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Seminar on Stochastic Analysis, Random Fields and Applications VII

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Preface

This volume contains the Proceedings of the *Seventh Seminar on Stochastic Analysis, Random Fields and Applications*, which took place at the Centro Stefano Franscini (Monte Verità) in Ascona (Ticino), Switzerland, from May 23 to 27, 2011. *All papers in this volume have been refereed.*

The seven editions of this conference have occurred with regularity at three-year intervals, starting in 1993. These conferences aimed to present the state of the art and attempted to anticipate research trends in the fundamentals of *stochastic analysis,* which lies at the interface between classical analysis and probability theory, as well as in adjacent fields and in applications. Traditional topics related to stochastic analysis include stochastic partial differential equations (SPDEs), Malliavin calculus, numerical approximations, stochastic control, optimal stopping, stochastic calculus related to singular diffusions and jump processes, fractional processes and rough paths. In recent years, new topics have appeared, such as Stein's method and statistics for stochastic processes, non-commutative probability theory, and stochastic transport problems. Stochastic analysis also includes numerous applications to science and engineering. In addition to the classical applications in physics (stochastic mechanics, quantum field theory, fluid dynamics) and mathematical finance, many specialists participate in active branches of engineering, such as robotics, and biology (brain modeling, Darwin evolution models, genomic analysis, etc.).

This time, our seminar emphasized innovations in SPDEs, in stochastic methods for deterministic partial differential equations (PDEs), stochastic numerics, statistics for stochastic processes and mathematical finance. There is clearly a wide overlap between these subjects.

In the topic of SPDEs, much research is oriented toward problems that originate in physical models (for instance, parabolic Anderson models, turbulence, stochastic porous media equations) and also in financial models (evolution of term structures for interest rates or forward prices in electricity markets). The driving noise can be Gaussian or Lévy. Extensions of rough paths to rough sheets were also presented, with the aim of solving the stochastic Burgers equation.

Stochastic methods in PDEs (and pseudo-differential equations) represent the natural bridge between stochastic processes in continuous time and classical analysis. Significant progress was presented, for instance on refinements of Malliavin calculus applied to the Boltzmann equation. Malliavin calculus is partly related to classical analysis on Wiener space, a topic in which important work has been done to extend to Wiener space the notion of function of bounded variation.

Stochastic numerics is a field which touches upon all of stochastic analysis. Significant work has been done to approximate stochastic PDEs and stochastic differential equations with Lévy noise, and important advances in simulating forward-backward stochastic differential equations were also presented; these relate classical techniques, such as Euler methods for discretizing stochastic differential equations, with improved Monte Carlo algorithms and quantization methods for evaluating (conditional) expectations.

Other theoretical work, which does not fall directly into the above categories, is stochastic calculus for non-semimartingales: this subject was approached via rough paths, as well as regularization and discretization methods.

One central application of stochastic analysis, that our conference has traditionally focused on, concerns applications to *mathematical finance*. Some talks emphasized modeling issues, particularly in models for electricity and insurance markets, or in the description of financial bubbles. Many contributions were devoted to portfolio management with regime switching, risk measures, affine processes and variance swap curve models. Here again, high-order numerical approximations were important.

During the conference, it has become our custom to offer a public lecture, which also constitutes a local cultural event. This time, the organizers invited Professor Nicolas Bouleau, who is a distinguished member of the stochastic analysis community, an expert in mathematical engineering and a pioneer in mathematical finance. In the current context of financial and economic crises, the speaker presented his thoughts on the impact of mathematics (and in particular stochastic analysis) on worldwide events. This public session was opened by Mr. Manuele Bertoli, the Minister of Education and Culture of the Ticino government, and was widely reported in local press. The text of Professor Bouleau's lecture appears in this volume.

Significant financial support for this meeting was provided by the Fonds National Suisse pour la Recherche Scientifique (Berne), the Centro Stefano Franscini (ETH-Zürich), and the Ecole Polytechnique Fédérale de Lausanne (EPFL). We take this opportunity to thank these institutions.

February 2013

Robert C. Dalang
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The editors wish to dedicate this volume to Esko Valkeila (1951–2012).

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Part I
Stochastic Analysis and Random Fields

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Recent Advances Related to SPDEs with Fractional Noise

Raluca M. Balan

Abstract. We review the literature related to stochastic partial differential equations with spatially-homogeneous Gaussian noise, and explain how one can introduce the structure of the fractional Brownian motion into the temporal component of the noise. The Hurst parameter H is assumed to be greater than 1/2. In the case of linear equations, we revisit the conditions for the existence of a mild solution. In the nonlinear case, we point out what are the difficulties due to the fractional component of the noise. These difficulties can be avoided in the case of equations with multiplicative noise, since in this case, the solution has a known Wiener chaos decomposition. Finally, this methodology is applied to the wave equation (in arbitrary dimension $d \geq 1$), driven by a Gaussian noise which has a spatial covariance structure given by the Riesz kernel.

Mathematics Subject Classification (2010). Primary 60H15; Secondary 60H07.

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1. Linear SPDEs

Consider a physical system whose space-time behavior is described by the equation $Lu = 0$, with some initial conditions, where L is a second-order pseudo-differential operator in $t > 0$ and $x \in \mathbb{R}^d$. For example, the solution $u(t,x)$ of the equation $\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u = 0$ with initial condition $u(0,x) = T(x)$ can be interpreted as the temperature of an insulated medium (of infinite volume) at time t and location x, assuming that at time 0, the medium had a temperature $T(x)$ at location x. When the system is subject to random perturbations, its evolution in space and time can be described by the equation:

$$Lu(t,x) = \dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d, \tag{1.1}$$

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where $\dot{W}$ represents the perturbing noise. The notation $\dot{W}(t,x)$ is standard, but at this point, it may seem ambiguous. The reason for this notation is explained in Remark 1.3 below.

The goal of this section is to give a mathematical meaning to the noise $\dot{W}$ and the solution of equation (1.1). The definition of the noise used in the present article is the one commonly encountered in stochastic analysis. Other definitions exist in the literature related to percolation models (see, e.g., Definition 3d1 of [38]).

The noise is a very important concept in engineering and physics, where it usually refers to anything that interferes with the state of a system. By analogy with the white light, which is a uniform mixture of all colors, the white noise has equal energy per cycle, and produces a flat spectrum over a defined frequency band. This analogy extends naturally to other "colors" of noise. In signal theory and audio engineering, the most commonly encountered example is the "power-law noise", which has a power spectral density per unit of bandwidth proportional to $1/f^{\alpha}$, where f is the frequency in Hertz (i.e., the number of cycles per second), and α is a parameter describing the color of the noise. As we will see below, a similar analogy exists in the mathematical theory of the noise.

Throughout this article, we assume that the noise $\dot{W}$ can be modeled by an isonormal Gaussian process $\{W(\varphi);\varphi \in C_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^d)\}$ with covariance

$$E[W(\varphi)W(\psi)] = J(\varphi,\psi),$$

for a non-negative-definite bilinear functional J (to be specified below). We endow $C_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^d)$ with the inner product

$$\langle \varphi,\psi\rangle_{\mathcal{H}} = J(\varphi,\psi),$$

and let $\mathcal{H}$ be the Hilbert space defined as the completion of $C_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^d)$ with respect to this inner product.

In the examples that we have in mind, some of the elements of $\mathcal{H}$ could be distributions in x. For instance, when J is given by formula (1.4) below, $\mathcal{H}$ contains the set of all functions $\mathbb{R}_+ \ni t \mapsto \varphi(t,\cdot) \in \mathcal{S}'(\mathbb{R}^d)$ such that the Fourier transform $\mathcal{F}\varphi(t,\cdot)$ is a function for all t, the map $(t,\xi) \mapsto \mathcal{F}\varphi(t,\cdot)(\xi)$ is measurable on $\mathbb{R}_+ \times \mathbb{R}^d$, and $\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} |\mathcal{F}\varphi(t,\cdot)(\xi)|^2 \mu(d\xi)dt < \infty$ (see [8]). Here, $\mathcal{S}'(\mathbb{R}^d)$ denotes the space of tempered distributions on $\mathbb{R}^d$.

The map $\varphi \mapsto W(\varphi) \in L^2(\Omega)$ is an isometry which can be extended to $\mathcal{H}$. More precisely, if $\varphi \in \mathcal{H}$, then there exists $(\varphi_n)_{n\geq 1} \subset C_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $\|\varphi_n - \varphi\|_{\mathcal{H}} \to 0$ as $n \to \infty$. By the isometry property, $\{W(\varphi_n)\}_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega)$. A classical argument shows that its limit does not depend on the sequence $(\varphi_n)_{n\geq 1}$. This limit is denoted by:

$$W(\varphi) = \int_0^{\infty}\int_{\mathbb{R}^d} \varphi(t,x)W(dt,dx).$$

The stochastic integral $W(\varphi)$ is well defined if and only if $\varphi \in \mathcal{H}$.

The following definition gives the rigorous meaning of the solution of equation (1.1).

Definition 1.1. Let w be the solution of the equation $Lu = 0$ with the same (deterministic) initial conditions as (1.1), and G be the fundamental solution of the equation $Lu = 0$. The process $\{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ defined by:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W(ds, dy) \tag{1.2}$$

is a **mild solution** of (1.1), provided that the stochastic integral on the right-hand side (RHS) of (1.2) is well defined for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

Remark 1.2. The approach used in the present article is known as the Walsh approach, due to the fundamental work [39]. It should be mentioned that there are some other approaches for the study of SPDEs in the literature. One of them uses the L^2-theory (initiated in [29] and [34]), which was extended later to an L^p-theory, in the seminal articles [27] and [28]. This approach has been recently used in [4] for the study of equations with fractional noise, and in [15] for equations with more general noise. Another approach is based on stochastic evolution equations in Hilbert spaces, as developed in the milestone monograph [14], in the case of the white noise in time (see also [26]). This approach requires the construction of an infinite-dimensional stochastic integral with respect to a Hilbert-space-valued stochastic process (called a Q-Wiener process). During the past decade, this approach has been used extensively for the study of equations with infinite-dimensional fractional noise (see [16, 19, 30, 32, 37] for a sample of relevant references). Although there exists a general technique of converting an SPDE into a stochastic evolution equation, the results obtained using different approaches are not easily comparable. The articles [17] and [12] fill an important gap in the literature by comparing the results obtained using the Walsh approach with those obtained using the L_p-theory, respectively the theory of stochastic evolution equations.

To see how Definition 1.1 works, suppose first that $\dot{W}$ is a "space-time white noise", which in the present article is modeled by a zero-mean Gaussian process $W = \{W(\varphi); \varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ with covariance functional

$$J(\varphi, \psi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t,x)\psi(t,x)dxdt.$$

Remark 1.3. In dimension $d = 1$, the process $\{W(\varphi); \varphi \in C_0^\infty(\mathbb{R}^2)\}$ has the same distribution as the generalized derivative $\dot{W}(\varphi) = \int \dot{\varphi}(t,x)W(t,x)dxdt$ of the Brownian sheet $\{W(t,x)\}_{(t,x)\in\mathbb{R}^2}$, where $\dot{\varphi}$ denotes the partial derivative $\frac{\partial^2 \varphi}{\partial t \partial x}$; see p. 260 of [18], or p. 285 of [39]. Hence, by a formal application of the integration by parts formula, one writes $\dot{W}(\varphi) = \int \varphi(t,x)\dot{W}(t,x)dxdt$, although the process $\{\dot{W}(t,x) = \frac{\partial^2 W}{\partial t \partial x}(t,x)\}_{t\geq 0, x\in\mathbb{R}}$ does not exist.

Note that $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}^d)$, and the stochastic integral on the RHS of (1.2) is well defined if and only if

$$\int_0^t \int_{\mathbb{R}^d} G^2(t-s, x-y)dyds < \infty. \tag{1.3}$$

For example, if $L = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$, then $G(t,x) = 1_{\{t>0\}}(2\pi t)^{-d/2}\exp(-\frac{|x|^2}{2t})$, and (1.3) holds if and only if $d = 1$.

A different model for the noise was introduced in [9], which may allow for the existence of a mild solution in higher dimensions. More precisely, this noise is defined as a zero-mean Gaussian process $W = \{W(\varphi); \varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ with covariance functional

$$J(\varphi, \psi) = \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x)\psi(t,y)f(x-y)dxdydt. \tag{1.4}$$

By the Bochner-Schwartz theorem, J is non-negative-definite if and only if the function f (the "spatial parameter" of the noise) is the Fourier transform in $\mathcal{S}'(\mathbb{R}^d)$ of a tempered measure μ.

In this case, for any functions φ, ψ in the space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$, we have:

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x)\psi(y)f(x-y)dxdy = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi),$$

where $\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\varphi(x)dx$ denotes the Fourier transform of φ.

Since $1_{[0,t]\times A} \in \mathcal{H}$ for any $t > 0$ and bounded Borel set $A \subset \mathbb{R}^d$, one can define a random field $\{W(t,x) = W(1_{[0,t]\times[0,x]})\}_{t>0, x\in\mathbb{R}^d}$, with covariance:

$$E[W(t,x)W(s,y)] = (t \wedge s)\int_{[0,x]}\int_{[0,y]} f(u-v)dudv.$$

If μ is finite, the spatial component of this covariance is in fact the covariance of a Gaussian process $\{X(x)\}_{x\in\mathbb{R}^d}$ defined as $X(x) = \int_{[0,x]} Y(u)du$, where $\{Y(x)\}_{x\in\mathbb{R}^d}$ is a stationary process with spectral measure μ. In this case, the derivative $\frac{\partial^d X}{\partial x_1 \ldots \partial x_d}(x)$ exists in $L^2(\Omega)$.

Note that when $d = 1$, the Gaussian process $\{W(t,x)\}_{x\in\mathbb{R}}$ has stationary increments (in space) for any $t > 0$, since for any $x, y, h \in \mathbb{R}$ with $x < y$, $W(t,y) - W(t,x)$ has the same distribution as $W(t, y+h) - W(t, x+h)$. To see this, note that

$$E|W(t,y+h) - W(t,x+h)|^2 = \int_{x+h}^{y+h}\int_{x+h}^{y+h} f(u-v)dudv$$

$$= \int_x^y \int_x^y f(u-v)dudv = E|W(t,y) - W(t,x)|^2.$$

The typical examples of measures μ are: (see Chapter V of [36])

(i) $\mu(d\xi) = |\xi|^{-\alpha}d\xi$ for some $\alpha \in (0,d)$, whose Fourier transform is the Riesz kernel $f(x) = c_{\alpha,d}|x|^{-(d-\alpha)}$;

(ii) $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2}d\xi$ for some $\alpha > 0$, whose Fourier transform is the Bessel kernel

$$f(x) = c_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} e^{-|x|^2/(4w)} dw.$$

In these examples, if $\varphi \in \mathcal{H}$, then $\varphi(t,\cdot)$ is a distribution which lies in the fractional Sobolev space $H_2^{-\alpha/2}(\mathbb{R}^d)$. Since in the case of the Bessel kernel, one does not have any restrictions on the range of the parameter α, and the density of the measure μ behaves as $|\xi|^{-\alpha}$ when $|\xi| \to \infty$, one may say that the noise induced by the Bessel kernel is the mathematical analogue of the power-law noise.

Further justification for this analogy comes from the fact that when $d = 1$ and $\mu(d\xi) = (1+|\xi|^2)^{-1}d\xi$, the Gaussian process $\{W(t,x)\}_{x\geq 0}$ has the same distribution as the process $\{X(x) = \int_0^x Y(s)ds, x \geq 0\}$, where $\{Y(t)\}_{t\geq 0}$ is an Ornstein–Uhlenbeck process (i.e., a rescaled Brownian motion). To see this, recall that $\{Y(t)\}_{t\geq 0}$ is the solution of the stochastic differential equation $dY(t) = -aY(t)dt + \sigma dB(t)$, where $a > 0, \sigma > 0$ and $\{B(t)\}_{t\geq 0}$ is a standard Brownian motion. If the initial value Y_0 has a normal distribution with mean 0 and variance $\frac{\sigma^2}{2a}$, then $\{Y(t)\}_{t\geq 0}$ is a zero-mean Gaussian process with covariance $f(t-s) := E[Y(t)Y(s)] = \frac{\sigma^2}{2a}e^{-a|t-s|}$ (see Example 6.8, Chapter 5 of [23]). In this case, $f = \mathcal{F}g$ where $g(\xi) = \frac{\sigma^2}{2\pi}(a^2+|\xi|^2)^{-1}$. When $a = 1$ and $\sigma^2 = 2\pi$, f coincides with the Bessel kernel of order 2 (see Exercise 2.3.4, Chapter 8 of [24]). The case of the colored noise with $\alpha = 2$ is known in the signal theory as the brown(ian) noise, a term which may be justified by the fact that $X(x)$ is an integral of a rescaled Brownian motion.

We have the following result.

Theorem 1.4. *Let J be the functional defined by* (1.4)*. Let $[0,T] \ni t \mapsto S(t,\cdot)$ be a deterministic function with values in the space of distributions with rapid decrease. Denote by $\mathcal{F}S(t,\cdot)$ the Fourier transform of $S(t,\cdot)$ in $\mathcal{S}'(\mathbb{R}^d)$. Suppose that S is non-negative, or*

$$\lim_{h\downarrow 0} \int_0^T \int_{\mathbb{R}^d} \sup_{t<r<t+h} |\mathcal{F}S(r,\cdot)(\xi) - \mathcal{F}S(t,\cdot)(\xi)|^2 \mu(d\xi)dt = 0.$$

If

$$I_T := \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}S(t,\cdot)(\xi)|^2 \mu(d\xi)dt < \infty,$$

then $S \in \mathcal{H}$ and $\|S\|_{\mathcal{H}}^2 = I_T$. (By convention, we set $S(t,\cdot) = 0$ if $t > T$.)

Assuming that the function $[0,t] \ni s \mapsto G(t-s, x-\cdot)$ satisfies the hypotheses of Theorem 1.4, we infer that equation (1.1) has a mild solution if and only if

$$\int_{\mathbb{R}^d} \int_0^t |\mathcal{F}G(s,\cdot)(\xi)|^2 ds\mu(d\xi) < \infty. \tag{1.5}$$

Example. Assume that $L = \frac{\partial}{\partial t} + (-\Delta)^k$, respectively $L = \frac{\partial^2}{\partial^2 t} + (-\Delta)^k$, for some $k > 0$. Then $\mathcal{F}G(t,\cdot)(\xi) = \exp(-t|\xi|^{2k})$, respectively $\mathcal{F}G(t,\cdot)(\xi) = \frac{\sin(t|\xi|^k)}{|\xi|^k}$. In both cases, one can show that condition (1.5) holds if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^{2k}} \mu(d\xi) < \infty. \tag{1.6}$$

Note that when $\mu(d\xi) = |\xi|^{-\alpha} d\xi$ for $\alpha \in (0,d)$, condition (1.6) holds if and only if $\alpha > d - 2k$.

To allow for a greater flexibility in the choice of the parameter α, various authors (e.g., [5, 19, 30, 37]) considered a noise whose temporal covariance is that of the *fractional Brownian motion* (fBm). The fBm was introduced by Kolmogorov in [25] (who called it "the Wiener spiral"), and is defined as a centered Gaussian process $(B_t)_{t\geq 0}$ with

$$E[B_t B_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) =: R_H(t,s),$$

for some $H \in (0,1)$ (the Hurst parameter). The paths of the fBm are γ-Hölder continuous for any $\gamma \in (0,H)$. Hence, when $H > 1/2$, the fBm paths are "smoother" than those of the Brownian motion. The fBm has numerous applications and has been used extensively in statistical analysis, as well as stochastic calculus.

When $H > 1/2$, the covariance R_H can be written as:

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} du dv,$$

where $\alpha_H = H(2H-1)$. In this case, we consider a zero-mean Gaussian process $W = \{W(\varphi); \varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ with covariance functional:

$$\begin{aligned} J(\varphi,\psi) &= \alpha_H \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^{2d}} \varphi(u,x)\psi(v,y)|u-v|^{2H-2} f(x-y) dx dy du dv \\ &= \alpha_H \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^d} \mathcal{F}\varphi(u,\cdot)(\xi)\overline{\mathcal{F}\psi(v,\cdot)(\xi)} |u-v|^{2H-2} \mu(d\xi) du dv. \end{aligned} \tag{1.7}$$

Remark 1.5. A detailed study of stochastic evolution equations driven by an infinite-dimensional fBm can be found in [37]. In the case of the equations with additive noise, the authors of [37] gave a necessary and sufficient condition for the existence of a mild solution, which is valid for both cases $H > 1/2$ and $H < 1/2$. However, in the case of equations with multiplicative noise, when the solution is interpreted using a Skorohod integral, the existence of the mild solution can be proved only in the case $H > 1/2$. The recent article [22] examined the stochastic heat equation with multiplicative fractional noise in time of Hurst index $H < 1/2$, but this noise is of the form $\frac{\partial W}{\partial t}(t,x)$, rather than the form $\frac{\partial^{d+1} W}{\partial t \partial x_1 \ldots \partial x_d}$ considered in the present article.

The following theorem was proved in [6] under weaker conditions.

Theorem 1.6. *Let J be the functional defined by* (1.7). *Let* $[0,T] \ni t \mapsto S(t,\cdot) \in \mathcal{S}'(\mathbb{R}^d)$ *be a deterministic function such that $\mathcal{F}S(t,\cdot)$ is a function, and the map $(t,\xi) \mapsto \mathcal{F}S(t,\cdot)(\xi)$ is measurable and bounded. If*

$$I_T = \alpha_H \int_{\mathbb{R}^d} \int_0^T \int_0^T \mathcal{F}S(u,\cdot)(\xi)\overline{\mathcal{F}S(v,\cdot)(\xi)}|u-v|^{2H-2}dudv\mu(d\xi) < \infty,$$

then $S \in \mathcal{H}$ and $\|S\|^2_{\mathcal{H}} = I_T$. (By convention, we set $S(t,\cdot) = 0$ if $t > T$.)

By Definition 1.1, equation (1.1) has a mild solution if and only if the stochastic integral on the RHS of (1.2) is well defined, i.e., $G(t-\cdot, x-\cdot) \in \mathcal{H}$. This is equivalent to:

$$I_t = \int_{\mathbb{R}^d} \int_0^t \int_0^t \mathcal{F}G(u,\cdot)(\xi)\mathcal{F}G(v,\cdot)(\xi)|u-v|^{2H-2}dudv\mu(d\xi) < \infty. \quad (1.8)$$

(To see this, note that if $G(t-\cdot, x-\cdot) \in \mathcal{H}$, then $I_t = \|G(t-\cdot, x-\cdot)\|^2_{\mathcal{H}} < \infty$. On the other hand, if $I_t < \infty$, then by applying Theorem 1.6 to the function $[0,t] \ni s \mapsto G(t-s, x-\cdot)$, we infer that $G(t-\cdot, x-\cdot) \in \mathcal{H}$.)

Example. (a) Using the same arguments as in Proposition 4.3 of [6] (in the case $k=1$), one can show that, if $L = \frac{\partial}{\partial t} + (-\Delta)^k$ for $k > 0$, condition (1.8) is equivalent to

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^{2k}}\right)^{2H} \mu(d\xi) < \infty. \quad (1.9)$$

This follows by providing upper and lower bounds for the inner integral *dudv* in the RHS of (1.8). For the upper bound, one uses the fact that

$$\alpha_H \int_0^t \int_0^t \varphi(u)\varphi(v)|u-v|^{2H-2}dudv \le b_H \left(\int_0^t |\varphi(u)|^{1/H}du\right)^{2H},$$

for any $\varphi \in L^{1/H}([0,t])$, which can be proved using Littlewood–Hardy inequality and Hölder inequality (see[31]). If f is the Riesz kernel of order $\alpha \in (0,d)$, then $\mu(d\xi) = |\xi|^{-\alpha}d\xi$ and condition (1.9) becomes $4Hk > d-\alpha$. On the other hand, when f is the covariance kernel of the fractional Brownian sheet with indices $H_1, \ldots, H_d \in (\frac{1}{2}, 1)$, i.e.,

$$f(x) = \prod_{i=1}^d (\alpha_{H_i}|x_i|^{2H_i-2}), \quad (1.10)$$

then $\mu(d\xi) = \prod_{i=1}^d (c_{H_i}|\xi_i|^{-(2H_i-1)})d\xi$ for some constants $c_{H_i} > 0$. In this case, condition (1.9) is equivalent to $4Hk > d - \sum_{i=1}^d (2H_i - 1)$.

(b) If $L = \frac{\partial^2}{\partial t^2} + (-\Delta)^k$ for $k > 0$, condition (1.8) is equivalent to

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^{2k}}\right)^{H+1/2} \mu(d\xi) < \infty. \quad (1.11)$$

This is proved again by providing upper and lower bounds for the integral *dudv* of (1.8). The idea is to write this integral in the spectral domain, using the fact that the Fourier transform of $\alpha_H|t|^{2H-2}$ in $\mathcal{S}'(\mathbb{R})$ is $c_H|\tau|^{-(2H-1)}$ (see Theorem 4.3 of [1]). When f is the Riesz kernel of order $\alpha \in (0,d)$, condition (1.11) becomes $(2H+1)k > d-\alpha$. When f is given by (1.10), condition (1.11) is equivalent to $(2H+1)k > d - \sum_{i=1}^{d}(2H_i-1)$.

Remark 1.7. In the recent article [2], it has been proved that (1.9) and (1.11) remain the necessary and sufficient conditions for the existence of the mild solution of the heat, respectively wave, equation, in the case $H < 1/2$.

2. Non-linear SPDEs

The linear equation considered in the previous section is not appropriate for many applications. Often, a more realistic model is the equation:

$$Lu(t,x) = \sigma(u(t,x))\dot{W}(t,x), \quad t>0, x \in \mathbb{R}^d. \tag{2.1}$$

Following an idea of [14], one would like to say that a process $\{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ satisfying the integral equation:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma(u(s,y))W(ds,dy)$$

is a solution of (2.1). But to do this, one has to give a meaning to the stochastic integral above, whose integrand is *random*, due to $Z(s,y) = \sigma(u(s,y))$.

When W is a space-time white noise, the following definition of stochastic integral was used in [39]. (This definition was generalized in [8] to the case of a spatially-homogeneous Gaussian noise and distribution-valued integrands.) If $\{Z(t,x)\}$ is a predictable and $L^2(\Omega)$-bounded random field, and $t \mapsto g(t,\cdot)$ is a (deterministic) function, then

$$\int_0^t \int_{\mathbb{R}^d} g(s,x)Z(s,x)W(ds,dx) = \int_0^t \int_{\mathbb{R}^d} g(s,x)W^Z(ds,dx),$$

where W^Z is the martingale measure: $W_t^Z(A) = \int_0^t \int_A Z(s,y)W(ds,dy)$.

The existence of the solution of (2.1) (with zero initial initial conditions) is obtained using a Picard's iteration scheme. Setting $u_0(t,x)=0$ and

$$u_{n+1}(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma(u_n(s,y))W(ds,dy), \quad \forall n \geq 0,$$

one shows that $\{u_n(t,x)\}_n$ converges in $L^2(\Omega)$, uniformly in (t,x). The martingale theory is a crucial tool in this development, leading to:

$$E|u_{n+1}(t,x)|^2 = \int_0^t \int_{\mathbb{R}^d} |G(t-s,x-y)|^2 |\sigma(u_n(s,y))|^2 dyds.$$

Since for $H \neq 1/2$, the fBm is not a semimartingale, the previous argument breaks down in the case of a Gaussian noise with covariance functional J given by (1.7). There are two methods that can be used to circumvent this difficulty: one is to exploit the sample path regularity of the fBm by defining a path-wise integral; the other is Malliavin calculus. The first method was used extensively in the literature related to stochastic differential equations (SDEs), but is not easily applicable to SPDEs. In this article, we use the Malliavin calculus, whose basic ingredient is an *isonormal Gaussian process*, i.e., a centered Gaussian process indexed by the elements of a Hilbert space, whose covariance preserves the inner product in this space (see [33]).

In our case, $\{W(\varphi)\}_{\varphi\in\mathcal{H}}$ is an isonormal Gaussian process. We consider the *divergence operator* δ : Dom $\delta \subset L^2(\Omega;\mathcal{H}) \to L^2(\Omega)$, defined as the adjoint of the *Mallliavin derivative* D (see [33]). For any $u \in$ Dom δ, we denote $\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x)W(\delta t, \delta x)$. By Proposition 1.3.1 of [33],

$$E|\delta(u)|^2 = E\|u\|^2_{\mathcal{H}} + E[\mathrm{Tr}(Du \circ Du)].$$

Whether the Picard's iteration actually diverges (or whether there exists a modified Picard's iteration which converges) remains an open problem.

3. SPDEs with multiplicative noise

There is a way around this difficulty, when $\sigma(x) = x$. Consider the equation:

$$Lu(t,x) = u(t,x)\dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d. \tag{3.1}$$

A candidate $\{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ for a solution of (3.1) should satisfy the equation:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)u(s,y)W(ds,dy), \tag{3.2}$$

where w is the solution of the equation $Lu = 0$ with some deterministic initial conditions.

Remark 3.1. To see what w looks like, we consider two examples. Suppose first that $L = \frac{\partial}{\partial t} - \mathcal{L}$, where $\mathcal{L}$ is the L^2-generator of a d-dimensional Lévy process, and the initial condition is $u(0,\cdot) = u_0$. Then $w(t,x) = (P_t u_0)(x)$, where $(P_t)_{t\geq 0}$ is the transition semigroup of the Lévy process. Assume next that $L = \frac{\partial^2}{\partial t^2} - \Delta$, $d \leq 2$, and the initial conditions are $u(0,\cdot) = u_0$, $\frac{\partial u}{\partial t}(0,\cdot) = u_1$. Then

$$w(t,x) = \frac{\partial}{\partial t}(G(t,\cdot) * u_0)(x) + (G(t,\cdot) * u_1)(x),$$

where $*$ denotes convolution and G is the fundamental solution of the equation $Lu = 0$:

$$G(t,x) = \frac{1}{2}1_{\{|x|<t\}} \quad \text{if } d = 1$$

$$G(t,x) = \frac{1}{2\pi}(t^2 - |x|^2)^{-1/2}1_{\{|x|<t\}} \quad \text{if } d = 2.$$

We now return to equation (3.2). The idea is to replace $u(s,y)$ in the RHS of (3.2) by

$$w(s,y) + \int_0^s \int_{\mathbb{R}^d} G(s-r, y-z)u(r,z)W(dr,dz),$$

and then iterate this procedure. Intuitively, the solution admits the series representation $u(t,x)=w(t,x)+\sum_{n\geq 1}J_n(t,x)$, where

$$J_n(t,x)=\int f_n(t_1,x_1,\ldots,t_n,x_n,t,x)W(dt_1,dx_1)\ldots W(dt_n,dx_n),$$

$$f_n(t_1,x_1,\ldots,t_n,x_n,t,x)=\prod_{i=1}^{n}G(t_{i+1}-t_i,x_{i+1}-x_i)w(t_1,x_1)1_{\{0<t_1<\cdots<t_n<t\}}, \quad (3.3)$$

$t_{n+1}=t$ and $x_{n+1}=x$. This idea, which originates in the seminal article [13], can be made rigorous in the present context using Malliavin calculus.

Remark 3.2. In the case of the heat equation with infinite-dimensional fractional noise, a similar chaos representation for the solution was proposed in [32], where it was conjectured that the solution admits a certain Feynman–Kac representation. This conjecture has been recently proved in [21].

We introduce the following definition.

Definition 3.3. A square-integrable adapted process $\{u(t,x);t\geq 0,x\in\mathbb{R}^d\}$ is a **solution** of equation (3.1) if for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, the process $v^{(t,x)}=G(t-\cdot,x-\cdot)u$ belongs to Dom δ and

$$u(t,x)=w(t,x)+\delta(v^{(t,x)})$$

Any random variable $F\in L^2(\Omega)$ admits the Wiener chaos representation $F=\sum_{n\geq 0}I_n(f_n)$, where I_n is the multiple Wiener integral of order n and $f_n\in\mathcal{H}^{\otimes n}$ (see [33]). In particular, $E|F|^2=\sum_{n\geq 0}n!\|\tilde{f}_n\|_{\mathcal{H}^{\otimes n}}$, $\tilde{f}_n$ being the symmetrization of f_n. Based on the heuristics above, a natural candidate for the solution is a square-integrable process $\{u(t,x);t\geq 0,x\in\mathbb{R}^d\}$ given by

$$u(t,x)=\sum_{n\geq 0}I_n(f_n(\cdot,t,x)), \quad (3.4)$$

provided that the kernel $f_n(\cdot,t,x)$ given by (3.3) lies in $\mathcal{H}^{\otimes n}$.

When $G(t,\cdot)$ is a distribution, one has to give a meaning to $f_n(\cdot,t,x)$ using its action on test functions. The second task is to find suitable conditions which ensure that $f_n(\cdot,t,x)$ lies in $\mathcal{H}^{\otimes n}$, by generalizing Theorem 1.6 to multiple variables. Third, one has to show that $u(t,x)$ given by (3.4) is a solution of (3.1). We refer the reader to Section 2 of [3] for the details.

Note that

$$E|u(t,x)|^2=\sum_{n\geq 1}E|I_n(f_n(\cdot,t,x))|^2=\sum_{n\geq 1}n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}},$$

where $\tilde{f}_n(\cdot,t,x)$ is the symmetrization of $f_n(t,x)$. If $f_n(\cdot,t,x)$ is a function,

$$\tilde{f}_n(t_1,x_1,\ldots,t_n,x_n,t,x)=\frac{1}{n!}\sum_{\rho\in S_n}f_n(t_{\rho(1)},x_{\rho(1)},\ldots,t_{\rho(n)},x_{\rho(n)},t,x), \quad (3.5)$$

where S_n denotes the set of all permutations of $\{1,\ldots,n\}$.

A sufficient condition for the existence of a solution $\{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ is that the series of (3.4) converges in $L^2(\Omega)$, i.e.,

$$I(t) := \sum_{n\geq 1} n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} < \infty. \tag{3.6}$$

We should point out that the question of the uniqueness of the solution remains an open problem.

4. The wave equation

The remaining part of this article is dedicated to verifying condition (3.6) in the case of the wave equation (in arbitrary dimension $d \geq 1$). Similar arguments work for the heat equation. For this, we assume that the covariance functional J is given by (1.7) with f being the Riesz kernel of order $\alpha \in (0,d)$, i.e.,

$$f(x) = c_{\alpha,d}|x|^{-(d-\alpha)} \quad \text{and} \quad \mu(d\xi) = |\xi|^{-\alpha}d\xi.$$

We consider the equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u(t,x)\dot{W}(t,x) \quad t>0, x\in\mathbb{R}^d$$

with initial conditions $u(0,\cdot)=1$ and $\frac{\partial u}{\partial t}u(0,\cdot)=0$. In this case, $w=1$ and

$$\mathcal{F}G(t,\cdot) = \frac{\sin(t|\xi|)}{|\xi|} \quad \text{for any } t>0, \xi\in\mathbb{R}^d.$$

Letting $\alpha_n(t) = (n!)^2\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}$, condition (3.6) becomes:

$$\sum_{n\geq 1}\frac{1}{n!}\alpha_n(t) < \infty. \tag{4.1}$$

We proceed to the calculation of $\alpha_n(t)$. Assume first that $d \leq 2$, so that $G(t,\cdot)$ is a function. Then

$$\alpha_n(t) = (n!)^2\alpha_H^n \int_{[0,t]^{2n}} d\mathbf{t}d\mathbf{s}\prod_{j=1}^n |t_j-s_j|^{2H-2}\int_{\mathbb{R}^{2nd}} d\mathbf{x}d\mathbf{y}\prod_{j=1}^n f(x_j-y_j)$$
$$\tilde{f}_n(t_1,x_1,\ldots,t_n,x_n,t,x)\tilde{f}_n(s_1,y_1,\ldots,s_n,y_n,t,x),$$

where $\mathbf{t}=(t_1,\ldots,t_n)$, $\mathbf{s}=(s_1,\ldots,s_n)$, $\mathbf{x}=(x_1,\ldots,x_n)$ and $\mathbf{y}=(y_1,\ldots,y_n)$.

When $d\geq 3$, $G(t,\cdot)$ is a distribution. To calculate the $\|\cdot\|^2_{\mathcal{H}^{\otimes n}}$-norm of $\tilde{f}_n(\cdot,t,x)$, we use a generalization of (1.7) to multiple variables (see Theorem 2.2 of [3]). In this case, $\alpha_n(t)$ can be computed using the fact that $\mathcal{F}G(t,\cdot)$ is a function:

$$\alpha_n(t) = (n!)^2\alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j-s_j|^{2H-2}\int_{\mathbb{R}^{nd}} \mathcal{F}\tilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\boldsymbol{\xi})$$
$$\overline{\mathcal{F}\tilde{f}_n(s_1,\cdot,\ldots,s_n,\cdot,t,x)(\boldsymbol{\xi})}\mu(d\xi_1)\ldots\mu(d\xi_n)d\mathbf{t}d\mathbf{s},$$

where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$. One can show that:

$$\mathcal{F}\tilde{f}_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)(\boldsymbol{\xi}) = \frac{1}{n!} e^{-i(\xi_1 + \cdots + \xi_n)\cdot x} \sum_{\rho \in S_n} \overline{\mathcal{F}G(u_1, \cdot)(\xi_{\rho(1)})}$$
$$\overline{\mathcal{F}G(u_2, \cdot)(\xi_{\rho(1)+\rho(2)})} \ldots \overline{\mathcal{F}G(u_n, \cdot)(\xi_{\rho(1)} + \cdots + \xi_{\rho(n)})} 1_{\{t_{\rho(1)} < \cdots < t_{\rho(n)}\}}$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ for all $1 \leq j \leq n$, and $t_{\rho(n+1)} = t$.

We obtain:

$$\alpha_n(t) = \alpha_H^n \int_{(0,t)^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_n(\mathbf{t}, \mathbf{s}) d\mathbf{t} d\mathbf{s}, \tag{4.2}$$

where $\mathbf{t} = (t_1, \ldots, t_n)$ and $\mathbf{s} = (s_1, \ldots, s_n)$. The function $\psi_n(\mathbf{t}, \mathbf{s})$ is given by:

$$\psi_n(\mathbf{t}, \mathbf{s}) = \int_{\mathbb{R}^{nd}} H_n(\mathbf{t}, \xi_1, \ldots, \xi_n) \overline{H_n(\mathbf{s}, \xi_1, \ldots, \xi_n)} \mu(d\xi_1) \ldots \mu_n(d\xi_n),$$

where

$$H_n(\mathbf{t}, \xi_1, \ldots, \xi_n) = \prod_{j=1}^n \mathcal{F}G(u_j, \ \cdot)(\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}),$$

if $t_{\rho(1)} < \cdots < t_{\rho(n)}$, and we denote $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ and $t_{\rho(n+1)} = t$.

The difficulty in the calculation of $\psi_n(\mathbf{t}, \mathbf{s})$ stems from the fact that the two n-tuples $\mathbf{t} = (t_1, \ldots, t_n)$ and $\mathbf{s} = (s_1, \ldots, s_n)$ are ordered by *different* permutations ρ and σ, so that $t_{\rho(1)} < \cdots < t_{\rho(n)}$ and $s_{\sigma(1)} < \cdots < s_{\sigma(n)}$.

Remark 4.1. A similar calculation of $\alpha_n(t)$ was carried out in [20], in the case of the stochastic heat equation with white noise in space. In this case, μ is the Lebesgue measure on $\mathbb{R}^d$ and $\psi_n(\mathbf{t}, \mathbf{s}) = (2\pi)^{-nd/2} \det[M(\mathbf{s}) + M(\mathbf{t})]^{-d/2}$, where $M(\mathbf{s})$ is the matrix with entries $(s_i \wedge s_j)_{1 \leq i,j \leq d}$. To separate the two terms (depending on $\mathbf{s}$ and $\mathbf{t}$), the authors of [20] use the inequality: $\det[M(\mathbf{s}) + M(\mathbf{t})] \geq \det M(\mathbf{s}) + \det M(\mathbf{t})$ (see page 294 of [20]). This technique is specific to the white noise in space and cannot be employed when $\mu(d\xi) = |\xi|^{-\alpha} d\xi$.

To avoid the problem of the two different permutations and separate the terms depending on $\mathbf{t}$ and $\mathbf{s}$, we use the Cauchy–Schwarz inequality. Although this method may not yield a sharp estimate, it seems to be the only way out of this difficulty for the present time. We obtain:

$$\psi_n(\mathbf{t}, \mathbf{s}) \leq \psi_n(\mathbf{t}, \mathbf{t})^{1/2} \psi_n(\mathbf{s}, \mathbf{s})^{1/2}. \tag{4.3}$$

To calculate $\psi_n(\mathbf{t}, \mathbf{t})$, let $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ for $j = 1, \ldots, n$. Using first the change of variable $\xi'_j = \xi_{\rho(j)}$, and then $\eta_j = \xi'_1 + \cdots + \xi'_j$, we get:

$$\psi_n(\mathbf{t}, \mathbf{t}) = \int_{\mathbb{R}^d} d\eta_1 \frac{\sin^2(u_1|\eta_1|)}{|\eta_1|^2} |\eta_1|^{-\alpha} \int_{\mathbb{R}^d} d\eta_2 \frac{\sin^2(u_2|\eta_2|)}{|\eta_2|^2} |\eta_2 - \eta_1|^{-\alpha} \ldots$$
$$\int_{\mathbb{R}^d} d\eta_n \frac{\sin^2(u_n|\eta_n|)}{|\eta_n|^2} |\eta_n - \eta_{n-1}|^{-\alpha}.$$

The inner integral above is estimated by the following lemma.

Lemma 4.2. *Assume that $d-2<\alpha<d$. Then,*

$$I := \int_{\mathbb{R}^d} \frac{\sin^2(t|\xi|)}{|\xi|^2} |\xi-\eta|^{-\alpha} d\xi \leq C_{\alpha,d} t^{\alpha-d+2}, \quad \textit{for any } t>0, \eta \in \mathbb{R}^d.$$

Proof. Using the change of variable $\xi' = t\xi$, we obtain,

$$I = t^{\alpha-d+2} \int_{\mathbb{R}^d} \frac{\sin^2(|\xi'|)}{|\xi'|^2} |\xi' - t\eta|^{-\alpha} d\xi'.$$

We claim that there exists a constant $C_{\alpha,d}>0$ such that:

$$I(a) := \int_{\mathbb{R}^d} \frac{\sin^2(|\xi|)}{|\xi|^2} |a-\xi|^{-\alpha} d\xi \leq C_{\alpha,d}, \quad \forall a \in \mathbb{R}^d.$$

To see this, we change the variable $a-\xi$ into ξ, and we write

$$I(a) = \int_{|\xi|\leq 1} \frac{\sin^2(|\xi-a|)}{|\xi-a|^2} |\xi|^{-\alpha} d\xi + \int_{|\xi|>1} \frac{\sin^2(|\xi-a|)}{|\xi-a|^2} |\xi|^{-\alpha} d\xi =: I_1(a) + I_2(a).$$

For $I_1(a)$, we use the fact that $|\frac{\sin x}{x}| \leq 1$ for any $x>0$. Hence

$$I_1(a) \leq \int_{|\xi|\leq 1} |\xi|^{-\alpha} d\xi = c_d \int_0^1 \lambda^{-\alpha+d-1} d\lambda = c_d \frac{1}{d-\alpha},$$

where $c_d>0$ is a constant depending on d. For $I_2(a)$, we use the fact that

$$\frac{\sin^2(t|\xi|)}{|\xi|^2} \leq 2(t^2+1) \frac{1}{1+|\xi|^2}, \quad \forall t>0, \forall \xi \in \mathbb{R}^d.$$

(see p. 81 of [35]). In our case, $t=1$. Hence

$$I_2(a) \leq 4 \int_{|\xi|>1} \frac{1}{1+|\xi-a|^2} |\xi|^{-\alpha} d\xi \leq 4 \sup_{a\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1+|\xi-a|^2} |\xi|^{-\alpha} d\xi.$$

Finally, we observe that $\alpha > d-2$ is equivalent to $\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} |\xi|^{-\alpha} d\xi < \infty$, which is in turn equivalent to: (see (5.5) of [7])

$$\sup_{a\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1+|\xi-a|^2} |\xi|^{-\alpha} d\xi < \infty. \qquad \square$$

Using Lemma 4.2 iteratively, we obtain that:

$$\psi_n(\mathbf{t},\mathbf{t}) \leq C_{\alpha,d}^n (u_1 \dots u_n)^{\alpha-d+2}. \tag{4.4}$$

From (4.3) and (4.4), it follows that:

$$\psi_n(\mathbf{t},\mathbf{s}) \leq C_{\alpha,d}^n [\beta(\mathbf{t})\beta(\mathbf{s})]^{(\alpha-d+2)/2} \tag{4.5}$$

where $\beta(\mathbf{t}) = \prod_{j=1}^n (t_{\rho(j+1)} - t_{\rho(j)})$ and $\beta(\mathbf{s}) = \prod_{j=1}^n (s_{\sigma(j+1)} - s_{\sigma(j)})$. Here ρ and σ are the permutations which arrange in increasing order the points $\{t_1,\dots,t_n\}$, respectively $\{s_1,\dots,s_n\}$.

From (4.2) and (4.5), we obtain:

$$\alpha_n(t) \leq (\alpha_H C_{\alpha,d})^n \int_{(0,t)^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} [\beta(\mathbf{t})\beta(\mathbf{s})]^{(\alpha-d+2)/2} d\mathbf{t} d\mathbf{s}. \tag{4.6}$$

Note that (4.6) is the wave equation analogue of the estimate (3.14) obtained in [20] for the heat equation with white noise in space (in dimension $d = 1$).

We proceed as in the proof of (3.15) of [20]. Using the Cauchy–Schwarz inequality and the fact that $\sup_{r \leq t} \int_0^t |r-s|^{2H-2} dr \leq c_H t^{2H-1}$, we obtain:

$$\alpha_n(t) \leq (\alpha_H C_{\alpha,d})^n \left\{ \int_{(0,t)^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \beta(\mathbf{t})^{\alpha-d+2} d\mathbf{t} d\mathbf{s} \right\}^{1/2}$$

$$\left\{ \int_{(0,t)^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \beta(\mathbf{s})^{\alpha-d+2} d\mathbf{t} d\mathbf{s} \right\}^{1/2}$$

$$= (\alpha_H C_{\alpha,d})^n \int_{(0,t)^n} \beta(\mathbf{t})^{\alpha-d+2} \left(\int_{(0,t)^n} \prod_{j=1}^n |t_j - s_j|^{2H-2} d\mathbf{s} \right) d\mathbf{t}$$

$$\leq (c_H \alpha_H C_{\alpha,d})^n t^{n(2H-1)} n! \int_{0<t_1<\cdots<t_n<t} [(t-t_n)\ldots(t_2-t_1)]^{\alpha-d+2} d\mathbf{t}.$$

To compute the last integral, we note that for any $h > -1$,

$$\int_{0<t_1<\cdots<t_n<t} [(t-t_n)\ldots(t_2-t_1)]^h d\mathbf{t} = \frac{1}{\Gamma((1+h)n+1)} \Gamma(1+h)^{n+1} t^{n(1+h)}.$$

Letting $h = \alpha - d + 2$, and using the fact that $\Gamma((1+h)n+1) \sim (n!)^{1+h}$, we obtain:

$$\alpha_n(t) \leq C_{\alpha,d,H}^n t^{n(2H+\alpha-d+2)} \frac{1}{(n!)^{\alpha-d+2}}, \tag{4.7}$$

where $C_{\alpha,d,H} > 0$ is a constant depending on α, d and H.

The major drawback of the previous method is that it yields an upper bound for $\alpha_n(t)$ in which the power of $n!$ does not depend on H. Clearly, from (4.7), we see that (4.1) holds if $\alpha > d - 2$. But this is the *same* condition as the one obtained for the existence of the mild solution in the case of the white noise in time (see Remark 5.2 of [7]). In fact, in the case of the wave equation, the restriction $\alpha > d - 2$ comes from Lemma 4.2, which is a key step in all these developments. One would hope that in the case of the fractional noise in time, the condition for the existence of the mild solution could be relaxed so that it would reflect the influence of the time parameter H. Unfortunately, this remains an open problem.

Remark 4.3. Similar calculations can be done for the heat equation:

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d \tag{4.8}$$

with deterministic initial condition $u(0,\cdot) = u_0$. In this case, $\mathcal{F}G(t,\cdot)(\xi) = e^{-t|\xi|^2/2}$ and instead of Lemma 4.2, one uses the following estimate:

$$\int_{\mathbb{R}^d} e^{-t|\xi|^2} |\xi - \eta|^{-\alpha} d\xi \leq C_{\alpha,d} t^{-(d-\alpha)/2},$$

for any $\alpha \in (0,d)$, where $C_{\alpha,d} > 0$ is a constant depending on α and d. Using the same procedure as above, this leads to the estimates: $\psi_n(\mathbf{t},\mathbf{s}) \leq C_{\alpha,d}^n[\beta(\mathbf{t})\beta(\mathbf{s})]^{-(d-\alpha)/4}$ and

$$\alpha_n(t) \leq (\alpha_H C_{\alpha,d})^n \int_{(0,t)^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2}[\beta(\mathbf{t})\beta(\mathbf{s})]^{-(d-\alpha)/4} d\mathbf{t}d\mathbf{s}.$$

(Note that the last inequality coincides with (3.14) of [20], in the case $d = 1$ and $\alpha = 0$.) We infer that

$$\alpha_n(t) \leq C_{\alpha,d,H}^n t^{n[2H-(d-\alpha)/2]}(n!)^{(d-\alpha)/2},$$

which results in the same problem mentioned above for the wave equation. Again, (4.1) holds if $\alpha > d-2$, but this last condition may not be optimal. It should be pointed out that the estimate (3.15) of [20] also does not contain the time parameter H in the power of $n!$, and therefore the existence result of [20] can be viewed as sub-optimal too. More precisely, in the case of the linear heat equation $u_t = \frac{1}{2}\Delta u + \dot{W}$ with white noise in space and $H > 1/2$, the condition for the existence of the mild solution is known to be $H > d/4$ (see Theorem 2.7 of [5]). Since $H < 1$, this means that d can be 1, 2 or 3. However, in the case of the equation (4.8) with white noise in space, Proposition 4.3 of [20] proved the existence (and uniqueness) of the mild solution only in the case $d = 1$, and for $d = 2$, only up to a time T_0. The necessary condition for the existence of a mild solution of the heat equation remains to be found.

