Applied and Numerical Harmonic Analysis

$$\widehat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma}\, dx$$

David V. Cruz-Uribe
Alberto Fiorenza

# Variable Lebesgue Spaces

## Foundations and Harmonic Analysis

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# Variable Lebesgue Spaces

## Foundations and Harmonic Analysis

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# Preface

This book represents the fruits of our collaboration on the variable Lebesgue spaces. Our work in this area stretches back over a decade. Its genesis is memorable: it began in Naples during an exceptionally cold January in 2002, shortly after the introduction of the euro. We worked through the details of a preprint we had received from our colleague Lars Diening which gave conditions for the maximal operator to be bounded on variable Lebesgue spaces defined on bounded sets. Our first task was to try to understand these previously unknown spaces. The more we worked with them, the more intrigued we became. Subsequently, we began to study the maximal operator on unbounded domains; a search for applications led us to the other classical operators of harmonic analysis and the interplay between weighted norm inequalities and variable Lebesgue spaces.

In 2007, the first author was invited to teach a graduate course on variable Lebesgue spaces at the University of Naples, Federico II, an invitation he gladly accepted. The notes for that course became the basis for this book. One problem, however, was that our knowledge of the field continued to evolve even as we tried to convert those notes into a final manuscript. Instead of writing we would stop to prove new theorems, leading to repeated revisions and expansions of the text. At this point, however, we think we have reached a reasonable place to stop: we have written (we believe) an introduction to variable Lebesgue spaces that will be useful for a wide audience. Simply put, we think we have finally gotten it right.

Many individuals have contributed directly and indirectly to this book. We want to acknowledge the late Christoph Neugebauer, who collaborated with us on our first paper on variable Lebesgue spaces and provided key insights. We want to thank our colleagues Lars Diening, Peter Hästö, Aleš Nekvinda and Stefan Samko, who freely shared with us preprints of their work. Their generosity kept us abreast of a very rapidly evolving field. We want to thank Jean Michel Rakotoson for his collegiality and for sharing with us his ideas and questions on variable Lebesgue spaces. We also want to thank our colleague Claudia Capone and the students who attended the variable Lebesgue space course for their patience as we tried for the first time to shape our knowledge into a coherent whole. We especially want to thank Carlo Sbordone, who first brought us together and has provided continuing support and

encouragement for our joint labors. And finally, we want to thank our wives and our children for patiently bearing with us as this book became a reality.

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# Contents

# Chapter 1
# Introduction

The variable Lebesgue spaces, as their name implies, are a generalization of the classical Lebesgue spaces, replacing the constant exponent $p$ with a variable exponent function $p(\cdot)$. The resulting Banach function spaces $L^{p(\cdot)}$ have many properties similar to the $L^p$ spaces, but they also differ in surprising and subtle ways. For this reason the variable Lebesgue spaces have an intrinsic interest, but they are also very important for their applications to partial differential equations and variational integrals with non-standard growth conditions. The past 20 years, and especially the past decade, have witnessed an explosive growth in the study of these and related spaces.

The goal of this book is to provide an introduction to the variable Lebesgue spaces. We first establish their structure and function space properties, paying special attention to the differences between bounded and unbounded exponents. Next, we develop the machinery of harmonic analysis on variable Lebesgue spaces. We first concentrate on the Hardy-Littlewood maximal operator, and then extend the Rubio de Francia theory of extrapolation to this setting. To do so we introduce the theory of Muckenhoupt $A_p$ weights and weighted norm inequalities. With these tools we can then study other operators, particularly convolution operators, singular integral operators and Riesz potentials. Finally, as an application of these results we give the essential properties of the variable Sobolev spaces.

In writing this book we had two different audiences in mind. First, we wanted to write an introduction suitable for researchers and students interested in learning about the variable Lebesgue spaces. At the same time, we hoped to create a useful reference for mathematicians already active in the area. For both audiences we have provided a coherent treatment of the material—in terms of notation, hypotheses and overall point of view—and thereby united results by many authors from a rapidly evolving field. We have also included a concise introduction of weights and weighted norm inequalities. These have become very important tools in the study of the variable Lebesgue spaces, and we have given a careful treatment of the key ideas needed to use them.

We have not, however, merely summarized existing work. We have included many new and previously unpublished results and new proofs of known results. Our

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goal was both to build upon earlier work and to fill in the gaps that inevitably arise in a rapidly developing field. At the end of each chapter we have included extensive notes, giving detailed references to the literature and summaries of additional topics not treated in the body. In this way we have made clear the history and current state of knowledge.

We are mindful, however, that the needs of these two audiences can conflict at times, since a comprehensive monograph aimed at specialists is usually not the best choice for students, and an introductory treatment may prove frustratingly slow for those who are already familiar with the material. In resolving this conflict we have always thought first of the needs of students, providing many details and exploring the underlying intuition. We ask the experts for their forbearance and hope they will remember their own experiences as graduate students.

In this chapter we provide a brief introduction to the variable Lebesgue spaces, recount some of their history (particularly their development before the "modern" period of research), and very briefly sketch some of the motivations for their study. Following this we give a more detailed summary of the contents of each chapter. In the last section we outline the minimum knowledge we expect of the reader and list the basic notation we will use throughout.

## 1.1 An Overview of Variable Lebesgue Spaces

To get a sense of the variable Lebesgue spaces, we begin with an elementary example. On the real line, consider the function $f(x) = |x|^{-1/3}$. The function $f$ is extremely well-behaved, but it is not in $L^p(\mathbb{R})$ for any $p$, $1 \leq p \leq \infty$. Given a single value of $p$ it either grows too quickly at the origin or decays too slowly at infinity.

To more fully describe the behavior of $f$ we must bring to bear two different $L^p$ spaces, for instance, $L^2$ and $L^4$. We can split up the domain of $f$ and say that $f \in L^2([-2,2])$ and $f \in L^4(\mathbb{R} \setminus [-2,2])$. The drawback of this approach is that for more complicated functions we need to introduce additional $L^p$ spaces or lose information. If we let

$$g(x) = |x|^{-1/3} + |x-1|^{-1/4},$$

then $g \in L^2([-2,2])$, or more generally in $L^p([-2,2])$ for any $p < 3$, but we have lost information about the local behavior of the singularity at $x = 1$. On the other hand, $g$ is no longer in $L^4(\mathbb{R} \setminus [-2,2])$: we have $g \in L^p(\mathbb{R} \setminus [-2,2])$ for $p > 4$. To capture this behavior we must subdivide the domain further, for example, writing $g \in L^2([-1,1/2])$, $g \in L^3([1/2,2])$ and $g \in L^{9/2}(\mathbb{R} \setminus [-1,2])$.

The variable Lebesgue spaces give a different approach: we leave the domain intact and instead allow the exponent to vary. Define the "exponent function"

$$p(x) = \frac{9|x| + 2}{2|x| + 1} = \frac{9}{2} - \frac{5/2}{2|x| + 1}.$$

Then $p(0) = 2$, $p(1) = 11/3$ and $p(x) \to 9/2$ as $|x| \to \infty$, and it is easy to see that

$$\int_{\mathbb{R}} |f(x)|^{p(x)}\,dx < \infty \quad \text{and} \quad \int_{\mathbb{R}} |g(x)|^{p(x)}\,dx < \infty.$$

In other words, the single variable exponent $p(\cdot)$ allows us to describe more precisely the behavior of each function. Moreover, we can distinguish between them at infinity by modifying the exponent function. For instance, if we let

$$q(x) = \frac{8|x| + 2}{2|x| + 1} = 4 - \frac{2}{2|x| + 1},$$

then

$$\int_{\mathbb{R}} |f(x)|^{q(x)}\,dx < \infty$$

and $|g(\cdot)|^{q(\cdot)}$ is locally integrable, but

$$\int_{\mathbb{R}} |g(x)|^{q(x)}\,dx = \infty.$$

These examples motivate the definition of the variable Lebesgue spaces. Given a set $\Omega$ and a measurable function $p(\cdot) : \Omega \to [1, \infty)$, we let $L^{p(\cdot)}(\Omega)$ be the set of functions $f$ such that

$$\int_{\Omega} |f(x)|^{p(x)}\,dx < \infty.$$

We can even incorporate $L^\infty$ into the definition: if we allow $p(\cdot)$ to be infinite on sets of positive measure and let $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$, then we can redefine $L^{p(\cdot)}(\Omega)$ as the set of functions such that

$$\rho_{p(\cdot)}(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty)} < \infty.$$

While this definition is very suggestive of the classical definition, we immediately encounter problems. If $p(\cdot)$ is a bounded function, then it is straightforward to show that with this definition $L^{p(\cdot)}(\Omega)$ is a vector space. But if $p(\cdot)$ is unbounded, then this is no longer the case: consider, for example, $\Omega = \mathbb{R}$ and $p(x) = 1 + |x|$. Then $\rho_{p(\cdot)}(1/2) < \infty$, so $1/2 \in L^{p(\cdot)}(\mathbb{R})$, but $\rho_{p(\cdot)}(1) = \infty$. Further, the "modular" $\rho_{p(\cdot)}$ does not immediately convert into a norm: unlike the case of constant exponents, we cannot put a power $1/p(x)$ on the outside of the integral.

The solution to this problem is similar to the approach taken in Orlicz spaces. Recall that given a Young function $\Phi : [0, \infty) \to [0, \infty)$, (i.e., a continuous, strictly increasing, convex function), the Orlicz space $L^\Phi(\Omega)$ consists of all functions $f$ such that for some $\lambda > 0$,

$$\rho_\Phi(f/\lambda) = \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx < \infty.$$

This becomes a Banach space when equipped with the Luxemburg norm:

$$\|f\|_{L^\Phi(\Omega)} = \inf\{\lambda > 0 : \rho_\Phi(f/\lambda) \le 1\}.$$

Similarly, we define $L^{p(\cdot)}(\Omega)$ to be the set of functions $f$ such that for some $\lambda > 0$, $\rho_{p(\cdot)}(f/\lambda) < \infty$, and define the norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\}.$$

When $p(\cdot)$ is a constant this immediately reduces to the usual norm on the classical Lebesgue spaces. With this norm $L^{p(\cdot)}(\Omega)$ becomes a Banach space and it has many properties in common with the classical Lebesgue spaces, especially when $p(\cdot)$ is bounded function. For example, in this case the variable Lebesgue spaces are separable, and the dual space of $L^{p(\cdot)}(\Omega)$ is isomorphic to $L^{p'(\cdot)}(\Omega)$, where the exponent $p'(\cdot)$ is defined pointwise by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

with the convention $1/\infty = 0$.

On the other hand, they diverge from the $L^p$ spaces in several critical ways. Most significantly, the variable Lebesgue spaces are not translation invariant: if $p(\cdot)$ is non-constant in $\mathbb{R}^n$, then there always exists $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ such that $g(x) = f(x+h)$ is not in $L^{p(\cdot)}(\mathbb{R}^n)$ . As a consequence the $L^{p(\cdot)}(\mathbb{R}^n)$ spaces are not rearrangement invariant Banach function spaces, and so a great deal of classical machinery is not applicable. In developing the theory of harmonic analysis on variable Lebesgue spaces much of the effort has gone into overcoming this problem.

When $p(\cdot)$ is unbounded, even more significant differences arise. In this case, $L^{p(\cdot)}(\Omega)$ is no longer separable and bounded functions of compact support are not dense. $L^{p'(\cdot)}(\Omega)$ is isomorphic to a proper subspace of $L^{p(\cdot)}(\Omega)^*$. In addition, it can happen that $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ when $\Omega$ has infinite measure. This is the case, for instance, if $\Omega = \mathbb{R}$ and $p(x) = 1 + |x|$. Further, when $p(\cdot)$ is unbounded, $L^{p(\cdot)}(\Omega)$ can contain unbounded functions whose singularities coincide with those of the exponent. For example, let $\Omega = (0, e^{-e})$ and $p(x) = \log\log(1/x)$. Then $f(x) = x^{\frac{-1}{2p(x)}}$ is unbounded and $f \in L^{p(\cdot)}(\Omega)$.

## 1.2 A Brief History of Variable Lebesgue Spaces

In this section we give an overview of the history of variable Lebesgue spaces. Recounting this history up to the mid 1990s is relatively straightforward, since relatively few mathematicians worked in this area. However, from that time the field has burgeoned, and we only note a few highlights; we make no pretense of being comprehensive. Unlike in subsequent chapters, where references are deferred to the last section, here we will give references to works as we discuss them.

It is generally accepted that the dividing line between the "early" and "modern" periods in the study of variable Lebesgue spaces is the foundational paper of Kováčik and Rákosník [219] from 1991. But the origin of the variable Lebesgue spaces predates their work by 60 years, since they were first studied by Orlicz [290] in 1931. For exponent functions $p(\cdot)$ such that $1 < p(x) < \infty$, he showed that if

$$\int_0^1 |f(x)|^{p(x)}\,dx < \infty,$$

then a necessary and sufficient condition on a function $g$ so that

$$\int_0^1 f(x)g(x)\,dx < \infty$$

is that for some $\lambda > 0$,

$$\int_0^1 \left(\frac{|g(x)|}{\lambda}\right)^{p'(x)}\,dx < \infty.$$

However, this paper is essentially the only contribution of Orlicz to the study of the variable Lebesgue spaces. (The one exception is an oblique reference in a paper with Musielak [275].) Instead, Orlicz turned his attention to the study of the spaces now called Orlicz spaces, which he also introduced in 1931 in a joint paper with Birnbaum [29]. (For the early history of these spaces, see Krasnosel'skiĭ and Rutickiĭ [222].)

The next step in the development of the variable Lebesgue spaces came two decades later in the work of Nakano [278,280] who developed the theory of modular spaces, sometimes referred to as Nakano spaces. A modular space is a topological vector space equipped with a "modular": a generalization of a norm. An important example of a modular space is the function space consisting of all functions $f$ such that for some $\lambda > 0$,

$$\int_\Omega \Phi(x, |f(x)|/\lambda)\,dx < \infty,$$

where $\Phi : \Omega \times [0,\infty) \to [0,\infty]$ is a function such that for almost every $x \in \Omega$, $\Phi(x,\cdot)$ behaves like a Young function. These spaces are referred to as Musielak-Orlicz spaces or generalized Orlicz spaces. (See [274].) They contain a number of function spaces as special cases. If $\Phi(x,t) = \Phi(t)$ is just a function of $t$, they are the Orlicz spaces. If $\Phi(x,t) = t^{p(x)}$, they are the variable Lebesgue spaces. And if $\Phi(x,t) = t^p w(x)$, they become the weighted Lebesgue spaces.

In [278], Nakano introduced the variable Lebesgue spaces as specific examples of modular spaces, and their properties were further developed in [280]. However, research in this area was focused on the topological properties of modular or

Musielak-Orlicz spaces, and the variable Lebesgue spaces were primarily considered interesting examples: see, for example, Yamamuro [349] and Portnov [292, 293]. One exception is the work of Hudzik [171–179] in the late 1970s. He introduced the generalized Sobolev spaces defined over Musielak-Orlicz spaces, and his work foreshadowed many of the results on variable Lebesgue spaces developed more recently.

The variable Lebesgue spaces reappeared independently in the Russian literature, where they were studied as spaces of interest in their own right. They were introduced by Tsenov [344] in 1961. He considered the problem of minimizing the integral

$$\int_0^1 |f(x) - \phi(x)|^{p(x)}\,dx,$$

where $f$ is continuous and $\phi$ is a polynomial of fixed degree. In 1979, Sharapudinov [329] developed the function space theory of the variable Lebesgue spaces on intervals on the real line, introducing the Luxemburg norm (though without reference to the Luxemburg norm, drawing instead on ideas of Kolmogorov [210]), and showing that when $p(\cdot)$ is bounded, $L^{p(\cdot)}([0,1])$ is separable and its dual space is $L^{p'(\cdot)}([0,1])$. In subsequent papers [330–332] he considered various other problems in analysis on the variable Lebesgue spaces. In [331] he was the first to consider questions that involved the regularity of the exponent function $p(\cdot)$, and introduced the local log-Hölder continuity condition,

$$|p(x) - p(y)| \le \frac{C_0}{-\log(|x-y|)}, \qquad |x-y| < \frac{1}{2}, \tag{1.1}$$

that has proved to be of critical importance in the theory of variable Lebesgue spaces. These spaces also appeared in the work of other Russian authors, for example, Kozlov [221].

The most influential work is due to Zhikov [353–356, 358–361], who beginning in 1986 applied the variable Lebesgue spaces to problems in the calculus of variations. Though, as we noted above, the "modern" period in the study of variable Lebesgue spaces is usually said to begin with the 1991 paper of Kováčik and Rákosník, Zhikov's work provides a bridge from the earlier period.

Zhikov was concerned with minimizing the functionals

$$F(u) = \int_\Omega f(x, \nabla u)\,dx$$

when the Lagrangian $f$ satisfies the non-standard growth condition

$$-c_0 + c_1|\xi|^p \le f(x,\xi) \le c_0 + c_2|\xi|^q,$$

where the $c_i$ are positive real constants and $0 < p < q$. A particular example of such a Lagrangian is $f(x,\xi) = |\xi|^{p(x)}$, where $p \le p(x) \le q$. The Euler-Lagrange equation associated to this functional is the $p(\cdot)$-Laplacian:

$$\triangle_{p(\cdot)}u = -\mathrm{div}(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u) = 0.$$

In the mid 1990s, functionals with non-standard growth and the $p(\cdot)$-Laplacian were also studied by Fan [111, 112] and Fan and Zhao [120, 121]. (These problems were developed in a different direction by Marcellini [252, 253].) Since then this field has expanded tremendously. For an overview of the study of partial differential equations with non-standard growth conditions, see the survey articles by Fan [113], Harjulehto, Hästö, Lê and Nuortio [157], and Mingione [263].

A very different problem also provided impetus for the study of the variable Lebesgue spaces. In the early 1990s, Samko and Ross [300, 320] (see also Samko [311, 314]) introduced a Riemann-Liouville fractional derivative of variable order,

$$D_a^{\alpha(\cdot)} f(x) = \frac{1}{\Gamma(1-\alpha(x))} \frac{d}{dx} \int_a^x (x-t)^{-\alpha(x)} f(t)\, dt,$$

and the corresponding variable Riesz potential,

$$I_a^{\alpha(\cdot)} f(x) = \frac{1}{\Gamma(\alpha(x))} \int_a^x (x-t)^{\alpha(x)-1} f(t)\, dt.$$

Investigating the behavior of these operators led naturally to the study of convolution and potential operators on the variable Lebesgue spaces: see Samko [312, 313] (also see Edmunds and Meskhi [103]).

Interest in the variable Lebesgue spaces has increased since the 1990s because of their use in a variety of applications. Foremost among these is the mathematical modeling of electrorheological fluids. These are fluids whose viscosity changes (often dramatically) when exposed to an electric field. See [147, 336] for a discussion of their physical properties and applications. Electrorheological fluids are understood experimentally, but a complete theoretical model is still lacking. In the study of fluid dynamics they are treated as non-Newtonian fluids; in one extensively studied model the energy is given by the integral

$$\int_\Omega |Du(x)|^{p(x)}\, dx,$$

where $Du$ is the symmetric part of the gradient of the velocity field and the exponent is a function of the electric field. Růžička [306, 307] introduced this model, and it was further developed by Acerbi and Mingione [3–5]. This problem was of considerable importance in spurring the development of the theory of variable Lebesgue and Sobolev spaces: see, for example, Diening and Růžička [88–91].

The variable Lebesgue spaces have also been used to model the behavior of other physical problems. Some examples include quasi-Newtonian fluids [360], the thermistor problem [361], fluid flow in porous media [15, 16], and magnetostatics [43].

The variable Lebesgue spaces have been applied to the study of image processing. As early as 1997, Blomgren *et al.* [30] suggested that in image reconstruction, a smoother image could be obtained by an interpolation technique that uses a variable exponent: the appropriate norm is

$$\int_{\Omega} |\nabla u(x)|^{p(\nabla u)}\, dx,$$

where the exponent $p(\cdot)$ decreases monotonically from two to one as $\nabla u$ increases. This approach and related ideas have been explored by a number of authors [1, 2, 32, 45, 46, 156, 236, 237, 348] in recent years.

In much of the work described above there was a need to extend the techniques and results of harmonic analysis to the variable Lebesgue spaces. It soon became clear that a central problem was determining conditions on an exponent $p(\cdot)$ so that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. The first major result was due to Diening [77], who showed that it is sufficient to assume that $p(\cdot)$ satisfies the local log-Hölder condition (1.1) and is constant outside of a large ball. This result was generalized in [62] (see also [42,56]), where it was shown that it was sufficient to assume that (1.1) holds and $p(\cdot)$ is log-Hölder continuous at infinity: there exists $p_\infty$ such that

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}.$$

Independently, Nekvinda [282] showed that it was sufficient to assume that $p(\cdot)$ satisfies a somewhat weaker integral decay condition. The log-Hölder conditions are the sharpest possible pointwise conditions (see Pick and Růžička [291] and [62]) but they are not necessary: see Nekvinda [284], Kopaliani [215] and Lerner [231]. Diening [79] has given a necessary and sufficient condition that is difficult to check but has important theoretical consequences. The importance of these results was reinforced by the work in [61], where it was shown that the theory of Rubio de Francia extrapolation could be extended to the variable Lebesgue spaces. This allows the theory of weighted norm inequalities to be used to prove that a multitude of operators (such as singular integrals) are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ whenever the maximal operator is.

## 1.3 The Organization of this Book

We begin in Chap. 2 with the function space properties of the variable Lebesgue spaces. We develop the properties of the modular $\rho_{p(\cdot)}$ and the norm, including versions of Hölder's and Minkowski's inequalities. We consider the various embeddings of variable Sobolev spaces into themselves and into the classical Lebesgue spaces. We next treat convergence in norm, in modular and in measure, prove that the spaces $L^{p(\cdot)}(\Omega)$ are complete, and determine conditions for various canonical sets of functions (such as bounded functions of compact support) to be dense. We discuss duality and show that $L^{p(\cdot)}(\Omega)^*$ is isomorphic to $L^{p'(\cdot)}(\Omega)$ if and only if $p(\cdot)$ is bounded. Finally, we give a generalization of the Lebesgue differentiation theorem.

Throughout this chapter we have proved results in the fullest generality possible, though some very technical theorems (such as the fine properties of convergence when $p(\cdot)$ is unbounded) are relegated to the notes in the last section. In particular, while many properties of the variable Lebesgue spaces are easier to prove when $p(\cdot)$ is bounded, we have worked out the theory in full generality to illuminate the precise differences between bounded and unbounded exponents.

In writing this chapter we have chosen to prove everything directly, following wherever possible the theory of the classical Lebesgue spaces. To illustrate key ideas we have included a number of concrete examples. There are other approaches to the properties of the variable Lebesgue spaces. The most common is to treat them as special cases of the Musielak-Orlicz spaces. This is the point of view adopted in the recent book by Diening, Harjulehto, Hästö and Růžička [82]. Similarly, many of the properties of variable Lebesgue spaces can be gotten by viewing them as particular examples of Banach function spaces (see Bennett and Sharpley [25]). These more abstract approaches have their advantages, but we believe strongly that our "nuts and bolts" approach has the singular advantage that any reader who works through the details will become intimately familiar with the variable Lebesgue spaces.

In the subsequent chapters we turn to the study of harmonic analysis on the variable Lebesgue spaces. For us this meant beginning with the theory of the Hardy-Littlewood maximal operator, which we do in Chaps. 3 and 4. In Chap. 3 we develop the theory assuming that the exponent function satisfies log-Hölder continuity conditions, both locally and at infinity. These conditions are not necessary, but they are versatile and have proved to be very important in many applications. Our main tool for working with the maximal operator is the Calderón-Zygmund decomposition using dyadic cubes. This is a relatively recent approach in the variable Lebesgue spaces, but one which has proved to be remarkably flexible. The earlier techniques used to study the maximal operator are described in the notes.

In Chap. 4 we continue to study the maximal operator, but now our focus is on weakening the log-Hölder continuity conditions, a problem which is still an area of active research. The proof of the boundedness of the maximal operator that we give in Chap. 3 makes it very easy to consider the conditions used locally and at infinity separately. We first give a weaker condition at infinity, introducing the $N_\infty$ condition of Nekvinda, and then constructing examples to show that this condition is still quite far from necessary. To understand the behavior of the maximal operator locally, we pause to introduce the theory of Muckenhoupt $A_p$ weights and weighted norm inequalities. These ideas have, somewhat surprisingly, proved to be both an important motivation and a useful tool for studying the variable Lebesgue spaces. For completeness we have given a self-contained presentation of this theory. Using these ideas we then consider a weaker local condition, introducing the $K_0$ condition of Kopaliani. This condition is necessary and sufficient for the maximal operator to be bounded on $L^{p(\cdot)}(\Omega)$ when $\Omega$ is bounded; an important open question is to find an analog of this condition to use at infinity. We conclude this chapter with a discussion (without proof) of a necessary and sufficient condition, due to Diening, for the maximal operator to be bounded. While not easy to apply in practice, it has important theoretical implications.

In Chap. 5 we study some other important operators from harmonic analysis. We begin with the theory of convolution operators and approximate identities on variable Lebesgue spaces. The fact that these spaces are not translation invariant means that Young's inequality fails to hold. However, if we assume sufficient regularity on the exponent $p(\cdot)$, then we prove that approximate identities converge pointwise, in measure, and in norm. Throughout this chapter we express this regularity in terms of boundedness properties of the maximal operator. To study other operators, we extend to the variable Lebesgue spaces one of the most powerful ideas in the theory of weighted norm inequalities: the Rubio de Francia theory of extrapolation. Using this, we show that if an operator satisfies weighted norm inequalities on the classical Lebesgue spaces, then it satisfies norm inequalities on variable Lebesgue spaces, assuming some regularity of the exponent $p(\cdot)$. We apply extrapolation to study two operators in detail: singular integral operators and Riesz potentials. These are important in applications and provide good models for using extrapolation to study other operators. While we state the basic properties of these operators that we use, we have chosen to omit these proofs and refer the reader to any of a number of standard reference works.

Finally, in Chap. 6 we develop the basic theory of variable Sobolev spaces, an essential tool in the study of the calculus of variations and partial differential equations in the variable Lebesgue spaces. We first prove their basic function space properties. We then consider the question of when smooth functions are dense in the variable Sobolev space $W^{1,p(\cdot)}(\Omega)$. We prove the Meyers-Serrin theorem assuming that $p(\cdot)$ is regular; we also give an example to show that some degree of regularity is required. As we did in Chap. 5, we state our regularity hypotheses in terms of the boundedness properties of the maximal operator. We next prove generalizations of the Poincaré inequality and the Sobolev embedding theorem to the variable Lebesgue spaces.

Unlike Chap. 2, our intention in Chap. 6 was not to give a comprehensive development of the theory of variable Sobolev spaces. Rather, we wanted to prove some essential properties to demonstrate the application of the ideas and techniques from the previous chapters. In particular, we deliberately omitted any discussion of questions related to the regularity of the boundary of $\Omega$.

We conclude every chapter with extensive references for the results in the body of the chapter. We have attempted to make these notes as comprehensive as possible, both to make clear the historical development of the field and to give proper credit to the many people who have contributed to it. We have also included in the notes discussions of additional topics to illustrate the many directions the field has evolved and to refer readers to results we felt were important but decided not to include in the text. In an appendix we have gathered together some open problems which we believe are important for future research on the variable Lebesgue spaces.

## 1.4 Prerequisites and Notation

Throughout this book we assume that the reader knows classical real analysis through the Lebesgue integral and some of the basic facts from functional analysis that are usually presented in a standard graduate analysis course. The books by Brezis [37], Royden [301] or Rudin [305] provide more than adequate background. In line with our aim of making this work accessible to students, we have generously scattered references throughout the text, particularly in Chap. 2, for the classical results we use.

Our treatment of the maximal operator in Chap. 3, weights and weighted norm inequalities in Chap. 4, and convolution operators and Rubio de Francia extrapolation in Chap. 5 is almost completely self-contained; hopefully it will provide a concise introduction for readers who are not familiar with these essential tools of harmonic analysis. The most important exception, however, is in our treatment of singular integral operators and Riesz potentials in Chap. 5. We state the principal results we use but we do not include any proofs. These sections will probably be more readily understood by readers who have had some prior exposure to this material. For readers who need more information, we refer to the books by Duoandikoetxea [96], García-Cuerva and Rubio de Francia [140] and Grafakos [143]. Similarly, in Chap. 6 we presume that the reader has some basic familiarity with weak derivatives and the classical Sobolev space theory; the works by Adams and Fournier [7], Maz'ja [260], Tartar [343] and Ziemer [363] contain far more information than is required. A brief introduction can be found in Gilbarg and Trudinger [142].

The following notation will be used throughout the text.

We will always use the convention that $0 \cdot \infty = 0$ and $1/\infty = 0$.

The dimension of the underlying space (i.e., $\mathbb{R}^n$) will be denoted by $n$; the variable $n$ is never used as an index or for enumeration. Points $x \in \mathbb{R}^n$ have their coordinates denoted by superscripts: $x = (x^1, \dots, x^n)$. The norm on $\mathbb{R}^n$ will be denoted by $|\cdot|$. For $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x)$ will denote the open ball centered at $x$. Given a ball $B$, $2B$ will denote the ball with the same center and twice the radius. If $Q \subset \mathbb{R}^n$ is a cube, then $\ell(Q)$ will denote the length of each edge.

A set $E \subset \mathbb{R}^n$ will always be assumed to be measurable. Given $E$, $\overline{E}$ will denote the closure of $E$ and $\partial E$ will denote the boundary of $E$: $\partial E = \overline{E} \cap \overline{(\mathbb{R}^n \setminus E)}$. The Lebesgue measure of $E$ will be denoted by $|E|$, and $\chi_E$ will be the characteristic function of $E$:

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

If $E$ is the empty set, then $\chi_E = 0$.

Measurable functions are real-valued and defined up to sets of measure zero: given two measurable functions $f$ and $g$, we will write $f = g$ if $f(x) = g(x)$ almost everywhere. If $f$ is a continuous function, the support of $f$, denoted $\mathrm{supp}(f)$, is the closure of the set $\{x \in \mathbb{R}^n : f(x) \neq 0\}$. If $f$ is a measurable function, let $A$ denote the union of all open sets $B$ such that $f(x) = 0$ for almost every $x \in B$. Then $\operatorname{supp}(f)$ is defined to be $\mathbb{R}^n \setminus A$. Given a measurable function $f$, the sign of $f$, $\operatorname{sgn} f$, is the function

$$\operatorname{sgn} f(x) = \begin{cases} 1 & f(x) > 0 \\ -1 & f(x) < 0 \\ 0 & f(x) = 0. \end{cases}$$

Given a measurable function $f$ and a set $E$, if $0 < |E| < \infty$, define

$$f_E = \fint_E f(y)\,dy = \frac{1}{|E|}\int_E f(y)\,dy.$$

Given a set $\Omega \subset \mathbb{R}^n$, $|\Omega| > 0$, (which will hereafter be referred to as a domain) define the classical Lebesgue spaces $L^p(\Omega)$ to be the Banach function spaces with norm

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p} & 1 \le p < \infty \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| & p = \infty. \end{cases}$$

If there is no confusion about the domain, we will write $\|f\|_p$.

Given $p$, $1 \le p \le \infty$, define the conjugate exponent $p'$ by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The symbol "′" will always be used to denote conjugate exponents and will never be used for differentiation. For derivatives of functions of one variable we will use the notation $Df = df/dx$.

Given an open set $\Omega$, let $C(\Omega) = C^0(\Omega)$ denote the set of functions that are continuous on $\Omega$, and $C(\overline{\Omega})$ the set of functions continuous up to the boundary. For $1 \le k \le \infty$, let $C^k(\Omega)$ denote the set of functions that have continuous partial derivatives of all orders less than or equal to $k$. If $k = \infty$, then $C^\infty(\Omega)$ is referred to as the set of smooth functions on $\Omega$. For $k \ge 0$, let $C_c^k(\Omega)$ be the set of all functions $f \in C^k(\Omega)$ such that $\operatorname{supp}(f)$ is compact and contained in $\Omega$. The Schwartz functions are the functions $f \in C^\infty(\mathbb{R}^n)$ such that $f$ and all its partial derivatives decay more quickly than $|x|^{-k}$ at infinity, for any $k > 0$.

For brevity we will omit the parentheses from expressions such as $[\log(x)]^a$ and instead write $\log(x)^a$.

Throughout the text the letters $C$ and $c$ will denote a constant whose value may depend on certain (implicitly) specified parameters and which may change even between lines in a single expression. To denote the dependence of the constant on parameters $X, Y, \dots$, we will write $C(X, Y, \dots)$.

# Chapter 2
# Structure of Variable Lebesgue Spaces

In this chapter we give a precise definition of the variable Lebesgue spaces and establish their structural properties as Banach function spaces. Throughout this chapter we will generally assume that $\Omega$ is a Lebesgue measurable subset of $\mathbb{R}^n$ with positive measure. Occasionally we will have to assume more, but we make it explicit if we do.

## 2.1 Exponent Functions

We begin with a fundamental definition.

**Definition 2.1.** Given a set $\Omega$, let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p(\cdot) : \Omega \to [1, \infty]$. The elements of $\mathcal{P}(\Omega)$ are called exponent functions or simply exponents. In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p(\cdot)$.

Some examples of exponent functions on $\Omega = \mathbb{R}$ include $p(x) = p$ for some constant $p$, $1 \leq p \leq \infty$, or $p(x) = 2 + \sin(x)$. Exponent functions can be unbounded: for instance, if $\Omega = (1, \infty)$, let $p(x) = x$, and if $\Omega = (0, 1)$, let $p(x) = 1/x$. We will consider these last two frequently, as they will provide good examples of the differences between bounded and unbounded exponent functions.

We define some notation to describe the range of exponent functions. Given $p(\cdot) \in \mathcal{P}(\Omega)$ and a set $E \subset \Omega$, let

$$p_-(E) = \operatorname*{ess\,inf}_{x \in E} p(x), \qquad p_+(E) = \operatorname*{ess\,sup}_{x \in E} p(x).$$

If the domain is clear we will simply write $p_- = p_-(\Omega)$, $p_+ = p_+(\Omega)$. As is the case for the classical Lebesgue spaces, we will encounter different behavior depending on whether $p(x) = 1$, $1 < p(x) < \infty$, or $p(x) = \infty$. Therefore, we define three canonical subsets of $\Omega$:

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$$\Omega_\infty^{p(\cdot)} = \{x \in \Omega : p(x) = \infty\},$$

$$\Omega_1^{p(\cdot)} = \{x \in \Omega : p(x) = 1\},$$

$$\Omega_*^{p(\cdot)} = \{x \in \Omega : 1 < p(x) < \infty\}.$$

Again, for simplicity we will omit the superscript $p(\cdot)$ if there is no possibility of confusion. Since $p(\cdot)$ is a measurable function, these sets are only defined up to sets of measure zero; however, in practice this will have no effect. Below, the value of certain constants will depend on whether these sets have positive measure; if they do we will use the fact that, for instance, $\|\chi_{\Omega_1^{p(\cdot)}}\|_\infty = 1$.

Given $p(\cdot)$, we define the conjugate exponent function $p'(\cdot)$ by the formula

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \qquad x \in \Omega,$$

with the convention that $1/\infty = 0$. Since $p(\cdot)$ is a function, the notation $p'(\cdot)$ can be mistaken for the derivative of $p(\cdot)$, but we will never use the symbol "$'$" in this sense.

The notation $p'$ will also be used to denote the conjugate of a constant exponent. The operation of taking the supremum/infimum of an exponent does not commute with forming the conjugate exponent. In fact, a straightforward computation shows that

$$\big(p'(\cdot)\big)_+ = \big(p_-\big)', \quad \big(p'(\cdot)\big)_- = \big(p_+\big)'.$$

For simplicity we will omit one set of parentheses and write the left-hand side of each equality as $p'(\cdot)_+$ and $p'(\cdot)_-$. We will always avoid ambiguous expressions such as $p'_+$.

Though the basic theory of variable Lebesgue spaces only requires that $p(\cdot)$ be a measurable function, in many applications in subsequent chapters we will often assume that $p(\cdot)$ has some additional regularity. In particular, there are two continuity conditions that are of such importance that we want to establish notation for them.

**Definition 2.2.** Given $\Omega$ and a function $r(\cdot) : \Omega \to \mathbb{R}$, we say that $r(\cdot)$ is locally log-Hölder continuous, and denote this by $r(\cdot) \in LH_0(\Omega)$, if there exists a constant $C_0$ such that for all $x,\ y \in \Omega$, $|x - y| < 1/2$,

$$|r(x) - r(y)| \le \frac{C_0}{-\log(|x - y|)}.$$

We say that $r(\cdot)$ is log-Hölder continuous at infinity, and denote this by $r(\cdot) \in LH_\infty(\Omega)$, if there exist constants $C_\infty$ and $r_\infty$ such that for all $x \in \Omega$,

$$|r(x) - r_\infty| \le \frac{C_\infty}{\log(e + |x|)}.$$

If $r(\cdot)$ is log-Hölder continuous locally and at infinity, we will denote this by writing $r(\cdot) \in LH(\Omega)$. If there is no confusion about the domain we will sometimes write $LH_0$, $LH_\infty$ or $LH$.

In practice we will often assume that $p(\cdot)$ or $1/p(\cdot)$ is contained in one of the log-Hölder continuity classes. In the latter case, if $p(\cdot)$ is unbounded at infinity we let $p_\infty = \infty$ and use the convention $1/p_\infty = 0$.

The next result is an immediate consequence of Definition 2.2.

**Proposition 2.3.** *Given a domain $\Omega$:*

1. *If $r(\cdot) \in LH_0(\Omega)$, then $r(\cdot)$ is uniformly continuous and $r(\cdot) \in L^\infty(E)$ for every bounded subset $E \subset \Omega$.*
2. *If $r(\cdot) \in LH_\infty(\Omega)$, then $r(\cdot) \in L^\infty(\Omega)$.*
3. *If $\Omega$ is bounded and $r(\cdot) \in L^\infty(\Omega)$, then $r(\cdot) \in LH_\infty(\Omega)$, with a constant $C_\infty$ depending on $\|r(\cdot)\|_\infty$, the diameter of $\Omega$, and its distance from the origin.*
4. *The inclusion $r(\cdot) \in LH_\infty(\Omega)$ is equivalent to the existence of a constant $C$ such that for all $x,\ y \in \Omega$, $|y| \geq |x|$,*

$$|r(x) - r(y)| \leq \frac{C}{\log(e + |x|)}.$$

5. *If $p_+ < \infty$, then $p(\cdot) \in LH_0(\Omega)$ is equivalent to assuming $r(\cdot) = 1/p(\cdot) \in LH_0(\Omega)$: in fact, given $x,\ y \in \Omega$,*

$$\left|\frac{p(x) - p(y)}{(p_+)^2}\right| \leq \left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \left|\frac{p(x) - p(y)}{(p_-)^2}\right|.$$

*Similarly, $p(\cdot) \in LH_\infty(\Omega)$ if and only if $r(\cdot) = 1/p(\cdot) \in LH_\infty(\Omega)$.*

Given two domains $\widetilde{\Omega} \subset \Omega$, we clearly have that if $p(\cdot) \in LH_0(\Omega)$, then $\tilde{p}(\cdot) = p(\cdot)\big|_{\widetilde{\Omega}} \in LH_0(\widetilde{\Omega})$, and similarly for the class $LH_\infty$. In applications, we will be concerned with the converse: given an exponent function in $LH(\widetilde{\Omega})$, can it be extended to a function in $LH(\Omega)$? The answer is yes as the next result shows.

**Lemma 2.4.** *Given a set $\Omega \subset \mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p(\cdot) \in LH(\Omega)$, there exists a function $\tilde{p}(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that:*

1. *$\tilde{p} \in LH$;*
2. *$\tilde{p}(x) = p(x)$, $x \in \Omega$;*
3. *$\tilde{p}_- = p_-$ and $\tilde{p}_+ = p_+$.*

*Remark 2.5.* It follows from the proof below that if we only have that $p(\cdot) \in LH_0(\Omega)$ or $LH_\infty(\Omega)$ we can extend it to a function in the same class on $\mathbb{R}^n$.

*Proof.* Since $p(\cdot)$ is bounded and uniformly continuous, by a well-known result it extends to a continuous function on $\overline{\Omega}$; denote this extension by $p(\cdot)$ as well. Then it is immediate that $p(\cdot) \in LH(\overline{\Omega})$, $p_-(\Omega) = p_-(\overline{\Omega})$, and $p_-(\Omega) = p_-(\overline{\Omega})$.

To extend $p(\cdot)$ from $\overline{\Omega}$ to all of $\mathbb{R}^n$ we first consider the case when $\overline{\Omega}$ is unbounded; the case when $\overline{\Omega}$ is bounded is simpler and will be sketched below. Define a new function $r(\cdot)$ by $r(x) = p(x) - p_\infty$. Then $r(\cdot)$ is still bounded (though no longer necessarily positive) and $r(\cdot) \in LH(\overline{\Omega})$.

We will extend $r(\cdot)$ to all of $\mathbb{R}^n$. If we define $\omega(t) = 1/\log(e/2t)$, $0 < t \leq 1/2$, and $\omega(t) = 1$ for $t \geq 1/2$, then a straightforward calculation shows that $\omega(t)/t$ is a decreasing function and $\omega(2t) \leq C\,\omega(t)$. Further, since $\log(e/2t) \approx \log(1/t)$, $0 < t < 1/2$, and since $r(\cdot)$ is bounded, $|r(x) - r(y)| \leq C\omega(|x-y|)$ for all $x, y \in \overline{\Omega}$. Therefore, there exists a function $\tilde{r}(\cdot)$ on $\mathbb{R}^n$ such that $\tilde{r}(x) = r(x)$, $x \in \overline{\Omega}$, and such that $\tilde{r}(\cdot) \in LH_0(\mathbb{R}^n)$, with a constant that depends only on $p(\cdot)$ and the $LH_0$ constant, and not on $\Omega$. For a proof, see Stein [339, Corollary 2.2.3, p. 175]. Briefly, and using the terminology of this reference, the function $\tilde{r}(\cdot)$ is defined as follows. Form the Whitney decomposition $\{Q_k\}$ of $\mathbb{R}^n \setminus \overline{\Omega}$ and let $\{\phi_k^*\}$ be a partition of unity subordinate to this decomposition. In each cube $Q_k$, fix a point $p_k \in \overline{\Omega}$ such that $\operatorname{dist}(p_k, Q_k) = \operatorname{dist}(\overline{\Omega}, Q_k)$. Then for $x \in \mathbb{R}^n \setminus \overline{\Omega}$,

$$\tilde{r}(x) = \sum_k r(p_k)\phi_k^*(x).$$

It follows immediately from this definition that for all $x \in \mathbb{R}^n$, $r_- \leq \tilde{r}(x) \leq r_+$. However, $\tilde{r}(\cdot)$ need not be in $LH_\infty$, so we must modify it slightly. To do so we need the following observation: if $f_1, f_2$ are functions such that $|f_i(x) - f_i(y)| \leq C\omega(|x-y|)$, $x, , y \in \mathbb{R}^n$, $i = 1, 2$, then $\min(f_1, f_2)$ and $\max(f_1, f_2)$ satisfy the same inequality. The proof of this observation consists of a number of very similar cases. For instance, suppose $\min(f_1(x), f_2(x)) = f_1(x)$ and $\min(f_1(y), f_2(y)) = f_2(y)$. Then

$$\begin{aligned} f_1(x) - f_2(y) &\leq f_2(x) - f_2(y) \leq C\omega(|x-y|), \\ f_2(y) - f_1(x) &\leq f_1(y) - f_1(x) \leq C\omega(|x-y|). \end{aligned}$$

Hence,

$$|\min(f_1(x), f_2(x)) - \min(f_1(y), f_2(y))| = |f_1(x) - f_2(y)| \leq C\omega(|x-y|).$$

It follows immediately from this observation that

$$s(x) = \max(\min(\tilde{r}(x), C_\infty/\log(e+|x|)), -C_\infty/\log(e+|x|))$$

is in $LH(\mathbb{R}^n)$. Therefore, if we define

$$\tilde{p}(x) = s(x) + p_\infty,$$

then (1)–(3) hold.

Finally, if $\Omega$ is bounded, we define $r(x) = p(x) - p_+$ and repeat the above argument essentially without change. □

## 2.2 The Modular

Intuitively, given an exponent function $p(\cdot) \in \mathcal{P}(\Omega)$, we want to define the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $f$ such that

$$\int_\Omega |f(x)|^{p(x)}\,dx < \infty.$$

There are problems with this approach, the most obvious being that it does not work when $\Omega_\infty$ has positive measure. To remedy them, we begin with the following definition.

**Definition 2.6.** Given $\Omega$, $p(\cdot) \in \mathcal{P}(\Omega)$ and a Lebesgue measurable function $f$, define the modular functional (or simply the modular) associated with $p(\cdot)$ by

$$\rho_{p(\cdot),\Omega}(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty)}.$$

If $f$ is unbounded on $\Omega_\infty$ or if $f(\cdot)^{p(\cdot)} \notin L^1(\Omega\setminus\Omega_\infty)$, we define $\rho_{p(\cdot),\Omega}(f) = +\infty$. When $|\Omega_\infty| = 0$, in particular when $p_+ < \infty$, we let $\|f\|_{L^\infty(\Omega_\infty)} = 0$; when $|\Omega \setminus \Omega_\infty| = 0$, then $\rho_{p(\cdot),\Omega}(f) = \|f\|_{L^\infty(\Omega_\infty)}$. In situations where there is no ambiguity we will simply write $\rho_{p(\cdot)}(f)$ or $\rho(f)$.

We will use the modular to define the space $L^{p(\cdot)}(\Omega)$ in the next section. In preparation, we give here its fundamental properties.

**Proposition 2.7.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$:*

1. *For all $f$, $\rho(f) \geq 0$ and $\rho(|f|) = \rho(f)$.*
2. *$\rho(f) = 0$ if and only if $f(x) = 0$ for almost every $x \in \Omega$.*
3. *If $\rho(f) < \infty$, then $f(x) < \infty$ for almost every $x \in \Omega$.*
4. *$\rho$ is convex: given $\alpha$, $\beta \geq 0$, $\alpha + \beta = 1$,*

$$\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g).$$

5. *$\rho$ is order preserving: if $|f(x)| \geq |g(x)|$ a.e., then $\rho(f) \geq \rho(g)$.*
6. *$\rho$ has the continuity property: if for some $\Lambda > 0$, $\rho(f/\Lambda) < \infty$, then the function $\lambda \mapsto \rho(f/\lambda)$ is continuous and decreasing on $[\Lambda, \infty)$. Further, $\rho(f/\lambda) \to 0$ as $\lambda \to \infty$.*

An immediate consequence of the convexity of $\rho$ is that if $\alpha > 1$, then $\alpha\rho(f) \leq \rho(\alpha f)$, and if $0 < \alpha < 1$, then $\rho(\alpha f) \leq \alpha\rho(f)$. We will often invoke this property by referring to the convexity of the modular.

*Proof.* Property (1) is immediate from the definition of the modular, and Properties (2), (3) and (5) follow from the properties of the $L^1$ and $L^\infty$ norms.

Property (4) follows since the $L^\infty$ norm is convex and since for almost every $x \in \Omega \setminus \Omega_\infty$, the function $t \mapsto t^{p(x)}$ is convex.

To prove (6), note that by Property (5), if $\lambda \geq \Lambda$, then $\rho(f/\lambda)$ is a decreasing function, and by the dominated convergence theorem (applied to the integral) it is continuous and tends to 0 as $\lambda \to \infty$. □

*Remark 2.8.* The modular does not satisfy the triangle inequality, i.e., $\rho(f+g) \leq \rho(f)+\rho(g)$. However, there is a substitute that is sometimes useful. For $1 \leq p < \infty$ and $a, b \geq 0$, $(a+b)^p \leq 2^{p-1}(a^p+b^p)$. Therefore, for almost every $x \in \Omega \setminus \Omega_\infty$,

$$|f(x)+g(x)|^{p(x)} \leq 2^{p(x)-1}(|f(x)|^{p(x)}+|g(x)|^{p(x)});$$

in particular, if $p_+ < \infty$,

$$\rho(f+g) \leq 2^{p_+-1}\big(\rho(f)+\rho(g)\big).$$

We will refer to this as the modular triangle inequality.

## 2.3 The Space $L^{p(\cdot)}(\Omega)$

The most basic property of the classical Lebesgue space $L^p$ is that it is a Banach space: a normed vector space that is complete with respect to the norm. Here we define $L^{p(\cdot)}(\Omega)$ and use the properties of the modular to show that it is a normed vector space; we defer the proof that it is complete until Sect. 2.7, after we establish the requisite convergence properties of the norm.

**Definition 2.9.** Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, define $L^{p(\cdot)}(\Omega)$ to be the set of Lebesgue measurable functions $f$ such that $\rho(f/\lambda) < \infty$ for some $\lambda > 0$. Define $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ to be the set of measurable functions $f$ such that $f \in L^{p(\cdot)}(K)$ for every compact set $K \subset \Omega$.

*Remark 2.10.* By Proposition 2.7, Property (3), if $f \in L^{p(\cdot)}(\Omega)$, then $f$ is finite almost everywhere.

Since we are dealing with measurable functions, we will adopt the usual convention that two functions are the same if they are equal almost everywhere; in particular, we will say $f \equiv 0$ if $f(x) = 0$ except on a set of measure 0.

In defining $L^{p(\cdot)}(\Omega)$ we do not restrict ourselves to a single value of $\lambda$: for instance, we do not take $L^{p(\cdot)}(\Omega)$ to be the set of all $f$ such that $\rho(f) < \infty$. We do so in order to make the space homogeneous when $p_+(\Omega \setminus \Omega_\infty) = \infty$.

*Example 2.11.* Let $\Omega = (1,\infty)$, $p(x) = x$, and $f(x) = 1$. Then $\rho(f) = \infty$, but for all $\lambda > 1$,

$$\rho(f/\lambda) = \int_1^\infty \lambda^{-x}\,dx = \frac{1}{\lambda\log(\lambda)} < \infty.$$

Similarly, if we let $\Omega = (0,1)$ and $p(x) = 1/x$, and again let $f(x) = 1$, then $\rho(f) < \infty$, but $\rho(f/\lambda) = \infty$ for all $\lambda < 1$.

However, this technicality is only necessary if $p(\cdot)$ is unbounded: more precisely, if $p_+(\Omega \setminus \Omega_\infty) < \infty$, then $L^{p(\cdot)}(\Omega)$ coincides with the set of functions such that $\rho(f)$ is finite.

**Proposition 2.12.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, then the property that $f \in L^{p(\cdot)}(\Omega)$ if and only if*

$$\rho(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega)} < \infty$$

*is equivalent to assuming that $p_- = \infty$ or $p_+(\Omega \setminus \Omega_\infty) < \infty$.*

*Proof.* We first assume that $p_- = \infty$ or $p_+(\Omega \setminus \Omega_\infty) < \infty$. Clearly, if $\rho(f) < \infty$, then $f \in L^{p(\cdot)}(\Omega)$. Conversely, if $f \in L^{p(\cdot)}(\Omega)$, then by Property (5) in Proposition 2.7 we have that $\rho(f/\lambda) < \infty$ for some $\lambda > 1$. But then

$$\rho(f) = \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|\lambda}{\lambda}\right)^{p(x)} dx + \lambda\|f/\lambda\|_{L^\infty(\Omega_\infty)} \le \lambda^{p_+(\Omega\setminus\Omega_\infty)}\rho(f/\lambda) < \infty.$$

Now suppose that $p_- < \infty$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$. We will construct a function $f$ such that $\rho(f) = \infty$ but $f \in L^{p(\cdot)}(\Omega)$. By the definition of the essential supremum, there exists a sequence of sets $\{E_k\}$ with finite measure such that:

1. $E_k \subset \Omega \setminus \Omega_\infty$,
2. $E_{k+1} \subset E_k$ and $|E_k \setminus E_{k+1}| > 0$,
3. $|E_k| \to 0$,
4. If $x \in E_k$, $p(x) \ge p_k > k$.

Define the function $f$ by

$$f(x) = \left(\sum_{k=1}^{\infty} \frac{1}{|E_k \setminus E_{k+1}|}\chi_{E_k\setminus E_{k+1}}(x)\right)^{1/p(x)}.$$

Then for any $\lambda > 1$,

$$\rho(f/\lambda) = \sum_{k=1}^{\infty} \fint_{E_k\setminus E_{k+1}} \lambda^{-p(x)}\,dx \le \sum_{k=1}^{\infty} \lambda^{-k} < \infty,$$

and the same computation shows that $\rho(f) = \infty$. □

*Remark 2.13.* The construction in the second half of the proof of Proposition 2.12 will be used frequently to prove that there are essential differences among the variable Lebesgue spaces that depend on whether $p_+(\Omega \setminus \Omega_\infty)$ is finite or infinite.

This ability to "pull" a constant out of the modular when $p_+ < \infty$ is very useful, and makes the study of variable Lebesgue spaces in this case much simpler. The proof of Proposition 2.12 is easily modified to prove the following inequalities.

**Proposition 2.14.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $p_+(\Omega \setminus \Omega_\infty) < \infty$, then for all $\lambda \geq 1$,*

$$\rho(\lambda f) \leq \lambda^{p_+(\Omega\setminus\Omega_\infty)}\rho(f).$$

*Moreover, if $p_+ < \infty$ and $\lambda \geq 1$, then*

$$\lambda^{p_-}\rho(f) \leq \rho(\lambda f) \leq \lambda^{p_+}\rho(f),$$

*and if $0 < \lambda < 1$, the reverse inequalities are true.*

**Theorem 2.15.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, $L^{p(\cdot)}(\Omega)$ is a vector space.*

*Proof.* Since the set of all Lebesgue measurable functions is itself a vector space, and since $0 \in L^{p(\cdot)}(\Omega)$, it will suffice to show that for all $\alpha, \beta \in \mathbb{R}$, not both 0, if $f, g \in L^{p(\cdot)}(\Omega)$, then $\alpha f + \beta g \in L^{p(\cdot)}(\Omega)$. By Property (5) in Proposition 2.7, there exists $\lambda > 0$ such that $\rho(f/\lambda), \rho(g/\lambda) < \infty$. Therefore, by Properties (1), (3) and (4) of the same proposition, if we let $\mu = \big(|\alpha| + |\beta|\big)\lambda$, then

$$\begin{aligned}\rho\left(\frac{\alpha f + \beta g}{\mu}\right) = \rho\left(\frac{|\alpha f + \beta g|}{\mu}\right) &\leq \rho\left(\frac{|\alpha|}{|\alpha| + |\beta|}\frac{|f|}{\lambda} + \frac{|\beta|}{|\alpha| + |\beta|}\frac{|g|}{\lambda}\right)\\ &\leq \frac{|\alpha|}{|\alpha| + |\beta|}\rho(f/\lambda) + \frac{|\beta|}{|\alpha| + |\beta|}\rho(g/\lambda) < \infty.\end{aligned}$$

□

On the classical Lebesgue spaces, if $1 \leq p < \infty$, then the norm is gotten directly from the modular:

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p}.$$

Such a definition obviously fails since we cannot replace the constant exponent $1/p$ outside the integral with the exponent function $1/p(\cdot)$. The solution is a more subtle approach which is similar to that used to define the Luxemburg norm on Orlicz spaces.

**Definition 2.16.** Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $f$ is a measurable function, define

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{\lambda > 0 : \rho_{p(\cdot),\Omega}(f/\lambda) \leq 1\big\}.$$

If the set on the right-hand side is empty we define $\|f\|_{L^{p(\cdot)}(\Omega)} = \infty$. If there is no ambiguity over the domain $\Omega$, we will often write $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^{p(\cdot)}(\Omega)}$.

By Property (6) of Proposition 2.7, $\|f\|_{L^{p(\cdot)}(\Omega)} < \infty$ for all $f \in L^{p(\cdot)}(\Omega)$; equivalently, $\|f\|_{L^{p(\cdot)}(\Omega)} = \infty$ when $f \notin L^{p(\cdot)}(\Omega)$. When $p(\cdot) = p$, $1 \le p \le \infty$, Definition 2.16 is equivalent to the classical norm on $L^p(\Omega)$: if $p < \infty$ and

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^p dx = 1,$$

then $\lambda = \|f\|_{L^p(\Omega)}$; the same is true if $p = \infty$.

Given two domains $\Omega$ and $\widetilde{\Omega}$, if $\widetilde{\Omega} \subset \Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, then $\tilde{p}(\cdot) = p(\cdot)\big|_{\tilde{\Omega}} \in \mathcal{P}(\widetilde{\Omega})$ and it is immediate from the definition of the norm that for $f \in L^{p(\cdot)}(\Omega)$,

$$\|f\|_{L^{\tilde{p}(\cdot)}(\tilde{\Omega})} = \|f\chi_{\tilde{\Omega}}\|_{L^{p(\cdot)}(\Omega)}.$$

Hereafter we will implicitly make these restrictions without comment and simply write $\|f\|_{L^{p(\cdot)}(\tilde{\Omega})}$, etc. Conversely, given $p(\cdot) \in \mathcal{P}(\widetilde{\Omega})$ and $f \in L^{p(\cdot)}(\widetilde{\Omega})$, we can extend both to $\Omega$ by defining $f(x) = 0$ for $x \in \Omega \setminus \widetilde{\Omega}$ and defining $p(\cdot)$ arbitrarily on $\Omega \setminus \widetilde{\Omega}$. If we do so, then $\|f\|_{L^{p(\cdot)}(\tilde{\Omega})} = \|f\|_{L^{p(\cdot)}(\Omega)}$. Moreover, if $p(\cdot) \in LH(\widetilde{\Omega})$, by Lemma 2.4 we may assume that $p(\cdot) \in LH(\Omega)$ as well.

**Theorem 2.17.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, the function $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ defines a norm on $L^{p(\cdot)}(\Omega)$.*

*Proof.* We will prove that $\|\cdot\|_{p(\cdot)}$ has the following properties:

1. $\|f\|_{p(\cdot)} = 0$ if and only if $f \equiv 0$;
2. (Homogeneity) for all $\alpha \in \mathbb{R}$, $\|\alpha f\|_{p(\cdot)} = |\alpha|\|f\|_{p(\cdot)}$;
3. (Triangle inequality) $\|f + g\|_{p(\cdot)} \le \|f\|_{p(\cdot)} + \|g\|_{p(\cdot)}$.

If $f \equiv 0$, then $\rho(f/\lambda) = 0 \le 1$ for all $\lambda > 0$, and so $\|f\|_{p(\cdot)} = 0$. Conversely, if $\|f\|_{p(\cdot)} = 0$, then for all $\lambda > 0$,

$$1 \ge \rho(f/\lambda) = \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx + \|f/\lambda\|_{L^\infty(\Omega_\infty)}.$$

We consider each term of the modular separately. It is immediate that we have $\|f\|_{L^\infty(\Omega_\infty)} \le \lambda$; hence, $f(x) = 0$ for almost every $x \in \Omega_\infty$. Similarly, if $\lambda < 1$, by Proposition 2.14 we have

$$1 \ge \lambda^{-p_-} \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\, dx.$$

Therefore, $\|f(\cdot)^{p(\cdot)}\|_{L^1(\Omega\setminus\Omega_\infty)} = 0$, and so $f(x) = |f(x)|^{p(x)} = 0$ for almost every $x \in \Omega \setminus \Omega_\infty$. Thus $f \equiv 0$ and we have proved (1).

To prove (2), note that if $\alpha = 0$, this follows from (1). Fix $\alpha \neq 0$; then by a change of variables,

$$\begin{aligned}\|\alpha f\|_{p(\cdot)} &= \inf\{\lambda > 0 : \rho(|\alpha| f/\lambda) \le 1\} \\ &= |\alpha| \inf\{\lambda/|\alpha| > 0 : \rho(f/(\lambda/|\alpha|)) \le 1\} \\ &= |\alpha| \inf\{\mu > 0 : \rho(f/\mu)) \le 1\} = |\alpha| \|f\|_{p(\cdot)}.\end{aligned}$$

Finally, to prove (3), fix $\lambda_f > \|f\|_{p(\cdot)}$ and $\lambda_g > \|g\|_{p(\cdot)}$; then $\rho(f/\lambda_f) \le 1$ and $\rho(g/\lambda_g) \le 1$. Now let $\lambda = \lambda_f + \lambda_g$. Then by Property (3) of Proposition 2.7,

$$\rho\left(\frac{f+g}{\lambda}\right) = \rho\left(\frac{\lambda_f}{\lambda}\frac{f}{\lambda_f} + \frac{\lambda_g}{\lambda}\frac{g}{\lambda_g}\right) \le \frac{\lambda_f}{\lambda}\rho(f/\lambda_f) + \frac{\lambda_g}{\lambda}\rho(g/\lambda_g) \le 1.$$

Hence, $\|f+g\|_{p(\cdot)} \le \lambda_f + \lambda_g$. If we now take the infimum over all such $\lambda_f$ and $\lambda_g$, we get the desired inequality. □

An immediate consequence of the order preserving property of the modular (Property (6) of Proposition 2.7) is that the norm itself is order preserving: if $|f(x)| \ge |g(x)|$ almost everywhere, then $\|f\|_{p(\cdot)} \ge \|g\|_{p(\cdot)}$.

Another elementary but useful property of the classical Lebesgue norm is that it is homogeneous in the exponent: more precisely, for $1 < s < \infty$, $\|f\|_{sp}^s = \||f|^s\|_p$. This property extends to variable Lebesgue spaces.

**Proposition 2.18.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $|\Omega_\infty| = 0$, then for all $s$, $1/p_- \le s < \infty$,*

$$\||f|^s\|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s.$$

*Proof.* This follows at once from the definition of the norm: since $|\Omega_\infty| = 0$, if we let $\mu = \lambda^{1/s}$,

$$\begin{aligned}\||f|^s\|_{p(\cdot)} &= \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|^s}{\lambda}\right)^{p(x)} dx \le 1\right\} \\ &= \inf\left\{\mu^s > 0 : \int_\Omega \left(\frac{|f(x)|}{\mu}\right)^{sp(x)} dx \le 1\right\} = \|f\|_{sp(\cdot)}^s.\end{aligned}$$

□

*Example 2.19.* If $|\Omega \setminus \Omega_\infty| = 0$, then $\|f\|_{p(\cdot)} = \|f\|_\infty$ and Proposition 2.18 is still true. However, if $|\Omega_\infty| > 0$ but $p(\cdot)$ is not identically infinite, then it need not hold. To see this, let $\Omega = [-1, 1]$, and define

$$p(x) = \begin{cases} 1 & -1 \le x \le 0 \\ \infty & 0 < x \le 1, \end{cases}$$

and

$$f(x) = \begin{cases} 1 & -1 \le x \le 0 \\ 2 & 0 < x \le 1. \end{cases}$$

Then

$$\rho_{p(\cdot)}(f^2/\lambda) = \int_{-1}^{0} \lambda^{-1}\,dx + 2^2\lambda^{-1} = 5\lambda^{-1},$$

and so $\|f^2\|_{p(\cdot)} = 5$. On the other hand, a similar computation shows that $\rho_{2p(\cdot)}(f/\lambda) = \lambda^{-2}+2\lambda^{-1}$; thus, if we solve the quadratic equation $\lambda^{-2}+2\lambda^{-1}-1 = 0$, we get that $\|f\|_{2p(\cdot)}^2 = (\sqrt{2}-1)^{-2} \neq 5$.

We conclude this section by considering more closely the relationship between the norm and the modular. Though the norm is defined as the infimum of the set $\{\lambda : \rho(f/\lambda) \le 1\}$, there may be an explicit value $\lambda$ for which the infimum is attained. For instance, in Example 2.11 we see that if $\Omega = (1,\infty)$, $p(x) = x$ and $f \equiv 1$, then the infimum of $\rho(f/\lambda)$ is attained when $\lambda$ is such that $\lambda\log(\lambda) = 1$. In fact, if $f$ is non-trivial, then the infimum is always attained. (If $f \equiv 0$, then clearly the infimum is zero and is not attained.) In Proposition 2.21 below we will prove that $\rho(f/\|f\|_{p(\cdot)}) \le 1$, so $\lambda = \|f\|_{p(\cdot)}$ is always an element of the set $\{\lambda : \rho(f/\lambda) \le 1\}$. However, even though the infimum is attained it is possible that $\rho(f/\|f\|_{p(\cdot)}) < 1$.

*Example 2.20.* Let $\Omega = (1,\infty)$ and $p(x) = x$. Then there exists a function $f \in L^{p(\cdot)}(\Omega)$ such that $\rho(f/\|f\|_{p(\cdot)}) < 1$.

*Proof.* We will construct a function $f$ such that $\rho(f) < 1$ but for any $\lambda < 1$, $\rho(f/\lambda) = \infty$. Then $\|f\|_{p(\cdot)} = 1$ and $\rho(f/\|f\|_{p(\cdot)}) = \rho(f) < 1$.

For $k \ge 2$ let $I_k = [k, k+k^{-2}]$ and define the function $f$ by

$$f(x) = \sum_{k=2}^{\infty} \chi_{I_k}(x).$$

Then

$$\rho(f) = \sum_{k=2}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - 1 < 1.$$

On the other hand, for any $\lambda < 1$,

$$\rho(f/\lambda) = \sum_{k=2}^{\infty} \int_{k}^{k+k^{-2}} \lambda^{-x}\,dx \ge \sum_{k=2}^{\infty} \frac{1}{\lambda^k k^2} = \infty.$$

□

This example can be adapted to any space such that $p_+(\Omega \setminus \Omega_\infty) = \infty$; otherwise, equality must hold.

**Proposition 2.21.** *Given* $\Omega$ *and* $p(\cdot) \in \mathcal{P}(\Omega)$*, if* $f \in L^{p(\cdot)}(\Omega)$ *and* $\|f\|_{p(\cdot)} > 0$*, then* $\rho(f/\|f\|_{p(\cdot)}) \leq 1$*. Further,* $\rho(f/\|f\|_{p(\cdot)}) = 1$ *for all non-trivial* $f \in L^{p(\cdot)}(\Omega)$ *if and only if* $p_+(\Omega \setminus \Omega_\infty) < \infty$*.*

*Proof.* Fix a decreasing sequence $\{\lambda_k\}$ such that $\lambda_k \to \|f\|_{p(\cdot)}$. Then by Fatou's lemma and the definition of the modular,

$$\rho(f/\|f\|_{p(\cdot)}) \leq \liminf_{k\to\infty} \rho(f/\lambda_k) \leq 1.$$

Now suppose that $p_+(\Omega \setminus \Omega_\infty) < \infty$ but assume to the contrary that $\rho(f/\|f\|_{p(\cdot)}) < 1$. Then for all $\lambda$, $0 < \lambda < \|f\|_{p(\cdot)}$, by Proposition 2.14,

$$\rho(f/\lambda) = \rho\left(\frac{\|f\|_{p(\cdot)}}{\lambda}\frac{f}{\|f\|_{p(\cdot)}}\right) \leq \left(\frac{\|f\|_{p(\cdot)}}{\lambda}\right)^{p_+(\Omega\setminus\Omega_\infty)} \rho\left(\frac{f}{\|f\|_{p(\cdot)}}\right).$$

Therefore, we can find $\lambda$ sufficiently close to $\|f\|_{p(\cdot)}$ such that $\rho(f/\lambda) < 1$. But by the definition of the norm, we must have $\rho(f/\lambda) \geq 1$. From this contradiction we see that equality holds.

Now suppose that $p_+(\Omega \setminus \Omega_\infty) = \infty$. Form the sets $\{E_k\}$ as in the proof of Proposition 2.12 and define the function $f$ by

$$f(x) = \left(\sum_{k=2}^{\infty} \frac{k^{-2}}{|E_k \setminus E_{k+1}|}\chi_{E_k\setminus E_{k+1}}(x)\right)^{1/p(x)}.$$

Then for all $\lambda < 1$,

$$\rho(f/\lambda) = \sum_{k=2}^{\infty} k^{-2} \fint_{E_k\setminus E_{k+1}} \lambda^{-p(x)}\,dx \geq \sum_{k=2}^{\infty} k^{-2}\lambda^{-k} = \infty.$$

On the other hand, essentially the same computation shows that

$$\rho(f) = \sum_{k=2}^{\infty} k^{-2} < 1.$$

Therefore, $f \in L^{p(\cdot)}(\Omega)$ and $\|f\|_{p(\cdot)} = 1$, but $\rho(f/\|f\|_{p(\cdot)}) < 1$. □

**Corollary 2.22.** *Fix* $\Omega$ *and* $p(\cdot) \in \mathcal{P}(\Omega)$*. If* $\|f\|_{p(\cdot)} \leq 1$*, then* $\rho(f) \leq \|f\|_{p(\cdot)}$*; if* $\|f\|_{p(\cdot)} > 1$*, then* $\rho(f) \geq \|f\|_{p(\cdot)}$*.*

*Proof.* If $\|f\|_{p(\cdot)} = 0$, then $f \equiv 0$ and so $\rho(f) = 0$. If $0 < \|f\|_{p(\cdot)} \leq 1$, then by the convexity of the modular (Property (4) of Proposition 2.7) and Proposition 2.21,

$$\rho(f) = \rho(\|f\|_{p(\cdot)}\, f/\|f\|_{p(\cdot)}) \leq \|f\|_{p(\cdot)}\rho(f/\|f\|_{p(\cdot)}) \leq \|f\|_{p(\cdot)}.$$

If $\|f\|_{p(\cdot)} > 1$, then $\rho(f) > 1$: for if $\rho(f) \leq 1$, then by the definition of the norm we would have $\|f\|_{p(\cdot)} \leq 1$. But then we have that

$$\rho\big(f/\rho(f)\big) = \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\rho(f)}\right)^{p(x)} dx + \rho(f)^{-1}\|f\|_{L^\infty(\Omega_\infty)}$$
$$\leq \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\rho(f)^{-1}\,dx + \rho(f)^{-1}\|f\|_{L^\infty(\Omega_\infty)} = 1.$$

It follows that $\|f\|_{p(\cdot)} \leq \rho(f)$. □

The previous result can be strengthened as follows.

**Corollary 2.23.** *Given* $\Omega$ *and* $p(\cdot) \in \mathcal{P}(\Omega)$, *suppose* $|\Omega_\infty| = 0$. *If* $\|f\|_{p(\cdot)} > 1$, *then*

$$\rho(f)^{1/p_+} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}.$$

*If* $0 < \|f\|_{p(\cdot)} \leq 1$, *then*

$$\rho(f)^{1/p_-} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_+}.$$

If $p(\cdot)$ is constant, Corollary 2.23 reduces to the identity

$$\|f\|_p = \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p}.$$

The first inequality makes sense if $p_+ = \infty$ and $\rho(f) = \infty$ provided we define $\infty^0 = 1$. The second inequality makes sense if $\|f\|_{p(\cdot)} = 0$, since in this case $\rho(f) = 0$; if $p_+ = \infty$, then we need to interpret $0^0$ as 1.

*Proof.* We prove the first pair of inequalities; the proof of the second is essentially the same. If $p_+ < \infty$, by Proposition 2.14,

$$\frac{\rho(f)}{\|f\|_{p(\cdot)}^{p_+}} \leq \rho\left(\frac{f}{\|f\|_{p(\cdot)}}\right) \leq \frac{\rho(f)}{\|f\|_{p(\cdot)}^{p_-}}.$$

By Proposition 2.21, $\rho(f/\|f\|_{p(\cdot)}) = 1$, so the desired result follows.

If $p_+ = \infty$, then $\rho(f)^{1/p_+} = 1$, so we only need to prove the right-hand inequality. By Corollary 2.22, $\rho(f) > 1$; hence, since $|\Omega_\infty| = 0$,

$$\rho\big(f/\rho(f)^{1/p_-}\big) = \int_\Omega \left(\frac{|f(x)|}{\rho(f)^{1/p_-}}\right)^{p(x)} dx \leq \int_\Omega |f(x)|^{p(x)}\rho(f)^{-1}\,dx = 1.$$

It follows that $\|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}$. □

*Remark 2.24.* If $|\Omega_\infty| > 0$, then Corollary 2.23 does not hold. Fix $p(\cdot)$ such that $p_- > 1$ and $|\Omega_\infty| > 0$, and take $f \in L^{p(\cdot)}(\Omega)$ such that $\operatorname{supp}(f) \subset \Omega_\infty$ and $\|f\|_{p(\cdot)} = \|f\|_{L^\infty(\Omega_\infty)} = \rho(f) \neq 1$. Then neither inequality comparing $\|f\|_{p(\cdot)}$ to $\rho(f)^{1/p_-}$ can hold in general.

As an application of the above results we will give an equivalent norm on $L^{p(\cdot)}(\Omega)$ that is usually referred to as the Amemiya norm.

**Proposition 2.25.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, define*

$$\|f\|_{p(\cdot)}^A = \inf\{\lambda > 0 : \lambda + \lambda\rho_{p(\cdot)}(f/\lambda)\}.$$

*Then for all $f \in L^{p(\cdot)}(\Omega)$,*

$$\|f\|_{p(\cdot)} \le \|f\|_{p(\cdot)}^A \le 2\|f\|_{p(\cdot)}.$$

*Proof.* Since both $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}^A$ are homogeneous, it will suffice to prove that if $\|f\|_{p(\cdot)} = 1$, then

$$1 \le \|f\|_{p(\cdot)}^A \le 2.$$

The second inequality is immediate: by the definition and Corollary 2.22,

$$\|f\|_{p(\cdot)}^A \le 1 + \rho(f) \le 1 + \|f\|_{p(\cdot)} = 2.$$

To prove the first inequality, note that if $\lambda \ge 1$, then

$$\lambda + \lambda\rho(f/\lambda) \ge \lambda \ge 1.$$

On the other hand, if $0 < \lambda < 1$, then arguing as in the proof of Proposition 2.14,

$$\lambda + \lambda\rho(f/\lambda) \ge \lambda^{1-p_-} \int_{\Omega\setminus\Omega_\infty} |f(x)|\,dx + \|f\|_{L^\infty(\Omega_\infty)} \ge \rho(f) = 1.$$

Therefore, if we take the infimum over all $\lambda > 0$ we get the desired inequality. □

## 2.4 Hölder's Inequality and the Associate Norm

In this section we show that the variable Lebesgue space norm satisfies a generalization of Hölder's inequality, and then use this to define an equivalent norm, the associate norm, on $L^{p(\cdot)}(\Omega)$. The classical Hölder's inequality is that for all $p$, $1 \le p \le \infty$, given $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\int_\Omega |f(x)g(x)|\,dx \le \|f\|_p\|g\|_{p'}.$$

This inequality is true for variable exponents with a constant on the right-hand side.

**Theorem 2.26.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\,dx \le K_{p(\cdot)}\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

*where*

$$K_{p(\cdot)} = \left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right)\|\chi_{\Omega_*}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty.$$

*Remark 2.27.* Each of the last three terms in the definition of $K_{p(\cdot)}$ is equal to 0 or 1, and at least one of them must equal 1. Therefore, if $p(\cdot)$ is not constant, $1 < K_{p(\cdot)} \le 4$.

*Proof.* If $\|f\|_{p(\cdot)} = 0$ or $\|g\|_{p'(\cdot)} = 0$, then $fg \equiv 0$ so there is nothing to prove. Therefore, we may assume that $\|f\|_{p(\cdot)}, \|g\|_{p'(\cdot)} > 0$; moreover, by homogeneity we may assume $\|f\|_{p(\cdot)} = \|g\|_{p'(\cdot)} = 1$.

We consider the integral of $|fg|$ on the disjoint sets $\Omega_\infty$, $\Omega_1$ and $\Omega_*$. If $x \in \Omega_\infty$, then $p(x) = \infty$ and $p'(x) = 1$, so

$$\begin{aligned}\int_{\Omega_\infty} |f(x)g(x)|\,dx &\le \|f\chi_{\Omega_\infty}\|_\infty \|g\chi_{\Omega_\infty}\|_1 \\ &= \|f\chi_{\Omega_\infty}\|_{p(\cdot)}\|g\chi_{\Omega_\infty}\|_{p'(\cdot)} \le \|f\|_{p(\cdot)}\|g\|_{p'(\cdot)} = 1.\end{aligned}$$

Similarly, if we reverse the roles of $p(\cdot)$ and $p'(\cdot)$, we have that

$$\int_{\Omega_1} |f(x)g(x)|\,dx \le 1.$$

To estimate the integral on $\Omega_*$ we use Young's inequality:

$$\begin{aligned}&\int_{\Omega_*} |f(x)g(x)|\,dx \\ &\quad\le \int_{\Omega_*} \frac{1}{p(x)}|f(x)|^{p(x)} + \frac{1}{p'(x)}|g(x)|^{p'(x)}\,dx \\ &\quad\le \frac{1}{p_-}\rho_{p(\cdot)}(f) + \frac{1}{p'(\cdot)_-}\rho_{p'(\cdot)}(g).\end{aligned}$$

Since

$$\frac{1}{p'(\cdot)_-} = \frac{1}{(p_+)'} = 1 - \frac{1}{p_+},$$

and by Proposition 2.21, $\rho_{p(\cdot)}(f),\ \rho_{p'(\cdot)}(g) \leq 1$, we have that

$$\int_{\Omega_*} |f(x)g(x)|\,dx \leq \frac{1}{p_-} + 1 - \frac{1}{p_+}.$$

Combining the above terms, and using the fact that each is needed precisely when the $L^\infty$ norm of the corresponding characteristic function equals 1, we have that

$$\int_\Omega |f(x)g(x)|\,dx$$
$$\leq \left(\left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right)\|\chi_{\Omega_*}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

which is the desired inequality. □

In the classical Lebesgue case, an immediate consequence of Hölder's inequality is that for $p,\ q,\ r$ such that $1 \leq p,\ q,\ r \leq \infty$, and $r^{-1} = p^{-1} + q^{-1}$, if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^r(\Omega)$ and

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

The same result holds in variable Lebesgue spaces; the proof again depends on Hölder's inequality, but is somewhat more complicated.

**Corollary 2.28.** *Given $\Omega$ and exponent functions $r(\cdot),\ q(\cdot) \in \mathcal{P}(\Omega)$ define $p(\cdot) \in \mathcal{P}(\Omega)$ by*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

*Then there exists a constant $K$ such that for all $f \in L^{q(\cdot)}(\Omega)$ and $g \in L^{r(\cdot)}(\Omega)$, $fg \in L^{p(\cdot)}(\Omega)$ and*

$$\|fg\|_{p(\cdot)} \leq K\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

*Proof.* Fix $p(\cdot),\ q(\cdot),\ r(\cdot)$ as in the statement of the theorem, and take $f \in L^{q(\cdot)}(\Omega)$ and $g \in L^{r(\cdot)}(\Omega)$. If $\|f\|_{q(\cdot)} = 0$ or if $\|g\|_{r(\cdot)} = 0$, then $fg \equiv 0$ so there is nothing to prove. Therefore, we may assume that these quantities are positive; further, by homogeneity we may assume that $\|f\|_{q(\cdot)} = \|g\|_{r(\cdot)} = 1$.

By the definition of $p(\cdot)$, $\Omega_\infty^{p(\cdot)} = \Omega_\infty^{q(\cdot)} \cap \Omega_\infty^{r(\cdot)}$. Therefore, we can define the exponent function $s(\cdot) \in \mathcal{P}(\Omega \setminus \Omega_\infty^{p(\cdot)})$ by

$$s(x) = \begin{cases} \frac{q(x)}{p(x)} & x \notin \Omega_\infty^{q(\cdot)} \cup \Omega_\infty^{r(\cdot)} \\ 1 & x \in \Omega_\infty^{r(\cdot)} \setminus \Omega_\infty^{q(\cdot)} \\ \infty & x \in \Omega_\infty^{q(\cdot)} \setminus \Omega_\infty^{r(\cdot)}. \end{cases}$$

Suppose for the moment that

$$|f(\cdot)|^{p(\cdot)} \in L^{s(\cdot)}(\Omega \setminus \Omega_\infty^{p(\cdot)}) \quad \text{and} \quad |g(\cdot)|^{p(\cdot)} \in L^{s'(\cdot)}(\Omega \setminus \Omega_\infty^{p(\cdot)}), \tag{2.1}$$

and $\||f(\cdot)|^{p(\cdot)}\|_{L^{s(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})}, \||g(\cdot)|^{p(\cdot)}\|_{L^{s'(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})} \leq 1$. Then by the generalized Hölder's inequality (Theorem 2.26),

$$\begin{aligned}
\rho_{p(\cdot)}(fg) &= \int_{\Omega\setminus\Omega_\infty^{p(\cdot)}} |f(x)|^{p(x)}|g(x)|^{p(x)}\,dx + \|fg\|_{L^\infty(\Omega_\infty^{p(\cdot)})} \\
&\leq K_{s(\cdot)}\||f(\cdot)|^{p(\cdot)}\|_{L^{s(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})}\||g(\cdot)|^{p(\cdot)}\|_{L^{s'(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})} \\
&\qquad + \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})}\|g\|_{L^\infty(\Omega_\infty^{r(\cdot)})} \\
&\leq K_{s(\cdot)} + \|f\|_{q(\cdot)}\|g\|_{r(\cdot)} \\
&= K_{s(\cdot)} + 1.
\end{aligned}$$

Then by the convexity of the modular (Property (4) of Proposition 2.7) $fg \in L^{p(\cdot)}(\Omega)$ and

$$\|fg\|_{p(\cdot)} \leq K_{s(\cdot)} + 1 = (K_{s(\cdot)} + 1)\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

Therefore, to complete the proof we need to show (2.1) and estimate the norms. We first consider $|f(\cdot)|^{p(\cdot)}$. Since $\|f\|_{q(\cdot)} = 1$, by Corollary 2.22, $\|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})} \leq \rho_{q(\cdot)}(f) \leq 1$. Further, $\Omega_\infty^{s(\cdot)} \subset \Omega_\infty^{q(\cdot)}$ and $\Omega \setminus \Omega_\infty^{s(\cdot)} \subset \Omega \setminus \Omega_\infty^{q(\cdot)}$, and on $\Omega_1^{s(\cdot)}$, $p(x) = q(x) < \infty$. Hence,

$$\begin{aligned}
\rho_{s(\cdot)}(f(\cdot)^{p(\cdot)}\chi_{\Omega\setminus\Omega_\infty^{p(\cdot)}}) &\leq \int_{\Omega\setminus\Omega_\infty^{s(\cdot)}} |f(x)|^{p(x)s(x)}\,dx + \||f(\cdot)|^{p(\cdot)}\|_{L^\infty(\Omega_{s(\cdot)}^\infty)} \\
&\leq \int_{\Omega\setminus\Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\,dx + \||f(\cdot)|^{p(\cdot)}\|_{L^\infty(\Omega_{q(\cdot)}^\infty)} \\
&\leq \int_{\Omega\setminus\Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\,dx + \|f\|_{L^\infty(\Omega_{q(\cdot)}^\infty)} \\
&\leq 1.
\end{aligned}$$

Therefore, by the definition of the norm, $\||f(\cdot)|^{p(\cdot)}\|_{L^{s(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})} \leq 1$. The same argument, with $s(\cdot)$ replaced by $s'(\cdot)$ and $q(\cdot)$ replaced by $r(\cdot)$ gives the corresponding bound for $|g(\cdot)|^{p(\cdot)}$. This completes the proof. □

*Remark 2.29.* It follows from the proof that we can take $K = K_{s(\cdot)} + 1$; by an abuse of notation we can write this as $K_{q(\cdot)/p(\cdot)} + 1$.

As a consequence of Corollary 2.28 we can generalize Theorem 2.26 to three or more exponents.

**Corollary 2.30.** *Given $\Omega$, suppose $p_1(\cdot), p_2(\cdot), \dots, p_k(\cdot) \in \mathcal{P}(\Omega)$ is a collection of exponents that satisfy*

$$\sum_{i=1}^{k} \frac{1}{p_i(x)} = 1, \qquad x \in \Omega.$$

*Then there exists a constant $C$, depending on the $p_i$, such that for all $f_i \in L^{p_i(\cdot)}(\Omega)$, $1 \le i \le k$,*

$$\int_\Omega |f_1(x) f_2(x) \cdots f_k(x)| \, dx \le C \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \cdots \|f_k\|_{p_k(\cdot)}.$$

*Proof.* We prove this by induction. When $k = 2$, this is just Theorem 2.26. Now suppose that for some $k \ge 2$ the inequality holds; we will prove it true for $k + 1$ exponents. Given exponents $p_1(\cdot), \dots, p_{k+1}(\cdot)$, define $r(\cdot)$ by

$$\frac{1}{r(x)} = \frac{1}{p_k(x)} + \frac{1}{p_{k+1}(x)}.$$

Fix functions $f_i \in L^{p_i(\cdot)}(\Omega)$; then by Corollary 2.28, $f_k f_{k+1} \in L^{r(\cdot)}(\Omega)$ and

$$\|f_k\|_{p_k(\cdot)} \|f_{k+1}\|_{p_{k+1}(\cdot)} \ge c \|f_k f_{k+1}\|_{r(\cdot)}.$$

Therefore, by our induction hypothesis applied to $p_1(\cdot), \dots, p_{k-1}(\cdot), r(\cdot)$,

$$\begin{aligned} &\|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \cdots \|f_{k+1}\|_{p_{k+1}(\cdot)} \\ &\qquad \ge c \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \cdots \|f_{k-1}\|_{p_{k-1}(\cdot)} \|f_k f_{k+1}\|_{r(\cdot)} \\ &\qquad \ge c \int_\Omega |f_1(x) \cdots f_{k+1}(x)| \, dx. \end{aligned}$$

□

In the classical Lebesgue space $L^p(\Omega)$, $1 \le p \le \infty$, the norm can be computed using the identity

$$\|f\|_p = \sup \int_\Omega f(x) g(x) \, dx,$$

where the supremum is taken over all $g \in L^{p'}(\Omega)$ with $\|g\|_{p'} \le 1$. Indeed, $g$ can be taken from any dense subset of $L^{p'}(\Omega)$—for example, $C_c(\Omega)$ if $p > 1$. A slightly weaker analog of this equality is true for variable Lebesgue spaces.

**Definition 2.31.** Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, and given a measurable function $f$, define

$$\|f\|'_{p(\cdot)} = \sup \int_\Omega f(x) g(x) \, dx, \tag{2.2}$$

where the supremum is taken over all $g \in L^{p'(\cdot)}(\Omega)$ with $\|g\|_{p'(\cdot)} \le 1$.

Temporarily denote by $M^{p(\cdot)}(\Omega)$ the set of all measurable functions $f$ such that $\|f\|'_{p(\cdot)} < \infty$.

**Proposition 2.32.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, the set $M^{p(\cdot)}(\Omega)$ is a normed vector space with respect to the norm $\|\cdot\|'_{p(\cdot)}$. Furthermore, the norm is order preserving: given $f, g \in M^{p(\cdot)}(\Omega)$ such that $|f| \leq |g|$, then $\|f\|'_{p(\cdot)} \leq \|g\|'_{p(\cdot)}$.*

*Proof.* It is immediate that $M^{p(\cdot)}(\Omega)$ is a vector space. The fact that $\|\cdot\|'_{p(\cdot)}$ is an order preserving norm is a consequence of the properties of integrals and supremums and the following equivalent characterization of $\|\cdot\|'_{p(\cdot)}$. First note that it is immediate from this definition that for all measurable functions $f$,

$$\|f\|'_{p(\cdot)} \leq \sup_{\|g\|_{p'(\cdot)} \leq 1} \left| \int_\Omega f(x)g(x)\,dx \right| \leq \sup_{\|g\|_{p'(\cdot)} \leq 1} \int_\Omega |f(x)g(x)|\,dx,$$

but in fact all of these are equal. To see this, it suffices to note that for any $g \in L^{p'(\cdot)}(\Omega)$, $\|g\|_{p'(\cdot)} \leq 1$, $|f(x)g(x)| = f(x)h(x)$, where $h(x) = \operatorname{sgn} f(x)|g(x)|$ and $\|h\|_{p'(\cdot)} \leq \|g\|_{p'(\cdot)} \leq 1$; hence,

$$\int_\Omega |f(x)g(x)|\,dx = \int_\Omega f(x)h(x)\,dx \leq \|f\|'_{p(\cdot)}.$$

□

*Remark 2.33.* As a consequence of the proof of Proposition 2.32 we get another version of Hölder's inequality:

$$\int_\Omega |f(x)g(x)|\,dx \leq \|f\|_{p(\cdot)} \|g\|'_{p'(\cdot)}.$$

In the next result we show that $M^{p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$ and that the norms $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|'_{p(\cdot)}$ are equivalent. We will refer to the norm $\|\cdot\|'_{p(\cdot)}$ as the associate norm on $L^{p(\cdot)}(\Omega)$.

**Theorem 2.34.** *Given $\Omega$, $p(\cdot) \in \mathcal{P}(\Omega)$, and a measurable $f$, then $f \in L^{p(\cdot)}(\Omega)$ if and only if $f \in M^{p(\cdot)}(\Omega)$; furthermore,*

$$k_{p(\cdot)} \|f\|_{p(\cdot)} \leq \|f\|'_{p(\cdot)} \leq K_{p(\cdot)} \|f\|_{p(\cdot)},$$

*where*

$$K_{p(\cdot)} = \left( \frac{1}{p_-} - \frac{1}{p_+} + 1 \right) \|\chi_{\Omega_*}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty,$$

$$\frac{1}{k_{p(\cdot)}} = \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_*}\|_\infty.$$

*Remark 2.35.* For every variable Lebesgue space we have that $K_{p(\cdot)} \leq 4$ and $k_{p(\cdot)} \geq 1/3$.

To motivate the proof of Theorem 2.34, recall the proof of (2.2) if $1 < p < \infty$. By Hölder's inequality, $\|f\|'_p \leq \|f\|_p$. To prove the reverse inequality, let

$$g(x) = \left(\frac{|f(x)|}{\|f\|_p}\right)^{p/p'} \operatorname{sgn} f(x).$$

Then $\|g\|_{p'} = 1$, and

$$\int_\Omega f(x)g(x)\,dx = \|f\|_p,$$

and so in fact the supremum is attained.

Our proof will be based on a similar but more complicated function $g$; first we need to prove a lemma.

**Lemma 2.36.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $\|f\chi_{\Omega_*}\|'_{p(\cdot)} \leq 1$ and $\rho(f\chi_{\Omega_*}) < \infty$, then $\rho(f\chi_{\Omega_*}) \leq 1$.*

*Proof.* Suppose to the contrary that $\rho(f\chi_{\Omega_*}) > 1$. Then by the continuity of the modular (Proposition 2.7, (6)) there exists $\lambda > 1$ such that $\rho(f\chi_{\Omega_*}/\lambda) = 1$. Let

$$g(x) = \left(\frac{|f(x)|}{\lambda}\right)^{p(x)-1} \operatorname{sgn} f(x)\chi_{\Omega_*}(x).$$

Then $\rho_{p'(\cdot)}(g) = \rho_{p(\cdot)}(f\chi_{\Omega_*}/\lambda) = 1$, so $\|g\|_{p'(\cdot)} \leq 1$. Therefore, by the definition of the associate norm,

$$\|f\chi_{\Omega_*}\|'_{p(\cdot)} \geq \int_\Omega f(x)\chi_{\Omega_*}(x)\,g(x)\,dx = \lambda\int_{\Omega_*}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx = \lambda\rho(f\chi_{\Omega_*}/\lambda) > 1.$$

This contradicts our hypothesis on $f$, so the desired inequality holds. □

*Proof of Theorem 2.34.* One implication is immediate: given $f \in L^{p(\cdot)}(\Omega)$, by Hölder's inequality for variable Lebesgue spaces (Theorem 2.26),

$$\|f\|'_{p(\cdot)} \leq K_{p(\cdot)}\|f\|_{p(\cdot)}.$$

To prove the converse, we will assume that

$$|\Omega_\infty^{p(\cdot)}|,\ |\Omega_1^{p(\cdot)}|,\ |\Omega_*^{p(\cdot)}| > 0.$$

If any of these sets has measure 0, then the proof can be readily adapted by omitting the terms associated with them. Further, by the definition of the norms we only have to prove this for non-negative functions $f$.

We will prove that if $\|f\|'_{p(\cdot)} \leq 1$ and $\rho_{p(\cdot)}(f\chi_{\Omega_*}) < \infty$, then

$$\rho_{p(\cdot)}(k_{p(\cdot)} f) \leq 1. \tag{2.3}$$

Given this, the desired inequality follows by an approximation argument. Fix any non-negative $f \in M^{p(\cdot)}(\Omega)$. By homogeneity we may assume that $\|f\|'_{p(\cdot)} = 1$. For each $k \geq 1$, define the sets

$$E_k = B_k(0) \cap \big(\Omega \setminus \Omega_* \cup \{x \in \Omega_* : p(x) < k\}\big),$$

and define the functions $f_k = \min(f, k)\chi_{E_k}$. Then $f_k \leq f$, so by Proposition 2.32, $\|f_k\|'_{p(\cdot)} \leq \|f\|'_{p(\cdot)} = 1$. Furthermore, the sequence $\{f_k\}$ increases to $f$ pointwise. Finally, $\rho(f_k\chi_{\Omega_*}) < \infty$, and so we can apply (2.3) with $f$ replaced by $f_k$. Therefore, by Fatou's lemma on the classical Lebesgue spaces and (2.3),

$$\rho_{p(\cdot)}(k_{p(\cdot)} f/\|f\|'_{p(\cdot)}) = \rho_{p(\cdot)}(k_{p(\cdot)} f) \leq \liminf_{k\to\infty} \rho_{p(\cdot)}(k_{p(\cdot)} f_k) \leq 1.$$

Thus, we have that

$$\|f\|_{p(\cdot)} \leq k_{p(\cdot)}^{-1} \|f\|'_{p(\cdot)}.$$

To complete the proof, fix $f$ with $\|f\|'_{p(\cdot)} \leq 1$ and $\rho(f\chi_{\Omega_*}) < \infty$; we will show that (2.3) holds. First note that by Proposition 2.32, $\|f\chi_{\Omega_*^{p(\cdot)}}\|'_{p(\cdot)} \leq 1$. Now fix $\epsilon$, $0 < \epsilon < 1$; then there exists a set $E_\epsilon \subset \Omega_\infty^{p(\cdot)}$ such that $0 < |E_\epsilon| < \infty$, and for each $x \in E_\epsilon$,

$$|f(x)| \geq (1-\epsilon)\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})}.$$

Now define the function $g_\epsilon$ by

$$g_\epsilon(x) = \begin{cases} k_{p(\cdot)}|f(x)|^{p(x)-1} \operatorname{sgn} f(x) & x \in \Omega_*^{p(\cdot)} = \Omega_*^{p'(\cdot)}, \\ k_{p(\cdot)} \operatorname{sgn} f(x) & x \in \Omega_1^{p(\cdot)} = \Omega_\infty^{p'(\cdot)}, \\ k_{p(\cdot)}|E_\epsilon|^{-1}\chi_{E_\epsilon}(x) \operatorname{sgn} f(x) & x \in \Omega_\infty^{p(\cdot)} = \Omega_1^{p'(\cdot)}. \end{cases}$$

We claim that $\rho_{p'(\cdot)}(g_\epsilon) \leq 1$, so $\|g_\epsilon\|_{p'(\cdot)} \leq 1$. To see this, note that

$$\begin{aligned} &\rho_{p'(\cdot)}(g_\epsilon/k_{p(\cdot)}) \\ &\qquad \leq \int_{\Omega_*^{p'(\cdot)}} |f(x)|^{p(x)}\,dx + \|\operatorname{sgn} f\|_{L^\infty(\Omega_\infty^{p'(\cdot)})} + |E_\epsilon|^{-1}\int_{\Omega_1^{p'(\cdot)}} \chi_{E_\epsilon}(x)\,dx \\ &\qquad = \int_{\Omega_*^{p(\cdot)}} |f(x)|^{p(x)}\,dx + \|\operatorname{sgn} f\|_{L^\infty(\Omega_1^{p(\cdot)})} + |E_\epsilon|^{-1}\int_{\Omega_\infty^{p(\cdot)}} \chi_{E_\epsilon}(x)\,dx. \end{aligned}$$

By Lemma 2.36, the first term on the right-hand side is dominated by 1; the second term equals 0 or 1, and the third term always equals 1. Therefore,

$$\rho_{p'(\cdot)}(g_\epsilon/k_{p(\cdot)}) \leq \|\chi_{\Omega_*^{p(\cdot)}}\|_\infty + \|\chi_{\Omega_1^{p(\cdot)}}\|_\infty + \|\chi_{\Omega_\infty^{p(\cdot)}}\|_\infty = \frac{1}{k_{p(\cdot)}}.$$

Since $k_{p(\cdot)} \leq 1$, by the convexity of the modular (Proposition 2.7),

$$\rho_{p'(\cdot)}(g_\epsilon) \leq k_{p(\cdot)} \rho_{p'(\cdot)}(g_\epsilon / k_{p(\cdot)}) \leq 1,$$

which is what we claimed to be true.

Furthermore, we have that

$$\begin{aligned}
&\int_\Omega f(x) g_\epsilon(x)\, dx \\
&\quad = k_{p(\cdot)} \int_{\Omega_*^{p(\cdot)}} |f(x)|^{p(x)}\, dx + k_{p(\cdot)} \int_{\Omega_1^{p(\cdot)}} |f(x)|\, dx + k_{p(\cdot)} \fint_{E_\epsilon} |f(x)|\, dx \\
&\quad \geq k_{p(\cdot)} \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)}\, dx + (1-\epsilon) k_{p(\cdot)} \|f\|_{L^\infty(\Omega_\infty)} \\
&\quad \geq (1-\epsilon) k_{p(\cdot)} \rho_{p(\cdot)}(f).
\end{aligned}$$

Therefore, by the definition of the associate norm, since $\|g_\epsilon\|_{p'(\cdot)} \leq 1$,

$$1 \geq \|f\|'_{p(\cdot)} \geq \int_\Omega f(x) g_\epsilon(x)\, dx \geq (1-\epsilon) k_{p(\cdot)} \rho_{p(\cdot)}(f).$$

Since $\epsilon > 0$ was arbitrary, again by the convexity of the modular we have that

$$1 \geq k_{p(\cdot)} \rho_{p(\cdot)}(f) \geq \rho_{p(\cdot)}(k_{p(\cdot)} f).$$

□

In the notation introduced above, given an exponent $p(\cdot)$, the Banach space $M^{p'(\cdot)}$ of measurable functions $f$ such that

$$\|f\|'_{p'(\cdot)} = \sup \left\{ \int_\Omega f(x) g(x)\, dx,\, g \in L^{p(\cdot)}(\Omega),\, \|g\|_{p(\cdot)} \leq 1 \right\} < \infty,$$

is called the associate space of $L^{p(\cdot)}(\Omega)$. As an immediate consequence of Theorem 2.34 we have the following result.

**Proposition 2.37.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, the associate space of $L^{p(\cdot)}(\Omega)$ is equal to $L^{p'(\cdot)}(\Omega)$, and $\|\cdot\|_{p'(\cdot)}$ and $\|\cdot\|'_{p'(\cdot)}$ are equivalent norms.*

Finally, as a corollary to Theorem 2.34 we prove a version of Minkowski's integral inequality for variable Lebesgue spaces.

**Corollary 2.38.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, let $f : \Omega \times \Omega \to \mathbb{R}$ be a measurable function (with respect to product measure) such that for almost every $y \in \Omega$, $f(\cdot, y) \in L^{p(\cdot)}(\Omega)$. Then*

$$\left\| \int_\Omega f(\cdot, y)\, dy \right\|_{p(\cdot)} \leq k_{p(\cdot)}^{-1} K_{p(\cdot)} \int_\Omega \|f(\cdot, y)\|_{p(\cdot)}\, dy. \tag{2.4}$$

*Proof.* If the right-hand side of (2.4) is infinite, then there is nothing to prove, so we may assume that this integral is finite. Define the function

$$g(x) = \int_\Omega f(x,y)\,dy,$$

and take any $h \in L^{p'(\cdot)}(\Omega)$, $\|h\|_{p'(\cdot)} \le 1$. Then by Fubini's theorem (see Royden [301]) and Hölder's inequality on the variable Lebesgue spaces (Theorem 2.26),

$$\begin{aligned}
\int_\Omega |g(x)h(x)|\,dx &\le \int_\Omega \int_\Omega |f(x,y)|\,dy\,|h(x)|\,dx \\
&= \int_\Omega \int_\Omega |f(x,y)h(x)|\,dx\,dy \\
&\le K_{p(\cdot)} \int_\Omega \|f(\cdot,y)\|_{p(\cdot)} \|h\|_{p'(\cdot)}\,dy \\
&\le K_{p(\cdot)} \int_\Omega \|f(\cdot,y)\|_{p(\cdot)}\,dy.
\end{aligned}$$

Therefore, we have that

$$\|g\|'_{p(\cdot)} \le K_{p(\cdot)} \int_\Omega \|f(\cdot,y)\|_{p(\cdot)}\,dy,$$

and inequality (2.4) follows by Theorem 2.34. □

## 2.5 Embedding Theorems

In this section we consider the embeddings of classical and variable Lebesgue spaces into one another. We begin by showing that every function in a variable Lebesgue space is locally integrable. To do so we prove a simple but useful lemma.

**Lemma 2.39.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $E \subset \Omega$ is such that $|E| < \infty$, then $\chi_E \in L^{p(\cdot)}(\Omega)$ and $\|\chi_E\|_{p(\cdot)} \le |E| + 1$.*

*Proof.* Fix $\lambda = |E| + 1$. Then

$$\begin{aligned}
\rho(\chi_E/\lambda) &= \int_{E\setminus\Omega_\infty} \lambda^{-p(x)}\,dx + \lambda^{-1}\|\chi_{E\cap\Omega_\infty}\|_\infty \\
&\le \lambda^{-p_-}|E| + \lambda^{-1}. \le \lambda^{-1}(|E|+1) = 1.
\end{aligned}$$

By the definition of the norm we get the desired result. □

*Remark 2.40.* If $|\Omega_\infty| = 0$, then by Corollary 2.23 we get a sharper bound that depends on $E$ and $p(\cdot)$:

$$\|\chi_E\|_{p(\cdot)} \leq \max\left(|E|^{1/p_-}, |E|^{1/p_+}\right).$$

**Proposition 2.41.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $f \in L^{p(\cdot)}(\Omega)$, then $f$ is locally integrable.*

*Proof.* Let $E \subset \Omega$ be a set of finite measure. Then by the generalized Hölder's inequality (Theorem 2.26) and Lemma 2.39,

$$\int_E |f(x)|\,dx \leq C\|f\|_{p(\cdot)}\|\chi_E\|_{p'(\cdot)} < \infty.$$

□

We now consider the embedding of $L^\infty(\Omega)$ into $L^{p(\cdot)}(\Omega)$. It follows from the proof of Lemma 2.39 that if $|\Omega\backslash\Omega_\infty| < \infty$, then $\chi_\Omega \in L^{p(\cdot)}(\Omega)$, which immediately implies that $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$. However, unlike in the case of classical Lebesgue spaces, this embedding can hold even if $|\Omega \setminus \Omega_\infty| = \infty$.

*Example 2.42.* Let $\Omega = (1, \infty)$ and $p(x) = x$. By Example 2.11, $1 \in L^{p(\cdot)}(\Omega)$, and so if $f \in L^\infty(\Omega)$,

$$\|f\|_{p(\cdot)} \leq \|f\|_\infty \|1\|_{p(\cdot)} < \infty.$$

More generally, we have the following characterization of when this embedding holds.

**Proposition 2.43.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ if and only if $1 \in L^{p(\cdot)}(\Omega)$, which in turn is true if and only if for some $\lambda > 1$,*

$$\int_{\Omega\backslash\Omega_\infty} \lambda^{-p(x)}\,dx < \infty. \tag{2.5}$$

*In particular, the embedding holds if $|\Omega| < \infty$ or if $1/p(\cdot) \in LH_\infty(\Omega)$ and $p(x) \to \infty$ as $|x| \to \infty$.*

*Proof.* We repeat the above argument: $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ if and only if $1 \in L^{p(\cdot)}(\Omega)$, and by the definition of $L^{p(\cdot)}(\Omega)$ and Proposition 2.7 this is true if and only if there exists $\lambda > 1$ such that

$$\rho(1/\lambda) = \int_{\Omega\backslash\Omega_\infty} \lambda^{-p(x)}\,dx + \lambda^{-1}\|1\|_{L^\infty(\Omega_\infty)} < \infty.$$

This in turn is equivalent to (2.5).

