Progress in Probability, Vol. 67, 55–84

General Upper and Lower Tail Estimates Using Malliavin Calculus and Stein's Equations

Richard Eden and Frederi Viens

To the memory of Prof. Paul Malliavin

Abstract. Following a strategy recently developed by Ivan Nourdin and Giovanni Peccati, we provide a general technique to compare the tail of a given random variable to that of a reference distribution, and apply it to all reference distributions in the so-called Pearson class. This enables us to give concrete conditions to ensure upper and/or lower bounds on the random variable's tail of various power or exponential types. The Nourdin-Peccati strategy analyzes the relation between Stein's method and the Malliavin calculus, and is adapted to dealing with comparisons to the Gaussian law. By studying the behavior of the solution to general Stein equations in detail, we show that the strategy can be extended to comparisons to a wide class of laws, including all Pearson distributions.

Mathematics Subject Classification (2010). Primary 60H07; Secondary 60G15, 60E15.

Keywords. Malliavin calculus, tail estimate, Stein's equation, Pearson distribution.

1. Introduction

In this article, following a strategy recently developed by Ivan Nourdin and Giovanni Peccati, we provide a general technique to compare the tail of a given random variable to that of a reference distribution, and apply it to all reference distributions in the so-called Pearson class, which enables us to give concrete conditions to ensure upper and/or lower bounds on the random variable's tail of power or exponential type. The strategy uses the relation between Stein's method and the

Both authors' research partially supported by NSF grant DMS-0907321.

Malliavin calculus. In this introduction, we detail the main ideas of this strategy, including references to related works; we also summarize the results proved in this article, and the methods used to prove them.

1.1. Stein's method and the analysis of Nourdin and Peccati

Stein's method is a set of procedures that is often used to measure distances between distributions of random variables. The starting point is the so-called Stein equation. To motivate it, recall the following result which is sometimes referred to as Stein's lemma. Suppose X is a random variable. Then $X \overset{\text{Law}}{=} Z \sim \mathcal{N}(0,1)$ if and only if

$$\mathbf{E}[f'(X) - Xf(X)] = 0 \tag{1.1}$$

for all continuous and piecewise differentiable functions f such that $\mathbf{E}[|f'(X)|] < \infty$ (see, e.g., [4, 5, 21]). If the above expectation is non-zero but close to zero, Stein's method can give us a way to express how close the law of X might be to the standard normal law, in particular by using the concept of Stein equation. For a given test function h, this is the ordinary differential equation $f'(x) - xf(x) = h(x) - \mathbf{E}[h(Z)]$ with continuous and piecewise differentiable solution f. As we will see in more detail and greater generality further below, if one is able to prove boundedness properties of f and f' for a wide class of test functions h, this can help evaluate the distance between the law of Z and laws of random variables that might be close to Z, including methods for proving convergence in distribution. This fundamental feature of Stein's method is described in many works; see [4] for a general introduction and review.

As a testament to the extraordinary versatility of Stein's method, recently Ivan Nourdin and Giovanni Peccati discovered a connection between Stein's method and the Malliavin calculus, with striking applications in a number of problems in stochastic analysis. Motivated by Berry–Esséen-type theorems for convergence of sequences of random variables in Wiener chaos, Nourdin and Peccati's first paper [8] on this connection considers an arbitrary square-integrable Malliavin-differentiable random variable X on a Wiener space, and associates the random variable

$$G := \langle DX; -DL^{-1}X \rangle \tag{1.2}$$

where D is the Malliavin derivative operator on the Wiener space, and L^{-1} is the pseudo-inverse of the generator of the Ornstein–Uhlenbeck semigroup (see Section 3.1 for precise definitions of these operators). One easily notes that if X is standard normal, then $G \equiv 1$ (Corollary 3.4 in [16]). Then by measuring the distance between G and 1 for an arbitrary X, one can measure how close the law of X is to the normal law. The connection to Stein's method comes from their systematic use of the basic observation that $\mathbf{E}[Gf(X)] = \mathbf{E}[Xf'(X)]$. It leads to the following simple and efficient strategy for measuring distances between the laws of X and Z. To evaluate, e.g., $\mathbf{E}[h(X)] - \mathbf{E}[h(Z)]$ for test functions h, one can:

1. write $\mathbf{E}[h(X)] - \mathbf{E}[h(Z)]$ using the solution of Stein's equation, as $\mathbf{E}[f'(X)] - \mathbf{E}[Xf(X)]$;
2. use their observation to transform this expression into $\mathbf{E}[f'(X)(1-G)]$;
3. use the boundedness and decay properties of f' (these are classically known from Stein's equation) to exploit the proximity of G to 1.

As we said, this strategy of relating Stein's method and the Malliavin calculus is particularly useful for analyzing problems in stochastic analysis. In addition to their study of convergence in Wiener chaos in [8], which they followed up with sharper results in [9], Nourdin and Peccati have implemented several other applications including: the study of cummulants on Wiener chaos [11], of fluctuations of Hermitian random matrices [12], and, with other authors, other results about the structure of inequalities and convergences on Wiener space, such as [3, 13, 14, 15]. In [16], it was pointed out that if ρ denotes the density of X, then the function

$$g(z) := \rho^{-1}(z) \int_z^\infty y\rho(y)\, dy, \tag{1.3}$$

which was originally defined by Stein in [21], can be represented as

$$g(z) = \mathbf{E}[G|X = z],$$

resulting in a convenient formula for the density ρ, which was then exploited to provide new Gaussian lower bound results for certain stochastic models, in [16] for Gaussian fields, and subsequently in [22] for polymer models in Gaussian and non-Gaussian environments, in [18] for stochastic heat equations, in [3] for statistical inference for long-memory stochastic processes, and multivariate extensions of density formulas in [1].

1.2. Summary of our results

Our specific motivation is drawn from the results in [22] which make assumptions on how G compares to 1 almost surely, and draw conclusions on how the tail of X, i.e., $\mathbf{P}[X > z]$, compares to the normal tail $\mathbf{P}[Z > z]$. By the above observations, these types of almost-sure assumptions are equivalent to comparing the deterministic function g to the value 1. For instance, one result in [22] can be summarized by saying that (under some additional regularity conditions) if $G \geq 1$ almost surely, i.e., if $g(z) \geq 1$ everywhere, then for some constant c and large enough z, $\mathbf{P}[X > z] > c\mathbf{P}[Z > z]$. This result, and all the ones mentioned above, concentrate on comparing laws to the standard normal law, which is done by comparing G to the constant 1, as this constant is the "G" for the standard normal Z.

In this paper, we find a framework which enables us to compare the law of X to a wide range of laws. Instead of assuming that g is comparable to 1, we only assume that it is comparable to a polynomial of degree less than or equal to 2. In [21], Stein had originally noticed that the set of all distributions such that their g is such a polynomial, is precisely the so-called Pearson class of distributions. They encompass Gaussian, Gamma, and Beta distributions, as well as the inverse-Gamma, and a number of continuous distributions with only finitely many moments, with prescribed power tail behavior. This means that one can hope to give precise criteria based on g, or via Malliavin calculus based on G, to guarantee upper and/or lower bounds on the tail $\mathbf{P}[X > z]$, with various Gaussian, exponential, or power-type behaviors. We achieve such results in this paper.

Specifically, our first set of results is in the following general framework. Let Z be a reference random variable supported on (a, b) where $-\infty \le a < b \le +\infty$, with a density ρ_* which is continuous on $\mathbf{R}$ and differentiable on (a, b). The function g corresponding to ρ_* is given as in (1.3), and we denote it by g_* (the subscripts $_*$ indicate that these are relative to our reference r.v.):

$$g_*(z) = \frac{\int_z^\infty y\rho_*(y)\,dy}{\rho_*(z)} \mathbf{1}_{(a,b)}(z). \tag{1.4}$$

We also use the notation

$$\Phi_*(z) = \mathbf{P}[Z > z]$$

for our reference tail. Throughout this article, for notational convenience, we assume that Z is centered (except when specifically stated otherwise in Section A.2 in the Appendix). Let X be Malliavin-differentiable, supported on (a, b), with its $G := \langle DX; -DL^{-1}X\rangle$ as in (1.2).

- (Theorem 3.4) Under mild regularity and integrability conditions on Z and X, if $G \ge g_*(X)$ almost surely, then for all $z < b$,

$$\mathbf{P}[X > z] \ge \Phi_*(z) - \frac{1}{Q(z)} \int_z^b (2y - z)\mathbf{P}[X > y]\,dy,$$

  where

$$Q(z) := z^2 - zg_*'(z) + g_*(z); \tag{1.5}$$

  typically Q is of order z^2 for large z.
- (Theorem 3.5) Under mild regularity and integrability conditions on Z and X, if $G \le g_*(X)$ almost surely, then for some constant c and all large enough $z < b$,

$$\mathbf{P}[X > z] \le c\Phi_*(z).$$

These results are generalizations of the work in [22], where only the standard normal Z was considered. They can be rephrased by referring to g as in (1.3), which coincides with $g(z) = \mathbf{E}[G|X = z]$, rather than G; this can be useful to apply the theorems in contexts where the definition of X as a member of a Wiener space is less explicit than the information one might have directly about g. We have found, however, that the Malliavin-calculus interpretation makes for efficient proofs of the above theorems.

The main application of these general theorems are to the Pearson class: Z such that its g_* is of the form $g_*(z) = \alpha z^2 + \beta z + \gamma$ in the support of Z. Assume $b = +\infty$, i.e., the support of Z is $(a, +\infty)$. Assume $\mathbf{E}\left[|Z|^3\right] < \infty$ (which is equivalent to $\alpha < 1/2$). Then the lower bound above can be made completely explicit, as can the constant c in the upper bound.

- (Corollary 4.8) Under mild regularity and integrability conditions on X [including assuming that there exists $c > 2$ such that $g(z) \leq z^2/c$ for large z], if $G \geq g_*(X)$ almost surely, then for any $c' < \frac{1}{1+2(1-\alpha)(c-2)}$ and all z large enough,

$$\mathbf{P}[X > z] \geq c'\Phi_*(z).$$

- (Corollary 4.7) Under mild regularity and integrability conditions on X, if $G \leq g_*(X)$ almost surely, then for any $c > (1-\alpha)/(1-2\alpha)$, and all z large enough,

$$\mathbf{P}[X > z] \leq c\Phi_*(z).$$

The results above can be used conjointly with asymptotically sharp conclusions when upper and lower bound assumptions on G are true simultaneously. For instance, we have the following, phrased using g's instead of G's.

- (Corollary 4.9, point 2) On the support $(a, +\infty)$, let $g_*(z) = \alpha z^2 + \beta z + \gamma$ and let $\bar{g}_*(z) = \bar{\alpha} z^2 + \bar{\beta} z + \bar{\gamma}$ with non-zero α and $\bar{\alpha}$. If for the Malliavin-differentiable X and its corresponding g, we have for all $z > a$, $g_*(z) \leq g(z) \leq \bar{g}_*(z)$, then there are constants c and $\bar{c}$ such that for large z,

$$cz^{-1-1/\alpha} \leq \mathbf{P}[X > z] \leq \bar{c}z^{-1-1/\bar{\alpha}}.$$

- (see Corollary 4.10) A similar result holds when $\alpha = \bar{\alpha} = 0$, in which $\mathbf{P}[X > z]$ compares to the Gamma-type tail $z^{-1-\gamma/\beta^2} \exp(-z/\beta)$.

The strategy used to prove these results is an analytic one, following the initial method of Nourdin and Peccati, this time using the Stein equation relative to the function g_* defined in (1.4) for a general reference r.v. Z:

$$g_*(x) f'(x) - xf(x) = h(x) - \mathbf{E}[h(Z)].$$

Our mathematical techniques are based on a careful analysis of the properties of g_*, its relation to the function Q defined in (1.5), and what consequences can be derived for the solutions of Stein's equation. The basic general theorems' proofs use a structure similar to that employed in [22]. The applications to the Pearson class rely heavily on explicit computations tailored to this case, which are facilitated via the identification of Q as a useful way to express these computations.