Remark 4.4. The difficulty with the calculation of $\alpha_n(t)$ is due mainly to the fractional component of the noise. In the case of the white noise in time, the calculation of $\alpha_n(t)$ is much simpler. In this case, for each $n \geq 1$, there is a *single* term which appears in the series (3.6), since

$$n!\|\tilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} = \|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}.$$

To see this, note that in the case of the white noise in time,

$$\alpha_n(t) = (n!)^2\alpha_H^n \int_{(0,t)^n} d\mathbf{t} \int_{\mathbb{R}^{2nd}} d\mathbf{x}d\mathbf{y} \prod_{j=1}^n f(x_j - y_j)$$
$$\tilde{f}_n(t_1,x_1,\ldots,t_n,x_n,t,x)\tilde{f}_n(t_1,y_1,\ldots,t_n,y_n,t,x).$$

Assume that $f_n(\cdot,t,x)$ is a function, and use the definition (3.5) of $\tilde{f}_n(\cdot,t,x)$. Since the formula (3.3) of $f_n(t_1,x_1,\ldots,t_n,x_n,t,x)$ contains the indicator of the set $\{0 < t_1 < \cdots < t_n < t\}$, it follows that

$$f_n(t_{\rho(1)},x_{\rho(1)},\ldots,t_{\rho(n)},x_{\rho(n)},t,x)f_n(t_{\sigma(1)},y_{\sigma(1)},\ldots,t_{\sigma(n)},y_{\sigma(n)},t,x) = 0,$$

for any $\rho,\sigma \in S_n$ with $\rho \neq \sigma$. Hence,

$$\alpha_n(t) = \alpha_H^n \sum_{\rho\in S_n} \int_{(0,t)^n} d\mathbf{t} \int_{\mathbb{R}^{2nd}} d\mathbf{x}d\mathbf{y} \prod_{j=1}^n f(x_j - y_j)$$
$$f_n(t_{\rho(1)},x_{\rho(1)},\ldots,t_{\rho(n)},x_{\rho(n)},t,x)f_n(t_{\rho(1)},y_{\rho(1)},\ldots,t_{\rho(n)},y_{\rho(n)},t,x).$$

Due to the symmetry of the integrand, this leads to:

$$\alpha_n(t) = n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}.$$

Remark 4.5. The recent article [21] considered the heat equation (4.8) driven by a Gaussian noise W with covariance functional given by (1.7), where f is given by (1.10). In this case, $\mu(d\xi) = \prod_{i=1}^d (c_{H_i}|\xi_i|^{-(2H_i-1)})d\xi$ for some constants $c_{H_i} > 0$. Under the condition $\sum_{i=1}^d (2H_i - 1) > d - 4H + 2$, the authors of [21] established Feynman–Kac (FK) formulas for both the weak solution (defined using the Stratonovich integral) and the mild solution (defined using the Skorohod integral, as in the present paper). The idea of the proof is to replace $\dot{W}$ by a regularization $\dot{W}^{\varepsilon,\delta}$, write down a FK formula for the solution $u^{\varepsilon,\delta}(t,x)$ of the heat equation with regularized noise, and then show that $u^{\varepsilon,\delta}(t,x)$ converges in $L^2(\Omega)$ to the solution $u(t,x)$, as $\varepsilon \to 0$ and $\delta \to 0$. This idea may be carried over to the wave equation, using a different type of FK formula for the wave equation (developed recently in [11]), but the details are non-trivial. On the other hand, a careful inspection of the proofs of [21] shows that the same FK formula for the heat equation continue to hold if one replaces f in (1.10) by the Riesz kernel $f(x) = c_{\alpha,d}|x|^{-(d-\alpha)}$, with $\alpha \in (0,d)$. In this case, the condition for the FK formula is $\alpha > d - 4H + 2$, which is *stronger* than the condition $\alpha > d - 2$ for the existence of the mild solution, obtained using the methods of the present article (see Remark 4.3). Note that when $d = 1$ and $\alpha = 0$ (i.e., the noise is white in space), the mild solution exists for any $H > 1/2$, but a Feynman–Kac formula can be written down only for $H > 3/4$. In particular, there is no Feynman–Kac formula for the equation driven by a space-time white noise (i.e., $d = 1$, $\alpha = 0$ and $H = 1/2$).

Remark 4.6. A different idea for finding an upper bound for $\alpha_n(t)$ would be to estimate first the $d\mathbf{t}d\mathbf{s}$ integral, as it was done in [6] in the case of the linear wave (or heat) equation. More precisely, using (4.2) and Fubini's theorem,

$$\alpha_n(t) = \int_{\mathbb{R}^{nd}} N_t^{(n)}(\xi_1,\ldots,\xi_n)\mu(d\xi_1)\ldots\mu(d\xi_n),$$

where

$$N_t^{(n)}(\boldsymbol{\xi}) = \alpha_H^n \int_{(0,t)^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} H(\mathbf{t},\boldsymbol{\xi})\overline{H(\mathbf{s};\boldsymbol{\xi})}d\mathbf{t}d\mathbf{s}$$

and $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_d)$. In particular, $\alpha_1(t) = \int_{\mathbb{R}^d} N_t^{(1)}(\xi)\mu(d\xi)$, where

$$N_t^{(1)}(\xi) = \alpha_H \int_0^t \int_0^t |r-s|^{2H-2}\mathcal{F}G(t-s,\cdot)(\xi)\overline{\mathcal{F}G(t-r,\cdot)(\xi)}dsdr.$$

The recent Theorem 4.3 of [1] showed that for any $t > 0$ and $\xi \in \mathbb{R}^d$,

$$c_{t,H}\left(\frac{1}{1+|\xi|^2}\right)^{H+1/2} \leq N_t^{(1)}(\xi) \leq C_{t,H}\left(\frac{1}{1+|\xi|^2}\right)^{H+1/2}, \tag{4.9}$$

where $C_{t,H} > 0$ is increasing in t. Therefore, $\alpha_1(t) < \infty$ if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{H+1/2} \mu(d\xi) < \infty, \tag{4.10}$$

which is the necessary and sufficient condition for the existence of the solution of the linear wave equation. (In [6], only the upper bound in (4.9) was proved, while for the necessity part, an indirect argument was used.) When $\mu(d\xi) = |\xi|^{-\alpha}$ for $\alpha \in (0, d)$, condition (4.10) becomes $\alpha > d - 2H - 1$.

In the case of arbitrary n, $N_t^{(n)}(\boldsymbol{\xi})$ is a sum of $(n!)^2$ terms, since each of $H(\mathbf{t}, \boldsymbol{\xi})$ and $H(\mathbf{s}, \boldsymbol{\xi})$ is a sum of $n!$ terms. Hence, $\alpha_n(t)$ is a sum of $(n!)^2$ terms. To see what these terms look like, consider the case $n = 2$. $N_t^{(2)}(\xi_1, \xi_2)$ is the sum of 4 terms, which correspond to the integrals over the regions $\{t_1 < t_2, s_1 < s_2\}$, $\{t_1 < t_2, s_2 < s_1\}$, $\{t_2 < t_1, s_1 < s_2\}$, and $\{t_2 < t_1, s_2 < s_1\}$. We examine the term $N_{t,2}^{(2)}(\xi_1, \xi_2)$, corresponding to the region $\{t_1 < t_2, s_2 < s_1\}$:

$$N_{t,2}^{(2)}(\xi_1, \xi_2) = \alpha_H^2 \int_{(0,t)^4} |t_1 - s_1|^{2H-2} |t_2 - s_2|^{2H-2} \phi_{\boldsymbol{\xi}}(t_1, t_2) \overline{\psi_{\boldsymbol{\xi}}(s_1, s_2)} d\mathbf{t} d\mathbf{s},$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$ and

$$\phi_{\boldsymbol{\xi}}(t_1, t_2) = \mathcal{F}G(t_2 - t_1; \cdot)(\xi_1) \mathcal{F}G(t - t_2; \cdot)(\xi_1 + \xi_2) 1_{\{t_1 < t_2\}}$$
$$\psi_{\boldsymbol{\xi}}(s_1, s_2) = \mathcal{F}G(s_1 - s_2; \cdot)(\xi_2) \mathcal{F}G(t - s_1; \cdot)(\xi_1 + \xi_2) 1_{\{s_2 < s_1\}}.$$

By the Cauchy–Schwarz inequality,

$$N_{t,2}^{(2)}(\xi_1, \xi_2) \leq \|\phi_{\boldsymbol{\xi}}\| \cdot \|\psi_{\boldsymbol{\xi}}\|,$$

where

$$\begin{aligned}
\|\phi_{\boldsymbol{\xi}}\|^2 &= \alpha_H^2 \int_{(0,t)^4} |t_1 - s_1|^{2H-2} |t_2 - s_2|^{2H-2} \phi_{\boldsymbol{\xi}}(t_1, t_2) \overline{\phi_{\boldsymbol{\xi}}(s_1, s_2)} d\mathbf{t} d\mathbf{s} \\
&= \alpha_H \int_0^t \int_0^t dt_2 ds_2 |t_2 - s_2|^{2H-2} \mathcal{F}G(t_2, \cdot)(\xi_1 + \xi_2) \overline{\mathcal{F}G(s_2, \cdot)(\xi_1 + \xi_2)} \\
&\quad \left(\alpha_H \int_0^{t_2} \int_0^{s_2} dt_1 ds_1 |t_1 - s_1|^{2H-2} \mathcal{F}G(t_2 - t_1, \cdot)(\xi_1) \overline{\mathcal{F}G(s_2 - s_1, \cdot)(\xi_1)}\right).
\end{aligned}$$

By the Cauchy–Schwarz inequality, the inner parenthesis above is bounded by $N_{t_2}^{(1)}(\xi_1)^{1/2} N_{s_2}^{(1)}(\xi_1)^{1/2}$, which in turn is bounded by $C_{t,H}(1 + |\xi_1|^2)^{-H-1/2}$, using (4.9) and the fact that $C_{t,H}$ is increasing in t. Therefore,

$$\begin{aligned}
\|\phi_{\boldsymbol{\xi}}\|^2 &\leq C_{t,H} \left(\frac{1}{1+|\xi_1|^2}\right)^{H+1/2} N_t^{(1)}(\xi_1 + \xi_2) \\
&\leq C_{t,H}^2 \left(\frac{1}{1+|\xi_1|^2}\right)^{H+1/2} \left(\frac{1}{1+|\xi_1 + \xi_2|^2}\right)^{H+1/2},
\end{aligned}$$

where we used (4.9) again for the second inequality above. Clearly, $\|\psi_{\boldsymbol{\xi}}\|^2$ satisfies a similar inequality with ξ_1 replaced by ξ_2. Putting together the 4 terms which make up $\alpha_2(t)$, we arrive at the conclusion that:

$$\alpha_2(t) \leq C_{t,H}^2 \sum_{\rho,\sigma \in S_2} \int_{\mathbb{R}^{2d}} \left(\frac{1}{1+|\xi_{\rho(1)}|^2}\right)^{\frac{H+1/2}{2}} \left(\frac{1}{1+|\xi_{\rho(1)}+\xi_{\rho(2)}|^2}\right)^{\frac{H+1/2}{2}}$$
$$\left(\frac{1}{1+|\xi_{\sigma(1)}|^2}\right)^{\frac{H+1/2}{2}} \left(\frac{1}{1+|\xi_{\sigma(1)}+\xi_{\sigma(2)}|^2}\right)^{\frac{H+1/2}{2}} \mu(d\xi_1)\mu(d\xi_2)$$
$$= C_{t,H}^2 \int_{\mathbb{R}^{2d}} \left[\sum_{\rho \in S_2} \left(\frac{1}{1+|\xi_{\rho(1)}|^2}\right)^{\frac{H+1/2}{2}} \left(\frac{1}{1+|\xi_{\rho(1)}+\xi_{\rho(2)}|^2}\right)^{\frac{H+1/2}{2}}\right]^2 \mu(d\xi_1)\mu(d\xi_2).$$

Using the inequality $(\sum_{i\in S} a_i)^2 \leq \text{card}(S) \sum_{i \in S} a_i^2$ for any finite set S and any positive numbers a_i, and the symmetry of the integrand, we obtain:

$$\alpha_2(t) \leq (2!)^2 C_{t,H}^2 \int_{\mathbb{R}^{2d}} \left(\frac{1}{1+|\xi_1|^2}\right)^{H+1/2} \left(\frac{1}{1+|\xi_1+\xi_2|^2}\right)^{H+1/2} \mu(d\xi_1)\mu(d\xi_2).$$

Using an argument similar to the proof of Lemma 8 of [10], one can show that (4.10) is equivalent to:

$$D_H := \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi+\eta|^2}\right)^{H+1/2} \mu(d\xi) < \infty.$$

Therefore, under (4.10), we have: $\alpha_2(t) \leq (2!)^2 C_{t,H}^2 D_H^2$. For general n, this method leads to the estimate:

$$\alpha_n(t) \leq (n!)^2 C_{t,H}^n D_H^n,$$

which is not powerful enough to yield the convergence of the series $\sum_{n\geq 1} \frac{\alpha_n(t)}{n!}$. This shows that the multiplicative case differs significantly from the additive case. In the multiplicative case, one has to control the $(n!)^2$ terms which appear in the development of $\alpha_n(t)$ to produce a finite series, whereas in the additive case, all that is required for the existence of the solution is that the first term $\alpha_1(t)$ of this series be finite.

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On Chaos Representation and Orthogonal Polynomials for the Doubly Stochastic Poisson Process

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Abstract. In an L_2-framework, we study various aspects of stochastic calculus with respect to the centered doubly stochastic Poisson process. We introduce an orthogonal basis via multilinear forms of the value of the random measure and we analyze the chaos representation property. We review the structure of non-anticipating integration for martingale random fields and in this framework we study non-anticipating differentiation. We present integral representation theorems where the integrand is explicitly given by the non-anticipating derivative.

Stochastic derivatives of anticipative nature are also considered: The Malliavin type derivative is put in relationship with another anticipative derivative operator here introduced. This gives a new structural representation of the Malliavin derivative based on simple functions. Finally we exploit these results to provide a Clark–Ocone type formula for the computation of the non-anticipating derivative.

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Keywords. Doubly stochastic Poisson process, Cox process, orthogonal polynomials, non-anticipating derivative, Malliavin derivative, Clark–Ocone formula, martingale random fields.

1. Introduction

The doubly stochastic Poisson process (DSPP) also known as the Cox process, was introduced in [8] as a generalization of the Poisson process in the sense that the intensity is stochastic. Models based on DSPP's are used in risk theory, in the

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study of ruin probabilities in insurance and insurance-linked securities pricing, and for stochastic volatility see, e.g., [2, 7, 18, 23].

For a given doubly stochastic Poisson process H with intensity α, we investigate some elements of stochastic calculus for $\tilde{H} := H - \alpha$, i.e., the *centered doubly stochastic Poisson process* (CDSPP) on a quite general Hausdorff topological space X. The stochastic intensity α is assumed non-atomic. The paper is dedicated to the study of the structure of L_2-spaces generated by the noise and the non-anticipating integration and differentiation schemes with stochastic integral representations in view. One foreseen application of such integral representations is in the study of backward stochastic differential equations and it is, at present, work in progress, see [17].

First we show that the observations of $\tilde{H} = H - \alpha$ give complete information on both H and α. Specifically, the σ-algebra generated by $\tilde{H}$ coincides with the one generated by H and α. With respect to the space $L_2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the CDSPP, we suggest an orthogonal system of polynomials which lead to a chaos expansion type of result. This orthogonal system is based on what we call α-multilinear forms. These prove to be key constructive elements in our proofs.

After this analysis on a general X, we specify the study to the time-space $X = (0, T] \times Z$ with the total ordering induced by time. Here we introduce an information structure associated to the CDSPP. We consider the filtration $\mathbb{G}$ generated by the CDSPP augmented by the knowledge of the whole intensity α. Note that, with respect to $\mathbb{G}$, the CDSPP is a stochastic measure with conditionally independent values. In this setup we study elements of stochastic integration and differentiation. We find a stochastic integral representation for all elements in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ by interpreting the CDSPP as a martingale random field (see, e.g., [16]) and applying the corresponding Itô stochastic integration scheme. The representation of $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ is explicit in the sense that the integrand is uniquely determined as the non-anticipating stochastic derivative $\mathscr{D}\xi$ with respect to the CDSPP. The non-anticipating derivative, introduced in [12] and later developed to include Lévy type random measures (see [13, 15]) and martingale random fields as integrators (see [16]) is defined by the linear operator adjoint to the Itô stochastic integral. A general formula for the calculus is here given in terms of limit of specific simple stochastic functions. In particular, non-anticipating derivatives are a natural tool to study mean-variance hedging, see [1, 12, 16].

When discussing non-anticipating differentiation, the connections with the well-known correspondent of the Clark–Ocone formulae have to be taken into account. A first study of chaos expansions in terms of iterated integrals for processes with conditionally independent increments can be found in [29]. Starting from this set up, a Malliavin derivative operator is defined. In the present paper we discuss explicitly the relationship between the orthogonal polynomials here suggested and the Itô type iterated integrals and we retrace the relationships between the non-anticipating derivative and the Clark–Ocone formula based on the Malliavin derivative operator given in the literature. Our study however takes a different approach to Malliavin calculus. In fact we introduce a new anticipative derivative operator D^a as a limit of specific simple stochastic functions. Because of its particular structure, it is immediate to see that the non-anticipative derivative $\mathscr{D}\xi$ at time t is the projection of D^a on the information $\mathcal{G}_t$. On the other side we prove that this operator coincides with the Malliavin derivative $D\xi$ as introduced in [29]. These arguments provide a new structural approach to the Malliavin derivative.

We have partially considered integration with respect to the smaller filtration generated by the CDSPP only. Based on the martingale structure the non-anticipative differentiation can be carried through. However, we remark that there is no structure of conditional independence in this case and the study of anticipative differentiation is rather different. The study of stochastic differentiation in this setting will be developed separately.

To conclude we remark that stochastic integral representations have been investigated in [4, 5, 10, 19, 20] for general point processes. Our contribution differs because we consider the filtration generated by the CDSPP, which is larger than the filtration generated by the DSPP alone.

The paper is organized as follows. Section 2 provides the basic information on DSPP and CDSPP on a general space X. Multilinear forms and chaos expansions are studied in Section 3. For $X = (0,T] \times Z$, stochastic non-anticipating integration and martingale random fields are discussed in Section 4. Section 5 presents the non-anticipating derivative $\mathscr{D}$. A review on iterated integrals and their connection with multilinear forms is detailed in Section 6. Finally Section 7 presents the anticipative derivatives D^a and D, their computation, and their relationship with the non-anticipating derivative $\mathscr{D}$ via a Clark–Ocone type formula.

2. The doubly stochastic Poisson process

Let X be a locally compact, second countable Hausdorff topological space. Under these conditions, there exists a complete, separable metric μ generating the topology on X. In particular this implies that X is σ-compact, i.e., that it admits representation as a countable union of compact sets, and that the topology on X has a countable basis consisting of precompact sets, i.e., sets with compact closure. We denote $\mathcal{B}_X$ the Borel σ-algebra of X and $\mathcal{B}_X^c$ the precompacts of $\mathcal{B}_X$. The stochastic elements considered in the paper are related to the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let α be a (positive) random measure on X. We assume that α is non-atomic, meaning that $\mathbb{P}\big(\alpha(\{x\}) = 0 \text{ for all } x \in X\big) = 1$. For later use in the study of the polynomials we also assume that

$$\mathbb{E}\big[e^{c\alpha(\Delta)}\big] < \infty, \quad c \in \mathbb{R},\ \Delta \in \mathcal{B}_X^c, \tag{2.1}$$

i.e., the moment generating function of $\alpha(\Delta)$ is well defined on the whole real line when Δ is precompact. This will in turn imply

$$\mathbb{E}\big[\alpha(\Delta)^k\big] < \infty \quad \text{for all } \Delta \in \mathcal{B}_X^c,\ k = 1, 2, \dots \tag{2.2}$$

We remark that condition (2.1) is satisfied if $X = [0,\infty) \times \mathbb{R}_0$ and $\alpha(dt,dz) = \nu(dz)dt$ with ν a Lévy measure on $\mathbb{R}_0$ and dt the Lebesgue measure on $[0,\infty)$, i.e., the case of pure jump Lévy processes.

Let us define

$$V(\Delta) := \mathbb{E}[\alpha(\Delta)], \quad \Delta \in \mathcal{B}_X.$$

We note that V is a non-atomic σ-finite measure (see, e.g., [21, Chapter 1.2]), which is finite at least on all precompact sets. The σ-algebra generated by α will be denoted $\mathcal{F}^\alpha$.

Let H be a random measure on X and let $\mathcal{F}^H_\Delta$ denote the σ-algebra generated by $H(\Delta')$, $\Delta' \in \mathcal{B}_X : \Delta' \subset \Delta$ (with $\Delta \in \mathcal{B}_X$). Set $\mathcal{F}^H$ to be the σ-algebra generated by all the values of H.

Definition 2.1. The random measure H is a doubly stochastic Poisson process (DSPP) if

A1) $\mathbb{P}\Big(H(\Delta) = k \,\Big|\, \alpha(\Delta)\Big) = \frac{\alpha(\Delta)^k}{k!} e^{-\alpha(\Delta)}$,

A2) $\mathcal{F}^H_{\Delta_1}$ and $\mathcal{F}^H_{\Delta_2}$ are conditionally independent given $\mathcal{F}^\alpha$ whenever Δ_1 and Δ_2 are disjoint sets.

In particular, the conditional independence 2.1 implies that

$$\mathbb{E}\Big[f\big(H(\Delta_1)\big) \,\Big|\, \mathcal{F}^H_{\Delta_2} \vee \mathcal{F}^\alpha\Big] = \mathbb{E}\Big[f\big(H(\Delta_1)\big) \,\Big|\, \mathcal{F}^\alpha\Big],$$

whenever $\Delta_1, \Delta_2 \in \mathcal{B}_X$, $\Delta_1 \cap \Delta_2 = \emptyset$ and for $f : \mathbb{R} \to \mathbb{R}$ such that the conditional expectation is well defined. From 2.1 we have

$$\mathbb{E}\Big[f\big(H(\Delta)\big) \,\Big|\mathcal{F}^\alpha\Big] = \sum_{k=0}^{\infty} f(k) \frac{\alpha(\Delta)^k}{k!} e^{-\alpha(\Delta)}, \quad \Delta \in \mathcal{B}_X, \tag{2.3}$$

and in particular

$$\mathbb{E}\Big[H(\Delta) \,\Big|\mathcal{F}^\alpha\Big] = \alpha(\Delta), \quad \Delta \in \mathcal{B}_X. \tag{2.4}$$

From the above formulae the following ones are obtained [19, Lemma 3a p. 23]:

$$\begin{aligned}
\mathbb{E}\Big[H(\Delta)\Big] &= \mathbb{E}\big[\alpha(\Delta)\big] = V(\Delta) \\
\mathrm{Var}\Big(H(\Delta)\Big) &= \mathbb{E}\big[\alpha(\Delta)\big] + \mathrm{Var}\big(\alpha(\Delta)\big) \\
\mathbb{E}\Big[H(\Delta)^2\Big] &= \mathbb{E}\big[\alpha(\Delta)\big] + \mathbb{E}\big[\alpha(\Delta)^2\big]
\end{aligned}$$

for $\Delta \in \mathcal{B}_X$ with $V(\Delta) < \infty$. In the case $V(\Delta) = \infty$, the above relationships hold but clearly $\mathrm{Var}(H(\Delta)) = \mathbb{E}[H(\Delta)^2] = \mathbb{E}[H(\Delta)] = \infty$.

Definition 2.2. The centered doubly stochastic Poisson process (CDSPP) is the signed random measure $\tilde{H} := H - \alpha$, ie

$$\tilde{H}(\Delta) := H(\Delta) - \alpha(\Delta), \quad \Delta \in \mathcal{B}_X.$$

We denote $\mathcal{F}^{\tilde{H}}$ the filtration generated by $\tilde{H}$. For any $\Delta \in \mathcal{B}_X$ with $V(\Delta) < \infty$, the conditional first moment is

$$\mathbb{E}\left[\tilde{H}(\Delta)\middle|\mathcal{F}^\alpha\right] = 0.$$

and the conditional second moment is

$$\mathbb{E}\left[\tilde{H}(\Delta)^2\middle|\mathcal{F}^\alpha\right] = \alpha(\Delta).$$

and thus

$$\mathbb{E}\big[\tilde{H}(\Delta)^2\big] = \mathrm{Var}\big(\tilde{H}(\Delta)\big) = \mathbb{E}\big[\alpha(\Delta)\big] = V(\Delta). \tag{2.5}$$

For the remaining conditional moments, the following recurrence formula holds:

Proposition 2.3.

$$\mathbb{E}\big[\tilde{H}(\Delta)^3\big|\mathcal{F}^\alpha\big] = \alpha(\Delta)$$

$$\mathbb{E}\big[\tilde{H}(\Delta)^n\big|\mathcal{F}^\alpha\big] = \alpha(\Delta) + \alpha(\Delta)\sum_{k=2}^{n-2}\binom{n}{k}\mathbb{E}\left[\tilde{H}(\Delta)^k\middle|\mathcal{F}^\alpha\right], \quad n \geq 4. \tag{2.6}$$

Proof. The formulae are obtained by induction for the Poisson distribution in [26, Section 3]. Those computations can easily be adapted to our case using (2.3). □

Corollary 2.4. *For $n \geq 4$, we have that*

$$\mathbb{E}\left[\tilde{H}(\Delta)^n\right] < \infty$$

if and only if

$$\mathbb{E}\left[\alpha(\Delta)^{n/2}\right] < \infty \quad \textit{for } n \textit{ even},$$

$$\mathbb{E}\left[\alpha(\Delta)^{(n-1)/2}\right] < \infty \quad \textit{for } n \textit{ odd}.$$

Proof. The result follows from an argument by induction using (2.6). □

Remark 2.5. We remark that, in view of Corollary 2.4, the assumption (2.1) is sufficient to ensure that $\tilde{H}(\Delta)$ has finite moments of all orders for $\Delta \in \mathcal{B}_X^c$.

For the arguments presented in the sequel it is crucial to investigate the relationship between the σ-algebras $\mathcal{F}^{\tilde{H}}$ and $\mathcal{F}^H \vee \mathcal{F}^\alpha$. While it is immediate to see that $\mathcal{F}^{\tilde{H}} \subseteq \mathcal{F}^H \vee \mathcal{F}^\alpha$, the opposite relationship is more delicate. Here after we introduce a dissecting system on X which is instrumental in the study of the considered random measures and associated structures. Recall that $\mathcal{B}_X^c$ is a ring generating the topology on X and that X is a Hausdorff topological space such that $X = \bigcup_{n=1}^\infty X_n$, where X_n, $n = 1, 2, \dots$ is a growing sequence of compacts. Hence $V(X_n) < \infty$. Denote $|\Delta| := \sup_{x,y\in\Delta} \mu(x, y)$, $\Delta \subset X$, where μ is the metric in X. Then $|X_n| < \infty$ for all n.

Being V non-atomic, for every n and $\epsilon_n > 0$, there exists a partition of X_n, i.e., a finite family of pairwise disjoint sets:

$$\Delta_{n,1}, \dots, \Delta_{n,K_n} \in \mathcal{B}_X^c : \quad X_n = \bigcup_{k=1}^{K_n} \Delta_{n,k} \tag{2.7}$$

such that $\sup_{k=1,\dots,K_n} V(\Delta_{n,k}) \le \epsilon_n$ and $\sup_{k=1,\dots,K_n} |\Delta_{n,k}| \le \epsilon_n$.

Let us consider a decreasing sequence $\epsilon_n \searrow 0$, $n \to \infty$. Then, based on (2.7), we give the following definition.

Definition 2.6. A *dissecting system* of X is the sequence of partitions of X:

$$\Delta_{n,1}, \dots, \Delta_{n,K_n+1}, \quad n = 1, 2, \dots \tag{2.8}$$

with $\bigcup_{k=1}^{K_n} \Delta_{n,k} = X_n$ from (2.7) and $\Delta_{n,K_n+1} := X \setminus X_n$, satisfying the *nesting property*:

$$\Delta_{n,k} \cap \Delta_{n+1,j} = \Delta_{n+1,j} \text{ or } \emptyset \tag{2.9}$$

for all $k = 1, \dots, K_n + 1$ and $j = 1, \dots, K_{n+1} + 1$.

We remark that, from (2.7) and (2.9), we have

$$\sup_{k=1,\dots,K_n} V(\Delta_{n,k}) \le \epsilon_n \to 0, \text{ and } \sup_{k=1,\dots,K_n} |\Delta_{n,k}| \le \epsilon_n \to 0 \;\; n \to \infty. \tag{2.10}$$

We can refer to, e.g., [21] and [9] for more on dissecting systems and partitions.

Lemma 2.7. *For any $\Delta \in \mathcal{B}_X$ such that $\alpha(\Delta) < \infty$ $\mathbb{P}$-a.s. we have that*

$$\sup_{k=1,\dots,K_n+1} \alpha(\Delta \cap \Delta_{n,k}) \longrightarrow 0, \; n \to \infty \; \mathbb{P}\text{-}a.s.$$

Proof. The sets

$$\tilde{\Delta}_{n,k} := \Delta \cap \Delta_{n,k}, \quad k = 1, \dots K_n, \; n = 1, 2, \dots$$

constitute a dissecting system of Δ. Note that $\alpha(\Delta_{k,n}) < \infty$ $\mathbb{P}$-a.s. for all k and n. Let $\tilde{\Omega}$ be the event where α is non-atomic and $\alpha(\Delta) < \infty$. Then $\mathbb{P}(\tilde{\Omega}) = 1$. From (2.8) we have

$$\alpha(\tilde{\Delta}_{n+1,j}, \omega) \le \sup_{k=1,\dots,K_n+1} \alpha(\tilde{\Delta}_{n,k}, \omega) \le \alpha(\Delta, \omega), \quad \omega \in \tilde{\Omega}$$

for all $j = 1, \dots, K_n + 1$. Hence, for every $n = 1, 2, \dots$, we have

$$\sup_{j=1,\dots,K_{n+1}+1} \alpha(\tilde{\Delta}_{n+1,j}, \omega) \le \sup_{k=1,\dots,K_{n+1}+1} \alpha(\tilde{\Delta}_{n,j}, \omega), \quad \omega \in \tilde{\Omega}.$$

We denote $A(\omega) := \lim_{n\to\infty} \sup_{k=1,\dots,K_{n+1}+1} \alpha(\tilde{\Delta}_{n,k}, \omega)$, $\omega \in \tilde{\Omega}$. Naturally $A(\omega) \ge 0$, but we need to prove $A(\omega) = 0$. We proceed by contradiction. Set $\tilde{\Omega}_0 := \{\omega \in \tilde{\Omega} | A(\omega) > 0\}$ and suppose $\mathbb{P}(\tilde{\Omega}_0) > 0$. For each n there exists a set $\tilde{\Delta}_{n,\delta(n)}$ such that $\alpha(\tilde{\Delta}_{n,\delta(n)}, \omega) \ge A(\omega) > 0$, $\omega \in \tilde{\Omega}_0$. Comparing $\tilde{\Delta}_{n,\delta(n)}$ with the sets $\tilde{\Delta}_{n-1,j}$, $j = 1, \dots, K_{n-1}+1$, we see that there is a set $\tilde{\Delta}_{n-1,\delta(n-1)}$ such that $\tilde{\Delta}_{n-1,\delta(n-1)} \supseteq \tilde{\Delta}_{n,\delta(n)}$.

Hence there exists a decreasing sequence of sets

$$\tilde{\Delta}_{n,\delta(n)}, \quad n = 1, 2, \ldots$$

such that for every n, $\tilde{\Delta}_{n,\delta(n)}$ is an element of the dissecting system of Δ and $0 < A(\omega) \leq \alpha(\tilde{\Delta}_{n,\delta(n)}, \omega)$ $(\omega \in \tilde{\Omega}_0)$. On the other side, from the property (2.10) of the dissecting system on X, and hence on Δ, the limit of a decreasing sequence of sets is either empty or a singleton. Thus we have

$$\lim_{n\to\infty} \alpha(\tilde{\Delta}_{n,\delta(n)}, \omega) = 0, \quad \omega \in \tilde{\Omega}_0,$$

since α is a non-atomic measure for $\omega \in \tilde{\Omega}_0$. This is a contradiction, and hence $A(\omega) = 0$ for all $\omega \in \tilde{\Omega}_0$. □

Theorem 2.8. *The following equality holds:*

$$\mathcal{F}^{\tilde{H}} = \mathcal{F}^H \vee \mathcal{F}^\alpha.$$

Proof. It is sufficient to show that $H(\Delta)$ and $\alpha(\Delta)$ are $\mathcal{F}^{\tilde{H}}$-measurable for any $\Delta \in \mathcal{B}_X^c$. Let $\Delta \in \mathcal{B}_X^c$ and recall its representation

$$\Delta = \bigcup_{k}^{K_n+1} \tilde{\Delta}_{n,k} = \bigcup_{k}^{K_n+1} \Big(\Delta \cap \Delta_{n,k}\Big), \quad n = 1, 2, \ldots,$$

as a pairwise disjoint union of sets obtained from the dissecting system (2.8) of X. Consider

$$g_n(\Delta) := \sum_{k=1}^{K_n+1} \mathrm{ceil}\Big(\tilde{H}(\tilde{\Delta}_{n,k})\Big) = \sum_{k=1}^{K_n+1} \mathrm{ceil}\Big(H(\tilde{\Delta}_{n,k}) - \alpha(\tilde{\Delta}_{n,k})\Big),$$

where $\mathrm{ceil}(y)$ is the smallest integer greater than y. The random variables $g_n(\Delta)$, $n = 1, \ldots,$ are clearly $\mathcal{F}^{\tilde{H}}$-measurable. From Lemma 2.7 there exists for $\mathbb{P}$-a.a. ω, a $N(\omega) \in \mathbb{N}$ such that $\sup_{k=1,\ldots,K_n+1} \alpha(\tilde{\Delta}_{n,j}, \omega) < 1$ for $n > N(\omega)$. Then we have

$$\lim_{n\to\infty} \mathrm{ceil}\Big(H(\tilde{\Delta}_{n,k}) - \alpha(\tilde{\Delta}_{n,k})\Big) = H(\tilde{\Delta}_{n,k}) \quad \mathbb{P}\text{-a.s.}$$

Thus

$$\lim_{n\to\infty} g_n(\Delta) = \lim_{n\to\infty} \sum_{k=1}^{K_n+1} \mathrm{ceil}\Big(\tilde{H}(\tilde{\Delta}_{n,k})\Big) = H(\Delta) \quad \mathbb{P}\text{-a.s.}$$

and $H(\Delta)$ is a pointwise limit of $\mathcal{F}^{\tilde{H}}$-measurable functions. Since $\alpha(\Delta) = H(\Delta) - \tilde{H}(\Delta)$, we also have that $\alpha(\Delta)$ is $\mathcal{F}^{\tilde{H}}$-measurable. □

Note that the initial assumption that α is $\mathbb{P}$-a.s. non-atomic is crucial for this result. On the other side we remark that the assumption (2.2) is here not required.

Theorem 2.8 can be regarded as an extension of a result proved for a time-changed Lévy processes with independent time-change in [27].

3. Multilinear forms, polynomials, and chaos expansions

In this section we construct a system of multilinear forms and show how they describe the intrinsic orthogonal structures in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Here and in the sequel we set $\mathcal{F} = \mathcal{F}^{\tilde{H}} = \mathcal{F}^H \vee \mathcal{F}^\alpha$, see Theorem 2.8.

Definition 3.1. For any group of pairwise disjoint sets $\Delta_1, \dots, \Delta_p \in \mathcal{B}_X^c$, an α-*multilinear form of order* p is a random variable of type

$$\beta \prod_{j=1}^{p} \tilde{H}(\Delta_j), \quad p \geq 1,$$

where β is an $\mathcal{F}^\alpha$-measurable random variables with finite moments of all orders. The 0-order α-multilinear forms are the $\mathcal{F}^\alpha$-measurable random variable with finite moments of all orders.

This definition is a generalization of the one given in [14, page 7]: A *p-order multilinear form* of the values $\tilde{H}(\Delta_j)$, $j = 1, \dots, p$, is a random variable of type

$$\prod_{j=1}^{p} \tilde{H}(\Delta_j), \quad p \geq 1. \tag{3.1}$$

The 0-order multilinear forms are the constants.

Note that any α-multilinear form is an element of $L_2(\Omega, \mathcal{F}, \mathbb{P})$. In fact, by assumption (2.2), the following holds:

$$\mathbb{E}\Big[\xi^2\Big] = \mathbb{E}\Big[\beta^2 \prod_{j=1}^{p} \mathbb{E}\big[\tilde{H}(\Delta_j)^2 \big| \mathcal{F}^\alpha\big]\Big] = \mathbb{E}\Big[\beta^2 \prod_{j=1}^{p} \alpha(\Delta_j)\Big] < \infty. \tag{3.2}$$

In the sequel we will consider multilinear forms on the sets (2.7)–(2.8) of the dissecting system of X.

The present section completes and extends to the CDSPP the results presented in [14] in which measure based multilinear forms were introduced for the study of stochastic calculus for Lévy stochastic measures. In that case the structure of independence of the random measure values was heavily exploited. In particular we stress that the space $\mathbb{H}^p$ via (3.3) here below is a substantial element of novelty and it is crucial for the forthcoming analysis.

Definition 3.2. For $p \geq 1$ we write $\mathbb{H}^p$ for the subspace in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the finite linear combinations of p-order α-multilinear form:

$$\sum_i \beta_i \prod_{j=1}^{p} \tilde{H}(\Delta_j^i). \tag{3.3}$$

Here above the sets Δ_j^i, $j = 1, \dots, p$, are pairwise disjoint and belong to the dissecting system (2.7)–(2.8) on X. The subspace $\mathbb{H}^0$ is the $\mathcal{F}^\alpha$-measurable random variables with finite variance.

Remark 3.3. We may consider the multipliers β in Definition 3.2 to be finite products of the form $\prod_{i=1}^n \alpha(\Delta_i)$ with Δ_i, $i = 1, \dots, n$ pairwise disjoint sets from the dissecting system (2.7)–(2.8).

Remark 3.4. Let $p \geq 1$. By definition, for any $\xi \in \mathbb{H}^p$ there exists a sequence $\{\xi_m\}_m$ such that $\xi_m \to \xi$, $m \to \infty$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$, with $\xi_m = \sum_{l=1}^{L_m} \xi_{ml}$ and ξ_{ml} p-order α-multilinear forms. Note that we can always choose approximating sequences where the ξ_{ml}, $l = 1, \dots, L_m$ are orthogonal.

Lemma 3.5. *For $p' \neq p''$, the subspaces $\mathbb{H}^{p'}$ and $\mathbb{H}^{p''}$ are orthogonal in $L_2(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof. We assume that $p'' > p'$. It is sufficient to prove the statement for $\xi' \in \mathbb{H}^{p'}$, $\xi'' \in \mathbb{H}^{p''}$ of type

$$\xi' = \beta' \prod_{i=1}^{p'} \tilde{H}(\Delta^i), \quad \xi'' = \beta'' \prod_{j=1}^{p''} \tilde{H}(\Delta^j)$$

where Δ^i, $i = 1, \dots, p'$ and Δ^j, $j = 1 \dots p'$ are two groups of pairwise disjoint sets (2.8) of the dissecting system of X. Note that, in view of the nesting property (2.9), there exists $n \in \mathbb{N}$ such that all the sets above can be represented in terms of finite disjoint unions of elements from the same nth partition (2.8)–(2.9). Thus we can represent ξ' and ξ'' by finite sums of p'-order and p''-order α-multilinear forms respectively over sets (2.8) in the same nth partition (2.9):

$$\xi' = \beta' \sum_k \prod_{i=1}^{p'} \tilde{H}(\Delta^i_{n,k})$$

$$\xi'' = \beta'' \sum_l \prod_{j=1}^{p''} \tilde{H}(\Delta^j_{n,l}).$$

To prove the statement it is then enough to verify that for all k, l,

$$\mathbb{E}\Big[\beta' \prod_{i=1}^{p'} \tilde{H}(\Delta^i_{n,k}) \beta'' \prod_{j=1}^{p''} \tilde{H}(\Delta^j_{n,l})\Big] = 0.$$

We remark that being $p'' > p'$, there is at least one set among $\Delta^j_{n,l}$, $j = 1, \dots, p''$ that is different from $\Delta^i_{n,k}$, $i = 1, \dots, p'$. Denote such a set by $\Delta^{\hat{j}}_{n,l}$. We have

$$\begin{aligned}
&\mathbb{E}\Big[\beta' \prod_{i=1}^{p'} \tilde{H}(\Delta^i_{n,k}) \beta'' \tilde{H}(\Delta^{\hat{j}}_{n,l}) \prod_{\substack{j=1 \\ j \neq \hat{j}}}^{p''} \tilde{H}(\Delta^j_{n,l})\Big] \\
&= \mathbb{E}\Big[\beta'\beta'' \mathbb{E}\Big[\prod_{i=1}^{p'} \tilde{H}(\Delta^i_{n,k}) \prod_{\substack{j=1 \\ j \neq \hat{j}}}^{p''} \tilde{H}(\Delta^j_{n,l}) \Big| \mathcal{F}^\alpha\Big] \mathbb{E}\Big[\tilde{H}(\Delta^{\hat{j}}_{n,l}) \Big| \mathcal{F}^\alpha\Big]\Big] = 0.
\end{aligned}$$

By this we end the proof. □

Definition 3.6. We write $\mathbb{H}_p$ for the subspaces of $L_2(\Omega, \mathcal{F}, \mathbb{P})$ defined by:

$$\mathbb{H}_p := \sum_{q=0}^{p} \oplus \mathbb{H}^q.$$

Namely the subspaces generated by the linear combinations of α-multilinear forms:

$$\sum_i \beta_i \prod_{j=1}^{p_i} \tilde{H}(\Delta_j^i), \quad p_i \le p.$$

We set

$$\mathbb{H} := \sum_{q=0}^{\infty} \oplus \mathbb{H}^q.$$

Lemma 3.7. *Let $\Delta', \Delta'' \in \mathcal{B}_X : \Delta' \cap \Delta'' = \emptyset$. Consider $\mathcal{F}_{\Delta'}$ and $\mathcal{F}_{\Delta''}$ as the σ-algebras generated by $\tilde{H}(\Delta)$, $\Delta \in \mathcal{B}_X : \Delta \subset \Delta'$ and $\Delta \subset \Delta''$, respectively. Let $\xi' \in \mathbb{H}_{p'}$ be $\mathcal{F}_{\Delta'}$-measurable and $\xi'' \in \mathbb{H}_{p''}$ be $\mathcal{F}_{\Delta''}$-measurable. The product $\xi'\xi''$ is measurable with respect to $\mathcal{F}_{\Delta' \cup \Delta''}$ and belongs to $\mathbb{H}_{p'+p''}$.*

Proof. If ξ' and ξ'' are of type (3.3), then clearly the product $\xi'\xi'' \in \mathbb{H}_{p'+p''}$ and it is $\mathcal{F}_{\Delta' \cup \Delta''}$-measurable. In the general case, ξ' and ξ'' are approximated in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ by sequences of elements ξ'_n and ξ''_n, $n = 1, 2, \dots$ of type (3.3):

$$\xi' = \lim_{n\to\infty} \xi'_n = \lim_{n\to\infty} \sum_i \beta'_{ni} \prod_{j=1}^{p'} \tilde{H}(\Delta^j_{n,i})$$

$$\xi'' = \lim_{n\to\infty} \xi''_n = \lim_{n\to\infty} \sum_k \beta'_{nk} \prod_{l=1}^{p''} \tilde{H}(\Delta^l_{n,k}).$$

Note that in view of the measurability assumptions we have $\Delta^j_{n,i} \subset \Delta'$, $j = 1, \dots, p'$ and $\Delta^j_{n,i} \subset \Delta''$, $j = 1, \dots, p''$ and also β'_{ni} are $\mathcal{F}^\alpha_{\Delta'}$-measurable, while β''_{nk} are $\mathcal{F}^\alpha_{\Delta''}$-measurable. Then it is easy to see that the statement holds. □

We remark that the result still holds true if we consider the σ-algebras $\mathcal{F}^H_{\Delta'} \vee \mathcal{F}^\alpha$ and $\mathcal{F}^H_{\Delta''} \vee \mathcal{F}^\alpha$, for $\Delta' \cap \Delta'' = \emptyset$.

The *polynomials* of the values of $\tilde{H}$ of degree p are the random variables ξ admitting representation as

$$\xi = \sum_{m=1}^{M} c_m \prod_{j=1}^{J_m} \tilde{H}(\Delta_j^m)^{p_j}, \quad c_m \in \mathbb{R},\ M, J_m, p_j \in \mathbb{N} \tag{3.4}$$

such that $\sum_{j=1}^{J_m} p_j \le p$, $m = 1, 2 \dots, M$ and $\Delta^m_j \in \mathcal{B}^c_X$, $j = 1, \dots, J_m$ are pairwise disjoint.

Theorem 3.8. *All the polynomials of values of $\tilde{H}$ of degree less or equal to p belong to the subspace $\mathbb{H}_p$.*

Proof. Let ξ be a polynomial of degree p as in (3.4). We proceed by induction. If $p = 0$ then $\xi \in \mathbb{H}_0$ and if $p = 1$ then $\xi \in \mathbb{H}_1$. Suppose the statement holds for $q < p$, we verify this for p. For each m, let us consider elements

$$\xi_m := \prod_{j=1}^{J_m} \tilde{H}(\Delta_j^m)^{p_j}, \quad \sum_{j=1}^{J_m} p_j \leq p.$$

If $p_j < p$ for $j = 1, \dots, J_m$

1. and $\sum_{j=1}^{J_m} p_j < p$ then the induction hypothesis holds and $\xi_m \in \mathbb{H}_p$,
2. and $\sum_{j=1}^{J_m} p_j = p$ with $J_m > 1$, then for any j we have $\tilde{H}(\Delta_j^m)^{p_j} \in \mathbb{H}_{p_j}$ by the induction hypothesis. Furthermore, being the sets disjoint, from Lemma 3.7 we have that $\prod_{j=1}^{m} \tilde{H}(\Delta_j^{J_m}) \in \mathbb{H}_{\sum p_j}$.

Hence we only have to verify the case $J_m = 1$. Namely

$$\xi = \tilde{H}(\Delta_j^m)^p \in \mathbb{H}_p, \quad \text{for } p > 1.$$

Set $\Delta := \Delta_j^m$. For all n, we can represent Δ in terms of the sets (2.8) of the dissecting system of X,

$$\Delta = \bigcup_{k=1}^{K_n+1} \left(\Delta \cap \Delta_{n,k}\right) := \bigcup_{k=1}^{K_n+1} \tilde{\Delta}_{n,k},$$

hence we have that

$$\xi = \tilde{H}(\Delta)^p = \left(\sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k}) \right)^p.$$

Let

$$\xi_n^{(1)} := \xi - \sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k})^p = \left(\sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k}) \right)^p - \sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k})^p$$

and

$$\xi_n^{(2)} := \sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k})^p.$$

For all n we have $\xi = \xi_n^{(1)} + \xi_n^{(2)}$ and thus $\xi = \lim_{n\to\infty} \xi_n^{(1)} + \xi_n^{(2)}$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Let us investigate $\xi_n^{(1)}$ and $\xi_n^{(2)}$ separately. First of all we note that $\xi_n^{(1)}$ is a polynomial, as in (3.4), with $p_j < p$ for all $j = 1, \dots, K_n + 1$. Thus $\xi_n^{(1)} \in \mathbb{H}_p$. Hence we have $\xi^{(1)} := \lim_{n\to\infty} \xi_n^{(1)} \in \mathbb{H}_p$ since $\mathbb{H}_p$ is closed in $L_2(\Omega, \mathcal{F}, \mathbb{P})$.

Consider the following limit in $L_2(\Omega, \mathcal{F}, \mathbb{P})$:

$$\begin{aligned}\xi^{(2)} := \lim_{n\to\infty} \xi_n^{(2)} &= \lim_{n\to\infty} \sum_{k=1}^{K_n+1} \Big(H(\tilde\Delta_{n,k}) - \alpha(\tilde\Delta_{n,k})\Big)^p \\ &= \lim_{n\to\infty} \sum_{k=1}^{K_n+1} \sum_{j=0}^{p} \binom{p}{j} H(\tilde\Delta_{n,k})^{p-j}(-1)^j \alpha(\tilde\Delta_{n,k})^j \\ &= \sum_{j=0}^{p} \lim_{n\to\infty} \sum_{k=1}^{K_n+1} \binom{p}{j} H(\tilde\Delta_{n,k})^{p-j}(-1)^j \alpha(\tilde\Delta_{n,k})^j. \end{aligned} \tag{3.5}$$

Since $\mathbb{P}\big(H(\{x\}) \neq 0, 1 \text{ for some } x \in X\big) = 0$ see [19, Theorem 1.3 page 19], we will ultimately have $H(\tilde\Delta_{n,k}) = 0$ or 1 $\mathbb{P}$-a.s. as $n \to \infty$. Thus for the first term ($j = 0$) in (3.5), using dominated convergence, we have:

$$\lim_{n\to\infty} \sum_{k=1}^{K_n+1} \big(H(\tilde\Delta_{n,k})\big)^p = \sum_{k=1}^{H(\Delta)} 1^p = H(\Delta) = \tilde H(\Delta) + \alpha(\Delta).$$

For the remaining terms ($j > 0$) in (3.5) the following estimate applies

$$\begin{aligned}\Big| \sum_{k=1}^{K_n+1} (-1)^j H(\tilde\Delta_{n,k})^p \alpha(\tilde\Delta_{n,k})^{p-j} \Big| &\leq \sum_{k=1}^{K_n+1} \mathbf{1}_{\{H(\tilde\Delta_{n,k})>0\}} H(\tilde\Delta_{n,k})^p \alpha(\tilde\Delta_{n,k})^{p-j} \\ &\leq \sup_k \alpha(\tilde\Delta_{n,k})^{p-j} \sum_{k=1}^{K_n} \mathbf{1}_{\{H(\tilde\Delta_{n,k})>0\}} H(\tilde\Delta_{n,k})^p \\ &\leq H(\tilde\Delta)^{p+1} \sup_k \alpha(\tilde\Delta_{n,k})^{p-j} \longrightarrow 0, \ n \to \infty, \end{aligned}$$

by Lemma 2.7. Thus $\xi^{(2)} = \tilde H(\Delta) + \alpha(\Delta) \in \mathbb{H}_1 \subseteq \mathbb{H}_p$. Hence $\xi = \xi^{(1)} + \xi^{(2)} \in \mathbb{H}_p$. □

The following statement is a direct consequence of the theorem above.

Corollary 3.9. All the polynomials of all degrees *of the values of* $\tilde H$ *belong to* $\mathbb{H}$.

Remark 3.10. We note that if the sets in (3.4) were not disjoint, then one could always represent the same polynomials via disjoint sets by applying the additivity of the measure $\tilde H$, but the degree would naturally change.

By assumption (2.1),

$$\mathbb{E}\big[e^{\sum_{j=1}^J c_j \tilde H(\Delta_j)}\big] \leq \prod_{j=1}^J \mathbb{E}\big[e^{2^J c_j \tilde H(\Delta_j)}\big] = \prod_{j=1}^J \mathbb{E}\big[e^{(e^{2^J c_j} - 1 - 2^J c_j)\alpha(\Delta_j)}\big] < \infty.$$

Following classical arguments via Fourier transforms (see, e.g., [25, Lemma 4.3.1 and Lemma 4.3.2]) one can see that the random variables

$$\exp\Big\{\sum\nolimits_{j=1}^J x_j \tilde H(\Delta_j)\Big\}, \quad j = 1, 2 \ldots, J;\ x = (x_1, \ldots, x_J) \in \mathbb{R}^J,$$

with Δ_j, $j = 1, \dots, J$ pairwise disjoint sets in $\mathcal{B}_X^c$, constitute a complete system in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. By Taylor approximations of the analytic extension on $\mathbb{C}^J$ we have that

$$\mathbb{E}\left[\left|\exp\left\{\sum\nolimits_{j=1}^{J} x_j \tilde{H}(\Delta_j)\right\} - \sum\nolimits_{p=0}^{q} \frac{\sum_{j=1}^{J} i x_j \tilde{H}(\Delta_j)^p}{p!}\right|^2\right] \longrightarrow 0,\ q \to \infty$$

(see, e.g., [3, Eq. (26.4)] for an estimate of the quantity here above justifying the convergence.) Hence we can conclude:

Lemma 3.11. *The polynomials of the values of $\tilde{H}(\Delta)$, $\Delta \in \mathcal{B}_X^c$ are dense in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.*

Theorem 3.12 (Chaos expansion). *The following equality holds:*

$$\mathbb{H} = L_2(\Omega, \mathcal{F}, \mathbb{P}).$$

Namely, any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ can be written as

$$\xi = \sum_{p=0}^{\infty} \xi_p, \quad \textit{where } \xi_p \in \mathbb{H}^p \textit{ for } p = 1, 2, \dots$$

Proof. The polynomials of the values of $\tilde{H}(\Delta)$ are dense in $L_2(\Omega, \mathcal{F}, \mathbb{P})$, see Lemma 3.11. By Theorem 3.8 and Corollary 3.9 all the polynomials are in $\mathbb{H}$. Since $\mathbb{H}$ is closed, we must have $L_2(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathbb{H}$. On the other side we recall that by construction $\mathbb{H} \subseteq L_2(\Omega, \mathcal{F}, \mathbb{P})$, see (3.2) and Definitions 3.2 and 3.6. □

Remark 3.13. Definitions 3.2 and 3.6 describe the spaces generated by α-multilinear forms. We can also consider analogous spaces generated only by the multilinear forms as in (3.1). However we have to stress that in this case the multilinear forms are not dense in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ with the only exception made when H is a Poisson random measure, i.e., if α is deterministic. Indeed write $\widetilde{\mathbb{H}}^p$ for the subspace in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the finite linear combinations of p-order multilinear forms:

$$\sum_i c_i \prod_{j=1}^{p} \tilde{H}(\Delta_j^i).$$

The sets Δ_j^i, $j = 1, \dots, p$, are pairwise disjoint and the c_i are constants. Set $\widetilde{\mathbb{H}}^0 = \mathbb{R}$ and $\widetilde{\mathbb{H}} := \sum_{p=0}^{\infty} \oplus \widetilde{\mathbb{H}}_p$. It is easily seen that $(\beta - \mathbb{E}[\beta]) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ is orthogonal to $\widetilde{\mathbb{H}}$ whenever β is $\mathcal{F}^\alpha$-measurable. There are also α-multilinear forms of higher orders that are orthogonal to $\widetilde{\mathbb{H}}$, one example is $(\mathbb{E}[\beta|\mathcal{F}_\Delta^\alpha] - \mathbb{E}[\beta])\tilde{H}(\Delta)$ $(\Delta \in \mathcal{B}_X^c)$. Thus, in general, $\widetilde{\mathbb{H}} \neq L_2(\Omega, \mathcal{F}, \mathbb{P})$. The case when H is a Poisson random measure was studied in [14] as a particular Lévy random field, and there we do have $\widetilde{\mathbb{H}} = L_2(\Omega, \mathcal{F}, \mathbb{P})$.