If $|\Omega| < \infty$, then the integral in (2.5) is clearly dominated by $|\Omega|$. If $1/p(\cdot) \in LH_\infty$ and $p(x) \to \infty$ as $|x| \to \infty$, then we have that

$$\frac{1}{p(x)} \leq \frac{C_\infty}{\log(e+|x|)}.$$

Therefore, for $\lambda > 1$ sufficiently large,

$$\int_{\Omega\setminus\Omega_\infty} \lambda^{-p(x)}\,dx \leq \int_{\Omega\setminus\Omega_\infty} \lambda^{-C_\infty^{-1}\log(e+|x|)}\,dx$$
$$\leq \int_{\Omega\setminus\Omega_\infty} (e+|x|)^{-C_\infty^{-1}\log(\lambda)}\,dx < \infty.$$

□

The smoothness condition $LH_\infty$ in Proposition 2.43 is in some sense sharp, as the next example shows.

*Example 2.44.* Let $\Omega = (e,\infty)$, and let $p(x) = \phi(x)\log(x)$, where $\phi$ is a decreasing function such that $\phi(x) \to 0$ as $x \to \infty$, and $p(\cdot)$ is increasing and $p(x) \to \infty$ as $x \to \infty$. Then $L^\infty(\Omega)$ is not contained in $L^{p(\cdot)}(\Omega)$.

A simple example of such a function $\phi$ is $\phi(x) \approx \log\log(x)^{-1}$.

*Proof.* We will show that for any $\lambda > 1$,

$$\int_e^\infty \lambda^{-p(x)}\,dx = \infty.$$

Fix $\lambda > 1$; since $\phi(x)$ decreases to 0, there exists $N > 0$ such that if $k \geq N$, then $\log(\lambda)\phi(e^{k+1}) < 1/2$. Then, since $p(\cdot)$ is increasing,

$$\int_e^\infty \lambda^{-p(x)}\,dx \geq \sum_{k\geq N}\int_{e^k}^{e^{k+1}} \lambda^{-p(x)}\,dx \geq \sum_{k\geq N} e^k\cdot\lambda^{-\phi(e^{k+1})\log(e^{k+1})}$$
$$\geq \sum_{k\geq N} e^k e^{-\phi(e^{k+1})\log(\lambda)(k+1)} \geq \sum_{k\geq N} e^k e^{-\frac{1}{2}(k+1)} = \infty.$$

□

As a consequence of Proposition 2.43 we can completely characterize the exponents $p(\cdot)$ and $q(\cdot)$ such that $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$. Unlike in the case of classical Lebesgue spaces, this embedding is possible even when $|\Omega| = \infty$.

**Theorem 2.45.** *Given $\Omega$ and $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, then $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ and there exists $K > 1$ such that for all $f \in L^{q(\cdot)}(\Omega)$, $\|f\|_{p(\cdot)} \leq K\|f\|_{q(\cdot)}$, if and only if:*

*1.* $p(x) \le q(x)$ *for almost every* $x \in \Omega$*;*
*2. There exists* $\lambda > 1$ *such that*

$$\int_D \lambda^{-r(x)}\,dx < \infty, \tag{2.6}$$

*where* $D = \{x \in \Omega : p(x) < q(x)\}$ *and* $r(\cdot)$ *is the defect exponent defined by*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

*Remark 2.46.* If $1/p(\cdot),\ 1/q(\cdot) \in LH_\infty(\Omega)$, then $1/r(\cdot) \in LH_\infty(\Omega)$ and arguing as we did in the proof of Proposition 2.43 we have that (2.6) holds if $r(x) \to \infty$ as $|x| \to \infty$.

*Proof.* Suppose first that Conditions (1) and (2) hold. By Proposition 2.43 we have that $1 \in L^{r(\cdot)}(\Omega)$. Therefore, by Corollary 2.28, given any $f \in L^{q(\cdot)}(\Omega)$,

$$\|f\|_{p(\cdot)} = \|1 \cdot f\|_{p(\cdot)} \le K\|1\|_{r(\cdot)}\|f\|_{q(\cdot)}.$$

To prove the converse, we will show that if either Condition (1) or (2) do not hold, then the embedding also does not hold.

Suppose first that Condition (1) does not hold. Then there exists a set $E \subset \Omega$, $|E| > 0$, such that if $x \in E$, $p(x) > q(x)$. We will construct $f \in L^{q(\cdot)}(\Omega) \setminus L^{p(\cdot)}(\Omega)$. There are two cases.

**Case 1:** $|\mathbf{\Omega}_\infty^{\mathbf{p}(\cdot)} \cap \mathbf{E}| > \mathbf{0}$. Since $q(\cdot)$ is finite on $E$, there exists a set $F \subset E \cap \Omega_\infty^{p(\cdot)}$, $0 < |F| < \infty$, and $r$, $1 < r < \infty$, such that if $x \in F$, $q(x) \le r$. Partition $F$ as the union of disjoint sets $F_j$, $j \ge 1$, such that $|F_j| = 2^{-j}|F|$ and define the function $f$ by

$$f(x) = \sum_{j=1}^{\infty} \left(\frac{3}{2}\right)^{j/r} \chi_{F_j}(x).$$

Then $f$ is unbounded, and so

$$\|f\|_{p(\cdot)} \ge \|f\chi_F\|_{p(\cdot)} = \|f\chi_F\|_\infty = \infty.$$

On the other hand, $f \in L^{q(\cdot)}(\Omega)$ since

$$\rho_{q(\cdot)}(f) = \int_F |f(x)|^{q(\cdot)}\,dx = \sum_{j=1}^{\infty} \int_{F_j} \left(\frac{3}{2}\right)^{jq(x)/r} dx$$
$$\le \sum_{j=1}^{\infty} \left(\frac{3}{2}\right)^{j} 2^{-j}|F| = 3|F| < \infty.$$

**Case 2:** $|\mathbf{\Omega}_{\infty}^{\mathbf{p}(\cdot)} \cap \mathbf{E}| = \mathbf{0}$. In this case, $1 \leq q(x) < p(x) < \infty$ almost everywhere on $E$. Therefore, there exists a set $F \subset E$, $0 < |F| < \infty$, and constants $\epsilon > 0$ and $r > 1$ such that if $x \in F$,

$$q(x) + \epsilon \leq p(x) \leq r < \infty.$$

In particular,

$$\frac{p(x)}{q(x)} \geq 1 + \frac{\epsilon}{r}.$$

Again partition $F$ into disjoint sets $F_j$, $|F_j| = 2^{-j}|F|$, and define $f$ by

$$f(x) = \sum_{j=1}^{\infty} \left(\frac{2^j}{j^2}\right)^{1/q(x)} \chi_{F_j}(x).$$

Then

$$\rho_{q(\cdot)}(f) = \sum_{j=1}^{\infty} 2^j j^{-2}|F_j| = |F| \sum_{j=1}^{\infty} j^{-2} < \infty.$$

On the other hand, since for $j \geq 4$, $2^j/j^2 \geq 1$,

$$\begin{aligned}\rho_{p(\cdot),F}(f) &= \sum_{j=1}^{\infty} \int_{F_j} \left(\frac{2^j}{j^2}\right)^{p(x)/q(x)} dx \\ &\geq \sum_{j=4}^{\infty} \left(\frac{2^j}{j^2}\right)^{1+\epsilon/r} |F_j| = |F| \sum_{j=4}^{\infty} 2^{\epsilon j/r} j^{-2(1+\epsilon/r)} = \infty.\end{aligned}$$

Since $p_+(F) \leq r < \infty$, by Proposition 2.12,

$$\|f\|_{L^{p(\cdot)}(\Omega)} \geq \|f\|_{L^{p(\cdot)}(F)} = \infty.$$

This completes the proof.

Now suppose that Condition (2) does not hold. Again there are two cases. Define the sets

$$D_\infty = \{x \in D : q(x) = \infty\}, \quad D_0 = \{x \in D : p(x) < q(x) < \infty\}.$$

Then (2.6) must fail to hold for all $\lambda > 1$ with $D$ replaced by $D_\infty$ or it fails to hold for all $\lambda > 1$ with $D$ replaced by $D_0$.

**Case 1:** Suppose first that for any $\lambda > 1$,

$$\int_{D_\infty} \lambda^{-r(x)}\, dx = \infty.$$

We will construct $f \in L^{q(\cdot)}(\Omega) \setminus L^{p(\cdot)}(\Omega)$. Let $f = \chi_{D_\infty}$; since $D_\infty \subset \Omega_\infty^{q(\cdot)}$, $\|f\|_{q(\cdot)} = \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})} = 1$, so $f \subset L^{q(\cdot)}(\Omega)$. On the other hand, by the definition of the defect exponent $r(\cdot)$, for $x \in D_\infty$, $p(x) = r(x)$. Hence, for all $\lambda > 1$

$$\rho_{p(\cdot)}(f/\lambda) = \int_{D_\infty} \lambda^{-r(x)}\,dx = \infty.$$

Since the same is obviously true for $\lambda \le 1$, it follows that $f \notin L^{p(\cdot)}(\Omega)$.

**Case 2:** Now suppose that for any $\lambda > 1$,

$$\int_{D_0} \lambda^{-r(x)}\,dx = \infty. \tag{2.7}$$

We will construct a sequence of functions $\{f_k\} \subset L^{q(\cdot)}(\Omega)$ such that $\|f_k\|_{q(\cdot)} \to 0$ as $k \to \infty$, but $\|f_k\|_{p(\cdot)} \ge 1$. It follows immediately that the embedding cannot hold.

Given (2.7), for any compact set $K \subset D_0$ and any $\lambda > 1$ we have that

$$\int_{D_0 \setminus K} \lambda^{-r(x)}\,dx = \infty.$$

Therefore, by the continuity of the integral we can construct a sequence of disjoint sets $D_j \subset D_0$, $j \ge 1$, such that

$$\int_{D_j} 2^{-jr(x)}\,dx = 1.$$

For each $k \ge 1$ define the function $f_k$ by

$$f_k(x) = \sum_{j>k} 2^{-j\frac{r(x)}{p(x)}} \chi_{D_j}(x).$$

Then

$$\rho_{p(\cdot)}(f_k) = \sum_{j>k} \int_{D_j} 2^{-jr(x)}\,dx = \sum_{j>k} 1 = \infty.$$

Thus $\|f_k\|_{p(\cdot)} \ge 1$. On the other hand, by the definition of the defect exponent $r(\cdot)$, we have that for $x \in D_0$,

$$q(x) - \frac{q(x)r(x)}{p(x)} = -r(x).$$

Hence,

$$\begin{aligned}\rho_{q(\cdot)}(2^k f_k) = \sum_{j>k} \int_{D_j} 2^{kq(x)} 2^{-j\frac{q(x)r(x)}{p(x)}}\,dx &\le \sum_{j>k} 2^{k-j} \int_{D_j} 2^{j\left(q(x)-\frac{q(x)r(x)}{p(x)}\right)}\,dx \\ &= \sum_{j>k} 2^{k-j} \int_{D_j} 2^{-jr(x)}\,dx = \sum_{j>k} 2^{k-j} = 1.\end{aligned}$$

Therefore, $\|f_k\|_{q(\cdot)} \le 2^{-k}$ and so $\|f_k\|_{q(\cdot)} \to 0$ as $k \to \infty$. □

As a corollary to the construction in the second half of the proof of Theorem 2.45 we have that the spaces $L^{p(\cdot)}(\Omega)$ are different for different exponent functions $p(\cdot)$.

**Corollary 2.47.** *Given* $\Omega$ *and* $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, *if there exists a set* $E \subset \Omega$, *such that* $|E| > 0$ *and* $p(x) \neq q(x)$, $x \in E$, *then the set* $\big(L^{p(\cdot)}(\Omega) \setminus L^{q(\cdot)}(\Omega)\big) \cup \big(L^{q(\cdot)}(\Omega) \setminus L^{p(\cdot)}(\Omega)\big)$ *is not empty.*

If $|\Omega^{p(\cdot)} \setminus \Omega_\infty^{p(\cdot)}| < \infty$, then condition (2.6) is true for any $\lambda > 1$, so a necessary and sufficient condition for the embedding $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is that $p(x) \leq q(x)$. Thus the next result is a corollary of Theorem 2.45. However, we give a direct proof of one implication since by doing so we get a sharper constant.

**Corollary 2.48.** *Given* $\Omega$ *and* $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, *suppose* $|\Omega \setminus \Omega_\infty^{p(\cdot)}| < \infty$. *Then* $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ *if and only if* $p(x) \leq q(x)$ *almost everywhere. Furthermore, in this case we have that*

$$\|f\|_{p(\cdot)} \leq (1 + |\Omega \setminus \Omega_\infty^{p(\cdot)}|)\|f\|_{q(\cdot)}. \tag{2.8}$$

*Proof.* We will assume that $p(x) \leq q(x)$ almost everywhere and prove (2.8). By the homogeneity of the norm, it will suffice to show that if $f \in L^{q(\cdot)}(\Omega)$, $\|f\|_{q(\cdot)} \leq 1$, then $\|f\|_{p(\cdot)} \leq 1 + |\Omega \setminus \Omega_\infty^{p(\cdot)}|$. By the definition of the norm,

$$1 \geq \rho_{q(\cdot)}(f) = \int_{\Omega \setminus \Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})}.$$

In particular, $|f(x)| \leq 1$ almost everywhere on $\Omega_\infty^{q(\cdot)}$. Further, since $p(x) \leq q(x)$, $\Omega_\infty^{p(\cdot)} \subset \Omega_\infty^{q(\cdot)}$ up to a set of measure zero. Therefore,

$$\begin{aligned}
\rho_{p(\cdot)}(f) &= \int_{\Omega \setminus \Omega_\infty^{q(\cdot)}} |f(x)|^{p(x)}\,dx + \int_{\Omega_\infty^{q(\cdot)} \setminus \Omega_\infty^{p(\cdot)}} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})} \\
&\leq |\{x \in \Omega \setminus \Omega_\infty^{q(\cdot)} : |f(x)| \leq 1\}| + \int_{\Omega \setminus \Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\,dx \\
&\qquad + |\Omega_\infty^{q(\cdot)} \setminus \Omega_\infty^{p(\cdot)}| + \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})} \\
&\leq |\Omega \setminus \Omega_\infty^{p(\cdot)}| + \rho_{q(\cdot)}(f) \\
&\leq |\Omega \setminus \Omega_\infty^{p(\cdot)}| + 1.
\end{aligned}$$

Hence, by the convexity of the modular,

$$\rho_{p(\cdot)}\left(\frac{f}{|\Omega \setminus \Omega_\infty^{p(\cdot)}| + 1}\right) \leq \frac{\rho_{p(\cdot)}(f)}{|\Omega \setminus \Omega_\infty^{p(\cdot)}| + 1} \leq 1,$$

and so $\|f\|_{p(\cdot)} \leq |\Omega \setminus \Omega_\infty^{p(\cdot)}| + 1$. □

*Remark 2.49.* A variant of this result is used in Chap. 3 to prove norm inequalities for the maximal operator: see Lemma 3.28 below.

Corollary 2.48 is commonly applied with the stronger hypothesis $|\Omega| < \infty$. In particular, as an immediate consequence we get the following relationship between the classical and variable Lebesgue spaces on bounded domains.

**Corollary 2.50.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $|\Omega| < \infty$. Then there exist constants $c_1,\ c_2 > 0$ such that*

$$c_1\|f\|_{p_-} \le \|f\|_{p(\cdot)} \le c_2\|f\|_{p_+}.$$

Finally, we give an embedding that will be very useful in applications. For $1 \le p < q < \infty$, define

$$L^p(\Omega) + L^q(\Omega) = \big\{f = g + h : g \in L^p(\Omega), h \in L^q(\Omega)\big\};$$

this is a Banach space with norm

$$\|f\|_{L^p(\Omega)+L^q(\Omega)} = \inf_{f=g+h}\{\|g\|_{L^p(\Omega)} + \|h\|_{L^q(\Omega)}\}.$$

**Theorem 2.51.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, then*

$$L^{p(\cdot)}(\Omega) \subset L^{p_+}(\Omega) + L^{p_-}(\Omega)$$

*and*

$$\|f\|_{L^{p_+}(\Omega)+L^{p_-}(\Omega)} \le 2\|f\|_{L^{p(\cdot)}(\Omega)}.$$

*Further, this embedding is proper if and only if $p(\cdot)$ is non-constant.*

*Proof.* By the homogeneity of the norms we may assume without loss of generality that $\|f\|_{p(\cdot)} = 1$. This implies that $\|f\|_{L^\infty(\Omega_\infty)} \le 1$. Decompose $f$ as $f_1 + f_2$, where

$$f_1 = f\chi_{\{x\in\Omega:|f(x)|\le 1\}}, \qquad f_2 = f\chi_{\{x\in\Omega\setminus\Omega_\infty:|f(x)|>1\}}. \tag{2.9}$$

If $p_+ < \infty$, $|\Omega_\infty| = 0$, so by Corollary 2.22,

$$\int_\Omega |f_1(x)|^{p_+}\,dx \le \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx \le \|f\|_{p(\cdot)} = 1,$$

$$\int_\Omega |f_2(x)|^{p_-}\,dx \le \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx \le \|f\|_{p(\cdot)} = 1.$$

Hence,

$$\|f\|_{L^{p_+}(\Omega)+L^{p_-}(\Omega)} \le \|f_1\|_{p_+} + \|f_2\|_{p_-} \le 2 = 2\|f\|_{p(\cdot)}.$$

If $p_+ = \infty$, then we argue as before for $f_2$ and for $f_1$ we note that $\|f_1\|_\infty \le 1 = \|f\|_{p(\cdot)}$.

Now assume that $p(\cdot)$ is non-constant. Then there exists $q$, $p_- < q < p_+$, such that $E = \{x \in \Omega : p(x) > q\}$ has positive measure. Then by (the proof of) Corollary 2.47, there exists a function $f \in L^{p_-}(\Omega) \subset L^{p_-}(\Omega) + L^{p_+}(\Omega)$ but $f \notin L^{p(\cdot)}(\Omega)$.

Conversely, if $p(\cdot)$ is constant then $p_- = p_+$ and equality clearly holds. □

*Remark 2.52.* In applying Theorem 2.51 we will often use the explicit decomposition $f = f_1 + f_2$ given by (2.9).

If we assume that the exponent $p(\cdot)$ is log-Hölder continuous at infinity, then we can give a different decomposition of $f$ that reflects this fact.

**Proposition 2.53.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$ and $p(\cdot) \in LH_\infty(\Omega)$. Then*

$$L^{p(\cdot)}(\Omega) \subset L^{p_\infty}(\Omega) + L^{p_-}(\Omega).$$

*Proof.* Fix $f \in L^{p(\cdot)}(\Omega)$. By homogeneity we may assume without loss of generality that $\|f\|_{p(\cdot)} = 1$. Decompose $f$ as $f_1 + f_2$ as in (2.9). Then $f_2 \in L^{p_-}(\Omega)$, so it will suffice to prove that $f_1 \in L^{p_\infty}(\Omega)$. Let $q(x) = \max(p(x), p_\infty)$; then $|f_1(x)|^{q(x)} \leq |f_1(x)|^{p(x)}$. Hence, by Proposition 2.12, $f_1 \in L^{q(\cdot)}(\Omega)$. By the definition of $q(\cdot)$,

$$\frac{1}{r(x)} = \frac{1}{p_\infty} - \frac{1}{q(x)} \leq \left| \frac{1}{p_\infty} - \frac{1}{p(x)} \right|.$$

Since $p(\cdot) \in LH_\infty(\Omega)$, by Theorem 2.45 and Remark 2.46, $L^{q(\cdot)}(\Omega) \subset L^{p_\infty}(\Omega)$. This completes the proof. □

## 2.6 Convergence in $L^{p(\cdot)}(\Omega)$

In this section we consider three types of convergence in the variable Lebesgue spaces: convergence in modular, in norm, and in measure.

**Definition 2.54.** Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, and given a sequence of functions $\{f_k\} \subset L^{p(\cdot)}(\Omega)$, we say that $f_k \to f$ in modular if for some $\beta > 0$, $\rho(\beta(f - f_k)) \to 0$ as $k \to \infty$. We say that $f_k \to f$ in norm if $\|f - f_k\|_{p(\cdot)} \to 0$ as $k \to \infty$.

In defining modular convergence it might seem more natural to assume that $\rho(f - f_k) \to \infty$. As in the definition of the norm, we introduce the constant $\beta$ to preserve the homogeneity of convergence: if $f_k \to f$ in modular, then we want $2f_k \to 2f$ in modular. With this alternative definition this is not always the case.

*Example 2.55.* Let $\Omega = (0, 1)$ and $p(x) = 1/x$. Let $f_k = \chi_{(0,1/k)}$. Then $\rho(f_k) = 1/k \to 0$, but for all $k$, $\rho(2f_k) = \infty$.

We can reformulate norm convergence in a way that highlights the connection with modular convergence.

**Proposition 2.56.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, the sequence $\{f_k\}$ converges to $f$ in norm if and only if for every $\beta > 0$, $\rho(\beta(f - f_k)) \to 0$ as $k \to \infty$. In particular, convergence in norm implies convergence in modular.*

*Proof.* Suppose first that $f_k \to f$ in norm. Fix $\beta > 0$. Then by the homogeneity of the norm,

$$\|\beta(f - f_k)\|_{p(\cdot)} = \beta\|f - f_k\|_{p(\cdot)} \to 0.$$

Hence, by Corollary 2.22, for all $k$ sufficiently large,

$$\rho(\beta(f - f_k)) \le \|\beta(f - f_k)\|_{p(\cdot)} \le 1,$$

and so $\rho(\beta(f - f_k)) \to 0$.

To prove the converse, fix $\lambda > 0$ and let $\beta = \lambda^{-1}$. Then for all $k$ sufficiently large, $\rho((f - f_k)/\lambda) \le 1$, and so $\|f - f_k\|_{p(\cdot)} \le \lambda$. Since this is true for any $\lambda$, $\|f - f_k\|_{p(\cdot)} \to 0$. □

While convergence in norm implies convergence in modular, the converse does not always hold.

*Example 2.57.* Let $\Omega = (1, \infty)$ and $p(x) = x$. Define $f \equiv 1$ and $f_k = \chi_{(1,k)}$. Then $f_k \to f$ in modular since

$$\rho((f - f_k)/2) = \int_k^\infty 2^{-x}\, dx \to 0$$

as $k \to \infty$. On the other hand, $f_k$ does not converge to $f$ in norm because for all $k \ge 1$,

$$\rho(f - f_k) = \int_k^\infty 1^x\, dx = \infty,$$

which in turn implies that $\|f - f_k\|_{p(\cdot)} \ge 1$.

This example can be generalized to any space $L^{p(\cdot)}(\Omega)$ such that $\Omega \setminus \Omega_\infty$ has positive measure and $p(\cdot)$ is unbounded on $\Omega \setminus \Omega_\infty$.

**Theorem 2.58.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, convergence in norm is equivalent to convergence in modular if and only if either $p_- = \infty$ or $p_+(\Omega \setminus \Omega_\infty) < \infty$.*

*Proof.* By Proposition 2.56, convergence in norm always implies convergence in modular. Therefore, we need only consider whether modular convergence implies norm convergence.

Suppose first that $p_- = \infty$. Then the modular and the norm are the same and the result is trivially true.

Now suppose that $p_- < \infty$ and $p_+(\Omega \setminus \Omega_\infty) < \infty$ and fix a sequence $\{f_k\}$ such that $f_k \to f$ in modular. Then there exist $\beta > 0$ such that $\rho(\beta(f - f_k)) \to 0$. Fix $\lambda$, $0 < \lambda < \beta^{-1}$. Then by Proposition 2.14,

$$\rho((f - f_k)/\lambda) \leq \left(\frac{1}{\beta\lambda}\right)^{p_+(\Omega\setminus\Omega_\infty)} \rho(\beta(f - f_k)).$$

Hence, for all $k$ sufficiently large we have that

$$\rho\left(\frac{f - f_k}{\lambda}\right) \leq 1.$$

Equivalently, for all such $k$, $\|f - f_k\|_{p(\cdot)} \leq \lambda$. Since $\lambda$ was arbitrary, $f_k \to f$ in norm.

Now suppose $p_- < \infty$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$. We will construct a sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ such that $\rho(f_k) \to 0$ but $\|f_k\|_{p(\cdot)} \geq 1/2$ for all $k$. Let $\{E_k\}$ be the sequence of sets constructed in the proof of Proposition 2.12. Define the function $f$ by

$$f(x) = \left(\sum_{k=1}^{\infty} \frac{1}{2^k |E_k \setminus E_{k+1}|} \chi_{E_k \setminus E_{k+1}}(x)\right)^{1/p(x)},$$

and for each $k$ let $f_k = f\chi_{E_k}$. Then for all $k \geq 1$,

$$\rho(f_k) = \sum_{j=k}^{\infty} \int_{E_j \setminus E_{j+1}} \frac{1}{2^j |E_j \setminus E_{j+1}|}\, dx = \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1};$$

hence, $f_k \in L^{p(\cdot)}(\Omega)$ and $\rho(f_k) \to 0$ as $k \to \infty$. On the other hand, for all $k \geq 1$,

$$\rho\left(\frac{f_k}{1/2}\right) = \sum_{j=k}^{\infty} \int_{E_j \setminus E_{j+1}} \frac{2^{p(x)}}{2^j |E_j \setminus E_{j+1}|}\, dx \geq \sum_{j=k}^{\infty} 2^{p_j - j} = \infty.$$

Thus, $\|f_k\|_{p(\cdot)} \geq 1/2$. This completes the proof. □

In the classical Lebesgue spaces the three ubiquitous convergence theorems are the monotone convergence theorem, Fatou's lemma, and the dominated convergence theorem. Versions of the first two are always true in variable Lebesgue spaces, but the third is only true when the exponent function is bounded. We prove each of these results in turn.

**Theorem 2.59.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be a sequence of non-negative functions such that $f_k$ increases to a function $f$ pointwise almost everywhere. Then either $f \in L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$, or $f \notin L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{p(\cdot)} \to \infty$.*

*Remark 2.60.* If $f \notin L^{p(\cdot)}(\Omega)$, we have defined $\|f\|_{p(\cdot)} = \infty$, so in every case we may write the conclusion as $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$.

Theorem 2.59 is sometimes referred to as the Fatou property of the norm. To avoid confusion with the variable Lebesgue space version of Fatou's lemma and to stress the parallels with the classical Lebesgue spaces, we will always refer to it as the monotone convergence theorem.

*Proof.* Since $\{f_k\}$ is an increasing sequence, so is $\{\|f_k\|_{p(\cdot)}\}$; thus, it either converges or diverges to $\infty$. If $f \in L^{p(\cdot)}(\Omega)$, since $f_k \leq f$, $\|f_k\|_{p(\cdot)} \leq \|f\|_{p(\cdot)}$; otherwise, since $f_k \in L^{p(\cdot)}(\Omega)$, $\|f_k\|_{p(\cdot)} < \infty = \|f\|_{p(\cdot)}$. In either case it will suffice to show that for any $\lambda < \|f\|_{p(\cdot)}$, for all $k$ sufficiently large $\|f_k\|_{p(\cdot)} > \lambda$.

Fix such a $\lambda$; by the definition of the norm, $\rho(f/\lambda) > 1$. Therefore, by the monotone convergence theorem on the classical Lebesgue spaces and the definition of the $L^\infty$ norm,

$$\begin{aligned}
\rho(f/\lambda) &= \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\|f\|_{L^\infty(\Omega_\infty)} \\
&= \lim_{k\to\infty} \left( \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f_k(x)|}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\|f_k\|_{L^\infty(\Omega_\infty)} \right) \\
&= \lim_{k\to\infty} \rho(f_k/\lambda).
\end{aligned}$$

(In this calculation we allow the possibility that $\rho(f/\lambda), \rho(f_k/\lambda) = \infty$.) Hence, for all $k$ sufficiently large, $\rho(f_k/\lambda) > 1$, and so $\|f_k\|_{p(\cdot)} > \lambda$. □

**Theorem 2.61.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ is such that $f_k \to f$ pointwise almost everywhere. If*

$$\liminf_{k\to\infty} \|f_k\|_{p(\cdot)} < \infty,$$

*then $f \in L^{p(\cdot)}(\Omega)$ and*

$$\|f\|_{p(\cdot)} \leq \liminf_{k\to\infty} \|f_k\|_{p(\cdot)}.$$

In the classical version of Fatou's lemma it is necessary to assume that each $f_k$ is non-negative. However, since we are taking the norm this hypothesis is not necessary in Theorem 2.61.

*Proof.* Define a new sequence

$$g_k(x) = \inf_{m\geq k} |f_m(x)|.$$

Then for all $m \geq k$, $g_k(x) \leq |f_m(x)|$, and so $g_k \in L^{p(\cdot)}(\Omega)$. Further, by definition $\{g_k\}$ is an increasing sequence and

$$\lim_{k\to\infty} g_k(x) = \liminf_{m\to\infty} |f_m(x)| = |f(x)|, \quad \text{a.e. } x \in \Omega.$$

Therefore, by Theorem 2.59,

$$\|f\|_{p(\cdot)} = \lim_{k\to\infty} \|g_k\|_{p(\cdot)} \le \lim_{k\to\infty} \Big( \inf_{m\ge k} \|f_m\|_{p(\cdot)} \Big) = \liminf_{k\to\infty} \|f_k\|_{p(\cdot)} < \infty,$$

and $f \in L^{p(\cdot)}(\Omega)$. □

**Theorem 2.62.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. If the sequence $\{f_k\}$ is such that $f_k \to f$ pointwise almost everywhere, and there exists $g \in L^{p(\cdot)}(\Omega)$ such that $|f_k(x)| \le g(x)$ almost everywhere, then $f \in L^{p(\cdot)}(\Omega)$ and $\|f - f_k\|_{p(\cdot)} \to 0$ as $k \to \infty$.*

*Further, if $p_+ = \infty$, then this result is always false.*

*Remark 2.63.* It follows at once from the triangle inequality that the dominated convergence theorem implies that $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$.

As an immediate corollary to the dominated convergence theorem we can give a stronger version of the monotone convergence theorem.

**Corollary 2.64.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_+ < \infty$, suppose the sequence of non-negative functions $f_k$ increases pointwise almost everywhere to a function $f \in L^{p(\cdot)}(\Omega)$. Then $\|f - f_k\|_{p(\cdot)} \to 0$.*

*Proof of Theorem 2.62.* Suppose first that $p_+ < \infty$. Then by Proposition 2.12,

$$|f(x) - f_k(x)|^{p(x)} \le 2^{p(x)-1}(|f(x)|^{p(x)} + |f_k(x)|^{p(x)}) \le 2^{p_+}|g(x)|^{p(x)} \in L^1(\Omega).$$

Therefore, by the classical dominated convergence theorem, $\rho(f - f_k) \to 0$ as $k \to 0$, and so by Theorem 2.58, $\|f - f_k\|_{p(\cdot)} \to 0$.

Now suppose that $p_+ = \infty$; then either $|\Omega_\infty| = 0$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$, or $|\Omega_\infty| > 0$. In the first case, let $f$ and $\{f_k\}$ be the functions constructed in the second half of the proof of Theorem 2.58. Then $f(\cdot)^{p(\cdot)} \in L^1(\Omega)$, so $f \in L^{p(\cdot)}(\Omega)$. Further, $f_k \le f$ and $f_k \to 0$ pointwise. However, $\|f_k\|_{p(\cdot)} \ge 1/2$, so the dominated convergence theorem does not hold.

If $|\Omega_\infty| > 0$, let $\{E_k\}$ be a sequence of sets such that for each $k$, $|E_k| > 0$ and $E_{k+1} \subset E_k \subset \Omega_\infty$, and $|E_k| \to 0$ as $k \to \infty$. Let $f_k = \chi_{E_k}$; then $f_k \le f_1$ and $f_k \to 0$ pointwise, but $\|f_k\|_{p(\cdot)} = \|f_k\|_\infty = 1$. □

As in the classical Lebesgue spaces, norm convergence need not imply that the sequence converges pointwise almost everywhere unless $p_- = \infty$.

*Example 2.65.* Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $|\Omega \setminus \Omega_\infty| > 0$, then there exists a sequence $\{f_k\}$ in $L^{p(\cdot)}(\Omega)$ such that $f_k \to 0$ in norm but not pointwise almost everywhere.

*Proof.* Since $|\Omega \setminus \Omega_\infty| > 0$, there exists a set $E \subset \Omega \setminus \Omega_\infty$ such that $0 < |E| < \infty$ and $p_+(E) < \infty$. Form a "dyadic" decomposition of $E$ as follows. Let $E = E_1^1 \cup E_2^1$, where the sets $E_1^1$ and $E_2^1$ are disjoint and have measure $|E|/2$. Repeat this decomposition. Then by induction, we get a collection of sets $\{E_j^i : i \geq 1, 1 \leq j \leq 2^i\}$ such that for each $i$, the sets $E_j^i$ are pairwise disjoint, $E = \bigcup_{j=1}^{2^i} E_j^i$, and $|E_j^i| = |E|/2^i$. Define the collection of functions $\{g_j^i\}$ by $g_j^i = \chi_{E_j^i}$. Then by Corollary 2.50,

$$\|g_j^i\|_{L^{p(\cdot)}(\Omega)} = \|g_j^i\|_{L^{p(\cdot)}(E)} \leq C\|g_j^i\|_{p_+(E)} = C(|E|/2^i)^{1/p_+(E)}. \tag{2.10}$$

Define the sequence $\{f_k\}$ by $\{g_1^1, g_2^1, g_1^2, g_2^2, g_3^2, g_4^2, \ldots\}$. Then (2.10) shows that $\|f_k\|_{p(\cdot)} \to 0$ as $k \to \infty$. On the other hand, given any point $x \in E$, for every $i$ there exists $j$ such that $x \in E_j^i$, so there exists an infinite number of functions $g_j^i$ such that $g_j^i(x) = 1$. Thus the sequence $\{f_k\}$ does not converge to 0 pointwise almost everywhere. □

Despite this example, we can always find a subsequence of a norm convergent sequence that converges pointwise almost everywhere. To show this we will consider the slightly stronger property of convergence in measure. Given a domain $\Omega$ and a sequence of functions $\{f_k\}$, recall that $f_k \to f$ in measure if for every $\epsilon > 0$, there exists $K > 0$ such that if $k \geq K$,

$$|\{x \in \Omega : |f(x) - f_k(x)| \geq \epsilon\}| < \epsilon.$$

If $\{f_k\}$ converges to $f$ in measure, then there exists a subsequence $\{f_{k_j}\}$ that converges to $f$ pointwise almost everywhere. (See Royden [301].) Norm convergence implies convergence in measure in the classical Lebesgue spaces, and the same is true for variable Lebesgue spaces.

**Theorem 2.66.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ converges to $f$ in norm, then it converges to $f$ in measure.*

*Proof.* Suppose to the contrary that there exists a sequence $\{f_k\}$ that converges to $f$ in norm but not in measure. Then by passing to a subsequence we may assume that there exists $\epsilon$, $0 < \epsilon < 1$, such that for all $k$,

$$|\{x \in \Omega : |f(x) - f_k(x)| \geq \epsilon\}| \geq \epsilon.$$

Denote the set on the left-hand side by $A_k$; since for each $k$ either $|A_k \cap \Omega_\infty| \geq \epsilon/2$ or $|A_k \setminus \Omega_\infty| \geq \epsilon/2$, by passing to another subsequence we may assume that one of these inequalities holds for all $k$.

If $|A_k \cap \Omega_\infty| \geq \epsilon/2$ for all $k$, then

$$\|f - f_k\|_{p(\cdot)} \geq \|(f - f_k)\chi_{\Omega_\infty}\|_{p(\cdot)} = \|f - f_k\|_{L^\infty(\Omega_\infty)} \geq \epsilon,$$

contradicting our assumption that $f_k$ converges to $f$ in norm. If $|A_k \setminus \Omega_\infty| \geq \epsilon/2$ for all $k$, then

$$\rho\left(\frac{f - f_k}{\epsilon^2/2}\right) \geq \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x) - f_k(x)|}{\epsilon^2/2}\right)^{p(x)} dx$$
$$\geq \int_{A_k\setminus\Omega_\infty} \left(\frac{2}{\epsilon}\right)^{p(x)} dx \geq \left(\frac{2}{\epsilon}\right)^{p-} |A_k \setminus \Omega_\infty| \geq 1.$$

Hence, $\|f - f_k\|_{p(\cdot)} \geq \epsilon^2/2 > 0$, again contradicting our assumption that $f_k$ converges to $f$ in norm. □

As an immediate corollary we get that every norm convergent sequence has a subsequence that converges pointwise almost everywhere. We record this fact as part of a somewhat stronger result which is a partial converse to the dominated convergence theorem.

**Proposition 2.67.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ converges in norm to $f \in L^{p(\cdot)}(\Omega)$. Then there exists a subsequence $\{f_{k_j}\}$ and $g \in L^{p(\cdot)}(\Omega)$ such that the subsequence converges pointwise almost everywhere to $f$, and for almost every $x \in \Omega$, $|f_{k_j}(x)| \leq g(x)$.*

*Proof.* By Theorem 2.66 we immediately have the existence of a subsequence $\{f_{k_j}\}$ that converges pointwise almost everywhere to $f$. Further, since convergent sequences are Cauchy sequences, we may choose the $k_j$ large enough that for each $j$, $\|f_{k_{j+1}} - f_{k_j}\|_{p(\cdot)} \leq 2^{-j}$. For simplicity, we will write $f_j$ instead of $f_{k_j}$.

For each $j$, define the function $h_j$ by

$$h_j(x) = \sum_{i=1}^{j-1} |f_{i+1}(x) - f_i(x)|.$$

Then $\{h_j\}$ is an increasing sequence and so converges pointwise to a function $h$. By our choice of the functions $f_j$,

$$\|h_j\|_{p(\cdot)} \leq \sum_{i=1}^{j-1} 2^{-i} \leq 1.$$

Hence, by the monotone convergence theorem (Theorem 2.59), $h \in L^{p(\cdot)}(\Omega)$. But then, for every $j$ and almost every $x \in \Omega$,

$$|f_j(x) - f_1(x)| \leq \sum_{i=1}^{j-1} |f_{i+1} - f_i(x)| = h_j(x) \leq h(x).$$

Thus, if we let $g = h + |f_1|$, we have that $g \in L^{p(\cdot)}(\Omega)$ and $|f_j(x)| \leq g(x)$ almost everywhere. □

We conclude this section by considering more carefully the relationship between convergence in norm, convergence in modular and convergence in measure.

**Theorem 2.68.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ is such that $\|f_k\|_{p(\cdot)} \to 0$ (or $\infty$), then the sequence $\rho(f_k) \to 0$ (or $\infty$). The converse holds if and only if $p_+(\Omega \setminus \Omega_\infty) < \infty$.*

*Proof.* Suppose first that $\|f_k\|_{p(\cdot)} \to 0$ (or $\infty$). Then the fact that $\rho(f_k) \to 0$ (or $\infty$) follows immediately from Corollary 2.22.

Now suppose that $p_+(\Omega \setminus \Omega_\infty) < \infty$. Given a sequence $\{f_k\}$ such that $\rho(f_k) \to 0$, there exists a sequence $\{a_k\}$ such that $a_k \leq 1$, $a_k \to 0$, but $a_k^{-p_+(\Omega\setminus\Omega_\infty)}\rho(f_k) \leq 1$. Then by Proposition 2.14,

$$\rho(f_k/a_k) \leq a_k^{-p_+(\Omega\setminus\Omega_\infty)}\rho(f_k) \leq 1.$$

Therefore, $\|f_k\|_{p(\cdot)} \leq a_k$ and so $\|f_k\|_{p(\cdot)} \to 0$.

If $\rho(f_k) \to \infty$, then the proof is nearly the same: there exists a sequence $\{a_k\}$ such that $a_k \geq 1$, $a_k \to \infty$ but such that, again by Proposition 2.14,

$$\rho(f_k/a_k) \geq a_k^{-p_+(\Omega\setminus\Omega_\infty)}\rho(f_k) > 1,$$

and so $\|f_k\|_{p(\cdot)} \geq a_k$.

Now suppose that $p_+(\Omega \setminus \Omega_\infty) = \infty$; we will show that neither convergence result holds. First, the example constructed in Theorem 2.58 shows that there is always a sequence $\{f_k\}$ such that $\rho(f_k) \to 0$ but $\|f_k\|_{p(\cdot)} \geq 1/2$. For the other case, form the sets $\{E_k\}$ as in the proof of Proposition 2.12 and define

$$f_k(x) = \left(\sum_{j=1}^{k} \frac{1}{|E_j \setminus E_{j+1}|}\chi_{E_j\setminus E_{j+1}}(x)\right)^{1/p(x)}.$$

Then arguing as in that proof, we have $\rho(f_k) = k$ but

$$\rho(f_k/2) = \sum_{j=1}^{k} \fint_{E_j\setminus E_{j+1}} 2^{-p(x)}\,dx \leq \sum_{j=1}^{\infty} 2^{-j} = 1.$$

Hence, $\rho(f_k) \to \infty$ but $\|f_k\|_{p(\cdot)} \leq 2$. □

**Theorem 2.69.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. Then for $f \in L^{p(\cdot)}(\Omega)$ and a sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$, the following are equivalent:*

*1. $f_k \to f$ in norm,*

2. *$f_k \to f$ in modular,*
3. *$f_k \to f$ in measure and for some $\gamma > 0$, $\rho(\gamma f_k) \to \rho(\gamma f)$.*

*Proof.* The equivalence of (1) and (2) was proved in Theorem 2.58; here we will prove the equivalence of (2) and (3).

To show that (2) implies (3), first note that by Theorem 2.66 norm convergence implies convergence in measure, so modular convergence also implies convergence in measure. To complete the proof of this implication we will show that convergence in modular implies that $\rho(\gamma f_k) \to \rho(\gamma f)$ for $\gamma = 1$.

We begin with an elementary inequality. By the mean value theorem, if $1 \le p < \infty$ and $a, b \ge 0$, then

$$|a^p - b^p| \le p \max(a^{p-1}, b^{p-1})|a - b| \le p(a^{p-1} + b^{p-1})|a - b|.$$

Therefore,

$$\begin{aligned} |\rho(f) - \rho(f_k)| &\le \int_\Omega \big||f(x)|^{p(x)} - |f_k(x)|^{p(x)}\big|\,dx \\ &\le p_+ \int_\Omega \big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\,dx. \end{aligned}$$

To estimate the right-hand side we write the domain of integration as $\Omega_1 \cup \Omega_*$. The integral on $\Omega_1$ is straightforward to estimate:

$$\begin{aligned} p_+ \int_{\Omega_1} &\big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\,dx \\ &= 2p_+ \int_{\Omega_1} |f(x) - f_k(x)|^{p(x)}\,dx \le 2p_+\rho(f - f_k). \end{aligned}$$

Since modular convergence and norm convergence are equivalent, by Proposition 2.56 the right-hand side tends to 0 as $k \to \infty$.

To estimate the integral on $\Omega_*$, fix $\epsilon$, $0 < \epsilon < 1/4$, and apply Young's inequality to get

$$\begin{aligned} p_+ &\int_{\Omega_*} \big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\,dx \\ &\le p_+ \int_{\Omega_*} \frac{\epsilon^{p'(x)}}{p'(x)}\big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)^{p'(x)}\,dx \\ &\qquad + p_+ \int_{\Omega_*} \frac{\epsilon^{-p(x)}}{p(x)}|f(x) - f_k(x)|^{p(x)}\,dx \\ &= I_1 + I_2. \end{aligned}$$

We estimate $I_1$ and $I_2$ separately. Since $p(x) > 1$ for all $x \in \Omega_*$,

$$I_2 \le p_+ \rho(\epsilon^{-1}(f - f_k)).$$

To estimate $I_1$ we need two additional inequalities: for $p > 0$ and $a, b > 0$, we have by elementary calculus that

$$a^p + b^p \le \max(1, 2^{1-p})(a+b)^p,$$
$$(a+b)^p \le \max(1, 2^{p-1})(a^p + b^p).$$

Hence, since $1 < p'(x) < \infty$ on $\Omega_*$,

$$\begin{aligned} I_1 &\le p_+ \int_{\Omega_*} \epsilon^{p'(x)} \max(1, 2^{2-p(x)})^{p'(x)} \big(|f(x)| + |f_k(x)|\big)^{p(x)}\, dx \\ &\le p_+ \int_{\Omega_*} (4\epsilon)^{p'(x)} \big(2|f(x)| + |f(x) - f_k(x)|\big)^{p(x)}\, dx \\ &\le 4\epsilon p_+ \int_{\Omega_*} 2^{p(x)-1} \big(2^{p(x)}|f(x)|^{p(x)} + |f(x) - f_k(x)|^{p(x)}\big)\, dx \\ &\le \epsilon p_+ 2^{2p_+ + 1} \rho(f) + p_+ \epsilon 2^{p_+ + 1} \rho(f - f_k). \end{aligned}$$

Combining this with the previous estimate, we see that

$$\begin{aligned} p_+ \int_{\Omega_*} &\big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\, dx \\ &\le \epsilon p_+ 2^{2p_+ + 1} \rho(f) + \epsilon p_+ 2^{p_+ + 1} \rho(f - f_k) + p_+ \rho(\epsilon^{-1}(f - f_k)). \end{aligned}$$

Therefore, by Proposition 2.56,

$$\begin{aligned} \limsup_{k\to\infty} p_+ \int_{\Omega_*} &\big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\, dx \\ &\le \epsilon p_+ 2^{2p_+ + 1} \rho(f). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $|\rho(f) - \rho(f_k)| \to 0$.

Now suppose that $f_k \to f$ in measure and that for some $\gamma > 0$, $\rho(\gamma f_k) \to \rho(\gamma f)$. Since we also have that $\gamma f_k \to \gamma f$ in measure, we may assume without loss of generality that $\gamma = 1$. Then for each $\epsilon$, $0 < \epsilon < 1$,

$$\begin{aligned} |\{x \in \Omega : |f(x) - f_k(x)|^{p(x)} > \epsilon\}| &\le |\{x \in \Omega : |f(x) - f_k(x)| > \epsilon^{1/p_-}\}| \\ &\le |\{x \in \Omega : |f(x) - f_k(x)| > \epsilon\}| \le \epsilon. \end{aligned}$$

Hence, $|f(\cdot) - f_k(\cdot)|^{p(\cdot)} \to 0$ in measure.

Further, arguing as we did above, we have that

$$\begin{aligned}
&\big||f(x)|^{p(x)} - |f_k(x)|^{p(x)}\big| \qquad (2.11)\\
&\quad \le p_+\big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\\
&\quad \le p_+|f(x)|^{p(x)-1}|f(x) - f_k(x)|\\
&\qquad + p_+ \max(1, 2^{p(x)-2})\\
&\qquad\quad \times \big(|f(x)|^{p(x)-1} + |f(x) - f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\\
&\quad \le p_+(2^{p_+} + 1)|f(x)|^{p(x)-1}|f(x) - f_k(x)| + p_+2^{p_+}|f(x) - f_k(x)|^{p(x)}.
\end{aligned}$$

Now fix $\epsilon$, $0 < \epsilon < 1$. Since $|f(\cdot)|^{p(\cdot)} \in L^1(\Omega)$, there exists $M \ge 1$ such that

$$|\{x : |f(x)|^{p(x)-1} > M\}| \le |\{x : |f(x)|^{p(x)} > M\}| \le \epsilon/2.$$

By inequality (2.11), since $f_k \to f$ and $|f(\cdot) - f_k(\cdot)|^{p(\cdot)} \to 0$ in measure, for all $k$ sufficiently large,

$$\begin{aligned}
&|\{x : \big||f(x)|^{p(x)} - |f_k(x)|^{p(x)}\big| > \epsilon\}|\\
&\quad \le |\{x : |f(x)|^{p(x)-1} > M\}|\\
&\qquad + |\{x : p_+(2^{p_+} + 1)M|f(x) - f_k(x)| > \epsilon/2\}|\\
&\qquad + |\{x : p_+2^{p_+}|f(x) - f_k(x)|^{p(x)} > \epsilon/2\}|\\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2p_+(2^{p_+} + 1)M} + \frac{\epsilon}{p_+2^{p_+ + 1}}\\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4}\\
&\quad = \epsilon.
\end{aligned}$$

Therefore, $|f_k(\cdot)|^{p(\cdot)} \to |f(\cdot)|^{p(\cdot)}$ in measure.

Now define

$$h_k(x) = 2^{p_+-1}|f_k(x)|^{p(x)} + 2^{p_+-1}|f(x)|^{p(x)} - |f(x) - f_k(x)|^{p(x)} \ge 0;$$

then $h_k \to 2^{p_+}|f(\cdot)|^{p(\cdot)}$ in measure. Therefore, by Fatou's lemma on the classical Lebesgue spaces with respect to convergence in measure (see Royden [301]),

$$\begin{aligned}
&2^{p_+}\int_\Omega |f(x)|^{p(x)}\,dx\\
&\quad \le \liminf_{k\to\infty}\int_\Omega 2^{p_+-1}|f_k(x)|^{p(x)} + 2^{p_+-1}|f(x)|^{p(x)} - |f(x) - f_k(x)|^{p(x)}\,dx.
\end{aligned}$$

Rearranging terms and using the fact that $\rho(f_k) \to \rho(f)$ we get that

$$\limsup_{k\to\infty} \int_\Omega |f(x) - f_k(x)|^{p(x)}\,dx \le 0.$$

Therefore, $f_k \to f$ in modular and our proof is complete. □

## 2.7 Completeness and Dense Subsets of $L^{p(\cdot)}(\Omega)$

In this section we prove that $L^{p(\cdot)}(\Omega)$ is a Banach space—that is, a complete normed vector space—and determine some canonical dense subsets of $L^{p(\cdot)}(\Omega)$. Since we proved that $L^{p(\cdot)}(\Omega)$ is a normed vector space in Sect. 2.3, to see that it is a Banach space we only have to show that it is complete.

Our proof of completeness follows one of the standard proofs for classical Lebesgue spaces and so makes heavy use of the convergence theorems proved in the previous section. We begin with a result that is of independent interest and is referred to as the Riesz-Fischer property.

**Theorem 2.70.** *Given $\Omega$ and $p(\cdot) \in L^{p(\cdot)}(\Omega)$, let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be such that*

$$\sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

*Then there exists $f \in L^{p(\cdot)}(\Omega)$ such that*

$$\sum_{k=1}^{i} f_k \to f$$

*in norm as $i \to \infty$, and*

$$\|f\|_{p(\cdot)} \le \sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)}.$$

*Proof.* Define the function $F$ on $\Omega$ by

$$F(x) = \sum_{k=1}^{\infty} |f_k(x)|,$$

and define the sequence $\{F_i\}$ by

$$F_i(x) = \sum_{k=1}^{i} |f_k(x)|.$$

Then the sequence $\{F_i\}$ is non-negative and increases pointwise almost everywhere to $F$. Further, for each $i$, $F_i \in L^{p(\cdot)}(\Omega)$, and its norm is uniformly bounded, since

$$\|F_i\|_{p(\cdot)} \leq \sum_{k=1}^{i} \|f_k\|_{p(\cdot)} \leq \sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

Therefore, by the monotone convergence theorem (Theorem 2.59), $F \in L^{p(\cdot)}(\Omega)$.

In particular, by Remark 2.10, $F$ is finite almost everywhere, so the sequence $\{F_i\}$ converges pointwise almost everywhere. Hence, if we define the sequence of functions $\{G_i\}$ by

$$G_i(x) = \sum_{k=1}^{i} f_k(x),$$

then this sequence also converges pointwise almost everywhere since absolute convergence implies convergence. Denote its sum by $f$.

Now let $G_0 = 0$; then for any $j \geq 0$, $G_i - G_j \to f - G_j$ pointwise almost everywhere. Furthermore,

$$\liminf_{i\to\infty} \|G_i - G_j\|_{p(\cdot)} \leq \liminf_{i\to\infty} \sum_{k=j+1}^{i} \|f_k\|_{p(\cdot)} = \sum_{k=j+1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

By Fatou's lemma (Theorem 2.61), if we take $j = 0$, then

$$\|f\|_{p(\cdot)} \leq \liminf_{i\to\infty} \|G_i\|_{p(\cdot)} \leq \sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

More generally, for each $j$ the same argument shows that

$$\|f - G_j\|_{p(\cdot)} \leq \liminf_{i\to\infty} \|G_i - G_j\|_{p(\cdot)} \leq \sum_{k=j+1}^{\infty} \|f_k\|_{p(\cdot)};$$

since the sum on the right-hand side tends to 0, we see that $G_j \to f$ in norm, which completes the proof. □

The completeness of $L^{p(\cdot)}(\Omega)$ now follows from the Riesz-Fischer property.

**Theorem 2.71.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, $L^{p(\cdot)}(\Omega)$ is complete: every Cauchy sequence in $L^{p(\cdot)}(\Omega)$ converges in norm.*

*Proof.* Let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be a Cauchy sequence. Choose $k_1$ such that $\|f_i - f_j\|_{p(\cdot)} < 2^{-1}$ for $i, j \geq k_1$, choose $k_2 > k_1$ such that $\|f_i - f_j\|_{p(\cdot)} < 2^{-2}$ for $i, j \geq k_2$, and so on. This construction yields a subsequence $\{f_{k_j}\}$, $k_{j+1} > k_j$, such that

$$\|f_{k_{j+1}} - f_{k_j}\|_{p(\cdot)} < 2^{-j}.$$

Define a new sequence $\{g_j\}$ by $g_1 = f_{k_1}$ and for $j > 1$, $g_j = f_{k_j} - f_{k_{j-1}}$. Then for all $j$ we get the telescoping sum

$$\sum_{i=1}^{j} g_i = f_{k_j};$$

further, we have that

$$\sum_{j=1}^{\infty} \|g_j\|_{p(\cdot)} \leq \|f_{k_1}\|_{p(\cdot)} + \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

Therefore, by the Riesz-Fischer property (Theorem 2.70), there exists $f \in L^{p(\cdot)}(\Omega)$ such that $f_{k_j} \to f$ in norm.

Finally, by the triangle inequality we have that

$$\|f - f_k\|_{p(\cdot)} \leq \|f - f_{k_j}\|_{p(\cdot)} + \|f_{k_j} - f_k\|_{p(\cdot)};$$

since $\{f_k\}$ is a Cauchy sequence, for $k$ sufficiently large we can choose $k_j$ to make the right-hand side as small as desired. Hence, $f_k \to f$ in norm. □

We now consider the question of dense subsets of $L^{p(\cdot)}(\Omega)$. To simplify our exposition, we will assume that all domains $\Omega$ are open.

**Theorem 2.72.** *Given an open set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose that $p_+ < \infty$. Then the set of bounded functions of compact support with* $\mathrm{supp}(f) \subset \Omega$ *is dense in $L^{p(\cdot)}(\Omega)$.*

*Proof.* Let $K_k$ be a nested sequence of compact subsets of $\Omega$ such that $\Omega = \bigcup_k K_k$. (For instance, let $K_k = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq 1/k\} \cap \overline{B_k(0)}$.) Fix $f \in L^{p(\cdot)}(\Omega)$ and define the sequence $\{f_k\}$ by

$$f_k(x) = \begin{cases} k & f_k(x) > k \\ f(x) & -k \leq f(x) \leq k \\ -k & f_k(x) < -k, \end{cases}$$

and let $g_k(x) = f_k(x)\chi_{K_k}(x)$. Since $f$ is finite almost everywhere, $g_k \to f$ pointwise almost everywhere; since $f \in L^{p(\cdot)}(\Omega)$ and $|g_k(x)| \leq |f(x)|$, $g_k \in L^{p(\cdot)}(\Omega)$. Therefore, since $p_+ < \infty$, by the dominated convergence theorem (Theorem 2.62), $g_k \to f$ in norm. □

As a corollary to Theorem 2.72 we get two additional dense subsets. Recall that $C_c(\Omega)$ denotes the set of all continuous functions whose support is compact and contained in $\Omega$. We define $S(\Omega)$ to be the collection of all simple functions, that is, functions whose range is finite: $s \in S(\Omega)$ if

$$s(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where the numbers $a_j$ are distinct and the sets $E_j \subset \Omega$ are pairwise disjoint. The family $S_0(\Omega)$ is the collection of $s \in S$ with the additional property that

$$\left| \bigcup_{j=1}^{n} E_j \right| < \infty.$$

**Corollary 2.73.** *Given an open set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. Then the sets $C_c(\Omega)$ and $S_0(\Omega)$ are dense in $L^{p(\cdot)}(\Omega)$.*

*Proof.* We will prove this for $C_c(\Omega)$; the proof for $S_0(\Omega)$ is identical. Fix $f \in L^{p(\cdot)}(\Omega)$ and fix $\epsilon > 0$; we will find a function $h \in C_c(\Omega)$ such that $\|f - h\|_{p(\cdot)} < \epsilon$.

By Theorem 2.72 there exists a bounded function of compact support, $g$, such that $\|f - g\|_{p(\cdot)} < \epsilon/2$. Let $\operatorname{supp}(g) \subset B \cap \Omega$ for some open ball $B$. Then since $p_+ < \infty$, $C_c(B \cap \Omega)$ is dense in $L^{p_+}(B \cap \Omega)$; thus there exists $h \in C_c(B \cap \Omega) \subset C_c(\Omega)$ such that

$$\|g - h\|_{L^{p_+}(\Omega)} = \|g - h\|_{L^{p_+}(B\cap\Omega)} < \frac{\epsilon}{2(1 + |B \cap \Omega|)}.$$

Therefore, by Corollary 2.48,

$$\|g - h\|_{L^{p(\cdot)}(\Omega)} = \|g - h\|_{L^{p(\cdot)}(B\cap\Omega)} \le (1 + |B \cap \Omega|)\|g - h\|_{L^{p_+}(B\cap\Omega)} < \epsilon/2,$$

and so

$$\|f - h\|_{p(\cdot)} \le \|f - g\|_{p(\cdot)} + \|g - h\|_{p(\cdot)} < \epsilon.$$

□

*Remark 2.74.* If $p_+ < \infty$, then the set $\bigcap_{p>1} L^p(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ since this intersection contains $C_c(\Omega)$. This observation will be useful in Chap. 5 below.

Theorem 2.72 need not be true if $p_+ = \infty$. This is clearly the case if $\Omega_\infty$ is open and $|\Omega_\infty| > 0$, since bounded functions of compact support with $\operatorname{supp}(f) \subset \Omega_\infty$ are not dense in $L^\infty(\Omega_\infty)$. But it still fails even if $p(\cdot)$ is simply unbounded. First, we will show that bounded functions are not dense, and then show that under certain conditions functions of compact support are not dense.

**Theorem 2.75.** *Given $\Omega$ open and $p(\cdot) \in \mathcal{P}(\Omega)$, if $p_+(\Omega \setminus \Omega_\infty) = \infty$, then bounded functions are not dense in $L^{p(\cdot)}(\Omega)$.*

*Remark 2.76.* It follows from Theorem 2.75 that if $p_+(\Omega \setminus \Omega_\infty) = \infty$, then $C_c(\Omega)$ and $S_0(\Omega)$ are not dense in $L^{p(\cdot)}(\Omega)$.

*Proof.* We will construct a function $f \in L^{p(\cdot)}(\Omega)$ that cannot be approximated by bounded functions. To do so we will modify the construction given in the proof of Proposition 2.12.

Since $p_+(\Omega \setminus \Omega_\infty) = \infty$, there exists an increasing sequence $\{p_i\}$, $p_i > i$, such that the sets

$$F_i = \{x \in \Omega \setminus \Omega_\infty : p_i < p(x) < p_{i+1}\}$$

have positive measure. For each $i$, choose $G_i \subset F_i$ such that

$$0 < |G_i| < \left(\frac{1}{2^i}\right)^{p_{i+1}} < 1.$$

Then for all $\lambda > 0$,

$$\rho(\chi_{G_i}/\lambda) = \int_{\Omega\setminus\Omega_\infty} \left(\frac{\chi_{G_i}(x)}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\|\chi_{G_i}\|_{L^\infty(\Omega_\infty)}$$
$$= \int_{G_i} \lambda^{-p(x)}\,dx \leq |G_i| \max(\lambda^{-p_i}, \lambda^{-p_{i+1}}).$$

Hence,

$$\|\chi_{G_i}\|_{p(\cdot)} \leq \inf\{\lambda > 0 : |G_i| \max(\lambda^{-p_i}, \lambda^{-p_{i+1}}) \leq 1\}$$
$$\leq \inf\{\lambda > 0 : |G_i| \leq \min(\lambda^{p_i}, \lambda^{p_{i+1}})\}$$
$$\leq \max(|G_i|^{1/p_i}, |G_i|^{1/p_{i+1}}) = |G_i|^{1/p_{i+1}} < 2^{-i}.$$

Now define the sets $\{E_k\}$ by

$$E_k = \bigcup_{i=k}^{\infty} G_i.$$

Then we have that

1. $E_k \subset \Omega \setminus \Omega_\infty$;
2. $E_{k+1} \subset E_k$ and $|E_k \setminus E_{k+1}| = |G_k| > 0$;
3. $|E_k| \to 0$ since
$$|E_k| = \sum_{i=k}^{\infty} |G_i| < \sum_{i=k}^{\infty} (2^{-i})^{p_{i+1}};$$
4. If $x \in E_k$, then $p(x) \geq p_k > k$;
5. $\|\chi_{E_k}\|_{p(\cdot)} \to 0$ since
$$\|\chi_{E_k}\|_{p(\cdot)} \leq \sum_{i=k}^{\infty} \|\chi_{G_i}\|_{p(\cdot)} < \sum_{i=k}^{\infty} 2^{-i}.$$

Properties (1)–(4) are exactly the properties from the proof of Proposition 2.12 used in the proof of Theorem 2.58 to construct the function $f$ and show that $f \in L^{p(\cdot)}(\Omega)$ and $\|f\chi_{E_k}\|_{p(\cdot)} \geq 1/2$; repeat this construction using these sets.

For any $h \in L^\infty(\Omega)$, by Property (5) fix $k$ sufficiently large such that

$$\|h\chi_{E_k}\|_{p(\cdot)} \leq \|h\|_{L^\infty}\|\chi_{E_k}\|_{p(\cdot)} < \frac{1}{4}.$$

Then by the triangle inequality we have that

$$\|f-h\|_{p(\cdot)} \geq \|(f-h)\chi_{E_k}\|_{p(\cdot)} \geq \|f\chi_{E_k}\|_{p(\cdot)} - \|h\chi_{E_k}\|_{p(\cdot)} \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Since $h$ is an arbitrary bounded function, we see that bounded functions are not dense in $L^{p(\cdot)}(\Omega)$. □

Intuitively, the next result shows that if $p(\cdot)$ is unbounded at the boundary of $\Omega$, then functions of compact support are not dense.

**Theorem 2.77.** *Given $\Omega$ open and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose that for every compact set $K \subset \Omega$, $p_+(\Omega \setminus K) = \infty$. Then functions with compact support and* supp$(f) \subset \Omega$ *are not dense in $L^{p(\cdot)}(\Omega)$.*

*Proof.* Define the sequence $K_k = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq 1/k\} \cap \overline{B_k(0)}$. Then the sets $K_k$ are compact, nested, and their union is $\Omega$. By our hypothesis there exists a sequence of disjoint sets $E_k \subset \Omega \setminus K_k$ such that $|E_k| > 0$ and $p_-(E_k) > k$. Let $E_k^* = E_k \setminus \Omega_\infty$ and $E_k^\infty = E_k \cap \Omega_\infty$. By passing to a subsequence and renumbering, we may assume without loss of generality that either $|E_k^\infty| > 0$ for every $k$ or $|E_k^*| > 0$ for every $k$. In the first case, define

$$f(x) = \sum_{k=1}^\infty \chi_{E_k^\infty}(x).$$

Since the sets $E_k^\infty$ are disjoint, $f \in L^\infty(\Omega_\infty) \subset L^{p(\cdot)}(\Omega)$. Further, given any function $g$ such that supp$(g)$ is compact and contained in $\Omega$, there exists $k_0$ such that supp$(g) \subset K_{k_0}$. But then,

$$\|f-g\|_{p(\cdot)} \geq \|\chi_{E_{k_0+1}^\infty}\|_{p(\cdot)} = \|\chi_{E_{k_0+1}^\infty}\|_\infty = 1.$$

Hence, functions of compact support are not dense.

If, on the other hand, $|E_k^*| > 0$ for every $k$, define

$$f(x) = \sum_{k=1}^\infty |E_k^*|^{-1/p(x)}\chi_{E_k^*}(x).$$

Then for any $\lambda > 1$,

$$\rho(f/\lambda) = \sum_{k=1}^{\infty} \int_{E_k^*} \lambda^{-p(x)}\,dx \leq \sum_{k=1}^{\infty} \lambda^{-k} < \infty.$$

Thus $f \in L^{p(\cdot)}(\Omega)$. But given $g$ as before,

$$\rho(f-g) \geq \sum_{k=k_0+1}^{\infty} \int_{E_k^*} f(x)^{p(x)}\,dx = \sum_{k=k_0+1}^{\infty} 1 = \infty.$$

Therefore, $\|f-g\|_{p(\cdot)} \geq 1$, so again functions of compact support are not dense in $L^{p(\cdot)}(\Omega)$. □

We conclude this section with an important characterization of the dense subsets of $L^{p(\cdot)}$. Recall that a Banach space is separable if it has a countable dense subset.

**Theorem 2.78.** *Given an open set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, then $L^{p(\cdot)}(\Omega)$ is separable if and only if $p_+ < \infty$.*

*Proof.* Suppose first that $p_+ < \infty$. Then the proof of separability is almost identical to the proof of Corollary 2.73 so we sketch only the key details. We can write

$$\Omega = \bigcup_{k=1}^{\infty} B_k(0) \cap \Omega.$$

Since $B_k(0) \cap \Omega$ is open, $L^{p_+}(B_k(0) \cap \Omega)$ is separable and so contains a countable dense subset. The union of all of these sets is a countable set contained in $L^{p(\cdot)}(\Omega)$. Arguing exactly as we did before we see that this set is also dense in $L^{p(\cdot)}(\Omega)$.

Now suppose that $p_+ = \infty$. We will show that no countable set is dense. If $|\Omega_\infty| > 0$, then this follows from the same construction that shows that $L^\infty(\Omega_\infty)$ is non-separable, since the restriction of any dense subset of $L^{p(\cdot)}(\Omega)$ will be dense in $L^\infty(\Omega_\infty)$. (See, for example, Brezis [37].)

Now let $|\Omega_\infty| = 0$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$, and suppose to the contrary that there exists a countable dense set $\{h_j\}$. Let the sets $E_k$ and the function $f$ be as in the proof of Theorem 2.75. Then for all $k$, $\|f\chi_{E_k}\|_{p(\cdot)} \geq 1/2$, so by Theorem 2.34, there exist functions $g_k \in L^{p'(\cdot)}(\Omega)$, $\|g_k\|_{p'(\cdot)} \leq 1$, and $\epsilon > 0$ such that

$$\int_\Omega f(x)\chi_{E_k(x)} g_k(x)\,dx \geq \epsilon.$$

By Hölder's inequality (Theorem 2.26), for each $j$,

$$\left| \int_\Omega h_j(x) g_k(x) \chi_{E_k}(x)\,dx \right| \leq C\|h_j\|_{p(\cdot)},$$

and so the sequence $\{\int h_j g_k \chi_{E_k}\,dx\}_k$ is bounded. Hence, it has a convergent subsequence, and so by a Cantor diagonalization argument we can find a subsequence of functions $\{g_{k_i}\chi_{E_{k_i}}\}_i$ such that for every $j$, the sequence $\{\int h_j g_{k_i}\chi_{E_{k_i}}\,dx\}_i$ converges and so is Cauchy.

From this fact we will see that for any $F \subset \Omega$ the sequence

$$\left\{\int_\Omega f(x)\chi_F(x)g_{k_i}(x)\chi_{E_{k_i}}(x)\,dx\right\}_i \tag{2.12}$$

is Cauchy. Fix $\epsilon > 0$ and let $h_j$ be such that $\|h_j - f\chi_F\|_{p(\cdot)} < \epsilon$. Then for all $i$ and $l$,

$$\begin{aligned}
&\left|\int_\Omega f(x)\chi_F(x)g_{k_i}(x)\chi_{E_{k_i}}(x)\,dx - \int_\Omega f(x)\chi_F(x)g_{k_l}(x)\chi_{E_{k_l}}(x)\,dx\right| \\
&\quad \leq \left|\int_\Omega \big(f(x)\chi_F(x) - h_j(x)\big)g_{k_i}(x)\chi_{E_{k_i}}(x)\,dx\right| \\
&\qquad + \left|\int_\Omega \big(f(x)\chi_F(x) - h_j(x)\big)g_{k_l}(x)\chi_{E_{k_l}}(x)\,dx\right| \\
&\qquad + \left|\int_\Omega h_j(x)\big(g_{k_i}(x)\chi_{E_{k_i}}(x) - g_{k_l}(x)\chi_{E_{k_l}}(x)\big)\,dx\right|.
\end{aligned}$$

By Hölder's inequality the first two terms are bounded by $C\epsilon$ and the last term is less than $\epsilon$ for all $i$ and $l$ sufficiently large. Thus the sequence (2.12) is Cauchy and so converges.

Since the sets $E_{k_i}$ are nested, we can define a sequence of measures on $E_1$ by

$$\mu_i(F) = \int_{E_1} f(x)\chi_F(x)g_{k_i}(x)\chi_{E_{k_i}}(x)\,dx, \qquad F \subset E_1.$$

Since (2.12) converges, there exists a set function $\mu$ such that

$$\mu(F) = \lim_{i\to\infty} \mu_i(F).$$

Since $|E_1| < \infty$, by the Hahn-Saks theorem $\mu$ is an absolutely continuous measure on $E_1$. (See Hewitt and Stromberg [169, ex. 19.68, p. 339].) Therefore, there exists $g \in L^1_{\mathrm{loc}}(E_1)$ such that

$$\mu(F) = \int_F g(x)\,dx.$$

We claim that $g \equiv 0$. To see this, note that since the sets $E_k$ are nested and $|E_k| \to 0$, $|\cap_i E_{k_i}| = 0$. Now fix any $i$ and let $F$ be such that $|F \cap E_{k_i}| = 0$. Then

$$\mu(F) = \lim_{i\to\infty} \mu_i(F) = 0.$$

This is true for all such sets $F$; in particular we can take $F$ to be the set where $g\chi_{E_1\setminus E_{k_i}}$ is positive or negative. Hence, we must have that $g \equiv 0$ on $E_1 \setminus E_{k_i}$. Since this is true for all $i$, $g \equiv 0$ on $E_1$. But then

$$0 = \mu(E_1) = \lim_{i\to\infty} \mu_i(E_1) = \lim_{i\to\infty} \int_\Omega f(x)\chi_{E_{k_i}}(x) g_{k_i}(x)\,dx \geq \epsilon,$$

which is a contradiction. Hence, $L^{p(\cdot)}(\Omega)$ is not separable. □

## 2.8 The Dual Space of a Variable Lebesgue Space

In this section we consider the dual space of $L^{p(\cdot)}(\Omega)$: that is, the Banach space $L^{p(\cdot)}(\Omega)^*$ of continuous linear functionals $\Phi : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ with norm

$$\|\Phi\| = \sup_{\|f\|_{p(\cdot)}\leq 1} |\Phi(f)|.$$

In the classical Lebesgue spaces, $L^{p'} \subset (L^p)^*$ (up to isomorphism), and equality holds if $p < \infty$. The behavior of the variable Lebesgue spaces is analogous if $p_+ < \infty$.

We will begin by constructing a large family of continuous linear functionals and showing that they are induced by elements of $L^{p'(\cdot)}(\Omega)$. Given a measurable function $g$, define the linear functional $\Phi_g$ on $L^{p(\cdot)}(\Omega)$ by

$$\Phi_g(f) = \int_\Omega f(x)g(x)\,dx.$$

**Proposition 2.79.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, and a measurable function $g$, then $\Phi_g$ is a continuous linear functional on $L^{p(\cdot)}(\Omega)$ if and only if $g \in L^{p'(\cdot)}(\Omega)$. Furthermore, $\|\Phi_g\| = \|g\|'_{p'(\cdot)}$, and so*

$$k_{p'(\cdot)}\|g\|_{p'(\cdot)} \leq \|\Phi_g\| \leq K_{p'(\cdot)}\|g\|_{p'(\cdot)}. \tag{2.13}$$

*Proof.* Given any measurable function $g$, it follows from the definitions that $\|\Phi_g\| = \|g\|'_{p'(\cdot)}$, and so by Theorem 2.34 (with the roles of $f$ and $g$ exchanged in the statement and $p(\cdot)$ replaced by $p'(\cdot)$), $\Phi_g$ is continuous if and only if $g \in L^{p'(\cdot)}(\Omega)$ and we get inequality (2.13). □

The linear mapping $g \mapsto \Phi_g$ provides a natural identification between $L^{p'(\cdot)}(\Omega)$ and a subspace of $L^{p(\cdot)}(\Omega)^*$. When $p(\cdot)$ is bounded, we get every element of the dual space in this way.

**Theorem 2.80.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, the following are equivalent:*

1. $p_+ < \infty$;
2. *The map $g \mapsto \Phi_g$ is an isomorphism: given any $g \in L^{p'(\cdot)}(\Omega)$, the functional $\Phi_g$ is continuous and linear; conversely, given any continuous linear functional $\Phi \in L^{p(\cdot)}(\Omega)^*$ there exists a unique $g \in L^{p'(\cdot)}(\Omega)$ such that $\Phi = \Phi_g$ and $\|g\|_{p'(\cdot)} \approx \|\Phi\|$.*

It follows from Theorem 2.80 that when $p_+ < \infty$ the dual space and the associate space of $L^{p(\cdot)}(\Omega)$ (see Proposition 2.37) coincide. In this case we will simply write $L^{p(\cdot)}(\Omega)^* = L^{p'(\cdot)}(\Omega)$; the isomorphism will be implicit.

As an immediate corollary to Theorem 2.80 we can characterize when $L^{p(\cdot)}(\Omega)$ is reflexive. (Recall that a Banach space $X$ is reflexive if $X^{**} = X$, with equality in the sense of isomorphism.)

**Corollary 2.81.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, $L^{p(\cdot)}(\Omega)$ is reflexive if and only if $1 < p_- \leq p_+ < \infty$.*

*Proof of Theorem 2.80.* Suppose first that $p_+ < \infty$. Fix $\Phi \in L^{p(\cdot)}(\Omega)^*$; we will find $g \in L^{p'(\cdot)}(\Omega)$ such that $\Phi = \Phi_g$. Note that by (2.13) we immediately get that $\|g\|_{p'(\cdot)} \approx \|\Phi\|$.

We initially consider the case when $|\Omega| < \infty$. Define the set function $\mu$ by $\mu(E) = \Phi(\chi_E)$ for all measurable $E \subset \Omega$. Since $\Phi$ is linear and $\chi_{E \cup F} = \chi_E + \chi_F$ if $E \cap F = \emptyset$, $\mu$ is additive. To see that it is countably additive, let

$$E = \bigcup_{j=1}^{\infty} E_j,$$

where the sets $E_j \subset \Omega$ are pairwise disjoint, and let

$$F_k = \bigcup_{j=1}^{k} E_j.$$

Then by Corollary 2.48,

$$\begin{aligned}\|\chi_E - \chi_{F_k}\|_{p(\cdot)} &\leq (1 + |\Omega|)\|\chi_E - \chi_{F_k}\|_{p_+} \\ &= (1 + |\Omega|)|E \setminus F_k|^{1/p_+}.\end{aligned}$$

Since $|E| < \infty$, $|E \setminus F_k|$ tends to 0 as $k \to \infty$; thus $\chi_{F_k} \to \chi_E$ in norm. Therefore, by the continuity of $\Phi$, $\Phi(\chi_{F_k}) \to \Phi(\chi_E)$; equivalently,

$$\sum_{j=1}^{\infty} \mu(E_j) = \mu(E),$$

and so $\mu$ is countably additive.

In other words $\mu$ is a measure on $\Omega$. Further, it is absolutely continuous: if $E \subset \Omega$, $|E| = 0$, then $\chi_E \equiv 0$, and so

$$\mu(E) = \Phi(\chi_E) = 0.$$

By the Radon-Nikodym theorem (see Royden [301]), absolutely continuous measures are gotten from $L^1$ functions. More precisely, there exists $g \in L^1(\Omega)$ such that

$$\Phi(\chi_E) = \mu(E) = \int_\Omega \chi_E(x) g(x)\, dx.$$

By the linearity of $\Phi$, for every simple function $f = \sum a_j \chi_{E_j}$, $E_j \subset \Omega$,

$$\Phi(f) = \int_\Omega f(x) g(x)\, dx.$$

By Corollary 2.73, the simple functions are dense in $L^{p(\cdot)}(\Omega)$, and so $\Phi$ and $\Phi_g$ agree on a dense subset. Thus, by continuity $\Phi = \Phi_g$, and so by Proposition 2.79, $g \in L^{p'(\cdot)}(\Omega)$.

Finally, to see that $g$ is unique, it is enough to note that if $g, \tilde{g} \in L^{p'(\cdot)}(\Omega)$ are such that $\Phi_g = \Phi_{\tilde{g}}$, then for all $f \in L^{p(\cdot)}(\Omega)$,

$$\int_\Omega f(x)(g(x) - \tilde{g}(x))\, dx = 0. \tag{2.14}$$

Since $|\Omega| < \infty$, by Corollary 2.50, $g - \tilde{g} \in L^{p'(\cdot)}(\Omega) \subset L^{p'(\cdot)_-}(\Omega) = L^{(p_+)'}(\Omega)$, and since (2.14) holds for all $f \in L^{p_+}(\Omega) \subset L^{p(\cdot)}(\Omega)$, by the duality theorem for the classical Lebesgue spaces, $g - \tilde{g} = 0$ almost everywhere.

We now consider the case when $|\Omega| = \infty$. Write

$$\Omega = \bigcup_{k=1}^\infty \Omega_k,$$

where for each $k$, $|\Omega_k| < \infty$ and $\overline{\Omega}_k \subset \Omega_{k+1}$. Given $\Phi \in L^{p(\cdot)}(\Omega)^*$, by restriction $\Phi$ induces a bounded linear functional on $L^{p(\cdot)}(\Omega_k)$ for each $k$. Therefore, by the above argument, there exists $g_k \in L^{p'(\cdot)}(\Omega_k)$ such that for all $f \in L^{p(\cdot)}(\Omega)$, $\operatorname{supp}(f) \subset \overline{\Omega}_k$,

$$\Phi(f) = \int_{\Omega_k} f(x) g_k(x)\, dx.$$

Further, $\|g_k\|_{p'(\cdot)} \leq k^{-1}_{p'(\cdot),\Omega_k} \|\Phi\| \leq 3\|\Phi\|$. Since the sets $\Omega_k$ are nested, we must have that for all $f$ with support in $\Omega_k$,

$$\int_{\Omega_k} f(x) g_k(x)\, dx = \int_{\Omega_{k+1}} f(x) g_{k+1}(x)\, dx.$$

Since the functions $g_k$ are unique, we must have that $g_k = g_{k+1}\chi_{\Omega_k}$. Therefore, we can define $g$ by $g(x) = g_k(x)$ for all $x \in \Omega_k$. Since $\operatorname{supp}(g_k) \subset \overline{\Omega}_k$, the sequence $|g_k|$ increases to $|g|$; hence, by the monotone convergence theorem for variable Lebesgue spaces (Theorem 2.59),

$$\|g\|_{p'(\cdot)} = \lim_{k\to\infty} \|g_k\|_{p'(\cdot)} \leq 3\|\Phi\| < \infty.$$

Thus $g \in L^{p'(\cdot)}(\Omega)$.

Now fix $f \in L^{p(\cdot)}(\Omega)$ and let $f_k = f\chi_{\Omega_k}$. Then $f_k \to f$ pointwise almost everywhere and $|f - f_k| \leq |f|$, so by the dominated convergence theorem in variable Lebesgue spaces (Theorem 2.62), $f_k \to f$ in norm. Further, $f_k g \to fg$ pointwise, and by Hölder's inequality for variable Lebesgue spaces (Theorem 2.26), $|f_k g| \leq |fg| \in L^1(\Omega)$. Therefore, by the classical dominated convergence theorem and the continuity of $\Phi$,

$$\int_\Omega f(x)g(x)\,dx = \lim_{k\to\infty} \int_{\Omega_k} f_k(x)g(x)\,dx$$
$$= \lim_{k\to\infty} \int_{\Omega_k} f_k(x)g_k(x)\,dx = \lim_{k\to\infty} \Phi(f_k) = \Phi(f).$$

Finally, since the restriction of $g$ to each $\Omega_k$ is uniquely determined, $g$ itself is the unique element of $L^{p'(\cdot)}(\Omega)$ with this property. This completes the proof of the first half of the theorem.

Now suppose that $p_+ = \infty$; we will show that there exists $\Phi \in L^{p(\cdot)}(\Omega)^*$ such that $\Phi \neq \Phi_g$ for any $g \in L^{p'(\cdot)}(\Omega)$.

If $|\Omega_\infty| > 0$, then we use the fact that $L^\infty(\Omega_\infty)^*$ contains (the isomorphic image of) $L^1(\Omega_\infty) = L^{p'(\cdot)}(\Omega_\infty)$ as a proper subset (see, for example, Brezis [37] or Dunford and Schwartz [95]); in other words there exists $\Phi \in L^\infty(\Omega_\infty)^*$ that is not induced by any element of $L^1(\Omega_\infty)$. By the Hahn-Banach theorem we can extend $\Phi$ to an element of $L^{p(\cdot)}(\Omega)^*$. This is clearly the desired element: if it were equal to $\Phi_g$ for some $g \in L^{p'(\cdot)}(\Omega)$, then its restriction to $L^{p(\cdot)}(\Omega_\infty)$ would be induced by $g\chi_{\Omega_\infty}$, contradicting our choice of $\Phi$.

Now assume that $|\Omega_\infty| = 0$ but $p_+(\Omega \setminus \Omega_\infty) = \infty$. We will prove that the desired $\Phi$ exists by contradiction. The proof starts as in the proof of Theorem 2.78. Suppose to the contrary that every $\Phi \in L^{p(\cdot)}(\Omega)^*$ is of the form $\Phi_g$, $g \in L^{p'(\cdot)}(\Omega)$. Fix sets $E_k$ and the function $f$ as constructed in the proof of Theorem 2.75. Then $f$ is non-negative, $\|f\|_{p(\cdot)} \leq 1$, $\|\chi_{E_k}\|_{p(\cdot)} \to 0$, and for every $k$, $\|f\chi_{E_k}\|_{p(\cdot)} \geq 1/2$. Therefore, by Theorem 2.34 there exist non-negative functions $g_k \in L^{p'(\cdot)}(\Omega)$, $\|g_k\|_{p'(\cdot)} \leq 1$, and $\epsilon > 0$ such that

$$\int_\Omega f(x)\chi_{E_k}(x)g_k(x)\,dx \geq \epsilon. \tag{2.15}$$

Without loss of generality we may assume that for all $k$, $g_k = g_k\chi_{E_k}$.

Define the sets

$$G_k = \left\{\Phi \in L^{p(\cdot)}(\Omega)^* : |\Phi(f\chi_{E_k})| < \epsilon/2\right\}.$$

Then we have that $L^{p(\cdot)}(\Omega)^* = \bigcup_k G_k$. To see this, fix $\Phi \in L^{p(\cdot)}(\Omega)^*$; by our original assumption there exists $g \in L^{p'(\cdot)}(\Omega)$ such that $\Phi = \Phi_g$. By Hölder's inequality (Theorem 2.26), $fg \in L^1(\Omega)$, and so by the classical dominated convergence theorem,

$$\lim_{k\to\infty} \Phi_g(f\chi_{E_k}) = \lim_{k\to\infty} \int_\Omega f(x)\chi_{E_k}(x)g(x)\,dx = 0.$$

Hence, for $k$ sufficiently large, $\Phi \in G_k$.

By definition, the sets $G_k$ are open in the weak* topology on $L^{p(\cdot)}(\Omega)^*$. Therefore, the collection $\{G_k\}$ is an open cover of the ball $B = \{\Phi \in L^{p(\cdot)}(\Omega)^* : \|\Phi\| \leq 4\}$. By the Banach-Alaoglu Theorem (see Brezis [37] or Conway [51]), $B$ is weak* compact, and so there exists $N > 0$ and a collection of indices $1 \leq k_1 < k_2 < \cdots < k_N$ such that $\{G_{k_i}\}_{i=1}^N$ is a finite subcover of $B$.

Define $\Phi_k \in L^{p(\cdot)}(\Omega)^*$ by

$$\Phi_k(h) = \Phi_{g_k}(h) = \int_\Omega h(x)\chi_{E_k} g_k(x)\,dx.$$

Since $\|g_k\|_{p'(\cdot)} \leq 1$, by Theorem 2.34, $\|\Phi_k\| \leq 4$ and so $\Phi_k \in B$. Let $k_i$ be such that $\Phi_k \in G_{k_i}$; then we have that $\Phi_k(f\chi_{E_{k_i}}) = |\Phi_k(f\chi_{E_{k_i}})| < \epsilon/2$. Since the sets $E_k$ are nested, for all $k \geq k_N$,

$$\begin{aligned}\int_{E_k} f(x)g_k(x)\,dx &= \int_\Omega f(x)\chi_{E_k}(x)g_k(x)\,dx \\ &\leq \int_\Omega f(x)\chi_{E_{k_i}} g_k(x)\,dx = \Phi_k(f\chi_{E_{k_i}}) < \epsilon/2.\end{aligned}$$

But this contradicts inequality (2.15). Therefore, our original supposition is false, and there exists $\Phi \in L^{p(\cdot)}(\Omega)^*$ not induced by any $g \in L^{p'(\cdot)}(\Omega)$. This completes our proof. □

## 2.9 The Lebesgue Differentiation Theorem

We conclude this chapter with a generalization of the Lebesgue differentiation theorem to variable Lebesgue spaces. In the classical case (see Grafakos [143]) if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \fint_{B_r(x)} f(y)\,dy = f(x).$$

Such points $x$ are referred to as Lebesgue points of the function $f$. This limit also holds if balls are replaced by cubes centered at $x$ or more generally by a nested sequence of balls or cubes whose intersection contains $x$. In particular, it holds for the sequence of dyadic cubes containing $x$. (See Sect. 3.2 below.) If $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, then by Proposition 2.41 $f$ is locally integrable, so the Lebesgue differentiation theorem holds for such $f$.

However, if $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$, then a stronger result holds (again see [143]): for almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \fint_{B_r(x)} |f(y) - f(x)|^p\,dy = 0.$$

An analog of this is true in the variable Lebesgue spaces.

**Proposition 2.82.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $|\Omega_\infty| = 0$, and $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$ there exists $\alpha > 0$ such that*

$$\lim_{r\to 0} \fint_{B_r(x)} \big|\alpha\big(f(y) - f(x)\big)\big|^{p(y)}\,dy = 0. \tag{2.16}$$

*If $p_+ < \infty$, then we can take $\alpha = 1$.*

*Proof.* Since this is a local result, it will suffice to fix a ball $B$ and prove (2.16) for almost every $x \in B$. Since $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, there exists $\lambda > 1$ such that

$$\int_B \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy < \infty.$$

Enumerate the rationals as $\{q_i\}$ and define $\beta_i = \big(2\lambda(|q_i| + 1)\big)^{-1}$. Then

$$\begin{aligned}\int_B \big|\beta_i\big(f(y) - q_i\big)\big|^{p(y)}\,dy &\le \int_B 2^{p(y)-1}\big(\big|\beta f(y)\big|^{p(y)} + |\beta q_i|^{p(y)}\big)\,dy \\ &\le \frac{1}{2}\int_B \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy + \int_B \left(\frac{|q_i|}{|q_i| + 1}\right)^{p(y)} dy < \infty.\end{aligned}$$

Therefore, by the classical Lebesgue differentiation theorem, for each $i$ and for almost every $x \in B$,

$$\lim_{r\to 0} \fint_{B_r(x)} \big|\beta_i\big(f(y) - q_i\big)\big|^{p(y)}\,dy = \big|\beta_i\big(f(x) - q_i\big)\big|^{p(x)}.$$

Since the countable union of sets of measure 0 again has measure 0, this limit holds for all $i$ and almost every $x \in B$. Fix such an $x$ and fix $\epsilon$, $0 < \epsilon < 1$. Then there exists $i$ such that

$$\big|\beta_i\big(f(x) - q_i\big)\big| < \epsilon.$$

Define $\alpha = \beta_i/2$. Then by Remark 2.8 we have that

$$\begin{aligned}
&\limsup_{r\to 0} \fint_{B_r(x)} \big|\alpha\big(f(y) - f(x)\big)\big|^{p(y)}\,dy \\
&\quad \leq \limsup_{r\to 0} \bigg(\fint_{B_r(x)} 2^{p(y)-1}\Big|\frac{\beta_i}{2}\big(f(y) - q_i\big)\Big|^{p(y)}\,dy \\
&\qquad\qquad + \fint_{B_r(x)} 2^{p(y)-1}\Big|\frac{\beta_i}{2}\big(f(x) - q_i\big)\Big|^{p(y)}\,dy\bigg) \\
&\quad \leq \frac{1}{2}\limsup_{r\to 0} \bigg(\fint_{B_r(x)} \big|\beta_i\big(f(y) - q_i\big)\big|^{p(y)}\,dy \\
&\qquad\qquad + \fint_{B_r(x)} \big|\beta_i\big(f(x) - q_i\big)\big|\,dy\bigg) \\
&\quad = \frac{1}{2}\Big(\big|\beta_i\big(f(x) - q_i\big)\big|^{p(x)} + \big|\beta_i\big(f(x) - q_i\big)\big|\Big) \\
&\quad < \epsilon.
\end{aligned}$$

The limit (2.16) follows at once.

Finally if $p_+ < \infty$, then the above proof can be readily modified to take $\alpha = \beta_i = 1$. □

*Remark 2.83.* When $p_+ < \infty$, by Theorem 2.58 the modular limit implies a limit of norms:

$$\lim_{r\to 0} \big\| |B_r(x)|^{-1/p(\cdot)}|f(\cdot) - f(x)| \big\|_{p(\cdot)} = 0.$$

## 2.10 Notes and Further Results

### *2.10.1 References*

As we discussed in Chap. 1, the variable Lebesgue spaces were considered by a number of authors independently and so many of the results in this chapter were probably discovered several times. In our treatment, we have primarily followed the work of Kováčik and Rákosník [219] and Diening [80]. (This work, Diening's habilitation thesis, has recently been expanded into a book, written jointly with Harjulehto, Hästö and Růžička [82].) The structure of variable Lebesgue spaces is also treated by Samko [313, 314] and Fan and Zhao [122]. A briefer overview, combined with an extensive bibliography, is given by Harjulehto and Hästö [150]. The structural parallels between the classical and variable Lebesgue spaces are clearest when $p_+ < \infty$, and this is the case frequently considered in the literature. Our approach has been to provide a unified treatment of bounded and unbounded exponents.

The local log-Hölder continuity condition $LH_0$ (Definition 2.2) first appeared in Sharapudinov [331] and later in Zhikov [358, 359, 361], Karapetyants and Ginzburg [189, 190], Ross and Samko [300], Samko [313], and Diening [77]. Since these papers this condition has become ubiquitous. The log-Hölder condition at infinity was introduced in [62]. Both log-Hölder conditions play a central role in harmonic analysis on variable Lebesgue spaces, as we will make clear in subsequent chapters.

The modular in Definition 2.6 is taken from [219]; for alternative definitions see Sect. 2.10.2 below. The variable Lebesgue space norm in Definition 2.16 is usually referred to as the Luxemburg norm, because it is analogous to the norm on Orlicz spaces (cf. [25]). However, it appeared in Musielak and Orlicz [275] in the more general context of modular spaces, and earlier in Nakano [280]. Independently it was defined by Sharapudinov [329], who based it on a more general result of Kolmogorov [210] about Minkowski functionals. For this reason, some authors refer to this norm as the Kolmogorov-Minkowski norm (e.g., [313]).

The extension theorem in Lemma 2.4 was first proved in [61]. A weaker version for functions in $LH_0$ appeared in [80] and for Lipschitz functions in [106]. The construction in the second half of the proof of Proposition 2.12 is due to Kováčik and Rákosník [219]; this construction and the variant of it in Theorem 2.75 play a major role in understanding the properties of variable Lebesgue spaces with unbounded exponents. A somewhat different and more general version of Proposition 2.18 (including the case $|\Omega_\infty| > 0$ and replacing the constant $s$ by a bounded function) is due independently to Samko [314] and Edmunds and Rakósník [106]; the simpler version given here was proved independently in [61]. Corollary 2.23 for $p_+ < \infty$ appeared in [122]; our version is adapted from Diening *et al.* [81]. Variants of this estimate have appeared elsewhere in the literature: see, for example, de Cicco *et al.* [73]. The proof of Proposition 2.25 is taken from Samko [313]. In the more general setting of modular spaces this was proved by Nakano [280] (who attributed this definition of the norm to Amemiya). See also Musielak [274] and Maligranda [244]. Independently, and both working in the more general setting of Musielak-Orlicz spaces, Fan [114] and Šragin [335] proved that the Amemiya norm is equal to the associate norm when $|\Omega_\infty| = 0$. (Šragin assumed that $|\Omega_\infty| = 0$. This result was also noted for modular spaces without proof by Hudzik and Maligranda [180, Remark 4].) For an application of the Amemiya norm, see [131].

Our proof of Hölder's inequality (Theorem 2.26) is taken from [219]. The generalized Hölder's inequality (Corollary 2.28) was proved by Diening [80] and earlier by Samko [313, 314] with the additional hypothesis that $r_+(\Omega \setminus \Omega_\infty^{r(\cdot)}) < \infty$. In the same papers, Samko also proved Corollary 2.30 and Minkowski's integral inequality (Corollary 2.38). His proof of Corollary 2.30 shows that the constant can be taken to be $\sum [p_i(\cdot)_-]^{-1}$.

The $L^\infty$ embedding in Proposition 2.43 was shown to us by Diening. Theorem 2.45 is due to Diening [77]; when $|\Omega| < \infty$ (i.e., Corollary 2.48) it was proved by Kováčik and Rákosník [219] and Samko [314]. A quantitative version when $p(\cdot)$ and $q(\cdot)$ are close was proved by Edmunds, Lang and Nekvinda [102]. The embedding in Theorem 2.51 was implicit in [67] and is explicit in Diening [80]. Proposition 2.53 and other, related embedding theorems were proved by Diening and Samko [92].

Our definition of modular convergence, Definition 2.54, is classical in the study of modular spaces; see Maligranda [244] or Musielak [274]. Diening [80] also uses this definition; both [219] and [122] assume $\beta = 1$ in the definition. The monotone convergence theorem for variable Lebesgue spaces (Theorem 2.59) was first stated without proof in [101]; a proof in the case $p_+ < \infty$ appeared in [58] and the full result was proved in [56]. Fatou's lemma and the dominated convergence theorem for variable Lebesgue spaces (Theorems 2.61 and 2.62) are new. The weak converse of the dominated convergence theorem, Proposition 2.67 is also new. For the converse in the case of the classical Lebesgue spaces see Brezis [37] or Lieb and Loss [238]. Theorem 2.68 for $p_+ < \infty$ is in [219] and implicit in [122]; our version is new. Theorem 2.69 is stated by Fan and Zhao [122] but the proof is only sketched. The complete proof was given in [60]; also see below.

The completeness of the variable Lebesgue spaces was proved by Kováčik and Rákosník [219] and Diening [80]; our proof is different and follows the proof in Bennett and Sharpley [25] for abstract Banach function spaces. Our approach also yields the Riesz-Fischer property (Theorem 2.70). Theorem 2.72 and Corollary 2.73 are in [219]. Theorem 2.75 is due to Kalyabin [187] and also to Edmunds, Lang and Nekvinda [101]. Theorem 2.77 is new; Harjulehto [149] gave a specific example of a space in which functions of compact support were not dense. Theorem 2.78 in the case $p_+ = \infty$ is new, but it depends critically on the construction from [219] and adapts an argument in [25].

Theorem 2.80 is proved in [219], but their proof depends on deeper results on Orlicz-Musielak spaces due to Hudzik [179] and Kozek [220]. Our proof is direct: when $p_+ < \infty$ we followed the proof for classical Lebesgue spaces in Royden [301], and for $p_+ = \infty$ we adapted an argument in Bennett and Sharpley [25]. A different proof of the characterization of reflexivity (Corollary 2.81) is due to Lukeš, Pick and Pokorný [242]: see Sect. 2.10.3 below.

The generalization of the Lebesgue differentiation theorem to the variable setting (Proposition 2.82) was proved by Harjulehto and Hästö [152] when $p_+ < \infty$. Our proof is a simple modification of theirs.

### 2.10.2 *Musielak-Orlicz Spaces and Modular Spaces*

The variable Lebesgue spaces are a particular example of a larger class of function spaces that also includes the classical and weighted Lebesgue spaces and Orlicz spaces as special cases. Given a set $\Omega$, let $\Phi : \Omega \times \mathbb{R}^+ \to [0, \infty]$ be such that for each $x \in \Omega$, the function $\Phi(x, \cdot)$ is non-decreasing, continuous and convex on the set where it is finite. Assume that $\Phi(x, 0) = 0$, $\Phi(x, t) > 0$ if $t > 0$, and $\Phi(x, t) \to \infty$ as $t \to \infty$. We also assume that for each $t \geq 0$, the function $\Phi(\cdot, t)$ is a measurable function.

Define the Musielak-Orlicz space $L^{\Phi(,)}(\Omega)$ to be the set of all functions $f$ such that for some $\lambda > 0$,

$$\rho_{\Phi(,)}(f) = \int_\Omega \Phi(x, |f(x)|/\lambda)\, dx < +\infty. \tag{2.17}$$

Then by arguments analogous to those above one can show that $L^{\Phi(,)}(\Omega)$ is a Banach function space with the norm

$$\|f\|_{L^{\Phi(,)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi(x, |f(x)|\lambda^{-1})\, dx \leq 1\right\}.$$

In this setting the norm is referred to as the Luxemburg norm. It is possible to define a so-called complementary function $\Psi$ which also generates a Musielak-Orlicz space. This space can be used to define the associate norm, which is also called the Orlicz norm. See [244, 274] for further details. Because the spaces $L^{\Phi(,)}$ generalize Orlicz spaces in the same way that $L^{p(\cdot)}$ generalizes the classical Lebesgue spaces, it makes sense to refer to $L^{\Phi(,)}$ as a variable Orlicz space, but this terminology has not been widely adopted.

Musielak-Orlicz spaces are themselves a special case of abstract Banach spaces called modular spaces. Given a set $X$ that is a real vector space, a convex modular is a function $\rho : X \to [0, \infty]$ such that:

1. $\rho(x) = 0$ if and only if $x = 0$;
2. $\rho(-x) = \rho(x)$ for all $x \in X$;
3. $\rho$ is convex;
4. The map $\lambda \mapsto \rho(\lambda x)$ is left-continuous.

If we let $X_\rho$ be the set of all $x \in X$ such that $\rho(\lambda^{-1}x) < \infty$ for some $\lambda > 0$, then this becomes a normed vector space with norm

$$\|x\|_{X_\rho} = \inf\{\lambda > 0 : \rho(\lambda^{-1}x) \leq 1\}.$$

For more further details, see [82, 244, 274].

The function $\rho_\Phi$ defined by (2.17) is a convex modular in this sense and $L^{\Phi(,)}$ is a modular space. In particular, if $p(\cdot) \in \mathcal{P}(\Omega)$, then (by Proposition 2.7) $\rho_{p(\cdot)}$ is a convex modular. Many of the classical Banach function spaces can also be viewed as Musielak-Orlicz spaces or as modular spaces. If let $\Phi(x, t) = t^p$, $1 \leq p < \infty$, we get the classical Lebesgue space $L^p(\Omega)$. If we let $\Phi(x, t) = t^p w(x)$, where $w$ is a positive, locally integrable function, then we get the weighted Lebesgue space $L^p(\Omega, w)$. If $\Phi(x, t) = \Phi(t)$, then we get the Orlicz spaces. For example, we can take $\Phi(t) = t^p \log(e+t)^a$, in which case $L^\Phi$ becomes the Zygmund space $L^p(\log L)^a$. (See Bennett and Sharpley [25].)

We can weaken the definition of modular by replacing (1) by

(1a) $\rho(0) = 0$;
(1b) If $\rho(\lambda x) = 0$ for all $\lambda > 0$, then $x = 0$.

Such functions $\rho$ are referred to as semi-modulars, and the theory of modular spaces readily extends to this setting. For example, if we let $\Phi(x,t) = \infty \cdot \chi_{(1,\infty)}(t)$ (letting $0 \cdot \infty = 0$), then (2.17) defines a semi-modular and we get $L^\infty(\Omega)$. We can extend this approach to get a very elegant definition of the variable Lebesgue spaces. Given $p(\cdot) \in \mathcal{P}(\Omega)$, define

$$\tilde{\rho}_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx,$$

with the convention that $t^\infty = \infty \cdot \chi_{(1,\infty)}(t)$. Then $\tilde{\rho}_{p(\cdot)}$ is a semi-modular. It is not equivalent to $\rho_{p(\cdot)}$: for example, if we let $\Omega = \mathbb{R}$, $p(x) = \infty$, and $f(x) = c > 0$, then $\rho_{p(\cdot)}(f) = c$, but $\tilde{\rho}_{p(\cdot)}(f) = 0$ if $0 < c \leq 1$ and $\tilde{\rho}_{p(\cdot)}(f) = \infty$ if $c > 1$. Nevertheless, the norm $\|\cdot\|_{X_{\tilde{\rho}}}$ is equivalent to $\|\cdot\|_{p(\cdot)}$: for all $f$,

$$\|f\|_{X_{\tilde{\rho}}} \leq \|f\|_{p(\cdot)} \leq 2\|f\|_{X_{\tilde{\rho}}}. \tag{2.18}$$

The whole theory of variable Lebesgue spaces can be developed from this perspective; it is done this way, for example, in [80, 82]. (A proof of (2.18) can be found in both.) This approach is extremely elegant and is also advantageous in some applications, since in certain limiting cases the space that appears naturally is a Musielak-Orlicz space. For instance, in Sect. 3.7.3 below, the behavior of the Hardy-Littlewood maximal operator is considered for functions $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}$, the Musielak-Orlicz space generated by $\Phi(x,t) = t^{p(x)} \log(e+t)^{q(x)}$. These are generalizations of the Zygmund spaces and were first considered in [59] and later by Mizuta and various co-authors [138, 166, 167, 243, 265, 267]. For another example generalizing the space $\exp L$, see Harjulehto and Hästö [153].

### 2.10.3 *Banach Function Spaces*

Another abstract approach to the variable Lebesgue spaces is that of Banach function spaces as defined by Bennett and Sharpley [25]. Let $\Omega \subset \mathbb{R}^n$ and let $\mathcal{M}$ be the set of all measurable functions with respect to Lebesgue measure. Given a mapping $\|\cdot\|_X : \mathcal{M} \to [0,\infty]$, the set

$$X = \{f \in \mathcal{M} : \|f\|_X < \infty\},$$

is a Banach function space if the pair $(X, \|\cdot\|_X)$ satisfies the following properties for all $f, g \in \mathcal{M}$:

1. $\|f\|_X = \||f|\|_X$ and $\|f\|_X = 0$ if and only if $f \equiv 0$;
2. $\|f + g\|_X \le \|f\|_X + \|g\|_X$;
3. For all $a \in \mathbb{R}$, $\|af\|_X = |a|\|f\|_X$;
4. $X$ is a complete normed vector space with respect to $\|\cdot\|_X$;
5. If $|f| \le |g|$ almost everywhere, then $\|f\|_X \le \|g\|_X$;
6. If $\{f_n\} \subset \mathcal{M}$ is a sequence such that $|f_n|$ increases to $|f|$ almost everywhere., then $\|f_n\|_X$ increases to $\|f\|_X$;
7. If $E \subset \Omega$ is a measurable set and $|E| < \infty$, then $\|\chi_E\|_X < \infty$;
8. $\int_E |f(x)|\,dx \le C_E\|f\|_X$ if $|E| < \infty$, where $C_E < \infty$ depends on $E$ and $X$, but not on $f$.

It follows at once from the results in this chapter that $\|\cdot\|_{p(\cdot)}$ is a Banach function space. This was first observed by Edmunds, Lang and Nekvinda [101] (see also Lukeš, Pick and Pokorný [242]). Many of the results proved in this chapter—especially the functional analytic ones on duality, separability, etc.—can be proved in this more general setting.