This article is structured as follows. Section 2 gives an overview of Stein's equations, and derives some fine properties of their solutions by referring to the function Q. These will be crucial in the proofs of our general upper and lower bound results, which are presented in Section 3 after an overview of the tools of Malliavin calculus which are needed in this article. Applications to comparisons with Pearson distributions, with a particular emphasis on tail behavior, including asymptotic results, are in Section 4. Section 5 is an Appendix containing the proofs of some technical lemmas and some details on Pearson distributions.

2. The Stein equation

2.1. Background and classical results

Characterization of the law of Z. As before, let Z be centered with a differentiable density on its support (a, b), and let g_* be defined as in (1.4). Nourdin and Peccati (Proposition 6.4 in [8]) collected the following results concerning this equation. If f is a function that is continuous and piecewise continuously differentiable, and if $\mathbf{E}[|f'(Z)|g_*(Z)] < \infty$, Stein (Lemma 1, p. 59 in [21]) proved that

$$\mathbf{E}[g_*(Z)f'(Z) - Zf(Z)] = 0$$

(compare this with (1.1) for the special case $Z \sim \mathcal{N}(0,1)$). Conversely, assume that

$$\int_0^b \frac{z}{g_*(z)}\,dz = \infty \qquad \text{and} \qquad \int_a^0 \frac{z}{g_*(z)}\,dz = -\infty. \tag{2.1}$$

If a random variable X has a density, and for any differentiable function f such that $x \mapsto |g_*(x)f'(x)| + |xf(x)|$ is bounded,

$$\mathbf{E}[g_*(X)f'(X) - Xf(X)] = 0 \tag{2.2}$$

then X and Z have the same law. In other words, under certain conditions, (2.2) can be used to characterize the law of a centered random variable X as being equal to that of Z.

Stein's equation, general case; distances between distributions. If h is a fixed bounded piecewise continuous function such that $\mathbf{E}[|h(Z)|] < \infty$, the corresponding *Stein equation* for Z is the ordinary differential equation in f defined by

$$h(x) - \mathbf{E}[h(Z)] = g_*(x)f'(x) - xf(x). \tag{2.3}$$

The utility of such an equation is apparent when we evaluate the functions at X and take expectations:

$$\mathbf{E}[h(X)] - \mathbf{E}[h(Z)] = \mathbf{E}[g_*(X)f'(X) - Xf(X)]. \tag{2.4}$$

The idea is that if the law of X is "close" to the law of Z, then the right side of (2.4) would be close to 0. Conversely, if the test function h can be chosen from specific classes of functions so that the left side of (2.4) denotes a particular notion of distance between X and Z, the closeness of the right-hand side of (2.4) to zero, in some uniform sense in the f's satisfying Stein's equation (2.3) for all the h's in that specific class of test functions, will imply that the laws of X and Z are close in the corresponding distance. For this purpose, it is typically crucial to establish boundedness properties of f and f' which are uniform over the class of test functions being considered.

For example, if $\mathcal{H} = \{h : ||h||_L + ||h||_\infty \leq 1\}$ where $||\cdot||_L$ is the Lipschitz seminorm, then the Fortet–Mourier distance $d_{FM}(X, Z)$ between X and Z is defined as

$$d_{FM}(X,Z) = \sup_{h\in\mathcal{H}} |\mathbf{E}[h(X)] - \mathbf{E}[h(Z)]|.$$

This distance metrizes convergence in distribution, so by using properties of the solution f of the Stein equation (2.4) for $h \in \mathcal{H}$, we can draw conclusions on the convergence in distribution of a sequence $\{X_n\}$ to Z. See [8] and [6] for details and other notions of distance between random variables.

Solution of Stein's equation. Stein (Lemma 4, p. 62 in [21]) proved that if (2.1) is satisfied, then his equation (2.3) has a unique solution f which is bounded and continuous on (a, b). If $x \notin (a, b)$, then

$$f(x) = -\frac{h(x) - \mathbf{E}\left[h(Z)\right]}{x} \tag{2.5}$$

while if $x \in (a, b)$,

$$f(x) = \int_a^x \left(h(y) - \mathbf{E}\left[h(Z)\right]\right) \frac{e^{\int_y^x \frac{z\,dz}{g_*(z)}}}{g_*(y)}\,dy. \tag{2.6}$$

2.2. Properties of solutions of Stein's equations

We assume throughout that ρ_* is differentiable on (a, b) and continuous on $\mathbf{R}$ (for which it is necessary that ρ_* be null on $\mathbf{R} - (a, b)$). Consequently, g_* is differentiable and continuous on (a, b). The next lemma records some elementary properties of g_*, such as its positivity and its behavior near a and b. Those facts which are not evident are established in the Appendix. All are useful in facilitating the proofs of other lemmas presented in this section, which are key to our article.

Lemma 2.1. *Let Z be centered and continuous, with a density ρ_* that is continuous on $\mathbf{R}$ and differentiable on its support (a, b), with a and b possibly infinite.*

1. $g_*(x) > 0$ *if and only if* $x \in (a, b)$;
2. g_* *is differentiable on* (a, b) *and* $[g_*(x)\rho_*(x)]' = -x\rho_*(x)$ *therein;*
3. $\lim_{x \to a} g_*(x)\rho_*(x) = \lim_{x \to b} g_*(x)\rho_*(x) = 0.$

A different expression for the solution f of Stein's equation (2.3) than the one given in (2.5), (2.6), which will be more convenient for our purposes, such as computing f' in the support of Z, was given by Schoutens [20] as stated in the next lemma.

Lemma 2.2. *For all $x \in (a, b)$,*

$$f(x) = \frac{1}{g_*(x)\rho_*(x)} \int_a^x \left(h(y) - \mathbf{E}\left[h(Z)\right]\right) \rho_*(y)\,dy. \tag{2.7}$$

If $x \notin [a, b]$, differentiating (2.5) *gives*

$$f'(x) = \frac{-xh'(x) + h(x) - \mathbf{E}\left[h(Z)\right]}{x^2} \tag{2.8}$$

while if $x \in (a, b)$, differentiating (2.7) *gives*

$$f'(x) = \frac{x}{[g_*(x)]^2\rho_*(x)} \int_a^x \left(h(y) - \mathbf{E}[h(Z)]\right) \rho_*(y)\,dy + \frac{h(x) - \mathbf{E}[h(Z)]}{g_*(x)}. \tag{2.9}$$

The proof of this lemma (provided in the Appendix for completeness) also gives us the next one.

Lemma 2.3. *Under our assumption of differentiability on* (a,b) *of* ρ_* *and hence of* g_**, Stein's condition* (2.1) *on* g_* *is satisfied.*

In Stein's equation (2.3), the test function $h = 1_{(-\infty,z]}$ lends itself to useful tail probability results since $\mathbf{E}[h(Z)] = \mathbf{P}[Z \leq z]$. From this point on, we will assume that $h = 1_{(-\infty,z]}$ with fixed $z > 0$, and that f is the corresponding solution of Stein's equation (we could denote the parametric dependence of f on z by f_z, but choose to omit the subscript to avoid overburdening the notation).

As opposed to the previous lemmas, the next two results, while still elementary in nature, appear to be new, and their proofs, which require some novel ideas of possibly independent interest, have been kept in the main body of this paper, rather than having them relegated to the Appendix. We begin with an analysis of the sign of f', which will be crucial to prove our main general theorems.

Lemma 2.4. *Suppose* $0 < z < b$. *If* $x \leq z$, *then* $f'(x) \geq 0$. *If* $x > z$, *then* $f'(x) \leq 0$.

Proof. The result follows easily from (2.8) when $x \notin [a,b]$: if $x < a$, then $f'(x) = (1 - \mathbf{E}[h(Z)])/x^2 \geq 0$, while if $x > b$, then $f'(x) = -\mathbf{E}[h(Z)]/x^2 \leq 0$. So now we can assume that $x \in (a,b)$. We will use the expression for the derivative f' given in (2.9).

Suppose $a < x \leq z$. Then $h(x) = 1$ and for any $y \leq x$, $h(y) = 1$ so

$$\begin{aligned} f'(x) &= \frac{x}{[g_*(x)]^2 \rho_*(x)} \int_a^x \left(1 - \mathbf{E}[h(Z)]\right) \rho_*(y)\, dy + \frac{1 - \mathbf{E}[h(Z)]}{g_*(x)} \\ &= \frac{1 - \mathbf{E}[h(Z)]}{[g_*(x)]^2 \rho_*(x)} \left(x \int_a^x \rho_*(y)\, dy + g_*(x)\rho_*(x) \right). \end{aligned}$$

Clearly, $f'(x) \geq 0$ if $x \geq 0$. Now define

$$n_1(x) := \int_a^x \rho_*(y)\, dy + \frac{g_*(x)\rho_*(x)}{x}.$$

We will show that $xn_1(x) \geq 0$ when $x < 0$. Since

$$\begin{aligned} n_1'(x) &= \rho_*(x) + \frac{x[g_*(x)\rho_*(x)]' - g_*(x)\rho_*(x)}{x^2} \\ &= \rho_*(x) + \frac{-x^2\rho_*(x) - g_*(x)\rho_*(x)}{x^2} = -\frac{g_*(x)\rho_*(x)}{x^2} \leq 0 \end{aligned}$$

then n_1 is nonincreasing on $(a,0)$ which means that whenever $a < x < 0$, $n_1(x) \leq \lim_{x \to a} n_1(x) = \lim_{x \to a} \frac{g_*(x)p(x)}{x} = 0$ since $\lim_{x \to a} g_*(x)\rho_*(x) = 0$. Therefore, $xn_1(x) \geq 0$ for $x < 0$. This completes the proof that $f'(x) \geq 0$ whenever $x \leq z$.

Finally, suppose that $z < x < b$ so $h(x) = 0$. Since $\mathbf{E}[h(Z)] = \mathbf{P}[Z \le z] = \int_a^z \rho_*(y)\,dy$,

$$\begin{aligned}
f'(x) &= \frac{x}{[g_*(x)]^2\rho_*(x)}\int_a^x h(y)\rho_*(y)\,dy \\
&\quad - \frac{x\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)}\int_a^x \rho_*(y)\,dy - \frac{\mathbf{E}[h(Z)]}{g_*(x)} \\
&= \frac{x}{[g_*(x)]^2\rho_*(x)}\int_a^z \rho_*(y)\,dy - \frac{x\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)}\int_a^x \rho_*(y)\,dy - \frac{\mathbf{E}[h(Z)]}{g_*(x)} \\
&= \frac{x}{[g_*(x)]^2\rho_*(x)}\mathbf{E}[h(Z)] - \frac{x\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)}\int_a^x \rho_*(y)\,dy - \frac{\mathbf{E}[h(Z)]}{g_*(x)} \\
&= \frac{\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)}\left(x - x\int_a^x \rho_*(y)\,dy - g_*(x)\rho_*(x)\right) \\
&= \frac{\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)} \cdot x n_2(x)
\end{aligned}$$

where

$$n_2(x) := 1 - \int_a^x \rho_*(y)\,dy - \frac{g_*(x)\rho_*(x)}{x} = 1 - n_1(x).$$

It is enough to show that $n_2(x) \le 0$ since $x > z \ge 0$. Since $n_2'(x) = -n_1'(x) \ge 0$, then $n_2(x) \le \lim_{x\to b} n_2(x) = 1 - \lim_{x\to b}\int_a^x \rho_*(y)\,dy - \lim_{x\to b}\frac{g_*(x)\rho_*(x)}{x} = 0$ because $\lim_{x\to b} g_*(x)\rho_*(x) = 0$. Therefore, $f'(x) \le 0$ if $x > z$, finishing the proof of the lemma. □

As alluded to in the previous subsection, of crucial importance in the use of Stein's method, is a quantitatively explicit boundedness result on the derivative of the solution to Stein's equation. We take this up in the next lemma.