4. Non-anticipating stochastic integration

In the sequel we consider $X = [0,T]\times Z$ for $T < \infty$ and Z a locally compact, second countable Hausdorff topological space. Being interested in integration, without loss of generality we assume that

$$\alpha\big(\{0\} \times Z\big) = 0 \quad \mathbb{P}\text{-a.s.}$$

Hence we can restrict the attention to $X = (0,T] \times Z$.

We chose a dissecting system of X to be given by partitions (2.7)–(2.8) of the form

$$\Delta_{n,k} = (s_{n,k}, u_{n,k}] \times B_{n,k}, \quad s_{n,k} < u_{n,k},\ B_{n,k} \in \mathcal{B}_Z^c, \tag{4.1}$$

for $n = 1, 2, \dots$ and $k = 1, 2, \dots, K_n$, see Definition 2.6. Here $\mathcal{B}_Z$ denotes the Borel σ-algebra on Z and $\mathcal{B}_Z^c$ the family of precompacts for the topology in Z. The set X is ordered with the natural ordering given by time in $[0,T]$. Two filtrations naturally appear in the present setting:

- $\mathbb{F} := \big\{\mathcal{F}_t, t \in [0,T]\big\}$ where $\mathcal{F}_t$ is generated by $\{\tilde{H}(\Delta) : \Delta \in \mathcal{B}_{[0,t]\times Z}\}$,
- $\mathbb{G} := \{\mathcal{G}_t, t \in [0,T]\}$ with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}^\alpha$.

Clearly we have that $\mathcal{F}_t \subseteq \mathcal{G}_t$, $\mathcal{F}_0$ is trivial but $\mathcal{G}_0 = \mathcal{F}^\alpha$ and $\mathcal{F}_T = \mathcal{G}_T = \mathcal{F}$.

We define a martingale random field as in [16], see in particular [16, Remark 2.3] for historical notes. We can also refer to the work of [28] and [6] as pioneering in the use of martingale random fields in stochastic calculus, though mostly related to Brownian sheet.

Hence we can see that the stochastic set function $\tilde{H}(\Delta)$, $\Delta \in \mathcal{B}_X$ is a martingale random field (with square integrable values) with respect to $\mathbb{F}$ and $\mathbb{G}$ as it satisfies the following properties:

1. $\tilde{H}$ has a σ-finite variance measure $V(\Delta) = E\big[\tilde{H}(\Delta)^2\big]$, $\Delta \in \mathcal{B}_X$, recall (2.5).
2. $\tilde{H}$ is additive, i.e., for pairwise disjoint sets $\Delta_1, \dots, \Delta_K$: $V(\Delta_k) < \infty$

$$\tilde{H}\Big(\bigcup_{k=1}^{K} \Delta_k\Big) = \sum_{k=1}^{K} \tilde{H}(\Delta_k)$$

   and σ-additive in L_2, i.e., for pairwise disjoint sets $\Delta_1, \Delta_2, \dots$: $V(\Delta_k) < \infty$

$$\tilde{H}\Big(\bigcup_{k=1}^{\infty} \Delta_k\Big) = \sum_{k=1}^{\infty} \tilde{H}(\Delta_k)$$

   with convergence in $L_2(\Omega, \mathcal{F}, \mathbb{P})$.
3. $\tilde{H}$ is adapted to $\mathbb{F}$ and $\mathbb{G}$.
4. $\tilde{H}$ has the *martingale property*. Consider $\Delta \subseteq (t,T] \times Z$. Then, from (2.4) we have:

$$\mathbb{E}\Big[\tilde{H}(\Delta)\,\Big|\,\mathcal{F}_t\Big] = \mathbb{E}\Big[\mathbb{E}\big[\tilde{H}(\Delta)\,|\,\mathcal{G}_t\big]\,\Big|\,\mathcal{F}_t\Big] = \mathbb{E}\Big[\mathbb{E}\big[\tilde{H}(\Delta)\,|\,\mathcal{F}^\alpha\big]\,\Big|\,\mathcal{F}_t\Big] = 0.$$

5. $\tilde{H}$ has conditionally orthogonal values. For any $\Delta_1, \Delta_2 \subseteq (t,T] \times Z$ such that $\Delta_1 \cap \Delta_2 = \emptyset$ and. Then, from 2.1, we have:
$$\begin{aligned}\mathbb{E}\Big[\tilde{H}(\Delta_1)\tilde{H}(\Delta_2)\,\Big|\,\mathcal{F}_t\Big] &= \mathbb{E}\Big[\mathbb{E}\big[\tilde{H}(\Delta_1)\tilde{H}(\Delta_2)\,\big|\,\mathcal{G}_t\big]\,\Big|\,\mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\big[\tilde{H}(\Delta_1)\big|\mathcal{F}^\alpha\big]\mathbb{E}\big[\tilde{H}(\Delta_2)\big|\mathcal{F}^\alpha\big]\Big|\mathcal{F}_t\Big] = 0.\end{aligned}$$

Given the martingale structure of the CDSPP $\tilde{H}$ with respect to the filtrations $\mathbb{G}$ and $\mathbb{F}$, we can construct a stochastic integration of Itô type according to the classical scheme, as retraced in [16]. Hereafter we consider $\mathbb{G}$ as the reference information flow. Recall that α is a positive random measure.

We define the $\mathbb{G}$-predictable σ-algebra $\mathcal{P}_\mathbb{G}$ as the σ-algebra generated by $\{F \times (s,u] \times B : \;\; F \in \mathcal{G}_s, s < u, B \in \mathcal{B}_Z\}$ and, as usual, we will say that a stochastic process ϕ is $\mathbb{G}$-*predictable* if the mapping $\phi = \phi(\omega,t,z)$, $\omega \in \Omega$, $(t,z) \in X$, is $\mathcal{P}_\mathbb{G}$-measurable. Hence we define
$$\|\phi\|_\Phi := \left(\mathbb{E}\left[\int_0^T \int_Z \phi^2(t,z)\,\alpha(dt,dz)\right]\right)^{1/2}$$
and set Φ to be the L_2-subspace of stochastic processes ϕ admitting a $\mathbb{G}$-predictable modification and such that $\|\phi\|_\Phi < \infty$.

Lemma 4.1. *$\mathcal{F}^\alpha \times \mathcal{B}_X \subset \mathcal{P}_\mathcal{G}$, and α is the $\mathbb{G}$-predictable compensator of H.*

We take the $\mathbb{G}$-predictable compensator to be as in [22], a predictable, locally integrable random measure such that $\mathbb{E}[H(\Delta)] = \mathbb{E}[\alpha(\Delta)]$.

Proof. For the first claim it is sufficient to show that $A \times (a,b] \times B$ with $A \in \mathcal{F}^\alpha$, $a < b$ and $B \in \mathcal{B}_Z^c$ is an element of $\mathcal{P}_\mathbb{G}$. Recall that $A \in \mathcal{G}_s$ for all s and the claim follows. Since $E[H(\Delta)] = \mathbb{E}[\alpha(\Delta)]$ for all $\Delta \in \mathcal{B}_X$, and α is $\mathbb{G}$-predictable, it is the $\mathbb{G}$-predictable compensator of H. □

The *non-anticipating stochastic integral* with respect to $\tilde{H}$ under $\mathbb{G}$ is the isometric operator I mapping:
$$I : \mathrm{dom}\, I \Longrightarrow L_2(\Omega, \mathcal{F}, \mathbb{P})$$
such that
$$I(\phi) := \int\limits_0^T \int\limits_Z \phi(t,z)\,\tilde{H}(dt,dz) := \sum_{k=1}^K \phi_k \tilde{H}\big(\Delta_k\big)$$
for any
$$\phi(t,z) = \sum_{k=1}^K \phi_k \mathbf{1}_{\Delta_k}(t,z) \tag{4.2}$$
with $\Delta_k = (s_k, u_k] \times B_k \in \mathcal{B}_X^c$ and ϕ_k a $\mathcal{G}_{s_k}$-measurable random variable such that $\|\phi\|_\Phi < \infty$.

In fact,

$$\mathbb{E}\big[I(\phi)^2\big] = \mathbb{E}\left[\left(\sum_{k=1}^{K} \phi_k \tilde{H}\big(\Delta_k\big)\right)^2\right] = \mathbb{E}\left[\sum_{k=1}^{K} \phi_k^2 \alpha\big(\Delta_k\big)\right] = \|\phi\|_\Phi^2. \tag{4.3}$$

Naturally the integrands are given by $\operatorname{dom} I \subseteq L_2(\Omega \times X)$, with $L_2(\Omega \times X) := L_2(\Omega\times X, \mathcal{F}\times\mathcal{B}_X, \mathbb{P}\times\alpha)$, which is the linear closure of the stochastic processes (4.2) and the integral is characterized in a standard manner exploiting the isometry (4.3).

Actually $\operatorname{dom} I = \Phi$. In fact the following result holds true.

Lemma 4.2. *For any $\phi \in \Phi$ there exists an approximating sequence of stochastic processes ϕ_n, $n = 1, 2, \ldots,$ of type* (4.2) *having the form:*

$$\phi_n(t, z) = \sum_{k=1}^{K_n} \mathbb{E}\left[\frac{1}{\alpha(\Delta_{n,k})} \int_{\Delta_{n,k}} \phi(\tau, \zeta)\, \alpha(d\tau, d\zeta) \,\Big|\mathcal{G}_{s_{n,k}}\right] \mathbf{1}_{\Delta_{n,k}}(t, z),$$

where $\Delta_k = (s_{n,k}, u_{n,k}] \times B_{n,k}$ are the sets (4.1) *of the dissecting system of $X = (0, T] \times Z$.*

Proof. The arguments of [16, Lemma 3.1] can be easily adapted to the present framework. □

Then, by isometry, it is clear that for any $\phi \in \Phi$, $I(\phi) = \lim_{n\to\infty} I(\phi_n)$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ where $\phi_n \in \Phi$ are processes of type (4.2) approximating ϕ in $L_2(\Omega\times X)$.

From the construction of the stochastic integral, it follows that for any $\phi \in \Phi$, the stochastic set function

$$\mu(\phi, \Delta) := \int_\Delta \phi(t, z)\, \tilde{H}(dt, dz), \quad \Delta \in \mathcal{B}_X, \tag{4.4}$$

is again a martingale random field [16, Remark 3.2] under $\mathbb{G}$ with variance measure

$$m(\phi, \Delta) := \mathbb{E}\left[\int_\Delta \phi^2(t, z)\, \alpha(dt, dz)\right], \quad \Delta \in \mathcal{B}_X.$$

Proposition 4.3. *Consider the $\mathcal{F}^\alpha$-measurable $\beta \in L^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ and $\phi \in \Phi$. Then*

$$\beta I(\phi) = I(\beta\phi)$$

if either side of the equality exists as an element of $L^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$.

Proof. Assume $\phi \in \Phi$ is of type (4.2) and β is bounded. Then, for every k, $\beta\phi_k$ is $\mathcal{G}_{s_k}$-measurable and

$$\beta I(\phi) = \sum_{k=1}^{K} \beta\phi_k \tilde{H}\big((s_k, u_k] \times B_k\big) = I(\beta\phi).$$

The general case follows by taking limits. □

The classical calculus rules hold: for any $\phi \in \Phi$ we have

$$\mathbb{E}\left[\int_\Delta \phi(t,z)\,\tilde{H}(dt,dz)\,|\mathcal{G}_t\right] = 0, \quad \Delta \in \mathcal{B}_{(s,T]\times Z},$$

and

$$\begin{aligned}\mathbb{E}&\left[\int_\Delta \phi_1(t,z)\tilde{H}(dt,dz)\int_\Delta \phi_2(t,z)\,\tilde{H}(dt,dz)\,|\mathcal{G}_s\right]\\ &= \mathbb{E}\left[\int_\Delta \phi_1(t,z)\phi_2(t,z)\,\alpha(dt,dz)\,|\mathcal{G}_s\right]\\ &= \int_\Delta \mathbb{E}\left[\phi_1(t,z)\phi_2(t,z)\ |\mathcal{G}_s\right]\alpha(dt,dz), \quad \Delta \in \mathcal{B}_{(s,T]\times Z},\end{aligned}$$

and in particular we have

$$\mathbb{E}\Big[I(\phi)^2\Big|\mathcal{F}^\alpha\Big] = \int_0^T\int_Z \mathbb{E}\big[\phi^2(t,z)\big|\mathcal{F}^\alpha\big]\,\alpha(dt,dz). \tag{4.5}$$

Remark 4.4. In the same way as for the case of information flow $\mathbb{G}$, we can define the $\mathbb{F}$-predictable σ-algebra $\mathcal{P}_\mathbb{F}$ and consider the associated $\mathbb{F}$-predictable random fields. Being any $\mathbb{F}$-predictable stochastic process also $\mathbb{G}$-predictable, the integration can be done in the same setting as above with the result that the corresponding stochastic functions of type (4.4) will be martingale random fields under $\mathbb{F}$. Clearly results as in Proposition 4.3 fail in general in this context. In fact β is $\mathcal{G}_0$-measurable, but not $\mathcal{F}_0$-measurable in general.

In the sequel we study integral representations with respect to the filtration $\mathbb{G}$ where the integrands are explicitly characterized via stochastic derivatives. Some of our results can be adapted to the case of the filtration $\mathbb{F}$. However the stochastic calculus is more delicate in the case of $\mathbb{F}$ and is a matter for future research.

5. Non-anticipating stochastic derivative and representation theorem

In this section we discuss stochastic differentiation in the context of non-anticipative calculus. We will use the terminology non-anticipating derivative to emphasize the fact that the operator introduced embeds the information flow associated with the framework as time evolves. This differs from other concepts of stochastic differentiation, as the Malliavin type derivative. We consider the relationships with anticipative derivatives in Section 7.

The *non-anticipating stochastic derivative* is the adjoint linear operator $\mathscr{D} = I^*$ of the stochastic integral:

$$\mathscr{D} : L_2(\Omega, \mathcal{F}, \mathbb{P}) \Longrightarrow \Phi.$$

We can see that the non-anticipating derivative can be computed as the limit

$$\mathscr{D}\xi = \lim_{n\to\infty} \phi_n$$

with convergence in Φ of the stochastic functions of type (4.2) given by

$$\phi_n(t,z) := \sum_{k=1}^{K_n} \mathbb{E}\left[\xi \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{s_{n,k}}\right] \mathbf{1}_{\Delta_{n,k}}(t,z)$$

where $\Delta_{n,k} = (s_{n,k}, u_{n,k}] \times B_{n,k}$ refers to the nth partition sets (4.1) in the dissecting system of $X = (0,T] \times Z$ (as per Definition 2.6). In fact we have the following result:

Theorem 5.1. *All $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ have representation*

$$\xi = \xi_0 + \int_0^T \int_Z \mathscr{D}_{t,z}\xi \, \tilde{H}(dt,dz).$$

Moreover $\mathscr{D}\xi_0 = 0$. In particular we have $\xi_0 = \mathbb{E}[\xi|\mathcal{F}^\alpha]$.

Proof. We proceed using arguments as those in [12, Theorem 2.1]. Set $\phi_{n,k} := \mathbb{E}\big[\xi \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\,|\mathcal{G}_{s_{n,k}}\big]$. First note that

$$\mathbb{E}\big[\big|\phi_{n,k}\tilde{H}(\Delta_{n,k})\big|^2\,|\mathcal{G}_{s_{n,k}}\big] \le \mathbb{E}\big[\xi^2\big|\mathcal{G}_{s_{n,k}}\big]$$

by application of the conditional Hölder inequality. Thus $\mathbb{E}\big[|\phi_{n,k}\tilde{H}(\Delta_{n,k})|^2\big] < \infty$. Moreover, we have that

$$\mathbb{E}\Big[\big(\xi - \phi_{n,k}\tilde{H}(\Delta_{n,k})\big)\psi\tilde{H}(\Delta_{n,k})\Big] = 0 \tag{5.1}$$

for all $\mathcal{G}_{s_{n,k}}$-measurable $\psi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$. In fact, we have

$$\begin{aligned}\mathbb{E}\Big[\Big(\xi - \phi_{n,k}\tilde{H}(\Delta_{n,k})\Big)\psi\tilde{H}(\Delta_{n,k})\Big|\mathcal{G}_{s_{n,k}}\Big]& \\ = \psi\mathbb{E}\big[\xi\tilde{H}(\Delta_{n,k})\big|\mathcal{G}_{s_{n,k}}\big] - \psi\phi_{n,k}\alpha(\Delta_{n,k}) = 0.&\end{aligned}$$

Then, from (5.1), we conclude that

$$\hat{\xi}_n := \sum_{k=1}^{K_n} \phi_{n,k}\tilde{H}(\Delta_{n,k}) = \int_0^T \int_Z \phi_n(s,z)\,\tilde{H}(ds,dz)$$

is the projection of ξ onto the subspace of the stochastic integrals of type (4.2). Moreover, $\hat{\xi} := \lim_{n\to\infty} \hat{\xi}_n$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ is the projection of ξ onto the subspace of all the stochastic integrals with respect to $\tilde{H}$. Indeed, for any integral $I(\psi)$ with $\psi \in \Phi$, and $\psi := \lim_{n\to\infty} \sum_{k=1}^{K_n} \psi_{n,k}\mathbf{1}_{\Delta_{n,k}} \in \Phi$, we have

$$\mathbb{E}\Big[\Big(\xi - \hat{\xi}\Big)I(\psi)\Big] = \lim_{n\to\infty} \sum_{k=1}^{K_n} \mathbb{E}\Big[\big(\xi - \phi_{n,k}\tilde{H}(\Delta_{n,k})\big)\psi_{n,k}\tilde{H}(\Delta_{n,k})\Big] = 0$$

(with convergence in $L_2(\Omega, \mathcal{F}, \mathbb{P})$). Denote by $\hat{\phi}$ the integrand representing $\hat{\xi}$, i.e.,

$$\hat{\xi} = \int_0^T \int_Z \hat{\phi}(s,z)\,\tilde{H}(ds,dz).$$

Then, by isometry, we have

$$\|\hat{\phi} - \phi_n\|_\Phi^2 = \|\hat{\xi} - \hat{\xi}_n\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}^2 \to 0,\ n \to \infty.$$

Hence $\hat{\phi} = \mathscr{D}\xi$. Moreover, being the difference $\xi^0 := \xi - \hat{\xi}$ orthogonal to all stochastic integrals, we have $\mathscr{D}\xi = 0$. In addition we also have that

$$\xi_0 = \mathbb{E}\big[\xi \big| \mathcal{G}_0\big] = \mathbb{E}\big[\xi \big| \mathcal{F}^\alpha\big].$$

By this we end the proof. □

Remark 5.2. Note that the non-anticipating derivative is continuous in L_2. Namely, if $\xi = \lim_{n\to\infty} \xi_n$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$, then

$$\mathscr{D}\xi = \lim_{n\to\infty} \mathscr{D}\xi_n \qquad \text{in } \Phi.$$

In fact $\|\mathscr{D}\xi - \mathscr{D}\xi_n\|_\Phi^2 \le \mathbb{E}\big[(\xi - \xi_n)^2\big] \longrightarrow 0$, $n \to \infty$.

Example 5.3. Let $\xi \in \mathbb{H}^p$ be a α-multilinear form $\xi = \beta \prod_{j=1}^p \tilde{H}(\Delta_j)$ with $\Delta_1 = (s_1, u_1] \times B_1, \Delta_2 = (s_2, u_2] \times B_2, \dots, \Delta_p = (s_p, u_p] \times B_p$ and $0 \le s_1 < u_1 \le s_2 < u_2 < \dots < u_p \le T$. Then

$$\mathscr{D}_{t,z}\xi = \beta \prod_{j=1}^{p-1} \tilde{H}(\Delta_j) \mathbf{1}_{\Delta_p}(t, z)$$

and

$$\xi = \int_{\Delta_p} \beta \prod_{j=1}^{p-1} \tilde{H}\big(\Delta_j\big)\, \tilde{H}(dt, dz).$$

Example 5.4. If $\beta \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ is $\mathcal{F}^\alpha$-measurable, then $\mathscr{D}\beta = 0$. In fact

$$\mathbb{E}\Big[\beta \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})} \Big| \mathcal{G}_{t_{n,k}}\Big] = 0$$

for all $\Delta_{n,k}$.

In general we have the following formula:

Proposition 5.5. *Let ξ be an α-multilinear form, $\xi = \beta \prod_{j=1}^p \tilde{H}(\Delta_j)$. Then*

$$\xi = \int_0^T \int_Z \mathscr{D}_{s,z}\xi\, \tilde{H}(ds, dz)$$

with

$$\mathscr{D}_{s,z}\xi = \beta \sum_{\substack{1\le i\le p \\ \Delta_i \subseteq \Delta'}} \mathbf{1}_{\Delta_i}(s, z) \prod_{j\ne i}^{p} \tilde{H}(\Delta_j \cap [0, s) \times Z) \tag{5.2}$$

Here the set Δ' is given by

$$\Delta' = \bigcup\nolimits_{j\notin\mathcal{I}} \Delta_j \tag{5.3}$$

where $\mathcal{I} = \big\{1 \le i \le p \,\big|\, \Delta_i \subset [0, t) \times Z$ and $\Delta_j \subset [t, T] \times Z$ for some $1 \le j \le p$ and $t \in [0, T]\big\}$.

To explain the set $\mathcal{I}$ in Proposition 5.5, in Example 5.3 we would have $\mathcal{I} = \{1, \dots, p-1\}$, corresponding to the sets $\Delta_1, \dots, \Delta_{p-1}$, i.e., the elements of the multilinear form that "are entirely before the last set".

Proof. Let ξ be a multilinear form of order $p \geq 1$, $\xi = \beta \prod_{j=1}^p \tilde{H}(\Delta_j)$. For simplicity assume $\Delta_j \cap \Delta_{n,k} = \emptyset$ or $\Delta_{n,k}$. Denote

$$\psi(n,k) = \mathbb{E}\Big[\beta \prod_{j=1}^{p} \tilde{H}(\Delta_j) \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})} \Big| \mathcal{G}_{t_{n,k}}\Big].$$

The computation of $\psi(n,k)$ is divided into three cases.

1. If $(\bigcup_{j=1}^P \Delta_j) \cap \Delta_{n,k} = \emptyset$ then $\psi(n,k) = 0$.
2. If there exists i such that $\Delta_i \subset (t_{n,k}, T] \times Z$ and $\Delta_i \cap \Delta_{n,k} = \emptyset$ then

$$\psi(n,k) = \mathbb{E}\Big[\beta \prod_{j \neq i} \tilde{H}(\Delta_j) \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})} \Big| \mathcal{G}_{t_{n,k}}\Big] \mathbb{E}\Big[\tilde{H}(\Delta_i) \Big| \mathcal{F}^\alpha\Big] = 0.$$

3. Neither case 1 or 2 is true. This implies that $\Delta_{n,k} \subset \Delta'$. By assumption there exists $1 \leq i \leq p$ such that $\Delta_i \cap \Delta_{n,k} = \Delta_{n,k}$. We have

$$\begin{aligned}
\psi(n,k) = \mathbb{E}\Big[\beta \prod_{j \neq i} \Big(\tilde{H}\big(\Delta_j \cap [0, t_{n,k}] \times Z\big) + \tilde{H}\big(\Delta_j \cap (t_{n,k}, T] \times Z\big)\Big) \\
\Big(\tilde{H}(\Delta_i \cap \Delta_{n,k}) + \tilde{H}(\Delta_i \cap \Delta_{n,k}^c\Big) \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})} \Big| \mathcal{G}_{t_{n,k}}\Big] \\
= \mathbb{E}\Big[\beta \prod_{j \neq i} \tilde{H}\big(\Delta_j \cap [0, t_{n,k}] \times Z\big) \Big| \mathcal{G}_{t_{n,k}}\Big] \\
= \beta \prod_{j \neq i} \tilde{H}\big(\Delta_j \cap [0, t_{n,k}] \times Z\big).
\end{aligned}$$

Thus

$$\psi(n,k) = \mathbf{1}_{\{\Delta_{n,k} \cap \Delta' \neq \emptyset\}}(n,k)\, \beta \prod_{\substack{j \\ \Delta_j \cap \Delta_{n,k} = \emptyset}} \tilde{H}\big(\Delta_j \cap [0, t_{n,k}] \times Z\big),$$

and with Δ' as above, $\mathscr{D}\xi$ is given by (5.2). Since $\mathbb{E}[\xi | \mathcal{F}^\alpha] = 0$ the representation is

$$\xi = \mathbb{E}\big[\xi \big| \mathcal{F}^\alpha\big] + I(\mathscr{D}\xi) = I(\mathscr{D}\xi). \qquad \square$$

The doubly stochastic Poisson process H is an example of a point process. For point processes in general, some integral representations have been developed in [4, 5, 10, 20], see also the survey [11]. Note that the filtration of reference in these studies is $\mathbb{F}^H = \{\mathcal{F}_t^H |, t \in [0,T]\}$. As an illustration consider [5, Theorem 8.8]:

Theorem 5.6. *Let $\xi \in L_2(\Omega, \mathcal{F}_T^H, \mathbb{P})$. Let Λ be the $\mathbb{F}^H$-predictable compensator of H. Then there exists an $\mathbb{F}^H$-predictable process ϕ such that*

$$\xi = \mathbb{E}[\xi] + \int_0^T \int_Z \phi(s,z)\big(H(ds,dz) - \Lambda(ds,dz)\big) \tag{5.4}$$

and $\mathbb{E}\big[\int_0^T \int_Z \phi(s,z)^2 \Lambda(ds,dz)\big] < \infty$.

We remark that Theorem 5.1 allows the representation of random variables that are $\mathcal{F}_T = \mathcal{F}_T^\alpha \vee \mathcal{F}_T^H$-measurable, which is a larger σ-algebra than $\mathcal{F}_T^H$. The function ϕ in (5.4) can be described in explicit terms depending on conditional expectations. This approach exploits the fact that the filtration $\mathbb{F}^H$ can be fully characterized by the jump times. This is not the case for the filtration $\mathbb{G}$ in which case we consider additional random noise such as the one generated by α. Theorem 5.1 provides an explicit characterization of the integrand in this setting.

6. Iterated integrals and chaos expansions

In this section, we revise the notion of Itô-type iterated integrals, with the intent to relate them with the α-multilinear forms of Section 3. With this in mind, the iterated integrals are developed without any symmetrization schemes. These iterated integrals will later help us connect the α-multilinear forms with the Malliavin-type derivatives developed in [29] using symmetrization schemes and multiple integrals. In particular, Theorem 6.4 resembles [29, Corollary 14], but our construction is better suited for an analysis starting from α-multilinear forms. Let

$$S_p := \Big\{(s_1, z_1 \dots s_p, z_p) \in ([0,T] \times Z)^p \Big| 0 \le s_1 \le s_2 \le \dots \le s_p \le T\Big\}$$

Denote Φ_α^p the set of $\mathcal{F}^\alpha$-measurable functions, $\phi : \Omega \times S_p \to \mathbb{R}$, such that

$$\|\phi\|_{\Phi_\alpha^p} := \Big(\mathbb{E}\Big[\int_{S_p} \phi^2(s_1, z_1, s_2 \dots s_p, z_p)\, \alpha(ds_1 dz_1) \dots \alpha(ds_p dz_p)\Big]\Big)^{\frac{1}{2}} < \infty. \tag{6.1}$$

For any $\phi \in \Phi_\alpha^p$, the *pth iterated integral* is defined as

$$J_p(T,\phi) := \int_0^T \int_Z \int_0^{s_p^-} \int_Z \dots \int_0^{s_2^-} \int_Z \phi(s_1, z_1 \dots, s_p, z_p)\, \tilde{H}(ds_1 dz_1) \dots \tilde{H}(ds_p dz_p),$$

and we set $J_p := \{J_p(T,\phi), \phi \in \Phi_\alpha^p\}$. From (4.3) and (4.5) we have

$$\mathbb{E}\Big[J_p^2(T,\phi)\Big] = \mathbb{E}\Big[\int_0^T \int_Z \mathbb{E}\Big[\Big(\int_0^{s_p} \int_Z \dots \int_0^{s_2} \int_Z \phi(s_1, z_1, \dots, s_p, z_p)$$

$$\tilde{H}(ds_1 dz_1) \dots \tilde{H}(ds_{p-1} dz_{p-1})\Big)^2 \Big| \mathcal{F}^\alpha\Big] \alpha(ds_p dz_p)\Big]$$

$$= \mathbb{E}\Big[\int_0^T\int_Z \Big(\int_0^{s_p}\int_Z \cdots \int_0^{s_2}\int_Z \phi^2(s_1, z_1, \dots, s_p, z_p)\, \alpha(ds_1, dz_1)\dots\alpha(ds_p dz_p)\Big]$$

$$= \big\|\phi\big\|^2_{\Phi^p_\alpha}. \tag{6.2}$$

The iterated integrals J_p are in correspondence with the space of α-multilinear forms $\mathbb{H}^p$ (see Definition 3.6). An example is instructive before considering the general case.

Example 6.1. Let ξ be a p-order α-multilinear form with pairwise disjoint time-intervals, i.e.,

$$\xi = \beta \prod_{j=1}^p \tilde{H}(\Delta_j),$$

with $\Delta_1 = (s_1, u_1] \times Z_1$, $\Delta_2 = (s_2, u_2] \times Z_2, \dots$ and $0 \le s_1 < u_1 a < \dots < s_p \le u_1$. Then

$$\xi = \beta \prod_{j=1}^p \tilde{H}(\Delta_j) = \beta \prod_{j=1}^p I\big(\mathbf{1}_{\Delta_j}\big)$$

$$= \int_0^T\int_Z \mathbf{1}_{\Delta_p}(s_p, z_p)\beta \prod_{j=1}^{p-1} I\big(\mathbf{1}_{\Delta_k}\big)\, \tilde{H}(ds_p, dz_p)$$

$$= \int_0^T\int_Z \mathbf{1}_{\Delta_p}(s_p, z_p) \int_0^{s_p}\int_Z \mathbf{1}_{\Delta_{p-1}}(s_{p-1}, z_{p-1})\beta$$

$$\prod_{j=1}^{p-2} I\big(\mathbf{1}_{\Delta_j}\big)\, \tilde{H}(ds_{p-1}, dz_{p-1})\, \tilde{H}(ds_p, dz_p)$$

$$= \int_0^T\int_Z\int_0^{s_p}\int_Z \cdots \int_0^{s_2}\int_Z \beta\mathbf{1}_{\Delta_p}(s_p, z_p)\dots\mathbf{1}_{\Delta_1}(s_1, z_1)\, \tilde{H}(ds_1 dz_1)\dots\tilde{H}(ds_p dz_p). \tag{6.3}$$

Next we expand this representation to the case when the sets are "overlapping in time". It is possible to investigate this using Itô's formula or symmetric functions, but instead we exploit the explicit result from Proposition 5.5.

Theorem 6.2. *If $\xi \in \mathbb{H}^p$, $p \ge 1$, then ξ can be represented as a pth iterated integral, i.e.,*

$$\xi = \int_0^T\int_Z\int_0^{s_p-}\int_Z \cdots \int_0^{s_2-}\int_Z \phi(s_p, z_p, \dots, s_1, z_1)\, \tilde{H}(ds_1 dz_1)\dots\tilde{H}(ds_p dz_p), \tag{6.4}$$

where $\phi \in \Phi^p_\alpha$. Furthermore we have

$$\big\|\xi\big\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} = \|\phi\|_{\Phi^p_a}. \tag{6.5}$$

Proof. First we prove this for the α-multilinear forms by induction. The result is true for α-multilinear forms of order $p = 1$. Consider $p \geq 2$. Assume, as induction hypothesis, that a representation of type (6.4) holds for all multilinear forms of order $p - 1$. Let

$$\xi' = \prod_{j=1}^{p-1} \tilde{H}(\Delta_j \cap [0, t) \times Z) \tag{6.6}$$

Being a $(p - 1)$-order α-multilinear for, it has representation

$$\xi' = \int_0^t \int_Z \cdots \int_0^{s_2-} \phi'_{p-1} \tilde{H}(ds_1, dz_1) \ldots \tilde{H}(dz_{p-1} ds_{p-1}),$$

with means of $\phi'_{p-1} \in \Phi_\alpha^{p-1}$. Denote this integral as $J_{p-1}(t, \phi'_{p-1})$.

Let ξ be an α-multilinear form of order p, $\xi = \beta \prod_{j=1}^p \tilde{H}(\Delta_j)$. From Proposition 5.5, we know that $\xi = I(\mathscr{D}\xi)$, with

$$\mathscr{D}_{s,z}\xi = \beta \sum_{\substack{1 \leq i \leq p \\ \Delta_i \subseteq \Delta'}} \mathbf{1}_{\Delta_i}(s, z) \prod_{j \neq i}^{p} \tilde{H}(\Delta_j \cap [0, s) \times Z).$$

Hence, by (6.6) we have

$$\begin{aligned}
\xi = I(\mathscr{D}\xi) &= \int_0^T \int_Z \Big[\beta \sum_{\substack{1 \leq i \leq p \\ \Delta_i \subseteq \Delta'}} \mathbf{1}_{\Delta_i}(s, z) \prod_{j \neq i}^{p} \tilde{H}(\Delta_j \cap [0, s) \times Z)\Big] \tilde{H}(ds, dz) \\
&= \int_0^T \int_Z \Big[\beta \sum_{\substack{1 \leq i \leq p \\ \Delta_i \subseteq \Delta'}} \mathbf{1}_{\Delta_i}(s, z) J_{p-1}(s, \phi_i)\Big] \tilde{H}(ds, dz) \\
&= \int_0^T \int_Z \int_0^{s_p-} \cdots \int_0^{s_2-} \int_Z \Big[\beta \sum_{\substack{1 \leq i \leq p \\ \Delta_i \subseteq \Delta'}} \mathbf{1}_{\Delta_i}(s_p, z_p) \phi_i(s_1, \ldots, s_{p-1}, z_{p-1})\Big] \\
&\qquad\qquad \tilde{H}(ds_1, dz_1) \ldots \tilde{H}(ds_p, dz_p)
\end{aligned}$$

which is an iterated integral of order p.

Any ξ in $\mathbb{H}^p$ is the limit in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ of a finite sums of multilinear forms of order p. Let ξ_n be such a sequence. Any finite sum of multilinear forms can be written as a pth iterated integral, let

$$\xi_n = J_p(\phi_n), \quad \phi_n \in \Phi_\alpha^p.$$

Since ξ_n is a Cauchy sequence in $L_2(\Omega, \mathcal{F}, \mathbb{P})$, ϕ_n is a Cauchy sequence in Φ_α^p by the isometry in (6.2). Hence there exists an unique $\phi \in \Phi_\alpha^p$ such that $\phi_n \to \phi$ as $n \to \infty$ and we must have $\xi = J_p(\phi)$. Finally, equation (6.5) follows directly from (6.2). □

Remark 6.3. From (6.3), we can see that if $\xi = \beta \prod_{j=1}^{p} \tilde{H}(\Delta_j)$ is an α-multilinear form, with $\beta \in \mathbb{R}$ then $\xi = J_p(\phi)$ with ϕ deterministic. For general $\xi \in \widetilde{\mathbb{H}}^p$ (Remark 3.13), we can use the same arguments as in Theorem 6.2 to conclude that $\xi = J_p(\phi)$, where $\phi \in \Phi_\alpha^p$ is deterministic. Thus $\widetilde{\mathbb{H}}^p$ is the space spanned by iterated integrals of order p with deterministic integrands.

Theorem 6.4 (Chaos expansion). *For any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$, there is unique sequence of integrands $\phi_p \in \Phi_\alpha^p$, $p = 1, 2, \dots$ such that the following representation holds:*

$$\xi = \mathbb{E}[\xi|\mathcal{F}^\alpha] + \sum_{p=1}^{\infty} J_p(\phi_p).$$

Proof. Theorem 3.12 shows that any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ has orthogonal decomposition

$$\xi = \sum_{p=0}^{\infty} \xi_p$$

with $\xi_p \in \mathbb{H}^p$, $p = 0, 1, \dots$. Any ξ_p, $p \geq 1$ can be written as a pth iterated integral by Theorem 6.2 and $\xi_0 = \mathbb{E}[\xi|\mathcal{G}_0] = \mathbb{E}[\xi|\mathcal{F}^\alpha]$ is the projection of ξ on $\mathbb{H}^0$. □

Directly from Theorem 6.4 we can formulate an integral representation theorem.

Corollary 6.5. *For any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ there exists a unique $\phi \in \Phi$ such that*

$$\xi = \mathbb{E}[\xi|\mathcal{F}^\alpha] + \int_0^T \int_Z \phi(s, z)\, \tilde{H}(ds, dz).$$

Note that this corollary is in line with classical stochastic integral representation theorems with respect to square integrable martingales as integrators. Corollary 6.5 offers no immediate way of computing the integrand ϕ since only the existence of the representation via the kernel functions of the iterated integrals is given. Corollary 6.5 deeply differs from Theorem 5.1 and the following Theorem 7.9. The last ones characterize the integrand ϕ in terms of derivative operators.

7. Anticipative stochastic derivatives and integral representations

Motivated by Clark–Ocone type formulae we study ways to compute the non-anticipating derivative and to have stochastic integral representations. We introduce a new anticipative derivative operator D^a, partially inspired by [15]. We study this operator in relation with a Malliavin-type derivative for processes with conditionally independent increments developed in [29].

Let $\mathcal{G}_{\Delta^c}$ be the minimal complete σ-algebra containing $\mathcal{F}^\alpha$ and the sets $\{\tilde{H}(\Delta')|\Delta' \subset \Delta^c\}$, where Δ^c is the complement of Δ.

Definition 7.1. For $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ we set $D^a_{s,z}\xi(n)$ as the element of $L_2(\Omega \times X)$ given by

$$D^a_{s,z}\xi(n) := \sum_{k=1}^{K_n} \mathbb{E}\Big[\xi \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big]\mathbf{1}_{\Delta_{n,k}}(s,z),$$

with the n referring to the nth partition of the dissecting system. Denote by $D^a\xi$ the limit in $L_2(\Omega \times X)$ (if it exists) given by

$$D^a\xi = \lim_{n\to\infty} D^a\xi(n). \tag{7.1}$$

We define $\mathbb{D}^a$ as the subset of $L_2(\Omega, \mathcal{F}, \mathbb{P})$ where the limit (7.1) exists and we define the norm:

$$\|\xi\|_{\mathbb{D}^a} := \Big(\mathbb{E}\Big[\int_0^T\int_Z (D^a_{s,z}\xi)^2\, \alpha(ds,dz)\Big]\Big)^{\frac{1}{2}} < \infty.$$

Then D^a is a linear operator, $D^a : \mathbb{D}^a \to L_2(\Omega \times X)$.

We remark that for any $\beta \in \mathbb{H}^0$ (recall Definition 3.2), $\beta \in \mathbb{D}^a$ and $D^a\beta = 0$.

Lemma 7.2. *For $p \geq 1$, let ξ be a p-order α-multilinear form, i.e., we have $\xi = \beta\prod_{j=1}^p \tilde{H}(\Delta_j)$. Then*

$$D^a_{s,z}\xi = \beta\sum_{i=1}^p \mathbf{1}_{\Delta_i}(s,z)\prod_{j\neq i}\tilde{H}(\Delta_j), \tag{7.2}$$

and

$$D^a\xi(n) = \sum_{i=1}^p\sum_{k=1}^{K_n}\frac{\alpha(\Delta_i \cap \Delta_{n,k})}{\alpha(\Delta_{n,k})}\prod_{j\neq i}\tilde{H}(\Delta_j)\mathbf{1}_{\{\Delta_{n,k}\cap\Delta_j=\emptyset\}}(k,j)\mathbf{1}_{\Delta_{n,k}}(s,z). \tag{7.3}$$

Furthermore

$$\|\xi\|_{\mathbb{D}^a} = \|D^a\xi\|_{L_2(\Omega\times X)} = \sqrt{p}\|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}.$$

Proof. For any n and $k = 1, \ldots, K_n$, denote

$$\psi(n,k) = \mathbb{E}\Big[\beta\prod_{j=1}^p\tilde{H}(\Delta_j)\frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big]$$

If $\Delta_{n,k} \cap \big(\bigcup_{j=1}^p \Delta_j\big) = \emptyset$ or if $\Delta_{n,k}$ intersects with *more than one* of the sets Δ_j's, then $\psi(n,k)$ is equal to zero by direct computation. If $\Delta_{n,k} \subset \Delta_i$ for some i, then

$$\begin{aligned}
\psi(n,k) &= \mathbb{E}\Big[\beta\prod_{j=1}^p\tilde{H}(\Delta_j)\frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big] \\
&= \beta\prod_{j\neq i}\tilde{H}(\Delta_j)\mathbb{E}\Big[\Big(\tilde{H}(\Delta_i\cap\Delta_{n,k}) + \tilde{H}(\Delta_i\setminus\Delta_{n,k})\Big)\frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big] \\
&= \beta\prod_{j\neq i}\tilde{H}(\Delta_j).
\end{aligned}$$

If $\Delta_i \subsetneq \Delta_{n,k}$ for some i and $\Delta_{n,k} \cap \Delta_j = \emptyset$ for all $j \neq i$, then

$$\psi(n,k) = \beta \frac{\alpha(\Delta_i)}{\alpha(\Delta_{n,k})} \prod_{j \neq i}^{p} \tilde{H}(\Delta_j).$$

Combining the above cases we conclude that

$$\psi(n,k) = \sum_{i=1}^{p} \frac{\alpha(\Delta_i \cap \Delta_{n,k})}{\alpha(\Delta_{n,k})} \prod_{j \neq i} \tilde{H}(\Delta_j) \mathbf{1}_{\{\Delta_{n,k} \cap \Delta_j = \emptyset\}}(k,j).$$

and (7.3) follows. Passing to the limit in $L_2(\Omega \times X)$ we have

$$D^a \xi = \lim_{n \to \infty} D^a \xi(n) = \beta \sum_{i=1}^{p} \mathbf{1}_{\Delta_i}(s,z) \prod_{j \neq i} \tilde{H}(\Delta_j).$$

Moreover

$$\begin{aligned} \|\xi\|^2_{\mathbb{D}^a} &= \mathbb{E}\Big[\int_0^T \int_Z \Big(\beta \sum_{i=1}^{p} \mathbf{1}_{\Delta_i}(s,z) \prod_{j \neq i} \tilde{H}(\Delta_j) \Big)^2 \alpha(ds,dz) \Big] \\ &= \mathbb{E}\Big[\beta^2 \sum_{i=1}^{p} \alpha(\Delta_i) \prod_{j \neq i} \tilde{H}(\Delta_j)^2 \Big] = \mathbb{E}\Big[\beta^2 p \prod_{j=1}^{p} \alpha(\Delta_j) \Big] = p \|\xi\|^2_{L_2(\Omega,\mathcal{F},\mathbb{P})}. \quad \square \end{aligned}$$

Comparing (7.2) and (7.3) we can see that, for the p-order α-multilinear form ξ, the following estimate holds for all n:

$$\|D^a \xi(n)\|_{L_2(\Omega \times X)} \leq \|D^a \xi\|_{L_2(\Omega \times X)} = \sqrt{p} \|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}. \tag{7.4}$$

The following statements are an immediate consequence in Lemma 7.2.

Corollary 7.3. *Let $p \geq 1$. Let ξ_1 and ξ_2 be orthogonal p-order α-multilinear forms. Then, for all n, $D^a \xi_1(n)$ and $D^a \xi_2(n)$ are orthogonal in $L_2(\Omega \times X)$. The same holds for $D^a \xi_1$ and $D^a \xi_2$.*

Corollary 7.4. *For $p_1 > p_2 \geq 1$, let $\xi_1 \in \mathbb{H}^{p_1}$ and $\xi_2 \in \mathbb{H}^{p_2}$ be α-multilinear forms. Then $D^a \xi_1$ and $D^a \xi_2$ are orthogonal in $L_2(\Omega \times X)$.*

Finally we have the following result:

Proposition 7.5. *For $p \geq 1$, if $\xi \in \mathbb{H}^p$ then $\xi \in \mathbb{D}^a$ with*

$$\|\xi\|_{\mathbb{D}^a} = \sqrt{p} \|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} < \infty.$$

Proof. Any $\xi \in \mathbb{H}^p \subset L_2(\Omega, \mathcal{F}, \mathbb{P})$ can be approximated by a sequence ξ_m, $m = 1, 2, \ldots$, of finite sums of α-multilinear forms of order p: $\lim_{m \to \infty} \|\xi_m - \xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} = 0$. First of all we observe that, from Remark 3.4, ξ_m can be represented as finite sums of orthogonal p-order α-multilinear forms. From Lemma 7.2 and Corollary 7.3 we can see that $D^a \xi_m$ is a Cauchy sequence in $L_2(\Omega \times X)$ with limit ϕ such that $\|\phi\|_{L_2(\Omega \times X)} = \sqrt{p} \|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}$.

We show that indeed $\phi = D^a\xi := \lim_{n\to\infty} D^a\xi(n)$ in $L_2(\Omega\times X)$. By application of Corollary 7.3 and (7.4) we have

$$\|D^a\xi_m(n)\|_{L_2(\Omega\times X)} \le \sqrt{p}\|\xi_m\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}. \tag{7.5}$$

Moreover we note that

$$\begin{aligned}
&\|D^a\xi_m(n) - D^a\xi(n)\|^2_{L_2(\Omega\times X)} \\
&\quad = \mathbb{E}\Big[\int_0^T\int_Z \sum_{k=1}^{K_n} \Big(\mathbb{E}\Big[(\xi_m-\xi)\frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big]\Big)^2 \mathbf{1}_{\Delta_{n,k}}(s,z)\,\alpha(ds,dz)\Big] \\
&\quad \le \mathbb{E}\Big[\sum_{k=1}^{K_n}\mathbb{E}\Big[(\xi_m-\xi)^2\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big]\mathbb{E}\big[\tilde{H}(\Delta_{n,k})^2\big|\mathcal{G}_{\Delta^c_{n,k}}\big]\frac{1}{\alpha(\Delta_{n,k})}\Big] \\
&\quad = K_n\|\xi_m-\xi\|^2.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\lim_{n\to\infty}\|\phi - D^a\xi(n)\|_{L_2(\Omega\times X)} \\
&\qquad \le \lim_{n\to\infty}\lim_{m\to\infty}\Big\{\|\phi - D^a\xi_m\|_{L_2(\Omega\times X)} + \|D^a\xi_m - D^a\xi_m(n)\|_{L_2(\Omega\times X)} \\
&\qquad\qquad + \|D^a\xi_m(n) - D^a\xi(n)\|_{L_2(\Omega\times X)}\Big\} = 0.
\end{aligned}$$

In fact

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\|D^a\xi_m(n) - D^a\xi(n)\|_{L_2(\Omega\times X)} \\
&\qquad \le \lim_{n\to\infty}\big\{\sqrt{K_n}\lim_{m\to\infty}\|\xi_m-\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}\big\} = 0
\end{aligned}$$

and by (7.5)

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\|D^a\xi_m - D^a\xi_m(n)\|_{L_2(\Omega\times X)} \\
&\quad \le \lim_{q\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\Big\{\|D^a\xi_m - D^a\xi_q(n)\|_{L_2(\Omega\times X)} \\
&\qquad\qquad + \|D^a\xi_q(n) + D^a\xi_m(n)\|_{L_2(\Omega\times X)}\Big\} \\
&\quad \le \lim_{q\to\infty}\|\phi - D^a\xi_q\|_{L_2(\Omega\times X)} + \sqrt{p}\lim_{q\to\infty}\lim_{m\to\infty}\|\xi_q-\xi_m\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} = 0. \qquad \square
\end{aligned}$$

The Malliavin calculus for processes with conditionally independent increments was developed in [29], this include the CDSPP. The results and developments therein are close to those of [24, Chapter 1]. We summarize some of those results with the aim of showing how these operators relate to the operator D^a and the non-anticipating derivative $\mathscr{D}$.

Let $f_p : L_2\big(\Omega\times((0,T]\times Z)^p\big)\to\mathbb{R}$ where f_p is $\mathcal{F}^\alpha\times\mathcal{B}_X$-measurable. Remark that f_p is *not* defined on Φ^p_α, which is a smaller space. We say that f_p is *simple* if

$$f_p = \sum_{i=1}^n \beta_i(\omega)\mathbf{1}_{\Delta_1}(s_1,z_1)\dots\mathbf{1}_{\Delta_p}(s_p,z_p)$$

where β_i, $i = 1, \dots, n$ is a bounded $\mathcal{F}^\alpha$-measurable random variable and the sets $\Delta_1, \dots, \Delta_p$ are pairwise disjoint. The multiple integrals of order p of a simple function are then

$$I_p(T, f_p) := \sum_i^n \beta_i \prod_{j=1}^p \tilde{H}(\Delta_j),$$

i.e., the multiple integrals of simple functions of order p are sums of α-multilinear forms of order p. These multiple integrals are extended to integrals of general $\mathcal{F}^\alpha \times \mathcal{B}_X$-measurable functions $f_p : L_2\big(\Omega \times (0,T] \times Z)^p\big) \to \mathbb{R}$ by taking limits of simple functions. We conclude that the space spanned by multiple integrals of order p on the functions above coincide with $\mathbb{H}^p$.

Any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ has representation (as per Theorem 6.4 and [29, Corollary 14])

$$\xi = \mathbb{E}\big[\xi \big| \mathcal{F}^\alpha\big] + \sum_{p=1}^\infty I_p(\tilde{f}_p),$$

by means of a sequence $\tilde{f}_p$, $p \geq 1$, of symmetric functions in $L_2\big(\Omega \times ((0,T] \times z)^p\big)$.