Here we give one such general result. We say that a function $f \in X$ has absolutely continuous norm if given any nested sequence of sets $\{E_k\}$ such that $|E_k| \to 0$, $\|f\chi_{E_k}\|_X \to 0$. The norm $\|\cdot\|_X$ is absolutely continuous if every function in $X$ has absolutely continuous norm. We define the associate space of $X$ to be the space $X'$ of functions $g$ such that

$$\|g\|_{X'} = \sup\left\{\int_\Omega |f(x)g(x)|\,dx : \|f\|_X \le 1\right\} < \infty.$$

Denoting by $X^*$ the dual space of $X$, then the following are equivalent [25]:

1. $\|\cdot\|_X$ is absolutely continuous;
2. $X$ is separable;
3. $X^* = X'$ (up to isomorphism).

As a corollary to Theorems 2.58 and 2.62 we have that the norm $\|\cdot\|_{p(\cdot)}$ is absolutely continuous if and only if $p_+ < \infty$. In proving this fact, as well as in proving separability and duality (Theorems 2.78 and 2.80) the construction from Proposition 2.12 played a central role.

The Banach space properties of the variable Lebesgue spaces have been considered by several authors. The subspace of functions in $L^{p(\cdot)}$, $p_+ = \infty$, that have absolutely continuous norm was examined by Edmunds, Lang and Nekvinda [101]. A Banach space $X$ is uniformly convex if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$, $\|x\|_X = \|y\|_X = 1$ and $\|x - y\|_X \ge \epsilon$, then $\|x + y\|_X \le 2 - \delta$. Lukeš, Pick and Pokorný [242] showed that the following are equivalent:

1. $1 < p_- \le p_+ < \infty$;
2. $L^{p(\cdot)}(\Omega)$ is reflexive;
3. $L^{p(\cdot)}(\Omega)$ and $L^{p'(\cdot)}(\Omega)$ have absolutely continuous norms;
4. $L^{p(\cdot)}(\Omega)$ is uniformly convex.

Earlier, the uniform convexity of $L^{p(\cdot)}(\Omega)$ was proved by Nakano [280] (when $\Omega = [0,1]$, see also [245]), Diening [80] and also by Fan and Zhao [122]; the uniform convexity of modular spaces was considered by Musielak [274]. In the same paper, Lukeš *et al*. characterized the exponents such that $L^{p(\cdot)}(\Omega)$ has the Radon-Nikodym and Daugavet properties. Dinca and Matei [93, 94] have considered the Gateaux derivative of the norm of $L^{p(\cdot)}(\Omega)$ and have also considered uniform convexity and the derivative of the norm for variable Sobolev spaces (see Chap. 6).

### *2.10.4 Alternative Definitions of the Modular*

In the framework we have adopted there are several equivalent definitions of the modular. One alternative is

$$\rho'_{p(\cdot)}(f) = \max\left(\int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx, \|f\|_{L^\infty(\Omega_\infty)}\right);$$

then $\rho'_{p(\cdot)}(f)$ is equivalent to $\rho_{p(\cdot)}(f)$ for all $f$, and the same results hold with minor modifications of the proof. This definition was used by Edmunds and Rákosník [106].

Another, more interesting alternative was considered by Samko [313] and developed systematically by Diening *et al*. [80, 82]. Modify the definition of the modular

$$\rho^*_{p(\cdot)}(f) = \int_{\Omega_*} \frac{1}{p(x)}|f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty)},$$

and use this to define the norm

$$\|f\|^*_{p(\cdot)} = \inf\{\lambda > 0 : \rho^*_{p(\cdot)}(f/\lambda) \le 1\}. \tag{2.19}$$

If $p_+ < \infty$, then it is immediate that

$$(p_+)^{-1}\rho^*_{p(\cdot)}(f) \le \rho_{p(\cdot)}(f) \le (p_-)^{-1}\rho^*_{p(\cdot)}(f),$$

and it follows that $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|^*_{p(\cdot)}$ are equivalent norms. However, it can be shown that this is the case even when $p_+ = \infty$.

One advantage of this definition is that Hölder's inequality follows with a universal constant. Indeed, the proof of Theorem 2.26 can be modified to show that

$$\int_\Omega |f(x)g(x)|\,dx \le 2\|f\|^*_{p(\cdot)}\|g\|^*_{p'(\cdot)}. \tag{2.20}$$

Furthermore, as Samko [313] pointed out, if in the definition of $\|\cdot\|^*_{p(\cdot)}$ we replace the constant 1 by $1/2$ on the right-hand side of (2.19), then the constant in (2.20) becomes 1. This phenomenon is exactly parallel to the behavior of the norm on Orlicz spaces and follows from the structure of the Luxemburg norm. See Miranda [264] or Greco, Iwaniec and Moscariello [145].

### 2.10.5 Variable Lebesgue Spaces and Orlicz Spaces

In certain applications where $p_- = 1$ and $|\Omega| < \infty$ (see, for instance Sect. 3.7.3 below) it is natural to ask if there is an embedding of $L^{p(\cdot)}(\Omega)$ into the Zygmund space $L \log L(\Omega)$: more precisely, when

$$\|f\|_{L \log L(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{2.21}$$

These embeddings were first studied by Hästö [163], and then in Futamura and Mizuta [136], Mizuta, Ohno and Shimomura [266], and also in [59]. They hold if $p(\cdot)$ satisfies a decay condition when $p(\cdot)$ is close to 1 in value. More precisely, let

$$\lambda(s) = 1 + \frac{\log\log(1/s)}{\log(1/s)}.$$

If for all $s > 0$ sufficiently small,

$$|\{x \in \Omega : p(x) \leq \lambda(s)\}| \leq Ks,$$

then (2.21) holds.

Necessary and sufficient conditions for the embeddings between Orlicz spaces and variable Lebesgue spaces can be gotten as special cases of a general theorem for Orlicz-Musielak spaces. Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, and given a Young function $\Phi$ and the corresponding Orlicz space $L^\Phi(\Omega)$, then $L^{p(\cdot)}(\Omega) \subset L^\Phi(\Omega)$ if and only if there exists $K > 1$ and $h \in L^1(\Omega)$ such that for all $x \in \Omega$ and $t > 0$,

$$\Phi(t) \leq Kt^{p(x)} + h(x).$$

Conversely, $L^\Phi(\Omega) \subset L^{p(\cdot)}(\Omega)$ if and only if there exists $K > 1$ and $g \in L^1(\Omega)$ such that

$$t^{p(x)} \leq K\Phi(t) + g(x).$$

This theorem is due to Ishii [182]; see also Hudzik [177], Kozek [220], or Musielak [274]. This result was used by Diening [77] to prove Theorem 2.45.

### 2.10.6 More on Convergence

Theorem 2.69 shows that convergence in norm, modular and measure are equivalent if $p_+ < \infty$. The relationship between these three kinds of convergence is more complicated when $p_+ = \infty$. As we showed in Proposition 2.56 and Theorem 2.66, convergence in norm always implies convergence in modular and convergence in measure. Conversely, convergence in modular implies convergence in norm exactly when $p_- = \infty$ or $p_+(\Omega \setminus \Omega_\infty) < \infty$ (Theorem 2.58), and the sequence of functions constructed in Theorem 2.66 also shows that convergence in measure never implies convergence in norm.

The relationship between convergence in modular and convergence in measure is more complicated. The proof of Theorem 2.69 can be generalized to prove the following results.

**Theorem 2.84.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, for each $M \geq 1$ let*

$$E_M = \{x \in \Omega \setminus \Omega_\infty : p(x) > M\}.$$

*Then the following are equivalent:*

1. *For any sequence $\{f_k\} \in L^{p(\cdot)}(\Omega)$ and $f \in L^{p(\cdot)}(\Omega)$, if $f_k \to f$ in modular, then $f_k \to f$ in measure and for every $\gamma > 0$ sufficiently small, $\rho(\gamma f_k) \to \rho(\gamma f)$;*
2. *$|E_M| \to 0$ as $M \to \infty$.*

**Theorem 2.85.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $|\Omega_\infty| = 0$, if $f \in L^{p(\cdot)}(\Omega)$ and $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ are such that $f_k \to f$ in measure and for some $\gamma$, $0 < \gamma < 1$, $\rho(\gamma f) < \infty$ and $\rho(\gamma f_k/3) \to \rho(\gamma f/3)$, then $f_k \to f$ in modular.*

For proofs and a complete discussion of the relationship between these three notions of convergence, see [60].

Beyond the three types of convergence, we can also consider weak convergence. A sequence $\{f_n\} \subset L^{p(\cdot)}(\Omega)$ converges weakly to $f \in L^{p(\cdot)}(\Omega)$ if for every $\Phi \in L^{p(\cdot)}(\Omega)^*$, $\Phi(f_n) \to \Phi(f)$. When $p_+ < \infty$, by Theorem 2.80, we have that $f_k \to f$ weakly in $L^{p(\cdot)}(\Omega)$ if for every $g \in L^{p'(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)^*$,

$$\int_\Omega f_k(x)g(x)\,dx \to \int_\Omega f(x)g(x)\,dx.$$

In the classical Lebesgue spaces, by the Radon-Riesz theorem, if $1 < p < \infty$, $f_k \to f$ weakly, and $\|f_k\|_p \to \|f\|_p$, then $f_k \to f$ in norm. This is also true in the variable Lebesgue spaces.

**Proposition 2.86.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $1 < p_- \leq p_+ < \infty$, if the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ converges weakly to $f \in L^{p(\cdot)}(\Omega)$, and if $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$, then $f_k \to f$ in norm.*

The proof is the same as in the classical case (see Hewitt and Stromberg [169]): it follows from the fact that with these hypotheses, $L^{p(\cdot)}(\Omega)$ is uniformly convex. (See Sect. 2.10.3.) For an example of the application of weak convergence in variable Lebesgue spaces, see Zecca [352] (which generalizes [146]).

### *2.10.7 Variable Sequence Spaces*

The sequence spaces $\ell^p$, $1 \leq p < \infty$, can be generalized to get a discrete version of the variable Lebesgue spaces. Given a function $p(\cdot) : \mathbb{N} \to [1, \infty)$, define $\ell^{p(\cdot)}$ to be the space of sequences $\alpha = \{a_k\}$ such that

$$\|\alpha\|_{\ell^{p(\cdot)}} = \inf\left\{\lambda > 0 : \sum_{k=1}^{\infty}\left(\frac{|a_k|}{\lambda}\right)^{p(k)} \leq 1.\right\}.$$

Arguing as above we can prove that $\ell^{p(\cdot)}$ is Banach space. These spaces were first considered by Orlicz [290] and Nakano [279] (see also [245]), and more recently by Edmunds and Nekvinda [104] and by Nekvinda [281, 283]. Diening [80] treats variable sequence spaces as a special case of the modular spaces, since the above definition of the norm is gotten from the definition of the norm on $L^{p(\cdot)}$ if we replace the underlying space by $\mathbb{N}$ and Lebesgue measure by counting measure.

Recently, Hästö has shown that the variable sequence spaces have applications to the study of operators on variable Lebesgue spaces. See [165] and Sect. 5.6.6 below.

# Chapter 3
# The Hardy-Littlewood Maximal Operator

In this chapter we begin the study of harmonic analysis on variable Lebesgue spaces. Our goal is to determine the behavior of some of the classical operators of harmonic analysis—approximate identities, singular integrals, and Riesz potentials. The foundation for this is the Hardy-Littlewood maximal operator, which is the subject of Chaps. 3 and 4.

In this chapter we first lay out the basic properties of the maximal operator and then prove the norm inequalities it satisfies on the classical Lebesgue spaces. To prove these results we will use the Calderón-Zygmund decomposition, an extremely versatile tool that has found countless applications, to prove the weak type inequalities and then use Marcinkiewicz interpolation to prove the strong type inequalities. We present this material in detail for two reasons. First, we will need to use it to extend these results to variable Lebesgue spaces. Second, we want to draw a contrast between the two settings: the proof in the variable case is significantly different since we are unable to use interpolation.

We then turn to our main goal, which is to extend the classical results to the variable Lebesgue spaces. Our central result, Theorem 3.16 below, shows that log-Hölder continuity is a sufficient condition for the maximal operator to be bounded on $L^{p(\cdot)}$. In Chap. 4 we will consider the ways in which this condition can be relaxed. We then examine pointwise and modular inequalities for the maximal operator. Finally, given the importance of interpolation in the classical case, we conclude this chapter with a digression: we consider interpolation on variable Lebesgue spaces and use this to prove that the set of exponents for which the maximal operator is bounded is convex.

## 3.1 Basic Properties

Given a locally integrable function $f$, the maximal function $Mf$ gives the largest average value of $f$ at each point. More precisely, we make the following definition.

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**Definition 3.1.** Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $Mf$, the Hardy-Littlewood maximal function of $f$, is defined for any $x \in \mathbb{R}^n$ by

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ that contain $x$ and whose sides are parallel to the coordinate axes.

There are several variations of the maximal operator, all of which are equivalent and are often used interchangeably. For example, the supremum could be taken over cubes centered at $x$; this is referred to as the centered maximal operator and is denoted by $M^c$. Clearly, $M^c f(x) \leq Mf(x)$. On the other hand, given any cube $Q$ containing $x$, there exists a cube $\tilde{Q}$ centered at $x$ and containing $Q$ such that $|\tilde{Q}| \leq 3^n|Q|$. Hence, $Mf(x) \leq 3^n M^c f(x)$. Similarly, the supremum could be taken over all cubes and not just those whose sides are parallel to the coordinate axes; again, this definition is pointwise equivalent to Definition 3.1. In many applications, it makes more sense to define the maximal operator by taking the supremum over all balls that contain $x$, or even over balls centered at $x$. By much the same argument as before these two operators are equivalent pointwise to one another. Further, given any ball $B$ there exist two cubes $Q_1$, $Q_2$, with the same center such that $Q_1 \subset B \subset Q_2$ and such that $|Q_2|/|Q_1| = n^{n/2}$; thus, the operators defined with centered balls and centered cubes are pointwise equivalent as well. Though we will generally use the maximal operator as given in Definition 3.1, we will occasionally use one of these equivalent definitions below.

Also, in Definition 3.1 and these alternate definitions, we did not specify if the cubes and balls were open or closed. Since the boundaries of both have measure zero, we get the same average if we replace a ball or cube by its closure. We will do this without comment; so, for instance, in Example 3.2 we will take our averages over (open) balls such that $x$ is contained on the boundary.

We can also consider the maximal operator for functions defined on some domain $\Omega$. There are two ways of doing this. First, given such a function, we extend it to a function on $\mathbb{R}^n$ by making it identically 0 on $\mathbb{R}^n \setminus \Omega$. Then to define the maximal operator of $f$ on $\Omega$, we would restrict the supremum to cubes $Q$ such that $|Q \cap \Omega| > 0$. On $\Omega$ this definition agrees with the one given above and is the approach we will use. Alternatively, we could modify the definition by restricting the supremum to cubes $Q$ (or balls $B$) contained in $\Omega$, or even cubes that are compactly contained in $\Omega$ (e.g., by assuming $2Q \subset \Omega$). This is no longer equivalent to the maximal operator defined above, but is dominated pointwise by it. In either case, therefore, when we prove norm inequalities for the maximal operator it will suffice to assume $\Omega = \mathbb{R}^n$.

The maximal operator is very difficult to compute exactly for most functions, but in certain cases it can be approximated easily. The following example and variations of it occur repeatedly in practice.

*Example 3.2.* In $\mathbb{R}^n$, let $f(x) = |x|^{-a}$, $0 < a < n$. Then $Mf(x) \approx |x|^{-a}$.

*Proof.* This equivalence is easier to see if we define the maximal operator using the supremum over balls containing the point. Fix a point $x \neq 0$; it is immediate that

$$Mf(x) \geq \fint_{B_{|x|}(0)} |y|^{-a}\,dy = \frac{c(n)}{|x|^n}\int_0^{|x|} r^{-a+n-1}\,dr = c(n,a)|x|^{-a}.$$

We will now show the reverse inequality. Let $B_r(x_0)$ be any ball containing $x$. Since $f$ is radially decreasing, if $r \geq |x|/4$, then

$$\begin{aligned}\fint_{B_r(x_0)} |f(y)|\,dy &\leq \fint_{B_r(0)} |f(y)|\,dy \\ &\leq \fint_{B_{|x|/4}(0)} |f(y)|\,dy \leq 4^n \fint_{B_{|x|}(0)} |f(y)|\,dy \leq C(n,a)|x|^{-a}.\end{aligned}$$

On the other hand, if $r < |x|/4$, then for every $y \in B_r((1 - r/|x|)x)$,

$$|y| \geq (1 - r/|x|)|x| - r = |x| - 2r \geq |x|/2.$$

Hence, $f(y) \leq 2^a|x|^{-a}$, and so

$$\fint_{B_r(x_0)} |f(y)|\,dy \leq \fint_{B_r((1-r/|x|)x)} |f(y)|\,dy \leq 2^a|x|^{-a}.$$

The desired inequality now follows from the definition of the maximal operator. □

**Proposition 3.3.** *The Hardy-Littlewood maximal operator has the following properties:*

1. *$M$ is sublinear: $M(f+g)(x) \leq Mf(x) + Mg(x)$, and $M$ is homogeneous: for all $\alpha \in \mathbb{R}$, $M(\alpha f)(x) = |\alpha| Mf(x)$.*
2. *For all $f$, $|f(x)| \leq Mf(x)$ almost everywhere.*
3. *If $f \in L^\infty(\mathbb{R}^n)$, then $Mf \in L^\infty(\mathbb{R}^n)$ and $\|Mf\|_\infty = \|f\|_\infty$.*
4. *If $f(x) \neq 0$ on a set of positive measure, then on any bounded set $\Omega$ there exists $\epsilon > 0$ such that $Mf(x) \geq \epsilon$, $x \in \Omega$.*
5. *If $f(x) \neq 0$ on a set of positive measure, then $Mf \notin L^1(\mathbb{R}^n)$.*

*Proof.* Property (1) follows immediately from the definition. Property (2) follows from the Lebesgue differentiation theorem (see Sect. 2.9): given any Lebesgue point $x$ of $f$, if we let $Q_r(x)$ be the cube of side length $r$ centered at $x$, then

$$Mf(x) \geq \lim_{r\to 0}\left|\fint_{Q_r(x)} f(y)\,dy\right| = |f(x)|.$$

Property (3) follows from Property (2) and the definition.

To prove Property (4): since $f(x) \neq 0$ on a set of positive measure, there exists a cube $Q_0$ centered at the origin such that

$$\int_{Q_0} |f(y)|\,dy > 0.$$

Let $Q$ be the smallest cube containing $\Omega$ and $Q_0$. Then for every $x \in \Omega$,

$$Mf(x) \geq \fint_Q |f(y)|\,dy \geq |Q|^{-1} \int_{Q_0} |f(y)|\,dy = \epsilon > 0.$$

Finally, to prove (5) let $Q_0$ be as above. For any $x \in \mathbb{R}^n \setminus Q_0$ let $Q_x$ be the cube centered at 0 with $\ell(Q_x) = |2x|$. Then

$$Mf(x) \geq \fint_{Q_x} |f(y)|\,dy \geq |2x|^{-n} \int_{Q_0} |f(y)|\,dy \notin L^1(\mathbb{R}^n).$$

□

## 3.2 The Calderón-Zygmund Decomposition

In this section we state and prove the classical norm inequalities for the Hardy-Littlewood maximal operator. The key idea is that while $Mf$ dominates $f$ pointwise, they still have comparable norms in $L^p$, $1 < p \leq \infty$, and even for $f \in L^1$ the size of $Mf$ can be controlled.

**Theorem 3.4.** *Given $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, for every $t > 0$,*

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq \frac{C_1}{t^p} \int_{\mathbb{R}^n} |f(x)|^p\,dx. \tag{3.1}$$

*Further, if $1 < p \leq \infty$, then*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C_2 \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.2}$$

*Remark 3.5.* From the proof of Theorem 3.4 we have explicit values for the constants: $C_1 = 3^n 4^{np}$ and $C_2 = [2p'12^n]^{1/p}$.

Inequality (3.1) is referred to as a weak $(p, p)$ inequality; when $p = 1$ it is a substitute for the fact that $M$ is never bounded on $L^1(\mathbb{R}^n)$. We can rewrite it in terms of $L^p$ norms: take the $p$-th root of both sides to get, for all $t > 0$,

$$t\|\chi_{\{x \in \mathbb{R}^n : Mf(x) > t\}}\|_p \leq C_1^{1/p} \|f\|_p. \tag{3.3}$$

For $p > 1$ the weak $(p, p)$ inequality follows from the strong $(p, p)$ inequality (3.2), since by Chebyshev's inequality we have that

$$t^p \int_{\mathbb{R}^n} \chi_{\{x\in\mathbb{R}^n : Mf(x)>t\}}(x)\, dx \leq \int_{\mathbb{R}^n} Mf(x)^p\, dx, \quad t > 0.$$

The original proof of Theorem 3.4 has two steps: first, prove the weak $(1, 1)$ inequality using the Vitali or Besicovitch covering lemma, and then prove the strong $(p, p)$ inequality using Marcinkiewicz interpolation. Calderón and Zygmund developed a more complicated decomposition in terms of dyadic cubes: in essence they showed that norm inequalities for the maximal operator follow from norm inequalities for the dyadic maximal operator (defined as the supremum over dyadic cubes). The advantage of their approach is its versatility; we will see this below when we extend Theorem 3.4 to the variable Lebesgue spaces.

To state the Calderón-Zygmund decomposition we begin with a definition.

**Definition 3.6.** Let $Q_0 = [0, 1)^n$, and let $\Delta_0$ be the set of all translates of $Q_0$ whose vertices are on the lattice $\mathbb{Z}^n$. More generally, for each $k \in \mathbb{Z}$, let $Q_k = 2^{-k} Q_0 = [0, 2^{-k})^n$, and let $\Delta_k$ be the set of all translates of $Q_k$ whose vertices are on the lattice $2^{-k}\mathbb{Z}^n$. Define the set of dyadic cubes $\Delta$ by

$$\Delta = \bigcup_{z\in\mathbb{Z}} \Delta_k.$$

The following properties of dyadic cubes are immediate consequences of the definition.

**Proposition 3.7.**

1. *For each $k \in \mathbb{Z}$, if $Q \in \Delta_k$, then $\ell(Q) = 2^{-k}$.*
2. *For each $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, there exists a unique cube $Q \in \Delta_k$ such that $x \in Q$.*
3. *Given any two cubes $Q_1, Q_2 \in \Delta$, either $Q_1 \cap Q_2 = \emptyset$, $Q_1 \subseteq Q_2$, or $Q_2 \subset Q_1$.*
4. *For each $k \in \mathbb{Z}$, if $Q \in \Delta_k$, then there exists a unique cube $\widetilde{Q} \in \Delta_{k-1}$ such that $Q \subset \widetilde{Q}$. ($\widetilde{Q}$ is referred to as the dyadic parent of $Q$.)*
5. *For each $k \in \mathbb{Z}$, if $Q \in \Delta_k$, then there exist $2^n$ cubes $P_i \in \Delta_{k+1}$ such that $P_i \subset Q$.*

Given the dyadic cubes we define the associated maximal operator.

**Definition 3.8.** Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define the dyadic maximal operator $M^d$ by

$$M^d f(x) = \sup_{\substack{Q\ni x\\ Q\in\Delta}} \fint_Q |f(y)|\, dy.$$

Clearly, $M^d f(x) \leq Mf(x)$. The reverse inequality does not hold pointwise: for example, if $f = \chi_{[0,1)^n}$, then for all $x$ not in the first quadrant (i.e., $x \notin [0, \infty)^n$), $M^d f(x) = 0$ but $Mf(x) > 0$. On the other hand, they are comparable in a weaker, measure theoretic sense. This is the substance of the next two results.

**Lemma 3.9.** *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is such that $\fint_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$, then for each $t > 0$ there exists a (possibly empty) set of disjoint dyadic cubes $\{Q_j\}$ such that*

$$E_t^d = \{x \in \mathbb{R}^n : M^d f(x) > t\} = \bigcup_j Q_j$$

*and*

$$t < \fint_{Q_j} |f(x)|\,dx \le 2^n t. \tag{3.4}$$

*Further, for almost every $x \in \mathbb{R}^n \setminus \bigcup_j Q_j$, $|f(x)| \le t$.*

Lemma 3.9 is referred to as the Calderón-Zygmund decomposition, and the cubes $\{Q_j\}$ are referred to as the Calderón-Zygmund cubes of $f$ at height $t$.

*Remark 3.10.* The condition that $\fint_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$ is satisfied if, for example, $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, since by Hölder's inequality,

$$\fint_Q |f(y)|\,dy \le |Q|^{-1/p} \|f\|_{L^p(\mathbb{R}^n)}.$$

*Remark 3.11.* The weaker hypothesis that $\fint_Q |f(y)|\,dy$ is bounded as $|Q| \to \infty$ is equivalent to the property that $Mf(x) < \infty$ almost everywhere. See [129] for further information.

*Proof of Lemma 3.9.* Fix $t > 0$; if $E_t^d$ is empty, then there are no dyadic cubes $Q$ such that $\fint_Q |f(y)|\,dy > t$ so we will let the collection $\{Q_j\}$ be the empty set. Otherwise, take $x \in E_t^d$. By the definition of the dyadic maximal operator, there exists $Q \in \Delta$ such that $x \in Q$ and

$$\fint_Q |f(y)|\,dy > t.$$

Since $\fint_Q |f(y)|\,dy \to 0$ as the size of $Q$ increases, if there is more than one dyadic cube with this property, then there must be a largest such cube. Denote it by $Q_x$. Since we can do this for every such $x$,

$$E_t^d \subset \bigcup_{x \in E_t^d} Q_x. \tag{3.5}$$

Conversely, given any other point $x' \in Q_x$,

$$M^d f(x') \ge \fint_{Q_x} |f(y)|\,dy > t,$$

and so $x' \in E_t^d$. Therefore, $Q_x \subset E_t^d$ and equality holds in (3.5).

Since $\Delta$ is countable, the set $\{Q_x : x \in E_t^d\}$ is at most countable. Re-index this set as $\{Q_j\}$. The cubes $Q_j$ are pairwise disjoint; for if there exist two different cubes that intersect, then by Proposition 3.7 one is contained in the other. However, this contradicts the way in which these cubes were chosen since each was supposed to be the largest such cube.

The left-hand inequality in (3.4) follows from our choice of the $Q_j$; furthermore, since each $Q_j$ was chosen to be the largest cube containing a point $x$ with this property, if we let $\widetilde{Q}_j$ be its dyadic parent,

$$t \geq \fint_{\tilde{Q}_j} |f(y)|\,dy \geq 2^{-n} \fint_{Q_j} |f(y)|\,dy.$$

Finally, for every $x \in \mathbb{R}^n \setminus E_t^d$, $M^d f(x) \leq t$. Therefore, for almost every such $x$, by the Lebesgue differentiation theorem (see Sect. 2.9),

$$|f(x)| = \lim_{\substack{x \in Q \in \Delta \\ |Q| \to 0}} \left| \fint_Q f(y)\,dy \right| \leq M^d f(x) \leq t.$$

□

*Remark 3.12.* As part of the proof of Lemma 3.9 we get that the $Q_j$ are the largest dyadic cubes with the property that $\fint_Q |f(y)|\,dy > t$, and any other dyadic cube with this property is contained in one of the $Q_j$. We refer to this property as the maximality of the Calderón-Zygmund cubes.

**Lemma 3.13.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $\fint_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$. Then for each $t > 0$, if $\{Q_j\}$ is the set of Calderón-Zygmund cubes of $f$ at height $t/4^n$,*

$$E_t = \{x \in \mathbb{R}^n : Mf(x) > t\} \subset \bigcup_j 3Q_j.$$

*Proof.* Fix $x \in E_t$; then there exists a cube $Q$ containing $x$ such that

$$\fint_Q |f(y)|\,dy > t.$$

Let $k \in \mathbb{Z}$ be such that $2^{-k-1} \leq \ell(Q) < 2^{-k}$. Then $Q$ intersects at most $M \leq 2^n$ dyadic cubes in $\Delta_k$; denote them by $P_1, \ldots, P_M$. Since $\ell(P_i) = 2^{-k} \leq 2\ell(Q)$, we have that

$$t < \fint_Q |f(y)|\,dy \leq |Q|^{-1} \sum_{i=1}^{M} \int_{P_i} |f(y)|\,dy \leq 2^n \sum_{i=1}^{M} \fint_{P_i} |f(y)|\,dy.$$

Therefore, there must exist at least one index $i$ such that

$$\fint_{P_i} |f(y)|\,dy > \frac{t}{2^n M} \geq \frac{t}{4^n}.$$

In particular, $P_i \subset E^d_{t/4^n}$; since it is a dyadic cube, by the maximality of the Calderón-Zygmund cubes, $P_i \subset Q_j$ for some $j$. Further, $P_i$ and $Q$ intersect, so $x \in Q \subset 3P_i \subset 3Q_j$. Since such a cube $Q_j$ exists for every $x \in E_t$, we get the desired inclusion. □

*Proof of Theorem* 3.4. We will first prove inequality (3.1) and then prove (3.2) for $1 < p < \infty$. We have already shown that the maximal operator is bounded on $L^\infty$: by Proposition 3.3 we have that $\|Mf\|_\infty = \|f\|_\infty$. Fix $p$, $1 \leq p < \infty$, and $f \in L^p(\mathbb{R}^n)$. For any $t > 0$, by Lemma 3.9, there exist the disjoint Calderón-Zygmund cubes $\{Q_j\}$ of $f$ at height $t/4^n$. By Lemma 3.13 and Hölder's inequality (when $p > 1$),

$$\begin{aligned}
|\{x \in \mathbb{R}^n : Mf(x) > t\}| &\leq \left|\bigcup_j 3Q_j\right| \\
&\leq \sum_{j=1}^{\infty} |3Q_j| \leq \sum_{j=1}^{\infty} 3^n |Q_j| \left(\frac{4^n}{t} \fint_{Q_j} |f(x)|\,dx\right)^p \\
&\leq \sum_{j=1}^{\infty} 3^n |Q_j| \frac{4^{np}}{t^p} \fint_{Q_j} |f(x)|^p\,dx \leq \frac{3^n 4^{np}}{t^p} \int_{\mathbb{R}^n} |f(x)|^p\,dx.
\end{aligned}$$

Now fix $p$, $1 < p < \infty$, and $f \in L^p(\mathbb{R}^n)$. For each $t > 0$ we can decompose $f$ as $f_0^t + f_1^t$, where

$$f_0^t = f\chi_{\{x\in\mathbb{R}^n : |f(x)| > t/2\}}, \quad f_1^t = f\chi_{\{x\in\mathbb{R}^n : |f(x)| \leq t/2\}}.$$

Since $\|f_1^t\|_\infty \leq t/2$, we have by Proposition 3.3 that

$$Mf(x) \leq Mf_0^t(x) + Mf_1^t(x) \leq Mf_0^t(x) + t/2.$$

Given a function $h \in L^p(\mathbb{R}^n)$,

$$\|h\|_p^p = p \int_0^\infty t^{p-1} |\{x \in \mathbb{R}^n : |h(x)| > t\}|\,dt. \tag{3.6}$$

(See [238, 305].) Therefore, by the weak $(1, 1)$ inequality and Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} Mf(x)^p \, dx &= p \int_0^\infty t^{p-1} |\{x \in \mathbb{R}^n : Mf(x) > t\}| \, dt \\
&\leq p \int_0^\infty t^{p-1} |\{x \in \mathbb{R}^n : Mf_0^t(x) > t/2\}| \, dt \\
&\leq 2p \cdot 12^n \int_0^\infty t^{p-2} \int_{\mathbb{R}^n} |f_0^t(x)| \, dx \, dt \\
&= 2p \cdot 12^n \int_0^\infty t^{p-2} \int_{\{x \in \mathbb{R}^n : |f(x)| > t/2\}} |f(x)| \, dx \, dt \\
&= 2p \cdot 12^n \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} t^{p-2} \, dt \, dx \\
&= 2p' \cdot 12^n \int_{\mathbb{R}^n} |f(x)|^p \, dx.
\end{aligned}$$

□

In the proof of Theorem 3.4 the argument that the strong $(p, p)$ inequality follows from the weak $(1, 1)$ inequality is a special case of the Marcinkiewicz interpolation theorem. (For a precise statement of this result, see Sect. 3.7.8 below.) Rather than apply Marcinkiewicz interpolation we have worked out the details since this is a key part of the classical proof that does not generalize to the variable Lebesgue spaces.

The proof breaks down in our use of (3.6). This inequality holds in the classical Lebesgue spaces because they are rearrangement invariant. Given a set $\Omega$ and measurable function $f$ on $\Omega$, the distribution function of $f$ is defined by

$$\mu_f(t) = |\{x \in \Omega : |f(x)| > t\}|.$$

Two functions $f$ and $g$ are said to be equimeasurable if for every $t > 0$, $\mu_f(t) = \mu_g(t)$. We say that $L^p(\Omega)$ is rearrangement invariant because given any two equimeasurable functions $f, g \in L^p(\Omega)$, $\|f\|_p = \|g\|_p$. Indeed, this follows at once from (3.6). However, this is not the case in a variable Lebesgue space as the next example shows.

*Example 3.14.* Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, assume $p(\cdot)$ is non-constant. Then there exist two equimeasurable functions $f, g$ such that $g \in L^{p(\cdot)}(\Omega)$ but $f \notin L^{p(\cdot)}(\Omega)$.

*Proof.* We consider two cases. First, suppose that $p(\cdot)$ is non-constant on $\Omega \setminus \Omega_\infty$. Then there exist values $p_0 < p_1 < \infty$ such that the sets

$$E = \{x \in \Omega \setminus \Omega_\infty : p(x) < p_0\}, \quad F = \{x \in \Omega \setminus \Omega_\infty : p(x) > p_1\}$$

have positive measure. Form two sequences $\{E_k\}$, $\{F_k\}$ of pairwise disjoint sets such that $E_k \subset E$, $F_k \subset F$, and for each $k \geq 1$, $|E_k| = |F_k| < \infty$. Define the functions

$$f(x) = \sum_{k=1}^{\infty} k^{-1/p_0} |E_k|^{-1/p(x)} \chi_{E_k}(x),$$

$$g(x) = \sum_{k=1}^{\infty} k^{-1/p_0} |F_k|^{-1/p(x)} \chi_{F_k}(x).$$

Then $f$ and $g$ are equimeasurable. Moreover,

$$\rho(g) = \sum_{k=1}^{\infty} \fint_{F_k} k^{-p(x)/p_0}\, dx \le \sum_{k=1}^{\infty} k^{-p_1/p_0} < \infty,$$

so $g \in L^{p(\cdot)}(\Omega)$. On the other hand, for any $\lambda > 1$,

$$\rho(f/\lambda) = \sum_{k=1}^{\infty} \fint_{E_k} k^{-p(x)/p_0} \lambda^{-p(x)}\, dx \ge \sum_{k=1}^{\infty} k^{-1} \lambda^{-p_0} = \infty,$$

so $f \notin L^{p(\cdot)}(\Omega)$.

Now suppose that $p(\cdot)$ is constant on $\Omega \setminus \Omega_\infty$ and $|\Omega_\infty| > 0$. We can immediately adapt the above argument. Fix $p_0 > p_+(\Omega \setminus \Omega_\infty)$; define $E$ as before and let $F = \Omega_\infty$. Then given $f$, $g$ as above, $\|g\|_{L^{p(\cdot)}(\Omega)} = \|g\|_{L^\infty(\Omega_\infty)} = 1$ while we again have $f \notin L^{p(\cdot)}(\Omega)$.

□

## 3.3 The Maximal Operator on Variable Lebesgue Spaces

We now turn to the central topic of this chapter: the behavior of the maximal operator on variable Lebesgue spaces. We can immediately show that the maximal operator is well-defined.

**Proposition 3.15.** *Given any $p(\cdot) \in L^{p(\cdot)}(\mathbb{R}^n)$, if $f \in L^{p(\cdot)}(\mathbb{R}^n)$, then $Mf$ is well defined and $Mf(x) < \infty$ almost everywhere.*

*Proof.* By Proposition 2.41 $f$ is locally integrable, so $Mf$ is well-defined. By Theorem 2.51 we can write $f = f_1 + f_2$ where $f_1 \in L^{p_+}$ and $f_2 \in L^{p_-}$. Then $Mf \le Mf_1 + Mf_2$, and by Theorem 3.4 the right-hand side is finite almost everywhere. □

We can, however, say a great deal more if we assume that the exponent $p(\cdot)$ has a certain amount of regularity. Recall the definition of log-Hölder continuity (Definition 2.2): given a set $\Omega \subset \mathbb{R}^n$ and a function $r(\cdot) : \Omega \to \mathbb{R}$, we say that $r(\cdot) \in LH_0(\Omega)$ if there exists a constant $C_0$ such that for all $x,\ y \in \Omega$, $|x-y| < 1/2$,

$$|r(x) - r(y)| \leq \frac{C_0}{-\log(|x-y|)}.$$

We say that $r(\cdot) \in LH_\infty(\Omega)$ if there exist constants $C_\infty$ and $r_\infty$ such that for all $x \in \Omega$,

$$|r(x) - r_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.$$

Finally, we define $LH(\Omega) = LH_0(\Omega) \cap LH_\infty(\Omega)$.

Using this definition we can state the main result of this chapter.

**Theorem 3.16.** *Given a set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $1/p(\cdot) \in LH(\Omega)$, then*

$$\|t\chi_{\{x:Mf(x)>t\}}\|_{L^{p(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{3.7}$$

*If in addition $p_- > 1$, then*

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{3.8}$$

*In both inequalities the constant depends on the dimension $n$, the log-Hölder constants of $1/p(\cdot)$, $p_-$, and $p_\infty$ (if this value is finite).*

*Remark 3.17.* If $p_+ < \infty$, then by Proposition 2.3 the hypothesis $1/p(\cdot) \in LH(\Omega)$ is equivalent to assuming $p(\cdot) \in LH(\Omega)$.

As we noted in Sect. 2.1, if $\Omega$ is bounded, then $1/p(\cdot)$ is automatically in $LH_\infty(\Omega)$ with a constant that depends on $\|1/p(\cdot)\|_\infty$, the diameter of $\Omega$ and its distance from the origin. Thus in this case it suffices to assume $1/p(\cdot) \in LH_0(\Omega)$ to conclude that the maximal operator is bounded. However, a close examination of the proof of Theorem 3.16 (see especially the estimate for the set where $|f| \leq 1$, p. 104) shows that we can sharpen the constant, and we record this as a separate result.

**Corollary 3.18.** *Given a bounded set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $1/p(\cdot) \in LH_0(\Omega)$, then (3.7) holds. If $p_- > 1$, then (3.8) holds. The constants depend on $n$, $p_-$, $p_+$, and $|\Omega|$.*

We will prove Theorem 3.16 and Corollary 3.18 in Sect. 3.4 below. Here we examine more closely the statement and hypotheses. First note that by Chebyshev's inequality, we have that for all $t > 0$, $\|t\chi_{\{x:Mf(x)>t\}}\|_{L^{p(\cdot)}(\Omega)} \leq \|Mf\|_{L^{p(\cdot)}(\Omega)}$; therefore, inequality (3.7) is of importance primarily when $p_- = 1$.

In the classical case the maximal operator is bounded on $L^p$ for both $p$ finite and $p = \infty$, so it is reasonable that Theorem 3.16 includes the case $p_+ = \infty$. However, as we shall see, the proof has many technical details that can be eliminated if we restrict to the special case $p_+ < \infty$. At the other end of the scale of Lebesgue spaces, by Proposition 3.3 the maximal operator is not bounded on $L^1$, so the restriction $p_- > 1$ makes sense for (3.8) to hold, and (3.7) is the appropriate replacement when $p_- = 1$, similar to the weak $(1,1)$ inequality. Indeed, if $p(x) = 1$ on some open ball $B = B_r(x_0)$, then (3.8) cannot hold. For in this case define

$$f(x) = \frac{\chi_B(x)}{|x - x_0|^n \log(|x - x_0|)^2} \in L^1(B);$$

then for $x \in B$,

$$Mf(x) \approx \frac{\chi_B(x)}{|x - x_0|^n |\log(|x - x_0|)|} \notin L^1(B).$$

Originally, we conjectured that if $p(x) > 1$ everywhere and is "far" from 1 except on a small set (for example, if $\Omega = (0, 1/2)$ and $p(x) = 1 + |\log(x)|^{-1}$), then the maximal operator could be bounded on $L^{p(\cdot)}$. However, this is never the case.

**Theorem 3.19.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if $p_- = 1$, then the maximal operator is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

*Remark 3.20.* Though the maximal operator is not bounded when $p_- = 1$, there are weaker results related to the so-called variable $L \log L$ spaces. See Sect. 3.7.3 below.

*Proof.* To show that the maximal operator is not bounded, we will construct a sequence of functions $\{f_k\}$ such that for all $k$, $f_k \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\|Mf_k\|_{p(\cdot)} \geq c(n)(k+1)\|f_k\|_{p(\cdot)}$. For each $k \geq 1$, choose $s_k$ such that

$$1 < s_k < n\left(n - \frac{1}{k+1}\right)^{-1}.$$

Since $p_- = 1$, for each $k$ the set

$$E_k = \{x : p(x) < s_k\}$$

has positive measure. If we apply the Lebesgue differentiation theorem (see Sect. 2.9) to the function $\chi_{E_k}$, then there exists a point $x_k \in E_k$ such that

$$\lim_{r \to 0+} \frac{|B_r(x_k) \cap E_k|}{|B_r(x_k)|} = 1.$$

In particular, there exists $R_k$, $0 < R_k < 1$, such that if $0 < r \leq R_k$, then

$$\frac{|B_r(x_k) \cap E_k|}{|B_r(x_k)|} > 1 - 2^{-n(k+1)}. \tag{3.9}$$

Let $B_k = B_{R_k}(x_k)$ and define

$$f_k(x) = |x - x_k|^{-n+\frac{1}{k+1}} \chi_{B_k \cap E_k}(x).$$

To show that $f_k \in L^{p(\cdot)}(\mathbb{R}^n)$, note that since $R_k < 1$ and $-n + \frac{1}{k+1} < 0$,

$$\rho(f_k) = \int_{B_k \cap E_k} |x - x_k|^{\left(-n+\frac{1}{k+1}\right)p(x)}\, dx \leq \int_{B_k \cap E_k} |x - x_k|^{\left(-n+\frac{1}{k+1}\right)s_k}\, dx < \infty.$$

To estimate the norm of $Mf_k$, we will use the equivalent definition of the maximal operator and consider averages over balls. Fix $x \in B_k \cap E_k$ and let $r = |x - x_k| \leq R_k$. Then

$$Mf_k(x) \geq \frac{1}{|B_r(x_k)|} \int_{B_r(x_k) \cap E_k} |y - x_k|^{-n+\frac{1}{k+1}}\, dy.$$

Let $\delta_k = 2^{-(k+1)}$; then

$$|\{y : \delta_k r < |y - x_k| < r\}| = (1 - 2^{-n(k+1)})|B_r(x_k)|.$$

Therefore, since $|x - x_k|^{-n+\frac{1}{k+1}}$ is radially decreasing and since by (3.9) $|B_r(x_k) \cap E_k| \geq (1 - 2^{-n(k+1)})|B_r(x_k)|$, we have that

$$\begin{aligned} Mf_k(x) &\geq \frac{1}{|B_r(x_k)|} \int_{B_r(x_k) \cap E_k} |y - x_k|^{-n+\frac{1}{k+1}}\, dy \\ &\geq c(n) r^{-n} \int_{\{\delta_k r < |y - x_k| < r\}} |y - x_k|^{-n+\frac{1}{k+1}}\, dy \\ &= c(n)(k+1)(1 - \delta_k^{\frac{1}{k+1}})|x - x_k|^{-n+\frac{1}{k+1}} \\ &\geq c(n)(k+1) f_k(x). \end{aligned}$$

Trivially, this inequality also holds if $x \notin B_k \cap E_k$; hence, we have shown that $\|Mf_k\|_{p(\cdot)} \geq c(n)(k+1)\|f_k\|_{p(\cdot)}$, and this completes the proof. □

The assumption that $1/p(\cdot) \in LH_0(\Omega)$ is a very weak regularity condition. But the following simple counter-example shows that some kind of regularity assumption is needed.

*Example 3.21.* Let $\Omega = \mathbb{R}$ and define

$$p(x) = \begin{cases} 2 & x < 0 \\ 3 & x \geq 0. \end{cases}$$

Then the strong and weak type inequalities do not hold in $L^{p(\cdot)}(\mathbb{R})$.

*Proof.* Let $f(x) = |x|^{-2/5}\chi_{(-1,0)}(x)$. Since $|x|^{-4/5}\chi_{(-1,0)} \in L^1(\mathbb{R})$, so by Proposition 2.12, $f \in L^{p(\cdot)}(\mathbb{R})$. On the other hand, $Mf \notin L^{p(\cdot)}(\mathbb{R})$: if $0 < x < 1$, then

$$Mf(x) \geq \frac{1}{2x}\int_{-x}^{x} |f(y)|\,dy = \frac{1}{2x}\int_0^x y^{-2/5}\,dy = \frac{5}{6}x^{-2/5} \notin L^3((0,1));$$

hence, $\rho(Mf) = \infty$, so again by Proposition 2.12, $Mf \notin L^{p(\cdot)}(\Omega)$. Further, from this inequality we get that for any $t > 0$,

$$t^3|\{x \in \mathbb{R} : Mf(x) > t\}| \geq t^3\left(\frac{5}{6t}\right)^{5/2};$$

therefore, by Corollary 2.23,

$$\|t\chi_{\{x:Mf(x)>t\}}\|_{p(\cdot)} \geq \rho\big(t\chi_{\{x:Mf(x)>t\}}\big)^{1/3} \geq t\left(\frac{5}{6t}\right)^{5/6}.$$

Since the right-hand side is unbounded as $t \to \infty$, (3.7) does not hold. □

*Remark 3.22.* The values 2 and 3 play no essential role in the construction and can be replaced by any $p_1 \neq p_2$. Further, a straightforward modification shows that we can take, for instance, $p_2 = \infty$.

The $LH_\infty$ can be viewed as a regularity condition at infinity. The need for some control at infinity is shown by the next example.

*Example 3.23.* Let $p(x) = 3 + \sin(x)$. Then the strong and weak type inequalities do not hold on $L^{p(\cdot)}(\mathbb{R})$.

*Proof.* For each $k \geq 1$, define the sets

$$A_k = \left[\frac{\pi}{4} + 2k\pi, \frac{3\pi}{4} + 2k\pi\right], \quad B_k = \left[\frac{5\pi}{4} + 2k\pi, \frac{7\pi}{4} + 2k\pi\right].$$

If we let $a = 3 + \sqrt{2}/2$ and $b = 3 - \sqrt{2}/2$, then if $x \in A_k$, $p(x) \geq a$ and if $x \in B_k$, $p(x) \leq b$. Define the function

$$f(x) = \sum_{k=1}^{\infty} |x|^{-1/3}\chi_{A_k}(x).$$

Since $a/3 > 1$,

$$\rho(f) = \sum_{k=1}^{\infty}\int_{A_k} |x|^{-p(x)/3}\,dx \leq \int_{\pi/4+2\pi}^{\infty} |x|^{-a/3}\,dx < \infty,$$

so by Proposition 2.12, $f \in L^{p(\cdot)}(\mathbb{R})$. On the other hand, for $x \in [2k\pi, 2(k+1)\pi]$,

$$Mf(x) \geq \frac{1}{2\pi} \int_{2k\pi}^{2(k+1)\pi} f(y)\,dy \geq c|x|^{-1/3}. \tag{3.10}$$

Therefore, since $b/3 < 1$,

$$\rho(Mf) \geq \sum_{k=1}^{\infty} c \int_{B_k} |x|^{-p(x)/3}\,dx$$
$$\geq \sum_{k=1}^{\infty} c \int_{B_k} |x|^{-b/3}\,dx \geq c \sum_{k=1}^{\infty} \left(\frac{7\pi}{4} + 2k\pi\right)^{-b/3} = \infty,$$

and so again by Proposition 2.12, $Mf \notin L^{p(\cdot)}(\mathbb{R})$.

To see that the weak type inequality does not hold, let $t_k = c(2(k+1)\pi)^{-1/3}$, where $c$ is the constant from (3.10). Then

$$\rho(t_k \chi_{\{x:Mf(x)>t_k\}}) \geq \sum_{j=1}^{k} \int_{B_j} t_k^{p(x)}\,dx$$
$$\geq \sum_{j=1}^{k} |B_j| \left[c(2(k+1)\pi)^{-1/3}\right]^b \geq ck(k+1)^{-b/3}.$$

Since $b/3 < 1$, we have that as $k \to \infty$,

$$\|t_k \chi_{\{x:Mf(x)>t_k\}}\|_{p(\cdot)} \to \infty.$$

□

Finally, we note that the $LH_0$ and $LH_\infty$ conditions are not necessary, but are sharp in the sense that no pointwise condition that decays more slowly to 0 suffices to guarantee that the maximal operator is bounded. See Examples 4.1 and 4.43 below.

## 3.4 The Proof of Theorem 3.16

We begin this section with a general discussion on proving norm inequalities in variable Lebesgue spaces; these remarks summarize our overall approach in the proof and are applicable to other operators as well. We then prove five lemmas. The first gives a geometric characterization of the $LH_0$ condition. The next two allow us to apply the $LH_\infty$ condition. As will be clear from their use in the proof, we use the $LH_0$ condition where the function $f$ is large, and the $LH_\infty$ condition where $f$ is small. The last two lemmas allow us to apply the Calderón-Zygmund decomposition in the variable Lebesgue spaces.

After these preliminaries we prove the strong type inequality (3.8). We do this in two different ways: we first give a proof assuming that $p_+ < \infty$, so that we may simply assume that $p(\cdot) \in LH$; we then prove it in full generality. We give two different proofs for both pedagogic and historical reasons. First, the proof when $p_+ = \infty$ has many technical details that obscure the main ideas of the proof. We believe that the reader will have a better understanding of the proof by first seeing the special case. Second, the proof when $p_+ < \infty$ uses a technique that played an important role in the original proofs of this result and which we believe is still of independent interest. On the other hand, the proof when $p_+ = \infty$ let us weaken our hypotheses; we will explore this in detail in Chap. 4. Finally, we prove the weak type inequality (3.7). Since the proof is very similar to but easier than the proof of the strong type inequality, we only describe the key details.

## *Proving Norm Inequalities*

In the classical Lebesgue spaces, proving that the maximal operator is bounded on $L^p(\Omega)$ is equivalent to showing the modular inequality

$$\int_\Omega Mf(x)^p\,dx \leq C\int_\Omega |f(x)|^p\,dx.$$

By Theorem 3.31 below, however, the modular inequality is never true in variable Lebesgue spaces. Therefore, we need to work directly with the norm inequality

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}.$$

Since the maximal operator is homogeneous—that is, $M(\alpha f)(x) = |\alpha|Mf(x)$—it is enough to prove this assuming that $\|f\|_{L^p(\Omega)} = 1$, (which by Corollary 2.22 implies that $\rho(f) \leq 1$). In turn, by the definition of the norm, it will suffice to show that there exists $\lambda > 0$ such that $\rho(Mf/\lambda) \leq 1$, since this implies that

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leq \lambda = \lambda\|f\|_{L^{p(\cdot)}(\Omega)}.$$

In the particular case that $p_+ < \infty$, by Proposition 2.14,

$$\rho(Mf/\lambda) = \int_\Omega \left(\frac{Mf(x)}{\lambda}\right)^{p(x)} dx \leq 1$$

if and only if there exists $C > 0$ such that

$$\int_\Omega Mf(x)^{p(x)}\,dx \leq C.$$

Therefore, though the proofs are formally the same, in the case $p_+ < \infty$ they are generally less complicated since we do not have to establish *a priori* the exact value of the constant $\lambda$.

A similar approach is used to prove the weak type inequalities

$$\|t\chi_{\{x:\Omega:Mf(x)>t\}}\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{L^{p(\cdot)}(\Omega)} :$$

we will show that there exists $\lambda > 0$ such that

$$\int_{\{x:Mf(x)>t\}} \left(\frac{t}{\lambda}\right)^{p(x)} dx \leq 1.$$

For both the strong and weak type inequalities we will use both the Calderón-Zygmund decomposition and the norm inequalities on the classical Lebesgue spaces. This is a key difference between the proof of the classical theorem and the proof for the variable Lebesgue spaces: we are unable to use the weak type inequalities to prove the strong type inequalities by interpolation.

## *Five Lemmas*

Our first lemma characterizes the $LH_0$ condition.

**Lemma 3.24.** *Given $r(\cdot) : \mathbb{R}^n \to [0,\infty)$ such that $r_+ < \infty$, the following are equivalent:*

*1. $r(\cdot) \in LH_0(\mathbb{R}^n)$;*
*2. There exists a constant $C$ depending on $n$ such that given any cube $Q$ and $x \in Q$,*

$$|Q|^{r(x)-r_+(Q)} \leq C \quad \text{and} \quad |Q|^{r_-(Q)-r(x)} \leq C.$$

*Proof.* Suppose $r(\cdot) \in LH_0(\mathbb{R}^n)$. We will prove the first inequality in (2); the proof of the second is identical. Fix $Q$; without loss of generality we may assume $Q$ is closed. If $\ell(Q) \geq (2\sqrt{n})^{-1}$, then

$$|Q|^{r(x)-r_+(Q)} \leq (2\sqrt{n})^{n(r_+-r_-)} = C(n, r(\cdot)).$$

If $\ell(Q) < (2\sqrt{n})^{-1}$, then for all $y \in Q$, $|x-y| < \sqrt{n}\ell(Q) < 1/2$. In particular, since $r(\cdot)$ is continuous, there exists $y \in Q$ such that $r(y) = r_+(Q)$. Therefore, by this estimate and the definition of $LH_0$,

$$\begin{aligned} |Q|^{r(x)-r_+(Q)} &\leq \big(n^{-1/2}|x-y|\big)^{-n|r(x)-r(y)|} \\ &\leq \exp\left(\frac{C_0\big(\log(n^{1/2}) - \log(|x-y|)\big)}{-\log(|x-y|)}\right) \leq C(n, r(\cdot)). \end{aligned}$$

Now suppose that (2) holds. Fix $x, y \in \mathbb{R}^n$ such that $|x-y| < 1/2$; then there exists a cube $Q$ such that $x, y \in Q$ and $\ell(Q) \le |x-y|$ (and so $|Q| < 1$). Combining the two inequalities in (2) we have that

$$C \ge |Q|^{r_-(Q)-r_+(Q)} \ge |Q|^{-|r(x)-r(y)|}$$
$$\ge |x-y|^{-n|r(x)-r(y)|} = \exp\big(-n|r(x)-r(y)|\log(|x-y|)\big).$$

If we take the logarithm we get that

$$|r(x)-r(y)| \le \frac{C}{-\log(|x-y|)},$$

where $C$ does not depend on $x, y$. Hence $r(\cdot) \in LH_0(\mathbb{R}^n)$. □

*Remark 3.25.* Lemma 3.24 is true (with different constants) if we replace cubes by balls. Details are left to the reader.

The second lemma lets us use the $LH_\infty$ condition to replace a variable exponent with a constant one, and vice versa. We will use this in the proof when $p_+ < \infty$.

**Lemma 3.26.** *Let $r(\cdot) : \mathbb{R}^n \to [0,\infty)$ be such that $r(\cdot) \in LH_\infty(\mathbb{R}^n)$ and $0 < r_\infty < \infty$, and let $R(x) = (e+|x|)^{-N}$, $N > n/r_-$. Then there exists a constant $C$ depending on n, $N$ and the $LH_\infty$ constant of $r(\cdot)$ such that given any set $E$ and any function $F$ with $0 \le F(y) \le 1$ for $y \in E$,*

$$\int_E F(y)^{r(y)}\,dy \le C\int_E F(y)^{r_\infty}\,dy + \int_E R(y)^{r_-}\,dy, \tag{3.11}$$

$$\int_E F(y)^{r_\infty}\,dy \le C\int_E F(y)^{r(y)}\,dy + \int_E R(y)^{r_-}\,dy. \tag{3.12}$$

*Proof.* We will prove (3.11); the proof of the second inequality is essentially the same. Write the set $E$ as $E_1 \cup E_2$, where $E_1 = \{x \in E : F(y) \le R(y)\}$ and $E_2 = \{x \in E : R(y) < F(y)\}$. Then

$$\int_{E_1} F(y)^{r(y)}\,dy \le \int_{E_1} R(y)^{r(y)}\,dy \le \int_{E_1} R(y)^{r_-}\,dy.$$

On the other hand, by the $LH_\infty$ condition,

$$R(y)^{-|r(y)-r_\infty|} = \exp\big(N\log(e+|y|)|r(y)-r_\infty|\big) \le \exp(NC_\infty). \tag{3.13}$$

Hence, since $F(y) \le 1$,

$$\int_{E_2} F(y)^{r(y)}\,dy \le \int_{E_2} F(y)^{r_\infty} F(y)^{-|r(y)-r_\infty|}\,dy$$
$$\le \int_{E_2} F(y)^{r_\infty} R(y)^{-|r(y)-r_\infty|}\,dy \le \exp(NC_\infty)\int_{E_2} F(y)^{r_\infty}\,dy.$$

□

*Remark 3.27.* The assumption that $N > n/r_-$ is only included to insure that the last integral in (3.11) and (3.12) is finite. If $|E| < \infty$, then we can take any $N > 0$.

The third lemma is a variation of the embedding theorems proved in Sect. 2.5. It does not directly involve the $LH_\infty$, but we will use it in conjunction with Proposition 2.43 which does use this condition. This result replaces Lemma 3.26 when $p_+ = \infty$.

**Lemma 3.28.** *Given $\Omega$ and $t(\cdot), u(\cdot) \in \mathcal{P}(\Omega)$, suppose $t(x) \le u(x)$ almost everywhere. Suppose $g \in L^{t(\cdot)}(\Omega)$ and $|g(x)| \le 1$ almost everywhere; then $g \in L^{u(\cdot)}(\Omega)$. Moreover, if $\|g\|_{t(\cdot)} \le 1$, then $\|g\|_{u(\cdot)} \le 1 + \|g\|_{t(\cdot)}$, and if $\|g\|_{t(\cdot)} \ge 1$, then $\|g\|_{u(\cdot)} \le 2\|g\|_{t(\cdot)}$.*

*Proof.* Fix $g \in L^{t(\cdot)}(\Omega)$, $|g(x)| \le 1$, and suppose first that $\|g\|_{t(\cdot)} \le 1$. Then by Corollary 2.22, $\rho_{t(\cdot)}(g) \le \|g\|_{t(\cdot)}$. Then, since $\Omega_\infty^{t(\cdot)} \subset \Omega_\infty^{u(\cdot)}$,

$$\rho_{u(\cdot)}(g) = \int_{\Omega\setminus\Omega_\infty^{u(\cdot)}} |g(x)|^{u(x)}\,dx + \|g\|_{L^\infty(\Omega_\infty^{u(\cdot)})}$$
$$\le \int_{\Omega\setminus\Omega_\infty^{t(\cdot)}} |g(x)|^{t(x)}\,dx + 1 \le \rho_{t(\cdot)}(g) + 1 \le \|g\|_{t(\cdot)} + 1.$$

Hence, by the convexity of the modular (Proposition 2.7), $\rho_{u(\cdot)}(g/(\|g\|_{t(\cdot)}+1)) \le 1$, and so $\|g\|_{u(\cdot)} \le \|g\|_{t(\cdot)} + 1$.

Now suppose $\|g\|_{t(\cdot)} \ge 1$. Let $h = g/\|g\|_{t(\cdot)}$. Then $|h(x)| \le 1$ and $\|h\|_{t(\cdot)} = 1$. Therefore, by what we have shown, $\|h\|_{u(\cdot)} \le \|h\|_{t(\cdot)} + 1 = 2$, and the desired inequality follows at once. □

The next two lemmas let us apply the Calderón-Zygmund decomposition in variable Lebesgue spaces. If $p_+ < \infty$, then we can apply the decomposition directly to any function in $L^{p(\cdot)}(\mathbb{R}^n)$.

**Lemma 3.29.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $p_+ < \infty$. Then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\fint_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$. In particular, the conclusion of Lemma 3.9 holds.*

*Proof.* Fix $f \in L^{p(\cdot)}(\mathbb{R}^n)$; by Theorem 2.51 we have that $f = f_1 + f_2$, where $f_1 \in L^{p_+}(\mathbb{R}^n)$ and $f_2 \in L^{p_-}(\mathbb{R}^n)$. Since $p_+ < \infty$, by Remark 3.10, as $|Q| \to \infty$,

$$\fint_Q f(x)\,dx = \fint_Q f_1(x)\,dx + \fint_Q f_2(x)\,dx \to 0.$$

□

The conclusion of Lemma 3.29 need not be true if $p_+ = \infty$. For example, let $p(x) = \infty$ or let $p(x) = 1 + |x|$. In either case $1/p(\cdot) \in LH_\infty$, and so by Proposition 2.43, $1 \in L^{p(\cdot)}(\mathbb{R}^n)$. On the other hand, for all cubes $Q$, $\fint_Q 1\,dy = 1$. However, we can always apply the Calderón-Zygmund decomposition to bounded functions of compact support. Even though such functions need not be dense (see Theorem 2.75) the following lemma lets us reduce to this case by an approximation argument.

**Lemma 3.30.** *Given a non-negative function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if the sequence $\{f_k\}$ is such that $f_k(x) \le f(x)$ and $\{f_k(x)\}$ increases to $f(x)$ for almost every $x$, then $\{Mf_k(x)\}$ increases to $Mf(x)$.*

*Proof.* Since $f_k \le f$, $Mf_k \le Mf$ as well. Since $\{f_k\}$ is an increasing sequence, so is $\{Mf_k\}$. Therefore, it will suffice to prove that for almost every $x$,

$$Mf(x) \le \lim_{k\to\infty} Mf_k(x).$$

Fix $x$ such that $Mf(x) < \infty$. Then for every $\epsilon > 0$, there exists a cube $Q$ containing $x$ such that

$$Mf(x) \le (1+\epsilon)\fint_Q f(y)\,dy.$$

Hence, by the monotone convergence theorem on classical Lebesgue spaces,

$$Mf(x) \le (1+\epsilon)\fint_Q f(y)\,dy = (1+\epsilon)\lim_{k\to\infty}\fint_Q f_k(y)\,dy \le (1+\epsilon)\lim_{k\to\infty} Mf_k(x).$$

Since $\epsilon > 0$ is arbitrary, we get the desired inequality. A similar argument holds if $x$ is such that $Mf(x) = \infty$. □

## *Proof of Inequality (3.8): The Case $p_+ < \infty$*

We begin the proof by making four reductions. First, we may assume that $\Omega = \mathbb{R}^n$. For if this case is true, given $p(\cdot) \in LH(\Omega)$ by Lemma 2.4 we can extend it to an exponent function in $LH(\mathbb{R}^n)$. Further, given any $f \in L^{p(\cdot)}(\Omega)$, we may assume that $f \equiv 0$ outside of $\Omega$. Thus

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \le \|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = C\|f\|_{L^{p(\cdot)}(\Omega)}.$$

Second, since $Mf = M(|f|)$, we may assume that $f$ is non-negative. Third, by homogeneity we may assume that $\|f\|_{p(\cdot)} = 1$. Then by Corollary 2.22,

$$\rho(f) = \int_{\mathbb{R}^n} f(x)^{p(x)}\,dx \le 1.$$

Finally, since $p_+ < \infty$, by Proposition 2.3 we have that our hypothesis $1/p(\cdot) \in LH(\mathbb{R}^n)$ is equivalent to $p(\cdot) \in LH(\mathbb{R}^n)$.

Decompose $f$ as $f_1 + f_2$, where

$$f_1 = f\chi_{\{x: f(x)>1\}}, \qquad f_2 = f\chi_{\{x: f(x)\leq 1\}};$$

then $\rho(f_i) \leq \|f_i\|_{p(\cdot)} \leq 1$. Further, since $Mf \leq Mf_1 + Mf_2$, it will suffice to show, for $i = 1, 2$, that $\|Mf_i\|_{p(\cdot)} \leq C(n, p(\cdot))$; since $p_+ < \infty$, as we pointed out above it will in turn suffice to show that

$$\rho(Mf_i) = \int_{\mathbb{R}^n} Mf_i(x)^{p(x)}\,dx \leq C.$$

**The estimate for $f_1$** Let $A = 4^n$, and for each $k \in \mathbb{Z}$ let

$$\Omega_k = \{x \in \mathbb{R}^n : Mf_1(x) > A^k\}.$$

Since $f_1 \in L^{p(\cdot)}(\mathbb{R}^n)$, by Proposition 3.15, $Mf_1(x) < \infty$ almost everywhere; similarly, without loss of generality we may assume $f_1$ is non-zero on a set of positive measure, and so by Proposition 3.3, $Mf(x) > 0$ for all $x$. Therefore, up to a set of measure 0, $\mathbb{R}^n = \bigcup_k \Omega_k \setminus \Omega_{k+1}$. Further, by Lemma 3.29 for each $k$ we can apply Lemma 3.9 to form the Calderón-Zygmund decomposition of $f$ at height $A^{k-1}$: pairwise disjoint cubes $\{Q_j^k\}_j$ such that

$$\Omega_k \subset \bigcup_j 3Q_j^k \quad \text{and} \quad \fint_{Q_j^k} f_1(y)\,dy > A^{k-1}.$$

From the second we get that

$$\fint_{3Q_j^k} f_1(y)\,dy > 3^{-n}A^{k-1}.$$

For each $k$, define the sets $E_j^k$ inductively: $E_1^k = (\Omega_k \setminus \Omega_{k+1}) \cap 3Q_1^k$, $E_2^k = ((\Omega_k \setminus \Omega_{k+1}) \cap 3Q_2^k) \setminus E_1^k$, $E_3^k = ((\Omega_k \setminus \Omega_{k+1}) \cap 3Q_3^k) \setminus (E_1^k \cup E_2^k)$, etc. Then the sets $E_j^k$ are pairwise disjoint for all $j$ and $k$ and $\Omega_k \setminus \Omega_{k+1} = \bigcup_j E_j^k$.

We now estimate as follows:

$$\begin{aligned}
\int_{\mathbb{R}^n} Mf_1(x)^{p(x)}\,dx &= \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} Mf_1(x)^{p(x)}\,dx \\
&\leq \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} [A^{k+1}]^{p(x)}\,dx \\
&\leq A^{2p_+}3^{np_+}\sum_{k,j}\int_{E_j^k}\left(\fint_{3Q_j^k} f_1(y)\,dy\right)^{p(x)} dx.
\end{aligned}$$

To estimate the last sum, note that since $f_1(x) = 0$ or $f_1(x) \geq 1$ almost everywhere, if we let $p_{jk} = p_-(3Q_j^k)$,

$$\int_{3Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy \leq \int_{3Q_j^k} f_1(y)^{p(y)}\,dy \leq 1. \tag{3.14}$$

Further, since $p(\cdot) \in LH_0(\mathbb{R}^n)$ and $p_+ < \infty$, by Lemma 3.24 there exists a constant $C$ depending on $p(\cdot)$ and $n$ such that for $x \in 3Q_j^k$,

$$|3Q_j^k|^{-p(x)} \leq C|3Q_j^k|^{-p_{jk}}. \tag{3.15}$$

Therefore, since for $x \in E_j^k \subset 3Q_j^k$, $p(x) \geq p_{jk} \geq p_-$, by (3.14), (3.15) and Hölder's inequality with exponent $p_{jk}/p_-$,

$$\begin{aligned}
&\sum_{k,j} \int_{E_j^k} \left( \fint_{3Q_j^k} f_1(y)\,dy \right)^{p(x)} dx \\
&\qquad \leq \sum_{k,j} \int_{E_j^k} |3Q_j^k|^{-p(x)} \left( \int_{3Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy \right)^{p(x)} dx \\
&\qquad \leq C \sum_{k,j} \int_{E_j^k} |3Q_j^k|^{-p_{jk}} \left( \int_{3Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy \right)^{p_{jk}} dx \\
&\qquad \leq C \sum_{k,j} \int_{E_j^k} \left( \fint_{3Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy \right)^{p_{jk}} dx \\
&\qquad \leq C \sum_{k,j} \int_{E_j^k} \left( \fint_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy \right)^{p_-} dx \\
&\qquad \leq C \sum_{k,j} \int_{E_j^k} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx \\
&\qquad \leq C \int_{\mathbb{R}^n} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx.
\end{aligned}$$

Since $p_- > 1$, by Theorem 3.4 the maximal operator is bounded on $L^{p_-}(\mathbb{R}^n)$. Hence,

$$\int_{\mathbb{R}^n} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx \leq C \int_{\mathbb{R}^n} f_1(x)^{p(x)}\,dx \leq C.$$

If we combine the above inequalities we get the desired result.

**The estimate for $f_2$** Since $0 \leq f_2(x) \leq 1$, we also have that $0 \leq Mf_2(x) \leq 1$. Since $p_- > 1$, if we let $R(x) = (e + |x|)^{-n}$, then by inequality (3.11),

$$\int_{\mathbb{R}^n} Mf_2(x)^{p(x)}\,dx \leq C \int_{\mathbb{R}^n} Mf_2(x)^{p_\infty}\,dx + \int_{\mathbb{R}^n} R(x)^{p_-}\,dx.$$

The second integral is a constant independent of $f$. To bound the first integral, since $p_\infty \geq p_- > 1$, by Theorem 3.4 and (3.12),

$$\begin{aligned}\int_{\mathbb{R}^n} Mf_2(x)^{p_\infty}\,dx &\leq C \int_{\mathbb{R}^n} f_2(x)^{p_\infty}\,dx \\ &\leq C \int_{\mathbb{R}^n} f_2(x)^{p(x)}\,dx + C \int_{\mathbb{R}^n} R(x)^{p_-}\,dx \leq C.\end{aligned}$$

Combining these inequalities we get the desired estimate for $f_2$. This completes the proof.

## *Proof of Inequality (3.8): The General Case*

The proof when $p_+ = \infty$ is similar to the proof when $p_+ < \infty$: we again write $f = f_1 + f_2$ and estimate each piece separately. The estimate for $f_1$ is similar but more complicated because we need to deal with the fact that the exponent function $p(\cdot)$ is unbounded and may in fact be infinite on a set of positive measure. The estimate for $f_2$ is different. It is possible to adapt the argument above, but doing so requires dividing $f_2$ into two pieces supported on $\Omega_\infty$ and $\mathbb{R}^n \setminus \Omega_\infty$, and the estimate for each of these pieces then depends on whether $p_\infty = \infty$ or $p_\infty < \infty$. We therefore take a different approach using Proposition 2.43. In those places where the argument is the same or similar, we will refer to the above proof for details.

As in the proof when $p_+ < \infty$, we may assume without loss of generality that $f$ is non-negative and $\|f\|_{p(\cdot)} = 1$. However, we can no longer apply Lemma 3.29 since $p_+ = \infty$. Instead, it will suffice to prove the desired inequality for functions that are bounded and have compact support. For if this case is true, given an arbitrary non-negative $f$, let $f_k(x) = \min(f(x), k)\chi_{B_k(0)}(x)$. Then the sequence $\{f_k\}$ increases pointwise to $f$ and by Lemma 3.30, $Mf_k$ increases pointwise to $Mf$. Therefore, by Fatou's lemma for variable Lebesgue spaces (Theorem 2.61),

$$\|Mf\|_{p(\cdot)} \leq \liminf_{k\to\infty} \|Mf_k\|_{p(\cdot)} \leq C \liminf_{k\to\infty} \|f_k\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Fix such a function $f$. Then by Corollary 2.22,

$$\rho(f) = \int_{\mathbb{R}^n \setminus \Omega_\infty} f(x)^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty)} \le 1.$$

Decompose $f$ as $f_1 + f_2$, where

$$f_1 = f\chi_{\{x: f(x)>1\}}, \qquad f_2 = f\chi_{\{x: f(x)\le 1\}}.$$

Then up to a set of measure zero, $\operatorname{supp}(f_1) \subset \mathbb{R}^n \setminus \Omega_\infty$ and $\rho(f_i) \le \|f_i\|_{p(\cdot)} \le 1$. As before, it will suffice to show that for $i = 1, 2$, $\|Mf_i\|_{p(\cdot)} \le C$. For $i = 1$ we will find a constant $\lambda_1 = \lambda_1(n, p(\cdot)) > 0$ such that

$$\rho(Mf_1/\lambda_1) \le 1.$$

When $i = 2$ we will show directly that $\|Mf_2\|_{p(\cdot)} \le C$.

**The estimate for $f_1$** Let $\lambda_1^{-1} = \alpha_1\beta_1\gamma_1$; the specific values of these constants will be fixed below. Then

$$\rho(\alpha_1\beta_1\gamma_1 Mf_1) = \int_{\mathbb{R}^n \setminus \Omega_\infty} [\alpha_1\beta_1\gamma_1 Mf_1(x)]^{p(x)}\,dx + \alpha_1\beta_1\gamma_1\|Mf_1\|_{L^\infty(\Omega_\infty)}. \tag{3.16}$$

We will show that each term on the right is bounded by $1/2$. To estimate the first, we use the same decomposition argument as before. Let $A = 4^n$ and define

$$\Omega_k = \{x \in \mathbb{R}^n \setminus \Omega_\infty : Mf_1(x) > A^k\}.$$

Since $f_1$ is bounded, $Mf_1(x) < \infty$ everywhere; as before we may assume without loss of generality that $Mf_1(x) > 0$. Hence, $\mathbb{R}^n \setminus \Omega_\infty = \bigcup_k \Omega_k \setminus \Omega_{k+1}$. Since $f_1$ also has compact support, by Lemma 3.9 for each $k$ there exists a collection of disjoint cubes $\{Q_j^k\}_j$ such that

$$\Omega_k \subset \bigcup_j 3Q_j^k \quad \text{and} \quad 3^n \fint_{3Q_j^k} f_1(y)\,dy \ge \fint_{Q_j^k} f_1(y)\,dy > A^{k-1}.$$

Form the sets $E_j^k$ that are pairwise disjoint for all $j$ and $k$ and are such that $\Omega_k \setminus \Omega_{k+1} = \bigcup_j E_j^k$. Let $\alpha_1 = A^{-2}3^{-n}$ and $p_{jk} = p_-(3Q_j^k)$; since $|3Q_j^k \cap \Omega_k| > 0$, $p_{jk} < \infty$. Then by Hölder's inequality,

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus\Omega_\infty} & [\alpha_1\beta_1\gamma_1 Mf_1(x)]^{p(x)}\,dx \\
&= \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} [\alpha_1\beta_1\gamma_1 Mf_1(x)]^{p(x)}\,dx \\
&\le \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} [\alpha_1\beta_1\gamma_1 A^{k+1}]^{p(x)}\,dx \\
&\le \sum_{k,j} \int_{E_j^k} \left(\beta_1\gamma_1 \fint_{3Q_j^k} f_1(y)\,dy\right)^{p(x)} dx \\
&\le \sum_{k,j} \int_{E_j^k} \left(\beta_1\gamma_1 \left(\fint_{3Q_j^k} f_1(y)^{p_{jk}/p_-}\,dy\right)^{p_-/p_{jk}}\right)^{p(x)} dx. \qquad (3.17)
\end{aligned}$$

Let $r(\cdot) = 1/p(\cdot)$; then $r(\cdot) \in LH_0(\mathbb{R}^n)$, $r_+ \le 1$, and $r_+(3Q_j^k) = 1/p_{jk}$. Therefore, by Lemma 3.24 we can choose $\beta_1 < 1$ such that

$$\beta_1|3Q_j^k|^{-p_-/p_{jk}} \le |3Q_j^k|^{-p_-/p(x)}.$$

Further, since $f_1(x) = 0$ or $f_1(x) \ge 1$, and $\operatorname{supp}(f_1) \subset \mathbb{R}^n \setminus \Omega_\infty$,

$$\int_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy \le \int_{3Q_j^k} f_1(y)^{p(y)}\,dy \le \rho(f_1) \le 1.$$

Therefore, since $x \in E_j^k \subset 3Q_j^k$, $p(x) \ge p_{jk}$, and assuming for the moment that $\gamma_1 < 1$,

$$\begin{aligned}
\sum_{k,j} \int_{E_j^k} & \left(\beta_1\gamma_1 \left(\fint_{3Q_j^k} f_1(y)^{p_{jk}/p_-}\,dy\right)^{p_-/p_{jk}}\right)^{p(x)} dx \\
&\le \sum_{k,j} \int_{E_j^k} |3Q_j^k|^{-p_-} \left(\gamma_1 \int_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy\right)^{p(x)p_-/p_{jk}} dx \\
&\le \sum_{k,j} \int_{E_j^k} |3Q_j^k|^{-p_-} \left(\gamma_1 \int_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy\right)^{p_-} dx \\
&\le \sum_{k,j} \int_{E_j^k} \gamma_1^{p_-} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx \\
&\le \int_{\mathbb{R}^n} \gamma_1^{p_-} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx.
\end{aligned}$$

Since $p_- > 1$, by Theorem 3.4 we can choose $\gamma_1 < 1$ such that

$$\int_{\mathbb{R}^n} \gamma_1^{p_-} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx \le \frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega_\infty} f_1(y)^{p(y)}\,dy \le \frac{1}{2}.$$

This gives us the desired estimate for the first term in (3.16).

We will now show that $\alpha_1\beta_1\gamma_1\|Mf_1\|_{L^\infty(\Omega_\infty)} \le 1/2$. Since $\alpha_1 \le 1/4$, it will suffice to show (after possibly making $\beta_1, \gamma_1$ smaller than the values chosen above) that

$$\beta_1\gamma_1\|Mf_1\|_{L^\infty(\Omega_\infty)} \le 2. \tag{3.18}$$

Fix $x \in \Omega_\infty$. Since $\operatorname{supp}(f_1) \subset \mathbb{R}^n \setminus \Omega_\infty$, when computing $Mf_1(x)$ we can restrict ourselves to cubes $Q$ containing $x$ such that $|Q \cap \Omega \setminus \Omega_\infty| > 0$. In particular, there exists such a cube that satisfies

$$Mf_1(x) \le 2\fint_Q f_1(y)\,dy.$$

Fix $r$, $p_-(Q) < r < \infty$; since $1/p(\cdot)$ is continuous, there exists a point $x_r \in Q \setminus \Omega_\infty$ such that $p(x_r) = r$. We claim that there exist $\beta_1, \gamma_1$ independent of $r$ such that

$$\left(\beta_1\gamma_1\fint_Q f_1(y)\,dy\right)^{p(x_r)} \le \fint_Q f_1(y)^{p(y)}\,dy \le \frac{1}{|Q|}.$$

The second inequality is immediate since $\rho(f_1) \le 1$. To prove the first, repeat the argument above, beginning with the estimate of the integral in (3.17) and replacing $p_-$ with 1, $3Q_j^k$ with $Q$, and $p_{jk}$ with $p_-(Q)$. This yields the desired inequality for $\beta_1, \gamma_1 > 0$ sufficiently small but not depending on our choice of $r$. Therefore, we have that

$$\beta_1\gamma_1\fint_Q f_1(y)\,dy \cdot |Q|^{1/r} \le 1.$$

Since this is true for all $r$ large, we can take the limit as $r \to \infty$ to get

$$\beta_1\gamma_1 Mf_1(x) \le 2\beta_1\gamma_1\fint_Q f_1(y)\,dy \le 2.$$

Since this estimate holds for almost all $x$, we have proved inequality (3.18). This completes the estimate for $f_1$.

**The estimate for $f_2$** Define the sets

$$E = \{x \in \mathbb{R}^n : p(x) \ge p_\infty\}, \quad F = \{x \in \mathbb{R}^n : p(x) < p_\infty\}.$$

Then

$$\|Mf_2\|_{p(\cdot)} \le \|Mf_2\chi_E\|_{p(\cdot)} + \|Mf_2\chi_F\|_{p(\cdot)} = \|Mf_2\|_{L^{p(\cdot)}(E)} + \|Mf_2\|_{L^{p(\cdot)}(F)}.$$

We will show that each term on the right-hand side is bounded by a constant independent of $f_2$. To estimate $\|Mf_2\|_{L^{p(\cdot)}(E)}$, note first that since $0 \le f_2(x) \le 1$, by Proposition 3.3, $Mf_2(x) \le 1$. Furthermore, since $f_2$ is bounded and has compact support, $f_2 \in L^{p_\infty}(\mathbb{R}^n)$; since $p_\infty > 1$, by Theorem 3.4, $Mf_2 \in L^{p_\infty}(\mathbb{R}^n)$. If $\|Mf_2\|_{L^{p_\infty}(E)} < 1$, then by Lemma 3.28 with $g = Mf_2$, $t(\cdot) = p_\infty$ and $u(\cdot) = p(\cdot)$,

$$\|Mf_2\|_{L^{p(\cdot)}(E)} \le \|Mf_2\|_{L^{p_\infty}(E)} + 1 < 2.$$

On the other hand, if $\|Mf_2\|_{L^{p_\infty}(E)} \ge 1$, then by Lemma 3.28 and Theorem 3.4,

$$\begin{aligned}\|Mf_2\|_{L^{p(\cdot)}(E)} &\le 2\|Mf_2\|_{L^{p_\infty}(E)} \le 2\|Mf_2\|_{L^{p_\infty}(\mathbb{R}^n)} \\ &\le C\|f_2\|_{L^{p_\infty}(\mathbb{R}^n)} \le C\big(\|f_2\|_{L^{p_\infty}(E)} + \|f_2\|_{L^{p_\infty}(F)}\big).\end{aligned}$$

We first estimate the norm of $f_2$ on $E$. By the definition of $E$, we can define the defect exponent $r(\cdot) \in \mathcal{P}(E)$ by

$$\frac{1}{p_\infty} = \frac{1}{p(x)} + \frac{1}{r(x)}.$$

Therefore, by the generalized Hölder's inequality (Corollary 2.28),

$$\begin{aligned}\|f_2\|_{L^{p_\infty}(E)} &\le K\|1\|_{L^{r(\cdot)}(E)}\|f_2\|_{L^{p(\cdot)}(E)} \\ &\le K\|1\|_{L^{r(\cdot)}(E)}\|f_2\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le K\|1\|_{L^{r(\cdot)}(E)}.\end{aligned}$$

Since $1/p(\cdot) \in LH_\infty(E)$, $r(\cdot) \to \infty$ as $|x| \to \infty$, and $1/r(\cdot) \in LH_\infty(E)$. Therefore, by Proposition 2.43, $1 \in L^{r(\cdot)}(E)$ and this bound is finite and independent of $f_2$.

To estimate the norm of $f_2$ on $F$, we apply Lemma 3.28 with $g = f_2 \in L^{p(\cdot)}(F)$, $t(\cdot) = p(\cdot)$ and $u(\cdot) = p_\infty$; since $\|f_2\|_{L^{p(\cdot)}(F)} \le \|f_2\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1$, we get that

$$\|f_2\|_{L^{p_\infty}(F)} \le \|f_2\|_{L^{p(\cdot)}(F)} + 1 \le 2.$$

Combining the above estimates we see that

$$\|Mf_2\|_{L^{p(\cdot)}(E)} \le C\big(K\|1\|_{L^{r(\cdot)}(E)} + 2\big) < \infty.$$

The estimate for $\|Mf_2\|_{L^{p(\cdot)}(F)}$ is very similar. Define the defect exponent $s(\cdot) \in \mathcal{P}(F)$ by

$$\frac{1}{p(x)} = \frac{1}{p_\infty} + \frac{1}{s(x)}.$$

Then by the generalized Hölder's inequality,

$$\|Mf_2\|_{L^{p(\cdot)}(F)} \le K\|1\|_{L^{s(\cdot)}(F)}\|Mf_2\|_{L^{p_\infty}(F)},$$

and again by Proposition 2.43, $\|1\|_{L^{s(\cdot)}(F)} < \infty$. Further, we can now argue as we did above to get

$$\begin{aligned}\|Mf_2\|_{L^{p_\infty}(F)} \leq \|Mf_2\|_{L^{p_\infty}(\mathbb{R}^n)} &\leq C\|f_2\|_{L^{p_\infty}(\mathbb{R}^n)} \\ &\leq C\big(\|f_2\|_{L^{p_\infty}(E)} + \|f_2\|_{L^{p_\infty}(F)}\big) \leq C < \infty.\end{aligned}$$

This yields the desired estimate for $\|Mf_2\|_{L^{p(\cdot)}(F)}$ and the proof is complete.

## *Proof of the Weak Type Inequality*

The proof of inequality (3.7) is nearly identical to the proof of the strong type inequality given in the previous section, so here we will only give the principal details and will refer back for those parts of the proof that remain essentially the same.

We begin by making the same reductions as before and writing $f = f_1 + f_2$. Then given any $t > 0$,

$$\begin{aligned}&\{x \in \mathbb{R}^n : Mf(x) > t\} \\ &\qquad \subset \{x \in \mathbb{R}^n : Mf_1(x) > t/2\} \cup \{x \in \mathbb{R}^n : Mf_2(x) > t/2\} \\ &\qquad = F_1 \cup F_2.\end{aligned}$$

Therefore, it will suffice to show that for $i = 1, 2$, $t\|\chi_{F_i}\|_{p(\cdot)} \leq C$. When $i = 2$, the argument is almost identical to that given above to estimate $\|Mf_2\|_{p(\cdot)}$, replacing $Mf_2$ by $\frac{1}{2}t\chi_{F_2}$. Since $f_2 \leq 1$, $Mf_2 \leq 1$, so the set $F_2$ is empty if $t > 2$, and $\frac{1}{2}t\chi_{F_2} \leq 1$ for $0 < t \leq 2$. If $p_\infty < \infty$, use the weak type inequality from Theorem 3.4 instead of the strong type and proceed as before. If $p_\infty = \infty$, use the inequality $t\chi_{F_2}(x) \leq 2Mf_2(x)$ and the fact that the maximal operator is bounded on $L^\infty$.

To estimate $\|\chi_{F_1}\|_{p(\cdot)}$ we will show that for some $\alpha_1, \beta_1, \gamma_1 > 0$,

$$\rho(\alpha_1\beta_1\gamma_1 t\chi_{F_1}) = \int_{F_1\setminus\Omega_\infty} [\alpha_1\beta_1\gamma_1 t]^{p(x)}\,dx + \alpha_1\beta_1\gamma_1 t\|\chi_{F_1}\|_{L^\infty(\Omega_\infty)} \leq 1.$$

We will show that each term in the middle is bounded by $1/2$ for suitable choice of $\alpha_1, \beta_1, \gamma_1$.

The estimate for the second term is immediate: since $t\chi_{F_1}(x) \leq Mf_1(x)$,

$$\alpha_1\beta_1\gamma_1 t\|\chi_{F_1}\|_{L^\infty(\Omega_\infty)} \leq \alpha_1\beta_1\gamma_1\|Mf_1\|_{L^\infty(\Omega_\infty)},$$

and the proof given above that the right-hand side is bounded does not depend on the fact that $p_- > 1$. Since the other hypotheses are the same, we get the desired bound.

To bound

$$\int_{F_1\setminus\Omega_\infty} [\alpha_1\beta_1\gamma_1 t]^{p(x)}\,dx$$

we apply Lemma 3.9 to find disjoint dyadic cubes $\{Q_j\}$ such that

$$F_1 \subset \bigcup_j 3Q_j \quad \text{and} \quad \fint_{Q_j} f_1(y)\,dy > 4^{-n}\frac{t}{2}.$$

Form disjoint sets $E_j$ such that $E_j \subset 3Q_j$ and $F_1 \setminus \Omega_\infty = \bigcup_j E_j$.

We can now argue as before, replacing $3Q_j^k$ by $Q_j$ and using the fact that $p_- \geq 1$ to get

$$\int_{F_1\setminus\Omega_\infty} [\alpha_1\beta_1\gamma_1 t]^{p(x)}\,dx \leq \sum_j \int_{E_j} \left(\gamma_1 \fint_{Q_j} f_1(y)^{p(y)}\,dy\right) dx.$$

Since $|E_j| \leq 3^n|Q_j|$, the cubes $Q_j$ are disjoint and $\operatorname{supp}(f_1) \subset \mathbb{R}^n \setminus \Omega_\infty$, so if we let $\gamma_1 = \frac{1}{2}3^{-n}$,

$$\begin{aligned}\sum_j \int_{E_j} \left(\gamma_1 \fint_{Q_j} f_1(y)^{p(y)}\,dy\right) dx &= \sum_j \gamma_1 \frac{|E_j|}{|Q_j|} \int_{Q_j} f_1(y)^{p(y)}\,dy \\ &\leq \sum_j \frac{1}{2} \int_{Q_j} f_1(y)^{p(y)}\,dy \leq \frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega_\infty} f_1(y)^{p(y)}\,dy \leq \frac{1}{2}.\end{aligned}$$

This completes the proof of inequality (3.7).

## 3.5 Modular Inequalities

In this section we consider a different approach to generalizing Theorem 3.4 for the Hardy-Littlewood maximal operator. In the classical Lebesgue spaces, norm inequalities are equivalent to modular inequalities: this suggests that we consider modular inequalities for the maximal operator in the variable Lebesgue spaces. In particular, if $p_+ < \infty$, then corresponding to inequalities (3.7) and (3.8) are the inequalities

$$\int_{\{x:Mf(x)>t\}} t^{p(x)}\,dx \leq C\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx, \tag{3.19}$$

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \leq C\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx. \tag{3.20}$$

By the definition of the norm, these modular inequalities imply the corresponding norm inequalities, so for them to be true we would need to make the same or stronger assumptions on the exponent function. In fact, these inequalities never hold unless $p(\cdot)$ is constant.

**Theorem 3.31.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $p_+ < \infty$. Then the modular inequalities (3.19) and (3.20) are true if and only if there is a constant $p_0$ such that $p(\cdot) = p_0$ almost everywhere.*

The proof of Theorem 3.31 depends on results from the theory of weighted norm inequalities in harmonic analysis, and so we defer it until the end of Sect. 4.3 (p. 158) below.

Weaker modular inequalities that include an error term, however, are true. A strong type modular inequality similar to (3.20) is actually implicit in the proof of Theorem 3.16. Here we give an alternative approach that also yields a weak type inequality. As a first step we prove a modular inequality that can be thought of as a modular Hölder's inequality for variable Lebesgue spaces. Results similar to this played a very important role in the original proofs of Theorem 3.16.

**Theorem 3.32.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $p_+ < \infty$ and $p(\cdot) \in LH(\mathbb{R}^n)$. Then for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|f\|_{p(\cdot)} \leq 1$, any cube $Q$ and any $x \in Q$,*

$$\left(\fint_Q |f(y)|\,dy\right)^{p(x)} \leq C\fint_Q |f(y)|^{p(y)}\,dy + \frac{C}{(e+|x|)^{(n+1)p_-}} + C\fint_Q \frac{dy}{(e+|y|)^{(n+1)p_-}} \leq C\fint_Q |f(y)|^{p(y)}\,dy + \frac{C}{(e+|x|)^n}. \tag{3.21}$$

*Proof.* We begin by proving the first inequality in (3.21). Fix $Q$ and $x \in Q$. Without loss of generality, we may assume that $f$ is non-negative. As in the proof of Theorem 3.16, decompose $f$ as $f_1 + f_2$, where

$$f_1 = f\chi_{\{x:f(x)>1\}}, \qquad f_2 = f\chi_{\{x:f(x)\leq 1\}}.$$

Then by Remark 2.8,

$$\left(\fint_Q |f(y)|\,dy\right)^{p(x)} \leq 2^{p_+-1}\left[\left(\fint_Q |f_1(y)|\,dy\right)^{p(x)} + \left(\fint_Q |f_2(y)|\,dy\right)^{p(x)}\right].$$

We will estimate each of these integrals in turn. The estimate for $f_1$ is very similar to the argument for $f_1$ in the proof of Theorem 3.16 when $p_+ < \infty$; hence, we only sketch the details. Since $f_1 = 0$ or $f_1 \geq 1$ and $\|f_1\|_{p(\cdot)} \leq 1$, we have that

$$\int_Q f_1(y)\,dy \le \int_Q f_1(y)^{p(y)}\,dy \le 1,$$

and so by Lemma 3.24 and Hölder's inequality (if $p_- > 1$),

$$\left(\fint_Q f_1(y)\,dy\right)^{p(x)} \le |Q|^{-p(x)}\left(\int_Q f_1(y)\,dy\right)^{p_-}$$
$$\le C|Q|^{-p_-}\left(\int_Q f_1(y)^{p(y)/p_-}\,dy\right)^{p_-} \le C\fint_Q f_1(y)^{p(y)}\,dy.$$

To prove the estimate for the integral of $f_2$, we adapt the proof of Lemma 3.26. Let $R(y) = (e+|y|)^{-n-1}$ and suppose first that

$$\fint_Q f_2(y)\,dy \le R(x).$$

Then by inequality (3.13),

$$\left(\fint_Q f_2(y)\,dy\right)^{p(x)} \le R(x)^{p(x)} \le R(x)^{p_\infty} R(x)^{-|p(x)-p_\infty|} \le CR(x)^{p_-}.$$

On the other hand, if

$$\fint_Q f_2(y)\,dy \ge R(x),$$

then by Hölder's inequality and (3.13),

$$\left(\fint_Q f_2(y)\,dy\right)^{p(x)} \le \left(\fint_Q f_2(y)\,dy\right)^{p_\infty}\left(\fint_Q f_2(y)\,dy\right)^{-|p(x)-p_\infty|}$$
$$\le \fint_Q f_2(y)^{p_\infty}\,dy \cdot R(x)^{-|p(x)-p_\infty|} \le C\fint_Q f_2(y)^{p_\infty}\,dy.$$

Since $0 \le f_2 \le 1$ we can apply Lemma 3.26 to get

$$\fint_Q f_2(y)^{p_\infty}\,dy \le C\fint_Q f_2(y)^{p(y)}\,dy + \fint_Q R(y)^{p_-}\,dy.$$

This completes the proof of the first inequality in (3.21). To prove the second, note that since $R(x)^{p_-} \le (e+|x|)^{-n}$ it will suffice to show that

$$\fint_Q R(y)^{p_-}\,dy \le \frac{C}{(e+|x|)^n}.$$

To do so we consider three cases. If $|x| > 2\sqrt{n}\ell(Q)$, then there exists a constant $C$ depending only on $n$ such that for every $y \in Q$, $R(y) \le CR(x)$. Hence,

$$\fint_Q R(y)^{p_-}\,dy \le CR(x)^{p_-}.$$

If $1 \le |x| \le 2\sqrt{n}\ell(Q)$, then, since $R \in L^{p_-}(\mathbb{R}^n)$,

$$\fint_Q R(y)^{p_-}\,dy \le C|x|^{-n}\int_{\mathbb{R}^n} R(y)^{p_-}\,dy \le C(e+|x|)^{-n}.$$

Finally if $|x| \le 1$, then, since $R(y) \le 1$,

$$\fint_Q R(y)^{p_-}\,dy \le 1 \le C(e+|x|)^{-n}.$$

This completes the proof of (3.21). □

We now state and prove our modular inequalities.

**Theorem 3.33.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $p_+ < \infty$ and $p(\cdot) \in LH(\mathbb{R}^n)$, suppose $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\|f\|_{p(\cdot)} \le 1$. If $p_- > 1$, then*

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le C\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx + C\int_{\mathbb{R}^n} \frac{dx}{(e+|x|)^{np_-}}. \tag{3.22}$$

*If $p_- = 1$, then for all $t > 0$,*

$$\int_{\{x:Mf(x)>t\}} t^{p(x)}\,dx \le C\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx + C\int_{\mathbb{R}^n} \frac{dx}{(e+|x|)^{n+1}}. \tag{3.23}$$

*Proof.* To prove inequality (3.22) we use the second estimate in Theorem 3.32 with the exponent $p(\cdot)$ replaced by $p(\cdot)/p_-$. Fix $x$ and fix any cube $Q$ containing $x$. Then

$$\left(\fint_Q |f(y)|\,dy\right)^{p(x)} \le \left(\fint_Q |f(y)|^{p(y)/p_-}\,dy + \frac{C}{(e+|x|)^n}\right)^{p_-}$$
$$\le CM(|f(\cdot)|^{p(\cdot)/p_-})(x)^{p_-} + \frac{C}{(e+|x|)^{np_-}}.$$

If we take the supremum over all such cubes $Q$, we get that

$$Mf(x)^{p(x)} \le CM(|f(\cdot)|^{p(\cdot)/p_-})(x)^{p_-} + \frac{C}{(e+|x|)^{np_-}}.$$

If we now integrate both sides of this inequality over all $x \in \mathbb{R}^n$, since $p_- > 1$ by Theorem 3.4 we have that

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le C\int_{\mathbb{R}^n} M(|f(\cdot)|^{p(\cdot)/p_-})(x)^{p_-}\,dx + C\int_{\mathbb{R}^n} \frac{dx}{(e+|x|)^{np_-}}$$
$$\le C\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx + C\int_{\mathbb{R}^n} \frac{dx}{(e+|x|)^{np_-}}.$$

This completes the proof of inequality (3.22).

To prove inequality (3.23) we use the first estimate in Theorem 3.32. Fix $t > 0$; then by Lemmas 3.9, 3.13 and 3.29, let $\{Q_j\}$ be the Calderón-Zygmund cubes of $f$ at height $t/4^n$: pairwise disjoint cubes such that

$$E = \{x : Mf(x) > t\} \subset \bigcup_j 3Q_j, \quad \text{and} \quad \fint_{Q_j} |f(y)|\,dy > t/4^n.$$

Form a sequence of pairwise disjoint sets $\{E_j\}$ as follows: $E_1 = E \cap 3Q_1$, $E_2 = (E\cap 3Q_2)\setminus E_1$, $E_3 = (E\cap 3Q_3)\setminus(E_1\cup E_2), \ldots$. Then $|E_j| \le |3Q_j|$, $E = \bigcup_j E_j$ and the $E_j$ are pairwise disjoint. Given these sets we have that

$$\begin{aligned}
&\int_{\{x:Mf(x)>t\}} t^{p(x)}\,dx \\
&\quad\le 4^{p_+}\sum_j \int_{E_j}\left(\fint_{Q_j}|f(y)|\,dy\right)^{p(x)} dx \\
&\quad\le C\sum_j \int_{E_j}\left(\fint_{Q_j}|f(y)|^{p(y)}\,dy + \frac{1}{(e+|x|)^{n+1}} + \fint_{Q_j}\frac{dy}{(e+|y|)^{n+1}}\right)dx \\
&\quad\le C\sum_j \left(\int_{Q_j}|f(y)|^{p(y)}\,dy + \int_{E_j}\frac{dx}{(e+|x|)^{n+1}} + \int_{Q_j}\frac{dy}{(e+|y|)^{n+1}}\right) \\
&\quad\le C\int_{\mathbb{R}^n}|f(y)|^{p(y)}\,dy + C\int_{\mathbb{R}^n}\frac{dy}{(e+|y|)^{n+1}}.
\end{aligned}$$

This completes the proof of inequality (3.23). □

We conclude this section with a different version of the modular weak type inequality. We can write the weak $(p, p)$ inequality (3.1) as a modular inequality in another way:

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \le C\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{t}\right)^p dx. \tag{3.24}$$

In light of Theorem 3.31 it is surprising that essentially this inequality holds in the variable Lebesgue spaces with almost no assumptions on $p(\cdot)$.

**Theorem 3.34.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if $|\Omega_\infty| < \infty$, then there exists a constant $C$ such that for all $t > 0$ and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\begin{aligned}|\{x \in \mathbb{R}^n : Mf(x) > t\}| &\le C\rho(4f/t) \\ &= C\int_{\mathbb{R}^n\setminus\Omega_\infty} \left(\frac{4|f(x)|}{t}\right)^{p(x)} dx + Ct^{-1}|\Omega_\infty|\|f\|_{L^\infty(\Omega_\infty)}.\end{aligned}$$

*Furthermore, if $|\Omega_\infty| = \infty$, then this inequality is false with any finite constant in the last term.*

*Remark 3.35.* If $p_+(\mathbb{R}^n \setminus \Omega_\infty) < \infty$, then by increasing the constant $C$ by a factor of $4^{p_+}$ we can replace the right-hand side by $C\rho(f/t)$. Also see Sect. 3.7.2 below.

*Proof.* Assume first that $|\Omega_\infty| < \infty$. Fix $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $t > 0$. We modify the decomposition used in the proof of the weak type inequality in Theorem 3.4: define

$$f_1 = f\chi_{\{x\in\mathbb{R}^n\setminus\Omega_\infty:|f(x)|>t/2\}}, \quad f_2 = f\chi_{\{x\in\mathbb{R}^n\setminus\Omega_\infty:|f(x)|\le t/2\}}, \quad f_3 = f\chi_{\Omega_\infty}.$$

By Proposition 3.3, $Mf_2(x) \le t/2$. Therefore,

$$\begin{aligned}&|\{x \in \mathbb{R}^n : Mf(x) > t\}| \\ &\qquad \le |\{x \in \mathbb{R}^n : Mf_1(x) + Mf_2(x) + Mf_3(x) > t\}| \\ &\qquad \le |\{x \in \mathbb{R}^n : Mf_1(x) + Mf_3(x) > t/2\}| \\ &\qquad \le |\{x \in \mathbb{R}^n : Mf_1(x) > t/4\}| + |\{x \in \mathbb{R}^n : Mf_3(x) > t/4\}|.\end{aligned}$$

We estimate the last two terms separately. Since $|4t^{-1}f_1| \ge 1$, by the weak $(p_-, p_-)$ inequality for the maximal operator (Theorem 3.4),

$$\begin{aligned}|\{x \in \mathbb{R}^n : Mf_1(x) > t/4\}| &= |\{x \in \mathbb{R}^n : M(4t^{-1}f_1)(x) > 1\}| \\ &\le |\{x \in \mathbb{R}^n : M((4t^{-1}|f_1|)^{p(\cdot)/p_-})(x) > 1\}| \\ &\le C\int_{\mathbb{R}^n} \left(\frac{4|f_1(x)|}{t}\right)^{p(x)} dx \\ &\le C\int_{\mathbb{R}^n\setminus\Omega_\infty} \left(\frac{4|f(x)|}{t}\right)^{p(x)} dx.\end{aligned}$$

Similarly, by the weak $(1,1)$ inequality,

$$|\{x \in \mathbb{R}^n : Mf_3(x) > t/4\}| \le \frac{C}{t}\int_{\mathbb{R}^n} |f_3(x)|\,dx \le Ct^{-1}|\Omega_\infty|\|f\|_{L^\infty(\Omega_\infty)}.$$

If we combine these inequalities we get the desired result.

Now suppose $|\Omega_\infty| = \infty$. Then $f = \chi_{\Omega_\infty} \in L^{p(\cdot)}(\mathbb{R}^n)$, and we have that $\|f\|_{p(\cdot)} = \|f\|_{L^\infty(\Omega_\infty)} = 1$. On the other hand, for all $t < 1$,

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| = \infty.$$

□

## 3.6 Interpolation and Convexity

We conclude this chapter by briefly considering the theory of interpolation on variable Lebesgue spaces and its application to the Hardy-Littlewood maximal operator. As we noted at the end of Sect. 3.2, the Marcinkiewicz interpolation theorem is central to the proof of norm inequalities for the maximal operator on the classical Lebesgue spaces, but it is unknown whether this result is true in the variable Lebesgue spaces. (See Sect. 3.7.8 below.)

However, other interpolation theorems are true in the variable Lebesgue spaces. Here we will prove an elementary result that holds for positive integral operators: that is, operators of the form

$$Tf(x) = \int_\Omega K(x,y)f(y)\,dy,$$

where $K : \Omega \times \Omega \to [0,\infty)$ is a non-negative, measurable function.

**Theorem 3.36.** *Given $\Omega$ and $p_i(\cdot), q_i(\cdot) \in \mathcal{P}(\Omega)$, $i = 1, 2$, suppose that for all $f \in L^{p_i(\cdot)}(\Omega)$ the positive integral operator $T$ satisfies*

$$\|Tf\|_{q_i(\cdot)} \le B_i \|f\|_{p_i(\cdot)}. \tag{3.25}$$

*For each $\theta$, $0 < \theta < 1$, define $p_\theta(\cdot), q_\theta(\cdot) \in \mathcal{P}(\Omega)$ by*

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)}, \qquad \frac{1}{q_\theta(x)} = \frac{\theta}{q_1(x)} + \frac{1-\theta}{q_2(x)}.$$

*Then for all $f \in L^{p_\theta(\cdot)}(\Omega)$,*

$$\|Tf\|_{q_\theta(\cdot)} \le C B_1^\theta B_2^{1-\theta} \|f\|_{p_\theta(\cdot)}, \tag{3.26}$$

*where the constant $C$ depends only on $q_1(\cdot)$, $q_2(\cdot)$ and $\theta$.*

*Remark 3.37.* It will follow from the proof of Theorem 3.36 that we can take $C$ to be the universal constant 48.