Lemma 2.5. *Recall the function*

$$Q(x) := x^2 - xg_*'(x) + g_*(x)$$

defined in (1.5), *for all $x \in \mathbf{R}$ except possibly at a and b. Assume that $g_*''(x) < 2$ for all x and that $\frac{x - g_*'(x)}{Q(x)}$ tends to a finite limit as $x \to a$ and as $x \to b$. Suppose $0 < z < b$. Then $f'(x)$ is bounded. In particular, if $a < x \le z$,*

$$0 \le f'(x) \le \frac{z}{[g_*(z)]^2\rho_*(z)} + \frac{1}{Q(0)} < \infty,$$

while if $b > x > z$,

$$-\infty < -\frac{1}{Q(z)} \le f'(x) \le 0. \tag{2.10}$$

To prove this lemma, we need two auxiliary results. The first one introduces and studies the function Q which we already encountered in the introduction, and which will help us state and prove our results in an efficient way. The second one shows the relation between Q, g_*, and the tail Φ_* of Z, under conditions which will be easily verified later on in the Pearson case.

Lemma 2.6.

1. *If* $x \notin (a,b)$, *then* $Q(x) = x^2 > 0$.
2. *If* g_* *is twice differentiable in* (a,b) *(for example, when* ρ_* *is twice differentiable), then* $Q'(x) = x\,(2 - g_*''(x))$.
3. *If moreover* $g_*''(x) < 2$ *in* (a,b), *a reasonable assumption as we shall see later when* Z *is a Pearson random variable, then* $\min_{(a,b)} Q = Q\,(0)$ *so that* $Q(x) \geq Q(0) = g_*(0) > 0$.

Lemma 2.7. *With the assumptions on* g_* *and* Q *as in Lemma* 2.5, *then for all* x,

$$\frac{\max\left(x - g_*'(x), 0\right)}{Q(x)} g_*(x)\rho_*(x) \leq \Phi_*(x) \tag{2.11}$$

and

$$\frac{\max\left(g_*'(x) - x, 0\right)}{Q(x)} g_*(x)\rho_*(x) \leq 1 - \Phi_*(x). \tag{2.12}$$

Moreover for $0 < x < b$, *we have*

$$\Phi_*(x) \leq \frac{1}{x} \cdot g_*(x)\rho_*(x) \tag{2.13}$$

while if $a < x < 0$, *then*

$$1 - \Phi_*(x) \leq \frac{1}{-x} \cdot g_*(x)\rho_*(x). \tag{2.14}$$

Proof of Lemma 2.5. If $x < a$ with $a > -\infty$, then $f'(x) = \frac{1-\mathbf{E}[h(Z)]}{x^2} \leq \frac{1-\mathbf{E}[h(Z)]}{a^2}$. If $x > b$ with $b < \infty$, then $f'(x) = -\frac{\mathbf{E}[h(Z)]}{x^2} \geq -\frac{\mathbf{E}[h(Z)]}{b^2}$. So now we only need to assume that $x \in (a,b)$.

Suppose $a < x \leq z$. Use $f'(x) \geq 0$ given in (2.9):

$$\begin{aligned} f'(x) &= \frac{1 - \mathbf{E}[h(Z)]}{[g_*(x)]^2 \rho_*(x)} \left(x \int_a^x \rho_*(y)\,dy + g_*(x)\rho_*(x) \right) \\ &\leq \frac{x}{[g_*(x)]^2\rho_*(x)} \left(1 - \Phi_*(x)\right) + \frac{1}{g_*(x)}\ . \end{aligned}$$

When $x \geq 0$, we can rewrite the upper bound as:

$$f'(x) \leq r(x) + \frac{1}{g_*(x)} \left[1 - \frac{x\Phi_*(x)}{g_*(x)\rho_*(x)} \right]$$

where

$$r(x) = \frac{x}{[g_*(x)]^2\rho_*(x)} = \frac{x}{g_*(x)[g_*(x)\rho_*(x)]}.$$

We can bound $r(x)$ above since

$$\begin{aligned} r'(x) &= \frac{[g_*(x)]^2\rho_*(x) - x\left[g_*(x)\left(-x\rho_*(x)\right) + g_*(x)\rho_*(x)g_*'(x)\right]}{[g_*(x)]^4\left[\rho_*(x)\right]^2} \\ &= \frac{g_*(x) + x^2 - xg_*'(x)}{[g_*(x)]^3\rho_*(x)} = \frac{Q(x)}{[g_*(x)]^3\rho_*(x)} > 0 \end{aligned}$$

so $r(x) \leq r(z)$. To bound $\left[1 - x\Phi_*(x)/\left(g_*(x)\rho_*(x)\right)\right]/g_*(x)$, use (2.11) of Lemma 2.7:

$$\begin{aligned} \frac{1}{g_*(x)}\left[1 - \frac{x\Phi_*(x)}{g_*(x)\rho_*(x)}\right] &\leq \frac{1}{g_*(x)}\left[1 - \frac{x}{g_*(x)\rho_*(x)} \cdot \frac{x - g_*'(x)}{Q(x)}g_*(x)\rho_*(x)\right] \\ &= \frac{1}{g_*(x)}\left[1 - \frac{x^2 - xg_*'(x)}{Q(x)}\right] \\ &= \frac{1}{g_*(x)} \cdot \frac{g_*(x)}{Q(x)} = \frac{1}{Q(x)} \leq \frac{1}{Q(0)}. \end{aligned}$$

Therefore,

$$f'(x) \leq \frac{z}{[g_*(z)]^2\rho_*(z)} + \frac{1}{Q(0)}.$$

When $x < 0$, we use (2.12) of Lemma 2.7:

$$\begin{aligned} f'(x) &\leq \frac{x}{[g_*(x)]^2\rho_*(x)}\left(1 - \Phi_*(x)\right) + \frac{1}{g_*(x)} \\ &\leq \frac{x}{[g_*(x)]^2\rho_*(x)} \cdot \frac{g_*'(x) - x}{Q(x)}g_*(x)\rho_*(x) + \frac{1}{g_*(x)} \\ &= \frac{1}{g_*(x)} \cdot \frac{xg_*'(x) - x^2}{Q(x)} + \frac{1}{g_*(x)} \\ &= \frac{1}{g_*(x)}\left[\frac{xg_*'(x) - x^2 + Q(x)}{Q(x)}\right] = \frac{1}{Q(x)} \leq \frac{1}{Q(0)} \\ &\leq \frac{z}{[g_*(z)]^2\rho_*(z)} + \frac{1}{Q(0)}. \end{aligned}$$

Now we prove (2.10) and so suppose $x > z > 0$. From the proof of Lemma 2.4,

$$\begin{aligned} f'(x) &= \frac{\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)}\left(x - x\int_a^x \rho_*(y)\,dy - g_*(x)\rho_*(x)\right) \\ &= \frac{\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)}\left(x\Phi_*(x) - g_*(x)\rho_*(x)\right). \end{aligned}$$

We conclude by again using (2.11) of Lemma 2.7, to get

$$\begin{aligned}
-f'(x) &\leq \frac{1}{[g_*(x)]^2\rho_*(x)}\left(g_*(x)\rho_*(x) - x\Phi_*(x)\right)\\
&\leq \frac{1}{[g_*(x)]^2\rho_*(x)}\left(g_*(x)\rho_*(x) - x\cdot\frac{x-g_*'(x)}{Q(x)}\cdot g_*(x)\rho_*(x)\right)\\
&= \frac{1}{g_*(x)}\left(1-\frac{x^2-xg_*'(x)}{Q(x)}\right) = \frac{1}{g_*(x)}\cdot\frac{g_*(x)}{Q(x)}\\
&= \frac{1}{Q(x)} \leq \frac{1}{Q(z)}.
\end{aligned}$$

□

Remark 2.8. Since $z > 0$, and $Q(z) > Q(0)$, Lemma 2.5 implies the following convenient single bound for any fixed $z > 0$, uniform for all $x \in (a,b)$:

$$|f'(x)| \leq \frac{z}{[g_*(z)]^2\rho_*(z)} + \frac{1}{Q(0)}.$$

Occasionally, this will be sufficient for some of our purposes. The more precise bounds in Lemma 2.5 will also be needed, however.

3. Main results

In order to exploit the boundedness of f', we adopt the technique pioneered by Nourdin and Peccati, to rewrite expressions of the form $\mathbf{E}[Xm(X)]$ where m is a function, using the Malliavin calculus. For ease of reference, we include here the requisite Malliavin calculus constructs. Full details can be found in [17]; also see [22, Section 2] for an exhaustive summary.

3.1. Elements of Malliavin calculus

We assume our random variable X is measurable with respect to an isonormal Gaussian process W, associated with its canonical separable Hilbert space H. For illustrative purposes, one may further assume, as we now do, that W is the standard white-noise corresponding to $H = L^2([0,1])$, which is constructed using a standard Brownian motion on $[0,1]$, also denoted by W, endowed with its usual probability space $(\Omega, \mathcal{F}, \mathbf{P})$. This means that the white noise W is defined by $W(f) = \int_0^1 f(s)\,dW(s)$ for any $f \in H$, where the stochastic integral is the Wiener integral of f with respect to the Wiener process W. If we denote $I_0(f) = f$ for any non-random constant f, then for any integer $n \geq 1$ and any symmetric function $f \in H^n$, we let

$$I_n(f) := n!\int_0^1\int_0^{s_1}\cdots\int_0^{s_{n-1}} f(s_1, s_2, \ldots, s_n)\,dW(s_n)\cdots dW(s_2)\,dW(s_1),$$

where this integral is an iteration of n Itô integrals. It is called the nth multiple Wiener integral of f w.r.t. W, and the set $\mathcal{H}_n := \{I_n(f) : f \in H^n\}$ is the nth Wiener chaos of W. Note that $I_1(f) = W(f)$, and that $\mathbf{E}[I_n(f)] = 0$ for all $n \geq 1$. Again, see [17, Section 1.2] for the general definition of I_n and $\mathcal{H}_n$ when W

is a more general isonormal Gaussian process. The main representation theorem of the analysis on Wiener space is that $L^2(\Omega, \mathcal{F}, \mathbf{P})$ is the direct sum of all the Wiener chaoses. In other words, $X \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ if and only if there exists a sequence of non-random symmetric functions $f_n \in H^n$ with $\sum_{n=0}^{\infty} \|f_n\|_{H^n}^2 < \infty$ such that $X = \sum_{n=0}^{\infty} I_n(f_n)$. Note that $\mathbf{E}[X] = f_0$. Moreover, the terms in this so-called Wiener chaos decomposition of X are orthogonal in $L^2(\Omega, \mathcal{F}, \mathbf{P})$, and we have the isometry property $\mathbf{E}\left[X^2\right] = \sum_{n=0}^{\infty} n! \|f_n\|_{H^n}^2$. We are now in a position to define the Malliavin derivative D.

Definition 3.1. Let $\mathbf{D}^{1,2}$ be the subset of $L^2(\Omega, \mathcal{F}, \mathbf{P})$ formed by those $X = \sum_{n=0}^{\infty} I_n(f_n)$ such that

$$\sum_{n=1}^{\infty} n\ n! \|f_n\|_{H^n}^2 < \infty.$$

The Malliavin derivative operator D is defined from $\mathbf{D}^{1,2}$ to $L^2(\Omega \times [0,1])$ by $DX = 0$ if $X = \mathbf{E}X$ is non-random, and otherwise, for all $r \in [0,1]$, by

$$D_r X = \sum_{n=1}^{\infty} n I_{n-1}(f_n(r, \cdot)).$$

This can be understood as a Fréchet derivative of X with respect to the Wiener process W. If $X = W(f)$ then $DX = f$. Of note is the chain-rule formula $D(F(X)) = F'(X) DX$ for any differentiable F with bounded derivative, and any $X \in \mathbf{D}^{1,2}$.