Denote the symmetrization of f_p by

$$\tilde{f}_p := \frac{1}{p!} \sum_\sigma f\big(s_{\sigma(1)}, z_{\sigma(1)}, \dots, s_{\sigma(p)}, z_{\sigma(p)}\big)$$

where σ is running over all permutations of $1, \dots, p$. Let $\phi_p \in \Phi_\alpha^p$ (see (6.1)) and $f_p = \mathbf{1}_{S_p} \phi_p$. Then the following equalities hold [29, Section 3]:

$$J_p(T, \phi_p) = I_p(T, f_p) = I_p(T, \tilde{f}_p) = p! J_p(T, \tilde{f}_p).$$

The *Malliavin derivative* $D : \mathbb{D}_{1,2} \to L_2(\Omega \times X)$ is given by

$$D_{s,z}\xi := \sum_{p=1}^\infty p I_{p-1}\big(\tilde{f}_p(\cdot, s, z)\big) \tag{7.6}$$

for all $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ with $\xi = \mathbb{E}[\xi | \mathcal{F}^\alpha] + \sum_{p=1}^\infty I_p(\tilde{f}_p)$, such that

$$\|\xi\|_{\mathbb{D}_{1,2}} := \Big(\sum_{p=1}^\infty p p! \|\tilde{f}_p\|^2_{\Phi_\alpha^p} \Big)^{\frac{1}{2}} < \infty.$$

Indeed, $\|D\xi\|^2_{L_2(\Omega \times X)} = \sum_{p=1}^\infty p p! \|\tilde{f}_p\|^2_{\Phi_\alpha^p}$.

Lemma 7.6. *For $p \geq 1$, let $\xi = \beta \prod_{j=1}^p \tilde{H}(\Delta_j)$ be an α-multilinear form. Then*

$$D_{s,z}\xi = \beta \sum_{i=1}^p \mathbf{1}_{\Delta_i}(s,z) \prod_{i \neq j} \tilde{H}(\Delta_j) \tag{7.7}$$

and

$$\|\xi\|_{\mathbb{D}_{1,2}} = \sqrt{p} \|\xi\|_{L_2(\Omega, \mathcal{F}, \mathbb{P})}.$$

Proof. Since $\xi = J_p(T, \beta \mathbf{1}_{\Delta_1} \dots \mathbf{1}_{\Delta_p})$, $\xi = I_p(\tilde{f}_p)$ with

$$\tilde{f} = \beta \frac{1}{p!} \sum_{\sigma} \mathbf{1}_{\Delta_1}(s_{\sigma(1)}, z_{\sigma(1)}) \dots \mathbf{1}_{\Delta_p}(s_{\sigma(p)}, z_{\sigma(p)}).$$

Thus, from (7.6), we have

$$\begin{aligned} D_{s,z}\xi = {} & \beta \frac{1}{p!} p \sum_{\sigma} \mathbf{1}_{\Delta_p}(s_{\sigma(p)}, z_{\sigma(p)}) \\ & \qquad I_{p-1}\Big(\mathbf{1}_{\Delta_1}(s_{\sigma(1)}, z_{\sigma(1)}) \dots \mathbf{1}_{\Delta_p}(s_{\sigma(p-1)}, z_{\sigma(p-1)})\Big)\Big|_{\substack{s_{\sigma(p)}=s \\ z_{\sigma(p)}=z}} \\ = {} & \frac{p}{p!} \beta \sum_{i=1}^{p} \mathbf{1}_{\Delta_i}(s,z)(p-1)! \prod_{j \neq i} \tilde{H}(\Delta_j) \\ = {} & \beta \sum_{i=1}^{p} \mathbf{1}_{\Delta_i}(s,z) \prod_{j \neq i} \tilde{H}(\Delta_j). \end{aligned}$$

Let us compute the norm of ξ in $\mathbb{D}_{1,2}$. Note that

$$\|\tilde{f}_p\|^2_{\Phi^p_\alpha} = \mathbb{E}\Big[\beta^2 \frac{1}{p!} \prod_{j=1}^{p} \alpha(\Delta_j)\Big].$$

Hence

$$\|\xi\|^2_{\mathbb{D}_{1,2}} = pp! \|\tilde{f}_p\|^2_{\Phi^p_\alpha} = p\mathbb{E}\Big[\beta^2 \prod_{j=1}^{p} \alpha(\Delta_j)\Big] = p\|\xi\|^2_{L_2(\Omega,\mathcal{F},\mathbb{P})}. \qquad \square$$

We observe that if $\xi \in \mathbb{H}^p$, $p \geq 1$, then by the closability of D, [29, Lemma 21], and Lemma 7.6 it follows that $\xi \in \mathbb{D}_{1,2}$ with

$$\|\xi\|_{\mathbb{D}_{1,2}} = \sqrt{p}\|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} < \infty.$$

Moreover, if β is $\mathcal{F}^\alpha$ measurable we have $D\beta = 0$ by [29, Proposition 25].

Recall that $L_2(\Omega, \mathcal{F}, \mathbb{P}) = \mathbb{H} = \sum_{p=0}^{\infty} \oplus \mathbb{H}^p$ (see Theorem 3.12).

Proposition 7.7. *If $\xi \in \mathbb{H}$ then $\xi \in \mathbb{D}^a$ if and only if*

$$\sum_{p=1}^{\infty} \sqrt{p}\|\xi_p\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} < \infty.$$

Here $\xi_0, \xi_1, \dots$ is the orthogonal decomposition of ξ in the chaos expansion of Theorem 3.12. *Indeed we have $\|D^a\xi\|_{\mathbb{D}^a} = \|D\xi\|_{\mathbb{D}_{1,2}} = \sum_{p=1}^{\infty} \sqrt{p}\|\xi_p\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}$.*

Proof. This is a direct application of Lemma 7.2, Corollary 7.4 for D^a and of Theorem 6.2, Lemma 7.6, and (7.7), for D. $\square$

We conclude that the spaces $\mathbb{D}^a$ and $\mathbb{D}_{1,2}$ coincide but are not equal to the whole of $L_2(\Omega, \mathcal{F}, \mathbb{P})$, i.e.,

$$\mathbb{D}^a = \mathbb{D}_{1,2} \subsetneq L_2(\Omega, \mathcal{F}, \mathbb{P}).$$

Moreover, we stress that for any $\xi \in \mathbb{D}^a$, there exists a sequence ξ_m, $m = 1, 2 \ldots$, of finite sums of α-multilinear forms approximating ξ. Then $D^a\xi_m$ and $D\xi_m$ are two identical converging sequences by Lemma 7.2 and Lemma 7.6. These two sequences must have the same limit in $L_2(\Omega \times X)$.

We summarize the above arguments into the following statement:

Theorem 7.8. *The operators D^a and D coincide, i.e., $\mathbb{D}^a = \mathbb{D}_{1,2}$ and*

$$D^a\xi = D\xi \quad \text{in } L_2(\Omega \times X).$$

After the above result we can also interpret the operator D^a as an alternative approach to describe the Malliavin derivative which shows the anticipative dependence of the operator on the information in a much more structural and explicit way than the classical approach via chaos expansions of iterated integrals.

The following theorem is a Clark–Ocone type result which provides an alternative way to compute the non-anticipating derivative in the integral representation of Theorem 5.1. Denote $\mathbb{E}[D^a\xi|\mathcal{G}]$ the stochastic process given by $\phi(s,z) = \mathbb{E}\big[D^a_{s,z}\xi|\mathcal{G}_{s-}\big]$, $s \in [0,T]$, $z \in Z$.

Theorem 7.9. *For any $\xi \in \mathbb{D}^a$ we have*

$$\mathbb{E}\big[D^a\xi\big|\mathcal{G}\big] = \mathbb{E}\big[D\xi\big|\mathcal{G}\big] = \mathscr{D}\xi \quad \mathbb{P}\times\alpha \text{ a.e.}$$

Proof. The first equality follows from Theorem 7.8. Assume $\xi \in \mathbb{H}^p$ is a p-order α-multilinear form, $\xi = \beta\prod_{j=1}^p \tilde{H}(\Delta_j)$. From (7.2),

$$\begin{aligned}\mathbb{E}\Big[D^a_{s,z}\xi\Big|\mathcal{G}_{s-}\Big] &= \beta\sum_{i=1}^p \mathbb{E}\Big[\prod_{j\neq i}\tilde{H}(\Delta_j)\Big|\mathcal{G}_{s-}\Big]\mathbf{1}_{\Delta_i}(s,z)\\ &= \beta\sum_{\substack{1\leq i\leq p\\ \Delta_i\subseteq\Delta'}}\mathbf{1}_{\Delta_i}(s,z)\prod_{j\neq i}\tilde{H}(\Delta_j\cap[0,s)\times Z) = \mathscr{D}_{s,z}\xi\end{aligned}$$

by comparing to (5.2). The set Δ' is as described in (5.3). By approximation we obtain the statement first for the general $\xi \in \mathbb{H}^p$ and then for $\xi \in \mathbb{H}$: $\xi = \lim_{q\to\infty}\sum_{p=0}^q \xi_p$ with $\xi_p \in \mathbb{H}^p$. □

Corollary 7.10. *For any $\xi \in L_2(\Omega,\mathcal{F},\mathbb{P})$ there exists a sequence $\xi_q \in \mathbb{D}^a$, $q = 1,\ldots$ such that $\xi_q \to \xi$ in $L_2(\Omega,\mathcal{F},\mathbb{P})$ and*

$$\mathscr{D}\xi_q = \mathbb{E}\big[D^a\xi_q|\mathcal{G}\big] \longrightarrow \mathscr{D}\xi \quad \text{as } q\to\infty \quad \text{in } \Phi.$$

Thus

$$\xi = \mathbb{E}[\xi|\mathcal{F}^\alpha] + \lim_{q\to\infty}\int_0^T\int_Z \mathbb{E}\big[D^a_{s,z}\xi_q|\mathcal{G}_{s-}\big]\,\tilde{H}(ds,dz)$$

with convergence in $L_2(\Omega,\mathcal{F},\mathbb{P})$.

Proof. Take ξ_q to be the projection of ξ on $\mathbb{H}_q = \sum_{p=0}^q \oplus\mathbb{H}^p$, this is $\xi_q = \mathbb{E}\big[\xi\big|\mathcal{F}^\alpha\big] + \sum_{p=1}^q \xi_p$, and apply Remark 5.2 and Theorem 7.9. □

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General Upper and Lower Tail Estimates Using Malliavin Calculus and Stein's Equations

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To the memory of Prof. Paul Malliavin

Abstract. Following a strategy recently developed by Ivan Nourdin and Giovanni Peccati, we provide a general technique to compare the tail of a given random variable to that of a reference distribution, and apply it to all reference distributions in the so-called Pearson class. This enables us to give concrete conditions to ensure upper and/or lower bounds on the random variable's tail of various power or exponential types. The Nourdin-Peccati strategy analyzes the relation between Stein's method and the Malliavin calculus, and is adapted to dealing with comparisons to the Gaussian law. By studying the behavior of the solution to general Stein equations in detail, we show that the strategy can be extended to comparisons to a wide class of laws, including all Pearson distributions.

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1. Introduction

In this article, following a strategy recently developed by Ivan Nourdin and Giovanni Peccati, we provide a general technique to compare the tail of a given random variable to that of a reference distribution, and apply it to all reference distributions in the so-called Pearson class, which enables us to give concrete conditions to ensure upper and/or lower bounds on the random variable's tail of power or exponential type. The strategy uses the relation between Stein's method and the

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Malliavin calculus. In this introduction, we detail the main ideas of this strategy, including references to related works; we also summarize the results proved in this article, and the methods used to prove them.

1.1. Stein's method and the analysis of Nourdin and Peccati

Stein's method is a set of procedures that is often used to measure distances between distributions of random variables. The starting point is the so-called Stein equation. To motivate it, recall the following result which is sometimes referred to as Stein's lemma. Suppose X is a random variable. Then $X \overset{\text{Law}}{=} Z \sim \mathcal{N}(0,1)$ if and only if

$$\mathbf{E}[f'(X) - Xf(X)] = 0 \tag{1.1}$$

for all continuous and piecewise differentiable functions f such that $\mathbf{E}[|f'(X)|] < \infty$ (see, e.g., [4, 5, 21]). If the above expectation is non-zero but close to zero, Stein's method can give us a way to express how close the law of X might be to the standard normal law, in particular by using the concept of Stein equation. For a given test function h, this is the ordinary differential equation $f'(x) - xf(x) = h(x) - \mathbf{E}[h(Z)]$ with continuous and piecewise differentiable solution f. As we will see in more detail and greater generality further below, if one is able to prove boundedness properties of f and f' for a wide class of test functions h, this can help evaluate the distance between the law of Z and laws of random variables that might be close to Z, including methods for proving convergence in distribution. This fundamental feature of Stein's method is described in many works; see [4] for a general introduction and review.

As a testament to the extraordinary versatility of Stein's method, recently Ivan Nourdin and Giovanni Peccati discovered a connection between Stein's method and the Malliavin calculus, with striking applications in a number of problems in stochastic analysis. Motivated by Berry–Esséen-type theorems for convergence of sequences of random variables in Wiener chaos, Nourdin and Peccati's first paper [8] on this connection considers an arbitrary square-integrable Malliavin-differentiable random variable X on a Wiener space, and associates the random variable

$$G := \langle DX; -DL^{-1}X \rangle \tag{1.2}$$

where D is the Malliavin derivative operator on the Wiener space, and L^{-1} is the pseudo-inverse of the generator of the Ornstein–Uhlenbeck semigroup (see Section 3.1 for precise definitions of these operators). One easily notes that if X is standard normal, then $G \equiv 1$ (Corollary 3.4 in [16]). Then by measuring the distance between G and 1 for an arbitrary X, one can measure how close the law of X is to the normal law. The connection to Stein's method comes from their systematic use of the basic observation that $\mathbf{E}[Gf(X)] = \mathbf{E}[Xf'(X)]$. It leads to the following simple and efficient strategy for measuring distances between the laws of X and Z. To evaluate, e.g., $\mathbf{E}[h(X)] - \mathbf{E}[h(Z)]$ for test functions h, one can:

1. write $\mathbf{E}[h(X)] - \mathbf{E}[h(Z)]$ using the solution of Stein's equation, as $\mathbf{E}[f'(X)] - \mathbf{E}[Xf(X)]$;
2. use their observation to transform this expression into $\mathbf{E}[f'(X)(1-G)]$;
3. use the boundedness and decay properties of f' (these are classically known from Stein's equation) to exploit the proximity of G to 1.

As we said, this strategy of relating Stein's method and the Malliavin calculus is particularly useful for analyzing problems in stochastic analysis. In addition to their study of convergence in Wiener chaos in [8], which they followed up with sharper results in [9], Nourdin and Peccati have implemented several other applications including: the study of cummulants on Wiener chaos [11], of fluctuations of Hermitian random matrices [12], and, with other authors, other results about the structure of inequalities and convergences on Wiener space, such as [3, 13, 14, 15]. In [16], it was pointed out that if ρ denotes the density of X, then the function

$$g(z) := \rho^{-1}(z) \int_z^\infty y\rho(y)\, dy, \tag{1.3}$$

which was originally defined by Stein in [21], can be represented as

$$g(z) = \mathbf{E}[G|X = z],$$

resulting in a convenient formula for the density ρ, which was then exploited to provide new Gaussian lower bound results for certain stochastic models, in [16] for Gaussian fields, and subsequently in [22] for polymer models in Gaussian and non-Gaussian environments, in [18] for stochastic heat equations, in [3] for statistical inference for long-memory stochastic processes, and multivariate extensions of density formulas in [1].

1.2. Summary of our results

Our specific motivation is drawn from the results in [22] which make assumptions on how G compares to 1 almost surely, and draw conclusions on how the tail of X, i.e., $\mathbf{P}[X > z]$, compares to the normal tail $\mathbf{P}[Z > z]$. By the above observations, these types of almost-sure assumptions are equivalent to comparing the deterministic function g to the value 1. For instance, one result in [22] can be summarized by saying that (under some additional regularity conditions) if $G \geq 1$ almost surely, i.e., if $g(z) \geq 1$ everywhere, then for some constant c and large enough z, $\mathbf{P}[X > z] > c\mathbf{P}[Z > z]$. This result, and all the ones mentioned above, concentrate on comparing laws to the standard normal law, which is done by comparing G to the constant 1, as this constant is the "G" for the standard normal Z.

In this paper, we find a framework which enables us to compare the law of X to a wide range of laws. Instead of assuming that g is comparable to 1, we only assume that it is comparable to a polynomial of degree less than or equal to 2. In [21], Stein had originally noticed that the set of all distributions such that their g is such a polynomial, is precisely the so-called Pearson class of distributions. They encompass Gaussian, Gamma, and Beta distributions, as well as the inverse-Gamma, and a number of continuous distributions with only finitely many moments, with prescribed power tail behavior. This means that one can hope to give precise criteria based on g, or via Malliavin calculus based on G, to guarantee upper and/or lower bounds on the tail $\mathbf{P}[X > z]$, with various Gaussian, exponential, or power-type behaviors. We achieve such results in this paper.

Specifically, our first set of results is in the following general framework. Let Z be a reference random variable supported on (a, b) where $-\infty \le a < b \le +\infty$, with a density ρ_* which is continuous on $\mathbf{R}$ and differentiable on (a, b). The function g corresponding to ρ_* is given as in (1.3), and we denote it by g_* (the subscripts $_*$ indicate that these are relative to our reference r.v.):

$$g_*(z) = \frac{\int_z^\infty y\rho_*(y)\,dy}{\rho_*(z)}\mathbf{1}_{(a,b)}(z). \tag{1.4}$$

We also use the notation

$$\Phi_*(z) = \mathbf{P}[Z > z]$$

for our reference tail. Throughout this article, for notational convenience, we assume that Z is centered (except when specifically stated otherwise in Section A.2 in the Appendix). Let X be Malliavin-differentiable, supported on (a, b), with its $G := \langle DX; -DL^{-1}X\rangle$ as in (1.2).

- (Theorem 3.4) Under mild regularity and integrability conditions on Z and X, if $G \ge g_*(X)$ almost surely, then for all $z < b$,

  $$\mathbf{P}[X > z] \ge \Phi_*(z) - \frac{1}{Q(z)}\int_z^b (2y - z)\mathbf{P}[X > y]\,dy,$$

  where

  $$Q(z) := z^2 - zg_*'(z) + g_*(z); \tag{1.5}$$

  typically Q is of order z^2 for large z.
- (Theorem 3.5) Under mild regularity and integrability conditions on Z and X, if $G \le g_*(X)$ almost surely, then for some constant c and all large enough $z < b$,

  $$\mathbf{P}[X > z] \le c\Phi_*(z).$$

These results are generalizations of the work in [22], where only the standard normal Z was considered. They can be rephrased by referring to g as in (1.3), which coincides with $g(z) = \mathbf{E}[G|X = z]$, rather than G; this can be useful to apply the theorems in contexts where the definition of X as a member of a Wiener space is less explicit than the information one might have directly about g. We have found, however, that the Malliavin-calculus interpretation makes for efficient proofs of the above theorems.

The main application of these general theorems are to the Pearson class: Z such that its g_* is of the form $g_*(z) = \alpha z^2 + \beta z + \gamma$ in the support of Z. Assume $b = +\infty$, i.e., the support of Z is $(a, +\infty)$. Assume $\mathbf{E}\left[|Z|^3\right] < \infty$ (which is equivalent to $\alpha < 1/2$). Then the lower bound above can be made completely explicit, as can the constant c in the upper bound.

- (Corollary 4.8) Under mild regularity and integrability conditions on X [including assuming that there exists $c > 2$ such that $g(z) \leq z^2/c$ for large z], if $G \geq g_*(X)$ almost surely, then for any $c' < \frac{1}{1+2(1-\alpha)(c-2)}$ and all z large enough,

$$\mathbf{P}[X > z] \geq c'\Phi_*(z).$$

- (Corollary 4.7) Under mild regularity and integrability conditions on X, if $G \leq g_*(X)$ almost surely, then for any $c > (1-\alpha)/(1-2\alpha)$, and all z large enough,

$$\mathbf{P}[X > z] \leq c\Phi_*(z).$$

The results above can be used conjointly with asymptotically sharp conclusions when upper and lower bound assumptions on G are true simultaneously. For instance, we have the following, phrased using g's instead of G's.

- (Corollary 4.9, point 2) On the support $(a, +\infty)$, let $g_*(z) = \alpha z^2 + \beta z + \gamma$ and let $\bar{g}_*(z) = \bar{\alpha} z^2 + \bar{\beta} z + \bar{\gamma}$ with non-zero α and $\bar{\alpha}$. If for the Malliavin-differentiable X and its corresponding g, we have for all $z > a$, $g_*(z) \leq g(z) \leq \bar{g}_*(z)$, then there are constants c and $\bar{c}$ such that for large z,

$$cz^{-1-1/\alpha} \leq \mathbf{P}[X > z] \leq \bar{c} z^{-1-1/\bar{\alpha}}.$$

- (see Corollary 4.10) A similar result holds when $\alpha = \bar{\alpha} = 0$, in which $\mathbf{P}[X > z]$ compares to the Gamma-type tail $z^{-1-\gamma/\beta^2} \exp(-z/\beta)$.

The strategy used to prove these results is an analytic one, following the initial method of Nourdin and Peccati, this time using the Stein equation relative to the function g_* defined in (1.4) for a general reference r.v. Z:

$$g_*(x) f'(x) - xf(x) = h(x) - \mathbf{E}[h(Z)].$$

Our mathematical techniques are based on a careful analysis of the properties of g_*, its relation to the function Q defined in (1.5), and what consequences can be derived for the solutions of Stein's equation. The basic general theorems' proofs use a structure similar to that employed in [22]. The applications to the Pearson class rely heavily on explicit computations tailored to this case, which are facilitated via the identification of Q as a useful way to express these computations.

This article is structured as follows. Section 2 gives an overview of Stein's equations, and derives some fine properties of their solutions by referring to the function Q. These will be crucial in the proofs of our general upper and lower bound results, which are presented in Section 3 after an overview of the tools of Malliavin calculus which are needed in this article. Applications to comparisons with Pearson distributions, with a particular emphasis on tail behavior, including asymptotic results, are in Section 4. Section 5 is an Appendix containing the proofs of some technical lemmas and some details on Pearson distributions.

2. The Stein equation

2.1. Background and classical results

Characterization of the law of Z. As before, let Z be centered with a differentiable density on its support (a, b), and let g_* be defined as in (1.4). Nourdin and Peccati (Proposition 6.4 in [8]) collected the following results concerning this equation. If f is a function that is continuous and piecewise continuously differentiable, and if $\mathbf{E}[|f'(Z)|g_*(Z)] < \infty$, Stein (Lemma 1, p. 59 in [21]) proved that

$$\mathbf{E}[g_*(Z)f'(Z) - Zf(Z)] = 0$$

(compare this with (1.1) for the special case $Z \sim \mathcal{N}(0,1)$). Conversely, assume that

$$\int_0^b \frac{z}{g_*(z)}\,dz = \infty \qquad \text{and} \qquad \int_a^0 \frac{z}{g_*(z)}\,dz = -\infty. \tag{2.1}$$

If a random variable X has a density, and for any differentiable function f such that $x \mapsto |g_*(x)f'(x)| + |xf(x)|$ is bounded,

$$\mathbf{E}[g_*(X)f'(X) - Xf(X)] = 0 \tag{2.2}$$

then X and Z have the same law. In other words, under certain conditions, (2.2) can be used to characterize the law of a centered random variable X as being equal to that of Z.

Stein's equation, general case; distances between distributions. If h is a fixed bounded piecewise continuous function such that $\mathbf{E}[|h(Z)|] < \infty$, the corresponding *Stein equation* for Z is the ordinary differential equation in f defined by

$$h(x) - \mathbf{E}[h(Z)] = g_*(x)f'(x) - xf(x). \tag{2.3}$$

The utility of such an equation is apparent when we evaluate the functions at X and take expectations:

$$\mathbf{E}[h(X)] - \mathbf{E}[h(Z)] = \mathbf{E}[g_*(X)f'(X) - Xf(X)]. \tag{2.4}$$

The idea is that if the law of X is "close" to the law of Z, then the right side of (2.4) would be close to 0. Conversely, if the test function h can be chosen from specific classes of functions so that the left side of (2.4) denotes a particular notion of distance between X and Z, the closeness of the right-hand side of (2.4) to zero, in some uniform sense in the f's satisfying Stein's equation (2.3) for all the h's in that specific class of test functions, will imply that the laws of X and Z are close in the corresponding distance. For this purpose, it is typically crucial to establish boundedness properties of f and f' which are uniform over the class of test functions being considered.

For example, if $\mathcal{H} = \{h : ||h||_L + ||h||_\infty \leq 1\}$ where $||\cdot||_L$ is the Lipschitz seminorm, then the Fortet–Mourier distance $d_{FM}(X, Z)$ between X and Z is defined as

$$d_{FM}(X,Z) = \sup_{h \in \mathcal{H}} |\mathbf{E}[h(X)] - \mathbf{E}[h(Z)]|.$$

This distance metrizes convergence in distribution, so by using properties of the solution f of the Stein equation (2.4) for $h \in \mathcal{H}$, we can draw conclusions on the convergence in distribution of a sequence $\{X_n\}$ to Z. See [8] and [6] for details and other notions of distance between random variables.

Solution of Stein's equation. Stein (Lemma 4, p. 62 in [21]) proved that if (2.1) is satisfied, then his equation (2.3) has a unique solution f which is bounded and continuous on (a,b). If $x \notin (a,b)$, then

$$f(x) = -\frac{h(x) - \mathbf{E}\left[h(Z)\right]}{x} \tag{2.5}$$

while if $x \in (a,b)$,

$$f(x) = \int_a^x \left(h(y) - \mathbf{E}\left[h(Z)\right]\right) \frac{e^{\int_y^x \frac{z\,dz}{g_*(z)}}}{g_*(y)}\,dy. \tag{2.6}$$

2.2. Properties of solutions of Stein's equations

We assume throughout that ρ_* is differentiable on (a,b) and continuous on $\mathbf{R}$ (for which it is necessary that ρ_* be null on $\mathbf{R} - (a,b)$). Consequently, g_* is differentiable and continuous on (a,b). The next lemma records some elementary properties of g_*, such as its positivity and its behavior near a and b. Those facts which are not evident are established in the Appendix. All are useful in facilitating the proofs of other lemmas presented in this section, which are key to our article.

Lemma 2.1. *Let Z be centered and continuous, with a density ρ_* that is continuous on $\mathbf{R}$ and differentiable on its support (a,b), with a and b possibly infinite.*

1. *$g_*(x) > 0$ if and only if $x \in (a,b)$;*
2. *g_* is differentiable on (a,b) and $[g_*(x)\rho_*(x)]' = -x\rho_*(x)$ therein;*
3. $\lim_{x\to a} g_*(x)\rho_*(x) = \lim_{x\to b} g_*(x)\rho_*(x) = 0.$

A different expression for the solution f of Stein's equation (2.3) than the one given in (2.5), (2.6), which will be more convenient for our purposes, such as computing f' in the support of Z, was given by Schoutens [20] as stated in the next lemma.

Lemma 2.2. *For all $x \in (a,b)$,*

$$f(x) = \frac{1}{g_*(x)\rho_*(x)} \int_a^x \left(h(y) - \mathbf{E}\left[h(Z)\right]\right) \rho_*(y)\,dy. \tag{2.7}$$

If $x \notin [a,b]$, differentiating (2.5) gives

$$f'(x) = \frac{-xh'(x) + h(x) - \mathbf{E}\left[h(Z)\right]}{x^2} \tag{2.8}$$

while if $x \in (a,b)$, differentiating (2.7) gives

$$f'(x) = \frac{x}{[g_*(x)]^2\rho_*(x)} \int_a^x \left(h(y) - \mathbf{E}[h(Z)]\right) \rho_*(y)\,dy + \frac{h(x) - \mathbf{E}[h(Z)]}{g_*(x)}. \tag{2.9}$$

The proof of this lemma (provided in the Appendix for completeness) also gives us the next one.

Lemma 2.3. *Under our assumption of differentiability on* (a,b) *of* ρ_* *and hence of* g_**, Stein's condition* (2.1) *on* g_* *is satisfied.*

In Stein's equation (2.3), the test function $h = 1_{(-\infty,z]}$ lends itself to useful tail probability results since $\mathbf{E}[h(Z)] = \mathbf{P}[Z \leq z]$. From this point on, we will assume that $h = 1_{(-\infty,z]}$ with fixed $z > 0$, and that f is the corresponding solution of Stein's equation (we could denote the parametric dependence of f on z by f_z, but choose to omit the subscript to avoid overburdening the notation).

As opposed to the previous lemmas, the next two results, while still elementary in nature, appear to be new, and their proofs, which require some novel ideas of possibly independent interest, have been kept in the main body of this paper, rather than having them relegated to the Appendix. We begin with an analysis of the sign of f', which will be crucial to prove our main general theorems.

Lemma 2.4. *Suppose* $0 < z < b$. *If* $x \leq z$, *then* $f'(x) \geq 0$. *If* $x > z$, *then* $f'(x) \leq 0$.

Proof. The result follows easily from (2.8) when $x \notin [a,b]$: if $x < a$, then $f'(x) = (1 - \mathbf{E}[h(Z)])/x^2 \geq 0$, while if $x > b$, then $f'(x) = -\mathbf{E}[h(Z)]/x^2 \leq 0$. So now we can assume that $x \in (a,b)$. We will use the expression for the derivative f' given in (2.9).

Suppose $a < x \leq z$. Then $h(x) = 1$ and for any $y \leq x$, $h(y) = 1$ so

$$\begin{aligned} f'(x) &= \frac{x}{[g_*(x)]^2 \rho_*(x)} \int_a^x (1 - \mathbf{E}[h(Z)])\, \rho_*(y)\, dy + \frac{1 - \mathbf{E}[h(Z)]}{g_*(x)} \\ &= \frac{1 - \mathbf{E}[h(Z)]}{[g_*(x)]^2 \rho_*(x)} \left(x \int_a^x \rho_*(y)\, dy + g_*(x)\rho_*(x) \right). \end{aligned}$$

Clearly, $f'(x) \geq 0$ if $x \geq 0$. Now define

$$n_1(x) := \int_a^x \rho_*(y)\, dy + \frac{g_*(x)\rho_*(x)}{x}.$$

We will show that $x n_1(x) \geq 0$ when $x < 0$. Since

$$\begin{aligned} n_1'(x) &= \rho_*(x) + \frac{x[g_*(x)\rho_*(x)]' - g_*(x)\rho_*(x)}{x^2} \\ &= \rho_*(x) + \frac{-x^2 \rho_*(x) - g_*(x)\rho_*(x)}{x^2} = -\frac{g_*(x)\rho_*(x)}{x^2} \leq 0 \end{aligned}$$

then n_1 is nonincreasing on $(a,0)$ which means that whenever $a < x < 0$, $n_1(x) \leq \lim_{x\to a} n_1(x) = \lim_{x \to a} \frac{g_*(x)p(x)}{x} = 0$ since $\lim_{x\to a} g_*(x)\rho_*(x) = 0$. Therefore, $x n_1(x) \geq 0$ for $x < 0$. This completes the proof that $f'(x) \geq 0$ whenever $x \leq z$.

Finally, suppose that $z < x < b$ so $h(x) = 0$. Since $\mathbf{E}[h(Z)] = \mathbf{P}[Z \le z] = \int_a^z \rho_*(y)\,dy$,

$$\begin{aligned} f'(x) &= \frac{x}{[g_*(x)]^2\rho_*(x)} \int_a^x h(y)\rho_*(y)\,dy \\ &\quad - \frac{x\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)} \int_a^x \rho_*(y)\,dy - \frac{\mathbf{E}[h(Z)]}{g_*(x)} \\ &= \frac{x}{[g_*(x)]^2\rho_*(x)} \int_a^z \rho_*(y)\,dy - \frac{x\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)} \int_a^x \rho_*(y)\,dy - \frac{\mathbf{E}[h(Z)]}{g_*(x)} \\ &= \frac{x}{[g_*(x)]^2\rho_*(x)}\mathbf{E}[h(Z)] - \frac{x\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)} \int_a^x \rho_*(y)\,dy - \frac{\mathbf{E}[h(Z)]}{g_*(x)} \\ &= \frac{\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)} \left(x - x\int_a^x \rho_*(y)\,dy - g_*(x)\rho_*(x) \right) \\ &= \frac{\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)} \cdot x n_2(x) \end{aligned}$$

where

$$n_2(x) := 1 - \int_a^x \rho_*(y)\,dy - \frac{g_*(x)\rho_*(x)}{x} = 1 - n_1(x).$$

It is enough to show that $n_2(x) \le 0$ since $x > z \ge 0$. Since $n_2'(x) = -n_1'(x) \ge 0$, then $n_2(x) \le \lim_{x\to b} n_2(x) = 1 - \lim_{x\to b} \int_a^x \rho_*(y)\,dy - \lim_{x\to b} \frac{g_*(x)\rho_*(x)}{x} = 0$ because $\lim_{x\to b} g_*(x)\rho_*(x) = 0$. Therefore, $f'(x) \le 0$ if $x > z$, finishing the proof of the lemma. □

As alluded to in the previous subsection, of crucial importance in the use of Stein's method, is a quantitatively explicit boundedness result on the derivative of the solution to Stein's equation. We take this up in the next lemma.

Lemma 2.5. *Recall the function*

$$Q(x) := x^2 - xg_*'(x) + g_*(x)$$

defined in (1.5), *for all* $x \in \mathbf{R}$ *except possibly at* a *and* b. *Assume that* $g_*''(x) < 2$ *for all* x *and that* $\frac{x-g_*'(x)}{Q(x)}$ *tends to a finite limit as* $x \to a$ *and as* $x \to b$. *Suppose* $0 < z < b$. *Then* $f'(x)$ *is bounded. In particular, if* $a < x \le z$,

$$0 \le f'(x) \le \frac{z}{[g_*(z)]^2\rho_*(z)} + \frac{1}{Q(0)} < \infty,$$

while if $b > x > z$,

$$-\infty < -\frac{1}{Q(z)} \le f'(x) \le 0. \tag{2.10}$$

To prove this lemma, we need two auxiliary results. The first one introduces and studies the function Q which we already encountered in the introduction, and which will help us state and prove our results in an efficient way. The second one shows the relation between Q, g_*, and the tail Φ_* of Z, under conditions which will be easily verified later on in the Pearson case.

Lemma 2.6.

1. *If* $x \notin (a,b)$, *then* $Q(x) = x^2 > 0$.
2. *If* g_* *is twice differentiable in* (a,b) *(for example, when* ρ_* *is twice differentiable), then* $Q'(x) = x\,(2 - g_*''(x))$.
3. *If moreover* $g_*''(x) < 2$ *in* (a,b), *a reasonable assumption as we shall see later when* Z *is a Pearson random variable, then* $\min_{(a,b)} Q = Q\,(0)$ *so that* $Q(x) \geq Q(0) = g_*(0) > 0$.

Lemma 2.7. *With the assumptions on* g_* *and* Q *as in Lemma* 2.5, *then for all* x,

$$\frac{\max\left(x - g_*'(x), 0\right)}{Q(x)} g_*(x)\rho_*(x) \leq \Phi_*(x) \tag{2.11}$$

and

$$\frac{\max\left(g_*'(x) - x, 0\right)}{Q(x)} g_*(x)\rho_*(x) \leq 1 - \Phi_*(x). \tag{2.12}$$

Moreover for $0 < x < b$, *we have*

$$\Phi_*(x) \leq \frac{1}{x} \cdot g_*(x)\rho_*(x) \tag{2.13}$$

while if $a < x < 0$, *then*

$$1 - \Phi_*(x) \leq \frac{1}{-x} \cdot g_*(x)\rho_*(x). \tag{2.14}$$

Proof of Lemma 2.5. If $x < a$ with $a > -\infty$, then $f'(x) = \frac{1-\mathbf{E}[h(Z)]}{x^2} \leq \frac{1-\mathbf{E}[h(Z)]}{a^2}$. If $x > b$ with $b < \infty$, then $f'(x) = -\frac{\mathbf{E}[h(Z)]}{x^2} \geq -\frac{\mathbf{E}[h(Z)]}{b^2}$. So now we only need to assume that $x \in (a,b)$.

Suppose $a < x \leq z$. Use $f'(x) \geq 0$ given in (2.9):

$$\begin{aligned} f'(x) &= \frac{1 - \mathbf{E}[h(Z)]}{[g_*(x)]^2 \rho_*(x)} \left(x \int_a^x \rho_*(y)\, dy + g_*(x)\rho_*(x) \right) \\ &\leq \frac{x}{[g_*(x)]^2 \rho_*(x)} \left(1 - \Phi_*(x)\right) + \frac{1}{g_*(x)}\ . \end{aligned}$$

When $x \geq 0$, we can rewrite the upper bound as:

$$f'(x) \leq r(x) + \frac{1}{g_*(x)} \left[1 - \frac{x\Phi_*(x)}{g_*(x)\rho_*(x)} \right]$$

where

$$r(x) = \frac{x}{[g_*(x)]^2 \rho_*(x)} = \frac{x}{g_*(x)[g_*(x)\rho_*(x)]}.$$

We can bound $r(x)$ above since

$$\begin{aligned} r'(x) &= \frac{[g_*(x)]^2\rho_*(x) - x\left[g_*(x)\left(-x\rho_*(x)\right) + g_*(x)\rho_*(x)g_*'(x)\right]}{[g_*(x)]^4\left[\rho_*(x)\right]^2} \\ &= \frac{g_*(x) + x^2 - xg_*'(x)}{[g_*(x)]^3\rho_*(x)} = \frac{Q(x)}{[g_*(x)]^3\rho_*(x)} > 0 \end{aligned}$$

so $r(x) \le r(z)$. To bound $\left[1 - x\Phi_*(x)/\left(g_*(x)\rho_*(x)\right)\right]/g_*(x)$, use (2.11) of Lemma 2.7:

$$\begin{aligned} \frac{1}{g_*(x)}\left[1 - \frac{x\Phi_*(x)}{g_*(x)\rho_*(x)}\right] &\le \frac{1}{g_*(x)}\left[1 - \frac{x}{g_*(x)\rho_*(x)} \cdot \frac{x - g_*'(x)}{Q(x)} g_*(x)\rho_*(x)\right] \\ &= \frac{1}{g_*(x)}\left[1 - \frac{x^2 - xg_*'(x)}{Q(x)}\right] \\ &= \frac{1}{g_*(x)} \cdot \frac{g_*(x)}{Q(x)} = \frac{1}{Q(x)} \le \frac{1}{Q(0)}. \end{aligned}$$

Therefore,

$$f'(x) \le \frac{z}{[g_*(z)]^2\rho_*(z)} + \frac{1}{Q(0)}.$$

When $x < 0$, we use (2.12) of Lemma 2.7:

$$\begin{aligned} f'(x) &\le \frac{x}{[g_*(x)]^2\rho_*(x)}\left(1 - \Phi_*(x)\right) + \frac{1}{g_*(x)} \\ &\le \frac{x}{[g_*(x)]^2\rho_*(x)} \cdot \frac{g_*'(x) - x}{Q(x)} g_*(x)\rho_*(x) + \frac{1}{g_*(x)} \\ &= \frac{1}{g_*(x)} \cdot \frac{xg_*'(x) - x^2}{Q(x)} + \frac{1}{g_*(x)} \\ &= \frac{1}{g_*(x)}\left[\frac{xg_*'(x) - x^2 + Q(x)}{Q(x)}\right] = \frac{1}{Q(x)} \le \frac{1}{Q(0)} \\ &\le \frac{z}{[g_*(z)]^2\rho_*(z)} + \frac{1}{Q(0)}. \end{aligned}$$

Now we prove (2.10) and so suppose $x > z > 0$. From the proof of Lemma 2.4,

$$\begin{aligned} f'(x) &= \frac{\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)}\left(x - x\int_a^x \rho_*(y)\,dy - g_*(x)\rho_*(x)\right) \\ &= \frac{\mathbf{E}[h(Z)]}{[g_*(x)]^2\rho_*(x)}\left(x\Phi_*(x) - g_*(x)\rho_*(x)\right). \end{aligned}$$

We conclude by again using (2.11) of Lemma 2.7, to get

$$\begin{aligned}
-f'(x) &\le \frac{1}{[g_*(x)]^2\rho_*(x)}\left(g_*(x)\rho_*(x) - x\Phi_*(x)\right) \\
&\le \frac{1}{[g_*(x)]^2\rho_*(x)}\left(g_*(x)\rho_*(x) - x\cdot\frac{x-g_*'(x)}{Q(x)}\cdot g_*(x)\rho_*(x)\right) \\
&= \frac{1}{g_*(x)}\left(1-\frac{x^2-xg_*'(x)}{Q(x)}\right) = \frac{1}{g_*(x)}\cdot\frac{g_*(x)}{Q(x)} \\
&= \frac{1}{Q(x)} \le \frac{1}{Q(z)}. \qquad \square
\end{aligned}$$

Remark 2.8. Since $z > 0$, and $Q(z) > Q(0)$, Lemma 2.5 implies the following convenient single bound for any fixed $z > 0$, uniform for all $x \in (a,b)$:

$$|f'(x)| \le \frac{z}{[g_*(z)]^2\rho_*(z)} + \frac{1}{Q(0)}.$$

Occasionally, this will be sufficient for some of our purposes. The more precise bounds in Lemma 2.5 will also be needed, however.

3. Main results

In order to exploit the boundedness of f', we adopt the technique pioneered by Nourdin and Peccati, to rewrite expressions of the form $\mathbf{E}[Xm(X)]$ where m is a function, using the Malliavin calculus. For ease of reference, we include here the requisite Malliavin calculus constructs. Full details can be found in [17]; also see [22, Section 2] for an exhaustive summary.

3.1. Elements of Malliavin calculus

We assume our random variable X is measurable with respect to an isonormal Gaussian process W, associated with its canonical separable Hilbert space H. For illustrative purposes, one may further assume, as we now do, that W is the standard white-noise corresponding to $H = L^2([0,1])$, which is constructed using a standard Brownian motion on $[0,1]$, also denoted by W, endowed with its usual probability space $(\Omega, \mathcal{F}, \mathbf{P})$. This means that the white noise W is defined by $W(f) = \int_0^1 f(s)\,dW(s)$ for any $f \in H$, where the stochastic integral is the Wiener integral of f with respect to the Wiener process W. If we denote $I_0(f) = f$ for any non-random constant f, then for any integer $n \ge 1$ and any symmetric function $f \in H^n$, we let

$$I_n(f) := n!\int_0^1\int_0^{s_1}\cdots\int_0^{s_{n-1}} f(s_1,s_2,\ldots,s_n)\,dW(s_n)\cdots dW(s_2)\,dW(s_1),$$

where this integral is an iteration of n Itô integrals. It is called the nth multiple Wiener integral of f w.r.t. W, and the set $\mathcal{H}_n := \{I_n(f) : f \in H^n\}$ is the nth Wiener chaos of W. Note that $I_1(f) = W(f)$, and that $\mathbf{E}[I_n(f)] = 0$ for all $n \ge 1$. Again, see [17, Section 1.2] for the general definition of I_n and $\mathcal{H}_n$ when W

is a more general isonormal Gaussian process. The main representation theorem of the analysis on Wiener space is that $L^2(\Omega, \mathcal{F}, \mathbf{P})$ is the direct sum of all the Wiener chaoses. In other words, $X \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ if and only if there exists a sequence of non-random symmetric functions $f_n \in H^n$ with $\sum_{n=0}^{\infty} \|f_n\|_{H^n}^2 < \infty$ such that $X = \sum_{n=0}^{\infty} I_n(f_n)$. Note that $\mathbf{E}[X] = f_0$. Moreover, the terms in this so-called Wiener chaos decomposition of X are orthogonal in $L^2(\Omega, \mathcal{F}, \mathbf{P})$, and we have the isometry property $\mathbf{E}\left[X^2\right] = \sum_{n=0}^{\infty} n! \|f_n\|_{H^n}^2$. We are now in a position to define the Malliavin derivative D.

Definition 3.1. Let $\mathbf{D}^{1,2}$ be the subset of $L^2(\Omega, \mathcal{F}, \mathbf{P})$ formed by those $X = \sum_{n=0}^{\infty} I_n(f_n)$ such that

$$\sum_{n=1}^{\infty} n \; n! \|f_n\|_{H^n}^2 < \infty.$$

The Malliavin derivative operator D is defined from $\mathbf{D}^{1,2}$ to $L^2(\Omega \times [0,1])$ by $DX = 0$ if $X = \mathbf{E}X$ is non-random, and otherwise, for all $r \in [0,1]$, by

$$D_r X = \sum_{n=1}^{\infty} n I_{n-1}(f_n(r, \cdot)).$$

This can be understood as a Fréchet derivative of X with respect to the Wiener process W. If $X = W(f)$ then $DX = f$. Of note is the chain-rule formula $D(F(X)) = F'(X) DX$ for any differentiable F with bounded derivative, and any $X \in \mathbf{D}^{1,2}$.

Definition 3.2. The generator of the Ornstein–Uhlenbeck semigroup L is defined as follows. Let $X = \sum_{n=1}^{\infty} I_n(f_n)$ be a centered r.v. in $L^2(\Omega)$. If $\sum_{n=1}^{\infty} n^2 n! |f_n|^2 < \infty$, then we define a new random variable LX in $L^2(\Omega)$ by $-LX = \sum_{n=1}^{\infty} n I_n(f_n)$. The pseudo-inverse of L operating on centered r.v.'s in $L^2(\Omega)$ is defined by the formula $-L^{-1}X = \sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n)$. If X is not centered, we define its image by L and L^{-1} by applying them to $X - \mathbf{E}X$.

As explained in the introduction, for $X \in \mathbf{D}^{1,2}$, the random variable $G := \left\langle DX; -DL^{-1}X \right\rangle_H$ plays a crucial role to understand how X's law compares to that of our reference random variable Z. The next lemma is the key to combining the solutions of Stein's equations with the Malliavin calculus. Its use to prove our main theorems relies heavily on the fact that these solutions have bounded derivatives.

Lemma 3.3 (Theorem 3.1 in [8], Lemma 3.5 in [22]). *Let $X \in \mathbf{D}^{1,2}$ be a centered random variable with a density, and $G = \langle DX; -DL^{-1}X \rangle_H$. For any deterministic, continuous and piecewise differentiable function m such that m' is bounded,*

$$\mathbf{E}[Xm(X)] = \mathbf{E}[m'(X)G].$$

3.2. General tail results

The main theoretical results of this paper compare the tails of any two random variables X and Z, as we now state in the next two theorems. In terms of their usage, Z represents a reference random variable in these theorems; this can be seen from the fact that we have a better control in the theorems' assumption on the g_* coming from Z than on the law of X. Also, we will apply these theorems to a Pearson random variable Z in the next section, while there will be no restriction on $X \in \mathbf{D}^{1,2}$ beyond the assumption of the theorems in the present section. We will see that all assumptions on Z in this section are satisfied when Z is a Pearson random variable.

Theorem 3.4. *Let Z be a centered random variable with a twice differentiable density over its support (a, b). Let g_* and Q be defined as in (1.4) and (1.5), respectively. Suppose that $g_*''(x) < 2$, and $\frac{x - g_*'(x)}{Q(x)}$ has a finite limit as $x \to a$ and $x \to b$. Let $X \in \mathbf{D}^{1,2}$ be a centered random variable with a density, and whose support (a, b_X) contains (a, b). Let G be as in (1.2). If $G \geq g_*(X)$ a.s., then for every $z \in (0, b)$,*

$$\mathbf{P}[X > z] \geq \Phi_*(z) - \frac{1}{Q(z)} \int_z^b (2x - z)\mathbf{P}[X > x]\, dx.$$

Proof. Taking expectations in *Stein's equation* (2.3), i.e., referring to (2.4), we have

$$\mathbf{P}[X \leq z] - \mathbf{P}[Z \leq z] = \mathbf{E}[g_*(X) f'(X) - X f(X)]$$

which is equivalent to

$$\mathbf{P}[X > z] - \Phi_*(z) = \mathbf{E}[X f(X) - g_*(X) f'(X)].$$

Since $g_*(X) \geq 0$ almost surely and $f'(x) \leq 0$ if $x > z$,

$$\begin{aligned}
&\mathbf{P}[X > z] - \Phi_*(z) \\
&\quad = \mathbf{E}[\mathbf{1}_{X \leq z} X f(X)] + \mathbf{E}[\mathbf{1}_{X > z} X f(X)] - \mathbf{E}[\mathbf{1}_{X \leq z} g_*(X) f'(X)] \\
&\qquad - \mathbf{E}[\mathbf{1}_{X > z} g_*(X) f'(X)] \\
&\quad \geq \mathbf{E}[\mathbf{1}_{X \leq z} X f(X)] + \mathbf{E}[\mathbf{1}_{X > z} X f(X)] - \mathbf{E}[\mathbf{1}_{X \leq z} g_*(X) f'(X)].
\end{aligned}$$

Let $m(x) = [f(a) - f(z)]\mathbf{1}_{x \leq a} + [f(x) - f(z)]\mathbf{1}_{a < x \leq z}$ where the first term is 0 if $a = -\infty$. Note that m is continuous and piecewise differentiable. The derivative is $m'(x) = f'(x)\mathbf{1}_{a < x \leq z}$ except at $x = a$ and $x = z$. We saw in Lemma 2.5 that f' is bounded. Therefore, since $X \in \mathbf{D}^{1,2}$, we can use Lemma 3.3 to conclude that

$$[f(a) - f(z)]\mathbf{E}[\mathbf{1}_{X \leq a} X] + \mathbf{E}[\mathbf{1}_{a < X \leq z} X(f(X) - f(z))] = \mathbf{E}[\mathbf{1}_{a < X \leq z} f'(X) G]$$

from which we derive

$$\mathbf{E}[\mathbf{1}_{X \leq z} X f(X)] - f(z)\mathbf{E}[\mathbf{1}_{X \leq z} X] = \mathbf{E}[\mathbf{1}_{X \leq z} f'(X) G].$$

Therefore,

$$\begin{aligned}
&\mathbf{P}[X > z] - \Phi_*(z) \\
&\quad \geq \{\mathbf{E}[\mathbf{1}_{X\leq z} f'(X)G] + f(z)\mathbf{E}[\mathbf{1}_{X\leq z}X]\} + \mathbf{E}[\mathbf{1}_{X>z}Xf(X)] \\
&\qquad - \mathbf{E}[\mathbf{1}_{X\leq z} g_*(X) f'(X)] \\
&\quad = \mathbf{E}[\mathbf{1}_{X\leq z} f'(X)(G - g_*(X))] + f(z)\mathbf{E}[\mathbf{1}_{X\leq z}X] + \mathbf{E}[\mathbf{1}_{X>z}Xf(X)] \\
&\quad \geq f(z)\mathbf{E}[\mathbf{1}_{X\leq z}X] + \mathbf{E}[\mathbf{1}_{X>z}Xf(X)] \\
&\quad = f(z)\mathbf{E}[\mathbf{1}_{X\leq z}X] + \mathbf{E}[\mathbf{1}_{X>z}Xf(X)] - f(z)\mathbf{E}[\mathbf{1}_{X>z}X] + f(z)\mathbf{E}[\mathbf{1}_{X>z}X] \\
&\quad = f(z)\mathbf{E}[X] + \mathbf{E}[\mathbf{1}_{X>z}X(f(X) - f(z))] \\
&\quad = \mathbf{E}[\mathbf{1}_{X>z}X(f(X) - f(z))]\,.
\end{aligned}$$

Write $f(X) - f(z) = f'(\xi)(X - z)$ for some random $\xi > z$ ($X > \xi$ also). Note that $f'(\xi) < 0$ since $\xi > z$. We have $\mathbf{P}[X > z] - \Phi_*(z) \geq \mathbf{E}[\mathbf{1}_{X>z} f'(\xi)X(X - z)]$. From Lemma 2.5,

$$f'(\xi) \geq -\frac{1}{Q(z)}$$

since from Lemma 2.6, Q is nondecreasing on $(0, b)$.

If we define $S(z) := \mathbf{P}[X > z]$, it is elementary to show (see [22]) that

$$\mathbf{E}[\mathbf{1}_{X>z}X(X - z)] \leq \int_z^b (2x - z)S(x)\,dx.$$

From $\mathbf{P}[X > z] - \Phi_*(z) \geq \mathbf{E}[\mathbf{1}_{X>z} f'(\xi)X(X - z)]$,

$$\mathbf{P}[X > z] \geq \Phi_*(z) - \frac{1}{Q(z)} \int_z^b (2x - z)S(x)\,dx$$

which is the statement of the theorem.