*Proof.* Fix $\theta$, $0 < \theta < 1$, and $f \in L^{p_\theta(\cdot)}(\Omega)$. Since $T$ is a positive integral operator, $|T(f)(x)| \le T(|f|)(x)$, so we may assume without loss of generality that $f$ is non-negative. Moreover, we may assume that $f$ is bounded and has compact support. This follows by an argument similar to the one used in the proof of Theorem 3.16 when $p_+ = \infty$. For if this case is true, given a non-negative function $f$, let $f_k(x) = \min(f(x), k)\chi_{B_k(0)}(x)$. By Fatou's lemma on the classical Lebesgue spaces,

$$Tf(x) \le \liminf_{k\to\infty} Tf_k(x).$$

Therefore, by the monotone convergence theorem on variable Lebesgue spaces (Theorem 2.59),

$$\|Tf\|_{p_\theta(\cdot)} \le \liminf_{k\to\infty} \|Tf_k\|_{p_\theta(\cdot)} \le C \liminf_{k\to\infty} \|f_k\|_{p_\theta(\cdot)} \le C\|f\|_{p_\theta(\cdot)}.$$

Finally, since for any $a > 0$, $T(af)(x) = aTf(x)$, we may assume that $\|f\|_{p_\theta(\cdot)} = 1$.

Given such a function $f$, by Theorem 2.34,

$$\|Tf\|_{q_\theta(\cdot)} \le k_{q_\theta(\cdot)}^{-1} \sup \int_\Omega Tf(x)g(x)\,dx,$$

where the supremum is taken over all $g \in L^{q_\theta'(\cdot)}(\Omega)$ with $\|g\|_{q_\theta'(\cdot)} \le 1$. Since $f$ is non-negative, we may also assume that $g$ is non-negative. Fix any such $g$; then it will suffice to prove that

$$\int_\Omega Tf(x)g(x)\,dx \le CB_1^\theta B_2^\theta,$$

where $C$ only depends on $p(\cdot)$.

Define the functions $f_i(x) = f(x)^{\frac{p_\theta(x)}{p_i(x)}}$. To make sense of this when the exponent functions are infinite, note that $\Omega_\infty^{p_\theta(\cdot)} \subset \Omega_\infty^{p_1(\cdot)} \cap \Omega_\infty^{p_2(\cdot)}$. Then for $x \in \Omega_\infty^{p_i(\cdot)}$ we define

$$\frac{p_\theta(x)}{p_i(x)} = \begin{cases} 1 & x \in \Omega_\infty^{p_\theta(\cdot)} \\ 0 & x \in \Omega_\infty^{p_i(\cdot)} \setminus \Omega_\infty^{p_\theta(\cdot)}. \end{cases}$$

We define the functions $g_i(x) = g(x)^{\frac{q_\theta'(x)}{q_i'(x)}}$ in the same way, here using that $\Omega_\infty^{q_\theta'(\cdot)} \subset \Omega_\infty^{q_1'(\cdot)} \cap \Omega_\infty^{q_2'(\cdot)}$ and the fact that given exponents $q_1(\cdot)$, $q_2(\cdot)$, the interpolation exponent between the conjugate exponents $q_1'(\cdot)$ and $q_2'(\cdot)$ is the same as $q_\theta'(\cdot)$, the conjugate of the interpolation exponent between $q_1(\cdot)$ and $q_2(\cdot)$.

We now claim that $\|f_i\|_{p_i(\cdot)}$, $\|g_i\|_{q_i'(\cdot)} \le 2$, $i = 1, 2$. We will show this for $f_1$; the proofs for the other three functions are identical. By Corollary 2.22, $\rho_{p_\theta(\cdot)}(f) \le \|f\|_{p_\theta(\cdot)} = 1$. In particular,

$$\int_{\Omega\setminus\Omega_\infty^{p_\theta(\cdot)}} |f(x)|^{p_\theta(x)}\,dx \le 1, \qquad \|f\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} \le 1. \tag{3.27}$$

For almost every $x \in \Omega_\infty^{p_1(\cdot)}$, $f_1(x) \leq 1$. If $x \in \Omega_\infty^{p_\theta(\cdot)}$, this follows from the second inequality in (3.27) and the fact that $p_\theta(x)/p_1(x) = 1$; for $x \in \Omega_\infty^{p_1(\cdot)} \setminus \Omega_\infty^{p_\theta(\cdot)}$ this follows since $p_\theta(x)/p_1(x) = 0$. Hence,

$$\begin{aligned}
\|f_1\|_{p_1(\cdot)} &= \inf\left\{\lambda > 0 : \int_{\Omega\setminus\Omega_\infty^{p_1(\cdot)}} \left(\frac{|f_1(x)|}{\lambda}\right)^{p_1(x)} dx + \lambda^{-1}\|f_1\|_{L^\infty(\Omega_\infty^{p_1(\cdot)})} \leq 1\right\} \\
&\leq \inf\left\{\lambda > 0 : \int_{\Omega\setminus\Omega_\infty^{p_1(\cdot)}} \left(\frac{|f_1(x)|}{\lambda}\right)^{p_1(x)} dx + \lambda^{-1} \leq 1\right\} \\
&= \inf\left\{\lambda > 1 : \int_{\Omega\setminus\Omega_\infty^{p_1(\cdot)}} \left(\frac{|f_1(x)|}{\lambda}\right)^{p_1(x)} dx + \lambda^{-1} \leq 1\right\} \\
&\leq \inf\left\{\lambda > 1 : \lambda^{-1}\int_{\Omega\setminus\Omega_\infty^{p_\theta(\cdot)}} |f(x)|^{p(x)}\, dx + \lambda^{-1} \leq 1\right\} \\
&\leq \inf\left\{\lambda > 1 : 2\lambda^{-1} \leq 1\right\} \\
&= 2.
\end{aligned}$$

By the definition of the exponents we have that $f(x) = f_1(x)^\theta f_2(x)^{1-\theta}$; a similar identity holds for $g$. Therefore, since the kernel $K$, $f$ and $g$ are non-negative, by Hölder's inequality with exponent $\theta^{-1}$, Theorem 2.26 and our hypothesis, we have that

$$\begin{aligned}
\int_\Omega Tf(x)g(x)\,dx &= \int_\Omega\int_\Omega K(x,y)f(y)g(x)\,dy\,dx \\
&= \int_\Omega\int_\Omega \big(K(x,y)f_1(y)g_1(x)\big)^\theta \big(K(x,y)f_2(y)g_2(x)\big)^{1-\theta}\,dy\,dx \\
&\leq \left(\int_\Omega\int_\Omega K(x,y)f_1(y)g_1(x)\,dy\,dx\right)^\theta \\
&\qquad \times \left(\int_\Omega\int_\Omega K(x,y)f_2(y)g_2(x)\,dy\,dx\right)^{1-\theta} \\
&= \left(\int_\Omega Tf_1(x)g_1(x)\,dx\right)^\theta \left(\int_\Omega Tf_2(x)g_2(x)\,dx\right)^{1-\theta} \\
&\leq \big(K_{q_1(\cdot)}\|Tf_1\|_{q_1(\cdot)}\|g_1\|_{q_1'(\cdot)}\big)^\theta \big(K_{q_2(\cdot)}\|Tf_2\|_{q_2(\cdot)}\|g_2\|_{q_2'(\cdot)}\big)^{1-\theta} \\
&\leq 2K_{q_1(\cdot)}^\theta K_{q_2(\cdot)}^{1-\theta} B_1^\theta B_2^{1-\theta}\|f_1\|_{p_1(\cdot)}^\theta \|f_2\|_{p_2(\cdot)}^{1-\theta} \\
&\leq 4K_{q_1(\cdot)}^\theta K_{q_2(\cdot)}^{1-\theta} B_1^\theta B_2^{1-\theta}.
\end{aligned}$$

This completes the proof. □

As an application of Theorem 3.36 we will show that the set of exponents $p(\cdot)$ such that the Hardy-Littlewood maximal operator is bounded is convex: that is, if the maximal operator is bounded on $L^{p_i(\cdot)}(\Omega)$, $i = 1, 2$, then it is bounded on $L^{p_\theta(\cdot)}(\Omega)$, where $p_\theta(\cdot)$ is defined by (3.25). This is trivially true in the classical Lebesgue spaces since the maximal operator is bounded on $L^p$ for all $p$, $1 < p \leq \infty$. If $1/p_1(\cdot), 1/p_2(\cdot) \in LH(\Omega)$, then so is $p_\theta(\cdot)$, so this follows from Theorem 3.16. The importance of this result is that there exist exponents $p(\cdot)$ which are not log-Hölder continuous but the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$. We will consider such exponents in detail in Chap. 4.

**Theorem 3.38.** *Given $\Omega$, the set of exponents $p(\cdot)$ such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\Omega)$ is convex.*

*Proof.* To apply Theorem 3.36 we must first show that the maximal operator can be approximated by positive integral operators. Let $\{E_k\}$ be a collection of bounded, pairwise disjoint sets, and for each $k$ let $Q_k$ be a cube such that $E_k \subset Q_k$. Define the kernel $K$ by

$$K(x, y) = \sum_k \chi_{E_k}(x)|Q_k|^{-1}\chi_{Q_k}(y); \tag{3.28}$$

then the operator $T$ with kernel $K$ satisfies

$$|Tf(x)| \leq \sum_k \fint_{Q_k} |f(y)|\,dy \cdot \chi_{E_k(x)} \leq \sum_k Mf(x)\chi_{E_k(x)} = Mf(x).$$

Therefore, if the maximal operator is bounded on $L^{p_i(\cdot)}(\Omega)$, $i = 1, 2$, then so is every integral operator $T$ of this form. Hence, by Theorem 3.36 they are all uniformly bounded on $L^{p_\theta(\cdot)}(\Omega)$.

To complete the proof, arguing as we did in the proof of Theorem 3.16 in the case $p_+ = \infty$, it will suffice to show that there exists $C > 1$ such that if $f$ is non-negative, bounded and has compact support, then $\|Mf\|_{p_\theta(\cdot)} \leq C\|Mf\|_{p_\theta(\cdot)}$. Fix such a function $f$; to prove this inequality we will show that there exists a positive integral operator $T$ with a kernel of the form (3.28) such that for almost every $x$,

$$Mf(x) \leq 4^n T(|f|)(x). \tag{3.29}$$

To do so we employ the same decomposition as was used for the function $f_1$ in the proof of Theorem 3.16 when $p_+ < \infty$. (This part of the argument did not depend on the fact that $p_+$ was finite.) Let $A = 4^n$ and define the sets $\Omega_k$, $Q_j^k$ and $E_j^k$ as before. Then for almost every $x$, there exists an integer $k$ such that $x \in \Omega_k \setminus \Omega_{k+1}$. Therefore, we have that there exists a set $E_j^k$ and a cube $Q_j^k$ such that

$$Mf(x) \leq A^{k+1} \leq A\fint_{Q_j^k} f(y)\,dy \cdot \chi_{E_j^k}(x).$$

Since the sets $E_j^k$ are pairwise disjoint, if we let $T$ be the integral operator with kernel

$$K(x, y) = \sum_{k,j} \chi_{E_j^k}(x)|Q_j^k|^{-1}\chi_{Q_j^k}(y),$$

we get an operator such that (3.29) holds. This completes the proof. □

## 3.7 Notes and Further Remarks

### *3.7.1 References*

The Hardy-Littlewood maximal operator was first introduced by Hardy and Littlewood [148] in one dimension and extended to $\mathbb{R}^n$ by Wiener [347]. They also proved the $L^p$ estimates in Theorem 3.4. The maximal operator came into prominence with the work of Calderón and Zygmund [40] on singular integrals; in this paper they also introduced the dyadic decomposition that bears their name (Lemmas 3.9 and 3.13). Our treatment is based on Duoandikoetxea [96]; see also García-Cuerva and Rubio de Francia [140] and Grafakos [143]. For a proof of the $L^p$ inequalities using covering lemmas, see Stein [339]. For the alternative definitions of the maximal operator on a domain $\Omega$, see [129]. Example 3.14 showing that the variable Lebesgue spaces are not rearrangement invariant is new.

The maximal operator on variable Lebesgue spaces was considered by several authors. Diening [77] was the first to find a sufficient condition for boundedness. He showed that when $p_+ < \infty$, the $LH_0$ condition is sufficient for the maximal operator to be bounded on bounded domains. As a corollary to this result he showed that the maximal operator is bounded on $\mathbb{R}^n$ if $p(\cdot)$ is constant outside of a (large) ball containing the origin. Nekvinda [282] proved the strong type inequality in Theorem 3.16 when $p_+ < \infty$ and with the $LH_\infty$ condition replaced by a somewhat more general condition. We will discuss his result in detail in Chap. 4. The $LH_\infty$ condition was introduced in [62, 63] and the strong type inequality in the case $p_+ < \infty$ was proved there. A simpler proof, again with $p_+ < \infty$ was given in [42] (see also [128]). The case when $p_+ = \infty$ and the idea of using $1/p(\cdot) \in LH$ is due to Diening [80]; see also [81]. A very different proof of the strong type inequality when $p_+ < \infty$, gotten by viewing $L^{p(\cdot)}$ from the perspective of abstract Banach function spaces, was given by Lerner and Pérez [235].

The formulation of weak type inequalities in Theorem 3.16 was first suggested in [86]. They were first proved in [42].

The discussion in Sect. 3.3 is an expanded version of [54]. Theorem 3.19, the necessity of the condition $p_- > 1$ for the maximal operator to be bounded, was first proved in [62] with the additional assumption that $p(\cdot)$ is upper semi-continuous. This hypothesis was removed by Diening [80] (see also [81]) and our proof follows his.

A version of the proof of Theorem 3.16 given in Sect. 3.4 first appeared in [56]. This proof draws upon ideas of Sawyer [325] from the theory of weighted norm inequalities. The second half of the proof when $p_+ = \infty$, where we use the two embedding theorems, is new though it is connected to the proof of Nekvinda [282]. Lemma 3.24 is due to Diening [77] and was a key part of his original proof of the boundedness of the maximal operator. Lemma 3.26, which is central to the application of the $LH_\infty$ condition, appeared in [56]; it is based on earlier versions found in [42, 58, 62, 63]. A variant of this inequality, with weaker hypotheses, was proved by Nekvinda [282]. Lemma 3.29, which lets us apply the Calderón-Zygmund decomposition to functions in $L^{p(\cdot)}$ when $p_+ < \infty$, is new. Lemma 3.30 is part of the "folklore" of harmonic analysis; for instance, it is implicit in [140]. A proof was given in [56].

Theorem 3.31 is due to Lerner [228]. This result highlights a significant difference between variable Lebesgue spaces and Orlicz spaces: in the latter both norm and modular inequalities are true for the maximal operator. See, for example, Kokilashvili and Krbec [197]. Versions of the pointwise inequalities in Theorem 3.32 and the modular strong type inequality in Theorem 3.33 were used in the original proofs of the boundedness of the maximal operator by Diening [77] and Nekvinda [282], and also in [42, 62]. A version of the weak type inequality in Theorem 3.33 for bounded domains is due to Harjulehto and Hästö [153]. A modular weak type inequality for the maximal operator similar to Theorem 3.34 was first proved in [62] with different hypotheses; see Sect. 3.7.2 below. The version given here is a generalization of a result due to Aguilar Cañestro and Ortega Salvador [8].

The interpolation result for positive integral operators in Theorem 3.36 was first proved in [55]. It is based on a result for classical Lebesgue spaces given by Bennett and Sharpley [25]. It is a special case of a more general interpolation theorem due to Musielak [274]: see Sect. 3.7.8 below. The convexity result in Theorem 3.38 was first proved for the case $p_+ < \infty$ using a different approach by Diening, Hästö and Nekvinda [86]. The full result was proved in [55]. The approximation of the maximal operator by positive integral operators used in the proof was implicit in Sawyer [325] and developed in detail by de la Torre [75].

### *3.7.2 More on Modular Inequalities*

The pointwise and strong type modular inequalities in Theorems 3.32 and 3.33 can be generalized to the Musielak-Orlicz spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$ defined in Sect. 2.10.2. This leads to proofs that the maximal operator is bounded on such spaces. See [138, 166, 267].

Diening [80, 82] showed that the $LH_0$ condition is necessary for Theorem 3.32 to hold. More precisely, he proved the following result.

**Proposition 3.39.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \le p_+ < \infty$, suppose that there exists $\beta > 0$ and $h(\cdot) \in L^\infty(\mathbb{R}^n)$ such that for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|f\|_{p(\cdot)} \le 1$, every cube $Q$ and every $x \in Q$,*

$$\left(\beta \fint_Q |f(y)|\,dy\right)^{p(x)} \leq \fint_Q |f(y)|^{p(y)}\,dy + h(x),$$

*and the same inequality holds with $p(\cdot)$ replaced by $p'(\cdot)$. Then $p(\cdot) \in LH_0(\mathbb{R}^n)$.*

The original version of the modular weak type inequality in Theorem 3.34 proved in [62] did not have a constant inside the modular, but removing it required a restriction on the exponent $p(\cdot)$. We include this proposition here because it was the first result to suggest a connection between variable Lebesgue spaces and the Muckenhoupt $A_p$ weights. We will explore this connection more closely in Chap. 4.

We begin with a definition. A non-negative function $u$ satisfies the $RH_\infty$ condition if there exists a constant $C$ such that for every cube $Q$ and almost every $x \in Q$,

$$u(x) \leq C \fint_Q u(y)\,dy.$$

Examples of such functions include $u(x) = |x|^r$, $r > 0$. (See [70].)

**Proposition 3.40.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $|\Omega_\infty| = 0$ and $1/p(\cdot) \in RH_\infty$, then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq C \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{t}\right)^{p(x)} dx.$$

### 3.7.3 L log L Inequalities in Variable Lebesgue Spaces

Even though the maximal operator does not map $L^1$ into $L^1$, there is an inequality which gives a sharp condition on $f$ for $Mf$ to be locally integrable. Wiener [347] (see also [143]) showed that given a locally integrable function $f$ and any cube $Q$,

$$\int_Q Mf(x)\,dx \leq 2|Q| + C \int_{\mathbb{R}^n} |f(x)| \log(e + |f(x)|)\,dx. \tag{3.30}$$

Stein [338] proved that the converse is also true: given a locally integrable function $f$ such that $\operatorname{supp}(f) \subset Q$ and $Mf \in L^1(Q)$, then

$$\int_Q |f(x)| \log(e + |f(x)|)\,dx < \infty. \tag{3.31}$$

For general exponents a modular inequality similar to (3.30) holds: given any $p(\cdot)$ such that $|\Omega_\infty| = 0$, then for any $\epsilon > 0$ there exists a constant $C$ such that for any cube $Q$,

$$\int_Q Mf(x)\,dx \le 2|Q| + C\int_{\mathbb{R}^n} |f(x)|^{p(x)} \log(e+|f(x)|)^{q(x)}\,dx,$$

where $q(x) = \max(\epsilon^{-1}(\epsilon + 1 - p(x)), 0)$. The key feature is that $q(x) = 1$ when $p(x) = 1$, and $\operatorname{supp}(q(\cdot)) \subset \{x : p(x) < 1+\epsilon\}$. This inequality was proved in [59].

Inequality (3.30) implies an Orlicz space inequality, and an analogous result holds in the scale of Musielak-Orlicz spaces when $p(\cdot)$ is log-Hölder continuous. Recall that in Sect. 2.10.2 we defined $L^{p(\cdot)}(\log L)^{q(\cdot)}$ to be the Musielak-Orlicz space generated by the function $\Phi(x,t) = t^{p(x)}\log(e+t)^{q(x)}$. Then for any $\epsilon > 0$, given a cube $Q$ and an exponent function $1/p(\cdot) \in LH_0(Q)$, there exists a continuous exponent function $q(\cdot)$ such that $q(x) = 1$ when $p(x) = 1$, $\operatorname{supp}(q(\cdot)) \subset \{x : p(x) < 1+\epsilon\}$, and

$$\|Mf\|_{L^{p(\cdot)}(Q)} \le C\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}}.$$

This embedding was proved by Diening *et al.* [81]. Previously, a somewhat weaker version, assuming that $p_+ < \infty$ and replacing the exponent $p(\cdot)$ on the right-hand side by an exponent $r(\cdot)$ that is slightly larger on the set $\{x : 1 < p(x) < 1+\epsilon\}$, was proved in [59]. The generalization of Stein's converse (3.31) is also considered in this paper.

A variation on this problem is to consider conditions on the exponent function $p(\cdot)$ that are sufficient for the embedding

$$\|Mf\|_{L^1(Q)} \le C\|f\|_{L^{p(\cdot)}(Q)}. \tag{3.32}$$

By (3.30) it suffices to find conditions on $p(\cdot)$ such that $L^{p(\cdot)}(Q)$ is contained in the Orlicz space $L\log L(Q)$. This question was first considered by Hästö [163], and later by Futamura and Mizuta [136], Mizuta, Ohno and Shimomura [266], and also in [59]. These results have been generalized to iterations of the maximal operator in [158]. For more details on this question, see Sect. 2.10.5 above.

### *3.7.4 The Fractional Maximal Operator*

Closely related to the Hardy-Littlewood maximal operator are the fractional maximal operators.

**Definition 3.41.** For each $\alpha$, $0 < \alpha < n$, given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ define the fractional maximal operator by

$$M_\alpha f(x) = \sup_{Q\ni x} |Q|^{\alpha/n} \fint_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ containing $x$ and whose sides are parallel to the coordinate axes.

The fractional maximal operator was introduced by Muckenhoupt and Wheeden [272]; it plays a role in estimates for Riesz potentials similar to the one that the Hardy-Littlewood maximal operator does for singular integrals.

The fractional maximal operator is not bounded on $L^p$; rather, it satisfies an "off-diagonal" estimate: for $1 < p \leq n/\alpha$, $M_\alpha : L^p \to L^q$, where

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}. \tag{3.33}$$

When $p = n/\alpha$ we define $q = \infty$. If $\alpha = 1$, $q$ equals the Sobolev exponent $p^* = \frac{np}{n-p}$. When $p = 1$ the fractional maximal operator satisfies a weak $(1, q)$ inequality:

$$|\{x \in \mathbb{R}^n : M_\alpha f(x) > t\}| \leq C \left( \frac{1}{t} \int_{\mathbb{R}^n} |f(x)|\, dx \right)^{\frac{n}{n-\alpha}}.$$

These inequalities can be proved in two different ways. First, the Calderón-Zygmund decomposition and the proof of Theorem 3.4 can be adapted to the fractional maximal operator. Marcinkiewicz interpolation is replaced by a more general, off-diagonal interpolation theorem (see Stein and Weiss [341]). Alternatively, norm inequalities for $M_\alpha$ can be proved using Theorem 3.4 and a pointwise inequality that is a consequence of Hölder's inequality:

$$M_\alpha f(x) \leq \|f\|_{L^p}^{\alpha/n} M f(x)^{p/q}. \tag{3.34}$$

(For a proof see [56].)

Theorem 3.16 can be extended to the fractional maximal operator.

**Theorem 3.42.** *Fix $\alpha$, $0 < \alpha < n$. Given a set $\Omega$, let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that $1/p(\cdot) \in LH(\Omega)$ and $p_+ \leq n/\alpha$. Define $q(\cdot)$ as in (3.33). Then for all $t > 0$,*

$$\|t\chi_{\{x : M_\alpha f(x) > t\}}\|_{L^{q(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{3.35}$$

*If in addition $p_- > 1$, then*

$$\|M_\alpha f\|_{L^{q(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}.$$

*In both inequalities the constant depends on the dimension $n$, the log-Hölder constants of $1/p(\cdot)$, $p_-$, $p_\infty$ and $\alpha$.*

The proof of Theorem 3.42 is essentially the same as the proof of Theorem 3.16; indeed the proof given of both in [56] was intended in part to show that a unified proof could be given for the Hardy-Littlewood and fractional maximal operators.

In Chap. 5 below (see Remark 5.51) we will sketch another proof of this result assuming $p_+ < n/\alpha$ using extrapolation. This case was first proved in [42]. The proof used Theorem 3.16 and a generalization of (3.34) to variable exponents.

**Proposition 3.43.** *Fix $\alpha$, $0 < \alpha < n$. Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_+ < n/\alpha$ and $p(\cdot) \in LH(\Omega)$, then for all $f \in L^{p(\cdot)}(\Omega)$, $\|f\|_{p(\cdot)} \leq 1$,*

$$M_\alpha f(x) \leq CMf(x)^{p(x)/q(x)} + CR(x)^{p_-/q_+},$$

*where $R(x) = (e + |x|)^{-N}$, $N > 0$, and the constant $C$ depends on $p(\cdot)$, $\alpha$ and $N$.*

In fact, a slightly weaker version of Proposition 3.43 was proved in [42] (see Propositions 3.1 and 3.2) but the version given here follows at once by adapting the ideas in the proof of Theorem 3.32.

Variable fractional maximal operators were considered by Kokilashvili and Samko [205] and Kokilashvili and Meskhi [198]. Given an exponent function $\alpha(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that $\alpha_- > 0$ and $\sup \alpha(x)p(x) < n$, they defined the operator

$$M_{\alpha(\cdot)} f(x) = \sup_{Q \ni x} |Q|^{\alpha(x)} ⨍_Q |f(y)|\, dy.$$

They proved weighted inequalities of the form

$$\|\Gamma(\cdot) M_{\alpha(\cdot)} f\|_{q(\cdot)} \leq C \|f\|_{p(\cdot)},$$

where $\Gamma(x) = (1 + |x|)^{\gamma(x)}$,

$$\gamma(x) = C_\infty \alpha(x) \left(1 - \frac{\alpha(x)}{n}\right),$$

and $C_\infty$ is the $LH_\infty$ constant of $p(\cdot)$.

### 3.7.5 Hardy Operators on Variable Lebesgue Spaces

The classical Hardy inequality states for that for $1 < p < \infty$, for any $f \in L^p([0, \infty))$,

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(y)\, dy \right|^p dx \leq (p')^p \int_0^\infty |f(x)|^p\, dx.$$

This inequality has a long history and many generalizations: see Kufner, Maligranda and Persson [224] and Opic and Kufner [289]. It can be restated in terms of the Hardy operator, the linear operator given by

$$Hf(x) = \frac{1}{x}\int_0^x f(y)\,dy$$

as

$$\|Hf\|_p \le p'\|f\|_p. \tag{3.36}$$

Since $|Hf(x)| \le Mf(x)$, (3.36) follows at once (though with a worse constant) from the $L^p$ inequalities for the maximal operator.

In a similar fashion we can prove norm inequalities for the Hardy operator on variable Lebesgue spaces. However, $H$ is bounded with assumptions on $p(\cdot)$ that are weaker than those that are needed for the maximal operator.

**Theorem 3.44.** *Given $p(\cdot) \in \mathcal{P}([0,\infty))$, suppose that $1 \le p_- \le p_+ < \infty$, $p(\cdot) \in LH_\infty([0,\infty))$, and $p(\cdot)$ is log-Hölder continuous at the origin:*

$$|p(x) - p(0)| \le \frac{C_0}{-\log(x)}, \qquad 0 < x < 1/2.$$

*Then $\|Hf\|_{L^{p(\cdot)}([0,\infty))} \le C\|f\|_{L^{p(\cdot)}([0,\infty))}$.*

Theorem 3.44 was proved by Diening and Samko [92] as a consequence of a more general result about integral operators with "nice" kernels. Their main results include variable exponent generalizations of $H$ and its adjoint $H^*$. Earlier, weaker versions of this result were proved by Harjulehto, Hästö and Koskenoja [154], Kokilashvili and Samko [208], and Mashiyev, Çekiç Mamedov and Ogras [258, 259]. Edmunds, Kokilashvili and Meskhi [98] also studied the compactness of these operators. Harman [159] has shown that a slightly weaker continuity condition at the origin is necessary for the Hardy operator to be bounded.

Hardy's inequality can also be generalized to higher dimensions. An elementary result appeared as part of the proof of the boundedness of the maximal operator in [62]. Higher dimensional results were studied in [154], and also by Samko [317, 318], Rafeiro and Samko [295, 296], and Mamedov and Harman [160, 247].

Like the maximal operator, the Hardy operator satisfies a modular inequality

$$\int_0^\infty |Hf(x)|^{p(x)}\,dx \le C\int_0^\infty |f(x)|^{p(x)}\,dx \tag{3.37}$$

only if $p(\cdot)$ is constant; this was proved by Sinnamon [333]. Somewhat surprisingly, however, if we restrict $f$ to non-negative, decreasing functions, then inequality (3.37) can hold for non-constant $p(\cdot)$. A characterization and examples were given by Boza and Soria [35]. Further results were proved by Neugebauer [286]. A modular inequality in higher dimensions, assuming $\|f\|_\infty \le 1$ and $p(\cdot)$ is a radial, increasing function, was proved in [62].

### 3.7.6 Other Maximal Operators

Several variants of the Hardy-Littlewood maximal operator occur in harmonic analysis and have been studied in the variable Lebesgue space setting. For instance, the maximal operator as originally defined by Hardy and Littlewood [148] was a "one-sided" maximal operator: for $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, define

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|\, dy.$$

Similarly, we define

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|\, dy.$$

One-sided maximal operators were studied by Sawyer [326] and then by Martín-Reyes, Ortega Salvador and de la Torre [255, 256]. (See also [52, 71].)

The one-sided maximal operators are bounded on variable Lebesgue spaces if the exponent $p(\cdot)$ satisfies one-sided log-Hölder continuity conditions. For simplicity we will consider $M^+$; the conditions for $M^-$ are analogous. We say $p(\cdot) \in LH_0^+(\mathbb{R})$ if

$$p(x) - p(y) \le \frac{C_0}{-\log(y-x)}, \qquad 0 < y - x < 1/2.$$

We say $p(\cdot) \in LH_\infty^+(\mathbb{R})$ if there exists a bounded, non-increasing function $q(\cdot) \in \mathcal{P}(\mathbb{R})$ such that

$$|p(x) - q(x)| \le \frac{C_\infty}{\log(e + |x|)}.$$

**Theorem 3.45.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R})$, suppose $1 < p_- \le p_+ < \infty$ and $p(\cdot) \in LH_0^+(\mathbb{R}) \cap LH_\infty^+(\mathbb{R})$. Then $\|M^+ f\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}$.*

Theorem 3.45 was proved by Nekvinda [285]. He actually has a somewhat more general result related to his condition for the Hardy-Littlewood maximal operator [282]. This result was proved by Edmunds, Kokilashvili and Meskhi [99] with the $LH_\infty^+$ condition replaced by the stronger condition that $p(\cdot)$ is constant outside of a large ball. Another, very different proof is due to Bernardis, Gogatishvili, Martín-Reyes, Ortega Salvador and Pick [28], who adapted the abstract Banach function space approach of Lerner and Pérez [235]. Modular weak type inequalities for one-sided maximal operators were proved by Aguilar Cañestro and Ortega Salvador [8].

In higher dimensions the strong maximal operator is defined by

$$M_S f(x) = \sup_{R \ni x} \fint_R |f(y)|\, dy,$$

where the supremum is taken over all rectangles (i.e., parallelepipeds) in $\mathbb{R}^n$ that contain $x$ and whose sides are parallel to the coordinate axes. Like the Hardy-Littlewood maximal operator, the strong maximal operator is bounded on $L^p(\mathbb{R}^n)$, $1 < p \leq \infty$. (See, for instance, [143].) However, Kopaliani [213] proved that the strong maximal operator is never bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ unless $p(\cdot)$ is constant. Kokilashvili and Meskhi [200] generalized this to the strong fractional maximal operator.

To understand this result, it is worth considering the proof in the classical Lebesgue setting. For clarity we consider only the case $\mathbb{R}^2$. Let $M_x$ denote the maximal operator acting on the first variable:

$$M_x f(x_0, y_0) = \sup_{I \ni x_0} \fint_I |f(x, y_0)|\, dx,$$

where the supremum is taken over all intervals $I$ that contain $x_0$. Define $M_y$, the maximal operator acting on the second variable, similarly. Then it is straightforward to show that

$$M_S f(x_0, y_0) \leq (M_x \circ M_y) f(x_0, y_0).$$

Since the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbb{R})$, $M_x$ and $M_y$ are bounded on $L^p(\mathbb{R}^2)$, and so $M_S$ is as well. However, $M_x$ and $M_y$ need not be bounded on $L^{p(\cdot)}(\mathbb{R}^2)$. The following counter-example is adapted from Nägele [276]. Let $\Omega = [0,1]\times[0,1]$, $\Omega_1 = [0,1/3]\times[0,1]$ and $\Omega_2 = [2/3,1]\times[0,1]$. Let $p(\cdot) \in \mathcal{P}(\Omega)$ be any exponent such that $p(x,y) = 2$, $(x,y) \in \Omega_1$ and $p(x,y) = 3$, $(x,y) \in \Omega_2$. Let $g \in L^2([0,1]) \setminus L^3([0,1])$, and define $f(x,y) = g(y)\chi_{\Omega_1}(x,y)$. Then

$$\int_\Omega f(x,y)^{p(x,y)}\, dxdy = \frac{1}{3}\int_0^1 g(y)^2\, dy < \infty,$$

so by Proposition 2.12, $f \in L^{p(\cdot)}(\Omega)$. On the other hand, for $x_0 \in (1/3, 1)$,

$$M_x f(x_0, y_0) \geq \int_0^1 |f(x, y_0)|\, dx = \int_0^{1/3} |g(y_0)|\, dx = \frac{1}{3}|g(y_0)|.$$

Therefore, if $M_x$ were bounded on $L^{p(\cdot)}(\Omega)$ we would have (again by Proposition 2.12) that

$$\int_0^1 |g(y)|^3\, dy \leq C \int_{\Omega_2} |f(x,y)|^{p(x,y)}\, dxdy \leq C \int_\Omega |M_x f(x,y)|^{p(x,y)}\, dxdy < \infty,$$

which contradicts our choice of $g$.

### 3.7.7 Decreasing Rearrangements

As we showed in Example 3.14, $L^{p(\cdot)}(\Omega)$ is not rearrangement invariant. Therefore, the decreasing rearrangement of $f$—that is,

$$f^*(t) = \inf\{s : \mu_f(s) \le t\}$$

(where the infimum of the empty set is defined to be $\infty$, and $\mu_f$ is the distribution function, see p. 87), the decreasing function on $[0, |\Omega|)$ that is equimeasurable with $f$—does not play the same role in the study $L^{p(\cdot)}(\Omega)$ as it does for classical Lebesgue spaces and other rearrangement invariant spaces. (For more on the theory of decreasing rearrangements see [25, 223, 297].)

Nevertheless, the decreasing rearrangement is still applicable. Given a measurable function $f$ with decreasing rearrangement $f^*$, let

$$f^{**}(t) = \fint_0^t f^*(s)\,ds = M(f^*)(t) = H(f^*)(t),$$

where $H$ is the Hardy operator (see Sect. 3.7.5). Kokilashvili and Samko [206] defined a rearrangement invariant version of $L^{p(\cdot)}$. Given $\Omega$ and $p(\cdot) \in \mathcal{P}([0, |\Omega|))$, $p_+ < \infty$, the space $\Lambda^{p(\cdot)}(\Omega)$ consists of all measurable functions $f$ on $\Omega$ such that

$$\|f\|_{\Lambda^{p(\cdot)}(\Omega)} = \|f^{**}\|_{L^{p(\cdot)}([0,|\Omega|))} < \infty.$$

This is a Banach function space; if $p(\cdot) = p > 1$, then, since the maximal operator is bounded on $L^p$ and $f^{**}(t) \approx (Mf)^*(t)$ (see [25]), $\Lambda^p(\Omega)$ equals $L^p(\Omega)$ with an equivalent norm.

Ephremidze, Kokilashvili and Samko [108] used the same ideas to generalize the classical Lorentz $L^{p,q}$ spaces. Given $\Omega$ and $p(\cdot), q(\cdot) \in \mathcal{P}([0, |\Omega|))$, $p_+, q_+ < \infty$, the space $L^{p(\cdot),q(\cdot)}(\Omega)$ consists of all measurable functions $f$ on $\Omega$ such that

$$\|f\|_{L^{p(\cdot),q(\cdot)}}(\Omega) = \|t^{1/p(t)-1/q(t)} f^*(t)\|_{L^{q(\cdot)}([0,|\Omega|))} < \infty.$$

If $p(\cdot)$ and $q(\cdot)$ are constant, then this becomes the classical Lorentz space $L^{p,q}(\Omega)$; if $p(\cdot) = q(\cdot)$ and the maximal operator is bounded on $L^{p(\cdot)}([0, |\Omega|))$, then we have that $L^{p(\cdot),p(\cdot)}(\Omega) = \Lambda^{p(\cdot)}(\Omega)$.

The decreasing rearrangement has also been used to describe properties of $L^{p(\cdot)}(\Omega)$. Let $\Omega$ be bounded, and let $L_a^{p(\cdot)}(\Omega)$ denote the collection of functions in $L^{p(\cdot)}(\Omega)$ with absolutely continuous norm (see Sect. 2.10.3). Edmunds, Lang and Nekvinda [101] showed that $L_a^{p(\cdot)}(\Omega)$ equals the closure of the set of bounded functions in $L^{p(\cdot)}(\Omega)$ if and only if for all $A > 1$,

$$\int_0^{|\Omega|} A^{p^*(t)}\,dt < \infty.$$

It is also possible to estimate the $L^{p(\cdot)}$ norm of $f$ in terms of $f^*$. Given $\Omega$ bounded and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $|\Omega_\infty| = 0$, define the increasing rearrangement of $p(\cdot)$ to be the increasing function $p^\dagger(t) = p^*(|\Omega| - t)$, $t \in [0, |\Omega|)$. Then it was shown in [131] that for all $f \in L^{p(\cdot)}(\Omega)$,

$$\frac{1}{2(1+|\Omega|)}\|f^*\|_{L^{p^\dagger(\cdot)}([0,|\Omega|))} \leq \|f\|_{L^{p(\cdot)}(\Omega)} \leq 2(1+|\Omega|)\|f^*\|_{L^{p^*(\cdot)}([0,|\Omega|))}.$$

In the same paper there are also examples of $f \in L^{p(\cdot)}(\Omega)$ such that the right-hand term is infinite.

Rakotoson [298] has shown that decreasing rearrangements are a Lipschitz functional on the variable Lebesgue spaces. More precisely, given a bounded set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p^\dagger(\cdot) \in LH_0((0,\epsilon))$ for some $\epsilon > 0$, then for all $f, g \in L^{p(\cdot)}(\Omega)$,

$$\||f^* - g^*|^*\|_{L^{p^\dagger(\cdot)}((0,|\Omega|))} \leq C\|f - g\|_{L^{p(\cdot)}(\Omega)}.$$

This extends to the variable Lebesgue spaces results of Chiti [47] and Sakai [308].

### *3.7.8 Real and Complex Interpolation*

As we remarked after the proof of Theorem 3.4, at the heart of the classical $L^p$ norm inequalities for the maximal operator is the Marcinkiewicz interpolation theorem, which allows us to pass from weak type to strong type inequalities.

**Proposition 3.46.** *Given a set $\Omega$ and a non-negative measure $\mu$, suppose the sublinear operator $T$ satisfies the weak $(p_i, p_i)$ inequalities,*

$$\mu(\{x \in \Omega : |Tf(x)| > t\}) \leq \frac{M_i}{t^{p_i}} \int_\Omega |f(x)|^{p_i}\, d\mu, \quad i = 1, 2,$$

*where $1 \leq p_1 < p_2 < \infty$, or if $p_2 = \infty$, satisfies $\|Tf\|_{L^\infty(\Omega,\mu)} \leq M_2\|f\|_{L^\infty(\Omega,\mu)}$. Then for all $p$, $p_1 < p < p_2$,*

$$\left(\int_\Omega |Tf(x)|^p\, d\mu\right)^{1/p} \leq C\left(\int_\Omega |f(x)|^p\, d\mu\right)^{1/p},$$

*where $C$ depends on $p$, $p_1$, $p_2$, $M_1$ and $M_2$.*

For a proof, see Duoandikoetxea [96] or Grafakos [143]. For the history of this result and its generalizations, see Maligranda [246].

It is an open question whether a version of Marcinkiewicz interpolation is true in the variable Lebesgue spaces. More precisely, suppose the sublinear operator $T$ satisfies

$$\|t\chi_{\{x:Mf(x)>t\}}\|_{L^{p_i(\cdot)}(\Omega)} \leq C\|f\|_{L^{p_i(\cdot)}(\Omega)}, \qquad i = 1, 2;$$

then is it true that

$$\|Tf\|_{L^{p_\theta(\cdot)}(\Omega)} \leq C\|f\|_{L^{p_\theta(\cdot)}(\Omega)},$$

where $p_\theta(\cdot)$ is defined by

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)} \tag{3.38}$$

for any $\theta$, $0 < \theta < 1$?

This question was first posed by Diening, Hästö and Nekvinda [86]. The proof of the Marcinkiewicz interpolation theorem and its generalizations (the so-called real interpolation methods—see Bennett and Sharpley [25]) depend in an essential way on the fact that the classical Lebesgue spaces are rearrangement invariant, and so cannot be extended to the variable Lebesgue spaces.

On the other hand, the Riesz-Thorin interpolation theorem (i.e., the complex interpolation method) extends naturally to the variable Lebesgue spaces.

**Theorem 3.47.** *Given $\Omega$ and $p_i(\cdot)$, $q_i(\cdot) \in \mathcal{P}(\Omega)$, $i = 1, 2$, suppose $T$ is a linear operator such that*

$$\|Tf\|_{L^{q_i(\cdot)}(\Omega)} \leq M_i\|f\|_{L^{p_i(\cdot)}(\Omega)}, \quad i = 1, 2.$$

*Then for each $\theta$, $0 < \theta < 1$,*

$$\|Tf\|_{L^{q_\theta(\cdot)}(\Omega)} \leq CM_1^\theta M_2^{1-\theta}\|f\|_{L^{p_\theta(\cdot)}(\Omega)},$$

*where $p_\theta(\cdot)$ is defined by (3.38) and $q_\theta(\cdot)$ is defined similarly.*

Theorem 3.47 was proved in [82] when $p_i(\cdot) = q_i(\cdot)$ but the more general result is proved in exactly the same way. The special case when $p_i(\cdot) = q_i(\cdot)$ and $(p_i)_+ < \infty$ was proved in [86]. Earlier, Musielak [274] gave a proof in the general setting of Musielak-Orlicz spaces. Karlovich and Lerner [195] proved it in the special case $p_1(\cdot) = p(\cdot)$, $p_2(\cdot) = p'(\cdot)$ and $\theta = 1/2$ (i.e., $p_\theta(\cdot) = 2$). Kopaliani [216] generalized this result by showing that one of the endpoint spaces could be replaced by $BMO$, the space of functions of bounded mean oscillation, or $H^1$, the real Hardy space. Theorem 3.36 is a special case of Theorem 3.47 for positive integral operators.

# Chapter 4
# Beyond Log-Hölder Continuity

In this chapter we continue our study of the Hardy-Littlewood maximal operator. In Chap. 3 we showed that the log Hölder continuity conditions $LH_0$ and $LH_\infty$ are sufficient for the maximal operator to be bounded. In this chapter we will show that they are not necessary, even though they are the best possible pointwise decay conditions. To find weaker sufficient conditions we build upon the proof of Theorem 3.16, which showed that $LH_0$ and $LH_\infty$ play distinct roles. Intuitively, the $LH_0$ condition controls the behavior of the maximal operator locally (where a function is large) and the $LH_\infty$ condition controls it at infinity (where a function is small). We will, therefore, study replacements for each condition separately. We first consider $LH_\infty$: here the proof of Theorem 3.16 itself, when examined closely, quickly yields a weaker sufficient condition. However, we will also construct an example to show that it is still not necessary. The problem of finding a necessary and sufficient condition to replace $LH_\infty$ in our proof remains open.

We are able to give a replacement for the $LH_0$ condition, but new techniques are required. Recall that in the proof of Theorem 3.16 we used it to apply Lemma 3.24, but this geometric property of an exponent $p(\cdot)$ is equivalent to $p(\cdot) \in LH_0$. Our new condition is necessary and sufficient for the maximal operator to be bounded on bounded domains. The proof of this result is quite technical; surprisingly, it requires machinery from the theory of Muckenhoupt $A_p$ weights, and we introduce this useful tool before we discuss controlling the maximal operator locally. These results suggest that there is a deep and subtle connection between weighted and variable Lebesgue spaces that needs to be explored.

We will conclude this chapter with a discussion of a very deep theoretical result that gives a necessary and sufficient condition for the maximal operator to be bounded on a variable Lebesgue space. This condition, which simultaneously replaces both $LH_0$ and $LH_\infty$, is difficult to check in practice, but it has several important theoretical consequences.

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## 4.1 Control at Infinity: The $N_\infty$ Condition

In this section we consider the behavior of the maximal operator at infinity, where the function $f$ is small. We first show that $LH_\infty$ is the best possible pointwise decay condition: if we replace $\log(e+|x|)^{-1}$ with any function that is substantially larger, the maximal operator need not be bounded.

To understand the following example, recall Example 3.23 which gave an exponent $p(\cdot)$ that oscillates at infinity such that the maximal operator is unbounded. Moreover, if $p(\cdot)$ has a "jump discontinuity" at infinity, then the maximal operator will also be unbounded: for instance, Example 3.21 can be modified to show that if $p(\cdot) \in \mathcal{P}(\mathbb{R})$ is such that for some $K > 0$, $p(x) = 2$ for $x < -K$ and $p(x) = 3$ for $x > K$, then $M$ is not bounded on $L^{p(\cdot)}(\mathbb{R})$. But even when $p(\cdot)$ is uniformly continuous at infinity, if its modulus of continuity is too large, then the maximal operator may be unbounded.

*Example 4.1.* Fix $p_\infty$, $1 < p_\infty < \infty$, and let $\phi : [0,\infty) \to [0,1)$ be such that $\phi(0+) = 0$, $\phi_+ < p_\infty - 1$, $\phi$ is decreasing on $[1,\infty)$, $\phi(x) \to 0$ as $x \to \infty$, and

$$\lim_{x\to\infty} \phi(x)\log(x) = \infty. \tag{4.1}$$

Define $p(\cdot) \in \mathcal{P}(\mathbb{R})$ by

$$p(x) = \begin{cases} p_\infty & x \le 0 \\ p_\infty - \phi(x) & x > 0. \end{cases}$$

Then $p(\cdot) \notin LH_\infty(\mathbb{R})$ and the maximal operator is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

*Remark 4.2.* The assumption that $\phi(0+) = 0$ is included only to preclude the possibility that $p(\cdot)$ has a jump discontinuity at the origin, thus preventing the maximal operator from being bounded (see Example 3.21). It otherwise plays no role in the construction.

*Proof.* It is immediate from (4.1) that $p(\cdot)$ does not satisfy the $LH_\infty(\mathbb{R})$ condition, so we only have to construct a function $f$ such that $f \in L^{p(\cdot)}(\mathbb{R})$ but $Mf \notin L^{p(\cdot)}(\mathbb{R})$. By inequality (4.1) we have that

$$\lim_{x\to\infty} \left(1 - \frac{p_\infty}{p(2x)}\right)\log(x) = -\infty,$$

which in turn implies that

$$\lim_{x\to\infty} x^{1-p_\infty/p(2x)} = 0.$$

Hence, we can form a sequence $\{c_n\} \subset (-\infty,-1)$ such that $c_{n+1} < 2c_n$ and

$$|c_n|^{1-p_\infty/p(2|c_n|)} \leq 2^{-n}.$$

Let $d_n = 2c_n$ and define the function $f$ by

$$f(x) = \sum_{n=1}^{\infty} |c_n|^{-1/p(|d_n|)} \chi_{(d_n,c_n)}(x).$$

Since $p_+ < \infty$, by Proposition 2.12 it will suffice to show that $\rho(f) < \infty$ and $\rho(Mf) = \infty$. First,

$$\begin{aligned}\rho(f) = \sum_{n=1}^{\infty} \int_{d_n}^{c_n} |c_n|^{-p(x)/p(|d_n|)}\,dx &= \sum_{n=1}^{\infty} \int_{d_n}^{c_n} |c_n|^{-p_\infty/p(|d_n|)}\,dx \\ &= \sum_{n=1}^{\infty} |c_n|^{1-p_\infty/p(|d_n|)} \leq \sum_{n=1}^{\infty} 2^{-n} = 1.\end{aligned}$$

On the other hand, if $x \in (|c_n|, |d_n|)$, then

$$Mf(x) \geq \frac{1}{2|d_n|} \int_{d_n}^{|d_n|} f(y)\,dy \geq \frac{1}{2|d_n|} \int_{d_n}^{c_n} |c_n|^{-1/p(|d_n|)}\,dy = \frac{1}{4}|c_n|^{-1/p(|d_n|)}.$$

Therefore, since $p(\cdot)$ is an increasing function on $(1, \infty)$ and $|c_n| \geq 1$,

$$\begin{aligned}\rho(Mf) &\geq \left(\frac{1}{4}\right)^{p_+} \sum_{n=1}^{\infty} \int_{|c_n|}^{|d_n|} |c_n|^{-p(x)/p(|d_n|)}\,dx \\ &\geq \left(\frac{1}{4}\right)^{p_+} \sum_{n=1}^{\infty} \int_{|c_n|}^{|d_n|} |c_n|^{-p(|d_n|)/p(|d_n|)}\,dx = \left(\frac{1}{4}\right)^{p_+} \sum_{n=1}^{\infty} 1 = \infty.\end{aligned}$$

□

*Example 4.3.* A family of functions that satisfy the hypotheses of Example 4.1 is

$$p(x) = \begin{cases} p_0 & x \in (-\infty, 0] \\ p_0 - \frac{x}{\log(e+1)^a} & x \in (0, 1) \\ p_0 - \frac{1}{\log(e+x)^a} & x \in [1, \infty), \end{cases}$$

where $p_0 > 2$ and $0 < a < 1$.

While Example 4.1 shows that $LH_\infty$ is sharp in terms of pointwise decay at infinity, the proof of Theorem 3.16 yields a weaker sufficient condition that controls, in some sense, the average rate of decay. In this proof (see p. 104) the

$LH_\infty$ condition was used to estimate the maximal operator of $f_2 = f\chi_{\{x:f(x)\le 1\}}$ by applying Proposition 2.43. More precisely, we needed the two embeddings $L^\infty(E) \subset L^{r(\cdot)}(E)$ and $L^\infty(F) \subset L^{s(\cdot)}(F)$, where

$$E = \{x \in \Omega : p(x) \ge p_\infty\}, \quad F = \{x \in \Omega : p(x) < p_\infty\},$$

and $r(\cdot)$ and $s(\cdot)$ are defined on $E$ and $F$ by

$$\frac{1}{p_\infty} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad \frac{1}{p(x)} = \frac{1}{p_\infty} + \frac{1}{s(x)}.$$

(Note that in this argument we had for simplicity extended $p(\cdot)$ to an exponent in $LH_\infty(\mathbb{R}^n)$ using Lemma 2.4; however, the proof does not require this extension and goes through if we everywhere replace $\mathbb{R}^n$ with $\Omega$.) By Proposition 2.43, a necessary and sufficient condition for both of these embeddings to hold is that the integrals

$$\int_{\{x\in E: r(x)<\infty\}} \lambda^{-r(x)}\, dx \quad \text{and} \quad \int_{\{x\in F: s(x)<\infty\}} \lambda^{-s(x)}\, dx$$

are both finite for some $\lambda > 1$. If we combine these two integrals we get the following definition and the above argument yields a generalization of Theorem 3.16.

**Definition 4.4.** Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, we say that $p(\cdot) \in N_\infty(\Omega)$ if there exist constants $\Lambda_\infty > 0$ and $p_\infty \in [1,\infty]$ such that

$$\int_{\Omega_+} \exp\left(-\Lambda_\infty \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|^{-1}\right) dx < \infty,$$

where

$$\Omega_+ = \left\{x \in \Omega : \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| > 0\right\}.$$

*Remark 4.5.* If $p_+ < \infty$, then we can rewrite the $N_\infty$ condition in a somewhat simpler form:

$$\int_{\Omega_+} \exp\big(-\Lambda_\infty |p(x) - p_\infty|^{-1}\big)\, dx < \infty,$$

where $\Omega_+ = \{x \in \Omega : |p(x) - p_\infty| > 0\}$. This follows at once by an argument essentially the same as the proof in Proposition 2.3 that if $p_+ < \infty$, then $p(\cdot) \in LH_\infty$ if and only if $1/p(\cdot) \in LH_\infty$. Further, if $\Omega$ is bounded, then the $N_\infty$ condition holds trivially (just like the $LH_\infty$ condition), since for every $\Lambda_\infty > 0$ the integral is bounded by $|\Omega_+|$.

*Remark 4.6.* If $p(\cdot) \in N_\infty(\Omega)$, then it follows at once from Definition 4.4 that $p'(\cdot) \in N_\infty(\Omega)$.

**Theorem 4.7.** *Given a set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $1/p(\cdot) \in LH_0(\Omega)$ and $p(\cdot) \in N_\infty(\Omega)$, then*

$$\|t\chi_{\{x:Mf(x)>t\}}\|_{L^{p(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{4.2}$$

*Furthermore, if $p_- > 1$, then for all $f \in L^{p(\cdot)}(\Omega)$,*

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}.$$

The proof of Theorem 4.7 can be adapted to prove a perturbation result that complements the convexity result in Theorem 3.38.

**Proposition 4.8.** *Given a set $\Omega$ and $p(\cdot) \in \Omega$ such that $p_- > 1$ and $1/p(\cdot) \in LH_0(\Omega)$, suppose there exists $q(\cdot) \in \mathcal{P}(\Omega)$ such that for some $\Lambda_\infty > 0$,*

$$\int_{\Omega_+} \exp\left(-\Lambda_\infty \left|\frac{1}{p(x)} - \frac{1}{q(x)}\right|^{-1}\right) dx < \infty, \tag{4.3}$$

*where $\Omega_+ = \{x \in \Omega : |p(x)^{-1} - q(x)^{-1}| > 0\}$. If the maximal operator is bounded on $L^{q(\cdot)}(\Omega)$, then it is bounded on $L^{p(\cdot)}(\Omega)$. Similarly, if $p_- = 1$ and $M$ satisfies the weak type inequality (4.2) on $L^{q(\cdot)}(\Omega)$, then it also satisfies it on $L^{p(\cdot)}(\Omega)$.*

*Proof.* The argument is essentially the same as the proof of Theorem 4.7 (and so the proof of Theorem 3.16). Decomposing $f$ as $f_1 + f_2$, the argument for $f_1$ goes through without change. To estimate $f_2$ we repeat the argument given above for Theorem 4.7, replacing $p_\infty$ by $q(\cdot)$; then given our hypothesis (4.3) we can prove the necessary embeddings of $L^\infty$; the rest of the proof then follows without change. □

We now consider the relationship between $LH_\infty$ and $N_\infty$.

**Proposition 4.9.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $1/p(\cdot) \in LH_\infty(\Omega)$, then $p(\cdot) \in N_\infty(\Omega)$. However, there exists $p(\cdot) \in \mathcal{P}(\mathbb{R})$, $p_+ < \infty$, such that $p(\cdot) \in LH_0(\mathbb{R}) \cap N_\infty(\mathbb{R})$ but $p(\cdot) \notin LH_\infty(\mathbb{R})$.*

*Proof.* If $1/p(\cdot) \in LH_\infty(\Omega)$, then it follows immediately from the definitions that $p(\cdot) \in N_\infty(\Omega)$:

$$\int_{\Omega_+} \exp\left(-\Lambda_\infty \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|^{-1}\right) dx \leq \int_{\mathbb{R}^n} \exp\left(\frac{-\Lambda_\infty}{C_\infty} \log(e + |x|)\right) dx < \infty.$$

The last inequality holds for all $\Lambda_\infty > nC_\infty$.

To show that $N_\infty(\mathbb{R})$ is not contained in $LH_\infty(\mathbb{R})$, fix $p_\infty > 1$ and define $\phi(\cdot)$ by

$$\phi(x) = \begin{cases} \frac{1}{k} - |e^{k^2} - x| & 0 \leq |e^{k^2} - x| \leq \frac{1}{k}, \quad 1 \leq k < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $p(x) = p_\infty + \phi(x)$; then

$$|p(e^{k^2}) - p_\infty| = \phi(e^{k^2}) = \frac{1}{k} \approx \frac{k}{\log(e + e^{k^2})},$$

so $p(\cdot) \notin LH_\infty(\mathbb{R})$. On the other hand, it follows immediately from the definition that $\phi(\cdot)$ is Lipschitz, so $p(\cdot) \in LH_0(\mathbb{R})$. Furthermore,

$$\begin{aligned}\int_{\Omega_+} \exp\left(-|p(x) - p_\infty|^{-1}\right) dx &= \sum_{k=1}^\infty \int_{\{|e^{k^2}-x|<1/k\}} e^{-1/\phi(x)}\, dx \\ &= 2\sum_{k=1}^\infty \int_0^{1/k} e^{-1/y}\, dy \le \sum_{k=1}^\infty \frac{2}{ke^k} < \infty.\end{aligned}$$

Hence, by Remark 4.5, $p(\cdot) \in N_\infty(\mathbb{R})$. □

If $p(\cdot) \in N_\infty(\Omega)$, then $p(\cdot)$ has a limit (in a weak sense) at infinity.

**Proposition 4.10.** *Given an unbounded set* $\Omega$ *and* $p(\cdot) \in \mathcal{P}(\Omega)$, *suppose* $p(\cdot) \in N_\infty$. *Then:*

1. $1/p(\cdot)$ *converges to* $1/p_\infty$ *at infinity in the sense that if we define*

$$\pi_k(x) = \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \chi_{\Omega\setminus B_k(0)}(x),$$

   *then* $\pi_k \to 0$ *in measure as* $k \to \infty$.
2. *If* $1/p(\cdot)$ *is uniformly continuous (e.g., if* $1/p(\cdot) \in LH_0(\Omega)$*), then* $1/p(x) \to 1/p_\infty$ *as* $|x| \to \infty$.

*Remark 4.11.* The conclusion in (1) is equivalent to

$$\lim_{R\to\infty} \left|\left\{x \in \Omega \setminus B_R(0) : \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| > \epsilon\right\}\right| = 0,$$

and the proof can be readily modified to show this directly.

*Remark 4.12.* It is not necessary for a limit at infinity to exist, even in the weak sense of Proposition 4.10, for the maximal operator to be bounded. See Sect. 4.6.3 below.

*Proof.* To prove (1), fix $\epsilon > 0$. Since $p(\cdot) \in N_\infty(\Omega)$, for all $k > 0$ sufficiently large,

$$\int_{\{x\in\Omega_+:|x|>k\}} \exp\left(-\Lambda_\infty\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|^{-1}\right) dx < \epsilon\, e^{-\epsilon^{-1}\Lambda_\infty}.$$

Thus,

$$\left|\left\{x \in \Omega_+ \setminus B_k(0) : \exp\left(-\Lambda_\infty\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|^{-1}\right) \geq e^{-\epsilon^{-1}\Lambda_\infty}\right\}\right| < \epsilon,$$

and this immediately implies that

$$|\{x \in \Omega : |\pi_k(x)| \geq \epsilon\}| < \epsilon.$$

Hence, $\pi_k \to 0$ in measure.

To prove (2), suppose to the contrary that there exists $p(\cdot) \in N_\infty$ such that $1/p(\cdot)$ is uniformly continuous, but there exists $\epsilon > 0$ and a sequence $\{x_k\}$ such that $x_k \to \infty$ while $|1/p(x_k) - 1/p_\infty| > \epsilon$. By passing to a subsequence we may assume that if $j \neq k$, $|x_j - x_k| > 1$. Since $1/p(\cdot)$ is uniformly continuous there exists $\delta$, $0 < \delta < 1$, such that for every $k$, if $|x_k - x| < \delta$, then $|1/p(x) - 1/p_\infty| > \epsilon/2$. But this contradicts (1) and so we must have that $1/p(\cdot)$ converges to $1/p_\infty$ pointwise as $|x| \to \infty$. □

Since the condition in Proposition 2.43 is necessary and sufficient for the embedding of $L^\infty$ into $L^{p(\cdot)}$, $N_\infty$ is the weakest condition we can use in the proof of Theorem 3.16 to control the maximal operator at infinity. This is analogous to the fact that $LH_0$ is the weakest condition we can use in this proof to control the maximal operator locally. It remains an open problem to find a weaker condition on an exponent $p(\cdot)$ and a generalization of this proof lets us show that

$$\|Mf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$$

for all $f$ such that $|f| \leq 1$. (For a general necessary and sufficient condition, see Theorem 4.63 below.) The problem of finding such a condition is interesting since, as the next example shows, there exist exponents $p(\cdot)$ that do not satisfy the $N_\infty$ condition but $M$ is bounded on $L^{p(\cdot)}$.

*Example 4.13.* On the real line, given $p_0 > 1$ and $0 < a < 1$, define

$$p(x) = p_0 + \frac{1}{\log(e + |x|)^a}.$$

Then $p(\cdot) \notin N_\infty$ but $M$ is bounded on $L^{p(\cdot)}(\mathbb{R})$.

*Remark 4.14.* Compare these exponent functions to the ones in Example 4.3. There, we needed both the larger modulus of continuity and the asymmetry of $p(\cdot)$ to show that the maximal operator is unbounded. In the proof of Example 4.13 we need both the radial symmetry and the monotonicity of the exponents. While the interplay of these two hypotheses in the construction of the example is clear, their deeper significance is not known.

The construction requires a lemma which is a generalization of Lemma 3.26 and which is proved in exactly the same way.

**Lemma 4.15.** *Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $p_+, q_+ < \infty$ and*

$$|p(x) - q(x)| \le \frac{C_\infty}{\log(e + |x|)}. \tag{4.4}$$

*Then given any function $F$ with $0 \le F(y) \le 1$ and any $N > n$, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} F(y)^{p(y)}\,dy \le C \int_{\mathbb{R}^n} F(y)^{q(y)}\,dy + C \int_{\mathbb{R}^n} (e + |y|)^{-Nq(y)}\,dy.$$

*The same inequality also holds with the roles of $p(\cdot)$ and $q(\cdot)$ reversed.*

*Proof of Example 4.13.* We will follow the proof of Theorem 3.16 in the case $p_+ < \infty$. Fix a function $f$ and write $f = f_1 + f_2$ as before. Since $p(\cdot)$ is Lipschitz, $p(\cdot) \in LH_0(\mathbb{R})$, and so the estimate for $f_1$ proceeds as before.

The heart of the proof is the estimate for $f_2$. To simplify notation we will write simply $f$ instead of $f_2$. Then we have that $0 \le f(x) \le 1$ and

$$\rho(f) = \int_{\mathbb{R}} f(x)^{p(x)}\,dx \le 1,$$

and we need to show that there is a constant $C$ such that

$$\int_{\mathbb{R}} Mf(x)^{p(x)}\,dx \le C.$$

Our first step is to replace $p(\cdot)$ with a discrete exponent function $q(\cdot)$. For $x \ge 1$, let $\alpha(x) = x\log(x^\beta) = \beta x \log(x)$, where we fix $\beta > 1$ so that

$$1 < \frac{\beta + 1}{\beta - 1} < p_0. \tag{4.5}$$

Define the set

$$E_0 = (-e^{\alpha(1)}, e^{\alpha(1)}) = (-1, 1),$$

and for $k \ge 1$ define

$$E_k = (-e^{\alpha(k+1)}, e^{\alpha(k+1)}) \setminus E_{k-1}.$$

Note that for all $k \ge 1$,

$$|E_k| = 2(e^{\alpha(k+1)} - e^{\alpha(k)}). \tag{4.6}$$

Now let $q_0 = q_1 = 1$, and for $k \ge 2$ let

$$q_k = p_0 + \frac{1}{\alpha(k-1)^a} = p_0 + \frac{1}{(\beta(k-1)\log(k-1))^a}.$$

Define $q(\cdot) \in \mathcal{P}(\mathbb{R})$ by

$$q(x) = \sum_{k=0}^{\infty} q_k \chi_{E_k}(x).$$

Since $\alpha(k-1) < \alpha(k) < \log(e + e^{\alpha(k)})$, for all $k \geq 1$, $q_k > p(e^{\alpha(k)})$, and therefore, since $p(\cdot)$ is a decreasing function, $q(x) \geq p(x)$ for all $x \in \mathbb{R}$.

To replace $p(\cdot)$ by $q(\cdot)$ we will apply Lemma 4.15. Since $p(\cdot)$ and $q(\cdot)$ are bounded, if $x \in E_0 \cup E_1$, (4.4) clearly holds. Now fix $x \in E_k$, $k \geq 2$. Then

$$\begin{aligned} q(x) - p(x) = &\left( \frac{1}{(\beta(k-1)\log(k-1))^a} - \frac{1}{(\log(|x|))^a} \right) \\ &+ \left( \frac{1}{(\log(|x|))^a} - \frac{1}{\log(e+|x|)^a} \right). \end{aligned}$$

It follows immediately from the mean value theorem that the second term is bounded by $C\log(e + |x|)^{-1}$, so we only need to estimate the first term. Since $x \in E_k$, by the monotonicity of the logarithm and again by the mean value theorem,

$$\begin{aligned} &\frac{1}{(\beta(k-1)\log(k-1))^a} - \frac{1}{(\log(|x|))^a} \\ &\quad \leq \frac{1}{(\beta(k-1)\log(k-1))^a} - \frac{1}{(\beta(k+1)\log(k+1))^a} \\ &\quad \leq \frac{2a\beta(\log(k-1)+1)}{(\beta(k-1)\log(k-1))^{a+1}} \leq \frac{C}{(k-1)^{1+a}} \leq \frac{C}{\log(e+|x|)}. \end{aligned} \tag{4.7}$$

The last inequality holds since if $x \in E_k$,

$$9(k-1)^{1+a} \geq (k+1)^{1+a} \geq c(a,\beta)\log(e+|x|).$$

Thus (4.4) holds for all $x$. Since $0 \leq f \leq 1$, by Proposition 3.3, $Mf \leq 1$. Hence, by Lemma 4.15,

$$\int_{\mathbb{R}} Mf(x)^{p(x)}\,dx \leq C \int_{\mathbb{R}} Mf(x)^{q(x)}\,dx + C \int_{\mathbb{R}} (e+|x|)^{-2q(x)}\,dx.$$

Since the second integral is finite, it remains to bound the first integral. We divide up the domain of integration:

$$\int_{\mathbb{R}} Mf(x)^{q(x)}\,dx = \int_{E_0 \cup E_1} Mf(x)^{q(x)}\,dx + \sum_{k=2}^{\infty} \int_{E_k} Mf(x)^{q(x)}\,dx.$$

The first integral is easy to estimate:

$$\int_{E_0\cup E_1} Mf(x)^{q(x)}\,dx \le |E_0\cup E_1| = 2e^{\alpha(2)}.$$

To estimate the sum for each $k \ge 2$ we define the following sets:

$$F_k = \bigcup_{j=0}^{k-2} E_j, \quad G_k = E_{k-1}\cup E_k\cup E_{k+1}, \quad H_k = \bigcup_{j=k+2}^{\infty} E_j,$$

for each $k$, $F_k \cup G_k \cup H_k = \mathbb{R}$. Then, since $q_+ < \infty$, by the modular triangle inequality (Remark 2.8),

$$\begin{aligned}
&\sum_{k=2}^{\infty}\int_{E_k} Mf(x)^{q(x)}\,dx \\
&\quad\le C\left(\sum_{k=2}^{\infty}\int_{E_k} M(f\chi_{F_k})(x)^{q_k}\,dx + \sum_{k=2}^{\infty}\int_{E_k} M(f\chi_{G_k})(x)^{q_k}\,dx\right.\\
&\qquad\left. + \sum_{k=2}^{\infty}\int_{E_k} M(f\chi_{H_k})(x)^{q_k}\,dx\right)\\
&\quad= C(I_1+I_2+I_3).
\end{aligned}$$

We estimate each summation in turn. We first consider $I_2$. By Theorem 3.4 and Remark 3.5, the maximal operator is bounded on $L^{q_k}(\mathbb{R})$ with a constant bounded by $Cq_k' \le Cp_0'$. Further, for $x\in G_k$, $q_k \ge p(x)$. Thus,

$$\begin{aligned}
I_2 = \sum_{k=2}^{\infty}\int_{E_k} M(f\chi_{G_k})(x)^{q_k}\,dx &\le C\sum_{k=2}^{\infty}\int_{G_k} f(x)^{q_k}\,dx \\
&\le C\int_{\mathbb{R}} f(x)^{p(x)}\left(\sum_{k=2}^{\infty}\chi_{G_k}(x)\right)dx \le 3C\int_{\mathbb{R}} f(x)^{p(x)}\,dx \le C.
\end{aligned}$$

Next we estimate $I_3$. For $x\in E_k$,

$$M(f\chi_{H_k})(x) = \sup_J \fint_J f(y)\chi_{H_k}(y)\,dy,$$

where the supremum is taken over all intervals $J$ such that $x\in J$ and $|J\cap H_k|>0$. These two conditions combined with (4.6) imply that

$$|J| \ge e^{\alpha(k+2)} - e^{\alpha(k+1)} = |E_{k+1}|/2.$$

For all $y \in H_k$, $q_k \geq p(y)$. Therefore, by Hölder's inequality,

$$M(f\chi_{H_k})(x)^{q_k} \leq \sup_J \fint_J \big(f(y)\chi_{H_k}(y)\big)^{q_k}\,dy$$
$$\leq \frac{2}{|E_{k+1}|}\int_{H_k} f(y)^{q_k}\,dy \leq \frac{2}{|E_{k+1}|}\int_{H_k} f(y)^{p(y)}\,dy \leq \frac{2}{|E_{k+1}|}.$$

Hence,

$$I_3 = \sum_{k=2}^{\infty}\int_{E_k} M(f\chi_{H_k})(x)^{q_k}\,dx \leq 2\sum_{k=2}^{\infty}\frac{|E_k|}{|E_{k+1}|}.$$

By the definition of the sets $E_k$,

$$\frac{|E_k|}{|E_{k+1}|} = \frac{e^{\alpha(k+1)} - e^{\alpha(k)}}{e^{\alpha(k+2)} - e^{\alpha(k+1)}} \leq \frac{1}{e^{\alpha(k+2)-\alpha(k+1)} - 1}.$$

Moreover,

$$\alpha(k+2) - \alpha(k+1) = \beta(k+2)\log(k+2) - \beta(k+1)\log(k+1)$$
$$\geq \beta(k+2)\log(k+1) - \beta(k+1)\log(k+1) = \beta\log(k+1). \quad (4.8)$$

Therefore, since $\beta > 1$,

$$I_3 \leq 2\sum_{k=2}^{\infty}\frac{1}{(k+1)^{\beta} - 1} < \infty.$$

Finally, we consider $I_1$; this estimate is the most delicate. Since the maximal operator is sublinear (Proposition 3.3), for each $k \geq 2$,

$$\int_{E_k} M(f\chi_{F_k})(x)^{q_k}\,dx \leq \int_{E_k}\left(\sum_{j=0}^{k-2} M(f\chi_{E_j})(x)\right)^{q_k} dx. \quad (4.9)$$

For $x \in E_k$ and $0 \leq j \leq k-2$,

$$M(f\chi_{E_j})(x) = \sup_J \fint_J f(y)\chi_{E_j}(y)\,dy,$$

where the supremum is taken over all intervals $J$ such that $x \in J$ and $|J \cap E_j| > 0$. Therefore,

$$|J| \geq |x| - e^{\alpha(j+1)} \geq |x| - e^{\alpha(k-1)}.$$

For all $y \in E_j$, $q_{j+1} \geq p(y)$, and so by Hölder's inequality with exponent $q_{j+1}$,

$$\begin{aligned} M(f\chi_{E_j})(x) &\leq \frac{1}{|x| - e^{\alpha(k-1)}} \int_{E_j} f(y)\, dy \\ &\leq \frac{1}{|x| - e^{\alpha(k-1)}} |E_j|^{1/q'_{j+1}} \left( \int_{E_j} f(y)^{q_{j+1}}\, dy \right)^{1/q_{j+1}} \\ &\leq \frac{1}{|x| - e^{\alpha(k-1)}} |E_j|^{1/q'_{j+1}} \left( \int_{E_j} f(y)^{p(y)}\, dy \right)^{1/q_{j+1}} \\ &\leq \frac{1}{|x| - e^{\alpha(k-1)}} |E_j|^{1/q'_{j+1}}. \end{aligned}$$

If we combine this inequality with (4.9), we get

$$I_1 \leq \sum_{k=2}^{\infty} \left( \int_{E_k} \left( \frac{1}{|x| - e^{\alpha(k-1)}} \right)^{q_k} dx \right) \left( \sum_{j=0}^{k-2} |E_j|^{1/q'_{j+1}} \right)^{q_k}.$$

We first evaluate the integral on the right-hand side:

$$\begin{aligned} &\int_{E_k} \left( \frac{1}{|x| - e^{\alpha(k-1)}} \right)^{q_k} dx \\ &\qquad \leq 2 \int_{e^{\alpha(k)}}^{\infty} \left( \frac{1}{x - e^{\alpha(k-1)}} \right)^{q_k} dx = \frac{2}{q_k - 1} \frac{1}{(e^{\alpha(k)} - e^{\alpha(k-1)})^{q_k - 1}}. \qquad (4.10) \end{aligned}$$

Next we estimate the sum. For $j \geq 2$, $|E_j| \leq 2e^{\alpha(j+1)}$. Since

$$q_{j+1} = p_0 + \frac{1}{\alpha(j)^a},$$

we have that

$$\frac{1}{q'_{j+1}} = 1 - \frac{\alpha(j)^a + p_0^{-1} - p_0^{-1}}{p_0 \alpha(j)^a + 1} = \frac{1}{p'_0} + \frac{p_0^{-1}}{p_0 \alpha(j)^a + 1}.$$

Therefore,

$$\frac{\alpha(j+1)}{q'_{j+1}} = \frac{\alpha(j+1)}{p'_0} + \frac{p_0^{-1} \alpha(j+1) \alpha(j)^{-a}}{p_0 + \alpha(j)^{-a}}.$$

Since $\alpha(\cdot)$ is an increasing function and $\alpha(j+1)\alpha(j)^{-a}$ is increasing for $j$ large and tends to $\infty$ as $j \to \infty$, for $2 \leq j \leq k-2$ and $k$ large,

$$|E_j|^{1/q'_{j+1}} \le Ce^{\alpha(j+1)/q'_{j+1}} \le Ce^{\alpha(k-1)/q'_{k-1}}.$$

A similar estimate holds for $j = 0, 1$ or when $k$ is small. Therefore,

$$\begin{aligned}\left(\sum_{j=0}^{k-2}|E_j|^{1/q'_{j+1}}\right)^{q_k} &\le C(k-1)^{q_k}\exp\left(\alpha(k-1)\frac{q_k}{q'_{k-1}}\right)\\ &= C(k-1)^{q_k}\exp\left(\alpha(k-1)\frac{q_k}{q'_k}\right)\exp\left(\alpha(k-1)\left(\frac{q_k}{q'_{k-1}}-\frac{q_k}{q'_k}\right)\right).\end{aligned}$$

We estimate the last exponential: by the mean value theorem (arguing as we did for (4.7)),

$$\begin{aligned}&\exp\left(\alpha(k-1)\left(\frac{q_k}{q'_{k-1}}-\frac{q_k}{q'_k}\right)\right)\\ &\qquad = \exp\left(\alpha(k-1)\left(\frac{q_{k-1}-q_k}{q_{k-1}}\right)\right)\\ &\qquad \le \exp\left(\frac{1}{p_0}\alpha(k-1)\left(\frac{1}{\alpha(k-2)^a}-\frac{1}{\alpha(k-1)^a}\right)\right)\\ &\qquad \le \exp\left(\frac{a\beta}{p_0}\alpha(k-1)\frac{\log(k-1)+1}{\alpha(k-2)^{1+a}}\right)\\ &\qquad \le C.\end{aligned}$$

Thus we have that

$$\begin{aligned}\left(\sum_{j=0}^{k-2}|E_j|^{1/q'_{j+1}}\right)^{q_k} &\le C(k-1)^{q_k}\exp\left(\alpha(k-1)\frac{q_k}{q'_k}\right)\\ &= C(k-1)^{q_k}\exp\left(\alpha(k-1)(q_k-1)\right). \qquad (4.11)\end{aligned}$$

We can now estimate $I_1$: by (4.10), (4.11) and (4.8),

$$\begin{aligned}I_1 &\le C\sum_{k=2}^{\infty}(k-1)^{q_k}\left(\frac{e^{\alpha(k-1)}}{e^{\alpha(k)}-e^{\alpha(k-1)}}\right)^{q_k-1}\\ &\le C\sum_{k=2}^{\infty}(k-1)^{q_k}\left(\frac{1}{e^{\alpha(k)-\alpha(k-1)}-1}\right)^{q_k-1} \le C\sum_{k=2}^{\infty}(k-1)^{q_k-\beta(q_k-1)}.\end{aligned}$$

By (4.5),

$$q_k - \beta(q_k - 1) < \beta - (\beta - 1)p_0 < \beta - (\beta + 1) = -1.$$

Therefore, the final series converges. This completes the estimate of $I_1$ and so completes our proof. □

## 4.2 A Useful Tool: Muckenhoupt $A_p$ Weights

In this section we pause in our study of the maximal operator on variable Lebesgue spaces to introduce some ideas from the theory of weighted norm inequalities. The theory of weights is closely intertwined with the properties of the Hardy-Littlewood maximal operator, so it is perhaps not surprising, at least in retrospect, that these results are directly applicable to studying the maximal operator on variable Lebesgue spaces. We will apply these ideas in Sects. 4.3 and 4.4. Furthermore, we will show in Chap. 5 that the theory of weights provides an elegant approach to studying the behavior of other operators on variable Lebesgue spaces.

By a weight we mean a non-negative, measurable function such that $0 < w(x) < \infty$ almost everywhere. For $1 < p < \infty$, a weight $w$ is in the Muckenhoupt class $A_p$—or simply, $w \in A_p$—if

$$[w]_{A_p} = \sup_Q \left( \fint_Q w(x)\,dx \right) \left( \fint_Q w(x)^{1-p'}\,dx \right)^{p-1} < \infty, \tag{4.12}$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes.

*Remark 4.16.* In the definition of $A_p$ weights the hypothesis $0 < w(x) < \infty$ implies that both integrals in (4.12) are positive, and therefore both must be finite: i.e., $w$ and $w^{1-p'}$ are locally integrable. We could weaken this hypothesis and instead assume that $w$ is not identically equal to 0 or infinity; (4.12) would still make sense if we used the convention that $0 \cdot \infty = 0$. However, this gains nothing: implicit in this more general definition is the fact that $w$ and $w^{1-p'}$ are locally integrable. See Remark 4.36 in the next section.

When $p = 1$ define the class $A_1$ to be the weights such that

$$[w]_{A_1} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)} < \infty, \tag{4.13}$$

where $M$ is the Hardy-Littlewood maximal operator. There are two equivalent definitions that will be useful in practice. First, $w \in A_1$ if for almost every $x$,

$$Mw(x) \le [w]_{A_1} w(x). \tag{4.14}$$

Alternatively, we have that $w \in A_1$ if for every cube $Q$,

$$\fint_Q w(y)\,dy \le [w]_{A_1} \operatorname*{ess\,inf}_{x\in Q} w(x). \tag{4.15}$$

Clearly, (4.13) and (4.14) are equivalent, and (4.14) implies (4.15). To see the reverse implication, suppose (4.15) holds and fix $x$ such that $Mw(x) > [w]_{A_1} w(x)$. Then there exists a cube $Q$ such that

$$\fint_Q w(y)\,dy > [w]_{A_1} w(x);$$

we may assume that the vertices of $Q$ have rational co-ordinates. Thus $x$ belongs to a subset of $Q$ with measure 0. Since there are a countable number of such cubes, the collection of all such $x$ must have measure 0, which shows that (4.14) holds almost everywhere.

The collection of all the $A_p$ weights is referred to as $A_\infty$:

$$A_\infty = \bigcup_{p\ge 1} A_p.$$

*Remark 4.17.* In the definition of $A_p$ weights we can substitute balls for cubes. In Sect. 3.1 we showed that the maximal operator can be defined using either balls or cubes, and the same reasoning applies here: given any ball $B$, there exist two cubes $Q_1,\ Q_2$ with the same center such that $Q_1 \subset B \subset Q_2$ and $|Q_2|/|Q_1| = n^{n/2}$, and a similar relationship holds with the roles of balls and cubes reversed.

The $A_p$ condition, $p > 1$, is in some sense a reverse Hölder inequality. To see this more clearly, we rewrite (4.12) as

$$\|w^{1/p}\chi_Q\|_p \|w^{-1/p}\chi_Q\|_{p'} \le [w]_{A_p}^{1/p} |Q|. \tag{4.16}$$

By Hölder's inequality, the left-hand side is always greater than or equal to $|Q|$ so (4.16) can be regarded as a reverse of this inequality. In Theorem 4.22 below we will show that the $A_p$ condition implies another, fundamental inequality that is more clearly a reverse Hölder inequality.

The definition of $A_p$ immediately yields four basic properties of these weights.

**Proposition 4.18.** *Given $p$, $1 \le p < \infty$, then:*

1. *If $p > 1$ and $w \in A_p$, then $w^{1-p'} \in A_{p'}$ and $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{p'-1}$.*
2. *If $p < q < \infty$, then $A_p \subset A_q$ and $[w]_{A_q} \le [w]_{A_p}$.*
3. *If $p > 1$ and $w_1,\ w_2 \in A_1$, then $w = w_1 w_2^{1-p} \in A_p$ and $[w]_{A_p} \le [w_1]_{A_1}[w_2]_{A_1}^{p-1}$.*
4. *If $w \in A_p$, then for any cube $Q$,*

$$\fint_Q w(x)\,dx \le [w]_{A_p} \exp\left(\fint_Q \log\big(w(x)\big)\,dx\right).$$

*Remark 4.19.* The converse of Property (3) is also true and is a very deep property of $A_p$ weights called the Jones factorization theorem. Property (4) is referred to as the reverse Jensen inequality for $A_p$ weights. The converse is also true: if this inequality holds with some constant, then $w \in A_p$ for some $p \ge 1$. The sharp constant is denoted $[w]_{A_\infty}$.

*Proof.* Property (1) follows from the definition and the fact that $(p-1)(p'-1) = 1$. Property (2) follows from Hölder's inequality: for $q > p > 1$, $p' - 1 > q' - 1$, and so for any cube $Q$,

$$\left(\fint_Q w(x)\,dx\right)\left(\fint_Q w(x)^{1-q'}\,dx\right)^{q-1} \le \left(\fint_Q w(x)\,dx\right)\left(\fint_Q w(x)^{1-p'}\,dx\right)^{p-1} \le [w]_{A_p}.$$

If $p = 1$, then by (4.15) for any cube $Q$,

$$\left(\fint_Q w(x)^{1-q'}\,dx\right)^{q-1} = \left(\fint_Q \left(w(x)^{-1}\right)^{q'-1}\,dx\right)^{\frac{1}{q'-1}}$$
$$\le \operatorname*{ess\,sup}_{x\in Q} w(x)^{-1} = \left(\operatorname*{ess\,inf}_{x\in Q} w(x)\right)^{-1} \le [w]_{A_1}\left(\fint_Q w(x)\,dx\right)^{-1};$$

hence,

$$\left(\fint_Q w(x)\,dx\right)\left(\fint_Q w(x)^{1-q'}\,dx\right)^{q-1} \le [w]_{A_1}.$$

Property (3) follows by an almost identical argument:

$$\left(\fint_Q w_1(x)w_2(x)^{1-p}\,dx\right)\left(\fint_Q w_1(x)^{1-p'}w_2(x)^{(1-p)(1-p')}\,dx\right)^{p-1}$$
$$\le [w_2]_{A_1}^{p-1}\left(\fint_Q w_1(x)\,dx\right)\left(\fint_Q w_2(x)\,dx\right)^{1-p}$$
$$\times [w_1]_{A_1}\left(\fint_Q w_1(x)\,dx\right)^{-1}\left(\fint_Q w_2(x)\,dx\right)^{p-1}$$
$$= [w_1]_{A_1}[w_2]_{A_1}^{p-1}.$$

To prove Property (4) first note that by (2) we may assume that $p > 1$. But then, for any cube $Q$,

$$[A_p] \geq \lim_{q\to\infty} \left(\fint_Q w(x)\,dx\right)\left(\fint_Q w(x)^{1-q'}\,dx\right)^{q-1}$$
$$= \left(\fint_Q w(x)\,dx\right)\exp\left(\fint_Q -\log\big(w(x)\big)\,dx\right).$$

The final equality is a well-known identity: see Rudin [305, p. 71]. □

The simplest examples of $A_p$ weights are the power weights.

*Example 4.20.* For all $a \in \mathbb{R}$, $w(x) = |x|^a \in A_1$ if and only if $-n < a \leq 0$, and for $1 < p < \infty$, $w(x) = |x|^a \in A_p$ if and only if $-n < a < (p-1)n$.

*Proof.* We first consider the case $p = 1$. Our computations are easier if we consider the maximal operator as defined in terms of balls containing the point $x$. Suppose first that $-n < a \leq 0$. Then, as we showed in Example 3.2, $Mw(x) \leq c(n,a)|x|^a$ for every $x \neq 0$, so by (4.14), $|x|^a \in A_1$.

On the other hand, if $a \leq -n$, then $|x|^a$ is not integrable on any ball containing the origin, and so for every $x \in \mathbb{R}^n$, $Mw(x) = \infty$. Similarly, if $a > 0$, then

$$Mw(x) \geq \sup_{R>|x|} \fint_{B_R(0)} |y|^a\,dy = c(n,a) \sup_{R>|x|} R^a = \infty.$$

Now suppose $1 < p < \infty$. If $-n < a \leq 0$, then by the previous case and by Proposition 4.18, Property (2), $w \in A_1 \subset A_p$. If $0 < a < n(p-1)$, then the same argument shows that $w(x)^{1-p'} = |x|^{-a(p'-1)} \in A_1 \subset A_{p'}$ since $a(p'-1) < n$. Hence, by Proposition 4.18, Property (1), $w = w^{(1-p')(1-p)} \in A_p$. Finally, if $a \leq -n$ or $a \geq (p-1)n$, then either $w$ or $w^{1-p'}$ is not integrable on any set containing the origin, so $w \notin A_p$. □

We now turn to a fundamental property of $A_p$ weights: the reverse Hölder inequality.

**Definition 4.21.** Given $s > 1$, a weight $w$ satisfies the reverse Hölder inequality with exponent $s$, denoted by $w \in RH_s$, if

$$[w]_{RH_s} = \sup_Q \frac{\left(\fint_Q w(x)^s\,dx\right)^{1/s}}{\fint_Q w(x)\,dx} < \infty.$$

**Theorem 4.22.** *Given $p$, $1 \leq p < \infty$, if $w \in A_p$, then there exists $s > 1$ such that $w \in RH_s$. The constants $s$ and $[w]_{RH_s}$ depend only on $p$, $n$ and $[w]_{A_p}$.*

The proof of Theorem 4.22 requires a definition and two lemmas.

**Definition 4.23.** Given a cube $Q$, let $\Delta(Q)$ be the set of cubes consisting of $Q$ and all the cubes contained in $Q$ gotten by repeatedly bisecting the sides of $Q$. We refer to $\Delta(Q)$ as the set of dyadic cubes relative to $Q$.

The dyadic cubes relative to a cube $Q$ have properties very similar to the properties of dyadic cubes enumerated in Proposition 3.7 and we will make use of these without comment. Our first lemma is a local version of the Calderón-Zygmund decomposition (Lemma 3.9).

**Lemma 4.24.** *Given a cube $Q$ and a function $f \in L^1(Q)$, then for any $t \geq \fint_Q |f(y)|\,dy$, there exists a (possibly empty) set of disjoint cubes $\{Q_j\} \subset \Delta(Q)$ such that*

$$t < \fint_{Q_j} |f(y)|\,dy \leq 2^n t,$$

*and for almost every $x \in Q \setminus \bigcup_j Q_j$, $|f(x)| \leq t$.*

The cubes $\{Q_j\}$ are called the Calderón-Zygmund cubes of $f$ relative to $Q$ at height $t$.

*Proof.* Fix $t$; if $t \geq \|f\|_{L^\infty(Q)}$, then there is no $Q' \in \Delta(Q)$ such that $\fint_{Q'} |f(y)|\,dy > t$ so we will let the collection $\{Q_j\}$ be the empty set. Otherwise, let $E_t^Q = \{x : |f(x)| > t\}$. By the Lebesgue differentiation theorem (applied with respect to the cubes $\Delta(Q)$, see Sect. 2.9) for almost every $x \in E_t^Q$ there exists a cube $Q_x \in \Delta(Q)$ containing $x$ such that

$$\fint_{Q_x} |f(y)|\,dy > t \geq \fint_Q |f(y)|\,dy.$$

If more than one cube in $\Delta(Q)$ has this property, let $Q_x$ be the largest one. The second inequality implies that $Q_x \neq Q$. Since $\Delta(Q)$ is countable, the set $\{Q_x : x \in E_t^Q\}$ is at most countable; re-index it as $\{Q_j\}$. The cubes $Q_j$ are pairwise disjoint: if two cubes in $\Delta(Q)$ intersect, then one is contained in the other, and so by the maximality of each $Q_j$ it cannot be contained in another such cube.

Furthermore, since $Q_j \neq Q$, its dyadic parent $\tilde{Q}_j$ is contained in $\Delta(Q)$, and so by the maximality of $Q_j$,

$$\fint_{Q_j} |f(y)|\,dy \leq 2^n \fint_{\tilde{Q}_j} |f(y)|\,dy \leq 2^n t.$$

Finally, by our choice of the cubes $Q_j$ the set $E_t^Q$ is contained, up to a set of measure 0, in $\bigcup_j Q_j$. This completes our proof. □

The second lemma shows that if $w \in A_p$, then the measure $w\,dx$ has homogeneity properties similar to the Lebesgue measure. For brevity, hereafter, given a measurable set $E$ let $w(E) = \int_E w(x)\,dx$.

**Lemma 4.25.** *If $w \in A_p$, then for any cube $Q$ and measurable set $E \subset Q$:*

1. *For any $\alpha$, $0 < \alpha < 1$, there exists $\beta$, $0 < \beta < 1$, such that if $|E| \geq \alpha|Q|$, then $w(E) \geq \beta w(Q)$.*
2. *For any $\gamma$, $0 < \gamma < 1$, there exists $\delta$, $0 < \delta < 1$, such that if $\gamma|Q| \geq |E|$, then $\delta w(Q) \geq w(E)$.*

*Proof.* To prove (1), fix $\alpha$ and fix $Q$ and $E$ such that $|E| \geq \alpha|Q|$. Then by Hölder's inequality and the definition of $A_p$,

$$\alpha \leq \frac{|E|}{|Q|} = \fint_Q \chi_E(x) w(x)^{1/p} w(x)^{-1/p}\,dx \leq \frac{w(E)^{1/p}}{|Q|^{1/p}} \left( \fint_Q w(x)^{1-p'}\,dx \right)^{1/p'}$$
$$\leq [w]_{A_p}^{1/p} \frac{w(E)^{1/p}}{|Q|^{1/p}} \left( \fint_Q w(x)\,dx \right)^{-1/p} = [w]_{A_p}^{1/p} \left( \frac{w(E)}{w(Q)} \right)^{1/p}.$$

Rearranging terms we get $w(E) \geq \alpha^p [w]_{A_p}^{-1} w(Q)$, which gives us the desired value of $\beta$.

Property (2) follows at once from Property (1). Fix $\gamma$ and sets $Q$ and $E$ such that $\gamma|Q| \geq |E|$. Then $|Q \setminus E| \geq (1-\gamma)|Q|$, and so with $\beta = (1-\gamma)^p [w]_{A_p}^{-1}$,

$$w(Q) - w(E) = w(Q \setminus E) \geq \beta w(Q).$$

Rearranging terms we get $(1-\beta)w(Q) \geq w(E)$, which gives us the desired value of $\delta$. □

*Remark 4.26.* As a corollary to Lemma 4.25 we get that $w\,dx$ is a doubling measure: there exists a constant $[w]_D$ such that given any cube $Q$, $w(2Q) \leq [w]_D w(Q)$. In fact, this remains true if we replace $2Q$ by any cube $Q'$ that contains $Q$ such that $\ell(Q') = 2\ell(Q)$.

*Proof of Theorem 4.22.* Fix a cube $Q$ and for $k \geq 0$ define the sequence $t_k = 2^{k(n+1)} \fint_Q w(x)\,dx = 2^{k(n+1)} t_0$. For each $k$ form the Calderón-Zygmund cubes $\{Q_j^k\}$ of $w$ relative to $Q$ at height $t_k$ (Lemma 4.24), and define the sets $\Omega_k = \bigcup_j Q_j^k$. It follows from the construction of these cubes that $\Omega_{k+1} \subset \Omega_k$, and in fact for any $i$ there exists $j$ such that $Q_i^{k+1} \subset Q_j^k$. Then for each $j$ and $k$ we have that

$$
\begin{aligned}
|\Omega_{k+1} \cap Q_j^k| &= \sum_{Q_i^{k+1} \subset Q_j^k} |Q_i^{k+1}| \\
&< t_{k+1}^{-1} \sum_{Q_i^{k+1} \subset Q_j^k} w(Q_i^{k+1}) \le t_{k+1}^{-1} w(Q_j^k) \le \frac{2^n t_k}{t_{k+1}} |Q_j^k| = \frac{1}{2} |Q_j^k|.
\end{aligned}
$$

Hence, by Property (2) of Lemma 4.25 with $\gamma = 1/2$, there exists $\delta > 0$ such that $w(\Omega_{k+1} \cap Q_j^k) \le \delta w(Q_j^k)$. If we sum over all $j$, we get $w(\Omega_{k+1}) \le \delta w(\Omega_k)$, and so by induction we have that $w(\Omega_k) \le \delta^k w(\Omega_0)$.

Similarly, we have that $|\Omega_k| \le 2^{-k} |\Omega_0|$, so

$$
\Big| \bigcap_k \Omega_k \Big| = \lim_{k \to \infty} |\Omega_k| = 0.
$$

For almost every $x \in Q \setminus \Omega_k$, $w(x) \le t_k$. Therefore, for $\epsilon > 0$ to be fixed below,

$$
\begin{aligned}
⨍_Q w(x)^{1+\epsilon}\,dx &= \frac{1}{|Q|} \int_{Q \setminus \Omega_0} w(x)^{1+\epsilon}\,dx + \frac{1}{|Q|} \sum_{k=0}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} w(x)^{1+\epsilon}\,dx \\
&\le \frac{t_0^\epsilon}{|Q|} \int_{Q \setminus \Omega_0} w(x)\,dx + \frac{1}{|Q|} \sum_{k=0}^{\infty} t_{k+1}^\epsilon w(\Omega_k) \\
&\le \frac{t_0^\epsilon}{|Q|} \int_{Q \setminus \Omega_0} w(x)\,dx + \frac{1}{|Q|} \sum_{k=0}^{\infty} 2^{(k+1)(n+1)\epsilon} t_0^\epsilon \delta^k w(\Omega_0).
\end{aligned}
$$

Fix $\epsilon > 0$ so that $2^{(n+1)\epsilon} \delta < 1$. Then the series converges and the final term is bounded by

$$
t_0^\epsilon ⨍_{Q \setminus \Omega_0} w(x)\,dx + C|Q|^{-1} t_0^\epsilon w(\Omega_0) \le C \left( ⨍_Q w(x)\,dx \right)^{1+\epsilon}.
$$

The constant $C$ depends only on $n$ and $\delta$, and so on $n$, $p$ and $[w]_{A_p}$. Since this estimate is independent of the cube $Q$ we have that $w \in RH_s$ with $s = 1 + \epsilon$. □

As a consequence of the proofs of Lemma 4.25 and Theorem 4.22 we get the following corollary which will be of central importance in the proof of Theorem 4.52 in Sect. 4.4 below.

**Corollary 4.27.** *Given a cube $Q$, suppose a weight $w$ satisfies Property* (1) *of Lemma 4.25 for some $\alpha$, $\beta > 0$ and for every Calderón-Zygmund cube of $w$ relative to $Q$. Then there exist constants $s$, $C > 1$ depending only on $n$, $\alpha$ and $\beta$, such that*

$$\left(\fint_Q w(x)^s\,dx\right)^{1/s} \le C\fint_Q w(x)\,dx.$$

*In particular, if this is true for every cube $Q$, $w \in RH_s$.*

The converse of Theorem 4.22 is also true.

**Theorem 4.28.** *Given $w \in RH_s$, $s > 1$, there exists $p$, $1 < p < \infty$, such that $w \in A_p$. The value of $p$ depends only on $s$, $[w]_{RH_s}$, and $n$.*

*Proof.* The proof requires a generalization of the $A_p$ and $RH_s$ conditions. Given a non-negative measure $\mu$, we say that a weight $w$ such that $0 < w(x) < \infty$ $\mu$-almost everywhere is in $A_p(\mu)$ for some $p$, $1 < p < \infty$, if

$$[w]_{A_p(\mu)} = \sup_Q \left(\frac{1}{\mu(Q)}\int_Q w(x)\,d\mu\right)\left(\frac{1}{\mu(Q)}\int_Q w(x)^{1-p'}\,d\mu\right)^{p-1} < \infty.$$

Similarly, we write $w \in RH_s(\mu)$ if it satisfies the reverse Hölder inequality

$$[w]_{RH_s(\mu)} = \sup_Q \frac{\left(\frac{1}{\mu(Q)}\int_Q w(x)^s\,d\mu\right)^{1/s}}{\frac{1}{\mu(Q)}\int_Q w(x)\,d\mu} < \infty.$$

Given $w \in A_p$, let $d\mu = w\,dx$. Then the $A_p$ condition can be rewritten as

$$\left(\int_Q w(x)\,dx\right)^{p'-1}\int_Q \left(w(x)^{-1}\right)^{p'} w(x)\,dx \le [w]_{A_p}^{p'-1}\left(\int_Q w(x)^{-1}w(x)\,dx\right)^{p'},$$

which in turn is equivalent to

$$\frac{1}{\mu(Q)}\int_Q \left(w(x)^{-1}\right)^{p'}\,d\mu \le [w]_{A_p}^{p'-1}\left(\frac{1}{\mu(Q)}\int_Q w(x)^{-1}\,d\mu\right)^{p'}.$$

It follows immediately that $w \in A_p$ if and only if $w^{-1} \in RH_{p'}(\mu)$. A similar argument shows that $w \in RH_s$ if and only if $w^{-1} \in A_{s'}(\mu)$. Furthermore, we can repeat the proofs of Lemmas 4.24 and 4.25, and Theorem 4.22 starting with the assumption $w^{-1} \in A_{s'}(\mu)$. The proofs go through as before since $\mu$ is a doubling measure: more precisely, we apply Remark 4.26 to replace $2^n$ by $[w]_D$ in the construction of the local Calderón-Zygmund cubes.

Therefore, given $w \in RH_s$ there exists some $p > 1$ such that $w^{-1} \in RH_{p'}(\mu)$, which in turn is equivalent to $w \in A_p$. This completes our proof. □

As a consequence of the reverse Hölder inequality we get that if $w \in A_p$ for some $p$, then there exists $s > 1$ such that $M_s w(x) = M(w^s)(x)^{1/s} \le CMw(x)$. The next result is a sharper version of this inequality that we will need in Sect. 4.3.

**Proposition 4.29.** *Given $w \in A_1$, if $s_0 = 1 + \frac{1}{2^{n+2}[w]_{A_1}}$, then for $1 < s \le s_0$ and for almost every $x$,*

$$M_s w(x) \le 2Mw(x) \le 2[w]_{A_1} w(x). \tag{4.17}$$

The proof of Proposition 4.29 requires an inequality that is the reverse of the weak $(1,1)$ inequality for the maximal operator (Theorem 3.4).

**Lemma 4.30.** *Given a function $f \in L^1_{loc}(\mathbb{R}^n)$, for every cube $Q$ and $t \ge \fint_Q |f(x)|\,dx$,*

$$|\{x \in Q : Mf(x) > t\}| \ge \frac{2^{-n}}{t} \int_{\{x\in Q: |f(x)|>t\}} |f(x)|\,dx.$$

*Proof.* Fix $t \ge \fint_Q |f(x)|\,dx$; if $t \ge \|f\|_{L^\infty(Q)}$, then this result is trivially true. Otherwise, by Lemma 4.24, let $\{Q_j\}$ be the Calderón-Zygmund cubes of $f$ relative to $Q$ at height $t$. Then for every $x \in Q_j$,

$$Mf(x) \ge \fint_{Q_j} |f(x)|\,dx > t,$$

and since the cubes $Q_j$ are disjoint and $|f(x)| \le t$ for almost every $x \notin \bigcup_j Q_j$,

$$\begin{aligned} |\{x \in Q : Mf(x) > t\}| &\ge \sum_j |Q_j| \\ &\ge \frac{2^{-n}}{t} \sum_{Q_j} \int_{Q_j} |f(x)|\,dx \ge \frac{2^{-n}}{t} \int_{\{x\in Q: |f(x)|>t\}} |f(x)|\,dx. \end{aligned}$$

□

*Proof of Proposition 4.29.* Fix $w \in A_1$. The second inequality in (4.17) is immediate. By Hölder's inequality and the definition of the maximal operator, to prove the first inequality it will suffice to show that for any cube $Q$ and $x_0 \in Q$,

$$\fint_Q w(x)^{s_0}\,dx \le 2Mw(x_0)^{s_0}.$$

Let $\epsilon = (2^{n+2}[w]_{A_1})^{-1}$, $s_0 = 1 + \epsilon$, and fix a cube $Q$ and $x_0 \in Q$. Then by the analog of the identity (3.6) for the measure $\chi_Q w\,dx$ (see [238, 305]) we have that

$$\begin{aligned}\fint_Q w(x)^{s_0}\,dx = \fint_Q w(x)^\epsilon w(x)\,dx &= \epsilon|Q|^{-1}\int_0^\infty t^{\epsilon-1} w(\{x\in Q : w(x)>t\})\,dt \\ &= \epsilon|Q|^{-1}\int_0^{Mw(x_0)} t^{\epsilon-1} w(\{x\in Q : w(x)>t\})\,dt \\ &\qquad + \epsilon|Q|^{-1}\int_{Mw(x_0)}^\infty t^{\epsilon-1} w(\{x\in Q : w(x)>t\})\,dt.\end{aligned}$$

The first term is easy to estimate:

$$\begin{aligned}&\epsilon|Q|^{-1}\int_0^{Mw(x_0)} t^{\epsilon-1} w(\{x\in Q : w(x)>t\})\,dt \\ &\quad\le \epsilon|Q|^{-1}w(Q)\int_0^{Mw(x_0)} t^{\epsilon-1}\,dt = \fint_Q w(y)\,dy\cdot Mw(x_0)^\epsilon \le Mw(x_0)^{1+\epsilon}.\end{aligned}$$

To estimate the second term we use Lemma 4.30 (on the function $w$) and (3.6) (on the function $(Mw)\chi_Q$):

$$\begin{aligned}\epsilon|Q|^{-1}&\int_{Mw(x_0)}^\infty t^{\epsilon-1} w(\{x\in Q : w(x)>t\})\,dt \\ &= \epsilon|Q|^{-1}\int_{Mw(x_0)}^\infty t^{\epsilon-1}\int_{\{x\in Q: w(x)>t\}} w(x)\,dx\,dt \\ &\le 2^n\epsilon|Q|^{-1}\int_0^\infty t^\epsilon|\{x\in Q : Mw(x)>t\}|\,dt \\ &= \frac{2^n\epsilon}{1+\epsilon}\fint_Q Mw(x)^{1+\epsilon}\,dx \\ &\le \frac{2^n\epsilon[w]_{A_1}^{1+\epsilon}}{1+\epsilon}\fint_Q w(x)^{1+\epsilon}\,dx.\end{aligned}$$

Combining these estimates we get

$$\fint_Q w(x)^{1+\epsilon}\,dx \le Mw(x_0)^{1+\epsilon} + \frac{2^n\epsilon[w]_{A_1}^{1+\epsilon}}{1+\epsilon}\fint_Q w(x)^{1+\epsilon}\,dx.$$

Since for all $x\ge 1$, $x^{(2^{n+2}x)^{-1}} \le x^{(8x)^{-1}} \le 2$,

$$\frac{2^n\epsilon[w]_{A_1}^{1+\epsilon}}{1+\epsilon} \le 2^n 2^{-n-2}[w]_{A_1}^{-1}[w]_{A_1}^{1+(2^{n+2}[w]_{A_1})^{-1}} \le \frac{1}{4}[w]_{A_1}^{(2^{n+2}[w]_{A_1})^{-1}} \le \frac{1}{2}.$$

Therefore, if we rearrange terms we get the desired inequality. □

## 4.3 Applications of Weights to the Maximal Operator

In this section we give several applications involving weights, the maximal operator, and variable Lebesgue spaces. We begin with two results which are weighted norm inequalities and so do not involve variable Lebesgue spaces, but which will be an important motivation for the results in Sect. 4.4. We then prove a theoretical characterization of those variable Lebesgue spaces on which the maximal operator is bounded; we will apply this result in the proof of Theorem 4.52 in the next section. Finally, we give the proof of Theorem 3.31 on modular inequalities for the maximal operator.