Definition 3.2. The generator of the Ornstein–Uhlenbeck semigroup L is defined as follows. Let $X = \sum_{n=1}^{\infty} I_n(f_n)$ be a centered r.v. in $L^2(\Omega)$. If $\sum_{n=1}^{\infty} n^2 n! |f_n|^2 < \infty$, then we define a new random variable LX in $L^2(\Omega)$ by $-LX = \sum_{n=1}^{\infty} n I_n(f_n)$. The pseudo-inverse of L operating on centered r.v.'s in $L^2(\Omega)$ is defined by the formula $-L^{-1}X = \sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n)$. If X is not centered, we define its image by L and L^{-1} by applying them to $X - \mathbf{E}X$.

As explained in the introduction, for $X \in \mathbf{D}^{1,2}$, the random variable $G := \left\langle DX; -DL^{-1}X \right\rangle_H$ plays a crucial role to understand how X's law compares to that of our reference random variable Z. The next lemma is the key to combining the solutions of Stein's equations with the Malliavin calculus. Its use to prove our main theorems relies heavily on the fact that these solutions have bounded derivatives.

Lemma 3.3 (Theorem 3.1 in [8], Lemma 3.5 in [22]). *Let $X \in \mathbf{D}^{1,2}$ be a centered random variable with a density, and $G = \langle DX; -DL^{-1}X \rangle_H$. For any deterministic, continuous and piecewise differentiable function m such that m' is bounded,*

$$\mathbf{E}[Xm(X)] = \mathbf{E}[m'(X)G].$$

3.2. General tail results

The main theoretical results of this paper compare the tails of any two random variables X and Z, as we now state in the next two theorems. In terms of their usage, Z represents a reference random variable in these theorems; this can be seen from the fact that we have a better control in the theorems' assumption on the g_* coming from Z than on the law of X. Also, we will apply these theorems to a Pearson random variable Z in the next section, while there will be no restriction on $X \in \mathbf{D}^{1,2}$ beyond the assumption of the theorems in the present section. We will see that all assumptions on Z in this section are satisfied when Z is a Pearson random variable.

Theorem 3.4. *Let Z be a centered random variable with a twice differentiable density over its support (a,b). Let g_* and Q be defined as in* (1.4) *and* (1.5)*, respectively. Suppose that $g_*''(x) < 2$, and $\frac{x-g_*'(x)}{Q(x)}$ has a finite limit as $x \to a$ and $x \to b$. Let $X \in \mathbf{D}^{1,2}$ be a centered random variable with a density, and whose support (a, b_X) contains (a,b). Let G be as in* (1.2)*. If $G \geq g_*(X)$ a.s., then for every $z \in (0,b)$,*

$$\mathbf{P}[X > z] \geq \Phi_*(z) - \frac{1}{Q(z)} \int_z^b (2x - z)\mathbf{P}[X > x]\, dx.$$

Proof. Taking expectations in *Stein's equation* (2.3), i.e., referring to (2.4), we have

$$\mathbf{P}[X \leq z] - \mathbf{P}[Z \leq z] = \mathbf{E}[g_*(X)f'(X) - Xf(X)]$$

which is equivalent to

$$\mathbf{P}[X > z] - \Phi_*(z) = \mathbf{E}[Xf(X) - g_*(X)f'(X)].$$

Since $g_*(X) \geq 0$ almost surely and $f'(x) \leq 0$ if $x > z$,

$$\begin{aligned}
&\mathbf{P}[X > z] - \Phi_*(z) \\
&\quad = \mathbf{E}[\mathbf{1}_{X \leq z} Xf(X)] + \mathbf{E}[\mathbf{1}_{X > z} Xf(X)] - \mathbf{E}[\mathbf{1}_{X \leq z} g_*(X)f'(X)] \\
&\qquad - \mathbf{E}[\mathbf{1}_{X > z} g_*(X)f'(X)] \\
&\quad \geq \mathbf{E}[\mathbf{1}_{X \leq z} Xf(X)] + \mathbf{E}[\mathbf{1}_{X > z} Xf(X)] - \mathbf{E}[\mathbf{1}_{X \leq z} g_*(X)f'(X)].
\end{aligned}$$

Let $m(x) = [f(a) - f(z)]\mathbf{1}_{x \leq a} + [f(x) - f(z)]\mathbf{1}_{a < x \leq z}$ where the first term is 0 if $a = -\infty$. Note that m is continuous and piecewise differentiable. The derivative is $m'(x) = f'(x)\mathbf{1}_{a<x\leq z}$ except at $x = a$ and $x = z$. We saw in Lemma 2.5 that f' is bounded. Therefore, since $X \in \mathbf{D}^{1,2}$, we can use Lemma 3.3 to conclude that

$$[f(a) - f(z)]\mathbf{E}[\mathbf{1}_{X \leq a} X] + \mathbf{E}[\mathbf{1}_{a < X \leq z} X(f(X) - f(z))] = \mathbf{E}[\mathbf{1}_{a<X\leq z} f'(X)G]$$

from which we derive

$$\mathbf{E}[\mathbf{1}_{X \leq z} Xf(X)] - f(z)\mathbf{E}[\mathbf{1}_{X \leq z} X] = \mathbf{E}[\mathbf{1}_{X \leq z} f'(X)G].$$

Therefore,

$$
\begin{aligned}
&\mathbf{P}[X > z] - \Phi_*(z) \\
&\quad \geq \{\mathbf{E}[\mathbf{1}_{X \leq z} f'(X) G] + f(z)\mathbf{E}[\mathbf{1}_{X \leq z} X]\} + \mathbf{E}[\mathbf{1}_{X > z} X f(X)] \\
&\qquad - \mathbf{E}[\mathbf{1}_{X \leq z} g_*(X) f'(X)] \\
&\quad = \mathbf{E}[\mathbf{1}_{X \leq z} f'(X)(G - g_*(X))] + f(z)\mathbf{E}[\mathbf{1}_{X \leq z} X] + \mathbf{E}[\mathbf{1}_{X > z} X f(X)] \\
&\quad \geq f(z)\mathbf{E}[\mathbf{1}_{X \leq z} X] + \mathbf{E}[\mathbf{1}_{X > z} X f(X)] \\
&\quad = f(z)\mathbf{E}[\mathbf{1}_{X \leq z} X] + \mathbf{E}[\mathbf{1}_{X > z} X f(X)] - f(z)\mathbf{E}[\mathbf{1}_{X > z} X] + f(z)\mathbf{E}[\mathbf{1}_{X > z} X] \\
&\quad = f(z)\mathbf{E}[X] + \mathbf{E}[\mathbf{1}_{X > z} X (f(X) - f(z))] \\
&\quad = \mathbf{E}[\mathbf{1}_{X > z} X (f(X) - f(z))] \,.
\end{aligned}
$$

Write $f(X) - f(z) = f'(\xi)(X - z)$ for some random $\xi > z$ ($X > \xi$ also). Note that $f'(\xi) < 0$ since $\xi > z$. We have $\mathbf{P}[X > z] - \Phi_*(z) \geq \mathbf{E}[\mathbf{1}_{X>z} f'(\xi) X (X - z)]$. From Lemma 2.5,

$$
f'(\xi) \geq -\frac{1}{Q(z)}
$$

since from Lemma 2.6, Q is nondecreasing on $(0, b)$.

If we define $S(z) := \mathbf{P}[X > z]$, it is elementary to show (see [22]) that

$$
\mathbf{E}[\mathbf{1}_{X>z} X (X - z)] \leq \int_z^b (2x - z) S(x)\, dx.
$$

From $\mathbf{P}[X > z] - \Phi_*(z) \geq \mathbf{E}[\mathbf{1}_{X>z} f'(\xi) X (X - z)]$,

$$
\mathbf{P}[X > z] \geq \Phi_*(z) - \frac{1}{Q(z)} \int_z^b (2x - z) S(x)\, dx
$$

which is the statement of the theorem.

Lastly the reader will check that the assumption that the supports of Z and X have the same left-endpoint is not a restriction: stated briefly, this assumption is implied by the assumption $G \geq g_*(X)$ a.s., because $G = g_X(X)$ and g_* (resp. g_X) has the same support as Z (resp. X). □

To obtain a similar upper bound result, we will consider only asymptotic statements for z near b, and we will need an assumption about the relative growth rate of g_* and Q near b. We will see in the next section that this assumption is satisfied for all members of the Pearson class with four moments, although that section also contains a modification of the proof below which is more efficient when applied to the Pearson class.

Theorem 3.5. *Assume all the conditions of Theorem* 3.4 *hold, except for the support of X, which we now assume is contained in (a, b). Assume moreover that there exists $c < 1$ such that $\limsup_{z \to b} g_*(z)/Q(z) < c$. If $G \leq g_*(X)$ a.s., then there exists z_0 such that $b > z > z_0$ implies*

$$
\mathbf{P}[X > z] \leq \frac{1}{1 - c} \Phi_*(z).
$$

Proof. From Stein's equation (2.3), and its application (2.4),

$$\mathbf{P}[X > z] - \Phi_*(z) = \mathbf{E}[Xf(X) - g_*(X)f'(X)].$$

Since $X \in \mathbf{D}^{1,2}$, in Lemma 3.3, we can let $m = f$ since f is continuous, differentiable everywhere except at $x = a$ and $x = b$, and from Lemma 2.5 has a bounded derivative. Therefore,

$$\begin{aligned}
&\mathbf{P}[X > z] - \Phi_*(z) \\
&\quad = \mathbf{E}[Gf'(X)] - \mathbf{E}[g_*(X)f'(X)] \\
&\quad = \mathbf{E}[f'(X)\,(G - g_*(X))] \\
&\quad = \mathbf{E}[\mathbf{1}_{X\le z}f'(X)\,(G - g_*(X))] + \mathbf{E}[\mathbf{1}_{X>z}f'(X)\,(G - g_*(X))] \\
&\quad \le \mathbf{E}[\mathbf{1}_{X>z}f'(X)\,(G - g_*(X))] \\
&\quad = \mathbf{E}[\mathbf{1}_{X>z}f'(X)\mathbf{E}\,[G|X]] - \mathbf{E}[\mathbf{1}_{X>z}f'(X)g_*(X)]
\end{aligned}$$

where the last inequality follows from the assumption $G - g_*(X) \le 0$ a.s. and if $X \le z$, then $f'(X) \ge 0$. By Proposition 3.9 in [8], $\mathbf{E}\,[G|X] \ge 0$ a.s. Since $f'(X) \le 0$ if $X > z$, then by the last statement in Lemma 2.5, and the assumption on the asymptotic behavior of g_*/Q, for z large enough,

$$\begin{aligned}
\mathbf{P}[X > z] - \Phi_*(z) &\le -\mathbf{E}[\mathbf{1}_{X>z}f'(X)g_*(X)] \\
&\le \mathbf{E}\left[\mathbf{1}_{X>z}\frac{g_*(X)}{Q\,(X)}\right] \\
&\le c\mathbf{P}[X > z].
\end{aligned}$$

The theorem immediately follows. □

4. Pearson distributions

By definition, the law of a random variable Z is a member of the Pearson family of distributions if Z's density ρ_* is characterized by the differential equation $\rho_*'(z)/\rho_*(z) = (a_1 z + a_0)/(\alpha z^2 + \beta z + \gamma)$ for z in its support (a, b), where $-\infty \le a < b \le \infty$. If furthermore $\mathbf{E}[Z] = 0$, Stein (Theorem 1, p. 65 in [21]) proved that g_* has a simple form: in fact, it is quadratic in its support. Specifically, $g_*(z) = \alpha z^2 + \beta z + \gamma$ for all $z \in (a, b)$ if and only if

$$\frac{\rho_*'(z)}{\rho_*(z)} = -\frac{(2\alpha + 1)z + \beta}{\alpha z^2 + \beta z + \gamma}. \tag{4.1}$$

The Appendix contains a description of various cases of Pearson distributions, which are characterized by their first four moments, if they exist. In this section, we will operate under the following.