Lastly the reader will check that the assumption that the supports of Z and X have the same left-endpoint is not a restriction: stated briefly, this assumption is implied by the assumption $G \geq g_*(X)$ a.s., because $G = g_X(X)$ and g_* (resp. g_X) has the same support as Z (resp. X). □

To obtain a similar upper bound result, we will consider only asymptotic statements for z near b, and we will need an assumption about the relative growth rate of g_* and Q near b. We will see in the next section that this assumption is satisfied for all members of the Pearson class with four moments, although that section also contains a modification of the proof below which is more efficient when applied to the Pearson class.

Theorem 3.5. *Assume all the conditions of Theorem* 3.4 *hold, except for the support of X, which we now assume is contained in (a, b). Assume moreover that there exists $c < 1$ such that $\limsup_{z\to b} g_*(z)/Q(z) < c$. If $G \leq g_*(X)$ a.s., then there exists z_0 such that $b > z > z_0$ implies*

$$\mathbf{P}[X > z] \leq \frac{1}{1 - c}\Phi_*(z).$$

Proof. From Stein's equation (2.3), and its application (2.4),

$$\mathbf{P}[X > z] - \Phi_*(z) = \mathbf{E}[Xf(X) - g_*(X)f'(X)].$$

Since $X \in \mathbf{D}^{1,2}$, in Lemma 3.3, we can let $m = f$ since f is continuous, differentiable everywhere except at $x = a$ and $x = b$, and from Lemma 2.5 has a bounded derivative. Therefore,

$$\begin{aligned}
&\mathbf{P}[X > z] - \Phi_*(z) \\
&\qquad = \mathbf{E}[Gf'(X)] - \mathbf{E}[g_*(X)f'(X)] \\
&\qquad = \mathbf{E}[f'(X)\,(G - g_*(X))] \\
&\qquad = \mathbf{E}[\mathbf{1}_{X\le z}f'(X)\,(G - g_*(X))] + \mathbf{E}[\mathbf{1}_{X>z}f'(X)\,(G - g_*(X))] \\
&\qquad \le \mathbf{E}[\mathbf{1}_{X>z}f'(X)\,(G - g_*(X))] \\
&\qquad = \mathbf{E}[\mathbf{1}_{X>z}f'(X)\mathbf{E}\,[G|X]] - \mathbf{E}[\mathbf{1}_{X>z}f'(X)g_*(X)]
\end{aligned}$$

where the last inequality follows from the assumption $G - g_*(X) \le 0$ a.s. and if $X \le z$, then $f'(X) \ge 0$. By Proposition 3.9 in [8], $\mathbf{E}\,[G|X] \ge 0$ a.s. Since $f'(X) \le 0$ if $X > z$, then by the last statement in Lemma 2.5, and the assumption on the asymptotic behavior of g_*/Q, for z large enough,

$$\begin{aligned}
\mathbf{P}[X > z] - \Phi_*(z) &\le -\mathbf{E}[\mathbf{1}_{X>z}f'(X)g_*(X)] \\
&\le \mathbf{E}\left[\mathbf{1}_{X>z}\frac{g_*(X)}{Q\,(X)}\right] \\
&\le c\mathbf{P}[X > z].
\end{aligned}$$

The theorem immediately follows. □

4. Pearson distributions

By definition, the law of a random variable Z is a member of the Pearson family of distributions if Z's density ρ_* is characterized by the differential equation $\rho_*'(z)/\rho_*(z) = (a_1 z + a_0)/(\alpha z^2 + \beta z + \gamma)$ for z in its support (a, b), where $-\infty \le a < b \le \infty$. If furthermore $\mathbf{E}[Z] = 0$, Stein (Theorem 1, p. 65 in [21]) proved that g_* has a simple form: in fact, it is quadratic in its support. Specifically, $g_*(z) = \alpha z^2 + \beta z + \gamma$ for all $z \in (a, b)$ if and only if

$$\frac{\rho_*'(z)}{\rho_*(z)} = -\frac{(2\alpha + 1)z + \beta}{\alpha z^2 + \beta z + \gamma}. \tag{4.1}$$

The Appendix contains a description of various cases of Pearson distributions, which are characterized by their first four moments, if they exist. In this section, we will operate under the following.

Assumption P1. Our Pearson random variable satisfies $\mathbf{E}\left[Z^2\right] < \infty$ and $z^3\rho_*(z) \to 0$ as $z \to a$ and $z \to b$.

Remark 4.1. This assumption holds as soon as $\mathbf{E}\left[Z^3\right] < \infty$, which, by Lemma A.2 in the Appendix, holds if and only if $\alpha < 1/2$. The existence of a second moment, by the same lemma, holds if and only if $\alpha < 1$.

4.1. General comparisons to Pearson tails

In preparation to stating corollaries to our main theorems, applicable to all Pearson Z's simultaneously, we begin by investigating the specific properties of g_* and Q in the Pearson case. Because $g_*(z) = \left(\alpha z^2 + \beta z + \gamma\right) \mathbf{1}_{(a,b)}(z)$, we have the following observations:

- Since $g_*''(z) = 2\alpha$ on (a, b), and $\alpha < 1$ according to Remark 4.1, it follows that $g_*''(z) < 2$.
- If $z \in (a, b)$, then

$$Q(z) = z^2 - zg_*'(z) + g_*(z) = (1-\alpha)\, z^2 + \gamma$$

  and so $Q(z) \geq Q(0) = \gamma = g_*(0) > 0$, where the last inequality is because g_* is strictly positive on the interior of its support, which always contains 0. This is a quantitative confirmation of an observation made earlier about the positivity of Q in the general case.
- As $z \to a$ and $z \to b$,

$$\frac{z - g_*'(z)}{Q(z)} = \frac{(1-2\alpha)\, z - \beta}{(1-\alpha)\, z^2 + \gamma}$$

  approaches a finite number in case a and b are finite. As $|z| \to \infty$, the above ratio approaches 0.
- We have $\mathbf{E}\left[Z^2\right] = \frac{\gamma}{1-\alpha}$. Again, this is consistent with $\gamma > 0$ and $\alpha < 1$.

Remark 4.2. The above observations collectively mean that all the assumptions of Theorem 3.4 are satisfied for our Pearson random variable Z, so we can state the following.

Proposition 4.3. *Let Z be a centered Pearson random variable satisfying Assumption* P1. *Let g_* be defined as in* (1.4). *Let $X \in \mathbf{D}^{1,2}$ be a centered random variable with a density, and whose support (a, b_X) contains (a, b). Suppose that $G \geq g_*(X)$ a.s. Then for every $z \in (0, b)$,*

$$\mathbf{P}[X > z] \geq \Phi_*(z) - \frac{1}{(1-\alpha)\, z^2 + \gamma} \int_z^b (2x - z)\mathbf{P}[X > x]\, dx. \tag{4.2}$$

We have a quantitatively precise statement on the relation between $\operatorname{Var}[X]$ and the Pearson parameters.

Proposition 4.4.

1. *Assume that the conditions of Proposition* 4.3 *hold, particularly that $G \geq g_*(X)$; assume the support (a, b) of g_* coincides with the support of X. Then*

$$\operatorname{Var}[X] \geq \frac{\gamma}{1-\alpha} = \operatorname{Var}[Z].$$

2. *If we assume instead that $G \leq g_*(X)$ a.s., then the inequality above is reversed.*

Proof. Since X has a density, we can apply Lemma 3.3 and let $m(x) = x$.

$$\begin{aligned} \mathrm{Var}[X] &= \mathbf{E}[Xm(X)] = \mathbf{E}[G] \geq \mathbf{E}[g_*(X)] \\ &= \mathbf{E}[\mathbf{1}_{a<X<b}\left(\alpha X^2 + \beta X + \gamma\right)] = \alpha \mathbf{E}[X^2] + \beta \mathbf{E}[X] + \gamma \\ (1-\alpha)\,\mathrm{Var}[X] &\geq \beta \cdot 0 + \gamma \end{aligned}$$

This proves point 1. Point 2 is done identically. □

In order to formulate results that are specifically tailored to tail estimates, we now make the following

Assumption P2. The right-hand endpoint of our Pearson distribution's support is $b = +\infty$

Remark 4.5. Assumption P2 leaves out Case 3 in the Appendix in our list of Pearson random variables, i.e., the case of Beta distributions. Therefore, inspecting the parameter values in the other 4 Pearson cases, we see that Assumption P2 implies $\alpha \geq 0$, and also implies that if $\alpha = 0$, then $\beta \geq 0$.

Remark 4.6. In most of the results to follow, we will assume moreover that $\alpha < \frac{1}{2}$. By Lemma A.2, this is equivalent to requiring $\mathbf{E}\left[|Z|^3\right] < \infty$, and more generally from the lemma, our Pearson distribution has moment of order m if and only if $\alpha < 1/(m-1)$. As mentioned, $\alpha < \frac{1}{2}$ thus implies Assumption P1. Consequently Theorem 3.5 implies the following.

Corollary 4.7. *Let Z be a centered Pearson random variable satisfying Assumption* P2 (*support of Z is* $(a, +\infty)$). *Assume $\alpha < 1/2$. Let g_* be defined as in* (1.4). *Let $X \in \mathbf{D}^{1,2}$ be a centered random variable with a density and support contained in $(a, +\infty)$. If $G \leq g_*(X)$ a.s., for any $K > \frac{1-\alpha}{1-2\alpha}$, there exists z_0 such that if $z > z_0$, then*

$$\mathbf{P}[X > z] \leq K\ \Phi_*(z).$$

Proof. Since

$$\frac{g_*(z)}{Q(z)} = \frac{\alpha z^2 + \beta z + \gamma}{(1-\alpha) z^2 + \gamma}$$

then $\limsup_{z\to\infty} g_*(z)/Q(z) = \alpha/(1-\alpha) < 1$ if and only if $\alpha < \frac{1}{2}$. Therefore, Theorem 3.5 applies in this case, and with the c defined in that theorem, we may take here any $c > \alpha/(1-\alpha)$, so that we may take any $K = 1/(1-c)$ as announced. □

The drawback of our general lower bound theorems so far is that their statements are somewhat implicit. Our next effort is to fix this problem in the specific case of a Pearson Z: we strengthen Proposition 4.3 so that the tail $\mathbf{P}[X > z]$ only appears in the left-hand side of the lower bound inequality, making the bound explicit. The cost for this is an additional regularity and integrability assumption, whose scope we also discuss.

Corollary 4.8. *Assume that the conditions of Proposition* 4.3 *hold; in particular, assume* $X \in \mathbf{D}^{1,2}$ *and* $G \geq \alpha X^2 + \beta X + \gamma$ *a.s. In addition, assume there exists a constant* $c > 2$ *such that* $\mathbf{P}[X > z] \leq z\rho(z)/c$ *holds for large* z *(where* ρ *is the density of* X*). Then for large* z*,*

$$\mathbf{P}[X > z] \geq \frac{(c-2)\,Q(z)}{(c-2)\,Q(z) + 2z^2}\Phi_*(z) \approx \frac{(c-2) - \alpha\,(c-2)}{c - \alpha\,(c-2)}\Phi_*(z).$$

The existence of such a $c > 2$ *above is guaranteed if we assume* $g(z) \leq z^2/c$ *for large* z*, where* $g(x) := \mathbf{E}[G|X = x]$ *(or equivalently,* g *defined in* (1.3)*). Moreover, this holds automatically if* $G \leq \bar{g}_*(X)$ *a.s. for some quadratic function* $\bar{g}_*(x) = \bar{\alpha}x^2 + \bar{\beta}x + \bar{\gamma}$ *with* $\bar{\alpha} < 1/2$*.*

Proof. Since $z > 0$, we can replace $2x - z$ by $2x$ in the integral of (4.2).

$$\begin{aligned}
F(z) &:= \int_z^\infty xS(x)\,dx \leq \frac{1}{c}\int_z^\infty x^2|S'(x)|\,dx \\
&= \frac{1}{c}\left(z^2S(z) - \lim_{x\to\infty} x^2S(x) + 2\int_z^\infty xS(x)\,dx\right) \leq \frac{1}{c}\left(z^2S(z) + 2F(z)\right) \\
F(z) &\leq \frac{1}{c-2}z^2S(z)\,.
\end{aligned}$$

Therefore

$$S(z) = \mathbf{P}[X > z] \geq \Phi_*(z) - \frac{2}{Q(z)}F(z) \geq \Phi_*(z) - \frac{2z^2}{(c-2)\,Q(z)}S(z)$$

$$S(z)\left[1 + \frac{2z^2}{(c-2)\,Q(z)}\right] \geq \Phi_*(z)$$

$$\begin{aligned}
S(z) &\geq \frac{(c-2)\,Q(z)}{(c-2)\,Q(z) + 2z^2}\Phi_*(z) \\
&\approx \frac{(c-2)\,(1-\alpha)}{(c-2)\,(1-\alpha) + 2}\Phi_*(z).
\end{aligned}$$

This proves the inequality of the corollary.

To prove the second statement, recall that Nourdin and Viens (Theorem 3.1 in [16]) showed that

$$g(X) = \frac{\int_X^\infty x\rho(x)\,dx}{\rho(X)}$$

P-a.s. They also noted that the support of ρ is an interval since $X \in \mathbf{D}^{1,2}$. Therefore,

$$\frac{z}{c}\rho(z) \geq \frac{1}{z}g(z)\rho(z) = \int_z^\infty \frac{x}{z}\rho(x)\,dx \geq \int_z^\infty \rho(x)\,dx$$

a.s. This finishes the proof of the corollary. □

4.2. Comparisons in specific scales

In this section and the next, we will always assume $X \in \mathbf{D}^{1,2}$ is a centered random variable with a density and with support (a, ∞), and we will continue to denote by g the function defined by $g(x) := \mathbf{E}[G|X = x]$, or equivalently, defined in (1.3).

We can exploit the specific asymptotic behavior of the tail of the various Pearson distributions, via Lemma A.1 in the Appendix, to draw sharp conclusions about X's tail. For instance, if g is comparable to a Pearson distribution's g_* with $\alpha \neq 0$, we get a power decay for the tail (Corollary 4.9 below), while if α is zero and β is not, we get comparisons to exponential-type or gamma-type tails (Corollary 4.10 below). In both cases, when upper and lower bounds on G occur with the same α on both sides, we get sharp asymptotics for X's tail, up to multiplicative constants.

Corollary 4.9. *Let $g_*(x) := \alpha x^2 + \beta x + \gamma$ and $\bar{g}_*(x) := \bar{\alpha} x^2 + \bar{\beta} x + \bar{\gamma}$ be two functions corresponding to Pearson distributions (e.g., via (1.4)) where $0 < \alpha \leq \bar{\alpha} < 1/2$.*

1. *If $g(x) \leq \bar{g}_*(x)$ for all $x \geq a$, then there is a constant $c_u\left(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\right) > 0$ such that for large z,*
$$\mathbf{P}[X > z] \leq \frac{c_u}{z^{1+1/\bar{\alpha}}}.$$
2. *If $g_*(x) \leq g(x) \leq \bar{g}_*(x)$ for all $x \geq a$, then there are constants $c_u\left(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\right) > 0$ and $c_l(\bar{\alpha}, \alpha, \beta, \gamma) > 0$ such that for large z,*
$$\frac{c_l}{z^{1+1/\alpha}} \leq \mathbf{P}[X > z] \leq \frac{c_u}{z^{1+1/\bar{\alpha}}}.$$

Proof. Let $\Phi_{*\alpha,\beta,\gamma}$ and $\Phi_{*\bar{\alpha},\bar{\beta},\bar{\gamma}}$ be the probability tails of the Pearson distributions corresponding to g_* and $\bar{g}_*$ respectively. We can prove Point 1 by using Corollary 4.7 and Lemma A.1. There is a constant $k_u\left(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\right) > 0$ such that, for any $K > \frac{1-\bar{\alpha}}{1-2\bar{\alpha}}$, for large z,
$$\mathbf{P}[X > z] \leq K \Phi_{*\bar{\alpha},\bar{\beta},\bar{\gamma}}(z) \leq K \cdot \frac{k_u}{z^{1+1/\bar{\alpha}}}.$$
The upper bound in Point 2 follows directly from Point 1 because of the condition $g(x) \leq \bar{g}_*(x)$. This same condition also allows us to give a lower bound for $\mathbf{P}[X > z]$. Fix any $c \in (2, 1/\bar{\alpha})$. By Corollary 4.8 and Lemma A.1, there is a constant $k_l(\alpha, \beta, \gamma) > 0$ such that for large z,
$$\mathbf{P}[X > z] \geq \frac{(c-2) - \alpha(c-2)}{c - \alpha(c-2)} \Phi_{*\alpha,\beta,\gamma}(z) \geq \frac{(c-2) - \alpha(c-2)}{c - \alpha(c-2)} \cdot \frac{k_l}{z^{1+1/\alpha}}. \qquad \square$$

Corollary 4.10. *Let $g_*(x) := (\beta x + \gamma)_+$ and $\bar{g}_*(x) = \left(\bar{\beta} x + \bar{\gamma}\right)_+$ be two functions corresponding to Pearson distributions (e.g., via (1.4)) where $\beta, \bar{\beta}, \gamma, \bar{\gamma} > 0$ and $a = -\gamma/\beta$.*

1. *If $g(x) \leq \bar{g}_*(x)$ for all x, then there is a constant $c_u\left(\bar{\beta}, \bar{\gamma}\right) > 0$ such that for large z,*
$$\mathbf{P}[X > z] \leq c_u z^{-1+\bar{\gamma}/\bar{\beta}^2} e^{-z/\bar{\beta}}.$$

2. *If* $g_*(x) \leq g(x) \leq \bar{g}_*(x)$ *for all* x*, then there are constants* $c_u\left(\bar{\beta},\bar{\gamma}\right) > c_l(\beta,\gamma) > 0$ *such that for large* z*,*

$$c_l\ z^{-1+\gamma/\beta^2} e^{-z/\beta} \leq \mathbf{P}[X > z] \leq c_u z^{-1+\bar{\gamma}/\bar{\beta}^2} e^{-z/\bar{\beta}}.$$

Proof. Let $\Phi_{*\beta,\gamma}$ and $\Phi_{*\bar{\beta},\bar{\gamma}}$ be as in the proof of the previous corollary, noting here that $\alpha = \bar{\alpha} = 0$. The proof of Point 1 is similar to the proof of Point 1 in Corollary 4.9. The upper bound in Point 2 follows from Point 1 above and Point 1 of Corollary 4.9. For the lower bound of Point 2, if we fix any $c > 2$, then by Corollary 4.8 and Lemma A.1, there is a constant $k_l(\beta,\gamma) > 0$ such that for large z,

$$\mathbf{P}[X > z] \geq \frac{c-2}{c}\Phi_{*\beta,\gamma} \geq \frac{c-2}{c}\ k_l\ z^{-1+\gamma/\beta^2} e^{-z/\beta}. \qquad \square$$

Remark 4.11. The above corollary improves on a recently published estimate: in [16, Theorem 4.1], it was proved that if the law of $X \in \mathbf{D}^{1,2}$ has a density and if $g(X) \leq \beta X + \gamma$ a.s. (with $\beta \geq 0$ and $\gamma > 0$), then for all $z > 0$, $\mathbf{P}[X > z] \leq \exp\left(-\frac{z^2}{2\beta z + 2\gamma}\right)$. Using $g_*(z) = (\beta z + \gamma)_+$, Point 1 in Corollary 4.10 gives us an asymptotically better upper bound, with exponential rate $e^{-z/\beta}$ instead of $e^{-z/2\beta}$. Our rate is sharp, since our upper bound has the same exponential asymptotics as the corresponding Pearson tail, which is a Gamma tail.

4.3. Asymptotic results

Point 2 of Corollary 4.10 shows the precise behavior, up to a possibly different leading power term which is negligible compared to the exponential, of any random variable in $\mathbf{D}^{1,2}$ whose function g is equal to a Pearson function up to some uncertainty on the γ value. More generally, one can ask about tail asymptotics for X when g is asymptotically linear, or even asymptotically quadratic. Asymptotic assumptions on g are not as strong as assuming bounds on g which are uniform in the support of X, and one cannot expect them to imply statements that are as strong as in the previous subsection. We now see that in order to prove tail asymptotics under asymptotic assumptions, it seems preferable to revert to the techniques developed in [22]. We first propose upper bound results for tail asymptotics, which follow from Point 1 of Corollary 4.9 and Point 1 of Corollary 4.10. Then for full asymptotics, Point 2 of each of these corollaries do not seem to be sufficient, while [22, Corollary 4.5] can be applied immediately. Recall that in what follows $X \in \mathbf{D}^{1,2}$ is centered, has a density, and support (a,∞), and g is defined by $g(x) := \mathbf{E}[G|X=x]$, or equivalently, by (1.3).

Proposition 4.12.

1. *Suppose* $\limsup_{z\to+\infty} g(z)/z^2 = \alpha \in (0,1/2)$*. Then*

$$\limsup_{z\to+\infty} \frac{\ln \mathbf{P}[X > z]}{\ln z} \leq -\left(1+\frac{1}{\alpha}\right).$$

2. *Suppose* $\limsup_{z\to+\infty} g(z)/z = \beta > 0$. *Then*

$$\limsup_{z\to+\infty} \frac{\ln \mathbf{P}[X>z]}{z} \le -\frac{1}{\beta}.$$

Proof. Fix $\varepsilon \in (0, 1/2-\alpha)$. Then $g(x) < (\alpha+\varepsilon)\, x^2$ if x is large enough. Therefore, there exists a constant $\gamma_\varepsilon > 0$ such that $g(x) < (\alpha+\varepsilon)\, x^2 + \gamma_\varepsilon$ for all x. Let Z_ε be the Pearson random variable for which $g_*(z) = (\alpha+\varepsilon)\, z^2 + \gamma_\varepsilon$. This falls under Case 5 in Appendix A.2, so its support is $(-\infty,\infty)$, which then contains the support of X. From Point 1 of Corollary 4.9, there is a constant c_ε depending on ε such that for z large enough,

$$\mathbf{P}[X>z] \le c_\varepsilon z^{-1-\frac{1}{\alpha+\varepsilon}}.$$

We then have

$$\begin{aligned}
\ln \mathbf{P}[X>z] &\le \ln c_\varepsilon - \left(1+\frac{1}{\alpha+\varepsilon}\right)\ln z,\\
\frac{\ln \mathbf{P}[X>z]}{\ln z} &\le \frac{\ln c_\varepsilon}{\ln z} - \left(1+\frac{1}{\alpha+\varepsilon}\right),\\
\limsup_{z\to\infty}\frac{\ln \mathbf{P}[X>z]}{\ln z} &\le -\left(1+\frac{1}{\alpha+\varepsilon}\right).
\end{aligned}$$

Since ε can be arbitrarily close to 0, Point 1 of the corollary is proved. The proof of Point 2 is entirely similar, following from Corollary 4.10, which refers to Case 2 of the Pearson distributions given in Appendix A.2. This corollary could also be established by using results from [22]. □

Our final result gives full tail asymptotics. Note that it is not restricted to linear and quadratic behaviors.

Theorem 4.13.

1. *Suppose* $\lim_{z\to+\infty} g(z)/z^2 = \alpha \in (0,1)$. *Then*

$$\lim_{z\to+\infty} \frac{\ln \mathbf{P}[X>z]}{\ln z} \le -\left(1+\frac{1}{\alpha}\right).$$

2. *Suppose* $\lim_{z\to+\infty} g(z)/z^p = \beta > 0$ *for some* $p \in [0,1)$. *Then*

$$\lim_{z\to+\infty} \frac{\ln \mathbf{P}[X>z]}{z^{2-p}} \le -\frac{1}{\beta(2-p)}.$$

Proof. Since for any $\varepsilon \in (0, \min(\alpha, 1-\alpha))$, there exists z_0 such that $z > z_0$ implies $(\alpha-\varepsilon)\, z^2 \le g(z) \le (\alpha+\varepsilon)\, z^2$, the assumptions of Points 2 and 4 (a) in [22, Corollary 4.5] are satisfied, and Point 1 of the Theorem follows easily. Point 2 of the Theorem follows identically, by invoking Points 3 and 4 (b) in [22, Corollary 4.5]. All details are left to the reader. □

Appendix

A.1. Proofs of lemmas

Proof of Lemma 2.1. *Proof of point* 1. If $0 \le x < b$, then clearly $g_*(x) > 0$. If $a < x < 0$, we claim that $g_*(x) > 0$ still. Suppose we have the opposite: $g_*(x) \le 0$. Then $\int_x^b y\rho_*(y)\,dy = g_*(x)\rho_*(x) \le 0$. Since $\int_a^x y\rho_*(y)\,dy < 0$, then $\int_a^b y\rho_*(y)\,dy < 0$, contradicting $\mathbf{E}[Z] = 0$. Thus, $g_*(x) \ge 0$ for all x, and $g_*(x) > 0$ if and only if $a < x < b$.

Proof of point 2. Trivial.

Proof of point 3. This is immediate since

$$\lim_{x\downarrow a} g_*(x)\rho_*(x) = \lim_{x\downarrow a}\int_x^b y\rho_*(y)\,dy = \mathbf{E}[Z]$$

and similarly for $\lim_{x\uparrow b} g_*(x)\rho_*(x) = -\mathbf{E}[Z]$. □

Proof of Lemma 2.2. It is easy to verify that (2.6) and (2.7) are solutions to Stein's equation (2.3). To show that they are the same, let $\varphi(z) := g_*(z)\rho_*(z) = \int_z^b w\rho_*(w)\,dw$ for $z \in (a,b)$. Then

$$\frac{\varphi'(z)}{\varphi(z)} = -\frac{z\rho_*(z)}{g_*(z)\rho_*(z)} = -\frac{z}{g_*(z)}.$$

Integrating over $(y,x) \subseteq (a,b)$ leads to

$$\int_y^x \frac{z}{g_*(z)}\,dz = \log\frac{\varphi(y)}{\varphi(x)} = \log\frac{g_*(y)\rho_*(y)}{g_*(x)\rho_*(x)} \tag{A.1}$$

and so

$$\frac{e^{\int_y^x \frac{z\,dz}{g_*(z)}}}{g_*(y)} = \frac{1}{g_*(y)}\cdot\frac{g_*(y)\rho_*(y)}{g_*(x)\rho_*(x)} = \frac{\rho_*(y)}{g_*(x)\rho_*(x)}.$$

The derivative formula (2.9) comes via an immediate calculation

$$\begin{aligned} f'(x) &= -\frac{[g_*(x)\rho_*(x)]'}{[g_*(x)\rho_*(x)]^2}\int_a^x (h(y) - \mathbf{E}[h(Z)])\,\rho_*(y)\,dy \\ &\quad + \frac{(h(x) - \mathbf{E}[h(Z)])\,\rho_*(x)}{g_*(x)\rho_*(x)} \\ &= -\frac{-x\rho_*(x)}{[g_*(x)\rho_*(x)]^2}\int_a^x (h(y) - \mathbf{E}[h(Z)])\,\rho_*(y)\,dy + \frac{h(x) - \mathbf{E}[h(Z)]}{g_*(x)} \\ &= \frac{x}{[g_*(x)]^2\rho_*(x)}\int_a^x (h(y) - \mathbf{E}[h(Z)])\,\rho_*(y)\,dy + \frac{h(x) - \mathbf{E}[h(Z)]}{g_*(x)}. \end{aligned}$$ □

Proof of Lemma 2.3. From (A.1) in the previous proof, we have

$$\begin{aligned} \int_0^b \frac{z}{g_*(z)}\,dz &= \lim_{x\nearrow b}\int_0^x \frac{z}{g_*(z)}\,dz = \lim_{x\nearrow b}\log\frac{g_*(0)\rho_*(0)}{g_*(x)\rho_*(x)} \\ &= \log[g_*(0)\rho_*(0)] - \lim_{x\nearrow b}\log[g_*(x)\rho_*(x)] = \infty \end{aligned}$$

and

$$\int_a^0 \frac{z}{g_*(z)}\,dz = \lim_{x\searrow a}\int_x^0 \frac{z}{g_*(z)}\,dz = \lim_{x\searrow a}\log\frac{g_*(x)\rho_*(x)}{g_*(0)\rho_*(0)}$$
$$= \lim_{x\searrow a}\log\left[g_*(x)\rho_*(x)\right] - \log\left[g_*(0)\rho_*(0)\right] = -\infty. \qquad \square$$

Proof of Lemma 2.7. We prove (2.11) first. It is trivially true if $x \notin [a,b]$, so suppose $x \in (a,b)$. Let

$$m(x) := \Phi_*(x) - \frac{x - g_*'(x)}{Q(x)} \cdot g_*(x)\rho_*(x).$$

By a standard calculus proof, we will show that $m'(x) \le 0$ so that $m(x) \ge \lim_{y\to b} m(y)$. The result follows after observing that $\lim_{y\to b} m(y) = 0$. This is true since $\lim_{y\to b} g_*(y)\rho_*(y) = 0$ and $\lim_{y\to b}\Phi_*(x) = 0$. Now we show that $m'(x) \le 0$.

$$\begin{aligned}
m' &= -\rho_* - g_*\rho_*\left[\frac{x-g_*'}{x^2 - xg_*' + g_*}\right]' - \frac{x-g_*'}{Q}\left[g_*\rho_*\right]' \\
&= -\rho_* - g_*\rho_*\left[\frac{\left(x\left(x-g_*'\right)+g_*\right)\left(1-g_*''\right) - \left(x-g_*'\right)\left(2x - xg_*''\right)}{Q^2}\right] \\
&\quad - \frac{x-g_*'}{Q}\left[-x\rho_*\right] \\
\frac{Q^2}{\rho_*}m' &= -Q^2 - g_*\left[\left(x-g_*'\right)\left(x - xg_*'' - 2x + xg_*''\right) + g_*\left(1-g_*''\right)\right] \\
&\quad + Qx\left(x-g_*'\right) \\
&= \left[-x^2\left(x-g_*'\right)^2 - 2xg_*\left(x-g_*'\right) - g_*^2\right] + xg_*\left(x-g_*'\right) - g_*^2\left(1-g_*''\right) \\
&\quad + \left[x^2\left(x-g_*'\right)^2 + xg_*\left(x-g_*'\right)\right] \\
&= -g_*^2 - g_*^2\left(1-g_*''\right) = -g_*^2\left(2-g_*''\right) \le 0.
\end{aligned}$$

To prove (2.12) (again, it suffices to prove this for $x \in (a,b)$), let

$$n(x) := 1 - \Phi_*(x) - \frac{g_*'(x) - x}{Q(x)} \cdot g_*(x)\rho_*(x) = 1 - m(x)$$

so $n'(x) = -m'(x) \ge 0$. n is then nondecreasing so $n(x) \ge \lim_{x\to a} n(x) = 0$.

Now we prove (2.13). If $x > 0$,

$$\Phi_*(x) = \int_x^\infty \rho_*(y)\,dy \le \frac{1}{x}\int_x^\infty y\rho_*(y)\,dy = \frac{1}{x}\cdot g_*(x)\rho_*(x).$$

On the other hand, if $x < 0$,

$$1 - \Phi_*(x) = \int_{-\infty}^x \rho_*(y)\,dy \le \frac{1}{x}\int_{-\infty}^x y\rho_*(y)\,dy = -\frac{1}{x}\cdot g_*(x)\rho_*(x).$$

This proves (2.14). $\square$

Proof of last bullet point on page 71. We replicate here a method commonly used to find a recursive formula for the moments. See for example [7] and [19]. Cross-multiplying the terms in (4.1), multiplying by x^r and integrating over the support gives us

$$\int_a^b \left[(2\alpha+1)z^{r+1}+\beta z^r\right]\rho_*(z)\,dz = -\int_a^b \left(\alpha z^{r+2}+\beta z^{r+1}+\gamma z^r\right)\rho_*'(z)\,dz$$

and so

$$\begin{aligned} &-(2\alpha+1)\mathbf{E}\left[Z^{r+1}\right]-\beta\mathbf{E}\left[Z^r\right] \\ &\quad= \left(\alpha z^{r+2}+\beta z^{r+1}+\gamma z^r\right)\rho_*(z)\Big|_a^b \\ &\qquad -\int_a^b \left[\alpha(r+2)z^{r+1}+\beta(r+1)z^r+\gamma r z^{r-1}\right]\rho_*(z)\,dz \\ &\quad= \alpha(r+2)\mathbf{E}\left[Z^{r+1}\right]+\beta(r+1)\mathbf{E}\left[Z^r\right]+\gamma r\mathbf{E}\left[Z^{r-1}\right]\end{aligned}$$

where we assumed that $z^{r+2}\rho_*(z)\to 0$ at the endpoints a and b of the support. For the case $r=1$, this reduces to $z^3\rho_*(z)\to 0$ at the endpoints a and b, which we are assuming. Therefore,

$$(2\alpha+1)\mathbf{E}\left[Z^2\right]+\beta\mathbf{E}\left[Z\right] = 3\alpha\mathbf{E}\left[Z^2\right]+2\beta\mathbf{E}\left[Z\right]+\gamma\mathbf{E}\left[Z^0\right].$$

Since $\mathbf{E}\left[Z\right]=0$ and $\mathbf{E}\left[Z^0\right]=1$, this gives $\mathbf{E}\left[Z^2\right]=\frac{\gamma}{1-\alpha}$. □

A.2. Examples of Pearson distributions

We present cases of Pearson distributions depending on the degree and number of zeroes of $g_*(x)$ as a quadratic polynomial in (a,b). The Pearson family is closed under affine transformations of the random variable, so we can limit our focus on the five special cases below. The constant C in each case represents the normalization constant. See Diaconis and Zabell [5] for a discussion of these cases.

- Case 1. If $\deg g_*(z)=0$, ρ_* can be (after an affine transformation) written in the form $\rho_*(z)=Ce^{-z^2/2}$ for $-\infty<z<\infty$. This is the standard normal density, and $C=\frac{1}{\sqrt{2\pi}}$. For this case, $g_*(z)\equiv 1$. Consequently, $Q(z)=z^2+1$. If $z>0$, the inequalities (2.11) and (2.13) of Lemma 2.7 can be written

$$\frac{z}{(z^2+1)\sqrt{2\pi}}e^{-z^2/2}\le\Phi_*(z)\le\frac{1}{z\sqrt{2\pi}}e^{-z^2/2},$$

  a standard inequality involving the tail of the standard normal distribution.
- Case 2. If $\deg g_*(z)=1$, ρ_* can be written in the form $\rho_*(z)=Cz^{r-1}e^{-z/s}$ for $0<z<\infty$, with parameters $r,s>0$. This is a Gamma density, and $C=\frac{1}{s^r\Gamma(r)}$. It has mean $\mu=rs>0$ and variance rs^2. If one wants to make Z centered, the density takes the form $\rho_*(z)=C(z+\mu)^{r-1}e^{-(z+\mu)/s}$ for $-\mu<z<\infty$. For this case, $g_*(z)=s(z+\mu)_+$.

- Case 3. If $\deg g_*(x) = 2$ and g_* has two real roots, ρ_* can be written in the form $\rho_*(x) = Cx^{r-1}(1-x)^{s-1}$ for $0 < x < 1$, with parameters $r, s > 0$. This is a Beta density, and $C = \frac{1}{\beta(r,s)}$. It has mean $\mu = \frac{r}{r+s} > 0$ and variance $\frac{rs}{(r+s)^2(r+s+1)}$. Centering the density gives $\rho_*(x) = C(x+\mu)^{r-1}(1-x-\mu)^{s-1}$ for $-\mu < x < 1-\mu$. For this case, $g_*(x) = \frac{(x+\mu)(1-x-\mu)}{r+s}$ when $-\mu < x < 1-\mu$ and 0 elsewhere.
- Case 4. If $\deg g_*(x) = 2$ and g_* has exactly one real root, ρ_* can be written in the form $\rho_*(x) = Cx^{-r}e^{-s/x}$ for $0 < x < \infty$, with parameters $r > 1$ and $s \geq 0$. The normalization constant is $C = \frac{s^{r-1}}{\Gamma(r-1)}$. If $r > 2$, it has mean $\mu = \frac{s}{r-2} \geq 0$. If $r > 3$, it has variance $\frac{s^2\Gamma(r-3)}{\Gamma(r-1)}$. Centering this density yields $\rho_*(x) = C(x+\mu)^{-r}e^{-s/(x+\mu)}$ for $-\mu < x < \infty$. For this case, $g_*(x) = \frac{(x+\mu)^2}{r-2}$ when $-\mu < x$ and 0 elsewhere.
- Case 5. If $\deg g_*(x) = 2$ and g_* has no real roots, ρ_* can be written in the form $\rho_*(x) = C\left(1+x^2\right)^{-r} e^{s\arctan x}$ for $-\infty < x < \infty$, with parameters $r > 1/2$ and $-\infty < s < \infty$. The normalization constant is $C = \frac{\Gamma(r)}{\sqrt{\pi}\Gamma(r-1/2)}\left|\frac{\Gamma(r-is/2)}{\Gamma(r)}\right|^2$. If $r > 1$, it has mean $\mu = \frac{s}{2(r-1)}$. If $r > 3/2$, it has variance $\frac{4(r-1)^2+s^2}{4(r-1)^2(2r-3)}$. The centered form of the density is $\rho_*(x) = C\left[1+(x+\mu)^2\right]^{-r} e^{s\arctan(x+\mu)}$. For this case, $g_*(x) = \frac{1+(x+\mu)^2}{2(r-1)}$. Using our original notation, $\alpha = \frac{1}{2(r-1)}$, $\beta = \frac{\mu}{r-1}$ and $\gamma = \frac{\mu^2+1}{2(r-1)}$.

A.3. Other lemmas

Lemma A.1. *Let Z be a centered Pearson random variable. Then there exist constants $k_u > k_l > 0$ depending only on α, β, γ such that when z is large enough, we have the following inequalities.*

1. *If $\alpha = 0$ and $\beta > 0$, when z is large enough,*
$$\frac{k_l}{z^{1-\gamma/\beta^2}e^{z/\beta}} \leq \Phi_*(z) \leq \frac{k_u}{z^{1-\gamma/\beta^2}e^{z/\beta}}.$$
2. *If $\alpha > 0$, when z is large enough,*
$$\frac{k_l}{z^{1+1/\alpha}} \leq \Phi_*(z) \leq \frac{k_u}{z^{1+1/\alpha}}.$$
3. *Assuming Z's support extends to $-\infty$, if $\alpha > 0$, when $z < 0$ and $|z|$ is large enough,*
$$\frac{k_l}{|z|^{1+1/\alpha}} \leq 1-\Phi_*(z) \leq \frac{k_u}{|z|^{1+1/\alpha}}.$$

Proof. For the proof of this lemma, which is presumably well known, but is included for completeness, we will use C for the normalization constant of each density to be considered.

In Point 1, let $\mu = \gamma/\beta > 0$. Then Z has support $(-\mu, \infty)$; see Case 2 in Appendix A.2. In its support, Z has $g_*(z) = \beta z + \gamma = \beta(z+\mu)$ and density

$$\rho_*(z) = C(z+\mu)^{\mu/\beta - 1} \exp\left(-\frac{z+\mu}{\beta}\right).$$

Note that

$$\begin{aligned} &\lim_{z\to\infty} z^{-\mu/\beta} e^{z/\beta} g_*(z) \rho_*(z) \\ &\quad = C\beta \lim_{z\to\infty} \frac{(z+\mu)^{\mu/\beta}}{z^{\mu/\beta}} \exp\left(\frac{z}{\beta} - \frac{z+\mu}{\beta}\right) = C\beta e^{-\mu/\beta}. \end{aligned}$$

From Lemma 2.7,

$$\frac{z-\beta}{z^2+\gamma} g_*(z) \rho_*(z) \le \Phi_*(z) \le \frac{1}{z} g_*(z) \rho_*(z)$$

so

$$C\beta e^{-\mu/\beta} \le \liminf_{z\to\infty} z^{1-\mu/\beta} e^{z/\beta} \Phi_*(z) \le \limsup_{z\to\infty} z^{1-\mu/\beta} e^{z/\beta} \Phi_*(z) \le C\beta e^{-\mu/\beta}.$$

Therefore, we can choose some constants $k_u(\beta,\gamma) > k_l(\beta,\gamma) > 0$ such that when z is large enough,

$$\frac{k_l}{z^{1-\mu/\beta} e^{z/\beta}} \le \Phi_*(z) \le \frac{k_u}{z^{1-\mu/\beta} e^{z/\beta}}.$$

To prove Point 2, we first show that $\lim_{z\to\infty} z^{1/\alpha} g_*(z) \rho_*(z)$ is a finite number K. We consider the cases $4\alpha\gamma - \beta^2 = 0$ and $4\alpha\gamma - \beta^2 > 0$ separately. We need not consider $4\alpha\gamma - \beta^2 < 0$ since it corresponds to Case 3 in Appendix A.2 for which the right endpoint of the support of Z is $b < \infty$ and so necessarily $\alpha < 0$.

Supose that $4\alpha\gamma - \beta^2 = 0$ and let $\mu = \frac{\beta}{2\alpha} > 0$. Then $\alpha z^2 + \beta z + \gamma = \alpha(z+\mu)^2$ has one real root and the support of Z is $(-\mu, \infty)$; see Case 4 in Appendix A.2. In its support, Z has $g_*(z) = \alpha(z+\mu)^2$ and density

$$\rho_*(z) = C(z+\mu)^{-2-1/\alpha} \exp\left(-\frac{s}{z+\mu}\right)$$

where $s = \mu/\alpha = \beta/(2\alpha^2)$. Therefore,

$$\lim_{z\to\infty} z^{1/\alpha} g_*(z) \rho_*(z) = C\alpha \lim_{z\to\infty} \frac{z^{1/\alpha}}{(z+\mu)^{1/\alpha}} \exp\left(-\frac{s}{z+\mu}\right) = C\alpha.$$

Now suppose that $\delta^2 := (4\alpha\gamma - \beta^2)/(4\alpha^2) > 0$ so $\alpha z^2 + \beta z + \gamma$ has two imaginary roots and the support of Z is $(-\infty, \infty)$. Letting $\mu = \beta/(2\alpha)$ allows us to write $g_*(z) = \alpha(z+\mu)^2 + \alpha\delta^2$ and the density of Z as

$$\rho_*(z) = C\left[(z+\mu)^2 + \delta^2\right]^{-1-\frac{1}{2\alpha}} \exp\left[\frac{\mu}{\alpha\delta} \arctan\left(\frac{z+\mu}{\delta}\right)\right],$$

a slight variation of the density in Case 5 in Appendix A.2. Note that in our present case,

$$\begin{aligned}
&\lim_{z\to\infty} z^{1/\alpha} g_*(z)\rho_*(z) \\
&\quad = C\alpha \lim_{z\to\infty} \frac{z^{1/\alpha}}{\left[(z+\mu)^2+\delta^2\right]^{\frac{1}{2\alpha}}} \exp\left[\frac{\mu}{\alpha\delta}\arctan\left(\frac{z+\mu}{\delta}\right)\right] \\
&\quad = C\alpha \exp\left[\frac{\mu\pi}{2\alpha\delta}\right].
\end{aligned}$$

From Lemma 2.7,

$$\frac{(1-2\alpha)z-\beta}{(1-\alpha)z^2+\gamma} g_*(z)\rho_*(z) \le \Phi_*(z) \le \frac{1}{z} g_*(z)\rho_*(z).$$

From these bounds we conclude

$$K\frac{1-2\alpha}{1-\alpha} \le \liminf_{z\to\infty} z^{1+1/\alpha}\Phi_*(z) \le \limsup_{z\to\infty} z^{1+1/\alpha}\Phi_*(z) \le K.$$

Therefore, when z is large enough,

$$\frac{k_l}{z^{1+1/\alpha}} \le \Phi_*(z) \le \frac{k_u}{z^{1+1/\alpha}}$$

for some constants $k_u(\alpha,\beta,\gamma) > k_l(\alpha,\beta,\gamma) > 0$.

To prove Point 3, we consider Case 5 again.

$$\begin{aligned}
&\lim_{z\to-\infty} |z|^{1/\alpha} g_*(z)\rho_*(z) \\
&\quad = C\alpha \lim_{y\to\infty} \frac{y^{1/\alpha}}{\left[(-y+\mu)^2+\delta^2\right]^{\frac{1}{2\alpha}}} \exp\left[\frac{\mu}{\alpha\delta}\arctan\left(\frac{-y+\mu}{\delta}\right)\right] \\
&\quad = C\alpha \exp\left[-\frac{\mu\pi}{2\alpha\delta}\right].
\end{aligned}$$

The conclusion follows similarly after using Lemma 2.7 when $z<0$:

$$\frac{(1-2\alpha)|z|-\beta}{(1-\alpha)|z|^2+\gamma} g_*(z)\rho_*(z) \le 1-\Phi_*(z) \le \frac{1}{|z|} g_*(z)\rho_*(z).$$ □

Lemma A.2. *Let Z be a centered Pearson random variable. If $\alpha \le 0$, all moments of positive order exist. If $\alpha > 0$, the moment of order m exists if and only if $m < 1+1/\alpha$.*

Proof. The random variables in Case 1 ($\alpha=\beta=0$) of Appendix A.2 are normal, while those in Case 3 ($\alpha<0$) have finite intervals for support. It suffices to consider the cases where $\alpha=0$ and $\beta>0$, and where $\alpha>0$. Let $m>0$. We will use the fact that $\mathbf{E}\left[|Z|^m\right]<\infty$ if and only if $\sum_{n=1}^{\infty} n^{m-1}\mathbf{P}\left[|Z|\ge n\right]<\infty$.

If $\alpha = 0$ and $\beta > 0$, and Z is supported over (a, ∞), then by Lemma A.1, $\mathbf{E}\left[|Z|^m\right] < \infty$ if and only if

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{n^{1+\gamma/\beta^2} e^{n/\beta}} < \infty,$$

which is always the case.

Now suppose $\alpha > 0$. Since $\mathbf{P}\left[|Z| \geq n\right] = \Phi_* (n) + 1 - \Phi_* (-n)$, then by Lemma A.1 again, $\mathbf{E}\left[|Z|^m\right] < \infty$ if and only if

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{n^{1+1/\alpha}} = \sum_{n=1}^{\infty} \frac{1}{n^{2+1/\alpha-m}} < \infty.$$

This is the case if and only if $2 + 1/\alpha - m > 1$, i.e., $m < 1 + 1/\alpha$. □

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Uniqueness and Absolute Continuity for Semilinear SPDE's

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Abstract. Given a weak solution of a semilinear stochastic partial differential equation, sufficient conditions for its uniqueness in law are presented. Moreover we characterize this law and prove that it is absolutely continuous with respect to the law of the process solving the corresponding linear stochastic partial differential equation, obtained neglecting the nonlinear term. The conditions imposed involve a $\mathbb{P}$-a.s. assumption on the solution process. This allows to avoid a boundeness or linear growth condition on the nonlinear term. Finally, we prove the equivalence of the laws.

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1. Introduction

Let a stochastic partial differential equation (SPDE) be written in abstract form as an infinite-dimensional stochastic differential equation (see, e.g., [1, 2]). In particular we consider a semilinear SPDE in the form

$$\begin{cases} dX_t = AX_t\ dt + F(t, X_t)\ dt + B\ dW_t, & 0 < t \leq T, \\ X_0 = x. \end{cases} \tag{1.1}$$

We assume that there exists a weak solution of Equation (1.1). First, we ask when it is unique; moreover, we want to characterize its law. To this end, we look at Equation (1.1) as a perturbation of the linear SPDE

$$\begin{cases} dZ_t = AZ_t\ dt + B\ dW_t, & 0 < t \leq T, \\ Z_0 = x \end{cases} \tag{1.2}$$

by adding the drift term F. This linear equation, the so-called Ornstein–Uhlenbeck equation, under suitable assumptions has a unique solution, which is a Gaussian measure on some infinite-dimensional space. This measure is the natural measure to put in relationship with the law of the solution to Equation (1.1).

The basic tool used in our proofs is the Girsanov transform. We follow the lines of Liptser and Shiryaev [9], who used Girsanov theorem avoiding the Novikov's condition (see [10])

$$\mathbb{E}\Big[e^{\frac{1}{2}\int_0^T \|B^{-1}F(t,Z_t)\|^2 dt}\Big] < \infty$$

(the notations will be specified in the next section) or other $\mathbb{E}[e^{\cdots}]$-conditions (see, e.g., [6, 8]).

However, notice that [9] deals only with the finite-dimensional setting, whereas we work with stochastic differential equations in infinite-dimensional Hilbert spaces. Moreover, our setting is a bit different from that of [9], since we focus our attention on semilinear SPDE's and therefore our assumptions involve mainly the nonlinear term F which characterize the change of drift between Equations (1.2) and (1.1). Indeed, as it is usual in the infinite-dimensional setting, the basic starting point in the analysis of a stochastic equation is the Ornstein–Uhlenbeck Equation (1.2), and Equation (1.1) can be seen as a perturbation of Equation (1.2) by adding the nonlinear term F.

As far as the presentation of the results is concerned, we give the basic assumptions in Section 2. Then, Section 3 deals with the uniqueness in law for Equation (1.1), Section 4 with the absolute continuity of the law of (1.1) with respect to the law of (1.2) and Section 5 with the equivalence of these laws. This latter result is the same obtained by Kozlov [7], but we present it here since it is the easy consequence of our previous results of Sections 3 and 4.

2. Mathematical setting

For our framework we refer the reader to [1], where many examples are given.

Let H and U be separable Hilbert spaces. We denote by $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$ the norm and scalar product in H, respectively; similarly for U.

We are given

- $A : D(A) \subset H \to H$ a linear unbounded operator, which is the infinitesimal generator of a C_0 semigroup $\{S_t\}_{\{t\geq 0\}}$ in H
- $F : [0,T] \times H \to H$ measurable mapping
- $B : U \to H$ linear bounded operator

W denotes a cylindrical Wiener process in U, defined on a stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$; $\mathbb{E}$ is the mathematical expectation with respect to the measure $\mathbb{P}$; $\mathbb{F}_T(Z)$ denotes the σ-algebra generated by the process $\{Z_t\}_{0\leq t\leq T}$.

To any separable metric space, we associate by default the Borel σ-algebra.

Given two probability measures μ and ν on the same measurable space, we write $\mu \prec \nu$ if μ is absolutely continuous with respect to ν, and $\mu \sim \nu$ is they are mutually absolutely continuous, that is they are equivalent.

Let us consider the mild solution of the Ornstein–Uhlenbeck equation

$$Z_t = S_t x + \int_0^t S_{t-s} B \, dW_s, \qquad t \in [0,T]. \tag{2.1}$$

We assume that this is the unique strong solution of the stochastic Equation (1.2); on the other hand, for the semilinear Equation (1.1) we assume that there exists a weak solution in the mild form

$$X_t = S_t x + \int_0^t S_{t-s} F(s, X_s) \, ds + \int_0^t S_{t-s} B \, dW_s, \qquad t \in [0,T].$$

Here, *weak* and *strong* solutions denote the kind of solutions from the probabilistic point of view (see, e.g., [9]). In this paper we always consider a finite final time T and the solutions are the mild ones.

Remark 2.1. We denote by $B^\star$ the adjoint operator of B. The linear Equation (1.2) has a unique mild solution if and only if

$$Q_t := \int_0^t S_r B B^\star S_r^\star \, dr$$

is a trace class operator in H for any $t \in [0,T]$ (see [1, 2]). Moreover, if there exists $\alpha > 0$ such that

$$\int_0^T t^{-\alpha} Tr(S_t B B^\star S_t^\star) dt < \infty,$$

then the solution process has an H-continuous version (see [1, Ch. 5]).

We denote by B^{-1} the pseudoinverse operator of B (see, e.g., [1, Appendix B]) and when $\mathrm{Im}(F) \subseteq \mathrm{Im}(B)$ we set

$$\Phi = B^{-1} F.$$

Now we list our assumptions

[A0] $\mathrm{Im}(F) \subseteq \mathrm{Im}(B)$ and $\exists\ \Psi : [0,T] \times H \to U$ measurable such that

$$B\Psi(t,x) = F(t,x), \qquad \text{for } t \in [0,T],\ x \in H.$$

[A1] for any $x \in H$ there exists a unique strong solution Z to the linear Equation (1.2) with paths in $C([0,T];H)$ $\mathbb{P}$-a.s.

