extended to the torus can be found in [24]. The boundedness on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$ fails for symbols in $S^0_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ with $\rho<1$, further when $m>0$ is small in $S^{-m}_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ with $\rho<1$ one can only get $L^p(\mathbb{R}^n)$ boundedness for finite intervals centered at $p=2$, this is a consequence of Fefferman's estimates and the work on multipliers of Hirschman (cf. [11]) and Wainger (cf. [22]). The obstruction for the boundedness on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$ of operators in $OPS^0_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ with $\rho<1$ is explained in a more general setting by the works of Richard Beals [3] and [4]. In order to illustrate our main results we recall the L^∞-BMO bound obtained by C. Fefferman.

Theorem A. *Let $\sigma(x,\xi)\in S^{-n\epsilon/2}_{1-\epsilon,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$, where $0\le\delta<1-\epsilon<1$. Then $\sigma(x,D)$ is a bounded operator from L^∞ into BMO.*

We will obtain in the periodic case a version of theorem A in terms of symbols with limited regularity and making use of a recent L^2 estimate on the torus by Ruzhansky and Turunen (cf. [17], Theorem 4.8.1). More specifically we will establish the following theorem. Δ^α_ξ denotes partial differences on the lattice $\mathbb{Z}^n$.

Main Theorem. *Let $0<\epsilon<1$ and $k\in\mathbb{N}$ with $k>\frac{n}{2}$, let $a:\mathbb{T}^n\times\mathbb{Z}^n\to\mathbb{C}$ be a symbol such that $|\Delta^\alpha_\xi a(x,\xi)|\le C_\alpha\langle\xi\rangle^{-\frac{n}{2}\epsilon-(1-\epsilon)|\alpha|}$, $|\partial^\beta_x a(x,\xi)|\le C_\beta\langle\xi\rangle^{-\frac{n}{2}\epsilon}$ for $|\alpha|,|\beta|\le k$, then $a(x,D)$ is bounded from $L^\infty(\mathbb{T}^n)$ into $BMO(\mathbb{T}^n)$.*

As a consequence of real interpolation and the L^2 estimate by Ruzhansky and Turunen (cf. [17]) we obtain:

Theorem. *Let $0<\epsilon<1$ and $k\in\mathbb{N}$ with $k>\frac{n}{2}$, let $a:\mathbb{T}^n\times\mathbb{Z}^n\to\mathbb{C}$ be a symbol such that $|\Delta^\alpha_\xi a(x,\xi)|\le C_\alpha\langle\xi\rangle^{-\frac{n}{2}\epsilon-(1-\epsilon)|\alpha|}$, $|\partial^\beta_x a(x,\xi)|\le C_\beta\langle\xi\rangle^{-\frac{n}{2}\epsilon}$, for $|\alpha|,|\beta|\le k$. Then $\sigma(x,D)$ is a bounded operator from $L^p(\mathbb{T}^n)$ into $L^p(\mathbb{T}^n)$ for $2\le p<\infty$.*

2. Basics on pseudo-differential calculus on the torus

In this section we recall some elements of the basic theory of pseudo-differential operators on the torus. We refer the reader to [16] and [17] for a more comprehensive account on this theory. The dual group of $\mathbb{T}^n$ being $\mathbb{Z}^n$ we shall need of some elements of calculus on finite differences, a classical reference for this topic is [12]. Standard references for the study of pseudodifferential operators on the Euclidean space are [19], [20], [23].

Definition 2.1 (Periodic functions). A function $f : \mathbb{R}^n \to \mathbb{C}$ is 2π-*periodic* if $f(x+k) = f(x)$ for every $x \in \mathbb{R}^n$ and $k \in 2\pi\mathbb{Z}^n$. We shall identify these functions with functions defined on $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n = \{x + 2\pi\mathbb{Z}^n : x \in \mathbb{R}^n\}$. The space of 2π-periodic m times continuously differentiable functions is denoted by $C^m(\mathbb{T}^n)$. The test functions are the elements of the space $C^\infty(\mathbb{T}^n) = \cap_m C^m(\mathbb{T}^n)$.

Definition 2.2 (Schwartz space $\mathcal{S}(\mathbb{Z}^n)$). Let us denote by $\mathcal{S}(\mathbb{Z}^n)$ the space of *rapidly decaying* functions $\phi : \mathbb{Z}^n \to \mathbb{C}$. That is, $\varphi \in \mathcal{S}(\mathbb{Z}^n)$ if for all $0 < M < \infty$ there exists a constant $C_{\varphi,M}$ such that

$$|\varphi(\xi)| \leq C_{\varphi,M}\langle\xi\rangle^{-M}$$

holds for all $\xi \in \mathbb{Z}^n$. The topology on $\mathcal{S}(\mathbb{Z}^n)$ is defined by the seminorms p_k, where $k \in \mathbb{N}_0$ and $p_k(\varphi) = \sup_{\xi\in\mathbb{Z}^n} \langle\xi\rangle^k|\varphi|$.

In order to define the class of symbols that we will use, let us recall the definition of the Fourier transform on the torus for a function f in $C^\infty(\mathbb{T}^n)$

$$(\mathcal{F}_{\mathbb{T}^n}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{T}^n} e^{-ix\xi}u(x)dx$$

where $dx = (2\pi)^{-n}dx$. One can prove that

$$\mathcal{F}_{\mathbb{T}^n} : C^\infty(\mathbb{T}^n) \to \mathcal{S}(\mathbb{Z}^n)$$

is a continuous bijection. The inverse $\mathcal{F}_{\mathbb{T}^n}^{-1} : \mathcal{S}(\mathbb{Z}^n) \to C^\infty(\mathbb{T}^n)$ is obtained in order to get the reconstruction formula of f in the form of a discrete integral or sum over the dual group $\mathbb{Z}^n$

$$f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{ix\xi}(\mathcal{F}_{\mathbb{T}^n}f)(\xi),$$

so that for $g \in \mathcal{S}(\mathbb{Z}^n)$

$$(\mathcal{F}_{\mathbb{T}^n}^{-1}g)(x) = \sum_{\xi\in\mathbb{Z}^n} e^{ix\xi}g(\xi).$$

We shall need a suitable notion of derivative on the lattice $\mathbb{Z}^n$. On the discrete group $\mathbb{Z}^n$ we define the partial difference operator. Let $\sigma : \mathbb{Z}^n \to \mathbb{C}$. Let $e_j \in \mathbb{N}^n, (e_j)_j = 1$, and $(e_j)_i = 0$ if $i \neq j$. We define the **partial difference operator** Δ_{ξ_j} by

$$\Delta_{\xi_j}\sigma(\xi) := \sigma(\xi + e_j) - \sigma(\xi) \quad \text{and} \quad \Delta_\xi^\alpha = \Delta_{\xi_1}^{\alpha_1} \cdots \Delta_{\xi_n}^{\alpha_n}$$

for $\alpha \in \mathbb{N}_0^{\mathbb{N}}$.

The operator above enjoys good properties: discrete Leibniz formula, summation by parts, discrete Taylor expansion, discrete fundamental theorem of calculus (cf. [17]). We just recall a formula for higher-order partial differences and the discrete Leibniz formula.

Proposition 2.3. *Let $\varphi : \mathbb{Z}^n \to \mathbb{C}$. We have*

$$\Delta_\xi^\alpha\varphi(\xi) = \sum_{\beta\leq\alpha}(-1)^{|\alpha-\beta|}\binom{\alpha}{\beta}\varphi(\xi+\beta).$$

Lemma 2.4 (Discrete Leibniz formula). *Let* $\phi, \psi : \mathbb{Z}^n \to \mathbb{C}$ *then*

$$\Delta_\xi^\alpha(\phi\psi)(\xi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left(\Delta_\xi^\beta \phi(\xi)\right) \psi^{\alpha-\beta}(\xi+\beta).$$

We now can define Hörmander's classes on the torus.

Definition 2.5. Let $m \in \mathbb{R}, 0 \leq \delta, \rho \leq 1$. We say that a function $a(x,\xi)$ which is smooth in x for all $\xi \in \mathbb{Z}^n$ belongs to the *toroidal symbol class* $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$, if the following inequalities hold

$$|\partial_x^\beta \Delta_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{2.1}$$

for every $x \in \mathbb{T}^n$, for every $\alpha, \beta \in \mathbb{N}_0^n$, and for all $\xi \in \mathbb{Z}^n$.

A countable family of seminorms can be associated in the following way, for each α, β we define

$$p^m_{\alpha,\beta}(a) = \sup\left\{ \frac{|\Delta_\xi^\alpha \partial_x^\beta a(x,\xi)|}{\langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}} : (x,\xi) \in \mathbb{T}^n \times \mathbb{Z}^n \right\}.$$

Then $\{p^m_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^n\}$ is a countable family of seminorms and defines a Fréchet topology on $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$.

Remark 2.6. When the symbol $a(x,\xi)$ has finite regularity with respect to the spatial variable we will keep the notation of $p^m_{\alpha,\beta}$ from the corresponding seminorms.

A corresponding operator is associated to a symbol $a(x,\xi)$ in $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ which will be called a periodic pseudo-differential operator or pseudo-differential operator on the torus $\mathbb{T}^n$

$$a(x,D)f(x) = \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi x\cdot\xi} a(x,\xi)(\mathcal{F}_{\mathbb{T}^n} f)(\xi), \tag{2.2}$$

which can also be written as

$$a(x,D)f(x) = \sum_{\xi \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i2\pi(x-y)\cdot\xi} a(x,\xi) f(y)dy. \tag{2.3}$$

The corresponding class of operators with symbols in $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ will be denoted by $OPS^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$.

Remark 2.7. Let $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ be a measurable function and $m \in \mathbb{R}$ such that $|a(x,\xi)| \leq C < \xi >^m$ for some constant $C > 0$. It is not hard to prove that $a(x,D)f$ is well defined for $f \in C^\infty(\mathbb{T}^n)$. Hence, we can associate a pseudo-differential operator to such a symbol. In particular, we will do so for symbols with limited regularity.

In order to employ some multipliers on the torus we shall need the following lemma.

Lemma 2.8. *Let* $m \in \mathbb{R}$, *if* $\sigma \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ *and* σ *only depends on the Fourier variable* ξ *then* $\sigma \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$.

Proof. For the partial differences of order 1, $|\alpha| = 1$, we have

$$\Delta_\xi^\alpha = \sigma(\xi + e_j) - \sigma(\xi).$$

Applying the mean value theorem we get $\eta \in [\xi, \xi + e_j]$ and a constant $C > 0$ such that

$$|\sigma(\xi + e_j) - \sigma(\xi)| \leq |\partial_j \sigma(\eta)| \leq \langle \eta \rangle^{m-(\rho-\delta)} \leq C \langle \xi \rangle^{m-(\rho-\delta)}.$$

The last inequality because in the interval $[\xi, \xi+e_j]$ we have $\langle \eta \rangle \leq C\langle \xi \rangle$. For higher-order differences and derivatives we apply Proposition 2.3 and induction. ☐

There exists a process to interpolate the second argument of symbols on $\mathbb{T}^n \times \mathbb{Z}^n$ in a smooth way to get a symbol defined on $\mathbb{T}^n \times \mathbb{R}^n$. We recall a few consequences of this process linking symbols on $\mathbb{T}^n \times \mathbb{Z}^n$ and $\mathbb{T}^n \times \mathbb{R}^n$ (cf. [17]).

Theorem 2.9. *Let* $0 \leq \delta \leq 1$, $0 < \rho \leq 1$. *The symbol* $\tilde{a} \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ *is a toroidal symbol if and only if there exists a Euclidean symbol* $a \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$ *such that* $\tilde{a} = a|_{\mathbb{T}^n \times \mathbb{Z}^n}$. *Moreover, this extended symbol* a *is unique modulo* $S^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n)$.

Theorem 2.10 (Equality of Operator Classes). *For* $0 \leq \delta \leq 1$, $0 < \rho \leq 1$ *we have*

$$OpS^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n) = OpS^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n).$$

A look at the proof (cf. [17]) of Theorem 2.9 shows us that a more general version is still valid for symbols with limited regularity as follows:

Corollary 2.11. *Let* $0 \leq \delta \leq 1$, $0 < \rho \leq 1$. *Let the function* $a : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{C}$ *satisfy*

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{2.4}$$

for every $x \in \mathbb{T}^n$, *for all* $\xi \in \mathbb{Z}^n$, *for every* $|\alpha| \leq N_1$ *and* $|\beta| \leq N_2$. *Then the restriction* $\tilde{a} = a|_{\mathbb{T}^n \times \mathbb{Z}^n}$ *satisfies*

$$|\Delta_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha,\beta} < \xi >^{m-\rho|\alpha|+\delta|\beta|}, \tag{2.5}$$

for every $x \in \mathbb{T}^n$, *for all* $\xi \in \mathbb{Z}^n$, *for every* $|\alpha| \leq N_1$ *and* $|\beta| \leq N_2$. *Conversely, every function* $\tilde{a} : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ *satisfying* (2.5) *for every* $|\alpha| \leq N_1$ *and* $|\beta| \leq N_2$ *is a restriction* $\tilde{a} = a|_{\mathbb{T}^n \times \mathbb{Z}^n}$ *of some function* $a : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{C}$ *satisfying* (2.4) *for every* $|\alpha| \leq N_1$ *and* $|\beta| \leq N_2$.

Remark 2.12. It is important to point out that in the previous three statements the restriction $\delta \leq \rho$ is not imposed and so $\delta > \rho$ is allowed.

Note that in the corollary above we lose the uniqueness we have in Theorem 2.9. Our main results will be stated for symbols with limited regularity, to do so we will introduce the following notation.

Definition 2.13. Let the function $a : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{C}$ satisfy (2.4). We will say that $a \in S^m_{\rho,\delta,N_1,N_2}(\mathbb{T}^n \times \mathbb{R}^n)$. If $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ satisfies (2.5) for N_1, N_2 we will say that $a \in S^m_{\rho,\delta,N_1,N_2}(\mathbb{T}^n \times \mathbb{Z}^n)$. The corresponding classes of operators will be denoted by $OpS^m_{\rho,\delta,N_1,N_2}(\mathbb{T}^n \times \mathbb{R}^n)$ and $OpS^m_{\rho,\delta,N_1,N_2}(\mathbb{T}^n \times \mathbb{Z}^n)$ respectively.

It is also possible to obtain a version of Theorem 2.10 for symbols with limited regularity:

Theorem 2.14. *For* $0 \leq \delta \leq 1$, $0 < \rho \leq 1$ *we have*

$$OpS^m_{\rho,\delta,N_1,N_2}(\mathbb{T}^n \times \mathbb{R}^n) = OpS^m_{\rho,\delta,N_1,N_2}(\mathbb{T}^n \times \mathbb{Z}^n).$$

Remark 2.15.

i) $\langle\xi\rangle^m \in S^m_{1,0}(\mathbb{T}^n \times \mathbb{Z}^n)$ for all $m \in \mathbb{R}$. This is a consequence of Lemma 2.8.
ii) Symbols $\sigma(x,\xi)$ in $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$ can be obtained from symbols in $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$ due to Theorem 2.9.
iii) It is possible to construct a symbol $\sigma(x,\xi)$ in $S^m_{1,\delta}(\mathbb{S}^1 \times \mathbb{Z})$ in the following way: consider $\gamma(x,\xi) = \exp(i|\xi|^\delta x) < \xi >^m$ for $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}, \xi \in \mathbb{Z}$, extend γ smoothly to $-\pi \leq x \leq \pi, \xi \in \mathbb{Z}$ with $\gamma(x,\xi) = 0$ if $\frac{\pi}{2} \leq |x| \leq \pi, \xi \in \mathbb{Z}$.

3. $L^p(\mathbb{T}^n)$ estimates

In order to get our $L^p(\mathbb{T}^n)$ bounds we shall interpolate between L^2 and $L^\infty - BMO$ bounds. The proposition below (cf. [17], Theorem 4.8.1) does not impose any regularity condition on the symbol in contrast with similar results on $\mathbb{R}^n$, the authors give a sufficient condition for $L^2(\mathbb{T}^n)$ boundedness:

Theorem 3.1. *Let* $k \in \mathbb{N}$ *and* $k > \frac{n}{2}$. *Let* $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ *be a symbol such that*

$$|\partial_x^\beta a(x,\xi)| \leq C_\beta, \ |\beta| \leq k \tag{3.1}$$

then

$$a(x,D) : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n).$$

Moreover, there exists a constant C *such that*

$$\|a(x,D)f\|_{L^2(\mathbb{T}^n)} \leq C \max\{C_\beta : |\beta| \leq k\} \|f\|_{L^2(\mathbb{T}^n)}. \tag{3.2}$$

The theorem above will be our L^2 starting point. We now prove some preparatory results.

Lemma 3.2. *Let* $b(x,\xi), c(\xi)$ *be symbols on* $\mathbb{T}^n \times \mathbb{Z}^n$. *Then the composition* $b(x,D)c(D)$ *possesses a symbol with exact representation* $b(x,\xi) \cdot c(\xi)$.

Proof. It is a direct consequence of the definition of pseudodifferential operator having into account that $\mathcal{F}_{\mathbb{T}^n}(c(D)f)(\xi) = c(\xi)\mathcal{F}_{\mathbb{T}^n}(f)(\xi)$. □

The following lemma is a consequence of Theorem 3.1:

Lemma 3.3. *Let* $k \in \mathbb{N}$ *and* $k > \frac{n}{2}$. *Let* $b : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ *be a symbol and* $m \in \mathbb{R}$, *if there exists a constant* $C_\beta > 0$ *such that for* $|\beta| \leq k$

$$|\partial_x^\beta b(x,\xi)| \leq C_\beta \langle\xi\rangle^m, \tag{3.3}$$

then

$$b(x,D)J^{-m} : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n),$$

where J^m denotes the Bessel potential with symbol $\langle\xi\rangle^m$. Moreover there exists a constant C such that

$$\|b(x,D)J^{-m}f\|_{L^2(\mathbb{T}^n)} \leq C \max_{|\beta|\leq k} \sup_{(x,\xi)} |\partial_x^\beta b(x,\xi)\langle\xi\rangle^{-m}| \, \|f\|_{L^2(\mathbb{T}^n)}.$$

Proof. We observe that by Lemma 3.2 the symbol of $b(x,D)J^{-m}$ is $b(x,\xi)\langle\xi\rangle^{-m}$, the proof follows now from Theorem 3.1 since

$$|\partial_x^\beta(b(x,\xi)\langle\xi\rangle^{-m})| \leq |\partial_x^\beta b(x,\xi)|\langle\xi\rangle^{-m} \leq CC_\beta\langle\xi\rangle^m\langle\xi\rangle^{-m},$$

for $|\beta|\leq k$. □

Lemma 3.4. *Let ϕ be a function in $C^\infty(\mathbb{T}^n)$ and $b : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ a symbol, then the commutator $[\phi, b(x,D)]$ is a pseudodifferential operator with symbol*

$$\theta(x,\xi) = \sum_{\eta\in\mathbb{Z}^n} e^{2\pi i x\cdot\eta}\widehat{\phi}(\eta)\left[b(x,\xi) - b(x,\xi+\eta)\right].$$

Proof. We identify ϕ with the multiplication operator (by ϕ) that will be denoted by M_ϕ. Then

$$M_\phi f(y) = \phi(y)f(y) = \sum_{\xi\in\mathbb{Z}^n} e^{i\xi\cdot y}\phi(y)\widehat{f}(\xi), = \int_{\mathbb{T}^n} e^{i\xi\cdot(y-z)}\phi(y)f(z)dz.$$

Now,

$$\begin{aligned}
b(x,D)\circ M_\phi f(x) &= \int_{\mathbb{R}^n}\sum_{\eta\in\mathbb{Z}^n} e^{i\eta\cdot(x-y)}b(x,\eta)T_\phi f(y)dy \\
&= \int_{\mathbb{T}^n}\sum_{\xi\in\mathbb{Z}^n}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n} e^{i\eta\cdot(x-y)}e^{i\xi\cdot(y-z)}b(x,\eta)\phi(y)f(z)dzdy.
\end{aligned}$$

We will use the following identity

$$e^{i\eta\cdot(x-y)}e^{i\xi\cdot(y-z)} = e^{i(x-y)\cdot(\eta-\xi)}e^{i(x-z)\cdot\xi},$$

thus,

$$\begin{aligned}
b(x,D)\circ M_\phi f(x) &= \int_{\mathbb{T}^n}\sum_{\xi\in\mathbb{Z}^n}\int_{\mathbb{T}^n}\sum_{\eta\in\mathbb{Z}^n} e^{i(x-y)\cdot(\eta-\xi)}e^{i(x-z)\cdot\xi}b(x,\eta)\phi(y)f(z)dzdy \\
&= \int c(x,\xi)e^{i(x-z)\cdot\xi}f(z)dz,
\end{aligned}$$

where

$$c(x,\xi) = \sum_{\eta\in\mathbb{Z}^n} e^{ix\cdot\eta}\widehat{\phi}(\eta)b(x,\xi+\eta)d\eta.$$

Hence, $c(x,\xi)$ is the symbol of $b(x,D)\circ M_\phi$. On the other hand the symbol of $M_\phi\circ b(x,D)$ is $\phi(x)\cdot b(x,\xi)$. □

The following lemma will be applied to analyze local $L^\infty(\mathbb{T}^n) - L^\infty(\mathbb{T}^n)$ bounds.

Lemma 3.5. *Let σ be a symbol on $\mathbb{T}^n \times \mathbb{Z}^n$, if $\eta : \mathbb{Z} \to \mathbb{C}$ is a function supported in $R \leq |z| \leq 3R$ with $R > 1$, then for every α there exist constants A and $C_{\alpha\beta}$ such that for all $\lambda \in \mathbb{R}$, $s \in \mathbb{N}$ and for every $(x,\xi) \in \mathbb{T}^n \times \mathbb{Z}^n$ we have*

$$|\Delta_\xi^\alpha(\sigma(x,\xi)\eta(s|\xi|))| \leq C_\alpha \max_{\beta\leq\alpha} |\Delta_\xi^\beta \sigma(x,\xi)| A^{|\lambda|}\langle\xi\rangle^\lambda s^\lambda. \tag{3.4}$$

Proof.

$$\begin{aligned}\Delta_\xi^\alpha(\sigma(x,\xi)\eta(s|\xi|)) &= \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left(\Delta_\xi^\beta\sigma(x,\xi)\right)\Delta_\xi^{\alpha-\beta}\eta(s|\xi|)\\ &= \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left(\Delta_\xi^\beta\sigma(x,\xi)\right)\sum_{\gamma\leq\alpha-\beta}(-1)^{|\alpha-\beta-\gamma|}\binom{\alpha-\beta}{\gamma}\eta(s|\xi+\gamma|).\end{aligned}$$

Hence we obtain

$$|\Delta_\xi^\alpha(\sigma(x,\xi)\eta(s|\xi|))| \leq C_\alpha \max_{\beta\leq\alpha}|\Delta_\xi^\beta\sigma(x,\xi)|\tilde{C}_\alpha \sum_{\gamma\leq\alpha-\beta}|\eta(s|\xi+\gamma|)|.$$

Taking into account that $\eta(s|\cdot|)$ is supported in $R \leq |\cdot| \leq 3R$ there exists a constant $A > 1$ such that

$$A^{-1}s^{-1} \leq \langle\xi\rangle \leq As^{-1},$$

thus for every $\lambda \in \mathbb{R}$

$$1 \leq A^{|\lambda|}\langle\xi\rangle^\lambda s^\lambda.$$

Therefore

$$|\Delta_\xi^\alpha(\sigma(x,\xi)\eta(s|\xi|))| \leq C_\alpha \max_{\beta\leq\alpha}|\Delta_\xi^\beta\sigma(x,\xi)|A^{|\lambda|}\langle\xi\rangle^\lambda s^\lambda. \qquad \square$$

The next lemma is a periodic version of a classical by Charles Fefferman ([10], page 415). It furnishes local $L^\infty(\mathbb{T}^n) - L^\infty(\mathbb{T}^n)$ bounds, that kind of boundedness joint with the application of suitable partitions of unity will be essential in our analysis in the spirit of Littlewood–Paley theory.

Lemma 3.6. *Let $0 < \epsilon < 1$ and $k \in \mathbb{N}$ with $k > \frac{n}{2}$, let $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ be a symbol supported in $|\xi| \leq 1$ or $R \leq |\xi| \leq 3R$ for some $R > 0$ and such that $|\Delta_\xi^\alpha a(x,\xi)| \leq C_\alpha\langle\xi\rangle^{-\frac{n}{2}\epsilon-(1-\epsilon)|\alpha|}$, for $|\alpha| \leq k$ then $a(x,D)$ is bounded from $L^\infty(\mathbb{T}^n)$ into $L^\infty(\mathbb{T}^n)$ moreover there exists a constant C independent of a and f such that*

$$\|a(x,D)f\|_\infty \leq CC(a)\|f\|_{L^\infty},$$

where $C(a) = \max\{C_0, C_\alpha : |\alpha| = k\}$.

Remark 3.7. The fact that the above estimate is independent of R will be crucial for us in order to apply dyadic decompositions.

Proof of Lemma 3.6. Let $a(x,\xi)$ be supported in

$$\{(x,\xi) \in \mathbb{T}^n \times \mathbb{Z}^n / R \leq |\xi| \leq 3R\},$$

for $R>1$ and such that $|\Delta_\xi^\alpha a(x,\xi)|\leq C_\alpha\langle\xi\rangle^{-\frac{n}{2}\epsilon-(1-\epsilon)|\alpha|}$, for $|\alpha|\leq k$. Applying Corollary 2.11 to the symbol a we obtain $\tilde{a}\in S^{-\frac{n}{2}}_{\rho,\delta,k,0}(\mathbb{T}^n\times\mathbb{R}^n)$ such that a and $\tilde{a}$ coincide in $\mathbb{T}^n\times\mathbb{Z}^n$. Then $\tilde{a}$ has the same support as a and we notice that

$$a(x,D)f(x)=\tilde{a}(x,D)f(x)=\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{T}^n}e^{2\pi i(x-y)\cdot\xi}\tilde{a}\left(x,\xi\right)f(y)dyd\xi$$

$$=\int\limits_{\mathbb{T}^n}\mathcal{F}_{\mathbb{R}^n}\tilde{a}(x,y-x)f(y)dy=(\mathcal{F}_{\mathbb{R}^n}\tilde{a}(x,\cdot)*f)(x).$$

We obtain

$$|\tilde{a}(x,D)f(x)|\leq\|\mathcal{F}_{\mathbb{R}^n}\tilde{a}(x,\cdot)\|_{L_1(\mathbb{R}^n)}\|f\|_{L^\infty(\mathbb{T}^n)},\ x\in\mathbb{T}^n.$$

It will be enough to prove that for every $x\in\mathbb{R}^n$

$$\|\mathcal{F}_{\mathbb{R}^n}\tilde{a}(x,\cdot)\|_{L^1}\leq CC(\tilde{a}).$$

We set $b=R^{\epsilon-1}$. By the Cauchy–Schwarz inequality we get

$$\int\limits_{|y|<b}|\mathcal{F}_{\mathbb{R}^n}\tilde{a}(x,y)|dy\leq Cb^{\frac{n}{2}}\Bigg(\int\limits_{|y|<b}|\mathcal{F}_{\mathbb{R}^n}\tilde{a}(x,y)|^2dy\Bigg)^{\frac{1}{2}}\leq Cb^{\frac{n}{2}}\Bigg(\int|\tilde{a}(x,\xi)|^2d\xi\Bigg)^{\frac{1}{2}}$$

$$\leq CC_0b^{\frac{n}{2}}\Bigg(\int\limits_{|\xi|\leq 3R}\langle\xi\rangle^{-n\epsilon}d\xi\Bigg)^{\frac{1}{2}}$$

$$\leq CC_0R^{(\epsilon-1)\frac{n}{2}}\Bigg(\int\limits_{|\xi|\leq 3R}\langle\xi\rangle^{-n\epsilon}d\xi\Bigg)^{\frac{1}{2}}\leq CC_0,\ \text{for all } R>1.$$

Now, since $k>\frac{n}{2}$ we get

$$\int\limits_{|y|\geq b}|\mathcal{F}_{\mathbb{R}^n}\tilde{a}(x,y)|dy\leq Cb^{\frac{n}{2}-k}\Bigg(\int\limits_{y\geq b}|y|^{2k}|\mathcal{F}_{\mathbb{R}^n}\tilde{a}(x,y)|^2dy\Bigg)^{\frac{1}{2}}$$

$$\leq Cb^{\frac{n}{2}-k}\Bigg(\int\limits_{\mathbb{R}^n}|\nabla_\xi^k\tilde{a}(x,\xi)|^2d\xi\Bigg)^{\frac{1}{2}}$$

$$\leq C\max_{|\alpha|=k}C_\alpha R^{(\epsilon-1)(\frac{n}{2}-k)}\Bigg(\int\limits_{R\leq|\xi|}\langle\xi\rangle^{-n\epsilon-2k(1-\epsilon)}d\xi\Bigg)^{\frac{1}{2}}$$

$$\leq C\max_{|\alpha|=k}C_\alpha R^{(\epsilon-1)(\frac{n}{2}-k)}O\left(R^{(1-\epsilon)(\frac{n}{2}-k)}\right)\leq C\max_{|\alpha|=k}C_\alpha.$$

Therefore

$$\|\mathcal{F}_{\mathbb{R}^n}\tilde{a}(x,\cdot)\|_{L^1}\leq CC(a),\ \text{for every}\ x\in\mathbb{T}^n.$$

The proof is similar for the other type of support. □

We are now ready to establish our main result which can be seen as a generalization of the Fefferman bounds (Theorem A) in the introduction but improving the estimation of the suitable indices in the case of the torus.

Theorem 3.8. *Let $0 < \epsilon < 1$ and $k \in \mathbb{N}$ with $k > \frac{n}{2}$, let $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ be a symbol such that $|\Delta_\xi^\alpha a(x,\xi)| \leq C_\alpha \langle\xi\rangle^{-\frac{n}{2}\epsilon-(1-\epsilon)|\alpha|}$, $|\partial_x^\beta a(x,\xi)| \leq C_\beta \langle\xi\rangle^{\frac{n}{2}\epsilon}$ for $|\alpha|, |\beta| \leq k$ then $a(x,D)$ is bounded from $L^\infty(\mathbb{T}^n)$ into $BMO(\mathbb{T}^n)$, moreover there exists a constant C independent of a and f such that*

$$\|a(x,D)f\|_{BMO} \leq C p_{k,k}^{-\frac{n}{2}\epsilon}(a)\|f\|_{L^\infty}.$$

We shall need the following lemma for symbols supported in $|\xi| > R$.

Lemma 3.9. *Let $0 < \epsilon < 1$ and $k \in \mathbb{N}$ with $k > \frac{n}{2}$, let $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ be a symbol supported in $|\xi| > R$ with $R > 1$ such that $|\Delta_\xi^\alpha a(x,\xi)| \leq C_\alpha \langle\xi\rangle^{-\frac{n}{2}\epsilon-(1-\epsilon)|\alpha|}$ and $|\partial_x^\beta a(x,\xi)| \leq C_\beta \langle\xi\rangle^{-\frac{n}{2}\epsilon}$, for $|\alpha|, |\beta| \leq k$ then $a(x,D)$ is bounded from $L^\infty(\mathbb{T}^n)$ into $BMO(\mathbb{T}^n)$. Moreover there exists a constant C independent of a and f such that*

$$\|a(x,D)f\|_{BMO} \leq C max\{p_{0,0}^{-\frac{n}{2}\epsilon}(a), p_{k,0}^{-\frac{n}{2}\epsilon}(a)\}\|f\|_{L^\infty}.$$

Proof. We begin by considering a function ϕ on $\mathbb{T}^n$, with $0 \leq \phi \leq 10$, $\phi \geq 1$ on $B(x_0, r) \subset \mathbb{T}^n$ and the Fourier transform $\widehat{\phi}$ verifying $\text{supp}(\widehat{\phi}) \subset \{\xi \in \mathbb{Z}^n : |\xi| \leq (C^{-1}r)^{\frac{1}{1-\epsilon}}\}$. We write

$$\phi(x) \cdot a(x,D)f(x) = a(x,D)(\phi f)(x) + [\phi, a(x,D)]\, f(x) = I + II. \tag{3.5}$$

In order to manage the term I let us consider the Bessel potential J^m, then $J^{-m} : H^{-m}(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ isomorphically. We decompose

$$a(x,D)(\phi f) = (a(x,D) \cdot J^m)\left(J^{-m} \cdot (\phi f)\right). \tag{3.6}$$

Now, since the symbol of $a(x,D) \cdot J^m$ satisfies the hypothesis of Lemma 3.3 with $m = \frac{n}{2}\epsilon$, then there exists a constant $C > 0$

$$\|a(x,D)(\phi f)\|_{L^2}^2 \leq C M_a^2 \cdot \|J^{-m}(\phi f)\|_{L^2}^2, \tag{3.7}$$

where $M_a = \max\limits_{|\beta| \leq k} \sup\limits_{(x,\xi)} |\partial_x^\beta a(x,\xi)\langle\xi\rangle^m|$.

Now

$$\|J^{-m}(\phi f)\|_{L^2}^2 = \|\phi f\|_{H^{-m}}^2,$$

since the Bessel potential J^{-m} is a positive operator (preserves positivity of functions) and a multiplier with symbol $J(\xi) = \langle\xi\rangle^{-m}$, we obtain

$$\begin{aligned} \|J^{-m}(\phi f)\|_{L^2}^2 &\leq \|f\|_{L^\infty}^2 \|J^{-m}(\phi)\|_{L^2}^2 \leq C_1 \|f\|_{L^\infty}^2 \|\phi\|_{H^{-m}}^2 \\ &\leq C_1 \|f\|_{L^\infty}^2 (C^{-1}r)^n \quad \leq C\|f\|_{L^\infty}^2 |B(x_0,r)|. \end{aligned}$$

Therefore

$$\|a(x,D)(\phi f)\|_{L^2}^2 \leq C M_a^2 \|f\|_{L^\infty}^2 |B(x_0,r)|. \tag{3.8}$$

Applying the Cauchy–Schwarz inequality, we have

$$\frac{1}{|B(x_0,\delta)|}\int_B |a(x,D)(\phi f)(x)|dx \leq \left(\frac{1}{|B|}\int_B |a(x,D)(\phi f)(x)|^2 dx\right)^{\frac{1}{2}}$$
$$\leq C M_a \|f\|_{L^\infty}. \tag{3.9}$$

This proves the estimate for I.

By Lemma 3.4 the commutator $[\phi, a(x,D)]$ appearing in II is a pseudodifferential operator $\theta(x,D)$ with symbol

$$\theta(x,\xi) = \sum_{\eta\in\mathbb{Z}^n} e^{ix\cdot\theta}\widehat{\phi}(\eta)\left[a(x,\xi) - a(x,\xi+\eta)\right].$$

We write

$$\theta(x,\xi) = \sum_{j=0}^{\infty}\theta_j(x,\xi),$$

with $\theta_j(x,\xi)$ supported in $|\xi| \sim 2^j r^{-1}$. Now

$$\theta(x,\xi) = \sum_{|\eta|\leq (C^{-1}r)^{2(\epsilon-1)}} e^{ix\cdot\eta}\widehat{\phi}(\eta)\left[a(x,\xi) - a(x,\xi+\eta)\right].$$

Hence and by Lemma 3.6 we obtain

$$\|\,[\phi, a(x,D)f]\,\|_{L^\infty} \leq \sum_{j=0}^{\infty}\|\theta_j(x,D)f\|_{L^\infty} \leq \sum_{j=0}^{\infty} C2^{-\frac{j}{2}}C(a)\|f\|_{L^\infty}$$
$$\leq CC(a)\|f\|_{L^\infty}, \tag{3.10}$$

where $C(a)$ is as in Lemma 3.6.
Since $\phi \geq 1$ on $B(x_0,r)$, using (3.9) and (3.10) into (3.5) we have

$$\frac{1}{|B(x_0,r)|}\int\limits_B |a(x,D)f(x)|dx \leq \frac{1}{|B(x_0,r)|}\int\limits_B |\phi(x)\cdot a(x,D)f(x)|dx$$
$$\leq CC(a)\|f\|_{L^\infty}. \qquad \square$$

Proof of Theorem 3.8. Let $f \in L^\infty(\mathbb{T}^n)$, $x_0 \in \mathbb{T}^n$, and $B = B(x_0,r) \subset \mathbb{T}^n$. We will show that there exist an integer j and a constant $C > 0$ independent of f and B, such that

$$\frac{1}{|B(x_0,r)|}\int\limits_B |\sigma(x,D)f(x) - g_B|dx \leq C\|\sigma\|_{j;S^h h()}\|f\|_{L^\infty}, \tag{3.11}$$

where we have denoted $g = \sigma(x,D)f$.

We decompose $\sigma(x,\xi)$ into two parts, $\sigma = \sigma^0 + \sigma^1$, with σ^0 supported in $|\xi| \leq 2r^{-1}$, σ^1 supported in $|\xi| \geq \frac{1}{2}r^{-1}$. One can obtain such a decomposition in the following way. Let $\beta = 1_A$ be the characteristic function for $A = \{z \in \mathbb{Z} : |z| \leq 1\}$ and then

$$\sigma^0(x,\xi) = \sigma(x,\xi)\beta(r|\xi|),$$

and put $\sigma^1 = \sigma - \sigma^0$.

In order to obtain (3.11) it will be enough to consider σ^0, the corresponding estimation for σ^1 is a consequence of Lemma 3.9. We can write

$$\partial_{x_k}\sigma^0(x,D)f(x)=\sigma'_x(x,D)f(x),$$

where σ'_x is the symbol

$$\sigma'_x(x,\xi)=\partial_{x_k}\sigma^0(x,\xi)+i\xi_k\sigma^0(x,\xi).$$

We shall use a partition of unity to study σ'_x

$$\sigma'_x(x,\xi)=\sum_{j=1}^{\infty}\rho_j(x,\xi),$$

with ρ_j supported in $|\xi|\sim 2^{-j}r^{-1}$. Such a partition of unity is obtained from $\eta:\mathbb{R}\to\mathbb{R}$ defined by

$$\eta(s)=\begin{cases}0 & ,\ \text{if}\ \ |s|\le 1\\ 1 & ,\ \text{if}\ \ |s|\ge 2.\end{cases}$$

Let $\rho(s)=\eta(s)-\eta(2^{-1}s)$. Then $\operatorname{supp}\rho=\{1\le|s|\le 4\}$. One can verify that

$$1=\eta(s)+\sum_{j=1}^{\infty}\rho(2^js)\,,s\in\mathbb{R}.$$

Now, set $s=r|\xi|$ then

$$1=\eta(r|\xi|)+\sum_{j=1}^{\infty}\rho(r2^j(|\xi|).$$

The support of η being $\{|s|>1\}$ and $r|\xi|\le 1$, we obtain

$$1=\sum_{j=1}^{\infty}\rho(r2^j|\xi|)$$

for $\{|\xi|\le r^{-1}\}$.

Since $\operatorname{supp}\sigma'=\operatorname{supp}\sigma^0$ we have

$$\sigma'_x(x,\xi)=\sum_{j=1}^{\infty}\rho(r2^j|\xi|)\cdot\sigma'_x(x,\xi).$$

Then one can choose

$$\rho_j(x,\xi)=\rho(r2^j|\xi|)\cdot\sigma'_x(x,\xi).$$

We will apply Lemma 3.6 to each term $\rho_j(x,\xi)$ and for the estimation of the derivatives Lemma 3.5, since $\sigma'_x=\partial_{x_k}\sigma^0+i\xi_k\sigma^0$ we consider first $\partial_{x_k}\sigma^0$ and choosing $\lambda=-1$ we obtain

$$|\Delta_\xi^\alpha(\partial_{x_k}\sigma^0(x,\xi)\rho(r2^j|\xi|))|\le C_\alpha\max_{\beta\le\alpha}|\Delta_\xi^\beta\partial_{x_k}\sigma^0(x,\xi)|A^{|\lambda|}\langle\xi\rangle^\lambda(r2^j)^\lambda,$$
$$\le C_\alpha\max_{\beta\le\alpha}\langle\xi\rangle^{-\frac{n}{2}\epsilon-(1-\epsilon)|\beta|}\langle\xi\rangle A\langle\xi\rangle^{-1}(r2^j)^{-1}\le C_\alpha\max_{\beta\le\alpha}\langle\xi\rangle^{-\frac{n}{2}\epsilon-(1-\epsilon)|\beta|}r^{-1}2^{-j}.$$

Therefore, there exists a constant C such that

$$\|\partial_{x_k}\sigma^0(x,D)f\|_{L^\infty} \le \sum_{j=0}^{\infty}\|\rho_j(x,D)f\|_{L^\infty} \le Cr^{-1}\sum_{j=0}^{\infty}2^{-j}\|f\|_{L^\infty} \le Cr^{-1}\|f\|_{L^\infty}.$$

Now, by the mean value theorem we have

$$|\sigma^0(x,D)f(x) - g_B| \le C\|f\|_{L^\infty}.$$

Then

$$\frac{1}{|B(x_0,r)|}\int_B |\sigma^0(x,D)f(x) - g_B|dx \le C\|\sigma\|_{l;S}\|f\|_{L^\infty}. \tag{3.12}$$

This proves (3.11) for σ^0. □

Interpolation between $L^2(\mathbb{T}^n)$ and $BMO(\mathbb{T}^n)$ estimates allows us to obtain the next $L^p(\mathbb{T}^n)$ boundedness.

Theorem 3.10. *Let* $0 < \epsilon < 1$ *and* $k \in \mathbb{N}$ *with* $k > \frac{n}{2}$, *let* $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ *be a symbol such that* $|\Delta_\xi^\alpha a(x,\xi)| \le C_\alpha\langle\xi\rangle^{-\frac{n}{2}\epsilon-(1-\epsilon)|\alpha|}$, $|\partial_x^\beta a(x,\xi)| \le C_\beta\langle\xi\rangle^{-\frac{n}{2}\epsilon}$, *for* $|\alpha|,|\beta| \le k$. *Then* $\sigma(x,D)$ *is a bounded operator from* $L^p(\mathbb{T}^n)$ *into* $L^p(\mathbb{T}^n)$ *for* $2 \le p < \infty$.

Proof. The boundedness on $L^2(\mathbb{T}^n)$ is a consequence of the hypothesis on the derivatives with respect to x and Theorem 3.1. The $L^\infty(\mathbb{T}^n) - BMO(\mathbb{T}^n)$ bound is a consequence of Theorem 3.8. The theorem follows then from real interpolation. □

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Multiplication Properties in Gelfand–Shilov Pseudo-differential Calculus

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Abstract. We consider modulation space and spaces of Schatten–von Neumann symbols where corresponding pseudo-differential operators map one Hilbert space to another. We prove Hölder–Young and Young type results for such spaces under dilated convolutions and multiplications. We also prove continuity properties for such spaces under the twisted convolution, and the Weyl product. These results lead to continuity properties for twisted convolutions on Lebesgue spaces, e.g., $L^p_{(\omega)}$ is a twisted convolution algebra when $1 \le p \le 2$ and appropriate weight ω.

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0. Introduction

The aim of the paper is to extend the results in [51] on various types of products in pseudo-differential calculus to include convenient Banach spaces of Gelfand–Shilov functions and distributions. The family of Banach spaces consists of (weighted) Lebesgue spaces, modulation spaces and spaces of Schatten–von Neumann symbols in the pseudo-differential calculus. The products concern the usual multiplication and convolution, twisted convolution and the Weyl product. Especially we establish continuity properties for Lebesgue and modulation spaces under the twisted convolution and the Weyl product, and prove Young type results for Schatten–von Neumann symbols under the ordinary multiplication and convolution.

We recall that the composition of two Weyl operators corresponds to the Weyl product of the two operator symbols on the symbol level, and the twisted convolution appears when Weyl product is conjugated by symplectic Fourier transform. (See Section 1 for the details.) Convolution and multiplication products appear when investigating Toeplitz operators (also known as localization operators) in the framework of pseudo-differential calculus. More precisely, each Toeplitz operator is equal to a pseudo-differential operator, where the symbol of the pseudo-differential operator is a convolution between the Toeplitz symbol and a rank one symbol, which is an ordinary multiplication on the Fourier transform side. We remark that Toeplitz operators might be convenient to use when approximating certain pseudo-differential operators (see, e.g., [7, 44]), and in computation of kinetic energy in mechanics (cf. [31]).

The most of the questions here were carefully investigated in [51] in the case when the involved spaces are defined by weights of polynomial type (see, e.g., [28] for notations concerning the usual function and distribution spaces, and Section 1 for other notations). In particular, all function and distribution spaces in [51] stay between $\mathscr{S}$ and $\mathscr{S}'$. In the present paper we use the framework in [51], and extend the results in [51] such that we permit general moderate weights. This implies that the function and distribution spaces can be arbitrary close to Gelfand–Shilov spaces of the form $\mathcal{S}_s^s$ and Σ_s^s when $s \geq 1$, and their duals.

In several questions we may use similar arguments as in [51], while new types of difficulties appear in other questions, when passing from the distribution theory for Schwartz functions in [51], to corresponding theory for Gelfand–Shilov functions.

In order to be more specific, let $\mathscr{H}_1$ and $\mathscr{H}_2$ be modulation spaces which are Hilbert spaces (see [52]). Also let $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$, $p \in [1, \infty]$, be the set of Schatten–von Neumann operators of order p from $\mathscr{H}_1$ to $\mathscr{H}_2$, and let $s_p^w(\mathscr{H}_1, \mathscr{H}_2)$ be the set of all distributions $a \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ such that the corresponding Weyl operators $\operatorname{Op}^w(a)$ belong to $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$.

In general it is not complicated to establish continuity properties for spaces of the form $s_p^w = s_p^w(\mathscr{H}_1, \mathscr{H}_2)$ under the Weyl product and twisted convolution, because such questions can easily be reformulated into questions of compositions for Schatten–von Neumann operators on the operator level. It is more complicated to find continuity relations for dilated multiplications and convolutions on the s_p^w spaces, because such products take complicated forms on the operator level. In this situation we use certain Fourier techniques, similar to those in [44, Section 3] and [51], to get convenient integral formulas. By making appropriate estimates on these formulas in combination with duality and interpolation, we establish Young type results for s_p^w spaces under such products.

For Lebesgue and general modulation spaces, the situation is different. In fact, in contrast to spaces of Schatten symbols, it is complicated to find certain results under the Weyl product and the twisted convolution, while finding Hölder–Young results under convolutions and multiplications are straightforward. For example, continuity properties for modulation spaces under the Weyl product have been investigated in, e.g., [23, 27, 30, 40, 51]. In Section 2 we extend these properties by enlarging the family of weights in the definition of modulation and Lebesgue spaces. In particular we prove that $L^2_{(\omega)}$ is an algebra under the twisted convolution, when $\omega(X) = e^{c|X|}$ and $c \geq 0$.

For further considerations we recall some definitions. Let $t \in \mathbf{R}$ be fixed and let $a \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then the *pseudo-differential operator* $\operatorname{Op}_t(a)$ with *symbol* a is the continuous operator on $\mathcal{S}_{1/2}(\mathbf{R}^d)$, defined by the formula

$$(\operatorname{Op}_t(a)f)(x) = (2\pi)^{-d} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi. \tag{0.1}$$

The definition of $\operatorname{Op}_t(a)$ extends to each $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$, and then $\operatorname{Op}_t(a)$ is continuous from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$. (Cf., e.g., [28] or Section 1.) If $t = 1/2$, then $\operatorname{Op}_t(a)$ is equal to the Weyl operator $\operatorname{Op}^w(a)$ for a. If instead $t = 0$, then the standard (Kohn–Nirenberg) representation $a(x, D)$ is obtained.

The modulation spaces were introduced by Feichtinger in [13], and developed further and generalized in [14, 16–18, 22]. We are especially interested in the modulation spaces $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ which are the sets of Gelfand–Shilov distributions on $\mathbf{R}^d$ whose short-time Fourier transform (STFT) belong to the weighted and mixed Lebesgue spaces $L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$ and $L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$ respectively. Here, and $p, q \in [1,\infty]$, and we refer to (1.26) and (1.27) below for the definition of the latter space norms. In contrast to [51], the weight function ω here is allowed to belong to $\mathscr{P}_E(\mathbf{R}^{2d})$, the set of all moderated functions on the phase (or time-frequency shift) space $\mathbf{R}^{2d}$. We remark that the family $\mathscr{P}_E$ contain all polynomial type weights. It follows that ω, p and q to some extent quantify the degrees of asymptotic decay and singularity of the distributions in $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$. (We refer to [15] for a modern description of modulation spaces.)

In the Weyl calculus of pseudo-differential operators, operator composition corresponds on the symbol level to the *Weyl product* $\#$, which on the symplectic Fourier transform side corresponds to the twisted convolution $*_\sigma$. Sometimes, the Weyl product is called the twisted product. A problem in this field is to find conditions on the weight functions ω_j and $p_j, q_j \in [1,\infty]$, for the mappings

$$(a_1, a_2) \mapsto a_1 \# a_2 \quad \text{and} \quad (a_1, a_2) \mapsto a_1 *_\sigma a_2$$

on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ to be uniquely extendable to continuous mappings from

$$\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d}) \quad \text{to} \quad \mathcal{M}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d}),$$

and from

$$\mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d}) \quad \text{to} \quad \mathcal{W}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d}).$$

Here the modulation spaces $\mathcal{M}^{p,q}_{(\omega)}$ and $\mathcal{W}^{p,q}_{(\omega)}$ are obtained by replacing the usual STFT with the symplectic STFT in the definition of modulation space norms. One part of such questions might be to find appropriate conditions on ω_j and $p_j, q_j \in [1,\infty]$ such that

$$\|a_1 *_\sigma a_2\|_{\mathcal{W}^{p_0,q_0}_{(\omega_0)}} \lesssim \|a_1\|_{\mathcal{W}^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{\mathcal{W}^{p_2,q_2}_{(\omega_2)}}, \tag{0.2}$$

when $a_j \in \mathcal{S}_{1/2}$, $j = 0, 1, 2$. Here and in what follows we let $A \lesssim B$ indicate $A \le cB$, for a suitable constant $c > 0$, and we write $A \asymp B$ when $A \lesssim B$ and $B \lesssim A$. Important contributions in this context can be found in [23, 27, 30, 40, 43, 52], where Theorem $0.3'$ in [27] and Theorem 6.4 in [52] seem to be the most general results so far (see also Theorem 2.2).

The result for twisted convolution on modulation spaces which corresponds to Theorem $0.3'$ in [27] and Theorem 6.4 in [52] is given by Theorem 0.1 below. Here the assumptions on the involved weight functions and Lebesgue exponents on the modulation spaces are

$$\omega_0(X,Y) \lesssim \omega_1(X-Y+Z,Z)\omega_2(Y-Z,X+Z), \quad X,Y,Z \in \mathbf{R}^{2d}, \tag{0.3}$$

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0} = 1-\Big(\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0}\Big) \tag{0.4}$$

and

$$0 \le \frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0} \le \frac{1}{p_j}, \frac{1}{q_j} \le \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0}, \quad j=0,1,2. \tag{0.5}$$

Theorem 0.1. *Let $\omega_0,\omega_1,\omega_2 \in \mathscr{P}_E(\mathbf{R}^{4d})$ satisfy* (0.3)*, and that $p_j,q_j \in [1,\infty]$ for $j=0,1,2$, satisfy* (0.4) *and* (0.5)*. Then the map $(a_1,a_2) \mapsto a_1 *_\sigma a_2$ on $\mathcal{S}(\mathbf{R}^{2d})$ extends uniquely to a continuous map from $\mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $\mathcal{W}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, and for some constant $C>0$, the bound* (0.2) *holds for every $a_1 \in \mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.*

In Section 2 we also consider the case when $p_j=q_j=2$, and the involved weights $\omega_j(X,Y)$ are independent of the Y-variable, i.e., $\omega_j(X,Y)=\omega_j(X)$. In this case, $\mathcal{W}^{2,2}_{(\omega_j)}$ agrees with $L^2_{(\omega_j)}$, and the condition (0.3) is reduced to

$$\omega_0(X_1+X_2) \lesssim \omega_1(X_1)\omega_2(X_2) \tag{0.6}$$

Hence, Theorem 0.1 shows that the map $(a_1,a_2) \mapsto a_1 *_\sigma a_2$ extends to a continuous mapping from $L^2_{(\omega_1)} \times L^2_{(\omega_2)}$ to $L^2_{(\omega_0)}$, and that

$$\|a_1 *_\sigma a_2\|_{L^2_{(\omega_0)}} \lesssim \|a_1\|_{L^2_{(\omega_1)}} \|a_2\|_{L^2_{(\omega_2)}}, \tag{0.7}$$

holds when $a_1 \in L^2_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^2_{(\omega_2)}(\mathbf{R}^{2d})$. Here and in what follows, $p' \in [1,\infty]$ denotes the conjugate exponent to $p \in [1,\infty]$, i.e., p and p' should satisfy $1/p+1/p'=1$. The latter property is extended in Section 2 to involve mixed weighted norm spaces of Lebesgue type. As a special case we obtain the following generalization of (0.7).

Theorem 0.2. *Let $\omega_j \in \mathscr{P}_E(\mathbf{R}^{2d})$, and let $p_j \in [1,\infty]$ for $j=0,1,2$ satisfy* (0.6) *and*

$$\max_{j=0,1,2}\Big(\frac{1}{p_j},\frac{1}{p_j'}\Big) \le \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0} \le 1.$$

Then the map $(a_1,a_2) \mapsto a_1 *_\sigma a_2$ extends uniquely to a continuous mapping from $L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $L^{p_0}_{(\omega_0)}(\mathbf{R}^{2d})$, and

$$\|a_1 *_\sigma a_2\|_{L^{p_0}_{(\omega_0)}} \lesssim \|a_1\|_{L^{p_1}_{(\omega_1)}} \|a_2\|_{L^{p_2}_{(\omega_2)}} \tag{0.8}$$

when $a_1 \in L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

Theorem 0.2 and its extensions are used in the end of Section 2 to extend the class of possible window functions in the definition of modulation space norms.

In Section 5 we establish Young type results for dilated multiplications and convolutions for the spaces $s_p^w(\omega_1,\omega_2) \equiv s_p^w(\mathscr{H}_1,\mathscr{H}_2)$, when $\mathscr{H}_j$ for $j=1,2$ is modulation space $M^{2,2}_{(\omega_j)}(\mathbf{R}^d) = M^2_{(\omega_j)}(\mathbf{R}^d)$ with appropriate weights ω_j. The involved Schatten exponents should satisfy the Young condition

$$\frac{1}{p_1}+\frac{1}{p_2} = 1+\frac{1}{r}, \qquad 1\le p_1,p_2,r\le\infty, \tag{0.9}$$

and the involved dilation factors should satisfy

$$(-1)^{j_1}t_1^{-2}+(-1)^{j_2}t_2^{-2}=1 \tag{0.10}$$

or

$$(-1)^{j_1}t_1^{2}+(-1)^{j_2}t_2^{2}=1, \tag{0.11}$$

when $j_1,j_2\in\{0,1\}$. The conditions for the involved weight functions are

$$\begin{aligned}\vartheta(X_1+X_2) &\lesssim \vartheta_{j_1,1}(t_1X_1)\vartheta_{j_2,2}(t_2X_2),\\ \omega(X_1+X_2) &\lesssim \omega_{j_1,1}(t_1X_1)\omega_{j_2,2}(t_2X_2),\end{aligned} \tag{0.12}$$

where

$$\omega_{0,k}(X)=\vartheta_{1,k}(X)=\omega_k(X), \quad \vartheta_{0,k}(X)=\omega_{1,k}(X)=\vartheta_k(X). \tag{0.13}$$

With these conditions we prove

$$\|a_{1,t_1}*a_{2,t_2}\|_{s_r^w(1/\omega,\vartheta)} \le C^d\|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)}\|a_2\|_{s_{p_2}^w(1/\omega_2,\vartheta_2)}, \tag{0.14}$$

$$\|a_{1,t_1}a_{2,t_2}\|_{s_r^w(1/\omega,\vartheta)} \le C^d\|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)}\|a_2\|_{s_{p_2}^w(1/\omega_2,\vartheta_2)}, \tag{0.15}$$

for admissible a_1 and a_2. Here and in what follows we set $a_{j,t}=a_j(t\,\cdot\,)$. More precisely, in Section 5 we prove the following two theorems, as well as multi-linear extensions of these results (cf. Theorems 0.3′ and 0.4′), which generalize Theorem 3.3, Theorem 3.3′ and Corollary 3.5 in [44] and corresponding results in [51]. In fact, these results in [44] follow by letting $\mathscr{H}_1=\mathscr{H}_2=L^2$ in Theorems 0.3′ and 0.4′.

Theorem 0.3. *Let $p_1,p_2,r\in[1,\infty]$ satisfy* (0.9)*, and let $t_1,t_2\in\mathbf{R}\backslash 0$ satisfy* (0.10)*, for some choices of $j_1,j_2\in\{0,1\}$. Also let $\omega,\omega_j,\vartheta,\vartheta_j\in\mathscr{P}_E(\mathbf{R}^{2d})$ for $j=1,2$ satisfy* (0.12) *and* (0.13)*. Then the map $(a_1,a_2)\mapsto a_{1,t_1}*a_{2,t_2}$ on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$, extends uniquely to a continuous mapping from*

$$s_{p_1}^w(1/\omega_1,\vartheta_1)\times s_{p_2}^w(1/\omega_2,\vartheta_2)$$

to $s_r^w(1/\omega,\vartheta)$. Furthermore, (0.14) *holds for some constant*

$$C=C_0^2|t_1|^{-2/p_1}|t_2|^{-2/p_2},$$

where C_0 is independent of $a_1\in s_{p_1}^w(1/\omega_1,\vartheta_1)$, $a_2\in s_{p_2}^w(1/\omega_2,\vartheta_2)$, t_1,t_2 and d.

Moreover, $\operatorname{Op}^w(a_{1,t_1}*a_{2,t_2})\ge 0$ when $\operatorname{Op}^w(a_j)\ge 0$ for each $1\le j\le 2$.

Theorem 0.4. *Let $p_1, p_2, r \in [1,\infty]$ satisfy* (0.9), *and let $t_1, t_2 \in \mathbf{R}\backslash 0$ satisfy* (0.11), *for some choices of $j_1, j_2 \in \{0,1\}$. Also let $\omega, \omega_j, \vartheta, \vartheta_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ for $j = 1,2$ satisfy* (0.12) *and* (0.13). *Then the map $(a_1, a_2) \mapsto a_{1,t_1} a_{2,t_2}$ on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$, extends uniquely to a continuous mapping from*

$$s^w_{p_1}(1/\omega_1, \vartheta_1) \times s^w_{p_2}(1/\omega_2, \vartheta_2)$$

to $s^w_r(1/\omega, \vartheta)$. Furthermore, (0.15) *holds for some constant*

$$C = C_0^2 |t_1|^{-2/p_1'} |t_2|^{-2/p_2'},$$

where C_0 is independent of $a_1 \in s^w_{p_1}(1/\omega_1, \vartheta_1)$, $a_2 \in s^w_{p_2}(1/\omega_2, \vartheta_2)$, t_1, t_2 and d.

Some preparations to the dilated convolution and multiplication results in Section 5 are given in Sections 3 and 4. In Section 3 we introduce the notion of Gelfand–Shilov and Beurling tempered (quasi-)Banach and Hilbert spaces, and prove certain properties. Especially we establish embedding properties between such spaces, modulation spaces and Gelfand–Shilov spaces. These embeddings are also used in [54], when establishing Schatten–von Neumann results for operators with Gelfand–Shilov kernels. Furthermore we investigate certain relations for bases in the Hilbert space case.

In Section 4 we consider dual properties for $s^w_p(\mathscr{H}_1, \mathscr{H}_2)$. Here $\mathscr{H}_1$ and $\mathscr{H}_2$ belong to a broad class of Hilbert spaces containing any $M^{2,2}_{(\omega)}$ space. More precisely, assume that $p < \infty$. Then we prove that the dual for $s^w_p(\mathscr{H}_1, \mathscr{H}_2)$ can be identified with $s^w_p(\mathscr{H}_1', \mathscr{H}_2')$ for appropriate Hilbert spaces $\mathscr{H}_1'$ and $\mathscr{H}_2'$ through a unique extension of the L^2 form on $\mathcal{S}_{1/2}$. (Cf. Theorem 4.8.) In the last part of Section 4 we show some properties on bases and Hilbert–Schmidt operators. We use these results to establish estimates for generalized gamma functions evaluated in integer points (cf. Example 3.6).

In the last section we apply the results in Section 5 to prove that the class of trace-class symbols is invariant under compositions with odd entire functions. Here we also show how Theorem 0.3 can be used to define Toeplitz operators with symbols in dilated s^w_p spaces, and that such operators fulfill certain Schatten–von Neumann properties.

1. Preliminaries

In this section we introduce some notations and discuss basic results. We start by recalling some facts concerning Gelfand–Shilov spaces. Thereafter we recall some properties about pseudo-differential operators. Especially we discuss the Weyl product and twisted convolution. Finally we recall some facts about modulation spaces. The proofs are in general omitted, since the results can be found in the literature.

We start by considering Gelfand–Shilov spaces. Let $0 < h, s \in \mathbf{R}$ be fixed. Then $\mathcal{S}_{s,h}(\mathbf{R}^d)$ consists of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha|+|\beta|}\alpha!^s\,\beta!^s}$$

is finite. Here the supremum should be taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$.

Obviously $\mathcal{S}_{s,h} \hookrightarrow \mathscr{S}$ is a Banach space which increases with h and s. Here and in what follows we use the notation $A \hookrightarrow B$ when the topological spaces A and B satisfy $A \subseteq B$ with continuous embeddings. Furthermore, if $s > 1/2$, or $s = 1/2$ and h is sufficiently large, then $\mathcal{S}_{s,h}$ contains all finite linear combinations of Hermite functions. Since such linear combinations are dense in $\mathscr{S}$, it follows that the dual $(\mathcal{S}_{s,h})'(\mathbf{R}^d)$ of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is a Banach space which contains $\mathscr{S}'(\mathbf{R}^d)$.

The *Gelfand–Shilov spaces* $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ are the inductive and projective limits respectively of $\mathcal{S}_{s,h}(\mathbf{R}^d)$. This implies that

$$\mathcal{S}_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d), \tag{1.1}$$

and that the topology for $\mathcal{S}_s(\mathbf{R}^d)$ is the strongest possible one such that the inclusion map from $\mathcal{S}_{s,h}(\mathbf{R}^d)$ to $\mathcal{S}_s(\mathbf{R}^d)$ is continuous, for every choice of $h > 0$. The space $\Sigma_s(\mathbf{R}^d)$ is a Fréchet space with semi norms $\|\cdot\|_{\mathcal{S}_{s,h}}$, $h > 0$. Moreover, $\Sigma_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s > 1/2$, and $\mathcal{S}_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s \geq 1/2$. From now on we assume that $s > 1/2$ when considering $\Sigma_s(\mathbf{R}^d)$, and $s \geq 1/2$ when considering $\mathcal{S}_s(\mathbf{R}^d)$

The *Gelfand–Shilov distribution spaces* $\mathcal{S}_s'(\mathbf{R}^d)$ and $\Sigma_s'(\mathbf{R}^d)$ are the projective and inductive limit respectively of $\mathcal{S}_s'(\mathbf{R}^d)$. This means that

$$\mathcal{S}_s'(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}'(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s'(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}'(\mathbf{R}^d). \tag{1.1$'$}$$

We remark that in [20, 29, 34] it is proved that $\mathcal{S}_s'(\mathbf{R}^d)$ is the dual of $\mathcal{S}_s(\mathbf{R}^d)$, and $\Sigma_s'(\mathbf{R}^d)$ is the dual of $\Sigma_s(\mathbf{R}^d)$ (also in topological sense).

For each $\varepsilon > 0$ and $s > 1/2$ we have

$$\begin{aligned} \mathcal{S}_{1/2}(\mathbf{R}^d) &\hookrightarrow \Sigma_s(\mathbf{R}^d) \hookrightarrow \mathcal{S}_s(\mathbf{R}^d) \hookrightarrow \Sigma_{s+\varepsilon}(\mathbf{R}^d) \\ \text{and} \quad \Sigma_{s+\varepsilon}'(\mathbf{R}^d) &\hookrightarrow \mathcal{S}_s'(\mathbf{R}^d) \hookrightarrow \Sigma_s'(\mathbf{R}^d) \hookrightarrow \mathcal{S}_{1/2}'(\mathbf{R}^d). \end{aligned} \tag{1.2}$$

The Gelfand–Shilov spaces are invariant under several basic transformations. For example they are invariant under translations, dilations, tensor products and under (partial) Fourier transformations.

From now on we let $\mathscr{F}$ be the Fourier transform which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. Here $\langle \cdot , \cdot \rangle$ denotes the usual scalar product on $\mathbf{R}^d$. The map $\mathscr{F}$ extends uniquely to homeomorphisms on $\mathscr{S}'(\mathbf{R}^d)$, $\mathcal{S}_s'(\mathbf{R}^d)$ and $\Sigma_s'(\mathbf{R}^d)$, and restricts to homeomorphisms on $\mathscr{S}(\mathbf{R}^d)$, $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$.

It follows from the following lemma that elements in Gelfand–Shilov spaces can be characterized by estimates of the form

$$|f(x)| \lesssim e^{-\varepsilon|x|^{1/s}} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-\varepsilon|\xi|^{1/s}}. \tag{1.3}$$

The proof is omitted, since the result can be found in, e.g., [4, 20].

Lemma 1.1. *Let $f \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$. Then the following is true:*

(1) *if $s \ge 1/2$, then $f \in \mathcal{S}_s(\mathbf{R}^d)$, if and only if* (1.3) *holds for some $\varepsilon > 0$;*
(2) *if $s > 1/2$, then $f \in \Sigma_s(\mathbf{R}^d)$, if and only if* (1.3) *holds for each $\varepsilon > 0$.*

Gelfand–Shilov spaces can also easily be characterized by Hermite functions. We recall that the Hermite function h_α with respect to the multi-index $\alpha \in \mathbf{N}^d$ is defined by

$$h_\alpha(x) = \pi^{-d/4}(-1)^{|\alpha|}(2^{|\alpha|}\alpha!)^{-1/2}e^{|x|^2/2}(\partial^\alpha e^{-|x|^2}).$$

The set $\{h_\alpha\}_{\alpha\in\mathbf{N}^d}$ is an orthonormal basis for $L^2(\mathbf{R}^d)$. In particular,

$$f = \sum_\alpha c_\alpha h_\alpha, \quad c_\alpha = (f, h_\alpha)_{L^2(\mathbf{R}^d)}, \tag{1.4}$$

and

$$\|f\|_{L^2} = \|\{c_\alpha\}_\alpha\|_{l^2} < \infty,$$

when $f \in L^2(\mathbf{R}^d)$. Here and in what follows, $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbf{R}^d)}$ denotes any continuous extension of the L^2 form on $\mathcal{S}_{1/2}(\mathbf{R}^d)$.

The Hermite expansions can also be used to characterize distributions and their test function spaces. More precisely, let $p \in [1, \infty]$ be fixed. Then it is well known that f here belongs to $\mathscr{S}(\mathbf{R}^d)$, if and only if

$$\|\{c_\alpha\langle\alpha\rangle^t\}_\alpha\|_{l^p} < \infty \tag{1.5}$$

for every $t \in \mathbf{R}$. Furthermore, for every $f \in \mathscr{S}'(\mathbf{R}^d)$, the expansion (1.4) still holds, where the sum converges in $\mathscr{S}'$, and (1.5) holds for some choice of $t \in \mathbf{R}$, which depends on f.

The following proposition, which can be found in, e.g., [21], shows that similar conclusion for Gelfand–Shilov spaces hold, after the estimate (1.5) is replaced by

$$\|\{c_\alpha e^{t|\alpha|^{1/2s}}\}_\alpha\|_{l^p} < \infty. \tag{1.6}$$

(Cf. formula (2.12) in [21].)

Proposition 1.2. *Let $p \in [1, \infty]$, $f \in \mathcal{S}_{1/2}'\mathbf{R}^d)$, $s_1 \ge 1/2$, $s_2 > 1/2$ and let c_α be as in* (1.4)*. Then the following is true:*

(1) *$f \in \mathcal{S}_{s_1}'(\mathbf{R}^d)$, if and only if* (1.6) *holds with $s = s_1$ for every $t < 0$. Furthermore,* (1.4) *holds where the sum converges in $\mathcal{S}_{s_1}'$;*
(2) *$f \in \Sigma_{s_2}'(\mathbf{R}^d)$, if and only if* (1.6) *holds with $s = s_2$ for some $t < 0$. Furthermore,* (1.4) *holds where the sum converges in Σ_{s_2}';*

(3) $f \in \mathcal{S}_{s_1}(\mathbf{R}^d)$, *if and only if* (1.6) *holds with* $s = s_1$ *for some* $t > 0$. *Furthermore,* (1.4) *holds where the sum converges in* $\mathcal{S}_{s_1}$*;*

(4) $f \in \Sigma_{s_2}(\mathbf{R}^d)$, *if and only if* (1.6) *holds with* $s = s_2$ *for every* $t > 0$. *Furthermore,* (1.4) *holds where the sum converges in* Σ_{s_2}.

Next we recall some properties in pseudo-differential calculus. Let $s \geq 1/2$, $a \in \mathcal{S}_s(\mathbf{R}^{2d})$, and $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\mathrm{Op}_t(a)$ in (0.1) is a linear and continuous operator on $\mathcal{S}_s(\mathbf{R}^d)$. For general $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$, the pseudo-differential operator $\mathrm{Op}_t(a)$ is defined as the continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$ with distribution kernel given by

$$K_{a,t}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)((1-t)x + ty, x - y). \tag{1.7}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathcal{S}_s'(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense, since the mappings

$$\mathscr{F}_2 \quad \text{and} \quad F(x,y) \mapsto F((1-t)x + ty, y - x) \tag{1.8}$$

are homeomorphisms on $\mathcal{S}_s'(\mathbf{R}^{2d})$. In particular, the map $a \mapsto K_{a,t}$ is a homeomorphism on $\mathcal{S}_s'(\mathbf{R}^{2d})$.

For any $K \in \mathcal{S}_s'(\mathbf{R}^{d_1+d_2})$, we let T_K be the linear and continuous mapping from $\mathcal{S}_s(\mathbf{R}^{d_1})$ to $\mathcal{S}_s'(\mathbf{R}^{d_2})$, defined by the formula

$$(T_K f, g)_{L^2(\mathbf{R}^{d_2})} = (K, g \otimes \overline{f})_{L^2(\mathbf{R}^{d_1+d_2})}. \tag{1.9}$$

It is well known that if $t \in \mathbf{R}$, then it follows from Schwartz kernel theorem that $K \mapsto T_K$ and $a \mapsto \mathrm{Op}_t(a)$ are bijective mappings from $\mathscr{S}'(\mathbf{R}^{2d})$ to the set of linear and continuous mappings from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ (cf., e.g., [28]).

In this context we remark that the maps $K \mapsto T_K$ and $a \mapsto \mathrm{Op}_t(a)$ are uniquely extendable to bijective mappings from $\mathcal{S}_s'(\mathbf{R}^{2d})$ to the set of linear and continuous mappings from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$. In fact, the asserted bijectivity for the map $K \mapsto T_K$ follows from the kernel theorem [32, Theorem 2.2], by Lozanov–Crvenković, Perišić and Taskovic. This kernel theorem corresponds to Schwartz kernel theorem in the usual distribution theory. The other assertion follows from the fact that the map $a \mapsto K_{a,t}$ is a homeomorphism on $\mathcal{S}_s'$.

In particular, for each $a_1 \in \mathcal{S}_s'(\mathbf{R}^{2d})$ and $t_1, t_2 \in \mathbf{R}$, there is a unique $a_2 \in \mathcal{S}_s'(\mathbf{R}^{2d})$ such that $\mathrm{Op}_{t_1}(a_1) = \mathrm{Op}_{t_2}(a_2)$. The relation between a_1 and a_2 is given by

$$\mathrm{Op}_{t_1}(a_1) = \mathrm{Op}_{t_2}(a_2) \qquad \Longleftrightarrow \qquad a_2(x,\xi) = e^{i(t_2-t_1)\langle D_x, D_\xi\rangle} a_1(x,\xi). \tag{1.10}$$

(Cf. [28].) Note here that the right-hand side makes sense, since it is equivalent to $\widehat{a}_2 = e^{i(t_2-t_1)\langle x,\xi\rangle}\widehat{a}_1$, and that the map $a \mapsto e^{it\langle x,\xi\rangle}a$ is continuous on $\mathcal{S}_s'$.

Let $t \in \mathbf{R}$ and $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$ be fixed. Then a is called a rank-one element with respect to t, if the corresponding pseudo-differential operator is of rank-one, i.e.,

$$\mathrm{Op}_t(a)f = (f, f_2)f_1, \qquad f \in \mathcal{S}_s(\mathbf{R}^d), \tag{1.11}$$

for some $f_1, f_2 \in \mathcal{S}_s'(\mathbf{R}^d)$. By straightforward computations it follows that (1.11) is fulfilled, if and only if $a = (2\pi)^{d/2} W_{f_1,f_2}^t$, where W_{f_1,f_2}^t is the t-Wigner distribution, defined by the formula

$$W^t_{f_2,f_1}(x,\xi) \equiv \mathscr{F}(f_2(x+t\,\cdot\,)\overline{f_1(x-(1-t)\,\cdot\,)})(\xi), \tag{1.12}$$

which takes the form

$$W^t_{f_2,f_1}(x,\xi) = (2\pi)^{-d/2}\int f_2(x+ty)\overline{f_1(x-(1-t)y)}e^{-i\langle y,\xi\rangle}\,dy,$$

when $f_1, f_2 \in \mathcal{S}_s(\mathbf{R}^d)$. By combining these facts with (1.10), it follows that

$$W^{t_2}_{f_2,f_1} = e^{i(t_2-t_1)\langle D_x,D_\xi\rangle}W^{t_1}_{f_2,f_1}, \tag{1.13}$$

for each $f_1, f_2 \in \mathcal{S}_s'(\mathbf{R}^d)$ and $t_1, t_2 \in \mathbf{R}$. Since the Weyl case is particularly important, we set $W^t_{f_2,f_1} = W_{f_2,f_1}$ when $t = 1/2$, i.e., W_{f_2,f_1} is the usual (cross-)Wigner distribution of f_1 and f_2.

For future references we note the link

$$\begin{aligned}(\operatorname{Op}_t(a)f,g)_{L^2(\mathbf{R}^d)} &= (2\pi)^{-d/2}(a,W^t_{g,f})_{L^2(\mathbf{R}^{2d})},\\ &\qquad a\in\mathcal{S}_s'(\mathbf{R}^{2d}) \quad\text{and}\quad f,g\in\mathcal{S}_s(\mathbf{R}^d)\end{aligned} \tag{1.14}$$

between pseudo-differential operators and Wigner distributions, which follows by straightforward computations (see also, e.g., [10, 11]).

Next we discuss the Weyl product, twisted convolution and related objects. Let $s \ge 1/2$ and let $a, b \in \mathcal{S}_s'(\mathbf{R}^{2d})$ be appropriate. Then the Weyl product $a\#b$ between a and b is the function or distribution which fulfills $\operatorname{Op}^w(a\#b) = \operatorname{Op}^w(a)\circ \operatorname{Op}^w(b)$, provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$. More general, if $t \in \mathbf{R}$, then the product $\#_t$ is defined by the formula

$$\operatorname{Op}_t(a\#_t b) = \operatorname{Op}_t(a)\circ\operatorname{Op}_t(b), \tag{1.15}$$

provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$.

The Weyl product can also, in a convenient way, be expressed in terms of the symplectic Fourier transform and twisted convolution. More precisely, let $s \ge 1/2$. Then the *symplectic Fourier transform* for $a \in \mathcal{S}_s(\mathbf{R}^{2d})$ is defined by the formula

$$(\mathscr{F}_\sigma a)(X) = \pi^{-d}\int a(Y)e^{2i\sigma(X,Y)}\,dY,$$

where σ is the symplectic form, given by

$$\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle, \qquad X=(x,\xi)\in\mathbf{R}^{2d},\ Y=(y,\eta)\in\mathbf{R}^{2d}.$$

We note that $\mathscr{F}_\sigma = T\circ(\mathscr{F}\otimes(\mathscr{F}^{-1}))$, when $(Ta)(x,\xi) = 2^d a(2\xi,2x)$. In particular, $\mathscr{F}_\sigma$ is continuous on $\mathcal{S}_s(\mathbf{R}^{2d})$, and extends uniquely to a homeomorphism on $\mathcal{S}_s'(\mathbf{R}^{2d})$, and to a unitary map on $L^2(\mathbf{R}^{2d})$, since similar facts hold for $\mathscr{F}$. Furthermore, $\mathscr{F}_\sigma^2$ is the identity operator.

Let $s \ge 1/2$ and $a, b \in \mathcal{S}_s(\mathbf{R}^{2d})$. Then the *twisted convolution* of a and b is defined by the formula

$$(a*_\sigma b)(X) = (2/\pi)^{d/2}\int a(X-Y)b(Y)e^{2i\sigma(X,Y)}\,dY. \tag{1.16}$$

The definition of $*_\sigma$ extends in different ways. For example, it extends to a continuous multiplication on $L^p(\mathbf{R}^{2d})$ when $p \in [1,2]$, and to a continuous map from $\mathcal{S}_s'(\mathbf{R}^{2d}) \times \mathcal{S}_s(\mathbf{R}^{2d})$ to $\mathcal{S}_s'(\mathbf{R}^{2d})$. If $a, b \in \mathcal{S}_s'(\mathbf{R}^{2d})$, then $a\#b$ makes sense if and only if $a *_\sigma \widehat{b}$ makes sense, and then

$$a\#b = (2\pi)^{-d/2} a *_\sigma (\mathscr{F}_\sigma b). \tag{1.17}$$

We also remark that for the twisted convolution we have

$$\mathscr{F}_\sigma(a *_\sigma b) = (\mathscr{F}_\sigma a) *_\sigma b = \check{a} *_\sigma (\mathscr{F}_\sigma b), \tag{1.18}$$

where $\check{a}(X) = a(-X)$ (cf. [42, 44, 45]). A combination of (1.17) and (1.18) gives

$$\mathscr{F}_\sigma(a\#b) = (2\pi)^{-d/2}(\mathscr{F}_\sigma a) *_\sigma (\mathscr{F}_\sigma b). \tag{1.19}$$

In the Weyl calculus it is in several situations convenient to use the operator A on $\mathcal{S}_s'(\mathbf{R}^{2d})$, defined by the formula

$$Aa(x,y) = (\mathscr{F}_2^{-1}a)((y-x)/2, -(x+y)), \quad a \in \mathcal{S}_s'(\mathbf{R}^{2d}). \tag{1.20}$$

Here and in what follows we identify operators with their distribution kernels. We note that $Aa(x,y)$ agrees with $(2\pi)^{d/2}K_a^w(-x,y)$, where K_a^w is the distribution kernel to the Weyl operator $\operatorname{Op}^w(a)$. If $a \in \mathcal{S}_s(\mathbf{R}^{2d})$, then Aa is given by

$$Aa(x,y) = (2\pi)^{-d/2} \int a((y-x)/2, \xi) e^{-i\langle x+y, \xi\rangle}\, dy.$$

In particular, the map $a \mapsto Aa$ is bijective from $\mathcal{S}_s'(\mathbf{R}^{2d})$ to the set of linear and continuous operators from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^d)$, since similar facts are true for the Weyl quantization.

The operator A is important when using the twisted convolution, because for each $a, b \in \mathcal{S}_s(\mathbf{R}^{2d})$ we have

$$A(a *_\sigma b) = Aa \circ Ab. \tag{1.21}$$

(See [19, 42, 44, 45].)

In the following lemma we list some facts about the operator A. The result is a consequence of Fourier's inversion formula, and the verifications are left for the reader.

Lemma 1.3. *Let $s \ge 1/2$, A be as above, $a, a_1, a_2, b \in \mathcal{S}_s'(\mathbf{R}^{2d})$, where at least two of a_1, a_2, b should belong to $\mathcal{S}_s'(\mathbf{R}^{2d})$, and set $U = Aa$. Then the following is true:*

(1) *$\check{U} = A\check{a}$, if $\check{a}(X) = a(-X)$;*

(2) *$J_{\mathscr{F}}U = A\mathscr{F}_\sigma a$, where $J_{\mathscr{F}}U(x,y) = U(-x,y)$;*

(3) *$A(\mathscr{F}_\sigma a) = (2\pi)^{d/2}\operatorname{Op}^w(a)$ and $(\operatorname{Op}^w(a)f, g) = (2\pi)^{-d/2}(Aa, \check{g} \otimes \overline{f})$ when $f, g \in \mathcal{S}_{1/2}(\mathbf{R}^d)$;*

(4) *the Hilbert space adjoint of Aa equals $A\widetilde{a}$, where $\widetilde{a}(X) = \overline{a(-X)}$. Furthermore,*

$$(a_1 *_\sigma a_2, b) = (a_1, b *_\sigma \widetilde{a}_2) = (a_2, \widetilde{a}_1 *_\sigma b),$$

$$(a_1 *_\sigma a_2) *_\sigma b = a_1 *_\sigma (a_2 *_\sigma b).$$

A linear and continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}_s'(\mathbf{R}^n)$ is called positive semi-definite (of order $s \ge 1/2$) when $(Tf,f)_{L^2} \ge 0$ for every $f \in \mathcal{S}_s(\mathbf{R}^d)$. We write $T \ge 0$ when T is positive semi-definite or order s. A distribution $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$ is called σ-positive (of order s) if Aa is a positive semi-definite operator. The set of all σ-positive distributions on $\mathbf{R}^{2d}$ is denoted by $\mathcal{S}_{s,+}'(\mathbf{R}^{2d})$. Since $\mathcal{S}_s$ increases with s and that $\mathcal{S}_{1/2}$ is dense in $\mathcal{S}_s$, it follows that

$$\mathcal{S}_{t,+}'(\mathbf{R}^{2d}) = \mathcal{S}_{s,+}'(\mathbf{R}^{2d}) \bigcap \mathcal{S}_t'(\mathbf{R}^{2d}), \qquad t \ge s.$$

The following result is an immediate consequence of Lemma 1.3.

Proposition 1.4. *Let $s \ge 1/2$ and $a \in \mathcal{S}_s'(\mathbf{R}^{2d})$. Then*

$$a \in \mathcal{S}_{s,+}'(\mathbf{R}^{2d}) \quad \Longleftrightarrow \quad Aa \ge 0 \text{ as operator} \quad \Longleftrightarrow \quad \operatorname{Op}^w(\mathscr{F}_\sigma a) \ge 0.$$

We refer to [44, 45] for more facts about σ-positive functions and distributions in the framework of tempered distributions.

In the end of Section 5 we also discuss continuity for Toeplitz operators. Let $s \ge 1/2$, $a \in \mathcal{S}_s(\mathbf{R}^{2d})$ and $h_1, h_2 \in \mathcal{S}_s(\mathbf{R}^d)$. Then the Toeplitz operator $\operatorname{Tp}_{h_1,h_2}(a)$, with symbol a, and window functions h_1 and h_2, is defined by the formula

$$(\operatorname{Tp}_{h_1,h_2}(a)f_1, f_2) = (a(2\,\cdot\,)W_{f_1,h_1}, W_{f_2,h_2}) \tag{1.22}$$

when $f_1, f_2 \in \mathcal{S}_s(\mathbf{R}^d)$. The definition of $\operatorname{Tp}_{h_1,h_2}(a)$ extends in several ways (cf., e.g., [6, 26, 42, 44, 46, 49, 50, 52]).

In several of these extensions as well as in Section 5, we interpret Toeplitz operators as pseudo-differential operators, using the fact that

$$\begin{aligned} \operatorname{Tp}_{h_1,h_2}(a) &= \operatorname{Op}_t(a * u) \quad \text{when} \\ u(X) &= (2\pi)^{-d/2} W_{h_2,h_1}^t(-X), \end{aligned} \tag{1.23}$$

h_1, h_2 are suitable window functions on $\mathbf{R}^d$ and a is an appropriate distribution on $\mathbf{R}^{2d}$. The relation (1.23) is well known when $t = 0$ or $t = 1/2$ (cf., e.g., [6, 8, 38, 42, 44, 46–48, 50]). For general t, (1.23) is an immediate consequence of the case $t = 1/2$, (1.13), and the fact that

$$e^{it\langle D_x, D_\xi\rangle}(a * u) = a * (e^{it\langle D_x, D_\xi\rangle} u),$$

which follows by integration by parts.

Next we discuss basic properties for modulation spaces, and start by recalling the conditions for the involved weight functions. Let $0 < \omega, v \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$. Then ω is called *moderate* or *v-moderate* if

$$\omega(x+y) \lesssim \omega(x)v(y), \quad x, y \in \mathbf{R}^d. \tag{1.24}$$

Here the function v is called *submultiplicative*, if (1.24) holds when $\omega = v$. We note that if (1.24) holds, then

$$v(-x)^{-1} \lesssim \omega(x) \lesssim v(x).$$

Furthermore, for such ω it follows that (1.24) is true when

$$v(x) = Ce^{c|x|},$$

for some positive constants c and C. In particular, if ω is moderate on $\mathbf{R}^d$, then

$$e^{-c|x|} \lesssim \omega(x) \lesssim e^{c|x|},$$

for some constant $c > 0$.

The set of all moderate functions on $\mathbf{R}^d$ is denoted by $\mathscr{P}_E(\mathbf{R}^d)$. Furthermore, if v in (1.24) can be chosen as a polynomial, then ω is called of polynomial type, or polynomially moderate. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. If $\omega(x,\xi) \in \mathscr{P}_E(\mathbf{R}^{2d})$ is constant with respect to the x-variable (ξ-variable), then we write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}_E(\mathbf{R}^{2d})$ or in $\mathscr{P}_E(\mathbf{R}^d)$ depending on the situation.

Let $\phi \in \mathcal{S}_s'(\mathbf{R}^d)$ be fixed. Then the *short-time Fourier transform* $V_\phi f$ of $f \in \mathcal{S}_s'(\mathbf{R}^d)$ with respect to the *window function* ϕ is the Gelfand–Shilov distribution on $\mathbf{R}^{2d}$, defined by

$$V_\phi f(x,\xi) \equiv (\mathscr{F}_2(U(f \otimes \phi)))(x,\xi) = \mathscr{F}(f\,\overline{\phi(\,\cdot\, - x)})(\xi),$$

where $(UF)(x,y) = F(y, y-x)$. If $f, \phi \in \mathcal{S}_s(\mathbf{R}^d)$, then it follows that

$$V_\phi f(x,\xi) = (2\pi)^{-d/2} \int f(y)\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle}\, dy.$$

We recall that the short-time Fourier transform is closely related to the Wigner distribution, because

$$W_{f,\phi} f(x,\xi) = 2^d e^{2i\langle x,\xi\rangle} V_{\check\phi} f(2x, 2\xi), \tag{1.25}$$

which follows by elementary manipulations. In particular, Toeplitz operators can be expressed by the formula

$$(\mathrm{Tp}_{h_1,h_2}(a) f_1, f_2) = (aV_{\check h_1} f_1, V_{\check h_2} f_2). \tag{1.22$'$}$$

Let $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$, $p,q \in [1,\infty]$ and $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)\setminus 0$ be fixed. Then the mixed Lebesgue space $L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{1,(\omega)}} < \infty$, and $L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{2,(\omega)}} < \infty$. Here

$$\|F\|_{L^{p,q}_{1,(\omega)}} = \Big(\int \Big(\int |F(x,\xi)\omega(x,\xi)|^p\, dx \Big)^{q/p} d\xi \Big)^{1/q}, \tag{1.26}$$

and

$$\|F\|_{L^{p,q}_{2,(\omega)}} = \Big(\int \Big(\int |F(x,\xi)\omega(x,\xi)|^q\, d\xi \Big)^{p/q} dx \Big)^{1/p}, \tag{1.27}$$

with obvious modifications when $p = \infty$ or $q = \infty$. We note that these norms might attain $+\infty$.

The *modulation spaces* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are the Banach spaces which consist of all $f \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$ such that $\|f\|_{M^{p,q}_{(\omega)}} < \infty$ and $\|f\|_{W^{p,q}_{(\omega)}} < \infty$ respectively. Here

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{1,(\omega)}}, \quad \text{and} \quad \|f\|_{W^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{2,(\omega)}}. \tag{1.28}$$

We remark that the definitions of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are independent of the choice of $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d) \setminus 0$ and different ϕ gives rise to equivalent norms. (See Proposition 1.5 below.) From the fact that

$$V_{\widehat{\phi}}\widehat{f}(\xi,-x) = e^{i\langle x,\xi\rangle} V_{\check{\phi}} f(x,\xi), \qquad \check{\phi}(x) = \phi(-x), \tag{1.29}$$

it follows that

$$f \in W^{q,p}_{(\omega)}(\mathbf{R}^d) \quad \Longleftrightarrow \quad \widehat{f} \in M^{p,q}_{(\omega_0)}(\mathbf{R}^d), \qquad \omega_0(\xi,-x) = \omega(x,\xi).$$

For convenience we set $M^p_{(\omega)} = M^{p,p}_{(\omega)}$, which agrees with $W^p_{(\omega)} = W^{p,p}_{(\omega)}$. Furthermore we set $M^{p,q} = M^{p,q}_{(\omega)}$ and $W^{p,q} = W^{p,q}_{(\omega)}$ when $\omega \equiv 1$.

The proof of the following proposition is omitted, since the results can be found in [5, 12, 13, 16–18, 22, 46–49, 52]. Here we recall that $p, p' \in [1,\infty]$ satisfy $1/p + 1/p' = 1$.

Proposition 1.5. *Let $p, q, p_j, q_j \in [1,\infty]$ for $j = 1,2$, and $\omega, \omega_1, \omega_2, v \in \mathscr{P}_E(\mathbf{R}^{2d})$ be such that $v = \check{v}$, ω is v-moderate and $\omega_2 \lesssim \omega_1$. Then the following is true:*

(1) *$f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if (1.28) holds for any $\phi \in M^1_{(v)}(\mathbf{R}^d) \setminus 0$. Moreover, $M^{p,q}_{(\omega)}$ is a Banach space under the norm in (1.28) and different choices of ϕ give rise to equivalent norms;*

(2) *if $p_1 \le p_2$ and $q_1 \le q_2$ then*

$$\Sigma_1(\mathbf{R}^d) \hookrightarrow M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d) \hookrightarrow M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d) \hookrightarrow \Sigma_1'(\mathbf{R}^d).$$

(3) *the L^2 product $(\,\cdot\,,\,\cdot\,)_{L^2}$ on $\mathcal{S}_{1/2}$ extends uniquely to a continuous map from $M^{p,q}_{(\omega)}(\mathbf{R}^n) \times M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$ to $\mathbf{C}$. On the other hand, if $\|a\| = \sup|(a,b)|$, where the supremum is taken over all $b \in \mathcal{S}_{1/2}(\mathbf{R}^d)$ such that $\|b\|_{M^{p',q'}_{(1/\omega)}} \le 1$, then $\|\cdot\|$ and $\|\cdot\|_{M^{p,q}_{(\omega)}}$ are equivalent norms;*

(4) *if $p, q < \infty$, then $\mathcal{S}_{1/2}(\mathbf{R}^d)$ is dense in $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and the dual space of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$, through the L^2-form $(\,\cdot\,,\,\cdot\,)_{L^2}$. Moreover, $\mathcal{S}_{1/2}(\mathbf{R}^d)$ is weakly dense in $M^{p',q'}_{(\omega)}(\mathbf{R}^d)$ with respect to the L^2-form.*

Similar facts hold if the $M^{p,q}_{(\omega)}$ spaces are replaced by $W^{p,q}_{(\omega)}$ spaces.

Proposition 1.5 (1) allows us be rather vague concerning the choice of $\phi \in M^1_{(v)} \setminus 0$ in (1.28). For example, if $C > 0$ is a constant and $\mathscr{A}$ is a subset of $\mathcal{S}_{1/2}'$, then $\|a\|_{M^{p,q}_{(\omega)}} \le C$ for every $a \in \mathscr{A}$, means that the inequality holds for some choice of $\phi \in M^1_{(v)} \setminus 0$ and every $a \in \mathscr{A}$. Evidently, a similar inequality is true for any other choice of $\phi \in M^1_{(v)} \setminus 0$, with a suitable constant, larger than C if necessary.

Remark 1.6. By Theorem 3.9 in [52] and Proposition 1.5 (2) it follows that

$$\bigcap_{\omega \in \mathscr{P}_E} M^{p,q}_{(\omega)}(\mathbf{R}^d) = \Sigma_1(\mathbf{R}^d), \qquad \bigcup_{\omega \in \mathscr{P}_E} M^{p,q}_{(\omega)}(\mathbf{R}^d) = \Sigma_1'(\mathbf{R}^d)$$

More generally, let $s \geq 1$, and let $\mathcal{P}$ be the set of all $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ such that

$$\omega(x+y,\xi+\eta) \lesssim \omega(x,\xi) e^{c(|y|^{1/s}+|\eta|^{1/s})},$$

for some $c > 0$. Then

$$\bigcap_{\omega \in \mathcal{P}} M^{p,q}_{(\omega)}(\mathbf{R}^d) = \Sigma_s(\mathbf{R}^d), \qquad \bigcup_{\omega \in \mathcal{P}} M^{p,q}_{(1/\omega)}(\mathbf{R}^d) = \Sigma_s'(\mathbf{R}^d)$$
$$\bigcup_{\omega \in \mathcal{P}} M^{p,q}_{(\omega)}(\mathbf{R}^d) = \mathcal{S}_s(\mathbf{R}^d) \quad \text{and} \quad \bigcap_{\omega \in \mathcal{P}} M^{p,q}_{(1/\omega)}(\mathbf{R}^d) = \mathcal{S}_s'(\mathbf{R}^d),$$

and that

$$\Sigma_s(\mathbf{R}^d) \hookrightarrow M^{p,q}_{(\omega)}(\mathbf{R}^d) \hookrightarrow \mathcal{S}_s(\mathbf{R}^d) \quad \text{and} \quad \mathcal{S}_s'(\mathbf{R}^d) \hookrightarrow M^{p,q}_{(1/\omega)}(\mathbf{R}^d) \hookrightarrow \Sigma_s'(\mathbf{R}^d).$$

(Cf. Proposition 4.5 in [9], Proposition 4. in [25], Corollary 5.2 in [35] or Theorem 4.1 in [41]. See also [52, Theorem 3.9] for an extension of these inclusions to broader classes of Gelfand–Shilov and modulation spaces.)

We refer to Example 3.4 below and to [51, Remark 1.4] for other examples on interesting modulation spaces.

We finish the section by giving some remarks on the symplectic short-time Fourier transform. The symplectic short-time Fourier transform of $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ with respect to the window function $\Phi \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ is defined by

$$\mathcal{V}_\Phi a(X,Y) = \mathscr{F}_\sigma\big(a\,\Phi(\,\cdot\,-X)\big)(Y), \quad X,Y \in \mathbf{R}^{2d}.$$

Let $\omega \in \mathscr{P}_E(\mathbf{R}^{4d})$. Then $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ and $\mathcal{W}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ denote the modulation spaces, where the symplectic short-time Fourier transform is used instead of the usual short-time Fourier transform in the definitions of the norms. It follows that any property valid for $M^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ or $W^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ carry over to $\mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ and $\mathcal{W}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ respectively. For example, for the symplectic short-time Fourier transform we have

$$\mathcal{V}_{\mathscr{F}_\sigma \Phi}(\mathscr{F}_\sigma a)(X,Y) = e^{2i\sigma(Y,X)}\mathcal{V}_\Phi a(Y,X), \tag{1.30}$$

(cf. (1.29)) which implies that

$$\mathscr{F}_\sigma \mathcal{M}^{p,q}_{(\omega)}(\mathbf{R}^{2d}) = \mathcal{W}^{q,p}_{(\omega_0)}(\mathbf{R}^{2d}), \qquad \omega_0(X,Y) = \omega(Y,X). \tag{1.31}$$

2. Twisted convolution on modulation spaces and Lebesgue spaces

In this section we discuss algebraic properties of the twisted convolution when acting on modulation spaces of the form $\mathcal{W}^{p,q}_{(\omega)}$. The most general result corresponds to Theorem 0.3′ in [27], which concerns continuity for the Weyl product on modulation spaces of the form $\mathcal{M}^{p,q}_{(\omega)}$. Thereafter we use this result to establish continuity properties for the twisted convolution when acting on weighted Lebesgue spaces. We will mainly follow the analysis in Section 2 in [51], and the proofs are similar.

In these investigations we need the following lemma, which is strongly related to Lemma 4.4 in [43] and Lemma 2.1 in [27]. The latter results were fundamental in the proofs of [43, Theorem 4.1] and for the Weyl product results in [27].

Lemma 2.1. *Let $s > 1/2$, $a_1 \in \mathcal{S}_s'(\mathbf{R}^{2d})$, $a_2 \in \mathcal{S}_s(\mathbf{R}^{2d})$, $\Phi_1, \Phi_2 \in \Sigma_s(\mathbf{R}^{2d})$ and $X, Y \in \mathbf{R}^{2d}$. Then the following is true:*

(1) *if $\Phi = \pi^d \Phi_1 \# \Phi_2$, then $\Phi \in \Sigma_s(\mathbf{R}^{2d})$, and the map*

$$Z \mapsto e^{2i\sigma(Z,Y)} (\mathcal{V}_{\Phi_1} a_1)(X - Y + Z, Z)\,(\mathcal{V}_{\Phi_2} a_2)(X + Z, Y - Z)$$

belongs to $L^1(\mathbf{R}^{2d})$, and

$$\begin{aligned} &\mathcal{V}_\Phi(a_1 \# a_2)(X,Y) \\ &\quad = \int e^{2i\sigma(Z,Y)} (\mathcal{V}_{\Phi_1} a_1)(X - Y + Z, Z)\,(\mathcal{V}_{\Phi_2} a_2)(X + Z, Y - Z)\,dZ; \end{aligned}$$

(2) *if $\Phi = 2^{-d} \Phi_1 *_\sigma \Phi_2$, then $\Phi \in \Sigma_s(\mathbf{R}^{2d})$, and the map*

$$Z \mapsto e^{2i\sigma(X,Z-Y)} (\mathcal{V}_{\Phi_1} a_1)(X - Y + Z, Z)\,(\mathcal{V}_{\Phi_2} a_2)(Y - Z, X + Z)$$

belongs to $L^1(\mathbf{R}^{2d})$, and

$$\begin{aligned} &\mathcal{V}_\Phi(a_1 *_\sigma a_2)(X,Y) \\ &\quad = \int e^{2i\sigma(X,Z-Y)} (\mathcal{V}_{\Phi_1} a_1)(X - Y + Z, Z)\,(\mathcal{V}_{\Phi_2} a_2)(Y - Z, X + Z)\,dZ. \end{aligned}$$

Proof. The L^1-continuity for the mapping in (1) and (2) follow immediately from Theorems 6.4 and 6.5 in [52]. The integral formula for $\mathcal{V}_\Phi(a_1 \# a_2)$ in (1) then follows by similar arguments as for the proof of [43, Lemma 4.4], based on repeated applications of Fourier's inversion formula. The details are left for the reader. This gives (1).

The integral formula $\mathcal{V}_\Phi(a_1 *_\sigma a_2)$ in (2) now follows from (1), (1.17), (1.18) and (1.30). The proof is complete. □

For completeness we also write down the following extension of Theorem 0.3′ in [27]. Here the involved weight functions should satisfy

$$\omega_0(X,Y) \lesssim \omega_1(X - Y + Z, Z)\omega_2(X + Z, Y - Z), \quad X, Y, Z \in \mathbf{R}^{2d}, \tag{2.1}$$

and the exponent $p_j, q_j \in [1,\infty]$ satisfy (0.4) and

$$0 \le \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0} \le \frac{1}{p_j}, \frac{1}{q_j} \le \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0}, \quad j = 0,1,2. \tag{2.2}$$

Theorem 2.2. *Let $\omega_j \in \mathscr{P}_E(\mathbf{R}^{4d})$ and $p_j, q_j \in [1,\infty]$, $j = 0,1,2$, satisfy* (0.4), (2.1) *and* (2.2). *Then the map $(a_1, a_2) \mapsto a_1 \# a_2$ on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ extends uniquely to a continuous map from $\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $\mathcal{M}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, and the bound*

$$\|a_1 \# a_2\|_{\mathcal{M}^{p_0,q_0}_{(\omega_0)}} \lesssim \|a_1\|_{\mathcal{M}^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{\mathcal{M}^{p_2,q_2}_{(\omega_2)}}, \tag{2.3}$$

holds for every $a_1 \in \mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

The proof of Theorem 2.2 is similar to the proof of [27, Theorem 0.3′], after Proposition 1.9 and Lemma 2.1 in [27] have been replaced by Theorem 4.19 in [52] and Lemma 2.1. The details are left for the reader.

We note that Theorem 0.1 is an immediate consequence of (1.31), (1.19) and Theorem 2.2. Another way to prove Theorem 0.1 is to use similar arguments as in the proof of Theorem 2.2, based on (2) instead of (1) in Lemma 2.1.

We are now able to state and prove mapping results for the twisted convolution on weighted Lebesgue spaces. We start with the extension of Theorem 0.2 from the introduction.

Theorem 0.2′. *Let $k \in \{1,2\}$, $\omega_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ and let $p_j, q_j \in [1,\infty]$ for $j = 0,1,2$ satisfy* (0.6) *and*

$$\max_{j=0,1,2}\left(\frac{1}{p_j}, \frac{1}{p_j'}, \frac{1}{q_j}, \frac{1}{q_j'}\right) \le \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0}, \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0} \le 1,$$

Then the map $(a_1, a_2) \mapsto a_1 *_\sigma a_2$ extends uniquely to a continuous mapping from $L^{p_1,q_1}_{k,(\omega_1)}(\mathbf{R}^{2d}) \times L^{p_2,q_2}_{k,(\omega_2)}(\mathbf{R}^{2d})$ to $L^{p_0,q_0}_{k,(\omega_0)}(\mathbf{R}^{2d})$, and

$$\|a_1 *_\sigma a_2\|_{L^{p_0,q_0}_{k,(\omega_0)}} \lesssim \|a_1\|_{L^{p_1,q_1}_{k,(\omega_1)}} \|a_2\|_{L^{p_2,q_2}_{k,(\omega_2)}} \tag{0.8$'$}$$

when $a_1 \in L^{p_1,q_1}_{k,(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^{p_2,q_2}_{k,(\omega_2)}(\mathbf{R}^{2d})$.

Remark 2.3. The condition of the Lebesgue exponents for Theorems 0.2 and 0.2′ in [51] should be the same as in Theorems 0.2 and 0.2′ respectively. In this context, the results here extend corresponding results in [51].

For the proof we need the following lemma.

Lemma 2.4. *Let $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ be such that $\omega(x,\xi) = \omega(x)$. Then*

$$M^2_{(\omega)}(\mathbf{R}^d) = W^2_{(\omega)}(\mathbf{R}^d) = L^2_{(\omega)}(\mathbf{R}^d).$$

Proof. It is obvious that $M^2_{(\omega)} = W^2_{(\omega)}$. We have to prove $M^2_{(\omega)} = L^2_{(\omega)}$. Let $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d) \setminus 0$, and let $v \in \mathscr{P}_E(\mathbf{R}^d)$ be such that ω is v-moderate. Then (1.24) and Parseval's formula give

$$\begin{aligned}\|f\|^2_{M^2_{(\omega)}} &\asymp \iint |f(y)\phi(y-x))\omega(x)|^2\, dxdy \\ &\lesssim \iint |f(y)\omega(y)|^2 |\phi(y-x)v(y-x)|^2\, dxdy \asymp \|f\|^2_{L^2_{(\omega)}}.\end{aligned}$$

Since $\mathcal{S}_{1/2}$ is dense in $L^2_{(\omega)}$ and in $M^2_{(\omega)}$ it follows that $L^2_{(\omega)} \hookrightarrow M^2_{(\omega)}$.

In order to prove the opposite inclusion we note that $\phi_1 v \lesssim \phi_2$, when $\phi, \phi_1 \in \mathscr{P}_E$ are the Gauss functions $\phi_1(x) = e^{-|x|^2}$ and $\phi_2(x) = e^{-|x|^2/2}$. Hence (1.24) and Parseval's formula give

$$\begin{aligned}\|f\|^2_{L^2_{(\omega)}} &\lesssim \iint |f(y)\phi_1(y-x)\omega(y)|^2\, dxdy \\ &\lesssim \iint |f(y)\phi_1(y-x)v(y-x)\omega(x)|^2\, dxdy \\ &\lesssim \iint |f(y)\phi_2(y-x)\omega(x)|^2\, dxdy \\ &= \iint |\mathscr{F}(f\phi_2(\,\cdot\,-x))(\xi)\omega(x)|^2\, dxd\xi \asymp \|f\|^2_{M^2_{(\omega)}}.\end{aligned}$$

Hence $M^2_{(\omega)} \hookrightarrow L^2_{(\omega)}$.

The result now follows by combining these embeddings, and the proof is complete. □

Proof of Theorem 0.2′. By duality we may assume that

$$\max\Big(\frac{1}{p_j}, \frac{1}{p_j'}, \frac{1}{q_j}, \frac{1}{q_j'}\Big)$$

is attained when $j = 0$. Since $\mathcal{W}^2_{(\omega)} = \mathcal{M}^2_{(\omega)} = L^2_{(\omega)}$ when $\omega(X,Y) = \omega(X)$, in view of Lemma 2.4, the result follows from Theorem 0.1 in the case $p_0 = p_1 = p_2 = 2$.

Next we consider the case when the Young conditions

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0} = 1 \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0} = 1 \tag{2.4}$$

are fulfilled.

First we consider the case when $p_2, q_2 < \infty$, and we let $a_1 \in L^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and that $a_2 \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then

$$\|a_1 *_\sigma a_2\|_{L^{p_0,q_0}_{(\omega_0)}} \le (2/\pi)^{d/2} \|\, |a_1| * |a_2|\, \|_{L^{p_0,q_0}_{(\omega_0)}} \lesssim \|a_1\|_{L^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{L^{p_2,q_2}_{(\omega_2)}}, \tag{2.5}$$

by Young's inequality and (0.6). The result now follows in this case from the fact that $\mathcal{S}_{1/2}$ is dense in $L^{p_2,q_2}_{(\omega_2)}$, when $p_2, q_2 < \infty$.

In the same way, the case $p_1, q_1 < \infty$ follows. It remain to consider when $p_1, q_1 < \infty$ and $p_2, q_2 < \infty$ are violated. By (2.4) we get $p_1 = q_2 = \infty$ and

$p_2 = q_1 = 1$, or $p_1 = q_2 = 1$ and $p_2 = q_1 = \infty$, and it follows that $Y \mapsto a_1(X-Y)a_2(Y)e^{2i\sigma(X,Y)} \in L^1_{(\omega_0)}$ when $a_1 \in L^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$, and that (2.5) holds. This proves the result when (2.4) is fulfilled.

Next we let p_j and q_j be general. Then we may assume that $p_1, q_1 < \infty$ or $p_2, q_2 < \infty$, since otherwise, the Young condition (2.4) must hold, which has already been considered.