Assumption P1. Our Pearson random variable satisfies $\mathbf{E}\left[Z^2\right] < \infty$ and $z^3\rho_*(z) \to 0$ as $z \to a$ and $z \to b$.

Remark 4.1. This assumption holds as soon as $\mathbf{E}\left[Z^3\right] < \infty$, which, by Lemma A.2 in the Appendix, holds if and only if $\alpha < 1/2$. The existence of a second moment, by the same lemma, holds if and only if $\alpha < 1$.

4.1. General comparisons to Pearson tails

In preparation to stating corollaries to our main theorems, applicable to all Pearson Z's simultaneously, we begin by investigating the specific properties of g_* and Q in the Pearson case. Because $g_*(z) = \left(\alpha z^2 + \beta z + \gamma\right) \mathbf{1}_{(a,b)}(z)$, we have the following observations:

- Since $g_*''(z) = 2\alpha$ on (a,b), and $\alpha < 1$ according to Remark 4.1, it follows that $g_*''(z) < 2$.
- If $z \in (a,b)$, then
$$Q(z) = z^2 - zg_*'(z) + g_*(z) = (1-\alpha)\, z^2 + \gamma$$
and so $Q(z) \geq Q(0) = \gamma = g_*(0) > 0$, where the last inequality is because g_* is strictly positive on the interior of its support, which always contains 0. This is a quantitative confirmation of an observation made earlier about the positivity of Q in the general case.
- As $z \to a$ and $z \to b$,
$$\frac{z - g_*'(z)}{Q(z)} = \frac{(1-2\alpha)\, z - \beta}{(1-\alpha)\, z^2 + \gamma}$$
approaches a finite number in case a and b are finite. As $|z| \to \infty$, the above ratio approaches 0.
- We have $\mathbf{E}\left[Z^2\right] = \frac{\gamma}{1-\alpha}$. Again, this is consistent with $\gamma > 0$ and $\alpha < 1$.

Remark 4.2. The above observations collectively mean that all the assumptions of Theorem 3.4 are satisfied for our Pearson random variable Z, so we can state the following.

Proposition 4.3. *Let Z be a centered Pearson random variable satisfying Assumption* P1. *Let g_* be defined as in* (1.4). *Let $X \in \mathbf{D}^{1,2}$ be a centered random variable with a density, and whose support (a, b_X) contains (a,b). Suppose that $G \geq g_*(X)$ a.s. Then for every $z \in (0,b)$,*

$$\mathbf{P}[X > z] \geq \Phi_*(z) - \frac{1}{(1-\alpha)\, z^2 + \gamma} \int_z^b (2x - z)\mathbf{P}[X > x]\, dx. \tag{4.2}$$

We have a quantitatively precise statement on the relation between $\mathrm{Var}[X]$ and the Pearson parameters.

Proposition 4.4.

1. *Assume that the conditions of Proposition* 4.3 *hold, particularly that $G \geq g_*(X)$; assume the support (a,b) of g_* coincides with the support of X. Then*
$$\mathrm{Var}[X] \geq \frac{\gamma}{1-\alpha} = \mathrm{Var}[Z].$$

2. *If we assume instead that $G \le g_*(X)$ a.s., then the inequality above is reversed.*

Proof. Since X has a density, we can apply Lemma 3.3 and let $m(x) = x$.

$$\begin{aligned}\mathrm{Var}[X] &= \mathbf{E}[Xm(X)] = \mathbf{E}[G] \ge \mathbf{E}[g_*(X)] \\ &= \mathbf{E}[\mathbf{1}_{a<X<b}\left(\alpha X^2 + \beta X + \gamma\right)] = \alpha \mathbf{E}[X^2] + \beta \mathbf{E}[X] + \gamma \\ (1-\alpha)\,\mathrm{Var}[X] &\ge \beta \cdot 0 + \gamma\end{aligned}$$

This proves point 1. Point 2 is done identically. □

In order to formulate results that are specifically tailored to tail estimates, we now make the following

Assumption P2. The right-hand endpoint of our Pearson distribution's support is $b = +\infty$

Remark 4.5. Assumption P2 leaves out Case 3 in the Appendix in our list of Pearson random variables, i.e., the case of Beta distributions. Therefore, inspecting the parameter values in the other 4 Pearson cases, we see that Assumption P2 implies $\alpha \ge 0$, and also implies that if $\alpha = 0$, then $\beta \ge 0$.

Remark 4.6. In most of the results to follow, we will assume moreover that $\alpha < \frac{1}{2}$. By Lemma A.2, this is equivalent to requiring $\mathbf{E}\left[|Z|^3\right] < \infty$, and more generally from the lemma, our Pearson distribution has moment of order m if and only if $\alpha < 1/(m-1)$. As mentioned, $\alpha < \frac{1}{2}$ thus implies Assumption P1. Consequently Theorem 3.5 implies the following.

Corollary 4.7. *Let Z be a centered Pearson random variable satisfying Assumption* P2 (*support of Z is* $(a, +\infty)$). *Assume $\alpha < 1/2$. Let g_* be defined as in* (1.4). *Let $X \in \mathbf{D}^{1,2}$ be a centered random variable with a density and support contained in $(a, +\infty)$. If $G \le g_*(X)$ a.s., for any $K > \frac{1-\alpha}{1-2\alpha}$, there exists z_0 such that if $z > z_0$, then*

$$\mathbf{P}[X > z] \le K\ \Phi_*(z).$$

Proof. Since

$$\frac{g_*(z)}{Q(z)} = \frac{\alpha z^2 + \beta z + \gamma}{(1-\alpha)\, z^2 + \gamma}$$

then $\limsup_{z\to\infty} g_*(z)/Q(z) = \alpha/(1-\alpha) < 1$ if and only if $\alpha < \frac{1}{2}$. Therefore, Theorem 3.5 applies in this case, and with the c defined in that theorem, we may take here any $c > \alpha/(1-\alpha)$, so that we may take any $K = 1/(1-c)$ as announced. □

The drawback of our general lower bound theorems so far is that their statements are somewhat implicit. Our next effort is to fix this problem in the specific case of a Pearson Z: we strengthen Proposition 4.3 so that the tail $\mathbf{P}[X > z]$ only appears in the left-hand side of the lower bound inequality, making the bound explicit. The cost for this is an additional regularity and integrability assumption, whose scope we also discuss.

Corollary 4.8. *Assume that the conditions of Proposition* 4.3 *hold; in particular, assume* $X \in \mathbf{D}^{1,2}$ *and* $G \geq \alpha X^2 + \beta X + \gamma$ *a.s. In addition, assume there exists a constant* $c > 2$ *such that* $\mathbf{P}[X > z] \leq z\rho(z)/c$ *holds for large* z *(where* ρ *is the density of* X*). Then for large* z*,*

$$\mathbf{P}[X > z] \geq \frac{(c-2)\,Q(z)}{(c-2)\,Q(z) + 2z^2}\Phi_*(z) \approx \frac{(c-2) - \alpha\,(c-2)}{c - \alpha\,(c-2)}\Phi_*(z).$$

The existence of such a $c > 2$ *above is guaranteed if we assume* $g(z) \leq z^2/c$ *for large* z*, where* $g(x) := \mathbf{E}[G|X = x]$ *(or equivalently,* g *defined in* (1.3)*). Moreover, this holds automatically if* $G \leq \bar{g}_*(X)$ *a.s. for some quadratic function* $\bar{g}_*(x) = \bar{\alpha}x^2 + \bar{\beta}x + \bar{\gamma}$ *with* $\bar{\alpha} < 1/2$*.*

Proof. Since $z > 0$, we can replace $2x - z$ by $2x$ in the integral of (4.2).

$$\begin{aligned} F(z) &:= \int_z^\infty xS(x)\,dx \leq \frac{1}{c}\int_z^\infty x^2|S'(x)|\,dx \\ &= \frac{1}{c}\left(z^2S(z) - \lim_{x\to\infty} x^2S(x) + 2\int_z^\infty xS(x)\,dx\right) \leq \frac{1}{c}\left(z^2S(z) + 2F(z)\right) \\ F(z) &\leq \frac{1}{c-2}z^2S(z). \end{aligned}$$

Therefore

$$S(z) = \mathbf{P}[X > z] \geq \Phi_*(z) - \frac{2}{Q(z)}F(z) \geq \Phi_*(z) - \frac{2z^2}{(c-2)\,Q(z)}S(z)$$

$$S(z)\left[1 + \frac{2z^2}{(c-2)\,Q(z)}\right] \geq \Phi_*(z)$$

$$\begin{aligned} S(z) &\geq \frac{(c-2)\,Q(z)}{(c-2)\,Q(z) + 2z^2}\Phi_*(z) \\ &\approx \frac{(c-2)\,(1-\alpha)}{(c-2)\,(1-\alpha) + 2}\Phi_*(z). \end{aligned}$$

This proves the inequality of the corollary.

To prove the second statement, recall that Nourdin and Viens (Theorem 3.1 in [16]) showed that

$$g(X) = \frac{\int_X^\infty x\rho(x)\,dx}{\rho(X)}$$

P-a.s. They also noted that the support of ρ is an interval since $X \in \mathbf{D}^{1,2}$. Therefore,

$$\frac{z}{c}\rho(z) \geq \frac{1}{z}g(z)\rho(z) = \int_z^\infty \frac{x}{z}\rho(x)\,dx \geq \int_z^\infty \rho(x)\,dx$$

a.s. This finishes the proof of the corollary. □

4.2. Comparisons in specific scales

In this section and the next, we will always assume $X \in \mathbf{D}^{1,2}$ is a centered random variable with a density and with support (a,∞), and we will continue to denote by g the function defined by $g(x) := \mathbf{E}[G|X=x]$, or equivalently, defined in (1.3).

We can exploit the specific asymptotic behavior of the tail of the various Pearson distributions, via Lemma A.1 in the Appendix, to draw sharp conclusions about X's tail. For instance, if g is comparable to a Pearson distribution's g_* with $\alpha \neq 0$, we get a power decay for the tail (Corollary 4.9 below), while if α is zero and β is not, we get comparisons to exponential-type or gamma-type tails (Corollary 4.10 below). In both cases, when upper and lower bounds on G occur with the same α on both sides, we get sharp asymptotics for X's tail, up to multiplicative constants.