**Definition 4.31.** Given a weight $w$ and $p$, $1 \leq p < \infty$, define the space $L^p(w)$ to be the collection of all measurable functions $f$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty.$$

The space $L^p(w)$ is referred to as a weighted Lebesgue space.

**Definition 4.32.** Given a cube $Q$, define the averaging operator $A_Q$ by

$$A_Q f(x) = \fint_Q f(y)\,dy\;\chi_Q(x).$$

For brevity, hereafter will often write $f_Q$ instead of $\fint_Q f(y)\,dy$.

**Proposition 4.33.** *Given $p$, $1 \leq p < \infty$, and a cube $Q$,*

$$\int_Q |A_Q f(x)|^p w(x)\,dx \leq C_0 \int_Q |f(x)|^p w(x)\,dx \tag{4.18}$$

*for all $f$ such that $f\chi_Q \in L^p(w)$ if and only if*

$$\left(\fint_Q w(x)\,dx\right)\left(\fint_Q w(x)^{1-p'}\,dx\right)^{p-1} \leq C_0 \tag{4.19}$$

*when $p > 1$ or*

$$\fint_Q w(x)\,dx \leq C_0 \operatorname*{ess\,inf}_{x\in Q} w(x)\,dx \tag{4.20}$$

*when $p = 1$. As a consequence, $w \in A_p$ if and only if the operators $A_Q$ are uniformly bounded on $L^p(w)$ for all $Q$.*

*Remark 4.34.* Proposition 4.33 remains true if we replace averages $A_Q$ over cubes with averages $A_B$ over balls. This follows by the same proof, using the fact, noted after the definition of $A_p$, that we can define this condition using balls instead of cubes.

*Proof.* Suppose first that (4.19) holds. By Hölder's inequality,

$$\int_Q |A_Q f(x)|^p w(x)\,dx = \left(\fint_Q |f(x)|w(x)^{1/p}w(x)^{-1/p}\,dx\right)^p \int_Q w(x)\,dx$$

$$\leq \int_Q |f(x)|^p w(x)\,dx \left(\fint_Q w(x)^{1-p'}\,dx\right)^{p-1} \fint_Q w(x)\,dx \leq C_0 \int_Q |f(x)|^p w(x)\,dx.$$

A similar argument yields (4.18) if $p = 1$ and (4.20) holds.

Now suppose that (4.18) holds. If $p > 1$, let $f = w^{1-p'}\chi_Q$. Then inequality (4.18) becomes

$$\int_Q w(x)\,dx \left(\fint_Q w(x)^{1-p'}\,dx\right)^p \leq C_0 \int_Q w(x)^{(1-p')p} w(x)\,dx,$$

which in turn immediately yields (4.19).

If $p = 1$, then for every $q > 1$, $|A_Q(f)(x)|^q \leq A_Q(|f|^q)(x)$, and so

$$\int_Q |A_Q f(x)|^q w(x)\,dx \leq C_0 \int_Q |f(x)|^q w(x)\,dx.$$

Therefore, by the above argument we have that for every $q > 1$,

$$\left(\fint_Q w(x)\,dx\right)\left(\fint_Q w(x)^{1-q'}\,dx\right)^{q-1} \leq C_0.$$

However, by a well-known identity (see Rudin [305, Example 4, p. 71])

$$\lim_{q\to 1}\left(\fint_Q w(x)^{1-q'}\,dx\right)^{q-1} = \operatorname*{ess\,sup}_{x\in Q}\left(w(x)^{-1}\right),$$

which yields (4.20). □

The next result shows the close connection between Muckenhoupt $A_p$ weights and the maximal operator, and is fundamental in the study of weighted norm inequalities.

**Theorem 4.35.** *Given $p$, $1 \leq p < \infty$, then $w \in A_p$ if and only if for every $f \in L^p(w)$ and every $t > 0$,*

$$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq \frac{C}{t^p}\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx. \tag{4.21}$$

*Furthermore, if $p > 1$, then $w \in A_p$ if and only if*

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx. \tag{4.22}$$

*In both cases the constant depends on $p$, $n$ and $[w]_{A_p}$.*

*Proof.* The proof of the sufficiency of the $A_p$ condition is very similar to the proof of Theorem 3.4, so we only give the main steps. Arguing as we did in the proof of Theorem 3.16 using Lemma 3.30, it will suffice to prove both inequalities assuming that $f$ is bounded and has compact support. We first prove inequality (4.21). When $p = 1$, fix $t > 0$ and form the Calderón-Zygmund cubes $\{Q_j\}$ of $f$ at height $t$ (Lemma 3.9). Then

$$\begin{aligned} w(\{x \in \mathbb{R}^n : Mf(x) > t\}) &\le \sum_j w(3Q_j) \le \frac{1}{t}\sum_j w(3Q_j) \fint_{Q_j} |f(x)|\,dx \\ &\le \frac{3^n}{t}\sum_j \int_{Q_j} |f(x)| Mw(x)\,dx \le \frac{3^n [w]_{A_1}}{t}\int_{\mathbb{R}^n} |f(x)| w(x)\,dx. \end{aligned}$$

When $p > 1$ the proof is essentially the same, using Hölder's inequality to get

$$\frac{1}{t^p}\left(\fint_{Q_j} |f(x)|\,dx\right)^p \le \frac{1}{t^p}\left(\fint_{Q_j} |f(x)|^p w(x)\,dx\right)\left(\fint_{Q_j} w(x)^{1-p'}\,dx\right)^{p-1}.$$

To prove inequality (4.22), note that since $w \in A_p$, $p > 1$, by Proposition 4.18, $w^{1-p'} \in A_{p'}$. Therefore, by Theorem 4.22, $w^{1-p'} \in RH_s$ for some $s > 1$. Fix $q < p$ such that $q' - 1 = s(p' - 1)$. Then for every cube $Q$,

$$\begin{aligned} &\left(\fint_Q w(x)\,dx\right)\left(\fint_Q w(x)^{1-q'}\,dx\right)^{q-1} \\ &\qquad \le [w^{1-p'}]_{RH_s}\left(\fint_Q w(x)\,dx\right)\left(\fint_Q w(x)^{1-p'}\,dx\right)^{p-1} \le [w^{1-p'}]_{RH_s}[w]_{A_p}. \end{aligned}$$

Hence, $w \in A_q$. Further, by Proposition 4.18, for any $r > p$, $w \in A_r$. By the argument above, we have that inequality (4.21) holds with $p$ replaced by either $q$ or $r$. Therefore, by the Marcinkiewicz interpolation theorem (Proposition 3.46, which remains true with Lebesgue measure replaced by the measure $w\,dx$: see [96]), we get (4.22).

Finally we prove that the $A_p$ condition is necessary for (4.21); recall that when $p > 1$, (4.22) implies (4.21). When $p = 1$, fix $s > \operatorname{ess\,inf}_{x\in Q} w(x)$. Then there exists a set $E \subset Q$, $|E| > 0$, such that $w(x) \le s$ for all $x \in E$. Let $f = \chi_E$; then for all $t$, $0 < t < |E|/|Q|$, and $x \in Q$,

$$Mf(x) \geq \frac{|E|}{|Q|} > t.$$

Hence, by inequality (4.21),

$$tw(Q) \leq tw(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq Cw(E).$$

Taking the supremum over all such $t$ we get

$$\frac{w(Q)}{|Q|} \leq C\frac{w(E)}{|E|} \leq Cs.$$

If we take the infimum over all such $s$, we get

$$\fint_Q w(x)\,dx \leq C \operatorname*{ess\,inf}_{x\in Q} w(x).$$

Since the constant is independent of $Q$, $w \in A_1$.

Now fix $p > 1$ and a cube $Q$, and let $f = w^{1-p'}\chi_Q$. Then for $x \in Q$,

$$Mf(x) \geq \fint_Q w(y)^{1-p'}\,dy,$$

so if we take $t > 0$ to be any value smaller than the right-hand side, we can argue as we did above using the weak $(p, p)$ inequality to get

$$\left(\fint_Q w(y)^{1-p'}\,dy\right)^p w(Q) \leq C \int_Q w(y)^{1-p'}\,dy.$$

The $A_p$ condition follows immediately. □

*Remark 4.36.* In the definition of $A_p$ weights we assumed that $0 < w(x) < \infty$ almost everywhere, which implies that $w$ and $w^{1-p'}$ are locally integrable. However, if we omit this hypothesis from the definition (see Remark 4.16 above), then we can still prove that this is the case.

We can modify the proof of the weak type inequality (4.21) so that it works using the more general definition of $A_p$ weights, assuming only that $w$ is not identically equal to 0 or infinity. Given that the weak type inequality holds for $w \in A_p$, $1 \leq p < \infty$, we can show that $0 < w(x) < \infty$ almost everywhere. Fix a cube $Q$ and a measurable set $E \subset Q$; then by modifying the argument for the necessity of the $A_1$ condition above we can show that

$$\left(\frac{|E|}{|Q|}\right)^p w(Q) \leq Cw(E). \tag{4.23}$$

Suppose that there exists a bounded set $E$ of positive measure such that $w(x) = 0$ for $x \in E$. Then given any cube $Q$ containing $E$, (4.23) implies that $w(Q) = 0$, so $w = 0$ almost everywhere on $Q$. Since we can take $Q$ arbitrarily large, we have that $w = 0$ almost everywhere. Similarly, suppose that $w(x) = \infty$ on a set $E$ of positive measure. Then given any cube $Q$ containing $E$ we must have $w(Q) = \infty$. But then (4.23) implies that $w(E) = \infty$ for every subset $E \subset Q$, and so $w(x) = \infty$ almost everywhere on $Q$. Again since we can take $Q$ arbitrarily large, we have that $w = \infty$ almost everywhere. Thus in either case we contradict our original assumption that $w$ is not 0 or infinite almost everywhere.

We now turn to applications of weights to the theory of the maximal operator on variable Lebesgue spaces. Our first result is a characterization of $L^{p(\cdot)}$ spaces on which the maximal operator is bounded. To state it, we first make a definition. Given $\Omega$, $p(\cdot) \in \mathcal{P}(\Omega)$, and an operator $T : L^{p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$, we define the operator norm of $T$ by

$$\|T\|_{L^{p(\cdot)}(\Omega)} = \sup_{\|f\|_{p(\cdot)} \leq 1} \|Tf\|_{L^{p(\cdot)}(\Omega)}.$$

**Theorem 4.37.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, the following are equivalent:*

1. $M : L^{p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$.
2. *For all $s > 1$, $M : L^{sp(\cdot)}(\Omega) \to L^{sp(\cdot)}(\Omega)$ and*

$$\lim_{s \to 1+} (s-1)\|M\|_{L^{sp(\cdot)}(\Omega)} = 0.$$

3. *There exists $r_0$, $0 < r_0 < 1$, such that if $r_0 < r < 1$, then*

$$M : L^{rp(\cdot)}(\Omega) \to L^{rp(\cdot)}(\Omega).$$

*Remark 4.38.* It will be clear from the proof that we can weaken Condition (2) by assuming that there exists $s_0 > 1$ such that $M$ is bounded on $L^{sp(\cdot)}(\Omega)$ if $1 < s < s_0$ and this limit holds. The proof that Condition (1) implies Condition (2) shows that if $M$ is bounded on $L^{sp(\cdot)}$ for some $s > 1$, then it is bounded on $L^{sp(\cdot)}$ for all larger values of $s$.

*Remark 4.39.* By Theorem 3.19 a necessary condition for $M$ to be bounded on $L^{rp(\cdot)}$ is that $rp_- > 1$, so in Condition (3) we must have that $r_0 \geq 1/p_-$.

*Proof.* We first prove that (1) implies (2). Fix $s > 1$ and let $r = 1/s$. Define $M_r f(x) = M(|f|^r)(x)^{1/r}$. Then by Hölder's inequality, $M_r f(x) \leq Mf(x)$, and so by Proposition 2.18,

$$\|Mf\|_{sp(\cdot)} = \|M_r(|f|^s)\|_{p(\cdot)}^{1/s} \leq \|M(|f|^s)\|_{p(\cdot)}^{1/s} \leq C^{1/s}\||f|^s\|_{p(\cdot)}^{1/s} = C^{1/s}\|f\|_{sp(\cdot)}.$$

Hence, $\|M\|_{L^{sp(\cdot)}(\Omega)} \leq \|M\|_{L^{p(\cdot)}(\Omega)}$ which yields Condition (2).

The implication (3) implies (1) is also straightforward. Fix $r$, $r_0 < r < 1$, and let $s = 1/r$. Again by Hölder's inequality, we have that $Mf(x) \leq M(|f|^s)(x)^{1/s} =$

$M_s f(x)$, and so by Proposition 2.18,

$$\|Mf\|_{p(\cdot)} \leq \|M(|f|^s)^{1/s}\|_{p(\cdot)} = \|M(|f|^s)\|_{rp(\cdot)}^r \leq C^r \||f|^s\|_{rp(\cdot)}^r = C^r\|f\|_{p(\cdot)}.$$

To prove that (1) implies (3), we first construct an $A_1$ weight using a powerful technique referred to as the Rubio de Francia iteration algorithm. Given $h \in L^{p(\cdot)}(\Omega)$, define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|_{L^{p(\cdot)}(\Omega)}^k},$$

where for $k \geq 1$, $M^k = M \circ M \circ \cdots \circ M$ denotes $k$ iterations of the maximal operator and $M^0 f = |f|$. This operator has the following properties:

(*a*) For all $x \in \Omega$, $|h(x)| \leq \mathcal{R}h(x)$;
(*b*) $\mathcal{R}$ is bounded on $L^{p(\cdot)}(\Omega)$ and $\|\mathcal{R}h\|_{p(\cdot)} \leq 2\|h\|_{p(\cdot)}$;
(*c*) $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \leq 2\|M\|_{L^{p(\cdot)}(\Omega)}$.

Property (*a*) follows immediately from the definition. Property (*b*) follows from the subadditivity of the norm:

$$\|\mathcal{R}h\|_{p(\cdot)} \leq \sum_{k=0}^{\infty} \frac{\|M^k h\|_{p(\cdot)}}{2^k \|M\|_{L^{p(\cdot)}(\Omega)}^k} \leq \|f\|_{p(\cdot)} \sum_{k=0}^{\infty} 2^{-k} = 2\|h\|_{p(\cdot)}.$$

Property (*c*) follows by the subadditivity and homogeneity of the maximal operator (Proposition 3.3):

$$\begin{aligned} M(\mathcal{R}h)(x) &\leq \sum_{k=0}^{\infty} \frac{M^{k+1}h(x)}{2^k \|M\|_{L^{p(\cdot)}(\Omega)}^k} \\ &\leq 2\|M\|_{L^{p(\cdot)}(\Omega)} \sum_{k=0}^{\infty} \frac{M^{k+1}h(x)}{2^{k+1}\|M\|_{L^{p(\cdot)}(\Omega)}^{k+1}} \leq 2\|M\|_{L^{p(\cdot)}(\Omega)} \mathcal{R}h(x). \end{aligned}$$

By Property (*c*), Proposition 4.29 and Hölder's inequality there exists $s_0 > 1$ such that for all $s$, $1 < s < s_0$,

$$M_s(\mathcal{R}h)(x) \leq M_{s_0}(\mathcal{R}h)(x) \leq 4\|M\|_{L^{p(\cdot)}(\Omega)} \mathcal{R}h(x).$$

Let $r_0 = 1/s_0$, fix $r$, $r_0 < r < 1$, and let $s = 1/r$. Then by Properties (*a*) and (*b*) and Proposition 2.18,

$$\begin{aligned} \|Mf\|_{rp(\cdot)} &= \|(Mf)^{1/s}\|_{p(\cdot)}^s = \|M_s(|f|^r)\|_{p(\cdot)}^s \leq \|M_s(\mathcal{R}(|f|^r))\|_{p(\cdot)}^s \\ &\leq 4^s \|M\|_{L^{p(\cdot)}(\Omega)}^s \|\mathcal{R}(|f|^r)\|_{p(\cdot)}^s \leq C\||f|^r\|_{p(\cdot)}^s = C\|f\|_{rp(\cdot)}. \end{aligned}$$

This proves (3).

Finally, we prove that (2) implies (1). Fix $\epsilon$, $0 < \epsilon < 1$, such that if $s = 1+\epsilon 2^{-n-2}$, then

$$\epsilon \| M \|_{L^{sp(\cdot)}(\Omega)} = 2^{n+2}(s-1)\| M \|_{L^{sp(\cdot)}(\Omega)} < 1/2.$$

We now define a different version of the Rubio de Francia algorithm: for $h \in L^{sp(\cdot)}(\Omega)$ let

$$\mathcal{R}_\epsilon h(x) = \sum_{k=0}^{\infty} \epsilon^k M^k h(x).$$

Then arguing as we did above we have that:

$(a')$ For all $x \in \Omega$, $|h(x)| \leq \mathcal{R}_\epsilon h(x)$;
$(c')$ $\mathcal{R}_\epsilon h \in A_1$ and $[\mathcal{R}_\epsilon h]_{A_1} \leq \epsilon^{-1}$.

Let $r = 1/s$ and fix $f \in L^{p(\cdot)}(\Omega)$. By Properties $(a')$ and $(c')$, and by Proposition 4.29 (which holds by our choice of $s$)

$$Mf(x)^{1/s} = M_s(|f|^r)(x) \leq M_s(\mathcal{R}_\epsilon(|f|^r))(x) \leq 2\epsilon^{-1}\mathcal{R}_\epsilon(|f|^r)(x).$$

Therefore, by Proposition 2.18,

$$\begin{aligned}
\|Mf\|_{p(\cdot)} &\leq \left(\frac{2}{\epsilon}\right)^s \|\mathcal{R}_\epsilon(|f|^r)^s\|_{p(\cdot)} \\
&= \left(\frac{2}{\epsilon}\right)^s \|\mathcal{R}_\epsilon(|f|^r)\|_{sp(\cdot)}^s \\
&\leq \left(\frac{2}{\epsilon}\right)^s \left(\sum_{k=0}^{\infty} \epsilon^k \|M^k(|f|^r)\|_{sp(\cdot)}\right)^s \\
&\leq \left(\frac{2}{\epsilon}\right)^s \left(\sum_{k=0}^{\infty} \epsilon^k \|M\|_{L^{sp(\cdot)}(\Omega)}^k \||f|^r\|_{sp(\cdot)}\right)^s \\
&\leq \left(\frac{2}{\epsilon}\right)^s \||f|^r\|_{sp(\cdot)}^s \left(\sum_{k=0}^{\infty} 2^{-k}\right)^s \\
&= \left(\frac{4}{\epsilon}\right)^s \|f\|_{p(\cdot)}.
\end{aligned}$$

Thus $M$ is bounded on $L^{p(\cdot)}(\Omega)$. This completes our proof. □

*Remark 4.40.* We will use the Rubio de Francia algorithm again in Chap. 5 to prove our extrapolation theorem, which in turn will let us prove that a wide variety of operators are bounded on variable Lebesgue spaces. See Sect. 5.4.

*Proof of Theorem* 3.31. The proof requires one lemma.

**Lemma 4.41.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $p_+ < \infty$. Then the family of weights $w_t(x) = t^{p(x)}$, $t > 0$, are in $A_q$ for some $q > 1$ with uniform constant if and only if there exists a constant $p_0$ such that $p(\cdot) = p_0$ almost everywhere.*

*Proof.* Suppose first that $p(\cdot) = p_0$ almost everywhere. Then the weights $w_t$ are constants, and so for any $q > 1$, $w_t \in A_q$ and $[w_t]_{A_q} = 1$.

Conversely, suppose that $p(\cdot)$ is such that for all $t > 0$, $w_t \in A_q$ for some $q > 1$ and $[w_t]_{A_q} \le K < \infty$. For $N > 1$ let $Q_N = [-N, N]^n$ and define

$$p_N = \fint_{Q_N} p(x)\,dx.$$

Then by Property (4) of Proposition 4.18 (the reverse Jensen inequality) applied to $w_t$, for every $t > 0$,

$$\fint_{Q_N} t^{p(x)-p_N}\,dx \le K.$$

Now suppose that $p(\cdot)$ is not constant. If $|\{x \in Q_N : p(x) > p_N\}| > 0$, then if we take the limit as $t \to \infty$, the left-hand side is unbounded, a contradiction. Similarly, if $|\{x \in Q_N : p(x) < p_N\}| > 0$, the left-hand side is unbounded as $t \to 0$. Therefore, $p(x) = p_N$ for almost every $x \in Q_N$. Since this is true for all $N$, we must have that $p(\cdot)$ equals a constant almost everywhere. □

*Remark 4.42.* In the hypotheses of Lemma 4.41 it would suffice to assume either $0 < t < 1$ or $t > 1$. For if $p(\cdot)$ is not constant, then both of the sets $\{x \in Q_N : p(x) < p_N\}$ and $\{x \in Q_N : p(x) > p_N\}$ must have positive measure. However, in certain cases the weights $w_t$ can behave (locally) like $A_p$ weights with uniform constant: see Lemma 4.56 below.

*Proof of Theorem 3.31.* Since the strong type modular inequality (3.20) implies the weak type modular inequality (3.19), it will suffice to show that

$$\int_{\{x:Mf(x)>t\}} t^{p(x)}\,dx \le C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx, \qquad t > 0, \tag{4.24}$$

holds for all $f$ if and only if $p(\cdot)$ equals a constant almost everywhere. If $p(\cdot) = p_0$ almost everywhere, then (4.24) follows at once from Theorem 3.4.

Now suppose that (4.24) holds; fix $t > 0$ and let $w_t(x) = t^{p(x)}$. By Lemma 4.41 it will suffice to show that $w_t \in A_q$ for some $q > 1$ with $[w_t]_{A_q}$ independent of $t$. Fix a cube $Q$ and $\alpha$, $0 < \alpha < 1$. Given any measurable set $E \subset Q$ with $\alpha|Q| < |E|$, define $f = t\chi_E$. Then for all $x \in Q$, $Mf(x) \ge t|E|/|Q|$; hence, if $s < t|E|/|Q|$, by (4.24) and the monotone convergence theorem on the classical Lebesgue spaces we have that

$$\int_Q \big(t|E|/|Q|\big)^{p(x)}\,dx \le \liminf_{s\to t|E|/|Q|} \int_Q s^{p(x)}\,dx \le C \int_E t^{p(x)}\,dx.$$

Hence,

$$\alpha^{p_+} w_t(Q) = \alpha^{p_+} \int_Q t^{p(x)}\, dx \le \int_Q \big(t|E|/|Q|\big)^{p(x)}\, dx \le C \int_E t^{p(x)}\, dx = C w_t(E).$$

Thus, $w_t$ satisfies Property (1) of Lemma 4.25 with this $\alpha$ and $\beta = \alpha^{p_+} C^{-1}$. Therefore, by Corollary 4.27 and Theorem 4.28, $w_t \in A_q$ for some $q > 1$, and $[w_t]_{A_q}$ depends only on $\alpha$ and $\beta$. This completes the proof. □

## 4.4 Local Control: The $K_0$ Condition

In this section we consider the behavior of the maximal operator locally, where the function $f$ is large. As we noted in the introduction to this chapter, the $LH_0$ condition is equivalent to the geometric property in Lemma 3.24. Thus, we will introduce a new condition and adapt our proof of Theorem 3.16 to use it. In doing so the theory of Muckenhoupt $A_p$ weights will play a central role.

Before introducing our new condition, however, we first want to show that, as was the case for the $LH_\infty$ condition, the $LH_0$ condition is not necessary but is the best possible pointwise decay condition. As we saw in Example 3.21, if $p(\cdot)$ has a jump discontinuity, then $M$ cannot be bounded. The following example shows that if we replace the discontinuity by a continuous function that is steeper than $|\log(x)|^{-1}$, then we have the same phenomenon.

*Example 4.43.* Fix $p_0$, $1 < p_0 < \infty$, and let $\phi : [0,\infty) \to [0,1]$ be such that $\phi$ is increasing, $\phi(0) = 0$, $\phi(x) \to 0$ as $x \to 0^+$, and

$$\lim_{x\to 0+} \phi(x)\log(x) = -\infty. \tag{4.25}$$

Let $\Omega = (-1,1)$ and define $p(\cdot) \in \mathcal{P}(\Omega)$ by

$$p(x) = \begin{cases} p_0 + \phi(x) & x \ge 0 \\ p_0 & x < 0. \end{cases}$$

Then $p(\cdot) \notin LH_0(\Omega)$ and the maximal operator is not bounded on $L^{p(\cdot)}(\Omega)$.

*Proof.* The construction of this example is very similar to the construction of Example 4.1. It is immediate from (4.25) that $p(\cdot)$ does not satisfy the $LH_0(\Omega)$ condition at the origin, so we only have to construct a function $f$ such that $f \in L^{p(\cdot)}(\Omega)$ but $Mf \notin L^{p(\cdot)}(\Omega)$. Intuitively, we will generalize Example 3.21 by showing that $f(x) = |x|^{-1/p(|x|)}\chi_{(-1,0)}(x)$ is in $L^{p(\cdot)}(\Omega)$ but $Mf$ is not. However, to simplify the calculations we replace this $f$ by a discrete analog.

By (4.25) we have that

$$\lim_{x\to 0+}\left(1-\frac{p_0}{p(x/2)}\right)\log(x) = -\infty;$$

equivalently,

$$\lim_{x\to 0+} x^{1-\frac{p_0}{p(x/2)}} = 0.$$

Hence, we can form a sequence $\{a_n\} \subset (-1, 0)$ such that $a_n/2 < a_{n+1}$ and

$$|a_n|^{1-p_0/p(|a_n|/2)} \le 2^{-n}.$$

Let $b_n = a_n/2$ and define the function $f$ by

$$f(x) = \sum_{n=1}^{\infty} |a_n|^{-1/p(|b_n|)}\chi_{(a_n,b_n)}(x).$$

Since $p_+ < \infty$, by Proposition 2.12 it will suffice to show that $\rho(f) < \infty$ and $\rho(Mf) = \infty$. First, we have that

$$\rho(f) = \sum_{n=1}^{\infty}\int_{a_n}^{b_n} |a_n|^{-p_0/p(|b_n|)}\,dx = \frac{1}{2}\sum_{n=1}^{\infty}|a_n|^{1-p_0/p(|b_n|)} \le \sum_{n=1}^{\infty} 2^{-n-1} < \infty.$$

On the other hand, if $x \in (|b_n|, |a_n|)$, then

$$Mf(x) \ge \frac{1}{2|a_n|}\int_{a_n}^{|a_n|} f(y)\,dy \ge \frac{1}{2|a_n|}\int_{a_n}^{b_n} |a_n|^{-1/p(|b_n|)}\,dy = \frac{1}{4}|a_n|^{-1/p(|b_n|)}.$$

Therefore, since $p(\cdot)$ is an increasing function and $|a_n| \le 1$,

$$\begin{aligned}\rho(Mf) &\ge \left(\frac{1}{4}\right)^{p_+}\sum_{n=1}^{\infty}\int_{|b_n|}^{|a_n|} |a_n|^{-p(x)/p(|b_n|)}\,dx \\ &\ge \left(\frac{1}{4}\right)^{p_+}\sum_{n=1}^{\infty}\int_{|b_n|}^{|a_n|} |a_n|^{-p(|b_n|)/p(|b_n|)}\,dx = \left(\frac{1}{4}\right)^{p_+}\sum_{n=1}^{\infty}\frac{1}{2} = \infty.\end{aligned}$$

□

*Example 4.44.* A particular family of exponent functions $p(\cdot)$ that satisfy the hypotheses of Example 4.43 is

$$p(x) = \begin{cases} 2 & x \in (-1, 0] \\ 2 + \frac{1}{\log(e/x)^a} & x \in (0, 1), \end{cases}$$

where $0 < a < 1$.

We now introduce a condition to replace $LH_0$.

**Definition 4.45.** Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, then $p(\cdot) \in K_0(\Omega)$ if there exists a constant $C_K$ such that for every cube $Q$,

$$\|\chi_Q\|_{L^{p(\cdot)}(\Omega)}\|\chi_Q\|_{L^{p'(\cdot)}(\Omega)} \le C_K|Q|. \tag{4.26}$$

*Remark 4.46.* Inequality (4.26) is trivially true for any cube $Q$ such that $|Q\cap\Omega|=0$.

Definition 4.45 is motivated by the definition of the $A_p$ weights given in the previous section, particularly in the alternate form (4.16): if we take $p(\cdot)$ constant in (4.26) and set $w \equiv 1$ in (4.16), the two definitions coincide. Furthermore, as the next proposition shows, the $K_0$ condition plays exactly the same role for averaging operators on variable Lebesgue spaces as the Muckenhoupt $A_p$ condition does for weighted Lebesgue spaces (Proposition 4.33).

**Proposition 4.47.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, for any cube $Q_0$ there exists a constant $C_1 > 0$ such that*

$$\|\chi_{Q_0}\|_{L^{p(\cdot)}(\Omega)}\|\chi_{Q_0}\|_{L^{p'(\cdot)}(\Omega)} \le C_1|Q_0| \tag{4.27}$$

*if and only if there exists $C_2 > 0$ such that for all $f \in L^{p(\cdot)}(\Omega)$, $\operatorname{supp}(f) \subset \Omega$,*

$$\|A_{Q_0}f\|_{L^{p(\cdot)}(\Omega)} \le C_2\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{4.28}$$

*As a consequence, $p(\cdot) \in K_0(\Omega)$ if and only if the operators $A_Q$ are uniformly bounded on $L^{p(\cdot)}(\Omega)$ for every cube $Q$.*

*Remark 4.48.* From the proof we have that $K_{p(\cdot)}^{-1}C_2 \le C_1 \le 2k_{p'(\cdot)}^{-1}C_2$, where $K_{p(\cdot)}$ and $k_{p'(\cdot)}$ are the constants from Theorem 2.34.

*Proof.* Fix $Q_0$ and let $f \in L^{p(\cdot)}(\Omega)$. Suppose first that (4.27) holds; then by Hölder's inequality (Theorem 2.26),

$$\begin{aligned}\|A_{Q_0}f\|_{L^{p(\cdot)}(\Omega)} &= |f_{Q_0}|\|\chi_{Q_0}\|_{L^{p(\cdot)}(\Omega)}\\ &\le K_{p(\cdot)}|Q_0|^{-1}\|f\|_{L^{p(\cdot)}(\Omega)}\|\chi_{Q_0}\|_{L^{p'(\cdot)}(\Omega)}\|\chi_{Q_0}\|_{L^{p(\cdot)}(\Omega)}\\ &\le K_{p(\cdot)}C_1\|f\|_{L^{p(\cdot)}(\Omega)}.\end{aligned}$$

Conversely, if (4.28) holds, then by Theorem 2.34 there exists a function $g \in L^{p(\cdot)}(\Omega)$, $\|g\|_{L^{p(\cdot)}(\Omega)} \le 1$, such that

$$\begin{aligned}\|\chi_{Q_0}\|_{L^{p(\cdot)}(\Omega)}\|\chi_{Q_0}\|_{L^{p'(\cdot)}(\Omega)} &\le 2k_{p'(\cdot))}^{-1}\|\chi_{Q_0}\|_{L^{p(\cdot)}(\Omega)}\int_\Omega \chi_{Q_0}(x)g(x)\,dx\\ &= 2k_{p'(\cdot)}^{-1}|Q_0|\|A_{Q_0}g\|_{L^{p(\cdot)}(\Omega)} \le 2k_{p'(\cdot)}^{-1}C_2|Q_0|\|g\|_{L^{p(\cdot)}(\Omega)} \le 2k_{p'(\cdot)}^{-1}C_2|Q_0|.\end{aligned}$$

□

*Remark 4.49.* Proposition 4.47 remains true if we replace the averages $A_Q$ with averages $A_B$ over balls, and replace the $K_0$ condition by the corresponding inequality for balls:

$$\|\chi_B\|_{L^{p(\cdot)}(\Omega)}\|\chi_B\|_{L^{p'(\cdot)}(\Omega)} \leq C|B|.$$

On $\mathbb{R}^n$ the two conditions are equivalent: this follows from the fact that given any ball $B$ it is contained in a cube $Q$ such that $|B| \leq C(n)|Q|$, and the same is true with the roles of balls and cubes reversed. This is the analog of the corresponding fact for weighted norm inequalities: see Remark 4.34. This observation will be useful in our discussion of convolution operators in Chap. 5.

As a corollary to Proposition 4.47 we get a necessary condition for the boundedness of the maximal operator.

**Corollary 4.50.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$ or if it satisfies the weak type inequality*

$$\|t\chi_{\{x:Mf(x)>t\}}\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}, \qquad t > 0,$$

*then $p(\cdot) \in K_0(\Omega)$.*

*Proof.* Since the strong type inequality implies the weak type inequality, it will suffice to show that the latter implies the $K_0$ condition. Fix $f \in L^{p(\cdot)}(\Omega)$ and a cube $Q$. Then for all $x \in Q$,

$$Mf(x) \geq \fint_Q |f(y)|\,dy,$$

and so for all $t < \fint_Q |f(y)|\,dy$,

$$t\|\chi_Q\|_{p(\cdot)} \leq \|t\chi_{\{x:Mf(x)>t\}}\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

If we take the supremum over all such $t$, we get that

$$\|A_Q f\|_{p(\cdot)} \leq \fint_Q |f(y)|\,dy\,\|\chi_Q\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Since the constant is independent of $f$ and $Q$, we have that all the averaging operators are uniformly bounded. Therefore, by Proposition 4.47, $p(\cdot) \in K_0(\Omega)$. □

By Theorem 4.35, $w \in A_p$ is both necessary and sufficient for the maximal operator to be bounded on $L^p(w)$. But, despite their similarities this is a fundamental difference between the $K_0$ and $A_p$ conditions: the $K_0$ condition is not sufficient for the maximal operator to be bounded on $L^{p(\cdot)}$, even if combined with the $LH_0$ condition.

*Example 4.51.* There exists $p(\cdot) \in K_0(\mathbb{R}) \cap LH_0(\mathbb{R})$ such that the maximal operator is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

*Proof.* Let $\phi$ be a $C^\infty$ function such that $\operatorname{supp}(\phi) \subset [-1/2, 1/2]$, $0 \le \phi(x) \le 1$, and $\phi(x) = 1$ if $x \in [-1/4, 1/4]$. Define the exponent $p(\cdot) \in \mathcal{P}(\mathbb{R})$ by

$$p(x) = 3 + \sum_{k=1}^{\infty} \phi(x - e^k).$$

Then $3 \le p(x) \le 4$, and $p(\cdot) \in LH_0(\mathbb{R})$.

To show that $p(\cdot) \in K_0(\mathbb{R})$, fix an interval $Q$. Since $p(\cdot) \in LH_0(\mathbb{R})$, if $|Q| \le 1$, then by the first half of the proof of Proposition 4.57 below, we have that

$$\|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{p'(\cdot)} \le C|Q|.$$

Now suppose that for some $j \ge 1$, $e^{j-1} < |Q| \le e^j$. For each $k \in \mathbb{N}$, define the intervals $A_k = [e^k - 1/2, e^k + 1/2]$. Since $\rho(\chi_{A_k}) = |A_k| = 1$, by Corollary 2.23, $\|\chi_{A_k}\|_{p(\cdot)} = 1$. The interval $Q$ intersects at most $j_0 \le j$ of these intervals; denote them by $A_{k_1}, \dots, A_{k_{j_0}}$, and let $P = Q \setminus \bigcup_{i=1}^{j_0} A_{k_i}$. Then, since for all $j \ge 1$, $j \le 2e^{(j-1)/3}$,

$$\|\chi_Q\|_{p(\cdot)} \le \sum_{i=1}^{j_0} \|\chi_{A_{k_i}}\|_{p(\cdot)} + \|\chi_P\|_{p(\cdot)} \le j_0 + |P|^{1/3} \le j + |Q|^{1/3} \le 3|Q|^{1/3}.$$

Essentially the same computation shows that $\|\chi_Q\|_{p'(\cdot)} \le 3|Q|^{2/3}$, so if we multiply these estimates we get that $p(\cdot) \in K_0(\mathbb{R})$.

To show that the maximal operator is not bounded on $L^{p(\cdot)}(\mathbb{R})$, define the sets $B_k = [e^k - 1/4, e^k + 1/4]$ and $C_k = [e^k - 3/2, e^k + 3/2] \setminus A_k$, and let

$$f(x) = \sum_{k=1}^{\infty} k^{-1/3} \chi_{B_k}(x).$$

Then

$$\rho(f) = \sum_{k=1}^{\infty} \int_{B_k} k^{-p(x)/3}\, dx = \frac{1}{2} \sum_{k=1}^{\infty} k^{-4/3} < \infty,$$

and so by Proposition 2.12, $f \in L^{p(\cdot)}(\mathbb{R})$. On the other hand, if $x \in C_k$,

$$Mf(x) \ge \fint_{A_k \cup C_k} f(y)\, dy = \frac{1}{3} \int_{B_k} f(y)\, dy = \frac{1}{6} k^{-1/3}.$$

Therefore,

$$\rho(Mf) \geq \sum_{k=1}^{\infty} \int_{C_k} Mf(x)^{p(x)}\,dx \geq \left(\frac{1}{6}\right)^3 \sum_{k=1}^{\infty} k^{-1} = \infty.$$

So again by Proposition 2.12, $Mf \notin L^{p(\cdot)}(\mathbb{R})$. This completes our proof. □

Example 4.51 shows that the $K_0$ condition does not let us control the maximal operator at infinity: like the $LH_0$ condition (cf. Example 3.23) it allows $p(\cdot)$ to oscillate at infinity and cause the maximal operator to be unbounded. However, it does let us control the maximal operator locally, and so it is a replacement for the $LH_0$ condition. The next theorem is the main result of this section.

**Theorem 4.52.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in K_0(\mathbb{R}^n) \cap N_\infty(\mathbb{R}^n)$. Then*

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{4.29}$$

Theorem 4.52 can be generalized to arbitrary domains $\Omega$; however, we must still assume that $p(\cdot) \in K_0(\mathbb{R}^n)$ and not simply in $K_0(\Omega)$. The reason for this is that there does not exist an extension theorem for the $K_0$ condition analogous to Lemma 2.4 for log Hölder continuity. We leave the statement and proof of the most general result to the reader; here we give one special case that we want to highlight.

**Corollary 4.53.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in K_0(\mathbb{R}^n)$. Then given any bounded set $\Omega$,*

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)},$$

*where the constant $C$ depends on $n$, $p(\cdot)$ and $|\Omega|$.*

*Remark 4.54.* By the symmetry of the definition, $p(\cdot) \in K_0(\mathbb{R}^n)$ if and only if $p'(\cdot) \in K_0(\mathbb{R}^n)$. Therefore, by Corollaries 4.50 and 4.53, if the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and $p_+ < \infty$, then it is bounded on $L^{p'(\cdot)}(\Omega)$ for any bounded set $\Omega$. This fact complements the conditions in Theorem 4.37; we will return to this "duality" property in Sect. 4.5 below.

For the proof of Theorem 4.52 we need two lemmas. (We will prove Corollary 4.53 immediately after we prove the theorem.) The first shows that the $K_0$ condition is actually sufficient for a modular inequality. This result should be compared to the negative result in Theorem 3.31.

**Lemma 4.55.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < \infty$, suppose $p(\cdot) \in K_0(\mathbb{R}^n)$. Let $f \in L^{p(\cdot)}(\mathbb{R}^n)$. If there exist a cube $Q$ and constants $c_1, c_2 > 0$ such that $\fint_Q |f(x)|\,dx \geq c_1$ and $\|f\|_{p(\cdot)} \leq c_2$, then there exists a constant $C$ depending only on $p(\cdot)$, $c_1$, $c_2$, such that*

$$\int_Q \left( \fint_Q |f(y)|\,dy \right)^{p(x)} dx \le C \int_Q |f(x)|^{p(x)}\,dx.$$

*Proof.* Since $p'(\cdot)_- = (p_+)' > 1$, by continuity there exists a constant $\alpha > 0$ such that

$$\int_Q \alpha^{p'(y)-1}\,dy = \int_Q |f(x)|\,dx.$$

Given this value $\alpha$, we have that

$$\begin{aligned}
&\int_Q \left( \fint_Q |f(y)|\,dy \right)^{p(x)} dx \\
&\qquad = \int_Q \left( \fint_Q \alpha^{p'(y)-1}\,dy \right)^{p(x)} dx \\
&\qquad = \left( \int_Q \alpha^{-p(x)} \left( \fint_Q \alpha^{p'(y)}\,dy \right)^{p(x)-1} dx \right) \cdot \fint_Q \alpha^{p'(y)}\,dy \\
&\qquad = \left( \fint_Q \left( \fint_Q \alpha^{p'(y)-p'(x)}\,dy \right)^{p(x)-1} dx \right) \cdot \int_Q \alpha^{p'(y)}\,dy.
\end{aligned}$$

We estimate the two terms on the right-hand side separately. The second is straightforward:

$$\begin{aligned}
\int_Q \alpha^{p'(y)}\,dy &= 2\alpha \int_Q |f(y)|\,dy - \int_Q \alpha^{p'(y)}\,dy \\
&\le 2\alpha \int_{\{y\in Q:2\alpha|f(y)|>\alpha^{p'(y)}\}} |f(y)|\,dy \\
&= 2\alpha \int_{\{y\in Q:(2|f(y)|)^{p(y)-1}>\alpha\}} |f(y)|\,dy \\
&\le 2 \int_Q 2^{p(y)-1}|f(y)|^{p(y)}\,dy \\
&\le 2^{p_+} \int_Q |f(y)|^{p(y)}\,dy.
\end{aligned}$$

To finish the proof we will show that the first term is bounded by a constant. By our hypotheses on $f$,

$$c_1 \le \fint_Q \alpha^{p'(y)-1}\,dy \le \max\left(\alpha^{p'(\cdot)_+-1}, \alpha^{p'(\cdot)_--1}\right),$$

so $\alpha \geq d_1 > 0$, where $d_1$ depends only on $p(\cdot)$ and $c_1$. By the generalized Hölder's inequality (Theorem 2.26) and Proposition 2.21 (since $p_+ < \infty$),

$$\begin{aligned}
\int_Q \alpha^{p'(y)-1}\,dy &= \int_Q |f(x)|\,dx \\
&\leq K_{p(\cdot)}\|f\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \\
&\leq c_2 K_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \\
&= c_2 K_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \int_Q \left(\frac{1}{\|\chi_Q\|_{p'(\cdot)}}\right)^{p'(y)} dy \\
&\leq \int_Q \left(\frac{d_2}{\|\chi_Q\|_{p'(\cdot)}}\right)^{p'(y)-1} dy,
\end{aligned}$$

where $d_2 > 0$ depends only on $p(\cdot)$ and $c_2$. Therefore,

$$\alpha \leq \frac{d_2}{\|\chi_Q\|_{p'(\cdot)}}.$$

For each $x \in Q$ partition $Q$ into $E_+(x) = \{y \in Q : p'(y) > p'(x)\}$ and $E_-(x) = Q \setminus E_+(x)$. Then

$$\begin{aligned}
&\int_Q \alpha^{p'(y)-p'(x)}\,dy \\
&\qquad = \int_{E_+(x)} \alpha^{p'(y)-p'(x)}\,dy + \int_{E_-(x)} \alpha^{p'(y)-p'(x)}\,dy \\
&\qquad \leq \int_{E_+(x)} \left(\frac{d_2}{\|\chi_Q\|_{p'(\cdot)}}\right)^{p'(y)-p'(x)} dy + \int_{E_-(x)} d_1^{p'(y)-p'(x)}\,dy \\
&\qquad \leq D_1\|\chi_Q\|_{p'(\cdot)}^{p'(x)} \int_Q \left(\frac{1}{\|\chi_Q\|_{p'(\cdot)}}\right)^{p'(y)} dy + D_2|Q| \\
&\qquad \leq D_1\|\chi_Q\|_{p'(\cdot)}^{p'(x)} + D_2|Q|,
\end{aligned}$$

where $D_1 = \max(1, d_2^{p'(\cdot)_+ - p'(\cdot)_-})$ and $D_2 = \max(1, d_1^{p'(\cdot)_- - p'(\cdot)_+})$.

Therefore, using the fact that $p(\cdot) \in K_0$ (with constant $C_K$) and again by Proposition 2.21,

$$\fint_Q \left(\fint_Q \alpha^{p'(y)-p'(x)}\,dy\right)^{p(x)-1} dx$$

$$\leq \fint_Q \left( D_1 \frac{\|\chi_Q\|_{p'(\cdot)}^{p'(x)}}{|Q|} + D_2 \right)^{p(x)-1} dx$$

$$\leq (2D_1)^{p_+-1} \int_Q \left( \frac{\|\chi_Q\|_{p'(\cdot)}}{|Q|} \right)^{p(x)} dx + (2D_2)^{p_+-1}$$

$$\leq (2D_1)^{p_+-1} C_K^{p_+} \int_Q \left( \frac{1}{\|\chi_Q\|_{p(\cdot)}} \right)^{p(x)} dx + (2D_2)^{p_+-1}$$

$$= (2D_1)^{p_+-1} C_K^{p_+} + (2D_2)^{p_+-1}.$$

This completes our proof. □

Our second lemma is a corollary to Lemma 4.55. It shows that there is a deep connection between the $K_0$ condition and the Muckenhoupt $A_p$ weights. Its subtlety can be seen by comparing this result with Lemma 4.41.

**Lemma 4.56.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $p(\cdot) \in K_0(\mathbb{R}^n)$. Then given any cube $Q_0$, and constants $b_1, b_2 > 0$, for any $t$ such that $b_1 \leq t \leq \frac{b_2}{\|\chi_{Q_0}\|_{p(\cdot)}}$, there exist constants $C_0$, $s > 1$ such that*

$$\left( \fint_{Q_0} t^{sp(x)}\, dx \right)^{1/s} \leq C_0 \fint_{Q_0} t^{p(x)}\, dx.$$

*The constants $s$ and $C_0$ depend only on $b_1$, $b_2$ and $p(\cdot)$.*

*Proof.* By Corollary 4.27, it suffices to show that there exist $\alpha, \beta$ such that the weights $w_t(x) = t^{p(x)}$ satisfy Property (1) of Lemma 4.25 on every cube $Q \subset Q_0$ with a constant depending only on $b_1$, $b_2$ and $p(\cdot)$. Let $\alpha = 1/2$. Fix such a cube $Q$ and let $E \subset Q$ be any measurable subset with $|E| \geq |Q|/2$. Define $f = t\chi_E$. Then

$$\fint_Q |f(y)|\, dy = \fint_Q t\chi_E(y)\, dy = t\frac{|E|}{|Q|} \geq \frac{b_1}{2},$$

and

$$\|f\|_{p(\cdot)} = t\|\chi_E\|_{p(\cdot)} \leq b_2 \frac{\|\chi_E\|_{p(\cdot)}}{\|\chi_{Q_0}\|_{p(\cdot)}} \leq b_2.$$

Therefore, $f$ satisfies the hypotheses of Lemma 4.55 with $c_1 = b_1/2$ and $c_2 = b_2$, and so there exists a constant $C$ depending only on $b_1$, $b_2$ and $p(\cdot)$ such that

$$\int_Q t^{p(x)}\, dx \leq \left( \frac{|Q|}{|E|} \right)^{p_+} \int_Q \left( \fint_Q |f(y)|\, dy \right)^{p(x)} dx$$

$$\leq 2^{p_+} C \int_Q |f(x)|^{p(x)}\, dx = 2^{p_+} C \int_E t^{p(x)}\, dx.$$

Hence, $2^{-p_+}C^{-1}w_t(Q) \leq w_t(E)$. Let $\beta = 2^{-p_+}C^{-1}$; since this holds for every such cube $Q$, our proof is complete. □

*Proof of Theorem 4.52.* We will not prove directly that $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Rather, we will prove one of the equivalent conditions given in Sect. 4.3. More precisely, by Theorem 4.37 and Remark 4.38 it will suffice to show that there exists $s_0 > 1$ such that for all $s$, $1 < s < s_0$, $M$ is bounded on $L^{sp(\cdot)}(\mathbb{R}^n)$ and

$$\|Mf\|_{L^{sp(\cdot)}(\mathbb{R}^n)} \leq C(n, p(\cdot))(s-1)^{-1/p_-}\|f\|_{sp(\cdot)}.$$

Fix $s$, $1 < s < s_0$; the precise value of $s_0$ will be determined below, but for now we will assume $s_0 \leq 2$ so that $s - 1 \leq 1$. Our proof follows the pattern of the proofs of Theorems 3.16 and 4.7, and we will refer to them for some details that are the same. Arguing as we did at the beginning of the proof of Theorem 3.16 (p. 98), we may make the same reductions. Therefore, we can assume that $f$ is a non-negative, bounded function of compact support, and $\|f\|_{sp(\cdot)} = 1$. Decompose $f$ as $f_1 + f_2$, where

$$f_1 = f\chi_{\{x:f(x)>1\}}, \qquad f_2 = f\chi_{\{x:f(x)\leq 1\}}.$$

Then $\|Mf\|_{sp(\cdot)} \leq \|Mf_1\|_{sp(\cdot)} + \|Mf_2\|_{sp(\cdot)}$, and we will estimate each term separately. The estimate of the second term is the same as the estimate for this term in the proof of Theorem 4.7 (p. 131). It follows from Definition 4.4 that $p(\cdot) \in N_\infty$ implies $sp(\cdot) \in N_\infty$: since $s > 1$,

$$\exp\left(-2\Lambda_\infty|sp(x) - sp_\infty|^{-1}\right) \leq \exp\left(-\Lambda_\infty|p(x) - p_\infty|^{-1}\right),$$

so $sp(\cdot)$ satisfies the $N_\infty$ condition with a constant independent of $s$. Therefore, there exists a constant $C$ depending only on $n$ and $p(\cdot)$ such that $\|Mf_2\|_{sp(\cdot)} \leq C\|f\|_{sp(\cdot)}$. Since $s - 1 \leq 1$, we thus have that

$$\|Mf_2\|_{sp(\cdot)} \leq C(s-1)^{-1/p_-}\|f\|_{sp(\cdot)}.$$

To complete the proof we will show the same estimate for $f_1$. It will suffice to prove that there exists $C > 1$ independent of $s$ such that

$$\int_{\mathbb{R}^n} Mf_1(x)^{sp(x)}\,dx \leq C(s-1)^{-1}; \tag{4.30}$$

for if this holds, since $p(x)/p_- \geq 1$ for all $x$ and $C(s-1)^{-1} > 1$,

$$\int_{\mathbb{R}^n}\left(\frac{Mf_1(x)}{(C(s-1)^{-1})^{1/p_-}}\right)^{sp(x)} dx \leq \left(\frac{s-1}{C}\right)^s \int_{\mathbb{R}^n} Mf_1(x)^{sp(x)}\,dx \leq 1,$$

and so $\|Mf_1\|_{sp(\cdot)} \leq C(s-1)^{-1/p_-}\|f\|_{sp(\cdot)}$.

To prove (4.30) we modify the decomposition argument used in the proof of Theorem 3.16 in the case when $p_+ < \infty$ (p. 99). Let $A = 4^n$ and for all integers $k \geq 0$ define

$$\Omega_k = \{x \in \mathbb{R}^n : Mf_1(x) > A^k\}.$$

Then, arguing as before,

$$\int_{\mathbb{R}^n} Mf_1(x)^{sp(x)}\,dx$$
$$= \int_{\{x:Mf_1(x)\leq 1\}} Mf_1(x)^{sp(x)}\,dx + \sum_{k=0}^{\infty} \int_{\Omega_k\setminus\Omega_{k+1}} Mf_1(x)^{sp(x)}\,dx.$$

The first term is easy to estimate: since $Mf_1(x) \leq 1$ and $f_1(x) > 1$ or $f_1(x) = 0$, and since $p_- > 1$, by Theorem 3.4 and Remark 3.5,

$$\begin{aligned}
\int_{\{x:Mf_1(x)\leq 1\}} Mf_1(x)^{sp(x)}\,dx &\leq \int_{\{x:Mf_1(x)\leq 1\}} Mf_1(x)^{sp_-}\,dx \\
&\leq (sp_-)'C(n) \int_{\mathbb{R}^n} f_1(x)^{sp_-}\,dx \\
&\leq (sp_-)'C(n) \int_{\mathbb{R}^n} f_1(x)^{sp(x)}\,dx \\
&\leq (p_-)'C(n)(s-1)^{-1}.
\end{aligned}$$

To estimate the second term, for each $k$ form the Calderón-Zygmund cubes $\{Q_j^k\}$ of $f_1$ at height $A^{k-1}$ (Lemma 3.9). Then

$$\Omega_k \subset \bigcup_j 3Q_j^k.$$

Let $E_j^k$ be defined as before, and let $F_j^k = Q_j^k \setminus \bigcup_i Q_i^{k+1}$. By the maximality of the Calderón-Zygmund cubes, the sets $F_j^k$ are pairwise disjoint for all $j$ and $k$. Further, $|F_j^k| \geq \frac{1}{2}|Q_j^k|$. To see this estimate we argue as we did in the proof of Theorem 4.22. By the properties of the Calderón-Zygmund cubes, for each $j$ and $k$,

$$\begin{aligned}
|\Omega_{k+1} \cap Q_j^k| = \sum_{Q_i^{k+1}} |Q_i^{k+1}| &\leq A^{-k} \sum_{Q_i^{k+1}} \int_{Q_i^{k+1}} f_1(y)\,dy \\
&\leq A^{-k} \int_{Q_j^k} f_1(y)\,dy \leq \frac{2^n A^{k-1}}{A^k}|Q_j^k| \leq \frac{1}{2}|Q_j^k|.
\end{aligned}$$

Thus, $|F_j^k| \geq \frac{1}{2}|Q_j^k|$.

Therefore, using again that $s \leq 2$, we have that

$$\sum_{k=0}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} Mf_1(x)^{sp(x)}\, dx$$
$$\leq \sum_{k=0}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} (A^{k+1})^{sp(x)}\, dx$$
$$\leq 3^{snp_+} A^{s2p_+} \sum_{k,j} \int_{E_j^k} \left( \fint_{3Q_j^k} f_1(y)\, dy \right)^{sp(x)} dx$$
$$\leq 2 \cdot 3^{n+2np_+} A^{4p_+} \sum_{k,j} \fint_{3Q_j^k} \left( \fint_{3Q_j^k} f_1(y)\, dy \right)^{sp(x)} dx \cdot |F_j^k|. \qquad (4.31)$$

To estimate (4.31) we need to apply Lemmas 4.55 and 4.56. Since $k \geq 0$, by the properties of the Calderón-Zygmund cubes,

$$\fint_{3Q_j^k} f_1(y)\, dy \geq 3^{-n} \fint_{Q_j^k} f_1(y)\, dy > 3^{-n} A^{k-1} \geq 12^{-n} > 0. \qquad (4.32)$$

On the other hand, since $f_1 > 1$ or $f_1 = 0$, by Proposition 2.18,

$$\|f_1\|_{p(\cdot)} \leq \|f_1^s\|_{p(\cdot)} = \|f_1\|_{sp(\cdot)}^s \leq 1. \qquad (4.33)$$

Therefore, since $p(\cdot) \in K_0(\mathbb{R}^n)$ with constant $C_K$,

$$\fint_{3Q_j^k} f_1(y)\, dy \leq |3Q_j^k|^{-1} \|f_1\|_{p(\cdot)} \|\chi_{3Q_j^k}\|_{p'(\cdot)} \leq \frac{C_K}{\|\chi_{3Q_j^k}\|_{p(\cdot)}}. \qquad (4.34)$$

In both (4.32) and (4.34) the bounds are independent of $j$ and $k$. Therefore, by Lemma 4.56 with $t = \fint_{3Q_j^k} f_1(y)\, dy$, there exists $s_0 > 1$ and a constant $C_0$ that depends only on $n$ and $p(\cdot)$ such that for all $s$, $1 < s < s_0$,

$$\left( \fint_{3Q_j^k} \left( \fint_{3Q_j^k} f_1(y)\, dy \right)^{sp(x)} dx \right)^{1/s} \leq C_0 \fint_{3Q_j^k} \left( \fint_{3Q_j^k} f_1(y)\, dy \right)^{p(x)} dx. \qquad (4.35)$$

Moreover, by Lemma 4.55,

$$\fint_{3Q_j^k} \left( \fint_{3Q_j^k} f_1(y)\, dy \right)^{p(x)} dx \leq C \fint_{3Q_j^k} f_1(x)^{p(x)}\, dx. \qquad (4.36)$$

By (4.35), (4.36), Theorem 3.4 and Remark 3.5 we can estimate (4.31) as follows:

$$\begin{aligned}
&\sum_{k,j} \fint_{3Q_j^k} \left( \fint_{3Q_j^k} f_1(y)\,dy \right)^{sp(x)} dx \cdot |F_j^k| \\
&\quad \le C_0^s \sum_{k,j} \left( \fint_{3Q_j^k} \left( \fint_{3Q_j^k} f_1(y)\,dy \right)^{p(x)} dx \right)^s \cdot |F_j^k| \\
&\quad \le C_0^s C_1^s \sum_{k,j} \left( \fint_{3Q_j^k} f_1(x)^{p(x)}\,dx \right)^s \cdot |F_j^k| \\
&\quad \le C_0^2 C_1^2 \sum_{k,j} \int_{F_j^k} M(f_1(\cdot)^{p(\cdot)})(x)^s\,dx \\
&\quad \le C_0^2 C_1^2 \int_{\mathbb{R}^n} M(f_1(\cdot)^{p(\cdot)})(x)^s\,dx \\
&\quad \le C(n, p(\cdot)) s' \int_{\mathbb{R}^n} f_1(x)^{sp(x)}\,dx \\
&\quad \le C(n, p(\cdot))(s-1)^{-1}.
\end{aligned}$$

Combining all of the above estimates we get (4.30); this completes our proof. □

*Proof of Corollary 4.53.* The proof is essentially the same as the proof of Theorem 4.52. The estimate for $f_1$ is the same, and the constant is independent of $\Omega$. To estimate $f_2$, instead of following the proof of Theorem 4.7, we instead use the argument from the proof of Corollary 3.18. This gives us a constant that depends on $n$, $p(\cdot)$ and $|\Omega|$. □

We conclude this section with three results on the relationship between log-Hölder continuity and the $K_0$ condition.

**Proposition 4.57.** *Given a set $\Omega$ and $p(\cdot) \in \Omega$, suppose $p_+ < \infty$. If $p(\cdot) \in LH(\Omega)$, then $p(\cdot) \in K_0(\Omega)$. In particular, if $\Omega$ is bounded and $p(\cdot) \in LH_0(\Omega)$, then $p(\cdot) \in K_0(\Omega)$.*

*Proof.* Since $p(\cdot) \in LH(\Omega)$ implies that the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, this follows from Corollary 4.50, at least when $p_- > 1$. However, we can give a direct proof. Fix any cube $Q$. Since $\rho_{p(\cdot)}(\chi_Q) = \rho_{p'(\cdot)}(\chi_Q) = |Q|$, by Corollary 2.22 we must have that $\|\chi_Q\|_{p(\cdot)}$ and $\|\chi_Q\|_{p'(\cdot)}$ are either both greater than or less than 1. If $\|\chi_Q\|_{p(\cdot)}, \|\chi_Q\|_{p'(\cdot)} \le 1$, by Corollary 2.23 and Lemma 3.24,

$$\|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{p'(\cdot)} \le |Q|^{1/p_+ + 1/p'(\cdot)_+} = |Q|^{1+1/p_+ - 1/p_-} \le C|Q|.$$

If $\|\chi_Q\|_{p(\cdot)}, \|\chi_Q\|_{p'(\cdot)} \geq 1$, then we estimate them using Lemma 3.26. Fix $N > 1$ such that

$$\int_{\mathbb{R}^n} (e + |x|)^{-Np_-}\,dx \leq 1/2.$$

By Proposition 2.21 and (3.11),

$$1 = \int_Q \|\chi_Q\|_{p(\cdot)}^{-p(x)}\,dx$$
$$\leq C\int_Q \|\chi_Q\|_{p(\cdot)}^{-p_\infty}\,dx + \int_Q \frac{dx}{(e+|x|)^{Np_-}} \leq C|Q|\|\chi_Q\|_{p(\cdot)}^{-p_\infty} + \frac{1}{2}.$$

Rearranging terms we get that $\|\chi_Q\|_{p(\cdot)} \leq C|Q|^{1/p_\infty}$. The same argument shows that $\|\chi_Q\|_{p'(\cdot)} \leq C|Q|^{1/p'_\infty}$ and so

$$\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \leq C|Q|^{1/p_\infty + 1/p'_\infty} = C|Q|.$$

□

One surprising feature of Proposition 4.57 is that even though $K_0$ cannot be used to control the maximal operator at infinity, which is where we use the $LH_\infty$ condition, in the proof we need to use $LH_\infty$ to show that log-Hölder continuity implies the $K_0$ condition. The next example shows that this hypothesis is in some sense necessary.

*Example 4.58.* On $\mathbb{R}$ let $p(\cdot)$ be a smooth, increasing function such that $p(x) = 2$ if $x \leq -1$ and $p(x) = 3$ if $x > 1$. Then $p(\cdot) \in LH_0(\mathbb{R}) \setminus LH_\infty(\mathbb{R})$. Let $Q = [-N, N]$, $N > 3$, and let $Q_- = [-N, -1]$, $Q_+ = [1, N]$. Then by Corollary 2.23,

$$\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \geq \|\chi_{Q_-}\|_{p(\cdot)}\|\chi_{Q_+}\|_{p'(\cdot)}$$
$$\geq |Q_-|^{1/p_+(Q_-)}|Q_+|^{1/p'(\cdot)_-(Q_+)} = |Q_-|^{1/2}|Q_+|^{2/3} \geq c|Q|N^{1/6}.$$

Since we can take $N$ arbitrarily large, $p(\cdot) \notin K_0(\mathbb{R})$.

While log-Hölder continuity implies the $K_0$ condition, the converse is not true. Example 4.51 gives an exponent function $p(\cdot) \in K_0 \setminus LH_\infty$. There also exist exponents $p(\cdot) \in K_0 \setminus LH_0$.

*Example 4.59.* Given $a$, $0 < a < 1$, let $I_a = (-e^{-3^{1/a}}, e^{-3^{1/a}}) \subset \mathbb{R}$. Then the exponent $p(\cdot) \in \mathcal{P}(I_a)$ defined by

$$\frac{1}{p(x)} = \frac{1}{2} + \frac{1}{\log(1/|x|)^a}$$

is in $K_0(I_a) \setminus LH_0(I_a)$.

If we extend the exponent $p(\cdot)$ in Example 4.59 to be a constant on $\mathbb{R} \setminus I_a$, then a straightforward modification of the proof shows that $p(\cdot) \in K_0(\mathbb{R}) \cap N_\infty(\mathbb{R})$. Therefore, by the definition of $K_0$ and Remark 4.6, $p'(\cdot) \in K_0(\mathbb{R}) \cap N_\infty(\mathbb{R})$. Hence, by Theorem 4.52 the maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R})$. For $x \in I_a$,

$$p'(x) = 2 + \frac{4}{\log(1/|x|)^a - 2}.$$

This exponent function should be compared to Example 4.44. As was the case with the examples we gave in Sect. 4.1, the symmetry and monotonicity of the exponent function play significant but not well understood roles in determining whether the maximal operator is bounded.

*Proof.* It is immediate that $p(\cdot) \notin LH_0(I_a)$, so we only need to show that $p(\cdot) \in K_0(I_a)$. We make two reductions. First, it will suffice to consider intervals $I \subset I_a$. For suppose $I \cap (\mathbb{R} \setminus I_a) \neq \emptyset$. If $I \cap I_a = \emptyset$, then the $K_0$ condition is vacuously true. Otherwise, let $I_0 = I_a \cap I$. If the $K_0$ condition holds for $I_0$, then

$$\|\chi_I\|_{L^{p(\cdot)}(I_a)} \|\chi_I\|_{L^{p'(\cdot)}(I_a)} = \|\chi_{I_0}\|_{L^{p(\cdot)}(I_a)} \|\chi_{I_0}\|_{L^{p'(\cdot)}(I_a)} \leq C_K |I_0| \leq C_K |I|.$$

Second, it will suffice to show that $p(\cdot)$ satisfies the $K_0$ condition for any interval $I \subset [0, e^{-3^{1/a}})$. For suppose this is the case and let $I \subset I_a$ be such that it intersects $(-e^{-3^{1/a}}, 0)$. If it is entirely contained in this interval, let $J = \{x : -x \in I\}$. Then $J \subset [0, e^{-3^{1/a}})$ and by the symmetry of $p(\cdot)$,

$$\|\chi_I\|_{p(\cdot)} \|\chi_I\|_{p'(\cdot)} = \|\chi_J\|_{p(\cdot)} \|\chi_J\|_{p'(\cdot)} \leq C_K |J| = C_K |I|.$$

Now suppose that $0 \in I$. Let $J$ be the smallest interval centered at the origin that contains $I$, and define $J_+ = J \cap [0, e^{-3^{1/a}})$. Then $|J_+| \leq |I| \leq 2|J_+|$ and again by the symmetry of $p(\cdot)$,

$$\begin{aligned}
\|\chi_I\|_{p(\cdot)} \|\chi_I\|_{p'(\cdot)} &\leq \|\chi_J\|_{p(\cdot)} \|\chi_J\|_{p'(\cdot)} \\
&= \|\chi_{J \setminus J_+} + \chi_{J_+}\|_{p(\cdot)} \|\chi_{J \setminus J_+} + \chi_{J_+}\|_{p'(\cdot)} \\
&\leq \big(\|\chi_{J \setminus J_+}\|_{p(\cdot)} + \|\chi_{J_+}\|_{p(\cdot)}\big)\big(\|\chi_{J \setminus J_+}\|_{p'(\cdot)} + \|\chi_{J_+}\|_{p'(\cdot)}\big) \\
&= 4\|\chi_{J_+}\|_{p(\cdot)} \|\chi_{J_+}\|_{p'(\cdot)} \\
&\leq 4C_K |J_+| \\
&\leq 4C_K |I|.
\end{aligned}$$

To prove that $p(\cdot)$ satisfies the $K_0$ condition for intervals contained in $I_a$, we will first show that $p(\cdot)$ satisfies it on any interval $I = (x, y) \subset [0, e^{-3^{1/a}})$ such that $y - x \leq x$. To do so we will actually show that $1/p(\cdot) \in LH_0(I)$ with constant depending only on $p(\cdot)$, for then, by Propositions 2.3 and 4.57, $p(\cdot) \in K_0(I)$. Given $z, w \in I$, $z < w$, we have that

$$w - z \le y - z \le 2x - z \le 2z - z = z.$$

Let $f(t) = \log(1/t)^{-a}$; then

$$Df(t) = at^{-1}\log(1/t)^{-a-1},$$
$$D^2 f(t) = -at^{-2}\log(1/t)^{-a-2}\big(\log(1/t) - (1+a)\big).$$

For $t < e^{-(1+a)}$, $D^2 f < 0$. Since for all $a$, $e^{-3^{1/a}} < e^{-(1+a)}$, we have that $Df$ is a decreasing function on $(0, e^{-3^{1/a}})$. Therefore, by the mean value theorem,

$$\begin{aligned} \left|\frac{1}{p(z)} - \frac{1}{p(w)}\right| &= |f(z) - f(w)| \\ &\le |Df(z)||z-w| \\ &= Df(z)\cdot a\,(Df(w-z))^{-1}\log(1/|w-z|)^{-a-1} \\ &\le Df(z)\cdot a Df(z)^{-1}\log(1/|w-z|)^{-a-1} \\ &\le \frac{a}{-\log(w-z)}. \end{aligned} \tag{4.37}$$

Finally, we will show that $p(\cdot)$ satisfies the $K_0$ condition for intervals $I = (x, y)$, where $y - x \ge x$. In this case it will suffice to show this for intervals of the form $J = (0, y)$. For if it were true in this special case, then

$$\begin{aligned} \|\chi_I\|_{p(\cdot)}\|\chi_I\|_{p'(\cdot)} &\le \|\chi_J\|_{p(\cdot)}\|\chi_J\|_{p'(\cdot)} \le C_K|J| \\ &= C_K y \le C_K(y-x) + C_K x \le 2C_K(y-x) = 2C_K|I|. \end{aligned}$$

Moreover, given $y \in (0, e^{-3^{1/a}})$ and $J = (0, y)$, it will suffice to show that there exists a constant $C$ (independent of $k$) such that

$$\|\chi_J\|_{p(\cdot)} \le C|J|^{1/p(y)}. \tag{4.38}$$

For then, since $p'(\cdot)$ is an increasing function, by Corollary 2.23,

$$\|\chi_J\|_{p(\cdot)}\|\chi_J\|_{p'(\cdot)} \le C|J|^{1/p(y)}|J|^{1/p'(\cdot)_+} = C|J|^{1/p(y)}|J|^{1/p'(y)} = C|J|.$$

To prove (4.38), fix $k \ge 3^{1/a}$ such that $e^{-k} < y \le e^{-k+1}$. Define

$$\lambda = |J|^{1/p(y)} \ge |J|^{1/p_-} = |J|^{5/6} \ge e^{-5k/6}.$$

Suppose there exists a constant $C > 1$ such that

$$\int_J \lambda^{-p(t)}\,dt \le C; \tag{4.39}$$

then by the convexity of the modular (Proposition 2.7) we have that

$$\rho_{p(\cdot)}\big((C|J|^{1/p(y)})^{-1}\chi_J\big) \le C^{-1}\int_J \lambda^{-p(t)}\,dt \le 1,$$

which in turn gives us (4.38).

To prove (4.39) we replace the integral by an infinite series: since $p(\cdot)$ is decreasing,

$$\begin{aligned}\int_J \lambda^{-p(t)}\,dt &\le \sum_{j=k}^{\infty}\int_{e^{-j}}^{e^{-j+1}} \lambda^{-p(t)}\,dt \\ &\le e\sum_{j=k}^{\infty} e^{-j}\lambda^{-p(e^{-j})} = e\sum_{j=k}^{\infty} e^{-(1-a)j}\exp\big(-aj + \log(1/\lambda)p(e^{-j})\big).\end{aligned}$$

Rewrite the exponent in the final term as

$$-aj + \log(1/\lambda)p(e^{-j}) = -aj + \log(1/\lambda)\frac{2}{2j^{-a}+1},$$

and define the function

$$h(x) = -ax + \log(1/\lambda)\frac{2}{2x^{-a}+1}.$$

We will show that $h$ is decreasing for all $x \ge k$. Taking the derivative, we have that

$$Dh = -a + \frac{4a\log(1/\lambda)}{x^{1-a}(2+x^a)^2};$$

hence, $Dh < 0$ if

$$\frac{\log(1/\lambda)}{x^{1-a}(2+x^a)^2} < 1/4.$$

Since $\lambda \ge e^{-5k/6}$ and $x \ge k \ge 3^{1/a} > 1$,

$$\frac{\log(1/\lambda)}{x^{1-a}(2+x^a)^2} \le \frac{5k}{6k^{1-a}(2+k^a)^2} \le \frac{5k^a}{6k^a(2+k^a)} \le \frac{1}{6} < \frac{1}{4}.$$

Given that $h$ is decreasing, we have that

$$\begin{aligned}
&\sum_{j=k}^{\infty} e^{-(1-a)j} \exp\big(-aj + \log(1/\lambda)p(e^{-j})\big) \\
&\qquad \le \exp\big(-ak + \log(1/\lambda)p(e^{-k})\big) \sum_{j=k}^{\infty} e^{-(1-a)j} \\
&\qquad \le Ce^{-k}\lambda^{-p(e^{-k})} \\
&\qquad \le C|J||J|^{-p(e^{-k})/p(y)} \\
&\qquad = C|J|^{1-p(e^{-k})/p(y)}.
\end{aligned}$$

To complete the proof of (4.39) we need to show that the last term is uniformly bounded. Since $|J| < 1$,

$$|J|^{1-p(e^{-k})/p(y)} = \left(|J|^{1/p(e^{-k})-1/p(y)}\right)^{p(e^{-k})} \le \left(|J|^{1/p(e^{-k})-1/p(y)}\right)^{p_-},$$

and by Corollary 3.24 the right-hand side would be uniformly bounded if $1/p(\cdot) \in LH_0((e^{-k}, e^{-k+1}))$ with a constant independent of $k$. We can show this by arguing as we did above. If $e^{-k} < z < w < e^{-k+1}$, then $w - z \le e^{-k}(e-1) \le e^{-k}(e+1) \le (e+1)z$, so by modifying inequality (4.37) we get

$$\begin{aligned}
\left|\frac{1}{p(z)} - \frac{1}{p(w)}\right| &\le |Df(z)||z-w| \\
&\le aDf\left(\frac{w-z}{e+1}\right)(Df(w-z))^{-1}\log(1/|w-z|)^{-a-1} \\
&= a(e+1)\left(\frac{1}{\log((e+1)/|w-z|)}\right)^{1+a} \\
&\le \frac{a(e+1)}{-\log(w-z)}.
\end{aligned}$$

Therefore, $1/p(\cdot) \in LH_0((e^{-k}, e^{-k+1}))$ with constant $a(e+1)$. This completes our proof. □

## 4.5 A Necessary and Sufficient Condition

We conclude this chapter with a discussion of a necessary and sufficient condition for the maximal operator to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Though not easy to check for a given exponent function $p(\cdot)$, this condition has very important theoretical consequences. To state it we first define a generalization of the averaging operators $A_Q$.

**Definition 4.60.** Let $\mathcal{Q} = \{Q_j\}$ be a collection of pairwise disjoint cubes. Given a locally integrable function $f$, define the averaging operator $\mathcal{A}_{\mathcal{Q}}$ by

$$\mathcal{A}_{\mathcal{Q}} f(x) = \sum_j A_{Q_j} f(x) = \sum_j \fint_{Q_j} f(y)\, dy\, \chi_{Q_j}(x).$$

**Definition 4.61.** Given $\Omega$ and an exponent $p(\cdot) \in \mathcal{P}(\Omega)$, then $p(\cdot) \in \mathcal{A}$ if there exists a constant $C_{\mathcal{A}}$ such that given any set $\mathcal{Q}$ of disjoint cubes and any function $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|\mathcal{A}_{\mathcal{Q}} f\|_{p(\cdot)} \le C_{\mathcal{A}} \|f\|_{p(\cdot)}$.

If the collection $\mathcal{Q}$ consists of a single cube, then $\mathcal{A}_{\mathcal{Q}} f = A_Q f$, so by Proposition 4.47, $p(\cdot) \in \mathcal{A}$ implies that $p(\cdot) \in K_0(\mathbb{R}^n)$. The converse is false: there exist $p(\cdot) \in K_0 \setminus \mathcal{A}$. The exponent $p(\cdot)$ in Example 4.51 is one such: in the construction we showed that if $\mathcal{Q} = \{(e^k - 3/2, e^k + 3/2)\}_{k=1}^{\infty}$, then $\mathcal{A}_{\mathcal{Q}}$ is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

The condition $\mathcal{A}$ can be thought of as a generalization of the Muckenhoupt $A_p$ condition in the following sense: for $1 < p < \infty$, the operators $\mathcal{A}_{\mathcal{Q}}$ are uniformly bounded on $L^p(w)$ if and only if $w \in A_p$. Necessity follows from Proposition 4.33, and the sufficiency from the fact that given any collection $\mathcal{Q}$, $|\mathcal{A}_{\mathcal{Q}} f(x)| \le Mf(x)$. An important difference, however, is that the $A_p$ condition is a "geometric" condition depending only on the behavior of $w$ on cubes, while the condition $\mathcal{A}$ requires testing the operators on all functions $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

The importance of the condition $\mathcal{A}$ is shown by the following two results.

**Theorem 4.62.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $p(\cdot) \in \mathcal{A}$, then the maximal operator satisfies the weak type inequality*

$$\|t\chi_{\{x: Mf(x) > t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

*Proof.* As in the proof of the weak type inequality in Theorem 3.16 (p. 106) we use the Calderón-Zygmund decomposition, but the argument is simpler. Just as in that proof, it suffices to prove it for bounded, non-negative functions of compact support. Fix such a function $f$ and fix $t > 0$. Since $f$ has compact support, the set $\{x : Mf(x) > t\}$ is bounded. Let $\{Q_j\}$ be the Calderón-Zygmund cubes of $f$ at height $t/4^n$ (Lemma 3.9). Since

$$\{x \in \mathbb{R}^n : Mf(x) > t\} \subset \bigcup_j 3Q_j,$$

by the Besicovitch-Morse covering lemma (see de Guzmán [74]), there exists a constant $N$ (depending on the dimension) and collections of pairwise disjoint cubes $\mathcal{Q}_k = \{5Q_i^k\} \subset \{5Q_j\}$, $1 \le k \le N$, such that

$$\{x \in \mathbb{R}^n : Mf(x) > t\} \subset \bigcup_{k=1}^{N} \bigcup_i 5Q_i^k.$$

Therefore, by the properties of Calderón-Zygmund cubes and since $p(\cdot) \in \mathcal{A}$,

$$\begin{aligned}\|t\chi_{\{x:Mf(x)>t\}}\|_{p(\cdot)} &\leq \sum_{k=1}^{N} \Big\| t \sum_i \chi_{5Q_i^k} \Big\|_{p(\cdot)} \\ &\leq \sum_{k=1}^{N} 5^n \Big\| \sum_i \fint_{5Q_i^k} f(y)\,dy\, \chi_{5Q_i^k} \Big\|_{p(\cdot)} \\ &\leq \sum_{k=1}^{N} 5^n \|\mathcal{A}_{\mathcal{Q}_k} f\|_{p(\cdot)} \\ &\leq 5^n N C_{\mathcal{A}} \|f\|_{p(\cdot)}.\end{aligned}$$

□

**Theorem 4.63.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $1 < p_- \leq p_+ < \infty$. Then the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $p(\cdot) \in \mathcal{A}$.*

Since $|\mathcal{A}_{\mathcal{Q}} f(x)| \leq Mf$, one implication is immediate. The heart of the matter is the converse, and the proof is very long and quite technical. Therefore, we refer the reader to [82] for the details.

Here we concentrate on two corollaries. First, the proof itself shows that $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if it is bounded on $L^{rp(\cdot)}(\mathbb{R}^n)$ for some $r$, $0 < r < 1$. (We proved this in a different way in Theorem 4.37.) Second, we have that even though $M$ is not a linear operator, the boundedness of $M$ implies the "dual" inequality.

**Corollary 4.64.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $1 < p_- \leq p_+ < \infty$. Then $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $M$ is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$.*

*Proof.* By symmetry it will suffice to prove that if $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then it is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$. In this case, by Theorem 4.63 the operators $\mathcal{A}_{\mathcal{Q}}$ are uniformly bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. By Theorem 2.34, Fubini's theorem and Hölder's inequality (Theorem 2.26), given $f \in L^{p'(\cdot)}(\mathbb{R}^n)$, there exists $h \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|h\|_{p(\cdot)} \leq 1$, such that

$$\begin{aligned}\|\mathcal{A}_{\mathcal{Q}} f\|_{p'(\cdot)} &\leq C \int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{Q}} f(x) h(x)\,dx \\ &= C \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \fint_{Q_j} f(y)\,dy \cdot h(x) \chi_{Q_j}(x)\,dx \\ &= C \int_{\mathbb{R}^n} f(y) \mathcal{A}_{\mathcal{Q}} h(y)\,dy \\ &\leq C \|f\|_{p'(\cdot)} \|\mathcal{A}_{\mathcal{Q}} h\|_{p(\cdot)} \\ &\leq C \|f\|_{p'(\cdot)}.\end{aligned}$$

Since the final constant is the same for all operators $\mathcal{A}_Q$, again by Theorem 4.63 we have that the maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$. □

*Remark 4.65.* It would be very interesting to have a proof of Corollary 4.64 that did not rely on Theorem 4.63.

## 4.6 Notes and Further Results

### *4.6.1 References*

In the study of the maximal operator on variable Lebesgue spaces, the $LH_0$ and $LH_\infty$ conditions were initially referred to as "almost necessary." This was primarily because of Example 4.43, due to Pick and Růžička [291]. Somewhat later, Example 4.1 appeared in [62]; its construction is a modification of that of Pick and Růžička. As we noted in Sect. 3.7.2 above, Diening [80, 82] showed that the $LH_0$ was necessary for the pointwise inequalities used in the proofs of the boundedness of the maximal operator. (See Proposition 3.39.)

The $N_\infty$ condition was introduced by Nekvinda [282] in the case $p_+ < \infty$ and he proved the strong type inequality in Theorem 4.7. (The notation is new.) Earlier, a related condition for variable sequence spaces was considered by Nakano [279] (see also [245]). The case $p_+ = \infty$ is in [80, 82]. This proof makes explicit the connection between embedding $L^\infty$ in a variable Lebesgue space and $N_\infty$ condition. Our proofs of the strong type inequality and the weak type inequality are new. Proposition 4.8 when $p_+ < \infty$ is implicit in Nekvinda [282]. Proposition 4.9 showing that there are functions in $N_\infty \setminus LH_\infty$ was given in [42]. Proposition 4.10 was known but does not appear to have been stated explicitly. Example 4.13 is derived from a more general construction by Nekvinda [284], who gave a sufficient condition on "almost" monotone, radial exponents for the maximal operator to be bounded. The exact statement of Lemma 4.15 is new, but as we noted it is closely related to Lemma 3.26; consult the references given for this lemma in Sect. 3.7.1. A more general version, using the $N_\infty$ condition instead of $LH_\infty$, is due to Nekvinda [282].

The literature on the theory of Muckenhoupt $A_p$ weights is vast and it is beyond the scope of this section to give a detailed history of these results. For almost all the results of Sect. 4.2 we have followed Duoandikoetxea [96] and García-Cuerva and Rubio de Francia [140]; also see Grafakos [144]. For the Jones factorization theorem and the reverse Jensen inequality, mentioned in Remark 4.19, see [140]. There is a close connection between the constants $[w]_{A_p}$ and $[w]_{A_\infty}$: see Sbordone and Wik [328]. Corollary 4.27 is implicit in the literature; in the folklore the conclusion of this result is often stated as $w$ is in $RH_s$ (or $A_\infty$) "locally." The sharp reverse Hölder inequality in Proposition 4.29 is due to Lerner, Ombrosi and Pérez [234] (see also [233]). Lemma 4.30 is due to Stein [338] (see also [140]).

Proposition 4.33 relating the averaging operators and $A_p$ weights was first stated without proof by Jawerth [185] (see also Berezhnoi [27]). Theorem 4.35 was originally proved by Muckenhoupt [271]; also see the references above on weights. Remark 4.36 is well-known but often just assumed; our proof is from [140]. Theorem 4.37 was proved by Lerner and Ombrosi [233] for more general maximal operators on abstract Banach function spaces. The equivalence of Conditions (1) and (3) in this result was proved earlier by Diening [79]. The Rubio de Francia iteration algorithm which is at the heart of the proof plays an important role in the study of weighted norm inequalities; see [69] for more information and references. Lemma 4.41 and the proof of Theorem 3.31 are due to Lerner [228].

The $K_0$ condition was first considered by Berezhnoi [27] in the more general setting of weighted Banach function spaces. He states Proposition 4.47 without proof (see his Lemma 2.1); our proof is adapted from Diening [80] (see also [82]). The importance of the $K_0$ condition for the study of variable Lebesgue spaces was recognized by Kopaliani [212]. He referred to the class as $A_{p(\cdot)}$ to emphasize its connections with $A_p$ weights; our notation is new. He proved Theorem 4.52 with the $N_\infty$ condition replaced by the stronger assumption that $p(\cdot)$ is constant outside a large ball centered at the origin. His proof relies in a central way on the necessary and sufficient condition in Theorem 4.63. He proved Lemma 4.55, again using ideas from Diening [79]. Lerner [231] gave a new proof of Theorem 4.52 which used $A_p$ weights. Our proofs of Lemmas 4.55 and 4.56 are based on his; our proof of Theorem 4.52 is new. Proposition 4.57 is due to Diening [80] (see also [82]): see Sect. 4.6.2 below. Our proof is new. Example 4.51 is also due to Diening [80]. A different example was constructed by Kopaliani [215]. Example 4.59, an exponent in $K_0 \setminus LH_0$, is new.

The results in Sect. 4.5 are all due to Diening [79] and he first made the connection between condition $\mathcal{A}$ and the Muckenhoupt $A_p$ weights; also see [80, 82]. Diening also showed that this characterization can be extended to general Musielak-Orlicz spaces. The generalized averaging operator $\mathcal{A}_Q$ was considered in the context of Banach function spaces by Berezhnoi [27] and was implicit in Kopaliani [211].

### 4.6.2 More on the $K_0$ Condition

A different characterization of exponents in $K_0$ was given by Diening [80] (see also [82]).

**Proposition 4.66.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and a cube $Q$, let $P_Q$ and $P'_Q$ be the harmonic means of $p(\cdot)$ and $p'(\cdot)$ on $Q$:*

$$\frac{1}{P_Q} = \fint_Q \frac{1}{p(x)}\,dx, \qquad \frac{1}{P'_Q} = \fint_Q \frac{1}{p'(x)}\,dx.$$

*Then* $p(\cdot) \in K_0(\mathbb{R}^n)$ *if and only if*

$$\|\chi_Q\|_{p(\cdot)} \approx |Q|^{1/P_Q} \quad \textit{and} \quad \|\chi_Q\|_{p'(\cdot)} \approx |Q|^{1/P'_Q}.$$

The means $P_Q$ and $P'_Q$ at first seem somewhat unnatural, but they play a central role in the proof of Theorem 4.63, and more generally in the approach to variable Lebesgue spaces adopted in [80, 82].

Because of its similarity with the Muckenhoupt $A_p$ condition, it was initially hoped that the $K_0$ condition would be sufficient for the maximal operator to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, but as Example 4.51 showed, this is not the case. Furthermore, Kopaliani [215] constructed an example of $p(\cdot) \in K_0(\mathbb{R}^n)$, $n \geq 2$, such that the maximal operator does not satisfy the weak type inequality on $L^{p(\cdot)}(\mathbb{R}^n)$. On the other hand, Lerner [231] proved that if $p(\cdot)$ is radial and decreasing, and $1 < p_- \leq p_+ < \infty$, then the maximal operator satisfies the weak type inequality if and only if $p(\cdot) \in K_0(\mathbb{R}^n)$. As was the case with Examples 4.13 and 4.59, the fact that $p(\cdot)$ is radial and monotone plays a deep and subtle role in the proof.

There is another significant difference between $K_0$ and the Muckenhoupt $A_p$ condition. As part of the proof of Theorem 4.35 we showed that if $w \in A_p$, then there exists $q < p$ such that $w \in A_q$. However, Kopaliani [215] gave an example (in fact, the same example mentioned above) of $p(\cdot) \in K_0(\mathbb{R}^n)$ such that for any $r$, $1/p_- < r < 1$, $rp(\cdot) \notin K_0(\mathbb{R}^n)$. If exponent $p(\cdot) \in K_0$ had this property, then we could modify the proof of Theorem 4.52 to prove directly that the maximal operator is bounded on $L^{p(\cdot)}$ without having to use Theorem 4.37. Moreover, by Theorem 4.37 and Corollary 4.50, if the maximal operator is bounded on $L^{p(\cdot)}$, then $p(\cdot)$ has this property.

There is another condition, besides $N_\infty$, that can be combined with $K_0$ to prove that the maximal operator is bounded. An exponent $p(\cdot)$ satisfies condition $\mathcal{G}$ if there exists a constant $C_\mathcal{G}$ such that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, and any collection $\mathcal{Q} = \{Q_j\}$ of disjoint cubes in $\mathbb{R}^n$,

$$\sum_j \|f\chi_{Q_j}\|_{p(\cdot)}\|g\chi_{Q_j}\|_{p'(\cdot)} \leq C_\mathcal{G}\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

If $p(\cdot)$ is constant, then $p(\cdot) \in \mathcal{G}$: this is an immediate consequence of Minkowski's inequality for series. This condition was introduced by Berezhnoi [27] (see also [26]). Kopaliani [217] proved the following result.

**Proposition 4.67.** *Given* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$*, if* $1 < p_- \leq p_+ < \infty$ *and* $p(\cdot) \in \mathcal{G}$*, then the following are equivalent:*

1. $p(\cdot) \in K_0(\mathbb{R}^n)$*;*
2. $M$ *satisfies the weak type inequality on* $L^{p(\cdot)}(\mathbb{R}^n)$*;*
3. $M$ *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$*.*

### *4.6.3 Discontinuous Exponents*

Though $p(\cdot) \in LH_0$ is not necessary for the maximal operator to be bounded on $L^{p(\cdot)}$, all of the examples considered in this chapter have been uniformly continuous and have had a limit at infinity, at least in the weak sense of Proposition 4.10. After proving Theorem 4.63, Diening [79] conjectured that there is an exponent $p(\cdot)$ that does not have a limit at infinity but such that the maximal operator is bounded on $L^{p(\cdot)}$. Lerner [229] proved this conjecture. More surprisingly, he also showed that there exists an exponent that is discontinuous at the origin but $M$ is bounded.

*Example 4.68.* Given $p_0 > 1$ and $\mu \in \mathbb{R}$, define $p(\cdot) \in \mathcal{P}(\mathbb{R})$ by

$$p(x) = p_0 + \mu \sin(\log\log(1 + \max(|x|, |x|^{-1}))), \qquad x \neq 0.$$

Then for $\mu$ sufficiently close to 0, the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R})$, but $p(\cdot)$ does not have a limit at 0 or infinity.

Moreover, the behavior at 0 and infinity can be separated, so that, for instance, $p(\cdot)$ is continuous at 0 but does not have a limit at infinity. Lerner derived this example as a special case of a general result. To state it, recall that a function $f \in L^1_{\mathrm{loc}}(\Omega)$ is a function of bounded mean oscillation (denoted by $f \in BMO(\Omega)$) if

$$\|f\|_* = \sup_Q \fint_{Q\cap\Omega} |f(x) - f_Q|\,dx < \infty,$$

where the supremum is taken over all cubes that intersect $\Omega$. The space $BMO$ was introduced in [186]; see also [96, 144]. A function $\phi$ is called a multiplier of $BMO(\Omega)$ if given any $f \in BMO(\Omega)$, $\phi f \in BMO(\Omega)$. Multipliers were characterized in [184, 277, 337].

**Proposition 4.69.** *Given $p_0 > 1$, and a non-negative multiplier $\phi$ of $BMO(\mathbb{R}^n)$, there exists a positive constant $\mu_0 = \mu_0(n, p_0, \phi)$ such that if $0 < |\mu| \leq \mu_0$ and $p(x) = p_0 - \mu\phi(x)$, then the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. If $\phi_- > 0$ and $q(x) = p_0 + \mu\phi(x)$, then $M$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$.*

The proof of this proposition relies on the theory of weighted norm inequalities, the close connection between $BMO$ functions and $A_p$ weights, and a characterization of the multipliers of $BMO$. Using this characterization, a lengthy but straightforward calculation shows that $\phi(x) = 2 + \sin(\log\log(1 + \max(|x|, x|^{-1})))$ is a multiplier of $BMO(\mathbb{R})$ which then gives us Example 4.68.

Examples built using the function $\sin(\log\log(x))$ have a long history in the study of Orlicz spaces and other problems, beginning with Lindberg [239]. See also [126, 134, 181, 226, 254, 342].

Proposition 4.69 was generalized by Kapanadze and Kopaliani [188]. Given a function $f \in BMO(\Omega)$, define its $BMO$ modulus by

$$\gamma(f, r) = \sup_{|Q| \le r} \fint_{Q \cap \Omega} |f(x) - f_Q| \, dx, \qquad r > 0.$$

A function $f$ belongs to $VMO(\Omega)$ (the space of functions of vanishing mean oscillation) if $\gamma(f, r) \to 0$ as $r \to 0$. More generally, given any non-negative convex, increasing function $\sigma : [0, \infty) \to [0, \infty)$ such that $\sigma(0+) = 0$, we say that $f \in BMO^{\sigma}(\Omega)$ if $\gamma(f, r) \le C\sigma(r)$, and $f \in VMO^{\sigma}(\Omega)$ if $\sigma(r)^{-1}\gamma(f, r) \to 0$ as $r \to 0$. These spaces were introduced in [324, 334].

**Proposition 4.70.** *Let $\sigma(r) = \min(|\log(r)|^{-1}, e)$. Given a bounded domain $\Omega$ with Lipschitz boundary, if $p(\cdot) \in \mathcal{P}(\Omega)$ is such that $1 < p_- \le p_+ < \infty$ and $p(\cdot) \in VMO^{\sigma}(\Omega)$, then the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$. If $p(\cdot) \in BMO^{\sigma}(\Omega)$, then there exists $p_0 > 1$ such that if $q(\cdot) = p_0 + p(\cdot)$, then $M$ is bounded on $L^{q(\cdot)}(\Omega)$.*

If $\phi$ is a multiplier of $BMO$, then $\phi \in BMO^{\sigma}$ with $\sigma$ defined as above, so the second part of Proposition 4.70 is a local generalization of Proposition 4.69. Further, this characterization of exponents $p(\cdot)$ such that $M$ is bounded on $L^{p(\cdot)}(\Omega)$ is close to optimal in this scale. For a proof see [82].

**Proposition 4.71.** *If $p(\cdot) \in K_0(\mathbb{R}^n)$ and $p_+ < \infty$, then $p(\cdot) \in BMO^{\sigma}(\mathbb{R}^n)$, where $\sigma(r) = \min(|\log(r)|^{-1}, e)$.*

## *4.6.4 Perturbation of Exponents*

By Theorem 4.37, if the maximal operator is bounded on $L^{p(\cdot)}$, then there exists $r_0$, $1/p_- \le r_0 < 1$ such that if $s > r_0$, then $M$ is bounded on $L^{sp(\cdot)}$. Given this, Diening, Hästö and Nekvinda [86] asked the following question.

**Question 4.72.** If the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is it bounded on $L^{sp(\cdot)}(\mathbb{R}^n)$ for all $s > 1/p_-$?

In a similar vein, motivated by Proposition 4.69, Lerner [229] asked the following two questions.

**Question 4.73.** If $M$ is bounded on $L^{a+p(\cdot)}(\mathbb{R}^n)$ for some $a > 0$, is $M$ bounded on $L^{p(\cdot)}(\mathbb{R}^n)$?

**Question 4.74.** If $p \in \mathcal{P}(\mathbb{R}^n)$ is such that $1 < p_- \le p_+ < \infty$ and $p(\cdot)$ is a multiplier of $BMO$, is $M$ bounded on $L^{p(\cdot)}(\mathbb{R}^n)$?

Using his results, Lerner showed that if the answer to Question 4.73 is positive, then so is the answer to Question 4.74. Question 4.74 should also be compared to the second half of Proposition 4.70. Question 4.73 was generalized in [86] to the following.

**Question 4.75.** If $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ is $M$ bounded on $L^{a+p(\cdot)}(\mathbb{R}^n)$ for all $a > 1 - p_-$?

The answer to Questions 4.72–4.74 is no: in [231], Lerner constructed a pointwise multiplier of $BMO$ that yields counterexamples for each question. Question 4.75 remains open for $a > 0$.

Kopaliani [215] asked similar questions for exponents $p(\cdot) \in K_0(\mathbb{R}^n)$. As we noted above, he showed that there exists $p(\cdot) \in K_0(\mathbb{R}^n)$, $1 < p_- \leq p_+ < \infty$, such that $rp(\cdot) \notin K_0$ for any $r$, $1/p_- < r < 1$. On the other hand, by Hölder's inequality, for $s > 1$, $|A_Q f(x)|^s \leq A_Q(|f|^s)(x)$. Hence, by Propositions 2.18 and 4.47, if $p(\cdot) \in K_0$, then the operators $A_Q$ are uniformly bounded on $L^{sp(\cdot)}$, so $sp(\cdot) \in K_0$. He also showed that if $1 - p_- < a < 0$, then $a + p(\cdot) \notin K_0$. It is not known if $a + p(\cdot) \in K_0$ for all $a > 0$. The best that can be said (by Propositions 4.70 and 4.71) is that there exists $a > 0$ such that this is true.

### 4.6.5 *Weighted Variable Lebesgue Spaces*

The theory of $A_p$ weights and weighted norm inequalities for the maximal operator and other operators is a well developed and active field of research. It was natural, therefore, that a parallel theory of weighted norm inequalities on variable Lebesgue spaces would be studied. There has been a great deal of work in this area but the outlines of the theory have only recently begun to emerge.

The first problem is to define a weighted norm inequality on variable Lebesgue spaces. In the classical Lebesgue spaces there are two (almost) equivalent approaches: treating the weight as a measure and as a multiplier. The inequalities for the maximal operator in Theorem 4.35 treat the weight as a measure: the inequality

$$\left(\int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}$$

is equivalent to saying that the maximal operator is bounded on $L^p(w)$: the Lebesgue space defined with respect to the measure $w\,dx$. Alternatively, we can rewrite this inequality with the weight as a multiplier and with the norm the standard $L^p$ norm:

$$\|(Mf)w^{1/p}\|_p \leq \|f w^{1/p}\|_p.$$

In this case the weight $w$ is usually replaced by a new weight $w^p$; the norm inequality then becomes $\|(Mf)w\|_p \leq C\|fw\|_p$, and the Muckenhoupt $A_p$ condition can be rewritten as

$$\sup_Q |Q|^{-1}\|w\chi_Q\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'} < \infty.$$

(Compare this to inequality (4.16).) In the classical Lebesgue spaces the first approach is more common, although inequalities for fractional maximal operators and Riesz potentials (see Sects. 3.7.4 and 5.5) are usually stated with weights as multipliers: see Muckenhoupt and Wheeden [272].

When $p_+ < \infty$ or $|\Omega_\infty| = 0$, both approaches yield equivalent definitions. To treat weights as measures, given $p(\cdot)$ and a weight $w$, we define the weighted modular

$$\rho_{p(\cdot),w}(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)} w(x)\,dx.$$

The space $L^{p(\cdot)}(w,\Omega)$ would then consist of all measurable functions $f$ such that $\rho_{p(\cdot),w}(f/\lambda) < \infty$ for some $\lambda > 0$. In this case the theory developed in Chap. 2 holds without significant changes. (Such a general framework is developed in [82].) It is immediate from the definition that $f \in L^{p(\cdot)}(w,\Omega)$ if and only if $f w(\cdot)^{1/p(\cdot)} \in L^{p(\cdot)}(\Omega)$.

One curious feature of defining weights as measures is that when $p_+ = \infty$ there exist non-trivial weights such that $L^{p(\cdot)}(w,\Omega) = L^{p(\cdot)}(\Omega)$. For example, if we let $\Omega = (0,1)$, $p(x) = 1/x$ and $w(x) = 2^{1/x}$, then it is immediate that these spaces are the same. Such weights are called non-effective weights; for a complete characterization, see [130]. The problem of characterizing when one weighted variable Lebesgue space embeds into another (with a different weight) has also been considered: see [22, 97].

If $|\Omega_\infty| > 0$, these two approaches to defining weighted variable Lebesgue spaces are no longer equivalent. If $w$ is positive almost everywhere, then $L^\infty(w) = L^\infty$; however, if $w$ tends to 0 at a point (e.g., if $w(x) = |x|$) then there exist unbounded functions such that $fw \in L^\infty$.

Both approaches have been adopted in the variable Lebesgue spaces, though weights as multipliers seem to be the predominant approach. Essentially all work has been done assuming $p_+ < \infty$, so the difference is more a matter of outlook than of substance.

Weighted norm inequalities for the maximal operator have been generalized to the variable Lebesgue spaces using both approaches. For weights as multipliers the main result is the following.

**Definition 4.76.** Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and a weight $w$, we say that $w \in A_{p(\cdot)}$ if

$$[w]_{A_{p(\cdot)}} = \sup_Q |Q|^{-1} \|w\chi_Q\|_{p(\cdot)} \|w^{-1}\chi_Q\|_{p'(\cdot)} < \infty.$$

**Theorem 4.77.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $p_+ < \infty$ and $p(\cdot) \in LH(\mathbb{R}^n)$. Then the following are equivalent:*

1. $w \in A_{p(\cdot)}$;
2. *The Hardy-Littlewood maximal operator satisfies the weak type inequality*

$$\sup_{t>0} \|t\chi_{\{x\in\mathbb{R}^n : Mf(x)>t\}} w\|_{p(\cdot)} \le C\|fw\|_{p(\cdot)};$$

3. *If $p_- > 1$, then the maximal operator satisfies the strong type inequality*

$$\|(Mf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.$$

*Moreover, the necessity of the $A_{p(\cdot)}$ condition for either the strong or weak type inequality is true without the assumption that $p(\cdot) \in LH(\mathbb{R}^n)$.*

Theorem 4.77 was proved in [64]. The argument is based on a proof of Theorem 4.35 due to Christ and Fefferman [48]. Central to this approach is the non-trivial fact that if $w \in A_{p(\cdot)}$, then $w(\cdot)^{p(\cdot)} \in A_\infty$. (See p. 143.) Somewhat earlier and independently, another proof was given in [57]. (See also [82].) This one depends heavily on the machinery used to prove Theorem 4.63.

Weights as measures for the maximal operator were considered by Diening and Hästö [85].

**Definition 4.78.** Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, define $A^\dagger_{p(\cdot)}$ to be the set of weights $w$ such that

$$\sup_Q |Q|^{-P_Q}\|w\chi_Q\|_1\|w^{-1}\chi_Q\|_{p'(\cdot)/p(\cdot)} < \infty,$$

where $P_Q$ is the harmonic mean of $p(\cdot)$ on $Q$ (see Proposition 4.66), and $\|\cdot\|_{p'(\cdot)/p(\cdot)}$ is defined using Definition 2.16 even if $p'(\cdot)/p(\cdot)$ is not greater than or equal to 1.

*Remark 4.79.* One advantage of the $A^\dagger_{p(\cdot)}$ condition is that it is straightforward to show that if $w \in A^\dagger_{p(\cdot)}$, then $w \in A_{p_+}$. See [85] for details.

**Theorem 4.80.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in LH(\mathbb{R}^n)$. Then the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(w,\mathbb{R}^n)$ if and only if $w \in A^\dagger_{p(\cdot)}$.*

It follows from Theorems 4.77 and 4.80, and the equivalence of weights as measures and weights as multipliers discussed above, that if $w \in A^\dagger_{p(\cdot)}$, then $w(\cdot)^{1/p(\cdot)} \in A_{p(\cdot)}$; Diening and Hästö also gave a direct proof of this fact when $p(\cdot) \in LH(\mathbb{R}^n)$.

There are three open questions related to these results. The first is whether they can be generalized to include the case $p_+ = \infty$, and in particular $|\Omega_\infty| > 0$. In this case the conditions on the weights will no longer be equivalent. Muckenhoupt [271] showed that $\|Mf\|_{L^\infty(w)} \leq C\|f\|_{L^\infty(w)}$ if and only if $w(x) > 0$ almost everywhere, but $\|(Mf)w\|_\infty \leq C\|fw\|_\infty$ if and only if $w^{-1} \in A_1$. When $p(\cdot) = \infty$, the $A_{p(\cdot)}$ condition yields this latter condition, so it seems reasonable to conjecture that Theorem 4.77 holds when $p_+ = \infty$, with the hypothesis $1/p(\cdot) \in LH(\mathbb{R}^n)$. On the other hand, it is not clear how to extend the definition of $A^\dagger_{p(\cdot)}$ weights to include the case $p_+ = \infty$.