[A2] for any $x \in H$ there exists a weak solution X to the semilinear Equation (1.1) with paths in $C([0,T];H)$ $\mathbb{P}$-a.s.

[A3i] $\mathbb{P}\{\int_0^T \|\Phi(s, X_s)\|_U^2 ds < \infty\} = 1.$

[A3ii] $\mathbb{P}\{\int_0^T \|\Phi(s, Z_s)\|_U^2 ds < \infty\} = 1.$

Moreover, we assume $\{\Phi(t, X_t)\}_t, \{\Phi(t, Z_t)\}_t$ to be progressively measurable processes.

Remark 2.2. (i) **[A0]** contains a compatibility condition between B and F. For instance, let B be a diagonal operator in $H = U$, i.e., $Be_j = b_j e_j$ for all $j \in \mathbb{N}$ ($\{e_j\}_j$ is a complete orthonormal system in H); when $b_h = 0$ for some h it is necessary that $\langle F(t,x), e_h \rangle_H = 0$.

(ii) If B is invertible there is only one Ψ fulfilling **[A0]** and this Ψ coincides with Φ. Otherwise, when there are many functions Ψ such that $B\Psi = F$, we focus the attention on the expression Φ given by the pseudoinverse operator, as explained in [4, §8].

Conditions **[A0]**, **[A1]**, **[A2]**, **[A3i]** are always assumed from now on; condition **[A3ii]** will appear only in the last section.

For sake of completeness, we state Girsanov theorem (see, e.g., [5] or [1, Ch. 10]).

Theorem 2.3. *Assume that* $\{\Gamma(t)\}_{t\geq 0}$ *is a* U*-valued* $\mathbb{F}_t$*-predictable process such that*

$$\mathbb{E}\Big[e^{\int_0^T \langle \Gamma_s, dW_s\rangle_U - \frac{1}{2}\int_0^T \|\Gamma_s\|_U^2 ds}\Big] = 1.$$

Then the process

$$\tilde{W}_t = W_t - \int_0^t \Gamma_s ds, \qquad 0 \leq t \leq T$$

is a U*-cylindrical Wiener process with respect to* $\{\mathbb{F}_t\}_t$ *on the probability space* $(\Omega, \mathbb{F}_T, \tilde{\mathbb{P}})$*, where*

$$d\tilde{\mathbb{P}} = e^{\int_0^T \langle \Gamma_s, dW_s\rangle_U - \frac{1}{2}\int_0^T \|\Gamma_s\|_U^2 ds}\, d\mathbb{P}.$$

3. Uniqueness in law

Proposition 3.1. *We assume* **[A0]***,* **[A1]***. If there exist two weak solutions*

$$\big(X, (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big) \quad \text{and} \quad \big(X', (\Omega', \mathbb{F}', \{\mathbb{F}'_t\}, \mathbb{P}'),\ W'\big)$$

to Equation (1.1) *with the same initial data* $x \in H$ *and with a.e. path in* $C([0,T]; H)$*, such that*

$$\mathbb{P}\{\textstyle\int_0^T \|\Phi(s, X_s)\|_U^2 ds < \infty\} = \mathbb{P}'\{\int_0^T \|\Phi(s, X'_s)\|_U^2 ds < \infty\} = 1,$$

then the laws of X *and* X' *are the same.*

Proof. For each $n \in \mathbb{N}$, given $Y \in C([0,T]; H)$ we define a truncation coefficient

$$\chi_t^n(Y) = \text{ indicator function of the set } \{\textstyle\int_0^t \|\Phi(s, Y_s)\|_U^2 ds < n\}$$

for $0 \leq t \leq T$.

We want to prove uniqueness in law for the semilinear Equation (1.1). To this end, we introduce an auxiliary equation, putting in front of the nonlinear term F the truncation coefficient χ^n:

$$dY_t^n = AY_t^n\, dt + \chi_t^n(Y^n) F(t, Y_t^n)\, dt + B\, dW_t^n, \quad Y_0^n = x. \tag{3.1}$$

We study this new equation, starting from the Ornstein–Uhlenbeck Equation (1.2). We consider the weak solution $\big(X, (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ to Equation (1.1) and the strong solution Z to Equation (1.2) with this stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P})$ and this Wiener process W.

We use the Girsanov transform and obtain that there exists a unique weak solution of Equation (3.1). Indeed, for any n Novikov condition holds true:

$$\mathbb{E}\Big[e^{\frac{1}{2}\int_0^T \chi^n_s(Z)\|\Phi(s,Z_s)\|^2_U ds}\Big] \le e^{\frac{1}{2}n} < \infty.$$

Therefore, the process ρ^n defined by

$$\rho^n_t = e^{\int_0^t \chi^n_s(Z)\langle \Phi(s,Z_s), dW_s\rangle_U - \frac{1}{2}\int_0^t \chi^n_s(Z)\|\Phi(s,Z_s)\|^2_U ds}$$

$(0 \le t \le T)$ is a martingale (see [10]). To highlight the dependence on Z and W we will often write ρ^n as $\rho^n(Z,W)$.

Then the Girsanov theorem can be applied: we define a new probability measure on $(\Omega, \mathcal{F}_T)$ by $d\tilde{\mathbb{P}}^n = \rho^n_T(Z,W)d\mathbb{P}$ and

$$\tilde{W}^n_t = W_t - \int_0^t \chi^n_s(Z)\Phi(s,Z_s)\,ds\ , \qquad t \in [0,T],$$

is a U-cylindrical Wiener process with respect to $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \tilde{\mathbb{P}}^n)$. Substituting into Equation (2.1) we get

$$Z_t = S_t x + \int_0^t S_{t-s}B\ dW_s = S_t x + \int_0^t S_{t-s}B\ d\tilde{W}^n_s + \int_0^t S_{t-s}\chi^n_s(Z)F(s,Z_s)\ ds$$

This means that $\big(Z,\ (\Omega,\mathbb{F},\{\mathbb{F}_t\},\tilde{\mathbb{P}}^n),\ \tilde{W}^n\big)$ is a weak solution of Equation (3.1).

Moreover, the law of the process solving (3.1) is absolutely continuous with respect to the law of the process solving (1.2). Indeed, let $D^n : C([0,T];H) \to \mathbb{R}$ be a measurable non-negative function such that $D^n(Z(\omega)) = \mathbb{E}[\rho^n_T(Z,W)|\ \mathbb{F}_T(Z)](\omega)$ for $\mathbb{P}$-a.e. ω. For any Borelian subset Λ of $C([0,T];H)$,

$$\begin{aligned}\mathcal{L}_{Y^n}(\Lambda) = \tilde{\mathbb{P}}^n\{Z\in\Lambda\} &= \int_{\{Z\in\Lambda\}} \rho^n_T(Z,W)\ d\mathbb{P}\\ &= \int_{\{Z\in\Lambda\}} \mathbb{E}\big[\rho^n_T(Z,W)|\ \mathbb{F}_T(Z)\big]\ d\mathbb{P} = \int_{\{Z\in\Lambda\}} D^n(Z)\ d\mathbb{P} = \int_\Lambda D^n(z)\ d\mathcal{L}_Z(z).\end{aligned}$$

Hence $\dfrac{d\mathcal{L}_{Y^n}}{d\mathcal{L}_Z}(Z) = D^n(Z)$ for $Z \in C([0,T];H)$, i.e.,

$$\frac{d\mathcal{L}_{Y^n}}{d\mathcal{L}_Z}(Z) = \mathbb{E}\Big[\rho^n_T(Z,W)|\ \mathbb{F}_T(Z)\Big] \qquad \mathbb{P}\text{-a.s.} \tag{3.2}$$

Uniqueness in law comes from the fact that the linear equation has a unique solution and the laws of these two equations can be related using the Girsanov transform also starting from Equation (3.1); this also shows that these two laws are equivalent (see also §3 in [4]).

Now, we want to relate the solutions of the auxiliary Equation (3.1) and of the semilinear Equation (1.1). Given the weak solution $\big(X,(\Omega,\mathbb{F},\{\mathbb{F}_t\},\mathbb{P}),\ W\big)$ to the semilinear Equation (1.1), we define

$$\tau^n = \inf\{t\in[0,T] : \chi^n_t(X) = 0\} \wedge T,$$

setting the infimum equal to $+\infty$ when the set is empty.

Then each path X solving equation (1.1) is a solution of (3.1) (with the Wiener process W) on the time interval $[0, \tau^n)$; otherwise, on the time interval $[\tau^n, T]$ Equation (3.1) evolves as the Ornstein–Uhlenbeck equation.

Therefore, we define (pathwise) a process solving Equation (3.1) *with the Wiener process* W, as

$$X_t^n = \begin{cases} X_t & \text{for } t \in [0, \tau^n], \\ \text{the solution of the linear equation} & \text{for } t \in [\tau^n, T]. \end{cases}$$

In particular, $X = X^n$ on the set $\{\tau^n = T\}$. Therefore

$$\mathbb{P}\{\|X - X^n\|_{C([0,T];H)} > 0\} \le \mathbb{P}\{\tau^n < T\}.$$

Moreover, $\{\tau^n = T\} \supseteq \{\chi_T^n(X) = 1\}$; hence

$$\mathbb{P}\{\tau^n < T\} \le \mathbb{P}\{\chi_T^n(X) = 0\} = \mathbb{P}\{\textstyle\int_0^T \|\Phi(s, X_s)\|_U^2 ds \ge n\} \longrightarrow 0 \quad \text{as } n \to \infty$$

by **[A3i]**. Summing up, we have got that

$$\lim_{n\to\infty} \mathbb{P}\{\|X - X^n\|_{C([0,T];H)} > 0\} = 0.$$

This means that the law $\mathcal{L}_{X^n}$ of the process X^n converges in the metric of total variation to the law $\mathcal{L}_X$ of the process X, as measures on $C([0,T];H)$:

$$\lim_{n\to\infty} \|\mathcal{L}_X - \mathcal{L}_{X^n}\|_{TV} = 0. \tag{3.3}$$

But $\mathcal{L}_{X^n} = \mathcal{L}_{Y^n}$ and is unique. Hence the limit law (which is the law of a process solving (1.1)) is unique. □

Henceforth, we denote by $\mathcal{L}_X$ and $\mathcal{L}_Z$ the unique laws for the solutions of Equations (1.1) and (1.2) respectively.

Remark 3.2. We have proved the uniqueness in law for the semilinear equation without any boundedness or sublinear growth assumption on the nonlinear term F. For instance, the requirements **[A2]** and **[A3i]** allow to consider nonlinear terms F with polynomial growth; indeed, it is enough to have

$$\exists\ c > 0, p > 0 : \ \|\Phi(s,x)\|_U \le c(1 + \|x\|_H^p) \qquad \forall s \in [0,T], x \in H$$

in order to apply our result. Some interesting examples in fluid dynamics, with $p = 2$, are given in [3].

4. Absolute continuity

We define

$$\chi_T(Y) = \text{ indicator function of the set } \left\{\textstyle\int_0^T \|\Phi(s, Y_s)\|_U^2 ds < \infty\right\}.$$

We consider Equations (1.1) and (1.2) with the same assumptions as for the uniqueness result of the previous section. We have the following result.

Proposition 4.1. *We assume* **[A0]**, **[A1]**, **[A2]** *and* **[A3i]** *and consider* $x \in H$. *Then uniqueness in law holds for Equation* (1.1) *and* $\mathcal{L}_X \prec \mathcal{L}_Z$. *Moreover,*

$$\frac{d\mathcal{L}_X}{d\mathcal{L}_Z}(Z) = \mathbb{E}\Big[e^{\mathcal{I}_T(Z) - \frac{1}{2}\int_0^T \|\Phi(s,Z_s)\|_U^2 ds}\Big| \, \mathbb{F}_T(Z)\Big], \tag{4.1}$$

$\mathbb{P}$-*a.s., where*

$$\mathcal{I}_T(Z) = \mathbb{P} - \lim_{n\to\infty} \chi_T(Z) \int_0^T \chi_s^n(Z)\langle \Phi(s,Z_s), dW_s\rangle_U.$$

Proof. For each n, in (3.2) we have obtained that $\mathcal{L}_{Y^n} \prec \mathcal{L}_Z$. Using (3.3), in the limit we get

$$\mathcal{L}_X \prec \mathcal{L}_Z.$$

We now consider the limit of the Radon–Nikodym derivative given by (3.2) as $n \to \infty$. The limit $\|\mathcal{L}_{Y^n} - \mathcal{L}_X\|_{TV} \to 0$ implies that $\mathcal{L}_{Y^n}$ (equivalently, $\tilde{\mathbb{P}}^n$) is a Cauchy sequence in the metric of total variation. Since $\|\tilde{\mathbb{P}}^n - \tilde{\mathbb{P}}^m\|_{TV} = \|\frac{d\tilde{\mathbb{P}}^n}{d\mathbb{P}} - \frac{d\tilde{\mathbb{P}}^m}{d\mathbb{P}}\|_{L^1(\mathbb{P})}$, this is the same as saying that

$$\frac{d\tilde{\mathbb{P}}^n}{d\mathbb{P}} = \rho_T^n(Z,W)$$

is a Cauchy sequence in the metric of $L^1(\mathbb{P})$. Therefore, the sequence of random variables

$$\rho_T^n(Z,W) = e^{\int_0^T \chi_s^n(Z)\langle\Phi(s,Z_s), dW_s\rangle_U - \frac{1}{2}\int_0^T \chi_s^n(Z)\|\Phi(s,Z_s)\|_U^2 ds}$$

converges in the norm of $L^1(\mathbb{P})$ to some limit, which we denote by $\rho_T(Z,W)$. We want to identify $\rho_T(Z,W)$.

Notice that if $\int_0^T \|\Phi(s,Z_s)\|_U^2 ds < \infty$ $\mathbb{P}$-a.s., then the stochastic integral in the exponent of $\rho_T^n(Z,W)$ would converge in probability to $\int_0^T \langle\Phi(s,Z_s), dW_s\rangle_U$ (see [9], Section 4.2.6) and the deterministic integral to $\int_0^T \|\Phi(s,Z_s)\|_U^2 ds$. Otherwise, we consider the random variable

$$\mathcal{I}_T^n(Z) := \chi_T(Z)\int_0^T \chi_s^n(Z)\Phi(s,Z_s)dW_s,$$

which converges in probability to the random variabile $\mathcal{I}_T(Z)$ (see [9]). It follows that

$$\chi_T(Z)\rho_T(Z,W) = \chi_T(Z)e^{\mathcal{I}_T(Z) - \frac{1}{2}\int_0^T \|\Phi(s,Z_s)\|_U^2 ds}$$

$\mathbb{P}$-a.s. Moreover, with some elementary calculations (see [4, §5]) we show that

$$\rho_T(Z,W) = e^{\mathcal{I}_T(Z) - \frac{1}{2}\int_0^T \|\Phi(s,Z_s)\|_U^2 ds}$$

$\mathbb{P}$-a.s. From this we obtain (4.1) as we did for (3.2). □

Remark 4.2. The random variable

$$e^{\mathcal{I}_T(Z) - \frac{1}{2}\int_0^T \|\Phi(s,Z_s)\|_U^2 ds}$$

is non-negative and it vanishes (a.s.) on the set $\{\int_0^T \|\Phi(s,Z_s)\|_U^2 ds = \infty\}$. It defines the Radon–Nikodym derivative $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$, where $\tilde{\mathbb{P}}$ is the limit of $\tilde{\mathbb{P}}^n$ in the total variation norm.

5. Equivalence of laws

Now, if we assume also **[A3ii]**, that is

$$\mathbb{P}\{\int_0^T \|\Phi(s,Z_s)\|_U^2 ds < \infty\} = 1,$$

it is clear from the previous section that $\chi_T(Z) = 1$ and

$$\mathcal{I}_T(Z) = \int_0^T \langle \Phi(s,Z_s), dW_s\rangle_U.$$

Thus, the Radon–Nikodym $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ derivative is given by

$$e^{\int_0^T \langle \Phi(s,Z_s), dW_s\rangle_U - \frac{1}{2}\int_0^T \|\Phi(s,Z_s)\|_U^2 ds}.$$

Under these assumptions, we have a first result.

Theorem 5.1. *We are given $x \in H$. We assume* **[A0]**, **[A1]**, **[A2]**, **[A3i]** *and* **[A3ii]**. *Then*

i) *the process $\rho = \rho(Z,W)$ given by*

$$\rho_t = e^{+\int_0^T \langle \Phi(s,Z_s), dW_s\rangle_U - \frac{1}{2}\int_0^T \|\Phi(s,Z_s)\|_U^2 ds}, \quad 0 \le t \le T,$$

is a positive $\{\mathbb{F}_t\}$-martingale; in particular

$$\mathbb{E}[\rho_t(Z,W)] = 1 \quad \textit{for any } t \in [0,T].$$

ii)
$$\tilde{W}_t = W_t - \int_0^t \Phi(s,Z_s)\, ds, \qquad t \in [0,T], \tag{5.1}$$

is a U-cylindrical Wiener process with respect to $\tilde{\mathbb{P}}$, where the probability measure $\tilde{\mathbb{P}}$ is defined on $(\Omega, \mathbb{F}_T)$ by

$$d\tilde{\mathbb{P}} = \rho_T(Z,W)\, d\mathbb{P}.$$

Proof. i) Notice that the exponential process $\rho(Z,W)$ is a positive local martingale and then, by Fatou lemma, a supermartingale. Since $\rho_0(Z,W) = 1$, it is enough to have $\mathbb{E}[\rho_T(Z,W)] = 1$ in order to prove that it is a martingale. But, $\rho_T(Z,W)$ is the $L^1(\mathbb{P})$-limit of $\rho_T^n(Z,W)$; since we already know from the proof of Proposition 3.1 that

$$\mathbb{E}[\rho_T^n(Z,W)] = 1 \qquad \text{for any } n = 1, 2, \dots$$

we get that $\mathbb{E}[\rho_T(Z,W)] = 1$.

ii) Given i), this is the Girsanov theorem. □

In this section we add condition **[A3ii]** to all the previous ones and therefore the results of Proposition 3.1 and Proposition 4.1 hold true. Moreover, we have that $\big(Z,\ (\Omega,\mathbb{F},\{\mathbb{F}_t\},\tilde{\mathbb{P}}),\ \tilde{W}\big)$ is a weak solution to Equation (1.1) and

$$\mathbb{P}\{X\in\Lambda\}=\mathcal{L}_X(\Lambda)=\tilde{\mathbb{P}}\{Z\in\Lambda\}$$

for any Borelian subset Λ of $C([0,T];H)$.

Now, by means of Theorem 5.1 we get the equivalence of the laws.

Proposition 5.2. *We assume* **[A0]**, **[A1]**, **[A2]**, **[A3i]** *and* **[A3ii]** *with $x\in H$ fixed. Then $\mathcal{L}_X\sim\mathcal{L}_Z$; in particular*

$$\frac{d\mathcal{L}_X}{d\mathcal{L}_Z}(Z)=\mathbb{E}\Big[e^{+\int_0^T\langle\Phi(s,Z_s),dW_s\rangle_U-\frac{1}{2}\int_0^T\|\Phi(s,Z_s)\|_U^2ds}\Big|\ \mathbb{F}_T(Z)\Big],$$

$$\frac{d\mathcal{L}_Z}{d\mathcal{L}_X}(X)=\mathbb{E}\Big[e^{-\int_0^T\langle\Phi(s,X_s),dW_s\rangle_U-\frac{1}{2}\int_0^T\|\Phi(s,X_s)\|_U^2ds}\Big|\ \mathbb{F}_T(X)\Big],$$

$\mathbb{P}$*-a.s.*

Proof. Thanks to **[A3ii]**, the Radon–Nikodym derivative $\rho_T=\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ is strictly positive; therefore there exists $\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}$, that is $\mathbb{P}\prec\tilde{\mathbb{P}}$. Thus $\mathbb{P}\sim\tilde{\mathbb{P}}$ and $\mathcal{L}_X\sim\mathcal{L}_Z$.

What remains to prove is the expression of the second Radon–Nikodym derivative. From $\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}=\big(\rho_T(Z,W)\big)^{-1}$ and (5.1), we get

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}=e^{-\int_0^T\langle\Phi(s,Z_s),d\tilde{W}_s\rangle_U-\frac{1}{2}\int_0^T\|\Phi(s,Z_s)\|_U^2ds}.$$

In particular,

$$\tilde{\mathbb{E}}\Big[e^{-\int_0^T\langle\Phi(s,Z_s),d\tilde{W}_s\rangle_U-\frac{1}{2}\int_0^T\|\Phi(s,Z_s)\|_U^2ds}\Big]=1,$$

where the mathematical expectation is with respect to the probability measure $\tilde{\mathbb{P}}$. This is written for the solution $\big(Z,(\Omega,\mathbb{F},\{\mathbb{F}_t\},\tilde{\mathbb{P}}),\tilde{W}\big)$ of Equation (1.1). Since there is uniqueness in law for Equation (1.1), if we consider the same relationship for the solution $\big(X,\ (\Omega,\mathbb{F},\{\mathbb{F}_t\},\mathbb{P}),\ W\big)$ we get

$$\mathbb{E}\Big[e^{-\int_0^T\langle\Phi(s,X_s),dW_s\rangle_U-\frac{1}{2}\int_0^T\|\Phi(s,X_s)\|_U^2ds}\Big]=1.\tag{5.2}$$

Now, let us start from Equation (1.1) and consider Equation (1.2) as a modification of Equation (1.1) by a change of drift. Thanks to (5.2), we can use Girsanov theorem and obtain $\frac{d\mathcal{L}_Z}{d\mathcal{L}_X}(X)$ as usual. □

Remark 5.3. The same results hold true in a more general setting. Here we have presented the basic one. In particular, we can deal with the covariance of the noise depending on time and on the unknown, and with F and B taking values in a space bigger than H if the semigroup $\{S_t\}$ is analytic (see, e.g., [11]) so that it has a regulazing effect in the terms $\int_0^t S_{t-s}B\ dW_s$, $\int_0^t S_{t-s}F(s,X_s)\ ds$ appearing in the mild solutions. Finally, the initial data x can be an $\mathbb{F}_0$-measurable H-valued random variable.

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Rate of Convergence of Wong–Zakai Approximations for Stochastic Partial Differential Equations

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Abstract. In this paper we show that the rate of convergence of Wong–Zakai approximations for stochastic partial differential equations driven by Wiener processes is essentially the same as the rate of convergence of the driving processes W_n approximating the Wiener process, provided the area processes of W_n also converge to those of W with that rate. We consider non-degenerate and also degenerate stochastic PDEs with time dependent coefficients.

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1. Introduction

Consider for each integer $n \geq 1$ the stochastic PDE

$$du_n(t,x) = (L_n u_n(t,x) + f_n)\, dt + (M_n^k u_n(t,x) + g_n^k)\, dW_n^k(t), \tag{1.1}$$

for $(t,x) \in H_T = (0,T] \times \mathbb{R}^d$, for a fixed $T > 0$, with initial condition

$$u_n(0,x) = u_{n0}(x), \quad x \in \mathbb{R}^d, \tag{1.2}$$

given on a probability space $(\Omega, \mathcal{F}, P)$, where L_n and M_n^k are second- and first-order differential operators in $x \in \mathbb{R}^d$, respectively for every $\omega \in \Omega$. The free terms, f_n and $g_n = (g_n^k)$ are random fields, and $W_n = (W_n^k)$ is a continuous d_1-dimensional stochastic process with finite variation over $[0,T]$, for $k = 1, \ldots, d_1$.

Unless otherwise stated we use the summation convention with respect to repeated indices throughout the paper. The summation convention is not used if the repeated index is the subscript n.

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The operators L_n, M_n^k are of the form

$$L_n = a_n^{ij}(t,x)D_iD_j + a_n^i(t,x)D_i + a_n(t,x),$$
$$M_n^k = b_n^{ik}(t,x)D_i + b_n^k(t,x),$$

where $a_n^{ij},\dots,b_n^k$ are real-valued bounded functions on $\Omega \times [0,T] \times \mathbb{R}^d$ for all $i,j = 1,\dots,d$, $k = 1,\dots,d_1$, and integers $n \geq 1$, $D_i = \frac{\partial}{\partial x^i}$ for $i = 1,2,\dots,d$, and x^i is the ith co-ordinate of $x \in \mathbb{R}^d$. The free terms, f_n, $g_n^1,\dots,g_n^{d_1}$ are real-valued functions on $\Omega \times [0,T] \times \mathbb{R}^d$ for each n. We assume that L_n is either uniformly elliptic or degenerate elliptic for all n.

Assume that the operators L_n, M_n^k, the free terms f_n, g_n^k and the initial data u_{n0} converge to some operators

$$L = a^{ij}(t,x)D_{ij} + a^i(t,x)D_i + a(t,x),$$
$$M^k = b^{ik}(t,x)D_i + b^k(t,x),$$

random fields f, g^k and initial data u_0 respectively, and $W_n(t)$ converges to a d_1-dimensional Wiener process in probability, uniformly in $t \in [0,T]$. Then under some smoothness conditions on the coefficients of L_n, L, M_n^k, M^k and on the data u_{0n}, f_n, g_n^k, u_0, f, g^k, and under some additional conditions on the convergence of the related area processes and on the growth of the auxiliary process B_n (defined in (2.1) and (2.2) below), the solution u_n to (1.1) converges in probability to a random field u that satisfies the stochastic PDE

$$du(t,x) = (Lu(t,x) + f)\,dt + (M^k u(t,x) + g^k) \circ dW^k(t), \quad (t,x) \in H_T$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d,$$

where '$\circ$' indicates the Stratonovich differential. (See, e.g., [8] and [9].) When M^k and g^k do not depend on the variable t, then

$$(M^k u(t,x) + g^k) \circ dW^k = \tfrac{1}{2}(M^k M^k u(t,x) + M^k g^k)\,dt + (M^k u(t,x) + g^k)\,dW^k.$$

One of the important questions in the analysis of approximation schemes is the estimation of the speed of convergence. In this paper we show that, if the continuous finite variation processes W_n and their area processes converge almost surely to a Wiener process W and to its area processes, respectively, with a given rate, then $u_n(t)$ converges almost surely with *essentially* the same rate. The results of this paper are motivated by a question about robustness of nonlinear filters for partially observed processes, $(X(t), Y(t))_{t\in[0,T]}$. For a large class of *signal and observation* models, the signal X and the observation Y are governed by stochastic differential equations with respect to Wiener processes, and a basic assumption is that the observation process is a non-degenerate Itô process. Thus the observation is modelled by a process, which has infinite (first) variation on any (small) finite interval. In practice, however, due to the smoothing effect of measurements, the "signal data" is a process which has finite variation on any finite interval. This process can be viewed as an approximation Y_n to Y, and it is natural to assume that Y_n and its area processes converge almost surely in the sup norm to Y and its area processes, with some speed. By a direct application of the main theorems of the present article one can show that the "robust filtering equation", with Y_n in place of Y, admits a unique solution p_n which converges almost surely with almost the same order to the conditional density of $X(t)$ given the observation $\{Y(s) : s \in [0, t]\}$. The filtering equations in case of correlated signal and observation noise are stochastic PDEs with coefficients depending on the observations. Thus approximating the observations we approximate also the differential operators in the stochastic PDEs. This is why we consider Equation (1.1) with random operators L_n and M_n^k depending also on n, the parameter of the approximation.

Our results improve and generalise the results of [12] and [17], where only half of the order of convergence of W_n is obtained for the order of convergence of u_n. Moreover, our conditions are weaker, and we prove the optimal rate also in the case of degenerate stochastic PDEs, which allows to get our rate of convergence result also in the case of degenerate signal and observation models.

Wong–Zakai approximations of stochastic PDEs were studied intensively in the literature. See, for example, [1]–[9], [12]–[14], [17]–[18], and the references therein. With the exception of [4], [12], [14] and [17] the papers above prove convergence results of Wong–Zakai approximations for stochastic PDEs with various generalities, but do not present rate of convergence estimates. Wong–Zakai type approximation results for semilinear and fully nonlinear SPDEs are obtained via rough path approach in [5]–[7].

In [4], the initial value problem (1.1)–(1.2) is considered with non-random coefficients and without free terms, when W_n are polygonal approximations to the Wiener process W. By the method of characteristics it is proved that $u_n(t, x)$ converges almost surely, uniformly in $(t, x) \in [0, T] \times \mathbb{R}^d$. Though the rate of convergence of u_n to u is not stated explicitly in [4], from the rate of convergence result proved in [4] for the characteristics, one can easily deduce that for every $\kappa < 1/4$ there exists a finite random variable ξ_κ such that almost surely $|u_n(t, x) - u(t, x)| \le \xi n^{-\kappa}$ for all $t \in [0, T]$ and $x \in \mathbb{R}^d$. We note that for polygonal approximations the almost sure order of convergence of W_n and its area processes are of order $\kappa < 1/2$, and thus by our paper the almost sure rate of convergence of the Wong–Zakai approximations is the same $\kappa < 1/2$, in Sobolev norms, and via Sobolev's embedding in the supremum norm as well. In [14] the rate of convergence of Wong–Zakai approximations of stochastic PDEs driven by Poisson random measures is investigated.

Let us conclude with introducing some notation used throughout the paper. All random objects are given on a fixed probability space $(\Omega, \mathcal{F}, P)$ equipped with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}$, such that $\mathcal{F}_0$ contains all the P-null sets of the complete σ-algebra $\mathcal{F}$. The σ-algebra of predictable subsets of $[0, \infty) \times \Omega$ is denoted by $\mathcal{P}$ and the σ-algebra of Borel subsets of $\mathbb{R}^d$ is denoted by $\mathcal{B}(\mathbb{R}^d)$. The notation $C_0^\infty = C_0^\infty(\mathbb{R}^d)$ stands for the space of real-valued smooth functions with compact support on $\mathbb{R}^d$. For an integer m we use the notation H^m for the Hilbert–Sobolev space $W_2^m(\mathbb{R}^d)$ of generalized functions on $\mathbb{R}^d$. For $m \ge 0$ it is the closure of C_0^∞ in the norm $|\cdot|_m$ defined by

$$|f|_m^2 = \sum_{|\alpha|\le m} \int_{\mathbb{R}^d} |D^\alpha f(x)|^2\,dx,$$

where $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_d^{\alpha_d}$ for multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \{0,1,\ldots\}^d$, and D^0 is the identity operator. For $m<0$, H^m is the closure of C_0^∞ in the norm

$$|f|_m = \sup_{g\in C_0^\infty, |g|_{-m}\le 1} (f,g)_0,$$

where $(f,g)_0$ denotes the inner product in $L_2 = H^0$. We define in the same way the Hilbert–Sobolev space $H^m = H^m(\mathbb{R}^l)$ of $\mathbb{R}^l$-valued functions $g = (g^1,\ldots,g^l)$ on $\mathbb{R}^d$, such that $|g|_m^2 = \sum_{k=1}^l |g^k|_m^2$. We use the notation $(\cdot\,,\cdot)_m$ for the inner product in H^m, and for $m=0$ we often use the notation $(\cdot\,,\cdot)$ instead of $(\cdot\,,\cdot)_0$. For $m\ge 0$ denote by $\langle\cdot\,,\cdot\rangle_m$ the duality product between H^{m+1} and H^{m-1}, based on the inner product $(\cdot\,,\cdot)_m$ in H^m. For real numbers A and B we set $A\vee B = \max\{A,B\}$ and $A\wedge B = \min\{A,B\}$. For sequences of random variables $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ the notation $a_n = o(b_n)$ means that there exist random variables ξ_n converging almost surely to zero such that almost surely $|a_n| \le \xi_n |b_n|$ for all n. The notation $a_n = O(b_n)$ means that there exists a random variable η such that almost surely $|a_n|\le \eta|b_n|$ for all n.

2. Formulation of the results

Let $W = (W(t))_{t\in[0,T]}$ be a d_1-dimensional Wiener martingale with respect to $\mathbb{F}$, and consider for every integer $n\ge 1$ an $\mathbb{R}^{d_1}$-valued $\mathcal{F}_t$-adapted continuous process $W_n = (W_n(t))_{t\in[0,T]}$ of finite variation. Define the area processes of W and W_n as

$$\begin{aligned} A^{ij}(t) &:= \frac{1}{2}\int_0^t W^i(s)\,dW^j(s) - W^j(s)\,dW^i(s), \quad i,j = 1,2,\ldots,d_1,\\ A_n^{ij}(t) &:= \frac{1}{2}\int_0^t W_n^i(s)\,dW_n^j(s) - W_n^j(s)\,dW_n^i(s), \quad i,j=1,2,\ldots,d_1, \end{aligned} \tag{2.1}$$

and also the process

$$B_n^{ij}(t) := \int_0^t (W^i(s) - W_n^i(s))\,dW_n^j(s), \qquad i,j = 1,\ldots,d_1, \tag{2.2}$$

that will play a crucial role. We denote by $\|q\|\,(t)$ the first variation of a process q over the interval $[0,t]$ for $t\le T$.

Let $\gamma > 0$ be a fixed real number and assume that the following conditions hold.

Assumption 2.1. For each $\kappa < \gamma$ almost surely

(i) $\sup_{t\le T} |W(t) - W_n(t)| = O(n^{-\kappa})$,

(ii) $\sup_{t\le T} \left|A^{ij}(t) - A_n^{ij}(t)\right| = O(n^{-\kappa}) \quad i\ne j$,

(iii) $||B_n^{ij}||(T) = o(\ln n)$ for all $i,j = 1,\ldots,d_1$.

The following remark is shown in [12].

Remark 2.2. Define the matrix-valued process $S_n = (S_n^{ij}(t))$, $t \in [0,T]$ by

$$\begin{aligned} S_n^{ij}(t) &= \int_0^t (W^i(s) - W_n^i(s))\, dW_n^j(s) - \frac{1}{2}\langle W^i, W^j\rangle(t) \\ &= \int_0^t (W^i(s) - W_n^i(s))\; dW_n^j(s) - \frac{1}{2}\delta_{ij} t, \qquad i,j = 1,2,\ldots,d_1, \end{aligned}$$

for each integer $n \geq 1$, where $\langle W^i, W^j\rangle$ denotes the quadratic covariation process of W^i and W^j, and $\delta_{ij} = 1$ for $i = j$ and it is zero otherwise. Then by Itô's formula for $q_n^{ij} := (W^i - W_n^i)(W^j - W_n^j)$ we have

$$S_n^{ij}(t) + S_n^{ji}(t) = q_n^{ij}(0) - q_n^{ij}(t) + R_n^{ij}(t) + R_n^{ji}(t)$$

with

$$R_n^{ij}(t) := \int_0^t (W^i(s) - W_n^i(s))\, dW^j(s).$$

Moreover, given Part (i) in Assumption 2.1, Part (ii) is equivalent to condition (ii′):

$$\sup_{t\leq T} |S_n^{ij}(t)| = O(n^{-\kappa}) \quad \text{(a.s.) for each } \kappa < \gamma, \text{ for } i,j = 1,\ldots,d_1. \tag{2.3}$$

Assumption 2.1 holds for a large class of approximations W_n of W. The main examples are the following.

Example. (Polygonal approximations) Set $W_n(t) = 0$ for $t \in [0, T/n)$ and

$$W_n(t) = W(t_{k-1}) + n(t - t_k)(W(t_k) - W(t_{k-1}))/T$$

for $t \in [t_k, t_{k+1})$, where $t_k := kT/n$ for integers $k \geq 0$.

Example. (Smoothing) Define

$$W_n(t) = \int_0^1 W(t - u/n)\, du, \quad t \geq 0,$$

where $W(s) := 0$ for $s < 0$.

One can prove, see [13], that these examples satisfy the conditions of Assumptions 2.1 with $\gamma = 1/2$.

Now we formulate the conditions on the operators L_n, M_n^k and their convergence to operators L and M^k. We fix an integer $m \geq 0$ and a real number $K \geq 0$.

Assumption 2.3 (Ellipticity). There exists a constant $\lambda \geq 0$ such that for each integer $n \geq 1$ for $dP \times dt \times dx$ almost all $(\omega, t, x) \in \Omega \times [0,T] \times \mathbb{R}^d$

$$a_n^{ij}(t,x) z^i z^j \geq \lambda |z|^2,$$

for all $z = (z^1, z^2, \ldots, z^d) \in \mathbb{R}^d$.

If $\lambda > 0$ then we need the following conditions on the regularity of the coefficients $\mathbf{a}_n = (a_n^{ij}, a_n^i, a_n : i,j = 1,\dots,d)$, $\mathbf{b}_n = (b_n^{ik}, b_n^k : i = 1,\dots,d; k = 1,\dots,d_1)$, $\mathbf{a} := (a^{ij}, a^i, a : i,j = 1,\dots,d)$, $\mathbf{b} := (b^{ik}, b^k : i = 1,\dots,d; k = 1,\dots,d_1)$ for all $n \geq 1$, and on the data u_{n0}, f_n and $g_n = (g_n^k)$, u_0, f, $g = (g^k)$.

Assumption 2.4. The coefficients $\mathbf{a}_n$, $\mathbf{b}_n$ and their derivatives in x up to order $m+4$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable functions, and they are in magnitude bounded by K. For each $n \geq 1$, f_n is an H^{m+3}-valued predictable process, $g_n = (g_n^k)$ is an $H^{m+4}(\mathbb{R}^{d_1})$-valued predictable process and u_{n0} is an H^{m+4}-valued $\mathcal{F}_0$-measurable random variable, such that for every $\varepsilon > 0$ almost surely

$$|u_{n0}|_{m+3} = O(n^\varepsilon), \qquad \int_0^T |f_n|_{m+3}^2 \, dt = O(n^\varepsilon), \qquad \sup_{t \leq T} |g_n(t)|_{m+4} = O(n^\varepsilon).$$

One knows, see Theorem 3.5 below, that if Assumption 2.3 with $\lambda > 0$ and Assumption 2.4 hold, then for each $n \geq 1$ there is a unique generalized solution u_n to (1.1)–(1.2).

Assumption 2.5. The coefficients $\mathbf{a}$ and $\mathbf{b}$ and their derivatives in x up to order $m+1$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega \times H_T$, and they are in magnitude bounded by K. The initial value u_0 is an H^{m+1}-valued $\mathcal{F}_0$-measurable random variable, f is an H^m-valued predictable processes and $g = (g^k)$ is an $H^{m+1}(\mathbb{R}^{d_1})$-valued predictable process such that almost surely

$$\int_0^T |f(t)|_m^2 \, dt + \sup_{t \leq T} |g(t)|_{m+1}^2 < \infty.$$

Assumption 2.6. We have

$$\sup_{H_T} |D^\alpha \mathbf{a}_n - D^\alpha \mathbf{a}| = O(n^{-\gamma}), \qquad \sup_{H_T} |D^\beta \mathbf{b}_n - D^\beta \mathbf{b}| = O(n^{-\gamma}),$$

for all $|\alpha| \leq (m-1) \vee 0$ and $|\beta| \leq m+1$, and

$$\int_0^T |f_n(t) - f(t)|_{m-1}^2 \, dt + \sup_{t \leq T} |g_n(t) - g(t)|_{m+1}^2 = O(n^{-2\gamma}).$$

Now we formulate our main result when $\lambda > 0$ in Assumption 2.3, and b_n^{ik}, b_n^k and g_n^k do not depend on $t \in [0,T]$.

Theorem 2.7. *Assume that* $\mathbf{b}_n$ *and* g_n *are independent of* t. *Let Assumptions* 2.1, 2.3 *with* $\lambda > 0$, 2.4, 2.5 *and* 2.6 *hold. Then almost surely*

$$\sup_{t \leq T} |u_n(t) - u(t)|_m^2 + \int_0^T |u_n(t) - u(t)|_{m+1}^2 \, dt = O(n^{-2\kappa}), \quad \textit{for any } \kappa < \gamma.$$

In the degenerate case, $\lambda = 0$, instead of Assumptions 2.4, 2.5 and 2.6 we need to impose stronger conditions.

Assumption 2.8.

(i) For each $n \geq 1$ there exist functions σ_n^{ir} on $\Omega \times H_T$, for $i = 1, \dots, d$ and $r = 1, \dots, p$, for some $p \geq 1$, such that $a_n^{ij} = \sigma_n^{ir}\sigma_n^{jr}$ for all $i, j = 1, \dots, d$.

(ii) The functions σ_n^{ir}, b_n^i and their derivatives in x up to order $m+6$, the functions a_n^i, a_n, b_n and their derivatives in x up to order $m+5$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega\times H_T$, and in magnitude are bounded by K for $i = 1, \dots, d$ and $r = 1, \dots, p$. For each $n \geq 1$, f_n is an H^{m+4}-valued predictable process, $g_n = (g_n^k)$ is an $H^{m+5}(\mathbb{R}^{d_1})$-valued predictable process and u_{n0} is an H^{m+4}-valued $\mathcal{F}_0$-measurable random variable, such that for every $\varepsilon > 0$

$$|u_{n0}|_{m+4} = O(n^{\varepsilon}), \quad \int_0^T |f_n|_{m+4}^2 \, dt = O(n^{\varepsilon}), \quad \sup_{t\leq T} |g_n(t)|_{m+5} = O(n^{\varepsilon}).$$

Assumption 2.9. The coefficients $\mathbf{a}$ and $\mathbf{b}$ and their derivatives in x up to order $m+2$ are $\mathcal{P}\times\mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega\times H_T$, and they are in magnitude bounded by K. The initial value u_0 is an H^{m+2}-valued $\mathcal{F}_0$-measurable random variable, f is an H^{m+2}-valued predictable process and $g = (g^k)$ is an $H^{m+3}(\mathbb{R}^{d_1})$-valued predictable process such that

$$\int_0^T |f(t)|_{m+2}^2 \, dt + \sup_{t\leq T} |g(t)|_{m+2}^2 < \infty.$$

Assumption 2.10. We have

$$\sup_{H_T} |D^{\alpha}\mathbf{a}_n - D^{\alpha}\mathbf{a}| = O(n^{-\gamma}), \quad \sup_{H_T} |D^{\beta}\mathbf{b}_n - D^{\beta}\mathbf{b}| = O(n^{-\gamma})$$

for all $|\alpha| \leq m$ and $|\beta| \leq m+1$, and

$$\int_0^T |f_n - f|_m^2 \, dt = O(n^{-2\gamma}), \quad \int_0^T |g_n - g|_{m+1}^2 \, dt = O(n^{-2\gamma}).$$

Remark 2.11. Notice that Assumption 2.8 (i) implies Assumption 2.3 with $\lambda = 0$.

Theorem 2.12. *Assume that* $\mathbf{b}_n$ *and* g_n *do not depend on* t. *Let Assumptions* 2.1, 2.8, 2.9 *and* 2.10 *hold. Then*

$$\sup_{t\leq T} |u_n - u|_m = O(n^{-\kappa}) \quad \text{a.s. for each } \kappa < \gamma.$$

Let us now consider the case when all the coefficients and free terms may depend on $t \in [0, T]$. We use the notation

$$h_n := \left(b_n^{ik}, b_n^k, g_n^k : i = 1, \dots, d, \ k = 1, \dots, d_1\right), \quad n \geq 1,$$
$$h := \left(b^{ik}, b^k, g^k : i = 1, \dots, d, \ k = 1, \dots, d_1\right).$$

We make the following assumption.

Assumption 2.13. There exist $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable bounded functions

$$\mathbf{b}_n^{(r)} = (b_n^{ik(r)}, \, b_n^{k(r)} : i = 1, \dots, d, \ k = 1, \dots, d_1), \quad r = 0, \dots, d_1, \ n \geq 1$$

and $H^0(\mathbb{R}^{d_1})$-valued bounded predictable processes $g_n^{(r)} = (g_n^{k(r)} : k = 1, \dots, d_1)$, such that

$$d(h_n(t), \varphi) = (h_n^{(0)}(t), \varphi)\, dt + (h_n^{(k)}(t), \varphi)\, dW_n^k(t), \quad n \geq 1,$$

where $h_n^{(r)} = (\mathbf{b}_n^{(r)}, g_n^{(r)})$. For $r = 0, \dots, d_1$ and $j = 1, \dots, d_1$ there exist $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable bounded functions

$$\mathbf{b}^{(r)} = (b^{ik(r)},\, b^{k(r)} : \quad i = 1, \dots, d,\, k = 1, \dots, d_1),$$
$$\mathbf{b}^{(jr)} = (b^{ik(jr)},\, b^{k(jr)} : i = 1, \dots, d,\, k = 1, \dots, d_1),$$

and $H^0(\mathbb{R}^{d_1})$-valued bounded predictable processes $g^{k(r)}$ and $g^{k(jr)}$, $k = 1, \dots, d_1$, such that

$$d(h(t), \varphi) = (h^{(0)}(t), \varphi)\, dt + (h^{(k)}(t), \varphi)\, dW_n^k(t),$$
$$d(h^{(j)}(t), \varphi) = (h^{(j0)}(t), \varphi)\, dt + (h^{(jk)}(t), \varphi)\, dW^k(t)$$

for $\varphi \in C_0^\infty(\mathbb{R}^d)$, for $j = 1, \dots, d_1$, where $h^{(r)} = (\mathbf{b}^{(r)}, g^{k(r)} : k = 1, \dots, d_1)$ and $h^{(jr)} = (\mathbf{b}^{(jr)}, g^{k(jr)} : k = 1, \dots, d_1)$, $r = 0, \dots, d_1$.

If Assumption 2.3 holds with $\lambda > 0$, then we impose the following conditions.

Assumption 2.14. For $n \geq 1$ the coefficients $\mathbf{b}_n^{(r)}$ and their derivatives in x up to order $m+3$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega \times H_T$, and they are bounded in magnitude by K for $r = 0, 1, \dots, d_1$. The functions $g_n^{k(0)}$ are H^{m+2}-valued, $g_n^{k(j)}$ are H^{m+3}-valued predictable processes, such that

$$\int_0^T |g_n^{k(0)}(t)|_{m+2}^2\, dt = O(n^\varepsilon), \quad \sup_{H_T} |g_n^{k(j)}|_{m+3} = O(n^\varepsilon)$$

for each $\varepsilon > 0$ and all $k, j = 1, \dots, d_1$.

Assumption 2.15. The coefficients $\mathbf{b}^{(r)}$, $\mathbf{b}^{(jr)}$ and their derivatives in x up to order $m+1$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega \times H_T$, and they are bounded in magnitude by K for $r = 0, 1, \dots, d_1$ and $j = 1, 2, \dots, d_1$. The functions $g^{k(r)}$ and $g^{k(jr)}$ are H^{m+1}-valued predictable processes, and are bounded in H^{m+1}, for $r = 0, 1, \dots, d_1$ and $j = 1, 2, \dots, d_1$.

Assumption 2.16. For $j = 1, 2, \dots, d_1$ we have

$$\sup_{H_T} |D^\alpha \mathbf{b}_n^{(j)} - D^\alpha \mathbf{b}^{(j)}| = O(n^{-\gamma}) \quad \text{for } |\alpha| \leq m,$$
$$\sup_{t \leq T} |g_n^{k(j)} - g^{k(j)}|_m = O(n^{-\gamma}) \quad \text{for } k = 1, \dots, d_1.$$

One knows, see [9], that under the assumptions above the limit u of u_n for $n \to \infty$ exists and satisfies

$$du(t,x) = (Lu(t,x) + f)\, dt + (M^k u(t,x) + g^k) \circ dW^k(t), \quad (t,x) \in H_T \tag{2.4}$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \tag{2.5}$$

where

$$\begin{aligned}(M^k u(t,x) + g^k) \circ dW^k =& \frac{1}{2}(M^k M^k u(t,x) + M^k g^k(t,x))\,dt \\ &+ (M^k u(t,x) + g^k(t,x))\,dW^k(t) \\ &+ \frac{1}{2}\sum_{k=1}^{d_1}(M^{k(k)}u(t,x) + g^{k(k)}(t,x))\,dt,\end{aligned}$$

with $M^{k(k)} := b^{ik(k)}(t,x)D_i + b^{k(k)}(t,x)$.

We have the following results on the rate of convergence.

Theorem 2.17. *Let Assumptions* 2.1, 2.3 *with* $\lambda > 0$, 2.4, 2.5, 2.6 *and* 2.13 *through* 2.16 *hold. Then for each* $\kappa < \gamma$

$$\sup_{t\le T}|u_n(t) - u(t)|_m^2 + \int_0^T |u_n(t) - u(t)|_{m+1}^2\,dt = O(n^{-2\kappa}),$$

where u *is the generalized solution of* (2.4)–(2.5).

Let us now consider the case when $\lambda = 0$ in Assumption 2.3.

Assumption 2.18. For $n \ge 1$ the coefficients $\mathbf{b}_n^{(r)}$ and their derivatives in x up to order $m+4$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega\times H_T$, and they are bounded in magnitude by K for $r = 0,1,\dots,d_1$. The functions $g_n^{k(0)}$ are H^{m+3}-valued, $g_n^{k(j)}$ are H^{m+4}-valued predictable processes, such that

$$\int_0^T |g^{k(0)}(t)|_{m+3}^2\,dt = O(n^{\varepsilon}), \quad \sup_{H_T}|g^{k(j)}|_{m+4} = O(n^{\varepsilon})$$

for each $\varepsilon > 0$ and all $k, j = 1,\dots,d_1$.

Assumption 2.19. The coefficients $\mathbf{b}^{(r)}$, $\mathbf{b}^{(jr)}$ and their derivatives in x up to order $m+2$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega\times H_T$, and they are bounded in magnitude by K for $r = 0,1,\dots,d_1$ and $j = 1,2,\dots,d_1$. The functions $g^{k(r)}$ and $g^{k(jr)}$ are H^{m+1}-valued predictable processes, and are bounded in H^{m+1}, for $r = 0,1,\dots,d_1$ and $k,j = 1,2,\dots,d_1$.

Assumption 2.20. For $j = 1,2,\dots,d_1$ we have

$$\sup_{H_T}|D^\alpha \mathbf{b}_n^{(j)} - D^\alpha \mathbf{b}^{(j)}| = O(n^{-\gamma}) \quad \text{for } |\alpha| \le m+1,$$
$$\sup_{t\le T}|g_n^{k(j)} - g^{k(j)}|_{m+1} = O(n^{-\gamma}) \quad \text{for } k = 1,\dots,d_1.$$

Theorem 2.21. *Let Assumption* 2.1, *Assumptions* 2.8 *through* 2.13, *and Assumptions* 2.18 *through* 2.20 *hold. Then*

$$\sup_{t\le T}|u_n - u|_m = O(n^{-\kappa}) \quad \textit{for each } \kappa < \gamma,$$

where u *is the generalized solution of* (2.4)–(2.5).

3. Auxiliaries

3.1. Existence, uniqueness and known estimates for solutions

Consider the equation

$$\begin{aligned} du(t,x) =& (\mathcal{L}u(t,x) + f(t,x))\,dt + (\mathcal{M}^k u(t,x) + g^k(t,x))\,dW^k(t) \\ &+ (\mathcal{N}^\rho u(t,x) + h^\rho(t,x))\,dB^\rho(t), \quad t \in (0,T], \quad x \in \mathbb{R}^d \end{aligned} \tag{3.1}$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \tag{3.2}$$

where $W = (W^1, \dots, W^{d_1})$ is a d_1-dimensional Wiener martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, and $B^1,\dots,B^{d_2}$ are real-valued adapted continuous processes of finite variation over $[0,T]$. The operators $\mathcal{L}$, $\mathcal{M}^k$ and $\mathcal{N}^\rho$ are of the form

$$\mathcal{L} = \mathfrak{a}^{ij} D_i D_j + \mathfrak{a}^i D_i + \mathfrak{a}, \quad \mathcal{M}^k = \mathfrak{b}^{ik} D_i + \mathfrak{b}^k, \quad \mathcal{N}^\rho = \mathfrak{c}^{i\rho} D_i + \mathfrak{c}^\rho,$$

where the coefficients $\mathfrak{a}^{ij}$, $\mathfrak{a}^i$, $\mathfrak{a}$, $\mathfrak{b}^{ik}$, $\mathfrak{b}^k$, $\mathfrak{c}^{i\rho}$ and $\mathfrak{c}^\rho$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable real-valued bounded functions defined on $\Omega \times [0,T] \times \mathbb{R}^d$ for all $i,j = 1,\dots,d$, $k = 1,\dots,d_1$ and $\rho = 1,\dots,d_2$. The free terms $f = f(t,\cdot)$, $g^k(t,\cdot)$ and $h^\rho = h^\rho(t,\cdot)$ are H^0-valued predictable processes, and u_0 is an H^1-valued $\mathcal{F}_0$-measurable random variable.