Corollary 4.9. *Let $g_*(x) := \alpha x^2+\beta x+\gamma$ and $\bar{g}_*(x) := \bar{\alpha}x^2+\bar{\beta}x+\bar{\gamma}$ be two functions corresponding to Pearson distributions (e.g., via (1.4)) where $0<\alpha\le\bar{\alpha}<1/2$.*

1. *If $g(x) \le \bar{g}_*(x)$ for all $x \ge a$, then there is a constant $c_u\left(\bar{\alpha},\bar{\beta},\bar{\gamma}\right)>0$ such that for large z,*
$$\mathbf{P}[X>z] \le \frac{c_u}{z^{1+1/\bar{\alpha}}}.$$
2. *If $g_*(x) \le g(x) \le \bar{g}_*(x)$ for all $x \ge a$, then there are constants $c_u\left(\bar{\alpha},\bar{\beta},\bar{\gamma}\right)>0$ and $c_l(\bar{\alpha},\alpha,\beta,\gamma)>0$ such that for large z,*
$$\frac{c_l}{z^{1+1/\alpha}} \le \mathbf{P}[X>z] \le \frac{c_u}{z^{1+1/\bar{\alpha}}}.$$

Proof. Let $\Phi_{*\alpha,\beta,\gamma}$ and $\Phi_{*\bar{\alpha},\bar{\beta},\bar{\gamma}}$ be the probability tails of the Pearson distributions corresponding to g_* and $\bar{g}_*$ respectively. We can prove Point 1 by using Corollary 4.7 and Lemma A.1. There is a constant $k_u\left(\bar{\alpha},\bar{\beta},\bar{\gamma}\right)>0$ such that, for any $K>\frac{1-\bar{\alpha}}{1-2\bar{\alpha}}$, for large z,
$$\mathbf{P}[X>z] \le K\Phi_{*\bar{\alpha},\bar{\beta},\bar{\gamma}}(z) \le K\cdot\frac{k_u}{z^{1+1/\bar{\alpha}}}.$$
The upper bound in Point 2 follows directly from Point 1 because of the condition $g(x) \le \bar{g}_*(x)$. This same condition also allows us to give a lower bound for $\mathbf{P}[X>z]$. Fix any $c \in (2,1/\bar{\alpha})$. By Corollary 4.8 and Lemma A.1, there is a constant $k_l(\alpha,\beta,\gamma)>0$ such that for large z,
$$\mathbf{P}[X>z] \ge \frac{(c-2)-\alpha(c-2)}{c-\alpha(c-2)}\Phi_{*\alpha,\beta,\gamma}(z) \ge \frac{(c-2)-\alpha(c-2)}{c-\alpha(c-2)}\cdot\frac{k_l}{z^{1+1/\alpha}}. \qquad \square$$

Corollary 4.10. *Let $g_*(x) := (\beta x+\gamma)_+$ and $\bar{g}_*(x) = \left(\bar{\beta}x+\bar{\gamma}\right)_+$ be two functions corresponding to Pearson distributions (e.g., via (1.4)) where $\beta,\bar{\beta},\gamma,\bar{\gamma}>0$ and $a=-\gamma/\beta$.*

1. *If $g(x) \le \bar{g}_*(x)$ for all x, then there is a constant $c_u\left(\bar{\beta},\bar{\gamma}\right)>0$ such that for large z,*
$$\mathbf{P}[X>z] \le c_u z^{-1+\bar{\gamma}/\bar{\beta}^2}e^{-z/\bar{\beta}}.$$

2. *If* $g_*(x) \leq g(x) \leq \bar{g}_*(x)$ *for all* x*, then there are constants* $c_u\left(\bar{\beta},\bar{\gamma}\right) > c_l(\beta,\gamma) > 0$ *such that for large* z*,*

$$c_l\ z^{-1+\gamma/\beta^2} e^{-z/\beta} \leq \mathbf{P}[X > z] \leq c_u z^{-1+\bar{\gamma}/\bar{\beta}^2} e^{-z/\bar{\beta}}.$$

Proof. Let $\Phi_{*\beta,\gamma}$ and $\Phi_{*\bar{\beta},\bar{\gamma}}$ be as in the proof of the previous corollary, noting here that $\alpha = \bar{\alpha} = 0$. The proof of Point 1 is similar to the proof of Point 1 in Corollary 4.9. The upper bound in Point 2 follows from Point 1 above and Point 1 of Corollary 4.9. For the lower bound of Point 2, if we fix any $c > 2$, then by Corollary 4.8 and Lemma A.1, there is a constant $k_l(\beta,\gamma) > 0$ such that for large z,

$$\mathbf{P}[X > z] \geq \frac{c-2}{c}\Phi_{*\beta,\gamma} \geq \frac{c-2}{c}\ k_l\ z^{-1+\gamma/\beta^2} e^{-z/\beta}. \qquad \square$$

Remark 4.11. The above corollary improves on a recently published estimate: in [16, Theorem 4.1], it was proved that if the law of $X \in \mathbf{D}^{1,2}$ has a density and if $g(X) \leq \beta X + \gamma$ a.s. (with $\beta \geq 0$ and $\gamma > 0$), then for all $z > 0$, $\mathbf{P}[X > z] \leq \exp\left(-\frac{z^2}{2\beta z + 2\gamma}\right)$. Using $g_*(z) = (\beta z + \gamma)_+$, Point 1 in Corollary 4.10 gives us an asymptotically better upper bound, with exponential rate $e^{-z/\beta}$ instead of $e^{-z/2\beta}$. Our rate is sharp, since our upper bound has the same exponential asymptotics as the corresponding Pearson tail, which is a Gamma tail.

4.3. Asymptotic results

Point 2 of Corollary 4.10 shows the precise behavior, up to a possibly different leading power term which is negligible compared to the exponential, of any random variable in $\mathbf{D}^{1,2}$ whose function g is equal to a Pearson function up to some uncertainty on the γ value. More generally, one can ask about tail asymptotics for X when g is asymptotically linear, or even asymptotically quadratic. Asymptotic assumptions on g are not as strong as assuming bounds on g which are uniform in the support of X, and one cannot expect them to imply statements that are as strong as in the previous subsection. We now see that in order to prove tail asymptotics under asymptotic assumptions, it seems preferable to revert to the techniques developed in [22]. We first propose upper bound results for tail asymptotics, which follow from Point 1 of Corollary 4.9 and Point 1 of Corollary 4.10. Then for full asymptotics, Point 2 of each of these corollaries do not seem to be sufficient, while [22, Corollary 4.5] can be applied immediately. Recall that in what follows $X \in \mathbf{D}^{1,2}$ is centered, has a density, and support (a,∞), and g is defined by $g(x) := \mathbf{E}[G|X = x]$, or equivalently, by (1.3).

Proposition 4.12.

1. *Suppose* $\limsup_{z\to+\infty} g(z)/z^2 = \alpha \in (0, 1/2)$*. Then*

$$\limsup_{z\to+\infty} \frac{\ln \mathbf{P}[X > z]}{\ln z} \leq -\left(1 + \frac{1}{\alpha}\right).$$

2. *Suppose* $\limsup_{z\to+\infty} g(z)/z = \beta > 0$. *Then*

$$\limsup_{z\to+\infty} \frac{\ln \mathbf{P}[X>z]}{z} \leq -\frac{1}{\beta}.$$

Proof. Fix $\varepsilon \in (0, 1/2 - \alpha)$. Then $g(x) < (\alpha+\varepsilon)x^2$ if x is large enough. Therefore, there exists a constant $\gamma_\varepsilon > 0$ such that $g(x) < (\alpha+\varepsilon)x^2 + \gamma_\varepsilon$ for all x. Let Z_ε be the Pearson random variable for which $g_*(z) = (\alpha+\varepsilon)z^2 + \gamma_\varepsilon$. This falls under Case 5 in Appendix A.2, so its support is $(-\infty,\infty)$, which then contains the support of X. From Point 1 of Corollary 4.9, there is a constant c_ε depending on ε such that for z large enough,

$$\mathbf{P}[X>z] \leq c_\varepsilon z^{-1-\frac{1}{\alpha+\varepsilon}}.$$

We then have

$$\begin{aligned}
\ln \mathbf{P}[X>z] &\leq \ln c_\varepsilon - \left(1+\frac{1}{\alpha+\varepsilon}\right)\ln z,\\
\frac{\ln \mathbf{P}[X>z]}{\ln z} &\leq \frac{\ln c_\varepsilon}{\ln z} - \left(1+\frac{1}{\alpha+\varepsilon}\right),\\
\limsup_{z\to\infty} \frac{\ln \mathbf{P}[X>z]}{\ln z} &\leq -\left(1+\frac{1}{\alpha+\varepsilon}\right).
\end{aligned}$$

Since ε can be arbitrarily close to 0, Point 1 of the corollary is proved. The proof of Point 2 is entirely similar, following from Corollary 4.10, which refers to Case 2 of the Pearson distributions given in Appendix A.2. This corollary could also be established by using results from [22]. □

Our final result gives full tail asymptotics. Note that it is not restricted to linear and quadratic behaviors.

Theorem 4.13.

1. *Suppose* $\lim_{z\to+\infty} g(z)/z^2 = \alpha \in (0,1)$. *Then*

$$\lim_{z\to+\infty} \frac{\ln \mathbf{P}[X>z]}{\ln z} \leq -\left(1+\frac{1}{\alpha}\right).$$

2. *Suppose* $\lim_{z\to+\infty} g(z)/z^p = \beta > 0$ *for some* $p \in [0,1)$. *Then*

$$\lim_{z\to+\infty} \frac{\ln \mathbf{P}[X>z]}{z^{2-p}} \leq -\frac{1}{\beta(2-p)}.$$

Proof. Since for any $\varepsilon \in (0, \min(\alpha, 1-\alpha))$, there exists z_0 such that $z > z_0$ implies $(\alpha-\varepsilon)z^2 \leq g(z) \leq (\alpha+\varepsilon)z^2$, the assumptions of Points 2 and 4 (a) in [22, Corollary 4.5] are satisfied, and Point 1 of the Theorem follows easily. Point 2 of the Theorem follows identically, by invoking Points 3 and 4 (b) in [22, Corollary 4.5]. All details are left to the reader. □

Appendix

A.1. Proofs of lemmas

Proof of Lemma 2.1. *Proof of point* 1. If $0 \le x < b$, then clearly $g_*(x) > 0$. If $a < x < 0$, we claim that $g_*(x) > 0$ still. Suppose we have the opposite: $g_*(x) \le 0$. Then $\int_x^b y\rho_*(y)\,dy = g_*(x)\rho_*(x) \le 0$. Since $\int_a^x y\rho_*(y)\,dy < 0$, then $\int_a^b y\rho_*(y)\,dy < 0$, contradicting $\mathbf{E}[Z] = 0$. Thus, $g_*(x) \ge 0$ for all x, and $g_*(x) > 0$ if and only if $a < x < b$.

Proof of point 2. Trivial.

Proof of point 3. This is immediate since

$$\lim_{x\downarrow a} g_*(x)\rho_*(x) = \lim_{x\downarrow a} \int_x^b y\rho_*(y)\,dy = \mathbf{E}[Z]$$

and similarly for $\lim_{x\uparrow b} g_*(x)\rho_*(x) = -\mathbf{E}[Z]$. □

Proof of Lemma 2.2. It is easy to verify that (2.6) and (2.7) are solutions to Stein's equation (2.3). To show that they are the same, let $\varphi(z) := g_*(z)\rho_*(z) = \int_z^b w\rho_*(w)\,dw$ for $z \in (a,b)$. Then

$$\frac{\varphi'(z)}{\varphi(z)} = -\frac{z\rho_*(z)}{g_*(z)\rho_*(z)} = -\frac{z}{g_*(z)}.$$

Integrating over $(y,x) \subseteq (a,b)$ leads to

$$\int_y^x \frac{z}{g_*(z)}\,dz = \log\frac{\varphi(y)}{\varphi(x)} = \log\frac{g_*(y)\rho_*(y)}{g_*(x)\rho_*(x)} \tag{A.1}$$

and so

$$\frac{e^{\int_y^x \frac{z\,dz}{g_*(z)}}}{g_*(y)} = \frac{1}{g_*(y)}\cdot\frac{g_*(y)\rho_*(y)}{g_*(x)\rho_*(x)} = \frac{\rho_*(y)}{g_*(x)\rho_*(x)}.$$

The derivative formula (2.9) comes via an immediate calculation

$$\begin{aligned} f'(x) &= -\frac{[g_*(x)\rho_*(x)]'}{[g_*(x)\rho_*(x)]^2}\int_a^x (h(y) - \mathbf{E}[h(Z)])\,\rho_*(y)\,dy \\ &\quad + \frac{(h(x) - \mathbf{E}[h(Z)])\,\rho_*(x)}{g_*(x)\rho_*(x)} \\ &= -\frac{-x\rho_*(x)}{[g_*(x)\rho_*(x)]^2}\int_a^x (h(y) - \mathbf{E}[h(Z)])\,\rho_*(y)\,dy + \frac{h(x) - \mathbf{E}[h(Z)]}{g_*(x)} \\ &= \frac{x}{[g_*(x)]^2\rho_*(x)}\int_a^x (h(y) - \mathbf{E}[h(Z)])\,\rho_*(y)\,dy + \frac{h(x) - \mathbf{E}[h(Z)]}{g_*(x)}. \end{aligned}$$ □