The second question is to what degree the hypotheses on $p(\cdot)$ can be relaxed. It is tempting to conjecture that the $A_{p(\cdot)}$ condition is sufficient, but if $w = 1$, then it becomes the $K_0$ condition which is not sufficient: see Example 4.51. We conjecture that on a bounded domain $\Omega$, $\|(Mf)w\|_{L^{p(\cdot)}(\Omega)} \leq C\|fw\|_{L^{p(\cdot)}(\Omega)}$ if and only if $w \in A_{p(\cdot)}$. More generally, Diening and Hästö [85] have conjectured that if the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then this inequality holds on $\mathbb{R}^n$.

The third question is whether the $A_{p(\cdot)}$ weights are the correct weights for other operators on the variable Lebesgue spaces. As we will discuss in Chap. 5, the Muckenhoupt $A_p$ weights are also the correct weights for a variety of operators in the classical Lebesgue setting so this conjecture is a natural one. Very little has been done on this question (but see below). Karlovich [191] considered this problem for the Cauchy integral on Carleson curves from the more general perspective of weighted Banach function spaces. More recently, Karlovich and Spitkovsky [196] proved that the $A_{p(\cdot)}$ condition is necessary and sufficient for this operator to be bounded on $L^{p(\cdot)}(\mathbb{R})$.

A number of results are known for maximal operators and other operators in the special case where the weights are power weights: weights of the form $\rho(x) = |x|^a$, $a \in \mathbb{R}$, or more generally,

$$\rho(x) = \prod_{k=1}^{m} |x - c_k|^{a_k},$$

or variable power weights of the form $\rho(x) = |x|^{a(x)}$ or $\rho(x) = (1 + |x|)^{a(x)}$. Variable power weights arise naturally when considering the Hardy operator and its variants (see Sect. 3.7.5 above), since depending on how the operator is defined the power of $x$ can be treated as either part of the operator or as a multiplier. Inequalities with power weights have also been proved for maximal operators, Riesz potentials and singular integral operators. The first results of this kind were proved by Samko [318], Kokilashvili and Samko [204, 205, 207, 208], and later by Samko, Shargorodsky and Vakulov [321–323] (here weights are treated as measures) and Mashiyev, Çekiç, Mamedov and Ogras [258]. Generalizations of power weights were given by Kokilashvili, Samko and Samko [201, 203], Samko, Samko and Vakulov [309, 310], and Kokilashvili and Samko [209]. Additional results were proved by Karlovich [192–194]. Kokilashvili, Samko and Samko [202] considered an $A_p$ type condition more restrictive than $A_{p(\cdot)}$ that included many kinds of power weights.

There is also a theory of two-weight norm inequalities for variable Lebesgue spaces: that is, conditions on pairs of weights $(u, v)$ such that an operator $T$ satisfies $\|uTf\|_{p(\cdot)} \le \|vf\|_{p(\cdot)}$. (There is an equivalent formulation if we treat the weights as measures.) In the classical Lebesgue spaces this is an area of ongoing research with various approaches and many partial results. For more information and extensive references, see [68, 69, 140].

In the variable Lebesgue spaces, some results are known for various operators: Hardy operators, maximal operators, Riesz potentials and singular integral operators. The conditions imposed on the weights are generalizations of the conditions from the classical Lebesgue spaces; they are often quite restrictive and vary from operator to operator. One interesting feature is that in some cases log-Hölder continuity is replaced by decay conditions in which the modular of continuity depends on the weight. For some recent results, see Kokilashvili and Meskhi [198, 199], Edmunds, Kokilashvili and Meskhi [98, 100], Mamedov and Harman [247, 248], Mamedov and Zeren [249–251], Asif, Kokilashvili and Meskhi [17], Bandaliev [20, 21], and [65]. Asif and Meskhi [18, 19] have also considered the problem of compactness for operators in the two-weight case. An extensive treatment of two-weight norm inequalities, with many references, can be found in the monograph by Meskhi [262].

Weighted modular inequalities have also been considered by several authors. Sinnamon [333] has shown that if the Hardy operator satisfies a modular strong type inequality with exponent $p(\cdot)$ (see (3.37)) with respect to the measure $w\,dx$, then $p(\cdot)$ is constant. However, Boza and Soria [35, 36] and Neugebauer [286] have both proved one-weight modular inequalities for the Hardy operator with non-constant exponents for non-negative, decreasing functions. Aguilar Cañestro and Ortega Salvador [8, 9] have proved two-weight modular weak type inequalities for the maximal operator, generalizing Theorem 3.34.

A few authors have considered applications of norm inequalities in weighted variable Lebesgue spaces. Diening and Hästö [85] state without proof a weighted Poincaré inequality. Edmunds and Rákosník [106] and Gao, Zhao and Zhang [139] proved weighted embedding theorems for variable Sobolev spaces (see Chap. 6 below). Edmunds, Kokilashvili and Meskhi [100] give an application to the norm convergence of Fourier series. And Boureanu [33] used weighted variable Lebesgue spaces to show the existence of weak solutions to an elliptic partial differential equation.

# Chapter 5
# Extrapolation in the Variable Lebesgue Spaces

In this chapter we consider some of the classical operators of harmonic analysis: convolution operators, singular integral operators, and Riesz potentials. Rather than treat each operator separately, we develop a general theory that builds upon the Rubio de Francia theory of extrapolation from the theory of weighted norm inequalities. The advantage of this approach is that it quickly yields sufficient conditions for these operators to be bounded on variable Lebesgue spaces; moreover, it can be applied to many other operators as well.

To motivate our approach we will first consider a model operator: convolution operators and approximate identities. We first prove the basic results of the theory to highlight their dependence on the fact that the classical Lebesgue spaces are translation invariant. We then develop those parts of the theory that remain true in the variable Lebesgue spaces. One result which does not extend to this setting is Young's inequality. For the positive results our key hypothesis on $p(\cdot)$ is that the maximal operator is bounded on $L^{p(\cdot)}$. Next we give an overview of the theory of extrapolation, both the classical version and more recent formulations, and then extend extrapolation to yield inequalities in the variable Lebesgue spaces. This result further illustrates the connection between weights and the variable Lebesgue spaces that we saw in Chap. 4. Our approach to extrapolation is influenced by recent work in this field. Though a more abstract formulation, it has the advantage of increased flexibility. Finally, we apply extrapolation to study two operators: singular integral operators and Riesz potentials. These examples illustrate how extrapolation and the theory of weighted norm inequalities can be combined to prove results in the variable setting.

## 5.1 Basic Properties of Convolutions

In this section we give the basic properties of convolution operators in the classical Lebesgue spaces. The results themselves will be necessary tools for our work, and the proofs will highlight the problems we encounter when attempting to generalize them to variable Lebesgue spaces.

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**Definition 5.1.** Given two locally integrable functions $f$ and $g$ defined on $\mathbb{R}^n$, their convolution is the function $f * g$ defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy,$$

wherever this integral is finite.

It is immediate from the definition that convolutions are linear: given functions $f, g, h$ and $a \in \mathbb{R}$, $f * (g+h) = f * g + f * h$ and $f * (ag) = (af) * g = a(f * g)$ whenever all the terms are finite. Further, by a change of variables, we have that $f * g = g * f$. The integrability of convolutions is given by the following proposition.

**Proposition 5.2.** *Given measurable functions $f$ and $g$, the following are true:*

1. *For all $p$, $1 \leq p \leq \infty$, if $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, then $f * g \in L^\infty(\mathbb{R}^n)$ and*
$$\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'}. \tag{5.1}$$
2. *For all $p$, $1 \leq p \leq \infty$, if $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in L^p(\mathbb{R}^n)$ and*
$$\|f * g\|_p \leq \|f\|_p \|g\|_1. \tag{5.2}$$
3. *Given $p, q, r$, $1 \leq p, q, r \leq \infty$, such that*
$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q},$$
*if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and*
$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{5.3}$$

Inequalities (5.2) and (5.3) are referred to as Minkowski's inequality and Young's inequality.

*Proof.* Inequality (5.1) follows from Hölder's inequality and the translation invariance of the classical Lebesgue spaces: for all $x$,

$$|f * g(x)| \leq \|f(x - \cdot)\|_p \|g\|_{p'} = \|f\|_p \|g\|_{p'}.$$

If $p < \infty$, inequality (5.2) follows by Minkowski's inequality, Fubini's theorem and translation invariance:

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |f*g(x)|^p\,dx\right)^{1/p} &= \left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\right|^p\,dx\right)^{1/p}\\ &\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x-y)g(y)|^p\,dx\right)^{1/p}dy\\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x-y)|^p\,dx\right)^{1/p}|g(y)|\,dy\\ &= \|f\|_p\|g\|_1.\end{aligned}$$

If $p=\infty$, inequality (5.2) reduces to inequality (5.1).

Finally, to prove inequality (5.3), fix $p$, $1\le p\le\infty$. Given a function $f\in L^p(\mathbb{R}^n)$ define the linear operator $T_f g = f*g$. Then inequalities (5.1) and (5.2) can be restated as $\|T_f g\|_\infty \le \|f\|_p\|g\|_{p'}$ and $\|T_f g\|_p \le \|f\|_p\|g\|_1$. Therefore, by the Riesz-Thorin interpolation theorem (see [143, 341] and Sect. 3.7.8), $\|T_f g\|_r \le \|f\|_p\|g\|_q$, which is the desired inequality. □

While Definition 5.1 is stated for functions defined on all of $\mathbb{R}^n$, we can readily adapt it to functions defined on a set $\Omega$ by setting $f$ and $g$ equal to 0 on $\mathbb{R}^n\setminus\Omega$. Hereafter we will do so without comment. Note that the function $f*g$ may be non-zero on $\mathbb{R}^n\setminus\Omega$, since $f(x-y)$ will be non-zero for points $x\in\mathbb{R}^n\setminus\Omega$. However, if both functions have compact support, then so does $f*g$. For example, if $\operatorname{supp}(f),\operatorname{supp}(g)\subset B_R(0)$, then $f(x-y)$ can only be non-zero if $|x-y|<R$, so $\operatorname{supp}(f*g)\subset B_{2R}(0)$.

A very important application of convolution is the technique of approximate identities, also known as mollification. Given a function $\phi$, for each $t>0$ let $\phi_t(x)=t^{-n}\phi(x/t)$. This normalization is such that if $\phi\in L^1(\mathbb{R}^n)$, then $\|\phi_t\|_1=\|\phi\|_1$. Define the radial majorant of $\phi$ to be the function

$$\Phi(x)=\sup_{|y|\ge|x|}|\phi(y)|.$$

The function $\Phi$ is radial and decreasing as $|x|$ increases; however, even if $\phi\in L^1(\mathbb{R}^n)$, $\Phi$ need not be integrable. For example on the real line, let $\phi(x)=|x-1|^{-1/2}\chi_{(-2,2)}$; then $\Phi(x)=\infty$ for all $x\in(-1,1)$. We will be particularly interested in $\phi$ such that $\Phi\in L^1$; this is the case, for example, if $\phi$ is bounded and has compact support. It follows from the definitions that

$$|\phi_t*f(x)|\le(\Phi_t*|f|)(x), \tag{5.4}$$

so in practice we can often replace $\phi$ by its radial majorant and assume that $f$ is non-negative.

**Definition 5.3.** Given $\phi \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$, the set $\{\phi_t\} = \{\phi_t : t > 0\}$ is called an approximate identity. If the radial majorant of $\phi$ is also in $L^1(\mathbb{R}^n)$, $\{\phi_t\}$ is called a potential type approximate identity.

The name "approximate identity" is motivated by the following theorem.

**Theorem 5.4.** *Given an approximate identity $\{\phi_t\}$, then for all $p$, $1 \leq p < \infty$, if $f \in L^p(\mathbb{R}^n)$, then $\|\phi_t * f - f\|_p \to 0$ as $t \to 0$. Further, if $\{\phi_t\}$ is a potential type approximate identity, then for all $p$, $1 \leq p \leq \infty$, $\phi_t * f(x) \to f(x)$ pointwise almost everywhere as $t \to 0$.*

The proof of Theorem 5.4 requires two lemmas. To state the first we need a definition.

**Definition 5.5.** Given a measurable function $f$ and $h \in \mathbb{R}^n$, define the translation operator $\tau_h$ by $\tau_h f(x) = f(x - h)$.

**Lemma 5.6.** *Given $p$, $1 \leq p < \infty$, then $L^p(\mathbb{R}^n)$ is mean continuous: if $f \in L^p(\mathbb{R}^n)$,*

$$\lim_{|h|\to 0} \|\tau_h f - f\|_p = 0.$$

*Proof.* If $g$ is a continuous function of compact support then this follows at once by uniform continuity. Given any $f \in L^p(\mathbb{R}^n)$, fix $\epsilon > 0$. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, there exists $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p < \epsilon$. Therefore, by the translation invariance of the classical Lebesgue spaces,

$$\limsup_{|h|\to 0} \|\tau_h f - f\|_p \leq \limsup_{|h|\to 0} \big(\|\tau_h g - g\|_p + \|\tau_h g - \tau_h f\|_p + \|f - g\|_p\big) < 2\epsilon.$$

Since $\epsilon$ is arbitrary the desired limit must hold. □

**Lemma 5.7.** *Let $\{\phi_t\}$ be a potential type approximate identity and let $\Phi$ be the radial majorant of $\phi$. Then for every locally integrable function $f$ and every $x$,*

$$\sup_{t>0} |\phi_t * f(x)| \leq C(n)\|\Phi\|_1 Mf(x).$$

*Proof.* By (5.4) and the discussion in Sect. 3.1, it will suffice to prove that given any non-negative $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, for all $t > 0$,

$$\Phi_t * f(x) \leq \|\Phi\|_1 Mf(x),$$

where here we take the maximal operator to be the supremum of averages over balls. For each $j, k \geq 1$ let $B_j^k = B_{j2^{-k}}(0)$. Since $\Phi$ is radial, we abuse notation and let $\Phi(|x|) = \Phi(x)$. Define the function $\Phi_k$ by

$$\Phi_k(x) = \sum_{j=1}^{\infty} \big(\Phi(j2^{-k}) - \Phi((j+1)2^{-k})\big)\chi_{B_j^k}(x) = \sum_{j=1}^{\infty} a_j^k\, \chi_{B_j^k}(x).$$

Since $\Phi$ is decreasing, $a_j^k \geq 0$. Let $A_j^k = B_j^k \setminus B_{j-1}^k$; then for $x \in A_j^k$,

$$\Phi_k(x) = \sum_{i=j}^{\infty} \big(\Phi(i2^{-k}) - \Phi((i+1)2^{-k})\big) = \Phi(j2^{-k}) \leq \Phi(x).$$

Since $\Phi \in L^1$ and is non-negative, it decreases to 0 as $|x| \to \infty$. Therefore, the middle sum converges. Further, $\{\Phi_k\}$ increases to $\Phi$ pointwise almost everywhere. Hence, by the monotone convergence theorem on $L^1(\mathbb{R}^n)$, if $f$ is non-negative, for each $t > 0$, $(\Phi_k)_t * f$ increases to $\Phi_t * f$ pointwise as $k \to \infty$. Therefore, it will suffice to prove that for all $k \geq 1$ and $t > 0$,

$$(\Phi_k)_t * f(x) \leq \|\Phi\|_1 Mf(x).$$

We first consider the case $t = 1$. Since for all $x$,

$$|B_j^k|^{-1}\chi_{B_j^k} * f(x) = \fint_{B_j^k} f(x-y)\,dy = \fint_{B_{j2^{-k}}(x)} f(y)\,dy \leq Mf(x),$$

we have that

$$\Phi_k * f(x) = \sum_j a_j^k |B_j^k| \cdot |B_j^k|^{-1}\chi_{B_j^k} * f(x) \leq \|\Phi_k\|_1 Mf(x) \leq \|\Phi\|_1 Mf(x).$$

We can now repeat this argument with $\Phi_k$ replaced by $(\Phi_k)_t$; since $\|(\Phi_k)_t\|_1 = \|\Phi_k\|_1$, we get the desired inequality for all $t > 0$. □

*Proof of Theorem 5.4.* Fix $p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Since $\int \phi(x)\,dx = 1$,

$$\phi_t * f(x) - f(x) = \int_{\mathbb{R}^n} \big(f(x-y) - f(x)\big)\phi_t(y)\,dy = \int_{\mathbb{R}^n} \big(\tau_{ty} f(x) - f(x)\big)\phi(y)\,dy.$$

Therefore, by Minkowski's inequality,

$$\|\phi_t * f - f\|_p \leq \int_{\mathbb{R}^n} \|\tau_{ty} f - f\|_p |\phi(y)|\,dy.$$

Fix $\epsilon > 0$; then by Lemma 5.6 there exists $\delta > 0$ such that if $|ty| < \delta$, $\|\tau_{ty} f - f\|_p < \epsilon$. Since $\phi$ is integrable, there exists $t > 0$ such that

$$\int_{\{y:|y|\geq\delta/t\}} |\phi(y)|\,dy < \epsilon.$$

Thus, again by the translation invariance of $L^p(\mathbb{R}^n)$,

$$\|\phi_t * f - f\|_p \le \epsilon \int_{\{|y|<\delta/t\}} |\phi(y)|\,dy + 2\|f\|_p \int_{\{|y|\ge\delta/t\}} |\phi(y)|\,dy = \big(\|\phi\|_1 + 2\|f\|_p\big)\epsilon.$$

Since $\epsilon$ is arbitrary, we have that $\phi_t * f \to f$ in norm.

To prove pointwise convergence, We first consider the case $p < \infty$. If $g$ is a continuous function of compact support, then $\phi_t * g(x)$ converges uniformly to $g(x)$. To prove this, note that since $g$ is uniformly continuous, $\|\tau_h g - g\|_\infty \to 0$ as $h \to 0$, so we can repeat the above proof for norm convergence using the $L^\infty$ norm.

Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, fix a sequence $\{g_k\} \subset C_c(\mathbb{R}^n)$ that converges to $f$ in norm. Then by Lemma 5.7 and Theorem 3.4, for all $s > 0$,

$$\begin{aligned}
&|\{x \in \mathbb{R}^n : \limsup_{t\to 0} |\phi_t * f(x) - f(x)| > s\}| \\
&\quad\le \limsup_{k\to\infty} \Big( |\{x \in \mathbb{R}^n : \limsup_{t\to 0} |\phi_t * f(x) - \phi_t * g_k(x)| > s/3\}| \\
&\qquad\quad + |\{x \in \mathbb{R}^n : \limsup_{t\to 0} |f(x) - g_k(x)| > s/3\}| \\
&\qquad\quad + |\{x \in \mathbb{R}^n : \limsup_{t\to 0} |\phi_t * g_k(x) - g_k(x)| > s/3\}| \Big) \\
&\quad\le \limsup_{k\to\infty} \Big( |\{x \in \mathbb{R}^n : C(n)\|\Phi\|_1 M(f - g_k)(x) > s/3\}| \\
&\qquad\quad + |\{x \in \mathbb{R}^n : M(f - g_k)(x) > s/3\}| \Big) \\
&\quad\le \limsup_{k\to\infty} |(C(n)\|\Phi\|_1 + 1) C s^{-p} \int_{\mathbb{R}^n} |f(x) - g_k(x)|^p\,dx \\
&\quad= 0.
\end{aligned}$$

Since this is true for all $s > 0$, we have that

$$\begin{aligned}
&|\{x \in \mathbb{R}^n : \limsup_{t\to 0} |\phi_t * f(x) - f(x)| > 0\}| \\
&\qquad\qquad \le \sum_{j=1}^{\infty} |\{x \in \mathbb{R}^n : \limsup_{t\to 0} |\phi_t * f(x) - f(x)| > 1/j\}| = 0.
\end{aligned}$$

Therefore, $\phi_t * f(x) \to f(x)$ almost everywhere.

Now suppose $p = \infty$; we will show that given any ball $B$, $\phi_t * f(x) \to f(x)$ for almost every $x \in B$. Write $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus 2B}$. Then $\phi_t * f(x) = \phi_t * f_1(x) + \phi_t * f_2(x)$, and since $f_1 \in L^1(\mathbb{R}^n)$, the first term converges pointwise to $f(x)$ for almost every $x \in B$. Furthermore, if the radius of $B$ is $r$, then for $x \in B$,

$$\limsup_{t\to 0} |\phi_t * f_2(x)| \le \limsup_{t\to 0} \int_{|y|>r} |f(x-y)||\phi_t(y)|\,dy$$
$$\le \limsup_{t\to 0} \|f\|_\infty \int_{\{|y|>r/t\}} |\phi(y)|\,dy = 0.$$

This completes the proof. □

## 5.2 Approximate Identities on Variable Lebesgue Spaces

The proofs in the previous section depend heavily on the translation invariance of the classical Lebesgue spaces. As we will see in Theorem 5.17 below, this property never holds in the variable Lebesgue spaces unless $p(\cdot)$ is constant. Therefore, it would not be completely unexpected if none of the properties of convolutions remain true in this setting. And some do fail, in particular, Young's inequality, which we will consider in more detail in the next section.

Nevertheless, many key properties of approximate identities are preserved if we assume that the exponent function $p(\cdot)$ has some regularity. Our main tool is Lemma 5.7, which shows that there is a close connection between potential type approximate identities and the maximal operator.

In the classical case, the norm convergence of an approximate identity is relatively straightforward to prove, but pointwise convergence requires a more sophisticated argument using the maximal operator. For variable Lebesgue spaces it is the opposite: pointwise convergence is an immediate consequence of the classical result for any $p(\cdot)$, but norm convergence requires additional hypotheses and the boundedness of the maximal operator. We therefore first consider pointwise convergence.

**Theorem 5.8.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, let $f \in L^{p(\cdot)}(\Omega)$. If $\{\phi_t\}$ is any potential type approximate identity, then for all $t > 0$, $\phi_t * f$ is finite almost everywhere, and $\phi_t * f \to f$ pointwise almost everywhere.*

*Proof.* By Theorem 2.51, write $f = f_1 + f_2$, where $f_1 \in L^{p_+}(\Omega)$ and $f_2 \in L^{p_-}(\Omega)$. Since $\phi_t * f = \phi_t * f_1 + \phi_t * f_2$, and $\phi_t \in L^1(\mathbb{R}^n)$, by Young's inequality (5.3) each term is finite almost everywhere, and the desired limit follows at once from Theorem 5.4. □

When $\Omega$ has finite measure, as a corollary to Theorem 5.8 we get that potential type approximate identities also converge in measure. If we assume that $p_+ < \infty$, then this is true for any open set $\Omega$.

**Theorem 5.9.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_+ < \infty$, let $f \in L^{p(\cdot)}(\Omega)$. If $\{\phi_t\}$ is any potential type approximate identity, then $\phi_t * f \to f$ in measure on $\Omega$.*

*Proof.* Fix $f \in L^{p(\cdot)}(\Omega)$. Since $p_+ < \infty$, by Theorem 2.72 there exists a sequence $\{g_k\} \subset L^{p(\cdot)}(\Omega)$ of bounded functions of compact support such that $g_k \to f$ in norm. Fix $\epsilon$, $0 < \epsilon < 1$; then for any $k$,

$$\begin{aligned} |\{x \in \Omega : |\phi_t * f(x) - f(x)| \geq \epsilon\}| \leq |\{x \in \Omega : |\phi_t * (f - g_k)(x)| \geq \epsilon/3\}| \\ + |\{x \in \Omega : |\phi_t * g_k(x) - g_k(x)| \geq \epsilon/3\}| \\ + |\{x \in \Omega : |g_k(x) - f(x)| \geq \epsilon/3\}|. \end{aligned}$$

By Theorem 2.69, $g_k \to f$ in measure. Therefore, for all $k$ sufficiently large, the last term is less than $\epsilon/3$. Again using the fact that $p_+ < \infty$, by Lemma 5.7 and Theorem 3.34 we have that

$$\begin{aligned} &|\{x \in \Omega : |\phi_t * (f - g_k)(x)| \geq \epsilon/3\}| \\ &\quad \leq |\{x \in \Omega : M(f - g_k)(x) \geq \epsilon/C\}| \leq C\epsilon^{-p_+} \int_\Omega |f(x) - g_k(x)|^{p(x)}\, dx. \end{aligned}$$

Again by Theorem 2.69, $g_k \to f$ in modular, so we may choose $k$ sufficiently large that the right-hand side is also less than $\epsilon/3$. Finally, given $k$, $g_k \in L^1(\Omega)$, and so $\phi_t * g_k \to g_k$ in $L^1$ norm and so in measure as $t \to 0$. Therefore, for all $t$ sufficiently close to 0,

$$|\{x \in \Omega : |\phi_t * g_k(x) - g_k(x)| \geq \epsilon/3\}| < \epsilon/3.$$

If we combine the three inequalities, we get that

$$|\{x \in \Omega : |\phi_t * f(x) - f(x)| \geq \epsilon\}| < \epsilon;$$

since $\epsilon > 0$ is arbitrary, $\phi_t * f \to f$ in measure on $\Omega$. □

If $\Omega$ has infinite measure, then the next example shows that we need the additional hypothesis that $p_+ < \infty$.

*Example 5.10.* Define $p(\cdot) \in \mathcal{P}(\mathbb{R})$ by $p(x) = 1 + |x|$, and define the function $f$ by

$$f(x) = \begin{cases} 1 & x \in [2n, 2n+1] \\ 0 & x \in (2n-1, 2n) \end{cases} \quad n \geq 1.$$

By Proposition 2.43, $L^\infty(\mathbb{R}) \subset L^{p(\cdot)}(\mathbb{R})$, so $f \in L^{p(\cdot)}(\mathbb{R})$. Let $\phi(x) = \chi_{(-1/2,1/2)}(x)$. Then for all $t$, $0 < t < 1$,

$$\phi_t * f(x) = t^{-1} \int_{x-t/2}^{x+t/2} f(y)\, dy,$$

and so if $x \in (2n + t/2, 2n + 1 - t/2)$, $\phi_t * f(x) = 1$; if $x \in (2n - 1 + t/2, 2n - t/2)$, $\phi_t * f(x) = 0$; and between these intervals the value of $\phi_t * f$ is gotten by linearly interpolating between 0 and 1. Thus, $\phi_t * f$ converges to $f$ pointwise almost everywhere. On the other hand, for any $\epsilon > 0$,

$$|\{x \in \mathbb{R} : |\phi_t * f(x) - f(x)| \geq \epsilon\}| = \infty,$$

and so $\phi_t * f$ does not converge to $f$ in measure.

We now consider the convergence in norm of approximate identities. To achieve this we need a stronger assumption on $p(\cdot)$.

**Theorem 5.11.** *Given an open set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\Omega)$. If $\{\phi_t\}$ is a potential type approximate identity, then*

$$\sup_{t>0} \|\phi_t * f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}, \tag{5.5}$$

*and $\phi_t * f \to f$ in norm on $L^{p(\cdot)}(\Omega)$. The constant $C$ in (5.5) depends on $n$, $p(\cdot)$, $\|M\|_{L^{p'(\cdot)}(\Omega)}$ and $\|\Phi\|_1$.*

*Remark 5.12.* The hypothesis $p_+ < \infty$ is redundant: by Theorem 3.19, if the maximal operator is bounded on $L^{p'(\cdot)}(\Omega)$, then $p'(\cdot)_- > 1$, and so $p_+ = (p'(\cdot)_-)' < \infty$. We include this fact in the statement for clarity.

*Remark 5.13.* If $\Omega$ is unbounded, by Theorem 3.16 it will suffice to assume that $p(\cdot) \in LH(\Omega)$; however, if $\Omega$ is bounded then by Corollary 3.18 it suffices to assume that $p(\cdot) \in LH_0(\Omega)$. This fact is often useful in applications.

*Proof.* Fix $f \in L^{p(\cdot)}(\Omega)$ and $t > 0$. Let $\Phi$ be the radial majorant of $\phi$. Then by (5.4) and Theorem 2.34 there exists $h \in L^{p'(\cdot)}(\Omega)$, $\|h\|_{p'(\cdot)} = 1$, such that

$$\|\phi_t * f\|_{p(\cdot)} \leq \|\Phi_t * |f|\|_{p(\cdot)} \leq 2k_{p(\cdot)}^{-1} \int_\Omega \Phi_t * |f|(x)h(x)\,dx.$$

Since $\Phi_t$ is a radial function, by Fubini's theorem, Theorem 2.26, Lemma 5.7 and our assumption on $p'(\cdot)$,

$$\begin{aligned}\int_\Omega (\Phi_t * |f|)(x)h(x)\,dx &= \int_\Omega |f(x)|\Phi_t * h(x)\,dx \\ &\leq C(n)\|\Phi\|_1 \int_\Omega |f(x)|Mh(x)\,dx \leq C(n)\|\Phi\|_1 K_{p(\cdot)}\|f\|_{p(\cdot)}\|Mh\|_{p'(\cdot)} \\ &\leq C\|M\|_{L^{p'(\cdot)}(\Omega)}\|f\|_{p(\cdot)}\|h\|_{p'(\cdot)} = C\|f\|_{p(\cdot)}.\end{aligned}$$

Since the constants do not depend on $t$, inequality (5.5) follows at once.

To prove that $\phi_t * f$ converges to $f$ in norm on $L^{p(\cdot)}(\Omega)$, we use an approximation argument similar to that in the proof of Theorem 5.9. Fix $\epsilon > 0$. By Theorem 2.72 there exists a function $g$, bounded with compact support and not identically zero, such that $\|f - g\|_{p(\cdot)} < \epsilon$. Then by (5.5),

$$\|\phi_t * f - f\|_{p(\cdot)} \leq \|\phi_t * (f-g)\|_{p(\cdot)} + \|\phi_t * g - g\|_{p(\cdot)} + \|f-g\|_{p(\cdot)}$$
$$\leq C\epsilon + \|\phi_t * g - g\|_{p(\cdot)}.$$

Since $\epsilon > 0$ is arbitrary, to complete the proof it will suffice to show that

$$\lim_{t\to 0} \|\phi_t * g - g\|_{p(\cdot)} = 0;$$

since $p_+ < \infty$, by Theorem 2.58 it will suffice to show that

$$\lim_{t\to 0} \int_\Omega |\phi_t * g(x) - g(x)|^{p(x)}\, dx = 0.$$

Let $g_0(x) = g(x)/(2\|\phi\|_1\|g\|_\infty)$; since $\|\phi\|_1 \geq 1$, $\|g_0\|_\infty \leq 1/2$. Furthermore,

$$|\phi_t * g_0(x)| \leq \int_\Omega |\phi_t(x-y)||g_0(y)|\, dy \leq \|g_0\|_\infty \int_\Omega |\phi_t(x-y)|\, dy \leq 1/2.$$

Therefore, $\|\phi_t * g_0 - g_0\|_\infty \leq 1$, and so

$$\lim_{t\to 0} \int_\Omega |\phi_t * g(x) - g(x)|^{p(x)}\, dx$$
$$= \lim_{t\to 0} \int_\Omega (2\|\phi\|_1\|g\|_\infty)^{p(x)} |\phi_t * g_0(x) - g_0(x)|^{p(x)}\, dx$$
$$\leq (2\|\phi\|_1\|g\|_\infty + 1)^{p_+} \lim_{t\to 0} \int_\Omega |\phi_t * g_0(x) - g_0(x)|^{p_-}\, dx.$$

Since $g_0 \in L^{p_-}(\Omega)$ and $1 \leq p_- < \infty$, by Theorem 5.4 the last term equals 0. This completes the proof. □

*Remark 5.14.* If we assume that $p_- > 1$ and the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, then inequality (5.5) follows immediately from Lemma 5.7.

Our proof of norm convergence required the assumption that $p_+ < \infty$, and in fact this hypothesis is necessary. If $\Omega_\infty$ is open and has positive measure, then this is straightforward to prove. Let $f \in L^{p(\cdot)}(\Omega)$ be discontinuous on $\Omega_\infty$, and let $\phi \in C_c^\infty(\mathbb{R}^n)$. By Lemma 6.15 below, $\phi_t * f$ is continuous. If $\phi_t * f \to f$ in $L^{p(\cdot)}$ norm, then $\|\phi_t * f - f\|_{L^\infty(\Omega_\infty)}$ converges to 0: i.e., $\phi_t * f$ converges uniformly to $f$ on $\Omega_\infty$, and so $f$ is continuous, a contradiction.

Moreover, if $|\Omega_\infty| = 0$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$, then we can weaken the hypothesis that the maximal operator is bounded and still show that $p_+ < \infty$ is necessary.

*Example 5.15.* Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p(\cdot) \in K_0(\Omega)$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$. Then there exists $f \in L^{p(\cdot)}(\Omega)$ and a potential type approximate identity $\{\phi_t\}$ such that $\phi_t * f$ does not converge to $f$ in norm.

*Proof.* We first construct the function $f$. Since $p_+(\Omega \setminus \Omega_\infty) = \infty$, there exists an increasing sequence of natural numbers $p_k \geq k$ such that the sets

$$F_k = \{x \in \Omega \setminus \Omega_\infty : p_k < p(x) < p_{k+1}\}$$

have positive measure. For each $k$ let $x_k \in F_k$ be a Lebesgue point of the function $\chi_{F_k}$. Since the sets $F_k$ are disjoint, the $x_k$ are distinct. We will show (possibly after passing to an infinite subsequence) that there exists a sequence of positive numbers $\{r_k\}$ such that the balls $B_k = B_{r_k}(x_k)$ are pairwise disjoint. By passing to a subsequence of $\{x_k\}$ we may assume that no point in the sequence is a limit point of the sequence. If the original sequence has no limit points, keep the entire sequence. Otherwise, fix a limit point and pass to a subsequence that converges to it. If the limit point is an element of the subsequence, create a new subsequence by eliminating this one point.

We now construct the sequence $\{r_k\}$ by induction. Since we have that for every $k$, $x_k$ is not a limit point of the sequence, there exists a sequence $\{s_k\}$, $s_k > 0$, such that for $j \neq k$, $x_j \notin \overline{B_{s_k}(x_k)}$. Let $r_1 = s_1$. Since $x_2 \notin \overline{B_{s_1}(x_1)}$, there exists $r_2$, $0 < r_2 \leq s_2$, such that $B_{r_1}(x_1)$ and $B_{r_2}(x_2)$ are disjoint and the points $x_i$, $i \geq 3$, are not contained in $\overline{B_{r_2}(x_2)}$. Now for $k \geq 2$, suppose we have positive radii $r_1, \dots, r_k$ such that $r_k \leq s_k$ and the balls $B_{r_i}(x_i)$ are pairwise disjoint. Then $x_{k+1} \notin \overline{B_{r_i}(x_i)}$, $1 \leq i \leq k$, so there exists $r_{k+1}$, $0 < r_{k+1} \leq s_{k+1}$ such that $B_{r_{k+1}}(x_{k+1})$ is disjoint from the balls $B_{r_i}(x_i)$, $1 \leq i \leq k$. Continuing this construction, by induction we get the desired sequence.

Possibly after making each $r_k$ smaller, we may assume that the sequence $r_k \to 0$ as $k \to \infty$, and that the balls $B_k^* = B_{((2c_0)^{1/n}+1)r_k}(x_k)$ are disjoint, where $c_0$ is such that for every ball $B$ the averaging operator satisfies $\|A_B h\|_{L^{p(\cdot)}(B)} \leq c_0 \|h\|_{L^{p(\cdot)}(B)}$. (Since $p(\cdot) \in K_0(\Omega)$, $c_0 \geq 1$ exists by Proposition 4.47 and Remark 4.49.) If we pass to a subsequence and relabel the sets $F_k$ so that $x_k \in F_k$, then by the Lebesgue differentiation theorem (see Sect. 2.9) we may also assume that

$$\frac{|F_k \cap B_k|}{|B_k|} \geq \frac{1}{2}.$$

We can now give the function $f$. Let $G_k = F_k \cap B_k$. Define the function

$$f(x) = \sum_{k=1}^{\infty} \big(|G_k|^{-1} \chi_{G_k}(x)\big)^{1/p(x)}.$$

By the definition of $F_k$, $p_-(G_k) > k$, and so we have that

$$\rho_{p(\cdot)}(f/2) = \sum_{k=1}^{\infty} \fint_{G_k} 2^{-p(x)}\,dx \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Thus $f \in L^{p(\cdot)}(\Omega)$.

The desired potential type approximate identity $\{\phi_t\}$ is given by $\phi(x) = |B_1(0)|^{-1}\chi_{B_1(0)}(x)$. Let $t_k = (2c_0)^{1/n} r_k$; then since the balls $B_k^*$ are disjoint, for any $x \in B_k$ and $y \in B_{t_k}(x)$, $f(y) = 0$ unless $y \in B_k$. Hence,

$$\phi_{t_k} * f(x) = \int_{B_{t_k}(x)} f(y)\, dy = \frac{1}{t_k^n |B_1(0)|} \int_{B_k} f(y)\, dy = (2c_0)^{-1} A_{B_k} f(x).$$

Therefore,

$$\|\phi_{t_k} * f\|_{L^{p(\cdot)}(B_k)} \leq (2c_0)^{-1} \|A_{B_k} f\|_{L^{p(\cdot)}(B_k)} \leq 2^{-1} \|f\|_{L^{p(\cdot)}(B_k)}.$$

On the other hand,

$$\int_{B_k} f(y)^{p(y)}\, dy = \int_{G_k} |G_k|^{-1}\, dy = 1,$$

and so, since $G_k \subset \Omega \setminus \Omega_\infty$, by Proposition 2.21,

$$\|f\|_{L^{p(\cdot)}(B_k)} = \|f\|_{L^{p(\cdot)}(G_k)} = 1.$$

But then for every $k$ we have that

$$\begin{aligned}\|f - \phi_{t_k} * f\|_{L^{p(\cdot)}(\Omega)} &\geq \|f - \phi_{t_k} * f\|_{L^{p(\cdot)}(B_k)} \\ &\geq \|f\|_{L^{p(\cdot)}(B_k)} - \|\phi_{t_k} * f\|_{L^{p(\cdot)}(B_k)} \geq 1/2.\end{aligned}$$

Since the sequence $t_k \to 0$ as $k \to \infty$, it follows that $\phi_t * f$ does not converge to $f$ in norm as $t \to 0$. □

*Remark 5.16.* The construction in Example 5.15 should be compared to the construction in the proof of Proposition 2.12 and the related constructions in Chap. 2.

## 5.3 The Failure of Young's Inequality

Before considering how to generalize the results in the previous section to other operators, we first discuss the failure of Young's inequality to hold on the variable Lebesgue spaces. As we noted above, the proof of Proposition 5.2 depends fundamentally on the fact that the classical Lebesgue spaces are translation invariant: for any $p$, if $f \in L^p(\mathbb{R}^n)$ then for any $h \in \mathbb{R}^n$, $\tau_h f \in L^p(\mathbb{R}^n)$ and $\|f\|_p = \|\tau_h f\|_p$. This property is never true on the variable Lebesgue spaces unless the exponent is constant.

**Theorem 5.17.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, each of the translation operators $\tau_h$, $h \in \mathbb{R}^n$, is a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $p(\cdot)$ is constant. Moreover, if $p(\cdot)$ is non-constant, there exists $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ such that $\tau_h f \notin L^{p(\cdot)}(\mathbb{R}^n)$.*

*Proof.* If $p(\cdot)$ is constant, then this is immediate. To prove the converse, suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ is such that for all $h \in \mathbb{R}^n$, $\|\tau_h f\|_{p(\cdot)} \leq C_h \|f\|_{p(\cdot)}$. Fix $h$ and a ball $B$. If $f \in L^{p(\cdot)}(B)$ and $f = 0$ on $\mathbb{R}^n \setminus B$, then $\tau_h f \in L^{p(\cdot)}(B+h)$, where $B + h = \{x + h : x \in B\}$, and $\|f\|_{L^{\tau_{-h} p(\cdot)}(B)} = \|\tau_h f\|_{L^{p(\cdot)}(B+h)}$. Hence, by our assumption on $\tau_h$,

$$\|f\|_{L^{\tau_{-h} p(\cdot)}(B)} \leq \|\tau_h f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_h \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = C_h \|f\|_{L^{p(\cdot)}(B)}.$$

Therefore, by Theorem 2.45, $\tau_{-h} p(x) \leq p(x)$ for almost every $x \in B$. If we replace $h$ by $-h$ and repeat the argument, we get the reverse inequality. Thus, $\tau_h p(x) = p(x)$ almost everywhere in $B$. Since $B$ and $h$ are arbitrary, this implies that $p(\cdot)$ is constant.

Given a non-constant $p(\cdot)$, to construct the desired function $f$, fix $h \in \mathbb{R}^n$ such that $\tau_h$ is not a bounded operator. Then there exists a sequence of functions $f_k \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $\|f_k\|_{p(\cdot)} \leq 1$ but $\|\tau_h f_k\|_{p(\cdot)} \geq 4^k$. If for some $k$, $\tau_h f_k \notin L^{p(\cdot)}(\mathbb{R}^n)$, we are done. Otherwise, let

$$f = \sum_{k=1}^{\infty} 2^{-k} |f_k|.$$

Then

$$\|f\|_{p(\cdot)} \leq \sum_{k=1}^{\infty} 2^{-k} \|f_k\|_{p(\cdot)} \leq 1,$$

but for every $k$, $f \geq 2^{-k} |f_k|$, and so

$$\|\tau_h f\|_{p(\cdot)} \geq 2^{-k} \|\tau_h f_k\|_{p(\cdot)} \geq 2^k.$$

Hence, $\|\tau_h f\|_{p(\cdot)} = \infty$ and $\tau_h f \notin L^{p(\cdot)}(\mathbb{R}^n)$. □

As a corollary to Theorem 5.17 we get that Lemma 5.6 does not hold in the variable Lebesgue spaces unless $p(\cdot)$ is constant.

**Corollary 5.18.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $p(\cdot)$ not constant, then $L^{p(\cdot)}(\mathbb{R}^n)$ is not mean continuous: there exists $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that*

$$\lim_{|h| \to 0} \|\tau_h f - f\|_{p(\cdot)} \neq 0. \tag{5.6}$$

*Proof.* Since for any $h \in \mathbb{R}^n$, $\tau_{h/2} \circ \tau_{h/2} f = \tau_h f$, $\tau_h$ is bounded if $\tau_{2^{-k}h}$ is bounded. Hence, by Theorem 5.17 we can find $h \in \mathbb{R}^n$ such that if $h_k = 2^{-k} h$, then $\tau_{h_k}$ is unbounded. Therefore, we can find functions $f_k \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|f_k\|_{p(\cdot)} \leq 1$, such that $\tau_{h_k} f_k \notin L^{p(\cdot)}(\mathbb{R}^n)$. Let

$$f = \sum_{k=1}^{\infty} 2^{-k}|f_k|;$$

then $f \in L^{p(\cdot)}(\mathbb{R}^n)$, but for any $k$, $f \geq 2^{-k}|f_k|$ and so $\tau_{h_k} f \notin L^{p(\cdot)}(\mathbb{R}^n)$. Hence, $\tau_{h_k} f - f \notin L^{p(\cdot)}(\mathbb{R}^n)$ and (5.6) does not hold. □

We can now show that Young's inequality is never true for non-constant exponents.

**Theorem 5.19.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the inequality*

$$\|f * g\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}\|g\|_1 \tag{5.7}$$

*is true for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ if and only if $p(\cdot)$ is constant.*

*Proof.* If $p(\cdot)$ is constant, then (5.7) becomes (5.2) in Proposition 5.2.

Now suppose that $p(\cdot)$ is not constant, but assume to the contrary that (5.7) holds for all $f$ and $g$. By Theorem 5.17 there exists $h \in \mathbb{R}^n$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $\tau_h f \notin L^{p(\cdot)}(\Omega)$. If we replace $f$ by $|f|/\|f\|_{p(\cdot)}$ we may assume $f$ is non-negative and $\|f\|_{p(\cdot)} = 1$. For each $N > 0$, let $g_N(x) = \min(f(x), N)\chi_{B_N(0)}$. Then $\|g_N\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \leq 1$. Further, since $g_N$ is a bounded function of compact support, $\tau_h g_N \in L^{p(\cdot)}(\mathbb{R}^n)$. Since $\tau_h g_N \to \tau_h f$ pointwise, by Fatou's lemma for the variable Lebesgue spaces (Theorem 2.61),

$$\infty = \|\tau_h f\|_{p(\cdot)} \leq \liminf_{N\to\infty} \|\tau_h g_N\|_{p(\cdot)}.$$

Therefore, for every $k \geq 1$ we can find $N_k$ such that $f_k = g_{N_k} \in L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{p(\cdot)} \leq 1$ but $\|\tau_h f_k\|_{p(\cdot)} \geq 2^k$.

Let $\phi$ be a bounded, non-negative function of compact support such that $\|\phi\|_1 = 1$. For every $t > 0$, let $\psi_{t,h}(x) = t^{-n}\phi((x-h)/t)$. Then by a change of variables,

$$\begin{aligned}\psi_{t,h} * f_k(x) &= t^{-n}\int_{\mathbb{R}^n} \phi\left(\frac{x-y-h}{t}\right) f_k(y)\,dy \\ &= t^{-n}\int_{\mathbb{R}^n} \phi\left(\frac{x-y}{t}\right) f_k(y-h)\,dy = \phi_t * (\tau_h f_k)(x).\end{aligned}$$

Since $\tau_h f_k \in L^{p(\cdot)}(\mathbb{R}^n)$, by Theorem 5.8, $\phi_t * (\tau_h f_k) \to \tau_h f_k$ pointwise almost everywhere. Therefore, by Fatou's lemma in variable Lebesgue spaces (Theorem 2.61) and (5.7),

$$\begin{aligned}2^k \leq \|\tau_h f_k\|_{p(\cdot)} &\leq \liminf_{t\to 0} \|\phi_t * (\tau_h f_k)\|_{p(\cdot)} \\ &= \liminf_{t\to 0} \|\psi_{t,h} * f_k\|_{p(\cdot)} \leq C\|f_k\|_{p(\cdot)}\|\phi\|_1 \leq C.\end{aligned}$$

Since this cannot be true for every $k$, we get a contradiction. Hence, inequality (5.7) holds if and only if $p(\cdot)$ is constant. □

We can prove a very weak version of Young's inequality as an immediate consequence of Lemma 5.7.

**Proposition 5.20.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Then for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and every non-negative, radially decreasing function $g \in L^1(\mathbb{R}^n)$,*

$$\|f * g\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}\|g\|_1.$$

However, even given the restrictive hypotheses of Proposition 5.20, Young's inequality does not hold for general exponents. In particular, inequality (5.1) may fail as the next example shows.

*Example 5.21.* Let $p(\cdot) \in \mathcal{P}(\mathbb{R})$ be a smooth function such that $p(x) = 2$ if $x \in \mathbb{R}^n \setminus [-2, 2]$, and $p(x) = 4$ on $[-1, 1]$. Define

$$f(x) = |x-3|^{-1/3}\chi_{[2,4]}, \qquad g(x) = |x|^{-2/3}\chi_{[-1,1]}.$$

Since $f^2 \in L^1(\mathbb{R})$, by Proposition 2.12, $f \in L^{p(\cdot)}(\mathbb{R})$. Similarly, since $p'(x) = 4/3$ on $[-1, 1]$ and $g^{4/3} \in L^1(\mathbb{R})$, $g \in L^{p'(\cdot)}(\mathbb{R})$. However, we do not have that

$$\|f * g\|_\infty \leq C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

since $f * g$ is unbounded in a neighborhood of 3. To show this, let $E_x = [2, 4] \cap [x-1, x+1]$. Then by Fatou's lemma on the classical Lebesgue spaces,

$$\begin{aligned}\liminf_{x\to 3} f * g(x) &= \liminf_{x\to 3} \int_{\mathbb{R}^n} |x-y|^{-2/3}|y-3|^{-1/3}\chi_{E_x}(y)\,dy \\ &\geq \int_{\mathbb{R}^n} \lim_{x\to 3} \big(|x-y|^{-2/3}|y-3|^{-1/3}\chi_{E_x}(y)\big)\,dy = \int_2^4 |y-3|^{-1}\,dy = \infty.\end{aligned}$$

## 5.4 Rubio de Francia Extrapolation

In this section we state and prove the main result of this chapter, an extension of the Rubio de Francia extrapolation theorem to variable Lebesgue spaces. To provide motivation and context for this result, we will first revisit the proof of Theorem 5.11 to show the connection with weighted norm inequalities, and we will then describe the classical extrapolation theorem and recent generalizations of it.

Given a potential-type approximate identity $\{\phi_t\}$, the heart of the proof of Theorem 5.11 was a duality argument which led to the following inequality (again,

$\Phi$ is the radial majorant of $\phi$ and $\Phi_t(x) = t^{-n}\Phi(x/t)$):

$$\int_\Omega (\Phi_t * |f|)(x)h(x)\,dx \le C\|\Phi\|_1 \int_\Omega |f(x)|Mh(x)\,dx. \tag{5.8}$$

Suppose for the moment that $h \in A_1$ (see p. 142). Then we would have that $Mh(x) \le [h]_{A_1} h(x)$, and so we could rewrite (5.8) as

$$\int_\Omega (\Phi_t * |f|)(x)h(x)\,dx \le C\|\Phi\|_1 [h]_{A_1} \int_\Omega |f(x)|h(x)\,dx. \tag{5.9}$$

We could then continue the proof, applying the generalized Hölder's inequality to get the desired conclusion. In other words: treating $h$ as a weight, if we had the weighted norm inequality (5.9), then we could prove that the convolution operators $\Phi_t * f$ are uniformly bounded on $L^{p(\cdot)}(\Omega)$. Furthermore, the same argument would work for any operator $T$ for which we had the same weighted norm inequality.

The problem with this heuristic argument is that in general $h$ is not an $A_1$ weight. The key to overcoming this difficulty is to use the Rubio de Francia iteration algorithm, introduced in Sect. 4.3 in the proof of Theorem 4.37. This iteration algorithm is also the connection with the Rubio de Francia extrapolation theorem, whose proof is the motivation for our interpretation of the proof of Theorem 5.11. The extrapolation theorem is one of the most profound results in the theory of weighted norm inequalities, remarkable for its power and simplicity.

**Theorem 5.22.** *Given an operator $T$, suppose that for some $p_0$, $1 \le p_0 < \infty$, and every $w \in A_{p_0}$, there exists a constant $C_{p_0}$ depending on $T$, $p_0$, $n$ and $[w]_{A_{p_0}}$ (but not on $w$ itself) such that for all $f \in L^{p_0}(w)$,*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x)\,dx \le C_{p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)\,dx. \tag{5.10}$$

*Then for every $p$, $1 < p < \infty$, and every $w \in A_p$, there exists a constant $C_p$ depending on $p_0$, $n$, $p$ and $[w]_{A_p}$, such that for all $f \in L^p(w)$,*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p} w(x)\,dx \le C_{p} \int_{\mathbb{R}^n} |f(x)|^{p} w(x)\,dx.$$

The proof of Theorem 5.22 is a more sophisticated version of the heuristic argument sketched above. Duality is used to pass from $L^p(w)$ to $L^{p_0}(w)$, and the iteration algorithm and the properties of $A_p$ weights are used to modify $w$ to get a weight in $A_{p_0}$ so that the hypothesis can be applied. The desired conclusion is then gotten by Hölder's inequality and the boundedness of the maximal operator. (See [69] for the details of this proof.)

To understand the power of the Rubio de Francia extrapolation theorem, consider the simplest case: $p_0 = 2$. Since for all $p > 1$, $1 \in A_p$, Theorem 5.22 asserts that to prove an operator is bounded on any $L^p$ space it is enough to prove that it is bounded on $L^2(w)$ for all $w \in A_2$. Our goal is to show that if an operator is bounded on $L^2(w)$ for all $w \in A_2$, then it is bounded on $L^{p(\cdot)}$ given very natural hypotheses on $p(\cdot)$.

Before we state an extrapolation theorem for variable Lebesgue spaces, we first need to give an abstract generalization of Theorem 5.22. It is an interesting and somewhat surprising feature of the proof of this result that the properties of the operator $T$ play no role. In fact, we can replace the hypothesis (5.10) with a more general inequality,

$$\int_{\mathbb{R}^n} F(x)^{p_0} w(x)\,dx \le C_0 \int_{\mathbb{R}^n} G(x)^{p_0} w(x)\,dx, \tag{5.11}$$

where $(F, G)$ are pairs of non-negative, measurable functions, and then conclude that the weighted $L^p$ inequality holds for these pairs $(F, G)$ as well. This may seem a superfluous generalization, but it allows the theory of extrapolation to be extended to prove a much wider class of results; we will make the utility of this approach clear below (see Corollaries 5.33 and 5.34).

However, in order to pass from operators to pairs of functions, we need to be more careful. In the proof we must assume that the *left*-hand side of (5.11) is finite. When dealing with an operator $T$ (i.e., with pairs $(F, G) = (|Tf|, |f|)$) we normally get this by assuming that the right-hand side is finite. However, when working with arbitrary pairs it is simpler to assume exactly what we need.

Therefore, hereafter we will let $\mathcal{F}$ denote a family of pairs of non-negative, measurable functions; given $p$, $q$, $1 \le p, q < \infty$, if for some $w \in A_q$ we write

$$\int_\Omega F(x)^p w(x)\,dx \le C_0 \int_\Omega G(x)^p w(x)\,dx, \qquad (F, G) \in \mathcal{F},$$

then we mean that this inequality holds for all pairs $(F, G) \in \mathcal{F}$ such that the left-hand side is finite, and that the constant may depend on $n$, $p$, $\Omega$ and $[w]_{A_q}$ but not on $w$. Using this convention we can now state the more general form of the Rubio de Francia extrapolation theorem.

**Theorem 5.23.** *Suppose that for some $p_0$, $1 \le p_0 < \infty$, the family $\mathcal{F}$ is such that for all $w \in A_{p_0}$,*

$$\int_\Omega F(x)^{p_0} w(x)\,dx \le C_{p_0} \int_\Omega G(x)^{p_0} w(x)\,dx, \qquad (F, G) \in \mathcal{F}.$$

*Then for every $p$, $1 < p < \infty$, and every $w \in A_p$,*

$$\int_\Omega F(x)^p w(x)\,dx \le C_p \int_\Omega G(x)^p w(x)\,dx, \qquad (F, G) \in \mathcal{F}.$$

To state our version of Rubio de Francia extrapolation for variable Lebesgue spaces, we extend this convention as follows: if we write

$$\|F\|_{L^{p(\cdot)}(\Omega)} \leq C_{p(\cdot)} \|G\|_{L^{p(\cdot)}(\Omega)}, \qquad (F, G) \in \mathcal{F},$$

then we mean that this inequality holds for all pairs such that the left-hand side is finite and the constant may depend on $n$, $p(\cdot)$ and $\Omega$.

**Theorem 5.24.** *Given $\Omega$, suppose that for some $p_0 \geq 1$ the family $\mathcal{F}$ is such that for all $w \in A_1$,*

$$\int_\Omega F(x)^{p_0} w(x)\, dx \leq C_0 \int_\Omega G(x)^{p_0} w(x)\, dx, \qquad (F, G) \in \mathcal{F}. \tag{5.12}$$

*Given $p(\cdot) \in \mathcal{P}(\Omega)$, if $p_0 \leq p_- \leq p_+ < \infty$ and the maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\Omega)$, then*

$$\|F\|_{p(\cdot)} \leq C_{p(\cdot)} \|G\|_{p(\cdot)}, \qquad (F, G) \in \mathcal{F}. \tag{5.13}$$

*Remark 5.25.* In applications the family $\mathcal{F}$ must be constructed by choosing an appropriate dense subset of $L^{p(\cdot)}(\Omega)$ such that the left and right-hand sides of both (5.12) and (5.13) are finite. The full result is then gotten via an approximation argument. We will consider this step in greater detail for specific operators in Sect. 5.5 below.

*Remark 5.26.* As was the case for Theorem 5.11, the hypothesis $p_+ < \infty$ is redundant: if $p_+ = \infty$, then $((p(\cdot)/p_0)')_- = 1$ and the maximal operator cannot be bounded on $L^{(p(\cdot)/p_0)'}(\Omega)$. We again include it for clarity.

*Remark 5.27.* For $p(\cdot) = p$ constant in Theorem 5.24, we can only get inequalities for $p \geq p_0$. This is different from the conclusion of Theorem 5.23 which yields inequalities for all $p > 1$. The reason for this is that we are making a different assumption on the weighted norm inequality: $w \in A_1$ versus $w \in A_{p_0}$. In the theory of extrapolation this is sometimes summarized by saying that one can only "go up" when extrapolating from $A_1$ weights.

Theorem 5.24 has two main hypotheses: the weighted norm inequality (5.12) and the boundedness of the maximal operator on $L^{(p(\cdot)/p_0)'}(\Omega)$. We will discuss the first condition in Sect. 5.5 since this will depend on the specific operator. The second condition is a natural one in light of Theorem 5.11 (when $p_0 = 1$ we are assuming that $M$ is bounded on $L^{p'(\cdot)}(\Omega)$) and the close connection between Muckenhoupt $A_p$ weights and the maximal operator. Our results in Chaps. 3 and 4 provide a variety of conditions on the exponent $p(\cdot)$ for this to hold. In particular, by Theorem 3.16 it suffices to assume that $p(\cdot) \in LH(\Omega)$, for in this case we have that $(p(\cdot)/p_0)' \in LH(\Omega)$. If $\Omega$ is bounded, then by Corollary 3.18 we may assume $p(\cdot) \in LH_0(\Omega)$.

In applications we are often given an exponent $p(\cdot)$ and need to find $p_0$ such that (5.12) holds. Even if we do not assume log-Hölder continuity (which lets us take any $p_0$ such that $1 < p_0 < p_-$), we always have a possible choice for $p_0$. If we assume that the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, then by Theorem 4.37, there exists $p_0 > 1$ such that $M$ is bounded on $L^{p(\cdot)/p_0}(\Omega)$, and so by Corollary 4.64, $M$ is bounded on $L^{(p(\cdot)/p_0)'}(\Omega)$. (We will apply this idea in Corollary 5.32 below.) Given this, it might seem more natural to simply assume in Theorem 5.24 that the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$. However, we want to include the case $p_0 = p_- = 1$ and in this case the maximal operator cannot be bounded on $L^{p(\cdot)}(\Omega)$ but may be bounded on $L^{p'(\cdot)}(\Omega)$.

There is another version of the extrapolation theorem which will be useful in applications. Certain operators (in particular, the Riesz potentials) are not bounded on the Lebesgue space $L^p(\mathbb{R}^n)$, but instead map $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ for some $q$. By using the corresponding weighted inequalities, we can use extrapolation to extend these results to variable Lebesgue spaces. In stating this theorem, we adopt for "off-diagonal inequalities" the same conventions for the family $\mathcal{F}$ as before.

**Theorem 5.28.** *Given $\Omega$, suppose that for some $p_0$, $q_0$, $1 \le p_0 \le q_0$, the family $\mathcal{F}$ is such that for all $w \in A_1$,*

$$\left(\int_\Omega F(x)^{q_0} w(x)\,dx\right)^{1/q_0} \le C_0 \left(\int_\Omega G(x)^{p_0} w(x)^{p_0/q_0}\,dx\right)^{1/p_0}, \qquad (F,G) \in \mathcal{F}. \tag{5.14}$$

*Given $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_0 \le p_- \le p_+ < \frac{p_0 q_0}{q_0 - p_0}$, define $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}. \tag{5.15}$$

*If the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}(\Omega)$, then*

$$\|F\|_{q(\cdot)} \le C_{p(\cdot)} \|G\|_{p(\cdot)}, \qquad (F,G) \in \mathcal{F}. \tag{5.16}$$

*Remark 5.29.* It follows immediately from the hypotheses of Theorem 5.28 that $q(\cdot) \in \mathcal{P}(\Omega)$, $q_+ < \infty$ and $q_- \ge q_0$.

*Remark 5.30.* Off-diagonal weighted norm inequalities are customarily written with the weights as multipliers instead of measures (see Sect. 4.6.5): in other words, we would let $W = w^{1/q_0}$ and write inequality (5.14) as

$$\left(\int_\Omega \big(F(x)W(x)\big)^{q_0}\,dx\right)^{1/q_0} \le \left(\int_\Omega \big(G(x)W(x)\big)^{p_0}\,dx\right)^{1/p_0}.$$

We have adopted this non-standard form because the proof of Theorem 5.28 is somewhat easier with our hypothesis. In applications this has no effect: see Remark 5.50 below.

*Proof of Theorems 5.24 and 5.28.* When $q_0 = p_0$ we have that Theorem 5.24 is a special case of Theorem 5.28, so we only have to prove the latter result.

Fix $p(\cdot)$, $q(\cdot) \in \mathcal{P}(\Omega)$ as in the hypotheses, and let $\bar{p}(x) = p(x)/p_0$ and $\bar{q}(x) = q(x)/q_0$. By assumption the maximal operator is bounded on $L^{\bar{q}'(\cdot)}(\Omega)$. Define an iteration algorithm $\mathcal{R}$ on $L^{\bar{q}'(\cdot)}(\Omega)$ by

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^{\bar{q}'(\cdot)}(\Omega)}},$$

where for $k \geq 1$, $M^k = M \circ M \circ \cdots \circ M$ denotes $k$ iterations of the maximal operator and $M^0 h = |h|$. Then arguing exactly as we did in the proof of Theorem 4.37, we have that:

$(a)$ For all $x \in \Omega$, $|h(x)| \leq \mathcal{R}h(x)$;
$(b)$ $\mathcal{R}$ is bounded on $L^{\bar{q}'(\cdot)}(\Omega)$ and $\|\mathcal{R}h\|_{\bar{q}'(\cdot)} \leq 2\|h\|_{\bar{q}'(\cdot)}$;
$(c)$ $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \leq 2\|M\|_{L^{\bar{q}'(\cdot)}(\Omega)}$.

Fix a pair $(F, G) \in \mathcal{F}$ such that $F \in L^{q(\cdot)}(\Omega)$ (i.e., so that the left-hand side of (5.16) is finite). By Proposition 2.18 and Theorem 2.34,

$$\|F\|^{q_0}_{q(\cdot)} = \|F^{q_0}\|_{\bar{q}(\cdot)} \leq k^{-1}_{p(\cdot)} \sup \int_\Omega F(x)^{q_0} h(x)\, dx,$$

where the supremum is taken over all non-negative $h \in L^{\bar{q}'(\cdot)}(\Omega)$ with $\|h\|_{\bar{q}'(\cdot)} = 1$. Fix any such function $h$; we will show that

$$\int_\Omega F(x)^{q_0} h(x)\, dx \leq C\|G\|^{q_0}_{p(\cdot)}$$

with the constant $C$ independent of $h$. First note that by Property $(a)$ we have that

$$\int_\Omega F(x)^{q_0}\, h(x)\, dx \leq \int_\Omega F(x)^{q_0}\, \mathcal{R}h(x)\, dx. \tag{5.17}$$

We want to apply our hypothesis (5.14) to the right-hand term in (5.17). To do so we have to show that it is finite. But by the generalized Hölder's inequality (Theorem 2.26), Property $(b)$ and Proposition 2.18,

$$\int_\Omega F(x)^{q_0} \mathcal{R}h(x)\, dx \leq K_{p(\cdot)} \|F^{q_0}\|_{\bar{q}(\cdot)} \|\mathcal{R}h\|_{\bar{q}'(\cdot)} \leq 2K_{p(\cdot)} \|F\|^{q_0}_{q(\cdot)} \|h\|_{\bar{q}'(\cdot)} < \infty.$$

Therefore, by Property $(c)$, (5.14) holds with $w = \mathcal{R}h$. Further, the constant $C_0$ only depends on $[\mathcal{R}h]_{A_1}$ and so is independent of $h$. Hence, by (5.14) and again by Theorem 2.26 and Proposition 2.18 we get

$$\begin{aligned}\int_\Omega F(x)^{q_0} \mathcal{R}h(x)\, dx &\leq C_0^{q_0} \left( \int_\Omega G(x)^{p_0} \mathcal{R}h(x)^{p_0/q_0}\, dx \right)^{q_0/p_0} \\ &\leq C_0^{q_0} \|G^{p_0}\|^{q_0/p_0}_{\bar{p}(\cdot)} \|(\mathcal{R}h)^{p_0/q_0}\|^{q_0/p_0}_{\bar{p}'(\cdot)} = C_0^{q_0} \|G\|^{q_0}_{p(\cdot)} \|(\mathcal{R}h)^{p_0/q_0}\|^{q_0/p_0}_{\bar{p}'(\cdot)}.\end{aligned}$$

To complete the proof of (5.16) we need to show that $\|(\mathcal{R}h)^{p_0/q_0}\|_{\bar{p}'(\cdot)}^{q_0/p_0}$ is bounded by a constant independent of $h$. By the definition of $q(\cdot)$,

$$\bar{p}'(x) = \frac{p(x)}{p(x) - p_0} = \frac{q_0}{p_0}\frac{q(x)}{q(x) - q_0} = \frac{q_0}{p_0}\bar{q}'(x).$$

Therefore, by Proposition 2.18 and Property $(b)$,

$$\|(\mathcal{R}h)^{p_0/q_0}\|_{\bar{p}'(\cdot)}^{q_0/p_0} = \|\mathcal{R}h\|_{\bar{q}'(\cdot)} \le 2\|h\|_{\bar{q}'(\cdot)} = 2.$$

This completes our proof. □

*Remark 5.31.* From the proof we have that $C_{p(\cdot)} = (2k_{p(\cdot)}^{-1})^{1/q_0}C_0$, and by assumption, $C_0$ depends on $\|M\|_{L^{(q(\cdot)/q_0)}(\Omega)'}$.

Theorem 5.28 has three immediate corollaries. The first combines it with classical extrapolation to get results for a larger class of exponents. The other two illustrate the power of defining extrapolation in terms of pairs of functions by proving general weak type and vector-valued inequalities.

**Corollary 5.32.** *Given $\Omega$, suppose that for some $p_0 \ge 1$ the family $\mathcal{F}$ is such that for all $w \in A_{p_0}$,*

$$\int_\Omega F(x)^{p_0} w(x)\,dx \le C_0 \int_\Omega G(x)^{p_0} w(x)\,dx, \qquad (F,G) \in \mathcal{F}. \tag{5.18}$$

*Given $p(\cdot) \in \mathcal{P}(\Omega)$, if $1 < p_- \le p_+ < \infty$ and the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, then*

$$\|F\|_{p(\cdot)} \le C_{p(\cdot)}\|G\|_{p(\cdot)}, \qquad (F,G) \in \mathcal{F}. \tag{5.19}$$

*Proof.* Since $M$ is bounded on $L^{p(\cdot)}(\Omega)$, by Theorem 4.37 and Corollary 4.64, there exists $r > 1$ such that $M$ is bounded on $L^{(p(\cdot)/r)'}(\Omega)$. Given (5.18), by Theorem 5.23 we have that for all $w \in A_r$,

$$\int_\Omega F(x)^r w(x)\,dx \le C_r \int_\Omega G(x)^r w(x)\,dx, \qquad (F,G) \in \mathcal{F}. \tag{5.20}$$

Therefore, by Theorem 5.24, inequality (5.19) holds. □

**Corollary 5.33.** *Given $\Omega$, suppose that for some $p_0$, $q_0$, $1 \le p_0 \le q_0$, the family $\mathcal{F}$ is such that for all $w \in A_1$,*

$$w(\{x \in \Omega : F(x) > t\}) \le C_0 \left(\frac{1}{t^{p_0}} \int_\Omega G(x)^{p_0} w(x)^{p_0/q_0}\,dx\right)^{q_0/p_0}, \quad (F,G) \in \mathcal{F}. \tag{5.21}$$

*Given* $p(\cdot) \in \mathcal{P}(\Omega)$ *such that* $p_0 \le p_- \le p_+ < \frac{p_0 q_0}{q_0 - p_0}$, *define* $q(\cdot)$ *by* (5.15)*. If the maximal operator is bounded on* $L^{(q(\cdot)/q_0)'}(\Omega)$*, then for all* $t > 0$*,*

$$\|t\chi_{\{x\in\Omega:F(x)>t\}}\|_{q(\cdot)} \le C_{p(\cdot)}\|G\|_{p(\cdot)}, \qquad (F,G) \in \mathcal{F}. \tag{5.22}$$

*Proof.* Define a new family $\tilde{\mathcal{F}}$ consisting of the pairs

$$(F_t, G) = (t\chi_{\{x\in\Omega:F(x)>t\}}, G), \qquad (F,G) \in \mathcal{F}, \quad t > 0.$$

Then we can restate (5.21) as follows: for every $w \in A_1$,

$$\|F_t\|_{L^{q_0}(w)} = tw(\{x \in \Omega : F(x) > t\})^{1/q_0} \le C_0^{1/q_0}\|G\|_{L^{p_0}(w^{p_0/q_0})}, \quad (F_t, G) \in \tilde{\mathcal{F}}.$$

Therefore, we can apply Theorem 5.28 to the family $\tilde{\mathcal{F}}$ to conclude that (5.16) holds for the pairs $(F_t, G) \in \tilde{\mathcal{F}}$, which is exactly (5.22). □

**Corollary 5.34.** *Given* $\Omega$*, suppose that for some* $p_0 \ge 1$ *the family* $\mathcal{F}$ *is such that for all* $w \in A_{p_0}$*, inequality (5.18) holds. Given* $p(\cdot) \in \mathcal{P}(\Omega)$*, if* $1 < p_- \le p_+ < \infty$ *and the maximal operator is bounded on* $L^{p(\cdot)}(\Omega)$*, then for every* $r$*,* $1 < r < \infty$*, and sequence* $\{(F_i, G_i)\} \subset \mathcal{F}$*,*

$$\Big\|\Big(\sum_i F_i^r\Big)^{1/r}\Big\|_{p(\cdot)} \le C_{p(\cdot)}\Big\|\Big(\sum_i G_i^r\Big)^{1/r}\Big\|_{p(\cdot)}. \tag{5.23}$$

*Proof.* Fix $r$, $1 < r < \infty$. We first reduce the proof to the special case of finite sums. For if this case holds, given any sequence $\{(F_i, G_i)\} \subset \mathcal{F}$, by Fatou's lemma for variable Lebesgue spaces (Theorem 2.61),

$$\begin{aligned}\Big\|\Big(\sum_i F_i^r\Big)^{1/r}\Big\|_{p(\cdot)} &\le \liminf_{N\to\infty}\Big\|\Big(\sum_{i=1}^N F_i^r\Big)^{1/r}\Big\|_{p(\cdot)} \\ &\le C_{p(\cdot)}\liminf_{N\to\infty}\Big\|\Big(\sum_{i=1}^N G_i^r\Big)^{1/r}\Big\|_{p(\cdot)} \le C_{p(\cdot)}\Big\|\Big(\sum_i G_i^r\Big)^{1/r}\Big\|_{p(\cdot)}.\end{aligned}$$

Now form a new family $\mathcal{F}_r$ that consists of the pairs of functions $(F_{r,N}, G_{r,N})$ defined by

$$F_{r,N}(x) = \left(\sum_{i=1}^N F_i(x)^r\right)^{1/r}, \quad G_{r,N}(x) = \left(\sum_{i=1}^N G_i(x)^r\right)^{1/r},$$

where $N > 1$ and $\{(F_i, G_i)\}_{i=1}^N \subset \mathcal{F}$. Arguing as in the proof of Corollary 5.32, inequality (5.20) holds. Thus, for all $(F_{r,N}, G_{r,N}) \in \mathcal{F}_r$,

$$\int_{\Omega} F_{r,N}(x)^r w(x)\,dx = \sum_{i=1}^{N} \int_{\Omega} F_i(x)^r w(x)\,dx$$
$$\leq C_0 \sum_{i=1}^{N} \int_{\Omega} G(x)^r w(x)\,dx = C_0 \int_{\Omega} F_{r,N}(x)^r w(x)\,dx.$$

Therefore, we can apply Corollary 5.32 with $p_0 = r$ to this family to get

$$\|F_{r,N}\|_{p(\cdot)} \leq C_{p(\cdot)} \|G_{r,N}\|_{p(\cdot)}, \qquad (F_{r,N}, G_{r,N}) \in \mathcal{F}_r.$$

But this is (5.23) for all finite sums, which is what we needed to prove. □

## 5.5 Applications of Extrapolation

The importance of Theorems 5.24 and 5.28 is that they give us a straightforward method for deducing norm inequalities on $L^{p(\cdot)}(\Omega)$ from weighted norm inequalities. There is a vast literature on weighted norm inequalities, and it is perhaps only a slight exaggeration to say that every operator in classical harmonic analysis satisfies a weighted norm inequality useful for extrapolation. (One notable exception is the Fourier transform: see Sect. 5.6.10 below.) A broad survey of weighted norm inequalities for a variety of operators is beyond the scope of this work, and for this we refer the reader to the references given in Sects. 4.6 and 5.6. In this section we will concentrate on two specific operators: singular integral operators and Riesz potentials. These are extremely important in a wide range of applications of harmonic analysis; furthermore, they are very good models for understanding the technical details involved in applying weighted norm inequalities and extrapolation.

In this section we assume that the reader has some knowledge of the operators we are considering. Therefore, for brevity we will give a few standard results concerning them without proof and refer the reader to one of several references for proofs and further details.

### *Singular Integrals*

The most fundamental singular integral is the Hilbert transform: given a function $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, define the Hilbert transform by the principal value integral

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\{|x-y|\geq\epsilon\}} \frac{f(y)}{x-y}\,dy.$$

This limit exists both in $L^p$ norm and pointwise almost everywhere. Intuitively, $Hf$ can be thought of as the convolution of $f$ with $(\pi x)^{-1}$; for this reason singular integrals are often referred to as singular convolution operators.

*Remark 5.35.* The definition of the Hilbert transform in terms of convolutions can be made precise by introducing the concept of a tempered distribution. We will not consider this level of generality. See [96, 143] for more information.

In higher dimensions, natural generalizations of the Hilbert transform are the Riesz transforms $R_j$, $1 \leq j \leq n$, defined by

$$R_j f(x) = \lim_{\epsilon \to 0} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\{|x-y|\geq\epsilon\}} \frac{x^j - y^j}{|x-y|^{n+1}} f(y)\, dy,$$

where $\Gamma$ is the Gamma function and if $x \in \mathbb{R}^n$ we write its coordinates as $(x^1, \dots, x^n)$. For both the Hilbert transform and the Riesz transforms the constants are chosen so that their Fourier transform are very simple: if $f$ is a Schwartz function, then

$$\widehat{Hf}(\xi) = -i\, \mathrm{sgn}(\xi) \hat{f}(\xi), \qquad \widehat{R_j f}(\xi) = -i \frac{\xi^j}{|\xi|} \hat{f}(\xi).$$

All of these operators can be treated as special cases of a general theory of singular integral operators.

**Definition 5.36.** Given a function $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, suppose that there exists a constant $C > 0$ such that:

1. $|K(x)| \leq \dfrac{C}{|x|^n}$, $x \neq 0$;
2. $|\nabla K(x)| \leq \dfrac{C}{|x|^{n+1}}$, $x \neq 0$;
3. for $0 < r < R$, $\left| \int_{\{r<|x|<R\}} K(x)\, dx \right| \leq C$;
4. $\lim_{\epsilon \to 0} \int_{\{\epsilon<|x|<1\}} K(x)\, dx$ exists.

Then for $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, define the singular integral operator $T$ with kernel $K$ by

$$Tf(x) = \lim_{\epsilon \to 0} \int_{\{|x-y|\geq\epsilon\}} K(x-y) f(y)\, dy, \tag{5.24}$$

wherever this limit exists.

The first hypothesis in Definition 5.36 guarantees that for each $\epsilon > 0$ the integral on the right-hand side of (5.24) exists. Furthermore, if $f$ has compact support and $x \notin \mathrm{supp}(f)$, then the limit exists and we have that

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\,dy.$$

It is not immediate whether and in what sense this limit exists for more general $f$. But the remaining hypotheses, which can be thought of as "cancellation" conditions on $K$, combine to yield the following. (For a proof, see [96, 140, 143].)

**Theorem 5.37.** *Given a singular integral with kernel $K$, the limit in (5.24) converges both in $L^p$ norm and pointwise almost everywhere. Furthermore, if $f \in L^1(\mathbb{R}^n)$, then for all $t > 0$,*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > t\}| \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|\,dx.$$

*If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then*

$$\|Tf\|_p \leq C\|f\|_p.$$

*Remark 5.38.* The hypotheses of Theorem 5.37 can be relaxed in various ways. Further, this result holds for a more general class of operators, referred to as Calderón-Zygmund operators, that are not (singular) convolution operators. With appropriate assumptions, everything we say below extends to this larger class, but we restrict ourselves to singular integrals for simplicity. See [96, 143] for more information.

Our main result for singular integrals extends Theorem 5.37 to the variable Lebesgue spaces .

**Theorem 5.39.** *Let $T$ be a singular integral operator with kernel $K$. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < \infty$, if $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then for all functions $f$ that are bounded and have compact support,*

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}, \tag{5.25}$$

*and $T$ extends to a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n)$. If $p_- = 1$ and $M$ is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$, then for all $t > 0$ and functions $f$ that are bounded and have compact support,*

$$\|t\chi_{\{x:|Tf(x)|>t\|}\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}. \tag{5.26}$$

We will prove Theorem 5.39 using Theorem 5.24. To do so we need a weighted norm inequality for singular integrals. For a proof, see [96, 140, 143].

**Theorem 5.40.** *Given a singular integral $T$ with kernel $K$, if $w \in A_1$, then for all $t > 0$,*

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|w(x)\,dx. \tag{5.27}$$

*Further, if $1 < p < \infty$ and $w \in A_p$, then*

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx. \tag{5.28}$$

*In both cases the constant depends on $p$, $n$, $T$ and $[w]_{A_p}$.*

Part of the proof of Theorem 5.40 is showing that if $f \in L^p(w)$, then $Tf$ is well-defined, since Definition 5.36 only defines $Tf$ for $f$ in the unweighted spaces $L^p(\mathbb{R}^n)$. We will examine this argument more closely since similar questions arise in weighted norm inequalities for other operators and also in the variable Lebesgue spaces.

Let $f$ be a bounded function of compact support. Then $f \in \bigcap_{p\ge 1} L^p$ and so by Theorem 5.37, $Tf$ is well-defined and $Tf \in \bigcap_{p>1} L^p$. Since $w$ is locally integrable, $f \in L^p(w)$. Moreover, $Tf \in L^p(w)$ for all $w \in A_p$, $1 < p < \infty$. To see this, fix $f$ and let $B$ be a ball with center $x_0$ such that $\operatorname{supp}(f) \subset B$, and let $2B$ be the ball with the same center and twice the radius. Then for $x \in \mathbb{R}^n \setminus 2B$ and $y \in B$, we have that $|x-y| \ge |x-x_0| - |x_0-y| \ge \frac{1}{2}|x-x_0|$, and so

$$\begin{aligned} |Tf(x)| &= \left| \int_B K(x-y) f(y)\,dy \right| \\ &\le C \int_B \frac{|f(y)|}{|x-y|^n}\,dy \le C(n) \fint_{B_{|x-x_0|}(x_0)} |f(y)|\,dy \le C(n) Mf(x). \end{aligned}$$

Therefore, since $p > 1$ and $w \in A_p$, by Theorem 4.35 we have that

$$\int_{\mathbb{R}^n\setminus 2B} |Tf(x)|^p w(x)\,dx \le C \int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx < \infty.$$

On the other hand, by Theorem 4.22 there exists $s > 1$ such that $w \in RH_s$, so $w^s \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Hence,

$$\int_{2B} |Tf(x)|^p w(x)\,dx \le \left( \int_{2B} |Tf(x)|^{ps'}\,dx \right)^{1/s'} \left( \int_{2B} w(x)^s\,dx \right)^{1/s} < \infty.$$

A similar argument shows that when $p = 1$, the left-hand side of (5.27) is finite.

Therefore, if $f$ is a bounded function of compact support, $Tf \in L^p(w)$, and the left-hand sides of inequalities (5.27) and (5.28) are finite. These inequalities can then be proved for such functions. Finally, we can use them to extend the definition of $T$ to arbitrary $f \in L^p(w)$: define $Tf$ as the limit of the sequence $\{Tf_j\}$, where $\{f_j\}$ is a sequence of bounded functions of compact support that converge to $f$ in norm on $L^p(w)$: since $w\,dx$ is an absolutely continuous measure, such functions are dense in $L^p(w)$, $1 \le p < \infty$. If $p > 1$, this limit exists in norm since by (5.28) the sequence $\{Tf_j\}$ is Cauchy; when $p = 1$, by (5.27) the limit exists in measure. In both cases, by passing to a subsequence we get that the limit exists pointwise almost everywhere.

*Proof of Theorem 5.39.* We first consider the case $p_- > 1$. Define the family $\mathcal{F}$ to be all pairs $(|Tf|, |f|)$ where $f$ is a bounded function of compact support. By Theorem 5.40, for all $w \in A_1 \subset A_{p_0}$, $T$ is bounded on $L^{p_0}(w)$. In particular, for all such $f$, $\|Tf\|_{L^{p_0}(w)} < \infty$. Therefore, by Corollary 5.32,

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$$

for every bounded function of compact support such that the left-hand side is finite. But this is the case for every such $f$: arguing exactly as we did above for $L^p(w)$, if $\operatorname{supp}(f) \subset B$, then, since $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|Tf\|_{L^{p(\cdot)}(\mathbb{R}^n \setminus 2B)} \leq C\|Mf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)} < \infty.$$

Furthermore, by Corollary 2.48 and Theorem 5.37,

$$\|Tf\|_{L^{p(\cdot)}(2B)} \leq (1 + |2B|)\|Tf\|_{L^{p_+}(2B)} \leq C\|f\|_{L^{p_+}(B)} < \infty.$$

This proves inequality (5.25) for bounded functions of compact support.

We now extend $T$ to a bounded operator on all of $L^{p(\cdot)}(\mathbb{R}^n)$. Fix $f \in L^{p(\cdot)}(\mathbb{R}^n)$; since $p_+ < \infty$, by Theorem 2.72 and Proposition 2.67 there exists a sequence $\{f_j\}$ of bounded functions of compact support that converge to $f$ in norm and pointwise almost everywhere. By inequality (5.25) the sequence $\{Tf_j\}$ is a Cauchy sequence, and so it converges in norm to some limit; we define $Tf$ to be this limit. Again by Proposition 2.67 we can pass to a subsequence and assume that $Tf_j \to Tf$ pointwise almost everywhere. Therefore, by Fatou's lemma for the variable Lebesgue spaces (Theorem 2.61),

$$\|Tf\|_{p(\cdot)} \leq \liminf_{j\to\infty} \|Tf_j\|_{p(\cdot)} \leq \liminf_{j\to\infty} C\|f_j\|_{p(\cdot)} = C\|f\|_{p(\cdot)}.$$

Inequality (5.26) is proved in almost exactly the same way as (5.25), except that we take $p_0 = 1$ and use Corollary 5.33 instead of Corollary 5.32. □

We can use Corollary 5.34 to prove vector-valued inequalities for singular integral operators.

**Theorem 5.41.** *Let $T$ be a singular integral operator with kernel $K$. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < \infty$, if $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then $T$ satisfies a vector-valued inequality on $L^{p(\cdot)}(\mathbb{R}^n)$: for each $r$, $1 < r < \infty$,*

$$\left\| \left( \sum_{i=1}^{\infty} |Tf_i|^r \right)^{1/r} \right\|_{p(\cdot)} \leq C \left\| \left( \sum_{i=1}^{\infty} |f_i|^r \right)^{1/r} \right\|_{p(\cdot)}. \tag{5.29}$$

*Proof.* Fix $r > 1$ and define the family $\mathcal{F}$ to consist of all pairs of functions $(|Tf|, |f|)$, where $f$ is a bounded function of compact support. Then by Theorem 5.40 and Corollary 5.34 we have that for any sequence $\{(|Tf_i|, |f_i|) \subset \mathcal{F}$,

$$\left\|\left(\sum_i |Tf_i|^r\right)^{1/r}\right\|_{p(\cdot)} \leq C_{p(\cdot)}\left\|\left(\sum_i |f_i|^r\right)^{1/r}\right\|_{p(\cdot)},$$

provided that the left-hand side is finite. This is the case if the sum is finite: by convexity and Theorem 5.39,

$$\left\|\left(\sum_{i=1}^N |Tf_i|^r\right)^{1/r}\right\|_{p(\cdot)} \leq \left\|\sum_{i=1}^N |Tf_i|\right\|_{p(\cdot)} \leq \sum_{i=1}^N \|Tf_i\|_{p(\cdot)} < \infty.$$

To prove the full result, fix any sequence $\{f_i\} \subset L^{p(\cdot)}(\mathbb{R}^n)$. By Theorem 2.72, for each $i$ there exists a sequence $\{f_{i,j}\}$ of bounded functions of compact support that converges to $f_i$ in norm. Moreover, from the proof of this result we have that $|f_{i,j}|$ increases pointwise to $|f_i|$. Arguing as we did in the proof of Theorem 5.39, by passing to a subsequence we may assume that $Tf_{i,j}$ converges pointwise to $Tf_i$. Therefore, for each $N > 1$ we have that

$$\left(\sum_{i=1}^N |Tf_{i,j}|^r\right)^{1/r} \to \left(\sum_{i=1}^N |Tf_i|^r\right)^{1/r}$$

pointwise as $j \to \infty$. Hence, by Fatou's Lemma in the variable Lebesgue spaces (Theorem 2.61) we have that

$$\begin{aligned}\left\|\left(\sum_{i=1}^N |Tf_i|^r\right)^{1/r}\right\|_{p(\cdot)} &\leq \liminf_{j\to\infty}\left\|\left(\sum_{i=1}^N |Tf_{i,j}|^r\right)^{1/r}\right\|_{p(\cdot)} \\ &\leq C\liminf_{j\to\infty}\left\|\left(\sum_{i=1}^N |f_{i,j}|^r\right)^{1/r}\right\|_{p(\cdot)} \leq C\left\|\left(\sum_{i=1}^N |f_i|^r\right)^{1/r}\right\|_{p(\cdot)}.\end{aligned}$$

Since this is true for all $N > 1$, inequality (5.29) follows from the monotone convergence theorem for the variable Lebesgue spaces (Theorem 2.59). □

The hypothesis that $p_+ < \infty$ in Theorem 5.39 is in some sense necessary. In the classical Lebesgue spaces singular integrals are not bounded on $L^\infty$ or $L^1$. For example, on the real line if we let $f(x) = \chi_{[0,1]}(x)$, then a simple computation shows that

$$Hf(x) = \frac{1}{\pi}\log\left(\frac{|x|}{|x-1|}\right),$$

so $Hf \notin L^\infty(\mathbb{R})$. Similarly, for $x > 0$ sufficiently large, $Hf(x) \geq \frac{c}{x-1}$, so $Hf \notin L^1(\mathbb{R})$. Nevertheless, it is tempting to conjecture that if $p_+ = \infty$ and $|\Omega_\infty| = 0$, or if $p_- = 1$ and $|\Omega_1| = 0$, then the Hilbert transform might be bounded on $L^{p(\cdot)}(\mathbb{R})$. But this is never the case, and the analogous restriction holds in higher dimensions.

**Theorem 5.42.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if the Riesz transforms $R_j$, $1 \le j \le n$, are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then $1 < p_- \le p_+ < \infty$.*

*Proof.* We first consider the case when $p_- = 1$. Suppose to the contrary that there exists $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $p_- = 1$ and all of the Riesz transforms are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. To derive a contradiction we repeat the construction in Theorem 3.19 to form sets $E_k$ and balls $B_k = B_{\rho_k}(x_k)$. (To avoid confusion with the Riesz transforms, we denote the radius by $\rho_k$ instead of $R_k$ as in the original.) Again define the function $f_k \in L^{p(\cdot)}(\mathbb{R}^n)$ by

$$f_k(x) = |x - x_k|^{-n+\frac{1}{k+1}} \chi_{B_k \cap E_k}(x).$$

(Recall that the fact that $p_- = 1$ is used to show that $\|f_k\|_{p(\cdot)} < \infty$.)

The $(n-1)$-dimensional hyperplanes parallel to the coordinate axes through the point $x_k$ divide $B_k$ into $2^n$ sectors $S_i$. Since

$$\|f_k\|_{L^{p(\cdot)}(B_k)} \le \sum_{i=1}^{2^n} \|f_k\|_{L^{p(\cdot)}(S_i)},$$

one of the terms on the right is greater than or equal to $2^{-n}\|f_k\|_{L^{p(\cdot)}(B_k)}$. Denote this sector by $S^+$, and let $S^-$ be the sector gotten by reflecting $S^+$ through each of the hyperplanes through $x_k$.

Define the operator $R = \sum_j \sigma_j R_j$, where for each $j$, $\sigma_j = \pm 1$ is chosen so that if $x = (x^1, \dots, x^n) \in S^+$ and $y = (y^1, \dots, y^n) \in S^-$, $\sigma_j(x^j - y^j) = |x^j - y^j| \ge 0$. By assumption, $R$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, so if we define $g_k = f_k \chi_{S^-}$, there exists a constant $C_0$ such that for all $k$,

$$\|Rg_k\|_{L^{p(\cdot)}(B_k)} \le C_0 \|g_k\|_{L^{p(\cdot)}(B_k)} \le C_0 \|f_k\|_{L^{p(\cdot)}(B_k)}. \tag{5.30}$$

We will derive a contradiction to this inequality. Fix $x \in S^+$ and let $r = |x - x_k| \le \rho_k$. Define

$$S_r^- = \{y \in S^- : |y - x_k| < r\}.$$

Then for all $y \in S_r^-$, $|x - y| \le |x - x_k| + |y - x_k| \le 2r$. Further, we have that

$$|x - y| \le n^{1/2} \max_j |x^j - y^j| \le n^{1/2} \sum_{j=1}^{n} |x^j - y^j|.$$

Since $g_k \in L^1(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(g_k)$, the integral defining each Riesz transform converges absolutely. Thus, we can estimate as follows:

$$\begin{aligned} Rg_k(x) &= c(n)\sum_{j=1}^{n}\int_{S^-\cap E_k} \frac{\sigma_j(x^j-y^j)}{|x-y|^{n+1}}|y-x_k|^{-n+\frac{1}{k+1}}\,dy \\ &\geq c(n)n^{-1/2}\int_{S^-\cap E_k}\frac{|y-x_k|^{-n+\frac{1}{k+1}}}{|x-y|^n}\,dy \\ &\geq c(n)n^{-1/2}\int_{S_r^-\cap E_k}\frac{|y-x_k|^{-n+\frac{1}{k+1}}}{|x-y|^n}\,dy \\ &\geq c(n)n^{-1/2}2^{-n}r^{-n}\int_{S_r^-\cap E_k}|y-x_k|^{-n+\frac{1}{k+1}}\,dy. \end{aligned}$$

By the choice of the radius $\rho_k$ (see (3.9), p. 90) we have that

$$|B_r(x_k)\cap E_k| \geq (1-2^{-n(k+1)})|B_r(x_k)|.$$

Since $2^n|S_r^-| = |B_r(x_k)|$, this implies that

$$(2^n-1)|S_r^-| + |S_r^-\cap E_k| \geq 2^n(1-2^{-n(k+1)})|S_r^-|;$$

simplifying, this yields

$$|S_r^-\cap E_k| \geq (1-2^{-nk})|S_r^-|. \tag{5.31}$$

Let $\delta_k = 2^{-k}$ and define $W_r = \{y\in S_r^- : \delta_k r < |y-x_k| < r\}$; then by (5.31), $|W_r| \leq |S_r^-\cap E_k|$. Since $|y-x_k|^{-n+\frac{1}{k+1}}$ is radially decreasing, we have, after integrating in polar coordinates, that

$$\begin{aligned} r^{-n}\int_{S_r^-\cap E_k}|y-x_k|^{-n+\frac{1}{k+1}}\,dy &\geq r^{-n}\int_{W_r}|y-x_k|^{-n+\frac{1}{k+1}}\,dy \\ &= c(n)(k+1)(1-\delta_k^{\frac{1}{k+1}})r^{-n+\frac{1}{k+1}} \geq 4^{-1}c(n)(k+1)|x-x_k|^{-n+\frac{1}{k+1}}. \end{aligned}$$

If we combine these inequalities, we get that for every $x\in S^+$,

$$Rg_k(x) \geq c(n)(k+1)f_k(x). \tag{5.32}$$

Therefore,

$$\begin{aligned} \|Rg_k\|_{L^{p(\cdot)}(B_k)} &\geq \|Rg_k\|_{L^{p(\cdot)}(S^+)} \\ &\geq c(n)(k+1)\|f_k\|_{L^{p(\cdot)}(S^+)} \geq 2^{-n}c(n)(k+1)\|f_k\|_{L^{p(\cdot)}(B_k)}. \end{aligned}$$

For all $k$ sufficiently large this inequality contradicts (5.30). This completes the proof when $p_- = 1$.

We now consider the case $p_+ = \infty$. This follows from the previous case by duality. We again proceed by contradiction: suppose $p(\cdot)$ is such that $p_+ = \infty$ and all the Riesz transforms are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Since $p'(\cdot)_- = 1$, we can repeat the construction above, everywhere replacing $p(\cdot)$ by $p'(\cdot)$. Thus for every $k$ we get a function $f_k \in L^{p'(\cdot)}(B_k)$ and a sector $S^+$ such that

$$0 < \|f_k\|_{L^{p'(\cdot)}(B_k)} \leq 2^n \|f_k\|_{L^{p'(\cdot)}(S^+)}.$$

Further, by its definition we have that $f_k \in L^q(\mathbb{R}^n)$ for some $q$, $1 < q < \infty$. By Theorem 2.34 there exists a non-negative function $h \in L^{p(\cdot)}(S^+)$, $\|h\|_{L^{p(\cdot)}(S^+)} \leq 1$, such that

$$\int_{S^+} f_k(x)h(x)\,dx \geq \frac{k_{p(\cdot)}}{2}\|f_k\|_{L^{p'(\cdot)}(S^+)}. \tag{5.33}$$

By Fatou's lemma on the classical Lebesgue spaces,

$$\liminf_{N\to\infty} \int_{S^+} f_k(x)\min(N, h(x))\,dx \geq \int_{S^+} f_k(x)h(x)\,dx,$$

so we may assume without loss of generality that (5.33) holds for $h$ a bounded function supported in $S^+$. In particular, we may assume that $h \in L^{q'}(\mathbb{R}^n)$.

Again following the construction above, define the operator $R$. Let $g_k = f_k\chi_{S^-}$; then by (5.32),

$$\int_{S^+} f_k(x)h(x)\,dx \leq C(n)(k+1)^{-1}\int_{S_+} Rg_k(x)h(x)\,dx.$$

By a duality argument in the classical Lebesgue spaces (see [143]), since $f_k \in L^q(\mathbb{R}^n)$ and $h \in L^{q'}(\mathbb{R}^n)$, for $1 \leq j \leq n$,

$$\int_{S^+} R_j g_k(x)h(x)\,dx = -\int_{\mathbb{R}^n} g_k(x)R_j h(x)\,dx.$$

It follows that the same identity holds with $R$ in place of $R_j$.

Since the Riesz transforms are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, so is $R$. By duality we get a bounded linear operator $R^*$ on $L^{p(\cdot)}(\mathbb{R}^n)^*$. (See Conway [51].) Since $L^{p'(\cdot)}(\mathbb{R}^n)$ is (isomorphic to) a closed subspace of $L^{p(\cdot)}(\mathbb{R}^n)^*$, by restriction we may assume that $R^*$ is a bounded operator on $L^{p'(\cdot)}(\mathbb{R}^n)$ and we have the relationship

$$R^*g_k(h) = \int_{\mathbb{R}^n} g_k(x)Rh(x)\,dx. \tag{5.34}$$

(Here, $R^*g_k$ is a linear functional in $L^{p(\cdot)}(\mathbb{R}^n)^*$; *a priori* we do not know that it can be identified with a function in $L^{p'(\cdot)}(\mathbb{R}^n)$.) By Hölder's inequality (Theorem 2.26) and our assumption on $p(\cdot)$ we have that

$$|R^*g_k(h)| \le K_{p(\cdot)}\|g_k\|_{p'(\cdot)}\|Rh\|_{p(\cdot)} \le C\|g_k\|_{p'(\cdot)}\|h\|_{L^{p(\cdot)}(S^+)} \le C\|f_k\|_{L^{p'(\cdot)}(B_k)}.$$

Therefore, combining the above inequalities we have that

$$\|f_k\|_{L^{p'(\cdot)}(B_k)} \le C(n, p(\cdot))(k+1)^{-1}\left|\int_{\mathbb{R}^n} g_k(x)Rh(x)\,dx\right|$$
$$= C(n, p(\cdot))(k+1)^{-1}|R^*g_k(h)| \le C(n, p(\cdot))(k+1)^{-1}\|f_k\|_{L^{p'(\cdot)}(B_k)}.$$

Since $\|f_k\|_{L^{p'(\cdot)}(\mathbb{R}^n)} > 0$, this inequality cannot hold for $k$ large. This is the desired contradiction, and so the Riesz transforms are not bounded $L^{p(\cdot)}(\mathbb{R}^n)$ when $p_+ = \infty$. This completes our proof. □

It is unknown whether the hypotheses on the boundedness of the maximal operator are necessary for Theorem 5.39 to be true. We can prove, however, that a slightly weaker condition is necessary.

**Theorem 5.43.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that all the Riesz transforms satisfy the weak type inequality* (5.26)*. Then $p(\cdot) \in K_0(\mathbb{R}^n)$.*

*Proof.* Our proof is a variant of the proof in Corollary 4.50 that the $K_0$ condition is necessary for the maximal operator to satisfy the weak type inequality. Define the operator $R = \sum_j R_j$; then $R$ also satisfies the weak type inequality (5.26). By Proposition 4.47 it will suffice to prove that this fact implies that the averaging operators $A_Q$ are uniformly bounded for all cubes $Q$.

Fix a cube $Q$, and a non-negative function $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Let $Q^+$ be the cube such that if $x \in Q^+$ and $y \in Q$, then $x^j - y^j \ge 0$. Then arguing as we did in the proof of Theorem 5.42, for all $x \in Q^+$,

$$R(f\chi_Q)(x) = c(n)\sum_{j=1}^n \int_Q \frac{x^j - y^j}{|x-y|^{n+1}} f(y)\,dy$$
$$\ge c(n)\int_Q \frac{f(y)}{|x-y|^n}\,dy \ge c(n)\fint_Q f(y)\,dy.$$

Therefore, for all $t$, $0 < t < c(n)\fint_Q f(y)\,dy$, by inequality (5.26) we have that

$$t\|\chi_{Q+}\|_{p(\cdot)} \le t\|\chi_{\{x\in\mathbb{R}^n : |R(f\chi_Q)(x)|>t\}}\|_{p(\cdot)} \le C\|f\chi_Q\|_{p(\cdot)}.$$

If we take the supremum over all such $t$, we get

$$\fint_Q f(y)\,dy\|\chi_{Q+}\|_{p(\cdot)} \le C\|f\chi_Q\|_{p(\cdot)}.$$

Let $f = \chi_Q$; then this becomes $\|\chi_{Q+}\|_{p(\cdot)} \le C\|\chi_Q\|_{p(\cdot)}$.

We can now repeat the above proof, replacing the operator $R$ by $-R$ and exchanging the roles of $Q^+$ and $Q$. Then we also get that $\|\chi_Q\|_{p(\cdot)} \leq C\|\chi_{Q^+}\|_{p(\cdot)}$. Combining all of these inequalities, we get that

$$\|A_Q f\|_{p(\cdot)} = \fint_Q f(y)\,dy\,\|\chi_Q\|_{p(\cdot)}$$

$$\leq C \fint_Q f(y)\,dy\,\|\chi_{Q^+}\|_{p(\cdot)} \leq C\|f\chi_Q\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

The constant $C$ is independent of $Q$, so we have the desired inequality and our proof is complete. □

## *Riesz Potentials*

We begin with a definition.

**Definition 5.44.** Given $\alpha$, $0 < \alpha < n$, define the Riesz potential $I_\alpha$, also referred to as the fractional integral operator with index $\alpha$, to be the convolution operator

$$I_\alpha f(x) = \gamma(\alpha, n) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy,$$

where

$$\gamma(\alpha, n) = \frac{\Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right)}{\pi^{n/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}.$$

Since $|x-y|^{\alpha-n}$ is locally integrable, if $f$ is a bounded function of compact support, then $I_\alpha f(x)$ converges absolutely. In fact, if we rewrite the kernel as

$$|x-y|^{\alpha-n}\chi_{\{|x-y|\leq 1\}} + |x-y|^{\alpha-n}\chi_{\{|x-y|>1\}},$$

then by applying Proposition 5.2 we get that $I_\alpha f$ converges for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < n/\alpha$: the first term is immediate; to estimate the second, note that since $p < n/\alpha$, the kernel is in $L^{p'}(\mathbb{R}^n)$.

The constant $\gamma(\alpha, n)$ is chosen so that if $f$ is a Schwartz function, then the Fourier transform of the Riesz potential is

$$\widehat{I_\alpha f}(\xi) = (2\pi|\xi|)^{-\alpha}\hat{f}(\xi).$$

(See Stein [339].)

The Riesz potentials are not bounded on $L^p(\mathbb{R}^n)$, but satisfy off-diagonal inequalities.

**Theorem 5.45.** *Given $\alpha$, $0 < \alpha < n$, and $p$, $1 \leq p < n/\alpha$, define $q > p$ by $1/p - 1/q = \alpha/n$. If $p = 1$, then for all $t > 0$,*

$$|\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}| \leq \left(\frac{C}{t}\int_{\mathbb{R}^n} |f(x)|\,dx\right)^q. \tag{5.35}$$

*If $p > 1$, then*

$$\|I_\alpha f\|_q \leq C\|f\|_p. \tag{5.36}$$

The Riesz potentials are well defined on the variable Lebesgue spaces. If $p_+ < n/\alpha$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$, then $I_\alpha f(x)$ converges for every $x$. To see this, apply Theorem 2.51 to write $f = f_1 + f_2$, where $f_1 \in L^{p_-}(\mathbb{R}^n)$ and $f_2 \in L^{p_+}(\mathbb{R}^n)$. Then by the above observation, $I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x)$ converges absolutely. Furthermore, we can extend Theorem 5.45 to the variable Lebesgue spaces.

**Theorem 5.46.** *Fix $\alpha$, $0 < \alpha < n$. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < n/\alpha$, define $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}.$$

*If there exists $q_0 > \frac{n}{n-\alpha}$ such that $M$ is bounded on $L^{(q(\cdot)/q_0)'}(\mathbb{R}^n)$, then*

$$\|I_\alpha f\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}. \tag{5.37}$$

*If $p_- = 1$ and if $M$ is bounded on $L^{(q(\cdot)/q_0)'}(\mathbb{R}^n)$ when $q_0 = \frac{n}{n-\alpha}$, then for every $t > 0$,*

$$\|t\chi_{\{x\in\mathbb{R}^n:|I_\alpha f(x)|>t\}}\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}. \tag{5.38}$$

To prove Theorem 5.46 we will use Theorem 5.28. To do so we need the theory of weighted norm inequalities for the Riesz potentials. This theory is very closely related to the theory of $A_p$ weights but is adapted to off-diagonal inequalities.

**Definition 5.47.** Given $\alpha$, $0 < \alpha < n$, and $p$, $1 < p < n/\alpha$, define $q$ by

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Then a weight $w$ satisfies the $A_{p,q}$ condition (denoted by $w \in A_{p,q}$) if

$$[w]_{A_{p,q}} = \sup_Q \left(\fint_Q w(x)\,dx\right)\left(\fint_Q w(x)^{-p'/q}\,dx\right)^{q/p'} < \infty.$$

When $p = 1$, let $A_{1,q} = A_1$.

The connection between $A_{p,q}$ and the Muckenhoupt $A_p$ classes is given by the following lemma whose proof is an immediate consequence of Definition 5.47.

**Lemma 5.48.** *Given $\alpha$, $0 < \alpha < n$, and $p$, $1 < p < n/\alpha$, a weight $w \in A_{p,q}$ if and only if $w \in A_r$, $r = 1 + q/p'$.*

The Riesz potentials satisfy weighted norm inequalities that are the analog of Theorem 5.45.