To formulate the notion of the solution we assume that the generalized derivatives in x, $D_j a^{ij}$, are also bounded functions on $\Omega \times H_T$ for all $i,j = 1,\dots,d$.

Definition 3.1. By a solution of (3.1)–(3.2) we mean an H^1-valued weakly continuous adapted process $u = (u(t))_{t\in[0,T]}$, such that

$$\begin{aligned} (u(t),\varphi) = (u_0,\varphi) &+ \int_0^t \{-(\mathfrak{a}^{ij} D_i u, D_j\varphi) + \big((\mathfrak{a}^i - \mathfrak{a}^{ij}_j) D_i u + \mathfrak{a}u + f, \varphi\big)\}\,ds \\ &+ \int_0^t (\mathcal{M}^k u + g^k, \varphi)\,dW^k + \int_0^t (\mathcal{N}^\rho u + h^\rho, \varphi)\,dB^\rho, \end{aligned}$$

holds for $t \in [0,T]$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$, where $\mathfrak{a}^{ij}_j = D_j \mathfrak{a}^{ij}$.

To present those existence and uniqueness theorems from the L_2-theory of stochastic PDEs which we use in this paper, we formulate some assumptions.

Assumption 3.2. There is a constant $\lambda \geq 0$ such that for all $n \geq 1$, $dP \times dt \times dx$ almost all $(\omega,t,x) \in \Omega \times H_T$ we have

$$(\mathfrak{a}^{ij} - \tfrac{1}{2}\mathfrak{b}^{ik}\mathfrak{b}^{jk}) z^i z^j \geq \lambda |z|^2 \quad \text{for all } z = (z^1,\dots,z^d) \in \mathbb{R}^d.$$

To formulate some further conditions on the smoothness of the coefficients and the data of (3.1)–(3.2) we fix an integer $m \geq 1$. We consider first the case $\lambda > 0$ in Assumption 3.2, and make the following assumptions.

Assumption 3.3. The coefficients $\mathfrak{a}^{ij}$, $\mathfrak{a}^i$, $\mathfrak{a}$, $\mathfrak{b}^{ik}$, $\mathfrak{b}^k$, $\mathfrak{c}^{i\rho}$, $\mathfrak{c}^\rho$ and their derivatives in $x \in \mathbb{R}^d$ up to order m are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable real functions on $\Omega \times H_T$ and in magnitude are bounded by K.

Assumption 3.4. The initial value u_0 is an H^m-valued random variable. The free terms $f = f(t)$, $g^k = g^k(t)$, $h^\rho = h^\rho(t)$ are predictable H^m-valued processes such that almost surely

$$\int_0^T |f(t)|_{m-1}^2\,dt < \infty, \quad \int_0^T |g(t)|_m^2\,dt < \infty, \quad \int_0^T |h^\rho(t)|_m\,d\|B^\rho\|(t) < \infty$$

for all $k = 1, \ldots, d_1$ and $\rho = 1, \ldots, d_2$, where $|g|_l^2 = \sum_k |g^k|_l^2$ and $|h|_l^2 = \sum_\rho |h^\rho|_l^2$ for ever $l \geq 0$.

Theorem 3.5. *Let Assumptions* 3.2 *with* $\lambda > 0$, 3.3 *and* 3.4 *hold. Then* (3.1)–(3.2) *has a unique generalized solution* u. *Moreover,* u *is an* H^m*-valued weakly continuous process, it is strongly continuous as an* H^{m-1}*-valued process,* $u(t) \in H^{m+1}$ *for* $P \times dt$ *a.e.* (ω, t), *and there exist constants* $\nu \geq 0$ *and* $C > 0$ *such that for every* $l \in [0, m]$

$$E \sup_{t \leq T} e^{-\nu V} |u|_l^2 + E \int_0^T e^{-\nu V} |u|_{l+1}^2\,dt$$
$$\leq C \left\{ |u_0|_l^2 + \int_0^T e^{-\nu V} (|f|_{l-1}^2 + |g|_l^2)\,dt + \int_0^T e^{-\nu V} |h^\rho|_l^2\,d\|B^\rho\| \right\},$$

where $V(t) = t + \sum_{\rho=1}^{d_2} \|B^\rho(t)\|$. *The constants* ν *and* C *depend only on* λ, K, d, d_1, d_2 *and* l.

In the degenerate case, i.e., when $\lambda = 0$ in Assumption 3.2, we need to impose somewhat stronger conditions in the other assumptions of the previous theorem.

Theorem 3.6. *Let Assumptions* 3.2 (*with* $\lambda = 0$), 3.3 *and* 3.4 *hold. Assume, moreover, that the derivatives in* $x \in \mathbb{R}^d$ *of* a^{ij} *up to order* $m \vee 2$ *are bounded real functions on* $\Omega \times [0, T] \times \mathbb{R}^d$ *for all* $i, j = 1, \ldots, d$, *and that* $g^k = g^k(t)$ *are* H^{m+1}*-valued predictable processes for all* $k = 1, \ldots, d_1$, *such that almost surely*

$$\int_0^T |g(t)|_{m+1}^2\,dt < \infty.$$

Then the conclusion of Theorem 3.5 *remains valid.*

Theorem 3.6 is a slight modification of [11, Theorem 3.1] and can be proved in the same way. We can prove Theorem 3.5 in the same fashion.

3.2. Inequalities in Sobolev spaces and a Gronwall-type lemma

In the following lemmas we present some estimates we use in the paper. We consider the differential operators

$$\mathcal{M} = b^i D_i + b^0, \qquad \mathcal{N} = c^i D_i + c^0, \qquad \mathcal{K} = d^i D_i + d^0,$$

and

$$\mathcal{L} = a^{ij} D_i D_j + a^i D_i + a^0,$$

where a^{ij}, a^i, a^0, b^i, b^0, c^i, c^0, d^i and d^0 are Borel functions defined on $\mathbb{R}^d$ for $i, j = 1, \ldots, d$. We fix an integer $l \geq 0$ and a constant K. Recall the notation

$(\cdot\,,\cdot) = (\cdot\,,\cdot)_0$ for the inner product in $H^0 \equiv L^2(\mathbb{R}^n)$, and $\langle\cdot\,,\cdot\rangle$ for the duality product between H^1 and H^{-1}.

Lemma 3.7.

(i) *Assume that b^0 and its derivatives up to order l, and b^i and their derivatives up to order $l \vee 1$ are real functions, in magnitude bounded by K. Then for a constant $C = C(K, l, d)$*

$$|(D^\alpha \mathcal{M} v, D^\alpha v)| \le C|v|_l^2, \tag{3.3}$$

$$|(D^\alpha \mathcal{M} v, D^\alpha u) + (D^\alpha \mathcal{M} u, D^\alpha v)| \le C|v|_l|u|_l \tag{3.4}$$

for all $u, v \in H^{l+1}$ and multi-indices α, $|\alpha| \le l$.

(ii) *Assume that b^0, c^0 and their derivatives up to order $l \vee 1$, b^i and c^i and their derivatives up to order $(l+1) \vee 2$ are real functions, in magnitude bounded by K. Then for a constant $C = C(K, l, d)$*

$$|(D^\alpha \mathcal{M}\mathcal{N} v, D^\alpha v) + (D^\alpha \mathcal{M} v, D^\alpha \mathcal{N} v)| \le C|v|_l^2,$$

for all $v \in H^{l+2}$ and multi-indices α, $|\alpha| \le l$.

(iii) *Assume that b^0, c^0, d^0 and their derivatives up to order $(l+1)\vee 2$, b^i, c^i, d^i and their derivatives up to order $(l+2) \vee 3$ are real functions, and in magnitude are bounded by K for $i = 1, \dots, d$. Then for a constant $C = C(K, l, d)$*

$$\begin{aligned} &|(D^\alpha \mathcal{M}\mathcal{N}\mathcal{K} v, D^\alpha v) + (D^\alpha \mathcal{M}\mathcal{N} v, D^\alpha \mathcal{K} v) \\ &\quad + (D^\alpha \mathcal{M}\mathcal{K} v, D^\alpha \mathcal{N} v) + (D^\alpha \mathcal{M} v, D^\alpha \mathcal{N}\mathcal{K} v)| \le C|v|_l^2, \end{aligned}$$

for all $v \in H^{l+3}$ and multi-indices α, $|\alpha| \le l$.

Proof. These and similar estimates are proved in [9]. For the sake of completeness and the convenience of the reader we present a proof here. We can assume that $v \in C_0^\infty(\mathbb{R}^n)$. Let us start with (i). Integrating by parts, we have

$$(\mathcal{M} D^\alpha v, D^\alpha v) = -(D^\alpha v, \mathcal{M} D^\alpha v) + (D^\alpha v, \bar{m} D^\alpha v),$$

where $\bar{m} := 2b^0 - \sum_{i=1}^d D_i b^i$. Therefore, by writing $[\mathcal{M}, D^\alpha] = D^\alpha \mathcal{M} - \mathcal{M} D^\alpha$,

$$\begin{aligned} (D^\alpha \mathcal{M} v, D^\alpha v) &= (\mathcal{M} D^\alpha v, D^\alpha v) + ([\mathcal{M}, D^\alpha] v, D^\alpha v) \\ &= \tfrac{1}{2}(D^\alpha v, \bar{m} D^\alpha v) + ([\mathcal{M}, D^\alpha] v, D^\alpha v) \le C|v|_l^2, \end{aligned}$$

by the regularity assumed on the coefficients. Let us write

$$p(v) := (D^\alpha \mathcal{M} v, D^\alpha v) = \tfrac{1}{2}(D^\alpha v, \bar{m} D^\alpha v) + ([\mathcal{M}, D^\alpha] v, D^\alpha v) =: q(v) + r(v).$$

Defining

$$\begin{aligned} 2a(u, v) &:= p(u+v) - p(u) - p(v) = (D^\alpha \mathcal{M} u, D^\alpha v) + (D^\alpha \mathcal{M} v, D^\alpha u), \\ 2b(u, v) &:= q(u+v) - q(u) - q(v) = (\bar{m} D^\alpha u, D^\alpha v), \end{aligned}$$

and

$$2c(u, v) := r(u+v) - r(u) - r(v) = ([\mathcal{M}, D^\alpha] u, D^\alpha v) + ([\mathcal{M}, D^\alpha] v, D^\alpha u),$$

we have

$$a(u,v) = b(u,v) + c(u,v) \tag{3.5}$$

and

$$|a(u,v)| \le |b(u,v)| + |c(u,v)| \le C|u|_l|v|_l,$$

which proves the second inequality in (3.3). The identity (3.5) applied with $u = Nv$ establishes that

$$\begin{aligned}&(D^\alpha \mathcal{M}\mathcal{N}v, D^\alpha v) + (D^\alpha \mathcal{M}v, D^\alpha \mathcal{N}v)\\ &\quad = (\bar{m}D^\alpha \mathcal{N}v, D^\alpha v) + ([\mathcal{M}, D^\alpha]\mathcal{N}v, D^\alpha v) + ([\mathcal{M}, D^\alpha]v, D^\alpha \mathcal{N}v).\end{aligned}$$

By the previous case,

$$|(D^\alpha \mathcal{M}\mathcal{N}v, D^\alpha v) + (D^\alpha \mathcal{M}v, D^\alpha \mathcal{N}v)| \le C|v|_l^2,$$

and (ii) is proved. For (iii), integrating by parts,

$$\begin{aligned}\widetilde{p}(v) &:= (D^\alpha \mathcal{M}\mathcal{N}v, D^\alpha v) + (D^\alpha \mathcal{M}v, D^\alpha \mathcal{N}v)\\ &= (D^\alpha \mathcal{N}v, \bar{m}D^\alpha v) + ([\mathcal{M}, D^\alpha \mathcal{N}]v, D^\alpha v) + ([\mathcal{M}, D^\alpha]v, D^\alpha \mathcal{N}v)\\ &= \widetilde{q}(v) + \widetilde{r}(v) + \widetilde{s}(v).\end{aligned}$$

By polarizing this last identity as above and letting $u = \mathcal{K}v$, we have

$$\begin{aligned}&(D^\alpha \mathcal{M}\mathcal{N}\mathcal{K}v, D^\alpha v) + (D^\alpha \mathcal{M}\mathcal{N}v, D^\alpha \mathcal{K}v) + (D^\alpha \mathcal{M}\mathcal{K}v, D^\alpha \mathcal{N}v) + (D^\alpha \mathcal{M}v, D^\alpha \mathcal{N}\mathcal{K}v)\\ &\quad = (D^\alpha \mathcal{N}\mathcal{K}v, \bar{m}D^\alpha v) + (D^\alpha \mathcal{N}v, \bar{m}D^\alpha \mathcal{K}v) + ([\mathcal{M}, D^\alpha \mathcal{N}]\mathcal{K}v, D^\alpha v)\\ &\qquad + ([\mathcal{M}, D^\alpha \mathcal{N}]v, D^\alpha \mathcal{K}v)([\mathcal{M}, D^\alpha]\mathcal{K}v, D^\alpha \mathcal{N}v) + ([\mathcal{M}, D^\alpha]v, D^\alpha \mathcal{N}\mathcal{K}v) \le C|v|_l^2,\end{aligned}$$

where in the last inequality we used (ii). Hence (iii) is proved. □

Lemma 3.8. *Assume that a^{ij}, b^0 and their derivatives up to order $l \vee 1$, b^i and their derivatives up to order $l \vee 2$, a^i, a^0 and their derivatives up to order l are real functions, in magnitude bounded by K for $i = 1, \dots, d$. Then for a constant $C = C(K, l, d)$*

$$|(D^\alpha \mathcal{M}v, D^\alpha \mathcal{L}v) + \langle D^\alpha v, D^\alpha \mathcal{M}\mathcal{L}v\rangle| \le C|v|_{l+1}^2,$$

for $v \in H^{l+2}$ and multi-indices $|\alpha| \le l$.

Proof. Let us check first the case $l = 0$. Denote by $\mathcal{M}^*$ the formal adjoint of $\mathcal{M}$. We have

$$|(\mathcal{M}v, \mathcal{L}v) + \langle v, \mathcal{M}\mathcal{L}v\rangle = |((\mathcal{M} + \mathcal{M}^*)v, \mathcal{L}v)| = |(-b_i^i v + 2b^0 v, \mathcal{L}v)| \le C|v|_1,$$

where $b_i^i = D_i b^i$. For the general case, let $|\alpha| \le l$ and write

$$\begin{aligned}&(D^\alpha \mathcal{M}v, D^\alpha \mathcal{L}v) + \langle D^\alpha v, D^\alpha \mathcal{M}\mathcal{L}v\rangle\\ &\quad = (D^\alpha Mv, D^\alpha Lv) + \langle D^\alpha v, MD^\alpha \mathcal{L}v\rangle + (D^\alpha v, [\mathcal{M}, D^\alpha]\mathcal{L}v)\\ &\quad = (D^\alpha Mv, D^\alpha \mathcal{L}v) + (M^* D^\alpha v, D^\alpha \mathcal{L}v) + (D^\alpha v, [\mathcal{M}, D^\alpha]Lv)\\ &\quad = ([M, D^\alpha]v, D^\alpha \mathcal{L}v) + (-b_i^i D^\alpha v + 2b^0 D^\alpha v, D^\alpha Lv) + (D^\alpha v, [\mathcal{M}, D^\alpha]\mathcal{L}v),\end{aligned}$$

from which the estimate follows. □

The next lemma is a standard fact for elliptic differential operators $\mathcal{L} = a^{ij}D_iD_j + a^iD_i + a^0$.

Lemma 3.9. *Assume there exists a constant* $\lambda > 0$ *such that*

$$a^{ij}(x)z^iz^j \geq \lambda\,|z|^2\,, \quad \text{for all } z, x \in \mathbb{R}^d,$$

and that the derivatives of a^i *and* a^0 *up to order* $(l-1)\vee 0$, *and the derivatives of* a^{ij} *up to order* $l \vee 1$ *are functions, bounded by* K, *for* $i, j = 1, \dots, d$. *Then there is a constant* $C = C(K\lambda, l, d)$ *such that*

$$(D^\alpha v, D^\alpha \mathcal{L}v) \leq C|v|_l^2 - \frac{\lambda}{2}|v|_{l+1}^2,$$

for all $v \in H^{l+2}$ *and multi-indices* $|\alpha| \leq l$.

In the next two lemmas we assume that there exist vector fields

$$\sigma^1 = (\sigma^{i1}(x)), \dots, \sigma^p = (\sigma^{ip}(x)),$$

such that $a^{ij} = \sigma^{ir}\sigma^{jr}$ for all $i, j = 1, \dots, d$. Set

$$\mathcal{N}^r := \sigma^{ir}D_i, \qquad r = 1, \dots, p,$$

and notice that if the σ^r are differentiable then we can write $\mathcal{L} = \sum_{r=1}^p (\mathcal{N}^r)^2 + \mathcal{N}^0$, where $\mathcal{N}^0 = \left(a^j - \sigma^{ir}(D_i\sigma^{jr})\right) D_j + a^0$.

Lemma 3.10. *Assume that the derivatives of* σ *up to order* $(l+1)\vee 2$ *and the derivatives of* a^i, a^0 *up to order* $l \vee 1$ *are functions, bounded by a constant* K *for* $i = 1, \dots, d$. *Then*

$$(D^\alpha \mathcal{L}v, D^\alpha v) \leq -\sum_{r=1}^p |D^\alpha \mathcal{N}^r v|_0^2 + C|v|_l^2,$$

and

$$|(D^\alpha \mathcal{L}v, D^\alpha u)| \leq \sum_{r=1}^p |D^\alpha \mathcal{N}^r v|_0^2 + C(|v|_l^2 + |u|_{l+1}^2),$$

for all $v, u \in H^{l+2}$ *and multi-indices* $|\alpha| \leq l$, *with a constant* $C = C(K, d, l, p)$.

Proof. By Lemma 3.7 (ii) and (iii),

$$\begin{aligned}(D^\alpha v, D^\alpha \mathcal{L}v) &= (D^\alpha v, D^\alpha \mathcal{N}^r\mathcal{N}^r v)_0 + (D^\alpha v, D^\alpha \mathcal{N}^0 v)_0 \\ &\leq -(D^\alpha \mathcal{N}^r v, D^\alpha \mathcal{N}^r v)_0 + C|v|_l^2,\end{aligned}$$

with a constant $C = C(K, d, l, p)$ and the first inequality of the statement follows. To get the second one we need only note that by interchanging differential operators and by integration by parts we have

$$\begin{aligned}|(D^\alpha \mathcal{N}^r\mathcal{N}^r v, D^\alpha u)| &\leq |(D^\alpha \mathcal{N}^r v, \mathcal{N}^r D^\alpha u)| + |([\mathcal{N}^r, D^\alpha]\mathcal{N}^r v, D^\alpha u)| + C|u|_{l+1}^2 \\ &\leq \sum\nolimits_{r=1}^p |D^\alpha \mathcal{N}^r v|_0^2 + C|u|_{l+1}^2, \\ |(D^\alpha \mathcal{N}^0 v, D^\alpha u)| &\leq C(|v|_l^2 + |u|_{l+1}^2),\end{aligned}$$

with constants $C = C(K, d, p, l)$. □

Lemma 3.11. *Assume that the derivatives of σ^i and b^i up to order $(l+2)\vee 3$ and the derivatives of a^i, a^0 and b^0 up to order $(l+1)\vee 2$ are functions, bounded by a constant K for $i=1,\dots,d$. Then*

$$|(D^\alpha \mathcal{M}\mathcal{L}v, D^\alpha v) + (D^\alpha \mathcal{L}v, D^\alpha \mathcal{M}v)| \le C\sum_{r=1}^{p} |\mathcal{N}^r v|_l^2 + C|v|_l^2,$$

with a constant $C = C(K,l,d,p)$ for all $v \in H^{l+3}$ and multi-indices $|\alpha| \le l$.

Proof. Put $\mathcal{N} = \mathcal{K} = \mathcal{N}^r$, $r=1,\dots,p$ in Lemma 3.7 (iii) and use (i) of the same lemma for $\mathcal{N}^r v$ to get

$$\begin{aligned}&|(D^\alpha \mathcal{M}\mathcal{N}^r\mathcal{N}^r v, D^\alpha v) + (D^\alpha \mathcal{M}v, D^\alpha \mathcal{N}^r\mathcal{N}^r v)|\\ &\quad \le C|v|_l^2 + 2|(D^\alpha \mathcal{M}\mathcal{N}^r v, D^\alpha \mathcal{N}^r v)| \le C|v|_l^2 + C\sum_{r=1}^{p} |\mathcal{N}^r v|_l^2,\end{aligned}$$

and apply Lemma 3.7 (ii) to obtain

$$|(D^\alpha \mathcal{M}\mathcal{N}^0 v, D^\alpha v) + (D^\alpha \mathcal{M}v, D^\alpha \mathcal{N}^0 v)| \le C|v|_l^2,$$

which prove the corollary. □

The following Gronwall-type lemma will be useful for our estimates in the next section.

Lemma 3.12. *Let y_n, m_n, Q_n and q_n be sequences of real-valued continuous $\mathcal{F}_t$-adapted stochastic processes given on the interval $[0,T]$, such that Q_n is a non-decreasing non-negative process and m_n is a local martingale starting from 0. Let δ, γ be some real numbers with $\delta < \gamma$. Assume that almost surely*

$$0 \le y_n(t) \le \int_0^t y_n(s)\,dQ_n(s) + m_n(t) + q_n(t),$$

holds for all $t \in [0,\tau_n]$ and integers $n \ge 1$, where

$$\tau_n = \inf\left\{t \ge 0 : y_n(t) \ge n^{-\delta}\right\} \wedge T.$$

Suppose that almost surely

$$Q_n(\tau_n) = o(\ln n), \quad \sup_{t\le\tau_n} q_n(t) = O(n^{-\gamma}),$$

$$d\langle m_n\rangle \le (y_n^2 + k_n y_n)\,dQ_n, \text{ on } t \in [0,\tau_n], \quad \int_0^{\tau_n} k_n(s)\,dQ_n = O(n^{-\gamma})$$

for a sequence of non-negative $\mathcal{F}_t$-adapted processes k_n. Then almost surely

$$\sup_{t\le T} y_n(t) = O(n^{-\kappa}), \quad \text{for each } \kappa < \gamma. \tag{3.6}$$

Proof. Let us assume first that $\gamma > 0$. The case $\delta = 0$ is a slight modification of [12, Lemma 3.8]. It can be proved in the same way by using a suitable generalization of Lemma 3.7 from [12] (see [10]). For $\delta < \gamma \in (0,\infty)$, we see that the conditions of the lemma are satisfied with $\gamma' = \gamma - \delta$,

$$y_n'(t) = \frac{y_n(t)}{n^{-\delta}}, \quad m_n'(t) = \frac{m_n(t)}{n^{-\delta}}, \quad q_n'(t) = \frac{q_n(t)}{n^{-\delta}}, \quad k_n'(t) = \frac{k_n(t)}{n^{-\delta}}, \tag{3.7}$$

in place of γ, y_n, m_n, q_n and k_n, with $\delta = 0$. Hence we have (3.6) for y_n' in place of y_n for each $\kappa < \gamma'$, which gives (3.6) in this case.

Suppose now that $\gamma \le 0$. Take $\bar\gamma \in (\delta, \gamma)$ and set $\gamma' := \gamma - \bar\gamma$, $\delta' := \delta - \bar\gamma$. Define y_n', m_n', q_n' and k_n' as in (3.7) with $\bar\gamma$ in place of δ. Notice that $\gamma' > 0$ and that the conditions of the lemma are satisfied by the processes y_n', m_n', q_n' and k_n' in place of y_n, m_n, q_n and k_n, with γ' and δ' instead of γ and δ. Hence (3.6) holds for y_n' and γ' in place of y_n and γ, which gives the lemma. □

Corollary 3.13. *Let σ_n be an increasing sequence of stopping times converging to infinity almost surely. Assume that the conditions of the previous lemma are satisfied with $\bar\tau_n = \inf\{t \ge 0 : y_n(t) \ge n^{-\delta}\} \wedge T \wedge \sigma_n$ in place of τ_n. Then its conclusion,* (3.6)*, still holds.*

Proof. The conditions of Lemma 3.12 are satisfied by the processes

$$y_n'(t) = y_n(t \wedge \sigma_n), \quad m_n'(t) = m_n(t \wedge \sigma_n), \quad q_n'(t) = q_n(t \wedge \sigma_n),$$
$$k_n'(t) = k_n(t \wedge \sigma_n),$$

in place of y_n, m_n, q_n and k_n and with $\tau_n' = \inf\left\{t \ge 0 : y_n'(t) \ge n^{-\delta}\right\}$ in place of τ_n. Hence

$$\sup_{t \le T} y_n'(t) = O(n^{-\kappa}), \quad \text{for each } \kappa < \gamma. \tag{3.8}$$

Define the set $\Omega_n := [\sigma_n \ge T]$ and note that since $\sigma_n \nearrow \infty$ almost surely, the set $\Omega' = \cup_{n\ge1}\Omega_n$ has full probability. It remains to prove that the random variable

$$\xi := \sup_{m\ge1} \sup_{t\le T} \frac{y_m(t)}{m^{-\kappa}}$$

is finite almost surely for all $\kappa < \gamma$. Indeed, take $\omega \in \Omega'$. Then $\omega \in \Omega_n$ for some $n(\omega) \ge 1$, hence $\sigma_m(\omega) \ge T$ for all $m \ge n(\omega)$ and, by (3.8),

$$\sup_{t\le T} y_m(t \wedge \sigma_m) = \sup_{t\le T} y_m(t) \le \zeta_\kappa m^{-\kappa},$$

for all $m \ge n(\omega)$. Since ζ_κ is finite almost surely, so it is ξ. □

4. The growth of the approximations

In this section we estimate solutions u_n of (1.1) for large n. We fix an integer $l \ge 0$, a constant $K \ge 0$, and make the following assumptions.

Assumption 4.1. The derivatives in x of the coefficients a_n^{ij}, a_n^i, a_n, b_n^k up to order $l+1$, and the derivatives in x of b_n^{ik} up to order $(l+1) \vee 2$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable real functions on $\Omega \times [0,T] \times \mathbb{R}^d$ and in magnitude are bounded by K, for all $i,j = 1,\dots,d$, $k = 1,\dots,d_1$, and all $n \ge 1$.

Assumption 4.2. For each $\varepsilon > 0$ almost surely

$$|u_{n0}|_l = O(n^\varepsilon), \quad \int_0^T \left(|f_n|_l^2 + |g_n|_{l+1}^2\right) \, dt = O(n^\varepsilon), \quad \sup_{t \le T} |g_n(t)|_l^2 = O(n^\varepsilon).$$

We will often use the notation notation $f \cdot V(t)$ for the integral

$$\int_0^t f(s) \, dV(s),$$

when V is a semimartingale and f is a predictable process such that the stochastic integral of f against dV over $[0,t]$ is well defined. We define

$$\eta_n(t) = \sup_{m \ge n} \sup_{s \le t} |W(s) - W_m(s)| \, .$$

Notice that Assumption 2.1 (i) clearly implies that $\eta_n(T) = O(n^{-\kappa})$ almost surely for each $\kappa < \gamma$.

First we study the case when b_n^{ik}, b_n^k and g_n^k do not depend on $t \in [0,T]$.

Theorem 4.3. *Assume that b_n^{ik}, b_n^k and g_n^k do not depend on t for $i = 1, \dots, d$, $k = 1, \dots, d_1$ and $n \ge 1$. Let Assumptions* 2.1 (i) *and* (iii)*,* 4.1*,* 4.2 *and* 2.3 *with $\lambda > 0$ hold. Then for every $\varepsilon > 0$ almost surely*

$$\sup_{t \le T} |u_n(t)|_l^2 + \int_0^T |u_n|_{l+1}^2 \, dt = O(n^\varepsilon).$$

Proof. Assume for the moment that $u_{n0} \in H^{l+1}$ almost surely. Recall that we are assuming that W_n^k is of bounded variation, $n \ge 1$, $k = 1, \dots, d_1$. Then by Theorem 3.5, under Assumptions 4.1, 4.2 and 2.3 with $\lambda > 0$ there is a unique generalized solution u_n of (1.1)–(1.2), and it is an H^{l+1}-valued weakly continuous process such that almost surely

$$\int_0^T |u_n(t)|_{l+2}^2 \, dt < \infty.$$

In particular,

$$\begin{aligned} &(u_n(t), \varphi)_0 \\ &= (u_{n0}, \varphi)_0 + \int_0^t \left[-(a_n^{ij} D_i u_n, D_j \varphi)_0 + \left((a_n^i - D_j a_n^{ij}) D_i u_n + a_n u_n + f_n, \varphi\right)_0 \right] ds \\ &\qquad + \int_0^t (b_n^{ik} D_i u_n + b_n^k u_n + g_n^k, \varphi)_0 \, dW_n^k \end{aligned}$$

holds for all $t \in [0,T]$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$. Substituting here $\sum_{|\alpha| \le l} (-1)^{|\alpha|} D^{2\alpha} \varphi$ in place of φ and integrating by parts, we get

$$\begin{aligned} &(u_n(t), \varphi)_l \\ &= (u_{n0}, \varphi)_l + \int_0^t \left[-(a_n^{ij} D_i u_n, D_j \varphi)_l + \left((a_n^i - D_j a_n^{ij}) D_i u_n + a_n u_n + f_n, \varphi\right)_l \right] ds \\ &\qquad + \int_0^t (b_n^{ik} D_i u + b_n^k u_n + g_n^k, \varphi)_l \, dW_n^k(s). \end{aligned}$$

Hence using Itô's formula in the triple $H^{m+1} \hookrightarrow H^m \equiv (H^m)^* \hookrightarrow H^{m-1}$, we have

$$\begin{aligned} d|u_n|_l^2 &= 2(u_n, L_n u_n + f_n)_l\,dt + 2(u_n, M_n^k u_n + g_n^k)_l\,dW_n^k \\ &= 2(u_n, L_n u_n + f_n)_l\,dt + 2(u_n, M_n^k u_n + g_n^k)_l\,dW^k \\ &\quad + 2(u_n, M_n^k u_n + g_n^k)_l\,d(W_n^k - W^k). \end{aligned}$$

Therefore, integrating by parts in the last term above we have

$$|u_n|_l^2 = \sum_{j=1}^{6} I_n^{(j)}, \tag{4.1}$$

where

$$\begin{aligned} I_n^{(1)} &= |u_{n0}|_l^2\,, \\ I_n^{(2)} &= 2(u_n, L_n u_n + f_n)_l \cdot t, \\ I_n^{(3)} &= 2(u_n, M_n^k u_n + g_n^k)_l \cdot W^k(t), \\ I_n^{(4)} &= 2(u_n, M_n^k u_n + g_n^k)_l (W_n^k - W^k)\Big|_0^t, \\ I_n^{(5)} &= 2\left(\{(L_n u_n + f_n, M_n^k u_n + g_n^k)_l + \langle u_n, M_n^k(L_n u_n + f_n)\rangle_l\}(W^k - W_n^k)\right)\cdot t, \\ I_n^{(6)} &= 2\left((M_n^j u_n + g^j, M_n^k u_n + g_n^k)_l + \langle u_n, M_n^k(M_n^j u_n + g_n^j)\rangle_l\right) \cdot B_n^{kj}(t), \end{aligned}$$

and $\langle\cdot\,,\cdot\rangle_l$ is the duality product between H^{l+1} and H^{l-1}, based on the inner product $(\cdot\,,\cdot)_l$ in H^l. Using Lemma 3.9 we obtain

$$I_n^{(2)} \le \left[C\,|u_n|_l^2 - \lambda|u_n|_{l+1}^2 + C|f_n|_{l-1}^2\right] \cdot t$$

with a constant $C = C(K, l, d, \lambda)$. The term $I_n^{(3)}$ is a continuous local martingale starting from 0, such that its quadratic variation, $\langle I_n^{(3)}\rangle$, satisfies, by Lemma 3.7 (i)

$$d\langle I_n^{(3)}\rangle \le C(|u_n|_l^4 + |u_n|_l^2|g_n|_l^2)\,dt, \tag{4.2}$$

and also by Lemma 3.7 (i) we have

$$I_n^{(4)}(t) \le C \sup_{0\le s\le t}\left(|u_n(s)|_l^2 + |g_n(s)|_l^2\right)\eta_n(t)$$

with a constant $C = C(K, d, d_1, l)$, where $|g_n|_l^2 = \sum_k |g_n^k|_l^2$. Using Lemmas 3.7 (ii) and 3.8, we have

$$I_n^{(5)}(t) \le C\left[|u_n|_{l+1}^2 + |f_n|_l^2 + |g_n|_{l+1}^2\right]\eta_n \cdot t,$$

and

$$I_n^{(6)}(t) \le C\left[|u_n|_l^2 + |g_n|_l^2\right] \cdot \|B_n\|(t),$$

with $\|B_n\| := \sum_{j,k} \|B_n^{jk}\|$ and a constant $C = C(K, d, d_1, l)$. Therefore, from (4.1),

$$\begin{aligned}|u_n(t)|_l^2 \le \int_0^t (|u_n(s)|_l^2 + |g_n|_l^2)\, dQ_n(s) + m_n(t) \\ - \int_0^t (\lambda - C\eta_n(s))\, |u_n(s)|_{l+1}^2\, ds + C\int_0^t (|f_n(s)|_l + |g_n|_{l+1}^2)(1+\eta_n(s))\, ds \\ + |u_0|_l^2 + C \sup_{0\le s\le t} \Big(|u_n(s)|_l^2 + |g_n(s)|_l^2\Big)\, \eta_n(t), \qquad (4.3)\end{aligned}$$

with $m_n = I_n^{(3)}$ and $Q_n(s) = C(s + \|B_n\|(s))$. Define

$$\sigma_n := \inf\{t \ge 0 : 2C\eta_n(t) \ge \lambda\},$$

and note that almost surely

$$\begin{aligned}y_n(t) := |u_n(t)|_l^2 + \frac{\lambda}{2}\int_0^t |u_n(s)|_{l+1}^2\, ds \le \int_0^t |u_n(s)|_l^2\, dQ_n(s) + m_n(t) + q_n(t) \\ \le \int_0^t y_n(s)\, dQ_n(s) + m_n(t) + q_n(t) \quad \text{for all } t \in [0, \sigma_n], \qquad (4.4)\end{aligned}$$

with $q_n = q_n^{(1)} + q_n^{(2)}$, where

$$\begin{aligned}q_n^{(1)}(t) = |u_{n0}|_l^2 + C\eta_n(t) \sup_{0\le s\le t} |g_n|_l^2 \\ + C\int_0^t (1+\eta_n)(|f_n(s)|_l^2 + |g_n|_{l+1}^2)\, ds + C\int_0^t |g_n|_l^2\, d\|B_n\|(s), \\ q_n^{(2)}(t) = C\eta_n(t) \sup_{0\le s\le t} |u_n(s)|_l^2.\end{aligned}$$

Due to Assumptions 2.1 (i) and (iii) and 4.2 we have

$$\sup_{t\le T} |q_n^{(1)}(t)| = O(n^\varepsilon) \quad \text{almost surely for each } \varepsilon > 0.$$

For a given $\kappa \in (0, \gamma)$ and any $\varepsilon \in (0, \kappa)$ take $\bar\varepsilon \in (\varepsilon, \kappa)$ and define

$$\tau_n = \inf\{t \ge 0 : |u_n(t)|_l^2 \ge n^{\bar\varepsilon}\}.$$

Then clearly

$$\sup_{t\le \tau_n} |q_n^{(2)}(t)| = O(n^\varepsilon) \quad \text{a.s. for each } \varepsilon > 0.$$

Thus

$$\sup_{t\le \tau_n} |q_n(t)| = O(n^\varepsilon) \quad \text{a.s. for each } \varepsilon > 0.$$

Now taking into account (4.2) and noting that $\sigma_n \nearrow \infty$, we finish the proof of the theorem by applying Corollary 3.13 to (4.4). Since our estimates do not depend on the norm of u_{n0} in H^{l+1} but on its norm in H^l, by a standard approximation argument we can relax the assumption that u_{n0} is almost surely in H^{l+1}. □

In the case when b_n^{ik}, b_n^k and g_n^k depend on t we make the following assumption.

Assumption 4.4. For each $n \geq 1$, $i = 1, 2, \dots, d$ and $k = 1, \dots, d_1$ there exist real-valued $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable functions $b_n^{ik(r)}$, $b_n^{k(r)}$ on $\Omega \times [0,T] \times \mathbb{R}^d$ and H^0-valued predictable processes $g_n^{k(r)}$ for $r = 0, 1, \dots, d_1$, such that almost surely

$$d(b_n^{ik}(t), \varphi) = (b_n^{ik(0)}(t), \varphi)\, dt + (b_n^{ik(p)}(t), \varphi)\, dW_n^p(t),$$
$$d(b_n^{k}(t), \varphi) = (b_n^{k(0)}(t), \varphi)\, dt + (b_n^{k(p)}(t), \varphi)\, dW_n^p(t),$$
$$d(g_n^{k}(t), \varphi) = (g_n^{k(0)}(t), \varphi)\, dt + (g_n^{k(p)}(t), \varphi)\, dW_n^p(t),$$

for all $i = 1, \dots, d$, $k = 1, \dots, d_1$, every $n \geq 1$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$. The functions $b_n^{ik(r)}$ together with their derivatives in x up to order $l \vee 1$ and the functions $b_n^{k(r)}$ together with their derivatives in x up to order l are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable functions, bounded by K for all $n \geq 1$ and $r = 0, 1, \dots, d_1$. Moreover, for each $\varepsilon > 0$

$$\int_0^T |g_n^{k(0)}|_{l-1}^2\, dt = O(n^\varepsilon), \quad \sup_{t \leq T} \sum_{p=1}^d |g_n^{k(p)}|_l^2 = O(n^\varepsilon) \quad \text{for } k = 1, \dots, d_1.$$

Theorem 4.5. *Let the assumptions of Theorem 4.3 together with Assumption 4.4 hold. Then we have the conclusion of Theorem 4.3.*

Proof. We can follow the proof of the previous theorem with minor changes. We need only add an additional term,

$$I_n^{(7)} = 2\{(W^k - W_n^k)(u_n, M_n^{k(0)} u_n + g^{k(0)})_l\} \cdot t + 2(u_n, M_n^{k(p)} u_n + g^{k(p)})_l \cdot B_n^{kp}(t),$$

to the right-hand side of (4.1), where for each $n \geq 1$,

$$M_n^{k(r)} = b_n^{ik(r)} D_i + b_n^{k(r)},$$

for $k = 1, \dots, d_1$ and $r = 0, \dots, d_1$. Clearly, $2(u_n, g^{k(0)})_l \leq |u_n|_{l+1}^2 + |g^k|_{l-1}^2$, and hence

$$\begin{aligned} &|2(W^k - W_n^k)(u_n, M_n^{k(0)} u_n + g^{k(0)})_l| \\ &\quad \leq \eta_n(|u_n|_{l+1}^2 + |g_n^{(0)}|_{l-1}^2 + C|u_n|_l^2), \end{aligned}$$

and, by Lemma 3.7 (i),

$$2|(u_n, M_n^{k(p)} u_n + g^{k(p)})_l| \leq C|u_n|_l^2 + |g^{k(p)}|_l^2$$

with a constant $C = C(K, d, d_1, l)$, where $|g_n^{(0)}|_{l-1}^2 = \sum_{k=1}^{d_1} |g_n^{k(0)}|_{l-1}^2$. Thus inequality (4.3) holds with the additional term

$$q_n^{(3)}(t) = \eta_n(t) \int_0^t |g_n^{(0)}(s)|_{l-1}^2\, ds + \int_0^t |g^{k(p)}(s)|_l^2\, d\|B^{kp}\|(s)$$

added to its left-hand side and with a constant $C = C(\lambda, K, d_1, l)$. Since due to Assumptions 4.4 and 2.1 (iii), for each $\varepsilon > 0$ we have

$$\sup_{t \leq T} q_n^{(3)}(t) = O(n^\varepsilon) \quad \text{almost surely},$$

we can finish the proof as in the proof of Theorem 4.3. □

Let us consider now the degenerate case, $\lambda = 0$ in Assumption 2.3.

Assumption 4.6. For each $n \geq 1$ there exist real-valued functions σ_n^{ip} on $\Omega \times H_T$ for $p = 1, 2, \dots, d_2$ such that $a_n^{ij} = \sigma_n^{ip}\sigma_n^{jp}$ for all $i, j = 1, \dots, d$. For all $n \geq 1$ the functions σ_n^{ip} and b_n^i and their derivatives in $x \in \mathbb{R}^d$ up to order $(l+2) \vee 3$, the functions a_n^i, a_n, b_n and their derivatives in x up to order $(l+1) \vee 2$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable functions, bounded by K, for all $i = 1, \dots, d$ and $p = 1, \dots, d_2$.

Assumption 4.7. Let Assumption 4.4 hold and assume that for each $\varepsilon > 0$

$$\int_0^T |g_n^{k(0)}(t)|_l^2 \, dt = O(n^\varepsilon) \quad \text{almost surely for each } k = 1, 2, \dots, d_1.$$

Theorem 4.8. *Let Assumptions* 2.1 (i) *and* (iii)*,* 2.3 (*with* $\lambda = 0$)*,* 4.2*,* 4.6 *and* 4.7 *hold. Then*

$$\sup_{t \leq T} |u_n(t)|_l^2 + \sum_{r=1}^{d_2} \int_0^T |N_n^r u_n|_l^2 \, ds = O(n^\varepsilon), \quad \text{for every } \varepsilon > 0 \text{ almost surely,}$$

where $N_n^r = \sigma_n^{ir} D_i$*, for* $r = 1, \dots, d_2$*.*

Proof. The proof follows the lines of that of Theorems 4.3 and 4.5, but instead of estimating $I_n^{(2)}$ and $I_n^{(5)}$ separately, we estimate their sum as follows. Note that

$$\begin{aligned} dI_n^{(5)} &= 2\{(L_n u_n + f_n, M_n^k u_n + g_n^k)_l + \langle u_n, M_n^k(L_n u_n + f_n)\rangle_l\}(W^k - W_n^k)\, dt \\ &= 2\{(L_n u_n, M_n^k u_n)_l + \langle u_n, M_n^k L_n u_n\rangle_l + (L_n u_n, g_n^k)_l\}(W^k - W_n^k)\, dt \\ &\quad + 2\{(f_n, M_n^k u_n)_l + (u_n, M_n^k f_n)_l\}(W^k - W_n^k)\, dt \\ &\quad + 2(f_n, g_n^k)_l (W^k - W_n^k)\, dt. \end{aligned}$$

Using Lemmas 3.11, 3.10, 3.7 (i) and (ii), we obtain

$$\begin{aligned} |(L_n u_n, M_n^k u_n)_l + \langle u_n, M_n^k L_n u_n\rangle_l| &\leq C \sum_{r=1}^{d_2} |N_n^r u_n|_l^2 + C|u_n|_l^2, \\ (u_n, L_n u_n + f_n)_l &\leq -\sum_{r=1}^{d_2} |N_n^r u_n|_l^2 + C|u_n|_l^2 + |f_n|_l^2, \\ |(f_n, M_n^k u_n)_l + (u_n, M_n^k f_n)_l| &\leq C\left(|u_n|_l^2 + |f_n|_l^2\right), \\ |(L_n u_n, g_n^k)_l| &\leq \sum_{r=1}^{d_2} |N_n^r u_n|^2 + C(|u_n|_l^2 + |g_n^k|_{l+1}^2) \end{aligned}$$

with a constant $C = C(K, l, d, d_2)$. Hence

$$dI_n^{(5)}(t) \leq C\eta_n(t) \left(\sum_{r=1}^{d_2} |N_n^r u_n|_l^2 + |g_n|_{l+1}^2 + |f_n|_l^2 + |u_n|_l^2 \right) dt,$$

and recalling that $I_n^{(2)}(t) = 2(u_n, L_n u_n + f_n)_l \cdot t$, we get

$$I_n^{(2)}(t) + I_n^{(5)}(t) \le C\left\{(\eta_n+1)(|u_n|_l^2 + |f_n|_l^2) + \eta_n\,|g_n|_{l+1}^2)\right\} \cdot t \\ + \{(C\eta_n - 2)\sum_{r=1}^{d_2} |N_n^r u_n|_l^2\} \cdot t.$$

To estimate $I_n^{(7)}$ we use that, by Lemma 3.7(i),

$$|2(W^k - W_n^k)\{(u_n, M_n^{k(0)} u_n + g_n^{k(0)})_l\}| \le C\eta_n(|u_n|_l^2 + |g_n^{(0)}|_l^2)$$

with a constant C and $|g_n^{(0)}|_l^2 = \sum_k |g_n^{k(0)}|_l^2$. Thus using the estimates for $I_n^{(1)}$, $I_n^{(3)}$, $I_n^{(4)}$ and $I_n^{(6)}$ given in the proof of Theorem 4.3, and defining

$$\sigma_n = \inf\{t \ge 0 : C\eta_n \ge 1\},$$

and

$$y_n(t) = |u_n(t)|_l^2 + \sum_{r=1}^{d_2} \int_0^t |K_r u_n|_l^2\, ds,$$

we get

$$y_n(t) \le \int_0^t y_n(s)\, dQ_n(s) + m_n(t) + q_n(t) \quad \text{almost surely for all } t \in [0, \sigma_n],$$

with

$$Q_n(s) = C\{(\eta_n+1)s + \|B_n\|(s) + \eta_n\}, \quad m_n = I_n^{(3)}, \quad q_n = q_n^{(1)} + q_n^{(2)} + \bar q_n^{(3)},$$

$$\bar q_n^{(3)} := \int_0^t |g_n^{(0)}(s)|_l^2 \sup_{r\le t} |W(r) - W_n(r)|\, ds + \int_0^t |g_n^{k(p)}(s)|_l^2\, d\|B_n^{kp}\|(s),$$

where $\|B_n\|(s) = \sum_{k,p} \|B_n^{kp}\|(s)$, $q_n^{(1)}$ and $q_n^{(2)}$ are defined in the proof Theorem 4.3, and C is a constant depending only on K, d, d_1, d_2 and l. Hence the proof is the same as that of Theorem 4.3. □

5. Rate of convergence results for SPDEs

Here we present two theorems on rate of convergence which provide us with a technical tool to prove our main results. Consider for each integer $n \ge 1$ the problem

$$du_n(t,x) = (\mathcal{L}_n u_n(t,x) + f_n(t,x))\, dt + (\mathcal{M}_n^k u_n(t,x) + g_n^k(t,x))\, dW^k(t) \\ + (\mathcal{N}_n^\rho u_n(t,x) + h_n^\rho(t,x))\, dB_n^\rho(t), \quad (t,x) \in H_T, \tag{5.1}$$

$$u_n(0,x) = u_{n0}(x) \quad x \in \mathbb{R}^d, \tag{5.2}$$

where $B_n = (B_n^\rho)$ is an $\mathbb{R}^{d_2}$-valued continuous adapted process of finite variation on $[0,T]$. The operators $\mathcal{L}_n$, $\mathcal{M}_n^k$ and $\mathcal{N}_n^\rho$ are of the form

$$\mathcal{L}_n = \mathfrak{a}_n^{ij}(t,x)D_{ij} + \mathfrak{a}_n^i(t,x)D_i + \mathfrak{a}_n(t,x),$$

$$\mathcal{M}_n^k = \mathfrak{b}_n^{ik}(t,x)D_i + \mathfrak{b}_n^k(t,x), \quad \mathcal{N}_n^\rho = \mathfrak{c}_n^{i\rho}(t,x)D_i + \mathfrak{c}_n^\rho(t,x)$$

where $\mathfrak{a}_n^{ij}$, $\mathfrak{a}_n^i$, $\mathfrak{a}_n$, $\mathfrak{b}_n^{ik}$, $\mathfrak{b}_n^k$, $\mathfrak{c}_n^{i\rho}$ and $\mathfrak{c}_n^\rho$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable real functions on $\Omega\times H_T$ for $i,j=1,\dots,d$, $k=1,\dots,d_1$, $\rho=1,\dots,d_2$ and $n\geq 1$. For each n the initial value u_{n0} is an H^0-valued $\mathcal{F}_0$-measurable random variable, f_n is an H^{-1}-valued predictable process and g_n^k and h_n^ρ are H^0-valued predictable processes for $k=1,\dots,d_1$ and $\rho=1,\dots,d_2$. We use the notation $|g_n|_r^2=\sum_k|g_n^k|_r^2$ and $|h_n|_r^2=\sum_\rho|h_n^\rho|_r^2$ for $r\geq 0$.

Let $l\geq 0$ be an integer and let $K\geq 0$, $\gamma>0$ be fixed constants.

We assume the *stochastic parabolicity condition.*

Assumption 5.1. There is a constant $\lambda\geq 0$ such that for all $n\geq 1$, $dP\times dt\times dx$ almost all $(\omega,t,x)\in\Omega\times H_T$ we have

$$(\mathfrak{a}_n^{ij}-\tfrac{1}{2}\mathfrak{b}_n^{ik}\mathfrak{b}_n^{jk})z^iz^j\geq\lambda|z|^2\quad\text{for all } z=(z^1,\dots,z^d)\in\mathbb{R}^d.$$

In the case when $\lambda>0$ we will use the following conditions.

Assumption 5.2. The coefficients $\mathfrak{a}_n^{ij}$, $\mathfrak{b}_n^{ik}$, $\mathfrak{c}_n^{i\rho}$ and their derivatives in x up to order $l\vee 1$ are bounded in magnitude by K for all $i,j=1,\dots,d$, $k=1,\dots,d_1$, $\rho=1,\dots,d_2$ and $n\geq 1$. The coefficients $\mathfrak{a}_n^i$, $\mathfrak{a}_n$, $\mathfrak{b}_n^k$, $\mathfrak{c}_n^\rho$ and their derivatives in x up to order l are bounded in magnitude by K for all $i=1,\dots,d$, $k=1,\dots,d_1$, $\rho=1,\dots,d_2$ and $n\geq 1$.

Assumption 5.3. We have $|u_{n0}|_l=O(n^{-\gamma})$,

$$\int_0^T|f_n(s)|_{l-1}^2\,ds=O(n^{-2\gamma}),\quad \int_0^T|g_n(s)|_l^2\,ds=O(n^{-2\gamma}),$$

$$\int_0^T|h_n^\rho(t)|_l^2\,d\|B^\rho\|(t)=O(n^{-2\gamma}),\qquad \sum_{\rho=1}^{d_2}\|B_n^\rho\|(T)=o(\ln n).$$

Let u_n be a generalized solution of (5.1)–(5.2) in the sense of Definition 3.1, such that u_n is an H^l-valued weakly continuous process, $u_n(t)\in H^{l+1}$ for $P\times dt$-almost every $(\omega,t)\in\Omega\times[0,T]$, and almost surely

$$\int_0^T|u_n(t)|_{l+1}^2\,dt<\infty.$$

Then for $n\to\infty$ we have the following result.