Proof of Lemma 2.3. From (A.1) in the previous proof, we have

$$\begin{aligned} \int_0^b \frac{z}{g_*(z)}\,dz &= \lim_{x\nearrow b}\int_0^x \frac{z}{g_*(z)}\,dz = \lim_{x\nearrow b}\log\frac{g_*(0)\rho_*(0)}{g_*(x)\rho_*(x)} \\ &= \log[g_*(0)\rho_*(0)] - \lim_{x\nearrow b}\log[g_*(x)\rho_*(x)] = \infty \end{aligned}$$

and

$$\int_a^0 \frac{z}{g_*(z)}\,dz = \lim_{x\searrow a}\int_x^0 \frac{z}{g_*(z)}\,dz = \lim_{x\searrow a}\log\frac{g_*(x)\rho_*(x)}{g_*(0)\rho_*(0)}$$
$$= \lim_{x\searrow a}\log\left[g_*(x)\rho_*(x)\right] - \log\left[g_*(0)\rho_*(0)\right] = -\infty. \qquad \square$$

Proof of Lemma 2.7. We prove (2.11) first. It is trivially true if $x \notin [a,b]$, so suppose $x \in (a,b)$. Let

$$m(x) := \Phi_*(x) - \frac{x - g_*'(x)}{Q(x)} \cdot g_*(x)\rho_*(x).$$

By a standard calculus proof, we will show that $m'(x) \leq 0$ so that $m(x) \geq \lim_{y\to b} m(y)$. The result follows after observing that $\lim_{y\to b} m(y) = 0$. This is true since $\lim_{y\to b} g_*(y)\rho_*(y) = 0$ and $\lim_{y\to b}\Phi_*(x) = 0$. Now we show that $m'(x) \leq 0$.

$$\begin{aligned}
m' &= -\rho_* - g_*\rho_*\left[\frac{x-g_*'}{x^2 - xg_*' + g_*}\right]' - \frac{x-g_*'}{Q}\left[g_*\rho_*\right]' \\
&= -\rho_* - g_*\rho_*\left[\frac{\left(x\left(x-g_*'\right)+g_*\right)\left(1-g_*''\right) - \left(x-g_*'\right)\left(2x - xg_*''\right)}{Q^2}\right] \\
&\quad - \frac{x-g_*'}{Q}\left[-x\rho_*\right] \\
\frac{Q^2}{\rho_*}m' &= -Q^2 - g_*\left[\left(x-g_*'\right)\left(x - xg_*'' - 2x + xg_*''\right) + g_*\left(1-g_*''\right)\right] \\
&\quad + Qx\left(x-g_*'\right) \\
&= \left[-x^2\left(x-g_*'\right)^2 - 2xg_*\left(x-g_*'\right) - g_*^2\right] + xg_*\left(x-g_*'\right) - g_*^2\left(1-g_*''\right) \\
&\quad + \left[x^2\left(x-g_*'\right)^2 + xg_*\left(x-g_*'\right)\right] \\
&= -g_*^2 - g_*^2\left(1-g_*''\right) = -g_*^2\left(2-g_*''\right) \leq 0.
\end{aligned}$$

To prove (2.12) (again, it suffices to prove this for $x \in (a,b)$), let

$$n(x) := 1 - \Phi_*(x) - \frac{g_*'(x) - x}{Q(x)} \cdot g_*(x)\rho_*(x) = 1 - m(x)$$

so $n'(x) = -m'(x) \geq 0$. n is then nondecreasing so $n(x) \geq \lim_{x\to a} n(x) = 0$.

Now we prove (2.13). If $x > 0$,

$$\Phi_*(x) = \int_x^\infty \rho_*(y)\,dy \leq \frac{1}{x}\int_x^\infty y\rho_*(y)\,dy = \frac{1}{x}\cdot g_*(x)\rho_*(x).$$

On the other hand, if $x < 0$,

$$1 - \Phi_*(x) = \int_{-\infty}^x \rho_*(y)\,dy \leq \frac{1}{x}\int_{-\infty}^x y\rho_*(y)\,dy = -\frac{1}{x}\cdot g_*(x)\rho_*(x).$$

This proves (2.14). $\square$

Proof of last bullet point on page 71. We replicate here a method commonly used to find a recursive formula for the moments. See for example [7] and [19]. Cross-multiplying the terms in (4.1), multiplying by x^r and integrating over the support gives us

$$\int_a^b \left[(2\alpha+1)z^{r+1} + \beta z^r\right] \rho_*(z)\, dz = -\int_a^b \left(\alpha z^{r+2} + \beta z^{r+1} + \gamma z^r\right) \rho_*'(z)\, dz$$

and so

$$\begin{aligned} -(2\alpha+1)\mathbf{E}\left[Z^{r+1}\right] - \beta\mathbf{E}\left[Z^r\right] &= \left(\alpha z^{r+2} + \beta z^{r+1} + \gamma z^r\right)\rho_*(z)\Big|_a^b \\ &\quad - \int_a^b \left[\alpha(r+2)z^{r+1} + \beta(r+1)z^r + \gamma r z^{r-1}\right]\rho_*(z)\, dz \\ &= \alpha(r+2)\mathbf{E}\left[Z^{r+1}\right] + \beta(r+1)\mathbf{E}\left[Z^r\right] + \gamma r\mathbf{E}\left[Z^{r-1}\right] \end{aligned}$$

where we assumed that $z^{r+2}\rho_*(z) \to 0$ at the endpoints a and b of the support. For the case $r = 1$, this reduces to $z^3\rho_*(z) \to 0$ at the endpoints a and b, which we are assuming. Therefore,

$$(2\alpha+1)\mathbf{E}\left[Z^2\right] + \beta\mathbf{E}\left[Z\right] = 3\alpha\mathbf{E}\left[Z^2\right] + 2\beta\mathbf{E}\left[Z\right] + \gamma\mathbf{E}\left[Z^0\right].$$

Since $\mathbf{E}\left[Z\right] = 0$ and $\mathbf{E}\left[Z^0\right] = 1$, this gives $\mathbf{E}\left[Z^2\right] = \frac{\gamma}{1-\alpha}$. □

A.2. Examples of Pearson distributions

We present cases of Pearson distributions depending on the degree and number of zeroes of $g_*(x)$ as a quadratic polynomial in (a, b). The Pearson family is closed under affine transformations of the random variable, so we can limit our focus on the five special cases below. The constant C in each case represents the normalization constant. See Diaconis and Zabell [5] for a discussion of these cases.

- Case 1. If $\deg g_*(z) = 0$, ρ_* can be (after an affine transformation) written in the form $\rho_*(z) = Ce^{-z^2/2}$ for $-\infty < z < \infty$. This is the standard normal density, and $C = \frac{1}{\sqrt{2\pi}}$. For this case, $g_*(z) \equiv 1$. Consequently, $Q(z) = z^2 + 1$. If $z > 0$, the inequalities (2.11) and (2.13) of Lemma 2.7 can be written

$$\frac{z}{(z^2+1)\sqrt{2\pi}} e^{-z^2/2} \leq \Phi_*(z) \leq \frac{1}{z\sqrt{2\pi}} e^{-z^2/2},$$

  a standard inequality involving the tail of the standard normal distribution.
- Case 2. If $\deg g_*(z) = 1$, ρ_* can be written in the form $\rho_*(z) = Cz^{r-1}e^{-z/s}$ for $0 < z < \infty$, with parameters $r, s > 0$. This is a Gamma density, and $C = \frac{1}{s^r\Gamma(r)}$. It has mean $\mu = rs > 0$ and variance rs^2. If one wants to make Z centered, the density takes the form $\rho_*(z) = C(z+\mu)^{r-1}e^{-(z+\mu)/s}$ for $-\mu < z < \infty$. For this case, $g_*(z) = s(z+\mu)_+$.

- Case 3. If $\deg g_*(x) = 2$ and g_* has two real roots, ρ_* can be written in the form $\rho_*(x) = Cx^{r-1}(1-x)^{s-1}$ for $0 < x < 1$, with parameters $r, s > 0$. This is a Beta density, and $C = \frac{1}{\beta(r,s)}$. It has mean $\mu = \frac{r}{r+s} > 0$ and variance $\frac{rs}{(r+s)^2(r+s+1)}$. Centering the density gives $\rho_*(x) = C(x+\mu)^{r-1}(1-x-\mu)^{s-1}$ for $-\mu < x < 1-\mu$. For this case, $g_*(x) = \frac{(x+\mu)(1-x-\mu)}{r+s}$ when $-\mu < x < 1-\mu$ and 0 elsewhere.
- Case 4. If $\deg g_*(x) = 2$ and g_* has exactly one real root, ρ_* can be written in the form $\rho_*(x) = Cx^{-r}e^{-s/x}$ for $0 < x < \infty$, with parameters $r > 1$ and $s \geq 0$. The normalization constant is $C = \frac{s^{r-1}}{\Gamma(r-1)}$. If $r > 2$, it has mean $\mu = \frac{s}{r-2} \geq 0$. If $r > 3$, it has variance $\frac{s^2\Gamma(r-3)}{\Gamma(r-1)}$. Centering this density yields $\rho_*(x) = C(x+\mu)^{-r}e^{-s/(x+\mu)}$ for $-\mu < x < \infty$. For this case, $g_*(x) = \frac{(x+\mu)^2}{r-2}$ when $-\mu < x$ and 0 elsewhere.
- Case 5. If $\deg g_*(x) = 2$ and g_* has no real roots, ρ_* can be written in the form $\rho_*(x) = C\left(1+x^2\right)^{-r} e^{s \arctan x}$ for $-\infty < x < \infty$, with parameters $r > 1/2$ and $-\infty < s < \infty$. The normalization constant is $C = \frac{\Gamma(r)}{\sqrt{\pi}\Gamma(r-1/2)} \left|\frac{\Gamma(r-is/2)}{\Gamma(r)}\right|^2$. If $r > 1$, it has mean $\mu = \frac{s}{2(r-1)}$. If $r > 3/2$, it has variance $\frac{4(r-1)^2+s^2}{4(r-1)^2(2r-3)}$. The centered form of the density is $\rho_*(x) = C\left[1+(x+\mu)^2\right]^{-r} e^{s\arctan(x+\mu)}$. For this case, $g_*(x) = \frac{1+(x+\mu)^2}{2(r-1)}$. Using our original notation, $\alpha = \frac{1}{2(r-1)}$, $\beta = \frac{\mu}{r-1}$ and $\gamma = \frac{\mu^2+1}{2(r-1)}$.

A.3. Other lemmas

Lemma A.1. *Let Z be a centered Pearson random variable. Then there exist constants $k_u > k_l > 0$ depending only on α, β, γ such that when z is large enough, we have the following inequalities.*

1. *If $\alpha = 0$ and $\beta > 0$, when z is large enough,*
$$\frac{k_l}{z^{1-\gamma/\beta^2}e^{z/\beta}} \leq \Phi_*(z) \leq \frac{k_u}{z^{1-\gamma/\beta^2}e^{z/\beta}}.$$
2. *If $\alpha > 0$, when z is large enough,*
$$\frac{k_l}{z^{1+1/\alpha}} \leq \Phi_*(z) \leq \frac{k_u}{z^{1+1/\alpha}}.$$
3. *Assuming Z's support extends to $-\infty$, if $\alpha > 0$, when $z < 0$ and $|z|$ is large enough,*
$$\frac{k_l}{|z|^{1+1/\alpha}} \leq 1 - \Phi_*(z) \leq \frac{k_u}{|z|^{1+1/\alpha}}.$$

Proof. For the proof of this lemma, which is presumably well known, but is included for completeness, we will use C for the normalization constant of each density to be considered.