**Theorem 5.49.** *Given $\alpha$, $0 < \alpha < n$, and $p$, $1 \leq p < n/\alpha$, define $q$ by $1/p - 1/q = \alpha/n$ and let $w \in A_{p,q}$. If $p = 1$, then for every $t > 0$,*

$$w(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \leq C \left( \frac{1}{t} \int_{\mathbb{R}^n} |f(x)| w(x)^{1/q} \, dx \right)^q .$$

*If $p > 1$, then*

$$\left( \int_{\mathbb{R}^n} |I_\alpha f(x)|^q w(x) \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)^{p/q} \, dx \right)^{1/p} .$$

*In both cases the constant depends on $n$, $p$, $\alpha$ and $[w]_{A_{p,q}}$.*

*Remark 5.50.* Weighted norm inequalities for Riesz potentials are customarily written with the weight as a multiplier instead of a measure (see Sect. 4.6.5). Define the weight $W = w^{1/q}$; then the $A_{p,q}$ condition is equivalent to

$$\sup_Q \left( \fint_Q W(x)^q \, dx \right)^{1/q} \left( \fint_Q W(x)^{-p'} \, dx \right)^{1/p'} < \infty,$$

and the strong type inequality becomes

$$\left( \int_{\mathbb{R}^n} |I_\alpha f(x) W(x)|^q \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x) W(x)|^p \, dx \right)^{1/p} .$$

See Remark 5.30 above.

*Proof of Theorem 5.46.* Fix $\alpha$, $p(\cdot)$ and $q(\cdot)$ as in the hypotheses. We will first prove (5.37). Since $q_0 > \frac{n}{n-\alpha}$, if we define $p_0$ by $1/p_0 - 1/q_0 = \alpha/n$, then $p_0 > 1$. Therefore, by Lemma 5.48 and Theorem 5.49, if $w \in A_1 \subset A_{1+q_0/p_0'}$, then $w \in A_{p_0,q_0}$ and

$$\left( \int_{\mathbb{R}^n} |I_\alpha f(x)|^{q_0} w(x) \, dx \right)^{1/q_0} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)^{p_0/q_0} \, dx \right)^{1/p_0}$$

holds for all functions $f$ that are bounded and have compact support. (Note that the right-hand side is finite since $w \in A_1$ is locally integrable. Hence the left-hand side is finite as well.)

Define the family $\mathcal{F}$ to be the pairs $(|I_\alpha f|, |f|)$ with $f$ a bounded function of compact support. Then by Theorem 5.28,

$$\|I_\alpha f\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)} \tag{5.39}$$

for all bounded functions of compact support for which the left-hand side is finite. This is always the case. Fix such a function $f$, and let $B$ be a ball of radius at least 1 centered at the origin such that $\operatorname{supp}(f) \subset B$. Then by Corollary 2.48 and Theorem 5.45,

$$\|I_\alpha f\|_{L^{q(\cdot)}(2B)} \leq (1+|2B|)\|I_\alpha f\|_{L^{q_+}(2B)} \leq C\|f\|_{L^{p_+}(B)} < \infty.$$

To estimate the norm of $I_\alpha f$ on $\mathbb{R}^n \setminus 2B$, note first that if $x \in \mathbb{R}^n \setminus 2B$ and $y \in B$, $|x-y| \geq |x| - |y| \geq |x|/2$. Therefore, for all such $x$,

$$|I_\alpha f(x)| \leq \gamma(\alpha,n)\int_B \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \leq C|x|^{\alpha-n},$$

and so

$$\|I_\alpha f\|_{L^{q(\cdot)}(\mathbb{R}^n\setminus 2B)} \leq C\||\cdot|^{\alpha-n}\|_{L^{q(\cdot)}(\mathbb{R}^n\setminus 2B)}.$$

Since $p_- > 1$, $q_- > \frac{n}{n-\alpha}$, and so

$$\int_{\mathbb{R}^n\setminus 2B} |x|^{(\alpha-n)q(x)}\,dx \leq \int_{\mathbb{R}^n\setminus 2B} |x|^{(\alpha-n)q_-}\,dx < \infty,$$

and so by Proposition 2.12, $\||\cdot|^{\alpha-n}\|_{L^{q(\cdot)}(\mathbb{R}^n\setminus 2B)} < \infty$.

Thus (5.39) holds for all bounded functions of compact support. Since $p_+ < n/\alpha < \infty$, by Theorem 2.72 there exists a sequence $\{f_k\}$ of such functions which converge to $f$ in norm and such that $|f_k| \leq |f|$; by Proposition 2.67 if we pass to a subsequence, we may assume that it also converges pointwise almost everywhere. By Fatou's lemma in the classical Lebesgue spaces,

$$|I_\alpha f(x)| \leq I_\alpha(|f|)(x) \leq \liminf_{k\to\infty} I_\alpha(|f_k|)(x).$$

Therefore, by Fatou's Lemma on the variable Lebesgue spaces (Theorem 2.61),

$$\|I_\alpha f\|_{q(\cdot)} \leq \liminf_{k\to\infty}\|I_\alpha(|f_k|)\|_{q(\cdot)} \leq C\liminf_{k\to\infty}\|f_k\|_{p(\cdot)} \leq \|f\|_{p(\cdot)}.$$

This completes our proof of inequality (5.37).

The proof of the weak type inequality (5.35) is almost exactly the same, except that since $q_0 = \frac{n}{n-\alpha}$ we have $p_0 = 1$ and use the weighted weak type inequality and Corollary 5.33. □

*Remark 5.51.* The argument used to prove Theorem 5.46 can also be used to prove the corresponding result for the fractional maximal operator, Theorem 3.42 in Sect. 3.7.4. This follows since Theorem 5.49 is also true with $I_\alpha$ replaced by the fractional maximal operator $M_\alpha$; the proof is nearly identical to the proof of

Theorem 4.35 for the Hardy-Littlewood maximal operator. However, while this approach yields a short proof of Theorem 3.42, it only works if $p_+ < n/\alpha$ and Theorem 3.42 is also true when $p_+ = n/\alpha$.

## 5.6 Notes and Further Results

### *5.6.1 References*

The properties of convolution operators given in Sect. 5.1, particularly Proposition 5.2, Theorem 5.4 and Lemma 5.7 can be found in Duoandikoetxea [96], Grafakos [143] and Stein [339]. The key inequality in Lemma 5.7 relating potential type approximate identities and the Hardy-Littlewood maximal operator was first discovered (in a special case) by Hardy and Littlewood [148].

Convolution operators on variable Lebesgue spaces were first considered by Sharapudinov [332] on the unit circle and by Samko [312] on bounded domains. Both of these authors prove versions of Theorem 5.11 with different assumptions on the approximate identity. Theorems 5.8 and 5.11 were proved by Diening [77] assuming that $1 < p_- \leq p_+ < \infty$ and the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$. (See Remark 5.14.) The general version of Theorem 5.8 was proved in [58]. Theorem 5.11 was also proved there assuming that $p(\cdot) \in LH(\Omega)$. This proof depended on a pointwise estimate for approximate identities; see Sect. 5.6.2 below. The proof of Theorem 5.11 given here is new. Theorem 5.9 and Example 5.15 are new.

The fact that variable Lebesgue spaces are never translation invariant (Theorem 5.17) was first proved by Kováčik and Rákosník [219]. Diening [77] proved a somewhat different version of Theorem 5.19; see also [82]. Our proof is adapted from his. Proposition 5.20 and Example 5.21 are from [58]. Weaker versions of Young's inequality were also considered by Samko [312] and in [58]; see also [82].

Rubio de Francia first proved the extrapolation theorem that bears his name (Theorem 5.22) in [302–304]. Since then there have been a number of proofs and extensions of this result. The approach adopted here in terms of pairs of functions was implicit in [72] and developed in [66]. For a comprehensive treatment of extrapolation with extensive references, see [69]. Theorems 5.24 and 5.28 were first proved in [61]; see [69] for a proof from the more general perspective of abstract Banach function spaces. Corollary 5.34 was proved in [61]; Corollaries 5.32 and 5.33 were implicit there and in [69].

The modern theory of singular integral operators goes back to the seminal papers by Calderón and Zygmund [40, 41]. For the properties of Hilbert and Riesz transforms and for Theorem 5.37, see Stein [339], García-Cuerva and Rubio de Francia [140], Duoandikoetxea [96] and Grafakos [143]. For Theorem 5.40 and for the theory of weighted norm inequalities for singular integrals in general, see the last three references; for Calderón-Zygmund operators, the generalization of singular integral operators, see the last two and Stein [340]. The strong type inequality in Theorem 5.39 was first proved by Diening and Růžička [88]; our proof is adapted from [61], as is the proof of Theorem 5.41; see also [69]. Theorems 5.42 and 5.43 are new; the idea behind their proofs is taken from the proof of the necessity of the $A_p$ condition for singular integrals in [140].

The Riesz potentials were introduced by M. Riesz [299]. For Theorem 5.45 and their Fourier transform properties, see Stein [339]. The $A_{p,q}$ weights were introduced by Muckenhoupt and Wheeden [272]; they also proved Theorem 5.49. Theorem 5.46 has been considered by several authors. Samko [312] proved a strong type inequality on a bounded domain $\Omega$ assuming $p(\cdot) \in LH_0(\Omega)$ using delicate kernel estimates. A similar result was proved by Edmunds and Meskhi [103] on $\Omega = (0, 1)$. Diening [78] proved the strong type inequality on $\mathbb{R}^n$ assuming $p(\cdot) \in LH_0(\mathbb{R}^n)$ and $p(\cdot)$ is constant outside of a large ball. However, his proof extends to the case $p(\cdot) \in LH(\mathbb{R}^n)$ with almost no change. The proof depends on a pointwise inequality relating the Riesz potential and the Hardy-Littlewood maximal operator: see Sect. 5.6.2 below. Theorem 5.46 was also proved for $p(\cdot) \in LH(\mathbb{R}^n)$ in [42]. The proof depended on norm inequalities for the fractional maximal operator (Theorem 3.42) and an inequality due to Welland [346] relating the Riesz potential and the fractional maximal operator. The proof given here is from [61]. The closely related questions of the range and invertibility of the Riesz potential were studied by Almeida [10] and Almeida and Samko [11].

Weighted inequalities on variable Lebesgue spaces for singular integrals and Riesz potentials have been considered by a number of authors. See the references in Sect. 4.6.5.

### *5.6.2 Pointwise Estimates*

As we noted in Remark 5.14, if $p_- > 1$, then the proof of Theorem 5.11 follows from the norm inequalities for the maximal operator if we assume $M$ is bounded on $L^{p(\cdot)}$. To include the case $p_- = 1$, the proof of this result in [58] avoided the maximal operator and instead generalized the pointwise inequalities for the maximal operator. (See Theorem 3.32.)

**Proposition 5.52.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$ and $p(\cdot) \in LH(\Omega)$. Let $\{\phi_t\}$ be a potential-type approximate identity and let $\Phi$ be the radial majorant of $\phi$. Then for $f \in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot)} \leq 1$, all $t > 0$ and $x \in \Omega$,*

$$|\phi_t * f(x)|^{p(x)} \leq C\left(\Phi_t * |f(\cdot)|^{p(\cdot)}\right)(x) + C\left(\Phi_t * R\right)(x) + CR(x),$$

*where $R(x) = (e + |x|)^{-n-1}$.*

Inequality (5.5) follows from this result and from Propositions 5.2 and 2.12.

A similar approach to proving norm inequalities for the Riesz potential was used by Diening [78], who proved a pointwise estimate relating $I_\alpha$ and the maximal operator.

**Proposition 5.53.** *Given $\alpha$, $0 < \alpha < n$, let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $1 < p_- \le p_+ < n/\alpha$ and $p(\cdot) \in LH(\mathbb{R}^n)$. Define $q(\cdot)$ by $1/p(x) - 1/q(x) = \alpha/n$. Then there exists a function $S \in L^\infty(\mathbb{R}^n) \cap L^{p_-}(\mathbb{R}^n)$ such that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{p(\cdot)} \le 1$,*

$$|I_\alpha f(x)|^{q(x)/p_-} \le CMf(x)^{p(x)/p_-} + S(x).$$

Given these hypotheses we also have that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, so the desired inequality for the Riesz potential follows by Proposition 2.12. Proposition 5.53 generalizes to the variable Lebesgue spaces a pointwise inequality due to Hedberg [168]. This result was generalized to certain Musielak-Orlicz spaces (see Sect. 2.10.2) by Futamura, Mizuta and Shimomura [138].

### *5.6.3 More on Approximate Identities*

Theorems 5.8 and 5.11 are true for another class of approximate identities. In [58] it was shown that if $p(\cdot) \in LH(\Omega)$ and $\phi \in L^{p'(\cdot)_+}(\Omega)$ has compact support, then both of these results hold. Proposition 5.52 is valid in this case only if $|f| \le 1$, but if $|f| = 0$ or $|f| \ge 1$, then a different argument shows that for all $t > 0$,

$$\int_\Omega |\phi_t * f(x)|^{p(x)} \le C < \infty.$$

If the hypothesis that $\phi$ has compact support is omitted, then these versions of Theorems 5.8 and 5.11 do not hold in general: see [58] for a counter-example.

In [58] it was conjectured that Theorem 5.11 was true for any $\phi \in L^1(\mathbb{R}^n)$ if $\phi$ satisfies the decay condition

$$|\phi(x-y) - \phi(x)| \le C\frac{|y|}{|x|^{n+1}}, \qquad |x| > 2|y|.$$

(This holds for example if $|\nabla\phi(x)| \le C|x|^{-n-1}$.) In the classical Lebesgue spaces this result is due to Zo [364]. It was proved for the variable Lebesgue spaces in [61] using extrapolation and weighted norm inequalities for vector-valued singular integrals.

All of these results were extended to the more general setting of the Musielak-Orlicz spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$ (see Sect. 2.10.2) by Maeda, Mizuta and Ohno [243].

### 5.6.4 Applications of Extrapolation

As we noted above, the theory of extrapolation can be applied to a much larger collection of operators than discussed here. In [61] extrapolation was used to prove norm inequalities for the vector-valued maximal operator, "rough" singular integrals, commutators, multipliers, and square functions.

In [69] extrapolation was used to show that so-called "admissible" wavelets form an orthonormal basis for $L^{p(\cdot)}(\mathbb{R}^n)$ provided that $1 < p_- \le p_+ < \infty$ and the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. A different proof of this result was given by Izuki [183]. (Wavelets on variable Lebesgue spaces have also been studied by Kopaliani [214].) Huang and Xu [170] applied extrapolation to study multilinear singular integrals and commutators on the variable Lebesgue spaces. Motos, Planells and Talavera [270] used it to study variable Lebesgue spaces of entire analytic functions.

The proof of Theorem 5.24 can be generalized to many other settings. As we noted above, in [69] it was proved as a corollary to a general extrapolation result on abstract Banach function spaces. Implicit in this is an extrapolation result for Musielak-Orlicz spaces provided the maximal operator satisfies appropriate boundedness conditions. Maeda, Mizuta and Ohno [243] proved a special case of this for the $L^{p(\cdot)}(\log L)^{q(\cdot)}$ spaces. Kokilashvili and Samko [209] proved an extrapolation theorem for weighted variable Lebesgue spaces defined on metric spaces. They used this to study norm inequalities for a variety of operators in this setting. Edmunds, Kokilashvili and Meskhi [99] generalized extrapolation to variable Lebesgue spaces on which the one-sided maximal operator is bounded (see Sect. 3.7.6). This yields norm inequalities for the so-called "one-sided" operators.

In Sect. 6.4 we will use extrapolation to prove the Sobolev embedding theorem for variable Sobolev spaces.

### 5.6.5 Sharp Maximal Operator Estimates

Another approach to norm inequalities on the variable Lebesgue spaces is to use the Fefferman-Stein sharp maximal operator. Given a locally integrable function $f$, define the sharp maximal operator by

$$M^{\#}f(x) = \sup_Q \fint_Q |f(y) - f_Q|\, dy,$$

where $f_Q = \fint_Q f(y)\, dy$ and the supremum is taken over all cubes $Q$ with sides parallel to the coordinate axes. The sharp maximal operator was introduced by C. Fefferman and Stein [125].

Given a function $f$, $f$ and $M^{\#}f$ are comparable in $L^p$ norm: for all $p$, $1 \le p < \infty$,

$$\|f\|_p \le C\|M^{\#}f\|_p,$$

and in fact this inequality is true if we replace the $L^p$ norm by the $L^p(w)$ norm for any $w \in A_\infty$. (This was proved in [125]; see also [69,96].) The same inequality also holds in the variable Lebesgue spaces.

**Theorem 5.54.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 \le p_- \le p_+ < \infty$, if the maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$, then*

$$\sup_{t>0} \|t\chi_{\{x\in\mathbb{R}^n:|f(x)|>t\}}\|_{p(\cdot)} \le C \sup_{t>0} \|t\chi_{\{x\in\mathbb{R}^n:M^\# f(x)>t\}}\|_{p(\cdot)}.$$

Moreover,

$$\|f\|_{p(\cdot)} \le C\|M^\# f\|_{p(\cdot)}.$$

The strong type inequality in Theorem 5.54 was originally proved by Diening and Růžička [88] (see also [82]). A different proof using extrapolation was given in [61]. The weak type inequality can be proved via extrapolation using the corresponding estimate in the classical Lebesgue spaces. (The weighted weak type inequality is implicit in the so-called good-$\lambda$ inequality used to prove the strong type inequality. It is proved explicitly in a more general setting in [69].)

The importance of Theorem 5.54 comes from a pointwise estimate due to Álvarez and Pérez [14] that uses a variant of the sharp maximal operator: for $\delta > 0$, let $M^\#_\delta f(x) = M^\#(|f|^\delta)(x)^{1/\delta}$.

**Lemma 5.55.** *Let $T$ be a singular integral operator with kernel $K$. Then for all $\delta$, $0 < \delta < 1$, there exists a constant $C(\delta)$ such that for all bounded functions of compact support,*

$$M^\#_\delta(Tf)(x) \le C(\delta)Mf(x).$$

Given Lemma 5.55 and Theorem 5.54, norm inequalities for singular integrals (Theorem 5.39) follow immediately from the corresponding inequalities for the maximal operator. Diening and Růžička [88] used this approach to prove Theorem 5.39.

This technique is extremely flexible, since there exist sharp function estimates like Lemma 5.55 for many operators. For example, Adams [6] proved that for all $\alpha$, $M^\#(I_\alpha f)(x) \le C_\alpha M_\alpha f(x)$, where $M_\alpha$ is the fractional maximal operator (see Sect. 3.7.4). Norm inequalities for Riesz potentials now follow from the corresponding inequalities for $M_\alpha$ (Theorem 3.42). Karlovich and Lerner [195] used a sharp function estimate to prove norm inequalities for commutators of singular integrals on variable Lebesgue spaces. For a summary of sharp function estimates, see [69].

### *5.6.6 Local to Global Estimates*

In practice, it is often much easier to prove norm inequalities on bounded domains than on all of $\mathbb{R}^n$. For example, estimates for the maximal operator on bounded domains require only one hypothesis (say $p(\cdot) \in LH_0$) and the argument is much simpler. (See Diening's original proof [77].) Given an operator $T$ one could naively try to take advantage of this by decomposing $\mathbb{R}^n$ into the union of disjoint cubes $\{Q_j\}$ of side-length 1, and then writing

$$\|Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \sum_j \|Tf\|_{L^{p(\cdot)}(Q_j)}.$$

Ideally, one can then prove an estimate of the form $\|Tf\|_{L^{p(\cdot)}(Q_j)} \leq C\|f\|_{L^{p(\cdot)}(Q_j)}$. Unfortunately, even if this were possible, the problem remains that

$$\sum_j \|f\|_{L^{p(\cdot)}(Q_j)}$$

need not be comparable to $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. However, a more sophisticated version of this technique works.

**Proposition 5.56.** *Given $p(\cdot) \in LH(\mathbb{R}^n)$, fix a set $\mathcal{Q} = \{Q_j\}$ of cubes of equal side-length, ordered so that if $i < j$,* $\operatorname{dist}(Q_i, 0) \leq \operatorname{dist}(Q_j, 0)$. *Define*

$$\|f\|_{p(\cdot),\mathcal{Q}} = \left(\sum_j \|f\|_{L^{p(\cdot)}(Q_j)}^{p_\infty}\right)^{1/p_\infty};$$

*then*

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \|f\|_{p(\cdot),\mathcal{Q}}.$$

Proposition 5.56 is due to Hästö [165]; the simple proof uses a characterization of variable sequence spaces (see Sect. 2.10.7) discovered independently by Nekvinda [281] and Nakano [279] (see also [245]).

To illustrate how this technique can be used, we sketch an alternate proof from [165] that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (Theorem 3.16). We first need to divide the maximal operator itself into two pieces, a "local" piece and a "global" one. Define

$$M_L f(x) = \sup_{\substack{Q \ni x \\ \ell(Q) \leq 1}} \fint_Q |f(y)|\,dy, \qquad M_G f(x) = \sup_{\substack{Q \ni x \\ \ell(Q) > 1}} \fint_Q |f(y)|\,dy.$$

Then $Mf(x) \leq M_L f(x) + M_G f(x)$, so to prove that the maximal operator is bounded on $L^{p(\cdot)}$ it will suffice to prove that each of these operators is bounded. The key estimate is for $M_L$. Fix a collection $\mathcal{Q} = \{Q_j\}$ of cubes with side-length 1. Then for each $j$ and $x \in Q_j$,

$$M_L f(x) \leq M(f\chi_{3Q_j})(x).$$

Therefore, if we assume that the maximal operator is bounded on bounded domains, by Proposition 5.56 (applied twice) we have that

$$\begin{aligned}\|M_L f\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\\ \leq C\Big(\sum_j \|M_L f\|_{L^{p(\cdot)}(Q_j)}^{p_\infty}\Big)^{1/p_\infty} &\leq C\Big(\sum_j \|M(f\chi_{3Q_j})\|_{L^{p(\cdot)}(Q_j)}^{p_\infty}\Big)^{1/p_\infty} \\ &\leq C\Big(\sum_j \|f\|_{L^{p(\cdot)}(3Q_j)}^{p_\infty}\Big)^{1/p_\infty} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

To estimate $\|M_G f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ we introduce an ancillary operator. Given $x$, let $Q_x$ be the cube centered at $x$ such that $\ell(Q_x) = 1$, and define $Af(x) = \int_{Q_x} |f(y)|\,dy$. Then a straightforward argument shows that $M_G f(x) \leq C(n)M_G(Af)(x)$. Further, using an embedding theorem similar to Proposition 2.53 (see also [92]) and the fact that $M$ is bounded on $L^\infty$ and $L^{p_\infty}$, one can show that $A$ maps $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)\cap L^{p_\infty}(\mathbb{R}^n)$ and $M_G$ maps $L^\infty(\mathbb{R}^n)\cap L^{p_\infty}(\mathbb{R}^n)$ into $L^{p(\cdot)}(\mathbb{R}^n)$. Combining these estimates shows that $\|M_G f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Depending on the operator, the decomposition argument required to use the local estimate will be more or less complicated. Nevertheless, this technique has some advantages over extrapolation. It can be used in those cases in which weighted norm inequalities are not available. It can also be extended to other kinds of spaces, such as weighted variable Lebesgue spaces, variable Sobolev spaces, and variable Morrey spaces. For examples of such applications, see [165].

### 5.6.7 The Variable Riesz Potential

The Riesz potential can be generalized by allowing the value of the index $\alpha$ to vary. Given $\Omega$, let $\alpha(\cdot) : \Omega \to (0, n)$ be a measurable function. Define the variable Riesz potential by

$$I_{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha(x)}}\,dy.$$

These operators were introduced by Samko; on the real line they have applications to the theory of fractional differentiation and integration. (See [314] and the references it contains.) If $\Omega$ is bounded and $p(\cdot) \in LH_0(\Omega)$, then an equivalent definition is gotten by replacing $\alpha(x)$ by $\alpha(y)$:

$$I_{\alpha(\cdot)} f(x) \approx \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha(y)}}\,dy.$$

A generalization of Theorem 5.46 holds on bounded domains.

**Proposition 5.57.** *Given a bounded domain $\Omega$ let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in LH_0(\Omega)$. Let $\alpha(\cdot) : \Omega \to (0, n)$ be a measurable function such that $\alpha_- > 0$, $\big(p(\cdot)\alpha(\cdot)\big)_+ < n$, and $\alpha(\cdot) \in LH_0(\Omega)$. Define $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha(x)}{n}.$$

*Then*

$$\|I_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}.$$

Proposition 5.57 was proved by Samko [312]. A similar result on $\Omega = (0, 1)$ was proved by Edmunds and Meskhi [103]. Weighted inequalities were proved for this operator by Kokilashvili and Samko [205] and Samko, Samko and Vakulov [309, 310].

Samko [319] asked whether these results could be extended to unbounded domains. Hästö [165] proved that Proposition 5.57 is false if $\Omega = \mathbb{R}^n$. He gave an example of a smooth index function $\alpha(\cdot)$ and exponent $p(\cdot)$ that are constant on $B_1(0)$ and on $\mathbb{R}^n \setminus B_3(0)$ and a function $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $I_{\alpha(\cdot)} f \notin L^{q(\cdot)}(\mathbb{R}^n)$. However, Hästö also showed that by redefining the operator at infinity, the desired inequalities hold. Given $\alpha(\cdot)$, define the new index function $\beta(x, y) = \min(\alpha(x), \alpha(y))$, and let

$$I^{\beta}_{\alpha(\cdot)} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\beta(x,y)}}\, dy.$$

**Proposition 5.58.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in LH(\mathbb{R}^n)$. Let $\alpha(\cdot) : \mathbb{R}^n \to (0, n)$ be a measurable function such that $\alpha_- > 0$, $\big(p(\cdot)\alpha(\cdot)\big)_+ < n$, and $\alpha(\cdot) \in LH(\mathbb{R}^n)$. Define $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha(x)}{n}.$$

*Then*

$$\|I^{\beta}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

### *5.6.8 Vector-Valued Maximal Operators*

Given $r$, $1 < r < \infty$, the vector-valued maximal operator is defined on vector valued functions $f = \{f_i\}$ by

$$\overline{M}_r f(x) = \left(\sum_i Mf_i(x)^r\right)^{1/r} = \|Mf_i(x)\|_{\ell^r}.$$

It follows from the weighted norm inequalities for the maximal operator (Theorem 4.35) and Corollary 5.34 that if the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then so is $\overline{M}_r$. This was first proved in [61].

It is a natural question whether this operator can be generalized to a variable vector-valued maximal operator $\overline{M}_{r(\cdot)}$ by replacing the $\ell^r$ norm with the norm on a variable sequence space $\ell^{r(\cdot)}$ (see Sect. 2.10.7). However, Diening, Hästö and Roudenko [87] gave a simple counter-example to show that this does not hold in general even when $p(\cdot) = p$ is constant. Let $r(\cdot)$ be such that there exist bounded sets $\Omega_k$, $k = 1, 2$, such that if $x \in \Omega_k$, $r(x) = r_k$, and $r_1 > r_2$. Fix a sequence $\{a_i\} \in \ell^{r_1} \setminus \ell^{r_2}$, and define $f_i = a_i \chi_{\Omega_1}$. Then for all $x \in \Omega_2$, $Mf_i(x) \geq ca_i$. Therefore, if $\overline{M}_{r(\cdot)}$ were bounded on $L^p(\mathbb{R}^n)$, then

$$\big\| \|a_i\|_{\ell^{r_2}} \big\|_{L^p(\Omega_2)} \leq \big\| \|Mf_i\|_{\ell^{r(\cdot)}} \big\|_{L^p(\mathbb{R}^n)} \leq C \big\| \|f_i\|_{\ell^{r(\cdot)}} \big\|_{L^p(\mathbb{R}^n)} = \big\| \|a_i\|_{\ell^{r_1}} \big\|_{L^p(\Omega_1)},$$

which implies that $\{a_i\} \subset \ell^{r_2}$, a contradiction.

### *5.6.9 Two Classical PDEs*

We can apply the results in Sect. 5.2 to show that the classical solutions to the Laplacian and the heat equation extend to the variable Lebesgue spaces. Recall the definition of the Poisson and Gauss-Weierstrass kernels: for $t > 0$ and $x \in \mathbb{R}^n$, let

$$P_t(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}, \qquad W_t(x) = t^{-n} e^{-\pi |x|^2/t^2}.$$

It is immediate from the definitions that $\{P_t\}$ and $\{W_t\}$ are potential type approximate identities.

**Proposition 5.59.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$, then $u(x,t) = P_t * f(x)$ is the solution of the boundary value problem*

$$\begin{cases} \triangle u(x,t) = 0, & (x,t) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

*where the second equality is understood in the sense that $u(x,t)$ converges to $f(x)$ as $t \to 0$ pointwise almost everywhere and in $L^{p(\cdot)}(\mathbb{R}^n)$ norm.*

**Proposition 5.60.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$. Given $f \in L^{p(\cdot)}(\mathbb{R}^n)$, define $w(x,t) = W_t * f(x)$ and $\bar{w}(x,t) = w(x, \sqrt{4\pi t})$. Then $\bar{w}$ is the solution of the initial value problem*

$$\begin{cases} \frac{\partial \bar{w}}{\partial t}(w,t) - \triangle \bar{w}(x,t) = 0, & (x,t) \in \mathbb{R}^{n+1}_+, \\ \bar{w}(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

*where the second equality is understood in the sense that* $\bar{w}(x,t)$ *converges to* $f(x)$ *as* $t \to 0$ *pointwise almost everywhere and in* $L^{p(\cdot)}(\mathbb{R}^n)$ *norm.*

*Proof.* We sketch the proof of Proposition 5.59; the proof of Proposition 5.60 is identical. First, we show that $u$ is a solution. By Theorem 2.51 write $f = f_1 + f_2$ with $f_1 \in L^{p_-}(\mathbb{R}^n)$ and $f_2 \in L^{p_+}(\mathbb{R}^n)$. Then by the solution of the Laplacian in the classical Lebesgue spaces (see [142]), $u_1 = P_t * f_1$ and $u_2 = P_t * f_2$ are solutions, and so $u = u_1 + u_2$ is also a solution. The identity $u(x,0) = f(x)$ follows from the fact that $\{P_t\}$ is a potential type approximate identity and Theorems 5.8 and 5.11. □

Propositions 5.59 and 5.60 were first proved in [58]. Sharapudinov [332] proved similar results on the unit circle.

### 5.6.10 The Fourier Transform

The Fourier transform of a function $f$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\,dx, \qquad \xi \in \mathbb{R}^n.$$

This integral converges if $f$ is a Schwartz function, or more generally is in $L^1(\mathbb{R}^n)$. The definition can be extended in a natural way to functions $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$. A key norm inequality for the Fourier transform is the Hausdorff-Young inequality: for $1 \le p \le 2$,

$$\|\hat{f}\|_{p'} \le \|f\|_p.$$

It is tempting to conjecture that this inequality extends to the variable Lebesgue spaces: for all $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 \le p_- \le p_+ \le 2$,

$$\|\hat{f}\|_{p'(\cdot)} \le \|f\|_{p(\cdot)}.$$

However, this is false, and a counter-example is straightforward to construct on the real line. For $-1 < a < 0$, let $f(x) = |x|^a$. Then the Fourier transform of $f$ is given by

$$\hat{f}(\xi) = \frac{-2\sin\left(\frac{\pi a}{2}\right)\Gamma(a+1)}{|2\pi\xi|^{a+1}}.$$

(See [109, 294].) In particular, let $a = -2/3$, so that $\hat{f}(\xi) = c|\xi|^{-1/3}$. Now let the exponent $p(\cdot) \in \mathcal{P}(\mathbb{R})$ be such that $p(\cdot)$ is smooth, $p(x) = 5/4$ on $(-1, 1)$, and $p(x) = 2$ on $\mathbb{R} \setminus (-2, 2)$. Then we have that $f \in L^{p(\cdot)}(\mathbb{R})$ but $\hat{f} \notin L^{p'(\cdot)}(\mathbb{R})$.

The failure of the Hausdorff-Young inequality for this function $f$ comes from the fact that the Fourier transform redistributes the mass of $f$ between the origin and infinity. In the study of weighted norm inequalities for the Fourier transform this is handled by replacing the weight $w(x)$ by $w(1/x)$. For example, Benedetto, Henig and Johnson [24] showed that if $1 \leq p \leq 2$ and $w \in A_2$ is an even function on $\mathbb{R}$ that is non-decreasing on $(0, \infty)$, then

$$\left(\int_{\mathbb{R}^n} |\hat{f}(x)|^{p'} w(1/x)\, dx\right)^{1/p'} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^{p/p'}\, dx\right)^{1/p}.$$

The corresponding approach for variable Lebesgue spaces would be to replace the exponent $p'(\cdot)$ with $q(x) = p'(1/x)$. (Such a transformation has been applied in another setting: see Samko and Vakulov [323].) In the above example, $\hat{f} \in L^{q(\cdot)}(\mathbb{R})$. However, we must also assume that $p(\cdot)$ is non-decreasing on $(0, \infty)$. For instance, in the above example if we take $p(3/2) < p(0)$, then we could choose $p(\cdot)$ so that $g(x) = |x - 3/2|^{-4/5}$ is in $L^{p(\cdot)}(\mathbb{R})$; then by the translation properties of the Fourier transform, $\hat{g}(\xi) = c|\xi|^{-1/5}$, and $\hat{g} \notin L^{q(\cdot)}(\mathbb{R})$. This suggests the following problem.

**Question 5.61.** Given $p(\cdot) \in \mathcal{P}(\mathbb{R})$ such that $1 \leq p_- \leq p_+ \leq 2$ and $p(\cdot)$ is even and non-decreasing on $(0, \infty)$, is it the case that there exists a constant $C > 0$ such that for all $f \in L^{p(\cdot)}(\mathbb{R})$, $\|\hat{f}\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}$, where $q(x) = p'(1/x)$?

# Chapter 6
# Basic Properties of Variable Sobolev Spaces

In this chapter we present the elementary theory of variable Sobolev spaces. Unlike Chap. 2, where we systematically developed a complete theory of variable Lebesgue spaces, our goal here is less ambitious. Our aim is to illustrate how the theorems and techniques given in previous chapters can be applied to the variable Sobolev spaces. Consequently, many of our results are not given in the fullest generality possible, and there are a number of results from the classical theory of Sobolev spaces that we do not discuss.

We begin with a review of basic terminology and definitions. We assume that the reader has some knowledge of weak derivatives and classical Sobolev spaces, and so we state some results without proof and refer the reader to some standard references. (Additional references are given in Sect. 6.5.) We then discuss three topics: the density of smooth functions, Poincaré inequalities, and the Sobolev embedding theorem. The majority of our positive results require some regularity assumption on the exponent function $p(\cdot)$; following our approach in Chap. 5, in most cases we will state these in terms of boundedness properties of the maximal operator. Throughout this section we assume that $\Omega$ is an open set; unless we specify otherwise, it may be either bounded or unbounded.

## 6.1 The Space $W^{k,p(\cdot)}(\Omega)$

In this section we define the variable Sobolev spaces and give their basic function space properties. We begin with some preliminary definitions. A multi-index $\alpha$ is a vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of non-negative integers. Define the length of $\alpha$ as

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Let $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$. Given another multi-index $\beta$, define

$$\alpha \pm \beta = (\alpha_1 \pm \beta_1, \dots, \alpha_n \pm \beta_n).$$

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If $\beta_j \le \alpha_j$ for all $j$, then we write $\beta \le \alpha$. For $\beta \le \alpha$, define the multi-index binomial coefficient by

$$\binom{\alpha}{\beta} = \prod_{j=1}^{n} \binom{\alpha_j}{\beta_j} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

Given $f \in C^\infty(\Omega)$, for each $j$, $1 \le j \le n$, let $\partial_j f = \partial f/\partial x^j$ be the $j$-th partial derivative, and for each $i > 0$ let $\partial_j^i f = \partial_j \circ \partial_j \circ \cdots \circ \partial_j f$ denote $i$ iterations of the $j$-th partial derivative. For each $\alpha$, $|\alpha| > 0$, define the differentiation operator $D^\alpha$ by

$$D^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f.$$

If $\alpha = (0, \dots, 0) = 0$ is the zero vector, let $D^\alpha f = f$. Define the gradient of $f$ to be the vector $\nabla f = (\partial_1 f, \dots, \partial_n f)$. When $n = 1$ we will write $Df$ for $df/dx$.

**Definition 6.1.** Given $\Omega$, a function $f \in L^1_{\mathrm{loc}}(\Omega)$ and a multi-index $\alpha$, $f$ has a weak derivative of order $\alpha$ if there exists a function $g_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ such that for every function $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega f(x) D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega g_\alpha(x)\phi(x)\, dx.$$

It can be shown that $g_\alpha$ is unique, and we write $D^\alpha f = g_\alpha$.

If a function $f$ is smooth, then by integration by parts we have that the classical derivatives of $f$ are also weak derivatives. More generally, we have the following result. For a proof, see [363].

**Proposition 6.2.** *Given $\Omega$, if the function $f \in L^1_{\mathrm{loc}}(\Omega)$ is absolutely continuous on almost every line segment in $\Omega$ parallel to the coordinate axes, and if its classical partial derivatives along these lines are locally integrable, then it is weakly differentiable and its weak and classical derivatives coincide.*

Like the classical derivatives, weak derivatives are linear: if $f$ and $g$ have weak derivatives of order $\alpha$ then $D^\alpha(f+g) = D^\alpha f + D^\alpha g$ and $D^\alpha(cf) = cD^\alpha f$.

In the classical case, the Sobolev space $W^{k,p}(\Omega)$ consists of all functions $f \in L^p(\Omega)$ such that for all multi-indices $\alpha$ with $|\alpha| \le k$, $D^\alpha f$ exists and belongs to $L^p(\Omega)$. $W^{k,p}_{\mathrm{loc}}(\Omega)$ is defined in the same way, replacing $L^{p(\cdot)}(\Omega)$ with $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$. We extend these definitions to the variable exponent setting.

**Definition 6.3.** Given $\Omega$, $p(\cdot) \in \mathcal{P}(\Omega)$, and an integer $k \ge 1$, define the variable Sobolev space $W^{k,p(\cdot)}(\Omega)$ to be the set of all $f \in L^{p(\cdot)}(\Omega)$ such that if $|\alpha| \le k$, then $D^\alpha f \in L^{p(\cdot)}(\Omega)$. Let

$$\|f\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{p(\cdot)}.$$

If there is no ambiguity about the domain we will often write $\|f\|_{k,p(\cdot)}$ instead of $\|f\|_{W^{k,p(\cdot)}(\Omega)}$. For brevity we will write $\|\nabla f\|_{p(\cdot)}$ instead of $\||\nabla f|\|_{p(\cdot)}$. The space $W^{k,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is the set of all $f$ such that $f \in W^{k,p(\cdot)}(A)$ for every open set $A$ such that $\overline{A}$ is compact and contained in $\Omega$.

If $1 \leq k < l$, then for all $p(\cdot)$, $W^{l,p(\cdot)}(\Omega) \subset W^{k,p(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$. The variable Sobolev spaces can also be embedded in the classical Sobolev spaces. (We will consider the question of embeddings again in Sect. 6.4 below.)

**Proposition 6.4.** *Given $\Omega$, $p(\cdot) \in \mathcal{P}(\Omega)$ and $k \geq 1$, $W^{k,p(\cdot)}(\Omega) \subset W^{k,p_-}_{\mathrm{loc}}(\Omega)$. If $|\Omega| < \infty$, then $W^{k,p(\cdot)}(\Omega) \subset W^{k,p_-}(\Omega)$.*

Proposition 6.4 follows at once from Corollary 2.48. This embedding is useful in proving local or pointwise properties of functions in $W^{k,p(\cdot)}(\Omega)$. For instance, we can use it to extend the chain rule to the variable Sobolev spaces.

**Lemma 6.5.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, if $f \in W^{k,p(\cdot)}(\Omega)$ and $\phi \in C_c^\infty(\Omega)$, then $\phi f \in W^{k,p(\cdot)}(\Omega)$ and for all multi-indices $\alpha$ with $|\alpha| \leq k$,*

$$D^\alpha(\phi f) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta\phi D^{\alpha-\beta} f. \tag{6.1}$$

*Proof.* Since $W^{k,p(\cdot)}(\Omega) \subset W^{k,1}_{\mathrm{loc}}(\Omega)$, the identity (6.1) follows from the chain rule for classical Sobolev spaces (see [142]). For each $\beta$, $D^\beta\phi \in C_c^\infty(\Omega)$ and $D^{\alpha-\beta} f \in L^{p(\cdot)}(\Omega)$. Hence, it follows from (6.1) that $D^\alpha(\phi f) \in L^{p(\cdot)}(\Omega)$ and so $\phi f \in W^{k,p(\cdot)}(\Omega)$. □

The variable Sobolev spaces are Banach spaces and have some of the same properties as $L^{p(\cdot)}(\Omega)$. This is the substance of the next three results.

**Theorem 6.6.** *Given $\Omega$, $p(\cdot) \in \mathcal{P}(\Omega)$ and $k \geq 1$, $W^{k,p(\cdot)}(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{W^{k,p(\cdot)}(\Omega)}$.*

*Proof.* It is immediate that $W^{k,p(\cdot)}(\Omega)$ is a vector space; since $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ is a norm (Theorem 2.17), $\|\cdot\|_{W^{k,p(\cdot)}(\Omega)}$ is also a norm. We will show that $W^{k,p(\cdot)}(\Omega)$ is complete. Let $\{f_j\} \subset W^{k,p(\cdot)}(\Omega)$ be a Cauchy sequence. Then for each $\alpha$, $0 \leq |\alpha| \leq k$, since $\|D^\alpha f_j - D^\alpha f_i\|_{p(\cdot)} \leq \|f_j - f_i\|_{k,p(\cdot)}$, the sequence $\{D^\alpha f_j\}$ is a Cauchy sequence in $L^{p(\cdot)}(\Omega)$. Since $L^{p(\cdot)}(\Omega)$ is complete (Theorem 2.71) there exists $g_\alpha \in L^{p(\cdot)}(\Omega)$ such that $D^\alpha f_j \to g_\alpha$ in norm. Let $g = g_0$.

We clam that $g \in W^{k,p(\cdot)}(\Omega)$ and $f_j \to g$ in $W^{k,p(\cdot)}$ norm. To prove this we will show that if $0 \leq |\alpha| \leq k$, $D^\alpha g = g_\alpha$. Fix $\alpha$; then for any $\phi \in C_c^\infty(\Omega)$, by the generalized Hölder's inequality (Theorem 2.26),

$$\left|\int_\Omega \big(g(x) - f_j(x)\big) D^\alpha \phi(x)\,dx\right| \\ \leq \int_\Omega |g(x) - f_j(x)||D^\alpha \phi(x)|\,dx \leq K_{p(\cdot)} \|g - f_j\|_{p(\cdot)} \|D^\alpha \phi\|_{p'(\cdot)}.$$

Since $D^\alpha \phi \in L^{p'(\cdot)}(\Omega)$, it follows that

$$\lim_{j\to\infty} \int_\Omega f_j(x) D^\alpha \phi(x)\,dx = \int_\Omega g(x) D^\alpha \phi(x)\,dx.$$

The same argument shows that

$$\lim_{j\to\infty} \int_\Omega D^\alpha f_j(x) \phi(x)\,dx = \int_\Omega g_\alpha(x) \phi(x)\,dx.$$

Therefore, by Definition 6.1, $D^\alpha g = g_\alpha$ and our proof is complete. □

For the next two theorems we recall some notation and define an auxiliary space. Given a Banach space $X$, let $X^*$ denote the dual space—the collection of bounded linear functionals on $X$. We write $X^{**} = (X^*)^*$. Let $M = M(k,n)$ be the number of multi-indices $\alpha$ such that $0 \leq |\alpha| \leq k$ and define the product space

$$L_M^{p(\cdot)}(\Omega) = \prod_{i=1}^M L^{p(\cdot)}(\Omega),$$

that is, $L_M^{p(\cdot)}(\Omega)$ is the cartesian product of $M$ (different) copies of $L^{p(\cdot)}(\Omega)$. Note that $L_M^{p(\cdot)}(\Omega)$ is a Banach space whose norm is the sum of the norms of each copy of $L^{p(\cdot)}(\Omega)$. (See Conway [51].)

**Theorem 6.7.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. Then $W^{k,p(\cdot)}(\Omega)$ is separable. If $1 < p_- \leq p_+ < \infty$, then $W^{k,p(\cdot)}(\Omega)$ is reflexive: $W^{k,p(\cdot)}(\Omega)^{**} = W^{k,p(\cdot)}(\Omega)$, with equality in the sense of isomorphism.*

*Proof.* If $p_+ < \infty$, $L^{p(\cdot)}(\Omega)$ is separable (Theorem 2.78), and therefore $L_M^{p(\cdot)}(\Omega)$ is separable. Similarly, if $1 < p_- \leq p_+ < \infty$, then $L^{p(\cdot)}(\Omega)$ is reflexive (Corollary 2.81), and therefore so is $L_M^{p(\cdot)}(\Omega)$. (For these properties of Cartesian products of Banach spaces, see [51, 95].)

Impose some linear order on the $M$ multi-indices $\alpha$ with $|\alpha| \leq k$, and associate to each $f \in W^{k,p(\cdot)}(\Omega)$ the vector $(D^\alpha f)_{|\alpha|\leq k}$. Then this map is an isometry from $W^{k,p(\cdot)}(\Omega)$ into $L_M^{p(\cdot)}(\Omega)$, and since $W^{k,p(\cdot)}(\Omega)$ is a Banach space (Theorem 6.6), its image in $L_M^{p(\cdot)}(\Omega)$ is closed. Therefore, whenever $L_M^{p(\cdot)}(\Omega)$ is separable or reflexive, so is $W^{k,p(\cdot)}(\Omega)$. □

**Theorem 6.8.** *Given* $\Omega$ *and* $p(\cdot) \in \mathcal{P}(\Omega)$*, suppose* $p_+ < \infty$*. Then* $\Phi \in W^{k,p(\cdot)}(\Omega)^*$ *if and only if there exists* $(v_\alpha)_{|\alpha|\le k} \in L_M^{p'(\cdot)}(\Omega)$ *such that for all* $f \in W^{k,p(\cdot)}(\Omega)$*,*

$$\Phi(f) = \Phi_{(v_\alpha)}(f) = \sum_{|\alpha|\le k} \int_\Omega D^\alpha f(x) v_\alpha(x)\, dx. \tag{6.2}$$

*Proof.* Given any $(v_\alpha)_{|\alpha|\le k} \in L_M^{p'(\cdot)}(\Omega)$, by the generalized Hölder's inequality (Theorem 2.26), $\Phi_{(v_\alpha)}$ defines a bounded linear functional on $W^{k,p(\cdot)}(\Omega)$. To see the converse, fix $\Phi \in W^{k,p(\cdot)}(\Omega)^*$. Since $W^{k,p(\cdot)}(\Omega)$ is isometric to a closed subspace of $L_M^{p(\cdot)}(\Omega)$, by an abuse of notation identify $W^{k,p(\cdot)}(\Omega)$ with its image and regard $\Phi$ as a linear functional on this subspace. By the Hahn-Banach theorem (see [51]) there exists $\tilde{\Phi} \in L_M^{p(\cdot)}(\Omega)^*$ such that $\tilde{\Phi} = \Phi$ on $W^{k,p(\cdot)}(\Omega)$. But we have that

$$L_M^{p(\cdot)}(\Omega)^* = \prod_{i=1}^{M} L^{p(\cdot)}(\Omega)^* = L_M^{p'(\cdot)}(\Omega),$$

where equality is understood in terms of isomorphism: for the first equality, see [51]; the second is a consequence of Theorem 2.80 since $p_+ < \infty$. Therefore, there exists $(v_\alpha)_{|\alpha|\le k} \in L_M^{p'(\cdot)}(\Omega)$ such that $\tilde{\Phi} = \Phi_{(v_\alpha)}$, and so by restriction we get (6.2). $\square$

*Remark 6.9.* In Sect. 2.10.3 we defined Banach function spaces. Though spaces of functions, the variable Sobolev spaces $W^{k,p(\cdot)}(\Omega)$ are not Banach function spaces since they do not satisfy all of the axioms. In particular, given $E \subset \Omega$, $|E| < \infty$, then $\chi_E \notin W^{k,p(\cdot)}(\Omega)$ since it is not an element of $W^{1,1}_{\text{loc}}(\Omega)$. (Cf. [363].)

## 6.2 Density of Smooth Functions

Since $W^{k,p(\cdot)}(\Omega)$ is a Banach space, it is natural to consider the question of dense subsets. If $p_+ < \infty$, it is separable and so there exists a countable dense subset. However, we would like to identify particular families of functions that are dense. Because weak derivatives coincide with classical derivatives for smooth functions, it is natural to consider the question of when such functions are dense. We begin by defining two subspaces of $W^{k,p(\cdot)}(\Omega)$.

**Definition 6.10.** Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, for $k \ge 1$ let $W_0^{k,p(\cdot)}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$, and let $H^{k,p(\cdot)}(\Omega)$ be the closure of $C^k(\Omega) \cap W^{k,p(\cdot)}(\Omega)$.

Arguing as we did above, we have that if $|\Omega| < \infty$, then $W_0^{k,p(\cdot)}(\Omega) \subset W_0^{k,p_-}(\Omega)$ and $H^{k,p(\cdot)}(\Omega) \subset H^{k,p_-}(\Omega)$. The next proposition is an immediate consequence of Theorems 6.6 and 6.7, and the fact that closed subspaces of a Banach space are themselves Banach spaces.

**Proposition 6.11.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, for $k \geq 1$, $W_0^{k,p(\cdot)}(\Omega)$ and $H^{k,p(\cdot)}(\Omega)$ are Banach spaces. If $p_+ < \infty$, then they are separable, and if $1 < p_- \leq p_+ < \infty$, they are reflexive.*

In the classical Sobolev spaces, by the Meyers-Serrin theorem we have that for $1 \leq p < \infty$, $H^{k,p}(\Omega) = W^{k,p}(\Omega)$. (See [142].) This is not the case in variable Sobolev spaces: as the next example shows, for this equality to hold we must assume some kind of regularity on the exponent $p(\cdot)$.

*Example 6.12.* In $\mathbb{R}^2$, let $B = B_1(0)$. Then there exists $p(\cdot) \in \mathcal{P}(B)$ such that $C^1(B) \cap W^{1,p(\cdot)}(B)$ is not dense in $W^{1,p(\cdot)}(B)$.

*Proof.* The coordinate axes divide $B$ into four regions: denote them by $Q_1, \dots, Q_4$, beginning with the upper-right quadrant and proceeding counter-clockwise. Fix $p_1, p_2$, $1 < p_1 < 2 < p_2 < \infty$, and define $p(\cdot) \in \mathcal{P}(B)$ by

$$p(x) = \begin{cases} p_2 & x \in Q_1 \cup Q_3 \\ p_1 & x \in Q_2 \cup Q_4. \end{cases}$$

For $x = (x^1, x^2) \in B$, define the function $f$ by

$$f(x) = \begin{cases} 1 & x \in Q_1 \\ x^2/|x| & x \in Q_2 \\ 0 & x \in Q_3 \\ x^1/|x| & x \in Q_4. \end{cases}$$

It is immediate that $f \in L^\infty(B) \subset L^1(B)$ and is continuous on $B \setminus \{0\}$. We will show that $f \in W^{1,p(\cdot)}(B)$ but $f$ cannot be approximated by functions in $C^1(B)$. For $x \in Q_2$,

$$\partial_1 f(x) = \frac{-x^1 x^2}{|x|^3}, \qquad \partial_2 f(x) = \frac{(x^1)^2}{|x|^3};$$

hence, by Proposition 6.2, the weak and classical derivatives of $f$ coincide. Moreover, since $p_1 < 2$, integrating in polar coordinates we get

$$\int_{Q_2} |\partial_1 f(x)|^{p_1}\,dx = \int_0^{\pi/2} \int_0^1 \frac{r^{2p_1}\big(\cos(\theta)\sin(\theta)\big)^{p_1}}{r^{3p_1}} r\,dr\,d\theta < \infty.$$

A similar computation holds for $\partial_2 f$ in $Q_2$, and again for both partial derivatives in $Q_4$. Therefore, by Proposition 2.12, $f \in W^{1,p(\cdot)}(B)$.

To show that $f$ cannot be approximated in norm by $C^1$ functions, fix $g \in C^1(B)$; then by the definition of $f$ and $p(\cdot)$,

$$\|f - g\|_{W^{1,p(\cdot)}(B)} \geq \max\big(\|g\|_{W^{1,p_2}(Q_3)}, \|1 - g\|_{W^{1,p_2}(Q_1)}\big).$$

Suppose $g(0) \geq 1/2$; we will show that there exists a constant $c(p_2) > 0$ such that $\|g\|_{W^{1,p_2}(Q_3)} \geq c(p_2)$. If $g(0) < 1/2$, then essentially the same argument shows that $\|1-g\|_{W^{1,p_2}}(Q_1) \geq c > 0$.

Let $\partial_r$ denote the radial derivative; then

$$|\partial_1 g(x)| + |\partial_2 g(x)| \geq |\nabla g(x)| \geq |\partial_r g(x)|.$$

Therefore,

$$\begin{aligned}\|g\|^{p_2}_{W^{1,p_2}(Q_3)} &\geq \int_{Q_3} |g(x)|^{p_2}\,dx + \int_{Q_3} |\partial_1 g(x)|^{p_2}\,dx + \int_{Q_3} |\partial_2 g(x)|^{p_2}\,dx \\ &\geq \int_{Q_3} |g(x)|^{p_2}\,dx + 2^{-p_2}\int_{Q_3} |\partial_r g(x)|^{p_2}\,dx.\end{aligned} \tag{6.3}$$

We estimate the right-hand side by converting to polar coordinates. Let $\Theta$ be the part of the boundary of $Q_3$ that lies on the unit circle. Define

$$S = \{\theta \in \Theta : g(r,\theta) > 1/4, 1/4 \leq r \leq 1/2\},$$

and define $T = \Theta \setminus S$. Let $|S|_1$, $|T|_1$ be their one-dimensional Hausdorff measure; then $|S|_1 + |T|_1 = \pi/2$. Suppose $|S|_1 \geq \pi/4$. Then

$$\int_{Q_3} |g(x)|^{p_2}\,dx \geq \int_S \int_{1/4}^{1/2} |g(r,\theta)|^{p_2} r\,dr\,d\theta \geq \frac{3|S|_1}{32\cdot 4^{p_2}} \geq \frac{3\pi}{128\cdot 4^{p_2}}.$$

On the other hand, suppose $|T|_1 \geq \pi/4$. If $\theta \in T$, then there exists $r_\theta$, $1/4 \leq r_\theta \leq 1/2$, such that $g(r_\theta,\theta) \leq 1/4$. Therefore, by the fundamental theorem of calculus and Hölder's inequality we have that

$$\begin{aligned}1/4 \leq |g(r_\theta,\theta) - g(0,0)| &\leq \int_0^{r_\theta} |\partial_r g(r,\theta)|\,dr \leq \int_0^{1/2} |\partial_r g(r,\theta)| r^{1/p_2} r^{-1/p_2}\,dr \\ &\leq \left(\int_0^{1/2} |\partial_r g(r,\theta)|^{p_2} r\,dr\right)^{1/p_2} \left(\int_0^{1/2} r^{-p_2'/p_2}\,dr\right)^{1/p_2'}.\end{aligned}$$

Since $p_2 > 2$, $p_2'/p_2 = p_2' - 1 < 1$, and so the second integral on the right-hand side is finite. If we evaluate it and rearrange terms, we get that

$$\begin{aligned}\int_{Q_3} |\partial_r g(x)|^{p_2}\,dx &\geq \int_T \int_0^{1/2} |\partial_r g(r,\theta)|^{p_2} r\,dr\,d\theta \\ &\geq \frac{(2-p_2')^{p_2/p_2'}|T|_1}{2^{p_2}} \geq \frac{(2-p_2')^{p_2/p_2'}\pi}{4\cdot 2^{p_2}}.\end{aligned}$$

In either case, therefore, the right-hand side of (6.3) is bounded away from zero. Since this bound holds for every $g \in C^1(B)$, $f$ cannot be approximated in norm by functions in $C^1(B)$. □

Example 6.12 shows that we need some kind of regularity on the exponent $p(\cdot)$ for $H^{k,p(\cdot)}(\Omega)$ to be equal to $W^{k,p(\cdot)}(\Omega)$. For example, as we will show below, we could assume that $p(\cdot) \in LH(\Omega)$. But in fact, it suffices to assume that $p(\cdot)$ is locally regular: for example, that $p(\cdot) \in LH_0(\Omega)$, even if $\Omega$ is unbounded. We will state our hypotheses more generally in terms of the boundedness of the maximal operator.

**Definition 6.13.** Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, the maximal operator is locally bounded in $L^{p(\cdot)}(\Omega)$ if for every open set $A$ such that $\overline{A}$ is compact and contained in $\Omega$, the maximal operator is bounded in $L^{p(\cdot)}(A)$.

Our first result extends the Meyers-Serrin theorem to the variable Sobolev spaces.

**Theorem 6.14.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$ and the maximal operator is locally bounded on $L^{p'(\cdot)}(\Omega)$. Then $H^{k,p(\cdot)}(\Omega) = W^{k,p(\cdot)}(\Omega)$.*

The proof requires one lemma about the regularity of convolutions whose proof is a straightforward application of calculus; see [142] for details.

**Lemma 6.15.** *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$, then $\phi * f \in C^\infty(\mathbb{R}^n)$, and for any multi-index $\alpha$, $D^\alpha(\phi * f) = (D^\alpha\phi) * f$.*

*Proof of Theorem 6.14.* Since $H^{k,p(\cdot)}(\Omega) \subset W^{k,p(\cdot)}(\Omega)$ we only have to prove the reverse inclusion. To do so, it will suffice to show that given any $f \in W^{k,p(\cdot)}(\Omega)$ and $\epsilon > 0$, there exists $g \in C^k(\Omega)$ such that $\|f - g\|_{k,p(\cdot)} < \epsilon$. (By the triangle inequality we get $g \in W^{k,p(\cdot)}(\Omega)$.)

For each $j \geq 1$ define the sets

$$\Omega_j = \{x \in \Omega : |x| < j, \mathrm{dist}(x, \partial\Omega) > 1/j\}.$$

Let $\Omega_0 = \Omega_{-1} = \emptyset$ and let

$$A_j = \Omega_{j+1} \setminus \overline{\Omega}_{j-1}.$$

The sets $A_j$ are open, $\overline{A}_j$ is compact, and given any $x \in \Omega$, $x$ is contained in a finite number of the sets $A_j$. Therefore, there exists a partition of unity subordinate to this cover. (See [7].) More precisely, there exists a collection of functions $\{\psi_j\}$ such that $\psi_j \in C_c^\infty(A_j)$ and for all $x \in \Omega$, $0 \leq \psi_j(x) \leq 1$, and

$$\sum_{j=1}^{\infty} \psi_j(x) = 1.$$

Further, by Lemma 6.5 for each $j$ we have that $\psi_j f \in W^{k,p(\cdot)}(\Omega)$ and $\operatorname{supp}(\psi_j f) \subset A_j$.

Let $\phi \in C_c^\infty(B_1(0))$ be non-negative and such that $\int \phi(x)\,dx = 1$. Since

$$\phi_t * (\psi_j f)(x) = \int_{A_j} \phi_t(x-y)\psi_j(y) f(y)\,dy,$$

this convolution is non-zero only if $|x-y| < t$ and $y \in A_j$. Therefore, if we let $0 < t < (j+1)^{-1} - (j+2)^{-1}$, then this is the case only if $(j+2)^{-1} < \operatorname{dist}(x, \partial\Omega) \leq (j-2)^{-1}$, the second inequality holding if $j \geq 3$. Hence, $\operatorname{supp}(\phi_t * (\psi_j f)) \subset \Omega_{j+2} \setminus \overline{\Omega}_{j-2} = B_j$. Note that $\overline{B}_j$ is compact and contained in $\Omega$.

Since the maximal operator is locally bounded on $L^{p'(\cdot)}(\Omega)$, it is bounded on $L^{p'(\cdot)}(B_j)$, and so by Theorem 5.11 there exists $t_j < (j+1)^{-1} - (j+2)^{-1}$ such that for all $\alpha$, $0 \leq |\alpha| \leq k$,

$$\begin{aligned} &\|\phi_{t_j} * D^\alpha(\psi_j f) - D^\alpha(\psi_j f)\|_{L^{p(\cdot)}(\Omega)} \\ &\qquad = \|\phi_{t_j} * D^\alpha(\psi_j f) - D^\alpha(\psi_j f)\|_{L^{p(\cdot)}(B_j)} < \frac{\epsilon}{2^j M}, \end{aligned}$$

where $M$ is the total number of multi-indices with $|\alpha| \leq k$. By Lemma 6.15, $\phi_{t_j} * D^\alpha(\psi_j f) = D^\alpha(\phi_{t_j} * (\psi_j f))$. Therefore,

$$\|\phi_{t_j} * (\psi_j f) - \psi_j f\|_{k,p(\cdot)} = \sum_{|\alpha| \leq k} \|\phi_{t_j} * D^\alpha(\psi_j f) - D^\alpha(\psi_j f)\|_{p(\cdot)} \leq 2^{-j}\epsilon.$$

Define the function $g$ by

$$g(x) = \sum_{j=1}^\infty \phi_{t_j} * (\psi_j f)(x).$$

Again by Lemma 6.15, each summand is in $C^\infty(\Omega)$. Further, if $x \in \Omega$, it is contained in a finite number of sets $B_j$, so only a finite number of terms of this series are non-zero at $x$. Therefore, it converges locally uniformly and $g \in C^\infty(\Omega) \subset C^k(\Omega)$. Finally, we have that

$$\begin{aligned} \|g - f\|_{k,p(\cdot)} &= \left\| \sum_{j=1}^\infty \left(\phi_{t_j} * (\psi_j f) - \psi_j f\right) \right\|_{k,p(\cdot)} \\ &\leq \sum_{j=1}^\infty \|\phi_{t_j} * (\psi_j f) - \psi_j f\|_{k,p(\cdot)} < \sum_{j=1}^\infty 2^{-j}\epsilon = \epsilon. \end{aligned}$$

This completes the proof. □

*Remark 6.16.* As a corollary to the proof of Theorem 6.14 we get that the smaller set $C^\infty(\Omega) \cap W^{k,p(\cdot)}(\Omega)$ is dense in $W^{k,p(\cdot)}(\Omega)$.

Our second density theorem shows that if $\Omega = \mathbb{R}^n$, then smooth functions of compact support are dense.

**Theorem 6.17.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that $p_+ < \infty$ and the maximal operator is locally bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$. Then $W_0^{k,p(\cdot)}(\mathbb{R}^n) = W^{k,p(\cdot)}(\mathbb{R}^n)$.*

*Proof.* Since $W_0^{k,p(\cdot)}(\mathbb{R}^n) \subset W^{k,p(\cdot)}(\mathbb{R}^n)$, we only need to prove the reverse inequality. Fix $f \in W^{k,p(\cdot)}(\mathbb{R}^n)$ and $\epsilon > 0$; we will show that there exists $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\|f - \phi\|_{k,p(\cdot)} < \epsilon$. By Theorem 6.14 (see Remark 6.16) there exists $g \in C^\infty(\mathbb{R}^n) \cap W^{k,p(\cdot)}(\mathbb{R}^n)$ such that $\|f - g\|_{k,p(\cdot)} < \epsilon/2$. Therefore, it remains to show that we can find $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\|g - \phi\|_{k,p(\cdot)} < \epsilon/2$.

For each $N > 1$ let $\nu_N \in C_c^\infty(\mathbb{R}^n)$ be such that:

1. $\operatorname{supp}(\nu_N) \subset B_N(0)$;
2. For all $x$, $0 \le \nu_N(x) \le 1$;
3. If $x \in B_{N/2}(0)$, $\nu_N(x) = 1$;
4. There exists $K > 0$ such that for all $|\alpha| \le k$, $\|D^\alpha \nu_N\|_\infty \le K$.

Let $\phi_N = g\nu_N$. Then $\phi_N \in C_c^\infty(\mathbb{R}^n)$. By the product rule, for any multi-index $\alpha$, $|\alpha| \le k$,

$$D^\alpha \phi_N = \nu_N D^\alpha g + \sum_{\substack{\beta \le \alpha \\ \beta \ne 0}} \binom{\alpha}{\beta} D^\beta \nu_N D^{\alpha-\beta} g.$$

Therefore, for all $x$,

$$|D^\alpha \phi_N(x)| \le C \sum_{|\alpha| \le k} |D^\alpha g(x)|,$$

and the sum on the right-hand side is in $L^{p(\cdot)}(\mathbb{R}^n)$. For all $x$ and any $\alpha$ with $|\alpha| > 0$, $D^\alpha \nu_N(x) \to 0$ as $N \to \infty$. Hence, $D^\alpha \phi_N \to \nu_N D^\alpha g$ pointwise as $N \to \infty$. Therefore, by the dominated convergence theorem for the variable Lebesgue spaces (Theorem 2.62), for all $\alpha$, $|\alpha| \le k$, $D^\alpha \phi_N \to D^\alpha g$ in $L^{p(\cdot)}(\mathbb{R}^n)$. Therefore, we can find $N$ sufficiently large that if we let $\phi = \phi_N$, $\|g - \phi\|_{k,p(\cdot)} < \epsilon/2$. This completes the proof. □

The hypothesis that $p_+ < \infty$ in Theorem 6.17 is necessary, as the next example shows.

*Example 6.18.* Let $p(x) = |x| + 1$. Then $p(\cdot) \in \mathcal{P}(\mathbb{R})$, and by Proposition 2.43, the function $f(x) = 1$ is in $L^{p(\cdot)}(\mathbb{R})$; since $Df = 0$, $f \in W^{1,p(\cdot)}(\mathbb{R})$. On the other hand, by Theorem 2.77 bounded functions of compact support are not dense in $L^{p(\cdot)}(\mathbb{R})$, and so $C_c^\infty(\mathbb{R})$ cannot be dense in $W^{1,p(\cdot)}(\mathbb{R})$.

## 6.3 The Poincaré Inequalities

In this section we give sufficient regularity conditions on the exponent $p(\cdot)$ for the Poincaré inequalities to hold in the variable Sobolev spaces. As before, these conditions are in terms of the boundedness of the maximal operator. Recall that given a bounded set $\Omega$ and a function $f$, $f_\Omega = \fint_\Omega f(y)\,dy$.

**Theorem 6.19.** *Given a bounded, convex set $\Omega$ with diameter $D$, let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\Omega)$. Then for all $f \in W^{1,p(\cdot)}(\Omega)$,*

$$\|f - f_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq C(n, p(\cdot))\|M\|_{L^{p'(\cdot)}(\Omega)} \frac{D^{n+1}}{|\Omega|} \|\nabla f\|_{L^{p(\cdot)}(\Omega)}.$$

If the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, then it is bounded on $L^{p(\cdot)}(A)$ for any $A \subset \Omega$ with the same constant. Therefore, Theorem 6.19 has the following corollary.

**Corollary 6.20.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\Omega)$. Then for every ball $B \subset \Omega$ with radius $R$, and for all $f \in W^{1,p(\cdot)}(B)$,*

$$\|f - f_B\|_{L^{p(\cdot)}(B)} \leq C(n, p(\cdot))\|M\|_{L^{p'(\cdot)}(\Omega)} R \|\nabla f\|_{L^{p(\cdot)}(B)}.$$

The second Poincaré inequality is restricted to functions in $W_0^{1,p(\cdot)}(\Omega)$.

**Theorem 6.21.** *Given a bounded set $\Omega$ with diameter $D$, let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\Omega)$. Then for all $f \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq C(n, p(\cdot))\|M\|_{L^{p'(\cdot)}(\Omega)} D \|\nabla f\|_{L^{p(\cdot)}(\Omega)}.$$

The proofs of Theorems 6.19 and 6.21 require three lemmas. Recall that $I_1$ is the Riesz potential with index 1: see Definition 5.44.

**Lemma 6.22.** *Let $\Omega$ be a bounded set with diameter $D$, and let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\Omega)$. Then for all $f \in L^{p(\cdot)}(\Omega)$,*

$$\|I_1 f\|_{L^{p(\cdot)}(\Omega)} \leq C(n, p(\cdot))\|M\|_{L^{p'(\cdot)}(\Omega)} D \|f\|_{L^{p(\cdot)}(\Omega)}. \tag{6.4}$$

*Remark 6.23.* If $p(\cdot) = p$, $1 \leq p < \infty$, then inequality (6.4) is always true. The proof is much simpler and does not involve the maximal operator: it is an immediate consequence of Proposition 5.2 and the constant is $C(n)D$.

*Remark 6.24.* By Remark 5.14, we could change the hypotheses of Lemma 6.22 to be that $p_- > 1$ and the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$; in this case we do not need to assume $p_+ < \infty$. We can therefore make the corresponding change in the hypotheses of Theorems 6.19 and 6.21, and Corollary 6.20.

*Proof.* Fix $f \in L^{p(\cdot)}(\Omega)$ and extend $f$ to equal 0 on $\mathbb{R}^n \setminus \Omega$. Then for all $x \in \Omega$,

$$I_1 f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-1}}\,dy = \int_{B_D(x)} \frac{f(y)}{|x-y|^{n-1}}\,dy = \Phi * f(x),$$

where $\Phi(x) = |x|^{1-n}\chi_{B_D(0)}$ is a positive, radially decreasing function in $L^1(\mathbb{R}^n)$. If we integrate in polar coordinates we get that

$$\int_{B_D(0)} \Phi(x)\,dx = \int_{|\theta|=1}\int_0^D r^{1-n}r^{n-1}\,dr\,d\theta = C(n)D.$$

Therefore, by Theorem 5.11,

$$\|I_1 f\|_{L^{p(\cdot)}(\Omega)} \le C(n, p(\cdot))\|M\|_{L^{p'(\cdot)}(\Omega)} D\|f\|_{L^{p(\cdot)}(\Omega)}.$$

□

**Lemma 6.25.** *Given a bounded convex set $\Omega$ with diameter $D$, for every $f \in W^{1,1}(\Omega)$,*

$$|f(x) - f_\Omega| \le \frac{D^n}{n|\Omega|} I_1(|\nabla f|)(x).$$

*Proof.* We first prove this for $f \in C^1(\Omega)$. Fix $f \in C^1(\Omega)$ and for $x, y \in \Omega$ let

$$\theta = \frac{y-x}{|y-x|}.$$

Since $\Omega$ is convex, by the fundamental theorem of calculus

$$f(x) - f(y) = -\int_0^{|y-x|} \nabla f(x+\theta r)\cdot\theta\,dr.$$

If we integrate in $y$ over $\Omega$, we get

$$f(x) - f_\Omega = \frac{-1}{|\Omega|}\int_\Omega\int_0^{|y-x|} \nabla f(x+\theta r)\cdot\theta\,dr\,dy.$$

Let $B = B_D(x)$; then $\Omega \subset B$. Hence, by changing between Cartesian and polar coordinates, we have that

$$\begin{aligned}
|f(x) - f_\Omega| &\le \frac{1}{|\Omega|}\int_B \int_0^{|y-x|} |\nabla f(x+\theta r)|\chi_\Omega(x+\theta r)\,dr\,dy \\
&\le \frac{1}{|\Omega|}\int_{|\theta|=1}\int_0^D \int_0^\rho |\nabla f(x+\theta r)|\chi_\Omega(x+\theta r)\,dr\,\rho^{n-1}\,d\rho\,d\theta \\
&\le \frac{1}{|\Omega|}\int_{|\theta|=1}\int_0^\infty \int_0^D |\nabla f(x+\theta r)|\chi_\Omega(x+\theta r)\,\rho^{n-1}\,d\rho\,dr\,d\theta \\
&= \frac{D^n}{n|\Omega|}\int_{|\theta|=1}\int_0^\infty |\nabla f(x+\theta r)|\chi_\Omega(x+\theta r) r^{1-n} r^{n-1}\,d\theta\,dr \\
&= \frac{D^n}{n|\Omega|}\int_\Omega \frac{|\nabla f(y)|}{|x-y|^{n-1}}\,dy \\
&= \frac{D^n}{n|\Omega|} I_1(|\nabla f|)(x).
\end{aligned}$$

To prove this inequality in general, fix $f \in W^{1,1}(\Omega)$. By Theorem 6.14 there exists a sequence $\{g_k\} \subset C^1(\Omega) \cap W^{1,1}(\Omega)$ that converges to $f$ in $W^{1,1}(\Omega)$ norm and so also in $L^1(\Omega)$. Therefore, if we pass to a subsequence we may assume that $g_k \to f$ pointwise almost everywhere. By Remark 6.23 we have that (6.4) holds when $p = 1$ which in turn implies that $I_1(|\nabla g_k|) \to I_1(|\nabla f|)$ in $L^1(\Omega)$ norm; by passing to another subsequence we may assume that it converges pointwise almost everywhere. Hence,

$$|f(x) - f_\Omega| = \lim_{k\to\infty} |g_k(x) - (g_k)_\Omega| \le \lim_{k\to\infty} \frac{D^n}{n|\Omega|} I_1(|\nabla g_k|)(x) = \frac{D^n}{n|\Omega|} I_1(|\nabla f|)(x).$$

□

**Lemma 6.26.** *Given a bounded set $\Omega$, then for every $f \in W_0^{1,1}(\Omega)$,*

$$|f(x)| \le C(n) I_1(|\nabla f|)(x).$$

*Proof.* We first prove this for $f \in C_c^\infty(\Omega)$. Since $f$ has compact support, given any $\theta \in \mathbb{R}^n$, $|\theta| = 1$, by the fundamental theorem of calculus,

$$f(x) = \int_0^\infty \nabla f(x - \theta t)\cdot\theta\,dt.$$

If we integrate over the unit sphere and change from polar to Cartesian coordinates, we get

$$\begin{aligned}
f(x) &= C(n)\int_{|\theta|=1}\int_0^\infty \nabla f(x-\theta t)\cdot\theta\,dt\,d\theta \\
&= C(n)\int_{\mathbb{R}^n} \nabla f(x-y)\frac{y}{|y|^n}\,dy = C(n)\int_{\mathbb{R}^n} \nabla f(y)\frac{x-y}{|x-y|^n}\,dy.
\end{aligned}$$

If we now take absolute values, this becomes

$$|f(x)| \leq C(n) \int_\Omega \frac{|\nabla f(x)|}{|x-y|^{n-1}}\, dy = C(n) I_1(|\nabla f|)(x).$$

To prove this inequality for $f \in W_0^{1,1}(\Omega)$, we argue exactly as we did at the end of the proof of Lemma 6.25, using the fact that $C_c^\infty(\Omega)$ is dense in $W_0^{1,1}(\Omega)$. □

*Proof of Theorems 6.19 and 6.21.* Fix $f \in W^{1,p(\cdot)}(\Omega)$; then by Proposition 6.4, we have that $f \in W^{1,p_-}(\Omega) \subset W^{1,1}(\Omega)$. Therefore, by Lemmas 6.22 and 6.25,

$$\begin{aligned}\|f - f_\Omega\|_{L^{p(\cdot)}(\Omega)} &\leq \frac{D^n}{n|\Omega|} \|I_1(|\nabla f|)\|_{L^{p(\cdot)}(\Omega)} \\ &\leq C(n, p(\cdot)) \|M\|_{L^{p'(\cdot)}(\Omega)} \frac{D^{n+1}}{|\Omega|} \|\nabla f\|_{L^{p(\cdot)}(\Omega)}.\end{aligned}$$

The proof of Theorem 6.21 is identical except that we use Lemma 6.26 in place of Lemma 6.25. □

## 6.4 Sobolev Embedding Theorems

A key result in the classical theory is the Sobolev embedding theorem: if $1 \leq p < n$, then $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, where $p^* = np/(n-p)$. This result can be extended to variable Sobolev spaces. However, as the next example shows, just as we had to do for the density theorems in Sect. 6.2, we must assume some regularity on the exponent $p(\cdot)$. To state our results we first make a definition.

**Definition 6.27.** Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_+ \leq n$, define the Sobolev exponent $p^*(\cdot) \in \mathcal{P}(\Omega)$ by

$$p^*(x) = \frac{np(x)}{n - p(x)}.$$

*Example 6.28.* In $\mathbb{R}^2$, let $B = B_1(0)$. Then there exists an exponent $p(\cdot) \in \mathcal{P}(B)$ and $f \in W^{1,p(\cdot)}(B)$ such that $f \notin L^{p^*(\cdot)}(B)$.

*Proof.* Fix $p_1$ and $p_2$, $1 < p_1 < p_2 < 2$, and let $\sigma = 2(p_2 - p_1)/p_1$. Define the region $A \subset B$ in terms of polar coordinates,

$$A = \{(r, \theta) \in B : |\theta| < r^\sigma\},$$

and define $p(\cdot) \in \mathcal{P}(B)$ by

$$p(x) = \begin{cases} p_1 & x \in B \setminus A \\ p_2 & x \in A. \end{cases}$$

Let $f(x) = |x|^{-\mu}$, where $\mu = (2-p_2)/p_1$. Note that by Proposition 6.2 its weak and classical partial derivatives are the same. We will show that $f \in W^{1,p(\cdot)}(B)$ but $f \notin L^{p^*(\cdot)}(B)$. Integrating in polar coordinates we get

$$\begin{aligned}\int_B |f(x)|^{p(x)}\,dx &= \int_A |f(x)|^{p_2}\,dx + \int_{B\setminus A} |f(x)|^{p_1}\,dx \\ &\leq \int_0^1 \int_{-r^\sigma}^{r^\sigma} r^{1-\mu p_2}\,d\theta\,dr + 2\pi \int_0^1 r^{1-\mu p_1}\,dr \\ &\leq 2\int_0^1 r^{1-\mu p_2+\sigma}\,dr + 2\pi \int_0^1 r^{1-\mu p_1}\,dr.\end{aligned}$$

Since

$$1 - \mu p_2 + \sigma = \frac{p_2^2 - p_1}{p_1} > 0 \quad \text{and} \quad 1 - \mu p_1 = p_2 - 1 > 0,$$

both integrals converge. Further, a straightforward computation shows that $|\nabla f(x)| = \mu|x|^{-\mu-1}$, and so

$$\begin{aligned}\int_B |\nabla f(x)|^{p(x)}\,dx &\leq 2|\mu|^{p_2} \int_0^1 \int_0^{r^\sigma} r^{1-p_2(\mu+1)}\,d\theta\,dr + 2\pi|\mu|^{p_1} \int_0^1 r^{1-p_1(\mu+1)}\,dr \\ &= 2|\mu|^{p_2} \int_0^1 r^{1-p_2(\mu+1)+\sigma}\,dr + 2\pi|\mu|^{p_1} \int_0^1 r^{1-p_1(\mu+1)}\,dr.\end{aligned}$$

Since

$$1 - p_2(\mu+1) + \sigma = \frac{p_2^2 - p_1}{p_1} - p_2 = \frac{p_2^2 - p_1 p_2 - p_1}{p_1} > -1$$

and

$$1 - p_1(\mu+1) = p_2 - p_1 - 1 > -1,$$

these integrals also converge. Thus, by Proposition 2.12, $f \in W^{1,p(\cdot)}(B)$.

On the other hand,

$$\int_B |f(x)|^{p^*(x)}\,dx > \int_A |f(x)|^{p_2^*}\,dx = 2\int_0^1 r^{1-\frac{2p_2}{2-p_2}\mu+\sigma}\,dr = 2\int_0^1 r^{-1}\,dr = \infty,$$

and so again by Proposition 2.12, $f \notin L^{p^*(\cdot)}(B)$. □

By assuming that the exponent $p(\cdot)$ is regular, we can extend the Sobolev embedding theorem to variable Sobolev spaces by using extrapolation theory.

**Theorem 6.29.** *Given* $\Omega$ *and* $p(\cdot) \in \mathcal{P}(\Omega)$ *such that* $p_+ < n$*, suppose that the maximal operator is bounded on* $L^{(p^*(\cdot)/n')'}(\Omega)$*. Then* $W_0^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$*, and*

$$\|f\|_{p^*(\cdot)} \leq C\|\nabla f\|_{p(\cdot)}.$$

*Remark 6.30.* The hypotheses of Theorem 6.29 allow $p_- = 1$, including the case $p(\cdot) = 1$. If $p(\cdot) = 1$, then $(p^*(\cdot)/n')' = \infty$ and by Proposition 3.3 the maximal operator is bounded on $L^\infty(\Omega)$. More generally, by Definition 6.27 we always have that $p^*(\cdot)_- \geq n'$, and so $(p^*(\cdot)/n')'$ is well-defined. Further, $((p^*(\cdot)/n')')_- > 1$, and so if we assume, for example, that $p(\cdot) \in LH(\Omega)$, then $1/(p^*(\cdot)/n')' \in LH(\Omega)$ and the maximal operator is again bounded. (We take the inverse since $(p^*(\cdot)/n')'$ may be unbounded.)

*Remark 6.31.* In the classical Sobolev spaces, the embedding theorem holds for functions $f \in W^{1,p}(\Omega)$ with additional assumptions on the boundary of $\Omega$. The same is true in variable Lebesgue spaces; see Sect. 6.5.7 below.

To apply extrapolation we need the corresponding weighted norm inequality.

**Lemma 6.32.** *Given* $\Omega$*, then for all* $p$*,* $1 \leq p < n$*,* $w \in A_1$*, and* $f \in C_c^\infty(\Omega)$*,*

$$\left(\int_\Omega |f(x)|^{p^*} w(x)\,dx\right)^{1/p^*} \leq C\left(\int_\Omega |\nabla f(x)|^p w(x)^{p/p_*}\,dx\right)^{1/p},$$

*where* $p^* = np/(n-p)$.

*Proof.* Fix $f \in C_c^\infty(\Omega)$. For each $j \in \mathbb{Z}$ let

$$\Omega_j = \{x \in \Omega : 2^j < |f(x)| \leq 2^{j+1}\},$$

and define the function $f_j$ by

$$f_j(x) = \begin{cases} |f(x)| - 2^j & x \in \Omega_j, \\ 2^j & x \in \Omega_i,\ i > j, \\ 0 & \text{otherwise.} \end{cases}$$

The function $f_j$ is weakly differentiable and $|\nabla f_j(x)| = |\nabla f(x)|\chi_{\Omega_j}$ almost everywhere. (See [142].) Further, if $x \in \Omega_j$, by Lemma 6.26,

$$I_1(|\nabla f_{j-1}|)(x) \geq c(n)|f_{j-1}(x)| = c(n)2^{j-1}. \tag{6.5}$$

Since a strong type inequality implies the corresponding weak type inequality, by Theorem 5.49, since $w \in A_1 \subset A_{p,p^*}$, we have that

$$w(\{x \in \Omega : |I_1 h(x)| > t\}) \leq C\left(\frac{1}{t^p}\int_\Omega |h(x)|w(x)^{p/p^*}\,dx\right)^{p^*/p}. \tag{6.6}$$

Therefore, by (6.5) and (6.6) with $h = |\nabla f_{j-1}|$, we have that

$$\begin{aligned}
\int_\Omega |f(x)|^{p^*} w(x)\,dx &= \sum_j \int_{\Omega_j} |f(x)|^{p^*} w(x)\,dx \\
&\leq \sum_j \int_{\Omega_j} 2^{(j+1)p^*} w(x)\,dx \\
&= 4^{p^*} c(n)^{-p^*} \sum_j \int_{\Omega_j} \left(c(n)2^{j-1}\right)^{p^*} w(x)\,dx \\
&\leq C \sum_j \int_{\{x\in\Omega : I_1(|\nabla f_{j-1}|)(x) > c(n)2^{j-1}\}} \left(c(n)2^{j-1}\right)^{p^*} w(x)\,dx \\
&\leq C \sum_j \left(\int_\Omega |\nabla f_{j-1}(x)|^p w(x)^{p/p_*}\,dx\right)^{p^*/p} \\
&\leq C \left(\sum_j \int_{\Omega_{j-1}} |\nabla f(x)|^p w(x)^{p/p_*}\,dx\right)^{p^*/p} \\
&\leq C \left(\int_\Omega |\nabla f(x)|^p w(x)^{p/p_*}\,dx\right)^{p^*/p}.
\end{aligned}$$

This completes the proof. □

*Proof of Theorem 6.29.* Define the family $\mathcal{F}$ to be all the pairs $(|f|, |\nabla f|)$ with $f \in C_c^\infty(\Omega)$. Fix $q(\cdot) = p^*(\cdot)$, and $q_0 = n'$. By assumption, the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}(\Omega)$. Therefore, by Theorem 5.28 and Lemma 6.32 (with $p = 1$), for all $f \in C_c^\infty(\Omega)$

$$\|f\|_{L^{p^*(\cdot)}(\Omega)} \leq C\|\nabla f\|_{L^{p(\cdot)}(\Omega)},$$

provided the left-hand side is finite. But this is always the case.

Now fix $f \in W_0^{1,p(\cdot)}(\Omega)$. Then there exists a sequence $\{g_k\} \subset C_c^\infty(\Omega)$ such that $g_k \to f$ in $W_0^{1,p(\cdot)}$ norm, and so $g_k \to f$ and $\nabla g_k \to \nabla f$ in $L^{p(\cdot)}$ norm. By Proposition 2.67, if we pass to a subsequence, we may assume that $g_k \to f$ pointwise almost everywhere. Hence, by Fatou's lemma in the variable Lebesgue spaces (Theorem 2.61),

$$\|f\|_{p^*(\cdot)} \leq \liminf_{k\to\infty} \|g_k\|_{p^*(\cdot)} \leq C \liminf_{k\to\infty} \|\nabla g_k\|_{p(\cdot)} \leq C\|\nabla f\|_{p(\cdot)}.$$

□

As a corollary to Theorem 6.29 we can prove an embedding theorem for $W_0^{k,p(\cdot)}(\Omega)$. To avoid cumbersome hypotheses, we will only consider the case when $p(\cdot)$ is log-Hölder continuous.

**Corollary 6.33.** *Given $\Omega$, $p(\cdot) \in \mathcal{P}(\Omega)$ and $k \geq 1$, suppose $p_+ < n/k$ and $p(\cdot) \in LH(\Omega)$. Define $p_k^*(\cdot) \in \mathcal{P}(\Omega)$ by*

$$p_k^*(x) = \frac{np(x)}{n - kp(x)}.$$

*Then for all $f \in W_0^{k,p(\cdot)}(\Omega)$,*

$$\|f\|_{L^{p_k^*(\cdot)}(\Omega)} \leq \sum_{|\alpha|=k} \|D^\alpha f\|_{L^{p(\cdot)}(\Omega)}.$$

*Proof.* We proceed by induction. If $k = 1$, then this follows from Theorem 6.29 and Remark 6.30: since $p(\cdot) \in LH(\Omega)$, the maximal operator is bounded on $L^{(p^*(\cdot)/n')'}(\Omega)$, and so

$$\|f\|_{L^{p^*(\cdot)}(\Omega)} \leq C\|\nabla f\|_{L^{p(\cdot)}(\Omega)} \leq C \sum_{|\alpha|=1} \|D^\alpha f\|_{L^{p(\cdot)}(\Omega)}.$$

Now suppose that the result is true for some $k$. Fix $f \in W^{k+1,p(\cdot)}(\Omega)$. For each multi-index $\alpha$, $|\alpha| = 1$, $D^\alpha f \in W^{k,p(\cdot)}(\Omega)$, and so

$$\sum_{|\alpha|=1} \|D^\alpha f\|_{L^{p_k^*(\cdot)}(\Omega)} \leq C \sum_{|\alpha|=1} \sum_{|\beta|=k} \|D^\beta D^\alpha f\|_{L^{p(\cdot)}(\Omega)} \leq C \sum_{|\alpha|=k+1} \|D^\alpha f\|_{L^{p(\cdot)}(\Omega)}. \quad (6.7)$$

The Sobolev exponent corresponding to $p_k^*(\cdot)$ is

$$\frac{np_k^*(x)}{n - p_k^*(x)} = \frac{np(x)}{n - (k+1)p(x)} = p_{k+1}^*(x).$$

Since $p(\cdot) \in LH(\Omega)$, $p_k^*(\cdot) \in LH(\Omega)$, so we can again apply Theorem 6.29 as we did above to get

$$\|f\|_{L^{p_{k+1}^*}(\Omega)} \leq \sum_{|\alpha|=1} \|D^\alpha f\|_{L^{p_k^*(\cdot)}(\Omega)}. \tag{6.8}$$

If we combine (6.7) and (6.8), we get the desired inequality. □

In the classical case the Sobolev embedding theorem requires $p < n$: if $p = n$, then $p^* = \infty$, but there exist unbounded functions in $W^{1,n}$. A similar phenomenon holds in the variable Lebesgue spaces if $p_+ = n$, even if $p(x) < n$ for all $x \in \Omega$.

*Example 6.34.* Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, and for $x = (x^1, x^2)$ let $p(x) = 1+x^2$. Then $p_+ = 2$ and $p^*(x) = 2(1+x^2)/(1-x^2)$. Let $f(x) = (2+x^2)^{1/(1+x^2)}$. Then by Proposition 2.12, $f \in L^{p(\cdot)}(\Omega)$. Further,

$$\partial_2 f(x) = \left( \frac{1}{(2+x^2)(1+x^2)} - \frac{\log(2+x^2)}{(1+x^2)^2} \right) f(x),$$

and so $\nabla f \in L^{p(\cdot)}(\Omega)$. On the other hand,

$$\int_\Omega f(x)^{p^*(x)}\,dx = \int_0^1 (2+x^2)^{2/(1-x^2)}\,dx^2 \geq \int_0^1 4^{1/(1-x^2)}\,dx^2 = \infty,$$

and again by Proposition 2.12 $f \notin L^{p^*(\cdot)}(\Omega)$.

In the classical Sobolev spaces, if $p > n$, then functions in $W^{1,p}(\Omega)$ are Hölder continuous. The analogous result is true in variable Sobolev spaces. If $p_- > n$ and $\Omega$ is bounded, then we could use the fact that $W^{1,p(\cdot)}(\Omega) \subset W^{1,p_-}(\Omega)$ to draw the same conclusion. However, by taking into account the variable exponent we can get a sharper result.

**Definition 6.35.** Given a function $\alpha(\cdot) : \Omega \to (0,1)$, a function $f$ is said to be $\alpha(\cdot)$-Hölder continuous on $\Omega$, denoted by $f \in C^{\alpha(\cdot)}(\Omega)$, if for every $x \in \Omega$ there exists $r = r(x)$, $0 < r < \operatorname{dist}(x, \partial\Omega)$, such that

$$\sup_{y \in B_r(x)} \frac{|f(x) - f(y)|}{|x-y|^{\alpha(x)}} < \infty.$$

**Theorem 6.36.** *Given a set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose that $1/p(\cdot) \in LH_0(\Omega)$ and for all $x \in \Omega$, $p(x) > n$. Let $\alpha(x) = 1 - n/p(x)$. If $f \in W^{1,p(\cdot)}(\Omega)$, then $f$ is continuous. Moreover, if $r = r(x) = \min(1/2, \operatorname{dist}(x, \partial\Omega))$, then for all $y \in B_r(x)$,*

$$|f(x) - f(y)| \leq C(n, p_-(B_r(x)))\|\nabla f\|_{L^{p(\cdot)}(\Omega)}|x-y|^{\alpha(x)}. \tag{6.9}$$

*Moreover, if $p_- > n$, then $f \in C^{\alpha(\cdot)}(\Omega)$.*

The proof requires one lemma.

**Lemma 6.37.** *Given a ball $B$ of radius $r < 1/4$, suppose $p(\cdot) \in \mathcal{P}(B)$ is such that $p_-(B) > n$. If $f \in L^{p(\cdot)}(B)$ and $x \in B$, then*

$$|I_1 f(x)| \leq C(n, p_-(B)))r^{1-n/p_-(B)} \|f\|_{L^{p(\cdot)}(B)}.$$

*The constant increases as $p_-(B)$ decreases.*

*Proof.* Let $d = 2r < 1/2$, and fix $x \in B$. Extend $f$ to a function that is zero outside of $B$. For each $k \geq 0$ define

$$A_k = \{y \in B : 2^{-(k+1)}d \leq |x - y| < 2^{-k}d\}.$$

Then $B \subset \bigcup_k A_k$, and so by the generalized Hölder's inequality (Theorem 2.26) and Corollary 2.23 (since $p'(\cdot)_+(B) = p_-(B)' < n'$),

$$\begin{aligned}
|I_1 f(x)| &\leq \int_B \frac{|f(y)|}{|x-y|^{n-1}}\,dy \\
&= \sum_{k=0}^{\infty} \int_{A_k} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \\
&\leq \sum_{k=0}^{\infty} (2^{-(k+1)}d)^{1-n} \int_{A_k} |f(y)|\,dy \\
&\leq \sum_{k=0}^{\infty} (2^{-(k+1)}d)^{1-n} \|f\|_{L^{p(\cdot)}(B)} \|\chi_{A_k}\|_{L^{p'(\cdot)}(B)} \\
&\leq \sum_{k=0}^{\infty} (2^{-(k+1)}d)^{1-n} \|f\|_{L^{p(\cdot)}(B)} |A_k|^{1/p'(\cdot)_+(B)}.
\end{aligned}$$

Since $1/p'(\cdot)_+(B) \leq 1$, a straightforward argument shows that

$$|A_k|^{1/p'(\cdot)_+(B)} \leq C(n)(2^{-k}d)^{n/p_-(B)'}.$$

Therefore, if we continue the above estimate, we have that

$$\begin{aligned}
&\sum_{k=0}^{\infty} (2^{-(k+1)}d)^{1-n} \|f\|_{L^{p(\cdot)}(B)} |A_k|^{1/p'(\cdot)_+(B)} \\
&\qquad \leq C(n) \sum_{k=0}^{\infty} (2^{-(k+1)}d)^{1-n/p_-(B)} \|f\|_{L^{p(\cdot)}(B)} \\
&\qquad \leq C(n, p_-(B))d^{1-n/p_-(B)} \|f\|_{L^{p(\cdot)}(B)} \\
&\qquad = C(n, p_-(B))r^{1-n/p_-(B)} \|f\|_{L^{p(\cdot)}(B)}.
\end{aligned}$$

The constant $C(n, p_-(B))$ depends on the sum of the geometric series in this estimate, which converges because $p_-(B) > n$. Further, this constant increases as $p_-(B)$ decreases towards $n$. □

*Proof of Theorem 6.36.* Fix $x \in \Omega$ and let $r = r(x)$. Since $\overline{B_r(x)} \subset \Omega$ and $1/p(\cdot)$ is continuous, $p_-(B_r(x)) > n$. Fix $y \in B_r(x)$ and let $B \subset B_r(x)$ be the smallest ball containing both $x$ and $y$. Denote its radius by $\rho$; since $r < 1/2$, $\rho < 1/4$. Therefore, by Lemmas 6.25 and 6.37,

$$\begin{aligned}|f(x) - f(y)| &\le |f(x) - f_B| + |f(y) - f_B| \\ &\le C(n)I_1(|\nabla f|\chi_B)(x) + C(n)I_1(|\nabla f|\chi_B)(y) \\ &\le C(n, p_-(B))\rho^{1-n/p_-(B)}\|f\|_{L^{p(\cdot)}(B)} \\ &\le C(n, p_-(B))\rho^{1-n/p_-(B)}\|f\|_{L^{p(\cdot)}(\Omega)}.\end{aligned}$$

Since $1/p(\cdot) \in LH_0(\Omega)$, by Lemma 3.24 and Remark 3.25,

$$\begin{aligned}\rho^{1-n/p_-(B)} &= c(n)|B|^{1/n-1/p_-(B)} \\ &\le c(n, p(\cdot))|B|^{1/n-1/p(x)} = c(n, p(\cdot))|x - y|^{\alpha(x)}.\end{aligned}$$

If we combine these two estimates, we get (6.9).

Finally, since $p_-(B) \ge p_-$, if $p_- > n$, we can substitute $p_-$ for $p_-(B)$ into the estimate from Lemma 6.37 and get a bound in (6.9) independent of $x$. Hence, we have that $f \in C^{\alpha(\cdot)}(\Omega)$. □

Finally, we give an example to show that in Theorem 6.36 the restriction that $p(x) > n$ is necessary. For simplicity we only consider the case when is $n$ large.

*Example 6.38.* For $n > 3$, there exists a bounded set $\Omega \subset \mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$ with $p(\cdot) \in LH_0(\Omega)$ and $p_- = n$, and an unbounded (and so discontinuous) function $f \in W^{1,p(\cdot)}(\Omega)$.

*Proof.* Fix $n > 3$ and let $a = \frac{2}{n-1}$ and $b = 1 - a > 0$. Fix $R$, $0 < R < e^{-1}$, such that the function

$$\sigma(r) = \frac{-1}{\log(r|\log(r)|^a)}$$

is positive and increasing on $(0, R)$. Let $\Omega = B_R(0)$ and define $p(\cdot) \in \mathcal{P}(\Omega)$ by $p(x) = n + \sigma(|x|)$. Then $p_- = p(0) = n$ and $p(\cdot) \in LH_0(\Omega)$.

Let $f(x) = |\log(|x|)|^b$; then $f$ is unbounded at the origin. It remains to show that $f \in W^{1,p(\cdot)}(\Omega)$. Integrating in polar coordinates, we have that

$$\int_\Omega |f(x)|^{p(x)}\,dx = C(n)\int_0^R |\log(r)|^{bp(r)} r^{n-1}\,dr$$
$$\leq C(n)\int_0^R |\log(r)|^{bp_+} r^{n-1}\,dr < \infty.$$

A straightforward computation shows that

$$|\nabla f(x)| = \frac{b}{r|\log(r)|^a},$$

and by Proposition 6.2 the weak and classical derivatives coincide. Hence, again changing to polar coordinates, since $an > 1$,

$$\int_\Omega |\nabla f(x)|^{p(x)}\,dx = C(n)\int_0^R \frac{1}{r|\log(r)|^{an}}\frac{1}{(r|\log(r)|^a)^{\sigma(r)}}\,dr$$
$$\leq C(n)\int_0^R \frac{dr}{r|\log(r)|^{an}} < \infty.$$

Thus, $f \in W^{1,p(\cdot)}(\Omega)$. □

## 6.5 Notes and Further Results

### *6.5.1 References*

For results on the classical Sobolev spaces we primarily followed Adams and Fournier [7], Gilbarg and Trudinger [142], Maz'ja [260], and Ziemer [363]. There is a vast literature on this subject: see also [76, 110, 161, 227, 343]. As we noted in the introduction to this chapter, our treatment of variable Sobolev spaces is deliberately brief. For a comprehensive treatment from a somewhat different perspective, see Diening *et al*. [82]. Two useful introductory articles are Kováčik and Rákosník [219] and Fan and Zhao [122].

The variable Sobolev spaces were first explicitly defined by Kováčik and Rákosník [219]. Somewhat earlier, a special case was introduced by Zhikov [357]. Much earlier, generalized Sobolev spaces defined using Musielak-Orlicz spaces (see Sect. 2.10.2) were treated in a series of papers by Hudzik [171–179]. Versions of many of the results in this chapter are implicit in his work.

The definition of weak derivatives and Proposition 6.2 are well-known: see, for instance, [363]. For the chain rule, Lemma 6.5, see [142]. The proofs of Theorems 6.6–6.8 are adapted from the proofs in the classical case given in [7].

For our definition of $W_0^{k,p(\cdot)}$ we have followed [219]. A somewhat different definition that is equivalent in many cases is used in [82] (see also Sect. 6.5.6 below.) The density of smooth functions in the variable Sobolev spaces was first considered by Zhikov [355,357]; he constructed Example 6.12 and our construction is based on his. Later, Zhikov [362] showed that this example is actually a particular case of a more general result. (See Sect. 6.5.5 below.) A proof of Theorem 6.14 was sketched in [58] assuming $1/p(\cdot) \in LH_0(\Omega)$; the proof given here is adapted from the proof in the classical case in [7]. (See also Diening [77] for a related result when $p_- > 1$.) The proof of Lemma 6.15 is well-known: see, for instance, [142]. Theorem 6.17 was first proved by Samko [315,316]. A different proof was outlined briefly in [58].

The Poincaré inequality (Theorem 6.19) was first proved in variable Sobolev spaces by Harjulehto and Hästö [151] with somewhat different assumptions on the exponents. (See Sect. 6.5.6 below.) They also gave an example to show that the exponent $p(\cdot)$ needs some regularity. Theorem 6.21 was first proved, with different hypotheses, in [219]. It is implicit in [106]. The versions given here are similar to those in [82] but have slightly stronger hypotheses. (See Sect. 6.5.6 below.) The proofs of Lemmas 6.25 and 6.26 are taken from [142].

The Sobolev embedding theorem (Theorem 6.29) has been studied extensively. It was first considered by Kováčik and Rákosník [219], who constructed Example 6.28. They also proved a version of Theorem 6.29 assuming $p(\cdot)$ is continuous on $\overline{\Omega}$ and with the weaker conclusion that $f \in L^{q(\cdot)}(\Omega)$, where $q(x) \le p(x) - \epsilon$ for some $\epsilon > 0$. Their results were generalized to certain kinds of discontinuous exponents by Alisoy, Çekiç and Mashiyev [44, 257]. Fan, Shen and Zhao [118] proved Theorem 6.29 if $p(\cdot)$ is a Lipschitz function; this was proved independently by Edmunds and Rákosník [106,107] who later proved that it was enough to assume that $p(\cdot)$ is Hölder continuous. Diening [78] proved it assuming that $p_- > 1$ and $p(\cdot) \in LH(\Omega)$; this was proved independently in [42]. Another proof that included the case $p_- = 1$ was given by Harjulehto and Hästö [153] for bounded domains; this was extended by Hästö [165] to unbounded domains using the technique of local to global estimates (Sect. 5.6.6). The proof given here first appeared in [69]. Lemma 6.32 is well-known if $p > 1$; when $p = 1$ it is due to Maz'ja [260]. Our proof of this lemma was implicit in [132] and is explicit in [69]. The decomposition argument used in the proof is often attributed to Long and Nie [241] but it is implicit in Maz'ja [260]. A similar argument was used in [153]. Example 6.34 is due to Fan and Zhao [122].

The variable Hölder classes (Definition 6.35) were first considered by Ross and Samko [300] and by Karapetyants and Ginzburg [189,190] in the study of fractional derivatives. Theorem 6.36 was first proved by Edmunds and Rákosník [106], and a slightly more general version with a very different proof was given by Almeida and Samko [12, 13]; our proof is adapted from theirs. A slightly weaker version based on the classical continuity results was proved by Harjulehto and Hästö [151]; they also gave a generalization based on capacity theory for variable Sobolev spaces. Example 6.38 is new; related examples are given in [106, 151].

### 6.5.2 An Alternative Definition of the Norm

There is an equivalent definition of the norm on $W^{k,p(\cdot)}(\Omega)$ that has been adopted by some authors: see, for instance [31, 82]. Given $\Omega$, $p(\cdot) \in \mathcal{P}(\Omega)$ and $k \geq 1$, let

$$\|f\|^*_{W^{k,p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \sum_{|\alpha|\leq k} \rho_{p(\cdot)}(D^\alpha/\lambda) \leq 1\right\}.$$

It is immediate that for each $\alpha$, $|\alpha| \leq k$, $\|D^\alpha f\|_{p(\cdot)} \leq \|f\|^*_{W^{k,p(\cdot)}}$, and so

$$\|f\|_{W^{k,p(\cdot)}(\Omega)} \leq M(k,n)\|f\|^*_{W^{k,p(\cdot)}},$$

where $M(k,n)$ is the number of multi-indices $\alpha$, $|\alpha| \leq k$. To see the opposite inequality, note that

$$\sum_{|\alpha|\leq k} \rho_{p(\cdot)}(D^\alpha/\lambda) = \int_{\Omega\setminus\Omega_\infty} \sum_{|\alpha|\leq k} \left(\frac{|D^\alpha f(x)|}{\lambda}\right)^{p(x)} dx + \sum_{|\alpha|\leq k} \lambda^{-1}\|D^\alpha f\|_{L^\infty(\Omega_\infty)}$$

$$\leq \int_{\Omega\setminus\Omega_\infty} \left(M(k,n) \sum_{|\alpha|\leq k} \frac{|D^\alpha f(x)|}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\left\|M(k,n) \sum_{|\alpha|\leq k} |D^\alpha f|\right\|_{L^\infty(\Omega_\infty)}.$$

Therefore, by the definition of the norms and the triangle inequality,

$$\|f\|^*_{W^{k,p(\cdot)}(\Omega)} \leq M(k,n)\left\|\sum_{|\alpha|\leq k} |D^\alpha f|\right\|_{p(\cdot)} \leq M(k,n)\|f\|_{W^{k,p(\cdot)}(\Omega)}.$$

### 6.5.3 Boundary Regularity

In our discussion of variable Sobolev spaces we have avoided the important but more delicate questions related to the regularity of the boundary of $\Omega$. However, these have been considered by many authors. The simplest assumption is that the domain $\Omega$ is bounded and has a Lipschitz boundary. Given this assumption, Diening [77] noted that Theorem 6.14 could be extended to show that $C^\infty(\overline{\Omega})$ is dense in $W^{k,p(\cdot)}(\Omega)$ provided $p(\cdot) \in LH_0(\Omega)$ and $p_- > 1$. This result was extended to include the case $p_- = 1$ in [58]. Similarly, Edmunds and Rákosník [106] proved their version of the Sobolev embedding theorem (Theorem 6.29) with $W_0^{1,p(\cdot)}(\Omega)$ replaced by $W^{1,p(\cdot)}(\Omega)$ provided $\Omega$ is bounded and has a Lipschitz boundary.