Therefore, by reasons of symmetry we may assume that $p_1, q_1 < \infty$, and we let $\mathcal{L}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ be the completion of $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then $\mathcal{L}^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ possess the (complex) interpolation property

$$(\mathcal{L}^{p_1,q_1}_{(\omega)}(\mathbf{R}^{2d}), (\mathcal{L}^{p_2,q_2}_{(\omega)}(\mathbf{R}^{2d}))_{[\theta]} = \mathcal{L}^{p,q}_{(\omega)}(\mathbf{R}^{2d}),$$
$$\text{when} \qquad \frac{1-\theta}{p_1} + \frac{\theta}{p_2} = \frac{1}{p}, \quad \frac{1-\theta}{q_1} + \frac{\theta}{q_2} = \frac{1}{q}$$

and $p_1, q_1 < \infty$. (Cf. Chapter 5 in [2].) Hence, by multi-linear interpolation between the case $p_0 = p_1 = p_2 = 2$ and the case (2.4) it follows that $\mathcal{L}^{p_1,q_1}_{(\omega_1)} *_\sigma \mathcal{L}^{p_2,q_2}_{(\omega_2)} \hookrightarrow \mathcal{L}^{p_0,q_0}_{(\omega_0)}$, and that (0.8)$'$ holds when $a_1, a_2 \in \mathcal{S}_{1/2}$.

The result now follows for general $a_1 \in L^{p_1,q_1}_{(\omega_1)}$ and $a_2 \in L^{p_2,q_2}_{(\omega)}$ by density arguments, where a_2 is first approximated by elements in $\mathcal{S}_{1/2}$ weakly, and thereafter a_1 is approximated by elements in $\mathcal{S}_{1/2}$ in the norm convergence. The proof is complete. □

Corollary 2.5. *Let $\omega_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ for $j = 0, 1, 2$ and $p, q \in [1, \infty]$ satisfy (0.6), and $q \le \min(p, p')$. Then the map $(a_1, a_2) \mapsto a_1 *_\sigma a_2$ extends uniquely to a continuous mapping from $L^p_{(\omega_1)}(\mathbf{R}^{2d}) \times L^q_{(\omega_2)}(\mathbf{R}^{2d})$ or $L^q_{(\omega_1)}(\mathbf{R}^{2d}) \times L^p_{(\omega_2)}(\mathbf{R}^{2d})$ to $L^p_{(\omega_0)}(\mathbf{R}^{2d})$.*

In particular, if $p \in [1, 2]$ and in addition ω_0 is submultitplicative, then $(L^p_{(\omega_0)}(\mathbf{R}^{2d}), *_\sigma)$ is an algebra.

We finish the section by using Theorem 0.2$'$ to prove that the class of permitted windows in the modulation space norms can be extended. More precisely we have the following.

Theorem 2.6. *Let $p, p_0, q, q_0 \in [1, \infty]$ and $\omega, v \in \mathscr{P}_E(\mathbf{R}^{2d})$ be such that $p_0, q_0 \le \min(p, p', q, q')$, $\check{v} = v$ and ω is v-moderate. Also let $f \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$. Then the following is true:*

(1) *if $\phi \in M^{p_0,q_0}_{(v)}(\mathbf{R}^d) \setminus 0$, then $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if $V_\phi f \in L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$. Furthermore, $\|f\| \equiv \|V_\phi f\|_{L^{p,q}_{1,(\omega)}}$ defines a norm for $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and different choices of ϕ give rise to equivalent norms;*

(2) *if $\phi \in W^{p_0,q_0}_{(v)}(\mathbf{R}^d) \setminus 0$, then $f \in W^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if $V_\phi f \in L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$. Furthermore, $\|f\| \equiv \|V_\phi f\|_{L^{p,q}_{2,(\omega)}}$ defines a norm for $W^{p,q}_{(\omega)}(\mathbf{R}^d)$, and different choices of ϕ give rise to equivalent norms.*

For the proof we note that (1.25) gives

$$\|W_{f,\check\phi}\|_{L^{p,q}_{k,(\omega_0)}} \asymp \|V_\phi f\|_{L^{p,q}_{k,(\omega)}}, \quad \text{when} \quad \omega_0(x,\xi)=\omega(2x,2\xi) \tag{2.6}$$

for $k=1,2$.

Finally, by Fourier's inversion formula it follows that if $f_1,g_2\in\mathcal S_{1/2}'(\mathbf R^d)$ and $f_2,g_1\in L^2(\mathbf R^d)$, then

$$W_{f_1,g_1} *_\sigma W_{f_2,g_2} = (\check f_2,g_1)_{L^2} W_{f_1,g_2}. \tag{2.7}$$

Proof of Theorem 2.6. We may assume that $p_0=q_0=\min(p,p',q,q')$. Assume that $\phi,\psi\in M^{p_0,q_0}_{(v)}(\mathbf R^d)\hookrightarrow L^2(\mathbf R^d)$, where the inclusion follows from the fact that $p_0,q_0\le 2$ and $v\ge c$ for some constant $c>0$. Since $\|V_\phi\psi\|_{L^{p_0,q_0}_{k,(v)}}=\|V_\psi\phi\|_{L^{p_0,q_0}_{k,(v)}}$ when $\check v=v$, the result follows if we prove that

$$\|V_\phi f\|_{L^{p,q}_{k,(\omega)}} \lesssim (\|\psi\|_{L^2})^{-2}\|V_\psi f\|_{L^{p,q}_{k,(\omega)}}\|V_\phi\psi\|_{L^{p_0,q_0}_{k,(v)}}, \tag{2.8}$$

for some constant C which is independent of $f\in\mathcal S_{1/2}'(\mathbf R^d)$ and $\phi,\psi\in M^{p_0,q_0}_{(v)}(\mathbf R^d)$. For reasons of homogeneity, it is then no restriction to assume that $\|\psi\|_{L^2}=1$.

If $p_1=p$, $p_2=p_0$, $q_1=q$, $q_2=q_0$, $\omega_0=\omega(2\,\cdot\,)$ and $v_0=v(2\,\cdot\,)$, then Theorem 0.2$'$ and (2.7) give

$$\begin{aligned}\|V_\phi f\|_{L^{p,q}_{k,(\omega)}} \asymp \|W_{f,\check\phi}\|_{L^{p,q}_{k,(\omega_0)}} &\asymp \|W_{f,\check\psi} *_\sigma W_{\psi,\check\phi}\|_{L^{p,q}_{k,(\omega_0)}}\\ &\lesssim \|W_{f,\check\psi}\|_{L^{p,q}_{k,(\omega_0)}}\|W_{\psi,\check\phi}\|_{L^{p_0,q_0}_{k,(v_0)}} \asymp \|V_\psi f\|_{L^{p,q}_{k,(\omega)}}\|V_\phi\psi\|_{L^{p_0,q_0}_{k,(v)}},\end{aligned}$$

and (2.8) follows. The proof is complete. □

3. Gelfand–Shilov tempered vector spaces

In this section we introduce the notion of Gelfand–Shilov and Beurling tempered (quasi-)Banach and Hilbert spaces, and establish embedding properties for such spaces. These results are applied in the next sections when discussing Schatten–von Neumann operators within the theory of pseudo-differential operators. The results are also applied in [54] where decomposition and Schatten–von Neumann properties for linear operators with Gelfand–Shilov kernels are established. We remark that some parts of the approach here are somewhat similar to the first part of Section 4 in [51], where related questions on tempered Hilbert spaces (with respect to Schwartz tempered distributions) are considered.

We start by introducing some notations on quasi-Banach spaces. A quasi-norm $\|\cdot\|_{\mathscr B}$ on a vector space $\mathscr B$ (over $\mathbf C$) is a non-negative and real-valued function $\|\cdot\|_{\mathscr B}$ on $\mathscr B$ which is non-degenerate in the sense

$$\|f\|_{\mathscr B}=0 \quad\Longleftrightarrow\quad f=0, \qquad f\in\mathscr B,$$

and fulfills

$$\begin{aligned}&\|\alpha f\|_{\mathscr B}=|\alpha|\cdot\|f\|_{\mathscr B}, && f\in\mathscr B,\ \alpha\in\mathbf C\\ \text{and}\qquad &\|f+g\|_{\mathscr B}\le D(\|f\|_{\mathscr B}+\|g\|_{\mathscr B}), && f,g\in\mathscr B,\end{aligned} \tag{3.1}$$

for some constant $D \ge 1$ which is independent of $f, g \in \mathscr{B}$. Then $\mathscr{B}$ is a topological vector space when the topology for $\mathscr{B}$ is defined by $\| \cdot \|_{\mathscr{B}}$, and $\mathscr{B}$ is called a quasi-Banach space if $\mathscr{B}$ is complete under this topology.

Let $\mathscr{B}$ be a quasi-Banach space such that

$$\mathcal{S}_{1/2}(\mathbf{R}^d) \hookrightarrow \mathscr{B} \hookrightarrow \mathcal{S}_{1/2}'(\mathbf{R}^d), \tag{3.2}$$

and that $\mathcal{S}_{1/2}(\mathbf{R}^d)$ is dense in $\mathscr{B}$. We let $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$ be the sets of all $f \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$ such that $\check{f} \in \mathscr{B}$ and $\overline{f} \in \mathscr{B}$ respectively. Then $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$ are quasi-Banach spaces under the quasi-norms

$$\|f\|_{\check{\mathscr{B}}} \equiv \|\check{f}\|_{\mathscr{B}} \quad \text{and} \quad \|f\|_{\mathscr{B}^\tau} \equiv \|\overline{f}\|_{\mathscr{B}}$$

respectively. Furthermore, $\mathcal{S}_{1/2}(\mathbf{R}^d)$ is dense in $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$, and (3.2) holds after $\mathscr{B}$ have been replaced by $\check{\mathscr{B}}$ or $\mathscr{B}^\tau$. Moreover, if $\mathscr{B}$ is a Banach (Hilbert) space, then $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$ are Banach (Hilbert) spaces.

The L^2-*dual* $\mathscr{B}'$ of $\mathscr{B}$ is the set of all $\varphi \in \mathcal{S}_{1/2}'(\mathbf{R}^d)$ such that

$$\|\varphi\|_{\mathscr{B}'} \equiv \sup |(\varphi, f)_{L^2(\mathbf{R}^d)}|$$

is finite. Here the supremum is taken over all $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$ such that $\|f\|_{\mathscr{B}} \le 1$. Let $\varphi \in \mathscr{B}'$. Since $\mathcal{S}_{1/2}$ is dense in $\mathscr{B}$, it follows from the definitions that the map $f \mapsto (\varphi, f)_{L^2}$ from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathbf{C}$ extends uniquely to a continuous mapping from $\mathscr{B}$ to $\mathbf{C}$.

From now on we assume that the (quasi-)Banach spaces $\mathscr{B}, \mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2, \dots$, and the Hilbert spaces $\mathscr{H}, \mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_2, \dots$ are "Gelfand–Shilov tempered" or "Beurling tempered" in the following sense.

Definition 3.1. Let $\mathscr{B}$ be a quasi-Banach space such that (3.2) is fulfilled.

(1) $\mathscr{B}$ is called *Beurling tempered*, or *B-tempered* (*of order* $s > 1/2$) on $\mathbf{R}^d$, if $\mathscr{B}, \mathscr{B}' \hookrightarrow \Sigma_s'(\mathbf{R}^d)$, and $\Sigma_s(\mathbf{R}^d)$ is dense in $\mathscr{B}$ and $\mathscr{B}'$;

(2) $\mathscr{B}$ is called *Gelfand–Shilov tempered*, or *GS-tempered* (*of order* $s \ge 1/2$) on $\mathbf{R}^d$, if $\mathscr{B}, \mathscr{B}' \hookrightarrow \mathcal{S}_s'(\mathbf{R}^d)$, and $\mathcal{S}_s(\mathbf{R}^d)$ is dense in both $\mathscr{B}$ and $\mathscr{B}'$;

(3) $\mathscr{B}$ is called *tempered* on $\mathbf{R}^d$, if $\mathscr{B}, \mathscr{B}' \hookrightarrow \mathscr{S}'(\mathbf{R}^d)$, and $\mathscr{S}(\mathbf{R}^d)$ is dense in $\mathscr{B}$ and $\mathscr{B}'$.

Remark 3.2. Let $\mathscr{B}$ be a quasi-Banach space such that (3.2) holds. Then it follows from (1.2) and the fact that $\mathcal{S}_{1/2}$ is dense in $\mathcal{S}_s$, Σ_s and $\mathscr{S}$, when $s > 1/2$, that the following is true:

(1) if $s > 1/2$ and $\mathscr{B}$ is GS-tempered of order s, then $\mathscr{B}$ is B-tempered of order s;

(2) if $s \ge 1/2$, $\varepsilon > 0$ and $\mathscr{B}$ is B-tempered of order $s+\varepsilon$, then $\mathscr{B}$ is GS-tempered of order s;

(3) if $s > 1/2$ and $\mathscr{B}$ is tempered, then $\mathscr{B}$ is GS- and B-tempered of order s.

We also note that Definition 3.1 (3) in the Hilbert space case might not be the same as [51, Definition 4.1]. In fact, $\mathscr{H}$ is a tempered Hilbert in the sense of Definition 3.1 (3), is the same as *both* $\mathscr{H}$ and $\mathscr{H}'$ are tempered in the sense of [51, Definition 4.1].

For future references we remark that $\check{\mathscr{B}}$ and $\mathscr{B}^\tau$ are GS-tempered (B-tempered) quasi-Banach spaces, when $\mathscr{B}$ is a GS-tempered (B-tempered) quasi-Banach space, and that similar facts hold when $\mathscr{B}$ is a Banach or Hilbert space.

In the following analogy of [51, Lemma 4.2] we establish basic properties in the Hilbert space case.

Lemma 3.3. *Let $s \ge 1/2$ ($s > 1/2$), and let $\mathscr{H}$ be a GS-tempered (B-tempered) Hilbert space of order s on $\mathbf{R}^d$, with L^2-dual $\mathscr{H}'$. Then the following is true:*

1. *$\mathscr{H}'$ is a GS-tempered (B-tempered) Hilbert space of order s, which can be identified with the dual space of $\mathscr{H}$ through the L^2-form;*
2. *there is a unique map $T_{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}'$ such that*
$$(f,g)_{\mathscr{H}} = (T_{\mathscr{H}}f,g)_{L^2(\mathbf{R}^d)}, \quad f,g \in \mathscr{H}; \tag{3.3}$$
3. *if $T_{\mathscr{H}}$ is the map in (2), $\{e_j\}_{j\in I}$ is an orthonormal basis in $\mathscr{H}$ and $\varepsilon_j = T_{\mathscr{H}}e_j$, then $T_{\mathscr{H}}$ is isometric, $\{\varepsilon_j\}_{j\in I}$ is an orthonormal basis in $\mathscr{H}'$ and*
$$(\varepsilon_j, e_k)_{L^2(\mathbf{R}^d)} = \delta_{j,k}.$$

Proof. We only prove the result when $\mathscr{H}$ is GS-tempered. The case when $\mathscr{H}$ is B-tempered follows by similar arguments and is left for the reader.

First assume that $f \in \mathscr{H}$, $g \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, and let $T_{\mathscr{H}}f \in \mathscr{H}'$ be defined by (3.3). Then $T_{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}'$ is isometric. Furthermore, since $\mathcal{S}_{1/2}$ is dense in $\mathscr{H}$, and the dual space of $\mathscr{H}$ can be identified with itself, under the scalar product of $\mathscr{H}$, the asserted duality properties of $\mathscr{H}'$ follow.

Let $\{e_j\}_{j\in I}$ be an arbitrary orthonormal basis in $\mathscr{H}$, and let $\varepsilon_j = T_{\mathscr{H}}e_j$. Then it follows that $\|\varepsilon_j\|_{\mathscr{H}'} = 1$ and

$$(\varepsilon_j, e_k)_{L^2} = (e_j, e_k)_{\mathscr{H}} = \delta_{j,k}.$$

Furthermore, if

$$f = \sum \alpha_j e_j, \quad \varphi = \sum \alpha_j \varepsilon_j, \quad g = \sum \beta_j e_j \quad \text{and} \quad \gamma = \sum \beta_j \varepsilon_j$$

are finite sums, and we set $(\varphi,\gamma)_{\mathscr{H}'} \equiv (f,g)_{\mathscr{H}}$, then it follows that $(\,\cdot\,,\,\cdot\,)_{\mathscr{H}'}$ defines a scalar product on such finite sums in $\mathscr{H}'$, and that $\|\varphi\|^2_{\mathscr{H}'} = (\varphi,\varphi)_{\mathscr{H}'}$. By continuity extensions it now follows that $(\varphi,\gamma)_{\mathscr{H}'}$ extends uniquely to each $\varphi,\gamma \in \mathscr{H}'$, and that the identity $\|\varphi\|^2_{\mathscr{H}'} = (\varphi,\varphi)_{\mathscr{H}'}$ holds. This proves the result. □

From now on the basis $\{\varepsilon_j\}_{j\in I}$ in Lemma 3.3 is called the *dual basis* of $\{e_j\}_{j\in I}$.

Example 3.4. Let $\mathscr{H}_1 = H^2_s(\mathbf{R}^d)$ and $\mathscr{H}_2 = M^2_{(\omega_0)}(\mathbf{R}^d)$ where $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$. Then $\mathscr{H}_1$ is a tempered Hilbert space with dual $\mathscr{H}_1' = H^2_{-s}(\mathbf{R}^d)$. The space

$\mathscr{H}_2$ is B-tempered (GS-tempered) of order s, when $s \le 1$ ($s < 1$), and $\mathscr{H}_2' = M^2_{(1/\omega_0)}(\mathbf{R}^d)$.

We note that if $\omega_s(x,\xi) = \langle \xi \rangle^s$, then $M^2_{(\omega_s,0)} = H^2_s$, and we refer to [51, Example 4.3] for more examples on tempered Hilbert spaces.

In several situations, an orthonormal basis $\{e_j\}$ in a GS- or B-tempered Hilbert space $\mathscr{H}$ might be orthogonal in $L^2(\mathbf{R}^d)$. The following proposition shows that this is sufficient for $\{e_j\}$ being orthogonal in the dual $\mathscr{H}'$ of $\mathscr{H}$.

Proposition 3.5. *Let $\mathscr{H}$ be GS- or B-tempered Hilbert space on $\mathbf{R}^d$, $\{e_j\}_{j=1}^\infty$ and $\{e_{0,j}\}_{j=1}^\infty$ be orthonormal bases for $\mathscr{H}$ and $L^2(\mathbf{R}^d)$ respectively, and let $\{\varepsilon_j\}_{j=1}^\infty \in \mathscr{H}'$ be the dual basis of $\{e_j\}_{j=1}^\infty$. Then the following is true:*

(1) *if $e_j = c_j e_{0,j}$ for every $j \ge 1$ and some $\{c_j\}_{j=1}^\infty \subseteq \mathbf{C}$, then $\varepsilon_j = (\overline{c_j})^{-1} e_{0,j}$;*

(2) *if $\varepsilon_j = c_j e_j$ for every $j \ge 1$ and some $\{c_j\}_{j=1}^\infty \subseteq \mathbf{C}$, then $c_j > 0$, and $\{c_j^{1/2} e_j\}_{j=1}^\infty$ is an orthonormal basis for $L^2(\mathbf{R}^d)$;*

(3) *if $e_j = c_j e_{0,j}$ and $\varepsilon_j = d_j e_{0,j}$ for every $j \ge 1$ and some $\{c_j\}_{j=1}^\infty \subseteq \mathbf{C}$ and $\{d_j\}_{j=1}^\infty \subseteq \mathbf{C}$, then*

$$c_j \overline{d_j} = \|e_{0,j}\|_{\mathscr{H}} \cdot \|e_{0,j}\|_{\mathscr{H}'} = \|e_j\|_{L^2} \|\varepsilon_j\|_{L^2} = 1.$$

Proof. (1) We have

$$\delta_{j,k} = (e_j, e_k)_{\mathscr{H}} = (e_j, \varepsilon_k)_{L^2} = c_j (e_{0,j}, \varepsilon_k)_{L^2},$$

giving that

$$\delta_{j,k} = c_k (e_{0,j}, \varepsilon_k)_{L^2} = (e_{0,j}, \overline{c_k} \varepsilon_k)_{L^2}. \tag{3.4}$$

Since $\{e_{0,j}\}_{j=1}^\infty$ is an orthonormal basis for L^2, it follows from (3.4) that $\overline{c_k} \varepsilon_k = e_{0,k}$, and (1) follows.

(2) We have

$$1 = (e_j, e_j)_{\mathscr{H}} = (e_j, \varepsilon_j)_{L^2} = \overline{c_j} (e_j, e_j)_{L^2} = \overline{c_j} \|e_j\|^2_{L^2},$$

giving that $c_j > 0$. Furthermore, if $f_j = c_j^{1/2} e_j$, then

$$(f_j, f_k)_{L^2} = (c_j/c_k)^{1/2} (e_j, \varepsilon_k)_{L^2} = (c_j/c_k)^{1/2} \delta_{j,k} = \delta_{j,k}.$$

Hence $\{f_j\}_{j=1}^\infty$ is an orthonormal basis of L^2.

(3) By applying appropriate norms on the identities $e_j = c_j e_{0,j}$ and, $\varepsilon_j = d_j e_{0,j}$ and using the fact that e_j, $e_{0,j}$ and ε_j are unit vectors in $\mathscr{H}$, L^2 and $\mathscr{H}'$ respectively, we get

$$1 = (e_j, \varepsilon_j)_{L^2} = c_j \overline{d_j} (e_{0,j}, e_{0,j})_{L^2} = c_j \overline{d_j},$$

and

$$|c_j| = \|e_j\|_{L^2} = 1/\|e_{0,j}\|_{\mathscr{H}} \quad \text{and} \quad |d_j| = \|\varepsilon_j\|_{L^2} = 1/\|e_{0,j}\|_{\mathscr{H}'}.$$

The assertion follows by combining these equalities. The proof is complete. □

Example 3.6. Let $\omega \in L^\infty_{\text{loc}}(\mathbf{R}^d)$ be positive. Then ω is called weakly sub-Gaussian type weight, if the following conditions are fulfilled:

- $e^{-\varepsilon|x|^2} \lesssim \omega(x) \lesssim e^{\varepsilon|x|^2}$, for every choice of $\varepsilon > 0$;
- for some fixed $R \ge 2$ and some constant $c > 0$, the relation

$$\omega(x+y)\omega(x-y) \asymp \omega(x)^2$$

holds when $Rc \le |x| \le c/|y|$.

The set of all weakly sub-Gaussian weights on $\mathbf{R}^d$ is denoted by $\mathscr{P}_Q^0(\mathbf{R}^d)$, and is a family of weights which contains $\mathscr{P}_E(\mathbf{R}^d)$. (Cf. Definition 1.1 in [52].)

Now consider the modulation spaces $M^2_{(\omega)}(\mathbf{R}^d)$, where $\omega \in \mathscr{P}_Q^0(\mathbf{R}^{2d})$ satisfies

$$\omega(x,\xi) = \omega_0(r) = \omega_0(r_1,\dots,r_d), \quad r_j = |(x_j,\xi_j)|, \quad j = 1,\dots,d$$

(i.e., $\omega(x,\xi)$ is rotation invariant under each coordinate pair (x_j,ξ_j)). Note that the window function $\phi(x)$ in the definition of modulation space norms with weights in $\mathscr{P}_Q^0$ is fixed and equal to the Gaussian $\pi^{-d/4}e^{-|x|^2/2}$. Then there is a constant $C > 0$ such that

$$C^{-1} \le \|h_\alpha\|_{M^2_{(\omega)}} \|h_\alpha\|_{M^2_{(1/\omega)}} \le C, \tag{3.5}$$

for every Hermite function h_α on $\mathbf{R}^d$.

In fact, if $\mathscr{H} = M^2_{(\omega)}$, then it follows from Theorem 4.17 in [52] that $\mathscr{H}' = M^2_{(1/\omega)}$ and that

$$\|f\|_{\mathscr{H}'} \asymp \|f\|_{M^2_{(1/\omega)}}, \quad f \in \mathscr{H}'.$$

The statement is now a consequence of Proposition 3.5, and the facts that $\{h_\alpha\}_{\alpha\in\mathbf{N}^d}$ and $\{h_\alpha/\|h_\alpha\|_{M^2_{(\omega)}}\}_{\alpha\in\mathbf{N}^d}$ are orthonormal bases for L^2 and $M^2_{(\omega)}$, respectively.

The relation (3.5) in combination with results in [52] can be used to establish estimates for generalized gamma functions in integer points, in a similar way as formula (30) in [24]. More precisely, let $\mathfrak{V}$ be the Bargmann transform, and let $A^2_{(\omega)}(\mathbf{C}^d)$ be the weighted Bargmann–Fock space of all entire functions F on $\mathbf{C}^d$ such that

$$\|F\|_{A^2_{(\omega)}} \equiv \pi^{-d/2}\left(\int_{\mathbf{C}^d} |F(z)\omega(2^{1/2}\overline{z})|^2 e^{-|z|^2}\,d\lambda(z)\right)^{1/2} < \infty.$$

(Cf., e.g., [1, 52].) Here we identify $x+i\xi$ in $\mathbf{C}^d$ with (x,ξ) in $\mathbf{R}^{2d}$, and $d\lambda(z)$ is the Lebesgue measure on $\mathbf{C}^d$. Then $(\mathfrak{V}h_\alpha)(z) = z^\alpha/(\alpha!)^{1/2}$, and the map $f \mapsto \mathfrak{V}f$ is isometric and bijective from $M^2_{(\omega)}$ to $A^2_{(\omega)}$, in view of Theorem 3.4 in [52]. Consequently, (3.5) is equivalent to

$$I_{(\omega_0)} \cdot I_{(1/\omega_0)} \asymp (\alpha!)^2, \tag{3.6}$$

where

$$I_{(\omega_0)} \equiv \pi^d \|z^\alpha\|^2_{A^2_{(\omega)}} = \int_{\mathbf{C}^d} |z^{2\alpha}|\omega_0(|z_1|,\dots,|z_d|)^2 e^{-|z|^2}\,d\lambda(z).$$

By writing $z_j = r_j e^{i\theta_j}$ in terms of polar coordinates for every $j = 1, \dots, d$, and taking r_j^2 and θ_j as new variables of integrations, we get

$$I_{(\omega_0)} \asymp \int_{[0,\infty)^d} t^\alpha \vartheta(t) e^{-\|t\|}\, dt,$$

where $\vartheta(t_1, \dots, t_d) = \omega_0(t_1^{1/2}, \dots, t_d^{1/2})^2$ and $\|t\| = t_1 + \dots + t_d$, $t \in [0,\infty)^d$. Hence (3.6) is equivalent to

$$\int_{[0,\infty)^d} t^\alpha \vartheta(t) e^{-\|t\|}\, dt \cdot \int_{[0,\infty)^d} t^\alpha \vartheta(t)^{-1} e^{-\|t\|}\, dt \asymp (\alpha!)^2. \tag{3.7}$$

In particular, the formula (30) in the remark after Theorem 3.7 in [24] hold for the broad class $\mathscr{P}_Q^0$ of weights on $\mathbf{R}^d$.

The following result concerns continuous embeddings of the form

$$M^2_{(\omega_t)}(\mathbf{R}^d) \hookrightarrow \mathscr{B}, \mathscr{B}' \hookrightarrow M^2_{(1/\omega_t)}(\mathbf{R}^d), \tag{3.8}$$

when $\mathscr{B}$ is a quasi-Banach space. Here $M^2_{(\omega_t)}$ belongs to the extended family of modulation spaces in [52] and the weights ω_t are given by

$$\omega_t(x,\xi) = e^{t(|x|^{1/s} + |\xi|^{1/s})}, \tag{3.9}$$

when $s \ge 1/2$ and $t \in \mathbf{R}$.

Proposition 3.7. *Let $s > 1/2$, and let ω_t be given by* (3.9)*. Then the following is true:*

(1) *if $\mathscr{B}$ is a GS-tempered quasi-Banach space on $\mathbf{R}^d$ of order s, then* (3.8) *hold for every $t > 0$;*
(2) *if $\mathscr{B}$ is a B-tempered quasi-Banach space on $\mathbf{R}^d$ of order s, then* (3.8) *hold for some $t > 0$.*

We first prove Proposition 3.7 in the case that $\mathscr{B} = \mathscr{H}$ is a Hilbert space. Thereafter, the general result will follow from this case and Proposition 3.8 below.

Proof of Proposition 3.7 *when $\mathscr{B} = \mathscr{H}$ is a Hilbert space.* By Remark 1.6 it follows that

$$\Sigma_s \hookrightarrow M^2_{(\omega_t)} \hookrightarrow \mathcal{S}_s, \quad \mathcal{S}_s' \hookrightarrow M^2_{(1/\omega_t)} \hookrightarrow \Sigma_s' \tag{3.10}$$

when $t > 0$.

If $\mathscr{H}$ is GS-tempered, then it follows from these embeddings that $M^2_{(\omega_t)} \hookrightarrow \mathscr{H}, \mathscr{H}'$ holds for every $t > 0$. Furthermore, by Theorem 4.17 in [52] it follows that $\mathcal{S}_{1/2}$ is dense in these Hilbert spaces, and that the dual of $M^2_{(\omega_t)}$ is given by $M^2_{(1/\omega_t)}$.

Now if $\mathscr{H}$ is GS-tempered, then (3.10) gives

$$M^2_{(\omega_t)} \hookrightarrow \mathcal{S}_s \hookrightarrow \mathscr{H}, \mathscr{H}' \hookrightarrow \mathcal{S}_s' \hookrightarrow M^2_{(1/\omega_t)}, \quad t > 0,$$

and (1) follows.

In order to prove (2) we note that Theorem 3.9 and its proof in [52] implies that the topology for Σ_s is given by the semi-norms

$$f \mapsto \|f\|_{(t)} \equiv \|f\|_{M^2_{(\omega_t)}}, \quad t > 0.$$

Hence

$$\|f\|_{\mathscr H} \lesssim \|f\|_{M^2_{(\omega_t)}} \quad \text{and} \quad \|\varphi\|_{\mathscr H'} \lesssim \|\varphi\|_{M^2_{(\omega_t)}}, \quad f,\varphi \in M^2_{(\omega_t)}$$

hold for some choice of $t = t_0 > 0$, since $\Sigma_s \hookrightarrow \mathscr H$ and $\Sigma_s \hookrightarrow \mathscr H'$. This gives $M^2_{(\omega_{t_0})} \hookrightarrow \mathscr H$ and $M^2_{(\omega_{t_0})} \hookrightarrow \mathscr H'$. The assertion (2) now follows from these embeddings and duality. The proof is complete. □

With reference to the Hilbert spaces which occur in Example 3.6 we say that a Hilbert space $\mathscr H$ is of *Hermite type*, if $\{h_\alpha/\|h_\alpha\|_{\mathscr H}\}_\alpha$ is an orthonormal basis for $\mathscr H$,

$$(S_\pi f)(x) \equiv f(x_{\pi(1)},\dots,x_{\pi(d)}) \in \mathscr H \quad \text{when} \quad f \in \mathscr H$$

for every permutation π on $\{1,\dots,d\}$, and that $\|S_\pi f\|_{\mathscr H} = \|f\|_{\mathscr H}$ for every $f \in \mathscr H$ an permutation π.

Proposition 3.8. *Let $\mathscr B_1, \mathscr B_2$ be quasi-Banach spaces which are continuously embedded in $\mathcal S'_{1/2}(\mathbf R^d)$. Then the following is true:*

(1) *if $s \ge 1/2$, $\mathcal S_s(\mathbf R^d) \hookrightarrow \mathscr B_1$ and $\mathscr B_2 \hookrightarrow \mathcal S'_s(\mathbf R^d)$, then there are GS-tempered Hilbert spaces $\mathscr H_1$ and $\mathscr H_2$ of order s and of Hermite type such that*

$$\mathcal S_s(\mathbf R^d) \hookrightarrow \mathscr H_1 \hookrightarrow \mathscr B_1 \quad \text{and} \quad \mathscr B_2 \hookrightarrow \mathscr H_2 \hookrightarrow \mathcal S'_s(\mathbf R^d)$$

hold. Furthermore, $\mathscr H_1$ and $\mathscr H_2$ can be chosen such that $\mathscr H_1 \hookrightarrow \mathcal S_{s/\gamma}(\mathbf R^d)$ and $\mathcal S'_{s/\gamma}(\mathbf R^d) \hookrightarrow \mathscr H_2$ for every $\gamma \in (0,1)$;

(2) *if $s > 1/2$, $\Sigma_s(\mathbf R^d) \hookrightarrow \mathscr B_1$ and $\mathscr B_2 \hookrightarrow \Sigma'_s(\mathbf R^d)$, then there are B-tempered Hilbert spaces $\mathscr H_1$ and $\mathscr H_2$ of order s and of Hermite type such that*

$$\Sigma_s(\mathbf R^d) \hookrightarrow \mathscr H_1 \hookrightarrow \mathscr B_1 \quad \text{and} \quad \mathscr B_2 \hookrightarrow \mathscr H_2 \hookrightarrow \Sigma'_s(\mathbf R^d)$$

hold. Furthermore, $\mathscr H_1$ and $\mathscr H_2$ can be chosen such that $\mathscr H_1 \hookrightarrow \Sigma_{s/\gamma}(\mathbf R^d)$ and $\Sigma'_{s/\gamma}(\mathbf R^d) \hookrightarrow \mathscr H_2$ for every $\gamma \in (0,1)$;

(3) *if $\mathscr S(\mathbf R^d) \hookrightarrow \mathscr B_1$ and $\mathscr B_2 \hookrightarrow \mathscr S'(\mathbf R^d)$, then there are tempered Hilbert spaces $\mathscr H_1$ and $\mathscr H_2$ of Hermite type such that*

$$\mathscr S(\mathbf R^d) \hookrightarrow \mathscr H_1 \hookrightarrow \mathscr B_1 \quad \text{and} \quad \mathscr B_2 \hookrightarrow \mathscr H_2 \hookrightarrow \mathscr S'(\mathbf R^d)$$

hold.

Proof. We only prove (1). The other assertions follow by similar arguments and are left for the reader.

In order to prove (1) it is no restriction to assume that $\mathcal S_s$ is dense $\mathscr B_1$ by replacing $\mathscr B_1$ with the completion of $\mathcal S_s$ under the quasi-norm $\|\cdot\|_{\mathscr B_1}$. Let $f \in \mathscr B_1$. Since $\mathscr B_1 \hookrightarrow \mathcal S'_{1/2}$, it follows that

$$f = \sum_\alpha c_\alpha h_\alpha,$$

where h_α is the Hermite function of order α and its coefficients

$$c_\alpha = c_\alpha(f) = (f, h_\alpha)_{L^2}$$

satisfies

$$\sum_\alpha |c_\alpha|^2 e^{-c|\alpha|} < \infty,$$

for every $c > 0$.

The fact that $\mathcal{S}_s$ is continuously embedded in $\mathscr{B}_1$ implies that for every integer $j > 0$ we have

$$\|f\|^2_{\mathscr{B}_1} \le C_j D^{-2j} \sum_\alpha |c_\alpha|^2 e^{|\alpha|^{1/2s}/j},$$

where the constant $D \ge 1$ is the same as in (3.1), and the constant $C_j \ge 1$ is independent of f (cf. formula (2.12) in [21]).

For every integer $j \ge 1$, let

$$N_j = \sup\{\, |\alpha|\, ;\, C_j j^2 e^{j^j - 1} e^{-|\alpha|^{1/2s}/j} \ge 1 \,\},$$

and define inductively

$$R_1 = N_1 \quad \text{and} \quad R_{j+1} = \max(R_j + 1, N_{j+1}), \quad j \ge 1.$$

Furthermore we set

$$I_0 = \{\, \alpha\, ;\, |\alpha| \le R_1 \,\} \quad \text{and} \quad I_j = \{\, \alpha\, ;\, R_j < |\alpha| \le R_{j+1} \,\},$$

and we let $m(\alpha) = \sup_{\alpha \in I_0} C_1 e^{|\alpha|^{1/2s}}$ when $\alpha \in I_0$, and $m(\alpha) = e^{j^j - 1} e^{2|\alpha|^{1/2s}/j}$ when $\alpha \in I_j$, $j \ge 1$. We note that R_j is finite and increases to ∞ as j tends to ∞.

Let $\mathscr{H}_1$ be the Hilbert space which consists of all $f \in \mathcal{S}_s'$ such that

$$\|f\|_{\mathscr{H}_1} \equiv \Big(\sum_\alpha |c_\alpha(f)|^2 m(\alpha) \Big)^{1/2}$$

is finite. We shall prove that $\mathscr{H}_1$ satisfies the requested properties. Since

$$\lim_{|\alpha| \to \infty} m(\alpha) e^{-c|\alpha|^{1/2s}} = 0$$

when $c > 0$, it follows that $\mathcal{S}_s$ is continuously embedded in $\mathscr{H}_1$. Furthermore, the fact that $m(\alpha) = m(\beta)$ when $|\alpha| = |\beta|$ implies that $f \mapsto S_\pi f$ is a unitary map on $\mathscr{H}_1$, for every permutation π on $\{1, \dots, d\}$.

It remains to prove that $\mathscr{H}_1$ is continuously embedded in $\mathscr{B}_1$ and in $\mathcal{S}_{s/\gamma}$ when $0 < \gamma < 1$. Let $f \in \mathcal{S}_s$, and let

$$f_j = \sum_{\alpha \in I_j} c_\alpha(f) h_\alpha, \quad j \ge 0.$$

Then

$$f = \sum_{j \ge 0} f_j, \quad c_\alpha(f_j) = \begin{cases} c_\alpha(f), & \alpha \in I_j \\ 0, & \alpha \notin I_j \end{cases} \qquad \text{and} \quad \|f\|^2_{\mathscr{H}_1} = \sum_{j \ge 0} \|f_j\|^2_{\mathscr{H}_1}.$$

This gives

$$\begin{aligned}\|f\|_{\mathscr{B}_1} &\lesssim \sum_j D^j \|f_j\|_{\mathscr{B}_1} \le \Big(\sum_{\alpha\in I_0} |c_\alpha|^2 m(\alpha)\Big)^{1/2} + \sum_{j\ge 1} \Big(C_j \sum_{\alpha\in I_j} |c_\alpha|^2 e^{|\alpha|^{1/2s}/j}\Big)^{1/2} \\ &\le \Big(\sum_{\alpha\in I_0} |c_\alpha|^2 m(\alpha)\Big)^{1/2} + \sum_{j\ge 1} \frac{1}{j}\Big(\sum_{\alpha\in I_j} |c_\alpha|^2 e^{2|\alpha|^{1/2s}/j}\Big)^{1/2} \\ &\le \|f_0\|_{\mathscr{H}_1} + \sum_{j\ge 1}\frac{1}{j}\|f_j\|_{\mathscr{H}_1}.\end{aligned}$$

Hence, by Cauchy–Schwartz inequality we get

$$\|f\|_{\mathscr{B}_1} \le \|f_0\|_{\mathscr{H}_1} + \Big(\sum_{j\ge 1}\frac{1}{j^2}\Big)^{1/2}\Big(\sum_{j\ge 1}\|f_j\|^2_{\mathscr{H}_1}\Big)^{1/2} \lesssim \Big(\sum_{j\ge 0}\|f_j\|^2_{\mathscr{H}_1}\Big)^{1/2} = \|f\|_{\mathscr{H}_1},$$

which proves that $\mathscr{H}_1 \hookrightarrow \mathscr{B}_1$.

The inclusion $\mathscr{H}_1 \hookrightarrow \mathcal{S}_{s/\gamma}$ when $\gamma > 1$ follows if we prove that

$$e^{c|\alpha|^{\gamma/2s}} \lesssim m(\alpha), \tag{3.11}$$

for every $c > 0$. We claim that there is a constant C_0 which is independent of $j \ge 1$ and α such that

$$e^{c|\alpha|^{\gamma/2s}} \le C_0 e^{j^j-1} e^{2|\alpha|^{1/2s}/j}. \tag{3.12}$$

In fact, by applying the logarithm, (3.12) follows if we prove that for $r = 1/2s \le 1$ and constants $m_1, m_2 > 0$, the function

$$h(u,v) = m_1 u^u + u^{-1} v^r - m_2 v^{\gamma r}$$

is bounded from below, when $u, v \ge 1$.

In order to prove this we let $0 < \gamma_1, \gamma_2 < 1$ be chosen such that $\gamma_1 > \gamma$ and $\gamma_1 + \gamma_2 = 1$. Then the inequality on arithmetic and geometric mean-values gives that $h_0(u,v) \lesssim h(u,v)$, where

$$h_0(u,v) = u^{\gamma_2 u - \gamma_1} v^{\gamma_1 r} - m_1' v^{\gamma r} = v^{\gamma r}(u^{\gamma_2 u-\gamma_1} v^{(\gamma_1-\gamma) r} - m_1'),$$

for some $m_1' > 0$. Since $\gamma_1 > \gamma$, it follows that $h_0(u,v)$ tends to infinity when $u + v \to \infty$ and $u, v \ge 1$. The fact that h_0 is continuous then implies that $h_0(u,v)$ and thereby $h(u,v)$ is bounded from below when $u, v \ge 1$, which proves that (3.12) holds.

This gives

$$e^{c|\alpha|^{\gamma/2s}} \le C_0 e^{j^j-1} e^{2|\alpha|^{1/2s}/j} = C_0 m(\alpha), \quad \alpha \in I_j,$$

and (3.11) follows, which proves the first part of (1).

It remains to prove that $\mathscr{H}_2$ exists with the asserted properties. The fact that $\mathscr{B}_2$ is continuously embedded in $\mathcal{S}_s'$ implies that for every $j \ge 1$, there is a constant $C_j \ge 1$ such that

$$\sum_\alpha |c_\alpha|^2 C_j^{-1} e^{-|\alpha|^{1/2s}/j} \le \|f\|^2_{\mathscr{B}_2}.$$

Let

$$m(\alpha) = \sum_{j\ge 1} j^{-2}e^{-j^j}C_j^{-1}e^{-|\alpha|^{1/2s}/j},$$

and let $\mathscr{H}_2$ be the set of all $f \in \mathcal{S}_s'$ such that

$$\|f\|_{\mathscr{H}_2} \equiv \Big(\sum_\alpha |c_\alpha|^2 m(\alpha)\Big)^{1/2}$$

is finite.

By the definition it follows that $\|f\|_{\mathscr{H}_2} \lesssim \|f\|_{\mathscr{B}_2}$ when $f \in \mathscr{B}_2$, giving that $\mathscr{B}_2$ is continuously embedded in $\mathscr{H}_2$. Furthermore, if $c > 0$, then it follows that

$$\lim_{|\alpha|\to\infty} \frac{e^{-c|\alpha|^{1/2s}}}{m(\alpha)} = 0,$$

which implies that $\mathscr{H}_2$ is a Hilbert space.

It remains to prove that $\mathcal{S}_{s/\gamma}' \hookrightarrow \mathscr{H}_2$ when $0 < \gamma < 1$, which follows if we prove that

$$m(\alpha) \lesssim e^{-c|\alpha|^{\gamma/2s}}, \tag{3.13}$$

for every $c > 0$. By the same arguments as in the last part of the proof we have

$$e^{-j^j}e^{-|\alpha|^{1/2s}/j} \le C_0 e^{-c|\alpha|^{\gamma/2s}},$$

where C_0 neither depends on j nor on α. This gives

$$m(\alpha) = \sum_{j\ge 1} j^{-2}e^{-j^j}e^{-|\alpha|^{1/2s}/j} \lesssim e^{-c|\alpha|^{\gamma/2s}} \sum_{j\ge 1}\frac{1}{j^2} \asymp e^{-c|\alpha|^{\gamma/2s}},$$

and (3.13) follows. The proof is complete. □

The end of the proof of Proposition 3.7. We only prove (1). The assertion (2) follows by similar arguments and is left for the reader.

Let $\mathscr{B}$ be a GS-tempered quasi-Banach space. By Proposition 3.8 there are GS-tempered Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $\mathscr{H}_1 \hookrightarrow \mathscr{B} \hookrightarrow \mathscr{H}_2$, and by the first part of the proof it follows that $M^2_{(\omega_t)} \hookrightarrow \mathscr{H}_j \hookrightarrow M^2_{(1/\omega_t)}$ for every $t > 0$, $j = 1, 2$. The result now follows by combining these inclusions. The proof is complete. □

4. Schatten–von Neumann classes and pseudo-differential operators

In this section we discuss Schatten–von Neumann classes of pseudo-differential operators from a Hilbert space $\mathscr{H}_1$ to another Hilbert space $\mathscr{H}_2$, or more generally, from a (quasi-)Banach space $\mathscr{B}_1$ to another (quasi-)Banach space $\mathscr{B}_2$. Schatten–von Neumann classes were introduced by R. Schatten in [36] in the case when $\mathscr{H}_1 = \mathscr{H}_2$ are Hilbert spaces. (See also [39].) The theory was thereafter extended in [3, 33, 37, 53] to the case when $\mathscr{H}_1$ is not necessarily equal to $\mathscr{H}_2$, and in [3, 33, 39], the theory was extended in such way that it includes linear operators from a Banach space $\mathscr{B}_1$ to another Banach space $\mathscr{B}_2$. Furthermore, the definitions and some of the results in [3, 33, 39] can easily be modified to permit $\mathscr{B}_1$ and $\mathscr{B}_2$ to be arbitrary quasi-Banach spaces.

We will mainly follow the organization in the second part of Section 4 in [51], and we remark that there are several similarities between the proofs in this section, and the proofs of analogous results in Section 4 in [51].

We start by recalling the definition of Schatten–von Neumann operators in the (quasi-)Banach space case. We remark however that this general setting is not needed for the main results in the present and next sections (e.g., Theorem 4.8 below). For the reader who is not interested in this general approach may therefore assume that the operators act on Hilbert spaces.

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be (quasi-)Banach spaces, and let T be a linear map from $\mathscr{B}_1$ to $\mathscr{B}_2$. For every integer $j \ge 1$, the *singular number* of T of order j is given by

$$\sigma_j(T) = \sigma_j(\mathscr{B}_1, \mathscr{B}_2, T) \equiv \inf \|T - T_0\|_{\mathscr{B}_1 \to \mathscr{B}_2},$$

where the infimum is taken over all linear operators T_0 from $\mathscr{B}_1$ to $\mathscr{B}_2$ with rank at most $j-1$. Therefore, $\sigma_1(T)$ equals $\|T\|_{\mathscr{B}_1 \to \mathscr{B}_2}$, and $\sigma_j(T)$ is non-negative which decreases with j.

For any $p \in (0, \infty]$ we set

$$\|T\|_{\mathscr{I}_p} = \|T\|_{\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)} \equiv \|\{\sigma_j(\mathscr{B}_1, \mathscr{B}_2, T)\}_{j=1}^\infty\|_{l^p}$$

(which might attain $+\infty$). The operator T is called a *Schatten–von Neumann operator* of order p from $\mathscr{B}_1$ to $\mathscr{B}_2$, if $\|T\|_{\mathscr{I}_p}$ is finite, i.e., $\{\sigma_j(\mathscr{B}_1, \mathscr{B}_2, T)\}_{j=1}^\infty$ should belong to l^p. The set of all Schatten–von Neumann operators of order p from $\mathscr{B}_1$ to $\mathscr{B}_2$ is denoted by $\mathscr{I}_p = \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$. We note that $\mathscr{I}_\infty(\mathscr{B}_1, \mathscr{B}_2)$ agrees with $\mathcal{B}(\mathscr{B}_1, \mathscr{B}_2)$, the set of linear and bounded operators from $\mathscr{B}_1$ to $\mathscr{B}_2$, and if $p < \infty$, then $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ is contained in $\mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$, the set of linear and compact operators from $\mathscr{B}_1$ to $\mathscr{B}_2$. The spaces $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ for $p \in (0, \infty]$ and $\mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$ are quasi-Banach spaces which are Banach spaces when $\mathscr{B}_1$, $\mathscr{B}_2$ are Banach spaces and $p \ge 1$. Furthermore, $\mathscr{I}_2(\mathscr{B}_1, \mathscr{B}_2)$ is a Hilbert space when $\mathscr{B}_1$ and $\mathscr{B}_2$ are Hilbert spaces. If $\mathscr{B}_1 = \mathscr{B}_2$, then the shorter notation $\mathscr{I}_p(\mathscr{B}_1)$ is used instead of $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$, and similarly for $\mathcal{B}(\mathscr{B}_1, \mathscr{B}_2)$ and $\mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$.

Now let $\mathscr{B}_3$ be an other Banach space (or quasi-Banach space) and let T_k and $T_{0,k}$ be linear and continuous operators from $\mathscr{B}_k$ to $\mathscr{B}_{k+1}$ such that the rank of $T_{0,k}$ is at most j_k for $k = 1, 2$. If T_0 is defined by

$$T_0 = T_{0,2} \circ T_1 + T_2 \circ T_{0,1} - T_{0,2} \circ T_{0,1} = T_{0,2} \circ T_1 + (T_2 - T_{0,2}) \circ T_{0,1},$$

then

$$T_2 \circ T_1 - T_0 = (T_2 - T_{0,2}) \circ (T_1 - T_{0,1}),$$

and it follows that the rank of T_0 is at most $j_1 + j_2$. Hence

$$\sigma_{j_1+j_2+1}(T_2 \circ T_1) \le \|T_2 \circ T_1 - T_0\| \le \|T_1 - T_{0,1}\| \cdot \|T_2 - T_{0,2}\|,$$

and by taking the infimum on the right-hand side of all possible $T_{0,1}$ and $T_{0,2}$ we get

$$\sigma_{j_1+j_2+1}(T_2 \circ T_1) \le \sigma_{j_1+1}(T_1)\sigma_{j_2+1}(T_2), \qquad j_1, j_2 \ge 0. \tag{4.1}$$

If $\mathscr{B}_j = \mathscr{H}_j$, $j = 1, 2, 3$, are Hilbert spaces, then (4.1) can be improved into

$$\sigma_{j+1}(T_2 \circ T_1) \le \sigma_{j+1}(T_1)\sigma_{j+1}(T_2), \quad j \ge 0. \tag{4.1$'$}$$

(Cf. [33, 39].)

In [33, 39], (4.1) is used to prove that if $p_1, p_2, r \in (0, \infty]$ satisfy the Hölder condition $1/p_1 + 1/p_2 = 1/r$, and $T_k \in \mathscr{I}_{p_k}(\mathscr{B}_k, \mathscr{B}_{k+1})$, then $T_2 \circ T_1 \in \mathscr{I}_r(\mathscr{B}_1, \mathscr{B}_3)$, and

$$\|T_2 \circ T_1\|_{\mathscr{I}_r(\mathscr{B}_1,\mathscr{B}_3)} \le C_r \|T_1\|_{\mathscr{I}_{p_1}(\mathscr{B}_1,\mathscr{B}_2)} \|T_2\|_{\mathscr{I}_{p_2}(\mathscr{B}_2,\mathscr{B}_3)}. \tag{4.2}$$

Here $C_r = 1$ when $\mathscr{B}_j$, $j = 1, 2, 3$ are Hilbert spaces, and $C_r = 2^{1/r}$ otherwise. In order to be self-contained we here give a proof of (4.2).

Let $T = T_2 \circ T_1$. Since $\sigma_{2j+2}(T) \le \sigma_{2j+1}(T)$, it follows by letting $j_1 = j_2 = j$ in (4.1) that

$$\begin{aligned}
\|T\|_{\mathscr{I}_r} &= \Big(\sum_{k\ge 0} \sigma_{k+1}(T)^r\Big)^{1/r} \le 2^{1/r}\Big(\sum_{j\ge 0}\sigma_{2j+1}(T)^r\Big)^{1/r} \\
&\le 2^{1/r}\Big(\sum_{j\ge 0}\sigma_{j+1}(T_1)^r \sigma_{j+1}(T_2)^r\Big)^{1/r} \\
&\le 2^{1/r}\Big(\sum_{j\ge 0}\sigma_{j+1}(T_1)^{p_1}\Big)^{1/p_1}\Big(\sum_{j\ge 0}\sigma_{j+1}(T_2)^{p_2}\Big)^{1/p_2} = 2^{1/r}\|T_1\|_{\mathscr{I}_{p_1}}\|T_2\|_{\mathscr{I}_{p_2}}.
\end{aligned}$$

This gives (4.2).

If $\mathscr{B}_j$, $j = 1, 2, 3$ are Hilbert spaces, then the same arguments, using (4.1)$'$ instead of (4.1), shows that (4.2) holds for $C_r = 1$. (Cf. [33] or Chapters 2 and 3 in [39].)

If $\mathscr{B}_1$ and $\mathscr{B}_2$ are Banach spaces, then we note that ranks and norms for the operators are invariant when passing from the operators to their adjoints. This implies that T belongs to $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$, if and only if the adjoint T^* of T belongs to $\mathscr{I}_p(\mathscr{B}_2', \mathscr{B}_1')$, and

$$\|T\|_{\mathscr{I}_p(\mathscr{B}_1,\mathscr{B}_2)} = \|T^*\|_{\mathscr{I}_p(\mathscr{B}_2',\mathscr{B}_1')}.$$

We recall that if $\mathscr{B}_1 = \mathscr{H}_1$ and $\mathscr{B}_2 = \mathscr{H}_2$ are Hilbert spaces, then there is an alternative way to compute the $\mathscr{I}_p$ norms. More precisely, let $\mathrm{ON}(\mathscr{H}_k)$, $k = 1, 2$, denote the family of orthonormal sequences in $\mathscr{H}_k$. If $T : \mathscr{H}_1 \to \mathscr{H}_2$ is linear, and $p \in (0, \infty]$, then it follows that

$$\|T\|_{\mathscr{I}_p(\mathscr{H}_1,\mathscr{H}_2)} = \sup\Big(\sum |(Tf_j, g_j)_{\mathscr{H}_2}|^p\Big)^{1/p}$$

(with obvious modifications when $p = \infty$), where the supremum is taken over all $\{f_j\} \in \mathrm{ON}(\mathscr{H}_1)$ and $\{g_j\} \in \mathrm{ON}(\mathscr{H}_2)$.

Let $\{e_j\}$ be an orthonormal basis in $\mathscr{H}_1$, and let $S \in \mathscr{I}_1(\mathscr{H}_1)$. Then the trace of S is defined as

$$\mathrm{tr}_{\mathscr{H}_1} S = \sum (Se_j, e_j)_{\mathscr{H}_1},$$

and we recall that this is independent of the choice of the orthonormal basis $\{e_j\}$. For each pairs of operators $T_1, T_2 \in \mathscr{I}_\infty(\mathscr{H}_1, \mathscr{H}_2)$ such that $T_2^* \circ T_1 \in \mathscr{I}_1(\mathscr{H}_1)$, the sesqui-linear form

$$(T_1, T_2) = (T_1, T_2)_{\mathscr{H}_1, \mathscr{H}_2} \equiv \operatorname{tr}_{\mathscr{H}_1}(T_2^* \circ T_1)$$

of T_1 and T_2 is well defined. We refer to [3, 33, 39, 53] for more facts about Schatten–von Neumann classes.

Next we define symbol classes whose corresponding pseudo-differential operators belongs to $\mathscr{I}_p$ for some $p \in (0, \infty]$. Therefore, let $\mathscr{B}_1, \mathscr{B}_2 \hookrightarrow \mathcal{S}_{1/2}'(\mathbf{R}^d)$ be GS- or B-tempered (quasi-)Banach spaces, $t \in \mathbf{R}$ be fixed and let $p \in (0,, \infty]$. Then we let $s_p^A(\mathscr{B}_1, \mathscr{B}_2)$ and $s_{t,p}(\mathscr{B}_1, \mathscr{B}_2)$ be the sets of all $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ such that $Aa \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ and $\operatorname{Op}_t(a) \in \mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ respectively. We also let $s_\sharp^A(\mathscr{B}_1, \mathscr{B}_2)$ and $s_{t,\sharp}(\mathscr{B}_1, \mathscr{B}_2)$ be the sets of all $a \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ such that $Aa \in \mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$ and $\operatorname{Op}_t(a) \in \mathcal{K}(\mathscr{B}_1, \mathscr{B}_2)$ respectively. These spaces are equipped by the (quasi-)norms

$$\|a\|_{s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)} \equiv \|\operatorname{Op}_t(a)\|_{\mathscr{I}_p(\mathscr{B}_1,\mathscr{B}_2)}, \qquad \|a\|_{s_p^A(\mathscr{B}_1,\mathscr{B}_2)} \equiv \|Aa\|_{\mathscr{I}_p(\mathscr{B}_1,\mathscr{B}_2)},$$
$$\|a\|_{s_{t,\sharp}(\mathscr{B}_1,\mathscr{B}_2)} \equiv \|a\|_{s_{t,\infty}(\mathscr{B}_1,\mathscr{B}_2)}, \qquad \|a\|_{s_\sharp^A(\mathscr{B}_1,\mathscr{B}_2)} \equiv \|a\|_{s_\infty^A(\mathscr{B}_1,\mathscr{B}_2)}.$$

Since the mappings $a \mapsto Aa$ and $a \mapsto \operatorname{Op}_t(a)$ are bijections from $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ to the set of linear and continuous operators from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$, it follows that $a \mapsto Aa$ and $a \mapsto \operatorname{Op}_t(a)$ restrict to isometric bijections from $s_p^A(\mathscr{B}_1, \mathscr{B}_2)$ and $s_{t,p}(\mathscr{B}_1, \mathscr{B}_2)$ respectively to $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$. Consequently, the properties for $\mathscr{I}_p(\mathscr{B}_1, \mathscr{B}_2)$ carry over to $s_p^A(\mathscr{B}_1, \mathscr{B}_2)$ and $s_{t,p}(\mathscr{B}_1, \mathscr{B}_2)$. In particular, if $\mathscr{B}_1 = \mathscr{H}_1$ and $\mathscr{B}_2 = \mathscr{H}_2$ are Hilbert spaces, then the elements in $s_1^A(\mathscr{H}_1, \mathscr{H}_2)$ of finite rank (i.e., elements of the form $a \in s_1^A(\mathscr{H}_1, \mathscr{H}_2)$ such that Aa is a finite rank operator), are dense in $s_\sharp^A(\mathscr{H}_1, \mathscr{H}_2)$ and in $s_p^A(\mathscr{H}_1, \mathscr{H}_2)$ when $p < \infty$. Similar facts hold for $s_{t,\sharp}(\mathscr{H}_1, \mathscr{H}_2)$ and $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$. Since the Weyl quantization is particularly important in our considerations we also set

$$s_p^w = s_{t,p} \quad \text{and} \quad s_\sharp^w = s_{t,\sharp}, \quad \text{when} \quad t = 1/2.$$

If $\omega_1, \omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$, then we use the shorter notation $s_p^A(\omega_1, \omega_2)$ instead of $s_p^A(M^2_{(\omega_1)}, M^2_{(\omega_2)})$. Furthermore we set $s_p^A(\omega_1, \omega_2) = s_p^A(\mathbf{R}^{2d})$ when $\omega_1 = \omega_2 = 1$. In the same way the notations for $s_\sharp^A$, $s_{t,p}$, s_p^w, $s_{t,\sharp}$ and $s_\sharp^w$ are simplified.

Remark 4.1. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces. Then, except for the Hilbert–Schmidt case ($p = 2$), it is in general a hard task to find simple characterizations for $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$. Important questions therefore concern of finding convenient embedding properties between Schatten–von Neumann classes and well-known function and distribution spaces. We refer to Remark 4.5 in [51], for examples on such embeddings.

Remark 4.2. Let $t, t_1, t_2 \in \mathbf{R}$, $p \in [1, \infty]$, $\mathscr{B}_1, \mathscr{B}_2$ be GS- or B-tempered quasi-Banach spaces on $\mathbf{R}^d$ and let $a, b \in \mathcal{S}_{1/2}'(\mathbf{R}^{2d})$. Then it follows by Fourier's inversion formula that the map $e^{it\langle D_x, D_\xi\rangle}$ is a homeomorphism on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ which extends uniquely to a homeomorphism on $\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$. Furthermore, by (1.10) it follows that $e^{i(t_2-t_1)\langle D_x,D_\xi\rangle}$ restricts to an isometric bijection from $s_{t_1,p}(\mathscr{B}_1,\mathscr{B}_2)$ to $s_{t_2,p}(\mathscr{B}_1,\mathscr{B}_2)$.

The following proposition links $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$, $s_p^A(\mathscr{B}_1,\mathscr{B}_2)$ and other similar spaces to each others. Here $a^\tau(x,\xi)=\overline{a(x,-\xi)}$ is the "torsion" of $a\in\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$.

Proposition 4.3. *Let $t\in\mathbf{R}$, $\mathscr{B}_1,\mathscr{B}_2$ be GS- or B-tempered quasi-Banach spaces in $\mathbf{R}^d$, $a\in\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$, and let $p\in(0,\infty]$. Then $s_p^w(\mathscr{B}_1,\check{\mathscr{B}}_2)=s_p^A(\mathscr{B}_1,\mathscr{B}_2)$, and the following conditions are equivalent:*

1. *$a\in s_p^w(\mathscr{B}_1,\mathscr{B}_2)$;*
2. *$\mathscr{F}_\sigma a\in s_p^w(\mathscr{B}_1,\check{\mathscr{B}}_2)$;*
3. *$\overline{a}\in s_p^w(\mathscr{B}_2',\mathscr{B}_1')$;*
4. *$a^\tau\in s_p^A(\mathscr{B}_1^\tau,\mathscr{B}_2^\tau)$;*
5. *$\check{a}\in s_p^w(\check{\mathscr{B}}_1,\check{\mathscr{B}}_2)$;*
6. *$\widetilde{a}\in s_p^w(\check{\mathscr{B}}_2',\check{\mathscr{B}}_1')$;*
7. *$e^{i(t-1/2)\langle D_\xi,D_x\rangle}a\in s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$.*

Proof. Let $a_1=\mathscr{F}_\sigma a$, $a_2=\overline{a}$, $a_3=a^\tau$, $a_4=\check{a}$ and $a_5=\widetilde{a}$. Then the equivalences follow immediately from Remark 4.2 and the equalities

$$
\begin{aligned}
(\operatorname{Op}^w(a)f,g) &= (\operatorname{Op}^w(a_1)f,\check{g}) = (f,\operatorname{Op}^w(a_2)g)\\
&= \overline{(\operatorname{Op}^w(a_3)(x,D)\overline{f},\overline{g})} = (\operatorname{Op}^w(a_4)\check{f},\check{g}) = (\check{f},\operatorname{Op}^w(a_5)\check{g}),
\end{aligned}
$$

when $a\in\mathcal{S}_{1/2}'(\mathbf{R}^{2d})$ and $f,g\in\mathcal{S}_{1/2}(\mathbf{R}^d)$. (Cf. [51, Proposition 4.7].) The proof is complete. □

In Remarks 4.4 and 4.5 below we list some properties for $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$ and $s_p^A(\mathscr{H}_1,\mathscr{H}_2)$. These properties follow from well-known Schatten–von Neumann results in [3, 39, 53], in combination with (1.15), (1.21), (4.2), and the fact that the mappings $a\mapsto\operatorname{Op}_t(a)$ and $a\mapsto Aa$ are isometric bijections from $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$ and $s_p^A(\mathscr{B}_1,\mathscr{B}_2)$ respectively to $\mathscr{I}_p(\mathscr{B}_1,\mathscr{B}_2)$.

Remark 4.4. Let $p,p_j,q,r\in(0,\infty]$, $t\in\mathbf{R}$, $\mathscr{B}_j$ be GS- or B-tempered quasi-Banach spaces on $\mathbf{R}^d$, and let $\mathscr{H}_j$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$, $j=1,\dots,4$. Also let $C_r=1$ when $\mathscr{B}_1,\dots,\mathscr{B}_4$ are Hilbert spaces, and $C_r=2^{1/r}$ otherwise. Then the following is true:

1. the sets $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$ and $s_{t,\sharp}(\mathscr{B}_1,\mathscr{B}_2)$, are quasi-Banach space which increases with the parameter p. If in addition $r\le p<\infty$ and $p_1\le p_2$, then $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)\hookrightarrow s_{t,\sharp}(\mathscr{B}_1,\mathscr{B}_2)$, $s_{t,r}(\mathscr{B}_1,\mathscr{B}_2)$ is dense in $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$ and in $s_{t,\sharp}(\mathscr{B}_1,\mathscr{B}_2)$, and

$$
\|a\|_{s_{t,p_2}(\mathscr{B}_1,\mathscr{B}_2)}\le\|a\|_{s_{t,p_1}(\mathscr{B}_1,\mathscr{B}_2)},\quad a\in s_{t,\infty}(\mathscr{B}_1,\mathscr{B}_2). \tag{4.3}
$$

Moreover, if in addition $p \ge 1$ and $\mathscr{B}_j$, $j=1,2$, are Banach spaces, then $s_{t,p}(\mathscr{B}_1,\mathscr{B}_2)$ and $s_{t,\sharp}(\mathscr{B}_1,\mathscr{B}_2)$ are Banach spaces;

(2) if $\mathscr{B}_j = \mathscr{H}_j$, $j=1,2$, then equality is attained in (4.3), if and only if a is a rank one element, and then $\|a\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)} = (2\pi)^{-d/2}\|f_1\|_{\mathscr{H}_1}\|f_2\|_{\mathscr{H}_2}$, when a is given by (1.12);

(3) if $1/p_1 + 1/p_2 = 1/r$, $a_1 \in s_{t,p_1}(\mathscr{B}_1,\mathscr{B}_2)$ and $a_2 \in s_{t,p_2}(\mathscr{B}_2,\mathscr{B}_3)$, then $a_2 \#_t a_1 \in s_{t,r}(\mathscr{B}_1,\mathscr{B}_3)$, and

$$\|a_2 \#_t a_1\|_{s_{t,r}(\mathscr{B}_1,\mathscr{B}_3)} \le C_r \|a_1\|_{s_{t,p_1}(\mathscr{B}_1,\mathscr{B}_2)} \|a_2\|_{s_{t,p_2}(\mathscr{B}_2,\mathscr{B}_3)}. \tag{4.4}$$

On the other hand, for any $a \in s_{t,r}(\mathscr{H}_1,\mathscr{H}_3)$, there are elements $a_1 \in s_{t,p_1}(\mathscr{H}_1,\mathscr{H}_2)$ and $a_2 \in s_{t,p_2}(\mathscr{H}_2,\mathscr{H}_3)$ such that $a = a_2 \#_t a_1$ and equality holds in (4.4);

(4) if $\mathscr{B}_1 \hookrightarrow \mathscr{B}_2$ with dense embeddings and $\mathscr{B}_3 \hookrightarrow \mathscr{B}_4$, then $s_{t,p}(\mathscr{B}_2,\mathscr{B}_3) \hookrightarrow s_{t,p}(\mathscr{B}_1,\mathscr{B}_4)$.

Similar facts hold when the $s_{t,p}$ spaces and the product $\#_t$ are replaced by s_p^A spaces and $*_\sigma$, respectively.

In the next remark we make some further conclusions on dual forms for $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ and $s_p^A(\mathscr{H}_1,\mathscr{H}_2)$ when $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces. Here the forms $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}$ and $(\,\cdot\,,\,\cdot\,)_{s_2^A(\mathscr{H}_1,\mathscr{H}_2)}$ are defined by the formulas

$$(a,b)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)} = (\operatorname{Op}_t(a),\operatorname{Op}_t(b))_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)}, \quad a,b \in s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)$$

and

$$(a,b)_{s_2^A(\mathscr{H}_1,\mathscr{H}_2)} = (Aa,Ab)_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)}, \qquad a,b \in s_2^A(\mathscr{H}_1,\mathscr{H}_2).$$

We also recall that $p' \in [1,\infty]$ is the conjugate exponent for $p \in [1,\infty]$, i.e., $1/p + 1/p' = 1$. Finally, the set l_0^∞ consists of all sequences in l^∞ which tends to zero at infinity, and l_0^1 consists of all sequences $\{\lambda_j\}_{j\in I}$ such that $\lambda_j = 0$ except for finite numbers of $j \in I$.

Remark 4.5. Let $p, p_j \in [1,\infty]$ for $1 \le j \le 2$, $t \in \mathbf{R}$, and let $\mathscr{H}_1, \mathscr{H}_2$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$. Then the following is true:

(1) the form $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}$ on $s_{t,1}(\mathscr{H}_1,\mathscr{H}_2)$ extends uniquely to a sesquilinear and continuous form on $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2) \times s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)$, and for every $a_1 \in s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ and $a_2 \in s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)$, it holds

$$\begin{aligned}
(a_1,a_2)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)} &= \overline{(a_2,a_1)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}},\\
|(a_1,a_2)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}| &\le \|a_1\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)}\|a_2\|_{s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)} \quad \text{and}\\
\|a_1\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)} &= \sup |(a_1,b)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}|,
\end{aligned}$$

where the supremum is taken over all $b \in s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)$ such that

$$\|b\|_{s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)} \le 1.$$

If in addition $p < \infty$, then the dual space of $s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ can be identified with $s_{t,p'}(\mathscr{H}_1,\mathscr{H}_2)$ through this form;

(2) if $a \in s_{t,\sharp}(\mathscr{H}_1,\mathscr{H}_2)$, then

$$\operatorname{Op}_t(a)f = \sum_{j=1}^{\infty} \lambda_j (f,f_j)_{\mathscr{H}_1} g_j, \tag{4.5}$$

holds for some $\{f_j\}_{j=1}^{\infty} \in \mathrm{ON}(\mathscr{H}_1)$, $\{g_j\}_{j=1}^{\infty} \in \mathrm{ON}(\mathscr{H}_2)$ and non-negative decreasing sequence $\lambda = \{\lambda_j\}_{j=1}^{\infty} \in l_0^{\infty}$, where the operator on the right-hand side of (4.5) convergences with respect to the operator norm. Moreover, $a \in s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$, if and only if $\lambda \in l^p$, and then

$$\|a\|_{s_{t,p}} = \|\lambda\|_{l^p}$$

and the operator on the right-hand side of (4.5) converges with respect to the norm $\|\cdot\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)}$;

(3) if $0 \le \theta \le 1$ is such that $1/p = (1-\theta)/p_1 + \theta/p_2$, then the (complex) interpolation formula

$$(s_{t,p_1}(\mathscr{H}_1,\mathscr{H}_2), s_{t,p_2}(\mathscr{H}_1,\mathscr{H}_2))_{[\theta]} = s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$$

holds with equality in norms.

Similar facts hold when the $s_{t,p}$ spaces are replaced by s_p^A spaces.

In the sequel we assume that $\mathscr{B}_j = \mathscr{H}_j$, $j \ge 0$, are Hilbert spaces. A problem with the form $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathscr{H}_1,\mathscr{H}_2)}$ in Remark 4.5 is the somewhat complicated structure. In the following we show that there is a canonical way to replace this form with $(\,\cdot\,,\,\cdot\,)_{L^2}$. We start with the following result concerning polar decomposition of compact operators.

Proposition 4.6. *Let $t \in \mathbf{R}$, $p \in [1,\infty]$, $\mathscr{H}_1$ and $\mathscr{H}_2$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$ and let $a \in s_{t,\sharp}(\mathscr{H}_1,\mathscr{H}_2)$ ($a \in s_{\sharp}^A(\mathscr{H}_1,\mathscr{H}_2)$). Then*

$$a \equiv \sum_{j\in I} \lambda_j W_{g_j,\varphi_j}^t \qquad \Big(a \equiv \sum_{j\in I} \lambda_j W_{\check{g}_j,\varphi_j}\Big) \tag{4.6}$$

(with norm convergence) for some orthonormal sequences $\{\varphi_j\}_{j\in I}$ in $\mathscr{H}_1'$ and $\{g_j\}_{j\in I}$ in $\mathscr{H}_2$, and a sequence $\{\lambda_j\}_{j\in I} \in l_0^{\infty}$ of non-negative real numbers which decreases to zero at infinity. Furthermore, $a \in s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)$ ($a \in s_p^A(\mathscr{H}_1,\mathscr{H}_2)$), if and only if $\{\lambda_j\}_{j\in I} \in l^p$, and

$$\|a\|_{s_{t,p}(\mathscr{H}_1,\mathscr{H}_2)} = (2\pi)^{-d/2}\|\{\lambda_j\}_{j\in I}\|_{l^p} \quad (\|a\|_{s_p^A(\mathscr{H}_1,\mathscr{H}_2)} = \|\{\lambda_j\}_{j\in I}\|_{l^p}).$$

Proof. By Remark 4.5 (2) it follows that if $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, then

$$\operatorname{Op}_t(a)f(x) = \sum_{j\in I} \lambda_j (f,f_j)_{\mathscr{H}_1} g_j \tag{4.7}$$

for some orthonormal sequences $\{f_j\}$ in $\mathscr{H}_1$ and $\{g_j\}$ in $\mathscr{H}_2$, and a sequence $\{\lambda_j\} \in l_0^{\infty}$ of non-negative real numbers which decreases to zero at infinity. Now let $\{\varphi_j\}_{j\in I}$ be an orthonormal sequence in $\mathscr{H}_1'$ such that $(\varphi_j, f_k)_{L^2} = \delta_{j,k}$. Then $(f, f_j)_{\mathscr{H}_1} = (f, \varphi_j)_{L^2}$, and the result follows from (4.7), and the fact that

$$\mathrm{Op}_t(W^t_{g_j,\varphi_j})f = (2\pi)^{-d/2}(f, \varphi_j)_{L^2} g_j = (2\pi)^{-d/2}(f, f_j)_{\mathscr{H}_1} g_j.$$

The proof is complete. □

We have now the following.

Proposition 4.7. *Let $p \in [1, \infty)$, and let $\mathscr{H}_1$ and $\mathscr{H}_2$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$. Then the following is true:*

(1) *$\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is dense in $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$, $s^A_p(\mathscr{H}_1, \mathscr{H}_2)$, $s_{t,\sharp}(\mathscr{H}_1, \mathscr{H}_2)$ and $s^A_\sharp(\mathscr{H}_1, \mathscr{H}_2)$;*

(2) *$\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is dense in $s_{t,\infty}(\mathscr{H}_1, \mathscr{H}_2)$ and $s^A_\infty(\mathscr{H}_1, \mathscr{H}_2)$ with respect to the weak* topology.*

Proof. By Proposition 4.6 it follows that any element in $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$, $s^A_p(\mathscr{H}_1, \mathscr{H}_2)$, $s_{t\sharp}(\mathscr{H}_1, \mathscr{H}_2)$ or in $s^A_\sharp(\mathscr{H}_1, \mathscr{H}_2)$ can be approximated in norm by finite sums of the forms in (4.6). The assertion (1) now follows from the facts that any φ_j and g_j can be approximated in norms by elements in $\mathcal{S}_{1/2}(\mathbf{R}^d)$, and that the map $(\varphi, g) \mapsto W^t_{g,\varphi}$ is continuous from $\mathcal{S}_{1/2}(\mathbf{R}^d) \times \mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$.

The assertion (2) now follows from (1) and the fact that $s_{t,1}$ is weakly dense in $s_{t,\infty}$, since $\mathscr{I}_1$ is weakly dense in $\mathscr{I}_\infty$. □

Next we prove that the duals for $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ and $s^A_p(\mathscr{H}_1, \mathscr{H}_2)$ can be identified with $s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')$ and $s^A_{p'}(\mathscr{H}_1', \mathscr{H}_2')$ respectively through the form $(\,\cdot\,,\,\cdot\,)_{L^2}$.

Theorem 4.8. *Let $t \in \mathbf{R}$, $p \in [1, \infty)$ and let $\mathscr{H}_1, \mathscr{H}_2$ be GS- or B-tempered Hilbert spaces on $\mathbf{R}^d$. Then the L^2 form on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ extends uniquely to a duality between $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ and $s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')$, and the dual of $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ can be identified with $s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')$ through this form. Moreover, if $\ell \in s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)^*$ and $a \in s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')$ are such that $\overline{\ell(b)} = (a, b)_{L^2}$ when $b \in s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$, then*

$$\|\ell\| = \|a\|_{s_{t,p'}(\mathscr{H}_1', \mathscr{H}_2')}.$$

The same is true if the $s_{t,p}(\mathscr{H}_1, \mathscr{H}_2)$ spaces are replaced by $s^A_p(\mathscr{H}_1, \mathscr{H}_2)$ spaces.

Proof. We only prove the assertion in the case $t = 1/2$. The general case follows by similar arguments and is left for the reader. Let $\ell \in s^w_p(\mathscr{H}_1, \mathscr{H}_2)^*$. Since the map $b \mapsto \mathrm{Op}^w(b)$ is an isometric bijection from $s^w_p(\mathscr{H}_1, \mathscr{H}_2)$ to $\mathscr{I}_p(\mathscr{H}_1, \mathscr{H}_2)$, it follows from Remark 4.5 (1) that for some $S \in \mathscr{I}_{p'}(\mathscr{H}_1, \mathscr{H}_2)$ and each orthonormal basis $\{f_j\} \in \mathrm{ON}(\mathscr{H}_1)$ we have

$$\begin{aligned} &\ell(b) = \mathrm{tr}_{\mathscr{H}_1}(S^* \circ \mathrm{Op}^w(b)) = \sum (\mathrm{Op}^w(b)f_j, Sf_j)_{\mathscr{H}_2} \quad \text{and} \\ &\|\ell\| = \|S\|_{\mathscr{I}_{p'}(\mathscr{H}_1, \mathscr{H}_2)}, \end{aligned} \tag{4.8}$$

when $b \in s^w_p(\mathscr{H}_1, \mathscr{H}_2)$.

Now let $b \in s_p^w(\mathscr{H}_1, \mathscr{H}_2)$ be an arbitrary finite rank element. Then

$$b = \sum \lambda_j W_{g_j,\varphi_j} \quad \text{and} \quad \|b\|_{s_p^w(\mathscr{H}_1,\mathscr{H}_2)} = (2\pi)^{-d/2} \|\{\lambda_j\}\|_{l^p},$$

for some orthonormal bases $\{\varphi_j\} \in \mathrm{ON}(\mathscr{H}_1')$ and $\{g_j\} \in \mathrm{ON}(\mathscr{H}_2)$, and some sequence $\{\lambda_j\} \in l_0^1$. We also let $\{f_j\} \in \mathrm{ON}(\mathscr{H}_1)$ be the dual basis of $\{\varphi_j\}$ and a the Weyl symbol of the operator $T_{\mathscr{H}_2} \circ S \circ T_{\mathscr{H}_1'}$. Then $a \in s_{p'}^w(\mathscr{H}_1, \mathscr{H}_2)$ and $\|a\|_{s_{p'}^w(\mathscr{H}_1,\mathscr{H}_2)} = \|\ell\|$. By straightforward computations we also get

$$\begin{aligned}\ell(b) &= \mathrm{tr}_{\mathscr{H}_1}(S^* \circ \mathrm{Op}^w(b)) = \sum (\mathrm{Op}^w(b) f_j, S f_j)_{\mathscr{H}_2} \\ &= (2\pi)^{-d/2} \sum \lambda_j (g_j, S f_j)_{\mathscr{H}_2} = (2\pi)^{-d/2} \sum \lambda_j (g_j, \mathrm{Op}^w(a)\varphi_j)_{L^2(\mathbf{R}^d)} \\ &= (2\pi)^{-d} \sum \lambda_j (W_{g_j,\varphi_j}, a)_{L^2(\mathbf{R}^{2d})} = (2\pi)^{-d} (b, a)_{L^2(\mathbf{R}^{2d})}.\end{aligned}$$

Hence $\ell(b) = (2\pi)^{-d}(b, a)_{L^2(\mathbf{R}^{2d})}$. The result now follows from these identities and the fact that the set of finite rank elements are dense in $s_p^w(\mathscr{H}_1, \mathscr{H}_2)$. The proof is complete. □

An interesting question is wether Theorem 4.8 still holds after the Hilbert spaces $\mathscr{H}_k$ and $\mathscr{H}_k'$ have been replaced by appropriate Banach spaces.

We finish the section by a side result on bases and Hilbert–Schmidt operators on GS- or B-tempered Hilbert spaces.

Proposition 4.9. *Let $\mathscr{H}_j$ be GS- or B-tempered Hilbert space on $\mathbf{R}^{d_j}$ for $j = 1, 2$, and let T be a linear and continuous map from $\mathscr{H}_1$ to $\mathscr{H}_2$. Also let $\mathscr{H} = \mathscr{H}_2 \otimes (\mathscr{H}_1')^\tau$ (Hilbert tensor product). If K_T is the kernel of T, then $T \in \mathscr{I}_2(\mathscr{H}_1, \mathscr{H}_2)$, if and only if $K_T \in \mathscr{H}$, and*

$$\|T\|_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)} = \|K_T\|_{\mathscr{H}}. \tag{4.9}$$

Proof. First assume that $T \in \mathscr{I}_2(\mathscr{H}_1, \mathscr{H}_2)$, and let $\{e_{j,k}\}_{k=1}^\infty$ be an orthonormal basis for $\mathscr{H}_j$ and set $\varepsilon_{j,k} = T_{\mathscr{H}_j} e_{j,k}$, $j = 1, 2$. Then $\{\varepsilon_{j,k}\}_{k=1}^\infty$ is an orthonormal basis for $\mathscr{H}_j'$,

$$T e_{1,k} = \sum_l \lambda_{k,l} e_{2,l},$$

for some $\{\lambda_{k,l}\}_{k,l=1}^\infty$ and

$$\|T\|^2_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)} = (T, T)_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)} = \mathrm{tr}_{\mathscr{H}_1}(T^* \circ T),$$

giving that

$$\|T\|^2_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)} = \sum_k \|T e_{1,k}\|^2_{\mathscr{H}_2} = \sum_{k,l} |\lambda_{k,l}|^2. \tag{4.10}$$

Now let $N_1, N_2 > 0$ be integers and set

$$H_{N_1,N_2}(x, y) = \sum_{k \le N_1} \sum_{l \le N_2} \lambda_{k,l} (e_{2,l} \otimes \overline{\varepsilon_{1,k}})(x, y) = \sum_{k \le N_1} \sum_{l \le N_2} \lambda_{k,l} e_{2,l}(x) \overline{\varepsilon_{1,k}(y)} \in \mathscr{H}.$$

We shall prove that H_{N_1,N_2} has a limit H in $\mathscr{H}$ as $N_1, N_2 \to \infty$, and that $H = K_T$.

Since $\{\overline{\varepsilon_{1,k}}\}_{k=1}^{\infty}$ is an orthonormal basis for $(\mathscr{H}_1')^\tau$, we get

$$\|H_{N_1,N_2}\|_{\mathscr{H}}^2 = \sum_{k\le N_1}\sum_{l\le N_2} |\lambda_{k,l}|^2.$$

Hence (4.10) and the fact that $T \in \mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)$ imply that the limits

$$H_N = \lim_{N_2\to\infty} H_{N,N_2} \quad \text{and} \quad H = \lim_{N\to\infty} H_N$$

exist in $\mathscr{H}$, and that

$$\|H\|_{\mathscr{H}}^2 = \sum_{k,l} |\lambda_{k,l}|^2 = \|T\|_{\mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)}^2. \tag{4.11}$$

In order to prove that $H = K_T$ we let

$$f = \sum_k c_k e_{1,k} \in \mathscr{H}_1 \quad \text{and} \quad g = \sum_l d_l \varepsilon_{2,l} \in \mathscr{H}_2'$$

be arbitrary, and we set

$$f_N = \sum_{k\le N} c_k e_{1,k} \quad \text{and} \quad g_N = \sum_{l\le N} d_l \varepsilon_{2,l}.$$

Then $\|f - f_N\|_{\mathscr{H}_1} \to 0$ and $\|g - g_N\|_{\mathscr{H}_2'} \to 0$ as $N \to \infty$. Furthermore,

$$(Tf_{N_1}, g_{N_2})_{L^2(\mathbf{R}^{d_2})} = \sum_{k\le N_1}\sum_{l\le N_2} \lambda_{k,l} c_k \overline{d_l} = (H, g_{N_2}\otimes \overline{f_{N_1}})_{L^2(\mathbf{R}^{d_2+d_1})}.$$

By letting $N_1, N_2 \to \infty$ we get

$$(Tf, g)_{L^2(\mathbf{R}^{d_2})} = (H, g\otimes \overline{f})_{L^2(\mathbf{R}^{d_2+d_1})}.$$

Hence $H = K_T$, and (4.9) follows.

If instead $K_T \in \mathscr{H}$, then it follows by similar arguments as in the first part of the proof that (4.9) and the first equality in (4.11) hold with $H = K_T$. Hence, the second inequality in (4.11) shows that $T \in \mathscr{I}_2(\mathscr{H}_1,\mathscr{H}_2)$. The proof is complete. □

5. Young inequalities for weighted Schatten–von Neumann classes

In this section we establish Young type results for dilated convolutions and multiplications on $s_p^w(\mathscr{H}_1,\mathscr{H}_2)$, when $\mathscr{H}_1$ and $\mathscr{H}_2$ are appropriate modulation spaces of Hilbert type. Especially we prove multi-linear versions of Theorems 0.3 and 0.4. We will mainly follow the analysis in Section 5 in [51], and the proofs are similar. However, in order to be self-contained we here present proofs which are slightly condensed, where, at the same time, some misprints have been corrected.

We need some preparations for stating the results. If we have N convolutions, then the corresponding conditions compared to (0.9) is

$$p_1{}^{-1} + \cdots + p_N{}^{-1} = N - 1 + r^{-1}, \qquad 1 \le p_1, \dots, p_N, r \le \infty. \tag{0.9$'$}$$

In the same way, (0.10) should be replaced by

$$(-1)^{j_1}t_1^{-2}+\cdots+(-1)^{j_N}t_N^{-2}=1, \tag{0.10$'$}$$

and (0.11) by

$$(-1)^{j_1}t_1^{2}+\cdots+(-1)^{j_N}t_N^{2}=1. \tag{0.11$'$}$$

The condition (0.12) of the involved weight functions is modified into

$$\begin{aligned}\vartheta(X_1+\cdots+X_N)&\lesssim \vartheta_{j_1,1}(t_1X_1)\cdots\vartheta_{j_N,N}(t_NX_N),\\ \omega(X_1+\cdots+X_N)&\lesssim \omega_{j_1,1}(t_1X_1)\cdots\omega_{j_N,N}(t_NX_N),\end{aligned} \tag{0.12$'$}$$

where

$$\omega_{0,k}(X)=\vartheta_{1,k}(X)=\omega_k(X),\quad \vartheta_{0,k}(X)=\omega_{1,k}(X)=\vartheta_k(X). \tag{0.13$'$}$$

With these conditions we shall essentially prove estimates of the form

$$\|a_{1,t_1}*\cdots*a_{N,t_N}\|_{s_r^w(1/\omega,\vartheta)}\le C^d\|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s_{p_N}^w(1/\omega_N,\vartheta_N)}, \tag{0.14$'$}$$

and

$$\|a_{1,t_1}\cdots a_{N,t_N}\|_{s_r^w(1/\omega,\vartheta)}\le C^d\|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s_{p_N}^w(1/\omega_N,\vartheta_N)}. \tag{0.15$'$}$$

Here and in what follows we let a_s and b_t be given by $a_s=a(t\,\cdot\,)$ and $b_t=b(t\,\cdot\,)$ when $a,b\in\mathcal S_{1/2}'$ and $t\in\mathbf R\setminus 0$, and $a_{j,t}$ be given by $a_{j,t}=a_j(t\,\cdot\,)$ when $a_j\in\mathcal S_{1/2}'$, $j\in\mathbf N$, and $t\in\mathbf R\setminus 0$.

Theorem 0.3$'$. *Let $p_1,\dots,p_N,r\in[1,\infty]$ satisfy (0.9)$'$, and let $t_1,\dots,t_N\in\mathbf R\setminus 0$ satisfy (0.10)$'$, for some choices of $j_1,\dots,j_N\in\{0,1\}$. Also let $\omega,\omega_j,\vartheta,\vartheta_j\in\mathscr P_E(\mathbf R^{2d})$ for $j=1,\dots,N$ satisfy (0.12)$'$ and (0.13)$'$.*

Then the map $(a_1,\dots,a_N)\mapsto a_{1,t_1}*\cdots*a_{N,t_N}$ on $\mathcal S_{1/2}(\mathbf R^{2d})$, extends uniquely to a continuous mapping from

$$s_{p_1}^w(1/\omega_1,\vartheta_1)\times\cdots\times s_{p_N}^w(1/\omega_N,\vartheta_N)$$

to $s_r^w(1/\omega,\vartheta)$. Furthermore, (0.14)$'$ holds for some constant

$$C=C_0^N|t_1|^{-2/p_1}\cdots|t_N|^{-2/p_N},$$

where C_0 is independent of $a_1\in s_{p_1}^w(1/\omega_1,\vartheta_1)$, …,$a_N\in s_{p_N}^w(1/\omega_N,\vartheta_N)$, $t_1,\dots,t_N$ and d.

Moreover, $\operatorname{Op}^w(a_{1,t_1}*\cdots*a_{N,t_N})\ge 0$ when $\operatorname{Op}^w(a_j)\ge 0$ for each $1\le j\le N$.

Theorem 0.4$'$. *Let $p_1,\dots,p_N,r\in[1,\infty]$ satisfy (0.9)$'$, and let $t_1,\dots,t_N\in\mathbf R\setminus 0$ satisfy (0.11)$'$, for some choices of $j_1,\dots,j_N\in\{0,1\}$. Also let $\omega,\omega_j,\vartheta,\vartheta_j\in\mathscr P_E(\mathbf R^{2d})$ for $j=1,\dots,N$ satisfy (0.12)$'$ and (0.13)$'$.*

Then the map $(a_1,\dots,a_N)\mapsto a_{1,t_1}\cdots a_{N,t_N}$ on $\mathcal S_{1/2}(\mathbf R^{2d})$, extends uniquely to a continuous mapping from

$$s_{p_1}^w(1/\omega_1,\vartheta_1)\times\cdots\times s_{p_N}^w(1/\omega_N,\vartheta_N)$$

to $s_r^w(1/\omega,\vartheta)$. Furthermore, (0.15)$'$ holds for some constant

$$C=C_0^N|t_1|^{-2/p_1'}\cdots|t_N|^{-2/p_N'},$$

where C_0 is independent of $a_1 \in s^w_{p_1}(1/\omega_1,\vartheta_1)$, ...,$a_N \in s^w_{p_N}(1/\omega_N,\vartheta_N)$, $t_1,\dots,t_N$ and d.

We need some preparations for the proofs. First we observe that the roles of multiplications and convolutions are essentially interchanged on the symplectic Fourier transform side, because

$$\mathscr{F}_\sigma(a_1 * \cdots * a_N) = \pi^{dN}(\mathscr{F}_\sigma a_1)\cdots(\mathscr{F}_\sigma a_N), \tag{5.1}$$

holds when $a_1,\dots,a_N \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Hence it follows immediately from Lemma 1.3 and Proposition 4.3 that Theorems 0.3′ and 0.4′ are equivalent to the following two propositions. Here the condition (0.13)′ should be replaced by

$$\omega_{0,k}(X) = \vartheta_{1,k}(-X) = \omega_k(X), \quad \vartheta_{0,k}(X) = \omega_{1,k}(-X) = \vartheta_k(X). \tag{5.2}$$

We also recall that $a \in \mathcal{S}'_{s,+}(\mathbf{R}^{2d})$, $s \ge 1/2$, if and only if the operator Aa is positive semi-definite (cf. Proposition 1.4).

Proposition 5.1. *Let $p_1,\dots,p_N,r \in [1,\infty]$ satisfy (0.9)′, and let $t_1,\dots,t_N \in \mathbf{R}\setminus 0$ satisfy (0.10)′, for some choices of $j_1,\dots,j_N \in \{0,1\}$. Also let $\omega,\omega_j,\vartheta,\vartheta_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ for $j=1,\dots,N$ satisfy (0.12)′ and (5.2). Then the continuity assertions in Theorem 0.3′ hold after the s^w_p spaces have been replaced by s^A_p spaces.*

Proposition 5.2. *Let $p_1,\dots,p_N,r \in [1,\infty]$ satisfy (0.9)′, and let $t_1,\dots,t_N \in \mathbf{R}\setminus 0$ satisfy (0.11), for some choices of $j_1,\dots,j_N \in \{0,1\}$. Also let $\omega,\omega_j,\vartheta,\vartheta_j \in \mathscr{P}_E(\mathbf{R}^{2d})$ for $j=1,\dots,N$ satisfy (0.12)′ and (5.2). Then the continuity assertions in Theorem 0.4′ hold after the s^w_p spaces have been replaced by s^A_p spaces.*

Moreover, if $s > 1/2$ and $a_j \in \mathcal{S}'_{s,+}(\mathbf{R}^{2d}) \cap s^A_{p_j}(1/\omega_j,\vartheta_j)$ for every $j = 1,\dots,N$, then $a_{1,t_1}\cdots a_{N,t_N} \in \mathcal{S}'_{s,+}(\mathbf{R}^{2d}) \cap s^A_r(1/\omega,\vartheta)$.

When proving Propositions 5.1 and 5.2 we need some technical lemmas, and start with the following classification of Hilbert modulation spaces.

Lemma 5.3. *Let $\omega \in \mathscr{P}_E(\mathbf{R}^{4d})$ be such that $\omega(x,y,\xi,\eta) = \omega(x,\xi)$, $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)\setminus 0$ and let $F \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$. Then $F \in M^2_{(\omega)}$, if and only if*

$$\|F\| \equiv \Big(\iiint |V_\phi(F(\,\cdot\,,y))(x,\xi)\omega(x,\xi)|^2\, dxdyd\xi \Big)^{1/2} < \infty. \tag{5.3}$$

Furthermore, $F \mapsto \|F\|$ in (5.3) defines a norm which is equivalent to any $M^2_{(\omega)}$ norm.

Proof. We may assume that $\|\phi\|_{L^2} = 1$. Let $\Phi = \phi\otimes\phi$, and let $\mathscr{F}_1 F$ denotes the partial Fourier transform of $F(x,y)$ with respect to the x variable. By Parseval's formula we get

$$
\begin{aligned}
\|F\|_{M^2_{(\omega)}}^2 &= \iiiint |(V_\Phi F)(x,y,\xi,\eta)\omega(x,\xi)|^2\,dxdyd\xi d\eta \\
&= \iint \Big(\iint |(\mathscr{F}\big(F\,\overline{\Phi(\,\cdot\,-(x,y))}\,\big)(\xi,\eta)\omega(x,\xi)|^2\,dyd\eta\Big)\,dxd\xi \\
&= \iint \Big(\iint |(\mathscr{F}_1\big(F(\,\cdot\,,z)\,\overline{\phi(\,\cdot\,-x)}\,\big)(\xi)\phi(z-y)\omega(x,\xi)|^2\,dydz\Big)\,dxd\xi \\
&= \iint \Big(\int |(\mathscr{F}_1\big(F(\,\cdot\,,z)\,\overline{\phi(\,\cdot\,-x)}\,\big)(\xi)\omega(x,\xi)|^2\,dz\Big)\,dxd\xi = \|F\|,
\end{aligned}
$$

where the right-hand side is the same as $\|F\|$ in (5.3). The proof is complete. □

We omit the proof of the next lemma, since the result follows immediately from [44, Lemma 3.2], and the fact that $\mathcal{S}_{1/2} \hookrightarrow \mathscr{S}$.

Lemma 5.4. *Let $s,t \in \mathbf{R}$ be such that $(-1)^j s^{-2} + (-1)^k t^{-2} = 1$, for some choice of $j,k \in \{0,1\}$, and that $a,b \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Also let $T_{j,z}$ for $j \in \{0,1\}$ and $z \in \mathbf{R}^d$ be the operator on $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$, defined by the formula*

$$
(T_{0,z}U)(x,y) = (T_{1,z}U)(y,x) = U(x-z, y+z), \quad U \in \mathcal{S}_{1/2}(\mathbf{R}^{2d}).
$$

Then

$$
A(a(s\,\cdot\,)*b(t\,\cdot\,)) = (2\pi)^{d/2}|st|^{-d}\int (T_{j,sz}(Aa))(s^{-1}\,\cdot\,)(T_{k,-tz}(Ab))(t^{-1}\,\cdot\,)\,dz. \tag{5.4}
$$

We note that for the involved spaces in Theorems 0.3′ and 0.4′, and Propositions 5.1 and 5.2 we have

$$
s_p^A(1/\omega,\vartheta) \hookrightarrow s_p^A(\mathbf{R}^{2d}) \hookrightarrow s_p^A(\omega,1/\vartheta), \quad \text{when} \quad \omega,\vartheta \ge c, \tag{5.5}
$$

for some constant $c > 0$, and similarly when s_p^A is replaced by s_p^w. This is an immediate consequence of Remark 4.4 (4) and the embeddings $M^{2,2}_{(\omega)} \hookrightarrow M^{2,2} = L^2 \hookrightarrow M^{2,2}_{(1/\omega)}$ which are valid when ω is bounded from below. In particular, if $C_B(\mathbf{R}^d)$ denotes the set of all continuous functions on $\mathbf{R}^d$, vanishing at infinity, then

$$
\begin{aligned}
s_1^A(1/\omega,\vartheta) \hookrightarrow s_1^A(\mathbf{R}^{2d}) \hookrightarrow C_B(\mathbf{R}^{2d}) \cap \mathscr{F}C_B(\mathbf{R}^{2d}) \cap L^2(\mathbf{R}^{2d}),& \\
\text{when} \quad \omega,\vartheta \ge c,&
\end{aligned} \tag{5.6}
$$

and similarly when s_1^A is replaced by s_1^w. Here the latter embedding follows from Propositions 1.5 and 1.9 in [45].

Proof of Proposition 5.1 *in the case $N=2$.* We only consider the case $j_1 = 1$ and $j_2 = 0$, i.e., $t^{-2} - s^{-2} = 1$ when $t_1 = s$ and $t_2 = t$. The other cases follow by similar arguments and are left for the reader. We start to prove the theorem in the case $p_1 = p_2 = r = 1$. By Propositions 4.6, 4.7 and a simple argument of approximations, it follows that we may assume that $a_1 = u$ and $a_2 = v$ are rank one elements in $\mathcal{S}_{1/2}$ and satisfy

$$\|u\|_{s_1^A(1/\omega_1,\vartheta_1)} \le C \quad \text{and} \quad \|v\|_{s_1^A(1/\omega_2,\vartheta_2)} \le C,$$

for some constant C. Then $Au = f_1 \otimes \overline{f_2}$, $Av = g_1 \otimes \overline{g_2}$ and

$$\|f_1\|_{M^2_{(\vartheta_1)}}\|f_2\|_{M^2_{(\omega_1)}} \lesssim \|u\|_{s_1^A(1/\omega_1,\vartheta_1)}, \quad \|g_1\|_{M^2_{(\vartheta_2)}}\|g_2\|_{M^2_{(\omega_2)}} \lesssim \|v\|_{s_1^A(1/\omega_2,\vartheta_2)},$$

for some vectors $f_1, f_2, g_1, g_2 \in \mathcal{S}_{1/2}$ such that

$$\|f_1\|_{M^2_{(\vartheta_1)}} \le C, \quad \|f_2\|_{M^2_{(\omega_1)}} \le C, \quad \|g_1\|_{M^2_{(\vartheta_2)}} \le C, \quad \|g_2\|_{M^2_{(\omega_2)}} \le C,$$

for some constant $C > 0$.

Set

$$F(x,z) = \overline{f_2(x/s+sz)}g_1(x/t+tz), \quad G(y,z) = \overline{f_1(y/s-sz)}g_2(y/t-tz).$$

It follows from (5.4) that

$$A(u_s * v_t)(x,y) = (2\pi)^{d/2}|st|^{-d}\int F(x,z)\overline{G(y,z)}\,dz.$$

This implies that

$$\begin{aligned}\|u_s * v_t\|_{s_1^A(1/\omega,\vartheta)} &\lesssim |st|^{-d}\int \|F(\,\cdot\,,z)\|_{M^2_{(\vartheta)}}\|G(\,\cdot\,,z)\|_{M^2_{(\omega)}}\,dz \\ &\lesssim |st|^{-d} I_1 \cdot I_2,\end{aligned} \tag{5.7}$$

where

$$\begin{aligned} I_1 &= \Big(\iiint |V_\phi(F(\,\cdot\,,z))(x,\xi)\vartheta(x,\xi)|^2\,dxdzd\xi\Big)^{1/2} \\ I_2 &= \Big(\iiint |V_\phi(G(\,\cdot\,,z))(x,\xi)\omega(x,\xi)|^2\,dxdzd\xi\Big)^{1/2}, \end{aligned} \tag{5.8}$$

for some $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)\setminus 0$. Hence, $I_1 \lesssim \|F\|_{M^2_{(\vartheta_0)}}$ and $I_2 \lesssim \|G\|_{M^2_{(\omega_0)}}$ by Lemma 5.3, when $\omega_0(x,y,\xi,\eta) = \omega(x,\xi)$ and $\vartheta_0(x,y,\xi,\eta) = \vartheta(x,\xi)$.

We need to estimate $\|F\|_{M^2_{(\vartheta_0)}}$ and $\|G\|_{M^2_{(\omega_0)}}$. In order to estimate $\|F\|_{M^2_{(\vartheta_0)}}$ we choose the window function $\Phi \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ as

$$\Phi(x,z) = \phi(x/s+sz)\phi(x/t+tz),$$

for some real-valued $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)$. By taking $(x_1/s+sz_1, x_1/t+tz_1)$ as new variables when evaluating $V_\Phi F$, and using $t^{-2}-s^{-2}=1$, it follows by straightforward computations that

$$\begin{aligned} V_\Phi F(x,z,\xi,\zeta) &= (2\pi)^{-d}\iint F(x_1,z_1)\Phi(x_1-x,z_1-z)e^{-i\langle x_1,\xi\rangle - i\langle z_1,\zeta\rangle}\,dx_1dz_1 \\ &= |st|^{-d}\overline{V_\phi f_2(s^{-1}x+sz, s^{-1}\xi-(st^2)^{-1}\zeta)}V_\phi g_1(t^{-1}x+tz, t^{-1}\xi-(s^2t)^{-1}\zeta). \end{aligned}$$

Furthermore, by (0.12), (5.2) and the fact that $t^{-2} - s^{-2} = 1$, we obtain

$$\vartheta(x,\xi) = \vartheta\big((t^{-2}x+z)-(s^{-2}x+z),(t^{-2}\xi-(st)^{-2}\zeta)-(s^{-2}\xi-(st)^{-2}\zeta)\big)$$
$$\lesssim \omega_1(s^{-1}x+sz,s^{-1}\xi-(st^2)^{-1}\zeta)\vartheta_2(t^{-1}x+tz,t^{-1}\xi-(s^2t)^{-1}\zeta)$$

A combination of these relations now gives

$$|V_\Phi F(x,z,\xi,\zeta)\vartheta(x,\xi)| \lesssim |st|^{-d} J_1 \cdot J_2, \tag{5.9}$$

where

$$J_1 = |V_\phi f_2(s^{-1}x+sz,s^{-1}\xi-(st^2)^{-1}\zeta)\omega_1(s^{-1}x+sz,s^{-1}\xi-(st^2)^{-1}\zeta)|$$

and

$$J_2 = |V_\phi g_1(t^{-1}x+tz,t^{-1}\xi-(s^2t)^{-1}\zeta)\vartheta_2(t^{-1}x+tz,t^{-1}\xi-(s^2t)^{-1}\zeta)|.$$

By applying the L^2 norm and taking

$$s^{-1}x+sz,\quad t^{-1}x+tz,\quad s^{-1}\xi-(st^2)^{-1}\zeta,\quad t^{-1}\xi-(s^2t)^{-1}\zeta$$

as new variables of integration we get

$$\|F\|_{M^2_{(\vartheta)}} \lesssim |st|^{-2d}\|f_2\|_{M^2_{(\omega_1)}}\|g_1\|_{M^2_{(\vartheta_2)}}. \tag{5.10}$$

By similar computations it also follows that

$$\|G\|_{M^2_{(\omega)}} \lesssim |st|^{-2d}\|f_1\|_{M^2_{(\vartheta_1)}}\|g_2\|_{M^2_{(\omega_2)}}. \tag{5.11}$$

Hence, a combination of Proposition 4.6, (5.7), (5.8), (5.10) and (5.11) gives

$$\|u_s * v_t\|_{s_1^A(1/\omega,\vartheta)} \lesssim |st|^{-d}\|f_1\|_{M^2_{(\vartheta_1)}}\|f_2\|_{M^2_{(\omega_1)}}\|g_1\|_{M^2_{(\vartheta_2)}}\|g_2\|_{M^2_{(\omega_2)}}$$
$$\lesssim |st|^{-d}\|u\|_{s_1^A(1/\omega_1,\vartheta_1)}\|v\|_{s_1^A(1/\omega_2,\vartheta_2)}.$$

This proves the result in the case $p_1 = p_2 = r = 1$.

Next we consider the case $p_1 = \infty$, which implies that $p_2 = 1$ and $r = \infty$. Let $a \in s_\infty^A(1/\omega_1,\vartheta_1)$ and let $b, c \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then

$$(a_s * b_t, c) = |s|^{-4d}(a, \widetilde{b}_{t_0} * c_{s_0}),$$

where $\widetilde{b}(X) = \overline{b(-X)}$, $s_0 = 1/s$ and $t_0 = t/s$. We claim that

$$\|\widetilde{b}_{t_0} * c_{s_0}\|_{s_1^A(\omega_1,1/\vartheta_1)} \lesssim |s^2/t|^{2d}\|b\|_{s_1^A(1/\omega_2,\vartheta_2)}\|c\|_{s_1^A(\omega,1/\vartheta)} \tag{5.12}$$

Admitting this for a while, it follows by duality, using Theorem 4.8 that

$$\|a_s * b_t\|_{s_\infty^A(1/\omega,\vartheta)} \lesssim |s^2/t|^{2d}s^{-4d}\|a\|_{s_\infty^A(1/\omega_1,\vartheta_1)}\|b\|_{s_1^A(1/\omega_2,\vartheta_2)},$$

which gives (0.14). The result now follows in the case $p_1 = r = \infty$ and $p_2 = 1$ from the fact that $\mathcal{S}_{1/2}$ is dense in $s_1^A(1/\omega_2,\vartheta_2)$. In the same way the result follows in the case $p_2 = r = \infty$ and $p_1 = 1$.

For general $p_1, p_2, r \in [1,\infty]$ the result follows by multi-linear interpolation, using Theorem 4.4.1 in [2] and Remark 4.5 (3).

It remains to prove (5.12) when $b, c \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. The condition (0.10) is invariant under the transformation $(t,s) \mapsto (t_0, s_0) = (t/s, 1/s)$. Let

$$\widetilde{\omega} = 1/\omega_1, \quad \widetilde{\vartheta} = 1/\vartheta_1, \quad \widetilde{\omega}_1 = 1/\omega,$$
$$\widetilde{\vartheta}_1 = 1/\vartheta, \quad \widetilde{\omega}_2 = \vartheta_2 \quad \text{and} \quad \widetilde{\vartheta}_2 = \omega_2.$$

If $X_1 = -(X+Y)/s$ and $X_2 = Y/s$, then it follows that

$$\omega(X_1+X_2) \lesssim \vartheta_1(-sX_1)\omega_2(tX_2), \quad \vartheta(X_1+X_2) \lesssim \omega_1(-sX_1)\vartheta_2(tX_2),$$

is equivalent to

$$\widetilde{\omega}(X+Y) \lesssim \widetilde{\vartheta}_1(-s_0X)\widetilde{\omega}_2(t_0Y), \qquad \widetilde{\vartheta}(X+Y) \lesssim \widetilde{\omega}_1(-s_0X)\widetilde{\vartheta}_2(t_0Y).$$

Hence, the first part of the proof gives

$$\begin{aligned}
\|\widetilde{b}_{t_0} * c_{s_0}\|_{s_1^A(\omega_1,1/\vartheta_1)} &= \|\widetilde{b}_{t_0} * c_{s_0}\|_{s_1^A(1/\widetilde{\omega},\widetilde{\vartheta})} \\
&\lesssim |s_0t_0|^{-2d}\|\widetilde{b}\|_{s_1^A(1/\widetilde{\omega}_2,\widetilde{\vartheta}_2)}\|\widetilde{c}\|_{s_1^A(1/\widetilde{\omega}_1,\widetilde{\vartheta}_1)} \\
&= |s_0t_0|^{-2d}\|\widetilde{b}\|_{s_1^A(1/\vartheta_2,\omega_2)}\|\widetilde{c}\|_{s_1^A(\omega,1/\vartheta)} \\
&= |s_0t_0|^{-2d}\|b\|_{s_1^A(1/\omega_2,\vartheta_2)}\|\widetilde{c}\|_{s_1^A(\omega,1/\vartheta)},
\end{aligned}$$

and (5.12) follows. The proof in the case $N=2$ is complete. □

We need the following lemma for the proof of Proposition 5.1 in the general case.