In Point 1, let $\mu = \gamma/\beta > 0$. Then Z has support $(-\mu, \infty)$; see Case 2 in Appendix A.2. In its support, Z has $g_*(z) = \beta z + \gamma = \beta(z+\mu)$ and density

$$\rho_*(z) = C(z+\mu)^{\mu/\beta - 1} \exp\left(-\frac{z+\mu}{\beta}\right).$$

Note that

$$\lim_{z\to\infty} z^{-\mu/\beta} e^{z/\beta} g_*(z)\rho_*(z) = C\beta \lim_{z\to\infty} \frac{(z+\mu)^{\mu/\beta}}{z^{\mu/\beta}} \exp\left(\frac{z}{\beta} - \frac{z+\mu}{\beta}\right) = C\beta e^{-\mu/\beta}.$$

From Lemma 2.7,

$$\frac{z-\beta}{z^2+\gamma} g_*(z)\rho_*(z) \le \Phi_*(z) \le \frac{1}{z} g_*(z)\rho_*(z)$$

so

$$C\beta e^{-\mu/\beta} \le \liminf_{z\to\infty} z^{1-\mu/\beta} e^{z/\beta}\Phi_*(z) \le \limsup_{z\to\infty} z^{1-\mu/\beta} e^{z/\beta}\Phi_*(z) \le C\beta e^{-\mu/\beta}.$$

Therefore, we can choose some constants $k_u(\beta,\gamma) > k_l(\beta,\gamma) > 0$ such that when z is large enough,

$$\frac{k_l}{z^{1-\mu/\beta} e^{z/\beta}} \le \Phi_*(z) \le \frac{k_u}{z^{1-\mu/\beta} e^{z/\beta}}.$$

To prove Point 2, we first show that $\lim_{z\to\infty} z^{1/\alpha} g_*(z)\rho_*(z)$ is a finite number K. We consider the cases $4\alpha\gamma - \beta^2 = 0$ and $4\alpha\gamma - \beta^2 > 0$ separately. We need not consider $4\alpha\gamma - \beta^2 < 0$ since it corresponds to Case 3 in Appendix A.2 for which the right endpoint of the support of Z is $b < \infty$ and so necessarily $\alpha < 0$.

Supose that $4\alpha\gamma - \beta^2 = 0$ and let $\mu = \frac{\beta}{2\alpha} > 0$. Then $\alpha z^2 + \beta z + \gamma = \alpha(z+\mu)^2$ has one real root and the support of Z is $(-\mu, \infty)$; see Case 4 in Appendix A.2. In its support, Z has $g_*(z) = \alpha(z+\mu)^2$ and density

$$\rho_*(z) = C(z+\mu)^{-2-1/\alpha} \exp\left(-\frac{s}{z+\mu}\right)$$

where $s = \mu/\alpha = \beta/(2\alpha^2)$. Therefore,

$$\lim_{z\to\infty} z^{1/\alpha} g_*(z)\rho_*(z) = C\alpha \lim_{z\to\infty} \frac{z^{1/\alpha}}{(z+\mu)^{1/\alpha}} \exp\left(-\frac{s}{z+\mu}\right) = C\alpha.$$

Now suppose that $\delta^2 := (4\alpha\gamma - \beta^2)/(4\alpha^2) > 0$ so $\alpha z^2 + \beta z + \gamma$ has two imaginary roots and the support of Z is $(-\infty, \infty)$. Letting $\mu = \beta/(2\alpha)$ allows us to write $g_*(z) = \alpha(z+\mu)^2 + \alpha\delta^2$ and the density of Z as

$$\rho_*(z) = C\left[(z+\mu)^2 + \delta^2\right]^{-1-\frac{1}{2\alpha}} \exp\left[\frac{\mu}{\alpha\delta} \arctan\left(\frac{z+\mu}{\delta}\right)\right],$$

a slight variation of the density in Case 5 in Appendix A.2. Note that in our present case,

$$\begin{aligned}&\lim_{z\to\infty} z^{1/\alpha} g_*(z)\,\rho_*(z)\\&\quad= C\alpha \lim_{z\to\infty} \frac{z^{1/\alpha}}{\left[(z+\mu)^2+\delta^2\right]^{\frac{1}{2\alpha}}} \exp\left[\frac{\mu}{\alpha\delta}\arctan\left(\frac{z+\mu}{\delta}\right)\right]\\&\quad= C\alpha \exp\left[\frac{\mu\pi}{2\alpha\delta}\right].\end{aligned}$$

From Lemma 2.7,

$$\frac{(1-2\alpha)\,z-\beta}{(1-\alpha)\,z^2+\gamma} g_*(z)\,\rho_*(z) \le \Phi_*(z) \le \frac{1}{z} g_*(z)\,\rho_*(z).$$

From these bounds we conclude

$$K\frac{1-2\alpha}{1-\alpha} \le \liminf_{z\to\infty} z^{1+1/\alpha}\Phi_*(z) \le \limsup_{z\to\infty} z^{1+1/\alpha}\Phi_*(z) \le K.$$

Therefore, when z is large enough,

$$\frac{k_l}{z^{1+1/\alpha}} \le \Phi_*(z) \le \frac{k_u}{z^{1+1/\alpha}}$$

for some constants $k_u(\alpha,\beta,\gamma) > k_l(\alpha,\beta,\gamma) > 0$.

To prove Point 3, we consider Case 5 again.

$$\begin{aligned}&\lim_{z\to-\infty} |z|^{1/\alpha} g_*(z)\,\rho_*(z)\\&\quad= C\alpha \lim_{y\to\infty} \frac{y^{1/\alpha}}{\left[(-y+\mu)^2+\delta^2\right]^{\frac{1}{2\alpha}}} \exp\left[\frac{\mu}{\alpha\delta}\arctan\left(\frac{-y+\mu}{\delta}\right)\right]\\&\quad= C\alpha \exp\left[-\frac{\mu\pi}{2\alpha\delta}\right].\end{aligned}$$

The conclusion follows similarly after using Lemma 2.7 when $z<0$:

$$\frac{(1-2\alpha)\,|z|-\beta}{(1-\alpha)\,|z|^2+\gamma} g_*(z)\,\rho_*(z) \le 1-\Phi_*(z) \le \frac{1}{|z|} g_*(z)\,\rho_*(z). \qquad \square$$

Lemma A.2. *Let Z be a centered Pearson random variable. If $\alpha \le 0$, all moments of positive order exist. If $\alpha > 0$, the moment of order m exists if and only if $m < 1 + 1/\alpha$.*

Proof. The random variables in Case 1 ($\alpha=\beta=0$) of Appendix A.2 are normal, while those in Case 3 ($\alpha<0$) have finite intervals for support. It suffices to consider the cases where $\alpha=0$ and $\beta>0$, and where $\alpha>0$. Let $m>0$. We will use the fact that $\mathbf{E}\left[|Z|^m\right] < \infty$ if and only if $\sum_{n=1}^{\infty} n^{m-1}\mathbf{P}\left[|Z|\ge n\right] < \infty$.

If $\alpha = 0$ and $\beta > 0$, and Z is supported over (a, ∞), then by Lemma A.1, $\mathbf{E}\left[|Z|^m\right] < \infty$ if and only if

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{n^{1+\gamma/\beta^2} e^{n/\beta}} < \infty,$$

which is always the case.

Now suppose $\alpha > 0$. Since $\mathbf{P}\left[|Z| \geq n\right] = \Phi_*(n) + 1 - \Phi_*(-n)$, then by Lemma A.1 again, $\mathbf{E}\left[|Z|^m\right] < \infty$ if and only if

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{n^{1+1/\alpha}} = \sum_{n=1}^{\infty} \frac{1}{n^{2+1/\alpha-m}} < \infty.$$

This is the case if and only if $2 + 1/\alpha - m > 1$, i.e., $m < 1 + 1/\alpha$. □

References

[1] H. Airault, P. Malliavin, and F. Viens, *Stokes formula on the Wiener space and n-dimensional Nourdin-Peccati analysis*. J. Funct. Anal., **258 (5)** (2009), 1763–1783.

[2] B. Bercu, I. Nourdin, and M. Taqqu, *Almost sure central limit theorems on the Wiener space*. Stoch. Proc. Appl., **120 (9)** (2010), 1607–1628.

[3] J.-C. Breton, I. Nourdin, and G. Peccati, *Exact confidence intervals for the Hurst parameter of a fractional Brownian motion*. Electron. J. Statist., **3** (2009), 416–425.

[4] L. Chen and Q.-M. Shao, *Stein's method for normal approximation*. In *An Introduction to Stein's Method*, pp. 1–59, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., Singapore Univ. Press, Singapore, 2005.

[5] P. Diaconis and S. Zabell, *Closed form summation for classical distributions: variations on a theme of De Moivre*. Statistical Science, **6 (3)** (1991), 284–302.

[6] R.M. Dudley, *Real Analysis and Probability*. 2nd ed., Cambridge University Press, Cambridge, 2003.

[7] W.P. Elderton and N.L. Johnson, *Systems of Frequency Curves*. Cambridge University Press, London, 1969.

[8] I. Nourdin and G. Peccati, *Stein's method on Wiener chaos*. Probability Theory and Related Fields, **145 (1)** (2009), 75–118.

[9] I. Nourdin and G. Peccati, *Stein's method and exact Berry–Esséen asymptotics for functionals of Gaussian fields*. Ann. Probab., **37 (6)** (2009), 2231–2261.

[10] I. Nourdin and G. Peccati, *Stein's method meets Malliavin calculus: a short survey with new estimates*. In press in *Recent Development in Stochastic Dynamics and Stochastic Analysis*, J. Duan, S. Luo and C. Wang, editors, pp. 191–227, World Scientific, 2009.

[11] I. Nourdin and G. Peccati, *Cumulants on the Wiener Space*. J. Funct. Anal., **258 (11)** (2010), 3775–3791.

[12] I. Nourdin and G. Peccati, *Universal Gaussian fluctuations of non-Hermitian matrix ensembles: from weak convergence to almost sure CLTs*. Alea, **7** (2010), 341–375.

[13] I. Nourdin, G. Peccati, and G. Reinert, *Invariance principles for homogeneous sums: universality of Gaussian Wiener chaos*. Ann. Probab., **38 (5)** (2010), 1947–1985.

[14] I. Nourdin, G. Peccati, and G. Reinert, *Second order Poincaré inequalities and CLTs on Wiener space*. J. Funct. Anal., **257 (2)** (2009), 593–609.

[15] I. Nourdin, G. Peccati, and A. Réveillac, *Multivariate normal approximation using Stein's method and Malliavin calculus*. Ann. I.H.P., **46 (1)** (2010), 45–58.

[16] I. Nourdin I and F. Viens, *Density formula and concentration inequalities with Malliavin calculus*. Electron. J. Probab., **14** (2009), 2287–2309.

[17] D. Nualart, *The Malliavin Calculus and Related Topics*. 2nd ed., Springer Verlag, Berlin, 2006.

[18] D. Nualart and Ll. Quer-Sardanyons, *Gaussian density estimates for solutions to quasi-linear stochastic partial differential equations*. Stoch. Proc. Appl., **119 (11)** (2009), 3914–3938.

[19] J.K. Ord, *Families of Frequency Distributions*, Hafner Pub. Co., New York, 1972.

[20] W. Schoutens, *Orthogonal polynomials in Stein's method*. Math. Anal. Appl., **253 (2)** (2001), 515–531.

[21] C. Stein, *Approximate Computation of Expectations*. Institute of Mathematical Statistics Lecture Notes – Monograph Series, **7**, Institute of Mathematical Statistics, Hayward, 1986.

[22] F. Viens, Stein's lemma, Malliavin Calculus, and tail bounds, with application to polymer fluctuation exponent. Stoch. Proc. Appl., **119 (10)** (2009), 3671–3698.

Richard Eden
Department of Mathematics
Ateneo de Manila University
Loyola Heights
Quezon City, Philippines 1108
e-mail: reden@math.admu.edu.ph

Frederi Viens
Department of Statistics
Purdue University
150 N. University St.
West Lafayette, IN 47907-2067, USA
e-mail: viens@purdue.edu