A generalization of the Lipschitz boundary condition that is well-suited to variable Lebesgue spaces is the John domain. A bounded set $\Omega$ is an $\alpha$-John domain, $\alpha > 0$, if there exists $x_0 \in \Omega$ such that given any $x \in \Omega$, there exists a rectifiable curve $\gamma_x : [0, \ell_x] \to \Omega$, parameterized according to arc length, with $\gamma_x(0) = x_0$ and $\gamma_x(\ell_x) = x$, such that $B_{\alpha^{-1}t}(\gamma_x(t)) \subset \Omega$. An unbounded set is an $\alpha$-John domain if it is the nested union of bounded $\alpha$-John domains. These domains were introduced in the study of variable Lebesgue spaces by Harjulehto and Hästö [151] and have been used by a number of authors since then: in particular they are used extensively in [82].

### 6.5.4 Extension Theorems

Closely related to boundary regularity is the question of extension theorems. Given $W^{k,p(\cdot)}(\Omega)$, an extension operator is a bounded linear map $E : W^{k,p(\cdot)}(\Omega) \to W^{k,p(\cdot)}(\mathbb{R}^n)$ such that for all $x \in \Omega$, $Ef(x) = f(x)$. The existence of an extension operator is closely related to the geometry of the boundary of $\Omega$. Unlike in the classical case, in the variable Sobolev spaces the exponent $p(\cdot) \in \mathcal{P}(\Omega)$ must also be extended to an exponent $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$; when $p(\cdot) \in LH(\Omega)$ this has been done using variants of Lemma 2.4. Edmunds and Rákosník [106] proved that if $\Omega$ is bounded and has a Lipschitz boundary, and $p(\cdot)$ is a Lipschitz function, then $p(\cdot)$ can be extended to a Lipschitz function on $\mathbb{R}^n$ and there is an extension operator from $W^{1,p(\cdot)}(\Omega)$ into $W^{1,p(\cdot)}(\Omega)$. Diening [78] generalized this result to exponents $p(\cdot) \in LH_0(\Omega)$. An elementary proof using extrapolation is given in [82], assuming $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 < p_- \le p_+ < \infty$ and the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$. An elementary extension theorem for cubes was proved in [127].

Extrapolation results exist for more general domains. To state one such we begin with two definitions. (For more details, see [7].)

**Definition 6.39.** Given a point $x \in \mathbb{R}^n$, a finite cone with vertex at $x$, $C_x$, is a set of the form

$$C_x = B_1 \cap \{x + t(y - x) : y \in B_2, 0 < t < 1\},$$

where $B_1$ is an open ball centered at $x$, and $B_2$ is an open ball which does not contain $x$.

**Definition 6.40.** A set $\Omega \subset \mathbb{R}^n$ has the uniform cone property if there exists a finite collection of open sets $\{U_j\}$ (not necessarily bounded) and an associated collection $\{C_j\}$ of finite cones such that the following hold:

1. There exists $\delta > 0$ such that

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\} \subset \bigcup_j U_j;$$

2. For every index $j$ and every $x \in \Omega \cap U_j$, $x + C_j \subset \Omega$.

An example of a domain with the uniform cone property is any domain whose boundary is locally a Lipschitz graph.

**Theorem 6.41.** *Given a set* $\Omega$ *with the uniform cone property, suppose* $p(\cdot) \in \mathcal{P}(\Omega)$ *is such that* $1 < p_- \leq p_+ < \infty$ *and* $p(\cdot) \in LH(\Omega)$*. Then for every* $k \geq 1$ *there exists a bounded linear extension operator*

$$E_k : W^{k,p(\cdot)}(\Omega) \to W^{k,p(\cdot)}(\mathbb{R}^n).$$

In the classical case Theorem 6.41 is due to Calderón [39] (see also [7]). In the variable Sobolev spaces it was proved in [61] as a (non-trivial) application of extrapolation theory.

Another extension theorem that applies to a large class of domains (the so-called $(\epsilon, \infty)$-domains) and eliminates the restriction $1 < p_- \leq p_+ < \infty$, is due to Fröschl [133] (see also [82]).

### 6.5.5 More on the Density of Smooth Functions

The density of smooth functions in $W^{k,p(\cdot)}(\Omega)$ is a delicate problem, but one which appears to be central to a deeper understanding of the variable Sobolev spaces as well as in applications: see, for example, Harjulehto *et al.* [155], Nuortio [287] and Proposition 6.48 below. In Example 6.12 we gave an example of an exponent with a jump discontinuity such that smooth functions were not dense. On the other hand, by Theorem 6.14 and Example 4.68, we have that there exist discontinuous exponents such that smooth functions are dense.

One approach to this problem is to give conditions on the modulus of continuity of $p(\cdot)$ that are sufficient for bounded functions to be dense. As we showed in Chap. 4, log-Hölder continuity is the best possible condition for the maximal operator, but this condition can be weakened and smooth functions are still dense in $W^{k,p(\cdot)}(\Omega)$.

**Proposition 6.42.** *Given* $\Omega$ *and* $p(\cdot) \in \mathcal{P}(\Omega)$*, suppose there exists a function* $\omega$ *such that* $|p(x) - p(y)| \leq \omega(|x - y|)$ *whenever* $|x - y| < 1/2$*. If* $\omega$ *satisfies*

$$\int_0^{1/2} t^{-1+\frac{n}{p_-}\omega(t)}\, dt = \infty, \tag{6.10}$$

*then* $C^\infty(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ *is dense in* $W^{1,p(\cdot)}(\Omega)$*.*

Proposition 6.42 is due to Zhikov [362]. Condition (6.10) holds, for example, if

$$\omega(t) = \frac{k \log\log(1/t)}{\log(1/t)}, \qquad k \leq p_-/n.$$

Note that by Example 4.43 this shows that there exist exponents $p(\cdot)$ such that the maximal operator is not bounded on $L^{p(\cdot)}(\Omega)$ but smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$.

This condition is very close to optimal: Hästö [162] constructed an exponent on $\Omega = B_1(0)$ in $\mathbb{R}^2$ for which the best modulus of continuity is

$$\omega(t) = \frac{(1+\epsilon)\log\log(1/t)}{\log(1/t)}, \qquad \epsilon > 0,$$

and smooth functions are not dense in $W^{1,p(\cdot)}(\Omega)$. (For a more general condition guaranteeing smooth functions are not dense, see [82].)

A different approach is due to Edmunds and Rákosník [105]. They proved Theorem 6.14 assuming a complicated monotony condition on $p(\cdot)$. Hästö [164] generalized this condition by showing that it could be combined with a weaker form of log-Hölder continuity. In the same paper he also considered the question of when continuous functions are dense in $W^{1,p(\cdot)}(\Omega)$. Similar results on the density of $C^\infty$ were derived by Fan, Wang and Zhao [119].

Finally, we note that on the real line, if $p_+ < \infty$, then $C^\infty(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ is always dense in $W^{1,p(\cdot)}(\Omega)$. This was proved by Harjulehto and Hästö [150].

### *6.5.6 More on the Poincaré Inequalities*

Theorems 6.19 and 6.21 hold with weaker hypotheses on the exponent $p(\cdot)$. In terms of regularity related to the maximal operator, Diening *et al.* [82] showed that both theorems are true for any $p(\cdot) \in \mathcal{P}(\Omega)$ provided $p(\cdot) \in \mathcal{A}$. (See Sect. 4.5 for the definition of the condition $\mathcal{A}$.) Note that this eliminates the assumption that $p_+ < \infty$ or $p_- > 1$ that our proofs required. Since the $K_0$ condition is necessary and sufficient for the maximal operator to be bounded on $L^{p(\cdot)}(\Omega)$ when $\Omega$ is bounded, we conjecture that these results hold if we instead assume that $p(\cdot) \in K_0$.

Even weaker conditions unrelated to the maximal operator are sufficient.

**Theorem 6.43.** *Given a bounded convex set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p(\cdot)$ is continuous on $\overline{\Omega}$. Then for all $f \in W^{1,p(\cdot)}(\Omega)$,*

$$\|f - f_\Omega\|_{p(\cdot)} \le C(n, p(\cdot), \Omega)\|\nabla f\|_{p(\cdot)}.$$

**Theorem 6.44.** *Given a bounded set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p(\cdot)$ is continuous on $\overline{\Omega}$. Then for all $f \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\|f\|_{p(\cdot)} \le C(n, p(\cdot), \Omega)\|\nabla f\|_{p(\cdot)}.$$

Theorem 6.43 is due to Harjulehto and Hästö [151]. They also proved a version when $p(\cdot)$ has no regularity but has limited oscillation: i.e., if $(p_-)^* \ge p_+$. Kováčik and Rákosník [219] proved Theorem 6.44. (See also [82].) Mercaldo *et al.* [261] have shown that Theorem 6.44 is true on Lipschitz domains if $p(\cdot)$ is discontinuous and takes on just two values: 2 and $p_0$, $1 < p_0 < 2$, on Lipschitz subsets of $\Omega$.

Theorem 6.44 can be extended to a subspace of $W^{1,p(\cdot)}(\Omega)$ that may be larger than $W_0^{1,p(\cdot)}(\Omega)$ Define $\mathring{W}^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$, or equivalently, let $\mathring{W}^{1,p(\cdot)}(\Omega)$ be the closure in $W^{1,p(\cdot)}(\Omega)$ of the set of all functions in $W^{1,p(\cdot)}(\Omega)$ with compact support. It is immediate that the proofs of Theorems 6.19 and 6.21 extend to this space. Similarly, Theorem 6.44 can be proved in this setting: see [82]. More recently, with the additional hypothesis that $p_- > 1$, Ciarlet and Dinca [50] gave a proof that did not rely on density arguments.

In the constant exponent case it is always the case that $\mathring{W}^{1,p}(\Omega) = W_0^{1,p}(\Omega)$. However, in the variable exponent case $\mathring{W}^{1,p}(\Omega)$ can be larger. Fan and Zhao [122] pointed out that an example of Zhikov (Example 6.12) can be adapted to show this. But with some regularity on $p(\cdot)$, the two are the same: Harjulehto [149] showed that if $C^\infty(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$ (see Sects. 6.2 and 6.5.5), then equality holds. (See also [82].)

When $\partial\Omega$ has some regularity (e.g., is Lipschitz) the space $\mathring{W}^{1,p(\cdot)}(\Omega)$ can also be defined as the space of all functions in $W^{1,p(\cdot)}(\Omega)$ that have zero trace—intuitively, functions that are zero on $\partial\Omega$. The existence of a bounded trace operator for classical Sobolev spaces has many applications in the study of PDEs. The trace operator in the variable Sobolev space has been studied by several authors: see Diening and Hästö [83,84] and Fan [115]. Liu [240] has considered the trace operator in weighted Sobolev spaces.

### 6.5.7 *More on the Sobolev Embedding Theorem*

The extension theorems discussed above let us extend the Sobolev embedding theorem to the spaces $W^{1,p(\cdot)}(\Omega)$. For example, the following result is an immediate consequence of Theorems 6.17, 6.29 and 6.41.

**Theorem 6.45.** *Given a set $\Omega$ with the uniform cone property, and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p(\cdot) \in LH(\Omega)$ and $1 < p_- \leq p_+ < n$, then $W^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$, and*

$$\|f\|_{p^*(\cdot)} \leq C\|\nabla f\|_{p(\cdot)}. \tag{6.11}$$

In the classical case, when $p = n$ inequality (6.11) is false: $W^{1,n}(\Omega)$ is not contained in $L^\infty(\Omega)$. (This is actually shown by the function in Example 6.38.) As a substitute, there are exponential integrability results: for example, if $\Omega$ is a bounded domain, then for all $f \in W_0^{1,n}(\Omega)$,

$$\int_\Omega \exp\left[\left(C_1 \frac{|f(x)|}{\|\nabla f\|_{L^n(\Omega)}}\right)^{n'}\right] dx \leq C_2|\Omega|. \tag{6.12}$$

(See [363].) This inequality implies that $f$ is in the Orlicz space $Exp(L^{n'})(\Omega)$.

This result can be generalized to the variable Sobolev spaces. Given a bounded domain $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p(\cdot) \in LH_0(\Omega)$ and $p_+ \leq n$, Harjulehto and Hästö [153] defined the Musielak-Orlicz space $L^{\Phi(,)}(\Omega)$ with

$$\Phi(x,t) = \frac{\max(\log(|t|),1)}{\Gamma(t/n')} \int_1^{p^*(x)} |t|^\sigma \, d\sigma$$

(see Sect. 2.10.2), and proved that if $f \in W^{1,p(\cdot)}(\Omega)$, then $f \in L^{\Phi(,)}(\Omega)$. The function $\Phi$ is such that if $p_+ < n$, then $L^{\Phi(,)}(\Omega) = L^{p^*(\cdot)}(\Omega)$, and if $p(\cdot) = n$, then $L^{\Phi(,)}(\Omega) = Exp(L^{n'})(\Omega)$.

Another approach when $p_+ = n$ using weighted variable Lebesgue spaces is due to Edmunds and Rákosník [106]. Given a bounded set $\Omega$ with a Lipschitz boundary, let $p(\cdot) \in \mathcal{P}(\Omega)$ be a Lipschitz function such that $p_+ = n$. Define the weight

$$w(x) = \min\big((n - p(x))^b, 1\big),$$

where $b > 4 - 1/n$. Then for all $f \in W^{1,p(\cdot)}(\Omega)$,

$$\|fw\|_{L^{p^*(\cdot)}(\Omega)} \leq C\|f\|_{1,p(\cdot)}.$$

A similar result was proved by Futamura and Mizuta [135].

Recently, Fan [116] showed that if $\Omega$ is an unbounded domain that satisfies the cone property, and if $p(\cdot) \in \mathcal{P}(\Omega)$ is Lipschitz, then

$$\|f\|_{L^{q(\cdot)}(\Omega)} \leq C\|f\|_{1,p(\cdot)},$$

where $q(\cdot)$ is such that $q_+ < \infty$ and $q(x) \leq p^*(x)$ for all $x \in \Omega$. For instance, we can take $q(x) = \min(p^*(x), N)$ for any $N > 1$.

Though Example 6.38 shows that in general functions in $W^{1,p(\cdot)}(\Omega)$ need not be continuous when $p_- = n$, Futamura and Mizuta [135] showed that they are log-Hölder continuous provided that $p(\cdot)$ decays to $n$ sufficiently slowly. For exponents that decay more quickly they proved an exponential integrability result similar to inequality (6.12).

In Theorem 6.29 the regularity of the exponent is given in terms of the boundedness properties of the maximal operator. The exponent $p(\cdot)$ in Example 6.28 has a jump discontinuity, so we know (see Example 3.21) that the maximal operator is not bounded on $L^{p(\cdot)}$. Diening, Hästö and Nekvinda [86] modified this example to produce an exponent $p(\cdot)$ that is continuous, but not uniformly continuous, such that the Sobolev embedding theorem fails.

On the other hand if $p(\cdot)$, $q(\cdot)$ are continuous on $\overline{\Omega}$ and $q(x) < p(x)$ for $x \in \overline{\Omega}$, then $\|f\|_{q(\cdot)} \leq C\|f\|_{1,p(\cdot)}$. (See Sect. 6.5.8 below.) Futamura, Mizuta and Shimomura [137, 269] considered uniformly continuous exponents that are not log-Hölder continuous, but gave embeddings into a Musielak-Orlicz space (see Sect. 2.10.2). In light of these results we have the following open question, a generalization of a question first posed in [86].

**Question 6.46.** Does there exist a set $\Omega$ and a uniformly continuous exponent $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Sobolev embedding theorem is false for $W_0^{1,p(\cdot)}(\Omega)$? Is there a uniform continuity condition weaker than $LH_0(\Omega)$ such that if an exponent satisfies it, then the Sobolev embedding theorem holds?

Finally, we note an application of rearrangement inequalities to the Sobolev embedding theorem. (See Sect. 3.7.7.) Given a bounded set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ with $p_- > 1$, suppose the increasing rearrangement $p^\dagger(\cdot)$ of $p(\cdot)$ is in $LH_0([0,\epsilon))$ for some $\epsilon > 0$. If $r(\cdot) \in \mathcal{P}(\Omega)$ is such that its decreasing rearrangement satisfies $r^*(0) < (p_-)^* = np_-/(n-p_-)$, then for all $f \in W_0^{1,p(\cdot)}(\Omega)$, $f \in L^{r(\cdot)}(\Omega)$. This embedding is similar to the suboptimal results in [219]; what is of interest is the extremely weak regularity assumptions on $p(\cdot)$: $p(\cdot)$ does not itself need to be continuous. For this and related results, see [131, 298].

### *6.5.8 Compact Embeddings*

In the classical Sobolev spaces, given a bounded set $\Omega$, if $p < n$ and $q < p^*$, then $W_0^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$: if $\{g_k\} \subset W^{1,p(\cdot)}(\Omega)$ is bounded in $W^{1,p(\cdot)}$ norm, then it has a subsequence that converges in $L^q$ norm. (This is referred to as the Rellich-Kondrachov theorem: see [7, 142].) This result extends to the variable Sobolev spaces.

**Proposition 6.47.** *Given a bounded set $\Omega$, suppose $p(\cdot) \in \mathcal{P}(\Omega)$ is such that $p_+ < n$ and $p(\cdot)$ is continuous on $\overline{\Omega}$. (e.g. $p(\cdot) \in LH_0(\Omega)$). Let $q(\cdot) \in \mathcal{P}(\Omega)$ be such that $q(x) \le p(x) - \epsilon$ for all $x \in \Omega$ and some $\epsilon > 0$. Then $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$.*

Proposition 6.47 was first proved by Kováčik and Rákosník [219]; other proofs are due to Fan and Zhao [122], Fan, Shen and Zhao [118], and Diening [78].

Proposition 6.47 has been generalized in various directions. Fan, Zhao and Zhao [123] proved a compact embedding theorem for radial functions in $W^{1,p(\cdot)}(\mathbb{R}^n)$ assuming $p(\cdot)$ is also radial. These results were generalized by Yao and Wang [350]. (They also generalize a compact embedding from a weighted Sobolev space given without proof by Fan and Han [117].) Ohno [288] examined compact embeddings into Musielak-Orlicz spaces.

In the classical case the embedding is not compact if $q = p^*$. However, in variable Sobolev spaces it is possible to have a compact embedding even if $q(x) = p^*(x)$ at some points in the domain. This was first considered by Kurata and Shioji [225] and later by Mizuta, Ohno, Shimomura and Shioji [268]. Gao, Zhao and Zhang [139] gave embeddings into a weighted variable Lebesgue space when $q(x) < p^*(x) + \epsilon$ for some $\epsilon > 0$.

### *6.5.9 Mean Continuity*

Functions in the classical Sobolev space are mean continuous: given $\Omega$ and any set $A$ such that $\overline{A} \subset \Omega$, then for all $h$, $|h| < \mathrm{dist}(A, \partial\Omega)$,

$$\|f - \tau_h f\|_{L^p(A)} \le C|h|\|\nabla f\|_{L^p(\Omega)}.$$

(See [142].) Since the variable Lebesgue spaces are not mean continuous (Corollary 5.18), it is not surprising that this result fails in the variable Sobolev spaces. There exists a smooth exponent $p(\cdot)$ on $\Omega = (0,1) \times (0,1)$ (though not continuous on $\overline{\Omega}$) and a function $f \in W^{1,p(\cdot)}(\Omega)$ such that $\tau_h f \notin L^{p(\cdot)}(A_h)$, where $h$ can be taken arbitrarily small and $A_h$ is compactly contained in $\Omega$. See [127].

This paper also contains a substitute result. Given a lower semi-continuous exponent $p(\cdot)$ and $h \in \mathbb{R}^n$, define $p_h(\cdot)$ by

$$p_h(x) = \inf\{p(x + ht) : 0 \le t \le 1\}.$$

**Proposition 6.48.** *Given a cube $Q$ and $p(\cdot) \in \mathcal{P}(Q)$, suppose $p(\cdot)$ is lower semi-continuous, $1 < p_- \le p_+ < \infty$, and that the maximal operator is locally bounded on $L^{p'(\cdot)}(Q)$. Given an open set $A$ such that $\overline{A} \subset Q$ and $h$, $|h| < \mathrm{dist}(A, \partial Q)$, then for all $f \in W^{1,p(\cdot)}(Q)$,*

$$\|f - \tau_h f\|_{L^{p_h(\cdot)}(A)} \le C|h|\|\nabla f\|_{L^{p(\cdot)}(Q)}.$$

In the proof of Proposition 6.48 the key property needed is that smooth functions are dense in $W^{1,p(\cdot)}(Q)$; our hypothesis is sufficient for this, but as we noted in Sect. 6.5.5, it also holds with other assumptions.

As a consequence of Proposition 6.48 we can generalize a characterization of $W^{1,p}(\Omega)$ in terms of the double integral of differences quotients due to Bourgain, Brezis and Mironescu [34]. They showed that if $\phi \in C_c^\infty$ is a radial function, then there exists a constant $K = K(p,n)$ such that for all $f \in W^{1,p}(\Omega)$,

$$\lim_{t\to 0} \int_\Omega \int_\Omega \left( \frac{|f(x) - f(y)|}{|x-y|} \right)^p \phi_t(x-y)\, dy\, dx = K\|\nabla f\|_p^p.$$

In [127] a weaker version of this result was proved.

**Proposition 6.49.** *Given $Q$ and $p(\cdot)$ as in Proposition 6.48, suppose $p(\cdot) \in C(Q)$. Let $\phi \in C_c^\infty$ be a radial function. On $Q \times Q$ define $q(x,y) = \inf\{p((1-t)x + ty) : 0 \le t \le 1\}$ and $F_t(x,y) = \frac{|f(x)-f(y)|}{|x-y|}\phi_t(x-y)^{1/q(x,y)}$. Then for all $f \in W^{1,p(\cdot)}(Q)$,*

$$c \limsup_{t\to 0} \|F_t\|_{L^{q(\cdot)}(Q\times Q)} \le \|\nabla f\|_{p(\cdot)} \le C \liminf_{t\to 0} \|F_t\|_{L^{q(\cdot)}(Q\times Q)}.$$

Mean continuity in mixed norm variable Lebesgue spaces was considered by Bandaliev and Abbasova [23]. Modular mean continuity along the lines of the Lebesgue differentiation theorem (see Sect. 2.9) was considered by Harjulehto and Hästö [152] using a capacity theory for variable Sobolev spaces developed in [155]. (See also [82]). These results were generalized to Musielak-Orlicz spaces by Futamura, Mizuta and Shimomura [137].

### 6.5.10 Gagliardo-Nirenberg Inequalities

In the classical Sobolev space $W^{k,p}(\mathbb{R}^n)$ it is possible to estimate the norms of the derivatives of order $j < k$ by the $L^p$ norms of $f$ and the derivatives of order $k$. For brevity, let

$$|f|_{j,p} = \sum_{|\alpha|=j} \|D^\alpha f\|_p;$$

then for $0 \le j \le k$, $\epsilon > 0$ sufficiently close to 0, and for $f \in W^{k,p}(\mathbb{R}^n)$, $1 \le p < \infty$,

$$|f|_{j,p} \le C\big(\epsilon |f|_{k,p} + \epsilon^{-j/(k-j)} \|f\|_p\big).$$

(See [7].) Zang and Fu [351] extended this inequality to the variable Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$:

$$|f|_{j,p(\cdot)} \le C\big(\epsilon |f|_{k,p(\cdot)} + \epsilon^{-j/(k-j)} \|f\|_{p(\cdot)}\big), \tag{6.13}$$

assuming that $p_- > 1$ and the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. A generalization, with the same hypotheses on $p(\cdot)$, was proved by Kopaliani and Chelidze [218]. In the case $p_- \ge 1$, a modular interpolation inequality was proved by Giannetti [141] assuming $p(\cdot) \in LH(\mathbb{R}^n)$. A weighted interpolation theorem, with very different hypotheses, was proved by Cianci and Nicolosi [49].

# Appendix A
# Appendix: Open Problems

In this appendix we present a collection of open problems that we believe will be fruitful for further research. Some of these were already mentioned in the text and they are all listed roughly in the order the relevant material appears in the book. We do not include any problems beyond the scope of this book; in particular, we do not include anything related to the calculus of variations or partial differential equations, two areas of intense activity. For these problems we refer the reader to the survey articles by Fan [113], Harjulehto *et al.* [157] and Mingione [263].

**Problem A.1.** *Let $\Omega$ be an unbounded set. Characterize the closure of bounded functions in $L^{p(\cdot)}(\Omega)$ when $p_+(\Omega \setminus \Omega_\infty) = \infty$. In particular, if $L^\infty(\Omega)$ is a subset of $L^{p(\cdot)}(\Omega)$, characterize its closure.*

By Theorem 2.75, if $p_+(\Omega \setminus \Omega_\infty) = \infty$ then bounded functions are never dense. If $\Omega$ is bounded, this question was studied by Edmunds, Lang and Nekvinda [101]. They showed that in this case the closure of the set of bounded functions is equal to the set of functions with absolutely continuous norm (see Sect. 2.10.3) if and only for every $A > 1$,

$$\int_0^{|\Omega|} A^{p^*(t)}\,dt < \infty,$$

where $p^*$ is the decreasing rearrangement of $p(\cdot)$.

**Problem A.2.** *Find "useful" dense subsets of $L^{p(\cdot)}(\Omega)$ when $p_+ = \infty$.*

By Theorem 2.75, if $p_+(\Omega \setminus \Omega_\infty) = \infty$, bounded functions are not dense, and by Theorem 2.77, functions with compact support are not dense if $p(\cdot)$ is unbounded on the boundary of $\Omega$. As we saw in Chap. 5, bounded functions of compact support play an important role in studying the operators of harmonic analysis when $p_+ < \infty$. Is there a useful substitute when $p_+ = \infty$? For instance, under what circumstances is the set

$$L^{p(\cdot)}(\Omega) \cap \bigcup_{1 \le p < \infty} L^p(\Omega)$$

dense in $L^{p(\cdot)}(\Omega)$?

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**Problem A.3.** *Characterize the dual space $L^{p(\cdot)}(\Omega)^*$ if $p_+ = \infty$.*

We have that $L^{p(\cdot)}(\Omega)^* = L^{p'(\cdot)}(\Omega)$ (up to isomorphism) if and only if $p_+ < \infty$ (Theorem 2.80). When $p_+ = \infty$, $L^{p'(\cdot)}(\Omega)$ is (isomorphic to) a closed subspace of $L^{p(\cdot)}(\Omega)^*$. When $p(\cdot)$ is constant the dual of $L^\infty$ is the set of finitely additive measures that are absolutely continuous with respect to Lebesgue measure (see [95]).

The solution to this problem may depend on whether $p(\cdot)$ is such that $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$. In this case, since the identity map $I : L^\infty(\Omega) \to L^{p(\cdot)}(\Omega)$ is bounded, any element of $L^{p(\cdot)}(\Omega)^*$ induces a bounded linear functional on $L^\infty(\Omega)$. The above characterization of $L^\infty(\Omega)^*$ may then become relevant.

**Problem A.4.** *Generalize the Marcinkiewicz interpolation theorem, or more generally, one of the real interpolation methods, to the scale of variable Lebesgue spaces.*

The problem of interpolation was discussed in Sect. 3.7.8. Given the close connection between Marcinkiewicz interpolation and rearrangements (see Sect. 3.7.7) one initial step would be to find a replacement in the variable Lebesgue spaces for the $L^p$ identity

$$\int_\Omega |f(x)|^p\,dx = p\int_0^\infty t^{p-1}|\{x \in \Omega : |f(x)| > t\}|\,dt.$$

Recall that this identity is central to proving the classical norm inequalities for the maximal operator (Theorem 3.4); therefore, it may be reasonable to assume boundedness of the maximal operator as a hypothesis when proving such a replacement.

A weaker version of this problem would be to prove a strong type norm inequality for the maximal operator (as in Theorem 3.16) as a consequence of the weak type inequality.

**Problem A.5.** *Explore the interaction between the symmetry of the exponent function and log-Hölder continuity in controlling the boundedness of the maximal operator.*

As we discussed in Chap. 4, while log-Hölder continuity is the sharpest possible pointwise decay condition on $p(\cdot)$ for the maximal operator to be bounded on $L^{p(\cdot)}$, Examples 4.1 and 4.43 depend on the exponent $p(\cdot)$ being non-symmetric. In Examples 4.13 and 4.59 we gave symmetric exponents that decay more slowly but the maximal operator is still bounded.

It is not clear what role symmetry plays. A first step would be to determine if Example 4.43 could be improved by showing there exists a symmetric function $p(\cdot) \in \mathcal{P}((-1,1))$ which has the specified decay and such that the maximal operator is not bounded on $L^{p(\cdot)}((-1,1))$. Or is it the case that symmetry, monotonicity and continuity are sufficient to prove the maximal operator is bounded? The construction of Example 4.59 will be relevant to this question.

**Problem A.6.** *Find a condition weaker than the $N_\infty$ condition that, together with the $K_0$ condition is necessary and sufficient for the maximal operator to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Theorem 4.63 gives a single necessary and sufficient condition—the condition $\mathcal{A}$, due to Diening—for the maximal operator to be bounded. While of great theoretical importance, it is difficult to check: in practice, one shows that an averaging operator is bounded by showing that the maximal operator is bounded.

Therefore, it would be interesting to have a different condition. As our approach in Chaps. 3 and 4 demonstrated, it is natural to consider two conditions, one to control the maximal operator locally (i.e., the $K_0$ condition), and one to control it at infinity. However, at this point it is unclear what this condition should look like. A more careful analysis of the maximal operator at infinity (i.e., the estimates for $f_2$ in our proofs) is needed.

**Problem A.7.** *Adapt the ideas from the theory of two weight norm inequalities to find necessary and sufficient conditions for the maximal operator to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

There are several approaches in the theory of two weight norm inequalities that may be applicable to the variable Lebesgue spaces. One would be to generalize the $K_0$ condition so that it also controls the maximal operator at infinity. For example, if we define

$$T_Q(x) = \frac{|Q|}{|Q| + |x - x_Q|^n} \approx M(\chi_Q)(x),$$

we could replace the $K_0$ condition with the "tail" condition

$$\sup_Q |Q|^{-1} \|T_Q\|_{p(\cdot)} \|T_Q\|_{p'(\cdot)} < \infty,$$

where the supremum is taken over all cubes. This is referred to as a tail condition since it depends not only on the local behavior of the norm but also its behavior at infinity. Clearly, this condition implies $p(\cdot) \in K_0(\mathbb{R}^n)$. Such conditions have played a role in the study of two-weight norm inequalities for the maximal operator. See, for example, Muckenhoupt and Wheeden [273] and Sawyer and Wheeden [327].

Another possibility is to adapt the "testing" condition for the maximal operator introduced by Sawyer [325] (see also [53, 140]). He showed that a necessary and sufficient condition for the inequality

$$\int_{\mathbb{R}^n} Mf(x)^p u(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx$$

is that for every cube $Q$,

$$\int_Q M(v^{1-p'}\chi_Q)(x)^p u(x)\, dx \le C \int_Q v(x)^{1-p'}\, dx.$$

The necessity of this condition comes from setting $f = v^{1-p'}\chi_Q$ in the norm inequality and restricting the integral on the left-hand side to $Q$.

The analogous testing condition for the variable Lebesgue spaces would be

$$\int_{\mathbb{R}^n} M(\chi_Q)(x)^{p(x)}\,dx \le C|Q|.$$

The domain of integration on the left-hand side must remain equal to $\mathbb{R}^n$: otherwise, since for $x \in Q$, $M(\chi_Q)(x) = 1$, the inequality would be trivial. Modular Sawyer-type conditions for weighted norm inequalities in the variable Lebesgue spaces on the real line have been considered by Kokilashvili and Meskhi [199], but with the additional assumption that $p(\cdot) \in LH_0(\mathbb{R})$ and $p(\cdot)$ is constant outside of a large ball. It may be better to express this condition in terms of the norm instead of the modular:

$$\|M(\chi_Q)\|_{p(\cdot)} \le C\|\chi_Q\|_{p(\cdot)}.$$

Moreover, by Corollary 4.64, if $1 < p_- \le p_+ < \infty$, then $M$ is bounded on $L^{p(\cdot)}$ if and only if it is bounded on $L^{p'(\cdot)}$; hence, it may be necessary to also assume the "dual" inequality,

$$\|M(\chi_Q)\|_{p'(\cdot)} \le C\|\chi_Q\|_{p'(\cdot)}.$$

To determine if both conditions are necessary, the exponent constructed in Example 4.51 may be relevant. One obstacle to this approach is that Sawyer's proof relies on Marcinkiewicz interpolation.

Finally, though it would not yield necessary conditions, another approach to generalizing the $K_0$ condition might be to emulate the "$A_p$ bump" conditions used in the theory of two-weight norm inequalities to variable Lebesgue spaces. See [69] for details and further references. Related ideas were used in [57] to prove weighted norm inequalities on variable Lebesgue spaces, Theorem 4.77. (See also [82].)

**Problem A.8.** *Given $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $1 < p_- \le p_+ < \infty$ and the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, give a direct proof that the maximal operator is bounded on $L^{p'(\cdot)}(\Omega)$.*

This result is true: see Corollary 4.64. However, the only known proof is indirect, passing through the characterization of the boundedness of the maximal operator in Theorem 4.63. Even in the classical Lebesgue spaces, this is known only as a consequence of the fact that the maximal operator is bounded on every $L^p$ space, $1 < p < \infty$.

By duality, it is very easy to show that if $M$ is bounded on $L^{p(\cdot)}$, then it satisfies the weak type inequality on $L^{p'(\cdot)}$. By Theorem 2.34 and the Fefferman-Stein inequality [124], there exists $g \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|g\|_{p(\cdot)} \le 1$, such that

$$\begin{aligned}\|t\chi_{\{x:Mf(x)>t\}}\|_{p'(\cdot)} &\le 2k_{p'(\cdot)}^{-1}\int_{\mathbb{R}^n} t\chi_{\{x:Mf(x)>t\}}(x)g(x)\,dx \\ &\le C\int_{\mathbb{R}^n} |f(x)|Mg(x)\,dx \le C\|f\|_{p'(\cdot)}\|Mg\|_{p(\cdot)} \le C\|f\|_{p'(\cdot)}.\end{aligned}$$

There are several possible approaches to this problem. One is to try to improve this duality argument, perhaps using the characterization of boundedness in Theorem 4.37. One can show using the bounds in Theorem 3.4 that if $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then, in the notation of that result, $(s-1)\|M\|_{\mathcal{B}(L^{sp(\cdot)})(\mathbb{R}^n)}$ is bounded for $s > 1$. However, it not clear how to sharpen this argument to get that this quantity goes to 0 as $s \to 1$.

Another, more abstract approach would be to work with the generalization of Lorentz-Shimogaki indices for non-rearrangement invariant Banach function spaces developed by Lerner and Pérez [235].

A third approach would be to use a recent paper by Lerner [232]. As a consequence of his more general results in Banach function spaces, he shows that $M$ is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$ if and only the sharp maximal operator $M^\#$ (see Sect. 5.6.5) satisfies the Fefferman-Stein inequality

$$\|f\|_{p(\cdot)} \le C\|M^\# f\|_{p(\cdot)}.$$

This inequality is true, but the only known proofs use the fact that maximal operator is bounded on $L^{p'(\cdot)}$.

**Problem A.9.** *Characterize the sets $\Omega$ and the exponents $p(\cdot) \in \mathcal{P}(\Omega)$ such that if the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, there exists an extension of $p(\cdot)$ to all of $\mathbb{R}^n$ such that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

When $1/p(\cdot) \in LH(\Omega)$, then by Lemma 2.4 such an extension always exists. However, given that log-Hölder continuity is not a necessary condition, and boundedness of the maximal operator is an ubiquitous assumption, more general conditions are desirable.

A related problem would be to determine conditions such that if $p(\cdot) \in K_0(\Omega)$, then $p(\cdot)$ can be extended to an exponent function in $K_0(\mathbb{R}^n)$.

**Problem A.10.** *If the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, determine whether it is bounded on $L^{a+p(\cdot)}(\mathbb{R}^n)$ for all $a > 0$.*

This is just a restatement of Question 4.75, which is discussed in Sect. 4.6.4.

**Problem A.11.** *Determine whether Theorems 4.77 and 4.80 can be extended to include the case $p_+ = \infty$.*

**Problem A.12.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, determine whether $w \in A_{p(\cdot)}$ is a sufficient condition for the maximal operator to satisfy $\|wMf\|_{p(\cdot)} \le C\|wf\|_{p(\cdot)}$.*

**Problem A.13.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \le p_+ < \infty$ and $p(\cdot) \in LH(\mathbb{R}^n)$, and given $w \in A_{p(\cdot)}$, determine whether a singular integral (as defined in Sect. 5.5) satisfies*

$$\|(Tf)w\|_{p(\cdot)} \le C\|fw\|_{p(\cdot)}.$$

*Explore the same question for other classical operators.*

The above three problems are just a restatement of questions discussed in Sect. 4.6.5.

**Problem A.14.** *Find the sharp constant in Theorem 4.77 in terms of the* $[w]_{A_{p(\cdot)}}$ *constant of the weight* $w$.

In the theory of weighted norm inequalities, the problem of finding sharp constants in terms of the $[w]_{A_p}$ constant of a Muckenhoupt weight $w$ has been of great interest. (See [68] and the references it contains.) For the maximal operator, this problem was solved by Buckley [38] (see also Lerner [230]). It would be of interest to have similar sharp bounds for weighted estimates in the variable Lebesgue spaces. A first step to this problem would be to determine if the examples given by Buckley can be adapted to give useful lower bounds.

**Problem A.15.** *Let* $\{\phi_t\}$ *be a potential type approximate identity. Characterize the exponents* $p(\cdot)$ *with* $p_+ = \infty$ *such that* $\phi_t * f \to f$ *in modular and in measure.*

When $p_+ < \infty$, then by Theorem 5.9 we always have convergence in measure, and by Theorem 2.56 convergence in modular is equivalent to convergence in norm, which is characterized in Theorem 5.11. However, when $p_+ = \infty$, Example 5.10 shows that if $\Omega$ is unbounded, then $\phi_t * f$ may not converge in measure. Similarly, Example 5.15 shows that norm convergence does not hold when $p_+ = \infty$. The convergence results in [60] (see also Sect. 2.10.6) are relevant to this question.

**Problem A.16.** *Given an approximate identity* $\{\phi_t\}$*, characterize the exponents* $p(\cdot)$ *such that* $\rho_{p(\cdot)}(\phi_t * f) \to \rho_{p(\cdot)}(f)$.

We conjecture that if $\phi$ has compact support, it will suffice to assume that $p(\cdot) \in LH_0$. Weaker conditions may also hold.

**Problem A.17.** *If* $p(\cdot)$ *is such that all the Riesz transforms are bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$*, then determine whether the maximal operator is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorems 5.42 and 5.43 together imply that if all the Riesz transforms are bounded, then $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in K_0(\mathbb{R}^n)$, so the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$ for any bounded set $\Omega$. In the weighted Lebesgue spaces the analogous result is true, since if the Riesz transforms are bounded on $L^p(w)$, then $w \in A_p$. (See [140].)

**Problem A.18.** *If* $p(\cdot)$ *is unbounded, find* $BMO$ *type estimates for singular integrals on* $L^{p(\cdot)}$.

Given Theorem 5.42, we cannot have that singular integrals are bounded on $L^{p(\cdot)}$ if $p_+ = \infty$. This corresponds to the fact that singular integrals are not bounded on $L^\infty$. However, it is known that if $f \in L^\infty$, then $Tf \in BMO$, the space of functions of bounded mean oscillation. This raises the question of whether comparable results exist for $f \in L^{p(\cdot)}(\mathbb{R}^n)$, provided that $L^\infty(\mathbb{R}^n) \subset L^{p(\cdot)}(\mathbb{R}^n)$.

**Problem A.19.** *Given $p(\cdot) \in \mathcal{P}(\mathbb{R})$ such that $1 \le p_- \le p_+ \le 2$ and $p(\cdot)$ is even and non-decreasing on $(0, \infty)$, determine whether the Fourier transform satisfies the generalized Hausdorff-Young inequality $\|\hat{f}\|_{q(\cdot)} \le C\|f\|_{p(\cdot)}$, where $q(x) = p'(1/x)$.*

This is a restatement of Question 5.61 discussed in Sect. 5.6.10.

**Problem A.20.** *Characterize the sets $\Omega$ and exponents $p(\cdot)$ such that smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$.*

This and related questions were discussed in Sect. 6.5.5.

**Problem A.21.** *Given a bounded set $\Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, determine if it is sufficient to assume that $p(\cdot) \in K_0(\Omega)$ for the Poincaré inequalities Theorems 6.19 and 6.21 to hold in $W^{1,p(\cdot)}(\Omega)$.*

This is a restatement of a question discussed in Sect. 6.5.6.

**Problem A.22.** *Determine whether there exist $\Omega$ and a uniformly continuous exponent $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Sobolev embedding theorem (Theorem 6.29) does not hold for $W_0^{1,p(\cdot)}(\Omega)$. Determine whether the embedding theorem is true if $LH_0(\Omega)$ is replaced by a weaker decay condition.*

This is a restatement of Question 6.46 discussed in Sect. 6.5.7.

**Problem A.23.** *Determine whether the Gagliardo-Nirenberg inequalities on the variable Sobolev spaces can be extended to the case $p_- = 1$.*

The Gagliardo-Nirenberg inequalities (6.13) discussed in Sect. 6.5.10 assume that $p_- > 1$. The reason for this is that their proofs require the maximal operator to be bounded on $L^{p(\cdot)}$. Since in the classical case they are true if $p = 1$, it is reasonable to conjecture that they hold in the variable case when $p_- = 1$. There are two possible approaches to this problem: one is to extend the modular inequality due to Giannetti [141] so that it could be used to prove norm inequalities. The other is to use extrapolation; this would require the assumption that $p_+ < \infty$ and that the maximal operator is bounded on $L^{p'(\cdot)}$. It would also require weighted Gagliardo-Nirenberg inequalities: for results of this kind, see [345].

**Problem A.24.** *Determine whether Riesz-Thorin interpolation (i.e., complex interpolation) can be extended to the scale of the variable Sobolev spaces.*

Complex interpolation holds for the variable Lebesgue spaces: see Sect. 3.7.8. Moreover, this problem is true for the classical Sobolev spaces (see [25]); therefore, it is reasonable to conjecture that it is also true for the variable Sobolev spaces.

# Bibliography


1. R. Aboulaich, S. Boujena, and E. El Guarmah. Sur un modèle non-linéaire pour le débruitage de l'image. *C. R. Math. Acad. Sci. Paris*, 345(8):425–429, 2007.
2. R. Aboulaich, D. Meskine, and A. Souissi. New diffusion models in image processing. *Comput. Math. Appl.*, 56(4):874–882, 2008.
3. E. Acerbi and G. Mingione. Regularity results for electrorheological fluids: the stationary case. *C. R. Math. Acad. Sci. Paris*, 334(9):817–822, 2002.
4. E. Acerbi and G. Mingione. Regularity results for stationary electro-rheological fluids. *Arch. Ration. Mech. Anal.*, 164(3):213–259, 2002.
5. E. Acerbi and G. Mingione. Gradient estimates for the $p(x)$-Laplacean system. *J. Reine Angew. Math.*, 584:117–148, 2005.
6. D. R. Adams. A note on Riesz potentials. *Duke Math. J.*, 42(4):765–778, 1975.
7. R. A. Adams and J. J. F. Fournier. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
8. M. Aguilar Cañestro and P. Ortega Salvador. Weighted weak type inequalities with variable exponents for Hardy and maximal operators. *Proc. Japan Acad. Ser. A Math. Sci.*, 82(8):126–130, 2006.
9. M. Aguilar Cañestro and P. Ortega Salvador. Weak-type inequalities and convergence of the ergodic averages in variable Lebesgue spaces with weights. *Proc. Roy. Soc. Edinburgh Sect. A*, 139(4):673–683, 2009.
10. A. Almeida. Inversion of the Riesz potential operator on Lebesgue spaces with variable exponent. *Fract. Calc. Appl. Anal.*, 6(3):311–327, 2003.
11. A. Almeida and S. Samko. Characterization of Riesz and Bessel potentials on variable Lebesgue spaces. *J. Funct. Spaces Appl.*, 4(2):113–144, 2006.
12. A. Almeida and S. Samko. Pointwise inequalities in variable Sobolev spaces and applications. *Z. Anal. Anwend.*, 26(2):179–193, 2007.
13. A. Almeida and S. Samko. Embeddings of variable Hajłasz-Sobolev spaces into Hölder spaces of variable order. *J. Math. Anal. Appl.*, 353(2):489–496, 2009.
14. J. Alvarez and C. Pérez. Estimates with $A_\infty$ weights for various singular integral operators. *Boll. Un. Mat. Ital. A (7)*, 8(1):123–133, 1994.
15. B. Amaziane, L. Pankratov, and A. Piatnitski. Nonlinear flow through double porosity media in variable exponent Sobolev spaces. *Nonlinear Anal. Real World Appl.*, 10(4):2521–2530, 2009.
16. S. N. Antontsev and S. I. Shmarev. A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions. *Nonlinear Anal.*, 60(3):515–545, 2005.


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17. M. Asif, V. Kokilashvili, and A. Meskhi. Boundedness criteria for maximal functions and potentials on the half-space in weighted Lebesgue spaces with variable exponents. *Integral Transforms Spec. Funct.*, 20(11–12):805–819, 2009.
18. M. Asif and A. Meskhi. On the essential norm for the Hilbert transforms in $L^{p(x)}$ spaces. *Georgian Math. J.*, 15(2):209–223, 2008.
19. M. Asif and A. Meskhi. Weighted estimates of a measure of noncompactness for maximal and potential operators. *J. Inequal. Appl.*, pages Art. ID 697407, 19, 2008.
20. R. A. Bandaliev. The boundedness of certain sublinear operator in the weighted variable Lebesgue spaces. *Czechoslovak Math. J.*, 60(135)(2):327–337, 2010.
21. R. A. Bandaliev. The boundedness of multidimensional Hardy operators in weighted variable Lebesgue spaces. *Lith. Math. J.*, 50(3):249–259, 2010.
22. R. A. Bandaliev. Embedding between variable exponent Lebesgue spaces with measures. *Azerbaijan J. Math.*, 2(1):119–125, 2012.
23. R. A. Bandaliev and M. M. Abbasova. On an inequality and $p(x)$-mean continuity in the variable Lebesgue space with mixed norm. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.*, 26(7, Math. Mech.):47–56, 2006.
24. J. J. Benedetto, H. P. Heinig, and R. Johnson. Fourier inequalities with $A_p$-weights. In *General inequalities, 5 (Oberwolfach, 1986)*, volume 80 of *Internat. Schriftenreihe Numer. Math.*, pages 217–232. Birkhäuser, Basel, 1987.
25. C. Bennett and R. Sharpley. *Interpolation of Operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.
26. E. I. Berezhnoĭ. Sharp estimates for operators on cones in ideal spaces. *Trudy Mat. Inst. Steklov.*, 204(Issled. po Teor. Differ. Funktsii Mnogikh Peremen. i ee Prilozh. 16):3–34, 1993.
27. E. I. Berezhnoĭ. Two-weighted estimations for the Hardy-Littlewood maximal function in ideal Banach spaces. *Proc. Amer. Math. Soc.*, 127(1):79–87, 1999.
28. A. L. Bernardis, A. Gogatishvili, F. J. Martín-Reyes, P. Ortega Salvador, and L. Pick. The one-sided $A_p$ conditions and local maximal operator. *Proc. Edinb. Math. Soc. (2)*, 55(1):79–104, 2012.
29. Z. W. Birnbaum and W. Orlicz. Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen. *Stud. Math.*, 3:1–67, 1931.
30. P. Blomgren, T. Chan, P. Mulet, and C. K. Wong. Total variation image restoration: numerical methods and extensions. In *Proceedings of the 1997 IEEE International Conference on Image Processing*, volume III, pages 384–387, 1997.
31. M. Bocea, M. Mihăilescu, and C. Popovici. On the asymptotic behavior of variable exponent power-law functionals and applications. *Ric. Mat.*, 59(2):207–238, 2010.
32. E. M. Bollt, R. Chartrand, S. Esedoḡlu, P. Schultz, and K. R. Vixie. Graduated adaptive image denoising: local compromise between total variation and isotropic diffusion. *Adv. Comput. Math.*, 31(1–3):61–85, 2009.
33. M.-M. Boureanu. Existence of solutions for an elliptic equation involving the $p(x)$-Laplace operator. *Electron. J. Differential Equations*, pages No. 97, 10 pp. (electronic), 2006.
34. J. Bourgain, H. Brezis, and P. Mironescu. Another look at Sobolev spaces. Menaldi, José Luis (ed.) *et al.*, Optimal control and partial differential equations. In honour of Professor Alain Bensoussan's 60th birthday. Proceedings of the conference, Paris, France, December 4, 2000. Amsterdam: IOS Press; Tokyo: Ohmsha. 439–455 (2001)., 2001.
35. S. Boza and J. Soria. Weighted Hardy modular inequalities in variable $L^p$ spaces for decreasing functions. *J. Math. Anal. Appl.*, 348(1):383–388, 2008.
36. S. Boza and J. Soria. Weighted weak modular and norm inequalities for the Hardy operator in variable $L^p$ spaces of monotone functions. *Rev. Mat. Complut.*, 25(2):459–474, 2012.
37. H. Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maî trise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].
38. S. M. Buckley. Estimates for operator norms on weighted spaces and reverse Jensen inequalities. *Trans. Amer. Math. Soc.*, 340(1):253–272, 1993.

39. A. P. Calderón. Lebesgue spaces of differentiable functions and distributions. In *Proc. Sympos. Pure Math., Vol. IV*, pages 33–49. American Mathematical Society, Providence, R.I., 1961.
40. A. P. Calderón and A. Zygmund. On the existence of certain singular integrals. *Acta Math.*, 88:85–139, 1952.
41. A. P. Calderón and A. Zygmund. On singular integrals. *Amer. J. Math.*, 78:289–309, 1956.
42. C. Capone, D. Cruz-Uribe, and A. Fiorenza. The fractional maximal operator and fractional integrals on variable $L^p$ spaces. *Rev. Mat. Iberoam.*, 23(3):743–770, 2007.
43. B. Çekiç, A. V. Kalinin, R. A. Mashiyev, and M. Avci. $l^{p(x)}(\omega)$-estimates for vector fields and some applications to magnetostatics problems. *J. Math. Anal. Appl.*, 389(2):838–851, 2012.
44. B. Çekiç, R. A. Mashiyev, and G. T. Alisoy. On the Sobolev-type inequality for Lebesgue spaces with a variable exponent. *Int. Math. Forum*, 1(25–28):1313–1323, 2006.
45. Y. Chen, W. Guo, Q. Zeng, and Y. Liu. A nonstandard smoothing in reconstruction of apparent diffusion coefficient profiles from diffusion weighted images. *Inverse Probl. Imaging*, 2(2):205–224, 2008.
46. Y. Chen, S. Levine, and M. Rao. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.*, 66(4):1383–1406 (electronic), 2006.
47. G. Chiti. Rearrangements of functions and convergence in Orlicz spaces. *Applicable Anal.*, 9(1):23–27, 1979.
48. M. Christ and R. Fefferman. A note on weighted norm inequalities for the Hardy-Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 87(3):447–448, 1983.
49. P. Cianci and F. Nicolosi. A weighted interpolation inequality with variable exponent of the Nirenberg-Gagliardo kind. *Nonlinear Anal.*, 71(12):5915–5929, 2009.
50. P. G. Ciarlet and G. Dinca. A Poincaré inequality in a Sobolev space with a variable exponent. *Chin. Ann. Math. Ser. B*, 32(3):333–342, 2011.
51. J. B. Conway. *A Course in Functional Analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
52. D. Cruz-Uribe. The class $A^+_\infty(g)$ and the one-sided reverse Hölder inequality. *Canad. Math. Bull.*, 40(2):169–173, 1997.
53. D. Cruz-Uribe. New proofs of two-weight norm inequalities for the maximal operator. *Georgian Math. J.*, 7(1):33–42, 2000.
54. D. Cruz-Uribe. The Hardy-Littlewood maximal operator on variable-$L^p$ spaces. In *Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003)*, volume 64 of *Colecc. Abierta*, pages 147–156. Univ. Sevilla Secr. Publ., Seville, 2003.
55. D. Cruz-Uribe. Interpolation of positive operators on variable Lebesgue spaces. *Math. Ineq. Appl.*, 15(3):639–644, 2012.
56. D. Cruz-Uribe, L. Diening, and A. Fiorenza. A new proof of the boundedness of maximal operators on variable Lebesgue spaces. *Boll. Unione Mat. Ital. (9)*, 2(1):151–173, 2009.
57. D. Cruz-Uribe, L. Diening, and P. Hästö. The maximal operator on weighted variable Lebesgue spaces. *Frac. Calc. Appl. Anal.*, 14(3):361–374, 2011.
58. D. Cruz-Uribe and A. Fiorenza. Approximate identities in variable $L^p$ spaces. *Math. Nachr.*, 280(3):256–270, 2007.
59. D. Cruz-Uribe and A. Fiorenza. $L \log L$ results for the maximal operator in variable $L^p$ spaces. *Trans. Amer. Math. Soc.*, 361(5):2631–2647, 2009.
60. D. Cruz-Uribe and A. Fiorenza. Convergence in variable Lebesgue spaces. *Publ. Mat.*, 54(2):441–459, 2010.
61. D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Pérez. The boundedness of classical operators on variable $L^p$ spaces. *Ann. Acad. Sci. Fenn. Math.*, 31(1):239–264, 2006.
62. D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. The maximal function on variable $L^p$ spaces. *Ann. Acad. Sci. Fenn. Math.*, 28(1):223–238, 2003. See also errata [63].
63. D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. Corrections to: "The maximal function on variable $L^p$ spaces" [Ann. Acad. Sci. Fenn. Math. **28** (2003), no. 1, 223–238; ]. *Ann. Acad. Sci. Fenn. Math.*, 29(1):247–249, 2004.

64. D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. Weighted norm inequalities for the maximal operator on variable Lebesgue spaces. *J. Math. Anal. Appl.*, 394(2):744–760, 2012.
65. D. Cruz-Uribe and F. Mamedov. On a general weighted Hardy type inequality in the variable exponent Lebesgue spaces. *Rev. Mat. Complut.*, 25(2):335–367, 2012.
66. D. Cruz-Uribe, J. M. Martell, and C. Pérez. Extrapolation from $A_\infty$ weights and applications. *J. Funct. Anal.*, 213(2):412–439, 2004.
67. D. Cruz-Uribe, J. M. Martell, and C. Pérez. Sharp two-weight inequalities for singular integrals, with applications to the Hilbert transform and the Sarason conjecture. *Adv. Math.*, 216(2):647–676, 2007.
68. D. Cruz-Uribe, J. M. Martell, and C. Pérez. Sharp weighted estimates for classical operators. *Adv. Math.*, 229:408–441, 2011.
69. D. Cruz-Uribe, J. M. Martell, and C. Pérez. *Weights, Extrapolation and the Theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.
70. D. Cruz-Uribe and C. J. Neugebauer. The structure of the reverse Hölder classes. *Trans. Amer. Math. Soc.*, 347(8):2941–2960, 1995.
71. D. Cruz-Uribe and C. J. Neugebauer. Weighted norm inequalities for the centered maximal operator on $\mathbf{R}^+$. *Ricerche Mat.*, 48(2):225–241 (2000), 1999.
72. D. Cruz-Uribe and C. Pérez. Two weight extrapolation via the maximal operator. *J. Funct. Anal.*, 174(1):1–17, 2000.
73. V. De Cicco, C. Leone, and A. Verde. Lower semicontinuity in SBV for integrals with variable growth. *SIAM J. Math. Anal.*, 42(6):3112–3128, 2010.
74. M. de Guzmán. *Differentiation of Integrals in $R^n$*. Lecture Notes in Mathematics, Vol. 481. Springer-Verlag, Berlin, 1975. With appendices by Antonio Córdoba, and Robert Fefferman, and two by Roberto Moriyón.
75. A. de la Torre. On the adjoint of the maximal function. In *Function spaces, differential operators and nonlinear analysis (Paseky nad Jizerou, 1995)*, pages 189–194. Prometheus, Prague, 1996.
76. F. Demengel and G. Demengel. *Espaces fonctionnels*. Savoirs Actuels (Les Ulis). [Current Scholarship (Les Ulis)]. EDP Sciences, Les Ulis, 2007. Utilisation dans la résolution des équations aux dérivées partielles. [Application to the solution of partial differential equations].
77. L. Diening. Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.*, 7(2):245–253, 2004.
78. L. Diening. Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. *Math. Nachr.*, 268:31–43, 2004.
79. L. Diening. Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. *Bull. Sci. Math.*, 129(8):657–700, 2005.
80. L. Diening. *Lebesgue and Sobolev Spaces with Variable Exponent*. Habilitation, Universität Freiburg, 2007.
81. L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta, and T. Shimomura. Maximal functions in variable exponent spaces: limiting cases of the exponent. *Ann. Acad. Sci. Fenn. Math.*, 34(2):503–522, 2009.
82. L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with Variable Exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.
83. L. Diening and P. Hästö. Variable exponent trace spaces. *Studia Math.*, 183(2):127–141, 2007.
84. L. Diening and P. Hästö. Further results on variable exponent trace spaces. In H. G. W. Begehr and F. Nicolosi, editors, *More progresses in analysis: Proceedings of the 5th International ISAAC Congress held at the University of Catania, Catania, July 25–30, 2005*, pages 101–106. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009.
85. L. Diening and P. Hästö. Muckenhoupt weights in variable exponent spaces. *Preprint*, 2010.

86. L. Diening, P. Hästö, and A. Nekvinda. Open problems in variable exponent Lebesgue and Sobolev spaces. In *FSDONA04 Proceedings (Drabek and Rakosnik (eds.); Milovy, Czech Republic*, pages 38–58. Academy of Sciences of the Czech Republic, Prague, 2005.
87. L. Diening, P. Hästö, and S. Roudenko. Function spaces of variable smoothness and integrability. *J. Funct. Anal.*, 256(6):1731–1768, 2009.
88. L. Diening and M. Růžička. Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. *J. Reine Angew. Math.*, 563:197–220, 2003.
89. L. Diening and M. Růžička. Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$. I. *J. Math. Anal. Appl.*, 298(2):559–571, 2004.
90. L. Diening and M. Růžička. Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$. II. *J. Math. Anal. Appl.*, 298(2):572–588, 2004.
91. L. Diening and M. Růžička. Non-Newtonian fluids and function spaces. In *NAFSA 8—Nonlinear analysis, function spaces and applications. Vol. 8*, pages 94–143. Czech. Acad. Sci., Prague, 2007.
92. L. Diening and S. Samko. Hardy inequality in variable exponent Lebesgue spaces. *Fract. Calc. Appl. Anal.*, 10(1):1–18, 2007.
93. G. Dinca and P. Matei. Geometry of Sobolev spaces with variable exponent and a generalization of the $p$-Laplacian. *Anal. Appl. (Singap.)*, 7(4):373–390, 2009.
94. G. Dinca and P. Matei. Geometry of Sobolev spaces with variable exponent: smoothness and uniform convexity. *C. R. Math. Acad. Sci. Paris*, 347(15–16):885–889, 2009.
95. N. Dunford and J. T. Schwartz. *Linear Operators. Part I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. General theory, With the assistance of William G. Bade and R. G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.
96. J. Duoandikoetxea. *Fourier Analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
97. D. E. Edmunds, A. Fiorenza, and A. Meskhi. On a measure of non-compactness for some classical operators. *Acta Math. Sin. (Engl. Ser.)*, 22(6):1847–1862, 2006.
98. D. E. Edmunds, V. Kokilashvili, and A. Meskhi. On the boundedness and compactness of weighted Hardy operators in spaces $L^{p(x)}$. *Georgian Math. J.*, 12(1):27–44, 2005.
99. D. E. Edmunds, V. Kokilashvili, and A. Meskhi. One-sided operators in $L^{p(x)}$ spaces. *Math. Nachr.*, 281(11):1525–1548, 2008.
100. D. E. Edmunds, V. Kokilashvili, and A. Meskhi. Two-weight estimates in $L^{p(x)}$ spaces with applications to Fourier series. *Houston J. Math.*, 35(2):665–689, 2009.
101. D. E. Edmunds, J. Lang, and A. Nekvinda. On $L^{p(x)}$ norms. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 455(1981):219–225, 1999.
102. D. E. Edmunds, J. Lang, and A. Nekvinda. Some $s$-numbers of an integral operator of Hardy type on $L^{p(\cdot)}$ spaces. *J. Funct. Anal.*, 257(1):219–242, 2009.
103. D. E. Edmunds and A. Meskhi. Potential-type operators in $L^{p(x)}$ spaces. *Z. Anal. Anwendungen*, 21(3):681–690, 2002.
104. D. E. Edmunds and A. Nekvinda. Averaging operators on $l^{\{p_n\}}$ and $L^{p(x)}$. *Math. Inequal. Appl.*, 5(2):235–246, 2002.
105. D. E. Edmunds and J. Rákosník. Density of smooth functions in $W^{k,p(x)}(\Omega)$. *Proc. Roy. Soc. London Ser. A*, 437(1899):229–236, 1992.
106. D. E. Edmunds and J. Rákosník. Sobolev embeddings with variable exponent. *Studia Math.*, 143(3):267–293, 2000.
107. D. E. Edmunds and J. Rákosník. Sobolev embeddings with variable exponent. II. *Math. Nachr.*, 246/247:53–67, 2002.
108. L. Ephremidze, V. Kokilashvili, and S. Samko. Fractional, maximal and singular operators in variable exponent Lorentz spaces. *Fract. Calc. Appl. Anal.*, 11(4):407–420, 2008.
109. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Tables of Integral Transforms. Vol. I.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. Based, in part, on notes left by Harry Bateman.
110. L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

111. X. Fan. The regularity of Lagrangians $f(x,\xi) = |\xi|^{\alpha(x)}$ with Hölder exponents $\alpha(x)$. *Acta Math. Sinica (N.S.)*, 12(3):254–261, 1996. A Chinese summary appears in Acta Math. Sinica **40** (1997), no. 1, 158.
112. X. Fan. Regularity of nonstandard Lagrangians $f(x,\xi)$. *Nonlinear Anal.*, 27(6):669–678, 1996.
113. X. Fan. $p(x)$-Laplacian equations. In *Topological methods, variational methods and their applications (Taiyuan, 2002)*, pages 117–123. World Sci. Publ., River Edge, NJ, 2003.
114. X. Fan. Amemiya norm equals Orlicz norm in Musielak-Orlicz spaces. *Acta Math. Sin. (Engl. Ser.)*, 23(2):281–288, 2007.
115. X. Fan. Boundary trace embedding theorems for variable exponent Sobolev spaces. *J. Math. Anal. Appl.*, 339(2):1395–1412, 2008.
116. X. Fan. Sobolev embeddings for unbounded domain with variable exponent having values across $N$. *Math. Inequal. Appl.*, 13(1):123–134, 2010.
117. X. Fan and X. Han. Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbf{R}^N$. *Nonlinear Anal.*, 59(1–2):173–188, 2004.
118. X. Fan, J. Shen, and D. Zhao. Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 262(2):749–760, 2001.
119. X. Fan, S. Wang, and D. Zhao. Density of $C^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with discontinuous exponent $p(x)$. *Math. Nachr.*, 279(1–2):142–149, 2006.
120. X. Fan and D. Zhao. Regularity of minimizers of variational integrals with continuous $p(x)$-growth conditions. *Chinese J. Contemp. Math.*, 17(4):327–336, 1996.
121. X. Fan and D. Zhao. Regularity of minimum points of variational integrals with continuous $p(x)$-growth conditions. *Chinese Ann. Math. Ser. A*, 17(5):557–564, 1996.
122. X. Fan and D. Zhao. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 263(2):424–446, 2001.
123. X. Fan, Y. Zhao, and D. Zhao. Compact imbedding theorems with symmetry of Strauss-Lions type for the space $W^{1,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 255(1):333–348, 2001.
124. C. Fefferman and E. M. Stein. Some maximal inequalities. *Amer. J. Math.*, 93:107–115, 1971.
125. C. Fefferman and E. M. Stein. $H^p$ spaces of several variables. *Acta Math.*, 129(3–4):137–193, 1972.
126. A. Fiorenza. An inequality for Jensen means. *Nonlinear Anal.*, 16(2):191–198, 1991.
127. A. Fiorenza. A mean continuity type result for certain Sobolev spaces with variable exponent. *Commun. Contemp. Math.*, 4(3):587–605, 2002.
128. A. Fiorenza. The maximal operator and the Sobolev theorem on variable exponent spaces. In *Proceedings of the Renato Caccioppoli Centenary Conference, Napoli, September 23–25, 2004*, volume 54(2) of *Richerche Mat.*, pages 571–578, 2005.
129. A. Fiorenza and M. Krbec. On the domain and range of the maximal operator. *Nagoya Math. J.*, 158:43–61, 2000.
130. A. Fiorenza and M. Krbec. A note on noneffective weights in variable Lebesgue spaces. *J. Funct. Spaces Appl.*, pages Art. ID 853232, 5, 2012.
131. A. Fiorenza and J. M. Rakotoson. Relative rearrangement and lebesgue spaces $L^{p(\cdot)}$ with variable exponent. *J. Math. Pures Appl. (9)*, 88(6):506–521, 2007.
132. B. Franchi, C. Pérez, and R. L. Wheeden. Sharp geometric Poincaré inequalities for vector fields and non-doubling measures. *Proc. London Math. Soc. (3)*, 80(3):665–689, 2000.
133. S. Fröschl. Fortsetzungen in Lebesgue und Sobolevräumen mit variablen Exponenten. Master's thesis, University of Freiburg, Germany, 2007.
134. N. Fusco and C. Sbordone. Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions. *Comm. Pure Appl. Math.*, 43(5):673–683, 1990.
135. T. Futamura and Y. Mizuta. Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent. *Math. Inequal. Appl.*, 8(4):619–631, 2005.
136. T. Futamura and Y. Mizuta. Maximal functions for Lebesgue spaces with variable exponent approaching 1. *Hiroshima Math. J.*, 36(1):23–28, 2006.
137. T. Futamura, Y. Mizuta, and T. Shimomura. Sobolev embeddings for Riesz potential space of variable exponent. *Math. Nachr.*, 279(13–14):1463–1473, 2006.

138. T. Futamura, Y. Mizuta, and T. Shimomura. Integrability of maximal functions and Riesz potentials in Orlicz spaces of variable exponent. *J. Math. Anal. Appl.*, 366(2):391–417, 2010.
139. J. Gao, P. Zhao, and Y. Zhang. Compact Sobolev embedding theorems involving symmetry and its application. *NoDEA Nonlinear Differential Equations Appl.*, 17(2):161–180, 2010.
140. J. García-Cuerva and J. L. Rubio de Francia. *Weighted Norm Inequalities and Related Topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985.
141. F. Giannetti. The modular interpolation inequality in Sobolev spaces with variable exponent attaining the value 1. *Math. Inequal. Appl.*, to appear, 2011.
142. D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
143. L. Grafakos. *Classical Fourier Analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition, 2008.
144. L. Grafakos. *Modern Fourier Analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition, 2008.
145. L. Greco, T. Iwaniec, and G. Moscariello. Limits of the improved integrability of the volume forms. *Indiana Univ. Math. J.*, 44(2):305–339, 1995.
146. P. Hajłasz and J. Onninen. On boundedness of maximal functions in Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.*, 29(1):167–176, 2004.
147. T. C. Halsey. Electrorheological fluids. *Science, New Series*, 258(5083):761–766, Oct. 30 1992.
148. G. H. Hardy and J. E. Littlewood. A maximal theorem with function-theoretic applications. *Acta Math.*, 54(1):81–116, 1930.
149. P. Harjulehto. Variable exponent Sobolev spaces with zero boundary values. *Math. Bohem.*, 132(2):125–136, 2007.
150. P. Harjulehto and P. Hästö. An overview of variable exponent Lebesgue and Sobolev spaces. In *Future trends in geometric function theory*, volume 92 of *Rep. Univ. Jyväskylä Dep. Math. Stat.*, pages 85–93. Univ. Jyväskylä, Jyväskylä, 2003.
151. P. Harjulehto and P. Hästö. A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces. *Rev. Mat. Complut.*, 17(1):129–146, 2004.
152. P. Harjulehto and P. Hästö. Lebesgue points in variable exponent spaces. *Ann. Acad. Sci. Fenn. Math.*, 29(2):295–306, 2004.
153. P. Harjulehto and P. Hästö. Sobolev inequalities with variable exponent attaining the values 1 and $n$. *Publ. Mat.*, 52(2):347–363, 2008.
154. P. Harjulehto, P. Hästö, and M. Koskenoja. Hardy's inequality in a variable exponent Sobolev space. *Georgian Math. J.*, 12(3):431–442, 2005.
155. P. Harjulehto, P. Hästö, M. Koskenoja, and S. Varonen. Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$. *J. Funct. Spaces Appl.*, 1(1):17–33, 2003.
156. P. Harjulehto, P. Hästö, V. Latvala, and O. Toivanen. Critical variable exponent functionals in image restoration. *Appl. Math. Let.*, to appear.
157. P. Harjulehto, P. Hästö, Ú. V. Lê, and M. Nuortio. Overview of differential equations with non-standard growth. *Nonlinear Anal.*, 72(12):4551–4574, 2010.
158. P. Harjulehto, P. Hästö, Y. Mizuta, and T. Shimomura. Iterated maximal functions in variable exponent Lebesgue spaces. *Manuscripta Math.*, 135(3–4):381–399, 2011.
159. A. Harman. On necessary condition for the variable exponent Hardy inequality. *J. Funct. Spaces Appl.*, 2012, 2012. Article ID 385925.
160. A. Harman and F. Mamedov. On boundedness of weighted Hardy operator in $L^{p(\cdot)}$ and regularity condition. *J. Inequal. Appl.*, pages Art. ID 837951, 14, 2010.
161. D. D. Haroske and H. Triebel. *Distributions, Sobolev Spaces, Elliptic Equations*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.
162. P. Hästö. Counter-examples of regularity in variable exponent Sobolev spaces. In *The p-harmonic equation and recent advances in analysis*, volume 370 of *Contemp. Math.*, pages 133–143. Amer. Math. Soc., Providence, RI, 2005.

163. P. Hästö. The maximal operator in Lebesgue spaces with variable exponent near 1. *Math. Nachr.*, 280(1–2):74–82, 2007.
164. P. Hästö. On the density of continuous functions in variable exponent Sobolev space. *Rev. Mat. Iberoam.*, 23(1):213–234, 2007.
165. P. Hästö. Local-to-global results in variable exponent spaces. *Math. Res. Letters*, 16(2):263–278, 2009.
166. P Hästö, Y Mizuta, T Ohno, and T Shimomura. Sobolev inequalities for Orlicz spaces of two variable exponents. *Glasgow Math. J.*, pages 1–14, 2009.
167. P. Hästö, Y. Mizuta, T. Ohno, and T. Shimomura. Sobolev inequalities for Orlicz spaces of two variable exponents. *Glasg. Math. J.*, 52(2):227–240, 2010.
168. L. I. Hedberg. On certain convolution inequalities. *Proc. Amer. Math. Soc.*, 36:505–510, 1972.
169. E. Hewitt and K. Stromberg. *Real and Abstract Analysis*. Springer-Verlag, New York, 1975. A modern treatment of the theory of functions of a real variable, Third printing, Graduate Texts in Mathematics, No. 25.
170. A.-w. Huang and J.-s. Xu. Multilinear singular integrals and commutators in variable exponent Lebesgue spaces. *Appl. Math. J. Chinese Univ. Ser. B*, 25(1):69–77, 2010.
171. H. Hudzik. A generalization of Sobolev spaces. I. *Funct. Approximatio Comment. Math.*, 2:67–73, 1976.
172. H. Hudzik. A generalization of Sobolev spaces. II. *Funct. Approximatio Comment. Math.*, 3:77–85, 1976.
173. H. Hudzik. On generalized Orlicz-Sobolev space. *Funct. Approximatio Comment. Math.*, 4:37–51, 1976.
174. H. Hudzik. On density of $C^\infty(\Omega)$ in Orlicz-Sobolev space $W^k_M(\Omega)$ for every open set $\Omega \subset \mathbf{R}^n$. *Funct. Approximatio Comment. Math.*, 5:113–128, 1977.
175. H. Hudzik. On imbedding theorems of Orlicz-Sobolev space $W^k_M(\Omega)$ into $C^m(\Omega)$ for open, bounded, and starlike $\Omega \subset \mathbf{R}^n$. *Comment. Math. Prace Mat.*, 20(2):341–363, 1977/78.
176. H. Hudzik. On problem of density of $C^\infty_0(\Omega)$ in generalized Orlicz-Sobolev space $W^k_M(\Omega)$ for every open set $\Omega \subset \mathbf{R}^n$. *Comment. Math. Prace Mat.*, 20(1):65–78, 1977/78.
177. H. Hudzik. On continuity of the imbedding operation from $W^k_{M_1}(\Omega)$ into $W^k_{M_2}(\Omega)$. *Funct. Approx. Comment. Math.*, 6:111–118, 1978.
178. H. Hudzik. Density of $C^\infty_0(\mathbf{R}^n)$ in generalized Orlicz-Sobolev space $W^k_M(\mathbf{R}^n)$. *Funct. Approx. Comment. Math.*, 7:15–21, 1979.
179. H. Hudzik. The problems of separability, duality, reflexivity and of comparison for generalized Orlicz-Sobolev spaces $W^k_M(\Omega)$. *Comment. Math. Prace Mat.*, 21(2):315–324, 1980.
180. H. Hudzik and L. Maligranda. Amemiya norm equals Orlicz norm in general. *Indag. Math. (N.S.)*, 11(4):573–585, 2000.
181. J. E. Hutchinson and M. Meier. A remark on the nonuniqueness of tangent cones. *Proc. Amer. Math. Soc.*, 97(1):184–185, 1986.
182. J. Ishii. On equivalence of modular function spaces. *Proc. Japan Acad.*, 35:551–556, 1959.
183. M. Izuki. Wavelets and modular inequalities in variable $L^p$ spaces. *Georgian Math. J.*, 15(2):281–293, 2008.
184. S. Janson. On functions with conditions on the mean oscillation. *Ark. Mat.*, 14(2):189–196, 1976.
185. B. Jawerth. Weighted inequalities for maximal operators: linearization, localization and factorization. *Amer. J. Math.*, 108(2):361–414, 1986.
186. F. John and L. Nirenberg. On functions of bounded mean oscillation. *Comm. Pure Appl. Math.*, 14:415–426, 1961.
187. G. A. Kalyabin. A nondensity criterion for $L^\infty(\mathbb{R}^n)$ in $L^{p(\cdot)}(\mathbb{R}^n)$. *Mat. Zametki*, 82(2):315–316, 2007.
188. E. Kapanadze and T. Kopaliani. A note on maximal operator on $L^{p(t)}(\Omega)$ spaces. *Georgian Math. J.*, 15(2):307–316, 2008.

189. N. K. Karapetyants and A. I. Ginzburg. Fractional integrals and singular integrals in the Hölder classes of variable order. *Integral Transform. Spec. Funct.*, 2(2):91–106, 1994.
190. N. K. Karapetyants and A. I. Ginzburg. Fractional integrodifferentiation in Hölder classes of arbitrary order. *Georgian Math. J.*, 2(2):141–150, 1995.
191. A. Yu. Karlovich. Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces. *J. Integral Equations Appl.*, 15(3):263–320, 2003.
192. A. Yu. Karlovich. Remark on the boundedness of the Cauchy singular integral operator on variable Lebesgue spaces with radial oscillating weights. *J. Funct. Spaces Appl.*, 7(3):301–311, 2009.
193. A. Yu. Karlovich. Maximal operators on variable Lebesgue spaces with weights related to oscillations of Carleson curves. *Math. Nachr.*, 283(1):85–93, 2010.
194. A. Yu. Karlovich. Singular integral operators on variable Lebesgue spaces with radial oscillating weights. In *Operator algebras, operator theory and applications*, volume 195 of *Oper. Theory Adv. Appl.*, pages 185–212. Birkhäuser Verlag, Basel, 2010.
195. A. Yu. Karlovich and A. K. Lerner. Commutators of singular integrals on generalized $L^p$ spaces with variable exponent. *Publ. Mat.*, 49(1):111–125, 2005.
196. A. Yu. Karlovich and I. M. Spitkovsky. The Cauchy singular integral operator on weighted variable Lebesgue spaces. *preprint*, 2012.
197. V. Kokilashvili and M. Krbec. *Weighted Inequalities in Lorentz and Orlicz Spaces.* World Scientific Publishing Co. Inc., River Edge, NJ, 1991.
198. V. Kokilashvili and A. Meskhi. Weighted criteria for generalized fractional maximal functions and potentials in Lebesgue spaces with variable exponent. *Integral Transforms Spec. Funct.*, 18(9–10):609–628, 2007.
199. V. Kokilashvili and A. Meskhi. Two-weighted inequalities for Hardy-Littlewood functions and singular integrals in $L^{p(\cdot)}$ spaces. *J. Math. Sci.*, 173(6):656–673, 2011.
200. V. Kokilashvili and A. Meskhi. Two-weighted norm inequalities for the double Hardy transforms and strong fractional maximal functions in variable exponent Lebesgue space. In *Spectral theory, function spaces and inequalities*, volume 219 of *Oper. Theory Adv. Appl.*, pages 105–124. Birkhäuser/Springer Basel AG, Basel, 2012.
201. V. Kokilashvili, N. Samko, and S. Samko. The maximal operator in variable spaces $L^{p(\cdot)}(\Omega,\rho)$ with oscillating weights. *Georgian Math. J.*, 13(1):109–125, 2006.
202. V. Kokilashvili, N. Samko, and S. Samko. The maximal operator in weighted variable spaces $L^{p(\cdot)}$. *J. Funct. Spaces Appl.*, 5(3):299–317, 2007.
203. V. Kokilashvili, N. Samko, and S. Samko. Singular operators in variable spaces $L^{p(\cdot)}(\Omega,\rho)$ with oscillating weights. *Math. Nachr.*, 280(9–10):1145–1156, 2007.
204. V. Kokilashvili and S. Samko. Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Proc. A. Razmadze Math. Inst.*, 129:145–149, 2002.
205. V. Kokilashvili and S. Samko. On Sobolev theorem for Riesz-type potentials in Lebesgue spaces with variable exponent. *Z. Anal. Anwendungen*, 22(4):899–910, 2003.
206. V. Kokilashvili and S. Samko. Singular integrals and potentials in some Banach function spaces with variable exponent. *J. Funct. Spaces Appl.*, 1(1):45–59, 2003.
207. V. Kokilashvili and S. Samko. Singular integrals in weighted Lebesgue spaces with variable exponent. *Georgian Math. J.*, 10(1):145–156, 2003.
208. V. Kokilashvili and S. Samko. Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Rev. Mat. Iberoamericana*, 20(2):493–515, 2004.
209. V. Kokilashvili and S. Samko. Operators of harmonic analysis in weighted spaces with non-standard growth. *J. Math. Anal. Appl.*, 352(1):15–34, 2009.
210. A. N. Kolmogorov. Zur Normierbarkeit eines allgemeinen topologischen linearen Räumes. *Studia Math.*, 5:29–33, 1934.
211. T. Kopaliani. On some structural properties of Banach function spaces and boundedness of certain integral operators. *Czechoslovak Math. J.*, 54(129)(3):791–805, 2004.
212. T. Kopaliani. Infimal convolution and Muckenhoupt $A_{p(\cdot)}$ condition in variable $L^p$ spaces. *Arch. Math. (Basel)*, 89(2):185–192, 2007.

213. T. Kopaliani. A note on strong maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces. *Proc. A. Razmadze Math. Inst.*, 145:43–46, 2007.
214. T. Kopaliani. Greediness of the wavelet system in $L^{p(t)}(\mathbb{R})$ spaces. *East J. Approx.*, 14(1):59–67, 2008.
215. T. Kopaliani. On the Muckenchaupt condition in variable Lebesgue spaces. *Proc. A. Razmadze Math. Inst.*, 148:29–33, 2008.
216. T. Kopaliani. Interpolation theorems for variable exponent Lebesgue spaces. *J. Funct. Anal.*, 257(11):3541–3551, 2009.
217. T. Kopaliani. A characterization of some weighted norm inequalities for maximal operators. *Z. Anal. Anwend.*, 29(4):401–412, 2010.
218. T. Kopaliani and G. Chelidze. Gagliardo-Nirenberg type inequality for variable exponent Lebesgue spaces. *J. Math. Anal. Appl.*, 356(1):232–236, 2009.
219. O. Kováčik and J. Rákosník. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, 41(116)(4):592–618, 1991.
220. A. Kozek. Convex integral functionals on Orlicz spaces. *Comment. Math. Prace Mat.*, 21(1):109–135, 1980.
221. S. M. Kozlov. Duality of a type of functionals. *Funktsional. Anal. i Prilozhen.*, 17(3):9–14, 1983.
222. M. A. Krasnosel′skiĭ and Ja. B. Rutickiĭ. *Convex Functions and Orlicz Spaces*. Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1961.
223. S. G. Kreĭn, Yu. Ī. Petunīn, and E. M. Semënov. *Interpolation of Linear Operators*, volume 54 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, R.I., 1982. Translated from the Russian by J. Szűcs.
224. A. Kufner, L. Maligranda, and L.-E. Persson. *The Hardy Inequality*. Vydavatelský Servis, Plzeň, 2007. About its history and some related results.
225. K. Kurata and N. Shioji. Compact embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ and its application to nonlinear elliptic boundary value problem with variable critical exponent. *J. Math. Anal. Appl.*, 339(2):1386–1394, 2008.
226. F. Leonetti. Weak differentiability for solutions to nonlinear elliptic systems with $p,q$-growth conditions. *Ann. Mat. Pura Appl. (4)*, 162:349–366, 1992.
227. G. Leoni. *A First Course in Sobolev Spaces*, volume 105 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.
228. A. K. Lerner. On modular inequalities in variable $L^p$ spaces. *Arch. Math. (Basel)*, 85(6):538–543, 2005.
229. A. K. Lerner. Some remarks on the Hardy-Littlewood maximal function on variable $L^p$ spaces. *Math. Z.*, 251(3):509–521, 2005.
230. A. K. Lerner. An elementary approach to several results on the Hardy-Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 136(8):2829–2833, 2008.
231. A. K. Lerner. On some questions related to the maximal operator on variable $L^p$ spaces. *Trans. Amer. Math. Soc.*, 362(8):4229–4242, 2010.
232. A. K. Lerner. Some remarks on the Fefferman-Stein inequality. *J. Anal. Math.*, 112:329–349, 2010.
233. A. K. Lerner and S. Ombrosi. A boundedness criterion for general maximal operators. *Publ. Mat.*, 54(1):53–71, 2010.
234. A. K. Lerner, S. Ombrosi, and C. Pérez. Sharp $A_1$ bounds for Calderón-Zygmund operators and the relationship with a problem of Muckenhoupt and Wheeden. *Int. Math. Res. Not. IMRN*, (6):Art. ID rnm161, 11, 2008.
235. A. K. Lerner and C. Pérez. A new characterization of the Muckenhoupt $A_p$ weights through an extension of the Lorentz-Shimogaki theorem. *Indiana Univ. Math. J.*, 56(6):2697–2722, 2007.
236. S. Levine. An adaptive variational model for image decomposition. In A. Rangarajan *et al.*, editor, *Energy Minimization Methods in Computer Vision and Pattern Recognition*, volume 3757 of *Lecture Notes in Computer Science*, pages 382–397. Springer-Verlag, Berlin, 2006.

237. F. Li, Z. Li, and L. Pi. Variable exponent functionals in image restoration. *Appl. Math. Comput.*, 216(3):870–882, 2010.
238. E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.
239. K. Lindberg. On subspaces of Orlicz sequence spaces. *Studia Math.*, 45:119–146, 1973.
240. Q. Liu. Compact trace in weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega; v_0, v_1)$. *J. Math. Anal. Appl.*, 348(2):760–774, 2008.
241. R. L. Long and F. S. Nie. Weighted Sobolev inequality and eigenvalue estimates of Schrödinger operators. In *Harmonic analysis (Tianjin, 1988)*, volume 1494 of *Lecture Notes in Math.*, pages 131–141. Springer, Berlin, 1991.
242. J. Lukeš, L. Pick, and D. Pokorný. On geometric properties of the spaces $L^{p(x)}$. *Rev. Mat. Complut.*, 24(1):115–130, 2011.
243. F.-Y. Maeda, Y. Mizuta, and T. Ohno. Approximate identities and Young type inequalities in variable Lebesgue-Orlicz spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$. *Ann. Acad. Sci. Fenn. Math.*, 35:405–420, 2010.
244. L. Maligranda. *Orlicz spaces and interpolation*, volume 5 of *Seminários de Matemática [Seminars in Mathematics]*. Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.
245. L. Maligranda. Hidegoro Nakano (1909–1974) – on the centenary of his birth. In M. Kato, L. Maligranda, and T. Suzuki, editors, *Banach and Function Spaces III: Proceedings of the International Symposium on Banach and Function Spaces 2009*, pages 99–171. Yokohama Publishers, 2011.
246. L. Maligranda. Józef Marcinkiewicz (1910–1940) – On the centenary of his birth. In M. Nawrocki and W. Wnuk, editors, *Marcinkiewicz Centenary Volume*, volume 95 of *Banach Center Publications*, pages 133–234. Institute of Mathematics, Polish Academy of Sciences, 2011.
247. F. Mamedov and A. Harman. On a weighted inequality of Hardy type in spaces $L^{p(\cdot)}$. *J. Math. Anal. Appl.*, 353(2):521–530, 2009.
248. F. Mamedov and A. Harman. On a Hardy type general weighted inequality in spaces $L^{p(\cdot)}$. *Integral Equations Operator Theory*, 66(4):565–592, 2010.
249. F. Mamedov and Y. Zeren. On a two-weighted estimation of the maximal operator in the Lebesgue space with variable exponent. *Ann. Mat. Pura Appl.*, 190(2):263–275, 2011.
250. F. Mamedov and Y. Zeren. Two-weight inequalities for the maximal operator in a Lebesgue space with variable exponent. *J. Math. Sci.*, 173(6):701–716, 2011.
251. F. Mamedov and Y. Zeren. On equivalent conditions for the general weighted Hardy type inequality in the space $L^{p(\cdot)}$. *Z. Anal. Anwend.*, 31(1):55–74, 2012.
252. P. Marcellini. Regularity and existence of solutions of elliptic equations with $p,q$-growth conditions. *J. Differential Equations*, 90(1):1–30, 1991.
253. P. Marcellini. Regularity for elliptic equations with general growth conditions. *J. Differential Equations*, 105(2):296–333, 1993.
254. P. Marcellini and G. Papi. Nonlinear elliptic systems with general growth. *J. Differential Equations*, 221(2):412–443, 2006.
255. F. J. Martín-Reyes. New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions. *Proc. Amer. Math. Soc.*, 117(3):691–698, 1993.
256. F. J. Martín-Reyes, P. Ortega Salvador, and A. de la Torre. Weighted inequalities for one-sided maximal functions. *Trans. Amer. Math. Soc.*, 319(2):517–534, 1990.
257. R. A. Mashiyev and B. Çekiç. Sobolev-type inequality for spaces $L^{p(x)}(\mathbb{R}^N)$. *Int. J. Contemp. Math. Sci.*, 2(9–12):423–429, 2007.
258. R. A. Mashiyev, B. Çekiç, F. Mamedov, and S. Ogras. Hardy's inequality in power-type weighted $L^{p(\cdot)}(0,\infty)$ spaces. *J. Math. Anal. Appl.*, 334(1):289–298, 2007.
259. R. A. Mashiyev, B. Çekiç, and S. Ogras. On Hardy's inequality in $L^{p(x)}(0,\infty)$. *JIPAM. J. Inequal. Pure Appl. Math.*, 7(3):Article 106, 5 pp. (electronic), 2006.
260. V. G. Maz'ja. *Sobolev Spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

261. A. Mercaldo, J. D. Rossi, S. Segura de León, and C. Trombetti. On the behaviour of solutions to the Dirichlet problem for the $p(x)$-Laplacian when $p(x)$ goes to 1 in a subdomain. *Differential Integral Equations*, 25(1–2):53–74, 2012.
262. A. Meskhi. *Measure of Non-compactness for Integral Operators in Weighted Lebesgue Spaces*. Nova Science Publishers, Hauppauge, New York, 2009.
263. G. Mingione. Regularity of minima: an invitation to the dark side of the calculus of variations. *Appl. Math.*, 51(4):355–426, 2006.
264. C. Miranda. *Istituzioni di Analisi Funzionale Lineare*, volume I. Unione Matematica Italiana, Pitagora Editrice, Bologna, 1978.
265. Y. Mizuta, E. Nakai, T. Ohno, and T. Shimomura. Hardy's inequality in Orlicz-Sobolev spaces of variable exponent. *Hokkaido Math. J.*, 40(2):187–203, 2011.
266. Y. Mizuta, T. Ohno, and T. Shimomura. Integrability of maximal functions for generalized Lebesgue spaces with variable exponent. *Math. Nachr.*, 281(3):386–395, 2008.
267. Y. Mizuta, T. Ohno, and T. Shimomura. Sobolev's inequalities and vanishing integrability for Riesz potentials of functions in the generalized Lebesgue space $L^{p(\cdot)}(\log L)^{q(\cdot)}$. *J. Math. Anal. Appl.*, 345(1):70–85, 2008.
268. Y. Mizuta, T. Ohno, T. Shimomura, and N. Shioji. Compact embeddings for Sobolev spaces of variable exponents and existence of solutions for nonlinear elliptic problems involving the $p(x)$-Laplacian and its critical exponent. *Ann. Acad. Sci. Fenn. Math.*, 35(1):115–130, 2010.
269. Y. Mizuta and T. Shimomura. Sobolev's inequality for Riesz potentials with variable exponent satisfying a log-Hölder condition at infinity. *J. Math. Anal. Appl.*, 311(1):268–288, 2005.
270. J. Motos, M. J. Planells, and C. F. Talavera. On variable exponent Lebesgue spaces of entire analytic functions. *J. Math. Anal. Appl.*, 388(2):775–787, 2012.
271. B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.
272. B. Muckenhoupt and R. L. Wheeden. Weighted norm inequalities for fractional integrals. *Trans. Amer. Math. Soc.*, 192:261–274, 1974.
273. B. Muckenhoupt and R. L. Wheeden. Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform. *Studia Math.*, 55(3):279–294, 1976.
274. J. Musielak. *Orlicz Spaces and Modular Spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.
275. J. Musielak and W. Orlicz. On modular spaces. *Studia Math.*, 18:49–65, 1959.
276. P. Nägele. Monotone operators in spaces with variable exponents. Master's thesis, Universität Freiburg, 2009.
277. E. Nakai and K. Yabuta. Pointwise multipliers for functions of bounded mean oscillation. *J. Math. Soc. Japan*, 37(2):207–218, 1985.
278. H. Nakano. *Modulared Semi-Ordered Linear Spaces*. Maruzen Co. Ltd., Tokyo, 1950.
279. H. Nakano. Modulared sequence spaces. *Proc. Japan Acad.*, 27:508–512, 1951.
280. H. Nakano. *Topology and Linear Topological Spaces*. Maruzen Co. Ltd., Tokyo, 1951.
281. A. Nekvinda. Equivalence of $l^{\{p_n\}}$ norms and shift operators. *Math. Inequal. Appl.*, 5(4):711–723, 2002.
282. A. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$. *Math. Inequal. Appl.*, 7(2):255–265, 2004.
283. A. Nekvinda. A note on maximal operator on $\ell^{\{p_n\}}$ and $L^{p(x)}(\mathbb{R})$. *J. Funct. Spaces Appl.*, 5(1):49–88, 2007.
284. A. Nekvinda. Maximal operator on variable Lebesgue spaces for almost monotone radial exponent. *J. Math. Anal. Appl.*, 337(2):1345–1365, 2008.
285. A. Nekvinda. A note on one-sided maximal operator in $L^{p(\cdot)}(\mathbb{R})$. *Math. Inequal. Appl.*, to appear.
286. C. J. Neugebauer. Weighted variable $L^p$ integral inequalities for the maximal operator on non-increasing functions. *Studia Math.*, 192(1):51–60, 2009.
287. M. Nuortio. *Parabolic nonlinear PDE with nonstandard growth: supersolutions and superparabolic functions*. PhD thesis, University of Oulu, Finland, 2011.

288. T. Ohno. Compact embeddings in the generalized Sobolev space $W_0^{1,p(\cdot)}(G)$ and existence of solutions for nonlinear elliptic problems. *Nonlinear Anal.*, 71(5–6):1534–1541, 2009.
289. B. Opic and A. Kufner. *Hardy-type Inequalities*, volume 219 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1990.
290. W. Orlicz. Über konjugierte Exponentenfolgen. *Stud. Math.*, 3:200–211, 1931.
291. L. Pick and M. Růžička. An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded. *Expo. Math.*, 19(4):369–371, 2001.
292. V. R. Portnov. Certain properties of the Orlicz spaces generated by the functions $M(x, w)$. *Dokl. Akad. Nauk SSSR*, 170:1269–1272, 1966.
293. V. R. Portnov. On the theory of Orlicz spaces which are generated by variable $N$-functions. *Dokl. Akad. Nauk SSSR*, 175:296–299, 1967.
294. A. D. Poularikas, editor. *The Transforms and Applications Handbook*. The Electrical Engineering Handbook Series. CRC Press, Boca Raton, FL, 1996.
295. H. Rafeiro and S. Samko. Hardy type inequality in variable Lebesgue spaces. *Ann. Acad. Sci. Fenn. Math.*, 34(1):279–289, 2009.
296. H. Rafeiro and S. Samko. Corrigendum to "Hardy type inequality in variable Lebesgue spaces" [mr2489027]. *Ann. Acad. Sci. Fenn. Math.*, 35(2):679–680, 2010.
297. J. M. Rakotoson. *Réarrangement Relatif*, volume 64 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Berlin, 2008.
298. J. M. Rakotoson. Lipschitz properties in variable exponent problems via relative rearrangement. *Chin. Ann. Math.*, 31B(6):991–1006, 2010.
299. M. Riesz. L'intégrale de Riemann-Liouville et le problème de Cauchy. *Acta Math.*, 81:1–223, 1949.
300. B. Ross and S. Samko. Fractional integration operator of variable order in the Hölder spaces $H^{\lambda(x)}$. *Internat. J. Math. Math. Sci.*, 18(4):777–788, 1995.
301. H. L. Royden. *Real Analysis*. Macmillan Publishing Company, New York, third edition, 1988.
302. J. L. Rubio de Francia. Factorization and extrapolation of weights. *Bull. Amer. Math. Soc. (N.S.)*, 7(2):393–395, 1982.
303. J. L. Rubio de Francia. A new technique in the theory of $A_p$ weights. In *Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982)*, pages 571–579. Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983.
304. J. L. Rubio de Francia. Factorization theory and $A_p$ weights. *Amer. J. Math.*, 106(3):533–547, 1984.
305. W. Rudin. *Real and Complex Analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
306. M. Růžička. *Electrorheological Fluids: Modeling and Mathematical Theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
307. M. Růžička. Modeling, mathematical and numerical analysis of electrorheological fluids. *Appl. Math.*, 49(6):565–609, 2004.
308. Y. Sakai. Spectral orders and differences. *Hokkaido Math. J.*, 13(3):271–276, 1984.
309. N. Samko, S. Samko, and B. Vakulov. Weighted Sobolev theorem in Lebesgue spaces with variable exponent. *J. Math. Anal. Appl.*, 335(1):560–583, 2007.
310. N. Samko, S. Samko, and B. Vakulov. Weighted Sobolev theorem in Lebesgue spaces with variable exponent: corrigendum [mr2340340]. *Armen. J. Math.*, 3(2):92–97, 2010.
311. S. Samko. Fractional integration and differentiation of variable order. *Anal. Math.*, 21(3):213–236, 1995.
312. S. Samko. Convolution and potential type operators in $L^{p(x)}(\mathbf{R}^n)$. *Integral Transform. Spec. Funct.*, 7(3–4):261–284, 1998.
313. S. Samko. Convolution type operators in $L^{p(x)}$. *Integral Transform. Spec. Funct.*, 7(1–2):123–144, 1998.
314. S. Samko. Differentiation and integration of variable order and the spaces $L^{p(x)}$. In *Operator theory for complex and hypercomplex analysis (Mexico City, 1994)*, volume 212 of *Contemp. Math.*, pages 203–219. Amer. Math. Soc., Providence, RI, 1998.

315. S. Samko. Density $C_0^\infty(\mathbf{R}^n)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbf{R}^n)$. *Dokl. Akad. Nauk*, 369(4):451–454, 1999.
316. S. Samko. Denseness of $C_0^\infty(\mathbf{R}^N)$ in the generalized Sobolev spaces $W^{M,P(X)}(\mathbf{R}^N)$. In *Direct and inverse problems of mathematical physics (Newark, DE, 1997)*, volume 5 of *Int. Soc. Anal. Appl. Comput.*, pages 333–342. Kluwer Acad. Publ., Dordrecht, 2000.
317. S. Samko. Hardy inequality in the generalized Lebesgue spaces. *Fract. Calc. Appl. Anal.*, 6(4):355–362, 2003.
318. S. Samko. Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces with variable exponent. *Fract. Calc. Appl. Anal.*, 6(4):421–440, 2003.
319. S. Samko. On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. *Integral Transforms Spec. Funct.*, 16(5–6):461–482, 2005.
320. S. Samko and B. Ross. Integration and differentiation to a variable fractional order. *Integral Transform. Spec. Funct.*, 1(4):277–300, 1993.
321. S. Samko, E. Shargorodsky, and B. Vakulov. Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators. II. *J. Math. Anal. Appl.*, 325(1):745–751, 2007.
322. S. Samko and B. Vakulov. A weighted Sobolev theorem for spatial and spherical potentials in Lebesgue spaces with variable exponents. *Dokl. Akad. Nauk*, 403(1):7–10, 2005.
323. S. Samko and B. Vakulov. Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators. *J. Math. Anal. Appl.*, 310(1):229–246, 2005.
324. D. Sarason. Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.*, 207:391–405, 1975.
325. E. T. Sawyer. A characterization of a two-weight norm inequality for maximal operators. *Studia Math.*, 75(1):1–11, 1982.
326. E. T. Sawyer. Weighted inequalities for the one-sided Hardy-Littlewood maximal functions. *Trans. Amer. Math. Soc.*, 297(1):53–61, 1986.
327. E. T. Sawyer and R. L. Wheeden. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.*, 114(4):813–874, 1992.
328. C. Sbordone and I. Wik. Maximal functions and related weight classes. *Publ. Mat.*, 38(1):127–155, 1994.
329. I. I. Sharapudinov. The topology of the space $\mathcal{L}^{p(t)}([0,1])$. *Mat. Zametki*, 26(4):613–632, 655, 1979.
330. I. I. Sharapudinov. Approximation of functions in the metric of the space $\mathcal{L}^{p(t)}([a,b])$ and quadrature formulas. In *Constructive function theory '81 (Varna, 1981)*, pages 189–193. Publ. House Bulgar. Acad. Sci., Sofia, 1983.
331. I. I. Sharapudinov. The basis property of the Haar system in the space $\mathcal{L}^{p(t)}([0,1])$ and the principle of localization in the mean. *Mat. Sb. (N.S.)*, 130(172)(2):275–283, 286, 1986.
332. I. I. Sharapudinov. On the uniform boundedness in $L^p$ $(p = p(x))$ of some families of convolution operators. *Mat. Zametki*, 59(2):291–302, 320, 1996.
333. G. Sinnamon. Four questions related to Hardy's inequality. In *Function spaces and applications (Delhi, 1997)*, pages 255–266. Narosa, New Delhi, 2000.
334. S. Spanne. Some function spaces defined using the mean oscillation over cubes. *Ann. Scuola Norm. Sup. Pisa (3)*, 19:593–608, 1965.
335. I. V. Šragin. The Amemiya norm in an Orlicz-Nakano space. *Kišinev. Gos. Univ. Učen. Zap.*, 91:91–102, 1967.
336. J. E. Stangroom. Electrorheological fluids. *Physics in Technology*, 14:290–296, 1983.
337. D. A. Stegenga. Bounded Toeplitz operators on $H^1$ and applications of the duality between $H^1$ and the functions of bounded mean oscillation. *Amer. J. Math.*, 98(3):573–589, 1976.
338. E. M. Stein. Note on the class $L \log L$. *Studia Math.*, 32:305–310, 1969.
339. E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
340. E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

341. E. M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
342. G. Talenti. Boundedness of minimizers. *Hokkaido Math. J.*, 19(2):259–279, 1990.
343. L. Tartar. *An Introduction to Sobolev Spaces and Interpolation Spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Berlin, 2007.
344. I. V. Tsenov. Generalization of the problem of best approximation of a function in the space $L^s$. *Uch. Zap. Dagestan. Gos. Univ.*, 7:25–37, 1961.
345. B. O. Turesson. *Nonlinear potential theory and weighted Sobolev spaces*, volume 1736 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.
346. G. V. Welland. Weighted norm inequalities for fractional integrals. *Proc. Amer. Math. Soc.*, 51:143–148, 1975.
347. N. Wiener. The ergodic theorem. *Duke Math. J.*, 5(1):1–18, 1939.
348. T. Wunderli. On time flows of minimizers of general convex functionals of linear growth with variable exponent in BV space and stability of pseudosolutions. *J. Math. Anal. Appl.*, 364(2):591–598, 2010.
349. S. Yamamuro. On conjugate spaces of Nakano spaces. *Trans. Amer. Math. Soc.*, 90:291–311, 1959.
350. J. Yao and X. Wang. Compact imbeddings between variable exponent spaces with unbounded underlying domain. *Nonlinear Anal.*, 70(10):3472–3482, 2009.
351. A. Zang and Y. Fu. Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces. *Nonlinear Anal.*, 69(10):3629–3636, 2008.
352. G. Zecca. The boundedness of the maximal operator on variable Sobolev spaces. *Rend. Acc. Sc. fis. mat Napoli*, LXXV:31–43, 2008.
353. V. V. Zhikov. Problems of convergence, duality, and averaging for a class of functionals of the calculus of variations. *Dokl. Akad. Nauk SSSR*, 267(3):524–528, 1982.
354. V. V. Zhikov. Questions of convergence, duality and averaging for functionals of the calculus of variations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 47(5):961–998, 1983.
355. V. V. Zhikov. Averaging of functionals of the calculus of variations and elasticity theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50(4):675–710, 877, 1986.
356. V. V. Zhikov. The Lavrent′ ev effect and averaging of nonlinear variational problems. *Differentsial′ nye Uravneniya*, 27(1):42–50, 180, 1991.
357. V. V. Zhikov. Passage to the limit in nonlinear variational problems. *Mat. Sb.*, 183(8):47–84, 1992.
358. V. V. Zhikov. On the homogenization of nonlinear variational problems in perforated domains. *Russian J. Math. Phys.*, 2(3):393–408, 1994.
359. V. V. Zhikov. On Lavrentiev's phenomenon. *Russian J. Math. Phys.*, 3(2):249–269, 1995.
360. V. V. Zhikov. Meyer-type estimates for solving the nonlinear Stokes system. *Differ. Uravn.*, 33(1):107–114, 143, 1997.
361. V. V. Zhikov. On some variational problems. *Russian J. Math. Phys.*, 5(1):105–116 (1998), 1997.
362. V. V. Zhikov. On the density of smooth functions in Sobolev-Orlicz spaces. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 310(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 35 [34]):67–81, 226, 2004.
363. W. P. Ziemer. *Weakly Differentiable Functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.
364. F. Zo. A note on approximation of the identity. *Studia Math.*, 55(2):111–122, 1976.

# Symbol Index

**Sets, functions and integration**



**Continuity and differentiability**



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**Properties of exponents**

**Norms and modulars**



**Function spaces**

**Operators**

**Weights and weight conditions**

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# Subject Index



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