Lemma 5.5. *Let $\rho, t_1, \dots, t_N \in \mathbf{R} \setminus 0$ fulfill $(0.10)'$ and $\rho^{-2} + (-1)^{j_N} t_N^{-2} = 1$. For $t_j' = t_j/\rho$ set*

$$\omega_0(X) = \inf \omega_{j_1,1}(t_1'X_1)\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1}) \quad \textit{and}$$
$$\vartheta_0(X) = \inf \vartheta_{j_1,1}(t_1'X_1)\cdots\vartheta_{j_{N-1},N-1}(t_{N-1}'X_{N-1}),$$

where the infima are taken over all $X_1,\dots,X_{N-1}$ such that $X = X_1+\cdots+X_{N-1}$. Then the following is true:

1. *$\omega_0, \vartheta_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$;*
2. *for each $X_1, \dots X_{N-1} \in \mathbf{R}^{2d}$ it holds*
$$\omega_0(X_1+\cdots+X_{N-1}) \le \omega_{j_1,1}(t_1'X_1)\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1}), \quad \textit{and}$$
$$\vartheta_0(X_1+\cdots+X_{N-1}) \le \vartheta_{j_1,1}(t_1'X_1)\cdots\vartheta_{j_{N-1},N-1}(t_{N-1}'X_{N-1});$$
3. *for each $X, Y \in \mathbf{R}^{2d}$ it holds*
$$\omega(X+Y) \lesssim \omega_0(\rho X)\omega_N(t_NY) \quad \textit{and} \quad \vartheta(X+Y) \lesssim \vartheta_0(\rho X)\vartheta_N(t_NY).$$

Proof. The assertion (2) follows immediately from the definitions of ω_0 and ϑ_0, and (3) is an immediate consequence of $(0.12)'$.

In order to prove (1) we assume that $X = X_1 + \cdots + X_{N-1}$. Since $\omega_{j_1,1} \in \mathscr{P}_E(\mathbf{R}^{2d})$, it follows that

$$\begin{aligned}\omega_0(X+Y) &\le \omega_{j_1,1}(t_1'(X_1+Y))\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1})\\ &\le \omega_{j_1,1}(t_1'X_1)\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1})v(Y),\end{aligned}$$

for some $v \in \mathscr{P}_E(\mathbf{R}^{2d})$. By taking the infimum over all representations $X = X_1 + \cdots + X_N$, the latter inequality becomes $\omega_0(X+Y) \le \omega_0(X)v(Y)$. This implies that $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$, and in the same way it follows that $\vartheta_0 \in \mathscr{P}_E(\mathbf{R}^{2d})$. The proof is complete. □

Proof of Proposition 5.1 *for general N.* We may assume that $N > 2$ and that the proposition is already proved for lower values on N. The condition on t_j is that $c_1 t_1^{-2} + \cdots + c_N t_N^{-2} = 1$, where $c_j \in \{\pm 1\}$. For symmetry reasons we may assume that $c_1 t_1^{-2} + \cdots + c_{N-1} t_{N-1}^{-2} = \rho^{-2}$, where $\rho > 0$. Let $t_j' = t_j/\rho$, ω_0 and ϑ_0 be the same as in Lemma 5.5, and let $r_1 \in [1,\infty]$ be such that $1/r_1 + 1/p_N = 1 + 1/r$. Then $c_1(t_1')^{-2} + \cdots + c_{N-1}(t_{N-1}')^{-2} = 1$, $r_1 \ge 1$ since $p_N \le r$, and

$$1/p_1 + \cdots + 1/p_{N-1} = N - 2 + 1/r_1.$$

By the induction hypothesis and Lemma 5.5 (2) it follows that

$$b = a_{1,t_1'} * \cdots * a_{N-1,t_{N-1}'} = \rho^{d(2N-4)}(a_{1,t_1} * \cdots * a_{N-1,t_{N-1}})(\cdot/\rho)$$

makes sense as an element in $s^A_{r_1}(1/\omega_0, \vartheta_0)$, and

$$\|b\|_{s^A_{r_1}(1/\omega_0,\vartheta_0)} \lesssim \prod_{j=1}^{N-1} |t_j'|^{-2d/p_j} \|a\|_{s^A_{p_j}(1/\omega_j,\vartheta_j)}.$$

Since $1/r_1 + 1/p_N = 1 + 1/r$, it follows from Lemma 5.5 (3) that $b_\rho * a_{N,t_N}$ makes sense as an element in $s^A_r(1/\omega,\vartheta)$, and

$$\begin{aligned}\|(a_{1,t_1} * \cdots * a_{N-1,t_{N-1}}) * a_{N,t_N}\|_{s^A_r(1/\omega,\vartheta)} &= \rho^{-d(2N-4)}\|b_\rho * a_{N,t_N}\|_{s^A_r(1/\omega,\vartheta)}\\ &\le C_1\|a_1\|_{s^A_{p_1}(1/\omega_1,\vartheta_1)}\cdots\|a_N\|_{s^A_{p_N}(1/\omega_N,\vartheta_N)},\end{aligned}$$

where

$$C_1 \asymp \rho^{d(4-2N-2/r_1)}|t_N|^{-2d/p_N}\prod_{j=1}^{N-1}|t_j'|^{-2d/p_j} = \prod_{j=1}^{N}|t_j|^{-2d/p_j}.$$

This proves the extension assertions. The uniqueness as well as the symmetry assertions follow from the facts that $\mathcal{S}_{1/2}$ is dense in s^A_p when $p < \infty$ and dense in s^A_∞ with respect to the weak* topology, and that at most one p_j is equal to infinity due to the Young condition. The proof is complete. □

Proof of Proposition 5.2. The continuity assertions follow by combining Proposition 4.3, Proposition 5.1 and (5.1).

When verifying the positivity statement we may argue by induction as in the proof of Proposition 5.1. This together with Proposition 1.4 and some simple arguments of approximation shows that it suffices to prove that $a_s b_t$ is positive semi-definite when $\pm s^2 \pm t^2 = 1$, $st \neq 0$, and $a, b \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ are σ-positive rank-one element.

For any $U \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$ we set

$$U_{0,z}(x,y) = U_{1,z}(-y,-x) = U(x+z, y+z).$$

Then Lemmas 1.3 and 5.4 give

$$A(a_s b_t)(x,y) = (2/\pi)^{d/2} |st|^{-d} \int (Aa)_{j,z/s}(sx,sy)(Ab)_{k,-z/t}(tx,ty)\, dz,$$

for some choice of $j, k \in \{0,1\}$. Since $a, b \in C_+$ are rank-one elements, it follows that the integrand is of the form $\phi_z(x) \otimes \overline{\phi_z(y)}$ in all these cases. Consequently, $A(a_s b_t)$ is a positive semi-definite operator. □

Remark 5.6. We note that the arguments and conclusions in Remark 5.7 in [51] holds after $\mathscr{P}$ has been replaced by $\mathscr{P}_E$.

6. Some consequences

In this section we explain some consequences of the results in previous section. We omit the proofs since they are the same as corresponding results in Section 6 in [51], after the weight class $\mathscr{P}$ has been replaced by $\mathscr{P}_E$. It follows for example from Proposition 5.2, that $s_1^A(1/v, v)$ is stable under composition with odd entire analytic functions, when v is submultiplicative,

Thereafter we explain how the definition of Toeplitz operators can be extended to include appropriate dilations of s_p^w as permitted Toeplitz symbols.

We start by considering compositions of elements in $s_1^A(1/v, v)$ with analytic functions. In these considerations we restrict ourself to the case when $v = \check{v} \in \mathscr{P}_E(\mathbf{R}^{2d})$ is submultiplicative. We note that each element in $s_1^A(1/v, v)$ is a continuous function which turns to zero at infinity, since (5.6) shows that $s_1^A(1/v, v) \hookrightarrow C_B(\mathbf{R}^{2d})$.

A part of these investigations concerns σ-positive functions and distributions, and it is convenient to let $C_+(\mathbf{R}^{2d})$ denote the set of all continuous functions on $\mathbf{R}^{2d}$, which are σ-positive (cf. [44]).

It follows that any product of odd numbers of elements in $s_1^A(1/v, v)$ are again in $s_1^A(1/v, v)$. In fact, assume that $a_1, \dots, a_N \in s_1^A(1/v, v)$, $|\alpha|$ is odd, and that $t_j = 1$. Then it follows from Theorem 5.2 that $a_1^{\alpha_1} \cdots a_N^{\alpha_N} \in s_1^A(1/v, v)$, and

$$\|a_1^{\alpha_1} \cdots a_N^{\alpha_N}\|_{s_1^A(1/v,v)} \le C_0^{d|\alpha|} \prod \|a_j\|_{s_1^A(1/v,v)}^{\alpha_j}, \tag{6.1}$$

for some constant C_0 which is independent of α and d.

Furthermore, if in addition $a_1, \dots, a_N$ are σ-positive, then the same is true for $a_1^{\alpha_1} \cdots a_N^{\alpha_N}$. The following result is an immediate consequence of these observations.

Proposition 6.1. *Let* $a_1,\dots,a_N\in s_1^A(1/v,v)$*, where* $v=\check v\in \mathscr P_E(\mathbf R^{2d})$ *is submultiplicative,* C_0 *is the same as in* (6.1)*, and let* $R_1,\dots,R_N>0$*. Also let* f,g *be odd analytic functions from the polydisc*

$$\{\, z\in \mathbf C^N\, ;\, |z_j|<C_0R_j\,\}$$

to $\mathbf C$*, with expansions*

$$f(z)=\sum_\alpha c_\alpha z^\alpha \quad \text{and} \quad g(z)=\sum_\alpha |c_\alpha| z^\alpha .$$

Then $f(a)=f(a_1,\dots,a_N)$ *is well defined and belongs to* $s_1^A(1/v,v)$*, and*

$$\|f(a)\|_{s_1^A(1/v,v)}\le g(C_0\|a_1\|_{s_1^A(1/v,v)},\dots,C_0\|a_N\|_{s_1^A(1/v,v)}).$$

If in addition $a_1,\dots,a_N\in C_+(\mathbf R^{2d})$*, then* $g(a)\in C_+(\mathbf R^{2d})$*.*

For rank one elements we also have the following generalization of [44, Proposition 4.10].

Proposition 6.2. *Let* $v,v_1\in \mathscr P_E(\mathbf R^{2d})$ *are even, submultiplicative and fulfill* $v_1=v(\,\cdot\,/\sqrt2\,)$*. Also let* $u\in s_\infty^w(1/\omega,\omega)$ *be an element of rank one, and let* $a(X)=|u(X/\sqrt2)|^2$*. Then* $a\in s_1^w(1/v_1,v_1)$*, and* $\operatorname{Op}^w(a)\ge 0$*.*

We finish the section by applying our results on Toeplitz operators (cf. (1.22)). The following result, parallel to Theorems 3.1 and 3.5 in [53], generalizes [46, Proposition 4.5].

Theorem 6.3. *Let* $p\in[1,\infty]$ *and* $\omega,\omega_0,\vartheta,\vartheta_j\in \mathscr P_E(\mathbf R^{2d})$ *for* $j=0,1,2$ *be such that*

$$\omega(X_1-X_2)\lesssim \omega_0(\sqrt2\,X_1)\vartheta_2(X_2),\quad \vartheta(X_1-X_2)\lesssim \vartheta_0(\sqrt2\,X_1)\vartheta_1(X_2).$$

Then the definition of $\operatorname{Tp}_{h_1,h_2}(a)$ *extends uniquely to each* $a\in \mathcal S_{1/2}'(\mathbf R^{2d})$ *and* $h_j\in M^2_{(\vartheta_j)}$ *for* $j=1,2$ *such that* $b=a(\sqrt2\,\cdot\,)\in s_p^w(1/\omega_0,\vartheta_0)$*, and*

$$\|\operatorname{Tp}_{h_1,h_2}(a)\|_{\mathscr I_p(M^2_{(1/\omega)},M^2_{(\vartheta)})}\lesssim \|a(\sqrt2\,\cdot\,)\|_{s_p^w(1/\omega_0,\vartheta_0)}\|h_1\|_{M^2_{(\vartheta_1)}}\|h_2\|_{M^2_{(\vartheta_2)}}.$$

Furthermore, if $h_1=h_2$ *and* $\operatorname{Op}^w(b)\ge 0$*, then* $\operatorname{Tp}_{h_1,h_2}(a)\ge 0$*.*

Proof. Since $W_{h_2,h_1}\in s_1^w(1/\vartheta_1,\vartheta_2)$, the result is an immediate consequence of (1.23) and Theorem 0.3. □

Appendix

In this appendix we prove basic results for pseudo-differential operators with symbols in modulation spaces, where the corresponding weights belong to $\mathscr{P}_E$. The arguments are in general similar as corresponding results in [47, 50].

The continuity results that we are focused on are especially Theorems A.1–A.3. Here Theorem A.1 is the extension of Feichtinger–Gröchenig's kernel theorem for modulation spaces with weights in $\mathscr{P}_E$. This result corresponds to Schwartz kernel theorem in distribution theory. The second result (Theorem A.2) concerns pseudo-differential operators with symbols in modulation spaces, which act on modulation spaces. Theorem A.3 gives necessary and sufficient conditions on symbols such that corresponding pseudo-differential operators are Schatten–von Neumann operators of certain degrees. Finally in Propositions A.4 and A.5 we establish preparatory results on Wigner distributions and pseudo-differential calculus in the context of modulation space theory.

Before stating the results we recall same facts on distribution kernels to linear operators in the background of Gelfand–Shilov spaces. Let $s \ge 1/2$ and let $K \in \mathcal{S}_s'(\mathbf{R}^{d_1+d_2})$. Then K gives rise to a linear and continuous operator $T = T_K$ from $\mathcal{S}_s(\mathbf{R}^{d_1})$ to $\mathcal{S}_s'(\mathbf{R}^{d_2})$, defined by the formula

$$Tf(x) = \langle K(x,\,\cdot\,), f\rangle, \tag{A.1}$$

which should be interpreted as (1.9) when $f \in \mathcal{S}_s(\mathbf{R}^{d_1})$ and $g \in \mathcal{S}_s(\mathbf{R}^{d_2})$.

Before establishing the corresponding result for modulation with weights in $\mathscr{P}_E$, we present appropriate conditions on the involved weights and Lebesgue exponent. The involved weights are related to each others by the formulas

$$\frac{\omega_2(x,\xi)}{\omega_1(y,\eta)} \lesssim \omega(x,y,\xi,-\eta), \quad x,\xi \in \mathbf{R}^{2d_2},\ y,\eta \in \mathbf{R}^{2d_1} \tag{A.2}$$

or

$$\frac{\omega_2(x,\xi)}{\omega_1(y,\eta)} \asymp \omega(x,y,\xi,-\eta), \quad x,\xi \in \mathbf{R}^{2d_2},\ y,\eta \in \mathbf{R}^{2d_1}, \tag{A.2$'$}$$

and

$$\omega(x,y,\xi,\eta) \asymp \ \omega_0((1-t)x+ty, t\xi-(1-t)\eta, \xi+\eta, y-x), \qquad x,y,\xi,\eta \in \mathbf{R}^d, \tag{A.3}$$

or equivalently,

$$\omega_0(x,\xi,\eta,y) \asymp \omega(x-ty, x+(1-t)y, \xi+(1-t)\eta, -\xi+t\eta), \qquad x,y,\xi,\eta \in \mathbf{R}^d. \tag{A.3$'$}$$

We note that (A.2) and (A.3) imply

$$\frac{\omega_2(x,\xi)}{\omega_1(y,\eta)} \lesssim \omega_0((1-t)x+ty, t\xi+(1-t)\eta, \xi-\eta, y-x), \tag{A.4}$$

and that (A.2)$'$ and (A.3) imply

$$\frac{\omega_2(x,\xi)}{\omega_1(y,\eta)} \asymp \omega_0((1-t)x+ty, t\xi+(1-t)\eta, \xi-\eta, y-x), \tag{A.4$'$}$$

The Lebesgue exponents of the modulation spaces should satisfy conditions of the form

$$1/p_1 - 1/p_2 = 1/q_1 - 1/q_2 = 1 - 1/p - 1/q, \quad q \le p_2, q_2 \le p, \tag{A.5}$$

or

$$p_1 \le p \le p_2, \quad q_1 \le \min(p,p') \quad \text{and} \quad q_2 \ge \max(p,p'). \tag{A.6}$$

Theorem A.1. *Let $t \in \mathbf{R}$, $\omega_j \in \mathscr{P}_E(\mathbf{R}^{2d_j})$ for $j=1,2$ and $\omega \in \mathscr{P}_E(\mathbf{R}^{2d_2+2d_1})$ be such that* (A.2)$'$ *holds. Also let T is a linear and continuous map from $\mathcal{S}_{1/2}(\mathbf{R}^{d_1})$ to $\mathcal{S}_{1/2}'(\mathbf{R}^{d_2})$. Then the following conditions are equivalent:*

(1) *T extends to a continuous mapping from $M^1_{(\omega_1)}(\mathbf{R}^{d_1})$ to $M^\infty_{(\omega_2)}(\mathbf{R}^{d_2})$;*

(2) *there is a unique $K \in M^\infty_{(\omega)}(\mathbf{R}^{d_2+d_1})$ such that* (A.1) *holds for every $f \in \mathcal{S}_{1/2}(\mathbf{R}^{d_1})$;*

(3) *if in addition $d_1 = d_2 = d$ and* (A.3) *holds, then there is a unique $a \in M^\infty_{(\omega_0)}(\mathbf{R}^{2d})$ such that $Tf = \operatorname{Op}_t(a)f$ when $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$.*

Furthermore, if (1)–(2) *are fulfilled, then $\|T\|_{M^1_{(\omega_1)} \to M^\infty_{(\omega_2)}} \asymp \|K\|_{M^\infty_{(\omega)}}$, and if in addition $d_1 = d_2$ and $T = \operatorname{Op}_t(a)$ in* (3)*, then $\|K\|_{M^\infty_{(\omega)}} \asymp \|a\|_{M^\infty_{(\omega_0)}}$.*

Theorem A.2. *Let $t \in \mathbf{R}$ and $p,q,p_j,q_j \in [1,\infty]$ for $j=1,2$, satisfy* (A.5)*. Also let $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ and $\omega_1,\omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$ satisfy* (A.4)*. If $a \in M^{p,q}_{(\omega)}(\mathbf{R}^{2d})$, then $\operatorname{Op}_t(a)$ from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}'(\mathbf{R}^d)$ extends uniquely to a continuous mapping from $M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d)$ to $M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d)$, and*

$$\|\operatorname{Op}_t(a)\|_{M^{p_1,q_1}_{(\omega_1)} \to M^{p_2,q_2}_{(\omega_2)}} \lesssim \|a\|_{M^{p,q}_{(\omega_0)}}. \tag{A.7}$$

Moreover, if in addition a belongs to the closure of $\mathcal{S}_{1/2}$ under the $M^{p,q}_{(\omega_0)}$ norm, then $\operatorname{Op}_t(a) : M^{p_1,q_1}_{(\omega_1)} \to M^{p_2,q_2}_{(\omega_2)}$ is compact.

Theorem A.3. *Let $t \in \mathbf{R}$ and $p,q,p_j,q_j \in [1,\infty]$ for $j=1,2$, satisfy* (A.6)*. Also let $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ and $\omega_1,\omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$ satisfy* (A.4)$'$*. Then*

$$M^{p_1,q_1}_{(\omega_0)}(\mathbf{R}^{2d}) \hookrightarrow s_{t,p}(\omega_1,\omega_2) \hookrightarrow M^{p_2,q_2}_{(\omega_0)}(\mathbf{R}^{2d}).$$

For the proofs we also need the following extensions of Propositions 4.1 and 4.8 in [49].

Proposition A.4. *Let $t \in \mathbf{R}$, and let $p_j,q_j,p,q \in [1,\infty]$ be such that $p \le p_j, q_j \le q$, for $j=1,2$, and*

$$1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p + 1/q. \tag{A.8}$$

Also let $\omega_1,\omega_2 \in \mathscr{P}_E(\mathbf{R}^{2d})$ and $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ be such that

$$\omega_0((1-t)x+ty, t\xi+(1-t)\eta, \xi-\eta, y-x) \lesssim \omega_1(x,\xi)\omega_2(y,\eta). \tag{A.9}$$

Then the map $(f_1,f_2)\mapsto W^t_{f_1,f_2}$ *from* $\mathcal S_{1/2}'(\mathbf R^d)\times \mathcal S_{1/2}'(\mathbf R^d)$ *to* $\mathcal S_{1/2}'(\mathbf R^{2d})$ *restricts to a continuous mapping from* $M^{p_1,q_1}_{(\omega_1)}(\mathbf R^d)\times M^{p_2,q_2}_{(\omega_2)}(\mathbf R^d)$ *to* $M^{p,q}_{(\omega_0)}(\mathbf R^{2d})$*, and*

$$\|W^t_{f_1,f_2}\|_{M^{p,q}_{(\omega_0)}}\lesssim \|f_1\|_{M^{p_1,q_1}_{(\omega_1)}}\|f_2\|_{M^{p_2,q_2}_{(\omega_2)}} \tag{A.10}$$

when $f_1,f_2\in \mathcal S_{1/2}'(\mathbf R^d)$.

Proposition A.5. *Let* $p\in[1,\infty]$, $\omega_j\in \mathscr P_E(\mathbf R^{2d_j})$, $j=1,2$, $\omega\in\mathscr P_E(\mathbf R^{2d_2+2d_1})$, *and let* T *be a linear and continuous operator from* $\mathcal S_{1/2}(\mathbf R^{d_1})$ *to* $\mathcal S_{1/2}'(\mathbf R^{d_2})$ *with distribution kernel* $K\in\mathcal S_{1/2}'(\mathbf R^{d_2+d_1})$. *Then the following is true:*

(1) *if* $d_1=d_2=d$ *and* $\omega_0\in\mathscr P_E(\mathbf R^{2d}\oplus\mathbf R^{2d})$ *satisfy* (A.3)$'$, $a\in\mathcal S_{1/2}'(\mathbf R^{2d})$ *and* $K=K_{a,t}$ *is given by* (1.7)*, then* $K\in M^p_{(\omega)}(\mathbf R^{2d})$*, if and only if* $a\in M^p_{(\omega_0)}(\mathbf R^{2d})$*, and*

$$\|K\|_{M^p_{(\omega)}}\asymp \|a\|_{M^p_{(\omega_0)}};$$

(2) *if* (A.2)$'$ *holds, then* $T\in\mathscr I_2(M^2_{(\omega_1)},M^2_{(\omega_2)})$*, if and only if* $K\in M^2_{(\omega)}(\mathbf R^{d_2+d_1})$*, and then*

$$\|T\|_{\mathscr I_2}\asymp \|K\|_{M^2_{(\omega)}}. \tag{A.11}$$

For the proofs we note that (A.9) is the same as

$$\omega_0(x,\xi,\eta,y)\lesssim \omega_1(x-ty,\xi+(1-t)\eta)\omega_2(x+(1-t)y,\xi-t\eta). \tag{A.9$'$}$$

Proof of Proposition A.4. We only prove the result when $p,q<\infty$. The straightforward modifications to the cases when $p=\infty$ or $q=\infty$ are left for the reader. Let $\phi_1,\phi_2\in\Sigma_1(\mathbf R^d)\setminus 0$, and let $\Phi=W^t_{\phi_1,\phi_2}$. Then Fourier's inversion formula gives

$$\begin{aligned}(V_\Phi(W^t_{f_1,f_2}))&(x,\xi,\eta,y)\\ &= e^{-i\langle y,\xi\rangle}F_1(x-ty,\xi+(1-t)\eta)\overline{F_2(x+(1-t)y,\xi-t\eta)},\end{aligned}$$

where $F_j=V_{\phi_j}f_j$. By applying the $L^{p,q}_{(\omega)}$-norm on the latter equality, and using (A.9)$'$, it follows from Minkowski's inequality that

$$\|W^t_{f_1,f_2}\|_{M^{p,q}_{(\omega_0)}}\lesssim \big(\|G_1*G_2\|_{L^r}\big)^{1/p}\le\Big(\int H(\eta)\,d\eta\Big)^{1/q},$$

where $G_j=|F_j\omega_j|^p$, $r=q/p\ge1$ and

$$H(\eta)=\Big(\int\Big(\int\Big(\int G_1(y-x,\eta-\xi)G_2(x,\xi)\,dx\Big)^r dy\Big)^{1/r}d\xi\Big)^r.$$

Now let $r_j,s_j\in[1,\infty]$ for $j=1,2$ be chosen such that

$$1/r_1+1/r_2=1/s_1+1/s_2=1+1/r.$$

Then Young's inequality gives

$$H(\eta)\le\Big(\int\|G_1(\,\cdot\,,\eta-\xi)\|_{L^{r_1}}\|G_2(\,\cdot\,,\xi)\|_{L^{r_2}}\,d\xi\Big)^r$$

Hence an other application of Young's inequality gives

$$\|W^t_{f_1,f_2}\|_{M^{p,q}_{(\omega_0)}} \lesssim \Big(\int H(\eta)\, d\eta \Big)^{1/q} \lesssim \big(\|G_1\|_{L^{r_1,s_1}} \|G_2\|_{L^{r_2,s_2}} \big)^{1/p}$$

By letting $p_j = pr_j$ and $q_j = qs_j$, the last inequality gives (A.10). The proof is complete. □

Proof of Proposition A.5. (1) Let $\Phi, \Psi \in \mathcal{S}_{1/2}(\mathbf{R}^{2d}) \setminus 0$ be such that

$$\Phi(x, y) = (\mathscr{F}_2\Psi)((1-t)x + ty, x - y).$$

Then it follows by straightforward applications of Fourier's inversion formula that

$$|(V_\Phi K_{a,t})(x, y, \xi, \eta)| \asymp |(V_\Psi a)((1-t)x + ty, t\xi - (1-t)\eta, \xi + \eta, y - x)|.$$

The assertion now follows by applying the $L^p_{(\omega)}$ norm on the last equality.

Next we prove (2). Let $\{f_j\} \in \mathrm{ON}(M^2_{\omega_1})$ and $\{h_k\} \in \mathrm{ON}(M^2_{\omega_2})$. Then

$$\|T\|^2_{\mathscr{I}_2} = \sum_{j,k} |(Tf_j, h_k)_{M^2_{(\omega_2)}})|^2 = \sum_{j,k} |(K, h_k \otimes \overline{f_j})_{M^2_{(\omega_2)} \otimes L^2}|^2 \tag{A.12}$$

Next we consider the operator $T'_\vartheta = I_{M^2_{(\omega_2)}} \otimes \mathcal{R}_{1/\vartheta}$, where $\vartheta(x, \xi) = \omega_1(x, -\xi)$, which acts from $M^2_{(\omega_2)} \otimes M^2_{(1/\vartheta)}$ to $M^2_{(\omega_2)} \otimes M^2_{(\vartheta)}$ (Hilbert tensor products). Then (A.12) gives

$$\begin{aligned}\|T\|^2_{\mathscr{I}_2} &= \sum_{j,k} |(T'_{\omega_0} K, h_k \otimes \overline{f_j})_{M^2_{(\omega_2)} \otimes M^2_{(\omega_1)}}|^2 \\ &= \|T'_{\omega_0} K\|^2_{M^2_{(\omega_2)} \otimes M^2_{(\omega_0)}} = \|K\|^2_{M^2_{(\omega_2)} \otimes M^2_{(1/\omega_0)}} = \|K\|^2_{M^2_{(\omega)}},\end{aligned}$$

and the result follows. The proof is complete. □

Proof of Theorem A.1. Let T be extendable to a continuous map from $M^1_{(\omega_1)}(\mathbf{R}^{d_1})$ to $M^\infty_{(\omega_2)}(\mathbf{R}^{d_2})$. It follows from [32, Theorem 2.2] and Remark 1.6 that (A.1) holds for some $K \in \mathcal{S}'_{1/2}(\mathbf{R}^{d_2+d_1})$. We shall prove that K belongs to $M^\infty_{(\omega)}$.

From the assumptions and Proposition 1.5 (3) it follows that

$$|(K, g \otimes \overline{f})_{L^2}| \lesssim \|f\|_{M^1_{(\omega_1)}} \|g\|_{M^1_{(1/\omega_2)}}, \tag{A.13}$$

when $f \in \mathcal{S}_{1/2}(\mathbf{R}^{d_1})$ and $g \in \mathcal{S}_{1/2}(\mathbf{R}^{d_2})$. By letting $\Phi = \overline{g} \otimes f$ be fixed, and replacing f and g with

$$f_{y,\eta} = e^{-i\langle \cdot, \eta \rangle} f(\cdot - y) \quad \text{and} \quad g_{x,\xi} = e^{i\langle \cdot, \xi \rangle} f(\cdot - x),$$

(A.13) takes the form

$$|(V_\Phi K)(x, y, \xi, \eta)| \lesssim \|f_{y,\eta}\|_{M^1_{(\omega_1)}} \|g_{x,\xi}\|_{M^1_{(1/\omega_2)}}. \tag{A.13$'$}$$

If $v \in \mathscr{P}_E$ is chosen such that ω_1 is v-moderate, and $\phi_1 \in \mathcal{S}_{1/2} \setminus 0$, then

$$\|f_{y,\eta}\|_{M^1_{(\omega_1)}} \asymp \iint |(V_{\phi_1} f)(z-y,\zeta+\eta)\omega_1(z,\zeta)|\, dzd\zeta$$
$$\lesssim \omega_1(y,-\eta)\|f\|_{M^1_{(v)}} \asymp \omega_1(y,-\eta).$$

In the same way we get

$$\|g_{x,\xi}\|_{M^1_{(1/\omega_2)}} \lesssim \omega_2(x,\xi)^{-1}.$$

If these estimates are inserted into (A.13)$'$, we obtain

$$|(V_\Phi K)(x,y,\xi,\eta)\omega(x,y,\xi,\eta)| \lesssim 1,$$

By taking the supremum of the left-hand side it follows that $\|K\|_{M^\infty_{(\omega)}} < \infty$. Hence $K \in M^\infty_{(\omega)}$, and we have proved that (1) implies (2).

By straightforward computations it also follows that (2) gives (1). The details are left for the reader.

The equivalence between (2) and (3) follows immediately from Proposition A.5. The proof is complete. □

Proof of Theorem A.2. The conditions on p_j and q_j implies that

$$p' \le p_1, q_1, p_2', q_2' \le q', \quad 1/p_1 + 1/p_2' = 1/q_1 + 1/q_2' = 1/p' + 1/q'.$$

Hence Proposition A.4, and (A.4) show that

$$\|W^t_{g,f}\|_{\widetilde{M}^{p',q'}_{(1/\omega)}} \lesssim \|f\|_{M^{p_1,q_1}_{(\omega_1)}} \|g\|_{M^{p_2',q_2'}_{(1/\omega_2)}}$$

when $f \in M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d)$ and $g \in M^{p_2',q_2'}_{(1/\omega_2)}(\mathbf{R}^d)$.

The continuity is now an immediate consequence of (1.14) and Proposition 1.5 (4), except for the case $p = q' = \infty$, which we need to consider separately.

Therefore assume that $p = \infty$, and $q = 1$, and let $a \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$. Then $p_1 = p_2$ and $q_1 = q_2$, and it follows from Proposition A.4 and the first part of the proof that $W^t_{g,f} \in M^{1,\infty}_{(1/\omega_0)}$, and that (A.7) holds. In particular,

$$|(\operatorname{Op}_t(a)f,g))| \lesssim \|f\|_{M^{p_1,q_1}_{(\omega_1)}} \|g\|_{M^{p_1',q_1'}_{(1/\omega_2)}},$$

and the result follows when $a \in \mathcal{S}_{1/2}$. The result now follows for general $a \in M^{\infty,1}_{(\omega_0)}$, by taking a sequence $\{a_j\}_{j\ge1}$ in $\mathcal{S}_{1/2}$, which converges narrowly to a. (For narrow convergence see Theorems 4.15 and 4.19, and Proposition 4.16 in [52]).

It remains to prove that if a belongs to the closure of $\mathcal{S}_{1/2}$ under $M^{p,q}_{(\omega_0)}$ norm, then $\operatorname{Op}_t(a) : M^{p_1,q_1}_{(\omega_1)} \to M^{p_2,q_2}_{(\omega_2)}$ is compact. As a consequence of Theorem A.3, it follows that $\operatorname{Op}_t(a_0)$ is compact when $a_0 \in \mathcal{S}_{1/2}$, since $\mathcal{S}_{1/2} \hookrightarrow M^1_{(\omega_0)}$ when $\omega_0 \in \mathscr{P}_E$, and that every trace-class operator is compact. The compactness of $\operatorname{Op}_t(a)$ now follows by approximating a with elements in $\mathcal{S}_{1/2}$. The proof is complete. □

Proof of Theorem A.3. The first embedding in

$$M^{\infty,1}_{(\omega_0)} \hookrightarrow s_{t,\infty}(\omega_1,\omega_2) \hookrightarrow M^{\infty}_{(\omega_0)}$$

follows from Theorem A.2, and the second one from Proposition 1.5 (2) and Theorem A.1.

By Propositions 1.5 (3) and 4.7, Theorem 4.8 and duality, the latter inclusions give

$$M^{1}_{(\omega_0)} \hookrightarrow s_{t,1}(\omega_1,\omega_2) \hookrightarrow M^{1,\infty}_{(\omega_0)},$$

and we have proved the result when $p = 1$ and when $p = \infty$. Furthermore, by Proposition A.5 we have $M^2_{(\omega_0)} = s_{t,2}(\omega_1,\omega_2)$, and the result also holds in the case $p = 2$. The result now follows for general p from these cases and interpolation. (See, e.g., Proposition 5.8 in [52].) The proof is complete. □

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Operator Invariance

Leon Cohen

Abstract. Linear time invariant (LTI) systems have produced a rich set of ideas including the concepts of convolution, impulse response function, causality, and stability, among others. We discuss how these concepts are generalized when we consider invariance other than time shift invariance. We call such systems *linear operator invariant systems* because the invariance is characterized by an operator. In the standard case of LTI systems the relation between input and output function in the Fourier domain is multiplication. We generalize this and show that multiplication still holds in the operator transform domain. Transforming back to the time domain defines *generalized convolution.*

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1. Introduction

Many problems in physics, engineering, and mathematics can be formulated as input-output relations that are characterized by a linear system operator $\mathbf{L}$ [6]. One writes

$$y(t) = \mathbf{L}x(t) \tag{1.1}$$

where $x(t)$ is said to be the input and $y(t)$ the output. An important class of system operators are those that are "linear time invariant" (LTI) systems which means that if the input is shifted by an amount t_0 then the output will be shifted by the same amount

$$y(t+t_0) = \mathbf{L}x(t+t_0) \qquad \text{linear time invariant system.} \tag{1.2}$$

Over the last hundred years the theory of linear time invariant systems has developed into a rich subject with many associated ideas including convolution, impulse

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response function, causality, and stability, among others. Our aim here is to discuss how other possible invariant systems may be formulated and to generalize the concepts associated with LTI systems.

The notation we use is that operators are capital bold face. All integrals go from $-\infty$ to ∞ unless noted otherwise. The commutator between two operators is denote by

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}. \tag{1.3}$$

Throughout the paper all functions will be supposed to belong to suitably regular function spaces in order that all performed operations makes sense.

2. How does time invariance arise?

Physically it is clear how "time invariance" can be imposed on Eq. (1.1) to get Eq. (1.2)[6]. To do it mathematically we need an operator that accomplishes the desired result. The time shift operator is $e^{it_0\mathbf{D}_t}$

$$e^{it_0\mathbf{D}_t} f(t) = f(t + t_0) \tag{2.1}$$

where

$$\mathbf{D}_t = \frac{1}{i}\frac{d}{dt} \tag{2.2}$$

and $f(t)$ is any function [2, 7]. Therefore, if we operate on Eq. (1.1) we have

$$e^{it_0\mathbf{D}_t} y(t) = e^{it_0\mathbf{D}_t} Lx(t) \tag{2.3}$$

giving

$$y(x + t_0) = e^{it_0\mathbf{D}_t} Lx(t). \tag{2.4}$$

To make this equal to Eq. (1.2) we have to be able to say that

$$e^{it_0\mathbf{D}_t}\mathbf{L} = \mathbf{L}e^{it_0\mathbf{D}_t} \tag{2.5}$$

and therefore the condition for time invariance is that

$$[e^{it_0\mathbf{D}_t}, \mathbf{L}] = 0. \tag{2.6}$$

Note that Eq. (2.6) implies that

$$[\mathbf{D}_t, \mathbf{L}] = 0 \tag{2.7}$$

as can be readily proven by differentiating Eq. (2.6) with regard to t_0 and subsequently setting $t_0 = 0$. If indeed it is the case that Eq. (2.7) is true then we have

$$\begin{aligned} y(x + t_0) &= e^{it_0\mathbf{D}_t}\mathbf{L}x(t) \\ &= \mathbf{L}e^{it_0\mathbf{D}_t}x(t) \\ &= \mathbf{L}x(t + t_0) \text{ if } [\mathbf{D}_t, \mathbf{L}] = 0 \text{ (time invariant system).} \end{aligned} \tag{2.8}$$

Generalized operator invariance. One way to generalize time invariance to arbitrary invariance is to associate a physical attribute by a Hermitian operator $\mathbf{A}$. We operate with $e^{ia_0\mathbf{A}}$ on Eq. (1.1), with a_0 real, to obtain

$$e^{ia_0\mathbf{A}}y(t) = \mathbf{L}e^{ia_0\mathbf{A}}x(t) \qquad \text{if } [\mathbf{A},\mathbf{L}] = 0 \qquad (\text{“}a\text{” invariant system}) \tag{2.9}$$

and therefore we say that for a system represented by $\mathbf{L}$, the system is invariant with respect to the physical quantity represented by $\mathbf{A}$ if $\mathbf{L}$, and $\mathbf{A}$ commute. We call such systems *linear operator invariant systems*

3. How does the system function arise?

The remarkable property of linear invariant systems is that if we know the response to a delta function then we explicitly know the response for any other input [1, 2]. In particular if

$$h(t-t') = \mathbf{L}\delta(t-t') \tag{3.1}$$

or

$$h(t) = \mathbf{L}\delta(t) \tag{3.2}$$

then

$$y(t) = \int_{-\infty}^{\infty} h(t-t')x(t')dt. \tag{3.3}$$

The function $h(t)$ is called the system function, impulse response function, or Green function. We give the standard proof of this classical result as it will help us to generalize to arbitrary invariance. Write

$$x(t) = \int_{-\infty}^{\infty} \delta(t-t')x(t')dt \tag{3.4}$$

and substitute into Eq. (1.1)

$$y(t) = \mathbf{L}x(t) = \mathbf{L}\int_{-\infty}^{\infty} \delta(t-t')x(t')dt. \tag{3.5}$$

Because of linearity we can put $\mathbf{L}$ inside the integral to obtain

$$y(t) = \int_{-\infty}^{\infty} \mathbf{L}\delta(t-t')x(t')dt = \int_{-\infty}^{\infty} h(t-t')x(t')dt \tag{3.6}$$

which is Eq. (3.3).

Generalization of system function. Notice that in Eq. (3.1) the impulse response is a function of $t-t'$ which is a characteristic of time invariance. For the general case, where the operator L is invariant with respect to A, we write

$$y(t) = \int_{-\infty}^{\infty} h(t,t')x(t')dt \tag{3.7}$$

and the issue becomes how do we obtain $h(t,t')$. We start with

$$y(t) = \int_{-\infty}^{\infty} \mathbf{L}\delta(t-t')x(t')dt \tag{3.8}$$

and therefore

$$h(t,t') = \mathbf{L}\,\delta(t-t'). \tag{3.9}$$

We now find the explicit expression for $h(t,t')$ that is connected to the operator **A** that represents the invariance. It is important to appreciate that even though **L** is operating on the delta function of $t-t'$ the result is not generally a function of $t-t'$; indeed that will only be the case for linear time invariant systems where the system operator commutes with $\mathbf{D}_t$.

There are an infinite number of representations of the delta functions, each associated with an Hermitian operator as we now explain. Suppose we have a complete and orthogonal set of functions that are the eigenfunctions of the eigenvalue problem for a Hermitian operator **A**

$$\mathbf{A}u(a,t) = au(a,t) \qquad \text{(continuous spectrum)} \tag{3.10}$$

$$\mathbf{A}u_n(t) = a_n(t)u_n(t) \qquad \text{(discrete spectrum)} \tag{3.11}$$

then [1, 7, 2]

$$\delta(t-t') = \int u^*(a,t')\,u(a,t)\,da \qquad \text{(continuous spectrum)} \tag{3.12}$$

$$\delta(t-t') = \sum_n u^*{}_n(t')\,u_n(t) \qquad \text{(discrete spectrum).} \tag{3.13}$$

We first consider the continuous spectrum case and subsequently, in Section 9.1, give the expressions for the discrete case. Substituting Eq. (3.12) into Eq. (3.9) we have

$$h(t,t') = \int u^*(a,t')\,\mathbf{L}u(a,t)\,da \tag{3.14}$$

and using Eq. (3.8) we obtain

$$y(t) = \iint u^*(a,t')\,\mathbf{L}u(a,t)\,x(t')dadt. \tag{3.15}$$

Now consider the issue of invariance which as discussed means that the operator **A** commutes with **L**. Since commuting operators have common eigenfunctions we can write

$$\mathbf{L}u(a,t) \;=\; L(a)\,u(a,t) \qquad \text{(if } \mathbf{A} \text{ commutes with } \mathbf{L}\text{)} \tag{3.16}$$

where $L(a)$ is a function of a that of course depends on **L**. Furthermore we can obtain $L(a)$ from

$$L(a) = \int u^*(a,t)\mathbf{L}u(a,t)dt. \tag{3.17}$$

Substituting into Eq. (3.17) into Eq. (3.14) results in

$$h(t,t') = \int u^*(a,t')\,L(a)u(a,t)\,da. \tag{3.18}$$

4. The transform domain

Since a complete set of functions has been brought in it is natural to examine the situation in the transform domain. If we have an input function, $x(t)$, then its transform, $X(a)$, in the a domain is given by [1, 7, 2]

$$X(a) = \int x(t)\, u^*(a,t)\, dt \tag{4.1}$$

where

$$x(t) = \int X(a)\, u(a,t)\, da. \tag{4.2}$$

$X(a)$ is called the transform of $x(t)$. For the output we write

$$Y(a) = \int y(t)\, u^*(a,t)\, dt \tag{4.3}$$

$$y(t) = \int Y(a)\, u(a,t)\, da. \tag{4.4}$$

We now derive the input-output relation in the transform domain. Multiply Eq. (3.7) by $u^*(a,t)$ and integrate with respect to t

$$\begin{aligned} Y(a) &= \int y(t)u^*(a,t)dt = \int u^*(a,t)h(t,t')x(t')dt'dt \\ &= \int u^*(a,t)h(t,t')u(a',t')X(a')dtdt'da'. \end{aligned} \tag{4.5}$$

Hence, if we take

$$H(a,a') = \int u^*(a,t)h(t,t')u(a',t')dtdt' \tag{4.6}$$

then

$$Y(a) = \int H(a,a')X(a')da'. \tag{4.7}$$

Also, by inverting Eq. (4.6) we have

$$h(t,t') = \int u(a,t)H(a,a')u^*(a',t')dada'. \tag{4.8}$$

Now we impose invariance. Multiply Eq. (3.16) by $u^*(a,t')$ and integrate to obtain

$$H(a,a') = \int u^*(a,t)\, u^*(a'',t')\, L(a'')u(a'',t)\, u(a',t')dtdt'da'' \tag{4.9}$$

which simplifies to

$$H(a,a') = \delta(a-a')\, L(a'). \tag{4.10}$$

Substituting this into Eq. (4.7) we have

$$Y(a) = L(a)X(a). \tag{4.11}$$

If we were dealing with the standard LTI case, then in Eq. (4.11) a would be frequency, and Y, L, and X would be the Fourier transforms of the output, system function, and input function respectively. In the frequency domain they are connected by multiplication. What we have shown is that multiplication in the transform domain holds for any invariant.

4.1. Generalized convolution

In the Fourier case, when we transform Eq. (4.11) back to the time domain we get convolution. Therefore when we transform Eq. (4.11) back to the time domain we will get what we call *generalized convolution*. Multiply Eq. (4.11) with $u(a,t)$ and integrate with respect to a to obtain

$$y(t) = \int L(a)X(a)u(a,t)da \tag{4.12}$$

where we have used Eq. (4.3). Now define

$$l(t) = \int L(a)\, u(a,t)\, da \tag{4.13}$$

with

$$L(a) = \int l(t)\, u^*(a,t)\, dt \tag{4.14}$$

and substitute into Eq. (4.12) to obtain

$$y(t) = \int l\,(t'')\, u^*(a,t')x(t')\, u^*(a,t'')\; u(a,t)dadt'dt''. \tag{4.15}$$

We write this as

$$y(t) = \int R(t,t',t'')x(t')\, l(t'')\, dt'dt'' \tag{4.16}$$

where

$$R(t,t',t'') = \int u(a,t)u^*(a,t')u^*(a,t'')da. \tag{4.17}$$

We call the left-hand side of Eq. (4.16) the generalized convolution between $x(t)$ and $l(t)$ corresponding to the operator $\mathbf{A}$. As we will see in the examples section this reduces to ordinary convolution for the Fourier case.

5. Relation between the system function and the invariance operator

We now address the following problem: Suppose we are given the system function $h(t,t')$ but not the system operator $\mathbf{L}$. How can we tell if a particular operator, $\mathbf{A}$, commutes with the unknown L operator or for invariance to hold? Operate on Eq. (3.7) with $e^{ia_0\mathbf{A}}$

$$e^{ia_0\mathbf{A}}y(t) = \int_{-\infty}^{\infty} e^{ia_0\mathbf{A}(t)}h(t,t')x(t')dt'. \tag{5.1}$$

For invariance, hence, we must have

$$e^{ia_0\mathbf{A}}y(t) = \int_{-\infty}^{\infty} h(t,t')e^{ia_0\mathbf{A}(t')}x(t')dt' \tag{5.2}$$

and therefore the criterion is that

$$\int_{-\infty}^{\infty} e^{ia_0\mathbf{A}(t)}h(t,t')x(t')dt' = \int_{-\infty}^{\infty} h(t,t')e^{ia_0\mathbf{A}(t')}x(t')dt'. \tag{5.3}$$

Note that Eq. (5.3) must be true for any function $x(t)$. An example of this will be given for scale in the examples section.

6. Series and parallel connections

Series connections and parallel connections of systems amount, from the mathematical point of view, to multiplications and sums of the corresponding linear operators respectively. We consider them in the following two subsections.

6.1. Series connection

Suppose the output, $y(t)$, of a linear system (system 1) is the input of another linear system, $z(t)$, (system 2). We write

$$y(t) = \mathbf{L}_1 x(t) \tag{6.1}$$

$$z(t) = \mathbf{L}_2 y(t) \tag{6.2}$$

and we want to be able to write that $z(t)$ is the output of a third system where the input is $x(t)$

$$z(t) = \mathbf{L}_3 x(t). \tag{6.3}$$

This is a series connection for system one and two. Clearly

$$\mathbf{L}_3 = \mathbf{L}_2\mathbf{L}_1. \tag{6.4}$$

We now obtain the relationship between the system functions. Writing

$$y(t) = \int_{-\infty}^{\infty} h_1(t,t')x(t')dt' \tag{6.5}$$

$$z(t) = \int_{-\infty}^{\infty} h_2(t,t')y(t')dt' \tag{6.6}$$

$$z(t) = \int_{-\infty}^{\infty} h_3(t,t')x(t')dt' \tag{6.7}$$

we have

$$z(t) = \int_{-\infty}^{\infty} h_2(t,t')y(t')dt' \tag{6.8}$$

$$= \int_{-\infty}^{\infty} h_2(t,t') \int_{-\infty}^{\infty} h_1(t',t'')x(t'')dt''dt' \tag{6.9}$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} h_2(t,t')h_1(t',t'')dt' \right\} x(t'')dt'' \tag{6.10}$$

and therefore

$$h_3(t,t') = \int_{-\infty}^{\infty} h_2(t,t'')h_1(t'',t')dt''. \tag{6.11}$$

In the Fourier domain

$$Z(a) = \int H_2(a,a')Y(a')da' \tag{6.12}$$

$$= \int H_2(a,a')H_1(a',a'')X(a'')da'da'' \tag{6.13}$$

and therefore

$$H_3(a,a') = \int H_2(a,a'')H_1(a'',a')da''. \tag{6.14}$$

If we impose invariance by taking

$$H_2(a,a'') = \delta(a-a'')\, L_2(a'') \tag{6.15}$$

$$H_1(a'',a') = \delta(a'-a'')\, L_1(a'') \tag{6.16}$$

then

$$H_3(a,a') = \int \delta(a-a'')\, L_2(a'')\delta(a'-a'')\, L_1(a'')da'' \tag{6.17}$$

$$= \delta(a-a')L_2(a')L_1(a') \tag{6.18}$$

and therefore

$$L_3(a) = L_2(a)L_1(a). \tag{6.19}$$

6.2. Parallel connection

In a parallel connection we have that the output is given by

$$z(t) = \mathbf{L}_2 x(t) + \mathbf{L}_1 x(t) \tag{6.20}$$

$$= (\mathbf{L}_2 + \mathbf{L}_1)x(t). \tag{6.21}$$

In this case

$$\mathbf{L}_3 = \mathbf{L}_2 + \mathbf{L}_1 \tag{6.22}$$

and the impulse response functions also add,

$$h_3(t,t') = h_2(t,t') + h_1(t',t'). \tag{6.23}$$

7. Causality

A causal system is where only the past times affect the output. In terms of the impulse response function that means

$$y(t) = \int_{-\infty}^{t} h(t,t')x(t')dt. \tag{7.1}$$

That is, the future values do not contribute which means that

$$\int_{t}^{\infty} h(t,t')x(t')dt = 0. \tag{7.2}$$

Since this must be true for any function we have that the condition for causality is

$$h(t,t') = 0 \qquad \text{for } t > t'. \tag{7.3}$$

8. Stability

The standard definition of stability is that for a bounded input there should be a bounded output. This is sometimes called the BIBO criterion. To obtain the condition on the system function for this to be the case we take the absolute value of Eq. (3.7)

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(t,t')x(t')dt' \right| \le \int_{-\infty}^{\infty} |h(t,t')|\, |x(t')|\, dt'. \tag{8.1}$$

Now, if we assume that the input $|x(t')|$ is bounded and its highest value is $x_{\max}$ then

$$|y(t)| \le x_{\max} \int_{-\infty}^{\infty} |h(t,t')|\, dt'. \tag{8.2}$$

Hence $y(t)$ will be bounded if the integral in Eq. (8.2) is a bounded function of t, that is

$$\int_{-\infty}^{\infty} |h(t,t')|\, dt' < \infty \qquad \text{(BIBO stable)}. \tag{8.3}$$

9. Summary

We summarize the basic ideas and results we have developed. We have two linear operators, the system operator, **L**, and the operator **A** which represents the invariant quantity we are interested

$y(t) = \mathbf{L}x(t)$ **L** : the system operator
A : operator that represents the invariant quantity

if $[\mathbf{A}, \mathbf{L}] = 0$ we have an "a" invariant system.

Note that the operators **L** and **A** operate on functions of time only. We assume that **A** is Hermitian and the eigenvalue problem is written as

$$\mathbf{A}\, u(a,t) \;=\; a\, u(a,t) \qquad \text{(continuous eigenvalues)}. \tag{9.1}$$

Since **L** commutes with **A** we can write that

$$\mathbf{L}u(a,t) \;=\; L(a)\,u(a,t) \tag{9.2}$$

where

$$L(a) = \int u^*(a,t)\mathbf{L}u(a,t)dt. \tag{9.3}$$

System function: The system function $h(t,t')$ is defined by way of

$$y(t) = \int_{-\infty}^{\infty} h(t,t')x(t')dt' \tag{9.4}$$

and is given by

$$h(t,t') = \int u^*(a,t')\,\mathbf{L}u(a,t)\,da = \int u^*(a,t')\;L(a)\,u(a,t)\,da. \tag{9.5}$$

For invariance with respect to the operator **A** $h(t,t')$ must satisfy

$$\int_{-\infty}^{\infty} e^{ia_0\mathbf{A}(t)}h(t,t')x(t')dt' = \int_{-\infty}^{\infty} h(t,t')e^{ia_0\mathbf{A}(t')}x(t')dt'. \tag{9.6}$$

In the transform domain

$$Y(a) = \int_{-\infty}^{\infty} H(a,a')X(a')da' \;=\; L(a)X(a) \tag{9.7}$$

where

$$H(a,a') = \int u^*(a,t)h(t,t')u(a',t')dtdt' = \delta(a-a')L(a). \tag{9.8}$$

Generalized convolution: The generalized convolution between $x(t)$ and $l(t)$ is

$$y(t) = \int R(t,t',t'')x(t')\,l(t'')dt'dt'' \tag{9.9}$$

where

$$R(t,t',t'') = \int u(a,t)u^*(a,t')u^*(a,t'')da. \tag{9.10}$$

9.1. Discrete case

For the discrete case one can write the equivalent equations straightforwardly. We write

$$\mathbf{A}\,u_n(t) \;=\; a_n\,u_n(t). \tag{9.11}$$

If **L** commutes with **A**

$$\mathbf{L}u_n(t) \;=\; L(a_n)\,u_n(t) \tag{9.12}$$

and

$$L(a_n) \;=\; \int u_n^*(t)\mathbf{L}u_n(t)dt. \tag{9.13}$$

System function: The system function $h(t,t')$ is

$$y(t) = \int_{-\infty}^{\infty} h(t,t')x(t')dt' \tag{9.14}$$

and is given by

$$h(t,t') = \sum_n u_n^*(t')\mathbf{L}u_n(t) = \sum_n u_n^*(t')\, L(a_n)u_n(t). \tag{9.15}$$

For invariance with respect to the operator $\mathbf{A}$ $h(t,t')$, the condition on the system function is:

$$\int_{-\infty}^{\infty} e^{ia_0\mathbf{A}(t)}h(t,t')x(t')dt' = \int_{-\infty}^{\infty} h(t,t')e^{ia_0\mathbf{A}(t')}x(t')dt'. \tag{9.16}$$

In the transform domain

$$Y_n(a) = \sum_m H_{nm}X_m = L(a_n)X_n(a) \tag{9.17}$$

where

$$X_n(a) = \int u_n^*(t)x(t)dt \tag{9.18}$$

$$Y_n(a) = \int u_n^*(t)y(t)dt \tag{9.19}$$

$$H_{nm} = \int u_n^*(t)h(t,t')u_m(t')dtdt' = \delta_{nm}L(a_n). \tag{9.20}$$

Generalized convolution: The generalized convolution between $x(t)$ and $l(t)$ is

$$y(t) = \int R(t,t',t'')x(t')\, l(t'')dt'dt'' \tag{9.21}$$

where

$$R(t,t',t'') = \sum_n u_n(t)u_n^*(t')u_n(t''). \tag{9.22}$$

10. Examples

10.1. Example 1: LTI

The linear time invariant case is obtained by taking

$$\mathbf{A} = \mathbf{D}_t = \frac{1}{i}\frac{d}{dt} \tag{10.1}$$

and "a" becomes frequency

$$a = \omega. \tag{10.2}$$

The eigenvalue problem

$$D_t u(\omega,t) = cu(\omega,t) \tag{10.3}$$

gives

$$u(\omega,t) = \frac{1}{\sqrt{2\pi}}e^{it\omega}. \tag{10.4}$$

Now consider

$$\mathbf{L}u(a,t) = \frac{1}{\sqrt{2\pi}}\mathbf{L}e^{it\omega} = \frac{1}{\sqrt{2\pi}}L(\omega)e^{it\omega}. \tag{10.5}$$

Using Eq. (9.5) we have

$$h(t,t') = \int u^*(\omega,t')\,L(\omega)\,u(\omega,t)\,d\omega \tag{10.6}$$

$$= \frac{1}{2\pi}\int L(\omega)\,e^{i(t-t')\omega}\,d\omega. \tag{10.7}$$

As expected we see that $h(t,t')$ is a function of $t-t'$ and therefore we may take

$$h(t,t') = h(t-t'). \tag{10.8}$$

Now consider

$$H(\omega,\omega') = \int u^*(\omega,t)h(t,t')u(\omega',t')dtdt' \tag{10.9}$$

$$= \frac{1}{2\pi}\int e^{i\omega t}\,h(t-t')e^{-i\omega' t'}\,dtdt' \tag{10.10}$$

$$= \delta(\omega-\omega')H(\omega) \tag{10.11}$$

which gives

$$H(\omega) = \frac{1}{\sqrt{2\pi}}\int e^{-i\omega t}\,h(t)dt. \tag{10.12}$$

Comparing with Eq. (10.7) we see that

$$L(\omega) = H(\omega). \tag{10.13}$$

Furthermore

$$Y(\omega) = H(\omega)X(\omega) \tag{10.14}$$

which is a standard result of LTI systems.

Now consider convolution. Using Eq. (9.10) we have

$$y(t) = \int R(t,t',t'')x(t'')\,l(t')dt'dt'' \tag{10.15}$$

where

$$R(t,t',t'') = \int u(\omega,t)u^*(\omega,t')u^*(\omega,t'')d\omega \tag{10.16}$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}\int e^{it\omega}e^{-it'\omega}e^{-it''\omega}d\omega \tag{10.17}$$

$$= \frac{1}{\sqrt{2\pi}}\delta(t-t'-t''). \tag{10.18}$$

Substituting into Eq. (10.15) we obtain

$$y(t) = \frac{1}{\sqrt{2\pi}}\int l(t-t')x(t')\,dt' \tag{10.19}$$

which is the standard convolution.

10.2. Example 2: Scale

In this section all functions of time are defined only for the positive time axis. Suppose we take a system function given by

$$h(t,t') = \frac{h(t/t')}{t'}. \tag{10.20}$$

Is this system function scale invariant? The scale operator, $\mathcal{C}$, is given by

$$\mathcal{C} \;=\; \frac{1}{2}\left(tD_t + D_t t\right) \;=\; \frac{1}{2i}\left(t\frac{d}{dt} + \frac{d}{dt}t\right) \tag{10.21}$$

and has the following fundamental property [3, 2],

$$e^{i\sigma\mathcal{C}}\, f(t) \;=\; e^{\sigma/2}\, f(\,e^{\sigma}t) \tag{10.22}$$

for real σ. Now we examine the criterion given by Eq. (9.6)

$$\int_0^\infty e^{i\sigma\mathcal{C}(t)} h(t,t')x(t')dt' = \int_0^\infty h(t,t')e^{i\sigma\mathcal{C}(t')}x(t')dt' \tag{10.23}$$

to see whether the system function is scale invariant. Starting with the left-hand side we have

$$\int_0^\infty e^{\sigma/2}h(\,e^{\sigma}t,t')x(t')dt' = \int_0^\infty e^{\sigma/2}\frac{h(\,e^{\sigma}t/t')}{t'}x(t')dt' \tag{10.24}$$

$$= \int_0^\infty e^{\sigma/2}\frac{h(t/t')}{t'}x(e^{\sigma}t')dt' \tag{10.25}$$

$$= \int_0^\infty \frac{h(t/t')}{t'}e^{i\sigma\mathcal{C}(t')}x(t')dt'. \tag{10.26}$$

Eq. (10.25) follows from Eq. (10.24) by a simple change of variables, $t' \to e^{-\sigma}t'$. Hence we have Eq. (10.23) and we have invariance with respect to $\mathcal{C}$. Explicitly, we have

$$e^{i\sigma\mathcal{C}(t)}y(t) = \int_0^\infty h(t,t')e^{i\sigma\mathcal{C}(t')}x(t')dt' \tag{10.27}$$

which reduces to

$$y(\,e^{\sigma}t) = \int_0^\infty h(t,t')x(\,e^{\sigma}t')dt'. \tag{10.28}$$

Since σ is arbitrary we may write

$$y(\,kt) = \int_0^\infty h(t,t')x(\,kt')dt' \quad , \qquad t \geq 0, \qquad k > 0. \tag{10.29}$$

10.2.1. Transform domain. We now consider the problem in the scale domain. For that we need the scale eigenfunctions and the scale transform. The eigenvalue problem $\mathcal{C}\,u(c,t) \;=\; c\,u(c,t)$ gives

$$u(c,t) \;=\; \frac{1}{\sqrt{2\pi}}\,\frac{e^{ic\ln t}}{\sqrt{t}} \quad , \qquad t \geq 0 \tag{10.30}$$

which are complete and orthogonal [3],

$$\int_0^\infty u^*(c',t)\,u(c,t)\,dt=\delta(c-c') \tag{10.31}$$

$$\int u^*(c,t')\,u(c,t)\,dc=\delta(t-t') \qquad t,t' \geq 0. \tag{10.32}$$

The transform pairs are given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int D(c)\,\frac{e^{ic\ln t}}{\sqrt{t}}\,dc \quad ; \quad t \geq 0 \tag{10.33}$$

and

$$D(c) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\,\frac{e^{-ic\ln t}}{\sqrt{t}}\,dt \tag{10.34}$$

where $D(c)$ is called the scale transform of $f(t)$.

In the transform domain

$$Y(c) = \int_{-\infty}^\infty H(c,c')X(c')dc' \tag{10.35}$$

where

$$Y(c) = \frac{1}{\sqrt{2\pi}} \int_0^\infty y(t)\,\frac{e^{-ic\ln t}}{\sqrt{t}}\,dt \tag{10.36}$$

$$X(c) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x(t)\,\frac{e^{-ic\ln t}}{\sqrt{t}}\,dt \tag{10.37}$$

$$H(c,c') = \int_0^\infty \int_0^\infty u^*(c,t)h(t,t')u(c',t')dtdt'. \tag{10.38}$$

We now evaluate $H(c,c')$

$$H(c,c') = \int_0^\infty \int_0^\infty u^*(c,t)\frac{h(t/t')}{t'}u(c',t')dtdt' \tag{10.39}$$

which reduces to

$$H(c,c') = \delta(c-c') \int_0^\infty \frac{e^{-ict}}{\sqrt{t}}h(t)dt. \tag{10.40}$$

However the scale transform of $h(t)$ is

$$H(c) = \int \frac{e^{-ict}}{\sqrt{t}}h(t)dt \tag{10.41}$$

and therefore

$$H(c,c') = \delta(c-c')H(c). \tag{10.42}$$

Convolution theorem. Using Eq. (9.5) we have

$$R(t,t',t'') = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}} \int \frac{e^{ic\ln t}}{\sqrt{t}}\frac{e^{-ic\ln t'}}{\sqrt{t'}}\frac{e^{-ic\ln t''}}{\sqrt{t''}}dc \tag{10.43}$$

which reduces to

$$R(t,t',t'') = \frac{1}{\sqrt{2\pi}}\delta(t-t't'') \tag{10.44}$$

and therefore

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty \delta(t - t't'')x(t')\, l(t'')\, dt'dt'' \qquad (10.45)$$

which evaluates to

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{t'} x(t')\, l(t/t')\, dt'. \qquad (10.46)$$

This is the convolution theorem for scale between the $x(t)$ and $l(t)$.

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Initial Value Problems in the Time-Frequency Domain

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Abstract. We transform an initial value problem for a stochastic differential equation to the time-frequency domain. The result is a deterministic time-frequency equation whose forcing term incorporates the given set of initial values. The structure and solution of the time-frequency equation reveal the spectral properties of the nonstationary random process solution to the stochastic differential equation. By applying our method to the Langevin equation, we obtain the exact time-frequency spectrum for an arbitrary initial value.

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1. Introduction

We consider the stochastic differential equation defined as

$$a_n \frac{d^n x(t)}{dt^n} + \cdots + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t), \tag{1}$$

where $a_0, \ldots, a_n$ are constant deterministic coefficients, the forcing term $f(t)$ is a nonstationary random process, and $x(t)$ is the nonstationary random process representing the solution. The processes $f(t)$ and $x(t)$ are referred to as the input and output signal, respectively. A variety of random phenomena can be modeled by using Eq. (1), such as vibrations of structures [1], electrical and electronic devices with noisy inputs [2], financial time series [3], and Brownian motion processes [4].

Since $x(t)$ is a nonstationary random process its properties vary with time, including the Fourier spectrum. This time variation can be represented by using time-frequency analysis [5, 6], a body of techniques for the spectral analysis of nonstationary signals, either deterministic, random, or chaotic [7]. Contrary to classical spectral analysis, where a unique connection between time and frequency exists, namely, the Fourier spectrum, time-frequency analysis provides an infinite number of representations of the time-varying spectrum of a signal. We consider the Wigner spectrum [8–10]

$$\overline{W}_x(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E[x^*(t-\tau/2)x(t+\tau/2)]e^{-i\tau\omega}d\tau,$$

obtained by taking the expected value E of the Wigner distribution [11, 12]

$$W_x(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x^*(t-\tau/2)x(t+\tau/2)e^{-i\tau\omega}d\tau.$$

The star sign indicates complex conjugation. Equation (1) can be transformed to the time-frequency domain of the Wigner spectrum [13–15]. The result of this transformation is a deterministic time-frequency equation, whose properties clarify the nature of the nonstationary random process $x(t)$. In [16] it is proved that the solution to the time-frequency equation corresponds to the Wigner spectrum of the solution to the equation in time. In this article we show how to write the time-frequency equation when we add to Eq. (1) the set of initial values given by

$$x(0) = x_0, \tag{2}$$

$$x^{(1)}(0) = x_1,$$

$$\vdots$$

$$x^{(n-1)}(0) = x_{n-1}, \tag{3}$$

where

$$x^{(k)}(t) = \frac{d^k x(t)}{dt^k}. \tag{4}$$

We consider both the case of deterministic and random initial values. We apply the developed method to the case of the Langevin equation, and we obtain the exact analytic Wigner spectrum.

The article is organized as follows. In Section 2 we first review the transformation to the time-frequency domain, and then we show how to add the initial values. In Section 3 we obtain the Wigner spectrum for the initial value problem of the Langevin equation.

2. Transformation to the time-frequency domain

We first rewrite Eq. (1) as

$$a_n \prod_{k=1}^{n} (D-\lambda_k)\, x(t) = f(t), \tag{5}$$

where $D = \frac{d}{dt}$, and the complex numbers $\lambda_1, \ldots, \lambda_n$ are the roots of the characteristic equation

$$a_n\lambda^n \ldots + a_1\lambda + a_0 = 0.$$

These roots are referred to as the poles. In general, we can write

$$\lambda_k = \alpha_k + i\beta_k,$$

where α_k and β_k are the real and imaginary parts of λ_k, respectively. Equation (5) can be transformed to the time-frequency equation for the Wigner spectrum given by [15]

$$\frac{|a_n|^2}{4^n} \prod_{k=1}^{n} (\partial_t - p_k)(\partial_t - p_k^*)\, \overline{W}_x(t,\omega) = \overline{W}_f(t,\omega), \tag{6}$$

where $\partial_t = \frac{\partial}{\partial t}$, and $p_1,\ p_1^*, \ldots, p_n,\ p_n^*$ are the time-frequency poles, defined as

$$p_k = 2\alpha_k + 2i\,(\beta_k - \omega)\,.$$

We see that the input of the time-frequency equation is the Wigner spectrum of the input $f(t)$, whereas the output $\overline{W}_x(t,\omega)$ is the Wigner spectrum of the output $x(t)$. Since no derivatives with respect to ω are present, this equation can be solved as an ordinary differential equation with respect to time. Furthermore, the Wigner spectrum is a deterministic quantity, and hence the time-frequency equation is deterministic. We also note that, although the Wigner spectrum is a nonlinear transformation, the time-frequency equation is still linear.

We now consider the initial value problem

$$\begin{aligned} a_n \prod_{k=1}^{n} (D - \lambda_k)\, x(t) &= f(t), \\ x(0) &= x_0, \\ x^{(1)}(0) &= x_1, \\ &\vdots \\ x^{(n-1)}(0) &= x_{n-1}, \end{aligned}$$

corresponding to Eq. (5) with the set of initial values given in Eqs. (2)–(3). We seek a time-frequency equation that can incorporate the initial values $x_0, \ldots, x_{n-1}$. First, we rewrite the initial value problem by using delta functions [17]

$$a_n \prod_{k=1}^{n} (D - \lambda_k)\, x(t) = f(t) + \sum_{k=0}^{n-1} b_k \delta^{(k)}(t), \tag{7}$$

where

$$b_k = a_{k+1}x_0 + a_{k+2}x_1 + \cdots + a_n x_{n-1-k}, \tag{8}$$

and

$$\delta^{(0)}(t) \equiv \delta(t).$$

Then, we set

$$f_0(t) = f(t) + \sum_{k=0}^{n-1} b_k \delta^{(k)}(t). \tag{9}$$

Substituting in Eq. (7), we have

$$a_n \prod_{k=1}^{n} (D - \lambda_k)\, x(t) = f_0(t).$$

After this change of variable we can apply the usual transformation given in Eq. (6), obtaining

$$\frac{|a_n|^2}{4^n} \prod_{k=1}^{n} (\partial_t - p_k)\,(\partial_t - p_k^*)\, \overline{W}_x(t,\omega) = \overline{W}_{f_0}(t,\omega), \tag{10}$$

which is the desired equation. A series of interesting facts about the Wigner spectrum of the input term $f_0(t)$ can be immediately derived.

Deterministic initial values. If, in Eq. (9), we indicate the input term due to the initial values by

$$f_I(t) = \sum_{k=0}^{n-1} b_k \delta^{(k)}(t), \tag{11}$$

then we can rewrite the input $f_0(t)$ as

$$f_0(t) = f(t) + f_I(t).$$

In general, due to the nonlinearity of the Wigner spectrum, it is

$$\overline{W}_{f_0}(t,\omega) \neq \overline{W}_f(t,\omega) + W_{f_I}(t,\omega).$$

We can in fact write

$$\overline{W}_{f_0}(t,\omega) = \overline{W}_f(t,\omega) + W_{f_I}(t,\omega) + 2\Re\left\{\overline{W}_{f,f_I}(t,\omega)\right\}, \tag{12}$$

where

$$\overline{W}_{f,f_I}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E\left[f^*(t-\tau/2) f_I(t+\tau/2)\right] e^{-i\tau\omega} d\tau$$

is the cross-Wigner spectrum. For the common case

$$E\left[f(t)\right] = 0 \tag{13}$$

we can instead write

$$\overline{W}_{f_0}(t,\omega) = \overline{W}_f(t,\omega) + W_{f_I}(t,\omega), \tag{14}$$

because

$$\begin{aligned}\overline{W}_{f,f_I}(t,\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} E\left[f^*(t-\tau/2) f_I(t+\tau/2)\right] e^{-i\tau\omega} d\tau,\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_I(t+\tau/2) E\left[f^*(t-\tau/2)\right] e^{-i\tau\omega} d\tau,\\ &= 0,\end{aligned}$$

which holds since the initial values are deterministic quantities.

Random initial values. When the initial values $x_0, \ldots, x_{n-1}$ are random variables, because of Eq. (8) also the coefficients $b_0, \ldots, b_{n-1}$ are random variables.

Consequently, $f_I(t)$ is a random process and we have to replace the Wigner distribution in Eq. (12) with the Wigner spectrum

$$\overline{W}_{f_0}(t,\omega) = \overline{W}_f(t,\omega) + \overline{W}_{f_I}(t,\omega) + 2\Re\left\{\overline{W}_{f,f_I}(t,\omega)\right\}.$$

A reasonable assumption in a physical system is to consider the mechanisms that generate the input $f(t)$ to be statistically independent from those that are responsible of the initial values. Therefore, the coefficients b_k and, consequently, the random process $f_I(t)$, are statistically independent from the input $f(t)$, and we can write

$$\begin{aligned}\overline{W}_{f,f_I}(t,\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} E\left[f^*(t-\tau/2)f_I(t+\tau/2)\right]e^{-i\tau\omega}d\tau,\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} E\left[f^*(t-\tau/2)\right]E\left[f_I(t+\tau/2)\right]e^{-i\tau\omega}d\tau.\end{aligned}$$

If we consider an input $f(t)$ with zero mean, as in Eq. (13), we obtain

$$\overline{W}_{f,f_I}(t,\omega) = 0,$$

and, consequently

$$\overline{W}_{f_0}(t,\omega) = \overline{W}_f(t,\omega) + \overline{W}_{f_I}(t,\omega).$$

3. Example

We consider the initial value problem of the Langevin equation [18], defined as

$$\frac{dx(t)}{dt} + \gamma x(t) = u(t)\xi(t),$$
$$x(0) = x_0,$$

where $\gamma > 0$, x_0 is real and deterministic, $u(t)$ is the step function defined as

$$u(t) = \begin{cases} 1, & t \geq 0,\\ 0, & t < 0,\end{cases}$$

and $\xi(t)$ is a white Gaussian noise with zero mean and autocorrelation function given by

$$R_\xi(t_1,t_2) = E\left[x(t_1)x(t_2)\right] = \delta(t_1 - t_2).$$

This equation belongs to the class defined in Eq. (1), with

$$a_0 = \gamma, \quad a_1 = 1.$$

The factored form is obtained straightforwardly

$$(D - \lambda_1)x(t) = u(t)\xi(t), \quad \text{where} \quad \lambda_1 = -\gamma.$$

To compute the Wigner spectrum of the solution $x(t)$, we first rewrite the initial value problem according to Eq. (7)

$$\frac{dx(t)}{dt} + \gamma x(t) = u(t)\xi(t) + b_0\delta(t).$$

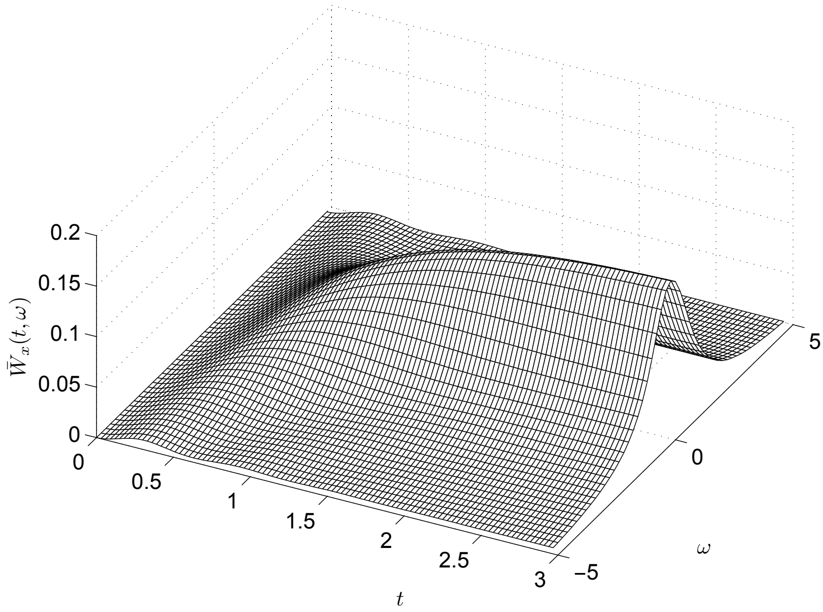

FIGURE 1. Wigner spectrum $\overline{W}_x(t,\omega)$ of the solution to the Langevin equation when $x_0 = 0$. The plot shows the evolution of the time-frequency spectrum when the initial value is zero. We observe a transient spectrum which eventually reaches a steady state corresponding to the classical power spectrum obtained with the assumption of wide sense stationarity.

From Eq. (8), it is

$$b_0 = x_0.$$

Substituting, we have

$$\frac{dx(t)}{dt} + \gamma x(t) = u(t)\xi(t) + x_0\delta(t).$$

Therefore, the modified input term which takes into account the initial values is

$$f_0(t) = u(t)\xi(t) + x_0\delta(t).$$

From Eq. (1) and Eq. (11) it is, respectively

$$f(t) = u(t)\xi(t),$$
$$f_I(t) = x_0\delta(t).$$

By applying the transformation of Eq. (10), we have

$$\frac{1}{4}\left(\partial_t - p_1\right)\left(\partial_t - p_1^*\right)\overline{W}_x(t,\omega) = \overline{W}_{f_0}(t,\omega), \tag{15}$$

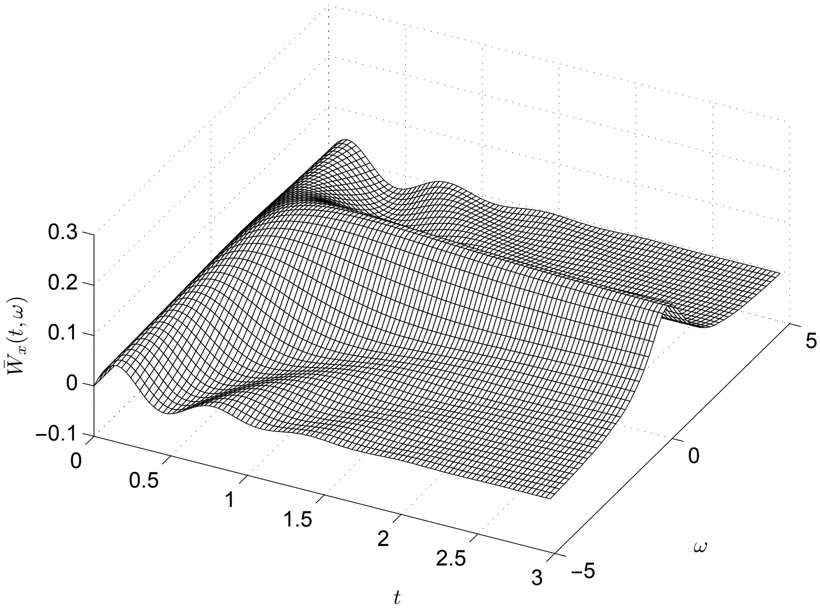

FIGURE 2. Wigner spectrum $\overline{W}_x(t,\omega)$ of the solution to the Langevin equation when $x_0 = 1$. The non-zero initial value interferes with the transient spectrum shown in Figure 1. As a result, for short times we observe a larger spread of the time-frequency spectrum. As in Figure 1, the time-frequency spectrum eventually reaches a steady state corresponding to the classical power spectrum.

where

$$p_1 = 2\alpha_1 + 2i\left(\beta_1 - \omega\right).$$

Since the real and imaginary parts of λ_1 are given by, respectively

$$\alpha_1 = -\gamma, \quad \beta_1 = 0,$$

it is

$$p_1 = -2\gamma - 2i\omega.$$

Moreover, $\xi(t)$ has zero mean, hence

$$E\left[f(t)\right] = E\left[u(t)\xi(t)\right] = u(t)E\left[\xi(t)\right] = 0.$$

Therefore, Eq. (14) holds. The Wigner spectrum of $f(t)$ is given by [15]

$$\overline{W}_f(t,\omega) = \frac{1}{2\pi}u(t).$$

Furthermore [5]

$$W_{f_I}(t,\omega) = \frac{1}{2\pi}x_0^2\delta(t).$$

Substituting these results in Eq. (14) and then in Eq. (15), we have

$$(\partial_t + 2\gamma + 2i\omega)\,(\partial_t + 2\gamma - 2i\omega)\,\overline{W}_x(t,\omega) = \frac{2}{\pi}\left(u(t) + x_0^2\delta(t)\right). \tag{16}$$

We solve this equation by taking the Laplace transform along the time axis [19]

$$(s + 2\gamma + 2i\omega)\,(s + 2\gamma - 2i\omega)\,\overline{W}_x(s,\omega) = \frac{2}{\pi}\left(\frac{1}{s} + x_0^2\right), \tag{17}$$

where

$$\overline{W}_x(s,\omega) = \int_0^{+\infty} \overline{W}_x(t,\omega)e^{st}dt.$$

Therefore

$$\overline{W}_x(s,\omega) = \frac{2}{\pi}\left(\frac{1}{s} + x_0^2\right)\frac{1}{(s + 2\gamma + 2i\omega)\,(s + 2\gamma - 2i\omega)}. \tag{18}$$

Inversion of this equation gives, for $\omega \neq 0$

$$\overline{W}_x(t,\omega) = u(t)\left[\frac{1}{2\pi}\frac{1}{\gamma^2+\omega^2}\left[1 - e^{-2\gamma t}\left(\cos 2\omega t + \gamma\frac{\sin 2\omega t}{\omega}\right)\right] + \frac{1}{\pi}x_0^2 e^{-2\gamma t}\frac{\sin 2\omega t}{\omega}\right], \tag{19}$$

and, for $\omega = 0$

$$\overline{W}_x(t,0) = u(t)\frac{1}{2\pi\gamma^2}\left[1 - e^{-2\gamma t}\left(1 + 2\gamma t - 4x_0^2\gamma^2 t\right)\right]. \tag{20}$$

In Figure 1 we show $\overline{W}_x(t,\omega)$ when $x_0 = 0$, whereas in Figure 2 we consider the case $x_0 = 1$. A brute force solution for random initial values is given in [20].

4. Conclusions

We have shown how to transform an initial value problem for a stochastic differential equation defined in the time domain, to the time-frequency domain of the Wigner spectrum. The result is a deterministic time-frequency equation which can be used to better understand the structure of the nonstationary random process obtained as the solution to the time equation. This problem has a fundamental interest is science and engineering, since the considered class of stochastic differential equations is the model for several random phenomena.

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Polycaloric Distributions and the Generalized Heat Operator

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Abstract. The aim of this paper is to introduce the notion of p-caloric distributions with respect to the generalized heat operator and to prove a representation formula. Based on the representation formula for p-caloric distributions and using the parametrix of the generalized heat operator we shall give two extensions of Poisson formula.

Finally, we shall define the generalized iterated heat operator of order $\lambda \in \mathbf{C}$, $\operatorname{Re} \lambda < 0$ by means of the kernel distribution of its parametrix.

Mathematics Subject Classification (2010). Primary 58J40, 35S10; Secondary 35K05, 35S05.

Keywords. Heat operator, parametrix, kernel distribution, pseudodifferential operators.

Introduction

The purpose of this work is to consider the iterated generalized heat equation $\left(\frac{\partial}{\partial t} - A(t)\right)^p u = 0$, in n-dimensional C^∞ manifold countable at infinity X. Here $A(t)$ is a pseudo-differential operator of order m in X, valued in $L(H)$, depending smoothly on t in $[0, T)$, where T is some real number > 0. H is a finite-dimensional Hilbert space (over $\mathbf{C}$), $L(H)$ is the space of (bounded) linear operators in H and u is an H-valued distribution in X.

The operator $L = \frac{\partial}{\partial t} - A(t)$ has been studied by several authors, in particular by Trèves in [8] under the name of generalized heat operator, which we shall adopt in the following.

We shall give in our paper a necessary and sufficient condition in order that an H-valued distribution u in X to belong to the kernel of the generalized iterated heat operator $L^p = \left(\frac{\partial}{\partial t} - A(t)\right)^p$, $p \geq 1$, i.e., u to be a polycaloric distribution.

Our condition can be read as follows: $u = u_0 + tu_1 + \cdots + t^{p-1}u_{p-1}$, where $u_0, u_1, \ldots, u_{p-1}$ are caloric distributions, i.e., $Lu_j = 0$, for all $0 \leq j \leq p-1$.

Let us remark that such decomposition theorems have been obtained by some authors in the frame of the theory of polyharmonic or polycaloric functions (see [1], [4], [6], [7]).

By using the parametrix of the generalized heat operator which has been obtained by Trèves in [8] we shall give two applications of the representation formula for polycaloric distributions.

One of these applications refer to the following initial value problem:

$$\begin{cases} L^p u = 0, & \text{in } X \times [0,T) \\ \dfrac{\partial^k u}{\partial t^k}|_{t=0} = w_k, & \text{in } X,\ 0 \leq k \leq p-1,\ p \geq 2. \end{cases}$$

Its solution will be defined by an expression which will be called the first extension for Poisson formula in the case of polycaloric distributions.

The other one application refers to the following initial value problem:

$$\begin{cases} L^p u = 0, & \text{in } X \times [0,T) \\ L^k u|_{t=0} = v_k, & \text{in } X,\ 0 \leq k \leq p-1,\ p \geq 2. \end{cases}$$

Then, the expression which will give its solution will be named the second extension for Poisson formula in the case of polycaloric distributions. Let us see that when $X = \mathbf{R}^n$, $A(t) = \Delta_x$ is the Laplace operator, then we recover similar results such as in Nicolescu [4].

In the end, when $X = \mathbf{R}^n$, by using the Schwartz kernel distribution of the parametrix of the generalized heat operator and a similar idea such as in [3], we shall define the generalized iterated heat operator of an arbitrary order $\lambda \in \mathbf{C}$, $\operatorname{Re}\lambda < 0$.

Some definitions and results from Trèves' book [8] concerning the generalized heat equation and its parametrix will be recalled in Section 1.

In Section 2 we shall introduce the notion of polycaloric distribution and we shall give a decomposition theorem for them. Based on the decomposition theorem in Section 2 and by using the parametrix of the generalized heat operator we shall give in Section 3, a first extension for the Poisson formula for 2-caloric distributions.

In Section 4 we shall give a first extension for the Poisson formula for p-caloric distributions.

A second extension for the Poisson formula for 2-caloric distributions will be given in Section 5.

A similar result as that in Section 5 but in the case of p-caloric distributions will be proved in Section 6.

Finally, in Section 7 we shall define the generalized iterated heat operator of an arbitrary order $\lambda \in \mathbf{C}$, $\operatorname{Re}\lambda < 0$, by using the Schwartz kernel distribution of the parametrix of the generalized heat operator.

1. Some preliminaries concerning the generalized heat equation and its parametrix

The notations and the namings that we shall use in the following are those of [8].

Let X be an n-dimensional C^∞ manifold countable at infinity, let $(\Omega, x_1, \ldots, x_n)$ be a local chart in X, let H be a finite-dimensional Hilbert space and let $L(H)$ be the space of (bounded) linear operators on H. The norm in H will be denoted by $|\cdot|_H$ whereas the operator norm in $L(H)$ will be denoted by $||\cdot||$. Our basic ingredient is a pseudodifferential operator of order m in X, $A(t)$-valued in $L(H)$, depending smoothly on t in $[0, T)$.

In fact, $A(t)$ is a matrix whose entries are scalar pseudodifferential operators in X, if one uses a basis in H.

Thus, in every local chart $(\Omega, x_1, \ldots, x_n)$ in X, $A(t)$ is given by

$$A_\Omega(t)u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} a_\Omega(x,t,\xi)\hat{u}(\xi)d\xi, \qquad u \in C_0^\infty(\Omega; H), \tag{1.1}$$

modulo regularizing operators, depending smoothly on t in $[0, T)$, where

$$a_\Omega(x,t,\xi) \text{ is a } C^\infty \text{ function of } t \in [0,T), \text{ valued in } S^m(\Omega; L(H)). \tag{1.2}$$

We have denoted by $S^m(\Omega; L(H))$ the space of classical symbols in Ω, valued in $L(H)$. We shall denote by $\Psi^m(\Omega; L(H))$ the corresponding class of operators.

Now, we shall consider the following Cauchy problem from Trèves [8].

$$\frac{dU(t)}{dt} - A(t) \cdot U(t) = 0, \text{ in } X \times [0,T) \tag{1.3}$$

$$U(t)|_{t=0} = I, \text{ in } X, \tag{1.4}$$

where $I : H \to H$ in the identity operator of H.

Let us mention that the equality (1.3) it means congruence modulo regularizing operators depending smoothly on t in $[0, T)$ and that the equality (1.4) it means congruence modulo regularizing operators.

We are interested in solving the initial value problem (1.3), (1.4). To this end let us make the following hypothesis:

(1.5) Let $(\Omega, x_1, \ldots, x_n)$ be a local chart in X. Suppose that there is a symbol $a_\Omega(x, t, \xi)$ satisfying (1.2) and defining $A_\Omega(t)$ by (1.1), congruent to $A(t)$ modulo regularizing operators in Ω depending smoothly on t in $[0, T)$, such that

(1.6) to every compact K of $\Omega \times [0, T)$ there is a compact $K' \subset C_- = \{z \in \mathbf{C}; \mathrm{Re}z < 0\}$ such that

(1.7) $zI - a_\Omega(x, t, \xi)/(1 + |\xi|^2)^{m/2} : H \to H$ is a bijection (hence a homeomorphism), for all (x, t) in K, ξ in $\mathbf{R}^n$ and z in $\mathbf{C} \setminus K'$.

Then, Trèves has proved in his book [8] the following theorem which states the existence and the "uniqueness" of the parametrix of the generalized heat equation (1.3).

Theorem 1.1. *Under the hypothesis* (1.5)–(1.7), *the initial value problem* (1.3)–(1.4) *has a solution* $U(t)$ *which is a function of* $t \in [0,T)$*-valued in*

$$\dot{\Psi}^0(X;L(H))(=\Psi^0(X;L(H))/\Psi^{-\infty}(X;L(H)).$$

There is a representative of the equivalence class $U(t)$ *with the following property. In each local chart* $(\Omega, x_1, \ldots, x_n)$ *the representative in question is equivalent to an element* $U_\Omega(t)$ *of* $\Psi^0(\Omega; L(H))$ *given by*

$$U_\Omega(t)u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} \mathcal{U}_\Omega(x,t,\xi)\hat{u}(\xi)d\xi, \quad u \in C_0^\infty(\Omega;H), \tag{1.8}$$

whose symbol $\mathcal{U}_\Omega$ *has the following properties*:

$$\mathcal{U}_\Omega : \Omega \times [0,T) \times \mathbf{R}^n \to L(H) \text{ is a } C^\infty \text{ function of } t \in [0,T). \tag{1.9}$$

(1.10) *To every compact* $\mathcal{K}$ *of* $\Omega \times [0,T)$, *to every pair of n-tuples* $\alpha, \beta \in \mathbf{Z}_+^n$, *and to every pair of integers* $r, N \geq 0$, *there is a constant* $C > 0$ *such that for all* (x,t) *in* $\mathcal{K}, \xi \in \mathbf{R}^n$

$$||\partial_x^\alpha \partial_\xi^\beta \partial_t^r \mathcal{U}_\Omega(x,t,\xi)|| \leq Ct^{-N}(1+|\xi|)^{(r-N)m-|\beta|}. \tag{1.11}$$

Any C^∞ *function of* t *in* $[0,T)$ *valued in the space of continuous linear mappings* $\mathcal{E}'(X;H) \to \mathcal{D}'(X;H)$ *which satisfies* (1.3), (1.4) *belongs to the equivalence class* $U(t)$.

Remark 1.2. From (1.11) it follows that $\mathcal{U}(x,t,\xi)$ belongs to $S^{-\infty}(\Omega; L(H))$, so, the operator (1.8) is regularizing; it means that the equivalence class $U(t)$ is zero. This generalizes the well-known property of the parametrix of the heat equation.

Remark 1.3. When $X = \mathbf{R}^n$, $A(t) = \Delta_x$, the Laplace operator in n variables (in which case $a_\Omega(x,t,\xi) = -|\xi|^2$), then equation (1.3)–(1.4) define the parametrix in the forward Cauchy problem for the heat equation $\dfrac{\partial U}{\partial t} - \Delta_x U = 0$.

In the same manner as above we can consider the following Cauchy problem:

$$\frac{dU(t)}{dt} - A(t)\cdot U(t) = 0, \text{ in } X \times [t', T), \tag{1.12}$$

$$U(t)|_{t=t'} = I, \text{ in } X, \tag{1.13}$$

$0 \leq t' \leq T$ is a fixed real number.

We shall denote by $U(t,t') \in \dot{\Psi}^0(X;L(H))$ the solution of the initial value problem (1.12)–(1.13).

In every local chart $(\Omega, x_1, \ldots, x_n)$ in X, this solution is equivalent with the pseudodifferential operator $U_\Omega(t,t') \in \Psi^0(\Omega; L(H))$, defined by

$$U_\Omega(t,t')u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} \mathcal{U}_\Omega(x,t,t',\xi)\hat{u}(\xi)d\xi, \quad u \in C_0^\infty(\Omega;H). \tag{1.14}$$

By means of the solution of the initial value problem (1.12)–(1.13) it can be solved the following nonhomogeneous Cauchy problem:

$$\frac{\partial u}{\partial t} - A(t)u = f, \text{ in } X \times [0,T) \tag{1.15}$$

$$u|_{t=0} = u_0, \text{ in } X, \tag{1.16}$$

where $f \in \mathcal{D}'(X \times [0,T); H)$, $u_0 \in \mathcal{D}'(X; H)$.

We mention that the equality (1.15) is understood as congruence modulo $C^\infty(X \times [0,T); H)$ whereas the equality in (1.16) it means congruence modulo $C^\infty(X; H)$.

In particular, if $f \in C^0([0,T); \mathcal{D}'(X; H))$, then the solution of the initial value problem (1.15)–(1.16) can be expressed such as

$$u(t) = U(t)u_0 + \int_0^t U(t,t')f(t')dt'. \tag{1.17}$$

where the equality in (1.17) it means congruence modulo $C^\infty(X; H)$.

2. Polycaloric distributions and their caloric component parts

Let $L = \dfrac{\partial}{\partial t} - A(t)$ be the generalized heat operator, where $A(\cdot) \in C^\infty([0,T); \dot{\Psi}^m(X; L(H))$ is a pseudodifferential operator of order m in X, valued in $L(H)$, depending smoothly on t in $[0,T)$.

Definition 2.1. Let p be any positive integer. We shall denote by $C_p = C_p(X \times [0,T); H)$ the subspace of $D'(X \times [0,T); H)$ consisting of the distributions u such that $L^p u = 0$ in $X \times [0,T)$. The elements of C_p will be called the polycaloric distributions of degree p or the p-caloric distributions with respect to the operator L.

When $p = 1$ we often refer to the elements of C_1 such as caloric distributions.

Remark 2.2. Let p be any positive integer and let $u \in C_p$ be a p-caloric distribution. Then $L^k u$ belongs to C_{p-k} for any $0 \le k \le p$. Moreover, u is a q-caloric distribution for any positive integer $q \ge p$.

First we give a lemma which we need in the proof of our main theorem.

Lemma 2.3. *Let u be a distribution in $D'(X \times [0,T); H)$. Then*

$$\begin{aligned} L^p(t^k u) = t^k L^p u &+ \binom{k}{1} t^{k-1} p L^{p-1} u \\ &+ \binom{k}{2} t^{k-2} p(p-1) L^{p-2} u \\ &+ \cdots + p(p-1)\cdots(p-k+1) L^{p-k} u, \end{aligned} \tag{2.1}$$

for any nonnegative integers k, p and for any t in $[0,T)$. By definition we put $L^{p-i}u = 0$, for any integers i, p such that $i > p$ and $L^0 u = u$.

Corollary 2.4. *Let u be a caloric distribution in C_1 (that is $Lu = 0$ in $X \times [0,T)$). Then $L^k(t^k u) = u$, $L^{k+1}(t^k u) = 0$ and $L^k(t^j u) = 0$, for any integers j, k such that $0 \leq j < k$.*

The proof of Lemma 2.3 can easily be done by induction.

Remark 2.5. For shorthand, we write $(L^p u)^{(0)} = L^p u$, $(L^p u)^{(i)} = p(p-1)\cdots(p-i+1)L^{p-i}u$ for any nonnegative integers i, p (we remember the convention that $L^{p-i}u = 0$ for $i > p$ and $L^0 u = u$). By means Newton's formula we can rewrite in a symbolic manner (2.1) as

$$L^p(t^k u) = (t + L^p u)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} t^{k-i}(L^p u)^{(i)}.$$

Remark 2.6. By (2.1), we can deduce that if $u \in D'(X \times [0,T); H)$ is a $(p-1)$-caloric distribution, $p \geq 1$, then tu is a p-caloric distribution. Generally, if $u \in D'(X \times [0,T); H)$ is a $(p-k)$-caloric distribution, $p \geq k$, then $t^k u$ is a p-caloric distribution.

Our main result concerning p-caloric distributions in $X \times [0,T)$, valued in H is contained in the following theorem.

Theorem 2.7 (Decomposition theorem for polycaloric distributions). Let us assume that u is a p-caloric distribution in $X \times [0,T)$, valued in H. Then there exist unique caloric distributions in $X \times [0,T)$, valued in H, $u_0, u_1, \dots, u_{p-1}$ such that

$$u = u_0 + tu_1 + \cdots + t^{p-1}u_{p-1}, \quad \text{in} \quad X \times [0,T). \tag{2.2}$$

Moreover, the distributions $u_0, u_1, \dots, u_{p-1}$ are given by the following formulae:

$$\begin{aligned}
u_0 &= u - \frac{1}{1!}tLu + \frac{1}{2!}t^2L^2u + \cdots + \frac{(-1)^{p-1}}{(p-1)!}t^{p-1}L^{p-1}u, \\
u_1 &= \frac{1}{1!}\left(Lu - \frac{1}{1!}tL^2u + \cdots + \frac{(-1)^{p-2}}{(p-2)!}t^{p-2}L^{p-1}u\right), \\
&\cdots\cdots\cdots\cdots \\
u_{p-2} &= \frac{1}{(p-2)!}\left(L^{p-2}u - \frac{1}{1!}tL^{p-1}u\right), \\
u_{p-1} &= \frac{1}{(p-1)!}L^{p-1}u.
\end{aligned} \tag{2.3}$$

Conversely, the sum in (2.2), with $u_0, u_1, \dots, u_{p-1}$ caloric distributions in $X \times [0,T)$, valued in H, defines a p-caloric distribution in $X \times [0,T)$, valued in H.

Remark 2.8. We shall call the caloric distributions $u_0, u_1, \dots, u_{p-1}$, in $X \times [0,T)$, valued in H, the caloric component parts of p-caloric distribution u.

Let us see that caloric component parts of a p-caloric distribution are expressed in a linear manner by means of that distribution and of the operators $L, L^2, \dots, L^{p-1}$.

Proof of Theorem 2.7. *The Existence of* (2.2). To prove existence of decomposition relation (2.2), consider the null space $C_k = \{u \in D'(X \times [0,T); H); L^k u = 0\}$ of L^k and define the multiplication operator by t_k, that is $T_k : D'(X \times [0,T); H) \to D'(X \times [0,T); H)$, $T_k u = t^k u$, for any distribution u in $D'(X \times [0,T); H)$. Then, it will be sufficient to show that

$$C_k = C_{k-1} + T_{k-1} C_1, \tag{2.4}$$

for any nonnegative integer k.

Indeed, we can easily deduce by induction that

$$C_p = C_1 + T_1 C_1 + \cdots + T_{p-1} C_1, \tag{2.5}$$

which is another way to express the decomposition relation (2.2). To prove (2.4) we split the proof into two parts.

$$C_k \supset C_{k-1} + T_{k-1} C_1. \tag{i}$$

As $C_{k-1} \subset C_k$ by Remark 2.2, we only need to show that $T_{k-1} C_1 \subset C_k$. But, from Lemma 2.3 and Corollary 2.4 it follows that $L^k(t^{k-1}u) = 0$, for any u in C_1 and any nonnegative integer k, as desired.

$$C_k \subset C_{k-1} + T_{k-1} C. \tag{ii}$$

Let us remark that we have the following decomposition relation

$$u = (I - [(k-1)!]^{-1} T_{k-1} L^{k-1}) u + [(k-1)!]^{-1} T_{k-1} L^{k-1} u. \tag{2.6}$$

It is evidently that $[(k-1)!]^{-1} L^{k-1} u$ belongs to C_1.

Therefore, we only need to show that the first term in the right-hand side of (2.6) is in C_{k-1}. But from Lemma 2.3 and Remark 2.2 it follows that

$$\begin{aligned} &L^{k-1}(u - [(k-1)!]^{-1} T_{k-1} L^{k-1} u) \\ &\quad = L^{k-1} u - [(k-1)!]^{-1} \\ &\quad \times \left(T_{k-1} L^{2k-2} + \begin{pmatrix} k-1 \\ 1 \end{pmatrix} T_{k-2} L^{2k-3} + \cdots + (k-1)! L^{k-1} \right) u = 0 \end{aligned}$$

as desired. Thus the existence of decomposition relation (2.2) follows.

The Uniqueness of (2.2). First we prove that for any distribution u in C_k the decomposition formula

$$u = v + T_{k-1} w_{k-1}, \quad v \in C_{k-1}, w_{k-1} \in C_1, \tag{2.7}$$

if it exists then it is unique. Then the uniqueness follows by induction.

Now let us assume that there exists the decomposition formula (2.7). Thus, by applying the operator L^{k-1} on both sides of (2.7), we have

$$\begin{aligned} L^{k-1} u &= L^{k-1} v + T_{k-1} L^{k-1} w_{k-1} \\ &\quad + \begin{pmatrix} k \\ 1 \end{pmatrix} (k-1) T_{k-2} L^{k-2} w_{k-1} + \cdots + (k-1)! w_{k-1}. \end{aligned} \tag{2.8}$$

Hence it follows that

$$w_{k-1} = [(k-1)!]^{-1} L^{k-1} u. \tag{2.9}$$

So, by (2.7), (2.9)

$$v = u - T_{k-1}w_{k-1} = u - [(k-1)!]^{-1}T_{k-1}L^{k-1}u. \tag{2.10}$$

Thus, by (2.9), (2.10) it follows the uniqueness of decomposition formula (2.7). To prove the converse part of Theorem 2.7 let us remark that from Corollary 2.4 it follows that

$$L^pT_jC_1 = 0, \tag{2.11}$$

for any nonnegative integers j, p such that $j < p$. So, by (2.11)

$$T_jC_1 \subset C_p, \tag{2.12}$$

for any nonnegative integers j, p, such that $j < p$, as desired. Now, let us take a p-caloric distribution u in C_p, and let $u_0, u_1, \dots, u_{p-1}$ be its caloric component parts in C_1. We want to express these caloric component parts by means of distribution u and the linear operators $L, L^2, \dots, L^{p-1}$.

To this end let us observe that by Remark 2.2, Lemma 2.3 and the existence part of Theorem 2.7, we have

$$Lu = v_0 + tv_1 + \cdots + t^{p-2}v_{p-2}, \quad \text{in} \quad X \times [0, T), \tag{2.13}$$

$$Lu = u_1 + 2tu_2 + \cdots + (p-1)t^{p-2}u_{p-1}, \text{ in } X \times [0, T), \tag{2.14}$$

where $v_0, \dots, v_{p-2}$ are suitable elements in C_1. By the uniqueness part of Theorem 2.7 and by (2.13), (2.14), we get

$$u_1 = v_0,\ u_2 = \frac{1}{2}v_1, \dots, u_{p-1} = \frac{1}{p-1}v_{p-2} \tag{2.15}$$

$$u_0 = u - \left(tv_0 + \frac{1}{2}t^2v_1 + \cdots + \frac{1}{p-1}t^{p-1}v_{p-2}\right). \tag{2.16}$$

Now let us denote by $v_0^i, v_1^i, \dots, v_{i-1}^i$ the caloric component parts of i-caloric distribution $L^{p-i}u$, in the hypothesis that u is a p-caloric distribution, $1 \le i \le p$. Then we can prove by induction that we get

$$\begin{aligned}
v_0^i &= L^{p-i}u - \frac{1}{1!}tL^{p-i+1}u + \frac{1}{2!}t^2L^{p-i+2}u + \cdots + \frac{(-1)^{i-1}}{(i-1)!}t^{i-1}L^{p-1}u,\\
v_1^i &= \frac{1}{1!}\left(L^{p-i+1}u - \frac{1}{1!}tL^{p-i+2}u + \cdots + \frac{(-1)^{i-2}}{(i-2)!}t^{i-2}L^{p-1}u\right),\\
v_2^i &= \frac{1}{2!}\left(L^{p-i+2}u - \frac{1}{1!}tL^{p-i+3}u + \cdots + \frac{(-1)^{i-3}}{(i-3)!}t^{i-3}L^{p-1}u\right),\\
&\dots\dots\dots\dots\dots\dots\\
v_{i-1}^i &= \frac{1}{(i-1)!}L^{p-1}u.
\end{aligned} \tag{2.17}$$

In particular, if we take $i = p$ in (2.17), then we get (2.3). This completes the proof of our theorem. □

Remark 2.9. We have given by Theorem 2.7 a necessary and sufficient condition for a distribution to belong to the kernel of the iterated of the generalized heat operator L^p, $p \geq 1$.

3. The first extension for poisson formula for 2-caloric distributions

Let us consider the following initial value problem:

$$\begin{cases} L^2 u = 0, \text{ in } X \times [0,T) \\ u(x,t)|_{t=0} = w_0, \text{ in } X \\ \dfrac{\partial u}{\partial t}|_{t=0} = w_1(x), \text{ in } X, \end{cases} \tag{3.1}$$

where $L = \dfrac{\partial}{\partial t} - A(t)$, $A(\cdot) \in C^\infty([0,T); \Psi^m(X; L(H))$ is the generalized heat operator, $w_0, w_1 \in D'(X;H)$ and $u \in C^\infty([0,T); D'(X;H))$.

We mention that in all rigor we must consider the equalities in (3.1) modulo $C^\infty(X \times [0,T); H)$ respectively modulo $C^\infty(X;H)$ and that the operator $A(t)$ is supposed to be properly supported pseudodifferential operator.

By using Theorem 2.7 we shall look for the solution of the initial value problem (3.1) in the form

$$u = u_0 + tu_1, \tag{3.2}$$

where $u_0, u_1 \in C^\infty([0,T); D'(X;H))$, $Lu_0 = 0$, $Lu_1 = 0$. Let us remark that u_0 is the solution for the Cauchy problem

$$\begin{cases} Lu = 0, \text{ in } X \times [0,T), \\ u(x,t)|_{t=0} = w_0(x), \text{ in } X. \end{cases} \tag{3.3}$$

So,

$$u_0(x,t) = U(t)w_0(x), \tag{3.4}$$

where $U(t)$ is the parametrix of the generalized heat operator. By (3.4), we have

$$\frac{\partial u_0}{\partial t}(x,t) = A(t)U(t)w_0(x). \tag{3.5}$$

From (3.2) we deduce

$$\frac{\partial u}{\partial t} = \frac{\partial u_0}{\partial t} + u_1 + t\frac{\partial u_1}{\partial t} = v + t\frac{\partial u_1}{\partial t}, \tag{3.6}$$

where we have denoted $v = \dfrac{\partial u_0}{\partial t} + u_1$. Thus,

$$\begin{aligned} Lv &= L\frac{\partial u_0}{\partial t} + Lu_1 = \frac{\partial^2 u_0}{\partial t^2} - A(t)\frac{\partial u_0}{\partial t} \\ &= \frac{\partial^2 u_0}{\partial t^2} + A'(t)u_0 - \frac{\partial}{\partial t}(A(t)u_0) = A'(t)u_0. \end{aligned} \tag{3.7}$$

because $Lu_0 = 0$, by hypothesis. Thus, by (3.1), (3.6) and (3.7) we deduce that v is the solution for the following Cauchy problem

$$\begin{cases} Lv = A'(t)U(t)w_0 \text{ in } X \times [0,T), \\ v(x,t)|_{t=0} = w_1(x), \text{ in } X. \end{cases} \tag{3.8}$$

So,

$$v(x,t) = U(t)w_1(x) + \int_0^t U(t,t')A'(t')U(t')w_0(x)dt', \tag{3.9}$$

where $U(t,t')$ is the solution for the Cauchy problem

$$\begin{cases} \frac{dU}{dt}(t) - A(t) \cdot U(t) = 0, \text{ in } X \times [t',T) \\ U(t)|_{t=t'} = I, \text{ in } X, \ 0 \le t' < T. \end{cases}$$

By (3.5), (3.6), (3.9), we get

$$\begin{aligned} u_1(x,t) = v(x,t) - \frac{\partial u_0}{\partial t} &= U(t)w_1(x) \\ &+ \int_0^t U(t,t')A'(t')U(t')w_0(x)dt' - A(t)U(t)w_0(x). \end{aligned} \tag{3.10}$$

So, by (3.2), (3.4) and (3.10) we get

$$\begin{aligned} u(x,t) &= U(t)w_0(x) + tU(t)w_1(x) \\ &\quad + t\int_0^t U(t,t')A'(t')U(t')w_0(x)dt' \\ &\quad - tA(t)U(t)w_0(x) \\ &= (I - tA(t))U(t)w_0(x) + tU(t)w_1(x) \\ &\quad + t\int_0^t U(t,t')A'(t')U(t')w_0(x)dt', \end{aligned} \tag{3.11}$$

which gives the solution for the Cauchy problem (3.1).

Remark 3.1. In particular, when we take $X = \mathbf{R}$ and $A(t) = \dfrac{\partial^2}{\partial x^2}$, then we get (10) in Chapter II in [4].

Remark 3.2. The fact that the "abstract" operator $A(t)$ is a pseudodifferential operator has a role in the above computations. For example $A'(t)$ is also a pseudodifferential operator with symbol $a'(x,t,\xi)$, where $a(x,t,\xi)$ is the symbol of $A(t)$.

4. The first extension for the Poisson formula for p-caloric distributions

We are interested in solving the following initial value problem:

$$\begin{cases} L^p u = 0, \text{ in } X \times [0,T) \\ \dfrac{\partial^k u}{\partial t^k}(x,t)|_{t=0} = w_k(x), \text{ in } X, \text{ for all } 0 \leq k \leq p-1, \end{cases} \tag{4.1}$$

where $L = \dfrac{\partial}{\partial t} - A(t)$, $A(\cdot) \in C^\infty([0,T); \Psi^m(X; L(H))$ is the generalized heat operator, $w_k \in D'(X;H)$, $0 \leq k \leq p-1$, $p \geq 2$ and $u \in C^\infty([0,T); D'(X;H))$.

We look for the solution for initial value problem (4.1) in the form

$$u(x,t) = u_0(x,t) + tu_1(x,t) + \cdots + t^{p-1}u_{p-1}(x,t), \tag{4.2}$$

where $u_k \in C^\infty([0,T); D'(X;H))$, $Lu_k = 0$, for all $0 \leq k \leq p-1$. By (4.1), (4.2) we get

$$\begin{cases} Lu_0 = 0, \text{ in } X \times [0,T) \\ u_0(x,t)|_{t=0} = w_0, \text{ in } X. \end{cases} \tag{4.3}$$

So, by (4.3)

$$u_0(x,t) = U(t)w_0(x), \tag{4.4}$$

where $U(\cdot) \in C^\infty([0,T); \Psi^0(X; L(H))$ is the parametrix of the generalized heat operator. In the following we try to obtain the caloric component parts u_k, $0 \leq k \leq p-1$ of u by iteration.

By (4.2) we derive by successive derivation with respect to t

$$\frac{\partial^l u}{\partial t^l}(x,t) = v_0(x,t) + tv_1(x,t) + \cdots + t^{p-1}v_{p-1}(x,t), \tag{4.5}$$

where $v_0, v_1, \ldots, v_{p-1}$ are not generally caloric distributions (because the operators $\dfrac{\partial}{\partial t}$ and $L = \dfrac{\partial}{\partial t} - A(t)$, do not commute). Thus, by (4.2), (4.5) we get

$$v_0(x,t) = \frac{\partial^l u_0}{\partial t^l}(x,t) + l\frac{\partial^{l-1}}{\partial t^{l-1}}u_1(x,t) + \cdots + l!u_l(x,t). \tag{4.6}$$

But, by (4.1), (4.5) it follows that

$$v_0(x,t)|_{t=0} = w_l(x). \tag{4.7}$$

Then we deduce by (4.6), (4.7) that v_0 is the solution of the initial value problem

$$\begin{cases} Lv(x,t) = F_{l-1}(x,t), \text{ in } X \times [0,T) \\ v(x,t)|_{t=0} = w_l(x), \text{ in } X, \end{cases} \tag{4.8}$$

where

$$F_{l-1}(x,t) = L\left(\frac{\partial^l u_0}{\partial t^l}(x,t) + l\frac{\partial^{l-1}u_1}{\partial t^{l-1}}(x,t) + \cdots + (l-1)!\frac{\partial u_{l-1}}{\partial t}(x,t)\right).$$

Thus, as in the same manner as in the preceding paragraph, we get

$$v_0(x,t) = U(t)w_l(x) + \int_0^t U(t,t')F_{l-1}(x,t')dt'. \tag{4.9}$$

Then, by (4.6), (4.9) the caloric component part of order l, u_l, $1 \leq l \leq p-1$ of u is given by the formula

$$u_l(x,t) = \frac{1}{l!}U(t)w_l(x) + \frac{1}{l!}\int_0^t U(t,t')F_{l-1}(x,t')dt' \tag{4.10}$$
$$- \frac{\partial u_{l-1}}{\partial t}(x,t) - \frac{1}{2!}\frac{\partial^2 u_{l-2}}{\partial t^2}(x,t) - \cdots - \frac{1}{l!}\frac{\partial^l u_0}{\partial t^l}(x,t),$$

for all $1 \leq l \leq p-1$.

Thus, by (4.4), (4.10) we can deduce by iteration the caloric component parts $u_0, u_1, \dots, u_{p-1}$ of u, which is the solution of the initial value problem (4.1).

Let us also remark that the initial conditions of problem (4.1) are satisfied. Indeed, by (4.2), (4.10) we get

$$\begin{aligned} \left.\frac{\partial^l u}{\partial t^l}\right|_{t=0} = \left.\frac{\partial^l u_0}{\partial t^l}\right|_{t=0} &+ 1!\begin{pmatrix} l \\ 1 \end{pmatrix}\left.\frac{\partial^{l-1} u_1}{\partial t^{l-1}}\right|_{t=0} \\ &+ \cdots + (l-1)!\begin{pmatrix} l \\ l-1 \end{pmatrix}\left.\frac{\partial u_{l-1}}{\partial t}\right|_{t=0} \\ &+ l! u_l|_{t=0} = w_l, \quad \text{for all } 0 \leq l \leq p-1. \end{aligned}$$

5. The second extension for the Poisson formula for 2-caloric distributions

Let us consider in the following the initial value problem:

$$\begin{cases} L^2 u = 0, \text{ in } X \times [0,T) \\ u(x,t)|_{t=0} = v_0(x), \text{ in } X \\ Lu(x,t)|_{t=0} = v_1(x), \text{ in } X, \end{cases} \tag{5.1}$$

where $L = \dfrac{\partial}{\partial t} - A(t)$ is the generalized heat operator, v_0, v_1 belong to $D'(X;H)$ and u is an unknown distribution in X, valued in H and depending smoothly on t in $[0,T)$. We mention that we must reason in all rigor modulo $C^\infty(X;H)$. By using Theorem 2.7 we shall look for the solution of the initial value problem (5.1) in the form

$$u(x,t) = u_0(x,t) + tu_1(x,t), \text{ in } X \times [0,T), \tag{5.2}$$

where u_0, u_1 belong to $C^\infty([0,T);D'(X;H))$ such that $Lu_0 = 0$, $Lu_1 = 0$ (that is u_0, u_1 are caloric distributions). By Lemma 2.3 and (5.2) we get

$$Lu(x,t) = u_1(x,t), \text{ in } X \times [0,T). \tag{5.3}$$

Moreover, by initial conditions in problem (5.1), we derive

$$Lu(x,t)|_{t=0} = u_1(x,t)|_{t=0} = v_1(x) \text{ in } X. \tag{5.4}$$

So, by (5.3), (5.4) we deduce that u_1 is a solution of the initial value problem.

$$\begin{cases} Lu = 0, \text{ in } X \times [0,T) \\ u(x,t)|_{t=0} = v_1(x), \text{ in } X. \end{cases} \tag{5.5}$$

Then, in accordance with the preceding reasonings we get

$$u_1(x,t) = U(t)v_1(x), \text{ in } X \times [0,T). \tag{5.6}$$

At the same time, by using (5.1), (5.2) we can deduce as well as in the preceding paragraphs that u_0 is a solution for the following Cauchy problem

$$\begin{cases} Lu = 0, \text{ in } X \times [0,T) \\ u(x,t)|_{t=0} = v_0(x), \text{ in } X. \end{cases} \tag{5.7}$$

So,

$$u_0(x,t) = U(t)v_0(x). \tag{5.8}$$

By availing ourselves of (5.6), (5.8) we reach the conclusion that the solution of the initial value problem (5.1) is given by

$$\begin{aligned} u(x,t) &= U(t)v_0(x) + tU(t)v_1(x) \\ &= U(t)(v_0(x) + tv_1(x)). \end{aligned} \tag{5.9}$$

Formula (5.9) can be thought as an extension for the Poisson formula. Let us remark that by preceding reasoning u given by (5.9) verify the equation in (5.1). Moreover it is easy to verify that u given by (5.9) satisfy the initial conditions in (5.1).

Remark 5.1. In particular, when $X = \mathbf{R}^n$, $A = \Delta_x$, the Laplace operator and v_0, v_1 are continuous functions in X, then we recover (4) in Chapter III, §1 in [4].

6. The second extension for the Poisson formula for *p*-caloric distributions

Let us look on the initial value problem:

$$\begin{cases} L^p u = 0, \text{ in } X \times [0,T) \\ u(x,t)|_{t=0} = v_0(x), \text{ in } X \\ L^k u(x,t)|_{t=0} = v_k(x), \text{ in } X, \text{for all} \\ \qquad\qquad 1 \le k \le p-1, \ p \ge 2, \end{cases} \tag{6.1}$$

where $L = \dfrac{\partial}{\partial t} - A(t)$ is the generalized heat operator, v_0, v_k, $1 \le k \le p-1$, $p \ge 2$ are distributions in X, valued in H and u is an unknown distribution in X, valued in H, depending smoothly on t in $[0,T)$. We look for a solution of initial value problem (6.1) in the form

$$u(x,t) = u_0(x,t) + tu_1(x,t) + \cdots + t^{p-1}u_{p-1}(x,t), \tag{6.2}$$

where u_j belongs to $C^\infty([0,T);D'(X;H))$, $Lu_j=0$, for all $1\le j\le p-1$, $p\ge 2$. By applying inductively the operator L on both sides of (6.2), we get

$$\begin{aligned} Lu &= u_1+2tu_2+\cdots+(p-1)t^{p-2}u_{p-1},\\ L^2u &= 2!u_2+3\cdot 2tu_3+\cdots+(p-1)(p-2)t^{p-3}u_{p-1},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ L^{p-2}u &= (p-2)!u_{p-2}+(p-1)!tu_{p-1},\\ L^{p-1}u &= (p-1)!u_{p-1}. \end{aligned} \tag{6.3}$$

By (6.1), (6.3) we get

$$u_{p-1}(x,t)|_{t=0}=\frac{1}{(p-1)!}v_{p-1}(x). \tag{6.4}$$

But $Lu_{p-1}=0$, in $X\times[0,T)$. So, u is a solution of the initial value problem

$$\begin{cases} Lu=0, \text{ in } X\in[0,T),\\ u(x,t)|_{t=0}=\dfrac{1}{(p-1)!}v_{p-1}(x), \text{ in } X. \end{cases} \tag{6.5}$$

Thus, u_{p-1} is given by

$$u_{p-1}(x,t)=\frac{1}{(p-1)!}U(t)v_{p-1}(x), \text{ in } X\times[0,T). \tag{6.6}$$

By (6.3), (6.6) and the fact that $Lu_{p-2}=0$, we deduce that u_{p-2} is a solution for the following Cauchy problem

$$\begin{cases} Lu=0, \text{ in } X\times[0,T),\\ u(x,t)|_{t=0}=\dfrac{1}{(p-2)!}v_{p-2}(x), \text{ in } X. \end{cases} \tag{6.7}$$

So,

$$u_{p-2}(x,t)=\frac{1}{(p-2)!}U(t)v_{p-2}(x), \text{ in } X\times[0,T). \tag{6.8}$$

In the same manner we generally get

$$u_{p-k-1}(x,t)=\frac{1}{(p-k-1)!}U(t)v_{p-k-1}(x) \text{ in } X\times[0,T), \tag{6.9}$$

for all $0\le k\le p-1$. By availing ourselves of (6.2) and (6.9) we reach the conclusion that the formal solution for the initial value (6.1) can be written as

$$u(x,t)=U(t)F(x,t), \text{ in } X\times[0,T), \tag{6.10}$$

where

$$F(x,t)=v_0(x)+\frac{t}{1!}v_1(x)+\cdots+\frac{t^{p-1}}{(p-1)!}v_{p-1}, \text{in } X\times[0,T). \tag{6.11}$$

By (6.10), (6.11) we get

$$u(x,t)|_{t=0}=U(t)F(x,t)|_{t=0}=v_0(x), \text{ in } X. \tag{6.12}$$

By using Lemma 2.3 and the definition of the parametrix of the generalized heat operator we get

$$L^k u(x,t) = U(t)F_k(x,t), \tag{6.13}$$

where

$$F_k(x,t) = v_k(x) + \frac{1}{1!}v_{k+1}(x) + \cdots + \frac{t^{p-k-1}}{(p-k-1)!}v_{p-1}(x), \text{ in } X \times [0,T). \tag{6.14}$$

Thus, by (6.13), we have

$$L^k u(x,t)|_{t=0} = v_k(x), \text{ in } X, \text{ for all } 0 \le k \le p-1. \tag{6.15}$$

So, by (6.2), (6.9), (6.12), (6.15) it follows that the distribution u is a solution of the initial value problem (6.1). The uniqueness of this solution follows from its construction, but it also can be verified directly.

Remark 6.1. When $X = \mathbf{R}^n$, $A(t) = \Delta_x$ is the Laplace operator and $v_l \in C^0(X)$, $0 \le l \le p-1$ are continuous functions in X, then we recover (8) in Chapter III, §2 in [4], as particular case.

7. The generalized iterated heat operator

In this paragraph we shall define the generalized iterated heat operator of an arbitrary order $\lambda \in \mathbf{C}$, $\operatorname{Re}\lambda < 0$, in the case $X = \mathbf{R}^n$ by using a similar idea as in [3] (see also [5]). To this end we shall use the kernel distribution of its parametrix.

Let us remember that if $L = \dfrac{\partial}{\partial t} - A(t)$ is the generalized heat operator and $U(t)$ is its parametrix, then

$$\frac{dU}{dt}(t) - A(t)U(t) = 0, \text{ in } X \times [0,T), \tag{7.1}$$

$$U(t)|_{t=0} = I, \text{ in } X, \tag{7.2}$$

modulo $\Psi^{-\infty}(X \times [0,T); L(H))$, respectively $\Psi^{-\infty}(X; L(H))$, where $I : H \to H$ is the identity operator of H. Moreover, $A(\cdot) \in C^\infty([0,T); \dot{\Psi}^m(X; L(H)))$ and $U(\cdot) \in C^\infty([0,T); \dot{\Psi}^0(X; L(H)))$. By Theorem 1.1 in each local chart $(\Omega, x_1, \ldots, x_n)$ of X we can write

$$\begin{aligned} U_\Omega(t)u(x) &= (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} \mathcal{U}_\Omega(x,t,\xi)\hat{u}(\xi)d\xi \\ &= \int_\Omega K(x,y,t)u(y)dy, \ u \in C_0^\infty(\Omega; H), \end{aligned} \tag{7.3}$$

where $\mathcal{U}_\Omega \in C^\infty([0,T)); S^{-\infty}(\Omega \times \mathbf{R}^n; L(H))$, $U_\Omega \in C^\infty([0,T); \Psi^{-\infty}(\Omega; L(H)))$, $K \in C^\infty(\Omega \times \Omega \times [0,T); L(H))$ (that is K is a function type distribution) and

$$K(x,y,t) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x-y,\xi\rangle} \mathcal{U}_\Omega(x,t,\xi)d\xi \tag{7.4}$$

represents the Schwartz kernel distribution of pseudodifferential operator $U_\Omega(t)$, $t \in [0,T)$. By (7.1) and (7.3) we get

$$\begin{gathered}\frac{\partial K}{\partial t} - A(t)K = 0, \text{ in } X \times X \times [0,T), \\ \text{modulo } C^\infty(X \times X \times [0,T); H).\end{gathered} \tag{7.5}$$

$$\int_{\mathbf{R}^n} K(x,y,t)u(y)dy|_{t=0} = u(x), \quad u \in C_0^\infty(X;H). \tag{7.6}$$

We want to associate to the Schwartz kernel distribution of the parametrix of the generalized heat operator a family of distributions, depending of a complex parameter λ in $\mathbf{C}$ such that if we take $\lambda = -k$, k a positive integer, then we get the iterated of the operator L.

To this end let us consider the generalized function

$$t_+^{\lambda-1} = \begin{cases} t^{\lambda-1}, \ t > 0 \\ 0, \ t \le 0, \ \lambda \in \mathbf{C}. \end{cases}$$

Let also $\delta(x,t)$ be the Dirac distribution in $\mathbf{R}^{n+1}$ and let $\Gamma(\lambda)$ be the Euler function of second kind.

Then we can prove the following theorem.

Theorem 7.1. *Let the generalized function*

$$L_\lambda(x,y,t) = \frac{t_+^{\lambda-1}}{\Gamma(\lambda)} K(x,y,t),$$

where we assume that $K = 0$, when $t \notin [0,T)$ and y is considered such as a parameter. Then, $L_\lambda(t,x,y)$ is an entire function of λ and for $\lambda = -k$, $k \in \mathbf{N}$, the support of this distribution is concentrated in $(0,y)$. Moreover, the following relations are true

$$L_\lambda(x,y,t)|_{\lambda=-k} = L^k \delta(x-y,t), \tag{7.7}$$

for any nonnegative integer k, where $L^0 = I$, $L^k = LL^{k-1}$, for all $k \ge 1$.

Proof. The idea of proof is to consider the generalized function $f_\lambda(x,y,t) = t_+^{\lambda-1}K(x,y,t)$ defined by the relation

$$\langle f_\lambda, \varphi\rangle = \langle t_+^{\lambda-1}K(x,y,t), \varphi(x,t)\rangle = \langle t_+^{\lambda-1}, \psi(t)\rangle, \tag{7.8}$$

where

$$\psi(t) = \langle K(x,y,t), \varphi(x,t)\rangle, \quad \varphi \in C_0^\infty(X \times \mathbf{R}; H) \tag{7.9}$$

and to prove that

$$\psi^{(k)}(0) = (-1)^k \langle L^k \delta(x-y,t), \ \varphi(x,t)\rangle \tag{7.10}$$

for all $k \ge 0$.

In order to prove (7.10) let us remark that the function type distribution $K(x,y,t)$ with respect to (x,t) (y is considered such as a parameter) satisfies the following relation

$$\int_{\mathbf{R}^n} K(x,y,t)u(x)dx\Big|_{t=0} = u(y),\ u \in C_0^\infty(X;H), \tag{7.11}$$

which is a consequence of the fact that $\mathcal{U}_\Omega(x,0,\xi) = I$, where $I : H \to H$ is the identity operators of H. Now, by an easy calculation and using (7.9) we obtain

$$\begin{aligned}\psi'(t) &= \left\langle \frac{\partial K}{\partial t}(x,y,t), \varphi(x,t)\right\rangle + \left\langle K(x,y,t), \frac{\partial \varphi}{\partial t}(x,t)\right\rangle \\ &= \langle A(t)K(x,y,t), \varphi(x,t)\rangle + \left\langle K(x,y,t), \frac{\partial \varphi}{\partial t}(x,t)\right\rangle \\ &= \left\langle K(x,y,t), \left[\frac{\partial}{\partial t} + A^*(t)\right]\varphi(x,t)\right\rangle,\end{aligned} \tag{7.12}$$

where $A^*(t)$ is the adjoint of the operator $A(t)$ as an $L(H)$-valued pseudodifferential operator in X.

So, by induction we see easily that

$$\text{(7.13)}\ \ \psi^{(k)}(t) = \left\langle K(x,y,t), \left[\frac{\partial}{\partial t} + A^*(t)\right]^k \varphi(x,t)\right\rangle,\ k \in \mathbf{N}. \tag{7.13}$$

Thus, by (7.11), (7.13) we get

$$\begin{aligned}\psi^{(k)}(0) = \left[\frac{\partial}{\partial t} + A^*(t)\right]^k \varphi(y,t)\Big|_{t=0} &= \left\langle \delta(x-y,t), \left[\frac{\partial}{\partial t} + A^*(t)\right]^k \varphi(x,t)\right\rangle \\ = (-1)^k \left\langle \left[\frac{\partial}{\partial t} - A(t)\right]^k \delta(x-y,t), \varphi(x,t)\right\rangle \\ = (-1)^k \langle L^k \delta(x-y,t), \varphi(x,t)\rangle,\end{aligned} \tag{7.14}$$

for all $k \in \mathbf{N}$.

Then, by (7.8), (7.10) and using the fact that $t_+^{\lambda-1}$ and $\Gamma(\lambda)$ are meromorphic functions of λ in $\mathbf{C}$ which have simple poles in $\lambda_k = -k$, for all nonnegative integers k with the residue $\frac{(-1)^k}{k!}\delta^{(k)}$ respectively $\frac{(-1)^k}{k!}$ (see [2]), the conclusion of the Theorem 7.1 follows. □

Now, let us consider the distribution

$$u_\lambda(x,t) = L_\lambda(x,y,t) * u(x,t), \tag{7.15}$$

where $u(x,t) \in D'(X \times \mathbf{R}; H)$ is a suitable chosen distribution such that the convolution in (7.15) is defined. Then the following relation

$$u_\lambda(x,t) = f(x,t), \quad f \in D'(X \times [0,T); H) \tag{7.16}$$

coincide when we take $\lambda = -k$, $k \geq 1$ with the equation

$$L^k u = f. \tag{7.17}$$

Indeed, by (7.7), (7.15) and (7.16) in which we take $y = 0$ and $\lambda = -k$ respectively we get

$$\begin{aligned} u_{-k}(x,t) &= L_{-k}(x,t) * u(x,t) = L^k \delta(x,t) * u(x,t) \\ &= L^k u(x,t) = f(x,t). \end{aligned}$$

Thus, the equation (7.16) can be used to define the iterated of an arbitrary order λ in $\mathbf{C}$ with $\operatorname{Re}\lambda < 0$ of the generalized heat operator $L = \dfrac{\partial}{\partial t} - A(t)$.

In the following let us recall that Trèves also has considered in his book [8] the initial value problem

$$\frac{dU}{dt}(t) - A(t)U(t) = 0, \quad \text{in} \quad X \times [t', T) \tag{7.18}$$

$$U(t)|_{t=t'} = I, \quad \text{in} \quad X, \tag{7.19}$$

where t' is any number such that $0 \leq t' < T$. The equality in (7.18) it means congruence modulo $C^\infty([t',T);\, \Psi^{-\infty}(X; L(H)))$ whereas the equality in (7.19) means congruence modulo $\Psi^{-\infty}(X; L(H))$. I is the identity operator of H.

Trèves has showed that a solution $U(t,t')$ of (7.18)–(7.19) is an equivalence class, which has a representation in any local chart $(\Omega, x_1, \ldots, x_n)$ in X which is equivalent to an operator $U_\Omega(t,t')$ defined by

$$U_\Omega(t,t')u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} \mathcal{U}_\Omega(x,t,t',\xi)\hat{u}(\xi)d\xi,$$

where $u \in C_0^\infty(\Omega; H)$ and

$$\mathcal{U}_\Omega(x,t,t',\xi) = (2\pi i)^{-1} \oint_\gamma e^{(t-t')\rho z} h_\Omega(x,t,\xi,z)dz,$$

$$\rho = \rho(\xi) = (1 + |\xi|^2)^{m/2},\ k_\Omega \in S^0(\Omega \times [0,T]; L(H)).$$

Here k_Ω is a suitable formal symbol of degree zero, valued in $L(H)$ depending holomorphically on the complex variable z is an open neighborhood of the integration contour γ provided (x,t) ranges in a compact subset of $\Omega \times [0,T)$.

Let us remark that we can write

$$U_\Omega(t,t')u(x) = \int_\Omega K(x,y,t,t')u(y)dy, \quad u \in C_0^\infty(\Omega; H),$$

where

$$K(x,y,t,t') = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x-y,\xi\rangle} \mathcal{U}_\Omega(x,t,t',\xi)d\xi.$$

By (7.18) and (7.19), we get

$$\frac{\partial K}{\partial t} - A(t)K = 0, \quad \text{in} \quad X \times [t', T)$$

$$\left.\int_{\mathbf{R}^n} K(x,y,t,t')u(y)dy\right|_{t=t'} = u(x), \quad u \in C_0^\infty(X; H).$$

Then we can state and prove by analogy with Theorem 7.1 the following theorem.

Theorem 7.2. *The generalized function*

$$L_\lambda(x,y,t,t') = \frac{(t-t')_+^{\lambda-1}}{\Gamma(\lambda)} K(x,y,t,t'),$$

where $K = 0$ for $t \notin [t', T)$ and where (y, t') are considered such as parameters, is an entire function of λ and for $\lambda = -k$, $k \in \mathbf{N}$ the support of this distribution is concentrated in (y, t'). In addition to the following relations are valid

$$L_\lambda(x,y,t,t')|_{\lambda=-k} = L^k \delta(x-y,t,t'), \quad k \in \mathbf{N},$$

where L^k denote the iterations of order k of the operator L.

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Smoothing Effect and Fredholm Property for First-order Hyperbolic PDEs

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Abstract. We give an exposition of recent results on regularity and Fredholm properties for first-order one-dimensional hyperbolic PDEs. We show that large classes of boundary operators cause an effect that smoothness increases with time. This property is the key in finding regularizers (parametrices) for hyperbolic problems. We construct regularizers for periodic problems for dissipative first-order linear hyperbolic PDEs and show that these problems are modeled by Fredholm operators of index zero.

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1. Introduction

In contrast to ODEs and parabolic PDEs, the Fredholm property and regularity behavior of hyperbolic problems are much less understood. In a recent series of papers [18, 20, 21], the latter two written jointly with Lutz Recke, we undertook a detailed analysis of this subject for first-order one-dimensional hyperbolic operators. The purpose of the present survey paper is to present some of our results and their extensions with emphasize on the smoothing phenomenon, construction of parametrices, and the Fredholmness of index zero.

An important step in local investigations of nonlinear differential equations (many ODEs and parabolic PDEs) is to establish the Fredholm solvability of their linearized versions. In the hyperbolic case this step is much more involved. Since the singularities of (semi-)linear hyperbolic equations propagate along characteristic curves, a solution cannot be more regular in the entire time-space domain than it is on the boundary. It can even be less regular which is known as the *loss-of-smoothness* effect. Therefore the Fredholm analysis of hyperbolic problems requires establishing an optimal regularity relation between the spaces of solutions and right-hand sides of the differential equations.

Proving a Fredholm solvability is typically based on the basic fact that any Fredholm operator is exactly a compact perturbation of a bijective operator. In the hyperbolic case, using the compactness argument gets complicated because of the lack of regularity over the *whole* time-space domain.

Our approach is based on the fact that for a range of boundary operators, solutions improve smoothness *dynamically*, more precisely, they eventually become k-times continuously differentiable for each particular k. We prove such kind of results in Section 2. Note that in some interesting cases the smoothing phenomenon was shown earlier in [10, 13, 23, 25].

This phenomenon allows us in Section 3 to work out a regularization procedure via construction of a parametrix. We here present a quite general approach to proving the Fredholmness for first-order dissipative hyperbolic PDEs and apply it to the periodic problems. Our Fredholm results cover non-strictly hyperbolic systems with discontinuous coefficients, but they are new even in the case of strict hyperbolicity and smooth coefficients.

From a more general perspective, the smoothing effect and Fredholmness properties play an important role in the study of the Hopf bifurcation and periodic synchronizations in nonlinear hyperbolic PDEs [2, 22] via the Implicit Function Theorem and Lyapunov–Schmidt procedure [7, 15] and averaging procedure [6, 32].

From the practical point of view, our techniques cover the so-called traveling-wave models from laser dynamics [24, 30] (describing the appearance of self-pulsations of lasers and modulation of stationary laser states by time periodic electric pumping), population dynamics [9, 14, 36], and chemical kinetics [3, 4, 5, 38] (describing mass transition in terms of convective diffusion and chemical reaction and analysis of chemical processes in counterflow chemical reactors).

2. Smoothing effect

Here we describe some classes of (initial-)boundary problems for first-order one-dimensional hyperbolic PDEs whose solutions improve their regularity in time.

Set

$$\Pi_T = \{(x,t) \,:\, 0 < x < 1, T < t < \infty\}.$$

We address the problem

$$(\partial_t + a(x,t)\partial_x + b(x,t))u = f(x,t), \tag{2.1}$$

$$u(x,0) = \varphi(x), \tag{2.2}$$

$$\begin{aligned} u_j(0,t) &= (Ru)_j(t), \quad 1 \le j \le m \\ u_j(1,t) &= (Ru)_j(t), \quad m < j \le n \end{aligned} \tag{2.3}$$

in the semi-strip Π_0 and the problem (2.1), (2.3) in the strip $\Pi_{-\infty}$. Here $u = (u_1,\dots,u_n)$, $f = (f_1,\dots,f_n)$, and $\varphi = (\varphi_1,\dots,\varphi_n)$ are vectors of real-valued functions, $b = \{b_{jk}\}_{j,k=1}^n$ and $a = \operatorname{diag}(a_1,\dots,a_n)$ are matrices of real-valued functions, and $0 \le m \le n$ are fixed integers. Furthermore, R is an operator mapping $C\left(\overline{\Pi}_0\right)^n$ into $C\left([0,\infty)\right)^n$, and similarly for R in $\Pi_{-\infty}$. In Sections 2.1–2.3 we give examples of R as representatives of some classes of boundary operators ensuring smoothing solutions.

In the domain under consideration we assume that

$$a_j > 0 \text{ for all } j \le m \quad \text{and} \quad a_j < 0 \text{ for all } j > m, \tag{2.4}$$

$$\inf_{x,t} |a_j| > 0 \text{ for all } j \le n, \tag{2.5}$$

and

$$\begin{aligned} &\text{for all } 1 \le j \ne k \le n \text{ there exists } p_{jk} \in C^1([0,1]\times\mathbb{R}) \\ &\text{such that } b_{jk} = p_{jk}(a_k - a_j). \end{aligned} \tag{2.6}$$

Note that all these conditions are not restrictive neither from the practical nor from the theoretical points of view. In particular, condition (2.4) is true in traveling-wave models of laser and population dynamics as well as chemical kinetics, where the functions u_j for $j \le m$ (respectively, $m+1 \le j \le n$) describe "species" traveling to the right (respectively, to the left). Condition (2.5) means that all characteristics of the system (2.1) are bounded and the system (2.1) is, hence, non-degenerate. Finally, the condition (2.6) is a kind of Levy condition usually appearing to compensate non-strict hyperbolicity where the coefficients a_j and a_k for some $j \ne k$ coincide at least at one point, say, (x_0, t_0). In this case the lower-order terms with the coefficients b_{jk} and b_{kj} contribute to the system at (x_0, t_0) longitudinally to characteristic directions (keeping responsibility for the propagation of singularities), while in the strictly hyperbolic case we have a qualitatively different transverse contribution at that point. The purpose of (2.6) is to suppress propagation of singularities through the non-diagonal lower-order terms of (2.1).

We will impose the following smoothness assumptions on the initial data: The entries of a, b, and f are C^∞-smooth in all their arguments in the respective domains, while the entries of φ are assumed to be continuous functions only.

Let us introduce the system resulting from (2.1)–(2.3) (resp., from (2.1), (2.3)) via integration along characteristic curves. For given $j \le n$, $x \in [0,1]$, and $t \in \mathbb{R}$, the jth characteristic of (2.1) passing through the point (x,t) is defined as the solution $\xi \in [0,1] \mapsto \omega_j(\xi; x, t) \in \mathbb{R}$ of the initial value problem

$$\partial_\xi \omega_j(\xi; x, t) = \frac{1}{a_j(\xi, \omega_j(\xi; x, t))}, \quad \omega_j(x; x, t) = t. \tag{2.7}$$

Define

$$c_j(\xi, x, t) = \exp \int_x^\xi \left(\frac{b_{jj}}{a_j}\right)(\eta, \omega_j(\eta; x, t))\, d\eta, \quad d_j(\xi, x, t) = \frac{c_j(\xi, x, t)}{a_j(\xi, \omega_j(\xi; x, t))}.$$

Due to (2.5), the characteristic curve $\tau = \omega_j(\xi; x, t)$ reaches the boundary of Π_T in two points with distinct ordinates. Let $x_j(x,t)$ denote the abscissa of that point whose ordinate is smaller. Straightforward calculations show that a C^1-map

$u : [0,1] \times [0,\infty) \to \mathbb{R}^n$ is a solution to (2.1)–(2.3) if and only if it satisfies the following system of integral equations

$$\begin{aligned} u_j(x,t) = (BSu)_j(x,t) \\ &- \int_{x_j(x,t)}^{x} d_j(\xi,x,t) \sum_{\substack{k=1 \\ k\neq j}}^{n} b_{jk}(\xi,\omega_j(\xi;x,t)) u_k(\xi,\omega_j(\xi;x,t)) d\xi \\ &+ \int_{x_j(x,t)}^{x} d_j(\xi,x,t) f_j(\xi,\omega_j(\xi;x,t)) d\xi, \quad j \le n, \end{aligned} \tag{2.8}$$

where

$$(Bu)_j(x,t) = c_j(x_j(x,t),x,t) u_j\left(x_j(x,t), \omega_j(x_j(x,t);x,t)\right), \tag{2.9}$$

$$(Su)_j(x,t) = \begin{cases} (Ru)_j(t) & \text{if } t > 0, \\ \varphi_j(x) & \text{if } t = 0. \end{cases} \tag{2.10}$$

Here B is a shifting operator from $\partial\Pi_0$ along characteristic curves of (2.1), while the operator S is used to denote the boundary operator on the whole $\partial\Pi_0$. Similarly, a C^1-map $u : [0,1] \times \mathbb{R} \to \mathbb{R}^n$ is a solution to (2.1), (2.3) if and only if it satisfies the system (2.8), where the definition of S is changed to $S = R$.

This motivates the following definition:

Definition 2.1.

(1) A continuous function u is called a continuous solution to (2.1)–(2.3) in $\overline{\Pi}_0$ if it satisfies (2.8) with S defined by (2.10).

(2) A continuous function u is called a continuous solution to (2.1), (2.3) in $\overline{\Pi}_{-\infty}$ if it satisfies (2.8) with $S = R$.

Existence results for (continuous) solutions to the problems under consideration are obtained in [1, 16, 17, 19].

Definition 2.2. A solution u to the problem (2.1)–(2.3) or (2.1), (2.3) is called *smoothing* if, for every $k \in \mathbb{N}$, there exists $T > 0$ such that $u_j \in C^k\left(\overline{\Pi}_T\right)$ for all $j \le n$.

For the initial-boundary value problem (2.1)–(2.3) Definition 2.2 reflects a dynamic nature of the smoothing property stating that the regularity of solutions increases in time. The fact that the regularity cannot be uniform in the entire domain is a straightforward consequence of the propagation of singularities along characteristic curves. Moreover, switching from C^k to C^{k+1}-regularity is jump-like; this phenomenon is usually observed in the situations when solutions of hyperbolic PDEs change their regularity (see, e.g., [25, 27, 29, 31]).

Note that, if the problem (2.1), (2.3) is subjected to periodic conditions in t, then Definition 2.2 implies that the smoothing solutions immediately meet the C^∞-regularity in the entire domain.

Definition 2.2 captures the general nature of the smoothing phenomenon for hyperbolic PDEs. A more precise information can be extracted from the proof of

Theorems 2.3, 2.4, and 2.5 below: Reaching the C^k-regularity for solutions needs only a C^k-regularity for a, b, and f. More exact regularity conditions for the boundary data, which also depend on k, can be derived from these proofs as well. These refinements are useful in some applications.

Definition 2.2 can be strengthened by admitting worse regularities for the initial data. One extension of this kind, when the initial data are strongly singular distributions concentrated at a finite number of points, can be found in [18]. In [18] we used a delta-wave solution concept. Another result in this direction [20, 21] concerns periodic problems and uses a variational setting of the problem (see also Theorem 3.2 (ii)). In [20, 21] we get an improvement of the solution regularity from being functionals to being functions.

In what follows we demonstrate the smoothing effect on generic examples of large classes of boundary operators and show which kinds of problems can be covered by our techniques. Our approach to establishing smoothing results is based on the consideration of the integral representation of the problems and the observation that the boundary and the integral parts of this representation have different influence on the regularity of solutions. Our main idea is to show that the integral part has a “self-improvement” property, while in many interesting cases the boundary part is not responsible for propagation of singularities. The latter contrasts to the case of the Cauchy problem where the solutions cannot be smoothing as the boundary term all the time “remembers” the regularity of the initial data. It is worthy to note that in the case of the problem (2.1)–(2.3) in Π_0 the domain of influence of the initial conditions is determined by both parts of the integral system and is in general infinite. This makes the smoothing effect non-obvious.

2.1. Classical boundary conditions

Here we specify conditions (2.3) to

$$\begin{aligned} u_j(0,t) = h_j(t), \quad 1 \le j \le m, \\ u_j(1,t) = h_j(t), \quad m < j \le n. \end{aligned} \tag{2.11}$$

and consider the problem (2.1), (2.2), (2.11).

Theorem 2.3. *Assume that the data a_j, b_{jk}, f_j, and h_j are smooth in all their arguments and φ_j are continuous functions. Assume also* (2.4), (2.5), *and* (2.6). *Then any continuous solution to the problem* (2.1), (2.2), (2.11) *is smoothing.*

Note that in the case of smooth classical boundary conditions (2.11), the domain of influence of the initial data $\varphi(x)$ on u_i for every $i \le n$ in general is unbounded (due to the lower-order terms in (2.1)). In spite of this, the influence of the initial data on the regularity of u becomes weaker and weaker in time causing the smoothing effect.

Proof. Suppose that u is a continuous solution to the problem (2.1)–(2.3) and show that the operator of the problem improves the regularity of u in time. The idea of the proof is similar to [18].

We start with an operator representation of u. To this end, introduce linear bounded operators $D, F : C\left(\overline{\Pi}_0\right)^n \to C\left(\overline{\Pi}_0\right)^n$ by

$$(Du)_j(x,t) = -\int_{x_j(x,t)}^{x} d_j(\xi,x,t) \sum_{\substack{k=1\\k\neq j}}^{n} b_{jk}(\xi,\omega_j(\xi;x,t))u_k(\xi,\omega_j(\xi;x,t))d\xi,$$

$$(Ff)_j(x,t) = \int_{x_j(x,t)}^{x} d_j(\xi,x,t) f_j(\xi,\omega_j(\xi;x,t))d\xi.$$

Note that Ff is a smooth function in x, t. In this notation the integral system (2.8) can be written as

$$u = BSu + Du + Ff. \tag{2.12}$$

It follows that

$$u = BSu + (DBS + D^2)u + (I + D)Ff. \tag{2.13}$$

In the first step we prove that the right-hand side of (2.13) restricted to $\overline{\Pi}_{T_1}$ for some $T_1 > 0$ is continuously differentiable in t. The $C^1\left(\overline{\Pi}_{T_1}\right)^n$-regularity of u will then follow from the fact that u given by (2.8) satisfies (2.1) in the distributional sense. By the assumption (2.5), we can fix a large enough $T_1 > 0$ such that the operator S in the right-hand side of (2.13) restricted to $\overline{\Pi}_{T_1}$ does not depend on φ and, hence, $Su = Ru = h$, where $h = (h_1, \dots, h_n)$. We therefore arrive at the equality

$$u|_{\overline{\Pi}_{T_1}} = Bh + DBh + D^2u + (I + D)Ff, \tag{2.14}$$

where $u|_{\overline{\Pi}_{T_1}}$ denotes the restriction of u to $\overline{\Pi}_{T_1}$. By the regularity assumption on a, b, f, and h, the function $Bh + DBh + (I + D)Ff$ is smooth. We have reduced the problem to show that the operator D^2 is smoothing, more specifically, that D^2u is C^1-smooth in t on $\overline{\Pi}_{T_1}$.

Notice that for $t \geq T_1$ the function $x_j(x,t)$ is a constant depending only on j. Below we therefore will drop the dependence of x_j on x and t. Fix a sequence $u^l \in C^1\left(\overline{\Pi}_0\right)^n$ such that

$$u^l \to u \text{ in } C\left(\overline{\Pi}_0\right)^n \text{ as } l \to \infty. \tag{2.15}$$

By convergence in $C(\Omega)^n$ here and below we mean the uniform convergence on any compact subset of Ω. Then $D^2u^l \to D^2u$ in $C\left(\overline{\Pi}_0\right)^n$ as well. It suffices to prove that $\partial_t\left[D^2u^l\right]$ converges in $C\left(\overline{\Pi}_{T_1}\right)^n$ as $l \to \infty$. Given $j \leq n$, consider the following expression for $\left(D^2u^l\right)_j(x,t)$, obtained by change of the order of integration:

$$\begin{aligned}&\left(D^2u^l\right)_j(x,t) \\ &= \sum_{\substack{k=1\\k\neq j}}^{n} \sum_{\substack{i=1\\i\neq k}}^{n} \int_{x_j}^{x}\int_{\eta}^{x} d_{jki}(\xi,\eta,x,t) b_{jk}(\xi,\omega_j(\xi;x,t)) u_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t)))d\xi d\eta\end{aligned} \tag{2.16}$$

with

$$d_{jki}(\xi,\eta,x,t) = d_j(\xi,x,t)d_k(\eta,\xi,\omega_j(\xi;x,t))b_{ki}(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t))).$$

It follows that

$$\begin{aligned}
&\partial_t \left[\left(D^2 u^l\right)_j (x,t)\right] \\
&= \sum_{\substack{k=1\\k\neq j}}^{n} \sum_{\substack{i=1\\i\neq k}}^{n} \int_{x_j}^{x}\int_{\eta}^{x} \partial_t \left[d_{jki}(\xi,\eta,x,t) b_{jk}(\xi,\omega_j(\xi;x,t))\right] u_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t)))d\xi d\eta \\
&\quad + \sum_{\substack{k=1\\k\neq j}}^{n} \sum_{\substack{i=1\\i\neq k}}^{n} \int_{x_j}^{x}\int_{\eta}^{x} d_{jki}(\xi,\eta,x,t) b_{jk}(\xi,\omega_j(\xi;x,t)) \qquad (2.17) \\
&\quad \times \partial_3\omega_k(\eta;\xi,\omega_j(\xi;x,t))\partial_t\omega_j(\xi;x,t)\partial_2 u_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t)))d\xi d\eta,
\end{aligned}$$

where $\partial_r g$ here and below denotes the derivative of g with respect to the rth argument. The first summand in the right-hand side converges in $C\left(\overline{\Pi}_{T_1}\right)$. Our task is therefore reduced to show the uniform convergence of all integrals in the second summand, whenever (x,t) varies on a compact subset of $\overline{\Pi}_{T_1}$. For this purpose we will transform the integrals as follows. Using (2.6) and the formulas

$$\partial_x\omega_j(\xi;x,t) = -\frac{1}{a_j(x,t)} \exp\int_{\xi}^{x} \left(\frac{\partial_t a_j}{a_j^2}\right)(\eta,\omega_j(\eta;x,t))d\eta, \qquad (2.18)$$

$$\partial_t\omega_j(\xi;x,t) = \exp\int_{\xi}^{x} \left(\frac{\partial_t a_j}{a_j^2}\right)(\eta,\omega_j(\eta;x,t))d\eta, \qquad (2.19)$$

we get

$$\begin{aligned}
&\int_{x_j}^{x}\int_{\eta}^{x} d_{jki}(\xi,\eta,x,t) b_{jk}(\xi,\omega_j(\xi;x,t)) \\
&\quad \times \partial_3\omega_k(\eta;\xi,\omega_j(\xi;x,t))\partial_t\omega_j(\xi;x,t)\partial_2 u_i^l(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t)))d\xi d\eta \\
&= \int_{x_j}^{x}\int_{\eta}^{x} d_{jki}(\xi,\eta,x,t)\partial_3\omega_k(\eta;\xi,\omega_j(\xi;x,t))\partial_t\omega_j(\xi;x,t) \\
&\quad \times b_{jk}(\xi,\omega_j(\xi;x,t))\left[\left(\partial_\xi\omega_k\right)(\eta;\xi,\omega_j(\xi;x,t))\right]^{-1}\left(\partial_\xi u_i^l\right)(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t)))d\xi d\eta \\
&= \int_{x_j}^{x}\int_{\eta}^{x} d_{jki}(\xi,\eta,x,t)\partial_t\omega_j(\xi;x,t)\left(a_k a_j p_{jk}\right)(\xi,\omega_j(\xi;x,t)) \\
&\quad \times \left(\partial_\xi u_i^l\right)(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t)))d\xi d\eta \\
&= \int_{x_j}^{x}\int_{\eta}^{x} \tilde{d}_{jki}(\xi,\eta,x,t)\left(\partial_\xi u_i^l\right)(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t)))d\xi d\eta \\
&= -\int_{x_j}^{x}\int_{\eta}^{x} \partial_\xi\tilde{d}_{jki}(\xi,\eta,x,t) u_i^l\left(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t))\right)d\xi d\eta \\
&\quad + \int_{x_j}^{x}\left[\tilde{d}_{jki}(\xi,\eta,x,t) u_i^l\left(\eta,\omega_k(\eta;\xi,\omega_j(\xi;x,t))\right)\right]_{\xi=\eta}^{\xi=x} d\eta.
\end{aligned}$$

Here

$$\tilde{d}_{jki}(\xi,\eta,x,t) = d_{jki}(\xi,\eta,x,t)\partial_t\omega_j(\xi;x,t)\left(a_k a_j p_{jk}\right)(\xi,\omega_j(\xi;x,t)).$$

Now, the desired convergence follows from (2.15).

In the second step we prove that there exists $T_2 > T_1$ such that $\partial_t u$ restricted to $\overline{\Pi}_{T_2}$ is C^1-smooth in t on $\overline{\Pi}_{T_2}$. Once this is done, we differentiate (2.1) with respect to t and get $\partial^2_{xt}u \in C\left(\overline{\Pi}_{T_2}\right)^n$; differentiating (2.1) with respect to x, we get $\partial^2_x u \in C\left(\overline{\Pi}_{T_2}\right)^n$. We will be able to conclude that $u \in C^2\left(\overline{\Pi}_{T_2}\right)^n$, as desired. To prove the existence of T_2, let $v = \partial_t u$. Differentiation of (2.1) formally in t leads to

$$(\partial_t + a_j\partial_x)v_j + \sum_{k=1}^n b_{jk}v_k + \sum_{k=1}^n \partial_t b_{jk}u_k + \partial_t a_j \partial_x u_j = \partial_t f_j.$$

Combining this with (2.1), we obtain

$$\begin{aligned}(\partial_t + a_j\partial_x)v_j &+ \sum_{k=1}^n b_{jk}v_k - \frac{\partial_t a_j}{a_j}v_j \\ &= \partial_t f_j - \sum_{k=1}^n \partial_t b_{jk}u_k + \frac{\partial_t a_j}{a_j}\left(\sum_{k=1}^n b_{jk}u_k - f_j\right) = G_j(f_j,\partial_t f_j,u).\end{aligned} \tag{2.20}$$

Here, for each $j \le n$, G_j is a certain linear function with smooth coefficients. Set

$$\tilde{c}_j(\xi,x,t) = \exp\int_x^\xi \left(\frac{b_{jj}}{a_j} - \frac{\partial_t a_j}{a_j^2}\right)(\eta,\omega_j(\eta;x,t))\,d\eta,\quad \tilde{d}_j(\xi,x,t) = \frac{\tilde{c}_j(\xi,x,t)}{a_j(\xi,\omega_j(\xi;x,t))}$$

and introduce three linear operators $\tilde{B},\tilde{D},\tilde{F} : C\left(\overline{\Pi}_0\right)^n \to C\left(\overline{\Pi}_0\right)^n$ by

$$\left(\tilde{B}u\right)_j(x,t) = \tilde{c}_j(x_j,x,t)u_j\left(x_j,\omega_j(x_j;x,t)\right),$$

$$\left(\tilde{D}u\right)_j(x,t) = -\int_{x_j}^x \tilde{d}_j(\xi,x,t)\sum_{\substack{k=1\\k\ne j}}^n b_{jk}(\xi,\omega_j(\xi;x,t))u_k(\xi,\omega_j(\xi;x,t))d\xi,$$

$$\left(\tilde{F}f\right)_j(x,t) = \int_{x_j}^x \tilde{d}_j(\xi,x,t)f_j(\xi,\omega_j(\xi;x,t))d\xi.$$

Similarly to the above, our starting point is that for any $T_2 \ge T_1$ the function v satisfies the following operator equation resulting from (2.20):

$$v|_{\overline{\Pi}_{T_2}} = \tilde{B}h' + \tilde{D}v + \tilde{F}G(f,\partial_t f,u),$$

and, hence, the equation

$$v|_{\overline{\Pi}_{T_2}} = \tilde{B}h' + \tilde{D}\tilde{B}h' + \tilde{D}^2 v + (I+\tilde{D})\tilde{F}G(f,\partial_t f,u), \tag{2.21}$$

where $G = (G_1,\dots,G_n)$ and $h' = (h'_1,\dots,h'_n)$. Again, due to the assumption (2.5), we can fix $T_2 > T_1$ such that the right-hand side of (2.21) does not depend on u and v in $\overline{\Pi}\setminus\Pi_{T_1}$. Due to Step 1, the function $(I+\tilde{D})\tilde{F}G(f,\partial_t f,u)$ then meets the C_t^1-regularity. Moreover, $\tilde{B}h' + \tilde{D}\tilde{B}h' \in C^\infty$. We are thus left to show that the operator $\tilde{D}^2$ is smoothing in the above sense. As $\tilde{D}$ is exactly the operator D with c_j and d_j replaced by the smooth functions $\tilde{c}_j$ and $\tilde{d}_j$, the desired smoothing property of $\tilde{D}^2$ follows from the proof of the smoothness of D^2 and the fact that $\tilde{D}^2 v$ in (2.21) does not depend on v in $\overline{\Pi} \setminus \Pi_{T_1}$.

Proceeding further by induction, assume that, given $r \geq 2$, there is $T_r > 0$ such that $u \in C^r\left(\overline{\Pi}_{T_r}\right)^n$ and prove that u meets the C^{r+1}-regularity in t on $\overline{\Pi}_{T_{r+1}}$ for some $T_{r+1} > T_r$. Set $w = \partial_t^r u$ and write our starting operator equation for w resulting from (2.1), (2.3) after r-times differentiation in t:

$$w|_{\overline{\Pi}_{T_{r+1}}} = \tilde{B}h^{(r)} + \tilde{D}\tilde{B}h^{(r)} + \tilde{D}^2 w + (I + \tilde{D})\tilde{F}\tilde{G}(f, \partial_t f, \ldots, \partial_t^r f, u, \partial_t u, \ldots, \partial_t^{r-1} u) \tag{2.22}$$

where $\tilde{G}$ is a vector of certain linear functions with smooth coefficients and the operators $\tilde{B}, \tilde{D}$, and $\tilde{F}$ are modified by $\tilde{c}_j(\xi, x, t)$ changing to

$$\tilde{c}_j(\xi, x, t) = \exp \int_x^\xi \left(\frac{b_{jj}}{a_j} - r\frac{\partial_t a_j}{a_j^2}\right)(\eta, \omega_j(\eta; x, t))\, d\eta.$$

Similarly to the above, fix $T_{r+1} > T_r$ such that the right-hand side of (2.22) does not depend on $u, \partial_t u, \ldots, \partial_t^{r-1} u$, and w in $\overline{\Pi} \setminus \Pi_{T_r}$. This ensures that the last two summands in (2.22) are C_t^1-functions. The first two summands are C_t^1-smooth by the regularity assumptions on the data. Finally, the $C^{r+1}\left(\overline{\Pi}_{T_{r+1}}\right)$-regularity of u follows from the previous steps of the proof and suitable differentiations of the system (2.1). □

Theorem 2.3 can be extended over the boundary operators of the following kind (both linear and nonlinear). Given $T > 0$, in the domain Π_T let us consider the problem (2.1)–(2.3) with $b_{jk} \equiv 0$ for all $j \neq k$ (i.e., the system (2.1) is decoupled) and with (2.2) replaced by $u(x, T) = \varphi(x)$ (the initial values are given at $t = T$). This entails that the domain of influence of φ now depends only on the boundary conditions. For the latter it is supposed that, for every $T > 0$ and $\varphi(x)$, the function $\varphi(x)$ has a bounded domain of influence on u. In other words, for any decoupled system (2.1), if $\varphi(x)$ has a singularity at some point $x \in [0, 1]$, then this singularity expands along a finite number of characteristic curves (we have a finite number of "reflections" from the boundary), and this number is bounded from above uniformly in $x \in [0, 1]$. This class of boundary operators is in detail described in [18], where the necessary and sufficient conditions for smoothing solutions are given. The results of [18] generalize the smoothing results obtained in [10, 13, 23, 25] for the system (2.1) with time-independent coefficients and (a kind of) Dirichlet boundary conditions.

2.2. Integral boundary conditions in age structured population models

Here we address another class of boundary operators admitting smoothing solutions. Though it covers a range of (partial) integral operators, we illustrate our smoothing result with an example from population dynamics.

Integral boundary conditions are usually used in continuous age structured population models to describe a fertility of populations. Let $u(x,t)$ denote the density of a population of age x at time t. Then the dynamics of u can be described by the following model (see, e.g., [11, 26, 36] and references therein):

$$(\partial_t + \partial_x + \mu)u = 0, \qquad (x,t) \in \overline{\Pi}_0, \tag{2.23}$$

$$u(x,0) = \varphi(x), \qquad x \in [0,1], \tag{2.24}$$

$$u(0,t) = h\left(\int_0^1 \gamma(x)u(x,t)\,dx\right), \qquad t \in \mathbb{R}, \tag{2.25}$$

where $\mu > 0$ is the mortality rate of the population and the functions h and γ describe the fertility of the population. Without losing potential applicability to the topic of population dynamics, h and γ are supposed to be C^∞-smooth functions. The integral in (2.25) is a kind of the so-called "partial" integral, since u depends not only on the variable of integration x, but also on the free variable t. Therefore the right-hand side of (2.25) is not smoothing. Nevertheless, it turns out that it is regular enough to contribute into smoothing solutions.

Theorem 2.4. *Assume that h and γ are C^∞-smooth functions and φ is a continuous function. Then any continuous solution to the problem* (2.23)–(2.25) *is smoothing.*

Proof. It suffices to show the smoothing property starting from large enough t. Therefore, we can use the notation:

$$(Ru)(t) = h\left(\int_0^1 \gamma(x)u(x,t)\,dx\right)$$

$$\omega(\xi;x,t) = t + \xi - x$$

$$c(\xi,x,t) = \tilde{c}(\xi,x,t) = e^{\mu(\xi-x)}$$

$$(Bu)(x,t) = (\tilde{B}u)(x,t) = e^{-\mu x}u(0,t-x),$$

the latter two being introduced for all large enough t. Integration along the characteristic curves implies that any continuous solution to (2.23)–(2.25) satisfies the operator equations $u = BRu$ and $u = Bu$ and, hence,

$$u = BRBu \tag{2.26}$$

whenever $t > T_1$, where T_1 is chosen to be so large that the operator BRB moves away from the initial boundary (the right-hand side of (2.26) does not depend on φ). Since

$$\begin{aligned}(BRBu)(t) &= e^{-\mu x}h\left(\int_0^1 \gamma(\xi)e^{-\mu\xi}u(0,t-x-\xi)\,d\xi\right)\\ &= e^{-\mu x}h\left(\int_{t-x-1}^{t-x} \gamma(t-x-\tau)e^{\mu(x-t+\tau)}u(0,\tau)\,d\tau\right),\end{aligned}$$

we obtain the C_t^1-smoothness of $BRBu$ and, hence, of u on $\overline{\Pi}_{T_1}$. The C^1-smoothness of u on $\overline{\Pi}_{T_1}$ now follows from (2.23).

Proceeding similarly to the proof of Theorem 2.3, in the second step we consider the following operator equation with respect to $v = \partial_t u$, obtained after differentiation of (2.23) and (2.25) with respect to t and integration along characteristic curves:

$$v|_{\overline{\Pi}_{T_2}} = B\partial_t RBv, \tag{2.27}$$

where

$$(\partial_t Rv)(t) = h'\left(\int_0^1 \gamma(x)u(x,t)\,dx\right)\int_0^1 \gamma(x)v(x,t)\,dx$$

and $T_2 > T_1$ is fixed to satisfy the property that the right-hand side of (2.27) does not depend on u and v in $\overline{\Pi}_0 \setminus \Pi_{T_1}$. It follows that

$$\begin{aligned} v|_{\overline{\Pi}_{T_2}} &= e^{-\mu x}h'\left(\int_0^1 \gamma(\xi)u(\xi,t-x)\,d\xi\right)\int_0^1 \gamma(\xi)e^{-\mu\xi}v(0,t-x-\xi)\,d\xi \\ &= e^{-\mu x}h'\left(\int_0^1 \gamma(\xi)u(\xi,t-x)\,d\xi\right)\int_{t-x-1}^{t-x} \gamma(t-x-\tau)e^{\mu(x-t+\tau)}v(0,\tau)\,d\tau. \end{aligned}$$

To conclude that $v \in C_t^1\left(\overline{\Pi}_{T_2}\right)^n$, it remains to note that u under the first integral in the right-hand side meets the C_t^1-regularity, while the second integral gives us the desired smoothing property.

In general, given T_r for $r \geq 2$, we choose $T_{r+1} > T_r$ by the argument as above and for $w = \partial_t^r u$ have the equation

$$w|_{\overline{\Pi}_{T_{r+1}}} = B\partial_t^r RBw, \tag{2.28}$$

where

$$\begin{aligned} (\partial_t^r Rw)(t) &= h'\left(\int_0^1 \gamma(x)u(x,t)\,dx\right)\int_0^1 \gamma(x)w(x,t)\,dx \\ &\quad + \frac{d^{r-1}}{dt^{r-1}}\left[h'\left(\int_0^1 \gamma(x)u(x,t)\,dx\right)\right]\int_0^1 \gamma(x)\partial_t u(x,t)\,dx \\ &\quad + \frac{d^{r-2}}{dt^{r-2}}\left[h'\left(\int_0^1 \gamma(x)u(x,t)\,dx\right)\int_0^1 \gamma(x)\partial_t u(x,t)\,dx\right]. \end{aligned}$$

Substituting the latter into (2.28) and changing variables under the integral of w similarly to the first two steps, we get the desired smoothing property for w. This completes the proof. □

2.3. Dissipative boundary conditions and periodic problems

Now we switch to boundary conditions having dissipative nature and fitting the smoothing property. A large class of dissipative boundary conditions for hyperbolic PDEs is described in [8].

To give an idea of the smoothing effect in this case, consider the following specification of (2.1):

$$\begin{aligned} u_j(0,t) &= h_j(z(t)), \quad 1 \leq j \leq m, \\ u_j(1,t) &= h_j(z(t)), \quad m < j \leq n, \end{aligned} \tag{2.29}$$

with

$$z(t) = (u_1(1,t), \dots, u_m(1,t), u_{m+1}(0,t), \dots, u_n(0,t)). \tag{2.30}$$

In the domain $\Pi_{-\infty}$ we address the problem (2.1), (2.29) subjected to periodic boundary conditions

$$u(x, t + 2\pi) = u(x,t). \tag{2.31}$$

The problems of this kind appear in laser dynamics and chemical kinetics (in Section 3 we investigate a traveling-wave model of kind (2.1), (2.29), (2.31) from laser dynamics). Within this section, using the standard notation for the (sub)spaces of continuous functions, we assume that the functions have additional property of 2π-periodicity in t. Write

$$h_j'(z) = \nabla_z h_j(z), \quad h'(z) = \left\{\partial_k h_j(z)\right\}_{j,k=1}^n.$$

Theorem 2.5. *Assume that a_j, b_{jk}, f_j, and h_j are C^N-smooth functions in all their arguments and the conditions* (2.4)–(2.6) *a re fulfilled. Moreover, the functions a_j, b_{jk}, f_j are supposed to be 2π-periodic in t. If*

$$\exp\left\{\int_x^{x_j} \left(\frac{b_{jj}}{a_j} - r\frac{\partial_t a_j}{a_j^2}\right)(\eta, \omega_j(\eta; x, t))\, d\eta\right\} \sum_{k=1}^n |\partial_k h_j(z)| < 1 \tag{2.32}$$

for all $j \le n$, $x \in [0,1]$, $t \in \mathbb{R}$, $z \in \mathbb{R}^n$, and $r = 0, 1, \dots, N$, then any continuous solution to the problem (2.1), (2.29), (2.31) *belongs to $C^N(\Pi_{-\infty})$.*

Proof. Any continuous solution to the problem (2.1), (2.29), (2.31) in $\Pi_{-\infty}$ fulfills (2.12) with $S = R$ and also satisfies the equation

$$u = Bu + Du + Ff \tag{2.33}$$

where the boundary conditions are not specified. Substituting (2.33) into (2.12), we obtain

$$u = BRu + (DB + D^2)u + (I + D)Ff. \tag{2.34}$$

We first show the bijectivity of $I - BR \in \mathcal{L}\left(C_t^1\left(\overline{\Pi}_{-\infty}\right)^n\right)$. On the account of (2.19) and the definition of B given by (2.9), we have

$$\begin{aligned}(BRu)_j(x,t) = c_j(x_j, x, t) h_j\left(z(\omega_j(x_j; x, t))\right) = c_j(x_j, x, t) h_j(0) \\ + \exp\left\{\int_x^{x_j}\left(\frac{b_{jj}}{a_j}\right)(\eta, \omega_j(\eta; x, t))\, d\eta\right\} \\ \times \int_0^1 h_j'\left(\alpha z(\omega_j(x_j; x, t))\right) d\alpha \cdot z(\omega_j(x_j; x, t))\end{aligned}$$

and

$$\begin{aligned}\partial_t\left[(BRu)_j(x,t)\right] = \partial_t c_j(x_j, x, t) h_j\left(z(\omega_j(x_j; x, t))\right) \\ + h_j'\left(z(\omega_j(x_j; x, t))\right) \cdot z'(\omega_j(x_j; x, t)) \\ \times \exp\left\{\int_x^{x_j}\left(\frac{b_{jj}}{a_j} - \frac{\partial_t a_j}{a_j^2}\right)(\eta, \omega_j(\eta; x, t))\, d\eta\right\},\end{aligned}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^n$. Taking into account (2.30), the bijectivity of $I - BR \in \mathcal{L}\left(C_t^1\left(\overline{\Pi}_{-\infty}\right)^n\right)$ now follows from the contractibility condition (2.32) with $r = 0, 1$ and from the proof of the C^k-regularity result for solutions of first-order hyperbolic PDEs given in [31].

Now we claim that the operators DB and D^2 in (2.34) are smoothing. The latter is smoothing by the proof in Theorem 2.3. Similar argument works also for DB. Indeed, by the definition of the operators D and B we have

$$\left(DBu^l\right)_j(x,t) = \sum_{\substack{k=1\\k\neq j}}^{n} \int_x^{x_j} d_j(\xi,x,t) b_{jk}(\xi,\omega_j(\xi;x,t)) c_k(x_k,\xi,\omega_j(\xi;x,t)) \times u_k^l(x_k,\omega_k(x_k;\xi,\omega_j(\xi;x,t)))d\xi, \tag{2.35}$$

where the sequence u^l is fixed to satisfy (2.15) with Π_0 replaced by $\Pi_{-\infty}$. To show that $\partial_t\left[DBu^l\right]$ converges uniformly on $\overline{\Pi}_{-\infty}$, we transform the integrals in (2.35) like to the case of D^2, that is, we differentiate (2.35) in t, use (2.6), and integrate by parts. In this way we get the smoothing property for DB. Turning back to the formula (2.34) and using in addition the fact that $(I + D)Ff$ is C^∞-smooth, we can rewrite (2.34) in the equivalent form

$$u = (I - BR)^{-1}\left[(DB + D^2)u + (I + D)Ff\right],$$

thereby reaching the C_t^1-regularity for u. Afterwards, the C^1-regularity of u is a straightforward consequence of the system (2.1).

Proceeding similarly to the proof of Theorem 2.3, we come to the formula for $v = \partial_t u$:

$$v = (I - \tilde{B}R_z')^{-1}\left[(\tilde{D}\tilde{B} + \tilde{D}^2)v + (I + \tilde{D})\tilde{F}G(f,\partial_t f,u)\right],$$

where $R_z'y = h'(z)y$. The property that $v \in C_t^1\left(\overline{\Pi}_{-\infty}\right)^n$ follows from the bijectivity of $I - \tilde{B}R_z' \in \mathcal{L}\left(C_t^1\left(\overline{\Pi}_{-\infty}\right)^n\right)$, which we have by condition (2.32) with $r = 1, 2$ and the C_t^1-regularity of $\tilde{D}\tilde{B} + \tilde{D}^2$ and $(I + \tilde{D})\tilde{F}G(f,\partial_t f,u)$. This entails $u \in C_t^2\left(\overline{\Pi}_{-\infty}\right)^n$. It follows by (2.1) that $u \in C^2\left(\overline{\Pi}_{-\infty}\right)^n$.

To complete the proof, we proceed by induction on the order of regularity of u. Assume that $u \in C^r\left(\overline{\Pi}_{-\infty}\right)^n$ for some $r \geq 2$ and prove that $u \in C^{r+1}\left(\overline{\Pi}_{-\infty}\right)^n$. Our starting formula for $w = \partial_t^r u$ is as follows:

$$\begin{aligned} w = (I - \tilde{B}R_z')^{-1}\Big[&(\tilde{D}\tilde{B} + \tilde{D}^2)w \\ &+ (I + \tilde{D})\tilde{F}\tilde{G}(f,\partial_t f,\ldots,\partial_t^r f,u,\partial_t u,\ldots,\partial_t^{r-1}u) \\ &+ \tilde{B}\partial_t^{r-1}R_z'z' + \tilde{B}\partial_t^{r-2}\left(R_z'z'\right)\Big], \end{aligned}$$

where $\partial_t^{r-1}R_z' = \left\{\partial_t^{r-1}(\partial_k h_j(z))\right\}_{j,k=1}^n$ and $\tilde{B}, \tilde{D}$, and $\tilde{F}$ are modified by $\tilde{c}_j(\xi,x,t)$ changing to $\tilde{c}_j(\xi,x,t) = \exp\int_x^\xi\left(\frac{b_{jj}}{a_j} - r\frac{\partial_t a_j}{a_j^2}\right)(\eta,\omega_j(\eta;x,t))\,d\eta$. By the regularity assumptions on the data and the induction assumption, the last three summands in the square brackets are C_t^1-functions. Using in addition our smoothing argument for $\tilde{D}\tilde{B}+\tilde{D}^2$ and the regularity properties of $(I-\tilde{B}R_z')^{-1}$, we arrive at the desired conclusion. □

3. Fredholm solvability of periodic problems

In [20, 21] we suggest an approach to establish the Fredholm property for first-order hyperbolic operators. This is done by construction an equivalent regularization in the form of a parametrix. The construction is, implicitly but essentially, based on the smoothing effect investigated in Section 2. Consider the first-order one-dimensional hyperbolic system

$$(\partial_t + a(x)\partial_x + b(x))u = f(x,t),\ x \in (0,1), \tag{3.36}$$

subjected to periodic conditions (2.31) and reflection boundary conditions

$$\begin{aligned} u_j(0,t) &= \sum_{k=m+1}^{n} r_{jk}^0 u_k(0,t),\ \ 1 \le j \le m, \\ u_j(1,t) &= \sum_{k=1}^{m} r_{jk}^1 u_k(1,t), \qquad m < j \le n. \end{aligned} \tag{3.37}$$

Here r_{jk}^0 and r_{jk}^1 are real numbers and the right-hand sides $f_j : [0,1]\times\mathbb{R} \to \mathbb{R}$ are supposed to be 2π-periodic with respect to t.

The main result of this section states that the system (3.36), (2.31), (3.37) is solvable if and only if the right-hand side is orthogonal to all solutions to the corresponding homogeneous adjoint system

$$-\partial_t u - \partial_x\left(a(x)u\right) + b^T(x)u = 0,\ x \in (0,1),$$

subjected to periodic conditions (2.31) and adjoint boundary conditions

$$\begin{aligned} a_j(0)u_j(0,t) &= -\sum_{k=1}^{m} r_{kj}^0 a_k(0)u_k(0,t), \qquad m < j \le n, \\ a_j(1)u_j(1,t) &= -\sum_{k=m+1}^{n} r_{kj}^1 a_k(1)u_k(1,t),\ \ 1 \le j \le m. \end{aligned} \tag{3.38}$$

We will present our result in three steps. First we introduce appropriate function spaces for solutions. Then we decompose the operator of the problem into two parts, only one being responsible for propagation of singularities. Finally, based on this decomposition and the smoothing property, we construct a parametrix thereby establishing the Fredholm solvability.

When choosing the function spaces, note that the problem (3.36), (2.31), (3.37) describes the so-called traveling-wave models from laser dynamics [24, 30]. From the physical point of view, it is desirable to allow discontinuities in the coefficients and the right-hand side of (3.36). This entails that the spaces of solutions should not be too small. On the other hand, they should not be too large, in order to admit embeddings into an algebra of functions with pointwise multiplication of its elements. The last property is important for potential applicability of our results to nonlinear problems, like describing such dynamic phenomena as Hopf bifurcation and periodic synchronizations. Finally, the solution spaces capable to capture the Fredholm solvability need to have optimal regularity with respect to the function spaces of the right-hand side.

We now describe the scale of spaces V^γ (for the solutions) and W^γ (for the right-hand side) meeting all these properties. For $\gamma \ge 0$, let W^γ denote the vector space of all locally integrable functions $f : [0,1] \times \mathbb{R} \to \mathbb{R}^n$ such that $f(x,t) = f(x, t+2\pi)$ for almost all $x \in (0,1)$ and $t \in \mathbb{R}$ and that

$$\|f\|^2_{W^\gamma} = \sum_{s\in\mathbb{Z}} (1+s^2)^\gamma \int_0^1 \left\| \int_0^{2\pi} f(x,t) e^{-ist}\, dt \right\|^2 dx < \infty. \tag{3.39}$$

Here and in what follows $\|\cdot\|$ is the Hermitian norm in $\mathbb{C}^n$. It is well known that W^γ is a Banach space with the norm (3.39); see, e.g., [12], [33, Chapter 5.10], and [35, Chapter 2.4].

Furthermore, for $\gamma \ge 1$ and $a \in L^\infty((0,1); \mathbb{M}_n)$, where $\mathbb{M}_n$ denotes the space of real $n \times n$ matrices, with ess inf $|a_j| > 0$ for all $j \le n$ we will work with the function spaces

$$U^\gamma = \left\{ u \in W^\gamma : \partial_x u \in W^0,\, \partial_t u + a\partial_x u \in W^\gamma \right\}$$

endowed with the norms

$$\|u\|^2_{U^\gamma} = \|u\|^2_{W^\gamma} + \|\partial_t u + a\partial_x u\|^2_{W^\gamma}.$$

Remark that the space U^γ depends on a and is larger than the space of all $u \in W^\gamma$ such that $\partial_t u \in W^\gamma$ and $\partial_x u \in W^\gamma$ (which does not depend on a). For $u \in U^\gamma$ there exist traces $u(0,\cdot), u(1,\cdot) \in L^2_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ (see [21]), and, hence, it makes sense to consider the closed subspaces in U^γ

$$V^\gamma = \{u \in U^\gamma : \text{(3.37) is fulfilled}\},$$
$$\tilde{V}^\gamma = \{u \in U^\gamma : \text{(3.38) is fulfilled}\}.$$

Our next task is to decompose the operator of our problem into two parts in order to single out the part, denoted below by $\mathcal{A}$, which is bijective and at the same time is responsible for the propagation of singularities. If this decomposition is optimal, then after a regularization procedure the other part becomes smoothing and therefore meets the compactness property. Let

$$b^0 = \operatorname{diag}(b_{11}, b_{22}, \ldots, b_{nn}) \quad \text{and} \quad b^1 = b - b^0$$

denote the diagonal and the off-diagonal parts of the coefficient matrix b, respectively.

Let us introduce operators $\mathcal{A} \in \mathcal{L}(V^\gamma; W^\gamma)$, $\tilde{\mathcal{A}} \in \mathcal{L}(\tilde{V}^\gamma; W^\gamma)$, and $\mathcal{B}, \tilde{\mathcal{B}} \in \mathcal{L}(W^\gamma)$ by

$$\begin{aligned} \mathcal{A}u &= \partial_t u + a\partial_x u + b^0 u, \\ \tilde{\mathcal{A}}u &= -\partial_t u - \partial_x(au) + b^0 u, \\ \mathcal{B}u &= b^1 u, \\ \tilde{\mathcal{B}}u &= (b^1)^T u. \end{aligned}$$

Remark that the operators $\mathcal{A}$, $\mathcal{B}$, and $\tilde{\mathcal{B}}$ are well defined for $a_j, b_{jk} \in L^\infty(0,1)$, while $\tilde{\mathcal{A}}$ is well defined under additional regularity assumptions with respect to the coefficients a_j, for example, for $a_j \in C^{0,1}([0,1])$. Note that the operator equation

$$\mathcal{A}u + \mathcal{B}u = f$$

is an abstract representation of the problem (3.36), (2.31), (3.37).

Finally, for $s \in \mathbb{Z}$ we introduce the complex $(n-m) \times (n-m)$ matrices

$$R_s = \left[\sum_{l=1}^{m} e^{is(\alpha_j(1)-\alpha_l(1))+\beta_j(1)-\beta_l(1)} r_{jl}^1 r_{lk}^0 \right]_{j,k=m+1}^{n},$$

where

$$\alpha_j(x) = \int_0^x \frac{1}{a_j(y)}\, dy, \quad \beta_j(x) = \int_0^x \frac{b_{jj}(y)}{a_j(y)}\, dy.$$

The following theorem states, first, that the pair of spaces (V^γ, W^γ) gives an optimal regularity trade-off between the spaces of solutions and right-hand sides and, second, that $\mathcal{A}$ meets the bijectivity property. The second desirable property for $\mathcal{A}$ of being an optimal operator responsible for propagation of singularities will be a consequence of our Fredholmness result.

Theorem 3.1 ([21]). *For every $c > 0$ there exists $C > 0$ such that the following is true: If*

$$a_j, b_{jj} \in L^\infty(0,1) \text{ and } \operatorname{ess\,inf} |a_j| \geq c \text{ for all } j = 1, \ldots, n, \tag{3.40}$$

$$\sum_{j=1}^{n} \|b_{jj}\|_\infty + \sum_{j=1}^{m} \sum_{k=m+1}^{n} |r_{jk}^0| + \sum_{j=m+1}^{n} \sum_{k=1}^{m} |r_{jk}^1| \leq \frac{1}{c},$$

and

$$|\det(I - R_s)| \geq c \text{ for all } s \in \mathbb{Z}, \tag{3.41}$$

then for all $\gamma \geq 1$ the operator $\mathcal{A}$ is an isomorphism from V^γ onto W^γ and

$$\|\mathcal{A}^{-1}\|_{\mathcal{L}(W^\gamma; V^\gamma)} \leq C.$$

Let

$$\langle f, u \rangle_{L^2} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \langle f(x,t), u(x,t) \rangle \, dxdt$$

denote the scalar product in the Hilbert space $L^2\left((0,1) \times (0,2\pi); \mathbb{R}^n\right)$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product in $\mathbb{R}^n$. As usual, by $BV(0,1)$ we denote the

Banach space of all functions $h : (0,1) \to \mathbb{R}$ with bounded variation, i.e., of all $h \in L^\infty(0,1)$ such that there exists $C > 0$ with

$$\left| \int_0^1 h(x)\varphi'(x)dx \right| \le C\|\varphi\|_{L^\infty(0,1)} \text{ for all } \varphi \in C_0^\infty(0,1). \tag{3.42}$$

The norm of h in $BV(0,1)$ is the sum of the norm of h in $L^\infty(0,1)$ and of the smallest possible constant C in (3.42). We are prepared to formulate the main result of this section.

Theorem 3.2 ([21]). *Suppose that conditions* (3.40) *and* (3.41) *are fulfilled for some* $c > 0$. *Suppose also that*

$$\begin{aligned} &\text{for all } j \neq k \text{ there is } p_{jk} \in BV(0,1) \text{ such that} \\ &a_k(x)b_{jk}(x) = p_{jk}(x)(a_j(x) - a_k(x)) \text{ for a.a. } x \in [0,1]. \end{aligned} \tag{3.43}$$

Then the following is true:

(i) *The operator* $\mathcal{A}+\mathcal{B}$ *is a Fredholm operator with index zero from* V^γ *into* W^γ *for all* $\gamma \ge 1$, *and* $\ker(\mathcal{A}+\mathcal{B}) = \{u \in V^\gamma : (\mathcal{A}+\mathcal{B})\,u = 0\}$ *does not depend on* γ.

(ii) (smoothing effect) *If* $a \in C^{0,1}([0,1];\mathbb{M}_n)$, *then* $\ker(\mathcal{A}+\mathcal{B})^* = \ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})$ *and*

$$\{(\mathcal{A}+\mathcal{B})u : u \in V^\gamma\} = \left\{ f \in W^\gamma : \langle f, u\rangle_{L^2} = 0 \text{ for all } u \in \ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}}) \right\},$$

where $\ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}}) = \{u \in \tilde{V}^\gamma : (\tilde{\mathcal{A}}+\tilde{\mathcal{B}})u = 0\}$ *does not depend on* γ.

Theorem 3.2 (ii) states that the kernel of the adjoint operator is actually defined on the classical function spaces. In other words, the kernel has much better regularity than ensured just by the formal definition of the adjoint operator. Here we encounter a smoothing effect for the solutions (of the adjoint hyperbolic problem), that are originally functionals. The proof of this effect in [20, 21] uses completely different techniques, based on a functional-analytic approach.

Finally, we outline the proof of Theorem 3.2 (i). As mentioned above, we construct a parametrix to the operator of the problem. By Theorem 3.1, the zero-order Fredholmness of the operator $\mathcal{A}+\mathcal{B} \in \mathcal{L}(V^\gamma; W^\gamma)$ is equivalent to the zero-order Fredholmness of the operator $I + \mathcal{B}\mathcal{A}^{-1} \in \mathcal{L}(W^\gamma)$. Furthermore, we use the following Fredholmness criterion (see [34, Theorem 5.5] or [37, Proposition 5.7.1]).

Lemma 3.3. *Let* I *denote the identity in a Banach space* W. *Suppose that* $\mathcal{D} \in \mathcal{L}(W)$ *and* $\mathcal{D}^2$ *is compact. Then* $I + \mathcal{D}$ *is Fredholm.*

Setting $\mathcal{D} = \mathcal{B}\mathcal{A}^{-1} \in \mathcal{L}(W^\gamma)$, we prove that $\mathcal{D}^2 \in \mathcal{L}(W^\gamma)$ is compact (while $\mathcal{D}$ alone can hardly be compact, being a type of a partial integral operator). This actually means that $\mathcal{D}^2$ has smoothing property. In fact, $\mathcal{D}^2$ is basically the same as the operator D^2, that we used in the proof of Theorem 2.3.

Since $I - \mathcal{D}^2 = (I - \mathcal{D})(I + \mathcal{D}) = (I + \mathcal{D})(I - \mathcal{D})$, the operator $I - \mathcal{D}$ is a parametrix of $I + \mathcal{D}$. It follows that the operator $\mathcal{A}+\mathcal{B}$ admits an equivalent regularization in the form of the right parametrix $\mathcal{A}^{-1}(I - \mathcal{B}\mathcal{A}^{-1})$.

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A Note on Wave-front Sets of Roumieu Type Ultradistributions

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Abstract. We study wave-front sets in weighted Fourier–Lebesgue spaces and corresponding spaces of ultradistributions. We give a comparison of these sets with the well-known wave-front sets of Roumieu type ultradistributions. Then we study convolution relations in the framework of ultradistributions. Finally, we introduce modulation spaces and corresponding wave-front sets, and establish invariance properties of such wave-front sets.

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Keywords. Wave-front sets, weighted Fourier–Lebesgue spaces, ultradistributions.

0. Introduction

This paper can be considered as a companion to [7], where the authors considered spaces of Beurling type ultradistributions mainly. The basic definitions and results are parallel to those given in [7]. However, apart from this, here one can find some new results. In particular, we focus our attention to convolution relations in the framework of ultradistributions, which is not considered in [7]. In that direction we extend some results from [11, 13]. The results of the present paper are mainly formulated both for Beurling and Roumieu type ultradistributions, and the proofs are presented for the Roumieu case only.

One of the main ingredients is the choice of test function space $\mathcal{S}^{(s)}(\mathbf{R}^d)$ for the analysis of wave-front sets both for Beurling and Roumieu type ultradistributions, see Subsection 0.2 for definitions. Although the space $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ is the test function space for Roumieu type ultradistributions, here we show that it is natural and necessary to use a smaller space in our approach to wave-front sets.

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The paper is organized as follows. In the present section, we recall the notion and elementary properties of weights, ultradistribution spaces and Fourier–Lebesgue type spaces. In Section 1 we introduce wave-front sets in Fourier–Lebesgue spaces in analogous way as it is done in [7, 12], and compare these sets with the well-known wave-front sets of Roumieu type ultradistributions, cf. [5, 10, 15]. In particular, in Proposition 1.3, we recover [7, Proposition 1.6 (1)]. In Section 2 we study convolutions in Fourier–Lebesgue spaces and extend corresponding results from [14] to spaces of ultradistributions. (Cf. Theorem 2.2 and Remark 2.3.) Finally, in Section 3 we introduce modulation spaces and the corresponding wave-front sets in the framework of ultradistributions. We prove Proposition 3.1 concerning the behavior of the short-time Fourier transform of a compactly supported ultradistributions of Roumieu type, cf. [7, Proposition 2.2]. This result is interesting in itself and it is used to show the invariance properties of the wave-front sets in Theorem 3.3. Here we give a complete proof to enlighten the role of the test function space $\mathcal{S}^{(s)}(\mathbf{R}^d)$ once again. Since modulation spaces and Fourier–Lebesgue spaces are locally the same, cf. Corollary 3.4, the results from Section 2 can be formulated in terms of modulation spaces. We left the technical details as an exercise to the reader.

We put $\mathbf{N} = \{0, 1, 2, \dots\}$, $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$, $\langle x \rangle = (1 + |x|^2)^{1/2}$, for $x \in \mathbf{R}^d$, and $A \lesssim B$ to indicate $A \le cB$ for a suitable constant $c > 0$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous and dense embedding of the topological vector space B_1 into B_2. The scalar product in L^2 is denoted by $(\cdot, \cdot)_{L^2} = (\cdot, \cdot)$ and $\complement A$ denotes the complement of the set $A \subset \mathbf{R}^d$.

0.1. Weights

In general, a weight function is a non-negative function in L^∞_{loc}.

Definition 0.1. Let ω, v be non-negative functions. Then

1. v is called *submultiplicative* if
$$v(x + y) \le v(x)v(y), \qquad \forall\ x, y \in \mathbf{R}^d;$$
2. ω is called *v-moderate* if
$$\omega(x + y) \lesssim v(x)\omega(y), \qquad \forall\ x, y \in \mathbf{R}^d.$$

Let $s > 1$. By $\mathscr{M}_{\{s\}}(\mathbf{R}^d)$ we denote the set of all weights which are moderate with respect to a weight v which satisfies $v \le Ce^{k|\,\cdot\,|^{1/s}}$ for some positive constants C and k. We refer to [3] for more facts about such weights.

Note that, if $\omega \in \mathscr{M}_{\{s\}}$ and v is even, then $1/v \lesssim \omega \lesssim v$, $\omega \neq 0$ everywhere and $1/\omega \in \mathscr{M}_{\{s\}}$.

0.2. Test function spaces and their duals

Next we introduce spaces of test functions and their duals in the context of spaces of ultradistributions. These test function spaces correspond to the spaces C_0^∞, $\mathscr{S}$ and C^∞ in the distribution theory in [5, 17]. We start by giving the definition of Gelfand–Shilov type spaces.

Definition 0.2. Let $s > 1$ and $A > 0$. The space $\mathcal{S}^s_A(\mathbf{R}^d)$ consists of all functions $\varphi \in C^\infty(\mathbf{R}^d)$ such that the norm

$$\|\varphi\|_{s,A} = \sup_{\alpha,\beta\in\mathbf{N}_0^d} \sup_{x\in\mathbf{R}^d} \frac{A^{|\alpha+\beta|}}{\alpha!^s\beta!^s} \langle x\rangle^{|\alpha|} |\varphi^{(\beta)}(x)|$$

is finite. The Gelfand–Shilov type spaces $\mathcal{S}^{(s)}(\mathbf{R}^d)$ and $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ are given by

$$\mathcal{S}^{(s)}(\mathbf{R}^d) = \bigcap_{A>0} \mathcal{S}^s_A(\mathbf{R}^d) \quad \mathcal{S}^{\{s\}}(\mathbf{R}^d) = \bigcup_{A>0} \mathcal{S}^s_A(\mathbf{R}^d),$$

with topologies given by the projective and the inductive limit, respectively.

We note that $\mathcal{S}_A(\mathbf{R}^d)$ is a Banach space, for every $A > 0$, and its dual is denoted by $(\mathcal{S}_A)'(\mathbf{R}^d)$. Then the Gelfand–Shilov type distribution spaces $(\mathcal{S}^{(s)})'(\mathbf{R}^d)$ and $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ are defined as

$$(\mathcal{S}^{(s)})'(\mathbf{R}^d) = \bigcup_{A>0} (\mathcal{S}^s_A)'(\mathbf{R}^d), \quad (\mathcal{S}^{\{s\}})'(\mathbf{R}^d) = \bigcap_{A>0} (\mathcal{S}^s_A)'(\mathbf{R}^d).$$

These spaces are the strong dual spaces of $\mathcal{S}^{(s)}(\mathbf{R}^d)$ and $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$, and are called the spaces of *tempered ultradistributions* of Beurling type and Roumieu type, respectively. If $s > t$, then

$$\begin{aligned}\mathcal{S}^{(t)}(\mathbf{R}^d) &\hookrightarrow \mathcal{S}^{\{t\}}(\mathbf{R}^d) \hookrightarrow \mathcal{S}^{(s)}(\mathbf{R}^d) \hookrightarrow \mathcal{S}^{\{s\}}(\mathbf{R}^d) \\ &\hookrightarrow (\mathcal{S}^{\{s\}})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}^{(s)})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}^{\{t\}})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}^{(t)})'(\mathbf{R}^d).\end{aligned}$$

In order to perform (micro)local analysis we use the following test function spaces on open sets, cf. [9].

Definition 0.3. Let X be an open set in $\mathbf{R}^d$. For a given compact set $K \subset X$, $s > 1$ and $A > 0$, we denote by $\mathcal{E}^s_{A,K}$ the space of all $\varphi \in C^\infty(X)$ such that the semi-norm

$$\|\varphi\|_{s,A,K} = \sup_{\beta\in\mathbf{N}_0^n} \sup_{x\in K} \frac{A^{|\beta|}}{\beta!^s} |\varphi^{(\beta)}(x)| \tag{0.1}$$

is finite.

The space of functions $\varphi \in C^\infty(X)$ such that (0.1) holds and $\operatorname{supp}\varphi \subseteq K$ is denoted by $\mathcal{D}^s_A(K)$.

Let $(K_n)_{n\in\mathbf{N}}$ be a sequence of compact sets such that $K_n \subset\subset K_{n+1}$, $n \in \mathbf{N}$, and $\bigcup_{n\in\mathbf{N}} K_n = X$. Then

$$\mathcal{E}^{(s)}(X) = \operatorname*{proj\,lim}_{n\to\infty} (\operatorname*{proj\,lim}_{A\to\infty} \mathcal{E}^s_{A,K_n}), \qquad \mathcal{E}^{\{s\}}(X) = \operatorname*{proj\,lim}_{n\to\infty} (\operatorname*{ind\,lim}_{A\to 0} \mathcal{E}^s_{A,K_n}),$$

$$\mathcal{D}^{(s)}(X) = \operatorname*{ind\,lim}_{n\to\infty} (\operatorname*{proj\,lim}_{A\to\infty} \mathcal{D}^s_A(K_n)), \qquad \mathcal{D}^{\{s\}}(X) = \operatorname*{ind\,lim}_{n\to\infty} (\operatorname*{ind\,lim}_{A\to 0} \mathcal{D}^s_A(K_n)).$$

From now on, we let $*$ denote (s) or $\{s\}$.

Obviously, $\mathcal{D}^*(X)$ is the subspace of $\mathcal{E}^*(X)$ whose elements are compactly supported.

The spaces of linear functionals over $\mathcal{D}^{(s)}(X)$ and $\mathcal{D}^{\{s\}}(X)$, denoted by $(\mathcal{D}^{(s)})'(X)$ and $(\mathcal{D}^{\{s\}})'(X)$ respectively, are called the spaces of *ultradistributions* of Beurling and Roumieu type respectively, while the spaces of linear functionals over $\mathcal{E}^{(s)}(X)$ and $\mathcal{E}^{\{s\}}(X)$, denoted by $(\mathcal{E}^{(s)})'(X)$ and $(\mathcal{E}^{\{s\}})'(X)$, respectively are called the spaces of *ultradistributions of compact support* of Beurling and Roumieu type respectively. We have

$$(\mathcal{E}^{\{s\}})'(X) \subseteq (\mathcal{E}^{(s)})'(X) \quad \text{and} \quad (\mathcal{E}^{*})'(X) \subseteq (\mathcal{E}^{*})'(\mathbf{R}^d) \subseteq (\mathcal{S}^{*})'(\mathbf{R}^d) \subseteq (\mathcal{D}^{*})'(\mathbf{R}^d).$$

0.3. Fourier–Lebesgue spaces

The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F} f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. It is a homeomorphism on $(\mathcal{S}^*)'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathcal{S}^*(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Let $q \in [1,\infty]$, $s > 1$ and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$. The (weighted) Fourier–Lebesgue space $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ is the inverse Fourier image of $L^q_{(\omega)}(\mathbf{R}^d)$, i.e., $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ consists of all $f \in (\mathcal{S}^*)'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathscr{F}L^q_{(\omega)}} \equiv \|\widehat{f} \cdot \omega\|_{L^q}. \tag{0.2}$$

is finite. If $\omega = 1$, then the notation $\mathscr{F}L^q$ is used instead of $\mathscr{F}L^q_{(\omega)}$. We note that if $\omega(\xi) = \langle\xi\rangle^s$, then $\mathscr{F}L^q_{(\omega)}$ is the Fourier image of the Bessel potential space H^q_s.

Remark 0.4. Whenever it is convenient we permit an x dependency for the weight ω in the definition of Fourier–Lebesgue spaces. More precisely, for each $\omega\in\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ we let $\mathscr{F}L^q_{(\omega)}$ be the set of all $f \in (\mathcal{S}^*)'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathscr{F}L^q_{(\omega)}} \equiv \|\widehat{f}\,\omega(x,\,\cdot\,)\|_{L^q}$$

is finite. Since ω is v-moderate it follows that different choices of x give rise to equivalent norms.

Next we introduce local Fourier–Lebesgue spaces of ultradistributions. Let X be an open set in $\mathbf{R}^d$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ and let $s > 1$. Then the *local* Fourier–Lebesgue space $\mathscr{F}L^q_{(\omega),\mathrm{loc}}(X)$ consists of all $f \in (\mathcal{S}^*)'(\mathbf{R}^d)$ such that $\varphi f \in \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ for each $\varphi \in \mathcal{D}^{(s)}(X)$. We note that $\mathscr{F}L^q_{(\omega),\mathrm{loc}}(X)$ is a Fréchet space under the topology, defined by the family of seminorms $f \mapsto \|\varphi f\|_{\mathscr{F}L^q_{(\omega)}}$, where $\varphi \in \mathcal{D}^{(s)}(X)$.

Furthermore,

$$\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d) \subseteq \mathscr{F}L^q_{(\omega),\mathrm{loc}}(\mathbf{R}^d) \subseteq \mathscr{F}L^q_{(\omega),\mathrm{loc}}(X). \tag{0.3}$$

In fact, if $f \in \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$, $\varphi \in \mathcal{D}^{(s)}(X)$ and if $v \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$ is chosen such that ω is v-moderate, then Young's inequality gives

$$\begin{aligned}\|\varphi f\|_{\mathscr{F}L^q_{(\omega)}} &= \|\mathscr{F}(\varphi f)\,\omega\|_{L^q} = (2\pi)^{-d/2}\|(\widehat{\varphi} * \widehat{f}\,)\,\omega\|_{L^q}\\ &\lesssim \||\widehat{\varphi}\,v| * |\widehat{f}\,\omega|\|_{L^q} \le C_\varphi\|\widehat{f}\,\omega\|_{L^q} = C_\varphi\|f\|_{\mathscr{F}L^q_{(\omega)}},\end{aligned}$$

where $C_\varphi = (2\pi)^{-d/2}\|\widehat{\varphi}\,v\|_{L^1}$. We claim that C_φ is finite. If $\varphi \in \mathcal{D}^{(s)}(X)$, then for every $N > 0$ we have

$$|\widehat{\varphi}(\xi)v(\xi)| \lesssim e^{-(N+k)|\xi|^{1/s}}e^{k|\xi|^{1/s}} = e^{-N|\xi|^{1/s}}. \tag{0.4}$$

It follows that $\|\widehat{\varphi}v\|_{L^p} < \infty$ for every $p \in [1,\infty]$. This proves that C_φ is finite and (0.3).

Let $q_1, q_2 \in [1,\infty]$ and $\omega_1, \omega_2 \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$. Then

$$\mathscr{F}L^{q_1}_{(\omega_1),\mathrm{loc}}(X) \subseteq \mathscr{F}L^{q_2}_{(\omega_2),\mathrm{loc}}(X), \quad \text{when} \quad q_1 \le q_2 \quad \text{and} \quad \omega_2 \lesssim \omega_1. \tag{0.5}$$

The inclusion in (0.5) is clear when $q_1 = q_2$ and $\omega_2 \lesssim \omega_1$. It remains to show that $\mathscr{F}L^q_{(\omega),\mathrm{loc}}$ increases with respect to q. Assume, without any loss of generality, that $f \in (\mathcal{E}^*)'(X)$, and that $\varphi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ is such that $\varphi \equiv 1$ in the neighborhood of $\operatorname{supp} f$. Choose $p \in [1,\infty]$ such that $1/q_1 + 1/p = 1/q_2 + 1$. Then, for a v-moderate weight ω, it follows from Young's inequality that

$$\|f\|_{\mathscr{F}L^{q_2}_{(\omega)}} \lesssim \|(\widehat{\varphi} * \widehat{f}\,)\omega\|_{L^{q_2}} \lesssim \|\widehat{\varphi}v\|_{L^p}\|\widehat{f}\omega\|_{L^{q_1}} = C\|f\|_{\mathscr{F}L^{q_1}_{(\omega)}},$$

for some constant C, and the result follows.

1. Wave-front sets of Fourier–Lebesgue type in some spaces of ultradistributions

In this section we introduce wave-front sets of Fourier–Lebesgue type in spaces of ultradistributions of Roumieu type, and refer to [7] for the Beurling case. The key difference between such wave-front sets can be realized through their relation with other types of wave-front sets of ultradistributions. The main result in this direction is Proposition 1.3, cf. [7, Proposition 1.6].

Let $s > 1$, $q \in [1,\infty]$, and $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone. If $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ we define

$$|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}} = |f|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}} \equiv \Big(\int_\Gamma |\widehat{f}(\xi)\omega(x,\xi)|^q\,d\xi\Big)^{1/q} \tag{1.1}$$

(with obvious interpretation when $q = \infty$). We note that $|\cdot|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}$ defines a semi-norm on $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ which might attain the value $+\infty$. Since ω is v-moderate it follows that different $x \in \mathbf{R}^d$ gives rise to equivalent semi-norms $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}$. Furthermore, if $\Gamma = \mathbf{R}^d \setminus 0$, $f \in \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ and $q < \infty$, then $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}$ agrees with the Fourier–Lebesgue norm $\|f\|_{\mathscr{F}L^q_{(\omega),x}}$ of f.

For the sake of notational convenience we set

$$\mathcal{B} = \mathscr{F}L^q_{(\omega)} = \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d), \quad \text{and} \quad |\cdot|_{\mathcal{B}(\Gamma)} = |\cdot|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}. \tag{1.2}$$

We let $\Theta_{\mathcal{B}}(f) = \Theta_{\mathscr{F}L^q_{(\omega)}}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that $|f|_{\mathcal{B}(\Gamma)} < \infty$, for some open conical neighborhood $\Gamma = \Gamma_\xi$ of ξ. We also let $\Sigma_{\mathcal{B}}(f)$ be the complement of $\Theta_{\mathcal{B}}(f)$ in $\mathbf{R}^d \setminus 0$. Then $\Theta_{\mathcal{B}}(f)$ and $\Sigma_{\mathcal{B}}(f)$ are open respectively closed subsets in $\mathbf{R}^d \setminus 0$, which are independent of the choice of $x \in \mathbf{R}^d$ in (1.1).

Definition 1.1. Let $s > 1$, $q \in [1, \infty]$, $\mathcal{B}$ be as in (1.2), and let X be an open subset of $\mathbf{R}^d$. If $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, then the wave-front set of ultradistribution $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$, $\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}_{\mathscr{F}L^q_{(\omega)}}(f)$ with respect to $\mathcal{B}$ consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{\mathcal{B}}(\varphi f)$ holds for each $\varphi \in \mathcal{D}^{(s)}(X)$ such that $\varphi(x_0) \neq 0$.

We note that $\mathrm{WF}_{\mathcal{B}}(f)$ is a closed set in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, and if $x \in \mathbf{R}^d$ is fixed and $\omega_0(\xi) = \omega(x, \xi)$, then $\mathrm{WF}_{\mathcal{B}}(f) = \mathrm{WF}_{\mathscr{F}L^q_{(\omega_0)}}(f)$, since $\Sigma_{\mathcal{B}}$ is independent of x.

If $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ is moderated with respect to a weight of polynomial growth at infinity and $f \in \mathcal{D}'(X)$,then $\mathrm{WF}_{\mathscr{F}L^q_{(\omega)}}(f)$ in Definition 1.1 is the same as the wave-front set introduced in [12, Definition 3.1]. Therefore, the information on regularity in the background of wave-front sets of Fourier–Lebesgue type in Definition 1.1 might be compared to the information obtained from the classical wave-front sets, cf. Example 4.9 in [12].

The following theorem shows that wave-front sets with respect to $\mathscr{F}L^q_{(\omega)}$ satisfy appropriate micro-local properties. It also shows that such wave-front sets are decreasing with respect to the parameter q, and increasing with respect to the weight ω.

Theorem 1.2. *Let $s > 1$, $q, r \in [1, \infty]$, X be an open set in $\mathbf{R}^d$ and $\omega, \vartheta \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ be such that*

$$r \leq q, \quad \text{and} \quad \vartheta(x, \xi) \lesssim \omega(x, \xi). \tag{1.3}$$

If $f \in (\mathcal{D}^{\{s\}})'(\mathbf{R}^d)$ and $\varphi \in \mathcal{D}^{(s)}(X)$ then

$$\mathrm{WF}_{\mathscr{F}L^q_{(\omega)}}(\varphi f) \subseteq \mathrm{WF}_{\mathscr{F}L^r_{(\vartheta)}}(f).$$

Proof. Note that it is enough to prove the theorem for $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$ since the statement only involves local assertions. Then it follows that $|\widehat{f}(\xi)\omega(\xi)| \lesssim e^{k|\xi|^{1/s}}$ for every $k > 0$, cf. [2]. Otherwise, the proof is similar to proofs of [12, Theorem 3.2] and [7, Theorem 1.2] and is therefore omitted. □

Next we compare the wave-front sets introduced in Definition 1.1 to the wave-front sets in spaces of ultradistributions given in [5, 10, 15]. In particular, we recover [7, Proposition 1.6 (1)].

Let $s > 1$ and let X be an open subset of $\mathbf{R}^d$. The ultradistribution $f \in (\mathcal{D}^{\{s\}})'(\mathbf{R}^d)$ is s-micro-regular at (x_0, ξ_0) if there exists $\varphi \in \mathcal{D}^{\{s\}}(X)$ such that

$\varphi(x) = 1$ in a neighborhood of x_0 and an open cone Γ which contains ξ_0 such that for some $C, k > 0$,

$$|\widehat{\varphi f}(\xi)| \le Ce^{-k|\xi|^{1/s}}, \quad \xi \in \Gamma. \tag{1.4}$$

The s-wave-front set of f, $\mathrm{WF}_{\{s\}}(f)$ is defined as the complement in $X \times \mathbf{R}^d \setminus 0$ of the set of all (x_0, ξ_0) where f is s-micro-regular, cf. [15, Definition 1.7.1] where the notation $\mathrm{WF}_s(f)$ is used. See also [10] and [5, Chapter 8.4].

Proposition 1.3. *Let $q \in [1, \infty]$, $s > 1$, and let $\omega_k(\xi) \equiv e^{k|\xi|^{1/s}}$ for $\xi \in \mathbf{R}^d$ and $k > 0$. If $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$ then*

$$\bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f) = \mathrm{WF}_{\{s\}}(f). \tag{1.5}$$

Proof. Recall that when k is fixed, the set $\mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f)$ is defined via $\varphi \in \mathcal{D}^{(s)}(X)$, $s > 1$, cf. Definition 1.1. Using the freedom of choice of k one can may show that the set $\bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f)$ is the complement of the set of points $(x_0, \xi_0) \in \mathbf{R}^{2d}$ for which there exists $k > 0$, $\varphi \in \mathcal{D}^{\{s\}}(X)$ such that $\varphi(x) = 1$ in a neighborhood of x_0 and an open cone Γ which contains ξ_0 such that

$$\Big(\int_\Gamma |\widehat{\varphi f}(\xi) e^{k|\xi|^{1/s}}|^q \, d\xi \Big)^{1/q} < \infty. \tag{1.6}$$

The assertion of the proposition is obviously true when $q = \infty$. Then $(x_0, \xi_0) \notin \mathrm{WF}_{\{s\}}(f)$ means $(x_0, \xi_0) \notin \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f)$ where $k > 0$ is the same as in (1.4). On the other hand, if $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$ is of the form $\omega_k(\xi) = e^{k|\xi|^{1/s}}$, $\xi \in \mathbf{R}^d$, for some $k > 0$, then, by Definition 1.1, $(x_0, \xi_0) \notin \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f)$ implies (1.4), that is

$$\mathrm{WF}_{\{s\}}(f) = \bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f).$$

Let $q \in [1, \infty)$ and assume that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f)$ for some $k > 0$. Then for any $\varepsilon > 0$ such that $k - \varepsilon > 0$ we have

$$\int_\Gamma |\widehat{\varphi f}(\xi) \omega_{k-\varepsilon}(\xi)|^q \, d\xi \le \sup_{\xi \in \Gamma} |\widehat{\varphi f}(\xi)|^q e^{kq|\xi|^{1/s}} \int_\Gamma e^{-\varepsilon|\xi|^{1/s}} \, d\xi < \infty,$$

which means that

$$(x_0, \xi_0) \notin \bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f) \quad \text{when} \quad (x_0, \xi_0) \notin \bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f).$$

Therefore

$$\mathrm{WF}_{\mathscr{F}L^q_{(\omega_{k-\varepsilon})}}(f) \subseteq \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f).$$

On the other hand, since the wave-front $\mathrm{WF}_{\mathscr{F}L^q_{(\omega)}}(f)$ is decreasing with respect to the parameter q, see Theorem 1.2, we have

$$\bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f) \subseteq \bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f), \quad q \in [1, \infty].$$

This completes the proof. □

2. Wave-front sets for convolutions in Fourier–Lebesgue spaces

In this section we prove that convolution properties, valid for standard wave-front sets of Hörmander type, also hold for the wave-front sets of Fourier–Lebesgue types, see [13, 14] for related results in the framework of tempered distributions.

We start with convolutions in Fourier–Lebesgue spaces, parallel to [13, Section 2].

Lemma 2.1. *Let $s>1$, $q,q_1,q_2\in[1,\infty]$ and let $\omega,\omega_1,\omega_2\in\mathscr{M}_{\{s\}}(\mathbf{R}^d)$ satisfy*

$$\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}\quad\text{and}\quad \omega(\xi)\lesssim\omega_1(\xi)\omega_2(\xi). \tag{2.1}$$

Then the convolution map $(f_1,f_2)\mapsto f_1*f_2$ from $\mathcal{S}^*(\mathbf{R}^d)\times\mathcal{S}^*(\mathbf{R}^d)$ to $\mathcal{S}^*(\mathbf{R}^d)$ extends to a continuous mapping from $\mathscr{F}L^{q_1}_{(\omega_1)}(\mathbf{R}^d)\times\mathscr{F}L^{q_2}_{(\omega_2)}(\mathbf{R}^d)$ to $\mathscr{F}L^{q}_{(\omega)}(\mathbf{R}^d)$. This extension is unique if $q_1<\infty$ or $q_2<\infty$.

The proof is omitted, since the arguments are the same as in the proof of [13, Lemma 2.1], taking into account that $\mathcal{S}^*$ is dense in $\mathscr{F}L^q_{(\omega)}$ when $q<\infty$.

Theorem 2.2. *Let $s>1$, $q,q_1,q_2\in[1,\infty]$ and let $\omega,\omega_1,\omega_2\in\mathscr{M}_{\{s\}}(\mathbf{R}^d)$ satisfy (2.1). If $f_1\in\mathscr{F}L^{q_1}_{(\omega_1),\mathrm{loc}}(\mathbf{R}^d)$, $f_2\in(\mathcal{D}^*)'(\mathbf{R}^d)$ and f_1 or f_2 have compact supports, then*

$$\mathrm{WF}_{\mathscr{F}L^q_{(\omega)}}(f_1*f_2)\subseteq\{\,(x+y,\xi)\,;\,x\in\operatorname{supp}f_1\ \text{and}\ (y,\xi)\in\mathrm{WF}_{\mathscr{F}L^{q_2}_{(\omega_2)}}(f_2)\,\}.$$

The proof of Theorem 2.2 is similar to the proof of [13, Proposition 2.2]. In order to be self-contained we present here a complete proof.

For the proof we let $B_\varepsilon(x)$ be the open ball in $\mathbf{R}^d$ with radius $\varepsilon>0$ and center at $x\in\mathbf{R}^d$.

Proof. We prove the assertion when $f_2\in(\mathcal{D}^{\{s\}})'(\mathbf{R}^d)$ and leave the other case to the reader. From the local property of the wave-front sets and the assumptions we may assume that both f_1 and f_2 have compact supports.

Let (x_0,ξ_0) be chosen such that $(y,\xi_0)\notin\mathrm{WF}_{\mathscr{F}L^{q_2}_{(\omega_2)}}(f_2)$ when $x\in\operatorname{supp}f_1$ and $x_0=x+y$, and let $F(x,t)=f_2(x-t)f_1(t)$. Since f_1 and f_2 have compact supports, it follows by the definition that for some conical neighborhood Γ_{ξ_0} of ξ_0, for some

$$y_1,\dots,y_n\in\mathbf{R}^d,\qquad r>0,\qquad r_0>0,$$

and suitable

$$\varphi\in\mathcal{D}^{(s)}(B_r(0)),\qquad \varphi_0\in\mathcal{D}^{(s)}(B_{r_0}(x_0)),\qquad\text{and}\quad \varphi_1\in\mathcal{D}^{(s)}(\mathbf{R}^d),$$

with $\varphi_0=1$ in a neighborhood of x_0, the following holds true:

1) $\sum_{j=1}^n\varphi(x-t-y_j)\varphi_1(t)\equiv1$ when $x\in\operatorname{supp}\varphi_0$ and t belongs to

$$\{\,t\in\mathbf{R}^d\,;\,(x,t)\in\operatorname{supp}F\ \text{for some}\ x\in\operatorname{supp}\varphi_0\,\}.$$

2) $|\varphi(\,\cdot\,-y_j)f_2|_{\mathscr{F}L^{q_2,\Gamma}_{(\omega_2)}}<\infty,\qquad 1\le j\le n.$

By Hölder's inequality we get

$$\begin{aligned}
|\varphi_0(f_1 * f_2)|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}} &\lesssim \sum_{j=1}^{n} |((\varphi_1 f_1) * (\varphi(\,\cdot\, - y_j) f_2)|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}} \\
&\lesssim \sum_{j=1}^{n} \|(\mathscr{F}(\varphi_1 f_1)\omega_1)\,(\mathscr{F}(\varphi(\,\cdot\, - y_j) f_2)\omega_2)\|_{L^q(\Gamma)} \\
&\lesssim \sum_{j=1}^{n} \|\mathscr{F}(\varphi_1 f_1)\omega_1\|_{L^{q_1}(\Gamma)} \|\mathscr{F}(\varphi(\,\cdot\, - y_j) f_2)\omega_2\|_{L^{q_2}(\Gamma)}.
\end{aligned}$$

Since the right-hand side is finite in the view of 2), it follows that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathscr{F}L^q_{(\omega)}}(f_1 * f_2)$, and the theorem is proved. □

Remark 2.3. Let $s > 1$, $\omega_1(\xi) = \omega_2(\xi) = e^{k|\xi|^{1/s}}$ and let $f_1 \in (\mathcal{D}^*)'(\mathbf{R}^d)$ and $f_2 \in (\mathcal{E}^*)'(\mathbf{R}^d)$. By letting k tend to infinity, it follows by a straightforward application of (1.5) and Theorem 2.2 that

$$\mathrm{WF}_{\{s\}}(f_1 * f_2) \subseteq \{\, (x+y, \xi)\, ;\, (x,\xi) \in \mathrm{WF}_{\{s\}}(f_1) \text{ and } (y,\xi) \in \mathrm{WF}_{\{s\}}(f_2)\,\}.$$

3. Invariance properties of wave-front sets with respect to modulation spaces

In this section we define wave-front sets with respect to modulation spaces, and show that they coincide with wave-front sets of Fourier–Lebesgue types. See [6, 7, 12] for related results.

3.1. Modulation spaces

In this subsection we consider properties of modulation spaces which will be used in microlocal analysis of ultradistributions.

Let $s > 1$. For a fixed non-zero window $\phi \in \mathcal{S}^*(\mathbf{R}^d)$ the short-time Fourier transform (STFT) of $f \in \mathcal{S}^*(\mathbf{R}^d)$ with respect to the window ϕ is given by

$$V_\phi f(x,\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y)\, \overline{\phi(y-x)}\, e^{-i\langle \xi, y\rangle}\, dy\,, \tag{3.1}$$

The map $(f,\phi) \mapsto V_\phi f$ from $\mathcal{S}^*(\mathbf{R}^d) \times \mathcal{S}^*(\mathbf{R}^d)$ to $\mathcal{S}^*(\mathbf{R}^{2d})$ extends uniquely to a continuous mapping from $(\mathcal{S}^*)'(\mathbf{R}^d) \times (\mathcal{S}^*)'(\mathbf{R}^d)$ to $(\mathcal{S}^*)'(\mathbf{R}^{2d})$ by duality.

Moreover, for a fixed $\phi \in \mathcal{S}^*(\mathbf{R}^d) \setminus 0$, $s \geq 1$, the following characterization of $\mathcal{S}^*(\mathbf{R}^d)$ holds:

$$f \in \mathcal{S}^*(\mathbf{R}^d) \quad \Longleftrightarrow \quad V_\phi f \in \mathcal{S}^*(\mathbf{R}^{2d}). \tag{3.2}$$

We refer to [4, 18] for the proof and more details on STFT in Gelfand–Shilov spaces.

The following proposition is an extension of [1, Proposition 4.2] where it has been proved in the context of Beurling type ultradistributions. Although the proof given in [7] can be easily modified for the Roumieu case, here we give another proof which uses the structural theorem for ultradistributions. This result plays an essential role in the proof of Theorem 3.3 which states that the wave-front sets of Fourier–Lebesgue and modulation space types are the same.

Proposition 3.1. *Let $s > 1$ and $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$. Then the STFT of f with respect to any window $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$, satisfies*

$$|V_\phi f(x,\xi)| \lesssim e^{-h|x|^{1/s}} e^{c|\xi|^{1/s}},$$

for every $h > 0$ and for every $c > 0$.

Proof. Let T_x and M_ξ (the translation and modulation operators) be given by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\xi f(t) = e^{i\langle \xi, t\rangle} f(t)\,. \tag{3.3}$$

Then

$$\partial^\alpha (M_\xi T_x \phi)(y) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (i\xi)^\beta M_\xi T_x (\partial^{\alpha-\beta}\phi)(y),$$

and for each term in the sum we have:

$$|M_\xi T_x (\partial^{\alpha-\beta}\phi)(y)| \le C_h (\alpha-\beta)!^s h^{|\alpha-\beta|} e^{-h|y-x|^{1/s}},$$

for every $h > 0$, see, e.g., [1, 2]. Furthermore, by [1, Theorem 4.1] it follows that every $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$ can be represented as

$$f = \sum_{\alpha \in \mathbf{N}^d} \partial^\alpha \mu_\alpha. \tag{3.4}$$

Here μ_α is a measure such that for every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) > 0$ we have $\operatorname{supp} \mu_\alpha \subseteq K_1$, $K \subset\subset K_1$, for some compact sets K and K_1, and

$$\int_K |d\mu_\alpha| \le C_\varepsilon \varepsilon^\alpha (\alpha!)^{-s}, \tag{3.5}$$

for a suitable constant $C_\varepsilon > 0$ which is independent of α. Furthermore, for each compact set K such that $\operatorname{supp} f \subset\subset K$, the μ_α can be chosen such that $\operatorname{supp} \mu_\alpha \subset K$.

This gives,

$$\begin{aligned}
|V_\phi f(x,\xi)| &\le \sum_{\alpha \in \mathbf{N}^d} |\langle \partial^\alpha \mu_\alpha, M_\xi T_x \phi\rangle| = \sum_{\alpha \in \mathbf{N}^d} |\langle \mu_\alpha, \partial^\alpha (M_\xi T_x \phi)\rangle| \\
&\le \sum_{\alpha \in \mathbf{N}^d} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |\xi^\beta| \int_{\mathbf{R}^d} (M_\xi T_x \partial_y^{\alpha-\beta}\phi)(y)| \, |d\mu_\alpha(y)| \\
&\le \sum_{\alpha \in \mathbf{N}^d} h^{|\alpha|} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |(\xi/h)^\beta| (\alpha-\beta)!^s \int_{\mathbf{R}^d} e^{-h|y-x|^{1/s}} \, |d\mu_\alpha(y)|.
\end{aligned}$$

Since $\sum_{\beta\le\alpha}\binom{\alpha}{\beta}=2^{|\alpha|}$, using (3.5) and the fact that $(\alpha-\beta)!\beta!\le\alpha!$ we get

$$
\begin{aligned}
|V_\phi f(x,\xi)| &\lesssim \sum_{\alpha\in\mathbf{N}^d} h^{|\alpha|}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}|(i\xi/h)^\beta|\frac{\alpha!^s}{\beta!^s}e^{-h|x|^{1/s}}\int_K e^{h|y|^{1/s}}\,|d\mu_\alpha(y)|\\
&\lesssim e^{-h|x|^{1/s}}\sum_{\alpha\in\mathbf{N}^d}(2|\varepsilon|h)^{|\alpha|}\sum_{\beta\le\alpha}\frac{|(i\xi/h)^\beta|}{\beta!^s}. \qquad (3.6)
\end{aligned}
$$

Let $c>0$ be arbitrary. Since $h>0$ can be chosen arbitrary, we have

$$
\sum_{\beta\in\mathbf{N}^d}\frac{|(i\xi/h)^\beta|}{\beta!^s}\le\Bigg(\sum_{\beta\in\mathbf{N}^d}\frac{|\xi/h|^{|\beta/s|}}{\beta!}\Bigg)^s=e^{s|\xi/h|^{1/s}}\le e^{c|\xi|^{1/s}},
$$

when h is chosen large enough. Hence, (3.6) gives

$$
|V_\phi f(x,\xi)|\le Ce^{-h\cdot|x|^{1/s}}e^{c|\xi|^{1/s}}\sum_{\alpha\in\mathbf{N}^d}(2|\varepsilon|h)^{|\alpha|}.
$$

The result now follows by choosing $|\varepsilon|<1/(2h)$. □

Now we recall the definition of modulation spaces. Let $s>1$, $\omega\in\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $p,q\in[1,\infty]$, and the window $\phi\in\mathcal{S}^{(s)}(\mathbf{R}^d)\setminus 0$ be fixed. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is the set of all ultra-distributions $f\in(\mathcal{S}^*)'(\mathbf{R}^d)$ such that

$$
\|f\|_{M^{p,q}_{(\omega)}}=\|f\|_{M^{p,q,\phi}_{(\omega)}}\equiv\|V_\phi f\|_{L^{p,q}_{(\omega)}}\equiv\|V_\phi f\,\omega\|_{L^{p,q}_1}<\infty. \qquad (3.7)
$$

Here $\|\cdot\|_{L^{p,q}_1}$ is the norm given by

$$
\|F\|_{L^{p,q}_1}\equiv\Big(\int_{\mathbf{R}^d}\Big(\int_{\mathbf{R}^d}|F(x,\xi)|^p\,dx\Big)^{q/p}d\xi\Big)^{1/q},
$$

when $F\in L^1_{\rm loc}(\mathbf{R}^{2d})$ (with obvious interpretation when $p=\infty$ or $q=\infty$). Furthermore, the modulation space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ consists of all $f\in(\mathcal{S}^*)'(\mathbf{R}^d)$ such that

$$
\|f\|_{W^{p,q}_{(\omega)}}=\|f\|_{W^{p,q,\phi}_{(\omega)}}\equiv\|V_\phi f\,\omega\|_{L^{p,q}_2}<\infty,
$$

where $\|\cdot\|_{L^{p,q}_2}$ is the norm given by

$$
\|F\|_{L^{p,q}_2}\equiv\Big(\int_{\mathbf{R}^d}\Big(\int_{\mathbf{R}^d}|F(x,\xi)|^q\,d\xi\Big)^{p/q}dx\Big)^{1/p},
$$

when $F\in L^1_{\rm loc}(\mathbf{R}^{2d})$.

If $s>1$, $p,q\in[1,\infty]$ and $\omega\in\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, then one can show that the spaces $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$, $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are locally the same, in the sense that

$$
\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)\bigcap(\mathcal{E}^*)'(\mathbf{R}^d)=M^{p,q}_{(\omega)}(\mathbf{R}^d)\bigcap(\mathcal{E}^*)'(\mathbf{R}^d)=W^{p,q}_{(\omega)}(\mathbf{R}^d)\bigcap(\mathcal{E}^*)'(\mathbf{R}^d).
$$

This follows by similar arguments as in [16] (and replacing the space of polynomially moderated weights $\mathscr{P}(\mathbf{R}^{2d})$ with $\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$). Later on we extend these properties in the context of wave-front sets and recover the equalities above.

The proof of the next proposition can be found in [1]. It concerns topological questions of modulation spaces, and properties of the adjoint of the short-time Fourier transform ${}^tV_\phi F$, Here we recall that

$$({}^tV_\phi F, f) \equiv (F, V_\phi f), \qquad f \in \mathcal{S}^*(\mathbf{R}^d),$$

when $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ and $F(x,\xi) \in L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$.

Proposition 3.2 ([1]). *Let $s > 1$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $p,q \in [1,\infty]$, and $\phi, \phi_1 \in \mathcal{S}^{(s)}(\mathbf{R}^d)$, with $(\phi,\phi_1)_{L^2} \neq 0$. Then the following is true:*

1. *the operator ${}^tV_\phi$ from $\mathcal{S}^{(s)}(\mathbf{R}^{2d})$ to $\mathcal{S}^{(s)}(\mathbf{R}^d)$ extends uniquely to a continuous operator from $L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ to $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and*
$$\|{}^tV_\phi F\|_{M^{p,q}_{(\omega)}} \lesssim \|V_{\phi_1}\phi\|_{L^1_{(v)}} \|F\|_{L^{p,q}_{(\omega)}}; \tag{3.8}$$
2. *$M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is a Banach space whose definition is independent on the choice of window $\phi \in \mathcal{S}^{(s)} \setminus 0$;*
3. *the set of windows can be extended from $\mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ to $M^1_{(v)}(\mathbf{R}^d) \setminus 0$.*

3.2. Wave-front sets with respect to modulation spaces

Next we define wave-front sets with respect to modulation spaces and show that they agree with corresponding wave-front sets of Fourier–Lebesgue types. As in [7], we show that [12, Theorem 6.1] holds if the weights of polynomial growth are replaced by more general submultiplicative weights.

Let $s > 1$, $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$, $\omega \in \mathscr{M}_{\{s\}}$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone and let $p,q \in [1,\infty]$. For any $f \in (\mathcal{S}^*)'(\mathbf{R}^d)$ we set

$$|f|_{\mathcal{B}(\Gamma)} = |f|_{\mathcal{B}(\phi,\Gamma)} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}(\mathbf{R}^d \times \Gamma)} \quad \text{when} \quad \mathcal{B} = M^{p,q}_{(\omega)} = M^{p,q}_{(\omega)}(\mathbf{R}^d). \tag{3.9}$$

Thus we define by $|\cdot|_{\mathcal{B}(\Gamma)}$ a semi-norm on $(\mathcal{S}^*)'(\mathbf{R}^d)$ which might attain the value $+\infty$. If $\Gamma = \mathbf{R}^d \setminus 0$ and $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$, then $|f|_{\mathcal{B}(\Gamma)} = \|f\|_{M^{p,q}_{(\omega)}}$.

We also set

$$|f|_{\mathcal{B}(\Gamma)} = |f|_{\mathcal{B}(\phi,\Gamma)} \equiv \Big(\int_{\mathbf{R}^d} \Big(\int_\Gamma |V_\phi f(x,\xi)\omega(x,\xi)|^q \, d\xi \Big)^{p/q} dx \Big)^{1/p}$$
$$\text{when} \quad \mathcal{B} = W^{p,q}_{(\omega)} = W^{p,q}_{(\omega)}(\mathbf{R}^d) \tag{3.10}$$

and note that similar properties hold for this semi-norm compared to (3.9).

Let $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $p,q \in [1,\infty]$, $f \in (\mathcal{S}^*)'(\mathbf{R}^d)$, and let $\mathcal{B} = M^{p,q}_{(\omega)}$ or $\mathcal{B} = W^{p,q}_{(\omega)}$. Then $\Theta_\mathcal{B}(f)$, $\Sigma_\mathcal{B}(f)$ and the wave-front set $\mathrm{WF}_\mathcal{B}(f)$ of f with respect to the modulation space $\mathcal{B}$ are defined in the same way as in Section 1, after replacing the semi-norms of Fourier–Lebesgue types in (1.1) with the semi-norms in (3.9) or (3.10) respectively.

The next result is proved in [12] in the case of tempered distributions and in [7] in the cases of ultradistributions of Beurling type.

Theorem 3.3. *Let $s > 1$, $p, q \in [1, \infty]$ and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$. If $f \in (\mathcal{D}^*)'(\mathbf{R}^d)$ then*

$$\mathrm{WF}_{\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)}(f) = \mathrm{WF}_{\mathcal{B}}(f), \tag{3.11}$$

where $\mathcal{B} = M^{p,q}_{(\omega)}(\mathbf{R}^d)$ or $\mathcal{B} = W^{p,q}_{(\omega)}(\mathbf{R}^d)$. In particular, $\mathrm{WF}_{\mathcal{B}}(f)$ is independent of p and $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ in (3.9) and (3.10).

The proof follows the same ideas as in the proof of [12, Theorem 6.1] and [7, Theorem 2.3]. The main part concerns proving that the wave-front sets of modulation types are independent of the choice of window $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$. Therefore we need the result of Proposition 3.1. Note also that the dual pairing between $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ and $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$ is well defined. Otherwise, the proof of Theorem 3.3 differs from the above-mentioned proofs in small technical details and it is therefore omitted.

Corollary 3.4. *Let $s > 1$, $p, q \in [1, \infty]$, and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$. If $f \in (\mathcal{E}^*)'(\mathbf{R}^d)$, then*

$$f \in \mathcal{B} \quad \Longleftrightarrow \quad \mathrm{WF}_{\mathcal{B}}(f) = \emptyset,$$

where $\mathcal{B}$ is equal to $\mathscr{F}L^q_{(\omega)}$, $M^{p,q}_{(\omega)}$ or $W^{p,q}_{(\omega)}$.

In particular, we recover [12, Corollary 6.2] and [16, Theorem 2.1 and Remark 4.6], see also [8].

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Ordinary Differential Equations in Algebras of Generalized Functions

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Abstract. A local existence and uniqueness theorem for ODEs in the special algebra of generalized functions is established, as well as versions including parameters and dependence on initial values in the generalized sense. Finally, a Frobenius theorem is proved. In all these results, composition of generalized functions is based on the notion of c-boundedness.

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1. Introduction

At the time of their introduction in the 1980s ([2], [3]), algebras of generalized functions in the Colombeau setting were primarily intended as a tool for treating nonlinear (partial) differential equations in the presence of singularities. Since then, many types of differential equations have been studied in the Colombeau setting (see [16], together with the references given therein, and the first part of [9] for a variety of examples). Nevertheless, the authors of [10] feel compelled to declare some 15 years later that "a refined theory of local solutions of ODEs is not yet fully developed" (p. 80). In fact, this state of affairs has not changed much since then. It is the purpose of this article to lay the foundations for such a theory, with composition of generalized functions based on the concept of c-boundedness.

As the basic object of study one may view the differential equation $\dot{u}(t) = F(t, u(t))$ with initial condition $u(\widetilde{t_0}) = \widetilde{x}_0$. Since $u(t)$ gets plugged into the second slot of F it is evident that one has to adopt a suitable concept of composition of generalized functions in order to give meaning to the right-hand side of the ODE, keeping in mind that in general, the composition of generalized functions is not defined.

One way of handling the composition $u \circ v$ of generalized functions u, v is to assume the left member u to be tempered (see [10, Subsection 1.2.3] for a definition). In this setting, a number of results on ODEs have been established, including a global existence and uniqueness theorem ([12, Theorem 3.1], [10, Theorem 1.5.2]). A more recent concept of composing generalized functions goes back to Aragona and Biagoni (cf. [1]): Here, the right member v is assumed to be *compactly bounded* (*c-bounded*) into the domain of u (see Section 2 for details); then the composition $u \circ v$ is defined as a generalized function. It is this latter approach we will adopt in this article. It seems to be suited better to local questions; moreover, the concept of c-boundedness permits an intrinsic generalization to smooth manifolds ([10, Subsection 3.2.4], contrary to that of tempered generalized distributions.

In a number of contributions, the notion of c-boundedness has already been taken as the basis for the treatment of generalized ODEs. The first instance, dating back to [15], served as a tool for an application to a problem in general relativity, see [10, Lemma 5.3.1] and the improved version in [6, Lemma 4.2]. Theorem 3.1 of [14] – where a theory of singular ordinary differential equations on differentiable manifolds is developed – provides a global existence and uniqueness result for autonomous ODEs on $\mathbb{R}^n$. Theorem 1.9 in [11] establishes existence of a solution assuming an L^1-bound (as a function of t, uniformly on $\mathbb{R}^n$ with respect to the second slot) on the representatives of F. Finally, the study of the Hamilton–Jacobi equation in the framework of generalized functions in [7] led to some local existence and uniqueness results for ODEs, in a setting adapted to this particular problem. We will discuss one of these Theorems in more detail in Section 3.

A special feature of the existence and uniqueness results 3.1 and 3.8 in Section 3 consists in their capacity to simultaneously allow generalized values both for $\widetilde{t}_0$ and $\widetilde{x}_0$ in the initial conditions, and to have, nevertheless, the domain of existence of the local solution equal to the one in the classical case.

The results of this article may be viewed as extending and refining the material of Chapter 5 of [4]. Section 2 makes available the necessary technical prerequisites. Local existence and uniqueness results for ODEs in the c-bounded setting are the focus of Section 3: Following the basic theorem handling the initial value problem mentioned above, two more statements are established covering ODEs with parameters and $\mathcal{G}$-dependence of the solution on initial values, respectively. Section 4, finally, presents a generalized version of the theorem of Frobenius, also in the c-bounded setting.

2. Notation and preliminaries

For subsets A, B of a topological space X, we write $A \subset\subset B$ if A is a compact subset of the interior B° of B. By $B_r(x)$ we denote the open ball with centre x and radius $r > 0$. We will make free use of the exponential law and the argument swap (flip), i.e., for functions $f : X \times Y \to Z$ we will write $f(x)(y) = f(x, y) = f^{\mathrm{fl}}(y, x) = f^{\mathrm{fl}}(y)(x)$.

Generally, the special Colombeau algebra can be constructed with real-valued or with complex-valued functions. For the purposes of the present article we consider the real version only. Concerning fundamentals of (special) Colombeau algebras, we follow [10, Subsection 1.2].

In particular, for defining the special Colombeau algebra $\mathcal{G}(U)$ on a given (non-empty) open subset U of $\mathbb{R}^n$, we set $\mathcal{E}(U) := \mathcal{C}^\infty(U,\mathbb{R})^{(0,1]}$ and

$$\mathcal{E}_M(U) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(U) \mid \forall K \subset\subset U \ \forall \alpha \in \mathbb{N}_0^n \ \exists N \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\},$$

$$\mathcal{N}(U) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(U) \mid \forall K \subset\subset U \ \forall \alpha \in \mathbb{N}_0^n \ \forall m \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}.$$

Elements of $\mathcal{E}_M(U)$ and $\mathcal{N}(U)$ are called *moderate* and *negligible functions*, respectively. By [10, Theorem 1.2.3], $(u_\varepsilon)_\varepsilon$ is already an element of $\mathcal{N}(U)$ if the above conditions are satisfied for $\alpha = 0$. $\mathcal{E}_M(U)$ is a subalgebra of $\mathcal{E}(U)$, $\mathcal{N}(U)$ is an ideal in $\mathcal{E}_M(U)$. The *special Colombeau algebra* on U is defined as

$$\mathcal{G}(U) := \mathcal{E}_M(U)/\mathcal{N}(U).$$

The class of a moderate net $(u_\varepsilon)_\varepsilon$ in this quotient space will be denoted by $[(u_\varepsilon)_\varepsilon]$. A generalized function on some open subset U of $\mathbb{R}^n$ with values in $\mathbb{R}^m$ is given as an m-tuple $(u_1,\dots,u_m) \in \mathcal{G}(U)^m$ of generalized functions $u_j \in \mathcal{G}(U)$ where $j = 1,\dots,m$.

$U \to \mathcal{G}(U)$ is a fine sheaf of differential algebras on $\mathbb{R}^n$.

The composition $v \circ u$ of two arbitrary generalized functions is not defined, not even if v is defined on the whole of $\mathbb{R}^m$ (i.e., if $u \in \mathcal{G}(U)^m$ and $v \in \mathcal{G}(\mathbb{R}^m)^l$). A convenient condition for $v \circ u$ to be defined is to require u to be "compactly bounded" (c-bounded) into the domain of v. Since there is a certain inconsistency in [10] concerning the precise description of c-boundedness (see [5, Section 2] for details) we include the explicit definition of this important property below. For a full discussion, see again [5, Section 2].

Definition 2.1. Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

(1) An element $(u_\varepsilon)_\varepsilon$ of $\mathcal{E}_M(U)^m$ is called *c-bounded from U into V* if the following conditions are satisfied:
   (i) There exists $\varepsilon_0 \in (0,1]$, such that $u_\varepsilon(U) \subseteq V$ for all $\varepsilon \leq \varepsilon_0$.
   (ii) For every $K \subset\subset U$ there exist $L \subset\subset V$ and $\varepsilon_0 \in (0,1]$ such that $u_\varepsilon(K) \subseteq L$ for all $\varepsilon \leq \varepsilon_0$.

   The collection of c-bounded elements of $\mathcal{E}_M(U)^m$ is denoted by $\mathcal{E}_M[U,V]$.

(2) An element u of $\mathcal{G}(U)^m$ is called *c-bounded from U into V* if it has a representative which is c-bounded from U into V. The space of all c-bounded generalized functions from U into V will be denoted by $\mathcal{G}[U,V]$.

Proposition 2.2. *Let $u \in \mathcal{G}(U)^m$ be c-bounded into V and let $v \in \mathcal{G}(V)^l$, with representatives $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$, respectively. Then the composition*

$$v \circ u := [(v_\varepsilon \circ u_\varepsilon)_\varepsilon]$$

is a well-defined generalized function in $\mathcal{G}(U)^l$.

Generalized functions can be composed with smooth classical functions provided they do not grow "too fast": The space of *slowly increasing smooth functions* is given by

$$\mathcal{O}_M(\mathbb{R}^n) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n) \,|\, \forall\, \alpha \in \mathbb{N}_0^n \;\exists\, N \in \mathbb{N}_0 \;\exists\, C > 0 : \\ |\partial^\alpha f(x)| \le C(1+|x|)^N \;\forall\, x \in \mathbb{R}^n\}.$$

Proposition 2.3. *If* $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)^m$ *and* $v \in \mathcal{O}_M(\mathbb{R}^m)$*, then*

$$v \circ u := [(v \circ u_\varepsilon)_\varepsilon]$$

is a well-defined generalized function in $\mathcal{G}(U)$*.*

We call $\mathcal{R} := \mathcal{E}_M/\mathcal{N}$ the ring of *generalized numbers*, where

$$\mathcal{E}_M := \{(r_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} \,|\, \exists\, N \in \mathbb{N} : \; |r_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\},$$
$$\mathcal{N} := \{(r_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} \,|\, \forall\, m \in \mathbb{N} : \; |r_\varepsilon| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}.$$

For $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ and $x_0 \in U$, the *point value of* u *at* x_0 is defined as the class of $(u_\varepsilon(x_0))_\varepsilon$ in $\mathcal{R}$.

On

$$U_M := \{(x_\varepsilon)_\varepsilon \in U^{(0,1]} \,|\, \exists\, N \in \mathbb{N} : \; |x_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}$$

we introduce an equivalence relation by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \;\Leftrightarrow\; \forall\, m \in \mathbb{N} : \; |x_\varepsilon - y_\varepsilon| = O(\varepsilon^m) \text{ as } \varepsilon \to 0$$

and denote by $\widetilde{U} := U_M/\sim$ the set of *generalized points*. For $U = \mathbb{R}$ we have $\widetilde{\mathbb{R}} = \mathcal{R}$. Thus, we have the canonical identification $\widetilde{\mathbb{R}^n} = \widetilde{\mathbb{R}}^n = \mathcal{R}^n$.

The set of *compactly supported points* is

$$\widetilde{U}_c := \{\widetilde{x} = [(\widetilde{x}_\varepsilon)_\varepsilon] \in \widetilde{U} \,|\, \exists\, K \subset\subset U \;\exists\, \varepsilon_0 \in (0,1] \;\forall\, \varepsilon \le \varepsilon_0 : \; x_\varepsilon \in K\}.$$

Obviously, for $u \in \mathcal{G}(U)$ and $\widetilde{x} \in \widetilde{U}_c$, $u(\widetilde{x})$ is a generalized number, the *generalized point value of* u *at* $\widetilde{x}$.

A point $\widetilde{x} \in \widetilde{U}_c$ is called *near-standard* if there exists $x \in U$ such that $x_\varepsilon \to x$ as $\varepsilon \to 0$ for one (thus, for every) representative $(x_\varepsilon)_\varepsilon$ of x. In this case we write $\widetilde{x} \approx x$.

Two generalized functions are equal in the Colombeau algebra if and only if their generalized point values coincide at all compactly supported points ([10, Theorem 1.2.46]). By [13], it is even sufficient to check the values at all near-standard points. We will need a slightly refined result which is easy to prove using the techniques of [10, Theorem 1.2.46] and [13]:

Proposition 2.4. *Let* $u \in \mathcal{G}(U \times V)$*. Then*

$$u = 0 \text{ in } \mathcal{G}(U \times V) \quad\Leftrightarrow\quad u(\,.\,,\widetilde{y}) = 0 \text{ in } \mathcal{G}(U) \text{ for all near-standard points } \widetilde{y} \in \widetilde{V}_c.$$

3. Local existence and uniqueness results for ODEs

In the first theorem of this section we give sufficient conditions to guarantee a (unique) solution of the local initial value problem

$$\dot{u}(t) = F(t, u(t)), \quad u(\widetilde{t}_0) = \widetilde{x}_0, \tag{1}$$

where I is an open interval in $\mathbb{R}$, U an open subset of $\mathbb{R}^n$, $F \in \mathcal{G}(I \times U)^n$, $\widetilde{t}_0 \in \widetilde{I}_c$ and $\widetilde{x}_0 \in \widetilde{U}_c$. A generalized function $u \in \mathcal{G}[J, U]$ (where J is some open subinterval of I) is called a (local) solution of (1) on J around $\widetilde{t}_0 \in \widetilde{I}_c$ with initial value $\widetilde{x}_0$ if the differential equation in (1) is satisfied in $\mathcal{G}(J)^n$ and the initial condition in (1) is satisfied in the set $\widetilde{U}$ of generalized points.

Reflecting our decision to employ the concept of c-boundedness to ensure the existence of compositions, a solution on some subinterval J of I will be a c-bounded generalized function from J into U satisfying (1). Due to the c-boundedness of u the requirement for $\widetilde{x}_0$ to be compactly supported in fact does not constitute a restriction.

Theorem 3.1 generalizes Theorem 5.2 of [4] insofar as the domain of existence of the local solution precisely equals the one in the classical case whereas the solution in [4] is only defined on a strictly smaller interval. Moreover, the present version establishes uniqueness with respect to the largest sensible target space (i.e., U), as opposed to the more restricted statement in [4].

Theorem 3.1. *Let I be an open subinterval of $\mathbb{R}$, U an open subset of $\mathbb{R}^n$, $\widetilde{t}_0$ a near-standard point in $\widetilde{I}_c$ with $\widetilde{t}_0 \approx t_0 \in I$, $\widetilde{x}_0 \in \widetilde{U}_c$ and $F \in \mathcal{G}(I \times U)^n$.*

Let α be chosen such that $[t_0 - \alpha, t_0 + \alpha] \subset\subset I$. Let $(\widetilde{x}_{0\varepsilon})_\varepsilon$ be a representative of $\widetilde{x}_0$ and $L \subset\subset U$, $\varepsilon_0 \in (0,1]$ such that $\widetilde{x}_{0\varepsilon} \in L$ for all $\varepsilon \le \varepsilon_0$. With $\beta > 0$ satisfying $L_\beta := L + \overline{B_\beta(0)} \subset\subset U$ set

$$Q := [t_0 - \alpha, t_0 + \alpha] \times L_\beta \qquad (\subset\subset I \times U).$$

Assume that F has a representative $(F_\varepsilon)_\varepsilon$ satisfying

$$\sup_{(t,x)\in Q} |F_\varepsilon(t, x)| \le a \qquad (\varepsilon \le \varepsilon_0) \tag{2}$$

for some constant $a > 0$. Then the following holds:

(i) *The initial value problem*

$$\dot{u}(t) = F(t, u(t)), \quad u(\widetilde{t}_0) = \widetilde{x}_0, \tag{3}$$

has a solution $u \in \mathcal{G}[J, W]$ where $J = (t_0 - h, t_0 + h)$ with $h = \min(\alpha, \frac{\beta}{a})$ and $W = L + B_\beta(0)$.

(ii) *Every solution of (3) in $\mathcal{G}[J, U]$ is already an element of $\mathcal{G}[J, W]$.*

(iii) *The solution of (3) is unique in $\mathcal{G}[J, U]$ if, in addition to (2),*

$$\sup_{(t,x)\in J\times W} |\partial_2 F_\varepsilon(t, x)| = O(|\log \varepsilon|) \tag{4}$$

holds.

Proof. Throughout the proof, it suffices to consider only values of ε not exceeding ε_0. Moreover, we can assume without loss of generality that

$$|\widetilde{t}_{0\varepsilon} - t_0| \le h/4 \quad \text{holds for all } \varepsilon \le \varepsilon_0. \tag{5}$$

(i) In a first step we fix ε and solve the (classical) initial value problem

$$\dot{u}_\varepsilon(t) = F_\varepsilon(t, u_\varepsilon(t)), \quad u_\varepsilon(\widetilde{t}_{0\varepsilon}) = \widetilde{x}_{0\varepsilon}, \tag{6}$$

on a suitable subinterval of $[t_0 - h, t_0 + h]$. To this end, set

$$\delta_\varepsilon := \sup\{|\widetilde{t}_{0\varepsilon'} - t_0| \mid 0 < \varepsilon' \le \varepsilon\} \quad \text{and} \quad J_\varepsilon := [t_0 - h + \delta_\varepsilon, t_0 + h - \delta_\varepsilon],$$

both for $\varepsilon \le \varepsilon_0$; note that $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. By this choice, we have $J_\varepsilon \subseteq [t_0 - h, t_0 + h]$. Indeed, from $t \in J_\varepsilon$ we infer $|t - \widetilde{t}_{0\varepsilon}| \le |t - t_0| + |t_0 - \widetilde{t}_{0\varepsilon}| \le h - \delta_\varepsilon + \delta_\varepsilon$. The solution u_ε of (6) now is obtained as the fixed point of the operator $T_\varepsilon : X_\varepsilon \to X_\varepsilon$ defined by

$$(T_\varepsilon f)(t) := \widetilde{x}_{0\varepsilon} + \int_{\widetilde{t}_{0\varepsilon}}^t F_\varepsilon(s, f(s))\,\mathrm{d}s \qquad (t \in J_\varepsilon)$$

where $X_\varepsilon := \{f : J_\varepsilon \to L_\beta \mid f \text{ is continuous}\}$ becomes a complete metric space when being equipped with the metric $d(f,g) := \|f - g\|_\infty = \sup_{t \in J_\varepsilon} |f(t) - g(t)|$. That T_ε in fact maps X_ε into X_ε is immediate from

$$|(T_\varepsilon f)(t) - \widetilde{x}_{0\varepsilon}| \le \left| \int_{\widetilde{t}_{0\varepsilon}}^t |F_\varepsilon(s, f(s))|\,\mathrm{d}s \right| \le a \cdot |t - \widetilde{t}_{0\varepsilon}| \tag{7}$$

by noting that $a \cdot |t - \widetilde{t}_{0\varepsilon}| \le ah \le \beta$ for $t \in J_\varepsilon$.

Now the existence of a fixed point of T_ε (hence, of a solution of (6)) follows from Weissinger's fixed point theorem ([17, §1], [8, I.1.6 (A5)]) by the following argument: A variant of [10, Lemma 3.2.47] referring only to the second slot (see [4, Remark 3.12] for an explicit version) yields a positive constant γ (depending on ε) such that $|F_\varepsilon(t,x) - F_\varepsilon(t,y)| \le \gamma \cdot |x - y|$ for all $(t,x), (t,y) \in Q$. From this we derive, by induction, $|(T_\varepsilon^k f)(t) - (T_\varepsilon^k g)(t)| \le \frac{\gamma^k}{k!}(t - \widetilde{t}_{0\varepsilon})^k \|f - g\|_\infty$ for $t \in [\widetilde{t}_{0\varepsilon}, t_0 + h - \delta_\varepsilon]$ and $k \in \mathbb{N}_0$. The case of $t \in [t_0 - h + \delta_\varepsilon, \widetilde{t}_{0\varepsilon}]$ being similar, we finally arrive at $\|T_\varepsilon^k f - T_\varepsilon^k g\|_\infty \le \frac{(h\gamma)^k}{k!} \|f - g\|_\infty$ which, due to $\sum_{k=0}^\infty \frac{(h\gamma)^k}{k!} = e^{h\gamma} < \infty$, suffices for an appeal to Weissinger's theorem. We obtain a solution u_ε of (6) on J_ε taking values in L_β. Moreover, $u_\varepsilon(t) \in W := L + B_\beta(0)$ for $t \in J_\varepsilon^\circ$ by (7).

If $\delta_\varepsilon = 0$ (i.e., if t_0 is standard) then u_ε is defined on $[t_0 - h, t_0 + h]$ and we set $\widetilde{u}_\varepsilon := u_\varepsilon$; by (7), $\widetilde{u}_\varepsilon(J) \subseteq W$. In the case $\delta_\varepsilon > 0$, Lemma 3.3 provides $\widetilde{u}_\varepsilon \in \mathcal{C}^\infty([t_0 - h, t_0 + h], W)$ being equal to u_ε on $\widetilde{J}_\varepsilon := [t_0 - h + 2\delta_\varepsilon, t_0 + h - 2\delta_\varepsilon]$. In both cases, $\widetilde{t}_{0\varepsilon} \in \widetilde{J}_\varepsilon$, $\widetilde{u}_\varepsilon(\widetilde{t}_{0\varepsilon}) = \widetilde{x}_{0\varepsilon}$ and $\dot{\widetilde{u}}_\varepsilon(t) = F_\varepsilon(t, \widetilde{u}_\varepsilon(t))$ holds on $\widetilde{J}_\varepsilon$.

In order to show that $(\widetilde{u}_\varepsilon)_\varepsilon$ is moderate on $J = (t_0 - h, t_0 + h)$ it suffices to establish the corresponding estimates on each $\widetilde{J}_{\varepsilon*}$ (with $\varepsilon^* \le \varepsilon_0$), allowing us to deal with u_ε rather than with $\widetilde{u}_\varepsilon$ for all $\varepsilon \le \varepsilon^*$. Thus, let $t \in \widetilde{J}_{\varepsilon^*}$ and $\varepsilon \le \varepsilon^*$. We have $u_\varepsilon(t) \in L_\beta$ and $|\dot{u}_\varepsilon(t)| \le a$. Via the moderateness estimates for $\partial_i F_\varepsilon$ $(i = 1, 2)$ we now obtain, by differentiating $\dot{u}_\varepsilon(t) = F_\varepsilon(t, u_\varepsilon(t))$, an estimate of the form

$$|\ddot{u}_\varepsilon(t)| \le |\partial_1 F_\varepsilon(t, u_\varepsilon(t))| + |\partial_2 F_\varepsilon(t, u_\varepsilon(t))| \cdot |\dot{u}_\varepsilon(t)| \le C\varepsilon^{-N}$$

with constants $C > 0$ and $N \in \mathbb{N}$ not depending on ε. The estimates for the higher-order derivatives of u_ε are now obtained inductively by differentiating the equation for $\ddot{u}_\varepsilon$.

Concerning c-boundedness of $(\widetilde{u}_\varepsilon)_\varepsilon$ from J into W let $J^\dagger := [t_0 - h', t_0 + h']$ with $\frac{h}{4} < h' < h$. For ε small enough as to satisfy $2\delta_\varepsilon \leq h - h'$, we have $J^\dagger \subseteq \widetilde{J}_\varepsilon$. (7) now yields $\widetilde{u}_\varepsilon(J^\dagger) = u_\varepsilon(J^\dagger) \subseteq L + \overline{B_{a(h'+\delta_\varepsilon)}} \subset\subset L + B_\beta(0)$.

Now that we have shown that the net $(\widetilde{u}_\varepsilon)_\varepsilon$ represents a member of $\mathcal{G}[J, W]$ ($\subseteq \mathcal{G}[J, U]$), it follows from the result established for fixed ε that the class of $(\widetilde{u}_\varepsilon)_\varepsilon$ is a solution of (3) on J in the sense specified at the beginning of this section: Due to the fact that equality in Colombeau spaces involves null estimates only on compact subsets of the domain, it indeed suffices that every $\widetilde{u}_\varepsilon$ satisfies the (classical) equation on $\widetilde{J}_\varepsilon$, taking into account $\delta_\varepsilon \to 0$.

(ii) Assume that $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}[J, U]$ satisfies $\dot{v}(t) = F(t, v(t))$ and $v(\widetilde{t}_0) = \widetilde{x}_0$. With $\widetilde{t}_{0\varepsilon}$, $\widetilde{x}_{0\varepsilon}$ and F_ε as in part (i) we have $v_\varepsilon(\widetilde{t}_{0\varepsilon}) = \widetilde{x}_{0\varepsilon} + \widetilde{n}_\varepsilon$ and $\dot{v}_\varepsilon(t) = F_\varepsilon(t, v_\varepsilon(t)) + n_\varepsilon(t)$ for some $(\widetilde{n}_\varepsilon)_\varepsilon \in \mathcal{N}^n$ and $(n_\varepsilon)_\varepsilon \in \mathcal{N}(J)^n$, respectively.

In order to show that $v \in \mathcal{G}[J, W]$ with $W = L + B_\beta(0)$ we again choose $J^\dagger = [t_0 - h', t_0 + h'] \subset\subset J$ with $\frac{h}{4} < h' < h$. Setting $\delta := \frac{a}{2}(h - h')$, we select $\varepsilon_1 (\leq \varepsilon_0)$ such that for all $\varepsilon \leq \varepsilon_1$, the three conditions $|\widetilde{n}_\varepsilon| < \frac{\delta}{3}$, $\int_{J^\dagger} |\widetilde{n}_\varepsilon(s)| \,\mathrm{d}s < \frac{\delta}{3}$ and $a|\delta_\varepsilon| < \frac{\delta}{3}$ are satisfied. Now for $\varepsilon \leq \varepsilon_1$, we claim that $|v_\varepsilon(t) - \widetilde{x}_{0\varepsilon}| \leq \frac{a}{2}(h + h')$ holds for all $t \in J^\dagger_+ := [\widetilde{t}_{0\varepsilon}, t_0 + h']$. If $|v_\varepsilon(t) - \widetilde{x}_{0\varepsilon}| < \frac{a}{2}(h + h')$ for all $t \in J^\dagger_+$, then we are done. Otherwise, choose t^* minimal in $J^\dagger_+$ with $|v_\varepsilon(t^*) - \widetilde{x}_{0\varepsilon}| = \frac{a}{2}(h + h')$. We demonstrate that, in fact, $t^* = t_0 + h'$. From the estimate

$$\begin{aligned}\frac{a}{2}(h + h') = |v_\varepsilon(t^*) - \widetilde{x}_{0\varepsilon}| &\leq |\widetilde{n}_\varepsilon| + \int_{\widetilde{t}_{0\varepsilon}}^{t^*} |\widetilde{n}_\varepsilon(s)| \,\mathrm{d}s + \int_{\widetilde{t}_{0\varepsilon}}^{t^*} |F_\varepsilon(t, \underbrace{v_\varepsilon(t)}_{\in L_\beta})| \,\mathrm{d}s \\ &\leq \frac{\delta}{3} + \frac{\delta}{3} + a|\delta_\varepsilon| + a(t^* - t_0) \\ &\leq \frac{a}{2}(h - h') + a(t^* - t_0)\end{aligned}$$

it readily follows that $t^* \geq t_0 + h'$, and thus $t^* = t_0 + h'$. Since, by a similar argument, $|v_\varepsilon(t) - \widetilde{x}_{0\varepsilon}| \leq \frac{a}{2}(h + h')$ holds also for all $t \in J^\dagger_- = [t_0 - h', \widetilde{t}_{0\varepsilon}]$ we finally arrive at

$$v_\varepsilon(J^\dagger) \subseteq L + \overline{B_{\frac{a}{2}(h+h')}(0)} \subset\subset L + B_\beta(0) = W.$$

This proves that v is c-bounded from J into W.

(iii) Let $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}[J, U]$ be another solution and $(n_\varepsilon)_\varepsilon \in \mathcal{N}^n$, $(\widetilde{n}_\varepsilon)_\varepsilon \in \mathcal{N}^n$ as above. By (ii), $v \in \mathcal{G}[J, W]$. As before let $J^\dagger := [t_0 - h', t_0 + h']$ (with $\frac{h}{4} < h' < h$) be a compact subinterval of J. Since both $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ are c-bounded from J into W, there exists a compact subset K of W such that $u_\varepsilon(J^\dagger) \subseteq K$ and $v_\varepsilon(J^\dagger) \subseteq K$ for ε sufficiently small. Moreover, we can assume $\delta_\varepsilon < h - h'$. Applying the second-slot version of [10, Lemma 3.2.47] to the function F_ε and some (fixed) compact set K' with $K \subset\subset K' \subset\subset W = L + B_\beta(0)$ yields a constant C' (only depending on

K') such that

$$|F_\varepsilon(t,x) - F_\varepsilon(t,y)| \le C' \sup_{(s,z)\in J^\dagger \times K'} (|F_\varepsilon(s,z)| + |\partial_2 F_\varepsilon(s,z)|) \cdot |x-y| \\ \le C'(a + C_1 |\log \varepsilon|) \cdot |x-y|$$

holds for all $t \in J^\dagger$ and all $x,y \in K$ (note that $J^\dagger \times K' \subseteq J \times W \subseteq Q$) where $C_1 > 0$ is the constant provided by (4). Therefore, for $t \in J^\dagger$ it follows that

$$|v_\varepsilon(t) - u_\varepsilon(t)| \le |\widetilde{y}_{0\varepsilon} - \widetilde{x}_{0\varepsilon}| + \left| \int_{\widetilde{t}_{0\varepsilon}}^t (|F_\varepsilon(s, v_\varepsilon(s)) - F_\varepsilon(s, u_\varepsilon(s))| + |n_\varepsilon(s)|)\,\mathrm{d}s \right| \\ \le |\widetilde{n}_\varepsilon| + \left| \int_{\widetilde{t}_{0\varepsilon}}^t |n_\varepsilon(s)|\mathrm{d}s \right| + C'(a + C_1|\log\varepsilon|) \cdot \left| \int_{\widetilde{t}_{0\varepsilon}}^t |v_\varepsilon(s) - u_\varepsilon(s)|\mathrm{d}s \right| \\ \le C_2\,\varepsilon^m + (C_3 + C_4|\log\varepsilon|) \cdot \left| \int_{\widetilde{t}_{0\varepsilon}}^t |v_\varepsilon(s) - u_\varepsilon(s)|\,\mathrm{d}s \right|$$

for suitable constants $C_2, C_3, C_4 > 0$ and arbitrary $m \in \mathbb{N}$. By Gronwall's Lemma, we obtain

$$\sup_{t\in J^\dagger} |v_\varepsilon(t) - u_\varepsilon(t)| \le C_2\,\varepsilon^m \cdot e^{(C_3 + C_4|\log\varepsilon|)\cdot|\int_{\widetilde{t}_{0\varepsilon}}^t 1\,\mathrm{d}s|} \le C_0\,\varepsilon^{m - h\,C_4}$$

for some constant $C_0 > 0$ (note that $|\widetilde{t}_{0\varepsilon} - t_0| \le h' + \delta_\varepsilon \le h$). This concludes the proof of the theorem. □

Remark 3.2.

(i) The proof of Theorem 3.1 establishes the following statement on the level of representatives: For any given representatives $(\widetilde{t}_{0\varepsilon})_\varepsilon$ of $\widetilde{t}_0 \in \widetilde{I}_c$ ($\widetilde{t}_{0\varepsilon} \to t_0 \in I$), $(\widetilde{x}_{0\varepsilon})_\varepsilon$ of $\widetilde{x}_0 \in \widetilde{U}_c$ and $(F_\varepsilon)_\varepsilon$ of $F \in \mathcal{G}(I \times U)^n$ satisfying (2) the following holds: If α, L, ε_0 and β are chosen as in Theorem 3.1 (including condition (5) as to ε_0), then u has a representative $(\widetilde{u}_\varepsilon)_\varepsilon$ that on every compact subinterval of J satisfies the classical initial value problem (6) for ε sufficiently small.

(ii) If $\widetilde{t}_0$ is standard, i.e., (without loss of generality) $\widetilde{t}_{0\varepsilon} = t_0 \in I$ for all ε, then $\delta_\varepsilon = 0$ and every u_ε exists (as a solution of (6)) even on $[t_0 - h, t_0 + h]$.

(iii) If $\widetilde{x}_0$ is standard, i.e., (without loss of generality) $\widetilde{x}_{0\varepsilon} = x_0 \in U$ for all ε, then $L := \{x_0\}$ yields $L_\beta = \overline{B_\beta(x_0)}$ as in the classical case.

Lemma 3.3.

(i) *Let $a < a_1 < a_2 < b_2 < b_1 < b$ and let U be a (non-empty) open subset of $\mathbb{R}^n$. Then for $f \in \mathcal{C}^\infty([a_1, b_1], U)$ being given, there exists $\widetilde{f} \in \mathcal{C}^\infty([a,b], U)$ with $\widetilde{f} = f$ on some open neighbourhood of $[a_2, b_2]$.*

(ii) *For any given positive δ, the function $\widetilde{f}$ can be chosen such as to satisfy $\widetilde{f}([a,b]) \subseteq f([a_1, b_1]) \cup B_\delta(f(a_1)) \cup B_\delta(f(b_1))$.*

Proof. (i) Choose $\delta > 0$ as to satisfy $\overline{B_\delta(f(a_1))} \cup \overline{B_\delta(f(a_2))} \subseteq U$. Choose $\eta > 0$ such that $f(t) \in B_\delta(f(a_1))$ holds for $t \in [a_1, a_1 + 2\eta]$ and $f(t) \in B_\delta(f(b_1))$ holds for $t \in [b_1 - 2\eta, b_1]$; without loss of generality we may assume $\eta < \frac{1}{3}\min(a_2 - a_1, b_1 - b_2)$.

Now let ψ be a smooth function with $0 \le \psi \le 1$ such that $\psi = 1$ on $[a_1 + 2\eta, b_1 - 2\eta]$ and $\psi = 0$ outside $(a_1 + \eta, b_1 - \eta)$. Then $\widetilde{f}$ defined on $[a, b]$ by

$$\widetilde{f}(t) := \begin{cases} f(a_1) & t \in [a, a_1 + \eta] \\ f(t)\psi(t) + f(a_1)(1 - \psi(t)) & t \in [a_1, a_2] \\ f(t) & t \in [a_1 + 2\eta, b_1 - 2\eta] \\ f(t)\psi(t) + f(b_1)(1 - \psi(t)) & t \in [b_2, b_1] \\ f(b_1) & t \in [b_1 - \eta, b] \end{cases}$$

satisfies all requirements since each of the five defining terms is smooth and on overlaps the two relevant terms give rise to the same values.

(ii) is clear from the proof of (i). □

Theorem 3.1 is distinguished from the related result [7, Theorem 4.5] by the following features: The existence statement (i) of Theorem 3.1 does not require logarithmic control of derivatives of F which, by contrast, is assumed in [7]; the domain interval of the solution in Theorem 3.1 equals the classical (open) one given by $(t_0 - h, t_0 + h)$ with $h = \min(\alpha, \frac{\beta}{a})$ while in [7] one has to take $h < \min(\alpha, \frac{\beta}{a})$; finally, the boundedness assumption on F in [7] refers to the whole open domain of F whereas in Theorem 3.1 it suffices to have boundedness of F on the (compact) subset Q. Generally, all existence and uniqueness results for ODEs in [7] are tailored for applications of the method of characteristics to the generalized Hamilton–Jacobi problem; hence the setting of [7] always includes initial conditions as parameters, necessitating the logarithmic growth condition even for existence results (compare Theorem 3.8 below).

The following three examples illustrate the significance of the boundedness assumption on F by displaying increasing obstacles against obtaining a generalized solution from the classical ones obtained for fixed ε, in the absence of condition (2).

Example 3.4. Let $F \in \mathcal{G}(\mathbb{R} \times \mathbb{R})$ be given by the representative $F_\varepsilon(t, x) := \frac{1}{\varepsilon}\big(2 - \frac{1}{1+x^2}\big)$, and let $t_0 = 0$ and $x_0 = 0$. Then F fails to satisfy condition (2) on any neighbourhood of (t_0, x_0). Nevertheless, there exists a unique global solution for every ε: Integrating $\dot{x}(t) = F_\varepsilon(t, x)$ yields $\frac{x}{2} + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}\, x) = \frac{1}{\varepsilon} t$. Setting $f(x) = \frac{x}{2} + \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}\, x)$, we obtain $u_\varepsilon(t) := f^{-1}(\frac{1}{\varepsilon} t)$ as the solution of the classical initial value problem. By Proposition 2.3, $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R})$. However, $(u_\varepsilon)_\varepsilon$ is not c-bounded. Hence, u_ε solves the differential equation for every ε but on any interval around 0, the generalized function $[(u_\varepsilon)_\varepsilon]$ is not a solution of the initial value problem in the setting of the c-bounded theory of ODEs since the composition $F(t, u(t))$ exists only componentwise on the level of representatives, yet not in the sense of Proposition 2.2.

Example 3.5. Let $F \in \mathcal{G}(\mathbb{R} \times \mathbb{R})$ be given by the representative $F_\varepsilon(t, x) := \frac{x}{\varepsilon}$, and let $t_0 = 0$ and $x_0 = 1$. Again, F does not satisfy condition (2) on any neighbourhood of (t_0, x_0). For each ε, there exists a unique (even global) solution $u_\varepsilon(t) = e^{\frac{t}{\varepsilon}}$. However, $(u_\varepsilon)_\varepsilon$ is not moderate on any neighbourhood of 0.

Example 3.6. Let $F \in \mathcal{G}\big(\mathbb{R} \times (\mathbb{R}\backslash\{-1\})\big)$ be defined by the representative $F_\varepsilon(t,x) := -\frac{t}{x+1} \cdot g(\varepsilon)$ where $g : (0,1] \to \mathbb{R}$ is a smooth map satisfying $g(\varepsilon) \to \infty$ for $\varepsilon \to 0$. Let $t_0 = 0$ and $x_0 = 0$. Then F violates condition (2) on any neighbourhood of (t_0, x_0). For every ε we obtain (unique) solutions $u_\varepsilon(t) = \sqrt{1 - g(\varepsilon)\, t^2} - 1$ that are defined, at most, on the open interval $(-1/\sqrt{g(\varepsilon)}, 1/\sqrt{g(\varepsilon)})$. Hence, there is not even a common domain. In this example, F failing to satisfy condition (2) leads to shrinking of the solutions' domains as $\varepsilon \to 0$. Note that this result is not a consequence of the rate of growth of $|F_\varepsilon(t,x)|$ on any compact set; rather, it only matters that $|F_\varepsilon(t,x)|$ does increase infinitely (as $\varepsilon \to 0$).

Theorem 3.1 can handle jumps as the following example shows.

Example 3.7. Let I be an open interval in $\mathbb{R}$ and U an open subset of $\mathbb{R}^n$. Consider the initial value problem

$$\dot{u}(t) = f(t, u(t)) \cdot (\iota H)(t) + g(t, u(t)), \quad u(t_0) = x_0, \tag{8}$$

where $f, g \in \mathcal{C}^\infty(I \times U, \mathbb{R}^n)$, $t_0 \in I$, $x_0 \in U$, and where ιH denotes the embedding of the Heaviside function H into the Colombeau algebra. If ρ is a mollifier (i.e., a Schwartz function on $\mathbb{R}$ satisfying $\int \rho(x)\,\mathrm{d}x = 1$ and $\int x^\alpha \rho(x)\,\mathrm{d}x = 0$ for all $\alpha \geq 1$), then a representative $(H_\varepsilon)_\varepsilon$ of ιH is given by $H_\varepsilon(t) = \int_{-\infty}^{t} \frac{1}{\varepsilon}\, \rho\left(\frac{s}{\varepsilon}\right)\,\mathrm{d}s$. Fix some $\alpha > 0$ such that $[t_0 - \alpha, t_0 + \alpha] \subseteq I$ and choose an open subset W of U with $x_0 \in W \subseteq \overline{W} \subset\subset U$. A short computation shows that $|H_\varepsilon(t)| \leq \|\rho\|_{L^1(\mathbb{R}^n)}$ for all t. Thus, $|f(t,x) \cdot H_\varepsilon(t) + g(t,x)| \leq a_1 \|\rho\|_{L^1(\mathbb{R}^n)} + a_2 =: a$ on $[t_0 - \alpha, t_0 + \alpha] \times \overline{W}$ for some constants $a_1, a_2 > 0$. Hence, by Theorem 3.1, the initial value problem (8) possesses a solution u in $\mathcal{G}[J, W]$ where $J := (t_0 - h, t_0 + h)$ and $h = \min\left(\alpha, \frac{\operatorname{dist}(x_0, \partial W)}{a}\right)$. Since the initial value problem also satisfies (4), the solution is unique in $\mathcal{G}[J, U]$.

Next, we turn our attention to generalized ODEs including parameters. In view of our goal to establish a Frobenius theorem in the present setting, we want the solution to be $\mathcal{G}$-dependent on the parameter.

Theorem 3.8. *Let I be an open subinterval of $\mathbb{R}$, U an open subset of $\mathbb{R}^n$, P an open subset of $\mathbb{R}^l$, $\widetilde{t}_0$ a near-standard point in $\widetilde{I}_c$ with $\widetilde{t}_0 \approx t_0 \in I$, $\widetilde{x}_0 \in \widetilde{U}_c$ and $F \in \mathcal{G}(I \times U \times P)^n$.*

Let α be chosen such that $[t_0 - \alpha, t_0 + \alpha] \subset\subset I$. Let $(\widetilde{x}_{0\varepsilon})_\varepsilon$ be a representative of $\widetilde{x}_0$ and $L \subset\subset U$, $\varepsilon_0 \in (0,1]$ such that $\widetilde{x}_{0\varepsilon} \in L$ for all $\varepsilon \leq \varepsilon_0$. With $\beta > 0$ satisfying $L_\beta := L + \overline{B_\beta(0)} \subset\subset U$ set

$$Q := [t_0 - \alpha, t_0 + \alpha] \times L_\beta \qquad (\subset\subset I \times U).$$

Assume that F has a representative $(F_\varepsilon)_\varepsilon$ satisfying

$$\sup_{(t,x,p) \in Q \times P} |F_\varepsilon(t,x,p)| \leq a \qquad (\varepsilon \leq \varepsilon_0) \tag{9}$$

for some constant $a > 0$ and that for all compact subsets K of P

$$\sup_{(t,x,p) \in Q \times K} |\partial_2 F_\varepsilon(t,x,p)| = O(|\log \varepsilon|). \tag{10}$$

Then the following holds: There exists $u \in \mathcal{G}[P\times J, W]$ *with* $J := [t_0-h, t_0+h]$, $h = \min\left(\alpha, \frac{\beta}{a}\right)$ *and* $W = L + B_\beta(0)$ *such that for all* $\widetilde{p} \in \widetilde{P}_c$ *the map* $u(\widetilde{p}, \,.\,) \in \mathcal{G}[J, W]$ *is a solution of the initial value problem*

$$\dot{u}(t) = F(t, u(t), \widetilde{p}), \quad u(\widetilde{t_0}) = \widetilde{x}_0.$$

The solution u *is unique in* $\mathcal{G}[P \times J, U]$*.*

Proof. Existence: Let $(\widetilde{t}_{0\varepsilon})_\varepsilon$ be a representative of $\widetilde{t}_0$. Proceeding as in the proof of Theorem 3.1, we set $\delta_\varepsilon := \sup\{|\widetilde{t}_{0\varepsilon'} - t_0| \,|\, 0 < \varepsilon' \leq \varepsilon\}$ and $J_\varepsilon := [t_0 - h + \delta_\varepsilon, t_0 + h - \delta_\varepsilon]$. For every $p \in P$ there exists a net of (classical) solutions $u_\varepsilon(p, \,.\,) : J_\varepsilon \to L_\beta$ of the initial value problem

$$\dot{u}_\varepsilon(t) = F_\varepsilon(t, u_\varepsilon(t), p), \quad u_\varepsilon(\widetilde{t}_{0\varepsilon}) = \widetilde{x}_{0\varepsilon} \qquad (\varepsilon \leq \varepsilon_0), \tag{11}$$

satisfying $u_\varepsilon(p, J_\varepsilon^\circ) \subseteq W$. By the classical Existence and Uniqueness Theorem for ODEs with parameter, the mappings $(p, t) \mapsto u_\varepsilon(p, t)$ are $\mathcal{C}^\infty$. Lemma 3.3 provides $\widetilde{u}_\varepsilon \in \mathcal{C}^\infty(P\times[t_0-h, t_0+h], W)$ being equal to u_ε on $\widetilde{J}_\varepsilon := [t_0-h+2\delta_\varepsilon, t_0+h-2\delta_\varepsilon]$.

In order to show that $(\widetilde{u}_\varepsilon)_\varepsilon$ is moderate on J it again suffices to establish the corresponding estimates for $(u_\varepsilon)_\varepsilon$. C-boundedness of $(\widetilde{u}_\varepsilon)_\varepsilon$ is shown as in the proof of Theorem 3.1.

The moderateness of $(u_\varepsilon)_\varepsilon$ will be shown in three steps: First we consider derivatives with respect to t, then only derivatives with respect to p and, finally, mixed derivatives.

The $\mathcal{E}_M$-estimates for $u_\varepsilon(p, t)$, $\partial_2 u_\varepsilon(p, t)$ and all its derivatives with respect to t are obtained in the same way as in the proof of Theorem 3.1.

Next, we consider the derivatives with respect to p. Differentiating the integral equation corresponding to the initial value problem (on the level of representatives) with respect to p yields

$$\partial_1 u_\varepsilon(p, t) = \int_{\widetilde{t}_{0\varepsilon}}^t \Big(\partial_2 F_\varepsilon\big(s, u_\varepsilon(p, s), p\big) \cdot \partial_1 u_\varepsilon(p, s) + \partial_3 F_\varepsilon\big(s, u_\varepsilon(p, s), p\big)\Big)\, \mathrm{d}s. \tag{12}$$

Let $K_1 \times K_2 \subset\subset P \times J$ and $(p, t) \in K_1 \times K_2$. By $u_\varepsilon(K_1 \times K_2) \subseteq L_\beta \subset\subset U$ and (10), we obtain

$$|\partial_1 u_\varepsilon(p, t)| \leq h\, C_1 \varepsilon^{-N_1} + \left| \int_{\widetilde{t}_{0\varepsilon}}^t C_2 |\log \varepsilon| \cdot |\partial_1 u_\varepsilon(p, s)|\, \mathrm{d}s \right|$$

for constants $C_1, C_2 > 0$ and some fixed $N \in \mathbb{N}$. By Gronwall's Lemma, it follows that

$$|\partial_1 u_\varepsilon(p, t)| \leq h\, C_1 \varepsilon^{-N_1} \cdot e^{|\int_{\widetilde{t}_{0\varepsilon}}^t C_2 |\log \varepsilon|\, \mathrm{d}s|} \leq (h\, C_1)\, \varepsilon^{-(N_1 + h\, C_2)}.$$

Differentiating (12) $i - 1$ times with respect to p ($i \in \mathbb{N}$) gives an integral formula for $\partial_1^i u_\varepsilon(p, t)$. Observe that in this formula $\partial_1^i u_\varepsilon(p, t)$ itself appears on the right-hand side only once, namely with $\partial_2 F_\varepsilon(s, u_\varepsilon(p, s), p)$ as coefficient, and that the remaining terms contain only ∂_1-derivatives of u_ε of order less than i. Thus, we may estimate the higher-order derivatives with respect to p inductively by differentiating equation (12) and applying Gronwall's Lemma.

Finally, it remains to handle the case of mixed derivatives. For arbitrary $i \in \mathbb{N}$ we have

$$\partial_1^i \partial_2 \, u_\varepsilon(p,t) = \frac{\partial^i}{\partial p^i} \frac{\partial}{\partial t} \left(\widetilde{x}_{0\varepsilon} + \int_{\widetilde{t}_{0\varepsilon}}^{t} F_\varepsilon\big(s, u_\varepsilon(p,s), p\big) \, \mathrm{d}s \right) = \frac{\partial^i}{\partial p^i} F_\varepsilon\big(t, u_\varepsilon(p,t), p\big). \tag{13}$$

By carrying out the i-fold differentiation on the right-hand side of equation (13), we obtain a polynomial expression in $\partial_2^k F_\varepsilon\big(t, u_\varepsilon(p,t), p\big)$, $\partial_3^k F_\varepsilon\big(t, u_\varepsilon(p,t), p\big)$ and $\partial_1^k u_\varepsilon(p,t)$ for $1 \leq k \leq i$ all of which satisfy the $\mathcal{E}_M$-estimates. The estimates for $\partial_1^i \partial_2^j \, u_\varepsilon(p,t)$ with $j \geq 2$ are now obtained inductively by differentiating equation (13) with respect to t.

Uniqueness: By Proposition 2.4, it suffices to show that for every near-standard point $\widetilde{p} \in \widetilde{P}_c$ the solution $u(\widetilde{p}, \,.\,)$ is unique in $\mathcal{G}[J,U]$. For a fixed near-standard point $\widetilde{p} = [(\widetilde{p}_\varepsilon)_\varepsilon] \in \widetilde{P}_c$, condition (10) implies the condition for uniqueness (4) in Theorem 3.1 with respect to $G_\varepsilon(t,x) := (F_\varepsilon(\,.\,,\,.\,,\widetilde{p}_\varepsilon))_\varepsilon$, yielding uniqueness of $u(\widetilde{p}, \,.\,)$ in $\mathcal{G}[J,U]$. □

Remark 3.9. Similarly to Remark 3.2 (i), a corresponding statement on the level of representatives can be extracted from the proof of the preceding theorem. Also (ii) and (iii) of Remark 3.2 apply.

Requiring also $\widetilde{x}_0$ in the initial condition in Theorem 3.8 to be near-standard, we even can prove $\mathcal{G}$-dependence of the solution on the initial values.

Theorem 3.10. *Let I be an open subinterval of $\mathbb{R}$, U an open subset of $\mathbb{R}^n$, P an open subset of $\mathbb{R}^l$, $\widetilde{t}_0$ a near-standard point in $\widetilde{I}_c$ with $\widetilde{t}_0 \approx t_0 \in I$, $\widetilde{x}_0$ a near-standard point in $\widetilde{U}_c$ with $\widetilde{x}_0 \approx x_0 \in U$ and $F \in \mathcal{G}(I \times U \times P)^n$.*

With $\alpha > 0$ and $\beta > 0$ satisfying $[t_0 - \alpha, t_0 + \alpha] \subset\subset I$ and $\overline{B_\beta(x_0)} \subset\subset U$, respectively, set

$$Q := [t_0 - \alpha, t_0 + \alpha] \times \overline{B_\beta(x_0)} \qquad (\subset\subset I \times U).$$

Assume that F has a representative $(F_\varepsilon)_\varepsilon$ satisfying

$$\sup_{(t,x,p) \in Q \times P} |F_\varepsilon(t,x,p)| \leq a \qquad (\varepsilon \leq \varepsilon_0) \tag{14}$$

for some constant $a > 0$ and $\varepsilon_0 \in (0,1]$ and that for all compact subsets K of P

$$\sup_{(t,x,p) \in Q \times K} |\partial_2 F_\varepsilon(t,x,p)| = O(|\log \varepsilon|). \tag{15}$$

Then the following holds: For fixed $h \in \left(0, \min\left(\alpha, \frac{\beta}{a}\right)\right)$ there exist open neighbourhoods J_1 of t_0 in $J := (t_0 - h, t_0 + h)$ and U_1 of x_0 in U and a generalized function $u \in \mathcal{G}[J_1 \times U_1 \times P \times J, B_\gamma(x_0)]$ with $\gamma \in (0,\beta)$ and $\beta - \gamma > 0$ sufficiently small, such that for all $(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}) \in \widetilde{J}_{1c} \times \widetilde{U}_{1c} \times \widetilde{P}_c$ the map $u(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}, \,.\,) \in \mathcal{G}[J, B_\gamma(x_0)]$ is a solution of the initial value problem

$$\dot{u}(t) = F(t, u(t), \widetilde{p}), \quad u(\widetilde{t}_1) = \widetilde{x}_1. \tag{16}$$

The solution u is unique in $\mathcal{G}[J_1 \times U_1 \times P \times J, B_\gamma(x_0)]$.

Proof. Existence: The basic strategy of the proof is to consider $(\widetilde{t}_0, \widetilde{x}_0)$ as part of the parameter and apply Theorem 3.8. However, we will have to cope with some technicalities.

Let $(\widetilde{t}_{0\varepsilon})_\varepsilon$ and $(\widetilde{x}_{0\varepsilon})_\varepsilon$ be representatives of $\widetilde{t}_0$ and $\widetilde{x}_0$, respectively. From now on, we always let $\varepsilon \le \varepsilon_0$. Let $\lambda \in (0,1)$ and set

$$\hat{I} := (-\lambda\alpha, \lambda\alpha), \quad I_1 := (t_0 - (1-\lambda)\alpha, t_0 + (1-\lambda)\alpha).$$

Choose $\mu \in \left(0, \frac{\beta}{3}\right)$, set $\gamma := \beta - 2\mu$ and define

$$\hat{U} := B_{\gamma+\mu}(0), \quad U_1 := B_\mu(x_0).$$

Then $\hat{I} + I_1 = (t_0 - \alpha, t_0 + \alpha) \subseteq I$ and $\hat{U} + U_1 = B_\beta(x_0) \subseteq U$. Hence, we may define $G_\varepsilon : \hat{I} \times \hat{U} \times (I_1 \times U_1 \times P) \to \mathbb{R}^n$ by

$$G_\varepsilon(t, x, (t_1, x_1, p)) := F_\varepsilon(t + t_1, x + x_1, p).$$

Obviously, $(G_\varepsilon)_\varepsilon$ is moderate and, therefore, $G := [(G_\varepsilon)_\varepsilon]$ is in $\mathcal{G}(\hat{I} \times \hat{U} \times (I_1 \times U_1 \times P))^n$. Now let $\delta \in (0, \lambda\alpha)$ and $\eta \in (0, \gamma - \mu)$. By assumptions (14) and (15), we obtain $|G_\varepsilon(t, x, (t_1, x_1, p))| \le a$ for all $(t, x, (t_1, x_1, p)) \in \overline{B_\delta(0)} \times \overline{B_\eta(0)} \times (I_1 \times U_1 \times P)$ and $|\partial_2 G_\varepsilon(t, x, (t_1, x_1, p))| = O(|\log\varepsilon|)$ for all $K \subset\subset I_1 \times U_1 \times P$ and $(t, x, (t_1, x_1, p)) \in \overline{B_\delta(0)} \times \overline{B_\eta(0)} \times K$. By Theorem 3.8, there exists $v \in \mathcal{G}[(I_1 \times U_1 \times P) \times \hat{J}, B_\eta(0)]$ with $\hat{J} := (-\hat{h}, \hat{h})$ and $\hat{h} = \min\left(\delta, \frac{\eta}{a}\right)$ such that for all $(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}) \in \widetilde{I}_{1c} \times \widetilde{U}_{1c} \times \widetilde{P}_c$ the map $v(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}, \,.\,) \in \mathcal{G}[\hat{J}, B_\eta(0)]$ is a solution of the initial value problem

$$\dot{v}(t) = G(t, v(t), (\widetilde{t}_1, \widetilde{x}_1, \widetilde{p})), \quad v(0) = 0. \tag{17}$$

The solution v is unique in $\mathcal{G}[(I_1 \times U_1 \times P) \times \hat{J}, \hat{U}]$.

By Remark 3.9, there exists a representative $(v_\varepsilon)_\varepsilon$ of v that satisfies the classical initial value problem for all $(t_1, x_1, p) \in I_1 \times U_1 \times P$ and ε sufficiently small. Let $\sigma \in \left[\frac{1}{2}, 1\right)$, $h := \sigma\hat{h}$ and $h_1 := \min((1-\sigma)\hat{h}, (1-\lambda)\alpha)$. Set $J := (t_0 - h, t_0 + h)$ and $J_1 := (t_0 - h_1, t_0 + h_1)$. Then $J_1 \subseteq J \subseteq \hat{J}$. We now define $u_\varepsilon : J_1 \times U_1 \times P \times J \to \mathbb{R}^n$ by

$$u_\varepsilon(t_1, x_1, p, t) := v_\varepsilon(t_1, x_1, p, t - t_1) + x_1.$$

The map u_ε is well defined since $J_1 \subseteq I_1$ and

$$|t - t_1| \le |t - t_0| + |t_0 - t_1| \le h + h_1 \le \sigma\hat{h} + (1-\sigma)\hat{h} = \hat{h}. \tag{18}$$

The moderateness of $(u_\varepsilon)_\varepsilon$ is an immediate consequence of the moderateness of $(v_\varepsilon)_\varepsilon$. By (18) and since $x_1 - x_0 \in B_\mu(0)$ for all $x_1 \in U_1$, it follows that

$$\begin{aligned} u_\varepsilon(J_1 \times U_1 \times P \times J) &\subseteq v_\varepsilon(I_1 \times U_1 \times P \times \hat{J}) + x_1 \subseteq B_\eta(0) + x_1 \\ &\subseteq B_\eta(x_0) - x_0 + x_1 \subseteq \overline{B_\eta(x_0) + B_\mu(0)} \subseteq B_\gamma(x_0), \end{aligned}$$

i.e., $u := [(u_\varepsilon)_\varepsilon]$ is an element of $\mathcal{G}[J_1 \times U_1 \times P \times J, B_\gamma(x_0)]$. Furthermore, the function $u_\varepsilon(\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon, \,.\,)$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t} u_\varepsilon(\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon, t) &= \frac{\partial}{\partial t}\Big(v_\varepsilon(\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon, t - \widetilde{t}_{1\varepsilon}) + \widetilde{x}_{1\varepsilon}\Big)\\ &= G_\varepsilon(t - \widetilde{t}_{1\varepsilon}, v_\varepsilon(\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon, t - \widetilde{t}_{1\varepsilon}), (\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon))\\ &= F_\varepsilon(t, v_\varepsilon(\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon, t - \widetilde{t}_{1\varepsilon}) + \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon) = F_\varepsilon(t, u_\varepsilon(\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon, t), \widetilde{p}_\varepsilon)\end{aligned}$$

and

$$u_\varepsilon(\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon, \widetilde{t}_{1\varepsilon}) = v_\varepsilon(\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}, \widetilde{p}_\varepsilon, 0) + \widetilde{x}_{1\varepsilon} = \widetilde{x}_{1\varepsilon}$$

for all $(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}) = ([(\widetilde{t}_{1\varepsilon})_\varepsilon], [(\widetilde{x}_{1\varepsilon})_\varepsilon], [(\widetilde{p}_\varepsilon)_\varepsilon]) \in \widetilde{J}_{1c} \times \widetilde{U}_{1c} \times \widetilde{P}_c$ and $t \in J$. Thus, $u(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}, \,.\,)$ is indeed a solution of the initial value problem (16).

Note that for any $h \in \Big(0, \min\big(\alpha, \frac{\beta}{a}\big)\Big)$ the constants λ, μ, δ, η, $\hat{h}$ and σ can be chosen within their required bounds such that all the necessary inequalities in the construction of $(u_\varepsilon)_\varepsilon$ are satisfied.

Uniqueness: By Proposition 2.4, it suffices to show that for every near-standard point $(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}) = ([(\widetilde{t}_{1\varepsilon})_\varepsilon], [(\widetilde{x}_{1\varepsilon})_\varepsilon], \widetilde{p}) \in \widetilde{J}_{1c} \times \widetilde{U}_{1c} \times \widetilde{P}_c$ the solution $u(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}, \,.\,)$ is unique in $\mathcal{G}[J, B_\gamma(x_0)]$: Let $(\widetilde{t}_{1\varepsilon}, \widetilde{x}_{1\varepsilon}) \to (t_1, x_1) \in J_1 \times U_1$ for $\varepsilon \to 0$. Assume that $w(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}) \in \mathcal{G}[J, B_\gamma(x_0)]$ is another solution of (16). For brevity's sake we simply write u and w in place of $u(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p})$ and $w(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p})$, respectively.

We will show that $w|_{(t_0-r,t_0+r)} = u|_{(t_0-r,t_0+r)}$ holds for any $r \in (0, h)$. Since $\mathcal{G}$ is a sheaf, the equality of w and u then also holds on J.

Now, let $r \in (0, h)$ and set $\rho := \frac{1}{2}(h - r)$. Define $\bar{w} : B_{r+\rho}(t_0 - t_1) \to B_{\gamma+\mu}(0)$ by $\bar{w}(t) := w(t + \widetilde{t}_1) - \widetilde{x}_1$. From $\widetilde{t}_{1\varepsilon} \to t_1$ as $\varepsilon \to 0$ it follows that $\bar{w}$ is well defined. Then, by the choice of ρ and Proposition 2.2, $\bar{w} \in \mathcal{G}[B_{r+\rho}(t_0 - t_1), B_{\gamma+\mu}(0)]$. Moreover, $\bar{w}$ is a solution of the initial value problem (17). Since $B_{r+\rho}(t_0 - t_1) \subseteq \hat{J}$ and solutions of (17) are unique in $\mathcal{G}[\hat{J}, B_{\gamma+\mu}(0)]$, it follows that $\bar{w} = v(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}, \,.\,)|_{B_{r+\rho}(t_0-t_1)}$. Noting that

$$w(t) = \bar{w}(t - \widetilde{t}_1) + \widetilde{x}_1 = v(\widetilde{t}_1, \widetilde{x}_1, \widetilde{p}, t - \widetilde{t}_1) + \widetilde{x}_1 = u(t),$$

we finally arrive at $w|_{(t_0-r,t_0+r)} = u|_{(t_0-r,t_0+r)}$. □

Remark 3.11. Concerning representatives, a remark analogous to 3.9 also applies to Theorem 3.10.

4. A Frobenius theorem in generalized functions

In this section, we will use the following notation: By $\mathbb{R}^{m\times n}$ we denote the space $\mathbb{R}^{mn}$, viewed as the space of $(m \times n)$-matrices over $\mathbb{R}$. A similar convention applies to $\mathcal{R}^{m\times n}$ and $\mathcal{G}(U)^{m\times n}$. For any $u \in \mathcal{G}(U)^m$ the derivative $\mathrm{D}u$ can be regarded as an element of $\mathcal{G}(U)^{m\times n}$.

Now we are ready to prove a generalized version of the Frobenius Theorem.

Theorem 4.1. *Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$ and $F \in \mathcal{G}(U \times V)^{m\times n}$. Let $\alpha > 0$ be chosen such that $\overline{B_\alpha(x_0)} \subset\subset U$. Let $(\widetilde{y}_{0\varepsilon})_\varepsilon$ be a representative of $\widetilde{y}_0$ and $L \subset\subset V$, $\varepsilon_0 \in (0,1]$ such that $\widetilde{y}_{0\varepsilon} \in L$ for all $\varepsilon \leq \varepsilon_0$. With $\beta > 0$ satisfying $L_\beta := L + \overline{B_\beta(0)} \subset\subset V$ set*

$$Q := \overline{B_\alpha(x_0)} \times L_\beta \qquad (\subset\subset U \times V).$$

Assume that F has a representative $(F_\varepsilon)_\varepsilon$ satisfying

$$\sup_{(x,y)\in Q} |F_\varepsilon(x,y)| \leq a \qquad (\varepsilon \leq \varepsilon_0) \tag{19}$$

for some constant $a > 0$ and

$$\sup_{(x,y)\in Q} |\partial_2 F_\varepsilon(x,y)| = O(|\log\varepsilon|). \tag{20}$$

Then the following are equivalent:

(A) *For all $(\widetilde{x}_0, \widetilde{y}_0) \in \widetilde{U}_c \times \widetilde{V}_c$ with $\widetilde{x}_0 \approx x_0 \in U$ the initial value problem*

$$\mathrm{D}u(x) = F(x, u(x)), \quad u(\widetilde{x}_0) = \widetilde{y}_0 \tag{21}$$

has a unique solution $u(\widetilde{x}_0, \widetilde{y}_0)$ in $\mathcal{G}[U(\widetilde{x}_0, \widetilde{y}_0), W]$, where $U(\widetilde{x}_0, \widetilde{y}_0)$ is an open neighbourhood of x_0 in U and $W = L + B_\beta(0)$.

(B) *The integrability condition is satisfied, i.e., the mapping*

$$(x, y, v_1, v_2) \mapsto \mathrm{D}F(x,y)(v_1, F(x,y)(v_1))(v_2) \tag{22}$$

is symmetric in $v_1, v_2 \in \mathbb{R}^n$ as a generalized function in $\mathcal{G}(U \times V \times \mathbb{R}^n \times \mathbb{R}^n)^m$.

Proof. We follow the line of argument of the classical proof based on the ODE theorem with parameters.

(A) $\Rightarrow$ (B): By Proposition 2.4, we only have to check the integrability condition (22) for all near-standard points $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{\mathbb{R}}^n_c$ and $(\widetilde{x}, \widetilde{y}) \in \widetilde{U}_c \times \widetilde{V}_c$: By (A), there exists a solution u of the initial value problem $\mathrm{D}u(x) = F(x, u(x))$, $u(\widetilde{x}) = \widetilde{y}$. Writing $\mathrm{D}u$ as $\mathrm{D}u = F \circ (\mathrm{id}, u)$, we obtain

$$\begin{aligned}
\mathrm{D}^2 u(\widetilde{x})(\widetilde{v}_1, \widetilde{v}_2) &= \big(\mathrm{D}^2 u(\widetilde{x})(\widetilde{v}_1)\big)(\widetilde{v}_2) = \big(\mathrm{D}(F \circ (\mathrm{id}, u))(\widetilde{x})(\widetilde{v}_1)\big)(\widetilde{v}_2) \\
&= \Big(\big(\mathrm{D}F(\widetilde{x}, u(\widetilde{x})) \circ (\mathrm{id}, \mathrm{D}u(\widetilde{x}))\big)(\widetilde{v}_1)\Big)(\widetilde{v}_2) \\
&= \Big(\mathrm{D}F\big(\widetilde{x}, u(\widetilde{x})\big)\big(\widetilde{v}_1, F(\widetilde{x}, u(\widetilde{x}))(\widetilde{v}_1)\big)\Big)(\widetilde{v}_2) = \mathrm{D}F(\widetilde{x}, \widetilde{y})\big(\widetilde{v}_1, F(\widetilde{x}, \widetilde{y})(\widetilde{v}_1)\big)(\widetilde{v}_2)
\end{aligned}$$

for all near-standard points $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{\mathbb{R}}^n_c$. The last expression is symmetric in $\widetilde{v}_1$ and $\widetilde{v}_2$ since, by Schwarz's Theorem, $\mathrm{D}^2 u(\widetilde{x})$ has this property.

(B) $\Rightarrow$ (A): Let $\widetilde{x}_0 = [(\widetilde{x}_{0\varepsilon})_\varepsilon]$ be a near-standard point in $\widetilde{U}_c$ with $\widetilde{x}_0 \approx x_0$ and let $\widetilde{y}_0 \in \widetilde{V}_c$.

Existence: Choose $\delta \in (0, \alpha)$ and set $\gamma := \alpha - \delta$. We can assume without loss of generality that $\widetilde{x}_{0\varepsilon} \in B_\delta(x_0)$ for all $\varepsilon \leq \varepsilon_0$. Then, for $t \in (-\gamma, \gamma)$ and $v \in B_1(0) \subseteq \mathbb{R}^n$, we have $\widetilde{x}_{0\varepsilon} + tv \in B_\alpha(x_0) \subseteq U$ and, thus, the function

$$\begin{array}{rccc}
G_\varepsilon : & (-\gamma, \gamma) \times V \times B_1(0) & \to & \mathbb{R}^m \\
& (t, y, v) & \mapsto & F_\varepsilon(\widetilde{x}_{0\varepsilon} + tv, y)(v)
\end{array}$$

is well defined. By Proposition 2.2, $G := [(G_\varepsilon)_\varepsilon]$ is a well-defined generalized function in $\mathcal{G}\left((-\gamma,\gamma)\times V\times B_1(0)\right)^m$. Now consider the initial value problem

$$\dot{f}(t) = G(t, f(t), v), \quad f(0) = \widetilde{y}_0, \tag{23}$$

with parameter $v \in B_1(0)$. Then the conditions of Theorem 3.8 are satisfied, i.e.,

$$|G_\varepsilon(t,y,v)| \le a \qquad \text{and} \qquad \partial_2 G_\varepsilon(t,y,v) = O(|\log\varepsilon|)$$

for all $(t,y,v) \in \overline{B_\eta(0)} \times L_\beta \times B_1(0)$ with $\eta \in (0,\gamma)$ fixed. From Theorem 3.8, it follows that there exists a generalized function $f \in \mathcal{G}[B_1(0)\times J, W]$ with $J := [-h,h]$, $h := \min\left(\eta, \frac{\beta}{a}\right)$ and $W := L + B_\beta(0)$ such that $f(v,\,.\,)$ is a solution of (23) for all $v \in B_1(0)$. Fix some $r \in (0,h)$ and $\lambda \in (0,1)$ and set

$$U(\widetilde{x}_0, \widetilde{y}_0) := B_{\lambda r}(x_0).$$

Assuming without loss of generality that $|x_0 - \widetilde{x}_{0\varepsilon}| < (1-\lambda)r$ for all $\varepsilon \le \varepsilon_0$, the function $u_\varepsilon(\widetilde{x}_0,\widetilde{y}_0) : U(\widetilde{x}_0,\widetilde{y}_0) \to W$ given by

$$u_\varepsilon(\widetilde{x}_0,\widetilde{y}_0)(x) := f_\varepsilon\left(\frac{1}{r}(x - \widetilde{x}_{0\varepsilon}), r\right)$$

is well defined. By Proposition 2.2, $u(\widetilde{x}_0,\widetilde{y}_0) := [(u_\varepsilon(\widetilde{x}_0,\widetilde{y}_0))_\varepsilon] \in \mathcal{G}[U(\widetilde{x}_0,\widetilde{y}_0), W]$. From now on, we will denote $u(\widetilde{x}_0,\widetilde{y}_0)$ simply by u.

The fact that u is indeed a solution of (21) follows from

$$\partial_1 f(v,t)(w) = F(x_0 + tv, f(v,t))(tw) \quad \text{in } \mathcal{G}((-h,h)\times B_1(0)\times\mathbb{R}^n)^m. \tag{24}$$

Assuming this to be true for the moment, we have

$$\begin{aligned}\mathrm{D}u(x)(\widetilde{w}) &= \left(\frac{\partial}{\partial x} f\left(\frac{x-\widetilde{x}_0}{r}, r\right)\right)(\widetilde{w}) = \partial_1 f\left(\frac{x-\widetilde{x}_0}{r}, r\right)\left(\frac{1}{r}\widetilde{w}\right) \\ &= F(x, u(x))(\widetilde{w})\end{aligned}$$

for all $\widetilde{w} \in \widetilde{\mathbb{R}}^n_c$. Applying Proposition 2.4 to the above equation, we obtain $\mathrm{D}u(x) = F(x,u(x))$ in $\mathcal{G}[U(\widetilde{x}_0,\widetilde{y}_0), W]$. Moreover, we observe that $f(0,\,.\,)$ is the (in $\mathcal{G}[(-h,h),W]$) constant function $t \mapsto \widetilde{y}_0$, and hence we obtain $u(\widetilde{x}_0) = f(\frac{1}{r}(\widetilde{x}_0 - \widetilde{x}_0), r) = \widetilde{y}_0$. Thus, u is indeed a solution of the initial value problem (21).

To complete the proof of existence, it remains to show (24): Consider the net $(k_\varepsilon)_\varepsilon$ given by $k_\varepsilon : (-h,h)\times B_1(0)\times\mathbb{R}^n \to \mathbb{R}^m$,

$$k_\varepsilon(t,v,w) := \partial_1 f_\varepsilon(v,t)(w) - F_\varepsilon(\widetilde{x}_{0\varepsilon} + tv, f_\varepsilon(v,t))(tw).$$

Note that, by Proposition 2.2, $k := [(k_\varepsilon)_\varepsilon]$ is a well-defined generalized function in $\mathcal{G}((-h,h)\times B_1(0)\times\mathbb{R}^n)^m$. Let $\widetilde{v} \in \widetilde{B_1(0)}_c$ and $\widetilde{w} \in \widetilde{\mathbb{R}}^n_c$. Differentiating $k(t,\widetilde{v},\widetilde{w})$ with respect to t, using the fact that $f(\widetilde{v},\,.\,)$ is a solution of (23) and setting $\widetilde{z} = (\widetilde{x}_0 + t\widetilde{v}, f(\widetilde{v},t))$, we obtain

$$\dot{k}(t,\widetilde{v},\widetilde{w}) = \partial_1 F(\widetilde{z})(t\widetilde{w},\widetilde{v}) + \partial_2 F(\widetilde{z})(\partial_1 f(\widetilde{v},t)(\widetilde{w}),\widetilde{v}) - \mathrm{D}F(\widetilde{z})(\widetilde{v}, F(\widetilde{z})(\widetilde{v}))(t\widetilde{w}).$$

Applying the integrability condition (B) to the last term on the right-hand side, we arrive at

$$\dot{k}(t,\widetilde{v},\widetilde{w}) = \left(\partial_2 F(\widetilde{x}_0 + t\widetilde{v}, f(\widetilde{v},t))^{\mathrm{fl}}(\widetilde{v})\right)\cdot k(t,\widetilde{v},\widetilde{w}). \tag{25}$$

Moreover, observe that $k(0,\widetilde{v},\widetilde{w}) = 0$ in $\widetilde{\mathbb{R}}^m$. Hence, $k(\,.\,,\widetilde{v},\widetilde{w})$ is a solution of a linear initial value problem. Setting $A_{\widetilde{v}}(t) := \partial_2 F(\widetilde{x}_0 + t\widetilde{v}, f(\widetilde{v},t))^{\mathrm{fl}}(\widetilde{v})$, it follows from (20) that

$$\sup_{t\in(-h,h)} |A_{\widetilde{v}}(t)| = O(|\log\varepsilon|).$$

By a Gronwall argument similar to the one in the uniqueness proof of Theorem 3.1 we infer that $k(\,.\,,\widetilde{v},\widetilde{w}) = 0$ is the only solution of (25). By Proposition 2.4, we conclude that $k = 0$ in $\mathcal{G}((-h,h)\times B_1(0)\times\mathbb{R}^n)^m$, thereby establishing the claim.

Uniqueness: Let $\bar{u} \in \mathcal{G}[B_{\lambda r}(x_0), W]$ be another solution of (21). We will show that $\bar{u}|_{B_s(x_0)} = u|_{B_s(x_0)}$ for all $s < \lambda r$. Since $\mathcal{G}$ is a sheaf, the equality then also holds on $B_{\lambda r}(x_0) = U(\widetilde{x}_0,\widetilde{y}_0)$.

Let $s \in (0,\lambda r)$ and let $\widetilde{v} = [(\widetilde{v}_\varepsilon)_\varepsilon] \in \widetilde{B_1(0)}_c$. Setting $\sigma := \frac{1}{3}(\lambda r - s)$, we define $g(\widetilde{v},\,.\,) : (-s-2\sigma, s+2\sigma) \to W$ by $g(\widetilde{v},t) := \bar{u}(\widetilde{x}_0 + t\widetilde{v})$. From $\widetilde{x}_{0\varepsilon} \to x_0$ as $\varepsilon \to 0$ it follows that $g(\widetilde{v},\,.\,)$ is well defined. Then, by the choice of σ and by Proposition 2.2, $g(\widetilde{v},\,.\,) \in \mathcal{G}[(-s-2\sigma,s+2\sigma),W]$. Moreover, $g(\widetilde{v},\,.\,)$ is a solution of (23) for $v = \widetilde{v}$. Since $(-s-2\sigma,s+2\sigma) \subseteq J$ and solutions of (23) are unique in $\mathcal{G}[J,W]$, it follows that $g(\widetilde{v},\,.\,) = f(\widetilde{v},\,.\,)|_{(-s-2\sigma,s+2\sigma)}$ for all $\widetilde{v} \in \widetilde{B_1(0)}_c$. By Proposition 2.4, $g : (v,t) \mapsto g(v,t)$ is equal to f on $(-s-2\sigma, s+2\sigma)$. Observe that for $c_1, c_2 > 0$ the generalized functions $(v,t) \mapsto f\Big(\frac{1}{c_1}v, c_1 t\Big)$ and $(v,t) \mapsto f\Big(\frac{1}{c_2}v, c_2 t\Big)$ are equal on the intersection of their domains. Hence, we obtain

$$\begin{aligned}\bar{u}(x) &= g\Big(\frac{1}{s+\sigma}(x-\widetilde{x}_0)\Big)(s+\sigma) = f\Big(\frac{1}{s+\sigma}(x-\widetilde{x}_0)\Big)(s+\sigma)\\ &= f\Big(\frac{1}{r}(x-\widetilde{x}_0)\Big)(r) = u(x),\end{aligned}$$

thereby establishing the claim. □

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Asymptotically Almost Periodic Generalized Functions

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Abstract. The paper introduces an algebra of asymptotically almost periodic generalized functions of Colombeau type and gives their main properties

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Keywords. Asymptotically almost periodic functions, asymptotically almost periodic distributions, Colombeau algebra, asymptotically almost periodic generalized functions.

1. Introduction

The algebra of generalized functions initiated by J.F. Colombeau, see [5] and [6], in connection with the problem of multiplication of Schwartz distributions, contains the space of Schwartz distributions [10]. These generalized functions are currently the subject of many scientific works, see [8] and [9].

An algebra of almost periodic generalized functions of Colombeau type has been introduced in [3], it contains classical almost period functions as well as almost periodic Schwartz distributions.

Asymptotically almost periodic functions were introduced and studied by M. Fréchet [7]. Asymptotically almost periodic Schwartz distributions have been introduced in [4].

The first aim of this paper is to introduce an algebra of asymptotically almost periodic generalized functions of Colombeau type containing Fréchet asymptotically almost period functions as well as asymptotically almost periodic Schwartz distributions. We also give some essential properties of these generalized functions. In a forthcoming paper, we will study some ordinary differential equations in this algebra of asymptotically almost periodic generalized functions.

2. Asymptotically almost periodic functions and distributions

This section recalls the main features of asymptotically almost periodic functions and distributions, for a more detailed study see [7] and [4].

Let $\mathcal{C}_b$ denote the space of bounded and continuous complex-valued functions on $\mathbb{R}$ endowed with the norm $\| \quad \|_\infty$ of uniform convergence on $\mathbb{R}$, $(\mathcal{C}_b, \| \quad \|_\infty)$ is a Banach algebra.

Definition 1. A continuous function f on $\mathbb{R}$ is called Bohr almost periodic function, if it satisfies any of the following equivalent conditions:

i) Given any sequence of real numbers $(h_n)_n$, one can extract a subsequence $(h_{n_k})_k$ such that the sequence $(f(. + h_{n_k}))_k$ converges uniformly on $\mathbb{R}$.

ii) For every $\varepsilon > 0$, the set

$$E\{\varepsilon, f\} = \left\{\tau \in \mathbb{R} : \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| < \varepsilon\right\},$$

is relatively dense in $\mathbb{R}$.

iii) For every $\varepsilon > 0$, there is a trigonometric polynomial P such that

$$\|f - P\|_\infty < \varepsilon.$$

The space of Bohr almost periodic functions on $\mathbb{R}$ is denoted by $\mathcal{C}_{\text{ap}}$.

Definition 2. The space of complex-valued continuous and bounded functions on $\mathbb{R}$ vanishing at infinity, is defined and denoted by

$$\mathcal{C}_{b_0}^+ := \left\{ f \in \mathcal{C}_b : \lim_{x \longrightarrow +\infty} f(x) = 0 \right\}.$$

Denote by $\mathbb{R}_+$ the half-line $[0, +\infty[$.

Definition 3. A complex-valued function f defined and continuous on $\mathbb{R}$ is called asymptotically almost periodic function, if there exist functions $g \in \mathcal{C}_{\text{ap}}$ and $h \in \mathcal{C}_{b_0}^+$, such that $f = g + h$ on $\mathbb{R}_+$, i.e.,

$$\mathcal{C}_{\text{aap}}(\mathbb{R}_+) := \left\{ f \in \mathcal{C}(\mathbb{R}) : \exists g \in \mathcal{C}_{\text{ap}}, \exists h \in \mathcal{C}_{b_0}^+, f = g + h \text{ on } \mathbb{R}_+ \right\}. \tag{1}$$

Remark 1. The decomposition $f = g + h$ is unique, so the functions g and h are called respectively the almost periodic part and the corrective part of f.

For asymptotic almost periodicity of Schwartz distributions see [4]. Let $\mathcal{D}$ denotes the space of test functions and $\mathcal{D}'$ the space of Schwartz distributions. If $h \in \mathbb{R}$ and $T \in \mathcal{D}'$, the translate of T by h, denoted by $\tau_h T$, is defined as

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h} \varphi \rangle, \varphi \in \mathcal{D},$$

where $\tau_{-h}\varphi(x) = \varphi(x+h)$.

Let $\mathcal{D}'_{L^\infty}$ be the space of bounded distributions.

Definition 4. A distribution $T \in \mathcal{D}'_{L^\infty}$ is said vanishing at infinity if

$$\forall \varphi \in \mathcal{D}, \lim_{h \to +\infty} \langle \tau_h T, \varphi \rangle = 0 \text{ in } \mathbb{C}.$$

Denote by $\mathcal{B}'_{0+}$ the space of bounded distributions vanishing at infinity.

Definition 5. A distribution $T \in \mathcal{D}'_{L^\infty}$ is said asymptotically almost periodic if there exist $R \in \mathcal{B}'_{\text{ap}}$ and $S \in \mathcal{B}'_{0+}$ such that $T = R + S$ on $\mathbb{R}_+$. The space of asymptotically almost periodic distributions is denoted by $\mathcal{B}'_{\text{aap}}(\mathbb{R}_+)$.

Remark 2. If $T \in \mathcal{B}'_{\text{aap}}(\mathbb{R}_+)$, the decomposition $T = R + S$ on $\mathbb{R}_+$ is unique.

Set $\mathcal{D}_+ := \{\varphi \in \mathcal{D}, \operatorname{supp} \varphi \subset \mathbb{R}_+\}$. We have the following characterizations of $\mathcal{B}'_{\text{aap}}(\mathbb{R}_+)$.

Theorem 1. *Let $T \in \mathcal{D}'_{L^\infty}$, the following assertions are equivalent:*

i) $T \in \mathcal{B}'_{\text{aap}}(\mathbb{R}_+)$.
ii) $T * \varphi \in \mathcal{C}_{\text{aap}}(\mathbb{R}_+), \forall \varphi \in \mathcal{D}_+$.
iii) $\exists k \in \mathbb{Z}_+, \exists (f_j)_j \subset \mathcal{C}_{\text{aap}}(\mathbb{R}_+) : T = \sum_{j \le k} f_j^{(j)}$ *on* $\mathbb{R}_+$.

3. Almost periodic generalized functions

Uniformly almost periodic functions have been introduced and studied by H. Bohr, see [2]. There exist three equivalent definitions of uniformly almost periodic functions, the first definition of H. Bohr, S. Bochner's definition and the definition based on the approximation property, see [1].

The Bochner's definition is more suitable for extension to distributions. L. Schwartz in [10] introduced the basic elements of almost periodic distributions.

In this section we recall the main properties of an algebra of almost periodic generalized functions generalizing trigonometric polynomials, classical almost periodic functions as well as almost periodic Schwartz distributions, for a detailed study see [3].

Let $I =]0,1]$ and

$$\mathcal{M}_{L^\infty} := \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \ |u_\varepsilon|_{k,\infty} = O(\varepsilon^{-m}), \ \varepsilon \longrightarrow 0 \right\}$$
$$\mathcal{N}_{L^\infty} := \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, \ |u_\varepsilon|_{k,\infty} = O(\varepsilon^{m}), \ \varepsilon \longrightarrow 0 \right\}.$$

Definition 6. The algebra of bounded generalized functions, denoted by $\mathcal{G}_{L^\infty}$, is defined by the quotient $\mathcal{G}_{L^\infty} = \frac{\mathcal{M}_{L^\infty}}{\mathcal{N}_{L^\infty}}$

Define

$$\begin{aligned} \mathcal{M}_{\text{ap}} &= \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{\text{ap}})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O(\varepsilon^{-m}), \varepsilon \longrightarrow 0 \right\} \\ \mathcal{N}_{\text{ap}} &= \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{\text{ap}})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O(\varepsilon^{m}), \varepsilon \longrightarrow 0 \right\}. \end{aligned} \tag{2}$$

The properties of $\mathcal{M}_{\text{ap}}$ and $\mathcal{N}_{\text{ap}}$ are summarized in the following proposition.

Proposition 2.

i) *The space* $\mathcal{M}_{\text{ap}}$ *is a subalgebra of* $(\mathcal{D}_{L^\infty})^I$.
ii) *The space* $\mathcal{N}_{\text{ap}}$ *is an ideal of* $\mathcal{M}_{\text{ap}}$.

The following definition introduces the algebra of almost periodic generalized functions.

Definition 7. The algebra of almost periodic generalized functions is the quotient algebra

$$\mathcal{G}_{\text{ap}} = \frac{\mathcal{M}_{\text{ap}}}{\mathcal{N}_{\text{ap}}}.$$

We have a characterization of elements of $\mathcal{G}_{\text{ap}}$ similar to the result (ii) of Theorem 2.2 for almost periodic distributions.

Theorem 3. *Let* $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}$, *the following assertions are equivalent:*

i) u *is almost periodic.*
ii) $u_\varepsilon * \varphi \in \mathcal{B}_{\text{ap}}, \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$.

Remark 3. The characterization (ii) does not depend on representatives.

Definition 8. Denote by Σ the set of functions $\rho \in \mathcal{S}$ satisfying $\int \rho(x)\, dx = 1$ and $\int x^k \rho(x)\, dx = 0,\ \forall\ k = 1, 2, \ldots$. Set $\rho_\varepsilon(.) := \frac{1}{\varepsilon}\rho\left(\frac{\cdot}{\varepsilon}\right),\ \varepsilon > 0$.

Proposition 4. *Let* $\rho \in \Sigma$, *the map*

$$\begin{array}{rccc} i_{\text{ap}}: & \mathcal{B}'_{\text{ap}} & \longrightarrow & \mathcal{G}_{\text{ap}} \\ & u & \longrightarrow & (u * \rho_\varepsilon)_\varepsilon + \mathcal{N}_{\text{ap}}, \end{array}$$

is a linear embedding which commutes with derivatives.

The space $\mathcal{B}_{\text{ap}}$ is embedded into $\mathcal{G}_{\text{ap}}$ canonically, i.e.,

$$\begin{array}{rccc} \sigma_{\text{ap}}: & \mathcal{B}_{\text{ap}} & \longrightarrow & \mathcal{G}_{\text{ap}} \\ & f & \longrightarrow & [(f)_\varepsilon] = (f)_\varepsilon + \mathcal{N}_{\text{ap}}. \end{array}$$

There is two ways to embed $f \in \mathcal{B}_{\text{ap}}$ into $\mathcal{G}_{\text{ap}}$. Actually, we have the same result.

Proposition 5. *The following diagram*

$$\begin{array}{ccc} \mathcal{B}_{\text{ap}} & \longrightarrow & \mathcal{B}'_{\text{ap}} \\ & \sigma_{\text{ap}} \searrow & \downarrow i_{\text{ap}} \\ & & \mathcal{G}_{\text{ap}} \end{array}$$

is commutative.

4. Asymptotically almost periodic generalized functions

In this section, we introduce the algebra of asymptotically almost periodic generalized functions of Colombeau type and we give their main properties.

Let $p \in [1, +\infty]$, the space

$$\mathcal{D}_{L^p} := \left\{ \varphi \in C^\infty : \varphi^{(j)} \in L^p, \forall j \in \mathbb{Z}_+ \right\}$$

endowed with the topology defined by the countable family of norms

$$|\varphi|_{k,p} = \sum_{j \leq k} \left\| \varphi^{(j)} \right\|_p, \forall k \in \mathbb{Z}_+,$$

is a differential Fréchet subalgebra of C^∞.

Definition 9. The space of infinitely differentiable asymptotically almost periodic functions on $\mathbb{R}$ is denoted and defined by

$$\mathcal{B}_{\text{aap}}(\mathbb{R}_+) := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in \mathcal{C}_{\text{aap}}(\mathbb{R}_+), \forall j \in \mathbb{Z}_+ \right\}.$$

We give some, easy to prove, properties of the space $\mathcal{B}_{\text{aap}}(\mathbb{R}_+)$.

Proposition 6. *We have*

i) $\mathcal{B}_{\text{aap}}(\mathbb{R}_+)$ *is a closed subalgebra of* $\mathcal{D}_{L^\infty}$ *stable by derivation.*
ii) *If* $T \in \mathcal{B}'_{\text{aap}}(\mathbb{R}_+)$ *and* $\varphi \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)$, *then* $\varphi T \in \mathcal{B}'_{\text{aap}}(\mathbb{R}_+)$.
iii) $\mathcal{B}_{\text{aap}}(\mathbb{R}_+) * L^1 \subset \mathcal{B}_{\text{aap}}(\mathbb{R}_+)$.
iv) $\mathcal{B}_{\text{aap}}(\mathbb{R}_+) = \mathcal{D}_{L^\infty} \cap \mathcal{C}_{\text{aap}}(\mathbb{R}_+)$.

Let $I =]0,1]$.

Definition 10. Define

$$\mathcal{M}_{\text{aap}} = \left\{ (u_\varepsilon)_\varepsilon \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O(\varepsilon^{-m}), \varepsilon \longrightarrow 0 \right\}$$
$$\mathcal{N}_{\text{aap}} = \left\{ (u_\varepsilon)_\varepsilon \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O(\varepsilon^{m}), \varepsilon \longrightarrow 0 \right\}.$$

The properties of $\mathcal{M}_{\text{aap}}$ and $\mathcal{N}_{\text{aap}}$ are summarized in the following proposition.

Proposition 7.

i) *The space* $\mathcal{M}_{\text{aap}}$ *is a subalgebra of* $\mathcal{B}_{\text{aap}}(\mathbb{R}_+)^I$.
ii) *The space* $\mathcal{N}_{\text{aap}}$ *is an ideal of* $\mathcal{M}_{\text{aap}}$.

Proof. i) It follows from the fact that $\mathcal{B}_{\text{aap}}(\mathbb{R}_+)$ is a differential algebra.

ii) Let $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{\text{aap}}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{\text{aap}}$, we have

$$\forall k \in \mathbb{Z}_+, \exists m' \in \mathbb{Z}_+, \exists c_1 > 0, \exists \varepsilon_0 \in I, \forall \varepsilon < \varepsilon_0, |v_\varepsilon|_{k,\infty} < c_1 \varepsilon^{-m'}.$$

Take $m \in \mathbb{Z}_+$, then for $m'' = m + m'$, $\exists c_2 > 0$ such that $|u_\varepsilon|_{k,\infty} < c_2 \varepsilon^{m''}$. Since the family of the norms $|\ \ |_{k,\infty}$ is compatible with the algebraic structure of $\mathcal{D}_{L^\infty}$, then $\forall k \in \mathbb{Z}_+$, $\exists c_k > 0$ such that

$$|u_\varepsilon v_\varepsilon|_{k,\infty} \leq c_k \ |u_\varepsilon|_{k,\infty} |v_\varepsilon|_{k,\infty},$$

consequently

$$|u_\varepsilon v_\varepsilon|_{k,\infty} < c_k c_2 \varepsilon^{m''} c_1 \varepsilon^{-m'} \leq C\varepsilon^m, \text{ where } C = c_1 c_2 c_k.$$

Hence $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{N}_{\text{aap}}$. □

Definition 11. The algebra of asymptotically almost periodic generalized functions is defined by the quotient algebra

$$\mathcal{G}_{\text{aap}}(\mathbb{R}_+) = \frac{\mathcal{M}_{\text{aap}}}{\mathcal{N}_{\text{aap}}}.$$

Recall the algebra of bounded generalized functions on $\mathbb{R}$,

$$\mathcal{G}_{L^\infty} := \frac{\mathcal{M}_{L^\infty}}{\mathcal{N}_{L^\infty}},$$

where

$$\mathcal{M}_{L^\infty} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \ \varepsilon \longrightarrow 0 \right\},$$

$$\mathcal{N}_{L^\infty} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, \ |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{m}\right), \ \varepsilon \longrightarrow 0 \right\}.$$

Remark 4. We have $\mathcal{G}_{\text{aap}}(\mathbb{R}_+) \subset \mathcal{G}_{L^\infty}$.

We give the characterization of elements of $\mathcal{G}_{\text{aap}}(\mathbb{R}_+)$ similar to the classical result for asymptotically almost periodic distributions.

Proposition 8. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}$, the following assertions are equivalent:*

i) *u is asymptotically almost periodic.*

ii) $u_\varepsilon * \varphi \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+), \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}_+$.

Proof. i) $\Longrightarrow$ ii) If $u \in \mathcal{G}_{\text{aap}}(\mathbb{R}_+)$, then for every $\varepsilon \in I$ we have $u_\varepsilon \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)$, the result (iii) of Proposition 6 gives $u_\varepsilon * \varphi \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+), \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}_+$.

ii) $\Longrightarrow$ i) Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{L^\infty}$ and $u_\varepsilon * \varphi \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+), \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}_+$, then from Theorem 1 (ii) it follows that $u_\varepsilon \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)$, it suffices to show that

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0,$$

which follows from the fact that $u \in \mathcal{M}_{L^\infty}$. If $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{L^\infty}$ and $u_\varepsilon * \varphi \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)$, $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}_+$, we obtain the same result, because $\mathcal{N}_{L^\infty} \subset \mathcal{M}_{L^\infty}$. □

Remark 5. The characterization (ii) does not depend on representatives.

Definition 12. Denote by Σ the subset of functions $\rho \in \mathcal{S}$ satisfying

$$\int_{\mathbb{R}} \rho(x)\, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}} x^k \rho(x)\, dx = 0, \forall k \geq 1.$$

Set $\rho_\varepsilon(.) = \frac{1}{\varepsilon}\rho\left(\frac{\cdot}{\varepsilon}\right), \varepsilon > 0$.

Proposition 9. *Let $\rho \in \Sigma$, the map*

$$\begin{array}{cccc} i_{\text{aap}}: & \mathcal{B}'_{\text{aap}}(\mathbb{R}_+) & \longrightarrow & \mathcal{G}_{\text{aap}}(\mathbb{R}_+) \\ & T & \longrightarrow & (T * \rho_\varepsilon)_\varepsilon + \mathcal{N}_{\text{aap}}, \end{array}$$

is a linear embedding which commutes with derivatives.

Proof. Let $T \in \mathcal{B}'_{\text{aap}}(\mathbb{R}_+)$, by characterization of asymptotically almost periodic distributions $\exists (f_\beta)_\beta \subset \mathcal{C}_{\text{aap}}(\mathbb{R}_+)$ such that $T = \sum\limits_{\beta \leq m} f_\beta^{(\beta)}$, so $\forall \alpha \in \mathbb{Z}_+$,

$$\begin{aligned} \left| \left(T^{(\alpha)} * \rho_\varepsilon \right)(x) \right| &\leq \sum_{\beta \leq m} \frac{1}{\varepsilon^{\alpha+\beta}} \int\limits_{\mathbb{R}} \left| f_\beta(x - \varepsilon y) \rho^{(\alpha+\beta)}(y) \right| dy \\ &\leq \sum_{\beta \leq m} \frac{1}{\varepsilon^{\alpha+\beta}} \left\| f_\beta \right\|_\infty \int_{\mathbb{R}} \left| \rho^{(\alpha+\beta)}(y) \right| dy, \end{aligned}$$

consequently, there exist $c > 0$ such that

$$\sup_{x \in \mathbb{R}} \left| \left(T^{(\alpha)} * \rho_\varepsilon \right)(x) \right| \leq \frac{c}{\varepsilon^{\alpha+m}},$$

hence, $\exists c' > 0$, such that

$$|T * \rho_\varepsilon|_{m',\infty} = \sum_{\alpha \leq m'} \sup_{x \in \mathbb{R}} \left| \left(T^{(\alpha)} * \rho_\varepsilon \right)(x) \right| \leq \frac{c'}{\varepsilon^{m+m'}}, \text{ where } c' = \sum_{\alpha \leq m'} \frac{c}{\varepsilon^\alpha},$$

which shows that $(T * \rho_\varepsilon)_\varepsilon \in \mathcal{M}_{\text{aap}}$. Let $(T * \rho_\varepsilon)_\varepsilon \in \mathcal{N}_{\text{aap}}$, then $\lim\limits_{\varepsilon \longrightarrow 0} T * \rho_\varepsilon = 0$ in $\mathcal{D}'_{L^\infty}$, but we have also $\lim\limits_{\varepsilon \longrightarrow 0} T * \rho_\varepsilon = T$ in $\mathcal{D}'_{L^\infty}$, this mean that i_{aap} is an embedding. The linearity of i_{aap} it results from the fact that the convolution is linear and that $i_{\text{aap}}\left(T^{(j)}\right) = \left(T^{(j)} * \rho_\varepsilon\right)_\varepsilon = (T * \rho_\varepsilon)_\varepsilon^{(j)} = \left(i_{\text{aap}}(T)\right)^{(j)}$. $\square$

The space $\mathcal{B}_{\text{aap}}(\mathbb{R}_+)$ is embedded into $\mathcal{G}_{\text{aap}}(\mathbb{R}_+)$ through $\mathcal{B}'_{\text{aap}}(\mathbb{R}_+)$, and canonically, i.e., by the map

$$\begin{array}{cccc} \sigma_{\text{aap}}: & \mathcal{B}_{\text{aap}}(\mathbb{R}_+) & \longrightarrow & \mathcal{G}_{\text{aap}}(\mathbb{R}_+) \\ & f & \longrightarrow & [(f)_\varepsilon] = (f)_\varepsilon + \mathcal{N}_{\text{aap}}. \end{array}$$

The following result shows that it is not important the way of embedding $\mathcal{B}_{\text{aap}}(\mathbb{R}_+)$.

Proposition 10. *The following diagram*

$$\begin{array}{ccc} \mathcal{B}_{\text{aap}}(\mathbb{R}_+) & \longrightarrow & \mathcal{B}'_{\text{aap}}(\mathbb{R}_+) \\ & \searrow^{\sigma_{\text{aap}}} & \downarrow i_{\text{aap}} \\ & & \mathcal{G}_{\text{aap}}(\mathbb{R}_+) \end{array}$$

is commutative.

Proof. We must show that $(f*\rho_\varepsilon - f)_\varepsilon \in \mathcal{N}_{\text{aap}}$. Let $f \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)$, by Taylor's formula and the fact that $\rho \in \Sigma$, we have

$$\begin{aligned}
&\|f*\rho_\varepsilon - f\|_\infty \\
&\quad \le \sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}}\sum_{k=0}^{m-1}\frac{(-\varepsilon y)^k}{k!}f^{(k)}(x)\,\rho(y)\,dy\right| + \sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}}\frac{(-\varepsilon y)^m}{m!}f^{(m)}(x-\theta\varepsilon y)\,\rho(y)\,dy\right| \\
&\quad \le 0+\varepsilon^m \sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\left|\frac{(-y)^m}{m!}f^{(m)}(x-\theta\varepsilon y)\,\rho(y)\,dy\right|.
\end{aligned}$$

Then $\exists C_m > 0$ such that

$$\|f*\rho_\varepsilon - f\|_\infty \le C_m \left\|f^{(m)}\right\|_\infty \|y^m\rho\|_1\,\varepsilon^m.$$

The same result can be obtained for all the derivatives of f. Hence $(f*\rho_\varepsilon - f)_\varepsilon \in \mathcal{N}_{\text{aap}}$. □

The algebra of tempered generalized functions on $\mathbb{C}$ is denoted $\mathcal{G}_{\mathcal{T}}(\mathbb{C})$, see [5].

Proposition 11. *Let $u \in \mathcal{G}_{\text{aap}}(\mathbb{R}_+)$ and $F \in \mathcal{G}_{\mathcal{T}}(\mathbb{C})$, then*

$$F \circ u = [(F\circ u_\varepsilon)_\varepsilon]$$

is a well-defined element of $\mathcal{G}_{\text{aap}}(\mathbb{R}_+)$.

Proof. It follows from the classical case of composition in context of Colombeau algebra, we have $F\circ u_\varepsilon \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)$ in view of the classical result of composition and convolution. □

The algebra of integrable generalized functions on $\mathbb{R}$ is denoted and defined by

$$\mathcal{G}_{L^1} := \frac{\mathcal{M}_{L^1}}{\mathcal{N}_{L^1}},$$

where

$$\mathcal{M}_{L^1} = \left\{(u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^1})^I, \forall k\in\mathbb{Z}_+, \exists m\in\mathbb{Z}_+, |u_\varepsilon|_{k,1} = O\left(\varepsilon^{-m}\right),\ \varepsilon\longrightarrow 0\right\},$$

$$\mathcal{N}_{L^1} = \left\{(u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^1})^I, \forall k\in\mathbb{Z}_+, \forall m\in\mathbb{Z}_+,\ |u_\varepsilon|_{k,1} = O\left(\varepsilon^{m}\right),\ \varepsilon\longrightarrow 0\right\}.$$

Proposition 12. *If $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{\text{aap}}(\mathbb{R}_+)$ and $v \in [(v_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^1}$, then the convolution $u*v$ given by*

$$(u*v)(x) = \left(\int_{\mathbb{R}} u_\varepsilon(x-y)\,v(y)\,dy\right)_\varepsilon + \mathcal{N}_{\text{aap}}$$

is a well-defined element of $\mathcal{G}_{\text{aap}}(\mathbb{R}_+)$.

Proof. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathrm{aap}}$ be a representative of u and $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{L^1}$ be a representative of v, then

$$\begin{aligned} \forall k &\in \mathbb{Z}_+, \exists m_1 \in \mathbb{Z}_+, \exists c_1 > 0, \exists \varepsilon_1 \in I, \forall \varepsilon \le \varepsilon_1, |u_\varepsilon|_{k,\infty} < c_1 \varepsilon^{-m_1}, \\ \forall k &\in \mathbb{Z}_+, \exists m_2 \in \mathbb{Z}_+, \exists c_2 > 0, \exists \varepsilon_2 \in I, \forall \varepsilon \le \varepsilon_2, |v_\varepsilon|_{k,1} < c_2 \varepsilon^{-m_2}. \end{aligned}$$

From (iii) of Proposition 6 for each $\varepsilon \in I$, $u_\varepsilon * v_\varepsilon$ is an infinitely differentiable asymptotically almost periodic function. Moreover, by the Young inequality there exist $c_3 > 0$ such that

$$\left\| (u_\varepsilon * v_\varepsilon)^{(j)} \right\|_\infty \le c_3 \left\| u_\varepsilon^{(j)} \right\|_\infty \|v_\varepsilon\|_1 ,$$

so there exist $c > 0$ such that $\forall k \in \mathbb{Z}_+$, we have

$$|u_\varepsilon * v_\varepsilon|_{k,\infty} \le c\, |u_\varepsilon|_{k,\infty} |v_\varepsilon|_{0,1} . \tag{3}$$

Consequently there exist $m = m_1 + m_2$ such that

$$|u_\varepsilon * v_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0,$$

which shows that $(u_\varepsilon * v_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathrm{aap}}$. The inequality 3 shows that $(u_\varepsilon * v_\varepsilon)_\varepsilon$ is independent on the representatives of $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$. □

It remains to prove that an asymptotically almost periodic generalized function is decomposed uniquely into an almost periodic generalized function and a generalized function vanishing at infinity.

Recall the space of infinitely differentiable almost periodic functions on $\mathbb{R}$,

$$\mathcal{B}_{\mathrm{ap}} := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in \mathcal{C}_{\mathrm{ap}}, \forall j \in \mathbb{Z}_+ \right\}.$$

The algebra of almost periodic generalized functions, denoted by $\mathcal{G}_{\mathrm{ap}}$, is defined as the quotient algebra

$$\mathcal{G}_{\mathrm{ap}} = \frac{\mathcal{M}_{\mathrm{ap}}}{\mathcal{N}_{\mathrm{ap}}},$$

where

$$\begin{aligned} \mathcal{M}_{\mathrm{ap}} &= \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{\mathrm{ap}})^I , \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0 \right\}, \\ \mathcal{N}_{\mathrm{ap}} &= \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{\mathrm{ap}})^I , \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{m}\right),\ \varepsilon \longrightarrow 0 \right\}. \end{aligned}$$

For more details on $\mathcal{G}_{\mathrm{ap}}$ see [3].

Definition 13. The space of infinitely differentiable bounded functions on $\mathbb{R}$ vanishing at infinity is defined and denoted by

$$\mathcal{B}_0^+ := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \lim_{x \longrightarrow +\infty} \varphi^{(j)}(x) = 0, \forall j \in \mathbb{Z}_+ \right\}.$$

Define

$$\begin{aligned} \mathcal{M}_0^+ &= \left\{ (u_\varepsilon)_\varepsilon \in \left(\mathcal{B}_0^+\right)^I , \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0 \right\}, \\ \mathcal{N}_0^+ &= \left\{ (u_\varepsilon)_\varepsilon \in \left(\mathcal{B}_0^+\right)^I , \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{m}\right),\ \varepsilon \longrightarrow 0 \right\}. \end{aligned}$$

Proposition 13. *The space* $\mathcal{M}_0^+$ *is a subalgebra of* $\left(\mathcal{B}_0^+\right)^I$ *and the space* $\mathcal{N}_0^+$ *is an ideal of* $\mathcal{M}_0^+$.

Proof. Easy. □

Definition 14. The algebra of bounded generalized functions vanishing at infinity, denoted by $\mathcal{G}_0^+$, is defined as the quotient algebra

$$\mathcal{G}_0^+ = \frac{\mathcal{M}_0^+}{\mathcal{N}_0^+}.$$

Let us show first the following result.

Proposition 14. *Let* $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}_{\text{ap}}$ *and* $w = [(w_\varepsilon)_\varepsilon] \in \mathcal{G}_0^+$, *then* $v + w \in \mathcal{G}_{\text{aap}}(\mathbb{R}_+)$.

Proof. Let $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}_{\text{ap}}$ and $w = [(w_\varepsilon)_\varepsilon] \in \mathcal{G}_0^+$, since $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{\text{ap}}$ and $(w_\varepsilon)_\varepsilon \in \mathcal{M}_0^+$, then $\forall k \in \mathbb{Z}_+, \exists m_1, m_2 \in \mathbb{Z}_+$, $\exists c_1 > 0, c_2 > 0$, such that

$$|v_\varepsilon|_{k,\infty} < c_1 \varepsilon^{-m_1}, \varepsilon \longrightarrow 0 \text{ and } |w_\varepsilon|_{k,\infty} < c_2 \varepsilon^{-m_2}, \varepsilon \longrightarrow 0,$$

consequently, $\forall k \in \mathbb{Z}_+, \exists m = m_1 + m_2$, $\exists c = c_1 + c_2$ such that

$$|v_\varepsilon + w_\varepsilon|_{k,\infty} \leq |v_\varepsilon|_{k,\infty} + |w_\varepsilon|_{k,\infty} < c\varepsilon^{-m}, \varepsilon \longrightarrow 0,$$

hence $(v_\varepsilon + w_\varepsilon)_\varepsilon \in \mathcal{M}_{\text{app}}$. If $(v_\varepsilon)_\varepsilon \in \mathcal{N}_{\text{ap}}$ and $(w_\varepsilon)_\varepsilon \in \mathcal{N}_0^+$, we will obtain the same result that $(v_\varepsilon + w_\varepsilon)_\varepsilon \in \mathcal{N}_{\text{app}}$. □

The following result is needed in the proof of the theorem.

Lemma 15. *If* $\varphi \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)$ *is such that*

$$\varphi \equiv \psi + \chi \text{ on } \mathbb{R}_+$$

and $\varphi^{(j)} \equiv f_j + g_j$ *on* $\mathbb{R}_+, j \in \mathbb{Z}_+$, *then* $\psi^{(j)} \equiv f_j$ *and* $\chi^{(j)} \equiv g_j$ *on* $\mathbb{R}_+$.

Proof. We have $\varphi^{(j)} \equiv \psi^{(j)} + \chi^{(j)}$ on $\mathbb{R}_+, \forall j \in \mathbb{Z}_+$,since $\varphi^{(j)} \in \mathcal{C}_{\text{aap}}(\mathbb{R}_+)$ and the decomposition of asymptotically almost periodic functions is unique, then $\forall j \in \mathbb{Z}_+$

$$\psi^{(j)} \equiv f_j \text{ and } \chi^{(j)} \equiv g_j \text{ on } \mathbb{R}_+.$$ □

The following result gives the decomposition of an asymptotically almost periodic generalized functions.

Theorem 16. *Let* $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{\text{aap}}(\mathbb{R}_+)$, *then there exist* $v = [(v_\varepsilon)_\varepsilon]$ *uniquely defined in* $\mathcal{G}_{\text{ap}}$ *and* $w = [(w_\varepsilon)_\varepsilon]$ *uniquely defined in* $\mathcal{G}_0^+$ *such that* $u = v + w$ *in* $\mathcal{G}_{\text{aap}}(\mathbb{R}_+)$.

Proof. Suppose that $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\text{aap}}(\mathbb{R}_+)$, so $\forall \varepsilon \in I, u_\varepsilon \in \mathcal{B}_{\text{aap}}(\mathbb{R}_+)$, then $\exists v_\varepsilon \in \mathcal{C}_{\text{ap}}$, $\exists w_\varepsilon \in \mathcal{C}_{b_0}^+$ such that $\forall \varepsilon \in I, u_\varepsilon = v_\varepsilon + w_\varepsilon$ on $\mathbb{R}_+$, by Lemma 15 we have $\forall \varepsilon \in I, \forall j \in \mathbb{Z}_+, u_\varepsilon^{(j)} = v_\varepsilon^{(j)} + w_\varepsilon^{(j)}$ on $\mathbb{R}_+$ and $v_\varepsilon \in \mathcal{B}_{\text{ap}}$ and $w_\varepsilon \in \mathcal{B}_0^+$. To have

$(v_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathrm{ap}}$ and $(w_\varepsilon)_\varepsilon \in \mathcal{M}_0^+$, it remains to show that $\forall k \in \mathbb{Z}_+, \exists m_1, m_2 \in \mathbb{Z}_+$, such that

$$|v_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m_1}\right), \varepsilon \longrightarrow 0 \text{ and } |w_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m_2}\right), \varepsilon \longrightarrow 0. \tag{4}$$

By hypothesis, $\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+$, such that

$$|u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0. \tag{5}$$

Let $(\alpha_n)_n$ be a sequence of real numbers which converges to $+\infty$, since for each $\varepsilon \in I$, $(v_\varepsilon)_\varepsilon$ is almost periodic function, then there exist $\left(\alpha_{n_k}^\varepsilon\right)_k$ a subsequence of $(\alpha_n)_n$ and an almost periodic function f_ε such that

$$\lim_{k\to+\infty} v_\varepsilon\left(\cdot + \alpha_{n_k}^\varepsilon\right) = f_\varepsilon \text{ exist uniformly.}$$

As $w_\varepsilon \in \mathcal{C}_{b_0}^+$, then $\lim_{k\to+\infty} w_\varepsilon\left(x + \alpha_{n_k}^\varepsilon\right) = 0$. So $\forall x \in \mathbb{R}$,

$$\lim_{k\to+\infty} u_\varepsilon\left(x + \alpha_{n_k}^\varepsilon\right) = \lim_{k\to+\infty} v_\varepsilon\left(x + \alpha_{n_k}^\varepsilon\right) = f_\varepsilon(x).$$

Hence, $\exists c > 0, \exists m \in \mathbb{Z}_+$ such that $|f_\varepsilon|_{0,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0$. On the other hand, thanks to the almost periodicity of f_ε, we have $\forall \varepsilon \in I, \forall x \in \mathbb{R}$,

$$\lim_{k\to+\infty} f_\varepsilon\left(x - \alpha_{n_k}^\varepsilon\right) = v_\varepsilon\left(x\right),$$

which gives $\exists m_1 \in \mathbb{Z}_+$ such that $|v_\varepsilon|_{0,\infty} = O\left(\varepsilon^{-m_1}\right), \varepsilon \longrightarrow 0$; this estimate and 5 give $\exists m_2 \in \mathbb{Z}_+$ such that $|w_\varepsilon|_{0,\infty} = O\left(\varepsilon^{-m_2}\right), \varepsilon \longrightarrow 0$; we obtain the estimate 4 for $k = 0$. The same procedure for the derivatives $u_\varepsilon^{(j)}, v_\varepsilon^{(j)}, w_\varepsilon^{(j)}$ gives the estimate 4 for every $k \in \mathbb{Z}_+$. Hence $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathrm{ap}}$ and $(w_\varepsilon)_\varepsilon \in \mathcal{M}_0^+$.

The same reasoning shows that if $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{aap}}(\mathbb{R}_+)$, then there exist $(v_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{ap}}$ and $(w_\varepsilon)_\varepsilon \in \mathcal{N}_0^+$, such that $\forall \varepsilon \in I, \forall j \in \mathbb{Z}_+, u_\varepsilon^{(j)} = v_\varepsilon^{(j)} + w_\varepsilon^{(j)}$ on $\mathbb{R}_+$.

Let us suppose that there exist another $\widetilde{v} = [(\widetilde{v}_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathrm{ap}}$ and $\widetilde{w} = [(\widetilde{w}_\varepsilon)_\varepsilon] \in \mathcal{G}_0^+$ such that $u = \widetilde{v} + \widetilde{w}$ in $\mathcal{G}_{\mathrm{aap}}(\mathbb{R}_+)$. We obtain that $(v_\varepsilon - \widetilde{v}_\varepsilon)_\varepsilon + (w_\varepsilon - \widetilde{w}_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{aap}}(\mathbb{R}_+)$. The same idea of the proof gives that $(v_\varepsilon - \widetilde{v}_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{aap}}$ and $(w_\varepsilon - \widetilde{w}_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{aap}}(\mathbb{R}_+)$. □

The theorem motivates the following definition.

Definition 15. If $u \in \mathcal{G}_{\mathrm{aap}}(\mathbb{R}_+)$ is such that $u = v + w$ in $\mathcal{G}_{\mathrm{aap}}(\mathbb{R}_+)$, where $v \in \mathcal{G}_{\mathrm{ap}}$ and $w \in \mathcal{G}_0^+$, then v and w are called respectively the almost periodic part and the corrective part of u.

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Wave Equations and Symmetric First-order Systems in Case of Low Regularity

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Abstract. We analyse an algorithm of transition between Cauchy problems for second-order wave equations and first-order symmetric hyperbolic systems in case the coefficients as well as the data are non-smooth, even allowing for regularity below the standard conditions guaranteeing well-posedness. The typical operations involved in rewriting equations into systems are then neither defined classically nor consistently extendible to the distribution theoretic setting. However, employing the nonlinear theory of generalized functions in the sense of Colombeau we arrive at clear statements about the transfer of questions concerning solvability and uniqueness from wave equations to symmetric hyperbolic systems and vice versa. Finally, we illustrate how this transfer method allows to draw new conclusions on unique solvability of the Cauchy problem for wave equations with non-smooth coefficients.

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1. Introduction

Theories for higher-order partial differential equations on the one hand and first-order systems of (pseudo)differential equations on the other hand are to a large extent developed in parallel, although elaborate mechanisms for rewriting the former into the latter do exist in terms of modern analysis (cf. [15, 27]). However, in general the transition methods require high-powered pseudodifferential operator techniques and, what is even more restrictive in special situations, do often require a certain smoothness of the coefficients (or symbols) to be mathematically meaningful in all their intermediate operations beyond mere formal manipulation. Nonlinear theories of generalized functions, in particular the differential algebras constructed in the sense of Colombeau, provide a means to embed distributions into a wider context where the transition between higher-order equations and first-order systems can be based on well-defined operations. Thus, Colombeau theory allows to rigorously address the question about the precise relation between generalized solutions to wave equations and those of corresponding hyperbolic first-order systems in case of non-smooth coefficients.

The theory of generalized solutions to linear hyperbolic first-order systems has been developed over 20 years and has achieved a spectrum of results on existence and uniqueness of solutions to the Cauchy problem, distributional limits and regularity of solutions, and symmetrizability (cf. [22, 23, 16, 24, 14, 6]).

On the other hand, generalized solutions of wave equations arising via the Laplace–Beltrami operator of a Lorentzian metric of low regularity have been studied in [28, 19, 8, 10, 13]. These investigations draw strong motivation from general relativity, in particular in the context of Chris Clarke's notion of generalized hyperbolicity [1, 2], which generalizes the classical notion of global hyperbolicity (i.e., the geometric condition necessary for global well-posedness of the Cauchy problem for wave equations). More precisely, local and global existence and uniqueness of generalized solutions have been established for a wide class of "weakly singular" space-time metrics which are described using the geometric theory of nonlinear generalized functions ([9, Chapter 3]).

In this paper we establish a rigorous way to translate Cauchy problems for wave equations into such for symmetric hyperbolic systems and vice versa in case of low regularity, thereby making results from either theory potentially available to the other. Also we show some of this potential by inspecting results on wave equations obtained from statements on first-order systems through a careful analysis of the translation process.

The plan of the paper is as follows. Section 2 introduces and briefly reviews the required basic notions from Colombeau's theory of generalized functions and numbers. In Section 3 we present an explicit method to transform a second-order wave equation with generalized function coefficients into a symmetric hyperbolic system of first order and describe in precise terms the relation between generalized solutions in either case. The main results here are summarized in Theorem 3.4 and in the simple Example 3.5 we illustrate what kind of difficulties from the pure distribution theoretic viewpoint are remedied by our result. As an application we show in Section 4 that solvability results on symmetric hyperbolic systems can be used to deduce in Theorem 4.3 new aspects on solvability of the Cauchy problem for wave equations with non-smooth coefficients.

2. Basic notions and spaces

Notation. We assume several notational conventions to keep calculations clearly laid out: We denote vector-valued functions by bold symbols, e.g., $\boldsymbol{v}$, and matrix-valued functions by bold and sans serif letters, e.g., R. We write v_i for the components of a vector $\boldsymbol{v}$ and R_{ij} for the components of a matrix R. The ith row respectively jth column of a matrix $\mathbf{R}$ is denoted by $\boldsymbol{R}_{i\cdot}$ or $\boldsymbol{R}_{\cdot j}$ respectively. The spatial gradient of a scalar function u shall be $\boldsymbol{u}' = \operatorname{grad} u = \partial_x u$, the spatial Hessian shall be $\mathbf{u}''$. We denote the Euclidean scalar product by $\langle \cdot, \cdot \rangle$.

Generalized functions. We will use variants of Colombeau algebras as presented in [3, 23, 9, 5]. Here, we recall their essential features: Let E be a locally convex topological vector space with a topology given by a family of semi-norms $\{p_j\}$ with j in some index set J. We define

$$\mathcal{M}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \forall j \in J \ \exists N \in \mathbb{N}_0 : p_j(u\varepsilon) = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\},$$
$$\mathcal{N}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \forall j \in J \ \forall m \in \mathbb{N}_0 : p_j(u\varepsilon) = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\},$$

the moderate respectively negligible subsets of $E^{(0,1]}$. Operations are induced from E by ε-wise application, so we have the (vector space) inclusion relation $\mathcal{N}_E \subseteq \mathcal{M}_E \subseteq E^{(0,1]}$. The generalized functions based on E are defined as the quotient space $\mathcal{G}_E := \mathcal{M}_E / \mathcal{N}_E$. If E is a differential algebra, then $\mathcal{N}_E$ is an ideal in $\mathcal{M}_E$ and therefore $\mathcal{G}_E$ is a differential algebra as well, called the Colombeau algebra based on E.

Let now U be an open subset of $\mathbb{R}^n$. If we choose $E = \mathcal{C}^\infty(U)$ with the topology of uniform convergence of all derivatives on compact sets, then we obtain the special Colombeau algebra on U, i.e., $\mathcal{G}_{\mathcal{C}^\infty(U)} = \mathcal{G}(U)$.

Moreover, we will also use the following three Sobolev spaces in this construction:

- $E = H^\infty(U) = \{u \in \mathcal{C}^\infty(\overline{U}) \mid \partial^\alpha u \in L^2(U) \, \forall \alpha \in \mathbb{N}_0^n\}$ with the family of norms
$$\|u\|_{H^k} = \Big(\sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^2} \Big)^{\frac{1}{2}} \quad k \in \mathbb{N}_0,$$
- $E = W^{\infty,\infty}(U) = \{u \in \mathcal{C}^\infty(\overline{U}) \mid \partial^\alpha u \in L^\infty(U) \, \forall \alpha \in \mathbb{N}_0^n\}$ with the family of norms
$$\|u\|_{W^{k,\infty}} = \max_{|\alpha| \le k} \|\partial^\alpha u\|_{L^\infty} \quad k \in \mathbb{N}_0,$$
- $E = \mathcal{C}^\infty(\overline{I} \times \mathbb{R}^n)$, where I is an open, bounded interval, equipped with the family of semi-norms
$$\|u\|_{k,K} = \max_{|\alpha| \le k} \|\partial^\alpha u\|_{L^\infty(\overline{I} \times K)},$$
where K is a compact subset of $\mathbb{R}^n$ and $k \in \mathbb{N}_0$.

To simplify notation, we denote the corresponding Colombeau algebras as in [12]:

$$\mathcal{G}_{L^2}(U) := \mathcal{G}_{H^\infty(U)} \qquad \mathcal{G}_{L^\infty}(U) := \mathcal{G}_{W^{\infty,\infty}(U)} \qquad \mathcal{G}(\overline{I} \times \mathbb{R}^n) := \mathcal{G}_{\mathcal{C}^\infty(\overline{I} \times \mathbb{R}^n)}.$$

Elements in $\mathcal{G}_E(U^n)$ are denoted by $u = [(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}_E(U^n)$. We restrict a Colombeau function $u(t,x)$ to an initial surface by taking $u|_{t=0} = [(u_\varepsilon(0,x))_\varepsilon]$. The ring of generalized numbers $\widetilde{\mathbb{R}}$ consists of elements $u = [(u_\varepsilon)_\varepsilon]$, where $u_\varepsilon \in \mathbb{R}$. Note that $\widetilde{\mathbb{R}}^n$ is a module over $\widetilde{\mathbb{R}}$, a fact we have to keep in mind when doing linear algebra.

Generalized functions in $\mathcal{G}(\mathbb{R}^n)$ are characterized by their generalized point values. In fact, considering only classical point values is insufficient as the following statement from [25] shows, cf. [9, Thm. 1.2.46]. Let $f \in \mathcal{G}(\mathbb{R}^n)$. The following are equivalent:

(i) $f = 0$ in $\mathcal{G}(\mathbb{R}^n)$, (ii) $f(\tilde{x}) = 0$ in $\widetilde{\mathbb{R}}$ for each $\tilde{x} \in \widetilde{\mathbb{R}}^n_c$.

Here $\widetilde{\mathbb{R}}^n_c$ denotes the set of compactly supported generalized points: A generalized point $x \in \widetilde{\mathbb{R}}^n$ is compactly supported if there exists $K \subseteq \widetilde{\mathbb{R}}^n$ compact and $\eta > 0$ such that $x_\varepsilon \in K$ for $\varepsilon < \eta$.

A matrix-valued generalized function $\mathbf{G} \in M_k(\mathcal{G}(\mathbb{R}^n))$ is called symmetric and nondegenerate if for any $\tilde{x} \in \widetilde{\mathbb{R}}^n_c$ the bilinear map $\mathbf{G}(\tilde{x}) : \widetilde{\mathbb{R}}^k \times \widetilde{\mathbb{R}}^k \to \widetilde{\mathbb{R}}$ is symmetric and nondegenerate, [9, Def. 5.1.2] Here, by nondegenerate we mean that $\boldsymbol{\xi} \in \widetilde{\mathbb{R}}^k$, $\mathbf{G}(\tilde{x})(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \, \forall \boldsymbol{\eta} \in \widetilde{\mathbb{R}}^k$ implies $\boldsymbol{\xi} = 0$. Apart from this pointwise definition, there exist equivalent characterizations of nondegeneracy, see [9, Theorem 3.2.74]. In particular, there always exists a representative entirely consisting of symmetric, nondegenerate matrices. If, in addition, there exists a representative of constant index j, we call $j = \nu(\mathbf{G})$ the index of $\mathbf{G}$. We call matrices in $M_k(\widetilde{\mathbb{R}})$ with $j = 0$ Riemannian metrics and such with $j = 1$ Lorentzian metrics. For concepts of linear algebra in $\widetilde{\mathbb{R}}^k$ we refer to [20]. Finally, we point out the following lemma on Lorentzian metrics (the proof of which is straightforward).

Lemma 2.1. *Let $\mathbf{G} \in M_{n+1}(\widetilde{\mathbb{R}})$ be of the form*

$$\mathbf{G} = \begin{pmatrix} -1 & \boldsymbol{g}^T \\ \boldsymbol{g} & \mathbf{R} \end{pmatrix},$$

with $\mathbf{R}$ a generalized Riemannian metric on $\widetilde{\mathbb{R}}^n$ and $\boldsymbol{g}$ a vector with entries in $\widetilde{\mathbb{R}}^n$, then $\mathbf{G}$ is Lorentzian.

3. Transformation between equations and systems

In this section we relate solutions of wave equations to solutions of corresponding symmetric first-order systems. To this end, consider a wave equation in $\mathcal{G}(\mathbb{R}^{n+1})$

$$-\partial_t^2 u + 2\sum_{i=1}^{n} g_i \partial_{x_i} \partial_t u + \sum_{i,j=1}^{n} R_{ij} \partial_{x_i} \partial_{x_j} u + a\partial_t u + \sum_{i=1}^{n} b_i \partial_{x_i} u + cu = f, \qquad (1)$$

with principal part derived from $\mathbf{G} := \begin{pmatrix} -1 & \boldsymbol{g}^T \\ \boldsymbol{g} & \mathbf{R} \end{pmatrix}$, a generalized Lorentzian metric. In fact, our arguments still hold in the more general case, where the matrix entry G_{00} is a strictly negative generalized function (i.e., $-G_{00} > \varepsilon^m$ on compact sets for some $m > 0$), upon dividing by $-G_{00}$. Here $\mathbf{R} = (R_{ij})$ is a positive definite, symmetric matrix of generalized functions, $\boldsymbol{g}$ and $\boldsymbol{b}$ are vectors with entries in $\mathcal{G}(\mathbb{R}^{n+1})$ and a, c, and f are generalized functions.

Next we rewrite the wave equation (1) as a first-order system. There exist several algorithms to obtain a hyperbolic first-order system. However, we employ an algorithm that also guarantees the symmetry of the system. Indeed by setting $\boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')^T$ we arrive at the system

$$-\partial_t \boldsymbol{w} + \sum_{i=1}^{n} \mathbf{A}_i \partial_{x_i} \boldsymbol{w} + \mathbf{B}\boldsymbol{w} = \boldsymbol{F} \tag{2}$$

in $\mathcal{G}(\mathbb{R}^{n+1})^{n+2}$. Here $\mathbf{S} = \mathbf{R}^{\frac{1}{2}}$ is constructed via ε-wise diagonalization. The so-constructed square root is a symmetric positive definite matrix with entries in $\mathcal{G}(\mathbb{R}^{n+1})$, as we will discuss below. The matrices $\mathbf{A}_i$, $\mathbf{B}$, and the vector $\boldsymbol{F}$ are given in the following way:

$$\begin{aligned} \mathbf{A}_i &= \begin{pmatrix} 0 & 0 & 0_{1\times n} \\ 0 & 2g_i & \boldsymbol{S}_{i\cdot} \\ 0_{n\times 1} & \boldsymbol{S}_{\cdot i} & 0_{n\times n} \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} 0 \\ f \\ 0_{n\times 1} \end{pmatrix} \\ \mathbf{B} &= \begin{pmatrix} 0 & 1 & 0_{1\times n} \\ c & a & (\operatorname{div}\mathbf{S})^T + (\boldsymbol{b} - \operatorname{div}\mathbf{S}^2)^T \mathbf{S}^{-1} \\ 0_{n\times 1} & 0_{n\times 1} & -(\partial_t \mathbf{S})\mathbf{S}^{-1} \end{pmatrix}. \end{aligned} \tag{3}$$

Here we have used the fact that $\operatorname{tr}(\mathbf{S}^2\mathbf{u}'') = \operatorname{div}(\mathbf{S}^2\boldsymbol{u}') - \langle \operatorname{div}\mathbf{S}^2, \boldsymbol{u}'\rangle$. A word on the notation is in order. For any symmetric matrix $\mathbf{S}$, we set the divergence $(\operatorname{div}\mathbf{S})_i = \sum_{j=1}^{n} \partial_{x_j} S_{ij}$, i.e., $\operatorname{div}\mathbf{S}$ is a vector, whose ith entry is simply the divergence of the ith row (or column) of the matrix $\mathbf{S}$.

Finally, we show that $\mathbf{S} = \mathbf{R}^{\frac{1}{2}}$ is a symmetric and positive definite matrix with generalized functions as entries. A representative of the matrix $\mathbf{S}$ is given by $\mathbf{S}_\varepsilon(t,x) = \mathbf{U}_\varepsilon(t,x)^T \mathbf{D}_\varepsilon(t,x)^{\frac{1}{2}} \mathbf{U}_\varepsilon(t,x)$ with

$$\mathbf{D}_\varepsilon(t,x)^{\frac{1}{2}} = \operatorname{diag}\Big(\sqrt{\lambda_{1,\varepsilon}(t,x)}, \ldots, \sqrt{\lambda_{n,\varepsilon}(t,x)}\Big),$$

where $\lambda_{1,\varepsilon}(t,x), \ldots, \lambda_{n,\varepsilon}$ are the eigenvalues of a symmetric and positive definite representative $(\mathbf{R}_\varepsilon)_\varepsilon$ of $\mathbf{R}$. Observe that $(\mathbf{D}_\varepsilon)_\varepsilon$ and $(\mathbf{U}_\varepsilon)_\varepsilon$ are not necessarily nets of matrices with smooth entries, however, the product $(\mathbf{U}_\varepsilon^T \mathbf{D}_\varepsilon^{\frac{1}{2}} \mathbf{U}_\varepsilon)_\varepsilon$ is smooth by the following lemma (where we denote by $S_n(\mathbb{R})$ and $S_n^+(\mathbb{R})$ the spaces of symmetric and positive definite symmetric matrices in $M_n(\mathbb{R})$).

Lemma 3.1. *The smooth map $f : S_n^+(\mathbb{R}) \to S_n^+(\mathbb{R})$ with $f(\mathbf{A}) = \mathbf{A}^2$ is a diffeomorphism.*

Proof. The map $f : S_n^+(\mathbb{R}) \to S_n^+(\mathbb{R})$ is bijective (e.g., [17, Prop. 6.8]) and we employ the inverse function theorem to conclude that f is a global diffeomorphism. Indeed since $S_n^+(\mathbb{R})$ is an open subset of $S_n(\mathbb{R})$ we may identify the tangent space $\mathrm{T}_{\mathbf{A}} S_n^+(\mathbb{R})$ for $\mathbf{A} \in S_n^+(\mathbb{R})$ with $S_n(\mathbb{R})$ and obtain $\mathrm{d}f(\mathbf{A})(\mathbf{B}) = \mathbf{AB} + \mathbf{BA}$. Now injectivity of $\mathrm{d}f(\mathbf{A})$ follows since if $\mathbf{AB} = -\mathbf{BA}$ and $\mathbf{B} \neq 0$ there is $0 \neq \lambda$ with $\mathbf{B}\boldsymbol{v} = \lambda\boldsymbol{v}$ for some $\boldsymbol{v} \neq 0$. But then $\mathbf{A}\boldsymbol{v}$ is an eigenvector to $-\lambda$ and by symmetry of $\mathbf{B}$ we have $\langle \boldsymbol{v}, A\boldsymbol{v}\rangle = 0$, contradicting positive definiteness of $\mathbf{A}$. □

From the wave equation to the first-order system. Assuming that we have a solution u to the wave equation (1), in the following lemma we will construct a solution to the first-order system (2). Basically, we define a vector $\boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')^T$ and rewrite the wave equation in terms of the three components of $\boldsymbol{w}$.

Lemma 3.2. *Let $u_0, u_1 \in \mathcal{G}(\mathbb{R}^n)$ and consider the second-order equation* (1) *with initial condition $(u, \partial_t u)|_{t=0} = (u_0, u_1)$. If $u \in \mathcal{G}(\mathbb{R}^{n+1})$ is a solution of* (1)*, then the vector $\boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')^T$ is a solution to the first-order system* (2) *with initial condition $\boldsymbol{w}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^T$.*

Proof. First we rewrite equation (1) in divergence form, i.e.,

$$-\partial_t^2 u + 2\langle \boldsymbol{g}, \partial_t \boldsymbol{u}'\rangle + \operatorname{div}(\mathbf{S}^2 \boldsymbol{u}') + a\partial_t u + \langle \boldsymbol{b} - \operatorname{div}\mathbf{S}^2, \boldsymbol{u}'\rangle + cu = f.$$

Introducing new variables $z := \partial_t u$ and $\boldsymbol{v} := \mathbf{S}\boldsymbol{u}'$, we may write

$$-\partial_t z + 2\langle \boldsymbol{g}, \boldsymbol{z}'\rangle + \operatorname{div}(\mathbf{S}\boldsymbol{v}) + az + \langle \boldsymbol{b} - \operatorname{div}\mathbf{S}^2, \mathbf{S}^{-1}\boldsymbol{v}\rangle + cu = f.$$

Hence we have

$$-\partial_t \boldsymbol{w} + \sum_{i=1}^{n} \begin{pmatrix} 0 & 0 & 0_{1\times n} \\ 0 & 2g_i & \boldsymbol{S}_{i\cdot} \\ 0_{n\times 1} & \boldsymbol{S}_{\cdot i} & 0_{n\times n} \end{pmatrix} \partial_{x_i}\boldsymbol{w}$$
$$+ \begin{pmatrix} 0 & 1 & 0_{1\times n} \\ c & a & (\operatorname{div}\mathbf{S})^T + (\boldsymbol{b} - \operatorname{div}\mathbf{S}^2)^T\mathbf{S}^{-1} \\ 0_{n\times 1} & 0_{n\times 1} & (\partial_t \mathbf{S})\mathbf{S}^{-1} \end{pmatrix} \boldsymbol{w} = \begin{pmatrix} 0 \\ f \\ 0_{n\times 1} \end{pmatrix},$$

where we have used that $\operatorname{div}(\mathbf{S}\boldsymbol{v}) = \sum_{i=1}^n \boldsymbol{S}_{i\cdot}\partial_{x_i}\boldsymbol{v} + (\operatorname{div}\mathbf{S})^T\boldsymbol{v}$. Note that the second equation in the above system is just the original wave equation written in new variables, whereas the other equations represent the transformation of variables. Finally, evaluating $\boldsymbol{w}$ at time $t = 0$ yields $\boldsymbol{w}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^T$. □

From the first-order system to the wave equation. We now look at the converse situation: Given a solution to the first-order system (2), we would like to prove existence for wave-type equations. Now, let $\mathbf{S}$ be a symmetric and invertible n-dimensional matrix with entries in $\mathcal{G}(\mathbb{R}^{n+1})$, let $\boldsymbol{g}$ and $\boldsymbol{b}$ be vectors with entries in $\mathcal{G}(\mathbb{R}^{n+1})$, and let a, c be generalized functions. Observe that the matrix $\mathbf{B}$ is not restricted by the structure of the term $(\operatorname{div}\mathbf{S})^T + (\boldsymbol{b} - \operatorname{div}\mathbf{S}^2)^T\mathbf{S}^{-1}$. Let $\tilde{\boldsymbol{b}}^T = (\operatorname{div}\mathbf{S})^T + (\boldsymbol{b} - \operatorname{div}\mathbf{S}^2)^T\mathbf{S}^{-1}$. Multiplication with $\mathbf{S}$ from the right gives

$$\tilde{\boldsymbol{b}}^T\mathbf{S} = (\operatorname{div}\mathbf{S})^T\mathbf{S} + (\boldsymbol{b} - \operatorname{div}\mathbf{S}^2)^T.$$

Bringing all terms except the one containing $\boldsymbol{b}$ to the other side results in

$$\tilde{\boldsymbol{b}}^T\mathbf{S} - (\operatorname{div}\mathbf{S})^T\mathbf{S} + (\operatorname{div}\mathbf{S}^2)^T = \boldsymbol{b}^T.$$

Finally, transposition leads to an equation for $\boldsymbol{b}$ entirely in terms of $\mathbf{S}$ and an arbitrarily chosen coefficient $\tilde{\boldsymbol{b}}$:

$$\boldsymbol{b} = \mathbf{S}^T\tilde{\boldsymbol{b}} - \mathbf{S}^T\operatorname{div}\mathbf{S} + (\operatorname{div}\mathbf{S}^2).$$

Lemma 3.3. *Let* $u_0, u_1 \in \mathcal{G}(\mathbb{R}^n)$*. If* $\boldsymbol{w} = (u, z, \boldsymbol{v})^T \in \mathcal{G}(\mathbb{R}^{n+1})^{n+2}$ *is a solution to the first-order system* (2) *with initial condition* $\boldsymbol{w}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^T$*, then* u *is a solution to* (1) *with initial condition* $(u, \partial_t u)|_{t=0} = (u_0, u_1)$*.*

Proof. Note that the first equation of our system is just $z = \partial_t u$. The last n equations read

$$\partial_t \boldsymbol{v} + \mathbf{S}\boldsymbol{z}' + (\partial_t \mathbf{S})\mathbf{S}^{-1}\boldsymbol{v} = 0,$$

which is the same as

$$\mathbf{S}\boldsymbol{z}' = \mathbf{S}\partial_t(\mathbf{S}^{-1}\boldsymbol{v})$$

since $\partial_t \boldsymbol{v} = \partial_t(\mathbf{S}\mathbf{S}^{-1}\boldsymbol{v}) = \mathbf{S}\partial_t(\mathbf{S}^{-1}\boldsymbol{v}) + (\partial_t\mathbf{S})\mathbf{S}^{-1}\boldsymbol{v}$. Multiplying by $\mathbf{S}^{-1}$ from the left and using that $z = \partial_t u$ we find

$$\partial_t(\boldsymbol{u}' - \mathbf{S}^{-1}\boldsymbol{v}) = 0.$$

By the initial condition $\boldsymbol{u}'|_{t=0} = \mathbf{S}^{-1}\boldsymbol{v}|_{t=0}$, we have $\boldsymbol{u}' = \mathbf{S}^{-1}\boldsymbol{v}$ for all t which is equivalent to $\boldsymbol{v} = \mathbf{S}\boldsymbol{u}'$. Hence, replacing z by $\partial_t u$ as well as $\boldsymbol{v}$ by $\mathbf{S}\boldsymbol{u}'$, the second equation of the system reads

$$\begin{aligned} &-\partial_t^2 u + 2\sum_{i=1}^n g_i \partial_{x_i}\partial_t u + a\partial_t u + \sum_{i=1}^n b_i \partial_{x_i} u + cu \\ &+ \sum_{i,j,k=1}^n \big(S_{ij}\partial_{x_i}(S_{jk}\partial_{x_k}u) + (\partial_{x_i}S_{ij})S_{jk}\partial_{x_k}u - \partial_{x_i}(S_{ij}S_{jk})\partial_{x_k}u\big) = f, \end{aligned}$$

which is the same as

$$-\partial_t^2 u + 2\sum_{i=1}^n g_i \partial_{x_i}\partial_t u + \sum_{i,j,k=1}^n S_{ik}S_{kj}\partial_{x_i}\partial_{x_j}u + a\partial_t u + \sum_{i=1}^n b_i\partial_{x_i}u + cu = f.$$

Moreover, the condition $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ is a direct consequence of the initial condition for the system. □

Equivalence. The content of Lemmas 3.2 and 3.3 can be summarized as follows. The problem of finding a solution to the Cauchy problem for the wave equation (1) is equivalent to the problem of finding a solution to the corresponding Cauchy problem for the first-order system (2). Uniqueness of solutions is preserved during the rewriting process as well, more precisely we have the following statement.

Theorem 3.4. *Given a wave equation* (1) *and the corresponding first-order system* (2)*. Let* $u_0, u_1 \in \mathcal{G}(\mathbb{R}^n)$*. Then for functions* $u \in \mathcal{G}(\mathbb{R}^{n+1})$ *and* $\boldsymbol{w} \in \mathcal{G}(\mathbb{R}^{n+1})^{n+2}$ *such that* $\boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')^T$ *the following are equivalent:*

(i) *The function* u *is the unique solution to the wave equation* (1) *with initial condition* $(u, \partial_t u)|_{t=0} = (u_0, u_1)$*.*
(ii) *The function* $\boldsymbol{w}$ *is the unique solution to the first-order system* (2) *with initial condition* $\boldsymbol{w}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^T$*.*

Proof. The translation of solutions between wave equations and first-order systems is an immediate consequence of Lemmas 3.2 and 3.3. To show uniqueness take the following considerations into account:

By Lemma 3.2, two distinct solutions to the initial value problem (1) would give rise to two distinct solutions to (2), since $u \mapsto \boldsymbol{w} = (u, \partial_t u, \mathbf{S}\boldsymbol{u}')^T$ is injective, thus contradicting unique solvability of the initial value problem for (2).

Suppose there were two distinct solutions $\boldsymbol{w}$ and $\widetilde{\boldsymbol{w}}$ to the initial value problem of (2) with $\boldsymbol{w}|_{t=0} = \widetilde{\boldsymbol{w}}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^T$. Then the first component of $\boldsymbol{w}$ and $\widetilde{\boldsymbol{w}}$ would give two distinct solutions to (1). But since the solution of (1) is unique, the first component of $\widetilde{\boldsymbol{w}}$ must be equal to the first component of $\boldsymbol{w}$. From the proof of Lemma 3.3 it is then clear that $\tilde{z} = z$ and $\tilde{\boldsymbol{v}} = \boldsymbol{v}$, hence $\widetilde{\boldsymbol{w}} = \boldsymbol{w}$. □

Theorem 3.4 guarantees that in the context of the differential algebra $\mathcal{G}$ the Cauchy problem for the second-order wave equation (1) is equivalent in fairly general circumstances to that for the corresponding first-order system (2–3) provided only the natural, and merely algebraic, consistency of initial data holds. This is not true in spaces of distributions. When the coefficients are of low regularity, the transformation process may fail at various places, e.g., we might end up with differential equations that do not make sense in any distribution space.

Example 3.5. Consider a wave equation used in linear acoustics: Let $p : \mathbb{R}^2 \to \mathbb{R}$ and let $c, \rho : \mathbb{R} \to [a, b]$ with $a > 0$. The acoustic wave equation reads

$$\partial_t^2 p - c^2 \rho \partial_x \left(\frac{1}{\rho} \partial_x p \right) = 0. \tag{4}$$

We define $\boldsymbol{w}^T = (p, \partial_t p, c\partial_x p)$ and formally obtain the symmetric hyperbolic system

$$-\partial_t w_1 + w_2 = 0, \tag{5a}$$

$$-\partial_t w_2 + c\partial_x w_3 - (c' + c \cdot (\ln \rho)')w_3 = 0, \tag{5b}$$

$$-\partial_t w_3 + c w_2 = 0. \tag{5c}$$

Equation (4) can be regarded as an equality in $L^2(\mathbb{R}^2)$, if we have $p \in H^2(\mathbb{R}^2)$, $\rho \in \mathrm{Lip}(\mathbb{R}^2)$ and $c \in L^\infty(\mathbb{R}^2)$. We then have $\partial_t p \in H^1(\mathbb{R}^2)$ and $c\partial_x p \in L^2(\mathbb{R}^2)$, since $c \in L^\infty(\mathbb{R}^2)$ and $\partial_x p \in H^1(\mathbb{R}^2)$. Thus $\boldsymbol{w} \in H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, but not better in general. Hence equation (5b) will in general not be defined on the level of distributions. For example, if $c(x) = 1 + H(x)$, then $c' w_3$ would be a product of δ with an L^2-function.

4. Existence and uniqueness for the Cauchy problems

Rewriting the wave equation (1) as the first-order system (2) via Theorem 3.4 allows to apply existence and uniqueness theorems for the latter to prove existence and uniqueness of a solution to the initial value problem for the wave equation. More precisely, let a vector $\boldsymbol{w}$ of generalized functions with representative $(\boldsymbol{w}_\varepsilon)_\varepsilon = ((u_\varepsilon, z_\varepsilon, \boldsymbol{v}_\varepsilon)^T)_\varepsilon$ be given, that is the unique solution of the Cauchy problem for (2) with initial data $\boldsymbol{w}|_{t=0} = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^T$, then Theorem 3.4 implies that

$(u_\varepsilon)_\varepsilon$ will be the unique generalized solution to the Cauchy problem for (1) with initial data $(u, \partial_t u)|_{t=0} = (u_0, u_1)$. In the following theorem we will give conditions on the coefficients of a wave equation (1) that guarantee the existence of a unique generalized solution to the corresponding first-order problem and, hence, existence and uniqueness of a generalized solution to the wave equation (1).

To this end, we are going to invoke the existence theory for symmetric hyperbolic systems developed in [21, 22, 4, 16, 11] and, in particular, the existence results of [14], which we will now briefly summarize. We start by recalling the essential asymptotic conditions. Let $U_T := (0,T) \times \mathbb{R}^n$. For a function g on U_T, we introduce its mixed L^1-L^∞-norm by $\|g\|_{L^{1,\infty}(U_T)} := \int_0^T \|g(s,\cdot)\|_{L^\infty(\mathbb{R}^n)}\,\mathrm{ds}$. A generalized function $f \in \mathcal{G}(U)$ is said to be of *local L^∞-log-type* ([21, Definition 1.1]) if it admits a representative $(f_\varepsilon)_\varepsilon$ such that for all K compact in U, we have $\|f_\varepsilon\|_{L^\infty(K)} = O(\log \frac{1}{\varepsilon})$ as $\varepsilon \to 0$; $f \in \mathcal{G}(U)$ is said to be of *L^∞-log-type* ([9, Definition 1.5.1]), if it admits a representative $(f_\varepsilon)_\varepsilon$ such that $\|f_\varepsilon\|_{L^\infty(U)} = O(\log \frac{1}{\varepsilon})$ as $\varepsilon \to 0$; $f \in \mathcal{G}_{L^\infty}(U_T)$ is said to be of *$L^{1,\infty}$-log-type* (cf. [4, Definition 2.1]) if it admits a representative $(f_\varepsilon)_\varepsilon$ such that $\|f_\varepsilon\|_{L^{1,\infty}(U_T)} = O(\log \frac{1}{\varepsilon})$ as $\varepsilon \to 0$.

Solution candidates to the Cauchy problem for symmetric hyperbolic systems with Colombeau generalized coefficients (2) are obtained as a net of solutions to the family of classical equations $-\partial_t \boldsymbol{w}_\varepsilon + \sum_{i=1}^n \mathbf{A}_{i,\varepsilon} \partial_{x_i} \boldsymbol{w}_\varepsilon + \mathbf{B}_{\boldsymbol{\varepsilon}} \boldsymbol{w}_\varepsilon = \boldsymbol{F}_\varepsilon$. By imposing additional asymptotic growth conditions in ε on the coefficient matrices, a Gronwall-type argument can be used to prove the moderateness of the family of smooth solutions, hence existence of generalized solutions. Uniqueness of generalized solutions amounts to stability of the family of smooth solutions under negligible perturbations of the data. For convenience of the reader, we combine results from [14], adjusted to the situation at hand, in the following theorem (cf. [14, Theorems 3.1, 3.2, and 3.4]).

Theorem 4.1. *Let $\mathbf{A}_i$, $\mathbf{B} \in M_{n+2}(\mathcal{G}_{L^\infty}(U_T))$, where $\mathbf{A}_i$ is symmetric. Then we have the following three results.*

A) *The Cauchy problem for the system (2) with initial data $\boldsymbol{w}_0 \in (\mathcal{G}(\mathbb{R}^n))^{n+2}$ and right-hand side $\boldsymbol{F} \in (\mathcal{G}(\overline{U}_T))^{n+2}$ has a unique solution $\boldsymbol{w} \in (\mathcal{G}(U_T))^{n+2}$ if*
   (i) *the spatial derivatives $\mathbf{A}_i'$ as well as $\frac{1}{2}(\mathbf{B} + \mathbf{B}^T)$, the symmetric part of the matrix $\mathbf{B}$, are of local L^∞-log-type,*
   (ii) *there exists some constant $R_{\mathbf{A}} > 0$ such that $\sup_{(t,x)} |\mathbf{A}_{i,\varepsilon}(t,x)| = O(1)$ on $(0,T) \times \{x \in \mathbb{R}^n : |x| > R_{\mathbf{A}}\}$ as $\varepsilon \to 0$.*

B) *The Cauchy problem for the system (2) with initial data $\boldsymbol{w}_0 \in (\mathcal{G}_{L^2}(\mathbb{R}^n))^{n+2}$ and right-hand side $\boldsymbol{F} \in (\mathcal{G}_{L^2}(U_T))^{n+2}$ has a unique solution $\boldsymbol{w} \in (\mathcal{G}_{L^2}(U_T))^{n+2}$ if the spatial derivatives $\mathbf{A}_i'$ as well as the symmetric part of the matrix $\mathbf{B}$ are of $L^{1,\infty}$-log-type.*

C) *Let initial data $\boldsymbol{w}_0 \in (\mathcal{G}_{L^\infty}(\mathbb{R}^n))^{n+2}$ and right-hand side $\boldsymbol{F} \in (\mathcal{G}_{L^\infty}(U_T))^{n+2}$ be given. If the spatial derivatives $\mathbf{A}_i'$ as well as the symmetric part of the matrix $\mathbf{B}$ are of L^∞-log-type, then there exists a unique solution $\boldsymbol{w} \in (\mathcal{G}(U_T))^{n+2}$ of (2) such that $\boldsymbol{w}|_{t=0} - \boldsymbol{w}_0 \in (\mathcal{N}(\mathbb{R}^n))^{n+2}$.*

Remark 4.2. When considering case C, the situation occurs that the initial data $\boldsymbol{w}_0$ is an element of the algebra $(\mathcal{G}_{L^\infty}(\mathbb{R}^n))^{n+2}$, whereas the restriction of the solution $\boldsymbol{w}$ to the initial surface, i.e., $\boldsymbol{w}|_{t=0}$ is in $(\mathcal{G}(\mathbb{R}^n))^{n+2}$. This issue can be resolved in the following way: Every representative $(f_\varepsilon)_\varepsilon$ of a generalized function $f \in \mathcal{G}_{L^\infty}(\mathbb{R}^n)$ is also moderate in the sense of $\mathcal{G}(\mathbb{R}^n)$; thus, $\mathcal{G}_{L^\infty}(\mathbb{R}^n)$ can be interpreted as a subset of $\mathcal{G}(\mathbb{R}^n)$ if we allow the difference of two representatives of f to be in the ideal $\mathcal{N}(\mathbb{R}^n)$ instead of $\mathcal{N}_{L^\infty}(\mathbb{R}^n)$. So, we obviously have that $\boldsymbol{w}|_{t=0} - \boldsymbol{w}_0 \in (\mathcal{N}(\mathbb{R}^n))^{n+2}$ but not necessarily in $(\mathcal{N}_{L^\infty}(\mathbb{R}^n))^{n+2}$. In other words, we consider the initial data to be in the algebra $(\mathcal{G}(\mathbb{R}^n))^{n+2}$ but additionally satisfying the moderateness estimates of $(\mathcal{G}_{L^\infty}(\mathbb{R}^n))^{n+2}$.

Finally, we are able to formulate an existence and uniqueness theorem for wave equations based on Theorems 3.4 and 4.1.

Theorem 4.3. *Consider the Cauchy problem*

$$-\partial_t^2 u + 2\sum_{i=1}^n g_i \partial_{x_i}\partial_t u + \sum_{i,j=1}^n R_{ij}\partial_{x_i}\partial_{x_j} u + a\partial_t u + \sum_{i=1}^n b_i \partial_{x_i} u + cu = f \tag{6}$$

and

$$(u, \partial_t u)|_{t=0} = (u_0, u_1)$$

with coefficients R_{ij}, g_i, a, b_i, c *in* $\mathcal{G}_{L^\infty}(U_T)$ *and* $\mathbf{R}$ *positive definite. Let, furthermore,* $\mathbf{S} = \mathbf{R}^{\frac{1}{2}}$, *where we take the square root via diagonalization of* $\mathbf{R}$. *Then we have the following three results.*

A) *The Cauchy problem* (6) *with initial data* $u_0, u_1 \in \mathcal{G}(\mathbb{R}^n)$ *and right-hand side* $f \in \mathcal{G}(U_T)$ *has a unique solution* $u \in \mathcal{G}(U_T)$ *if*
   (i) *the lower-order coefficients* a, c, $\boldsymbol{b}$, *as well as* $\mathbf{S}$, *the derivative* $\mathrm{d}\mathbf{S}$, *the inverse* $\mathbf{S}^{-1}$ *and* $\mathbf{g}'$ *are of local* L^∞*-log-type,*
   (ii) *there exists* $R_{\mathbf{S},\boldsymbol{g}} > 0$ *such that* $\sup_{(t,x)} |\boldsymbol{g}_\varepsilon(t,x)| = O(1)$ *and* $\sup_{(t,x)} |\mathbf{S}_\varepsilon(t,x)| = O(1)$ *on* $(0,T) \times \{x \in \mathbb{R}^n | \, |x| > R_{\mathbf{S},\boldsymbol{g}}\}$ *as* $\varepsilon \to 0$.

B) *The Cauchy problem* (6) *with initial data* $u_0, u_1 \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ *and right-hand side* $f \in \mathcal{G}_{L^2}(U_T)$ *has a unique solution* $u \in \mathcal{G}_{L^2}(U_T)$ *if the lower-order coefficients* a, c, $\boldsymbol{b}$, *as well as* $\mathbf{S}$, *the derivative* $\mathrm{d}\mathbf{S}$, *the inverse* $\mathbf{S}^{-1}$ *and* $\mathbf{g}'$ *are of* $L^{1,\infty}$*-log-type.*

C) *Let initial data* $u_0, u_1 \in \mathcal{G}_{L^\infty}(\mathbb{R}^n)$ *and right-hand side* $f \in \mathcal{G}_{L^\infty}(U_T)$ *be given. If the lower-order coefficients* a, c, $\boldsymbol{b}$, *as well as* $\mathbf{S}$, *the derivative* $\mathrm{d}\mathbf{S}$, *the inverse* $\mathbf{S}^{-1}$ *and* $\mathbf{g}'$ *are of* L^∞*-log-type, then there exists a unique solution* $u \in \mathcal{G}(U_T)$ *of the wave equation* (6) *such that* $(u, \partial_t u)|_{t=0} - (u_0, u_1) \in (\mathcal{N}(\mathbb{R}^n))^2$.

Proof. We start with the proof for case A. We rewrite the wave equation into the corresponding symmetric hyperbolic system with $\mathbf{A}_i$, $\mathbf{B}$ and $\boldsymbol{F}$ as in (3). Clearly the coefficients of the hyperbolic system are in $\mathcal{G}_{L^\infty}$ since the coefficients of the wave equation are. From condition (i) and the structure of (3) we obtain that $\mathbf{A}_i'$ and $\frac{1}{2}(\mathbf{B} + \mathbf{B}^T)$ – the symmetric part of $\mathbf{B}$ – are locally of L^∞-log-type. Since by condition (ii) u_0 and u_1 are in $\mathcal{G}(\mathbb{R}^n)$, and $\mathbf{S}$ has entries in $\mathcal{G}_{L^\infty}(U_T)$, the initial data for the system $\boldsymbol{w}_0 = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^T$ is in $(\mathcal{G}(\mathbb{R}^n))^{n+2}$. Furthermore, $f \in \mathcal{G}(U_T)$, thus $F = (0, f, 0) \in (\mathcal{G}(U_T))^{n+2}$. The matrix $\mathbf{S}$ and the vector $\boldsymbol{g}$ satisfy condition (ii). Thus, there exists a constant $R_{\mathbf{S},\boldsymbol{g}} > 0$ such that $\mathbf{A}$, which depends only on $\mathbf{S}$ and $\boldsymbol{g}$, is $O(1)$ on $(0,T) \times \{x \in \mathbb{R}^n | \, |x| > R_{\mathbf{S},\boldsymbol{g}}\}$ Summing up, all conditions of Theorem 4.1, case A are satisfied, and we can apply the theorem to obtain a solution $\boldsymbol{w}$ to the initial value problem of the hyperbolic system. Theorem 3.4 guarantees that the first component u of $\boldsymbol{w}$ is the unique solution to the Cauchy problem for the wave equation.

The proofs for cases B [resp. C] follow the same pattern. Again, we rewrite the wave equation into its corresponding symmetric hyperbolic first-order system and obtain matrices $\mathbf{A}_i$ and $\mathbf{B}$ in $M_{n+2}(\mathcal{G}_{L^\infty}(U_T))$. By condition (i) we have that $\mathbf{A}_i'$ and the symmetric part of $\mathbf{B}$ are $L^{1,\infty}$-log-type [resp. L^∞-log-type]. Since by condition (ii) the initial data u_0 and u_1 are in $\mathcal{G}_{L^2}(\mathbb{R}^n)$ [resp. $\mathcal{G}_{L^\infty}(\mathbb{R}^n)$], and $\mathbf{S}$ has entries in $\mathcal{G}_{L^\infty}(U_T)$, the initial data for the system $\boldsymbol{w}_0 = (u_0, u_1, \mathbf{S}\boldsymbol{u}_0')^T$ is in $(\mathcal{G}_{L^2}(\mathbb{R}^n))^{n+2}$ [resp. $(\mathcal{G}_{L^\infty}(\mathbb{R}^n))^{n+2}$]. We also have $f \in \mathcal{G}_{L^2}(U_T)$ [resp. $\mathcal{G}_{L^\infty}(U_T)$], thus $\boldsymbol{F} = (0, f, 0) \in (\mathcal{G}_{L^2}(U_T))^{n+2}$ [resp. $(\mathcal{G}_{L^\infty}(U_T))^{n+2}$]. Altogether we can apply Theorem 4.1, case B [resp. C] and obtain a solution $\boldsymbol{w}$ to the initial value problem for the hyperbolic system. Finally, Theorem 3.4 guarantees its first component u is the unique solution to the Cauchy problem for the wave equation, and we are done. $\square$

The asymptotic estimates on the coefficients required in Theorem 4.3 are less restrictive than those supposed in the (local) existence results for the initial value problem for the wave equation on "weakly singular" Lorentzian manifolds, cf. [8, Theorem 3.1] and [10, Theorem 3.1]. In particular, the conditions of Theorem 4.3 A) are general enough to cover the Laplace–Beltrami operator of metrics in the Geroch–Traschen class, which is was not possible previously. The relevance of this class, introduced in [7], comes from the fact that it is viewed as the "maximal reasonable" class of metrics that allows for the definition of the Riemann curvature tensor as a distribution (see also [18]). We finish this paper by deriving an existence result for the wave operator of such metrics.

A Lorentzian metric $\mathbf{g}_0$ on a smooth manifold M belongs to the Geroch–Traschen class if $\mathbf{g}_0$ and its inverse $\mathbf{g}_0^{-1}$ belong to $H^1_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M)$ and, furthermore, if $|\det \mathbf{g}_0| \geq C > 0$ almost everywhere on compact sets. Since we are only interested in a local existence result we may work in a fixed chart and moreover cut off the metric $\mathbf{g}_0$ outside some ball such that it is constant there. We then regularize $\mathbf{g}_0$ via componentwise convolution with a mollifier to obtain a generalized metric. More precisely, denoting the components of $\mathbf{g}_0$ by $g_{0,ij}$ we will write g^ε_{ij} for their smoothing, i.e., $g^\varepsilon_{ij} = g_{0,ij} * \psi_\varepsilon$, with $(\psi_\varepsilon)_\varepsilon$ being a model delta net (i.e., $\psi_\varepsilon(x) = \varepsilon^{-(n+1)}\rho(x/\varepsilon)$ for some fixed test function ρ with unit integral). For a more sophisticated way of smoothing metrics of the Geroch–Traschen class see [26]. We denote by $\mathbf{g} = [(\mathbf{g}_\varepsilon)_\varepsilon]$ the resulting generalized Lorentzian metric on $\mathbb{R}^{n+1}$. It is then clear that in general $\mathrm{d}\mathbf{g}_\varepsilon \neq O(1)$ (otherwise $\mathbf{g} \in W^{1,\infty}_{\mathrm{loc}}$), hence condition (B) of [8, Theorem 3.1] is violated as well as condition (i) of [10, Theorem 3.1].

However, logarithmically rescaling the mollifier (i.e., setting $\psi_\varepsilon(x) = \gamma_\varepsilon^{n+1}\rho(\gamma_\varepsilon x)$ with $\gamma_\varepsilon = \log\frac{1}{\varepsilon}$) allows to apply Theorem 4.3 A).

Corollary 4.4 (The Wave Equation for Geroch–Traschen Metrics). *Let* $\mathbf{g}$ *be a generalized metric on* $\mathbb{R}^{n+1}$ *obtained as the smoothing of a metric of Geroch–Traschen class as above with a logarithmically rescaled mollifier. Then the Cauchy problem for the wave equation*

$$\Box_{\mathbf{g}} u = 0 \qquad (u, \partial_t u)\,|_{t=0} = (u_0, u_1) \in \mathcal{G}(\mathbb{R}^n)$$

has a unique solution in $u \in \mathcal{G}(U_T)$ *for arbitrary* $T > 0$.

Sketch of proof. We check that the conditions of Theorem 4.3 A) hold. Indeed a, $\boldsymbol{b}$ and c vanish as well as f. Furthermore R_{ij} and g_i are given by the components of $\mathbf{g}$ which belong to $\mathcal{G}_{L^\infty}(U_T)$ due to the cut off applied to the metric $\mathbf{g}_0$. Condition A)(ii) holds true again due to the cut off and local boundedness of $\mathbf{g}_0$. As for condition A)(i), S and its inverse S^{-1} are even locally uniformly bounded. Finally, due to the logarithmic rescaling of the mollifier the derivatives of $\mathbf{g}$ are of local L^∞-log-type and we are done. $\Box$

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Concept of Delta-shock Type Solutions to Systems of Conservation Laws and the Rankine–Hugoniot Conditions

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Abstract. To solve nonlinear systems of conservation laws, we need a proper concept of weak solution. The aim of this paper is to explain how to derive integral identities for defining δ-shock type solutions in the sense of Schwartzian distributions. We consider two types of systems to compare our definitions. System (1.3) is a standard system admitting delta-shocks and our definition is given by the identities (2.9). System (3.1) is non-typical, and in addition to the identities (3.8), we need to use relation (3.7). We restrict ourselves to the consideration of δ-shocks concentrated only on the surface of codimension 1. Our approach can be used to derive integral identities for other type systems.

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1. Generalized solution to a system of conservation laws

1.1. L^∞-generalized solutions

Consider the Cauchy problem:

$$\begin{cases} U_t + \big(F(U)\big)_x = 0, & \text{in} \quad \mathbb{R} \times (0, \infty), \\ \qquad\qquad U = U^0, & \text{in} \quad \mathbb{R} \times \{t = 0\}, \end{cases} \tag{1.1}$$

where $F : \mathbb{R}^m \to \mathbb{R}^m$, $U^0 : \mathbb{R} \to \mathbb{R}^m$ are given vector-functions, $U = U(x,t) = (u_1, \dots, u_m)$, $t \geq 0$. As is well known, even in the case of smooth (and, certainly, in the case of discontinuous) initial data $U^0(x)$, in general, there does not exist any smooth and global in time solution of (1.1). As noted in [4, 11.1.1.], "the great

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difficulty in this subject is discovering a proper notion of weak solution for the initial problem (1.1)". "We must devise some way to interpret a less regular function as somehow "solving" this initial-value problem" [4, 3.4.1.a.]. But it is well known that a partial differential equation may not make sense even if U is differentiable. "However, observe that if we *temporarily* assume U is smooth, we can as follows rewrite, so that the resulting expression does not directly involve the derivatives of U" [4, 3.4.1.a.]. "The idea is to multiply the partial differential equation in (1.1) by a smooth function φ and then to integrate by parts, thereby transferring the derivatives onto φ" [4, 3.4.1.a.;11.1.1.]. Following this suggestion, we shall derive an integral identity which gives the definition of an L^∞-*generalized solution* of the Cauchy problem (1.1): $U \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}^m\big)$ is called a *generalized solution* of the Cauchy problem (1.1) if the integral identity

$$\int_0^\infty \int \Big(U \cdot \widetilde{\varphi}_t + F(U) \cdot \widetilde{\varphi}_x\Big)\,dx\,dt + \int U^0(x) \cdot \widetilde{\varphi}(x,0)\,dx = 0 \qquad (1.2)$$

holds for all compactly supported smooth test vector-functions $\widetilde{\varphi} : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$, where $\cdot$ is the scalar product of vectors, and $\int f(x)\,dx$ denotes the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$. "This identity, which we derived supposing U to be a smooth solution makes sense if U is merely bounded" [4, 11.1.1.].

1.2. Generalized solution with the δ-function singularities

There are "nonclassical" situations where, in contrast to Lax's and Glimm's classical results, the Cauchy problem (1.1) *either does not possess a weak L^∞-solution or possesses it for some particular initial data.* In order to solve the Cauchy problem in these cases, it is necessary to seek solutions in the class of singular functions called *δ-shocks.* Roughly speaking, a *δ-shock* is a solution such that its components *contain Dirac delta functions.* The theory of δ-shocks has been intensively developed in the last twenty years (see [2], [8]–[10], [12] and the references therein). In numerous papers δ-shocks are extensively studied in the *system of zero-pressure gas dynamics*:

$$\rho_t + \nabla \cdot (\rho U) = 0, \qquad (\rho U)_t + \nabla \cdot (\rho U \otimes U) = 0, \qquad (1.3)$$

where $\rho \geq 0$ is density, U is velocity, $\otimes$ is the tensor product of vectors. System (1.3) can be considered to describe the formation of large-scale structures of the universe [11], for modeling the formation and evolution of traffic jams. For modeling dusty gases one can use *zero-pressure gas dynamics with the energy conservation law* [9], [10]. δ-Shocks arise in the model of *non-classical shallow water flows* [3], in the model of *granular gases* [5]. It was observed in [7] that two component system of nonlinear chromatography

$$\Big(u_j + \frac{a_j u_j}{1 - u_1 + u_2}\Big)_t + (u_j)_x = 0, \quad u_j \geq 0, \quad x \geq 0, \quad t \geq 0, \qquad (1.4)$$

admits δ-shock wave type solutions, where a_j is a constant, $j = 1, 2$.

Since a "nonclassical" δ-shock wave type solution *does not satisfy the standard integral identities* of the type (1.2), it is necessary to understand *in which sense* it may satisfy a *nonlinear* system of partial differential equations, i.e., to find an *appropriate definition of this weak solution. Unfortunately, using the above-cited instruction from the Evans' book* [4, 3.4.1.a.], *it is impossible to derive integral identities* to define δ- and δ'-shock wave type solutions. Indeed, as can be seen from (1.1), if by integrating by parts we transfer the derivatives onto a test function φ, under the integral sign there still remain nonlinear terms $F(U)$ *undefined in the distributional sense* since components of U may contain the Dirac measures and their derivatives. To solve these problems, there are several approaches. For example, δ-shock can be defined as a *measure-valued solution*, in the Colombeau sense, or by using a *nonconservative product*.

Our approach here is to derive integral identities for defining δ-shock wave type solution in the sense of Schwartzian distributions. In Section 2, we derive integral identities for the case of multidimensional zero-pressure gas dynamics (1.3); and in Section 3, for a class of systems of conservation laws (3.1). System (3.1) is a substantial generalization of nonlinear chromatography (1.4). The fact that systems of the type (3.1) can admit delta-shocks gives a new perspective in the theory of singular solutions to systems of conservation laws. Using the above definitions one can derive the corresponding Rankine–Hugoniot conditions (see Theorems 2.1, 3.1).

2. Zero-pressure gas dynamics (1.3)

Let $\Gamma = \{(x,t) : S(x,t) = 0\}$ be a hypersurface of codimension 1 in $\{(x,t) : x \in \mathbb{R}^n, t \in [0,\infty)\} \subset \mathbb{R}^{n+1}$, $S \in C^\infty(\mathbb{R}^n \times [0,\infty))$, with $\nabla S(x,t)\big|_{S=0} \neq 0$ for any fixed t. Let $\Gamma_t = \{x \in \mathbb{R}^n : S(x,t) = 0\}$ be a moving surface in $\mathbb{R}^n$. Denote by ν the unit space normal to Γ_t pointing (in the positive direction) from $\Omega_t^- = \{x \in \mathbb{R}^n : S(x,t) < 0\}$ to $\Omega_t^+ = \{x \in \mathbb{R}^n : S(x,t) > 0\}$ such that $\nu_j = \frac{S_{x_j}}{|\nabla S|}$, $j = 1, \dots, n$. The time component of the normal vector $-G = \frac{S_t}{|\nabla S|}$ is the *velocity of the wave front* Γ_t along the space normal ν. Since we consider the evolution of a wave front Γ_t along the normal direction,

$$U_\delta = \nu G = -\frac{S_t \nabla S}{|\nabla S|^2} \tag{2.1}$$

is the δ-shock velocity. According to the above-cited papers, the *δ-shock for system* (1.3) *is a pair of distributions*

$$(U,\rho), \quad \text{where} \quad \rho(x,t) = \widehat{\rho}(x,t) + e(x,t)\delta(\Gamma), \tag{2.2}$$

$U \in L^\infty\big(\mathbb{R}^n \times (0,\infty); \mathbb{R}^n\big)$, $\widehat{\rho} \in L^\infty\big(\mathbb{R}^n \times (0,\infty); \mathbb{R}\big)$, $e \in C(\Gamma)$. The delta function concentrated on the surface Γ is introduced as the functional [6, 5.3.(1),(2)]:

$$\big\langle \delta(S), \varphi(x,t) \big\rangle = \int_{-\infty}^{\infty} \int_{\Gamma_t} \varphi(x,t)\, d\Gamma_t\, dt = \int_\Gamma \frac{\varphi(x,t)\, d\Gamma}{\sqrt{1+G^2}}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}). \tag{2.3}$$

We will consider the *δ-shock type initial data*

$$(U^0(x), \rho^0(x);\, U_\delta^0(x),\, x \in \Gamma_0), \quad \text{where} \quad \rho^0(x) = \widehat{\rho}^0(x) + e^0(x)\delta(\Gamma_0), \tag{2.4}$$

$U^0 \in L^\infty\big(\mathbb{R}^n;\mathbb{R}^n\big)$, $\widehat{\rho}^0 \in L^\infty\big(\mathbb{R}^n;\mathbb{R}\big)$, $e^0 \in C(\Gamma_0)$, $U^0_\delta(x)\big|_{\Gamma_0}$ is the *initial velocity* of the δ-shock wave, $\Gamma_0 = \{x : S^0(x) = 0\}$ is the initial position of the wave front.

For some calculations we need the following formulas. If $f(x,t)$ is a differentiable function, then according to [6, 12.6.(15),(16)], we have

$$\frac{\partial}{\partial t}\big(f\delta(S)\big) = \frac{\delta f}{\delta t}\delta(S) - Gf\delta'(S), \quad \frac{\partial}{\partial x_j}\big(f\delta(S)\big) = \frac{\delta f}{\delta x_j}\delta(S) + \nu_j f\delta'(S), \tag{2.5}$$

$$\frac{\delta f}{\delta t} \stackrel{\text{def}}{=} \frac{\partial f}{\partial t} + G\frac{\partial f}{\partial \nu} = \frac{\partial f}{\partial t} + U_\delta \cdot \nabla f, \quad \frac{\delta f}{\delta x_j} \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_j} - \nu_j \frac{\partial \widetilde{f}}{\partial \nu}, \quad j = 1,\dots,n, \tag{2.6}$$

are the δ-derivatives with respect to the time and space variables [6, 5.2.(15),(16)] and $\frac{\partial f}{\partial \nu} = \nu \cdot \nabla f$ is the normal derivative, $-G = \frac{S_t}{|\nabla S|}$.

Now we *temporarily* assume that system (1.3) admits a δ-shock type solution of the form (2.2), where U and $\widehat{\rho}$ are piecewise smooth. Since $\rho \in \mathcal{D}'(\mathbb{R}^n \times [0,\infty))$, in view of the first equation in (1.3), we conclude that $\rho U \in \mathcal{D}'(\mathbb{R}^n \times [0,\infty))$. Suppose that $\rho U = A(U,\widehat{\rho},e) + B(U,\widehat{\rho},e)\delta(-S(x,t))$, where $A(U,\widehat{\rho},e) = (A_1,\dots,A_n)$ is piecewise smooth and $B(U,\widehat{\rho},e) = (B_1,\dots,B_n)$ is continuous. Here the distribution $B_k\delta(S)$ is called the *simple layer* and acts according to the rule: $\langle B\delta(S),\varphi\rangle = \int_\Gamma \frac{B(x,t)\varphi(x,t)}{\sqrt{1+G^2}}\,d\Gamma$, $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$. Substituting the expression for ρU and $\rho(x,t) = \widehat{\rho}(x,t) + e(x,t)\delta(-S(x,t))$ into the first equation in (1.3) and using (2.5), (2.6), we find that the coefficient of $\delta'(-S)$ is equal to $eG - \sum_{j=1}^n B_j\nu_j = 0$. To satisfy the last relation, we choose $B_j = eG\nu_j$. Thus, in view of (2.1), $B_j = eU_\delta$, and

$$\rho U = A(U,\widehat{\rho},e) + eU_\delta\delta(-S(x,t)) \in \mathcal{D}'(\mathbb{R}^n \times [0,\infty)). \tag{2.7}$$

According to the second equation in (1.3), $\rho U \otimes U \in \mathcal{D}'(\mathbb{R}^n \times [0,\infty))$. Let us suppose that $\rho U \otimes U = C(U,\widehat{\rho},e) + D(U,\widehat{\rho},e)\delta(-S(x,t))$, $C(U,\widehat{\rho},e) = (C_{ij})$ is a piecewise smooth tensor-function and $D(U,\widehat{\rho},e) = (D_{ij})$ is a continuous tensor-function. Next, substituting the last expression for $\rho U\otimes U$ and (2.7) into the second equation in (1.3) and using (2.5), (2.6), we find that the coefficient of $\delta'(-S)$ is equal to $eU_{\delta i}G - \sum_{j=1}^n D_{ij}\nu_j = 0$, $i = 1,2,\dots,n$. To satisfy this relation, we choose $D_{ij} = eU_{\delta i}G\nu_j = eU_{\delta i}U_{\delta j}$. That is, $D = eU_\delta \otimes U_\delta$. Hence,

$$\rho U \otimes U = C(U,\widehat{\rho},e) + eU_\delta \otimes U_\delta\delta(-S(x,t)) \in \mathcal{D}'(\mathbb{R}^n \times [0,\infty)). \tag{2.8}$$

Formulas (2.7), (2.8) imply that $A(U,\widehat{\rho},e) = \widehat{\rho}U$, $C(U,\widehat{\rho},e) = \widehat{\rho}U\otimes U$ if $S(x,t) \neq 0$.

In view of (1.3),

$$\big\langle \rho_t + \nabla\cdot(\rho U),\varphi\big\rangle = 0, \quad \big\langle (\rho U)_t + \nabla\cdot(\rho U\otimes U),\varphi\big\rangle = 0, \quad \forall\,\varphi \in \mathcal{D}(\mathbb{R}^n \times [0,\infty)).$$

Integrating the last relations by parts, using (2.2), (2.7), (2.8), and taking into account that $A(U,\widehat{\rho},e) = \widehat{\rho}U$, $C(U,\widehat{\rho},e) = \widehat{\rho}U\otimes U$, we obtain

$$\begin{aligned}\big\langle \rho,\varphi_t\big\rangle + \sum_{j=1}^n \big\langle \rho U_j,\varphi_{x_j}\big\rangle = {}& \int_0^\infty\!\!\int \widehat{\rho}\Big(\varphi_t + U\cdot\nabla\varphi\Big)\,dx\,dt + \int \widehat{\rho}^0(x)\varphi(x,0)\,dx \\ &+ \big\langle \delta(S), e(\varphi_t + U_\delta\cdot\nabla\varphi)\big\rangle + \big\langle \delta(S), e\varphi\big\rangle\big|_{t=0} = 0,\end{aligned}$$

$$\big\langle \rho U, \varphi_t \big\rangle + \sum_{j=1}^{n} \big\langle \rho U U_j, \varphi_{x_j} \big\rangle = \int_0^\infty \int \widehat{\rho} U \Big(\varphi_t + U \cdot \nabla \varphi \Big)\, dx\, dt$$

$$+ \int U^0(x) \widehat{\rho}^0(x) \varphi(x,0)\, dx + \big\langle \delta(S), e U_\delta(\varphi_t + U_\delta \cdot \nabla\varphi) \big\rangle + \big\langle \delta(S), e U_\delta \varphi \big\rangle \big|_{t=0} = 0,$$

where in view of (2.3), (2.6), we have $\langle \delta(S), e(\varphi_t + U_\delta \cdot \nabla\varphi) \rangle = \int_\Gamma e \frac{\delta\varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}}$, $\langle \delta(S), e\varphi \rangle\big|_{t=0} = \int_{\Gamma_0} e^0(x)\varphi(x,0)\, d\Gamma_0$, $\langle \delta(S), eU_\delta(\varphi_t + U_\delta \cdot \nabla\varphi) \rangle = \int_\Gamma e U_\delta \frac{\delta\varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}}$, $\langle \delta(S), eU_\delta \varphi \rangle\big|_{t=0} = \int_{\Gamma_0} e^0(x) U_\delta^0(x) \varphi(x,0)\, d\Gamma_0$. Thus, *we derive integral identities.*

Generalization of the above identities gives the following definition.

Definition 2.1 ([12, Definition 9.1]). A pair of distributions (U, ρ) and a hypersurface Γ, where $\rho(x,t) = \widehat{\rho}(x,t) + e(x,t)\delta(\Gamma)$ and $U \in L^\infty\big(\mathbb{R}^n \times (0,\infty); \mathbb{R}^n\big)$, $\widehat{\rho}, \in L^\infty\big(\mathbb{R}^n \times (0,\infty); \mathbb{R}\big)$, $e, \in C(\Gamma)$, is called a *δ-shock wave type solution* of the Cauchy problem (1.3), (2.4) if the integral identities

$$\begin{aligned}
&\int_0^\infty \int \widehat{\rho} \Big(\varphi_t + U \cdot \nabla\varphi \Big)\, dx\, dt + \int_\Gamma e \frac{\delta\varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}} \\
&\qquad + \int \widehat{\rho}^0(x) \varphi(x,0)\, dx + \int_{\Gamma_0} e^0(x)\varphi(x,0)\, d\Gamma_0 = 0, \\
&\int_0^\infty \int \widehat{\rho} U \Big(\varphi_t + U \cdot \nabla\varphi \Big)\, dx\, dt + \int_\Gamma e U_\delta \frac{\delta\varphi}{\delta t} \frac{d\Gamma}{\sqrt{1+G^2}} \\
&\qquad + \int U^0(x) \widehat{\rho}^0(x) \varphi(x,0)\, dx + \int_{\Gamma_0} e^0(x) U_\delta^0(x) \varphi(x,0)\, d\Gamma_0 = 0,
\end{aligned} \tag{2.9}$$

hold for all $\varphi \in \mathcal{D}(\mathbb{R}^n \times [0,\infty))$. Here $-G = \frac{S_t}{|\nabla S|}$; $\int f(x)\, dx$ is the improper integral $\int_{\mathbb{R}^n} f(x)\, dx$; U_δ is the δ-shock velocity defined by (2.1); $\frac{\delta\varphi}{\delta t}$ is the δ-derivative (2.6).

Using Definition 2.1, we derive the *Rankine–Hugoniot conditions* for (1.3).

Theorem 2.1 ([12, Theorem 9.1; (9.19)]). *Let us assume that $\Omega \subset \mathbb{R}^n \times (0,\infty)$ is a region cut by a smooth hypersurface $\Gamma = \big\{(x,t) : S(x,t) = 0\big\}$ into left- and right-hand parts $\Omega^\mp = \{(x,t) : \mp S(x,t) > 0\}$. Let (U, ρ, H), Γ be a δ-shock wave type solution of system* (1.3) (*in the sense of Definition* 2.1), *and suppose that U, ρ, H are smooth in $\Omega^\pm$ and have one-sided limits $U^\pm$, $\widehat{\rho}^\pm$, $H^\pm$ on Γ. Then the* Rankine–Hugoniot conditions for the δ-shock

$$\begin{aligned}
\frac{\delta e}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta) &= \big([\rho U] - [\rho] U_\delta\big) \cdot \nu, \\
\frac{\delta (eU_\delta)}{\delta t} + \nabla_{\Gamma_t} \cdot (eU_\delta \otimes U_\delta) &= \big([\rho U \otimes U] - [\rho U] U_\delta\big) \cdot \nu,
\end{aligned} \tag{2.10}$$

hold on the discontinuity hypersurface Γ, where $\big[f(U,\rho)\big] = f(U^-, \rho^-,) - f(U^+, \rho^+)$ is the jump of the function $f(U,\rho)$ across the discontinuity hypersurface Γ, $\nabla_{\Gamma_t} = \big(\frac{\delta}{\delta x_1}, \dots, \frac{\delta}{\delta x_n}\big)$, δ-derivatives $\frac{\delta}{\delta t}$ and $\frac{\delta}{\delta x_j}$ are defined by (2.6).

3. New class of systems of conservation laws admitting δ-shocks

Consider the system of conservation laws

$$(u_j)_t + \big(u_j\, f_j(\mu_1 u_1 + \cdots + \mu_n u_n)\big)_x = 0, \qquad x \in \mathbb{R}, \quad t \ge 0, \tag{3.1}$$

where $f_j(\cdot)$ is a smooth function, μ_j is a constant, $j = 1, 2, \dots, n$. This class includes some *Temple type systems*, the system of *nonlinear chromatography* (see [7]); the system for *isotachophoresis* [1, (1.1.2), (1.1.3)]. A δ-shock type solution for system (3.1) is a vector-distribution $u = (u_1, \dots, u_n)$, where

$$u_j(x,t) = \widehat{u}_j(x,t) + e_j(x,t)\delta(\Gamma), \qquad j = 1, 2, \dots, n, \tag{3.2}$$

where $\widehat{u}_j \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}\big)$, $e_j \in C(\Gamma)$, $\Gamma = \{(x,t) : S(x,t) = 0\}$ is the discontinuity curve in the half-plane $\{(x,t) : x \in \mathbb{R}, t \ge 0\}$, $\delta(\Gamma)$ ($\equiv \delta(S)$) is the delta function (2.3) concentrated on Γ, $-G = \frac{S_t}{|S_x|}$. For system (3.1), we will use the *δ-shock type initial data*

$$u^0 = \big(u_1^0, \dots, u_n^0\big), \tag{3.3}$$

where

$$u_j^0(x) = \widehat{u}_j^0(x) + e_j^0 \delta(\Gamma_0),$$

and $\widehat{u}_j^0 \in L^\infty\big(\mathbb{R}; \mathbb{R}\big)$, e^0 is a constant.

(1) Since u_j is given, by (3.2), in order to define the term $f_j(\mu_1 u_1 + \cdots + \mu_n u_n)$ in the sense of Schwartzian distributions, we need to assume that $\sum_{j=1}^n \mu_j e_j(x,t) = 0$. In view of this condition, $\mu_1 u_1 + \cdots + \mu_n u_n = \mu_1 \widehat{u}_1 + \cdots + \mu_n \widehat{u}_n$.

(2) Let us *temporarily* assume that system (3.1) admits a δ-shock of the form (3.2), where $\widehat{u}_j$ is piecewise smooth. Since $(u_j)_t \in \mathcal{D}'(\mathbb{R} \times [0,\infty))$, in view of (3.1), the flux-functions $u_j\, f_j(u_1 + \cdots + u_n)$ will necessarily be distributions. Suppose that $u_j\, f_j(\mu_1 u_1 + \cdots + \mu_n u_n) = A_j(\widehat{u}) + B_j(\widehat{u}, e)\delta(-S(x,t))$, where $A_j(\widehat{u})$ is piecewise smooth and $B_j(\widehat{u}, e)$ is continuous, $j = 1, \dots, n$. Substituting the last relation and (3.2) into system (3.1) and using formulas (2.5), (2.6), we find that the coefficient of $\delta'(\Gamma)$ is $e_j \frac{S_t}{|S_x|} + B_j \frac{S_x}{|S_x|} = 0$, i.e., $B_j = -\frac{S_t}{S_x} e_j$, $j = 1, \dots, n$. Thus,

$$u_j\, f_j(\mu_1 u_1 + \cdots + \mu_n u_n) = A_j(\widehat{u}) + u_\delta(x,t) e_j \delta(-S(x,t)), \qquad j = 1, \dots, n, \tag{3.4}$$

where

$$u_\delta(x,t)\big|_\Gamma = G\nu = -S_t/S_x\big|_\Gamma \tag{3.5}$$

is the *velocity* of the δ-shock, $\nu = \frac{S_x}{|S_x|}$. It is clear that if $S(x,t) \ne 0$, then

$$u_j\, f_j(\mu_1 u_1 + \cdots + \mu_n u_n) = \widehat{u}_j\, f_j(\mu_1 \widehat{u}_1 + \cdots + \mu_n \widehat{u}_n) = A_j(\widehat{u}).$$

Consider the expression $\big\langle (u_j)_t + \big(u_j\, f_j(\mu_1 u_1 + \dots + \mu_n u_n)\big)_x, \varphi\big\rangle = 0$, $\varphi \in \mathcal{D}(\mathbb{R}\times[0,\infty))$. Integrating it by parts and using (3.2), (3.4), (2.3), (2.6), we obtain

$$\begin{aligned}
&\int_0^\infty \int \big(\widehat{u}_j\varphi_t + A_j(\widehat{u})\varphi_x\big)\,dx\,dt + \big\langle \delta(\Gamma), e_j\,\big(\varphi_t + u_\delta\varphi_x\big)\big\rangle \\
&\qquad + \int \widehat{u}_j^0(x)\varphi(x,0)\,dx + \big\langle \delta(\Gamma_0), e_j^0\varphi(x,0)\big\rangle \qquad (3.6)\\
&= \int_0^\infty \int \widehat{u}_j\Big(\varphi_t + f_j(\mu_1\widehat{u}_1 + \dots + \mu_n\widehat{u}_n)\varphi_x\Big)\,dx\,dt + \int \widehat{u}_j^0(x)\varphi(x,0)\,dx \\
&\qquad + \int_\Gamma e_j(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+G^2}} + e_j^0\varphi(x,0)\big|_{\Gamma_0} = 0, \quad \forall\,\varphi\in\mathcal{D}(\mathbb{R}\times[0,\infty)),
\end{aligned}$$

$j = 1,2,\dots,n$, where $\int_\Gamma \cdot\, dl$ is the line integral over the arc Γ. Under our assumptions relations (3.6) *constitute the integral identities.*

Now we suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a set of curves $\gamma_i = \{(x,t) : S_i(x,t) = 0\}$ of the class C^1, $(S_i)_x \neq 0$, $i \in I$, and I is a finite set. Let I_0 be a subset of I such that the arcs γ_k for $k \in I_0$ start from points of the x-axis and let $\Gamma_0 = \{x_k^0 : k \in I_0\}$ be the set of initial points of the arcs γ_k, $k \in I_0$. We consider the initial data (3.3), where $\sum_{j=1}^n \mu_j e_j^0 = 0$, and $\widehat{u}_j^0 \in L^\infty\big(\mathbb{R};\mathbb{R}\big)$; $e_j^0\delta(\Gamma_0) \overset{\text{def}}{=} \sum_{k\in I_0} e_{j;k}^0\delta(x - x_k^0)$, $e_{j;k}^0$ is a constant; $k \in I_0$; $j = 1,2,\dots,n$. Generalization of (3.6) implies

Definition 3.1 ([13]). A distribution $u(x,t) = (u_1(x,t),\dots,u_n(x,t))$ and a set of curves Γ, where $u_j(x,t) = \widehat{u}_j(x,t) + e_j(x,t)\delta(\Gamma)$, $\widehat{u}_j \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$, $e_j(x,t)\delta(\Gamma) \overset{\text{def}}{=} \sum_{i\in I} e_{j;i}(x,t)\delta(\gamma_i)$, $e_{j;i} \in C(\gamma_i)$, $i \in I$, $j = 1,2,\dots,n$, and

$$\sum_{j=1}^n \mu_j e_j(x,t) = 0, \qquad (3.7)$$

is called a *δ-shock wave type solution* of the Cauchy problem (3.1), (3.3) if

$$\begin{aligned}
&\int_0^\infty \int \widehat{u}_j\Big(\varphi_t + f_j(\mu_1\widehat{u}_1 + \dots + \mu_n\widehat{u}_n)\varphi_x\Big)\,dx\,dt + \int \widehat{u}_j^0(x)\varphi(x,0)\,dx \qquad (3.8)\\
&\quad + \sum_{i\in I}\int_{\gamma_i} e_{j;i}(x,t)\frac{\delta\varphi(x,t)}{\delta t}\frac{dl}{\sqrt{1+u_\delta^2}} + \sum_{k\in I_0} e_{j;k}^0\varphi(x_k^0,0) = 0,\ j = 1,2,\dots,n,
\end{aligned}$$

hold for all $\varphi \in \mathcal{D}(\mathbb{R}\times[0,\infty))$.

If $\Gamma = \{\gamma_i : i \in I\}$, where $\gamma_i = \{(x,t) : x = \phi_i(t)\}$, $\phi_i(t) \in C^1$, $i \in I$, and $\dot{(\cdot)} = \frac{d}{dt}(\cdot)$, then $\frac{\delta\varphi}{\delta t}\big|_{\gamma_i} = \varphi_t(\phi_i(t),t) + \dot{\phi}_i(t)\varphi_x(\phi_i(t),t) = \frac{d\varphi(\phi_i(t),t)}{dt}$. The assumption (3.7) implies that for $t = 0$ we have $\sum_{j=1}^n \mu_j e_j^0 = 0$. Since $u_j = \widehat{u}_j + e_j\delta(\Gamma)$, $j = 1,2,\dots,n$, we need the condition (3.7) to define the relation $f_j(\mu_1 u_1 + \dots + \mu_n u_n)$ in the sense of Schwartzian distributions.

Using Definition 3.1, similarly to [12, Theorem 2.1.], we derive the *Rankine–Hugoniot conditions* for (3.1).

Theorem 3.1 ([13]). *Let us assume that* $\Omega \subset \mathbb{R}_+ \times (0,\infty)$ *is a region cut by a smooth curve* $\Gamma = \{(x,t) : S(x,t) = 0\}$ *into the left- and right-hand parts* $\Omega_\mp$. *Let* $u = (u_1,\dots,u_n)$ *and let* Γ *be a* δ*-shock wave type solution of system* (3.1) *such that* $u_j(x,t) = \widehat{u}_j(x,t) + e_j(x,t)\delta(\Gamma)$ *are smooth in* $\Omega_\pm$ *and have one-sided limits* $\widehat{u}_{j\pm}$ *on* Γ, $j = 1,\dots,n$. *Then the* Rankine–Hugoniot conditions for the δ-shock

$$\begin{aligned} \frac{\delta e_j(x,t)}{\delta t} &= \Big([u_j\, f_j(\mu_1 u_1 + \cdots + \mu_n u_n)]_\Gamma - \big[u_j\big]_\Gamma u_\delta\Big)\frac{S_x}{|S_x|}\Big|_\Gamma, \\ u_\delta(x,t) &= \frac{\sum_{j=1}^n \mu_j [u_j\, f_j(\mu_1 u_1 + \cdots + \mu_n u_n)]}{\sum_{j=1}^n \mu_j [u_j]}\Big|_\Gamma, \quad j = 1,\dots,n, \end{aligned} \tag{3.9}$$

hold along Γ, *where* $u_\delta(x,t)$ *is the velocity* (3.5) *of the* δ*-shock. If* $\Gamma = \{(x,t) : x = \phi(t)\}$, $\phi(t) \in C^1(0,+\infty)$, *relations* (3.9) *take the form*

$$\begin{aligned} \dot{e}_j(t) &= \big([u_j\, f_j(\mu_1 u_1 + \cdots + \mu_n u_n)] - [u_j]\dot{\phi}(t)\big)\big|_{x=\phi(t)}, \\ \dot{\phi}(t) &= \frac{\sum_{j=1}^n \mu_j [u_j\, f_j(\mu_1 u_1 + \cdots + \mu_n u_n)]}{\sum_{j=1}^n \mu_j [u_j]}\Big|_{x=\phi(t)}, \quad j = 1,\dots,n. \end{aligned} \tag{3.10}$$

It is easy to see that (3.10) implies (3.7).

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Classes of Generalized Functions with Finite Type Regularities

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Abstract. We introduce and analyze spaces and algebras of generalized functions which correspond to Hölder, Zygmund, and Sobolev spaces of functions. The main scope of the paper is the characterization of the regularity of distributions that are embedded into the corresponding space or algebra of generalized functions with finite type regularities.

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1. Introduction

In this paper we develop regularity theory in generalized function algebras parallel to the corresponding theory within distribution spaces. We consider subspaces or subalgebras in algebras of generalized functions which correspond to the classical Sobolev spaces $W^{k,p}$, Zygmund spaces C^s_*, and Hölder spaces $\mathcal{H}^{k,\tau}$. We refer to [2, 6, 14] for the theory of generalized function algebras and their use in the study of various classes of equations.

It is known that the elements of algebras of generalized functions are represented by nets $(f_\varepsilon)_\varepsilon$ of smooth functions, with appropriate growth as $\varepsilon \to 0$, that the spaces of Schwartz's distributions are embedded into the corresponding algebras, and that the algebra of regular generalized functions corresponding to the space of smooth functions is $\mathcal{G}^\infty$ (cf. [14, 22]). Intuitively, these algebras are obtained through regularization of distributions (convolving them with delta nets) and factorization of an appropriate algebra of moderate nets of smooth functions

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with respect to an ideal of negligible nets, as Colombeau did [2] with his algebra $\mathcal{G}(\mathbb{R}^d)$ (in this way the name Colombeau algebras has appeared). By construction distributions are included in the corresponding Colombeau algebras and their natural linear operations are preserved.

The main goal of this paper is to find out natural conditions with respect to the growth order in ε which characterize generalized function spaces and algebras with finite type regularities. Actually, our main task is to seek optimal definitions for such generalized function spaces, since we would like to have backward information on the regularity properties of Schwartz distributions that are embedded into the corresponding space of generalized functions. Sobolev and Zygmund type spaces are very suitable for this purpose. Particularly, the Zygmund type spaces are useful in this respect, since we can almost literary transfer classical properties of these spaces into their generalized versions.

One can find many articles in the literature where local and microlocal properties of generalized functions in generalized function algebras have been considered; besides the quoted monographs, we refer to the papers [4]–[7], [9, 10, 15, 18, 22]. The motivation of this article came partly from the papers [9, 7], where Zygmund type algebras of generalized functions were studied and used in the qualitative analysis of certain hyperbolic problems. We shall define new classes of generalized functions that are also intrinsically connected with the classical Zygmund spaces.

Note that in our paper [19] we have studied regularity properties of distributions T in terms of growth properties of regularizing sequences $T * \delta_n$ with respect to the parameter $n \in \mathbb{N}$ and various seminorms. Some ideas from that paper are implicitly employed in Section 4 of the present article, where we reinterpret them in the setting of the Colombeau algebra.

The paper is organized as follows. Preliminaries are given in Section 2. In Section 3 we define our new spaces of generalized functions with finite type regularities, the spaces $\mathcal{G}^{k,-s}$, which correspond to local versions of the Zygmund spaces C^r_*. They are subspaces of the Colombeau algebra $\mathcal{G}(\Omega)$. Then, in Section 4, we investigate the role of these new classes of generalized functions in the regularity analysis of distributions; we characterize the regularity properties of those distributions that, after embedding or association, belong to one of these classes. Our main result is Theorem 1 of Section 4, we show that the intersection of $\mathcal{G}^{k,-s}(\Omega)$ with the embedded image of $\mathcal{D}'(\Omega)$ is precisely $\iota(C^{k-s}_{*,\mathrm{loc}}(\Omega))$. As a consequence, we obtain a quick proof of the important regularity theorem for $\mathcal{G}^\infty$ ([14]). Theorems 2 and 3 deal with the analysis of the regularity of a distribution via strong versions of association. Finally, Section 5 is devoted to the study of some generalized function spaces and algebras that are very helpful in *global* regularity analysis. The global classes that we introduce are capable of recovering the embedded image of the classical Zygmund and Hölder spaces of functions. We shall also compare our global Zygmund type generalized function spaces with the one proposed in [9, 7].

2. Preliminaries and notation

We denote by Ω an open subset of $\mathbb{R}^d$. We consider the families of local Sobolev seminorms $||\rho||_{W^{m,p}(\omega)} = \sup\{||\rho^{(\alpha)}||_{L^p(\omega)};\ |\alpha| \leq m\}$, where $m \in \mathbb{N}_0$, $p \in [1,\infty]$, and ω runs over all open subsets of Ω with compact closure ($\omega \subset\subset \Omega$). The local Sobolev space is then denoted as $W^{m,p}_{\text{loc}}(\Omega)$. In case ω is replaced by Ω, we obtain the family of norms $||\cdot||_{W^{m,p}(\Omega)}, m \in \mathbb{N}_0$.

Let $\mathcal{E}(\Omega)$ be the space of smooth functions in Ω. The spaces of moderate nets and negligible nets $\mathcal{E}_{L^p_{\text{loc}},M}(\Omega)$ and $\mathcal{N}_{L^p_{\text{loc}}}(\Omega)$ consist, resp., of nets $(f_\varepsilon)_{\varepsilon\in(0,1)} = (f_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)^{(0,1)}$ with the properties

$$\begin{aligned} &(\forall m \in \mathbb{N}_0)(\forall \omega \subset\subset \Omega)(\exists a \in \mathbb{R})(||f_\varepsilon||_{W^{m,p}(\omega)} = O(\varepsilon^a)) \\ \text{and}\ \ &(\forall m \in \mathbb{N}_0)(\forall \omega \subset\subset \Omega)(\forall b \in \mathbb{R})(||f_\varepsilon||_{W^{m,p}(\omega)} = O(\varepsilon^b)) \end{aligned} \tag{2.1}$$

(big O and small o are the Landau symbols). Note that, $p \in [1,\infty]$,

$$\mathcal{E}_M(\Omega) := \mathcal{E}_{L^\infty_{\text{loc}},M}(\Omega) = \mathcal{E}_{L^p_{\text{loc}},M}(\Omega),\ \ \mathcal{N}(\Omega) = \mathcal{N}_{L^\infty_{\text{loc}}}(\Omega) = \mathcal{N}_{L^p_{\text{loc}}}(\Omega).$$

We obtain the Colombeau algebra of generalized functions as a quotient:

$$\mathcal{G}(\Omega) = \mathcal{G}_{L^p_{\text{loc}}}(\Omega) = \mathcal{E}_{L^p_{\text{loc}},M}(\Omega)/\mathcal{N}_{L^p_{\text{loc}}}(\Omega),\ p \in [1,\infty].$$

The embedding of the Schwartz distribution space $\mathcal{E}'(\Omega)$ into $\mathcal{G}(\Omega)$ is realized through the sheaf homomorphism $\mathcal{E}'(\Omega) \ni T \mapsto \iota(T) = [(T * \phi_\varepsilon|\Omega)_\varepsilon] \in \mathcal{G}(\Omega)$, where the fixed net of mollifiers $(\phi_\varepsilon)_\varepsilon$ is defined by $\phi_\varepsilon = \varepsilon^{-d}\phi(\cdot/\varepsilon),\ \varepsilon < 1$, and $\phi \in \mathcal{S}(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} \phi(t)dt = 1,\ \int_{\mathbb{R}^d} t^\alpha \phi(t)dt = 0,\ |\alpha| > 0.$$

This sheaf homomorphism [6], extended over $\mathcal{D}'$, gives the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$. We also use the notation ι for the mapping from $\mathcal{E}'(\Omega)$ into $\mathcal{E}_M(\Omega)$, $\iota(T) = (T * \phi_\varepsilon|\Omega)_\varepsilon$. Throughout this article, ϕ will *always* be fixed and satisfy the above condition over its moments.

The generalized algebra of "regular generalized functions" $\mathcal{G}^\infty(\Omega)$ is defined in [14] as the quotient of the algebras $\mathcal{E}^\infty_M(\Omega)$ and $\mathcal{N}(\Omega)$, where $\mathcal{E}^\infty_M(\Omega)$ consists of those nets $(f_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)^{(0,1)}$ with the property

$$(\forall \omega \subset\subset \Omega)(\exists a \in \mathbb{R})(\forall \alpha \in \mathbb{N})(\sup_{|\alpha|\leq m} ||f^{(\alpha)}_\varepsilon(x)||_{L^\infty(\omega)} = O(\varepsilon^a)). \tag{2.2}$$

Observe that $\mathcal{G}^\infty$ is a subsheaf of $\mathcal{G}$; it has a similar role as C^∞ in $\mathcal{D}'$.

2.1. Hölder–Zygmund spaces

We will employ the Hölder–Zygmund spaces [8, 12, 24]. We now collect some background material about these spaces. We start with Hölder spaces. Let $k \in \mathbb{N}_0$ and $\tau \in (0,1)$, then the global Hölder space $\mathcal{H}^{k,\tau}(\mathbb{R}^d)$ [8, Chap. 8] consists of those C^k functions such that

$$||f||_{\mathcal{H}^{k,\tau}(\mathbb{R}^d)} = ||f||_{W^{k,\infty}(\mathbb{R}^d)} + \sup_{|\alpha|=k, x\neq y, x,y\in\mathbb{R}^d} \frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|}{|x-y|^\tau} < \infty. \tag{2.3}$$

The definition of the local space $\mathcal{H}^{k,\tau}_{\text{loc}}(\Omega)$ is clear.

There are several ways to introduce the global Zygmund space $C^r_*(\mathbb{R}^d)$ [8, 11, 24]. When $r = k + \tau, k \in \mathbb{N}_0, \tau \in (0,1)$, we have the equality $C^r_*(\mathbb{R}^d) = \mathcal{H}^{k,\tau}(\mathbb{R}^d)$, but the Zygmund spaces are actually defined for all $r \in \mathbb{R}$. They are usually introduced via either a dyadic Littlewood–Paley resolution [24] or a continuous Littlewood–Paley decomposition of the unity [8]. We follow the slightly more flexible approach from [16] via generalized (continuous) Littlewood–Paley pairs (a dyadic version can be found in [24, p. 7, Thm. 1.7]). Let $r \in \mathbb{R}$. We say that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ form a *generalized Littlewood–Paley* pair (of order r) if they satisfy the following compatibility conditions:

$$(\exists \sigma > 0, \eta \in (0,1))(|\hat{\varphi}(\xi)| > 0 \text{ for } |\xi| \leq \sigma \quad \text{and } |\hat{\psi}(\xi)| > 0 \text{ for } \eta\sigma \leq |\xi| \leq \sigma) \tag{2.4}$$

and

$$\int_{\mathbb{R}^d} t^\alpha \psi(t)dt = 0 \text{ for } |\alpha| \leq [r]. \tag{2.5}$$

When $r < 0$, the vanishing requirement over the moments is dropped. Then, $C^r_*(\mathbb{R}^d)$ is the space of all distributions $T \in \mathcal{S}'(\mathbb{R}^d)$ satisfying:

$$\|T\|_{C^r_*(\mathbb{R}^d)} := ||T * \varphi||_{L^\infty(\mathbb{R}^d)} + \sup_{y<1} y^{-r}||T * \psi_y||_{L^\infty(\mathbb{R}^d)} < \infty. \tag{2.6}$$

The definition and the norm (2.6) are independent of the choice of the pair (φ, ψ) as long as (2.4) and (2.5) hold [16]. When $r = k + \tau, k \in \mathbb{N}_0, \tau \in (0,1)$, the norms (2.3) and (2.6) are equivalent. A distribution $T \in \mathcal{D}'(\Omega)$ is said to belong to $C^r_{*,\mathrm{loc}}(\Omega)$ if for all $\rho \in \mathcal{D}(\Omega)$ we have $\rho T \in C^r_*(\mathbb{R}^d)$.

3. Classes of generalized functions with finite type regularities

In this paper we are interested in nets $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ such that for given $k \in \mathbb{N}$ there exists $s > 0$ such that ($p \in [1, \infty]$)

$$(\forall \omega \subset\subset \Omega)(||f_\varepsilon||_{W^{k,p}(\omega)} = O(\varepsilon^{-s}),\ \varepsilon \to 0). \tag{3.1}$$

Observe that (3.1) is closely related to (2.2). When $p = \infty$, such nets will be the representatives of, roughly speaking, $C^{k-s}_{*,\mathrm{loc}}$-generalized functions.

Definition 1. Let $s \in \mathbb{R}$, $k \in \mathbb{N}_0$, and $p \in [1, \infty]$.

(i) A net $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ is said to belong to $\mathcal{E}^{k,-s}_{L^p_{\mathrm{loc}},M}(\Omega)$ if (3.1) holds.

(ii) A generalized function $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ is said to belong to $\mathcal{G}^{k,-s}_{L^p_{\mathrm{loc}}}(\Omega)$ if $(f_\varepsilon)_\varepsilon \in \mathcal{E}^{k,-s}_{L^p_{\mathrm{loc}},M}(\Omega)$.

(iii) We set $\mathcal{G}^{\infty,-s}_{L^p_{\mathrm{loc}}}(\Omega) = \bigcap_{k\in\mathbb{N}} \mathcal{G}^{k,-s}_{L^p_{\mathrm{loc}}}(\Omega)$ and $\mathcal{E}^{\infty,-s}_{L^p_{\mathrm{loc}},M}(\Omega) = \bigcap_{k\in\mathbb{N}} \mathcal{E}^{k,-s}_{L^p_{\mathrm{loc}},M}(\Omega)$.

(iv) When $p = \infty$, we write $\mathcal{G}^{k,-s}(\Omega) = \mathcal{G}^{k,-s}_{L^\infty_{\mathrm{loc}}}(\Omega)$ and $\mathcal{E}^{k,-s}_M(\Omega) = \mathcal{E}^{k,-s}_{L^\infty_{\mathrm{loc}},M}(\Omega)$.

We list some properties of these classes of generalized functions in the next proposition. Their proofs follow immediately from Definition 1.

Proposition 1. *Let* $s \in \mathbb{R}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, *and* $p \in [1, \infty]$.

(i) $\mathcal{G}^{k,-s}_{L^p_{\mathrm{loc}}}(\Omega)$ *are vector spaces.*

(ii) $\mathcal{G}^{k,-s}_{L^p_{\mathrm{loc}}}(\Omega) \subseteq \mathcal{G}^{k_1,-s_1}_{L^p_{\mathrm{loc}}}(\Omega)$ *if* $k \geq k_1$ *and* $s \leq s_1$.

(iii) *Let* $P(D)$ *be a differential operator of order* $m \leq k$ *with constant coefficients. Then* $P(D) : \mathcal{G}^{k,-s}_{L^p_{\mathrm{loc}}}(\Omega) \to \mathcal{G}^{k-m,-s}_{L^p_{\mathrm{loc}}}(\Omega)$.

The intuitive idea behind these notions is to measure the regularity of the net in terms of the two parameters k and s: As the parameters k increases and s decreases, the net becomes more regular. Furthermore, it should be noticed that f belongs to the algebra of smooth generalized functions $\mathcal{G}^{\infty}(\Omega)$ if and only if $(\forall \omega \subset\subset \Omega)(\exists s)(f_{|\omega} \in \mathcal{G}^{\infty,-s}(\omega))$.

4. Characterization of local regularity through association

In this section we characterize local regularity of distributions via either embedding in our classes $\mathcal{G}^{k,-s}_{L^p_{\mathrm{loc}}}(\Omega)$ or association with its elements.

Recall that we say that the net $(f_\varepsilon)_\varepsilon \in \mathcal{E}^{(0,1)}(\Omega)$, or the generalized function $f = [(f_\varepsilon)_\varepsilon]$, is (distributionally) associated to the distribution T if $\lim_{\varepsilon \to 0} f_\varepsilon = T$ in the weak topology of $\mathcal{D}'(\Omega)$, that is,

$$(\forall \rho \in \mathcal{D}(\Omega))(\langle T - f_\varepsilon, \rho \rangle = o(1),\ \varepsilon \to 0). \tag{4.1}$$

We then write $(f_\varepsilon)_\varepsilon \sim T$, or $f \sim T$. In many cases, the rate of approximation in (4.1) may be much better than just $o(1)$; one can often profit from the knowledge of such an additional useful asymptotic information. Let $R : (0, 1] \to \mathbb{R}_+$ be a positive function such that $R(\varepsilon) = o(1)$, $\varepsilon \to 0$. We write $T - f_\varepsilon = O(R(\varepsilon))$ in $\mathcal{D}'(\Omega)$ if

$$(\forall \rho \in \mathcal{D}(\Omega))(\langle T - f_\varepsilon, \rho \rangle = O(R(\varepsilon)),\ \varepsilon < 1).$$

We begin with the following standard proposition. It gives the characterization of the embedding of $W^{k,p}_{\mathrm{loc}}$. The case $p = \infty$ motivates our main results of this section.

Proposition 2. *Let* $k \in \mathbb{N}_0$ *and* $p \in (1, \infty]$.

(a) $\iota(W^{k,p}_{\mathrm{loc}}(\Omega)) = \iota(\mathcal{D}'(\Omega)) \cap \mathcal{G}^{k,0}_{L^p_{\mathrm{loc}}}(\Omega)$.

(b) *More generally, if* $(f_\varepsilon)_\varepsilon \sim T \in \mathcal{D}'(\Omega)$ *and* $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}^{k,0}_{L^p_{\mathrm{loc}}}(\Omega)$, *then* $T \in W^{k,p}_{\mathrm{loc}}(\Omega)$.

Proof. It is enough to show part b). We have that for every $|\alpha| \leq k$, $((f_\varepsilon^{(\alpha)})_{|\omega})_\varepsilon$ is weakly precompact in $L^p(\omega)$ if $p < \infty$, resp. weakly* precompact in $L^\infty(\omega)$. The rest follows from the distributional convergence of $f_\varepsilon^{(\alpha)}$ to $T^{(\alpha)}$. □

We now formulate and prove the main results of this section. We focus on the case $p = \infty$ of the classes of generalized functions defined in Section 3.

4.1. Characterization of $C^r_{*,\mathrm{loc}}$ – the role of $\mathcal{G}^{k,-s}$

The next important theorem provides the precise characterization of those distributions that belong to $\mathcal{G}^{k,-s}(\Omega)$, they turn out to be elements of a Zygmund space. We only consider the case $s > 0$; otherwise, one has $\mathcal{G}^{k,-s}(\Omega) \cap \iota(\mathcal{D}'(\Omega)) = \{0\}$.

Theorem 1. *Let $s > 0$. We have $\mathcal{G}^{k,-s}(\Omega) \cap \iota(\mathcal{D}'(\Omega)) = \iota(C^{k-s}_{*,\mathrm{loc}}(\Omega))$.*

Before giving the proof of Theorem 1, we would like to discuss two corollaries of it. It is worth reformulating Theorem 1 in order to privilege the role of the Zygmund space.

Corollary 1. *Let $r \in \mathbb{R}$. If k is any non-negative integer such that $k > r$, then*

$$\iota(C^r_{*,\mathrm{loc}}(\Omega)) = \mathcal{G}^{k,r-k}(\Omega) \cap \iota(\mathcal{D}'(\Omega)).$$

Corollary 1 can be used to give a striking proof of Oberguggenberger's regularity result [14] for the smooth algebra $\mathcal{G}^\infty(\Omega)$:

Corollary 2. *We have $\iota(\mathcal{D}'(\Omega)) \cap \mathcal{G}^\infty(\Omega) = \iota(C^\infty(\Omega))$.*

Proof. One inclusion is obvious. By localizing, it suffices to show that if $T \in \mathcal{E}'(\Omega)$ and $\iota(T) \in \mathcal{G}^{\infty,-s}(\Omega)$ for some $s \in \mathbb{R}_+ \setminus \mathbb{N}$, then $T \in C^\infty(\Omega)$. Given any $k > 0$, write $r = k - s$. Corollary 1 yields $T \in C^{k-s}_*(\Omega)$. Since this can be done for all k, we conclude that $f \in C^\infty(\Omega)$. □

Proof of Theorem 1. Observe that the statement of Theorem 1 is a local one. Thus, it is enough to show that $\mathcal{G}^{k,-s}(\Omega) \cap \iota(\mathcal{E}'(\Omega)) = \iota(C^{k-s}_*(\mathbb{R}^d) \cap \mathcal{E}'(\Omega))$. So, further on, in this proof we assume $T \in \mathcal{E}'(\Omega)$. Let us show the reverse inclusion. The partial derivatives continuously act on the Zygmund spaces [8] as $\partial^m : C^\beta_*(\mathbb{R}^d) \mapsto C^{\beta-|m|}_*(\mathbb{R}^d)$. Thus, if $T \in C^{k-s}_*(\mathbb{R}^d) \cap \mathcal{E}'(\Omega)$ then $T^{(\alpha)} \in C^{-s}_*(\mathbb{R}^d) \cap \mathcal{E}'(\Omega)$ for all $|\alpha| \le k$. We can then apply [16, Lemm. 5.2] to each $T^{(\alpha)}$ (with $\theta = \check{\phi}$ in [16, Lemm. 5.2, Eq. (5.7)]) and conclude

$$||T^{(\alpha)} * \phi_\varepsilon||_{L^\infty(\mathbb{R}^d)} \le C\varepsilon^{-s} ||T^{(\alpha)}||_{C^{-s}_*(\mathbb{R}^d)}.$$

Thus $((T * \phi_\varepsilon)_{|\Omega})_\varepsilon \in \mathcal{E}^{k,-s}_M(\Omega)$.

Assume now that $((T * \phi_\varepsilon)_{|\Omega})_\varepsilon \in \mathcal{E}^{k,-s}_M(\Omega)$. We first show that actually $(T * \phi_\varepsilon)_\varepsilon \in \mathcal{E}^{k,-s}_M(\mathbb{R}^d)$. Indeed, let $\operatorname{supp} T \subset \omega_1 \subset\subset \omega_2 \subset\subset \Omega$. It suffices to prove that for every multi-index $\alpha \in \mathbb{R}^d$

$$\sup_{x \in \mathbb{R}^d \setminus \omega_2} \left| (T^{(\alpha)} * \phi_\varepsilon)(x) \right| = O(1), \ 0 < \varepsilon \le 1. \tag{4.2}$$

Let A be the distance between $\overline{\omega}_1$ and $\partial\omega_2$. Find r such that

$$(\forall \rho \in \mathcal{E}(\mathbb{R}^d))(|\langle T^{(\alpha)}, \rho\rangle| < C\|\rho\|_{W^{r,\infty}(\omega_1)}).$$

Setting $\rho(u) = \phi_\varepsilon(x-u)$ and using the fact that ϕ is rapidly decreasing, we obtain,

$$\sup_{x \in \mathbb{R}^d \setminus \omega_2} \left| (T^{(\alpha)} * \phi_\varepsilon)(x) \right| < \tilde{C} \sup_{x \in \mathbb{R}^d \setminus \omega_2} \sup_{u \in \omega_1} (\varepsilon + |x-u|)^{-r-d} \le \tilde{C} A^{-r-d},$$

which yields (4.2). Next, set $g_\varepsilon = \varepsilon^s(T * \phi_\varepsilon)$. Then, $(T * \phi_\varepsilon)_\varepsilon \in \mathcal{E}_M^{k,-s}(\mathbb{R}^d)$ precisely tells us that $(g_\varepsilon)_\varepsilon$ is a bounded net in the space $C_b^k(\mathbb{R}^d)$, the Banach space of k-times continuously differentiable functions that are globally bounded together with all their partial derivatives of order $\leq k$. Since the inclusion mapping $C_b^k(\mathbb{R}^d) \mapsto C_*^k(\mathbb{R}^d)$ is obviously continuous, we obtain that $(g_\varepsilon)_\varepsilon$ is a bounded net in the Zygmund space $C_*^k(\mathbb{R}^d)$. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that (ϕ, ψ) forms a generalized Littlewood–Paley pair of order k (cf. (2.4) and (2.5)). Then $T * \phi \in L^\infty(\mathbb{R}^d)$ and

$$\sup_{y\in(0,1]} y^{-k}||g_\varepsilon * \psi_y||_{L^\infty(\mathbb{R}^d)} = \sup_{y\in(0,1]} \varepsilon^s y^{-k}||T * \phi_\varepsilon * \psi_y||_{L^\infty(\mathbb{R}^d)} = O(1), \quad \varepsilon \in (0,1].$$

Setting $\varepsilon = y$ and $\psi_1 = \phi * \psi$ in the previous estimate and noticing that (ϕ, ψ_1) is again a Littlewood–Paley pair, we obtain

$$\sup_{y\in(0,1]} y^{s-k}||T * (\psi_1)_y||_{L^\infty(\mathbb{R}^d)} = O(1),$$

which in turn implies that $T \in C_*^{k-s}(\mathbb{R}^d)$. □

4.2. Regularity via association

We now move to regularity analysis through association. Theorem 1 can be also used to recover the following general form of Corollary 2, originally obtained in [17].

Theorem 2. *Let $T \in \mathcal{D}'(\Omega)$ and let $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ be associated to it. Assume that $f \in \mathcal{G}^\infty(\Omega)$. If $(f_\varepsilon)_\varepsilon$ approximates T with convergence rate:*

$$(\exists b > 0)(T - f_\varepsilon = O(\varepsilon^b) \text{ in } \mathcal{D}'(\Omega)). \tag{4.3}$$

Then $T \in C^\infty(\Omega)$.

Proof. Since the hypotheses and the conclusion of Theorem 2 are local statements, we may assume that $T \in \mathcal{E}'(\Omega)$ and there exists an open subset $\omega \subset\subset \Omega$ such that

$$\operatorname{supp} T, \operatorname{supp} f_\varepsilon \subset \omega, \ \varepsilon \in (0,1). \tag{4.4}$$

We will show that $T \in \mathcal{D}(\Omega)$. Our assumption now becomes $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M^{\infty,-s}(\Omega)$ for some $s > 0$. The support condition (4.4), the rate of convergence (4.3), and the equivalence between weak and strong boundedness on $\mathcal{E}'(\Omega)$ (Banach–Steinhaus theorem) yield

$$(\exists r \in \mathbb{N})(\exists C > 0)(\forall \rho \in \mathcal{E}(\Omega))(\forall t \in (0,1))(|\langle T - f_t, \rho\rangle| \leq Ct^b\|\rho\|_{W^{r,\infty}(\Omega)}). \tag{4.5}$$

Let β be an arbitrary positive number. Then, by $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M^{\infty,-s}(\Omega)$ and (4.5), given any $k \in \mathbb{N}$, we can find positive constants C_1 and C_2 (depending only on k, ϕ) such that

$$||T * \phi_\varepsilon||_{W^{k,\infty}(\omega)} \leq C_1 t^{-s} + C_2 t^b \varepsilon^{-d-r-k}, \quad \varepsilon, t \in (0,1).$$

Find $\eta > 0$ such that $\eta s/b < 1/2$. Setting $t = \varepsilon^{k\eta/b}$, we obtain

$$||T * \phi_\varepsilon||_{W^{k,\infty}(\omega)} \leq C_1 \varepsilon^{-k/2} + C_2 \varepsilon^{\eta k-d-r-k}, \quad \varepsilon \in (0,1).$$

We can now choose k such that $\beta < \min\{k/2, \eta k - d - r\}$; the conclusion from the previous estimate is that $((T * \phi_\varepsilon)_{|\omega})_\varepsilon \in \mathcal{E}_M^{k,\beta-k}(\omega)$, and hence, by Corollary 1, $T \in C^{\beta}_{*,\mathrm{loc}}(\omega)$. Since β was arbitrary, it follows that $T \in C^\infty(\mathbb{R}^d)$. □

We now discuss other sufficient criteria for regularity. The ensuing result is directly motivated by Proposition 2. We relax the growth constrains in it, and, by requesting an appropriate rate of convergence, we obtain two sufficient conditions for regularity of distributions.

Theorem 3. *Let $T \in \mathcal{D}'(\Omega)$ and let $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ be a generalized function associated to it. Furthermore, let $k \in \mathbb{N}$. Assume that either of following pair of conditions hold:*

(i) *$f \in \mathcal{G}^{k,-a}(\Omega)$, $\forall a > 0$, namely,*

$$(\forall a > 0)(\forall \omega \subset\subset \Omega)(\forall \alpha \in \mathbb{N}^d, |\alpha| \le k)(\sup_{x\in\omega} |f_\varepsilon^{(\alpha)}(x)| = O(\varepsilon^{-a})), \tag{4.6}$$

and the convergence rate of $(f_\varepsilon)_\varepsilon$ to T is as in (4.3).

(ii) *$f \in \mathcal{G}^{k,-s}(\Omega)$ for some $s > 0$, and there is a rapidly decreasing function $R : (0,1] \to \mathbb{R}_+$, i.e., $(\forall a > 0)(\lim_{\varepsilon\to 0} \varepsilon^{-a} R(\varepsilon) = 0)$, such that*

$$T - f_\varepsilon = O(R(\varepsilon)) \quad in\ \mathcal{D}'(\Omega). \tag{4.7}$$

Then, $T \in C^{k-\eta}_{*,\,\mathrm{loc}}(\Omega)$ for every $\eta > 0$.

Proof. By localization, it suffices again to assume that $T \in \mathcal{E}'(\Omega)$ and there exists an open subset $\omega \subset\subset \Omega$ such that (4.4) holds. The proof is analogous to that of Theorem 2.

(i) In view of the Banach–Steinhaus theorem, the conditions (4.3) and (4.4) imply (4.5). Thus, with $C_2 = C||\phi||_{W^{r,\infty}(\mathbb{R}^d)}$,

$$\begin{aligned} ||T * \phi_\varepsilon||_{W^{k,\infty}(\omega)} &\le C_2 t^b \varepsilon^{-d-r-k} + \|f_t * \phi_\varepsilon\|_{W^{k,\infty}(\omega)}, \\ &\le C_2 t^b \varepsilon^{-d-r-k} + \|\phi\|_{L^1(\mathbb{R}^d)} \|f_t\|_{W^{k,\infty}(\omega)}, \qquad t, \varepsilon \in (0,1). \end{aligned}$$

By (4.6), given any $a > 0$, there exists $M = M_a > 0$ such that

$$||T * \phi_\varepsilon||_{W^{k,\infty}(\omega)} \le C_2 t^b \varepsilon^{-d-r-k} + M t^{-a}, \quad t, \varepsilon \in (0,1).$$

By taking $t = \varepsilon^{(k+r+d)/b}$, it follows that

$$||T * \phi_\varepsilon||_{W^{k,\infty}(\omega)} \le C_2 + M \varepsilon^{-a(k+r+d)/b}, \quad \varepsilon \in (0,1).$$

If we take sufficiently small a, we conclude that $(T * \phi_\varepsilon)_\varepsilon \in \mathcal{E}_M^{k,-\eta}(\omega)$ for all $\eta > 0$, and the assertion follows from Theorem 1.

(ii) The relation (4.7), the fact that R is rapidly decreasing, and the Banach–Steinhaus theorem imply

$$(\exists r \in \mathbb{N})(\forall a > 0)(\forall \rho \in \mathcal{E}^r(\Omega))(|\langle T - f_\varepsilon, \rho\rangle| = O(\varepsilon^a)).$$

As in part (i), we have

$$||T * \phi_\varepsilon||_{W^{k,\infty}(\omega)} \le C t^a \varepsilon^{-d-r-k} + \|\phi\|_{L^1(\mathbb{R}^d)} \|f_t\|_{W^{k,\infty}(\omega)}, \quad t, \varepsilon \in (0,1).$$

for some constant $C = C_a$. Since $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M^{k,-s}$, there is another constant $C = C_{s,a} > 0$ such that

$$||T * \phi_\varepsilon||_{W^{k,\infty}(\omega)} \leq Ct^a \varepsilon^{-d-r-k} + Ct^{-s}, \quad t, \varepsilon \in (0,1).$$

Setting $t = \varepsilon^{(k+r+d)/a}$, we have

$$||T * \phi_\varepsilon||_{W^{k,\infty}(\omega)} \leq C + C\varepsilon^{-s(k+r+d)/a}, \quad \varepsilon \in (0,1).$$

Thus, taking large enough $a > 0$, one establishes $\iota(T) \in \mathcal{G}^{k,-\eta}(\omega)$ for all $\eta > 0$. The conclusion $T \in C_*^{k-\eta}(\mathbb{R}^d)$ follows once again from Theorem 1. □

The hypotheses (4.3) and (4.7) are essential parts of (i) and (ii) in Theorem 3; we illustrate that fact in the next two examples.

Example 1. Consider the generalized function $f = [(|\log\varepsilon|^{-d}\phi(\cdot\,|\log\varepsilon|))_\varepsilon]$. Clearly, $f \in \mathcal{G}^{\infty,-s}(\mathbb{R}^d)$, $\forall s > 0$. Moreover, $f \sim \delta$, the Dirac delta distribution. The conclusion of Theorem 3 fails in this example because the rate of convergence is too slow.

Example 2. Let $T \in \mathcal{E}'(\Omega)$ and $k > 2s > 0$. Suppose that $T \in C_*^{k-2s}(\mathbb{R}^d)$ but $T \notin C_*^{k-s}(\mathbb{R}^d)$. By Theorem 1, $\iota(T) \in \mathcal{G}^{k,-2s}(\mathbb{R}^d)$. However, the conclusion of Theorem 3 fails for T because the approximation rate is actually much slower than (4.7).

For distributions $T \in \mathcal{E}'(\Omega)$, part (i) of Theorem 3 is applicable to the regularization net $f_\varepsilon = (T * \phi_\varepsilon)_{|\Omega}$; however, for this particular case Theorem 1 provides the same conclusion.

5. Global Zygmund-type spaces and algebras

Let $r \in \mathbb{R}$, Hörmann ([9]) defined the Zygmund-type space of generalized functions $\tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$ via representatives $(u_\varepsilon)_\varepsilon$ satisfying, for each $\alpha \in \mathbb{N}_0^d$,

$$\|u_\varepsilon^{(\alpha)}\|_{L^\infty(\mathbb{R}^d)} = \begin{cases} O(1), & 0 \leq |\alpha| < r, \\ O(\log(1/\varepsilon)), & |\alpha| = r \in \mathbb{N}_0 \\ O(\varepsilon^{r-|\alpha|}), & |\alpha| > r. \end{cases} \quad \text{as } \varepsilon \to 0, \tag{5.1}$$

We shall propose in this section several other Zygmund-type classes of generalized functions. Since we are interested in global properties, it appears that the most natural framework to define them is the algebra $\mathcal{G}_{L^\infty}(\mathbb{R}^d)$, defined below in Subsection 5.1, and not the usual Colombeau algebra $\mathcal{G}(\mathbb{R}^d)$. Otherwise, the definitions would depend on representatives, and more seriously, some global properties that are intrinsically encoded in such spaces would be totally lost. Therefore, we have decided to study first $\mathcal{G}_{L^\infty}(\mathbb{R}^d)$. Subsection 5.2 is devoted to Zygmund-type classes of generalized functions and global regularity results. In Subsection 5.3, we introduce Hölder-type classes of generalized functions.

5.1. The algebra $\mathcal{G}_{L^\infty}(\mathbb{R}^d)$

The globally L^∞-based algebra of generalized functions is defined as follows [14, 13]. First consider the algebra

$$\mathcal{E}_{L^\infty,M}(\mathbb{R}^d) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^d);\ (\forall\alpha \in \mathbb{N}_0)(\exists a \in \mathbb{R})(||u_\varepsilon^{(\alpha)}||_{L^\infty(\mathbb{R}^d)} = O(\varepsilon^a))\Big\}$$

and the ideal

$$\mathcal{N}_{L^\infty}(\mathbb{R}^d) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^d);\ (\forall\alpha \in \mathbb{N}_0)(\forall b \in \mathbb{R})(||u_\varepsilon^{(\alpha)}||_{L^\infty(\mathbb{R}^d)} = O(\varepsilon^b))\Big\} -$$

The algebra $\mathcal{G}_{L^\infty}(\mathbb{R}^d)$ is defined as the quotient

$$\mathcal{G}_{L^\infty}(\Omega) = \mathcal{E}_{L^\infty,M}(\mathbb{R}^d)/\mathcal{N}_{L^\infty}(\mathbb{R}^d).$$

The natural class of distributions that can be embedded into $\mathcal{G}_{L^\infty}(\mathbb{R}^d)$ is the Schwartz distribution space of the so-called bounded distributions [21]. More precisely, this space is given by

$$\mathcal{D}'_{L^\infty}(\mathbb{R}^d) = \bigcup_{m\in\mathbb{N}_0} W^{-m,\infty}(\mathbb{R}^d) = \bigcup_{s\in\mathbb{R}} C^s_*(\mathbb{R}^d).$$

It is the dual of the test function space [21] $\mathcal{D}_{L^1}(\mathbb{R}^d) = \bigcap_{m\in\mathbb{N}} W^{m,1}(\mathbb{R}^d)$. Clearly, $\iota : \mathcal{D}'_{L^\infty}(\mathbb{R}^d) \mapsto \mathcal{G}_{L^\infty}(\mathbb{R}^d)$ given as usual by $\iota(T) = [(T * \phi_\varepsilon)_\varepsilon]$ provides a natural embedding. On the other hand, the embedding does not extend to $\mathcal{S}'(\mathbb{R}^d)$, and more interestingly, as long as $\iota(T) \in \mathcal{G}_{L^\infty}(\mathbb{R}^d)$ for a tempered distribution, it is forced to belong to $\mathcal{D}'_{L^\infty}(\mathbb{R}^d)$.

Theorem 4. *Let $T \in \mathcal{S}'(\mathbb{R}^d)$. If $(T * \phi_\varepsilon)_\varepsilon \in \mathcal{E}_{L^\infty,M}(\mathbb{R}^d)$, then $T \in \mathcal{D}'_{L^\infty}(\mathbb{R}^d)$.*

Proof. Because of the Schwartz characterization [21, Chap. VI] of $\mathcal{D}'_{L^\infty}(\mathbb{R}^d)$, it would be enough to show that, for each $\rho \in \mathcal{S}(\mathbb{R}^d)$, $T * \rho \in C_b(\mathbb{R}^d)$, the Banach space of continuous and bounded functions. In order to show so, we will use the vector-valued Tauberian theory for class estimates developed in [20, Chap. 7] (see also [3, 23]). Define the vector-valued distribution $\mathbf{T}$ whose action on test functions $\rho \in \mathcal{S}(\mathbb{R}^d)$ is given by $\langle \mathbf{T}, \rho\rangle = T * \check{\rho}$. Therefore, we must show that $\mathbf{T} \in \mathcal{S}'(\mathbb{R}^d, C_b(\mathbb{R}^d))$. Since T is tempered, there exists $N \in \mathbb{N}$ such that $\mathbf{T}$ takes values in the Banach space X consisting of continuous functions g on $\mathbb{R}^d$ such that $\|g\|_X := \sup_{t\in\mathbb{R}^d}(1+|t|)^{-N}\,|g(t)| < \infty$. Clearly, the inclusion mapping $C_b(\mathbb{R}^d) \mapsto X$ is continuous. On the other hand, we have the local class estimate

$$\|(\mathbf{T} * \phi_\varepsilon)(x)\|_{L^\infty(\mathbb{R}^d)} = \sup_{\xi\in\mathbb{R}^d} |(T * \phi_\varepsilon)(x + \xi)| = \|T * \phi_\varepsilon\|_{L^\infty(\mathbb{R}^d)} = O(\varepsilon^{-a}),$$

for some $a > 0$. Thus, in view of the [20, Thm. 7.9], we obtain the desired conclusion $\mathbf{T} \in \mathcal{S}'(\mathbb{R}^d, C_b(\mathbb{R}^d))$. □

Let us note that $\mathcal{E}_{L^\infty,M}(\mathbb{R}^d) \subset \mathcal{E}_M(\mathbb{R}^d)$ is a differential subalgebra and $\mathcal{N}_{L^\infty}(\mathbb{R}^d) \subset \mathcal{N}(\mathbb{R}^d)$. There is a canonical differential algebra mapping $\mathcal{G}_{L^\infty}(\mathbb{R}^d) \to \mathcal{G}(\mathbb{R}^d)$; however, this mapping is not injective. Hence $\mathcal{G}_{L^\infty}(\mathbb{R}^d)$ cannot be seen as a differential subalgebra of $\mathcal{G}(\mathbb{R}^d)$.

Example 3. This example shows that the canonical mapping $\mathcal{G}_{L^\infty}(\mathbb{R}^d) \to \mathcal{G}(\mathbb{R}^d)$ is not injective. Equivalently, we find a net $(u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^d) \cap \mathcal{E}_{L^\infty,M}(\mathbb{R}^d)$ which does not belong to $\mathcal{N}_{L^\infty}(\mathbb{R}^d)$. Let $\rho \in \mathcal{D}(\mathbb{R}^d)$ be non-trivial and supported by the ball with center at the origin and radius 1/2. Consider the net of smooth functions

$$u_\varepsilon(x) = \sum_{n=0}^{\infty} \frac{\chi_{[(n+1)^{-1},1]}(\varepsilon)}{(n+1)^2}\rho(x - 2ne_1),$$

where $\chi_{[(n+1)^{-1},1]}$ is the characteristic function of the interval $[1/(n+1), 1]$ and $e_1 = (1, 0, \ldots, 0)$. Then, clearly $(u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^d)$ because on compact sets it identically vanishes for small enough ε. On the other hand,

$$||u_\varepsilon||_{W^{m,\infty}} = ||\rho||_{W^{m,\infty}} \sum_{\frac{1}{\varepsilon}-1\leq n}^{\infty} \frac{1}{(n+1)^2} = \varepsilon\, ||\rho||_{W^{m,\infty}} + O(\varepsilon^2) \text{ as } \varepsilon \to 0.$$

Thus, the net satisfies all the requirements.

5.2. Global Zygmund classes

We come back to Hörmann's Zygmund class of generalized functions. We slightly modify his definition. Given $r \in \mathbb{R}$,, define $\tilde{\mathcal{G}}^r_*(\mathbb{R}^d)$ as the space of those $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}(\mathbb{R}^d)$ such that $(u_\varepsilon)_\varepsilon$ satisfies (5.1). Originally [9], the "tilde" did not appear in the notation but since we will introduce a new definition, which is intrinsically related to the classical definition of Zygmund spaces, we leave the notation $\mathcal{G}^r_*(\mathbb{R}^d)$ for our space.

Definition 2. Let $r \in \mathbb{R}$ and let $\varphi, \psi \in \mathcal{S}'(\mathbb{R}^d)$ be a pair satisfying (2.4) and (2.5) (i.e., a generalized Littlewood–Paley pair). The space $\mathcal{G}^r_*(\mathbb{R}^d) = \mathcal{G}^{r,0}_*(\mathbb{R}^d)$, called the Zygmund space of generalized functions of 0-growth order, consists of those $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}(\mathbb{R}^d)$ such that

$$||u_\varepsilon||_{C^r_*(\mathbb{R}^d)} = ||u_\varepsilon * \varphi||_{L^\infty(\mathbb{R}^d)} + \sup_{0<y<1} y^{-r}||u_\varepsilon * \psi_y||_{L^\infty(\mathbb{R}^d)} = O(1). \tag{5.2}$$

Moreover, $\mathcal{G}^{r,-s}_*(\mathbb{R}^d)$, the Zygmund space of generalized functions of $-s$-growth order, consists of those $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}(\mathbb{R}^d)$ such that $[(\varepsilon^s u_\varepsilon)_\varepsilon] \in \mathcal{G}^r_*(\mathbb{R}^d)$.

Observe that Definition 2 is independent of the choice of representatives. The main properties of these spaces are summarized in the next theorem. In particular, we show the embedding of the ordinary Zygmund spaces of functions and characterize those distributions which, after embedding, belong to our generalized Zygmund classes.

Theorem 5. *The following properties hold:*

(i) $\iota(C^r_*(\mathbb{R}^d)) = \mathcal{G}^r_*(\mathbb{R}^d) \cap \iota(\mathcal{D}'_{L^\infty}(\mathbb{R}^d))$.
(ii) $\mathcal{G}^{r,-s}_*(\mathbb{R}^d) \cap \iota(\mathcal{D}'_{L^\infty}(\mathbb{R}^d)) \subset \iota(C^{r-s}_*(\mathbb{R}^d))$.
(iii) $\mathcal{G}^{r_1,-s}_*(\mathbb{R}^d) \subset \mathcal{G}^{r,-s}_*(\mathbb{R}^d)$ *if* $r_1 > r$; $P(D)\mathcal{G}^{r,-s}_*(\mathbb{R}^d) \subset \mathcal{G}^{r-m,-s}_*(\mathbb{R}^d)$, *where* $P(D)$ *is a differential operator with constant coefficients and order* m.

(iv) *If* $r_1 + r_2 > 0$, *then*

$$\mathcal{G}_*^{r_1,-s_1}(\mathbb{R}^d)\cdot\mathcal{G}_*^{r_2,-s_2}(\mathbb{R}^d)\subset\mathcal{G}_*^{p,-s_1-s_2}(\mathbb{R}^d),\quad p=\min\{r_1,r_2\}.$$

In particular, $\mathcal{G}_*^{r,-s}(\mathbb{R}^d)$ *is an algebra if* $s=0$ *and* $r>0$.

Proof. (i) and (ii). We first show that $\iota(C_*^r(\mathbb{R}^d))\subset\mathcal{G}_*^r(\mathbb{R}^d)$. Let $u\in C_*^r(\mathbb{R}^d)$ and $u_\varepsilon=u*\phi_\varepsilon$. Obviously,

$$||u_\varepsilon*\varphi||_{L^\infty(\mathbb{R}^d)}\le\|u*\varphi\|_{L^\infty(\mathbb{R}^d)}\,\|\phi\|_{L^1(\mathbb{R}^d)}$$

and

$$\sup_{y<1}y^{-r}||u_\varepsilon*\psi_y||_{L^\infty(\mathbb{R}^d)}\le||\phi||_{L^1(\mathbb{R}^d)}\sup_{0<y<1}y^{-r}||u*\psi_y||_{L^\infty(\mathbb{R}^d)}.$$

Let us now prove the inclusion $\mathcal{G}_*^{r,-s}(\mathbb{R}^d)\cap\iota(\mathcal{D}'_{L^\infty}(\mathbb{R}^d))\subset\iota(C_*^{r-s}(\mathbb{R}^d))$. The proof is similar to the last part of the proof of Theorem 1. So, let $u=[(u*\phi_\varepsilon)_\varepsilon]\in\mathcal{G}_*^{r,-s}(\mathbb{R}^d)$, where $u\in\mathcal{D}'_{L^\infty}(\mathbb{R}^d)$. We have freedom of choice for the Littlewood–Paley pair in (5.2). Let then $\psi\in\mathcal{S}(\mathbb{R}^d)$ be such that (ϕ,ψ) forms a generalized Littlewood–Paley pair of order $\max\{r,r-s\}$ (cf. (2.4) and (2.5)). We have $u*\phi\in L^\infty(\mathbb{R}^d)$; on the other hand, setting $\varepsilon=y<1$ and observing that $(\phi*\psi)_y=\phi_y*\psi_y$, one obtains that

$$\sup_{0<y<1}y^{s-r}||u*(\phi*\psi)_y||_{L^\infty(\mathbb{R}^d)}=\sup_{0<y<1}y^{s-r}||u*\phi_y*\psi_y||_{L^\infty(\mathbb{R}^d)}<\infty.$$

Noticing that $(\phi,\phi*\psi)$ is again a generalized Littlewood–Paley pair of order $r-s$, we conclude $T\in C_*^{r-s}(\mathbb{R}^d)$.

(iii) The first part is clear. The second part follows from the fact [8] that $P(D)$ continuously maps the classical Zygmund space $C_*^r(\mathbb{R}^d)$ into $C_*^{r-m}(\mathbb{R}^d)$.

(iv) It is a consequence of [8, Prop. 8.6.8]. Actually, we have, by this proposition, that there exists $\varepsilon_0\in(0,1)$ and $K=K(r_1,r_2)$, which does not depend on ε, such that

$$||\varepsilon^{s_1+s_2}u_{1,\varepsilon}u_{2,\varepsilon}||_{C_*^p(\mathbb{R}^d)}\le K||\varepsilon^{s_1}u_{1,\varepsilon}||_{C_*^{r_1}(\mathbb{R}^d)}||\varepsilon^{s_2}u_{2,\varepsilon}||_{C_*^{r_2}(\mathbb{R}^d)},\ \varepsilon\le\varepsilon_0.\qquad\square$$

Remark 1. As in the case of multiplication of continuous functions, we have that $[((u_1u_2)*\phi_\varepsilon)_\varepsilon]\neq[(u_1*\phi_\varepsilon)_\varepsilon][(u_2*\phi_\varepsilon)_\varepsilon]$ but these products are associated.

In analogy with Definition 1, we can also introduce some other classes of generalized functions. They are now closely related to the classical global Zygmund spaces.

Definition 3. Let $s\in\mathbb{R}$ and $k\in\mathbb{N}_0$.

(i) A net $(f_\varepsilon)_\varepsilon\in\mathcal{E}_{L^\infty,M}(\mathbb{R}^d)$ is said to belong to $\mathcal{E}^{k,-s}_{L^\infty,M}(\mathbb{R}^d)$ if

$$\|f_\varepsilon\|_{W^{k,\infty}(\mathbb{R}^d)}=O(\varepsilon^{-s}).$$

(ii) A generalized function $f=[(f_\varepsilon)_\varepsilon]\in\mathcal{G}_{L^\infty}(\mathbb{R}^d)$ is said to belong to $\mathcal{G}^{k,-s}_{L^\infty}(\mathbb{R}^d)$ if $(f_\varepsilon)_\varepsilon\in\mathcal{E}^{k,-s}_{L^\infty,M}(\mathbb{R}^d)$.

Definition 3 does not depend on the choice of representatives. The next theorem characterizes those distributions that belong to $\mathcal{G}_{L^\infty}(\mathbb{R}^d)$ and gives an inclusion relation for Hörmann class $\tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$. Let us mention that a version of Proposition 1 also holds for $\mathcal{G}_{L^\infty}^{k,-s}(\mathbb{R}^d)$.

Theorem 6. *Let $r \in \mathbb{R}$ and $s > 0$.*

(i) *We have $\mathcal{G}_{L^\infty}^{k,-s}(\mathbb{R}^d) \cap \iota(\mathcal{D}'_{L^\infty}(\Omega)) = \iota(C_*^{k-s}(\mathbb{R}^d))$.*
(ii) *Given any integer $k > r$, we have $\iota(C_*^r(\mathbb{R}^d)) = \mathcal{G}_{L^\infty}^{k,r-k}(\mathbb{R}^d) \cap \iota(\mathcal{D}'_{L^\infty}(\Omega))$.*
(iii) *There holds*

$$\tilde{\mathcal{G}}_*^r(\mathbb{R}^d) \subset \bigcap_{k>r} \mathcal{G}_{L^\infty}^{k,r-k}(\mathbb{R}^d).$$

Proof. The property (iii) follows directly from the definitions. Observe that (i) and (ii) are equivalent. On the other hand, a straightforward modification of the proof of Theorem 1 yields (i), we leave the details of such a modification to the reader. □

The following remarks make some partial comparisons between our definition and Hörmann's definition [9]. We also formulate an open question.

Remark 2. Clearly, if $u = [(u_\varepsilon)_\varepsilon] \in \tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$, then $[(u_\varepsilon * \phi_\varepsilon)_\varepsilon] \in \tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$ but the opposite does not hold, in general. However, $u = [(u_\varepsilon)_\varepsilon]$ and $u = [(u_\varepsilon * \phi_\varepsilon)_\varepsilon]$ are equal in the sense of generalized distributions, which means that

$$\langle u_\varepsilon * \phi_\varepsilon - u_\varepsilon, \theta\rangle = o(\varepsilon^p) \text{ for every } p \text{ and every } \theta \in \mathcal{D}(\mathbb{R}^d).$$

Remark 3. Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_*^r(\mathbb{R}^d)$. We show that $[(u_\varepsilon * \phi_\varepsilon)_\varepsilon] \in \tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$. For this, we will make use of Lemma 8.6.5 of [8], which asserts that given $\kappa \in \mathcal{S}(\mathbb{R}^d)$, there exist constants $K_{r,\alpha}$, $\alpha \in \mathbb{N}_0$, such that for all $v \in C_*^r(\mathbb{R}^d)$ and $0 < y \leq 1$, there holds

$$\|(v * \kappa_y)^{(\alpha)}\|_{L^\infty(\mathbb{R}^d)} \leq \begin{cases} K_{r,\alpha}\|v\|_{C_*^r(\mathbb{R}^d)}, & 0 \leq |\alpha| < r, \\ K_{r,\alpha}\|v\|_{C_*^r(\mathbb{R}^d)}(1+\log(1/y)), & |\alpha| = r \in \mathbb{N}_0, \\ K_{r,\alpha}\|v\|_{C_*^r(\mathbb{R}^d)}(y^{r-|\alpha|}), & |\alpha| > r, \end{cases} \tag{5.3}$$

where, as usual, $\kappa_y = y^{-d}\kappa(\cdot/y)$. Thus, if we employ (5.3) with $v = u_\varepsilon$, $\kappa = \phi$, and $y = \varepsilon$, together with the fact that $\|u_\varepsilon\|_{C_*^r(\mathbb{R}^d)}$ is uniformly bounded with respect to ε, we obtain at once $[(u_\varepsilon * \phi_\varepsilon)_\varepsilon] \in \tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$, as claimed. At this point we should mention that the precise relation between the spaces $\mathcal{G}_*^r(\mathbb{R}^d)$ and $\tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$ is still unknown; therefore, we can formulate an *open question: Find the precise inclusion relation between these two spaces.*

Remark 4. As seen from the given assertions, our Zygmund generalized function spaces are suitable for the analysis of pseudodifferential operators.

5.3. Hölder-type spaces and algebras of generalized functions

We end this article by dealing with Hölderian-type classes of generalized functions. We will employ the norm (2.3).

Definition 4. Let $k \in \mathbb{N}_0$, $s \in \mathbb{R}$, $\tau \in (0,1]$ and let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}(\mathbb{R}^d)$. It is said that $u \in \mathcal{G}^{k,\tau,-s}_{L^\infty}(\mathbb{R}^d)$ if

$$||u_\varepsilon||_{\mathcal{H}^{k,\tau}(\mathbb{R}^d)} = O(\varepsilon^{-s}). \tag{5.4}$$

Recall [8] the classical situation. Let $k \in \mathbb{N}_0$, then $\mathcal{H}^{k,1}(\mathbb{R}^d) \subsetneq C^{k+1}_*(\mathbb{R}^d)$; but if $\tau \in (0,1)$, then $\mathcal{H}^{k,\tau}(\mathbb{R}^d) = C^{k+\tau}_*(\mathbb{R}^d)$. In our context, we have,

Proposition 3. *If $r = k+\tau, \tau \in (0,1)$, then $\mathcal{G}^{r,s}_*(\mathbb{R}^d) = \mathcal{G}^{k,\tau,s}(\mathbb{R}^d)$.*

Proof. There exists $C > 0$ such that for every $\varepsilon < 1$,

$$C^{-1}||\varepsilon^s u_\varepsilon||_{C^{k+\tau}_*(\mathbb{R}^d)} \leq ||\varepsilon^s u_\varepsilon||_{\mathcal{H}^{k,\tau}(\mathbb{R}^d)} \leq C||\varepsilon^s u_\varepsilon||_{C^{k+\tau}_*(\mathbb{R}^d)},$$

as follows from the equivalence between the norms (2.3) and (2.6). This implies the assertion. □

Because of Proposition 3, we will consider below only the case $\mathcal{G}^{k,1,s}(\mathbb{R}^d)$.

Proposition 4. *Let $k \in \mathbb{N}_0$ and $s \in \mathbb{R}$.*

(i) $\iota(\mathcal{H}^{k,1}(\mathbb{R}^d)) = \mathcal{G}^{k,1,0}(\mathbb{R}^d) \cap \iota(\mathcal{D}'_{L^\infty}(\mathbb{R}^d))$.
(ii) $\mathcal{G}^{k,1,s}(\mathbb{R}^d) \subsetneq \mathcal{G}^{k+1,s}_*(\mathbb{R}^d)$.
(iii) $\mathcal{G}^{k_1,1,s}(\mathbb{R}^d) \subset \mathcal{G}^{k,1,s}(\mathbb{R}^d)$ *if* $k_1 > k$.
(iv) *Let $P(D)$ be a differential operator of order $m < k$ with constant coefficients. Then $P(D) : \mathcal{G}^{k,\tau,s}(\mathbb{R}^d) \to \mathcal{G}^{k-m,\tau,s}(\mathbb{R}^d)$.*
(v) *Concerning the multiplication, we have*

$$\mathcal{G}^{k_1,1,s}(\mathbb{R}^d) \cdot \mathcal{G}^{k_2,1,s}(\mathbb{R}^d) \subset \mathcal{G}^{p,1,2s}(\mathbb{R}^d),$$

where $p = \min\{k_1, k_2\}$. In particular, $\mathcal{G}^{k_1,1,s}(\mathbb{R}^d)$ is an algebra if and only if $s = 0$.

Proof. The proofs of assertions (ii), (iii), (iv) and (v) are clear. We will prove (i). The direct inclusion follows from the definition. Suppose that $T \in \mathcal{D}'_{L^\infty}(\mathbb{R}^d)$ is such that $[(T_\varepsilon)_\varepsilon] \in \mathcal{G}^{k,1,0}(\mathbb{R}^d)$ where $T_\varepsilon = T * \phi_\varepsilon$. By assumption $\{T^{(\alpha)}_\varepsilon; 0 < \varepsilon < 1\}$ is a bounded and equicontinuous net of functions on any compact set in $\mathbb{R}^d$, for every $|\alpha| \leq k$. Thus, by the Arzelà–Ascoli theorem, it has a convergent subsequence for every $|\alpha| \leq k$ and, by diagonalization, there exists a sequence $(T_{\varepsilon_n})_n$ and $T \in C^k(\mathbb{R}^d)$ such that $T^{(\alpha)}_{\varepsilon_n} \to T^{(\alpha)}$ uniformly on any compact set $K \subset \mathbb{R}^d$. That $||T||_{W^{k,\infty}(\mathbb{R}^d)} < \infty$ follows now easily. Let $|\alpha| = k$. For every $x, y \in \mathbb{R}^d$, $x \neq y$,

$$\frac{|T^{(\alpha)}(x) - T^{(\alpha)}(y)|}{|x-y|^\tau} = \lim_{n\to\infty} \frac{|T^{(\alpha)}_{\varepsilon_n}(x) - T^{(\alpha)}_{\varepsilon_n}(y)|}{|x-y|^\tau} \leq C,$$

since

$$\sup_{x,y\in\mathbb{R}^d,\ x\neq y} \lim_{n\to\infty} \frac{|T^{(\alpha)}_{\varepsilon_n}(x) - T^{(\alpha)}_{\varepsilon_n}(y)|}{|x-y|^\tau} \leq \sup_{x,y\in\mathbb{R}^d,\ x\neq y} \sup_{n\in\mathbb{N}} \frac{|T^{(\alpha)}_{\varepsilon_n}(x) - T^{(\alpha)}_{\varepsilon_n}(y)|}{|x-y|^\tau}$$
$$\leq \sup_{x,y\in\mathbb{R}^d,\ x\neq y,\varepsilon<1} \frac{|T^{(\alpha)}_{\varepsilon}(x) - T^{(\alpha)}_{\varepsilon}(y)|}{|x-y|^\tau} \leq C,$$

and the assertion follows. □

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The Wave Equation with a Discontinuous Coefficient Depending on Time Only: Generalized Solutions and Propagation of Singularities

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Abstract. This paper is devoted to the investigation of propagation of singularities in hyperbolic equations with non-smooth coefficients, using the Colombeau theory of generalized functions. As a model problem, we study the Cauchy problem for the one-dimensional wave equation with a discontinuous coefficient depending on time. After demonstrating the existence and uniqueness of generalized solutions in the sense of Colombeau to the problem, we investigate the phenomenon of propagation of singularities, arising from delta function initial data, for the case of a piecewise constant coefficient. We also provide an analysis of the interplay between singularity strength and propagation effects. Finally, we show that in case the initial data are distributions, the Colombeau solution to the model problem is associated with the piecewise distributional solution of the corresponding transmission problem.

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1. Introduction

This paper is devoted to propagation of singularities in linear hyperbolic partial differential equations with non-smooth coefficients in the framework of the Colombeau theory of generalized functions [2, 3]. Such equations involve a nonlinear interaction of coefficient singularities with those in the solution which emanate

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from the initial data. In the past years, this problem has been investigated intensively in a series of papers, in which the coefficients and the solution have been viewed as elements of the Colombeau algebra of generalized functions.

So far, microlocal elliptic regularity is well understood for general equations and systems in this framework [6, 8, 18]. Further, generalized Fourier integral operators have been introduced in order to describe the wave front set of generalized solutions [7, 9, 10, 11]. These methods have allowed to describe the propagation of singularities in scalar first-order hyperbolic equations with generalized function coefficients [5, 12, 16]. As a new phenomenon, the Colombeau wave front set has been shown to possess a more refined structure than the corresponding distributional wave front set [22]. For a survey of these methods and relations to various classical approaches we refer to [15].

In the special situation of hyperbolic equations and systems with piecewise constant coefficients (with possibly jumps across smooth hypersurfaces), the problem may be interpreted as a transmission problem and can be solved classically by a piecewise distributional solution. In case of systems or higher-order equations, an incoming singularity will split at the interface into a refracted and a reflected wave. While in many cases it has been shown that the Colombeau solution is associated with the distributional solution of the transmission problem [13, 20], the task of identifying the refracted and reflected singularity by means of the Colombeau wave front set – directly in the generalized solution without considering the associated distribution – has remained open.

The purpose of this paper is to demonstrate – in a simple model problem – that the splitting of singularities at an interface can indeed be observed in the Colombeau solution. At the same time, the effect depends on the interplay of the scale of regularization of coefficients and data. We consider the Cauchy problem for the one-dimensional wave equation with a discontinuous coefficient depending on time

$$\begin{aligned} &\partial_t^2 u - c(t)^2 \partial_x^2 u = 0, \qquad t > 0,\ x \in \mathbb{R}, \\ &u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \qquad x \in \mathbb{R}. \end{aligned} \tag{1.1}$$

We will seek solutions in the Colombeau algebra $\mathscr{G}([0,\infty) \times \mathbb{R})$ of generalized functions, which will be defined in Section 2 below. The Colombeau algebras are constructed as factor algebras of nets of smooth functions, depending on a regularization parameter. They are differential algebras; the space of distributions can be imbedded as a subspace using regularization by convolution with a Friedrichs mollifier. The initial data as well as the coefficient will be taken from the algebra $\mathscr{G}(\mathbb{R})$, which contains the space $\mathscr{D}'(\mathbb{R})$ of distributions.

As a model situation, we consider the case when $c(t) = c_0 + (c_1 - c_0)H(t-1)$, $u_0 = 0$ and $u_1 = \delta$, where $c_0, c_1 > 0$, $c_0 \neq c_1$, H is the Heaviside function and δ is the delta function. The singularity of the initial data propagates from the origin in the two characteristic directions until it hits the coefficient discontinuity at time $t = 1$. As the coefficient has a discontinuity in time, the singular support is expected to split at time $t = 1$ into a transmitted and a refracted part. In the classical setting, the piecewise distributional solution exhibits this splitting. We will demonstrate the occurrence of the splitting effect in the Colombeau setting as well, using the generalized singular support. However, in the Colombeau setting the effect depends on the scale in terms of the regularization parameter. If the initial data and the coefficient are regularized on the same scale, both the transmitted and the refracted ray will be shown to belong to the Colombeau singular support. If the coefficient is regularized by means of a slow scale mollifier, only the transmitted ray belongs to the Colombeau singular support.

In order to be able to observe this effect, we have to prove a new existence result for the Cauchy problem (1.1) which does not restrict the choice of scale for the regularization of the coefficient. The equation can be rewritten as a one-dimensional first-order hyperbolic system with discontinuous coefficients in the Colombeau algebra of generalized functions. This transformation will be used in the existence and regularity results. We also demonstrate that the Colombeau solution has the piecewise solution of the transmission problem as its associated distribution. That is, the nets defining the Colombeau solution converge to the piecewise distributional solution. This latter argument relies on energy estimates.

The paper is organized as follows: we recall the definition and basic properties of the Colombeau algebra $\mathscr{G}$ in Section 2. In Section 3, we show that problem (1.1) is uniquely solvable in the Colombeau algebra $\mathscr{G}([0,\infty)\times\mathbb{R})$ without restriction on the scale of regularization of the coefficient (Theorem 3.1). The problem of propagation of singularities is addressed in Section 4 (Theorem 4.1). The question whether the regularity of the coefficient affects that of the solution is discussed in Section 5 (Theorem 5.1). In Section 6, we show that the Colombeau generalized solutions to problem (1.1) with arbitrary distributions as initial data admit the piecewise distributional solutions of the corresponding transmission problem as associated distributions (Theorem 6.1).

2. The Colombeau theory of generalized functions

We will employ the *special Colombeau algebra* denoted by $\mathcal{G}^s$ in [14], which was called the *simplified Colombeau algebra* in [1]. However, here we will simply use the letter $\mathcal{G}$ instead. Let us briefly recall the definition and basic properties of the algebra $\mathcal{G}$ of generalized functions. For more details, see [14].

Let Ω be a non-empty open subset of $\mathbb{R}^d$. Let $\mathcal{C}^\infty(\Omega)^{(0,1]}$ be the differential algebra of all maps from the interval $(0,1]$ into $\mathcal{C}^\infty(\Omega)$. Thus each element of $\mathcal{C}^\infty(\Omega)^{(0,1]}$ is a family $(u^\varepsilon)_{\varepsilon\in(0,1]}$ of real-valued smooth functions on Ω. The subalgebra $\mathcal{E}_M(\Omega)$ is defined by all elements $(u^\varepsilon)_{\varepsilon\in(0,1]}$ of $\mathcal{C}^\infty(\Omega)^{(0,1]}$ with the property that, for all $K\Subset\Omega$ and $\alpha\in\mathbb{N}_0^d$, there exists $p\geq 0$ such that

$$\sup_{x\in K}|\partial_x^\alpha u^\varepsilon(x)| = O(\varepsilon^{-p}) \quad \text{as } \varepsilon\downarrow 0. \tag{2.1}$$

The ideal $\mathcal{N}(\Omega)$ is defined by all elements $(u^\varepsilon)_{\varepsilon\in(0,1]}$ of $\mathcal{C}^\infty(\Omega)^{(0,1]}$ with the property that, for all $K \Subset \Omega$, $\alpha \in \mathbb{N}_0^d$ and $q \geq 0$,

$$\sup_{x\in K} |\partial_x^\alpha u^\varepsilon(x)| = O(\varepsilon^q) \quad \text{as } \varepsilon \downarrow 0.$$

The algebra $\mathcal{G}(\Omega)$ *of generalized functions* is defined by the quotient space

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega).$$

The Colombeau algebra on a closed half-space $[0,\infty)\times\mathbb{R}$ is constructed in a similar way.

We use capital letters for elements of $\mathcal{G}$ to distinguish generalized functions from distributions and denote by $(u^\varepsilon)_{\varepsilon\in(0,1]}$ a representative of $U \in \mathcal{G}$. Then for any U, $V \in \mathcal{G}$ and $\alpha \in \mathbb{N}_0^d$, we can define the partial derivative $\partial^\alpha U$ to be the class of $(\partial^\alpha u^\varepsilon)_{\varepsilon\in(0,1]}$ and the product UV to be the class of $(u^\varepsilon v^\varepsilon)_{\varepsilon\in(0,1]}$. Also, for any $U =$ class of $(u^\varepsilon(t,x))_{\varepsilon\in(0,1]} \in \mathcal{G}([0,\infty)\times\mathbb{R})$, we can define its restriction $U|_{t=0} \in \mathcal{G}(\mathbb{R})$ to the line $\{t = 0\}$ to be the class of $(u^\varepsilon(0,x))_{\varepsilon\in(0,1]}$.

Remark 2.1. The algebra $\mathcal{G}(\Omega)$ contains the space $\mathcal{E}'(\Omega)$ of compactly supported distributions. In fact, the map

$$f \mapsto \text{class of } (f * \rho_\varepsilon\,|_\Omega)_{\varepsilon\in(0,1]}$$

defines an imbedding of $\mathcal{E}'(\Omega)$ into $\mathcal{G}(\Omega)$, where

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)$$

and ρ is a fixed element of $\mathcal{S}(\mathbb{R}^d)$ such that $\int \rho(x)\,dx = 1$ and $\int x^\alpha \rho(x)\,dx = 0$ for any $\alpha \in \mathbb{N}_0^d$, $|\alpha| \geq 1$. This can be extended in a unique way to an imbedding of the space $\mathcal{D}'(\Omega)$ of distributions. Moreover, this imbedding turns $\mathcal{C}^\infty(\Omega)$ into a subalgebra of $\mathcal{G}(\Omega)$.

Definition 2.2. A generalized function $U \in \mathcal{G}(\Omega)$ is said to be *associated with a distribution* $w \in \mathcal{D}'(\Omega)$ if it has a representative $(u^\varepsilon)_{\varepsilon\in(0,1]} \in \mathcal{E}_M(\Omega)$ such that

$$u^\varepsilon \to w \quad \text{in } \mathcal{D}'(\Omega) \quad \text{as } \varepsilon \downarrow 0.$$

We write $U \approx w$ and call w the *associated distribution* of U provided U is associated with w.

Regularity theory for linear equations has been based on the subalgebra $\mathcal{G}^\infty(\Omega)$ of *regular generalized functions* in $\mathcal{G}(\Omega)$ introduced in [21]. It is defined by all elements which have a representative $(u^\varepsilon)_{\varepsilon\in(0,1]}$ with the property that, for all $K \Subset \Omega$, there exists $p \geq 0$ such that, for all $\alpha \in \mathbb{N}_0^d$,

$$\sup_{x\in K} |\partial_x^\alpha u^\varepsilon(x)| = O(\varepsilon^{-p}) \quad \text{as } \varepsilon \downarrow 0.$$

We observe that all derivatives of u^ε have locally the same order of growth in $\varepsilon > 0$, unlike elements of $\mathcal{E}_M(\Omega)$. This subalgebra $\mathcal{G}^\infty(\Omega)$ has the property $\mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{C}^\infty(\Omega)$, see [21, Theorem 25.2]. Hence, for the purpose of describing the regularity of generalized functions, $\mathcal{G}^\infty(\Omega)$ plays the same role for $\mathcal{G}(\Omega)$ as

$\mathcal{C}^\infty(\Omega)$ does in the setting of distributions. The $\mathcal{G}^\infty$-singular support (denoted by $\operatorname{sing\,supp}_{\mathcal{G}^\infty}$) of a generalized function is defined as the complement of the largest open set on which the generalized function is regular in the above sense.

We end this section by recalling the notion of *slow scale nets*. A net $(r^\varepsilon)_{\varepsilon\in(0,1]}$ is called a slow scale net if

$$|r^\varepsilon|^p = O(\varepsilon^{-1}) \quad \text{as } \varepsilon \downarrow 0$$

for every $p \geq 0$. A *positive slow scale net* is a slow scale net $(r^\varepsilon)_{\varepsilon\in(0,1]}$ such that $r^\varepsilon > 0$ for all $\varepsilon \in (0,1]$. We refer to [17] for a detailed discussion of slow scale nets.

Example 2.3. Let φ be a fixed element of $\mathcal{C}_0^\infty(\mathbb{R})$ such that φ is symmetric, $\varphi' \geq 0$ on $[-1,0]$, $\operatorname{supp}\varphi \subset [-1,1]$ and $\int_{\mathbb{R}} \varphi(x)\,dx = 1$. Put $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)/\varepsilon$. Then $U \in \mathcal{G}(\mathbb{R})$ defined by the class of $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ is associated with the delta function δ, and $\operatorname{sing\,supp}_{\mathcal{G}^\infty} U = \{0\}$. On the other hand, if $U \in \mathcal{G}(\mathbb{R})$ is defined as the class of $(\varphi_{h(\varepsilon)})_{\varepsilon\in(0,1]}$, where $(1/h(\varepsilon))_{\varepsilon\in(0,1]}$ is a positive slow scale net, then it is associated with the delta function again, but $\operatorname{sing\,supp}_{\mathcal{G}^\infty} U = \emptyset$. More generally, for any distribution $f \in \mathcal{D}'(\Omega)$, there exists a generalized function $U \in \mathcal{G}^\infty(\Omega)$ which is associated with f, see, e.g., [4]. Thus, any distribution on Ω can be interpreted as an element of $\mathcal{G}^\infty(\Omega)$ in the sense of association.

3. Existence and uniqueness of generalized solutions

We rewrite problem (1.1) in the form

$$\begin{aligned} &\partial_t^2 U - C^2 \partial_x^2 U = 0 && \text{in } \mathcal{G}([0,\infty)\times\mathbb{R}),\\ &U|_{t=0} = U_0, \quad \partial_t U|_{t=0} = U_1 && \text{in } \mathcal{G}(\mathbb{R}) \end{aligned} \tag{3.1}$$

in the space of generalized functions, where C is an element of $\mathcal{G}([0,\infty)\times\mathbb{R})$. In the Colombeau setting, existence and uniqueness follows, for example, from results in [19, 20], provided the coefficient C is of logarithmic type, i.e., satisfies bounds of type $O(\log|\varepsilon|)$ in (2.1). When the coefficient depends on time only, this hypothesis is not required, as we are going to show in the following existence and uniqueness theorem for problem (3.1).

Theorem 3.1. *Assume that $C \in \mathcal{G}([0,\infty)\times\mathbb{R})$ has a representative $(c^\varepsilon(t))_{\varepsilon\in(0,1]}$ independent of x and satisfying the following two conditions:*

(i) *there exist two constants c_0, $c_1 > 0$ such that, for any $\varepsilon \in (0,1]$ and $t \geq 0$,*

$$c_1 \geq c^\varepsilon(t) \geq c_0 > 0;$$

(ii) *for any $\varepsilon \in (0,1]$,*

$$\int_0^\infty |(c^\varepsilon)'(t)|\,dt < \infty.$$

Then for any initial data U_0, $U_1 \in \mathcal{G}(\mathbb{R})$, problem (3.1) *has a unique solution $U \in \mathcal{G}([0,\infty)\times\mathbb{R})$.*

Proof. Put $V = \partial_t U - C\partial_x U$ and $W = \partial_t U + C\partial_x U$. Then problem (3.1) can be rewritten as the Cauchy problem for a first-order hyperbolic system

$$\begin{aligned} (\partial_t + C\partial_x)V &= MV - MW && \text{in } \mathcal{G}([0,\infty)\times\mathbb{R}),\\ (\partial_t - C\partial_x)W &= MW - MV && \text{in } \mathcal{G}([0,\infty)\times\mathbb{R}),\\ V|_{t=0} &= V_0 = U_1 - (C|_{t=0})U_0' && \text{in } \mathcal{G}(\mathbb{R}),\\ W|_{t=0} &= W_0 = U_1 + (C|_{t=0})U_0' && \text{in } \mathcal{G}(\mathbb{R}), \end{aligned} \tag{3.2}$$

where $M = C'/(2C) \in \mathcal{G}([0,\infty)\times\mathbb{R})$. If problem (3.2) has a unique solution (V,W), then so does problem (3.1). To prove the existence of a solution (V,W), let $(v^\varepsilon, w^\varepsilon)$ be the unique $\mathcal{C}^\infty$-solution to the Cauchy problem

$$\begin{aligned} (\partial_t + c^\varepsilon(t)\partial_x)v^\varepsilon &= \mu^\varepsilon(t)v^\varepsilon - \mu^\varepsilon(t)w^\varepsilon, && t>0,\ x\in\mathbb{R},\\ (\partial_t - c^\varepsilon(t)\partial_x)w^\varepsilon &= \mu^\varepsilon(t)w^\varepsilon - \mu^\varepsilon(t)v^\varepsilon, && t>0,\ x\in\mathbb{R},\\ v^\varepsilon|_{t=0} &= v_0^\varepsilon = u_1^\varepsilon - c^\varepsilon(0)(u_0^\varepsilon)', && x\in\mathbb{R},\\ w^\varepsilon|_{t=0} &= w_0^\varepsilon = u_1^\varepsilon + c^\varepsilon(0)(u_0^\varepsilon)', && x\in\mathbb{R}, \end{aligned} \tag{3.3}$$

where $(u_0^\varepsilon)_{\varepsilon\in(0,1]}$, $(u_1^\varepsilon)_{\varepsilon\in(0,1]}$, $(c^\varepsilon)_{\varepsilon\in(0,1]}$ and $(\mu^\varepsilon)_{\varepsilon\in(0,1]}$ are representatives of U_0, U_1, C and M, respectively, such that c^ε is as in the statement and $\mu^\varepsilon = (c^\varepsilon)'/(2c^\varepsilon)$. For the existence of such $(v^\varepsilon, w^\varepsilon)$, see [20]. If we show that $(v^\varepsilon)_{\varepsilon\in(0,1]}$ and $(w^\varepsilon)_{\varepsilon\in(0,1]}$ belong to $\mathcal{E}_M([0,\infty)\times\mathbb{R})$, their equivalence classes in $\mathcal{G}([0,\infty)\times\mathbb{R})$ will form a solution of problem (3.2). To show that the zeroth derivatives of v^ε and w^ε satisfy estimate (2.1), consider the characteristic curves $\gamma_+^\varepsilon(t,x,\tau)$ and $\gamma_-^\varepsilon(t,x,\tau)$ passing through (t,x) at time $\tau = t$ which satisfy

$$\begin{aligned} \partial_\tau\gamma_+^\varepsilon(t,x,\tau) &= c^\varepsilon(\tau), & \gamma_+^\varepsilon(t,x,t) &= x,\\ \partial_\tau\gamma_-^\varepsilon(t,x,\tau) &= -c^\varepsilon(\tau), & \gamma_-^\varepsilon(t,x,t) &= x. \end{aligned}$$

Along these characteristic curves, v^ε and w^ε are respectively calculated as

$$v^\varepsilon(t,x) = v_0^\varepsilon(\gamma_+^\varepsilon(t,x,0)) + \int_0^t \mu^\varepsilon(s)(v^\varepsilon - w^\varepsilon)(s,\gamma_+^\varepsilon(t,x,s))\,ds, \tag{3.4}$$

$$w^\varepsilon(t,x) = w_0^\varepsilon(\gamma_-^\varepsilon(t,x,0)) + \int_0^t \mu^\varepsilon(s)(w^\varepsilon - v^\varepsilon)(s,\gamma_-^\varepsilon(t,x,s))\,ds. \tag{3.5}$$

For each $T > 0$, we define K_T as the trapezoidal region with corners $(0,-\xi)$, $(T,-\xi+c_1T)$, $(T,\xi-c_1T)$, $(0,\xi)$. Using (3.4) and (3.5), we see that

$$\|v^\varepsilon\|_{L^\infty(K_T)} \le \|v_0^\varepsilon\|_{L^\infty(K_0)} + \int_0^T |\mu^\varepsilon(s)|(\|v^\varepsilon\|_{L^\infty(K_s)} + \|w^\varepsilon\|_{L^\infty(K_s)})\,ds,$$

$$\|w^\varepsilon\|_{L^\infty(K_T)} \le \|w_0^\varepsilon\|_{L^\infty(K_0)} + \int_0^T |\mu^\varepsilon(s)|(\|w^\varepsilon\|_{L^\infty(K_s)} + \|v^\varepsilon\|_{L^\infty(K_s)})\,ds.$$

We add these two inequalities and apply Gronwall's inequality to get

$$\|v^\varepsilon\|_{L^\infty(K_T)} + \|w^\varepsilon\|_{L^\infty(K_T)} \le (\|v_0^\varepsilon\|_{L^\infty(K_0)} + \|w_0^\varepsilon\|_{L^\infty(K_0)})\exp\left(2\int_0^T |\mu^\varepsilon(s)|\,ds\right).$$

On the right-hand side, the terms involving v_0^ε and w_0^ε are of order $O(\varepsilon^{-p})$ for some $p \geq 0$. The exponential term is uniformly bounded in ε by the condition (ii) on c^ε. Hence, the zeroth derivatives of v^ε and w^ε satisfy estimate (2.1) on K_T. We can obtain analogous estimates for all derivatives of v^ε and w^ε by differentiating the equations and using the same argument. Thus, $(v^\varepsilon)_{\varepsilon\in(0,1]}$ and $(w^\varepsilon)_{\varepsilon\in(0,1]}$ belong to $\mathcal{E}_M([0,\infty)\times\mathbb{R})$.

For the proof of uniqueness, we only need to obtain the zero-order estimates (by Lemma 1.2.3 in [14]), which follow along the same line as above. □

We remark that condition (ii) in Theorem 3.1 can be weakened to the requirement that $\exp\left(\int_0^\infty |\mu^\varepsilon(t)|\,dt\right) = O(\varepsilon^{-p})$ as $\varepsilon \downarrow 0$ for some $p \geq 0$.

4. Propagation of singularities

In this section we study the phenomenon of propagation of singularities in the generalized solution to problem (1.1) with $c(t) = c_0 + (c_1 - c_0)H(t-1)$, $u_0 \equiv 0$ and $u_1 = \delta$, where $c_0, c_1 > 0$, $c_0 \neq c_1$, H is the Heaviside function and δ is the delta function. Let us begin by regularizing the coefficient and initial data. Let φ be as in Example 2.3 and put

$$c^\varepsilon(t) = (c * \varphi_\varepsilon)(t) = c_0 + (c_1 - c_0)\int_{\mathbb{R}} H(t-1-\varepsilon s)\varphi(s)\,ds.$$

Then $c^\varepsilon \to c$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \downarrow 0$, and the family $(c^\varepsilon)_{\varepsilon\in(0,1]}$ belongs to $\mathcal{E}_M([0,\infty)\times\mathbb{R})$. We define $C \in \mathcal{G}([0,\infty)\times\mathbb{R})$ as the class of $(c^\varepsilon)_{\varepsilon\in(0,1]}$. Then $\operatorname{sing\,supp}_{\mathcal{G}^\infty} C = \{t = 1\}$. We also define $U_0 \equiv 0$ and $U_1 \in \mathcal{G}(\mathbb{R})$ as the class of $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$, where φ_ε again is a mollifier as in Example 2.3. (Actually, the mollifier need not be the same as the one chosen for the regularization of the coefficient c.) Thus we interpret problem (1.1) with $c(t) = c_0 + (c_1 - c_0)H(t-1)$, $u_0 \equiv 0$ and $u_1 = \delta$ as problem (3.1) with C, U_0 and U_1 defined above. The existence and uniqueness of a generalized solution is then ensured by Theorem 3.1. As may be seen from the following theorem, the splitting of the singularities occurs at points of discontinuity of the coefficient (see Figure 1).

Theorem 4.1. *Let C, U_0 and U_1 be as above and let $U \in \mathcal{G}([0,\infty)\times\mathbb{R})$ be the solution to problem* (3.1). *Then it holds that*

$$\begin{aligned}\operatorname{sing\,supp}_{\mathcal{G}^\infty} U = &\left\{(t,x) \mid x = \pm\int_0^t c(s)\,ds,\ t \geq 0\right\} \\ &\cup \left\{(t,x) \mid x = \pm\left(2\int_0^1 c(s)\,ds - \int_0^t c(s)\,ds\right),\ t \geq 1\right\}.\end{aligned} \tag{4.1}$$

Proof. For the sake of presentation, we first assume that $c_1 > c_0$. To show that assertion (4.1) holds, we first calculate the $\mathcal{G}^\infty$-singular support of the solution (V, W) to problem (3.2) with $V_0 = U_1$ and $W_0 \equiv 0$. As may be seen from the proof

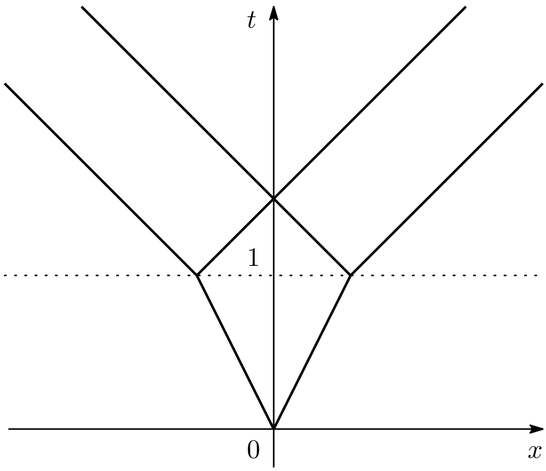

FIGURE 1. The $\mathcal{G}^\infty$-singular support of the solution U

of Theorem 3.1, the solution (V, W) has a representative $(v^\varepsilon, w^\varepsilon)_{\varepsilon\in(0,1]}$, which satisfies the integral equations (3.4) with $v_0^\varepsilon = \varphi_\varepsilon$ and (3.5) with $w_0^\varepsilon = 0$:

$$v^\varepsilon(t,x) = \varphi_\varepsilon(\gamma_+^\varepsilon(t,x,0)) + \int_0^t \mu^\varepsilon(s)(v^\varepsilon - w^\varepsilon)(s,\gamma_+^\varepsilon(t,x,s))\,ds,$$
$$w^\varepsilon(t,x) = \int_0^t \mu^\varepsilon(s)(w^\varepsilon - v^\varepsilon)(s,\gamma_-^\varepsilon(t,x,s))\,ds,$$

where

$$\gamma_+^\varepsilon(t,x,s) = \gamma_+^\varepsilon(t,x,0) + \int_0^s c^\varepsilon(\tau)\,d\tau, \qquad \gamma_+^\varepsilon(t,x,t) = x,$$
$$\gamma_-^\varepsilon(t,x,s) = \gamma_-^\varepsilon(t,x,0) - \int_0^s c^\varepsilon(\tau)\,d\tau, \qquad \gamma_-^\varepsilon(t,x,t) = x.$$

The solution $(v^\varepsilon, w^\varepsilon)$ is obtained by iteration, see [21]. From this, we can check that $v^\varepsilon(t,x) \geq 0$ and $w^\varepsilon(t,x) \leq 0$ for $t \geq 0$ and $x \in \mathbb{R}$. Using this fact, we get

$$v^\varepsilon(t,x) \geq \varphi_\varepsilon(\gamma_+^\varepsilon(t,x,0)) \geq 0, \tag{4.2}$$
$$\begin{aligned} w^\varepsilon(t,x) &\leq -\int_0^t \mu^\varepsilon(s)v^\varepsilon(s,\gamma_-^\varepsilon(t,x,s))\,ds \\ &\leq -\int_0^t \mu^\varepsilon(s)\varphi_\varepsilon(\gamma_+^\varepsilon(s,\gamma_-^\varepsilon(t,x,s),0))\,ds \\ &= -\int_0^t \mu^\varepsilon(s)\varphi_\varepsilon\left(\gamma_-^\varepsilon(t,x,0) - 2\int_0^s c^\varepsilon(\tau)\,d\tau\right)ds \leq 0. \end{aligned} \tag{4.3}$$

Step 1. We first prove that

$$\operatorname{sing\,supp}_{\mathcal{G}^\infty} V = \left\{(t,x) \mid x = \int_0^t c(s)\,ds,\ t \ge 0\right\}. \tag{4.4}$$

It is easy to check that $V \in \mathcal{G}([0,\infty)\times\mathbb{R})$ vanishes off $\{(t,x) \mid x = \int_0^t c(s)\,ds,\ t \ge 0\}$ and thus is $\mathcal{G}^\infty$-regular there. We will show first that $\{(t,x) \mid x = \int_0^t c(s)\,ds,\ t \ge 0\}$ is contained in $\operatorname{sing\,supp}_{\mathcal{G}^\infty} V$.

Fix $t > 0$ arbitrarily, and consider

$$A_{t,\varepsilon} = \left|\frac{v^\varepsilon(t,\gamma_+^\varepsilon(0,\varepsilon,t)) - v^\varepsilon(t,\gamma_+^\varepsilon(0,0,t))}{\gamma_+^\varepsilon(0,\varepsilon,t) - \gamma_+^\varepsilon(0,0,t)}\right|.$$

By inequality (4.2), we have $v^\varepsilon(t,\gamma_+^\varepsilon(0,0,t)) \ge \varphi(0)/\varepsilon > 0$. Noting that $v^\varepsilon(t,x) = 0$ for $x \ge \gamma_+^\varepsilon(0,\varepsilon,t)$, we get $v^\varepsilon(t,\gamma_+^\varepsilon(0,\varepsilon,t)) = 0$. Furthermore, $\gamma_+^\varepsilon(0,\varepsilon,t) - \gamma_+^\varepsilon(0,0,t) = \varepsilon$. Hence, we see that $A_{t,\varepsilon} \ge \varphi(0)/\varepsilon^2$. The mean value theorem shows that there exists $x_1^\varepsilon \in (\gamma_+^\varepsilon(0,0,t), \gamma_+^\varepsilon(0,\varepsilon,t))$ such that

$$|\partial_x v^\varepsilon(t,x_1^\varepsilon)| \ge \frac{\varphi(0)}{\varepsilon^2}.$$

Repeat this process to find that for any $n \ge 2$ there exists $x_n^\varepsilon \in (x_{n-1}^\varepsilon, \gamma_+^\varepsilon(0,\varepsilon,t))$ such that

$$|\partial_x^n v^\varepsilon(t,x_n^\varepsilon)| \ge \frac{\varphi(0)}{\varepsilon^{n+1}}.$$

Since $x_n^\varepsilon \to \int_0^t c(s)\,ds$ as $\varepsilon \downarrow 0$, assertion (4.4) holds.

Step 2. We next prove that

$$\operatorname{sing\,supp}_{\mathcal{G}^\infty} W = \left\{(t,x) \mid x = 2\int_0^1 c(s)\,ds - \int_0^t c(s)\,ds,\ t \ge 1\right\}. \tag{4.5}$$

The proof is similar. We can easily check that W equals 0 outside $\{(t,x) \mid x = 2\int_0^1 c(s)\,ds - \int_0^t c(s)\,ds,\ t \ge 1\}$ and thus is $\mathcal{G}^\infty$-regular there. We will show below that $\{(t,x) \mid x = 2\int_0^1 c(s)\,ds - \int_0^t c(s)\,ds,\ t \ge 1\} \subset \operatorname{sing\,supp}_{\mathcal{G}^\infty} W$.

Fix $t > 1$ arbitrarily. Let $\varepsilon < t-1$, and consider

$$B_{t,\varepsilon} = \left|\frac{w^\varepsilon(t,\gamma_-^\varepsilon(1,\int_0^1 c^\varepsilon(\tau)\,d\tau,t)) - w^\varepsilon(t,\gamma_-^\varepsilon(1-\varepsilon,-\varepsilon+\int_0^{1-\varepsilon} c^\varepsilon(\tau)\,d\tau,t))}{\gamma_-^\varepsilon(1,\int_0^1 c^\varepsilon(\tau)\,d\tau,t) - \gamma_-^\varepsilon(1-\varepsilon,-\varepsilon+\int_0^{1-\varepsilon} c^\varepsilon(\tau)\,d\tau,t)}\right|.$$

We see that, for $s \ge 0$,

$$\gamma_-^\varepsilon\left(1,\int_0^1 c^\varepsilon(\tau)\,d\tau,s\right) = 2\int_0^1 c^\varepsilon(\tau)\,d\tau - \int_0^s c^\varepsilon(\tau)\,d\tau. \tag{4.6}$$

Using this, inequality (4.3) and $\operatorname{supp}\mu^\varepsilon \subset [1-\varepsilon,1+\varepsilon]$, we get

$$w^\varepsilon\left(t,\gamma_-^\varepsilon\left(1,\int_0^1 c^\varepsilon(\tau)\,d\tau,t\right)\right) \le -\int_{1-\varepsilon}^{1+\varepsilon} \mu^\varepsilon(s)\varphi_\varepsilon\left(2\int_s^1 c^\varepsilon(\tau)\,d\tau\right)ds. \tag{4.7}$$

Choose $a \in (0,2c_1)$ so that $\varphi(a) > 0$. Then, for $1-\varepsilon \le 1-(a\varepsilon)/(2c_1) \le s \le 1+(a\varepsilon)/(2c_1) \le 1+\varepsilon$, we have $|2\int_s^1 c^\varepsilon(\tau)\,d\tau| \le a\varepsilon$. Using this and noting that

$\varphi' \geq 0$ on $[-1, 0]$ and the symmetry of φ, we find that $\varphi_\varepsilon(2\int_s^1 c^\varepsilon(\tau)\,d\tau) \geq \varphi_\varepsilon(a\varepsilon)$ for $1-(a\varepsilon)/(2c_1) \leq s \leq 1+(a\varepsilon)/(2c_1)$. Therefore, by inequality (4.7), we obtain

$$w^\varepsilon\left(t, \gamma_-^\varepsilon\left(1, \int_0^1 c^\varepsilon(\tau)\,d\tau, t\right)\right) \leq -\int_{1-(a\varepsilon)/(2c_1)}^{1+(a\varepsilon)/(2c_1)} \mu^\varepsilon(s)\varphi_\varepsilon(a\varepsilon)\,ds$$
$$= -\frac{1}{2}\log\frac{c^\varepsilon(1+(a\varepsilon)/(2c_1))}{c^\varepsilon(1-(a\varepsilon)/(2c_1))}\cdot\frac{\varphi(a)}{\varepsilon},$$

where

$$\beta = \frac{c^\varepsilon(1+(a\varepsilon)/(2c_1))}{c^\varepsilon(1-(a\varepsilon)/(2c_1))} = \frac{c_0 + (c_1-c_0)\int_{-\infty}^{a/(2c_1)}\varphi(y)\,dy}{c_0 + (c_1-c_0)\int_{-\infty}^{-a/(2c_1)}\varphi(y)\,dy}$$

is independent of ε and greater than 1. Thus, we obtain

$$w^\varepsilon\left(t, \gamma_-^\varepsilon\left(1, \int_0^1 c^\varepsilon(\tau)\,d\tau, t\right)\right) \leq -\frac{1}{2}\log\beta\cdot\frac{\varphi(a)}{\varepsilon} < 0.$$

We also see that, for $s \geq 0$,

$$\gamma_-^\varepsilon\left(1-\varepsilon, -\varepsilon + \int_0^{1-\varepsilon} c^\varepsilon(\tau)\,d\tau, s\right) = -\varepsilon + 2\int_0^{1-\varepsilon} c^\varepsilon(\tau)\,d\tau - \int_0^s c^\varepsilon(\tau)\,d\tau, \quad (4.8)$$

on which $w^\varepsilon \equiv 0$. Furthermore, by (4.6) and (4.8), we have

$$\gamma_-^\varepsilon\left(1, \int_0^1 c^\varepsilon(\tau)\,d\tau, t\right) - \gamma_-^\varepsilon\left(1-\varepsilon, -\varepsilon + \int_0^{1-\varepsilon} c^\varepsilon(\tau)\,d\tau, t\right)$$
$$= \varepsilon + 2\int_{1-\varepsilon}^1 c^\varepsilon(\tau)\,d\tau \leq (1+2c_1)\varepsilon.$$

Hence, $B_{t,\varepsilon} \geq (\log\beta\cdot\varphi(a))/(2(1+2c_1)\varepsilon^2)$. By the mean value theorem, there exists $x_1^\varepsilon \in (\gamma_-^\varepsilon(1-\varepsilon, -\varepsilon + \int_0^{1-\varepsilon} c^\varepsilon(\tau)\,d\tau, t), \gamma_-^\varepsilon(1, \int_0^1 c^\varepsilon(\tau)\,d\tau, t))$ such that

$$|\partial_x w^\varepsilon(t, x_1^\varepsilon)| \geq \frac{\log\beta\cdot\varphi(a)}{2(1+2c_1)\varepsilon^2}.$$

Repeating this process gives that for any $n \geq 2$, there exists $x_n^\varepsilon \in (\gamma_-^\varepsilon(1-\varepsilon, -\varepsilon + \int_0^{1-\varepsilon} c^\varepsilon(\tau)\,d\tau, t), x_{n-1}^\varepsilon)$ such that

$$|\partial_x^n w^\varepsilon(t, x_n^\varepsilon)| \geq \frac{\log\beta\cdot\varphi(a)}{2(1+2c_1)^n\varepsilon^{n+1}}.$$

Since $x_n^\varepsilon \to 2\int_0^1 c(s)\,ds - \int_0^t c(s)\,ds$ as $\varepsilon \downarrow 0$, assertion (4.5) holds.

Step 3. We finally prove that assertion (4.1) holds. Similarly to Steps 1 and 2, we can show that

$$\operatorname{sing\,supp}_{\mathcal{G}^\infty} V = \left\{(t,x) \mid x = -\left(2\int_0^1 c(s)\,ds - \int_0^t c(s)\,ds\right),\ t \geq 1\right\}, \quad (4.9)$$

$$\operatorname{sing\,supp}_{\mathcal{G}^\infty} W = \left\{(t,x) \mid x = -\int_0^t c(s)\,ds,\ t \geq 0\right\}, \quad (4.10)$$

if $V_0 \equiv 0$ and $W_0 = U_1$. By the definitions of V and W, the $\mathcal{G}^\infty$-singular support of U coincides with the union of (4.4), (4.5), (4.9) and (4.10). Thus, assertion (4.1) follows.

The case $c_0 > c_1$ can be treated by the same arguments, using a change of sign in U_1 and invoking the linearity of the equation. The proof of Theorem 4.1 is now complete. □

We remark that assertion (4.1) in Theorem 4.1 still holds when $U_1 \in \mathcal{G}(\mathbb{R})$ is given by the class of $(\varphi_\varepsilon^n)_{\varepsilon\in(0,1]}$ with $n \in \mathbb{N}$, i.e., U_1 is any power of the delta function.

5. Slow scale coefficients and regularity along the refracted ray

We here discuss how the regularity of the coefficient C affects that of the solution U to problem (3.1). As mentioned in Example 2.3, one may regularize the piecewise constant propagation speed $c(t) = c_0 + (c_1 - c_0)H(t-1)$ in such a way that the corresponding element $C \in \mathcal{G}(\mathbb{R})$ belongs to $\mathcal{G}^\infty(\mathbb{R})$. In fact, it suffices to use a mollifier $\varphi_{h(\varepsilon)}$, where $(1/h(\varepsilon))_{\varepsilon\in(0,1]}$ is a positive slow scale net. It has been shown in [18] that a bounded element C of $\mathcal{G}(\mathbb{R})$ belongs to $\mathcal{G}^\infty(\mathbb{R})$ if and only if all derivatives satisfy slow scale bounds. Consider problem (3.1) with $U_0 \equiv 0$ and U_1 given by the class of $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ as in Theorem 4.1. Then as may be seen from the following theorem, the refracted rays

$$\begin{aligned}\Gamma_- &= \left\{(t,x) \mid x = 2\int_0^1 c(s)\,ds - \int_0^t c(s)\,ds,\ t>1\right\}\\ &= \{(t,x) \mid x = c_0 - c_1(t-1),\ t>1\}\\ \Gamma_+ &= \left\{(t,x) \mid x = -2\int_0^1 c(s)\,ds + \int_0^t c(s)\,ds,\ t>1\right\}\\ &= \{(t,x) \mid x = -c_0 + c_1(t-1),\ t>1\}\end{aligned}$$

do not belong to $\operatorname{sing\,supp}_{\mathcal{G}^\infty} U$, if C belongs to $\mathcal{G}^\infty(\mathbb{R})$ (see Figure 2).

Theorem 5.1. *Let C, U_0 and U_1 be as above and let $U \in \mathcal{G}([0,\infty)\times\mathbb{R})$ be the solution to problem* (3.1). *If $C \in \mathcal{G}^\infty(\mathbb{R})$, then*

$$\operatorname{sing\,supp}_{\mathcal{G}^\infty} U = \left\{(t,x) \mid x = \pm\int_0^t c(s)\,ds,\ t\geq 0\right\}.$$

Proof. As can be seen from the proof of Theorem 4.1, it suffices to consider problem (3.2) when $V_0 = U_1$ and $W_0 \equiv 0$:

$$\begin{aligned}(\partial_t + C\partial_x)V &= MV - MW \qquad &&\text{in } \mathcal{G}([0,\infty)\times\mathbb{R}),\\ (\partial_t - C\partial_x)W &= MW - MV \qquad &&\text{in } \mathcal{G}([0,\infty)\times\mathbb{R}),\\ V|_{t=0} = U_1, \quad W|_{t=0} &= 0 &&\text{in } \mathcal{G}(\mathbb{R}).\end{aligned} \tag{5.1}$$

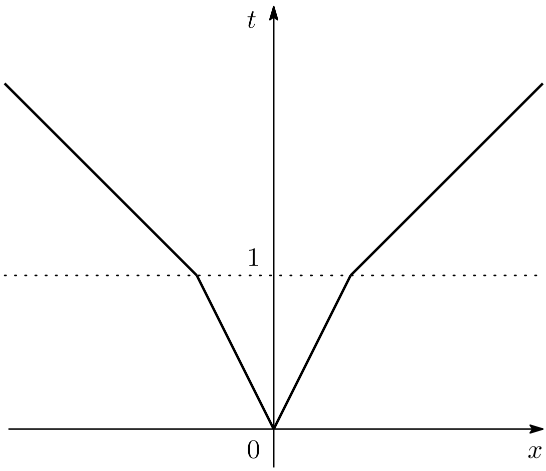

FIGURE 2. The $\mathcal{G}^\infty$-singular support of the solution U with slow scale coefficient

The same argument as in the proof of Theorem 4.1 shows that

$$\operatorname{sing\,supp}_{\mathcal{G}^\infty} V = \left\{ (t,x) \mid x = \int_0^t c(s)\,ds,\ t \geq 0 \right\}$$

holds. Thus, it remains to show that W is $\mathcal{G}^\infty$-regular in a neighborhood of Γ_-.

A simple calculation shows that the commutator of $\partial_t - C(t)\partial_x$ and $\partial_t + C(t)\partial_x$ is given by

$$[\partial_t - C(t)\partial_x, \partial_t + C(t)\partial_x] = 2C'(t)\partial_x.$$

Using this commutator relation and the second line of system (5.1), we have

$$\begin{aligned}(\partial_t - C\partial_x)(\partial_t + C\partial_x)W &= (\partial_t + C\partial_x)(\partial_t - C\partial_x)W + 2C'\partial_x W \\ &= (\partial_t + C\partial_x)\big(M(W-V)\big) + 2C'\partial_x W \\ &= M(\partial_t + C\partial_x)W - M(\partial_t + C\partial_x)V + M'(W-V) + 2C'\partial_x W.\end{aligned}$$

From the first line of system (5.1),

$$(\partial_t + C\partial_x)V = M(V-W).$$

Further,

$$\partial_x W = \frac{1}{2C}\Big((\partial_t + C\partial_x)W - (\partial_t - C\partial_x)W\Big) = \frac{1}{2C}\Big((\partial_t + C\partial_x)W + M(V-W)\Big).$$

Together with $C'/(2C) = M$ this leads to

$$(\partial_t - C\partial_x)(\partial_t + C\partial_x)W = 3M(\partial_t + C\partial_x)W + (M^2 - M')(V-W).$$

Let K_T be as in Theorem 4.1. Then the representatives $(v^\varepsilon, w^\varepsilon)$ of (V,W) are of order ε^{-N} on K_T for some N. Further, M and its derivatives are slow scale by assumption; hence $|(\mu^\varepsilon)^2 - (\mu^\varepsilon)'| \leq \lambda_1(\varepsilon)$ for some slow scale net $(\lambda_1(\varepsilon))_{\varepsilon\in(0,1]}$.

From integration along characteristics and Gronwall's inequality as in the proof of Theorem 3.1, we get

$$\|(\partial_t + c^\varepsilon \partial_x) w^\varepsilon\|_{L^\infty(K_T)} \leq \lambda_1(\varepsilon)\varepsilon^{-N} T \exp\left(\int_0^T 3|\mu^\varepsilon(s)|\,ds\right) = \mathcal{O}(\lambda_1(\varepsilon)\varepsilon^{-N})$$

since $\int_0^T |\mu^\varepsilon(s)|\,ds$ is bounded by construction. Recursively we find that

$$(\partial_t - C\partial_x)(\partial_t + C\partial_x)^k W = (2k+1)M(\partial_t + C\partial_x)^k W + L(V, W, M, C)$$

where L depends linearly on V, W, $(\partial_t + C\partial_x)W, \ldots, (\partial_t + C\partial_x)^{k-1}W$ and may contain derivatives of C, M and their products and powers. The linearity of L in the first variables and the slow scale property of C, M and their derivatives yield by induction that

$$\|(\partial_t + c^\varepsilon \partial_x)^k w^\varepsilon\|_{L^\infty(K_T)} = \mathcal{O}(\lambda_k(\varepsilon)\varepsilon^{-N})$$

for some slow scale net $(\lambda_k(\varepsilon))_{\varepsilon\in(0,1]}$.

For the remainder of the proof we may use the fact – already shown in Theorem 4.1 – that V vanishes on Γ_-, hence

$$(\partial_t - C\partial_x)W = MW, \quad (\partial_t - C\partial_x)^2 W = M(\partial_t - C\partial_x)W + M'W, \ldots$$

along Γ_-. Again the higher-order directional derivatives of W in the direction $\partial_t - C\partial_x$ can be estimated inductively and bounded by some slow scale net times ε^{-N}. Similarly, all mixed derivatives $(\partial_t \pm C\partial_x)^k(\partial_t \mp C\partial_x)^\ell W$ can be estimated by reduction to previously computed terms. In conclusion, all derivatives of W can be bounded by some slow scale net times ε^{-N}, hence are of order $\mathcal{O}(\varepsilon^{-N-1})$. Thus W is $\mathcal{G}^\infty$ along Γ_-. □

We remark that the result can be generalized in various ways. For example, the integral bound on $|\mu^\varepsilon|$ can be replaced by the requirement that the net $\exp\left(\int_0^\infty |\mu^\varepsilon(t)|\,dt\right)$ is slow scale. The proof is expected to go through for arbitrary initial data U_1 whose $\mathcal{G}^\infty$-singular support is $\{0\}$.

6. Associated distributions

We consider again the wave equation in one space dimension with propagation speed $c = c(t)$ depending on time, i.e.,

$$\partial_t^2 u - c(t)^2 \partial_x^2 u = 0 \tag{6.1}$$

with initial conditions

$$u|_{t=0} = u_0, \qquad \partial_t u|_{t=0} = u_1. \tag{6.2}$$

If $c(t)$ is piecewise constant, say $c(t) = c_0 + (c_1 - c_0)H(t-1)$, and $u_0, u_1 \in \mathcal{D}'$, this problem has a unique solution

$$u \in \mathcal{C}^1([0,\infty) : \mathcal{D}'(\mathbb{R})) \cap \big(\mathcal{C}^2((0,1) : \mathcal{D}'(\mathbb{R})) \oplus \mathcal{C}^2((1,\infty) : \mathcal{D}'(\mathbb{R}))\big). \tag{6.3}$$

Thus the problem is interpreted as a transmission problem across a discontinuity at $t = 1$; the transmission condition is that the solution should be a continuously differentiable function of time with values in $\mathcal{D}'(\mathbb{R})$. It is simply obtained by solving the wave equation for $t < 1$ and for $t > 1$ and taking the terminal values at $t = 1$ as initial values for $t > 1$.

On the other hand, imbedding u_0, u_1 and c into the Colombeau algebra by convolution with suitable compactly supported mollifiers (concerning c, we use a mollifier as in Example 2.3), problem (6.1), (6.2) has a unique solution $U \in \mathcal{G}([0,\infty)\times\mathbb{R})$. We are going to show that the Colombeau solution is associated with the piecewise distributional solution.

Theorem 6.1. *Let $c(t)$ be a piecewise constant, strictly positive function and let $u_0, u_1 \in \mathcal{D}'(\mathbb{R})$. Then the corresponding generalized solution $U \in \mathcal{G}([0,\infty)\times\mathbb{R})$ is associated with the piecewise distributional solution u satisfying* (6.3).

Proof. We first assume that $u_0 \in \mathcal{C}^2(\mathbb{R})$ and $u_1 \in \mathcal{C}^1(\mathbb{R})$ and that both have compact support. Fix some $T > 1$. Let $(u^\varepsilon)_{\varepsilon\in(0,1]}$ be a representative of the generalized solution that vanishes for x outside some compact set, independently of t, $0 \le t \le T$. This and the smoothness imply that u^ε belongs to $\mathcal{C}^\infty\big([0,T) : H^\infty(\mathbb{R})\big)$. Thus we can use energy estimates. Multiplying the wave equation $u^\varepsilon_{tt} - c^\varepsilon(t)^2 u^\varepsilon_{xx} = 0$ by u^ε_t and integrating by parts we get

$$\int_{\mathbb{R}} \big(u^\varepsilon_{tt}u^\varepsilon_t + (c^\varepsilon)^2 u^\varepsilon_x u^\varepsilon_{xt}\big)\,dx = 0.$$

Observing that

$$\frac{1}{2}\frac{d}{dt}(c^\varepsilon u^\varepsilon_x)^2 = (c^\varepsilon)^2 u^\varepsilon_x u^\varepsilon_{xt} + c^\varepsilon (c^\varepsilon)'(u^\varepsilon_x)^2$$

we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \big(|u^\varepsilon_t|^2 + (c^\varepsilon)^2|u^\varepsilon_x|^2\big)\,dx = \int_{\mathbb{R}} c^\varepsilon(c^\varepsilon)'|u^\varepsilon_x|^2\,dx$$

for $t \in [0,T)$ and thus the energy estimate

$$\int_{\mathbb{R}} \big(|u^\varepsilon_t|^2 + (c^\varepsilon)^2|u^\varepsilon_x|^2\big)\,dx \le \int_{\mathbb{R}} \big(|u^\varepsilon_1|^2 + (c^\varepsilon)^2|u^\varepsilon_{0x}|^2\big)\,dx + 2\int_0^t |c^\varepsilon(c^\varepsilon)'| \int_{\mathbb{R}} |u^\varepsilon_x|^2\,dx\,dt.$$

By assumption, $c^\varepsilon(c^\varepsilon)'$ is bounded in $L^1(0,T)$, and so Gronwall's inequality shows that $\int_{\mathbb{R}} |u^\varepsilon_x(t,x)|^2\,dx$ and hence also $\int_{\mathbb{R}} |u^\varepsilon_t(t,x)|^2\,dx$ remain bounded on $[0,T]$, uniformly in ε. In particular, the family $(u^\varepsilon)_{\varepsilon\in(0,1]}$ is bounded in $\mathcal{C}\big([0,T) : H^1(\mathbb{R})\big)$ as well as in $\mathcal{C}^1\big([0,T) : L^2(\mathbb{R})\big)$. By construction, the supports of the functions u^ε are contained in a common bounded set. Thus the first property implies that $(u^\varepsilon(\cdot,t))_{\varepsilon\in(0,1]}$ is relatively compact in $L^2(\mathbb{R})$ for every t. The second property shows that $(u^\varepsilon)_{\varepsilon\in(0,1]}$ is an equicontinuous subset of $\mathcal{C}\big([0,T) : L^2(\mathbb{R})\big)$. By Ascoli's theorem, the net $(u^\varepsilon)_{\varepsilon\in(0,1]}$ is relatively compact in $\mathcal{C}\big([0,T) : L^2(\mathbb{R})\big)$.

Taking x-derivatives in equation (6.1) we can apply the same argument to u^ε_x and conclude that both $\int_{\mathbb{R}} |u^\varepsilon_{xt}(t,x)|^2\,dx$ and $\int_{\mathbb{R}} |u^\varepsilon_{xx}(t,x)|^2\,dx$ remain bounded on $[0,T]$, uniformly in ε. By the differential equation (6.1), the same is true of

$\int_{\mathbb{R}} |u^\varepsilon_{tt}(t,x)|^2\,dx$. By the same argument as above, $(u^\varepsilon_t)_{\varepsilon\in(0,1]}$ is relatively compact in $\mathcal{C}\big([0,T) : L^2(\mathbb{R})\big)$ and so $(u^\varepsilon)_{\varepsilon\in(0,1]}$ is relatively compact in $\mathcal{C}^1\big([0,T) : L^2(\mathbb{R})\big)$.

There exists a subsequence $(u^{\varepsilon_k})_k$ such that

$$\lim_{k\to\infty} u^{\varepsilon_k} = \overline{u} \in \mathcal{C}^1\big([0,T) : L^2(\mathbb{R})\big) \subset \mathcal{C}^1([0,T) : \mathcal{D}'(\mathbb{R})).$$

But on every compact subinterval of $(0,1)$ and of $(1,T)$, c^ε is identically equal to c_0 or c_1, respectively, when ε is sufficiently small. This implies that $\overline{u}$ is a distributional solution of the wave equation (6.1) on both strips. Since $\overline{u} \in \mathcal{C}\big([0,T) : D'(\mathbb{R})\big)$, so is $\overline{u}_{xx}$. From the equation, we get that

$$\overline{u} \in \big(\mathcal{C}^2((0,1) : \mathcal{D}'(\mathbb{R})) \oplus \mathcal{C}^2((1,\infty) : \mathcal{D}'(\mathbb{R}))\big).$$

In other words, $\overline{u}$ is the unique piecewise distributional solution to (6.1), (6.2). Consequently, the whole net $(u^\varepsilon)_{\varepsilon\in(0,1]}$ converges to $u = \overline{u}$.

If $u_0 \in \mathcal{C}^2(\mathbb{R})$ and $u_1 \in \mathcal{C}^1(\mathbb{R})$ do not have compact support, we consider an arbitrary rectangle $[-R,R]\times[0,T]$ and take a cut-off function $\chi(x)$ identically equal to 1 on a neighborhood of $[-R - T\max_{0\le t\le T} c(t), R + T\max_{0\le t\le T} c(t)] \times [0,T]$. By the derivation above, the Colombeau solution with initial data χu_0, χu_1 is associated with the piecewise distributional solution with the corresponding initial data. But both solutions coincide with the corresponding solutions without cut-off on the rectangle $[-R,R]\times[0,T]$, by finite propagation speed. This shows that the association result holds without the assumption of compact support.

In the next to last step, take $u_0, u_1 \in \mathcal{D}'(\mathbb{R})$ and assume that they are distributions of finite order. Write $u_0 = u_0^- + u_0^+$, where u_0^- has its support bounded from the right, and u_0^+ has support bounded from the left, and similarly for u_1. There is an integer n such that $u_0^+ * I_n \in \mathcal{C}^2(\mathbb{R})$ and $u_1^+ * I_n \in \mathcal{C}^1(\mathbb{R})$, where I_n is the n-fold convolution of the Heaviside function with itself; a similar assertion holds for $u_0^- * \check{I}_n$ and $u_0^- * \check{I}_n$. We may assume without loss that $u_0^- = u_1^- = 0$. Let u and U be the piecewise distributional solution and the Colombeau solution with initial data u_0, u_1, respectively. Let $\widetilde{u}$ and $\widetilde{U}$ be the piecewise distributional solution and the Colombeau solution with initial data $u_0 * I_n$ and $u_1 * I_n$, respectively. By the previous step, $\widetilde{U}$ is associated with $\widetilde{u}$. But $\partial_x^n \widetilde{U} = U$ and $\partial_x^n \widetilde{u} = u$, since both satisfy the wave equation with initial data u_0, u_1 in the appropriate settings. Therefore, U is associated with u as well.

Finally, if $u_0, u_1 \in \mathcal{D}'(\mathbb{R})$ are arbitrary distributions, we use the fact that they are locally of finite order and argue by cut-off and finite propagation speed as above. □

Remark 6.2. The compactness argument in the first step of the proof can be set up in various spaces. For example, the uniform boundedness of $\int_{\mathbb{R}} |u^\varepsilon_{xt}(t,x)|^2\,dx$ on $[0,T]$ implies the equicontinuity of $(u^\varepsilon)_{\varepsilon\in(0,1]}$ in $\mathcal{C}\big([0,T) : H^1(\mathbb{R})\big) \subset \mathcal{C}\big([0,T) : \mathcal{C}(\mathbb{R})\big)$. Hence $(u^\varepsilon)_{\varepsilon\in(0,1]}$ is actually relatively compact in $\mathcal{C}\big([0,T) : \mathcal{C}(\mathbb{R})\big) = \mathcal{C}\big(\mathbb{R}\times[0,T)\big)$.

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Generalized Solutions of Abstract Stochastic Problems

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Abstract. The Cauchy problem $u'(t) = Au(t) + B\mathbb{W}(t), t \geq 0,\ u(0) = \zeta$ with singular white noise $\mathbb{W}$ and A not necessarily generating a C_0-semigroup is studied in spaces of distributions. Spaces of generalized with respect to both time variable t and random variable ω are built. Existence and uniqueness of generalized solutions in the obtained spaces is proved.

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1. Introduction

Modeling of different evolution processes subject to random perturbations gives rise to the Cauchy problems for operator-differential equations with white noise as an inhomogeneity. The basic one among them is the Cauchy problem

$$X'(t) = AX(t) + B\mathbb{W}(t), \quad t \in [0, \tau), \tau \leq \infty, \qquad X(0) = \zeta, \tag{1.1}$$

where A is a linear closed operator in a Hilbert space H, $B \in \mathcal{L}(\mathbb{H}, H)$ and $\{\mathbb{W}(t), t \geq 0\}$ is an $\mathbb{H}$-valued white noise. Because of irregularity of the white noise $\mathbb{W}$ the problem is usually reduced to an integral Itô type equation with the "primitive" of $\mathbb{W}$, i.e., some Wiener process [1, 2]. To make the problem have solutions one has to impose restrictive conditions on A, or consider less restrictive concept of solution (weak, mild, generalized).

In our work we study existence of generalized solutions of (1.1) introducing appropriate spaces of distributions. Our aim is construction of spaces of distributions where the problem has solutions in the case when A is not the generator

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of a C_0-semigroup, i.e., the Cauchy problem for the corresponding deterministic homogeneous equation is not well posed and $\mathbb{W}$ is the singular white noise.

This work unites the following two approaches to the concept of generalized solution of problem (1.1). The first one (see [2, 3]) uses spaces $\mathcal{D}'(L_2(\Omega; H))$ of distributions over $\mathcal{D}$ with values in the space $L_2(\Omega; H)$ of H-valued random variables with finite second moment, where $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space. We call them distributions with respect to t. This approach allows to obtain solutions in the case when A does not generate a C_0-semigroup, but is the generator of an integrated semigroup. It does not allow to consider the equation with the cylindrical white nose as the later does not belong to $\mathcal{D}'(L_2(\Omega; \mathbb{H}))$. One has to replace it with a "regularized" version called Q-white noise. The second approach developed in [4, 5] uses spaces of H-valued stochastic distributions $(\mathcal{S})_{-\rho}(H)$, $\rho \in [0; 1]$, i.e., distributions with respect to random variable $\omega \in \Omega$. One can consider equation with $\mathbb{W}(t)$ being a singular cylindrical white noise since it belongs to $(\mathcal{S})_{-\rho}(\mathbb{H})$ for any $t \in \mathbb{R}$ and is infinitely differentiable, but within this approach solutions of the Cauchy problem can be obtained only in the case when A generates a C_0-semigroup and if the initial value ζ belongs to $(\mathrm{dom} A)$ – the domain of A in $(\mathcal{S})_{-\rho}(H)$.

Taking in account these circumstances we introduce spaces of generalized functions in both t and ω thus making it possible to obtain solutions of the problem (1.1) with the generator of an integrated semigroup with no restrictions on ζ. The spaces of $\mathbb{R}$-valued distributions in both t and ω were introduced in [6]. In our work we build spaces $(\mathbb{S})'(H)$ of H-valued distributions in t and ω and introduce its extension – the space $(\mathbb{S})'^a(H)$ of distributions growing exponentially in t, where we obtain the existence result.

In Section 2 we review the necessary definitions, including definition of spaces $(\mathcal{S})_{-\rho}(H)$, considering for simplicity the case $\rho = 0$. Defining it we use an alternative approach, different from the one used in [4, 5], as it can then be used to define $(\mathbb{S})'(H)$ and $(\mathbb{S})'^a(H)$ which we do in Section 3.

2. Definitions

2.1. Integrated semigroups

Let A be a closed linear operator in a Banach space H.

Definition 2.1. A strongly continuous family $V = \{V(t),\ t \geq 0\}$ of linear bounded operators in H is called an n-times integrated semigroup with the generator A if

$$V(t)A\zeta = AV(t)\zeta,\ \zeta \in \mathrm{dom}\, A, \quad V(t)\zeta = A\int_0^t V(s)\zeta ds + \frac{t^n}{n!}\zeta,\ \zeta \in H\,.$$

The semigroup V is called exponentially bounded if $\|V(t)\| \leq Me^{at}$, $t \geq 0$ for some $M > 0, a \in \mathbb{R}$.

Integrated semigroups were defined via certain "semigroup property" in [7] where the notion of the infinitesimal generator of a semigroup was introduced.

The definition we give here is an equivalent one in the case of densely defined operator A (see details and examples in [8, 9]).

2.2. Wiener processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathbb{H}$ be a Hilbert space and Q be a linear symmetric positive trace class operator with a system of eigenvectors $\{e_i\}$, forming a basis of $\mathbb{H}$, such that $Qe_i = \sigma_i^2 e_i$, $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$.

Definition 2.2. Stochastic process $W_Q = \{W_Q(t), t \geq 0\}$ with values in $\mathbb{H}$ is called a Q-Wiener process, if

- (W1) $W_Q(0) = 0$ a.s.;
- (W2) W_Q has independent increments;
- (W3) the increments $W_Q(t) - W_Q(s)$ are normally distributed with mean zero and covariance operator equal to $(t-s)Q$;
- (W4) the trajectories of W_Q are continuous a.s.

A Q-Wiener process is a generalization of the Brownian motion $\{\beta(t), t \geq 0\}$, where $\beta(t) = \beta(\omega, t), \omega \in \Omega$, which is defined via conditions (W1)–(W4) in the case $\mathbb{H} = \mathbb{R}$ $(Q = I)$. A finite-dimensional Brownian motion has form $\sum_{j=1}^{n} \beta_j(t) e_j$, where $\{e_j\}$ is an orthonormal basis in $\mathbb{R}^n$ and β_j are independent Brownian motions. When passing to infinite dimensions, to avoid divergency in $\mathbb{H}$, one has to consider a regularized sum

$$W_Q(t) := \sum_{j=1}^{\infty} \sigma_j \beta_j(t) e_j, \; t \geq 0 , \qquad W_Q(t) \in L_2(\Omega; \mathbb{H}) ,$$

with $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$, which happens to be an $\mathbb{H}$-valued Q-Wiener process.

The formal series $\sum_{j=1}^{\infty} \beta_j(t) e_j =: W(t)$ is called a *cylindrical Wiener process.*

2.3. Spaces of abstract distributions. White noise in spaces of abstract distributions

For any Banach space $\mathcal{X}$ by $\mathcal{D}'(\mathcal{X})$ we denote the space of all $\mathcal{X}$-valued distributions over the space of test functions $\mathcal{D}$. In contrary to the $\mathbb{R}$-valued Schwartz distributions they are called *abstract distributions.* By $\mathcal{D}'_0(\mathcal{X})$ we denote the subspace of distributions having supports in $[0, \infty)$.

Let $\mathbb{H}$ now be a separable Hilbert space and W_Q be an $\mathbb{H}$-valued Q-Wiener process. Since W_Q is a continuous $L_2(\Omega; \mathbb{H})$-valued function of $t \geq 0$, we can define *Q-white noise* $\mathbb{W}_Q \in \mathcal{D}'_0(L_2(\Omega; \mathbb{H}))$ as the generalized derivative of W_Q continued as zero for $t < 0$, i.e., by the following equality:

$$\langle \mathbb{W}_Q, \theta \rangle := -\int_0^{\infty} W_Q(t) \theta'(t)\, dt = \int_0^{\infty} \theta(t)\, dW_Q(t) , \quad \theta \in \mathcal{D} . \qquad (2.1)$$

The first integral in (2.1) is understood as Bochner integral of an $L_2(\Omega; \mathbb{H})$-valued function, the second one – as an abstract Itô integral with respect to the Wiener process. The equality of the integrals follows from the Itô formula.

We will use convolution of distributions defined as follows (see [10]).

Definition 2.3. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces, such that there exists a continuous bilinear operation $(u,v)\mapsto uv\in\mathcal{Z}$ defined on $\mathcal{X}\times\mathcal{Y}$. For any $G\in\mathcal{D}_0'(\mathcal{X})$ and $F\in\mathcal{D}_0'(\mathcal{Y})$ the convolution $G*F\in\mathcal{D}_0'(\mathcal{Z})$ is defined by the equality

$$\langle G*F\,,\theta\rangle := \big\langle (g*f)^{(n+m)},\theta\big\rangle = (-1)^{n+m}\int_0^\infty (g*f)(t)\theta^{(n+m)}(t)\,dt\,,\quad \theta\in\mathcal{D}\,,$$

where $g:\mathbb{R}\to\mathcal{X}$, $f:\mathbb{R}\to\mathcal{Y}$ are continuous functions such that

$$\langle G,\theta\rangle = (-1)^n\int_0^\infty g(t)\theta^{(n)}(t)\,dt,\ \langle F,\theta\rangle = (-1)^m\int_0^\infty f(t)\theta^{(m)}(t)\,dt,$$

$(g*f)(t) := \int_0^t g(t-s)f(s)ds\,.$

This definition uses the fact that all distributions in $\mathcal{D}_0'(\mathcal{X})$ have locally (on any bounded interval in $\mathbb{R}$) a finite order, i.e., can be represented as a generalized derivative of some order of a continuous $\mathbb{H}$-valued function with support in $[0;\infty)$. In case of space $\mathcal{S}_0'(\mathcal{X})$ and the subspace $\mathcal{S}_0'^{a}(\mathcal{X})$ consisting of exponentially (with type a) bounded elements of $\mathcal{D}_0'(\mathcal{X})$ this fact holds true globally.

Note that in the particular case when G is a regular distribution, i.e., $\langle G,\theta\rangle = \int_0^\infty G(t)\theta(t)\,dt$, it holds $\langle G*F,\theta\rangle = \int_0^\infty G(t)\langle F(\cdot),\theta(t+\cdot)\rangle\,dt$.

2.4. Spaces of abstract stochastic distributions. Singular white noise

The theory of stochastic distributions uses the white noise probability space. It is the triple $(\mathcal{S}',\mathcal{B}(\mathcal{S}'),\mu)$, where $\mathcal{B}(\mathcal{S}')$ is the Borel σ-field of $\mathcal{S}'$ – Schwartz space of tempered distributions, μ is the centered Gaussian, or white noise measure, on $\mathcal{B}(\mathcal{S}')$ satisfying the equality

$$\int_{\mathcal{S}'} e^{i\langle\omega\,,\,\theta\rangle}d\mu(\omega) = e^{-\frac{1}{2}|\theta|_0^2}\,,\quad \theta\in\mathcal{S}$$

where $|\cdot|_0$ is the norm of $L_2(\mathbb{R})$. Existence of such measure is stated by the Bochner–Minlos theorem (see, e.g., [11, 12]).

Construction of spaces of *abstract stochastic distributions* [11, 12]) is analogous to the construction of the Gelfand triple $\mathcal{S}\subset L_2(\mathbb{R})\subset\mathcal{S}'$. Its central element is the space (L^2) of all functions of $\omega\in\mathcal{S}'$ which are square integrable with respect to measure μ. Hermite functions $\xi_k(x) = \pi^{-\frac{1}{4}}\big((k-1)!\big)^{-\frac{1}{2}}\ e^{-\frac{x^2}{2}}h_{k-1}(x)$ (where $h_k(x) = (-1)^k e^{\frac{x^2}{2}}(d/dx)^k e^{-\frac{x^2}{2}}$, are Hermite polynomials) are the eigenfunctions of the differential operator $\hat{D} = -\frac{d^2}{dt^2}+t^2+1$ with $\hat{D}\xi_k = (2k)\xi_k\,, k\in\mathbb{N}$ and form an orthonormal basis of $L_2(\mathbb{R})$. Stochastic Hermite polynomials $\mathbf{h}_\alpha(\omega) := \prod_k h_{\alpha_k}(\langle\omega\,,\,\xi_k\rangle)\,,\omega\in\mathcal{S}'$, where $\alpha\in\mathcal{T}$ (the set of all finite multi-indices) form an orthogonal basis of (L^2) with

$$(\mathbf{h}_\alpha\,,\,\mathbf{h}_\beta)_{(L^2)} = \alpha!\delta_{\alpha,\beta}\,,\qquad \alpha! := \prod_k \alpha_k!\,.$$

They are the eigenfunctions of the second quantization operator $\Gamma(\hat{D})$. It holds $\Gamma(\hat{D})\mathbf{h}_\alpha = \prod_k (2k)^{\alpha_k}\mathbf{h}_\alpha =: (2\mathbb{N})^\alpha\mathbf{h}_\alpha$.

The space of test functions $(\mathcal{S})$ is a countably Hilbert space $(\mathcal{S}) = \cap_{p\in\mathbb{N}}(\mathcal{S}_p)$ with the projective limit topology, where

$$(\mathcal{S}_p) = \Big\{\varphi = \sum_{\alpha\in\mathcal{T}} \varphi_\alpha \mathbf{h}_\alpha \in (L^2) \,:\, \sum_{\alpha\in\mathcal{T}} \alpha!|\varphi_\alpha|^2(2\mathbb{N})^{2p\alpha} < \infty\Big\}$$

with the norm $|\cdot|_p$, generated by the scalar product

$$(\varphi,\psi)_p = (\Gamma(\hat{D})^p\varphi, \Gamma(\hat{D})^p\psi)_{(L^2)} = \sum_{\alpha\in\mathcal{T}} \alpha!\varphi_\alpha\psi_\alpha(2\mathbb{N})^{2p\alpha}\,.$$

Its adjoint space $(\mathcal{S})'$ is called the space of stochastic (Hida) distributions (random variables). We have $(\mathcal{S})' = \cup_{p\in\mathbb{N}}(\mathcal{S}_{-p})$ with the inductive limit topology, where $(\mathcal{S}_{-p})$ is the adjoint of $(\mathcal{S}_p)$. The space $(\mathcal{S}_{-p})$ can be identified with the space of all formal expansions $\Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha$, satisfying $\sum_{\alpha\in\mathcal{T}} \alpha!|\Phi_\alpha|^2(2\mathbb{N})^{-2p\alpha} < \infty\,$, with scalar product

$$(\Phi,\Psi)_{-p} = (\Gamma(\hat{D})^{-p}\varphi, \Gamma(\hat{D})^{-p}\psi)_{(L^2)} = \sum_{\alpha\in\mathcal{T}} \alpha!\Phi_\alpha\Psi_\alpha(2\mathbb{N})^{-2p\alpha}\,.$$

Denote the corresponding norm by $|\cdot|_{-p}$. We have:

$$\langle\Phi,\varphi\rangle = \sum_{\alpha\in\mathcal{T}} \alpha!\Phi_\alpha\varphi_\alpha \;\text{ for }\; \Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha\mathbf{h}_\alpha \in (\mathcal{S})'\,,\; \varphi = \sum_{\alpha\in\mathcal{T}} \varphi_\alpha\mathbf{h}_\alpha \in (\mathcal{S})\,.$$

Thus we have the following Gelfand triple:

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})'\,.$$

Define $(\mathcal{S})'(\mathbb{H})$, the space of $\mathbb{H}$-valued generalized random variables over $(\mathcal{S})$ as the space of linear continuous operators $\Phi : (\mathcal{S}) \to \mathbb{H}$ with the topology of uniform convergence on bounded subsets of $(\mathcal{S})$. Denote the action of $\Phi \in (\mathcal{S})'(\mathbb{H})$ on $\varphi \in (\mathcal{S})$ by $\Phi[\varphi]$. The structure of $(\mathcal{S})'(\mathbb{H})$ is due to the next proposition (see the proof in [13]).

Proposition 2.4. *Any* $\Phi \in (\mathcal{S})'(\mathbb{H})$ *can be extended to a bounded operator from* $(\mathcal{S}_p)$ *to* $\mathbb{H}$ *for some* $p \in \mathbb{N}$.

The space $(\mathcal{S})$ is a nuclear countably Hilbert space since for any $p \in \mathbb{N}$ the embedding $I_{p,p+1} : (\mathcal{S}_{p+1}) \hookrightarrow (\mathcal{S}_p)$ is a Hilbert–Schmidt operator. From this fact and proposition 2.4 it follows

Corollary 2.5. *Any* $\Phi \in (\mathcal{S})'(\mathbb{H})$ *is a Hilbert–Schmidt operator from* $(\mathcal{S}_p)$ *to* $\mathbb{H}$ *for some* $p \in \mathbb{N}$.

For any $\Phi \in (\mathcal{S})'(\mathbb{H})$ denote by Φ_j the linear functional defined on $\varphi \in (\mathcal{S})$ by $\langle\Phi_j,\varphi\rangle := (\Phi[\varphi], e_j)$. Let p be such that Φ is Hilbert–Schmidt from $(\mathcal{S}_p)$ to $\mathbb{H}$. Then all $\Phi_j\,, j \in \mathbb{N}$ belong to the corresponding space $(\mathcal{S}_{-p})$, thus we have

$$\Phi_j = \sum_{\alpha\in\mathcal{T}} \Phi_{\alpha,j}\mathbf{h}_\alpha\,, \quad \sum_{\alpha\in\mathcal{T}} \alpha!|\Phi_{\alpha,j}|^2(2\mathbb{N})^{-2p\alpha} < \infty\,.$$

For the Hilbert–Schmidt norm of $\Phi : (\mathcal{S}_p) \to \mathbb{H}$ we obtain:

$$\|\Phi\|^2_{\mathrm{HS},p} = \sum_{\alpha\in\mathcal{T}} \left\| \Phi\left[\frac{\mathbf{h}_\alpha}{(\alpha!)^{\frac12}(2\mathbb{N})^{p\alpha}}\right]\right\|^2 = \sum_{\alpha\in\mathcal{T}}\sum_{j=1}^{\infty}\left|\left\langle \Phi_j, \frac{\mathbf{h}_\alpha}{(\alpha!)^{\frac12}(2\mathbb{N})^{p\alpha}}\right\rangle\right|^2$$
$$= \sum_{\alpha\in\mathcal{T},j\in\mathbb{N}} \alpha!|\Phi_{\alpha,j}|^2(2\mathbb{N})^{-2p\alpha}\,.$$

Denote by $(\mathcal{S}_{-p})(\mathbb{H})$ the space of all Hilbert–Schmidt operators acting from $(\mathcal{S}_p)$ to $\mathbb{H}$. It is a separable Hilbert space with an orthogonal basis consisting of operators $\mathbf{h}_\alpha \otimes e_j$, $\alpha \in \mathcal{T}$, $j \in \mathbb{N}$, defined by

$$(\mathbf{h}_\alpha \otimes e_j)\varphi := \big(\mathbf{h}_\alpha, \varphi\big)_{(L^2)} e_j\,, \quad \varphi \in (\mathcal{S}_p)\,.$$

It follows from Corollary 2.5 that $(\mathcal{S})'(\mathbb{H}) = \cup_{p\in\mathbb{N}}(\mathcal{S}_{-p})(\mathbb{H})$ and any $\Phi \in (\mathcal{S})'(\mathbb{H})$ has the following decomposition:

$$\Phi = \sum_{j\in\mathbb{N}} \Phi_j e_j = \sum_{\alpha\in\mathcal{T},j\in\mathbb{N}} \Phi_{\alpha,j}(\mathbf{h}_\alpha\otimes e_j) = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha\,,$$

where $\Phi_j = (\Phi[\cdot], e_j) \in (\mathcal{S}_{-p})$ for some $p \in \mathbb{N}$, $\Phi_\alpha = \sum_{j\in\mathbb{N}} \Phi_{\alpha,j} e_j \in \mathbb{H}$. For the corresponding norm we have

$$\|\Phi\|^2_{-p} = \sum_{j\in\mathbb{N}} |\Phi_j|^2_{-p} = \sum_{\alpha\in\mathcal{T},j\in\mathbb{N}} \alpha!|\Phi_{\alpha,j}|^2(2\mathbb{N})^{-2p\alpha} = \sum_{\alpha\in\mathcal{T}} \alpha!\|\Phi_\alpha\|^2(2\mathbb{N})^{-2p\alpha} < \infty\,.$$

We evidently have

$$(\mathcal{S}_{-p_1})(\mathbb{H}) \subseteq (\mathcal{S}_{-p_2})(\mathbb{H}) \quad \text{for} \quad p_1 < p_2\,,$$

and

$$\|\Phi\|_{-p_1} \geq \|\Phi\|_{-p_2} \quad \text{for all } \Phi \in (\mathcal{S}_{-p_1})(\mathbb{H})$$

To define singular white noise in these spaces first define a sequence of independent Brownian motions $\{\beta_j(t)\}$. Let $n : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection with the property $n = n(i,j) \geq ij$. As it was done in [4, 5], we use the Fourier coefficients of the decomposition of Brownian motion $\beta(t)$ in $(L_2)(\mathbb{R})$:

$$\beta(t,\omega) = \langle \omega, \mathbf{1}_{[0,t]}\rangle = \left\langle \omega, \sum_{i=1}^{\infty}\int_0^t \xi_i(s)ds\,\xi_i\right\rangle = \sum_{i=1}^{\infty}\int_0^t \xi_i(s)\,ds\,\mathbf{h}_{\epsilon_i}\,,$$

where $\epsilon_i := (0, 0, \ldots, \underset{i}{1}, 0, \ldots)$. Defining $\beta_j(t) = \sum_{i=1}^{\infty}\int_0^t \xi_i(s)\,ds\,\mathbf{h}_{\epsilon_{n(i,j)}}$, we obtain the next decomposition for the Wiener process $W(t), t \geq 0$:

$$W(t) = \sum_{j=1}^{\infty}\beta_j(t)\,e_j = \sum_{i,j\in\mathbb{N}}\int_0^t \xi_i(s)ds(\mathbf{h}_{\epsilon_{n(i,j)}}\otimes e_j) = \sum_{n=1}^{\infty}\int_0^t \xi_{i(n)}(s)ds(\mathbf{h}_{\epsilon_n}\otimes e_{j(n)}).$$

Its derivative with respect to t is called singular $\mathbb{H}$-valued white noise. It has the following decomposition:

$$\mathbb{W}(t) = \sum_{i,j\in\mathbb{N}} \xi_i(t)(\mathbf{h}_{\epsilon_{n(i,j)}}\otimes e_j) = \sum_{n=1}^{\infty}\xi_{i(n)}(t)(\mathbf{h}_{\epsilon_n}\otimes e_{j(n)}).$$

By the well-known estimates of the Hermite functions:

$$\left\|\int_0^t \xi_i(s)\,ds\,e_j\right\|_{\mathbb{H}}^2 = \left|\int_0^t \xi_i(s)\,ds\right|^2 = O(i^{-\frac{3}{2}})\,, \quad |\xi_i(t)| = O(i^{-1/4}),$$

we obtain

$$\begin{aligned}\|W(t)\|_{-1}^2 &= \sum_{i,j\in\mathbb{N}} \left|\int_0^t \xi_i(s)\,ds\right|^2 \big(2n(i,j)\big)^{-2} \le C \sum_{i,j\in\mathbb{N}} i^{-7/2} j^{-2} < \infty\,,\\ \|\mathbb{W}(t)\|_{-1}^2 &= \sum_{i,j\in\mathbb{N}} |\xi_i(t)|^2 \big(2n(i,j)\big)^{-2} \le C \sum_{i,j\in\mathbb{N}} i^{-5/2} j^{-2} < \infty\,,\end{aligned}$$

implying $W(t) \in (\mathcal{S})'(\mathbb{H})$ and $\mathbb{W}(t) \in (\mathcal{S})'(\mathbb{H})$ for all $t \ge 0$.

3. Spaces of generalized functions in t and ω

To define the spaces of generalized $\mathbb{H}$-valued functions of t and ω we take the tensor product of Hilbert spaces $L_2(\mathbb{R}) \otimes (L^2)$ to be the central element of the corresponding Gelfand triple. The set $\{\xi_i \otimes \mathbf{h}_\alpha\,, i \in \mathbb{N}, \alpha \in \mathcal{T}\}$ is the orthogonal basis of $L_2(\mathbb{R}) \otimes (L^2)$ with

$$\big(\xi_i \otimes \mathbf{h}_\alpha, \xi_k \otimes \mathbf{h}_\beta\big)_{L_2(\mathbb{R})\otimes(L^2)} = \alpha!\delta_{i,k}\delta_{\alpha,\beta}\,, \quad \big[\hat{D} \otimes \Gamma(\hat{D})\big]\xi_i \otimes \mathbf{h}_\alpha = 2i(2\mathbb{N})^\alpha \xi_i \otimes \mathbf{h}_\alpha$$

Denote by $(\mathbb{S}_{p,q})$, where $p, q \in \mathbb{N}$, the subspace of $L_2(\mathbb{R}) \otimes (L^2)$ consisting of all $\theta \in L_2(\mathbb{R}) \otimes (L^2)$ having decomposition

$$\theta = \sum_{i,\alpha} \theta_{i,\alpha}(\xi_i \otimes \mathbf{h}_\alpha) = \sum_\alpha \theta_\alpha \mathbf{h}_\alpha\,, \quad \theta_\alpha = \sum_{i\in\mathbb{N}} \theta_{i,\alpha}\xi_i \in L_2(\mathbb{R}) \tag{3.1}$$

such that

$$\begin{aligned}|\theta|_{p,q}^2 := \|\hat{D}^p \otimes \Gamma(\hat{D})^q\theta\|_{L_2(\mathbb{R})\otimes(L^2)}^2 &= \sum_{i,\alpha} \alpha!|\theta_{i,\alpha}|^2(2i)^{2p}(2\mathbb{N})^{2q\alpha}\\ &= \sum_\alpha \alpha!|\theta_\alpha|_p^2(2\mathbb{N})^{2q\alpha} < \infty\,.\end{aligned}$$

It is a Hilbert space with the scalar product

$$(\theta,\eta)_{p,q} = \big(\hat{D}^p \otimes \Gamma(\hat{D})^q\theta, \hat{D}^p \otimes \Gamma(\hat{D})^q\eta\big)_{L_2(\mathbb{R})\otimes(L^2)}\,. \tag{3.2}$$

We evidently have $(\mathbb{S}_{p_1,q_1}) \subseteq (\mathbb{S}_{p_2,q_2})$ when $p_1 \ge p_2$ and $q_1 \ge q_2$. Denote by $(\mathbb{S})$ the countably Hilbert space $\cap_{p,q}(\mathbb{S}_{p,q})$ with the projective limit topology. It is a nuclear space since for any p and q there exist $p' > p$ and $q' > q$ such that the embedding $(\mathbb{S}_{p',q'}) \hookrightarrow (\mathbb{S}_{p,q})$ is Hilbert–Schmidt.

Let $(\mathbb{S}_{-p,-q})$ be the conjugate of $(\mathbb{S}_{p,q})$. It consists of all formal expansions $\psi = \sum_{i,\alpha} \psi_{i,\alpha}(\xi_i \otimes \mathbf{h}_\alpha)$ with

$$|\psi|_{-p,-q}^2 := \|\hat{D}^{-p} \otimes \Gamma(\hat{D})^{-q}\psi\|_{L_2(\mathbb{R})\otimes(L^2)}^2 = \sum_{i,\alpha} \alpha!|\psi_{i,\alpha}|^2(2i)^{-2p}(2\mathbb{N})^{-2q\alpha} < \infty\,.$$

Denote by $(\mathbb{S})'$ the conjugate of $(\mathbb{S})$. We have $(\mathbb{S})' = \cup_{p,q}(\mathbb{S}_{-p,-q})$ with the inductive limit topology.

Denote by $(\mathbb{S})'(\mathbb{H})$ the space of all linear continuous operators $\Phi : (\mathbb{S}) \to \mathbb{H}$ with the topology of uniform convergence on bounded subsets of $(\mathbb{S})$. A set $M \subset (\mathbb{S})$ is called bounded if for any sequence $\{\varphi_n\} \subseteq M$ and any $\{\varepsilon_n\} \subset \mathbb{R}$ if $\varepsilon_n \to 0$, then $\varepsilon_n \varphi_n \to 0$ in $(\mathbb{S})$. We will denote by $\Phi[\varphi]$ the action of $\Phi \in (\mathbb{S})(\mathbb{H})$ on $\varphi \in (\mathbb{S})$.

Similarly to Proposition 2.4 one can prove the next proposition:

Proposition 3.1. *Any $\Phi \in (\mathbb{S})'(\mathbb{H})$ is a bounded operator from $(\mathbb{S}_{p,q})$ to $\mathbb{H}$ for some $p, q \in \mathbb{N}$.*

Since $(\mathbb{S})$ is nuclear, it follows

Corollary 3.2. *Any $\Phi \in (\mathbb{S})'(\mathbb{H})$ is a Hilbert–Schmidt operator from $(\mathbb{S}_{p,q})$ to $\mathbb{H}$ for some $p, q \in \mathbb{N}$.*

For any $\Phi \in (\mathbb{S})'(\mathbb{H})$ denote by Φ_j the linear functional, defined by $\langle \Phi_j, \varphi\rangle := (\Phi[\varphi], e_j)$ for all $\varphi \in (\mathbb{S})$. Let p and q are such that Φ is Hilbert–Schmidt from $(\mathbb{S}_{p,q})$ to $\mathbb{H}$. Then all Φ_j , $j \in \mathbb{N}$, belong to corresponding space $(\mathbb{S}_{-p,-q})$, and therefore can be represented in the form

$$\Phi_j = \sum_{i\in\mathbb{N},\alpha\in\mathcal{T}} \Phi_{i,\alpha,j}(\xi_i \otimes \mathbf{h}_\alpha)\,, \qquad \sum_{i\in\mathbb{N},\alpha\in\mathcal{T}} \alpha!|\Phi_{i,\alpha,j}|^2(2i)^{-2p}(2\mathbb{N})^{-2q\alpha} < \infty\,.$$

Denote by $\|\Phi\|_{-p,-q}$ the Hilbert–Schmidt norm of $\Phi : (\mathbb{S}_{p,q}) \to \mathbb{H}$. Then we have:

$$\begin{aligned}\|\Phi\|^2_{-p,-q} &= \sum_{i\in\mathbb{N},\alpha\in\mathcal{T}} \left\| \Phi\left[\frac{\xi_i \otimes \mathbf{h}_\alpha}{(\alpha!)^{\frac{1}{2}}(2i)^p(2\mathbb{N})^{q\alpha}}\right]\right\|^2 \\ &= \sum_{i\in\mathbb{N},\alpha\in\mathcal{T}} \sum_{j=1}^{\infty} \left|\left\langle \Phi_j, \frac{\xi_i \otimes \mathbf{h}_\alpha}{(\alpha!)^{\frac{1}{2}}(2i)^p(2\mathbb{N})^{q\alpha}}\right\rangle\right|^2 \\ &= \sum_{i\in\mathbb{N},\alpha\in\mathcal{T},j\in\mathbb{N}} \alpha!|\Phi_{i,\alpha,j}|^2(2i)^{-2p}(2\mathbb{N})^{-2p\alpha}\,.\end{aligned}$$

Denote by $(\mathbb{S}_{-p,-q})(\mathbb{H})$ the space of Hilbert–Schmidt operators acting from $(\mathbb{S}_{p,q})$ to $\mathbb{H}$. It is a separable Hilbert space with the orthogonal basis consisting of operators $\xi_i \otimes \mathbf{h}_\alpha \otimes e_j$, $\alpha \in \mathcal{T}$, $i, j \in \mathbb{N}$, defined by

$$(\xi_i \otimes \mathbf{h}_\alpha \otimes e_j)\theta := \big(\xi_i \otimes \mathbf{h}_\alpha, \theta\big)_{L_2(\mathbb{R})\otimes(L^2)} e_j\,, \quad \varphi \in (\mathbb{S}_{p,q})\,.$$

It follows from Corollary 3.2 that $(\mathbb{S})'(\mathbb{H}) = \cup_{p,q\in\mathbb{N}}(\mathbb{S}_{-p,-q})(\mathbb{H})$, any $\Phi \in (\mathbb{S})'(\mathbb{H})$ can be represented as

$$\begin{aligned}\Phi = \sum_{j\in\mathbb{N}} \Phi_j e_j &= \sum_{i,j\in\mathbb{N},\,\alpha\in\mathcal{T}} \Phi_{i,\alpha,j}(\xi_i \otimes \mathbf{h}_\alpha \otimes e_j) \\ &= \sum_{i\in\mathbb{N},\alpha\in\mathcal{T}} \Phi_{i,\alpha}(\xi_i \otimes \mathbf{h}_\alpha) = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha\,,\end{aligned} \tag{3.3}$$

where $\Phi_j = (\Phi[\cdot], e_j) \in (\mathbb{S}_{-p,-q})$ for some $p, q \in \mathbb{N}$, $\Phi_{i,\alpha} = \sum_{j\in\mathbb{N}} \Phi_{i,\alpha,j}\, e_j \in \mathbb{H}$, $\Phi_\alpha = \sum_{i,j\in\mathbb{N}} \Psi_{i,\alpha,j}(\xi_i \otimes e_j) \in \mathcal{S}_{-p}(\mathbb{H}) \subset \mathcal{S}'(\mathbb{H})$ and we have

$$\|\Phi\|^2_{-p,-q} = \sum_{j\in\mathbb{N}} |\Phi_j|^2_{-p,-q} = \sum_{i\in\mathbb{N},\,\alpha\in\mathcal{T},\,j\in\mathbb{N}} \alpha!|\Phi_{i,\alpha,j}|^2(2i)^{-2p}(2\mathbb{N})^{-2q\alpha}$$
$$= \sum_{i\in\mathbb{N},\,\alpha\in\mathcal{T}} \alpha!\|\Phi_{i,\alpha}\|^2(2i)^{-2p}(2\mathbb{N})^{-2q\alpha} = \sum_{\alpha\in\mathcal{T}} \alpha!\|\Phi_\alpha\|^2_{-p}(2\mathbb{N})^{-2q\alpha} < \infty\,.$$

We evidently have

$$(\mathbb{S}_{-p_1,-q_1})(\mathbb{H}) \subseteq (\mathbb{S}_{-p_2,-q_2})(\mathbb{H}) \quad \text{when} \quad p_1 \le p_2\,,\ q_1 \le q_2$$

and

$$\|\Phi\|_{-p_1,-q_1} \ge \|\Phi\|_{-p_2,-q_2} \quad \text{for all } \Phi \in (\mathbb{S}_{-p_1,-q_1})(\mathbb{H})$$

Example. ($\mathbb{H}$-valued cylindrical Wiener process and singular white noise in $(\mathbb{S})'(\mathbb{H})$) We can consider the cylindrical Wiener process defined in Section 2 as an element of $(\mathbb{S})'(\mathbb{H})$ defined by

$$W = \sum_{k,i,j\in\mathbb{N}} W_{k,i,j}(\xi_k \otimes \mathbf{h}_{\epsilon_{n(i,j)}} \otimes e_j)$$

Where $W_{k,i,j} = \int_{\mathbb{R}} \xi_k(t) \int_{[0\wedge t, 0\vee t]} \xi_i(s)\, ds\, dt$ are the Fourier coefficients of functions $\int_{[0\wedge t,0\vee t]} \xi_i(s)\, ds$ with respect to the basis $\{\xi_k\}$ in $L_2(\mathbb{R})$. We have

$$|W_{k,i,j}| \le \|\xi_k\|_{L_1(\mathbb{R})}\|\xi_i\|_{L_1(\mathbb{R})} = O(k^{-\frac{3}{4}} i^{-\frac{3}{4}})\,,$$

Consequently $\|W\|_{-0,-1} < \infty$. For the $\mathbb{H}$-valued singular white noise we obtain decomposition

$$\mathbb{W} = \sum_{i,j\in\mathbb{N}} (\xi_i \otimes \mathbf{h}_{\epsilon_{n(i,j)}} \otimes e_j)\,,$$

it follows that $\|\mathbb{W}\|_{-1,-1} < \infty$. So, W and $\mathbb{W}$ belong to $(\mathbb{S})'(\mathbb{H})$.

Denote by D_t the operator of differentiation with respect to t. For $\theta = \sum_{i,\alpha} \theta_{i,\alpha}(\xi_i \otimes \mathbf{h}_\alpha) \in (\mathbb{S})$ define

$$D_t\theta = \sum_{i,\alpha} \theta_{i,\alpha}(\xi_i' \otimes \mathbf{h}_\alpha)\,.$$

By the equality $\xi_i' = \sqrt{\frac{i-1}{2}}\xi_{i-1} - \sqrt{\frac{i}{2}}\xi_{i+1}$, $i \in \mathbb{N}$ for Hermite functions it follows

$$D_t\theta = \sum_{i,\alpha} \left(\sqrt{\frac{i}{2}}\theta_{i+1,\alpha} - \sqrt{\frac{i-1}{2}}\theta_{i-1,\alpha} \right) (\xi_i \otimes \mathbf{h}_\alpha)\,.$$

It is easy to see that D_t is well defined and is a continuous operator in $(\mathbb{S})$.

For $\Phi \in (\mathbb{S})'(\mathbb{H})$ we define differentiation with respect to t in a natural way: $D_t\Phi[\theta] := -\Phi[D_t\theta]$, $\theta \in (\mathbb{S})$. One can easily check that the equality $D_tW = \mathbb{W}$ is valid in $(\mathbb{S})'(\mathbb{H})$.

Let $B \in \mathcal{L}(\mathbb{H}, H)$. Define its action in $(\mathbb{S})'(\mathbb{H})$ by

$$B\Phi := \sum_{i\in\mathbb{N},\alpha\in\mathcal{T}} B\Phi_{i,\alpha}(\xi_i \otimes \mathbf{h}_\alpha)$$

for all $\Phi \in (\mathbb{S})'(\mathbb{H})$ given by (3.3). Let A be a linear closed operator in H with domain dom$A$. Denote its domain in $(\mathbb{S})'(H)$ by (dom$A$) and define it as the set of all $\Phi \in (\mathbb{S})'(H)$ having form (3.3) with $\Phi_{i,\alpha} \in \text{dom}A$ for all $i \in \mathbb{N}, \alpha \in \mathcal{T}$ and such that

$$\sum_{i\in\mathbb{N},\, \alpha\in\mathcal{T}} \alpha!\|A\Phi_{i,\alpha}\|^2(2i)^{-2p}(2\mathbb{N})^{-2q\alpha} < \infty$$

for some $p, q \in \mathbb{N}$.

The above-defined space $(\mathbb{S})'(\mathbb{H})$ of $\mathbb{H}$-valued generalized with respect to t and ω functions is not appropriate for solving the Cauchy problem for the equation with the generator of an exponentially bounded integrated semigroup. Now we define a suitable extension of this space.

Denote by $\mathcal{S}^a$ $(a \geq 0)$ the space of all $\phi \in \mathcal{S}$ such that

$$\forall p \in \mathbb{N}\ \exists C_p > 0\ \forall t \in \mathbb{R} \quad |\phi^{(p)}(t)| \leq C_p e^{-a|t|}\,. \tag{3.4}$$

Let $(\mathbb{S})^a$ be the space of all $\theta \in (\mathbb{S})$ having form (3.1) with $\theta_\alpha \in \mathcal{S}^a$ for any $\alpha \in \mathcal{T}$.

Lemma 3.3. *$(\mathbb{S})^a$ is a closed subspace of $(\mathbb{S})$.*

Proof. Let $\theta_n = \sum_\alpha \theta_{n,\alpha}\mathbf{h}_\alpha \in (\mathbb{S})^a$ with $\theta_{n,\alpha} \in \mathcal{S}, n \in \mathbb{N}, \alpha \in \mathcal{T}$, be a sequence, convergent to $\theta = \sum_\alpha \theta_\alpha \mathbf{h}_\alpha$ with $\theta_\alpha \in \mathcal{S}$ in the space $(\mathbb{S})$. We evidently have $\theta_{n,\alpha} \to \theta_\alpha$ in $\mathcal{S}$ for all $\alpha \in \mathcal{T}$. As it is proved in [14] (Chapter IV, Section 2) condition (3.4) is equivalent to the following:

$$\forall p \in \mathbb{N}\ \exists C_p > 0\ \forall k \in \mathbb{N} \quad \|\phi\|_{k,p} = \max_{x\in\mathbb{R}} |x^k \phi^{(p)}(x)| \leq C_p \left(\frac{k}{ae}\right)^k. \tag{3.5}$$

Since this set of norms defines the equivalent topology in $\mathcal{S}$, we have $\|\theta_{n,\alpha}\|_{k,p} \to \|\theta_\alpha\|_{k,p}$ for any k and p. Thus condition (3.5) holds for all θ_α, implying $\theta \in (\mathbb{S})^a$. □

By the properties of nuclear spaces, $(\mathbb{S})^a$ being a closed subspace of a countably Hilbert space is also a nuclear countably Hilbert space under the same set of scalar products. We will denote by $(\mathbb{S}_{p,q})^a$ the completion of $(\mathbb{S})^a$ with respect to the norm, generated by scalar product (3.2). We have $(\mathbb{S})^a = \cap_{p,q}(\mathbb{S}_{p,q})^a$. Denote by $(\mathbb{S})'^a(\mathbb{H})$ the space of all continuous linear operators acting from $(\mathbb{S})^a$ to $\mathbb{H}$. We have $(\mathbb{S})'(\mathbb{H}) = (\mathbb{S})'^0(\mathbb{H}) \subseteq (\mathbb{S})'^a(\mathbb{H})$. It is easy to see that Proposition 3.1 and Corollary 3.2 remain valid for $(\mathbb{S})'^a(\mathbb{H})$. Differentiation in t and action of linear bounded operators and linear closed operators, defined in $\mathbb{H}$, can be defined in $(\mathbb{S})'^a(\mathbb{H})$ in the same manner as it is done above in $(\mathbb{S})'(\mathbb{H})$.

For $\Phi \in (\mathbb{S})'^a(\mathbb{H})$ having form (3.3) we define the support of Φ by supp$\Phi :=$ $\cup_\alpha \text{supp}\Phi_\alpha$ and denote by $(\mathbb{S})_0'^a(\mathbb{H})$ the subspace of all $\Phi \in (\mathbb{S})'^a(\mathbb{H})$ with supp$\Phi \subseteq$

$[0;+\infty)$. Consider the generalized random process $\mathbb{W}_0$ defined by

$$\mathbb{W}_0 = \sum_{k,i,j\in\mathbb{N}} \mathbb{W}^0_{k,i,j}(\xi_k \otimes \mathbf{h}_{\epsilon_{n(i,j)}} \otimes e_j)\,,$$

where $\mathbb{W}^0_{k,i,j} = \int_0^\infty \xi_k(t)\xi_i(t)dt$ is the kth Fourier coefficient of $\xi_i(t)\chi_{[0;\infty)}(t)$. We will call it the white noise with support $[0;\infty)$. It evidently belongs to $(\mathbb{S})'(\mathbb{H})$.

Now let us give the setting of the Cauchy problem (1.1) in the constructed spaces. Let H_A, H be Hilbert spaces. For any $\Phi \in (\mathbb{S})'^a_0(H_A)$ all $\Phi_\alpha \in \mathcal{S}'^a_0(H_A)$ have the same finite order since $\Phi \in (\mathbb{S})^a_{-p,-q}(H_A)$ for some p and q. Let $P \in \mathcal{S}'^a_0(\mathcal{L}(H_A;H))$ – the space of $\mathcal{L}(H_A;H)$-valued distributions over the space of test functions $\mathcal{S}^a$. It also has finite order, therefore we can define the convolution $P*\Phi$ by the equality

$$P*\Phi = \sum_\alpha (P*\Phi_\alpha)\mathbf{h}_\alpha\,,$$

where $P*\Phi_\alpha$ for all $\alpha \in \mathcal{T}$ is understood in the sense of Definition 2.3. The convolution is well defined and $P*\Phi \in \mathbb{S}'^a_0(H)$. For any $\Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha \in (\mathbb{S})'^a(H)$ define

$$\delta_t \otimes \Phi := \sum_{i\in\mathbb{N},\alpha\in\mathcal{T}} \xi_i(t)\Phi_\alpha(\xi_i \otimes \mathbf{h}_\alpha)\,,$$

(using the Fourier series decomposition $\delta_t = \sum_{i\in\mathbb{N}} \xi_i(t)\xi_i$ in $\mathcal{S}'$).

We will call $X \in (\mathbb{S})'^a(H)$ a solution of the Cauchy problem (1.1) with $\zeta \in (\mathcal{S})'(H)$ if it satisfies the equation

$$D_t X - AX = \delta \otimes \zeta + B\mathbb{W}_0\,, \tag{3.6}$$

where $\delta = \delta_0$.

Let H_A further be the space dom$A$ with the scalar product $(x,y)_A = (x,y) + (Ax,Ay)$. Then we have $P := \delta' \otimes I - \delta \otimes A \in \mathcal{S}'^a_0(\mathcal{L}(H_A;H))$. Since the left-hand side of equation (3.6) is equal to $(\delta' \otimes I - \delta \otimes A) * X$, it follows that the problem has a unique solution if and only if the distribution has the convolution inverse, i.e., there exists a distribution $G \in \mathcal{S}'^a_0(\mathcal{L}(H,H_A))$ such that

$$P*G = \delta \otimes I_H \quad \text{and} \quad G*P = \delta \otimes I_{H_A}\,.$$

It is proved in [8] that this is the case when A is the generator of an exponentially bounded integrated semigroup $\{V(t); t \geq 0\}$ in H. Let $\|V(t)\| \leq Me^{at}\,,\ t \geq 0$ and define $\mathcal{V}[\phi]x = \int_0^\infty \phi(t)V(t)x\,dt$, $\phi \in \mathcal{S}^a,\ x \in H$. Then $\mathcal{V} \in \mathcal{S}'^a_0(\mathcal{L}(H,H_A))$ by the properties of integrated semigroups. The convolution inverse of P is given by $G[\phi]x = (-1)^n \int_0^\infty \phi^{(n)}(t)V(t)x\,dt$, $\phi \in \mathcal{S}^a,\ x \in H$ and we get

$$X = G*(\delta\otimes\zeta + B\mathbb{W}_0) = D^n_t V(t)\zeta + D^n_t(V * B\mathbb{W}_0)\,. \tag{3.7}$$

By exponential boundedness of V it belongs to $(\mathbb{S})'^a(H_A)$ and we obtain

Theorem 3.4. *Let A be the generator of an n-times integrated exponentially bounded semigroup $\{V(t); t \geq 0\}$ in H, $\overline{\mathrm{dom}A} = H$, then for any $\zeta = \sum_{\alpha\in\mathcal{T}} \zeta_\alpha \mathbf{h}_\alpha \in (\mathcal{S})'(H)$ there exists a unique $X \in (\mathbb{S})'^a(H_A)$ which is a solution of the problem* (3.6)*. It is given by the formula* (3.7)*.*

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Nonhomogeneous First-order Linear Malliavin Type Differential Equation

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Abstract. In this paper we solve a nonhomogeneous first-order linear equation involving the Malliavin derivative operator with stochastic coefficients by use of the chaos expansion method. We prove existence and uniqueness of a solution in a certain weighted space of generalized stochastic distributions and represent the obtained solution in the Wiener-Itô chaos expansion form.

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1. Introduction

This paper is devoted to study of generalized stochastic processes which have series expansion representation form given in terms of an orthogonal polynomial basis, defined on an infinite-dimensional space. In particular, we focus on a Hilbert space of square integrable processes defined on a Gaussian white noise probability space where the orthogonal basis is constructed using the Hermite polynomials and the Hermite functions. We provide definition of stochastic generalized random variable spaces over a space of square integrable random variables by adding certain weights in the convergence condition of the series expansion. Introduced by Hida (see [1]) and further developed by many authors (see [2], [3], [7], [10], [12] and references therein), white noise analysis was applied to solving different classes of stochastic differential equations ([5], [8], [14]).

This paper deals with the Malliavin derivative, one of three main operators of the Malliavin stochastic calculus, an infinite-dimensional differential calculus of variations in the white noise setting. Recall, the Skorokhod integral represents an extension of the Itô integral from a set of adapted processes to a set of nonanticipating processes. Its adjoint operator is known as the Malliavin derivative. Operators of Malliavin calculus are widely used in solving stochastic differential equations. In particular, Malliavin differential operator found place in stochastic differential equations connected to optimal control problems and problems in financial mathematics. We give a more general definition of the Malliavin derivative than in [10], [11]. We allow values in the Kondratiev space of stochastic distributions $(S)_{-\rho}$, $\rho \in [0,1]$ and thus obtain a larger domain for the derivative operator. For basic results related to the Malliavin derivative we refer to [2], [7], [10], [12] and for its applications we refer to [3], [4], [9], [11], [13].

Furthermore, as a description of the chaos expansion method, we solve a nonhomogeneous linear stochastic differential equations involving the Malliavin derivative. We provide a general method of solving, using the Wiener-Itô chaos decomposition form, also known as the propagator method. This method gives good framework and opportunity for solving many classes of stochastic equations (see [7], [8], [9]). The problem is based on finding an appropriate, large enough space of generalized functions where a solution of a considered equation exists.

The paper is organized in the following manner: In Section 2 we provide the basic notation used throughout the paper, followed by the survey on chaos expansions of generalized stochastic processes and S'-valued generalized stochastic processes. The Malliavin derivative is defined on a set of generalized stochastic processes and the characterization of its domain is stated. In Section 3 we apply the chaos expansion method in order to solve a nonhomogeneous first-order linear Malliavin type differential equation with singular coefficients, represented in the form

$$\mathbb{D}u = c \otimes u + h, \quad Eu = \widetilde{u}_0,$$

for $c \in \mathcal{S}'(\mathbb{R})$, $h \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$, $\widetilde{u}_0 \in X$ and E is the expectation.

2. Notions and notations

Let the basic probability space $(\Omega, \mathcal{F}, P)$ be the Gaussian white noise probability space $(S'(\mathbb{R}), \mathcal{B}, \mu)$, where $S'(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the sigma-algebra generated by the weak topology on Ω and μ denotes the white noise measure given by the Bochner–Minlos theorem. The Bochner–Minlos theorem states the existence of a Gaussian probability measure given by the integral transform of the characteristic function

$$C(\phi) = \int_{S'(\mathbb{R})} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}, \quad \phi \in \mathcal{S}(\mathbb{R}),$$

where $\langle\omega,\phi\rangle$ denotes the usual dual paring between a tempered distribution ω and a rapidly decreasing function ϕ.

Let $\{\xi_k,\ k \in \mathbb{N}\}$ be the family of Hermite functions and $\{h_k,\ k \in \mathbb{N}_0\}$ the family of Hermite polynomials. It is well known that the space of rapidly decreasing functions $S(\mathbb{R}) = \bigcap_{l\in\mathbb{N}_0} S_l(\mathbb{R})$, where $S_l(\mathbb{R}) = \{\varphi = \sum_{k=1}^{\infty} a_k\, \xi_k :\ \|\varphi\|_l^2 = \sum_{k=1}^{\infty} a_k^2 (2k)^l < \infty\}$, $l \in \mathbb{N}_0$, and the space of tempered distributions $S'(\mathbb{R}) =$

$\bigcup_{l\in\mathbb{N}_0} S_{-l}(\mathbb{R})$, where $S_{-l}(\mathbb{R}) = \{f = \sum_{k=1}^{\infty} b_k\, \xi_k \,:\, \|f\|_{-l}^2 = \sum_{k=1}^{\infty} b_k^2 (2k)^{-l} < \infty\}$, $l \in \mathbb{N}_0$.

Let $(L)^2 = L^2(S'(\mathbb{R}), \mathcal{B}, \mu)$, and $H_\alpha(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle\omega, \xi_k\rangle)$, $\alpha \in \mathcal{I}$ be the Fourier–Hermite orthogonal basis of $(L)^2$, where $\mathcal{I}$ denotes the set of sequences of nonnegative integers which have only finitely many nonzero components $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, 0 \ldots)$. In particular, for the kth unit vector $\varepsilon^{(k)} = (0, \ldots, 0, 1, 0, \ldots)$, the sequence of zeros with the number 1 as the kth component, $H_{\varepsilon^{(k)}}(\omega) = \langle\omega, \xi_k\rangle$, $k \in \mathbb{N}$. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$. Let $a = (a_k)_{k\in\mathbb{N}}$, $a_k \geq 1$, $a^\alpha = \prod_{k=1}^{\infty} a_k^{\alpha_k}$, $\frac{a^\alpha}{\alpha!} = \prod_{k=1}^{\infty} \frac{a_k^{\alpha_k}}{\alpha_k!}$ and $(2\mathbb{N}a)^\alpha = \prod_{k=1}^{\infty} (2k\, a_k)^{\alpha_k}$. Note that $\sum_{\alpha\in\mathcal{I}} (2\mathbb{N})^{-p\alpha} < \infty$ if $p > 0$ and $\sum_{\alpha\in\mathcal{I}} a^{-p\alpha} < \infty$ if $p > 1$. Let $\rho \in [0, 1]$.

The space of *Kondratiev stochastic test functions modified by the sequence* a, denoted by $(Sa)_\rho = \bigcap_{p\in\mathbb{N}_0} (Sa)_{\rho,p}$, $p \in \mathbb{N}_0$, is the projective limit of spaces

$$(Sa)_{\rho,p} = \Big\{ f = \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha \in L^2(\mu) \,:\, \|f\|_{(Sa)_{\rho,p}}^2 = \sum_{\alpha\in\mathcal{I}} (\alpha!)^{1+\rho}\, b_\alpha^2\, (2\mathbb{N}\, a)^{p\alpha} < \infty \Big\}.$$

The space of *Kondratiev stochastic generalized functions modified by the sequence* a, denoted by $(Sa)_{-\rho} = \bigcup_{p\in\mathbb{N}_0} (Sa)_{-\rho,-p}$, $p \in \mathbb{N}_0$, is the inductive limit of the spaces

$$(Sa)_{-\rho,-p} = \Big\{ F = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha \,:\, \|F\|_{(Sa)_{-\rho,-p}}^2 = \sum_{\alpha\in\mathcal{I}} (\alpha!)^{1-\rho}\, c_\alpha^2 (2\mathbb{N}\, a)^{-p\alpha} < \infty \Big\}.$$

The action of a generalized function $F \in (Sa)_{-\rho}$ onto a test function $f \in (Sa)_\rho$ is given by $\ll F, f \gg = \sum_{\alpha\in\mathcal{I}} \alpha!\, c_\alpha b_\alpha$. The *generalized expectation* of F is defined as $E_\mu(F) = \ll F, 1 \gg = c_0$. It is considered to be the zero coefficient in the chaos expansion of a generalized function F in orthogonal basis $\{H_\alpha\}_{\alpha\in\mathcal{I}}$. In particular, if $F \in L^2(\mu)$ it coincides with usual expectation.

For $a_k = 1$, $k \in \mathbb{N}$ these spaces reduce to the spaces of Kondratiev stochastic test functions $(S)_\rho$ and the Kondratiev stochastic generalized functions $(S)_{-\rho}$ respectively. For all $\rho \in [0, 1]$ we have a Gel'fand triplet

$$(Sa)_\rho \subseteq L^2(\mu) \subseteq (Sa)_{-\rho}.$$

In particular, the largest space of the Kondratiev stochastic distributions modified by the sequence a is obtained for $\rho = 1$ and is denoted by $(Sa)_{-1}$. In [4] we introduced the Gaussian type of these spaces and solve equations related to them.

2.1. Generalized stochastic processes

Let $I \subset \mathbb{R}$ and X be a Banach space of functions on I endowed with $\|\cdot\|_X$ and X' its dual. The most common examples used in applications are Schwartz spaces $S(\mathbb{R})$ and $S'(\mathbb{R})$, the Sobolev spaces $X = W_0^{1,2}(\mathbb{R})$ and $X' = W^{-1,2}(\mathbb{R})$.

Definition 2.1. Generalized stochastic processes are elements of tensor product space $X \otimes (S)_{-\rho}$.

Theorem 2.1 ([14]). *Let X be a Banach space endowed with $\|\cdot\|_X$. Generalized stochastic processes as elements of $X \otimes (S)_{-\rho}$ have a chaos expansion of the form*

$$u = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes H_\alpha, \quad f_\alpha \in X, \alpha \in \mathcal{I} \tag{1}$$

and there exists $p \in \mathbb{N}_0$ such that

$$||u||^2_{X\otimes(S)_{-\rho,-p}} = \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_X \, (\alpha!)^{1-\rho} \, (2\mathbb{N})^{-p\alpha} < \infty.$$

Remark 2.1. Generalized stochastic processes as elements of $X \otimes (Sa)_{-1}$ have a chaos expansion of the form (1) and there exists $p \in \mathbb{N}_0$ such that

$$\|u\|^2_{X\otimes(Sa)_{-1,-p}} = \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_X \, (2\mathbb{N}\, a)^{-p\alpha} < \infty.$$

Recall that $(S)_{-1}$ is nuclear and thus $(X \otimes (S)_1)' \cong X' \otimes (S)_{-1}$. In a similar manner one can consider processes as elements of $X' \otimes (S)_{-1}$. Note that $X' \otimes (S)_{-1}$ is isomorphic to the space of linear bounded mappings $X \to (S)_{-1}$.

Definition 2.2. Singular generalized stochastic processes are linear and continuous mappings from X into the space of generalized stochastic functions $(S)_{-1}$, i.e., elements of $\mathcal{L}(X, (S)_{-1})$.

Theorem 2.2 ([14]). *Let $X = \bigcap_{k=0}^\infty X_k$ be a nuclear space endowed with a family of seminorms $\{\|\cdot\|_k; k \in \mathbb{N}_0\}$ and let $X' = \bigcup_{k=0}^\infty X_{-k}$ be its topological dual. Singular generalized stochastic processes as elements of $X' \otimes (S)_{-\rho}$ have a chaos expansion of the form*

$$u = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes H_\alpha, \quad f_\alpha \in X_{-k}, \, \alpha \in \mathcal{I},$$

where $k \in \mathbb{N}_0$ does not depend on $\alpha \in \mathcal{I}$, and there exists $p \in \mathbb{N}_0$ such that

$$||u||^2_{X'\otimes(S)_{-\rho,-p}} = \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_{-k} \, (\alpha!)^{1-\rho} \, (2\mathbb{N})^{-p\alpha} < \infty.$$

With the same notation as in (1) we will denote by $Eu = f_{(0,0,0,\dots)}$ the generalized expectation of the process u.

Example 2.1. Brownian motion is an element of $C^\infty(\mathbb{R}) \otimes (L)^2$ and it is defined by the chaos expansion $B_t(\omega) = \sum_{k=1}^\infty \int_0^t \xi_k(s) ds \, H_{\varepsilon^{(k)}}(\omega)$. Singular white noise $W_t(\cdot)$ is defined by the chaos expansion $W_t(\omega) = \sum_{k=1}^\infty \xi_k(t) H_{\epsilon^{(k)}}(\omega)$, and it is an element of the space $C^\infty(\mathbb{R}) \otimes (S)_{-1,-p}$ for $p > \frac{5}{12}$ and for all t. It is integrable and the relation $\frac{d}{dt} B_t = W_t$ holds in the $(S)_{-1}$ sense (see [2]).

2.2. Schwartz space-valued generalized random processes

In [15] and [16] a general setting of S'-valued generalized random process is provided. $S'(\mathbb{R})$-valued generalized random processes are elements of $\widetilde{X}\otimes(S)_{-\rho}$, where $\widetilde{X} = X \otimes S'(\mathbb{R})$, and are given by chaos expansions of the form

$$f = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \otimes H_\alpha = \sum_{\alpha\in\mathcal{I}} b_\alpha \otimes H_\alpha = \sum_{k\in\mathbb{N}} c_k \otimes \xi_k,$$

where $b_\alpha = \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \in X \otimes S'(\mathbb{R})$, $c_k = \sum_{\alpha\in\mathcal{I}} a_{\alpha,k} \otimes H_\alpha \in X \otimes (S)_{-\rho}$ and $a_{\alpha,k} \in X$. Thus, for some $p, l \in \mathbb{N}_0$,

$$\|f\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-\rho,-p}} = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \|a_{\alpha,k}\|^2_X \,(\alpha!)^{1-\rho}(2k)^{-l}(2\mathbb{N})^{-p\alpha} < \infty.$$

2.3. The Malliavin derivative within chaos expansion

We provide now the definition of the Malliavin derivative which is an extension of the classical definition of this operator from the space of random processes to the space of generalized stochastic processes ([8], [10], [12]).

Definition 2.3. Let a generalized stochastic process $u \in X \otimes (S)_{-\rho}$ be of the form (1). If there exists $p \in \mathbb{N}_0$ such that

$$\sum_{\alpha\in\mathcal{I}} |\alpha|^{1+\rho}(\alpha!)^{1-\rho}\|f_\alpha\|^2_X(2\mathbb{N})^{-p\alpha} < \infty \tag{2}$$

then the Malliavin derivative of u is defined by

$$\mathbb{D}u = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k \, f_\alpha \otimes \xi_k \otimes H_{\alpha-\epsilon^{(k)}}.$$

Operator $\mathbb{D}$ is also called the stochastic gradient of a generalized stochastic process u. The set of processes u such that (2) is satisfied is the domain of the Malliavin derivative and is denoted by $Dom(\mathbb{D})_{-\rho}$. A process $u \in Dom(\mathbb{D})_{-\rho}$ is called Malliavin differentiable process.

Theorem 2.3. *The Malliavin derivative of a process $u \in X \otimes (S)_{-\rho}$ is a linear and continuous mapping*

$$\mathbb{D}: \; Dom(\mathbb{D})_{-\rho,-p} \subseteq X \otimes (S)_{-\rho,-p} \to X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-\rho,-p},$$

for $l > p+1$ and $p \in \mathbb{N}_0$.

Proof. We use the property $(\alpha-\varepsilon^{(k)})! = \frac{\alpha!}{\alpha_k}$, for $k \in \mathbb{N}$ in the proof of this theorem. Assume that a generalized process u is of the form (1) such that it satisfies (2) for some $p \geq 0$. Then we have

$$\begin{aligned}
&\|\mathbb{D}u\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-\rho,-p}} \\
&\quad= \sum_{\alpha\in\mathcal{I}} \Big\| \sum_{k\in\mathbb{N}} \alpha_k\, f_\alpha \otimes \xi_k \Big\|^2_{X\otimes(S)_{-l}(\mathbb{R})} (2\mathbb{N})^{-p\alpha+p\varepsilon^{(k)}} (\alpha-\varepsilon^{(k)})!^{1-\rho} \\
&\quad\leq \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} \alpha_k^2 (\alpha-\varepsilon^{(k)})!^{1-\rho} \|f_\alpha\|^2_X (2\mathbb{N})^{-p(\alpha-\varepsilon^{(k)})} (2k)^{-l} \\
&\quad= \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} \alpha_k^2 \left(\frac{\alpha!}{\alpha_k}\right)^{1-\rho} \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} (2k)^{-(l-p)} \\
&\quad\leq C \sum_{\alpha\in\mathcal{I}} \left(\sum_{k=1}^{\infty} \alpha_k\right)^{1+\rho} (\alpha!)^{1-\rho} \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} \\
&\quad= C \sum_{\alpha\in\mathcal{I}} |\alpha|^{1+\rho} (\alpha!)^{1-\rho} \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty,
\end{aligned}$$

where $C = \sum_{k=1}^{\infty} (2k)^{-(l-p)} < \infty$ for $l > p+1$. □

When $\rho = 1$ the result of the previous theorem reduces to the corresponding one in [4].

3. Nonhomogeneous first-order linear equation

We consider now a nonhomogeneous linear Malliavin differential type equation

$$\begin{cases} \mathbb{D}u = c \otimes u + h, \\ Eu = \widetilde{u}_0, \end{cases} \tag{3}$$

where $c \in S'(\mathbb{R})$, h is a S'-valued generalized stochastic process and $\widetilde{u}_0 \in X$.

Note that in a special case for $h = 0$ the equation (3) reduces to the corresponding homogeneous equation $\mathbb{D}u = c \otimes u$ satisfying condition $Eu = \widetilde{u}_0$. To be precise, in this case we obtain the generalized eigenvalue problem for the Malliavin derivative operator, which was solved in [4]. Moreover, it was proved that in a special case, obtained solution coincide with the stochastic exponential. Additionally, putting $c = 0$, the initial equation (3) transforms into the first-order differential equation with the Malliavin derivative operator $\mathbb{D}u = h$, $Eu = \widetilde{u}_0$, which was recently solved in [6].

The method we will use to solve this equation is a very general and useful tool of Wiener-Itô chaos expansions, also known as the propagator method. With this method we reduce the stochastic differential equation to an infinite system of deterministic equations, which can be solved by induction on length of multi-index. Summing up all coefficients of the expansion and proving convergence in an appropriate weight space, one obtains the solution of the initial stochastic differential equation. This method is applied in several papers: [4], [5], [6], [7], [8], [9], [14], [15].

Denote by $r = r(\alpha) = \min\{k \in \mathbb{N} : \alpha_k \neq 0\}$, for nonzero multi-index $\alpha \in \mathcal{I}$. Then the first nonzero component of α is the rth component α_r, i.e., $\alpha = (0,0,\ldots,0,\alpha_r,\ldots,\alpha_m,0,0,\ldots)$. Denote by $\alpha_{\varepsilon^{(r)}}$ the multi-index with all components equal to the corresponding components of α, except the rth, which is $\alpha_r - 1$. We call $\alpha_{\varepsilon^{(r)}}$ the *representative* of α and write

$$\alpha = \alpha_{\varepsilon^{(r)}} + \varepsilon^{(r)}, \quad \alpha \in \mathcal{I},\ |\alpha| > 0.$$

Note that $\alpha_{\varepsilon^{(r)}}$ is of the length $|\alpha| - 1$.

For example, the first nonzero component of $\alpha = (0,0,2,1,0,5,0,0,\ldots)$ is its third component. It follows that $r = 3$, $\alpha_r = 2$ and the representative of α is $\alpha_{\varepsilon^{(r)}} = \alpha - \varepsilon^{(3)} = (0,0,1,1,0,5,0,0,\ldots)$.

The set $\mathcal{K}_\alpha = \{\beta \in \mathcal{I} : \alpha = \beta + \varepsilon^{(j)}, \text{ for some } j \in \mathbb{N}\}$, $\alpha \in \mathcal{I}$, $|\alpha| > 0$ is a nonempty set, because $\alpha_{\varepsilon^{(r)}} \in \mathcal{K}_\alpha$. Moreover, if $\alpha = n\varepsilon^{(r)}$, $n \in \mathbb{N}$ then $\mathrm{Card}(\mathcal{K}_\alpha) = 1$. In all other cases $\mathrm{Card}(\mathcal{K}_\alpha) > 1$. For example if $\alpha = (0,1,3,0,0,5,0,\ldots)$, then the set $\mathcal{K}_\alpha$ has three elements

$$\mathcal{K}_\alpha = \{\alpha_{\varepsilon^{(2)}} = (0,0,3,0,0,5,0,\ldots),\ (0,1,2,0,0,5,0,\ldots),\ (0,1,3,0,0,4,0,\ldots)\}.$$

For $\alpha \in \mathcal{I}$ such that $\mathrm{Card}(\mathcal{K}_\alpha) > 1$, we denote by r_1 the smallest $k \in \mathbb{N}$ such that $\alpha_{\varepsilon^{(r)}} = \varepsilon^{(r_1)} + \alpha_{\varepsilon^{(r_1)}}$, i.e., $\alpha_{\varepsilon^{(r_1)}}$ is the representative of $\alpha_{\varepsilon^{(r)}}$ and is of length $|\alpha| - 2$. Then $\mathcal{K}_{\alpha_{\varepsilon^{(r)}}} = \{\beta_1 \in \mathcal{I} : \alpha_{\varepsilon^{(r)}} = \beta_1 + \varepsilon^{(k_1)}, \text{ for some } k_1 \in \mathbb{N}\}$. Further on if, $\mathrm{Card}(\mathcal{K}_{\alpha_{\varepsilon^{(r)}}}) > 1$ then we denote by r_2 the smallest $k \in \mathbb{N}$ such that $\alpha_{\varepsilon^{(r_1)}} = \varepsilon^{(r_2)} + \alpha_{\varepsilon^{(r_2)}}$ and so on. Note that $\mathcal{K}_{\alpha_{\varepsilon^{(r_1)}}} = \{\beta_2 \in \mathcal{I} : \alpha_{\varepsilon^{(r_1)}} = \beta_2 + \varepsilon^{(k_2)}, \text{ for some } k_2 \in \mathbb{N}\}$. With such a procedure we decompose $\alpha \in \mathcal{I}$ recursively by new representatives of previous representatives and we obtain sequence of $\mathcal{K}$-sets. Thus, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, 0, \ldots) \in \mathcal{I}$, $|\alpha| = s + 1$ there exists an increasing family of integers $1 \leq r \leq r_1 \leq r_2 \leq \cdots \leq r_s \leq m$, $s \in \mathbb{N}$ such that $\alpha_{\varepsilon^{(r_s)}} = (0,0,\ldots)$ and every multi-index α is decomposed by recurrent sum of representatives

$$\begin{aligned}
\alpha &= \varepsilon^{(r)} + \alpha_{\varepsilon^{(r)}} \\
&= \varepsilon^{(r)} + \varepsilon^{(r_1)} + \alpha_{\varepsilon^{(r_1)}} \\
&\qquad \vdots \\
&= \varepsilon^{(r)} + \varepsilon^{(r_1)} + \cdots + \varepsilon^{(r_s)} + \alpha_{\varepsilon^{(r_s)}}, \quad \alpha_{\varepsilon^{(r_s)}} = (0,0,\ldots).
\end{aligned} \tag{4}$$

For example, if $\alpha = (0,0,2,0,0,1,0,\ldots)$, then $r = 3$, $\alpha_r = 2$, $\alpha_{\varepsilon^{(r)}} = (0,0,1,0,0,1,0,\ldots)$, $r_1 = 3$, $\alpha_{r_1} = 1$, $\alpha_{\varepsilon^{(r_1)}} = (0,0,0,0,0,1,0,\ldots)$, $r_2 = 6$, $\alpha_{r_2} = 1$, $\alpha_{\varepsilon^{(r_2)}} = (0,0,0,\ldots)$, and thus $\alpha = \varepsilon^{(r)} + \varepsilon^{(r_1)} + \varepsilon^{(r_2)} + \alpha_{\varepsilon^{(r_2)}}$. Clearly, $s = |\alpha| - 1 = 2$.

Theorem 3.1. *Let* $c = \sum_{k=1}^{\infty} c_k\, \xi_k \in S'(\mathbb{R})$ *and let* $h \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ *having the representation* $h = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha,k} \otimes \xi_k \otimes H_\alpha$, *such that the coefficients* $h_{\alpha,k} \in X$

satisfy

$$\begin{cases} \text{for } |\alpha| = 1 & \frac{1}{\alpha_r} h_{\alpha_{\varepsilon(r)},r} = \frac{1}{\beta_k} h_{\beta,k}, \qquad \beta \in \mathcal{K}_\alpha, \\ \text{for } |\alpha| = 2 & \frac{1}{\alpha_r} h_{\alpha_{\varepsilon(r)},r} + \frac{1}{\alpha_r \alpha_{r_1}} c_r h_{\alpha_{\varepsilon(r_1)},r_1} = \frac{1}{\beta_k} h_{\beta,k} + \frac{1}{\beta_k \beta_{k_1}} c_k h_{\beta_1,k_1}, \\ & \qquad \beta \in \mathcal{K}_\alpha, \quad \beta_1 \in \mathcal{K}_{\alpha_{\varepsilon(r)}}, \\ \vdots & \end{cases} \tag{5}$$

for all possible decompositions of α.
If $c_k \geq \frac{1}{2k}$, *for all* $k \in \mathbb{N}$ *then equation* (3) *has a unique solution in* $X \otimes (Sc)_{-1}$. *The chaos expansion of the generalized stochastic process, which represents the unique solution of* (3) *is given in the form*

$$\begin{aligned} u = {} & u^{\text{hom}} + u^{\text{nhom}} \\ = {} & \sum_{\alpha \in \mathcal{I}} u_\alpha^{\text{hom}} \otimes H_\alpha + \sum_{\alpha \in \mathcal{I}, |\alpha|>0} u_\alpha^{\text{nhom}} \otimes H_\alpha \\ = {} & \widetilde{u}_0 \otimes \sum_{\alpha \in \mathcal{I}} \frac{c^\alpha}{\alpha!} H_\alpha + \sum_{\substack{\alpha \in \mathcal{I} \\ |\alpha|>0}} \left(\frac{1}{\alpha_r} h_{\alpha_{\varepsilon(r)},r} + \frac{1}{\alpha_r \alpha_{r_1}} c_r h_{\alpha_{\varepsilon(r_1)},r_1} \right. \\ & \left. + \frac{1}{\alpha_r \alpha_{r_1} \alpha_{r_2}} c_r c_{r_1} h_{\alpha_{\varepsilon(r_2)},r_2} + \cdots + \frac{1}{\alpha!} c_r c_{r_1} \ldots c_{r_{s-1}} h_{0,r_s} \right) \otimes H_\alpha, \end{aligned} \tag{6}$$

i.e., as a superposition of a homogeneous part, denoted by u^{hom}, *and its nonhomogeneous part denoted by* u^{nhom}. *The second sum on right-hand side of* (6) *runs through* $\alpha \in \mathcal{I}$ *represented in the recursive form* (4).

Proof. We are looking for the solution u in the form $u = \sum_{\alpha \in \mathcal{I}} u_\alpha \otimes H_\alpha$, where the coefficients $u_\alpha \in X$, $\alpha \in \mathcal{I}$ are to be found. From $Eu = \widetilde{u}_0$ it follows $u_{(0,0,\ldots)} = \widetilde{u}_0$ and thus, $u = \widetilde{u}_0 + \sum_{\alpha \in \mathcal{I}, |\alpha|>0} u_\alpha \otimes H_\alpha$. We use the chaos expansion method and transform the initial equation (3) to an equivalent system of deterministic equations. Thus,

$$\begin{aligned} & \mathbb{D}\left(\widetilde{u}_0 + \sum_{\substack{\alpha \in \mathcal{I} \\ |\alpha|>0}} u_\alpha \otimes H_\alpha \right) \\ & = \left(\sum_{k \in \mathbb{N}} c_k \xi_k \right) \otimes \left(\sum_{\alpha \in \mathcal{I}} u_\alpha \otimes H_\alpha \right) + \sum_{\alpha \in \mathcal{I}} \left(\sum_{k \in \mathbb{N}} h_{\alpha,k} \otimes \xi_k \right) \otimes H_\alpha \\ & \sum_{\substack{\alpha \in \mathcal{I} \\ |\alpha|>0}} \left(\sum_{k \in \mathbb{N}} \alpha_k u_\alpha \otimes \xi_k \right) \otimes H_{\alpha - \varepsilon^{(k)}} \end{aligned}$$

$$= \sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} c_k u_\alpha \otimes \xi_k \right) \otimes H_\alpha + \sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \xi_k \right) \otimes H_\alpha$$

$$\sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} (\alpha_k+1) u_{\alpha+\varepsilon^{(k)}} \otimes \xi_k \right) \otimes H_\alpha$$

$$= \sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} (c_k u_\alpha + h_{\alpha,k}) \otimes \xi_k \right) \otimes H_\alpha .$$

Due to the uniqueness of the chaos expansion of a generalized process in the orthogonal basis $\xi_k \otimes H_\alpha$, $\alpha \in \mathcal{I}$ and $k \in \mathbb{N}$, we transform (3) into a family of deterministic equations

$$(\alpha_k + 1)\, u_{\alpha+\varepsilon^{(k)}} = c_k u_\alpha + h_{\alpha,k}, \quad \text{for all } \alpha \in \mathcal{I},\, k \in \mathbb{N}. \tag{7}$$

The solution u_α, $\alpha \in \mathcal{I}$ is obtained by induction with respect to the length of multi-indices α. Recall, from $Eu = \widetilde{u}_0$ we obtained $u_{(0,0,\dots)} = \widetilde{u}_0$.

Starting with $|\alpha| = 0$, i.e., $\alpha = (0,0,\dots)$, the equations in (7) reduce to

$$u_{\varepsilon^{(k)}} = c_k u_{(0,0,\dots)} + h_{(0,0,\dots),k}, \quad k \in \mathbb{N} \tag{8}$$

and we obtain the coefficients u_α for α of length one. In particular we have the system

$$\begin{cases} u_{(1,0,0,0,\dots)} = c_1 \widetilde{u}_0 + h_{0,1} \\ u_{(0,1,0,0,\dots)} = c_2 \widetilde{u}_0 + h_{0,2} \\ u_{(0,0,1,0,\dots)} = c_3 \widetilde{u}_0 + h_{0,3} \\ u_{(0,0,0,1,0,\dots)} = c_4 \widetilde{u}_0 + h_{0,4} \\ \quad\vdots \end{cases} . \tag{9}$$

Note, u_α for $|\alpha| = 1$ are obtained as a superposition of a homogeneous part, represented in terms of $\widetilde{u}_0$, and a nonhomogeneous part, expressed in terms of $h_{\alpha_{\varepsilon^{(r)}},r} = h_{(0,0,\dots),r}$, $r \in \mathbb{N}$.

Next, for $|\alpha| = 1$ multi-indices are of the form $\alpha = \varepsilon^{(k)}$, $k \in \mathbb{N}$ and several cases occur. For $k = 1$, $\alpha = \varepsilon^{(1)} = (1,0,0,\dots)$ the system (7) transforms into

$$\begin{cases} u_{(2,0,0,0,\dots)} = \frac{1}{2} c_1 u_{(1,0,0,\dots)} + \frac{1}{2} h_{(1,0,0,\dots),1} \\ u_{(1,1,0,0,\dots)} = c_2 u_{(1,0,0,\dots)} + h_{(1,0,0,\dots),2} \\ u_{(1,0,1,0,\dots)} = c_3 u_{(1,0,0,\dots)} + h_{(1,0,0,\dots),3} \\ u_{(1,0,0,1,0,\dots)} = c_4 u_{(1,0,0,\dots)} + h_{(1,0,0,\dots),4} \\ \quad\vdots \end{cases} .$$

Now we replace expressions $u_{\varepsilon^{(k)}}$, $k \in \mathbb{N}$ with equalities (8), obtained in the previous step, and receive

$$\begin{cases} u_{(2,0,0,0,\dots)} = \frac{1}{2}c_1^2\widetilde{u}_0 + \frac{1}{2}c_1 h_{0,1} + \frac{1}{2}h_{(1,0,0,\dots),1} \\ u_{(1,1,0,0,\dots)} = c_1c_2\widetilde{u}_0 + c_2 h_{0,1} + h_{(1,0,0,\dots),2} \\ u_{(1,0,1,0,\dots)} = c_1c_3\widetilde{u}_0 + c_3 h_{0,1} + h_{(1,0,0,\dots),3} \\ u_{(0,0,0,1,0,\dots)} = c_1c_4\widetilde{u}_0 + c_4 h_{0,1} + h_{(1,0,0,\dots),4} \\ \vdots \end{cases}. \tag{10}$$

Continuing, for $k = 2$, $\alpha = \varepsilon^{(2)} = (0,1,0,0,\dots)$ the equations in (7) transform to

$$\begin{cases} u_{(1,1,0,0,\dots)} = c_1 u_{(0,1,0,0,\dots)} + h_{(0,1,0,0,\dots),1} \\ u_{(0,2,0,0,\dots)} = \frac{1}{2}c_2 u_{(0,1,0,0,\dots)} + \frac{1}{2}h_{(0,1,0,0,\dots),2} \\ u_{(0,1,1,0,\dots)} = c_3 u_{(0,1,0,0,\dots)} + h_{(0,1,0,0,\dots),3} \\ u_{(0,1,0,1,0,\dots)} = c_4 u_{(0,1,0,0,\dots)} + h_{(0,1,0,0,\dots),4} \\ \vdots \end{cases}, \tag{11}$$

and then after substitution (9) for (11) we obtain

$$\begin{cases} u_{(1,1,0,0,\dots)} = c_1c_2\widetilde{u}_0 + c_1 h_{0,2} + h_{(0,1,0,0,\dots),1} \\ u_{(0,2,0,0,\dots)} = \frac{1}{2}c_2^2\widetilde{u}_0 + \frac{1}{2}c_2 h_{0,2} + \frac{1}{2}h_{(0,1,0,0,\dots),2} \\ u_{(0,1,1,0,\dots)} = c_2c_3\widetilde{u}_0 + c_3 h_{0,2} + h_{(0,1,0,0,\dots),3} \\ u_{(0,1,0,1,0,\dots)} = c_2c_4\widetilde{u}_0 + c_4 h_{0,2} + h_{(0,1,0,0,\dots),4} \\ \vdots \end{cases}. \tag{12}$$

For $k = 3$, $\alpha = \varepsilon^{(3)} = (0,0,1,0,0,\dots)$ the system (7) reduces to

$$\begin{cases} u_{(1,0,1,0,0,\dots)} = c_1c_3\widetilde{u}_0 + c_1 h_{0,3} + h_{(0,0,1,0,0,\dots),1} \\ u_{(0,1,1,0,0,\dots)} = c_2c_3\widetilde{u}_0 + c_2 h_{0,3} + h_{(0,0,1,0,0,\dots),2} \\ u_{(0,0,2,0,\dots)} = \frac{1}{2}c_3^2\widetilde{u}_0 + \frac{1}{2}c_3 h_{0,3} + \frac{1}{2}h_{(0,0,1,0,0,\dots),3} \\ u_{(0,0,1,1,0,\dots)} = c_3c_4\widetilde{u}_0 + c_4 h_{0,3} + h_{(0,0,1,0,0,\dots),4} \\ \vdots \end{cases}. \tag{13}$$

Continuing with the same procedure we obtain the unknown coefficients u_α of length two. Further on, we will express all multi-indices, which have two different representations of the form $\alpha = \varepsilon^{(k)} + \varepsilon^{(k_1)}$, for $k \neq k_1$, $k, k_1 \in \mathbb{N}$ in terms of their representatives.

For example, multi-index

$$(1,1,0,0,\dots) = \varepsilon^{(1)} + (0,1,0,\dots) = \varepsilon^{(2)} + (1,0,0,\dots)$$

has two different representations of the form $\alpha = \varepsilon^{(k)} + \alpha_{\varepsilon^{(k)}}$ and thus the coefficient $u_{(1,1,0,0,\dots)}$ appears in systems (10) and (12). Thus, the additional condition

$$c_2 h_{0,1} + h_{(1,0,0,\dots),2} = c_1 h_{0,1} + h_{(0,1,0,\dots),1}$$

has to hold in order to have a solvable system.

Moreover, we express $\alpha = (1,1,0,0,\dots)$ in terms of a sequence of successive representatives, i.e., $r = 1$, $\alpha_{\varepsilon^{(r)}} = (0,1,0,0,\dots)$, $r_1 = 2$ and $\alpha_{\varepsilon^{(r_1)}} = (0,0,0,\dots)$. Thus, the element $u_{(1,1,0,0,\dots)}$ is given in the form $u_{(1,1,0,0,\dots)} = c_1 c_2 u_0 + h_{(0,1,0,\dots),1} + c_1 h_{0,2}$ obtained in (12). Also, the element $u_{(1,0,1,0,0,\dots)}$ appears in equalities (10) and (13) and we obtained additional condition

$$c_3 h_{0,1} + h_{(1,0,0,\dots),3} = c_1 h_{0,3} + h_{(0,0,1,0,\dots),1}$$

which need to be satisfied in order to have a unique u_α. Multi-index $\alpha = (1,0,1,0,0,\dots)$ can be decomposed in terms of a sequence of successive representatives as follows $\alpha = \varepsilon^{(r)} + \alpha_{\varepsilon^{(r)}}$, where $r = 1$, $\alpha_{\varepsilon^{(r)}} = (0,0,1,0,\dots)$ and $\alpha_{\varepsilon^{(r)}} = \varepsilon^{(r_1)} + \alpha_{\varepsilon^{(r_1)}}$, for $r_1 = 3$ and $\alpha_{\varepsilon^{(r_1)}} = (0,0,0,\dots)$. We use the form (13) to represent $u_{(1,0,1,0,0,\dots)}$ in terms of its representatives decomposition. Moreover, the element $u_{(0,1,1,0,0,\dots)}$ appears in equalities (12) and (13), and it follows that also the condition

$$c_3 h_{0,2} + h_{(0,1,0,0,\dots),3} = c_2 h_{0,3} + h_{(0,0,1,0,\dots),2}$$

has to be satisfied, and so on.

In this step we obtained forms of the coefficients u_α of length two, with validity of the additional condition

$$h_{\alpha_{\varepsilon^{(r)}},r} + c_r h_{(0,0,\dots),r_1} = h_{\beta,j} + c_j\, h_{(0,0,\dots),k}, \tag{14}$$

where $\alpha = \varepsilon^{(r)} + \varepsilon^{(r_1)} + (0,0,\dots)$, $1 \le r \le r_1$, $r, r_1 \in \mathbb{N}$ and all $\beta \in \mathcal{I}$ such that $\alpha = \beta + \varepsilon^{(j)}$ for $j \ge r$, and $\beta = (0,0,\dots) + \varepsilon^{(k)}$, for some $k \in \mathbb{N}$. Note that condition (14) corresponds to condition (5) for $|\alpha| = 2$. The coefficients u_α of length two are represented as a superposition of a homogeneous part, expressed in terms of $\widetilde{u}_0$ and a nonhomogeneous part expressed as a linear combination of $h_{\alpha,k}$ for α of length one and product of c_k, $k \in \mathbb{N}$ and $h_{\alpha,k}$ for α of length zero, i.e., in terms of representatives $\frac{1}{\alpha_r} h_{\alpha_{\varepsilon^{(r)}},r} + \frac{1}{\alpha!} c_r h_{0,r_1}$ for $\alpha = \varepsilon^{(r)} + \alpha_{\varepsilon^{(r)}}$, $\alpha_{\varepsilon^{(r)}} = \varepsilon^{(r_1)}$, $1 \le r \le r_1$.

For $|\alpha| = 2$ from system of equations (7) and results (10), (12), (13),..., calculated in the previous step, we obtain u_α, for α of length three. Different combinations for multi-indices of length two occur. If we choose $\alpha = (1,1,0,0,\dots)$ then the system (7) transforms into the system

$$\begin{cases} u_{(2,1,0,0,\dots)} = \frac{1}{2} c_1 u_{(1,1,0,0,\dots)} + \frac{1}{2} h_{(1,1,0,0,\dots),1} \\ u_{(1,2,0,0,\dots)} = \frac{1}{2} c_2 u_{(1,1,0,0,\dots)} + \frac{1}{2} h_{(1,1,0,0,\dots),2} \\ u_{(1,1,1,0,\dots)} = c_3 u_{(1,1,0,0,\dots)} + h_{(1,1,0,0,\dots),3} \\ u_{(1,1,0,1,0,\dots)} = c_4 u_{(1,1,0,0,\dots)} + h_{(1,1,0,0,\dots),4} \\ \qquad \vdots \end{cases}.$$

We substitute equalities for u_α of length two, obtained in the previous step in terms of their representatives decomposition, and transform the system to a more elegant one. In particular, we use expression in (12) for the element $u_{(1,1,0,0,\dots)}$ and obtain the system of equations

$$\begin{cases} u_{(2,1,0,0,\dots)} = \frac{1}{2}c_1^2c_2\widetilde{u}_0 + \frac{1}{2}c_1^2h_{0,2} + \frac{1}{2}c_1h_{(0,1,0,0,\dots),1} + \frac{1}{2}h_{(1,1,0,0,\dots),1} \\ u_{(1,2,0,0,\dots)} = \frac{1}{2}c_1c_2^2\widetilde{u}_0 + \frac{1}{2}c_1c_2h_{0,2} + \frac{1}{2}c_2h_{(0,1,0,\dots),1} + \frac{1}{2}h_{(1,1,0,0,\dots),2} \\ u_{(1,1,1,0,\dots)} = c_1c_2c_3\widetilde{u}_0 + c_1c_3h_{0,2} + c_3h_{(0,1,0,0,\dots),1} + h_{(1,1,0,0,\dots),3} \\ u_{(1,1,0,1,0,\dots)} = c_1c_2c_4\widetilde{u}_0 + c_1c_4h_{0,2} + c_4h_{(0,1,0,0,\dots),1} + h_{(1,1,0,0,\dots),4} \\ \vdots \end{cases}. \tag{15}$$

For $\alpha = (1,0,1,\dots)$ the system (7) transforms to the system

$$\begin{cases} u_{(2,0,1,0,0,\dots)} = \frac{1}{2}c_1^2c_3\widetilde{u}_0 + \frac{1}{2}c_1^2h_{0,3} + \frac{1}{2}c_1h_{(0,0,1,0,0,\dots),1} + \frac{1}{2}h_{(1,0,1,0,\dots),1} \\ u_{(1,1,1,0,0,\dots)} = c_1c_2c_3\widetilde{u}_0 + c_1c_2h_{0,3} + c_2h_{(0,0,1,0,\dots),1} + h_{(1,0,1,0,0,\dots),2} \\ u_{(1,0,2,0,\dots)} = \frac{1}{2}c_1c_3^2\widetilde{u}_0 + \frac{1}{2}c_1c_3h_{0,3} + \dfrac{1}{2}c_3h_{(0,0,1,0,0,\dots),1} + \frac{1}{2}h_{(1,0,1,0,0,\dots),3} \\ u_{(1,0,1,1,0,\dots)} = c_1c_3c_4\widetilde{u}_0 + c_1c_4h_{0,3} + c_4h_{(0,0,1,0,0,\dots),1} + h_{(1,0,1,0,0,\dots),4} \\ \vdots \end{cases} \tag{16}$$

and for $\alpha = (1,0,0,1,\dots)$ we obtain the system

$$\begin{cases} u_{(2,0,0,1,0,0,\dots)} = \frac{1}{2}c_1^2c_4\widetilde{u}_0 + \frac{1}{2}c_1^2h_{0,4} + \frac{1}{2}c_1h_{(0,0,0,1,0,0,\dots),1} + \frac{1}{2}h_{(1,0,1,0,\dots),1} \\ u_{(1,1,0,1,0,0,\dots)} = c_1c_2c_4\widetilde{u}_0 + c_1c_2h_{0,4} + c_2h_{(0,0,0,1,0,\dots),1} + h_{(1,0,0,1,0,0,\dots),2} \\ u_{(1,0,1,1,0,\dots)} = c_1c_3c_4\widetilde{u}_0 + c_1c_3h_{0,4} + c_3h_{(0,0,0,1,0,0,\dots),1} + h_{(1,0,0,1,0,0,\dots),3} \\ u_{(1,0,0,2,0,\dots)} = \frac{1}{2}c_1c_4^2\widetilde{u}_0 + \frac{1}{2}c_1c_4h_{0,4} + \frac{1}{2}c_4h_{(0,0,0,1,0,0,\dots),1} + \frac{1}{2}h_{(1,0,0,1,0,0,\dots),4} \\ \vdots \end{cases}.$$

We continue with multi-indices $\alpha = (0,1,1,0,0,\dots)$ and $\alpha = (0,1,0,1,0,\dots)$ and transform the system (7) respectively to the systems

$$\begin{cases} u_{(1,1,1,0,0,\dots)} = c_1c_2c_3\widetilde{u}_0 + c_1c_2h_{0,3} + c_1h_{(0,0,1,0,0,\dots),2} + h_{(0,1,1,0,\dots),1} \\ u_{(0,2,1,0,0,\dots)} = \frac{1}{2}c_2^2c_3\widetilde{u}_0 + \frac{1}{2}c_2^2h_{0,3} + \frac{1}{2}c_2h_{(0,0,1,0,\dots),2} + \frac{1}{2}h_{(0,1,1,0,0,\dots),2} \\ u_{(0,1,2,0,\dots)} = \frac{1}{2}c_2c_3^2\widetilde{u}_0 + \frac{1}{2}c_2c_3h_{0,3} + \frac{1}{2}c_3h_{(0,0,1,0,0,\dots),1} + h_{(0,1,1,0,0,\dots),3} \\ u_{(0,1,1,1,0,\dots)} = c_2c_3c_4\widetilde{u}_0 + c_2c_4h_{0,3} + c_4h_{(0,0,1,0,0,\dots),2} + h_{(0,1,1,0,0,\dots),4} \\ \vdots \end{cases} \tag{17}$$

and

$$\begin{cases} u_{(1,1,0,1,0,0,\dots)} = c_1c_2c_4\widetilde{u}_0 + c_1c_2h_{0,4} + \frac{1}{2}c_1h_{(0,0,0,1,0,0,\dots),2} + h_{(0,1,0,1,0,\dots),1} \\ u_{(0,2,0,1,0,0,\dots)} = \frac{1}{2}c_2^2c_4\widetilde{u}_0 + \frac{1}{2}c_2^2h_{0,4} + \frac{1}{2}c_2h_{(0,0,0,1,0,\dots),2} + \frac{1}{2}h_{(0,1,0,1,0,0,\dots),2} \\ u_{(0,1,1,1,0,\dots)} = c_2c_3c_4\widetilde{u}_0 + c_2c_3h_{0,4} + c_3h_{(0,0,0,1,0,0,\dots),2} + h_{(0,1,0,1,0,0,\dots),3} \\ u_{(0,1,0,2,0,\dots)} = \frac{1}{2}c_2c_4^2\widetilde{u}_0 + \frac{1}{2}c_2c_4h_{0,4} + \frac{1}{2}c_4h_{(0,0,0,1,0,0,\dots),2} + \frac{1}{2}h_{(0,1,0,1,0,0,\dots),4} \\ \vdots \end{cases}.$$

For multi-indices $\alpha = (2,0,0,0,\dots)$ and $\alpha = (0,2,0,0,\dots)$ the system (7) transforms respectively into

$$\begin{cases} u_{(3,0,0,0,\dots)} = \frac{1}{6}c_1^3\widetilde{u}_0 + \frac{1}{6}c_1^2h_{0,1} + \frac{1}{6}c_1h_{(1,0,0,0,\dots),1} + \frac{1}{3}h_{(2,0,0,0,\dots),1} \\ u_{(2,1,0,0,\dots)} = \frac{1}{2}c_1^2c_2\widetilde{u}_0 + \frac{1}{2}c_1c_2h_{0,1} + \frac{1}{2}c_2h_{(1,0,0,0,\dots),1} + h_{(2,0,0,0,\dots),2} \\ u_{(2,0,1,0,\dots)} = \frac{1}{2}c_1^2c_3\widetilde{u}_0 + \frac{1}{2}c_1c_3h_{0,1} + \frac{1}{2}c_3h_{(1,0,0,0,\dots),1} + \frac{1}{2}h_{(2,0,0,0,\dots),3} \\ u_{(2,0,0,1,0,\dots)} = \frac{1}{2}c_1^2c_4\widetilde{u}_0 + \frac{1}{2}c_2c_4h_{0,1} + \frac{1}{2}c_4h_{(0,0,0,1,0,0,\dots),1} + \frac{1}{2}h_{(2,0,0,0,\dots),4} \\ \vdots \end{cases} \tag{18}$$

and

$$\begin{cases} u_{(1,2,0,0,0,\dots)} = \frac{1}{2}c_1c_2^2\widetilde{u}_0 + \frac{1}{2}c_1c_2h_{0,2} + \frac{1}{2}c_1h_{(0,1,0,0,\dots),2} + h_{(0,2,0,0,\dots),1} \\ u_{(0,3,0,0,0,\dots)} = \frac{1}{6}c_2^3\widetilde{u}_0 + \frac{1}{6}c_2^2h_{0,2} + \frac{1}{6}c_2h_{(0,1,0,\dots),2} + \frac{1}{3}h_{(0,2,0,0,\dots),2} \\ u_{(0,2,1,0,0,\dots)} = \frac{1}{2}c_2^2c_3\widetilde{u}_0 + \frac{1}{2}c_2c_3h_{0,2} + \frac{1}{2}c_3h_{(0,1,0,0,\dots),2} + \frac{1}{2}h_{(0,2,0,0,\dots),3} \\ u_{(0,2,0,1,0,\dots)} = \frac{1}{2}c_2^2c_4\widetilde{u}_0 + \frac{1}{2}c_2c_4h_{0,2} + \frac{1}{2}c_4h_{(0,1,0,0,\dots),2} + \frac{1}{2}h_{(0,2,0,0,\dots),4} \\ \vdots \end{cases}.$$

Combining with the previous results, we obtain u_α, for $|\alpha| = 3$. Further on, we will express all multi-indices, which have several different representations of the form $\alpha = \varepsilon^{(k)} + \varepsilon^{(k_1)} + \varepsilon^{(k_2)}$, for $k, k_1, k_2 \in \mathbb{N}$ in terms of theirs representatives. Two different representations of $u_{(2,1,0,0,\dots)}$ appear in the systems (15) and (18), so the additional condition

$$\frac{1}{2}c_1^2h_{0,2}+\frac{1}{2}c_1h_{(0,1,0,\dots),1}+\frac{1}{2}h_{(1,1,0,0,\dots),1} = \frac{1}{2}c_1c_2h_{0,1}+\frac{1}{2}c_2h_{(1,0,0,\dots),1}+h_{(2,0,0,\dots),2}$$

follows. We express the element $u_{(2,1,0,0,\dots)}$ in form of the representatives. Clearly, recursive decomposition of multi-index $(2,1,0,0,\dots)$ is given by

$$\begin{aligned} (2,1,0,0,\dots) &= \varepsilon^{(r)} + (1,1,0,0,\dots) = \varepsilon^{(r)} + \varepsilon^{(r_1)} + (0,1,0,0,\dots) \\ &= \varepsilon^{(r)} + \varepsilon^{(r_1)} + \varepsilon^{(r_2)} + (0,0,\dots), \end{aligned}$$

for $r = 1$, $r_1 = 1$ and $r_2 = 2$. Thus, $u_{(2,1,0,0,\dots)} = \frac{1}{2}c_1^2c_2\widetilde{u}_0 + \frac{1}{2}h_{(1,1,0,0,\dots),1} + \frac{1}{2}c_1h_{(0,1,0,0,\dots),1} + \frac{1}{2}c_1^2h_{0,2}$.

Since the coefficient $u_{(1,1,1,0,0,\dots)}$ appears in three equations (15), (16) and (17), we receive another conditions

$$\begin{aligned} & c_1c_3h_{0,2}+c_3h_{(0,1,0,0,\dots),1}+h_{(1,1,0,0,\dots),3}\\ &\quad = c_1c_2h_{0,3}+c_2h_{(0,0,1,0,\dots),1}+h_{(1,0,1,0,0,\dots),2}\\ &\quad = c_1c_2h_{0,3}+c_1h_{(0,0,1,0,0,\dots),2}+h_{(0,1,1,0,\dots),1}\end{aligned}$$

and express $\alpha=(1,1,1,0,0,\dots)$ by its representatives, $r=1, r_1=2, r_3=2$. The representation of $u_{(1,1,1,0,0,\dots)}$ is given by (15), i.e., $u_{(1,1,1,0,0,\dots)}=c_1c_2c_3\widetilde{u}_0+h_{(0,1,1,0,\dots),1}+c_1h_{(0,0,1,0,\dots),2}+c_1c_2h_{0,3}$. Note that previous conditions correspond to conditions (5).

We proceed by the same procedure for all multi-index lengths to obtain u_α in the form

$$\begin{aligned} u_\alpha = \widetilde{u}_0\frac{c_1^{\alpha_1}}{\alpha_1!}\cdot\frac{c_2^{\alpha_2}}{\alpha_2!}\cdots+\Bigg(&\frac{1}{\alpha_r}h_{\alpha_{\varepsilon^{(r)}},r}+\frac{1}{\alpha_r\alpha_{r_1}}c_rh_{\alpha_{\varepsilon^{(r_1)}},r_1}+\cdots\\ &\cdots+\frac{1}{\alpha_r\alpha_{r_1}\dots\alpha_{r_s}}c_rc_{r_1}\dots c_{r_{s-1}}h_{0,r_s}\Bigg)\end{aligned}$$

and thus the form of the solution (6).

In general, we decompose multi-index α recurrently by the representatives. To be precise, in the firs step we have $\alpha=\varepsilon^{(r_1)}+\alpha_{\varepsilon^{(r_1)}}$. Then, in the next step we find the representative of $\alpha_{\varepsilon^{(r_1)}}$, i.e., $\alpha_{\varepsilon^{(r_1)}}=\varepsilon^{(r_2)}+\alpha_{\varepsilon^{(r_2)}}$ and so on…

The coefficients u_α are obtained in the form

$$u_\alpha = \underbrace{\prod_{i\in\mathbb{N}}\frac{c_i^{\alpha_i}}{\alpha_i!}\,\widetilde{u}_0}_{\text{homogeneous part}}+\underbrace{\left[\frac{1}{\alpha_r}h_{\alpha_{\varepsilon^{(r)}},r}+\frac{1}{\alpha_r\alpha_{r_1}}c_rh_{\alpha_{\varepsilon^{(r_1)}},r_1}+\cdots+\frac{1}{\alpha!}c_rc_{r_1}\dots c_{r_{s-1}}h_{0,r_s}\right]}_{\text{nonhomogeneous part}}$$

for the decomposition $\alpha=\varepsilon^{(r)}+\sum\limits_{1\le j\le s}\varepsilon^{(r_j)}+(0,0,\dots)$, $1\le r\le r_1\le\cdots\le r_s$, i.e., $\alpha_{\varepsilon^{(r_j)}}=\alpha-\varepsilon^{(r)}-\sum\limits_{1\le i\le j-1}\varepsilon^{(r_i)}$, $1\le j\le s$, where $|\alpha|=s+1$.

It remains to prove the convergence of the solution (6) in the space $X\otimes(Sc)_{-1}$, i.e., to prove that, for some $p>0$

$$\|u\|^2_{X\otimes(Sc)_{-1}}=\sum_{\alpha\in\mathcal{I}}\|u_\alpha\|^2_X\,(2\mathbb{N}c)^{-p\alpha}<\infty.$$

Let $h\in X\otimes S_{-p}(\mathbb{R})\otimes(S)_{-1,-p}$. Then, there exists $p>0$ such that

$$\|h\|^2_{X\otimes S_{-p}(\mathbb{R})\otimes(S)_{-1,-p}}=\sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\|h_{\alpha,k}\|^2_X\,(2k)^{-p}\,(2\mathbb{N})^{-p\alpha}<\infty.$$

Note that for $\widetilde{u}_0 \in X$ we have $\|\widetilde{u}_0\|_X = \|\widetilde{u}_0\|_{X\otimes(S)_{-1,-q}}$ for all $q > 0$. Then, the convergence follows from

$$\begin{aligned}\|u\|^2_{X\otimes(Sc)_{-1,-p}} &= \sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_X (2\mathbb{N}c)^{-p\alpha}\\ &\le 2\sum_{\alpha\in\mathcal{I}} \|u^{\text{hom}}_\alpha\|^2_X (2\mathbb{N}c)^{-p\alpha} + 2\sum_{\substack{\alpha\in\mathcal{I}\\|\alpha|>0}} \|u^{\text{nhom}}_\alpha\|^2_X (2\mathbb{N}c)^{-p\alpha}\\ &= 2A + 2B < \infty.\end{aligned}$$

From assumption $c_k \ge \frac{1}{2k}$, for all $k \in \mathbb{N}$, it follows that $\sum_{\alpha\in\mathcal{I}}(2\mathbb{N}c)^{-p\alpha} < \infty$ if $p > 0$. Then, for $p > 3$, we have

$$\begin{aligned}A &= \sum_{\alpha\in\mathcal{I}} \|u^{\text{hom}}_\alpha\|^2_X (2\mathbb{N}c)^{-p\alpha}\\ &= \sum_{\alpha\in\mathcal{I}} \|\widetilde{u}_0\|^2_X \frac{c^{2\alpha}}{(\alpha!)^2} (2\mathbb{N}c)^{-p\alpha}\\ &\le \|\widetilde{u}_0\|^2_X \sum_{\alpha\in\mathcal{I}} c^{2\alpha}(2\mathbb{N})^{-p\alpha}c^{-p\alpha}\\ &\le \|\widetilde{u}_0\|^2_X \sum_{\alpha\in\mathcal{I}} c^{-(p-2)\alpha} \sum_{\alpha\in\mathcal{I}}(2\mathbb{N})^{-p\alpha} < \infty.\end{aligned}$$

For $p > 3$ the convergence of the second part B follows from

$$\begin{aligned}B &= \sum_{\substack{\alpha\in\mathcal{I}\\|\alpha|>0}} \|u^{\text{nhom}}_\alpha\|^2_X (2\mathbb{N}c)^{-p\alpha}\\ &= \sum_{\substack{\alpha\in\mathcal{I},|\alpha|>0,\\ \alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}}} \left\| \frac{1}{\alpha_r} h_{\alpha_{\varepsilon^{(r)}},r} + \frac{1}{\alpha_r\alpha_{r_1}} c_r h_{\alpha_{\varepsilon^{(r_1)}},r_1} + \cdots + \frac{1}{\alpha!} c_r c_{r_1}\dots c_{r_{s-1}} h_{0,r_s}\right\|^2_X\\ &\qquad \times (2\mathbb{N}c)^{-p(\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)})}\\ &\le \sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}} \frac{|\alpha|}{\alpha_r^2}\left(\|h_{\alpha_{\varepsilon^{(r)}},r}\|^2_X + c_r^2\|h_{\alpha_{\varepsilon^{(r_1)}},r_1}\|^2_X + \cdots + c_r^2 c_{r_1}^2\dots c_{r_{s-1}}^2\|h_{0,r_s}\|^2_X\right)\\ &\qquad \times (2rc)^{-p}\,(2\mathbb{N}c)^{-p\alpha_{\varepsilon^{(r)}}}\\ &\le \sum_{\alpha\in\mathcal{I},|\alpha|>0} c^{2\alpha}\,(2\mathbb{N})^{\alpha}\left(\|h_{\alpha_{\varepsilon^{(r)}},r}\|^2_X + \|h_{\alpha_{\varepsilon^{(r_1)}},r_1}\|^2_X + \cdots + \|h_{0,r_s}\|^2_X\right)\\ &\qquad \times (2r)^{-p}\,c^{-p\alpha}(2\mathbb{N})^{-p\alpha_{\varepsilon^{(r)}}}\\ &\le \left(\sum_{\alpha\in\mathcal{I}} c^{-(p-2)\alpha}\right)\cdot \sum_{\alpha\in\mathcal{I}}\sum_{r\in\mathbb{N}} \|h_{\alpha,r}\|^2_X\,(2r)^{-p}\,(2\mathbb{N})^{-(p-1)\alpha} < \infty,\end{aligned}$$

where we have used the facts that $|\alpha| \le (2\mathbb{N})^{\alpha}$ and $(2\mathbb{N})^{p\varepsilon^{(r)}}(2\mathbb{N})^{-p\alpha} \le 1$ for all $\alpha \in \mathcal{I}$, $r \in \mathbb{N}$. With this statement we complete the proof. □

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