Theorem 5.4. *Let Assumptions* 5.2, 5.3 *and* 5.1 *with* $\lambda>0$ *hold. Then*

$$\sup_{t\leq T}|u_n(t)|_l^2+\int_0^T|u_n(t)|_{l+1}^2\,dt=O(n^{-2\kappa})\quad \textit{a.s. for } \kappa<\gamma. \tag{5.3}$$

Proof. By the definition of the generalized solution

$$\begin{aligned}&(u_n(t),\varphi)=(u_{n0},\varphi)\\&+\int_0^t\big[-(\mathfrak{a}_n^{ij}D_iu_n(s),D_j\varphi)+\big((\mathfrak{a}_n^i-(D_j\mathfrak{a}_n^{ij}))D_iu_n(s)+\mathfrak{a}_nu_n(s)+f_n(s),\varphi\big)\big]\,ds\\&+\int_0^t(\mathcal{M}^ku_n(s)+g_n^k(s),\varphi)\,dW^k(s)+\int_0^t(\mathcal{N}_n^\rho u_n(s)+h_n^\rho(s),\varphi)\,dB_n^\rho(s)\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. By Itô's formula

$$|u_n(t)|_0^2 = |u_{n0}|_0^2 + \int_0^t \mathcal{I}_n(s)\,ds + \int_0^t \mathcal{J}_n^k(s)\,dW^k(s) + \int_0^t \mathcal{K}_n^\rho(s)\,dB^\rho(s),$$

with

$$\begin{aligned}\mathcal{I}_n = &-2(\mathfrak{a}_n^{ij}D_iu_n, D_ju_n) + 2\big((\mathfrak{a}_n^i - \mathfrak{a}_{nj}^{ij})D_iu_n + \mathfrak{a}_nu_n + f_n, u_n\big)\\ &+ (\mathfrak{b}_n^{ik}D_iu_n + \mathfrak{b}_n^ku_n + g_n^k, \mathfrak{b}_n^{jk}D_ju_n + \mathfrak{b}_n^ku_n + g_n^k),\\ \mathcal{J}_n^k = &\,2(\mathcal{M}_n^ku_n + g_n^k, u_n), \qquad \mathcal{K}_n^\rho = 2(\mathcal{N}_n^\rho u_n + h_n^\rho, u_n),\end{aligned}$$

where $\mathfrak{a}_{nj}^{ij} := D_j\mathfrak{a}_n^{ij}$. By Assumption 5.1

$$-2(\mathfrak{a}_n^{ij}D_iu_n, D_ju_n) + (\mathfrak{b}_n^{ik}D_iu_n, \mathfrak{b}_n^{jk}D_ju_n) \le -2\lambda\sum_{i=1}^d |D_iu_n|^2.$$

Hence by standard estimates and Lemma 3.7 (i),

$$\mathcal{I}_n \le -\lambda|u_n|_1^2 + C(|u_n|^2 + |f_n|_{-1}^2 + |g_n|^2), \qquad |\mathcal{K}_n^\rho| \le C(|u_n|^2 + |h_n^\rho|^2),$$

$$|\mathcal{J}_n^k|^2 \le C(|u_n|^4 + |g_n^k|^2|u_n|^2),$$

with a constant $C = C(K,\lambda,d,d_1)$. Consequently, almost surely

$$\begin{aligned}|u_n(t)|^2 + \lambda\int_0^t |u_n|_1^2\,ds \le |u_{n0}|^2 + \int_0^t |u_n|^2\,dQ_n + \int_0^t \mathcal{J}_n^k\,dW^k \\ + C\int_0^t \left(|f_n|_{-1}^2 + |g_n|^2\right)\,ds + C\int_0^t |h_n^\rho|^2\,d\|B_n^\rho\|\end{aligned} \tag{5.4}$$

for all $t \in [0,T]$ and $n \ge 1$, where $Q_n(s) = C\left(s + \sum_{\rho=1}^{d_2}\|B_n^\rho\|(s)\right)$. By Assumption 5.3 we have

$$Q_n(T) = o(\ln n), \quad |u_{n0}|^2 + \int_0^T |f_n|_{-1}^2\,ds + \int_0^T |h_n^\rho(s)|^2\,d\|B_n^\rho\|(s) = O(n^{-2\gamma})$$

almost surely. Notice also that for

$$m_n(t) := \int_0^t \mathcal{J}_n^k\,dW^k(s) \tag{5.5}$$

we have

$$d\langle m_n\rangle = \sum_{k=1}^{d_1} |\mathcal{J}_n^k|^2\,dt \le Cd_1(|u_n(t)|^4 + \gamma_n|g_n|^2|u_n(t)|^2)\,dQ_n,$$

where $\gamma_n(t) = dt/dQ_n(t)$. Due to Assumption 5.3

$$\int_0^T \gamma_n|g_n|^2\,dQ_n = \int_0^T |g_n|^2\,ds = O(n^{-2\gamma}) \quad \text{almost surely.}$$

Hence applying Lemma 3.12 with

$$y_n(t) := |u_n|^2 + \lambda \int_0^t |u_n|_1^2 \, ds,$$
$$q_n(t) := |u_{n0}|^2 + C \int_0^t (|f_n|_{-1}^2 + |g_n|^2) \, ds + C \int_0^t |h_n^\rho|^2 \, d\|B_n^\rho\|,$$

and with m_n defined in (5.5), from (5.4) we get (5.3) for $l = 0$.

Assume now that $l \geq 1$ and let α be a multi-index such that $1 \leq |\alpha| \leq l$. Then $\alpha = \beta + \gamma$ for some multi-index γ of length 1, and by definition of the generalized solution we get

$$\begin{aligned}
&(D^\alpha u_n(t), \varphi) = (D^\alpha u_{n0}, \varphi) \\
&\quad - \int_0^t \left[(D^\alpha \mathfrak{a}_n^{ij} D_i u_n, D_j \varphi) + \left(D^\beta \{ (\mathfrak{a}_n^i - \mathfrak{a}_{nj}^{ij}) D_i u_n + \mathfrak{a}_n u + f_n \}, D^\gamma \varphi \right) \right] ds \\
&\quad + \int_0^t (D^\alpha \mathcal{M}^k u_n + D^\alpha g^k, \varphi) \, dW^k(s) + \int_0^t (D^\alpha \mathcal{N}_n^\rho u_n + D^\alpha h_n^\rho, \varphi) \, dB^\rho(s).
\end{aligned}$$

Hence by Itô's formula

$$|D^\alpha u_n(t)|_0^2 = |D^\alpha u_{n0}|_0^2 + \int_0^t \mathcal{I}_n^\alpha \, ds + \int_0^t \mathcal{K}_n^{\rho\alpha} \, dB^\rho + m_n^\alpha(t),$$

with

$$m_n^\alpha(t) = \int_0^t \mathcal{J}_n^{k\alpha} \, dW^k, \qquad \mathcal{J}_n^{k\alpha} = 2(D^\alpha \mathcal{M}_n^k u_n + D^\alpha g_n^k, D^\alpha u_n),$$

$$\begin{aligned}
\mathcal{I}_n^\alpha = &-2(D^\alpha \mathfrak{a}_n^{ij} D_i u_n, D_j D^\alpha u_n) - 2\left(D^\beta \{ (\mathfrak{a}_n^i - \mathfrak{a}_{nj}^{ij}) D_i u_n + \mathfrak{a}_n u + f_n \}, D^\alpha D^\gamma u_n \right) \\
&+ (D^\alpha \{ \mathfrak{b}_n^{ik} D_i u_n + \mathfrak{b}^k u_n + g^k \}, D^\alpha \{ \mathfrak{b}_n^{jk} D_j u_n + \mathfrak{b}_n^k \mathfrak{b}_n^k u_n + g^k \}),
\end{aligned}$$
$$\mathcal{K}_n^{\rho\alpha} = 2(D^\alpha \mathcal{N}_n^\rho u_n + D^\alpha h_n^\rho, D^\alpha u_n).$$

Due to Assumptions 5.1 and 5.2 we get

$$\begin{aligned}
&-2(D^\alpha \mathfrak{a}_n^{ij} D_i u_n, D^\alpha D_j u_n) + (D^\alpha \mathfrak{b}_n^{ik} D_i u_n, D^\alpha \mathfrak{b}_n^{jk} D_j u_n) \\
&\quad \leq -\lambda \sum_{i=1}^d |D^\alpha D_i u_n|^2 + C|u_n|_l^2
\end{aligned}$$

with a constant $C = C(K, d, d_1, l)$. Thus by standard estimates

$$\mathcal{I}_n^\alpha \leq -\frac{\lambda}{2} \sum_{i=1}^d |D^\alpha D_i u_n|^2 + C(|u_n|_l^2 + |f_n|_{l-1}^2 + |g_n|_l^2),$$

and by Lemma 3.7 (i),

$$|\mathcal{K}_n^{\rho\alpha}| \leq C(|u_n|_l^2 + |h_n^\rho|_l^2), \qquad |\mathcal{J}_n^{k\alpha}|^2 \leq C(|u_n|_l^4 + |g_n^k|_l^2 |u_n|_l^2),$$

with a constant $C = C(K, \lambda, d, d_1, l)$. Consequently, almost surely

$$|D^\alpha u_n(t)|_0^2 + \frac{\lambda}{2}\int_0^t \sum_{i=1}^d |D^\alpha D_i u_n|^2\,ds \tag{5.6}$$

$$\leq |D^\alpha u_{n0}|^2 + C\int_0^t |u_n|_l^2\,dV_n + C\int_0^t (|f_n|_{l-1}^2 + |g_n|_l^2)\,ds + C\int_0^t |h_n^\rho|_l^2\,d\|B_n^\rho\| + m_n^\alpha(t)$$

for every α, such that $1 \leq |\alpha| \leq l$. By virtue of (5.4) this inequality holds also for $|\alpha| = 0$. Thus summing up inequality (5.6) over all multi-indices α with $|\alpha| \leq l$ we get almost surely

$$\begin{aligned} y_n(t) := |u_n(t)|_l^2 + \frac{\lambda}{2}\int_0^t |u_n|_{l+1}^2\,ds &\leq \int_0^t |u_n|_l^2\,dQ_n + M_n(t) + q_n(t) \\ &\leq \int_0^t y_n\,dQ_n + M_n(t) + q_n(t) \end{aligned}$$

for all $t \in [0, T]$ and $n \geq 1$, with

$$M_n(t) = \sum_{|\alpha|\leq l} m_n^\alpha(t),$$

$$q_n(t) = |u_{n0}|_l^2 + C\int_0^t (|f_n|_{l-1}^2 + |g_n|_l^2)\,ds + C\int_0^t |h_n^\rho|_l^2\,d\|B_n^\rho\|(s),$$

and a constant $C = C(K, \lambda, d, d_1, l)$. Clearly,

$$d\langle m_n^\alpha\rangle = \sum_{k=1}^{d_1} |\mathcal{J}_n^{k\alpha}|^2\,dt \leq C(|u_n|_l^4 + \gamma_n |g_n|_l^2 |u_n|_l^2)\,dQ_n,$$

so

$$d\langle M_n\rangle \leq C(|u_n|_l^4 + \gamma_n|g_n|_l^2|u_n|_l^2)\,dQ_n \leq C(y_n^2 + \gamma_n|g_n|_l^2 y_n)\,dQ_n$$

with constants $C = C(K, \lambda, d, d_1, l)$. Hence we finish the proof of the lemma by using Assumption 5.3 and applying Lemma 3.12. □

In the degenerate case, i.e., when $\lambda = 0$ in Assumption 5.1, we need to replace Assumptions 5.2 and 5.3 by somewhat stronger assumptions in order to have the conclusion of the previous lemma.

Assumption 5.5. The coefficients $\mathfrak{a}_n^{ij}$ and their derivatives in x up to order $l \vee 2$, the coefficients $\mathfrak{b}_n^{ik}$, $\mathfrak{a}_n^i$, $\mathfrak{c}_n^{i\rho}$ and their derivatives in x up to order $l \vee 1$, and the coefficients $\mathfrak{a}_n$, $\mathfrak{b}_n^k$, $\mathfrak{c}_n^\rho$ and their derivatives in x up to order l are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable real functions, in magnitude bounded by K for all $i, j = 1, \ldots, d$, $k = 1, \ldots, d_1$, $\rho = 1, \ldots, d_2$ and $n \geq 1$.

Assumption 5.6. We have $|u_{n0}|_l = O(n^{-\gamma})$,

$$\int_0^T |f_n|_l^2\, ds = O(n^{-2\gamma}), \qquad \int_0^T |g_n|_{l+1}^2\, ds = O(n^{-2\gamma}),$$

$$\int_0^t |h_n(t)|_l^2\, d\|B^\rho\|(t) = O(n^{-2\gamma}), \quad \sum\nolimits_{\rho=1}^{d_2} \|B_n^\rho\|(T) = o(\ln n).$$

Theorem 5.7. *Let Assumptions* 5.5*,* 5.6 *and* 5.1 (*with* $\lambda = 0$) *hold. Let* u_n *be an* H^{l+1}*-valued weakly continuous generalized solution of* (5.1)–(5.2)*. Then*

$$\sup\nolimits_{t\le T} |u_n(t)|_l = O(n^{-\kappa}) \quad \textit{a.s. for each } \kappa < \gamma.$$

Proof. Let α be a multi-index such that $|\alpha| \le l$. Then, as in the proof of the previous theorem, by Itô's formula we have

$$|D^\alpha u_n(t)|^2 = |D^\alpha u_{n0}|^2 + \int_0^t \mathcal{I}_n^\alpha\, ds + \int_0^t \mathcal{K}_n^{\rho\alpha}\, dB^\rho + m_n^\alpha(t), \tag{5.7}$$

with

$$m_n^\alpha(t) = \int_0^t \mathcal{J}_n^{k\alpha}\, dW^k, \quad \mathcal{J}_n^{k\alpha} = 2(D^\alpha \mathcal{M}_n^k u_n + D^\alpha g_n^k, D^\alpha u_n),$$

$$\begin{aligned}\mathcal{I}_n^\alpha =\ & -2(D^\alpha \mathfrak{a}_n^{ij} D_i u_n, D_j D^\alpha u_n) - 2\big(D^\beta\{(\mathfrak{a}_n^i - \mathfrak{a}_{nj}^{ij}) D_i u_n + \mathfrak{a}_n u + f_n\}, D^\alpha D^\gamma u_n\big)\\ &+ (D^\alpha \mathfrak{b}_n^{ik} D_i u_n + D^\alpha \mathfrak{b}_n^k u_n + D^\alpha g_n^k, D^\alpha \mathfrak{b}_n^{jk} D_j u_n + D^\alpha \mathfrak{b}_n^k u_n + D^\alpha g_n^k),\\ \mathcal{K}_n^{\rho\alpha} =\ & 2(D^\alpha \mathcal{N}_n^\rho u_n + D^\alpha h_n^\rho, D^\alpha u_n),\end{aligned}$$

where β and γ are multi-indices such that $\alpha = \beta + \gamma$ and $|\gamma| = 1$ if $|\alpha| \ge 1$. By [15, Lemma 2.1] and [15, Remark 2.1],

$$\mathcal{I}_n^\alpha \le C(|u_n|_l^2 + |f_n|^2 + |g_n|_{l+1}^2),$$

and by Lemma 3.7 (i),

$$|\mathcal{K}_n^{\rho\alpha}| \le C(|u_n|_l^2 + |h_n^\rho|_l^2)$$

with a constant $C = C(K, d, d_1, l)$. Thus from (5.7) we get

$$\begin{aligned}&|D^\alpha u_n(t)|^2\\ &\le |D^\alpha u_{n0}|^2 + \int_0^t |u_n|_l^2\, dQ_n + C\int_0^t (|f_n|_l^2 + |g_n|_{l+1}^2)\, ds + C\int_0^t |h_n^\rho|_l^2\, d\|B_n^\rho\| + m_n^\alpha(t)\end{aligned}$$

for $|\alpha| \le l$, where $Q_n(s) = C\left(s + \sum_{\rho=1}^{d_2} \|B_n^\rho\|(s)\right)$. Summing up these inequalities over α, $|\alpha| \le l$, we obtain

$$y_n(t) := |u_n(t)|_l^2 \le \int_0^t |u_n|_l^2\, dQ_n + M_n(t) + q_n(t)$$

for all $t \in [0, T]$ and $n \ge 1$, where

$$M_n(t) = \sum_{|\alpha|\le l} m_n^\alpha(t), \quad q_n(t) = |u_{n0}|_l^2 + C\int_0^t (|f_n|_l^2 + |g_n|_{l+1}^2)\, ds + C\int_0^t |h_n^\rho|_l^2\, d\|B_n^\rho\|,$$

and $C = C(K, d, d_1, l)$ is a constant. Hence the rest of the proof is the same as that in the proof of the previous theorem. □

6. Proof of the main theorems

To prove our main results we look for processes r_n such that

$$\sup_{t\le T}|r_n(t)|_m = O(n^{-\kappa}) \quad \text{a.s. for each } \kappa<\gamma, \tag{6.1}$$

and $v_n := u - u_n - r_n$ solves a suitable Cauchy problem of the type (5.1)–(5.2), satisfying the conditions of Theorem 5.4 or Theorem 5.7, so that we could get for each $\kappa < \gamma$

$$\sup_{t\le T}|v_n|_m = O(n^{-\kappa}) \quad \text{a.s. for each } \kappa<\gamma.$$

6.1. Proof of Theorem 2.7

We will carry out the strategy above in several steps, formulated as lemmas below. By a well-known result, see, e.g., [16], u is an H^{m+1}-valued strongly continuous process, and

$$\sup_{t\le T}|u|^2_{m+1} + \int_0^T |u|^2_{m+2}\,dt < \infty \quad \text{almost surely.} \tag{6.2}$$

Moreover, we can apply Theorem 4.3 with $l = m+3$ to get

$$\sup_{t\le T}|u_n|^2_{m+3} + \int_0^T |u_n|^2_{m+4}\,dt = O(n^{\varepsilon}) \quad \text{a.s. for any } \varepsilon>0. \tag{6.3}$$

Notice that $u - u_n$ satisfies

$$\begin{aligned} d(u-u_n) &= \{L_n(u-u_n) + \bar f_n\}\,dt + \{M^k_n(u-u_n) + \bar g^k_n\}\,dW^k \\ &\quad + \tfrac12\{M^kM^ku + M^kg^k\}\,dt + (M^k_nu_n + g^k_n)\,d(W^k - W^k_n), \end{aligned} \tag{6.4}$$

with

$$\bar f_n := f - f_n + (L-L_n)u, \quad \bar g^k_n := g^k - g^k_n + (M^k - M^k_n)u.$$

Notice also that due to (6.2) and Assumption 2.6 we have

$$\int_0^T (|\bar f_n|^2_{m-1} + |\bar g_n|^2_m)\,dt = O(n^{-2\gamma}). \tag{6.5}$$

Next we rewrite Equation (6.4) as an equation for

$$w_n = u - u_n - z_n, \quad \text{where } z_n = (M^k_nu_n + g^k_n)(W^k - W^k_n).$$

Note that by (6.3) and by our assumptions we have for each $\kappa<\gamma$

$$\sup_{t\le T}|z_n(t)|^2_m + \int_0^T |z_n(t)|^2_{m+1}\,dt = O(n^{-2\kappa}) \quad \text{almost surely.} \tag{6.6}$$

Set $\mathcal{L}_n := L_n + \frac12 M^k_nM^k_n$ and recall the definition of S^{kl}_n in Remark 2.2.

Lemma 6.1. *The process w_n solves*

$$dw_n = (\mathcal{L}_n w_n + F_n)\,dt + (M_n^k w_n + G_n^k)\,dW^k \\ - M_n^k(M_n^l u_n + g_n^l)\,dS_n^{kl}, \tag{6.7}$$

where $G_n^k = \bar{g}_n^k + M_n^k z_n$ and

$$F_n = \bar{f}_n + \tfrac{1}{2}(M^k M^k - M_n^k M_n^k)u + \tfrac{1}{2}(M^k g^k - M_n^k g_n^k) \\ - (M_n^k(L_n u_n + f_n))(W^k - W_n^k) + \mathcal{L}_n z_n.$$

Proof. By using Itô's formula one can easily verify that

$$dz_n = M_n^k(L_n u_n + f_n)(W^k - W_n^k)\,dt + M_n^k(M_n^l u_n + g_n^l)(W^k - W_n^k)\,dW_n^l \\ + (M_n^k u_n + g_n^k)\,d(W^k - W_n^k). \tag{6.8}$$

Hence

$$\begin{aligned} dw_n &= \{L_n(u - u_n) + \bar{f}_n + \tfrac{1}{2}M^k M^k u + \tfrac{1}{2}M^k g^k \\ &\quad - M_n^k(L_n u_n + f_n)(W^k - W_n^k)\}\,dt \\ &\quad + \{M_n^k(u - u_n) + \bar{g}_n^k\}\,dW^k - M_n^k(M_n^l u_n + g_n^l)(W^k - W_n^k)\,dW_n^l \\ &= \{\mathcal{L}_n(u - u_n) + \bar{f}_n + \tfrac{1}{2}(M^k M^k - M_n^k M_n^k)u + \tfrac{1}{2}(M^k g^k - M_n^k g_n^k)\}\,dt \\ &\quad - (M_n^k L_n u_n + f_n)(W^k - W_n^k))\,dt \\ &\quad + (M_n^k w_n + G_n^k)\,dW^k - M_n^k(M_n^l u_n + g_n^l)\,dS_n^{kl} \\ &= (\mathcal{L}_n w_n + F_n)\,dt + (M^k v_n + G_n^k)\,dW^k - M_n^k(M_n^l u_n + g_n^l)\,dS_n^{kl}, \end{aligned}$$

The lemma is proved. □

It is easy to show that due to (6.5), (6.3), (6.2), Assumptions 2.4, 2.5, 2.6, and 2.1 (i) we have

$$\int_0^T (|F_n|_{m-1}^2 + |G_n|_m^2)\,dt = O(n^{-2\kappa}) \quad \text{(a.s.) for each } \kappa < \gamma. \tag{6.9}$$

We rewrite the last term in the right-hand side of (6.7) into symmetric and antisymmetric parts as follows:

$$\begin{aligned} &M_n^k(M_n^l u_n + g_n^l)\,dS_n^{kl} \\ &\quad = \tfrac{1}{2}(M_n^k M_n^l + M_n^l M_n^k)u_n\,dS_n^{kl} + \tfrac{1}{2}(M_n^k M_n^l - M_n^l M_n^k)u_n\,dS_n^{kl} \\ &\qquad + \tfrac{1}{2}M_n^k g_n^l\,d(S_n^{kl} + S_n^{lk}) + \tfrac{1}{2}(M_n^l g_n^k - M_n^k g_n^l)\,dS_n^{lk} \\ &\quad = \tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)\,d(S_n^{kl} + S_n^{lk}) + \tfrac{1}{2}\left([M_n^l, M_n^k]u_n + M_n^k g_n^l - M_n^l g_n^k\right)\,dS_n^{kl}, \end{aligned}$$

where $[A, B] = BA - AB$. Thus using Remark 2.2 we get

$$\begin{aligned} &M_n^k(M_n^l u_n + g_n^l)\,dS_n^{kl} \\ &\quad = -\tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)\,dq_n^{kl} + \tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)\,d(R_n^{kl} + R_n^{lk}) \\ &\qquad + \tfrac{1}{2}\left([M_n^l, M_n^k]u_n + M_n^k g_n^l - M_n^l g_n^k\right)\,dS_n^{kl} \end{aligned}$$

$$= -\tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)\,dq_n^{kl} + \tfrac{1}{2}\left(M_n^l \diamond M_n^k u_n + M^k g_n^l + M_n^l g_n^k\right)\,dR_n^{kl}$$
$$+ \tfrac{1}{2}\left([M_n^l, M_n^k]u_n + M_n^k g_n^l - M_n^l g_n^k\right)\,dS_n^{kl}, \tag{6.10}$$

where we use the notation $A \diamond B = BA + AB$ for linear operators A and B. Thus Equation (6.7) can be rewritten as follows.

Lemma 6.2. *The process* w_n *solves*

$$dw_n = (\mathcal{L}_n w_n + F_n)\,dt + (\tfrac{1}{2}[M_n^l, M_n^k]w_n + H_n^{kl})\,dS_n^{kl} + (M_n^k w_n + \bar{G}_n^k)\,dW^k$$
$$+ \tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)\,dq_n^{kl} + \tfrac{1}{2}\left([M^k, M^l]u + M^l g^k - M^k g^l\right)\,dS_n^{kl}, \tag{6.11}$$

where

$$H_n^{kl} = \tfrac{1}{2}[M_n^l, M_n^k](M_n^r u_n + g_n^r)(W^r - W_n^r)$$
$$+ \tfrac{1}{2}([M_n^k, M_n^l] - [M^k, M^l])u + \tfrac{1}{2}(M_n^l g_n^k - M^l g^k + M^k g^l - M_n^k g_n^l),$$
$$\bar{G}_n^k = G_n^k - \tfrac{1}{2}\left(M_n^k \diamond M_n^l u_n + M_n^l g_n^k + M_n^k g_n^l\right)(W^l - W_n^l).$$

Proof. Plugging (6.10) into (6.7) we get

$$dw_n = (\mathcal{L}_n w_n + F_n)\,dt + (M_n^k w_n + \bar{G}_n^k)\,dW^k$$
$$+ \tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)\,dq_n^{kl} - \tfrac{1}{2}\{[M_n^l, M_n^k]u_n + M_n^k g_n^l - M_n^l g_n^k\}\,dS_n^{kl}$$
$$= (\mathcal{L}_n w_n + F_n)\,dt + (M_n^k w_n + \bar{G}_n^k)\,dW^k$$
$$+ \tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)\,dq_n^{kl} + \left(\tfrac{1}{2}[M_n^l, M_n^k]w_n + M_n^l g_n^k - M_n^k g_n^l\right)\,dS_n^{kl}$$
$$- \tfrac{1}{2}[M_n^l, M_n^k]u\,dS_n^{kl} + \tfrac{1}{2}[M_n^l, M_n^k](M_n^r u_n + g_n^r)(W^r - W_n^r)\,dS_n^{kl}$$
$$= (\mathcal{L}_n w_n + F_n)\,dt + (\tfrac{1}{2}[M_n^l, M_n^k]w_n + H_n^{kl})\,dS_n^{kl} + (M_n^k w_n + \bar{G}_n^k)\,dW^k$$
$$+ \tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)\,dq_n^{kl} + \tfrac{1}{2}\{[M^k, M^l]u + M^l g^k - M^k g^l\}\,dS_n^{kl}. \qquad \square$$

In the same way as (6.9) is proved, we can easily get

$$\int_0^T |\bar{G}_n|_{m-1}^2\,dt = O(n^{-2\kappa}), \quad \sup_{t\le T}|H_n^{kl}(t)|_m^2 = O(n^{-2\kappa}) \quad \text{for } \kappa < \gamma \tag{6.12}$$

almost surely for all $k, l = 1, \dots, d_1$. Finally we rewrite (6.11) as an equation for $v_n = w_n - r_n$, where

$$r_n = \tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)q_n^{kl} + \tfrac{1}{2}\{[M^k, M^l]u + M^k g^l - M^l g^k\}S_n^{kl}.$$

Notice that by (6.3) and Remark 2.2, r_n satisfies (6.6) in place of z_n.

Lemma 6.3. *The process* v_n *solves*

$$dv_n = (\mathcal{L}_n v_n + \widetilde{F}_n)\,dt + (\tfrac{1}{2}[M_n^k, M_n^l]v_n + \widetilde{H}_n^{kl})\,dS_n^{kl} + (M_n^k v_n + \widetilde{G}_n^k)\,dW^k$$
$$- \tfrac{1}{2}M_n^k M_n^l(M_n^r u_n + g_n^r)(W^k - W_n^k)\,dB_n^{lr}, \tag{6.13}$$

where B_n^{lr} *is as in* (2.2) *and*

$$\tilde{F}_n = F_n + \mathcal{L}_n r_n - \tfrac{1}{2}M_n^k M_n^l (L_n u_n + f_n) q_n^{kl} - \tfrac{1}{2}[M^k, M^l](\mathcal{L}u + \tfrac{1}{2}M^r g^r + f)S_n^{kl},$$
$$\widetilde{G}_n^k = \bar{G}_n + M_n^k r_n - \tfrac{1}{2}[M^r, M^l](M^k u + g^k)S_n^{rl},$$
$$\widetilde{H}_n^{kl} = H_n^{kl} + \tfrac{1}{2}[M_n^l, M_n^k] r_n.$$

Proof. Indeed,

$$\begin{aligned} dv_n &= (\mathcal{L}_n v_n + F_n)\,dt + (\tfrac{1}{2}[M_n^l, M_n^k]v_n + \tilde{H}_n^{kl})\,dS_n^{kl} + (M_n^k v_n + \bar{G}_n^k)\,dW^k \\ &\quad + \mathcal{L}_n r_n\,dt + M_n^k r_n\,dW^k \\ &\quad - \tfrac{1}{2}M_n^k M_n^l (L_n u_n + f_n) q_n^{kl}\,dt - \tfrac{1}{2}M_n^k M_n^l (M_n^r u_n + g_n^r) q_n^{kl}\,dW_n^r \\ &\quad - \tfrac{1}{2}[M_n^k, M_n^l](\mathcal{L}u + \tfrac{1}{2}M^r g^r + f)S_n^{kl}\,dt - \tfrac{1}{2}[M_n^k, M_n^l](M^r u + g^r)S_n^{kl}\,dW^r \\ &= (\mathcal{L}v_n + \tilde{F}_n)\,dt + (\tfrac{1}{2}[M_n^l, M_n^k]v_n + \tilde{H}_n^{kl})\,dS_n^{kl} + (M_n^k v_n + \tilde{G}_n^k)\,dW^k \\ &\quad - \tfrac{1}{2}M_n^k M_n^l (M_n^r u_n + g_n^r)(W^k - W_n^k)\,dB_n^{lr}. \end{aligned}$$ □

Making use of (6.9), (6.12) and (2.3), we easily obtain that for $\kappa < \gamma$

$$\int_0^T (|\tilde{F}_n|_{m-1}^2 + |\tilde{G}_n^k|_m^2)\,dt = O(n^{-2\kappa}), \quad \sup_{t\le T} |\tilde{H}_n^{kl}(t)|_m^2 = O(n^{-2\gamma}) \tag{6.14}$$

almost surely for $k, l = 1, \ldots, d_1$. Hence we finish the proof of the theorem by applying Theorem 5.4 with $l = m$ to Equation (6.13) and using (6.1) for z_n and r_n. □

6.2. Proof of Theorem 2.12

We follow the proof of Theorem 2.7 with the necessary changes. By a well-known theorem on degenerate stochastic PDEs from [15], u is an H^{m+2}-valued weakly continuous process, and by Theorem 4.8 with $l = m + 4$ we have

$$\sup_{t\le T} |u_n|_{m+4}^2 = O(n^{\varepsilon}) \quad \text{a.s. for } \varepsilon > 0. \tag{6.15}$$

Clearly, $u - u_n$ satisfies Equation (6.4), and

$$\int_0^T (|\bar{f}_n|_m^2 + |\bar{g}_n|_{m+1}^2)\,dt = O(n^{-2\gamma}). \tag{6.16}$$

Moreover, Lemmas 6.1, 6.2 and 6.3 remain valid, and due to (6.15), (6.16) and our assumptions, we have for each $\kappa < \gamma$

$$\int_0^T (|F_n|_m^2 + |G_n|_{m+1}^2)\,dt = O(n^{-2\kappa}),$$

$$\int_0^T |\bar{G}_n|_{m+1}^2\,dt = O(n^{-2\kappa}), \quad \sup_{t\le T} |H_n^{kl}(t)|_m^2 = O(n^{-2\kappa}),$$

$$\int_0^T |\tilde{F}_n|_m^2 + |\tilde{G}_n|_{m+1}^2\,dt = O(n^{-2\kappa}), \quad \sup_{t\le T} |\tilde{H}_n^{kl}(t)|_m^2 = O(n^{-2\kappa})$$

almost surely for $k, l = 1, \dots, d_1$. Note also that r_n and z_n satisfy (6.1). Hence we finish the proof of the theorem by applying Theorem 5.7 with $l = m$ to Equation (6.13). □

Now we prove our main results in the case when the coefficients and the free terms depend on t.

6.3. Proof of Theorem 2.17

We follow the proof of Theorem 2.7 with the necessary changes. As before, (6.2) and (6.3) hold. Now $u - u_n$ satisfies Equation (6.4) with an additional term,

$$\frac{1}{2}\sum_{k=1}^{d_1}\left(M^{k(k)}u + g^{k(k)}\right)dt,$$

added to the right-hand side of (6.4). Thus to get the analogue of Lemma 6.1 we set

$$N_n = \frac{1}{2}\sum_{k=1}^{d_1} M_n^{k(k)}, \quad \bar{\mathcal{L}}_n = \mathcal{L}_n + N_n = L_n + \frac{1}{2}\Big(M_n^k M_n^k + \sum_{k=1}^{d_1} M_n^{k(k)}\Big),$$

$$M_n^{k(l)} = b_n^{ik(l)}D_i + b_n^{k(l)}, \quad M^{kl} = b^{ik(l)}D_i + b^{k(l)}$$

for $k = 1, \dots, d_1$, $l = 0, \dots, d_1$ and $n \geq 1$. Then for

$$w_n = u - u_n - z_n, \quad z_n = (M_n^k u_n + g_n^k)(W^k - W_n^k) \tag{6.17}$$

the corresponding lemma reads as follows.

Lemma 6.4. *The process w_n solves*

$$\begin{aligned} dw_n &= (\bar{\mathcal{L}}_n w_n + \bar{F}_n)\,dt + (M_n^k w_n + G_n^k)\,dW^k \\ &\quad - M_n^k(M_n^l u_n + g_n^l)\,dS_n^{kl} - (M_n^{k(l)}u_n + g_n^{k(l)})\,dS_n^{kl}, \end{aligned}$$

where

$$\begin{aligned} \bar{F}_n = F_n &+ \frac{1}{2}\sum_{k=1}^{d_1}(M^{k(k)} - M_n^{k(k)})u + \frac{1}{2}\sum_{k=1}^{d_1}(g^{k(k)} - g_n^{k(k)}) \\ &- (M^{k(0)}u_n + g^{k(0)})(W^k - W_n^k) + N_n z_n, \end{aligned} \tag{6.18}$$

and F_n and G_n are defined in Lemma 6.1.

Proof. We need only notice that for z_n Equation (6.8) holds with a new term,

$$(M_n^{k(0)}u_n + g^{k(0)})(W^k - W_n^k)\,dt + (M_n^{k(l)}u_n + g_n^{k(i)})(W^k - W_n^k)\,dW_n^l,$$

added to its right-hand side. □

Hence we get the following modification of Lemma 6.2.

Lemma 6.5. *The process w_n solves*

$$
\begin{aligned}
dw_n &= (\bar{\mathcal{L}}_n w_n + \bar{F}_n)\,dt + (\tfrac{1}{2}([M_n^l, M_n^k] + M_n^{k(l)})w_n + \bar{H}_n^{kl})\,dS_n^{kl} \\
&\quad + (M_n^k w_n + \bar{G}_n^k)\,dW^k + \tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)\,dq_n^{kl} \\
&\quad + \tfrac{1}{2}\Big(([M^k, M^l] - M^{k(l)})u + M^l g^k - M^k g^l - g^{k(l)}\Big)\,dS_n^{kl}, \qquad (6.19)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{H}_n^{kl} &= H_n^{kl} + \tfrac{1}{2}M_n^{k(l)}(M_n^r u_n + g_n^r)(W^r - W_n^r) \\
&\quad + \tfrac{1}{2}(M^{k(l)} - M_n^{k(l)})u + \tfrac{1}{2}(g^{k(l)} - g_n^{k(l)}), \qquad (6.20)
\end{aligned}
$$

and H_n^{kl} and $\bar{G}_n^k$ are defined in Lemma 6.2.

Now we rewrite Equation (6.19) as an equation for $\bar{v}_n := w_n - \bar{r}_n$, where

$$
\begin{aligned}
\bar{r}_n &= r_n - \tfrac{1}{2}(M^{k(l)}u + g^{k(l)})S_n^{kl} \qquad (6.21)\\
&= \tfrac{1}{2}M_n^k(M_n^l u_n + g_n^l)q_n^{kl} + \tfrac{1}{2}(([M^k, M^l] - M^{k(l)})u + M^l g^k - M^k g^l - g^{k(l)})S_n^{kl}.
\end{aligned}
$$

To this end we set

$$\tilde{f} = f + \tfrac{1}{2}M^k g^k + \tfrac{1}{2}\sum_{k=1}^{d_1} g^{k(k)},$$

and notice that

$$
\begin{aligned}
&dM_n^k(M_n^l u_n + g_n^l) \\
&\quad = M_n^k M_n^l(L_n u_n + f_n)\,dt + M_n^k M_n^l(M_n^j u_n + g_n^j)\,dW_n^j + T_n^{kl0}\,dt + T_n^{klj}\,dW_n^j,
\end{aligned}
$$

where

$$
\begin{aligned}
T_n^{kl0} &= (M_n^{k(0)}M_n^l + M_n^k M_n^{l(0)})u_n + M_n^{k(0)}g_n^l + M_n^k g_n^{l(0)}, \\
T_n^{klj} &= (M_n^{k(j)}M_n^l + M_n^k M_n^{l(j)})u_n + M_n^{k(j)}g_n^l + M_n^k g_n^{l(j)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
d[M^k, M^l]u &= [M^k, M^l](\mathcal{L}u + f + \tfrac{1}{2}M^j g^j)\,dt + [M^k, M^l](M^j + g^j)\,dW^j \\
&\quad + P^{kl0}\,dt + P^{klj}\,dW^j,
\end{aligned}
$$

$$d(-M^{k(l)}u + M^l g^k - M^k g^l - g^{k(l)}) = U^{kl0}\,dt + U^{klj}\,dW^j,$$

where $\mathcal{L} = L + \frac{1}{2}M^k M^k$,

$$
\begin{aligned}
P^{kl0} &= [M^k, M^{l(0)}]u + [M^{k(0)}, M^l]u + \sum_{j=1}^{d_1}[M^{k(j)}, M^{l(j)}]u \\
&\quad + ([M^k, M^{l(j)}] + [M^{k(j)}, M^l])(M^j u + g^j) \\
&\quad + \tfrac{1}{2}\sum_{j=1}^{d_1}[M^k, M^l](M^{j(j)}u + g^{j(j)}), \\
P^{klj} &= ([M^k, M^{l(j)}] + [M^{k(j)}, M^l])u,
\end{aligned}
$$

$$\begin{aligned}
U^{kl0} = &-M^{k(l0)}u - M^{k(l)}(\bar{\mathcal{L}}u + \tilde{f}) - M^{k(lj)}(M^j u + g^j)\\
&+ M^{l(0)}g^k + M^l g^{k(0)} + M^{l(j)}g^{k(j)}\\
&- M^{k(0)}g^l - M^k g^{l(0)} - M^{k(j)}g^{l(j)} - g^{k(l0)},\\
U^{klj} = &-M^{k(lj)}u - M^{k(l)}(M^j u + g^j) + M^{l(j)}g^k + M^l g^{k(j)}\\
&- M^{k(j)}g^l - M^k g^{l(j)} - g^{k(lj)}.
\end{aligned}$$

Let $\tilde{F}_n$, $\tilde{G}^k_n$ and $\tilde{H}^{kl}_n$ be defined now as in Lemma 6.3, but with F_n and G^k_n replaced there by $\bar{F}_n$ and $\bar{G}^k_n$ in (6.18) and (6.20), respectively.

Thus we have the following modification of Lemma 6.3.

Lemma 6.6. *The process* $\bar{v}_n = w_n - \bar{r}_n$ *solves*

$$\begin{aligned}
d\bar{v}_n = (\bar{\mathcal{L}}_n\bar{v}_n + \hat{F}_n)\,dt &+ (\tfrac{1}{2}([M^k_n, M^l_n] + M^{k(l)})\bar{v}_n + \hat{H}^{kl}_n)\,dS^{kl}_n\\
&+ (M^k_n v_n + \hat{G}^k_n)\,dW^k - \tfrac{1}{2}T^{klj}(W^k - W^k_n)\,dB^{lr}_n,
\end{aligned} \tag{6.22}$$

where

$$\begin{aligned}
\hat{F}_n &= \tilde{F}_n + N_n r_n - \tfrac{1}{2}\bar{\mathcal{L}}_n(M^{k(l)}u + g^{k(l)})S^{kl}_n\\
&\quad - \tfrac{1}{2}T^{kl0}q^{kl}_n - \tfrac{1}{2}(P^{kl0}_n + U^{kl0}_n)S^{kl}_n,\\
\hat{G}^k_n &= \widetilde{G}_n - \tfrac{1}{2}M^k_n(M^{j(l)}u + g^{j(l)})S^{jl}_n - \tfrac{1}{2}(P^{jlk} + U^{jlk})S^{jl}_n\\
\hat{H}^{kl}_n &= \tilde{H}^{kl}_n + \tfrac{1}{2}M^{k(l)}r_n - \tfrac{1}{4}([M^l_n, M^k_n] + M^{kl}_n)(M^{k(l)}u + g^{k(l)})S^{kl}_n.
\end{aligned}$$

We can verify that (6.14) holds with $\hat{F}_n$, $\hat{G}^k_n$ and $\hat{H}^{kl}$ in place of $\tilde{F}_n$, $\tilde{G}^k_n$ and $\tilde{H}^{kl}$, respectively. We can also see that z_n and $\bar{r}_n$ satisfy (6.6). Hence we finish the proof by applying Theorem 5.4 with $l = m$ to Equation (6.22).

6.4. Proof of Theorem 2.21

We get Lemma 6.6 in the same way as Lemma 6.3 is proved, and we can also see that

$$\int_0^T (|\hat{F}_n|^2_m + |\hat{G}^k_n|^2_{m+1})\,dt = O(n^{-2\kappa}), \quad \sup_{t\le T}|\hat{H}^{kl}_n(t)|^2_m = O(n^{-2\kappa})$$

for each $\kappa < \gamma$, almost surely for $k, l = 1, \dots, d_1$, where $\hat{F}_n$, $\hat{G}^k_n$ and $\hat{H}^{kl}_n$ are defined in Lemma 6.3. We can also verify that for z_n and $\bar{r}_n$, defined in (6.17) and (6.21), we have

$$\sup_{t\le T}|z_n(t)|_m + \sup_{t\le T}|\bar{r}_n(t)|_m = O(n^{-\kappa}) \quad \text{for each } \kappa < \gamma.$$

Hence we obtain the theorem by applying Theorem 5.7 with $l = m$ to Equation (6.22).

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Weak Approximations for SDE's Driven by Lévy Processes

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Abstract. In this article we briefly survey recent advances in some simulation methods for Lévy driven stochastic differential equations. We give a brief description of each method and extend the one jump scheme method for some subordinated models like the NIG process. Simulations of all the presented methods are performed and compared.

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1. Introduction

Let X be an $\mathbb{R}^N$-valued adapted stochastic process, unique solution of the stochastic differential equation (SDE) with jumps

$$X_t(x) = x + \int_0^t \tilde{V}_0(X_s(x))ds + \int_0^t V(X_s(x))dB_s + \int_0^t h(X_{s-}(x))dZ_s, \ t \in [0,1], \tag{1.1}$$

with smooth coefficients $\tilde{V}_0 : \mathbb{R}^N \to \mathbb{R}^N$, $V = (V_i^{(j)})_{j=1,\dots,N}^{i=1,\dots,d} : \mathbb{R}^N \to \mathbb{R}^N \otimes \mathbb{R}^d$, $h : \mathbb{R}^N \to \mathbb{R}^N \otimes \mathbb{R}^d$ whose derivatives of any order (≥ 1) are bounded. Here B denotes an d-dimensional standard Brownian motion and Z denotes an d-dimensional Lévy process with Lévy triplet $(\gamma, 0, \nu)$ such that all of its moments are finite unless stated otherwise.

In this report, we numerically compare and evaluate two types of discrete approximation schemes for X in order to estimate $\mathbb{E}[f(X_1)]$ for smooth functions f. More precisely, we find a discretization scheme $(X_{t_j}^{(n)}(x))_{j=0}^n$ for a partition

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$0 = t_0 < t_1 < \cdots < t_n = 1$ such that

$$|\mathbb{E}[f(X_1(x))] - \mathbb{E}[f(X_1^{(n)}(x))]| \leq \frac{C(f,x)}{n^m}, \tag{1.2}$$

for some $m \in \mathbb{N}$ and a positive constant $C(f,x)$.

In such a case, we say that such a scheme is an mth order discretization scheme for X. The actual simulation to estimate $\mathbb{E}[f(X_1(x))]$ is carried out using Monte Carlo methods. That is, one computes $\frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} f(X_1^{(n),i}(x))$ where $X_1^{(n),i}(x), i = 1, \dots, N_{MC}$ denotes N_{MC} i.i.d copies of $X_1^{(n)}(x)$. Therefore, using the law of large numbers, the final error of the estimate is of the order $O(\frac{1}{\sqrt{N_{MC}}} + \frac{1}{n^m})$. Then the optimal choice of n is $O(n^m) = O(\sqrt{N_{MC}})$.

From this result, we can see that there is a reduction in computation time if one can obtain a scheme with a high value of m even if the computational cost increases linearly with m. In this light, we want to address in this article, the performance of some competing approximation schemes for jump driven sde's of the type (1.1) in the infinite activity case (i.e., $\nu(\mathbb{R}^d) = \infty$) as it is in the case in many financial models.

The first simulation proposal is to simulate all the jump times and their corresponding jump sizes in the case that Z is a finite activity process (i.e., $\nu(\mathbb{R}^d) < \infty$) (see, e.g., [3]). This becomes impossible in the infinite activity case as the number of jumps in any interval is infinite a.s. and therefore instead one may simulate all jump times for jump sizes bigger than a fixed small parameter ε. Following a proposal by Asmussen and Rosiński [2], the small jumps are replaced by an independent Brownian motion with variance given by $\int_{|y|\leq\varepsilon} |y|^2 \nu(dy)$. It has been shown in [2] and [6] that when approximating the small jumps by Gaussian variables, the convergence rates, which are measured by either the Kolmogorov distance between laws of processes at a fixed time or the mean square of the supremum of the error during a finite and fixed interval, are significantly improved (see also [12]). In our context of weak approximation (1.2), we would also like that the approximation of small jumps should be accurate. When drift and/or continuous diffusion components appear in the stochastic differential equation then one naturally faces an optimality problem. That is, how to match the computational effort done on the jump part with efficient approximation schemes for the drift and the Brownian part of the equation between jump times.

This issue was addressed in the article [8]. The method introduced in that article will be one of the methods that we will use in our comparison. In that method one uses all the jumps of size bigger than ε, the jump times become time partition points and one approximates the effect of the drift up to a high order of accuracy and there is no continuous diffusion part. In [8] it is proven that the rate of convergence is fast but the calculation time may be long.

On another article [14], the authors take a different point of view. Instead of using a random time partition points given by the jumps times corresponding to jump sizes bigger than ε, a fixed time partition is used and an approximation for the increments of the Lévy process is used. In this approximation, one uses an approximation with at most a finite number of jumps per interval. The maximum number of jumps is set by the user and therefore this becomes a limitation on the computation time by the part of the user. This approach which, in some sense, goes in the reverse direction of the scheme in [8] assumes that an approximation for the drift and the continuous diffusion part have been set and one tries to find a simple approximation of the increment of the Lévy process so as to match the computational effort invested in the Brownian and drift part of the equation. In order to introduce this method, one needs to explain the framework of the operator splitting method in its stochastic form.

This method is well known as a numerical method for partial differential equations. The idea is to use it, finding stochastic representations for the approximating splitting therefore providing new simulations methods based on composition of flows which parallel the composition of semigroups. This idea has been successfully used for stochastic differential equations driven by Brownian motion (see [9] and [10]).

Nevertheless, it should be noted that the performance of every estimation scheme depends on the activity level of small jump of the driving Lévy process Z, which is measured by the Blumenthal–Getoor index

$$\varrho = \inf\{p \geq 0 : \int_{\|x\|\leq 1} \|x\|^p \nu(dx) < \infty\}.$$

Since the Lévy measure ν satisfying $\int_{\|x\|\leq 1} \|x\|^2 \nu(dx) < \infty$, the index $\varrho \in [0, 2]$.

The goal of the present article is to give a non-technical introduction to these schemes and to present a throughout simulation study in order to assess the properties of the approximation schemes described above. Therefore we refer the reader for the proofs to the corresponding articles and we only give here the intuition behind the schemes.

In order to give the reader an idea of what are the technical conditions that need to be satisfied to obtain a new scheme, in Section 2.4.5, we deal with one case that was not treated in [14]. The case we study corresponds to a normal subordinated model. In this case, we have that $\int |y|\nu(dy) = \infty$. We will verify the main two conditions needed in order to establish the weak rate for the approximation method which follows from the main Theorems 4.1, 4.3 and 5.1 in [14].

2. Approximation schemes

In this section, we define the approximation schemes for equation (1.1) which we will compare in this paper. For proofs we refer the reader to the corresponding theoretical articles. We strive here for understanding and intuition of these schemes.

2.1. Euler's scheme

The Euler scheme is the most natural approximation scheme. Its programming flow is as follows. We denote $t_i^n = i/n, i = 1, \ldots, n$.

1. Generate a sequence of independent random variables ΔZ_i^n, $i = 0, \ldots, n-1$, which have the same distribution as $Z_{1/n}$.
2. Generate a sequence of independent random variables ΔB_i^n, $i = 0, \ldots, n-1$, which have the same distribution as $B_{1/n}$.
3. $\bar{X}_0 = x$ and for $i = 0, \ldots, n-1$,

$$\bar{X}_{(i+1)/n} = \bar{X}_{i/n} + \frac{1}{n}\tilde{V}_0(\bar{X}_{i/n}(x)) + V(\bar{X}_{i/n}(x))\Delta B_i^n + h(\bar{X}_{i/n}(x))\Delta Z_i^n.$$

Various articles and results have been written on this scheme. The main problem with this scheme is that it assumes that one can simulate the Brownian increment and the Lévy increment with the same computational effort. This is hardly the case in general, as the law of Lévy processes is generally given through their characteristic function. Therefore in general, an inversion procedure is needed. For more on this direction, see [7].

This simulation scheme is an approximation scheme of order 1 under sufficient conditions on the Lévy measure and it has been proven in, e.g., [11].

2.2. Jump-size adapted discretization schemes

The purpose of this section is to introduce a simulation method which uses all the jumps associated with the Lévy process whose norm are bigger than a certain fixed value ε. As the number of this type of jumps is finite on finite intervals then this approximation process defines a compound Poisson process. Therefore its simulation may be possible if we assume that the jump distribution can be simulated. The main drawback of the method is that it may take long time to compute as ε becomes small. On the other hand, it is a very accurate method. For further details, we refer the reader to [8].

To introduce the method, suppose that $V = \tilde{V}_0 = 0$ and Z is a d-dimensional Lévy process without diffusion component. That is,

$$Z_t = \gamma t + \int_0^t \int_{|y|\le 1} y\hat{N}(dy, ds) + \int_0^t \int_{|y|>1} yN(dy, ds), \quad t \in [0, 1].$$

Here, $\gamma \in \mathbb{R}^d$ and N is a Poisson random measure on $\mathbb{R}^d \times [0, \infty]$ with intensity ν satisfying $\int (1 \wedge |y|^2)\nu(dy) < \infty$. $\hat{N}(dy, ds) = N(dy, ds) - \nu(dy)ds$ denotes the compensated version of N.

Consider a family of measurable functions $(\chi_\varepsilon)_{\varepsilon>0} : \mathbb{R}^d \to [0, 1]$ such that $\int_{\mathbb{R}^d} \chi_\varepsilon(y)\nu(dy) < \infty$ for all $\varepsilon > 0$, and $\lim_{\varepsilon\to 0} \chi_\varepsilon(y) = 0$, for all $y \neq 0$. This function will serve as the localization function for the jumps which will be simulated. Therefore, unless explicitly mentioned otherwise, we will usually take $\chi_\varepsilon(y) = 1(|y| > \varepsilon)$.

We assume that the associated Lévy measure ν satisfies that

$$\nu(\mathbb{R}^d) = \infty, \quad \int_{\mathbb{R}^d} |y|^2 \nu(dy) < \infty.$$

Let N_ε be a Poisson random measure with intensity $\chi_\varepsilon \nu \times ds$ and $\hat{N}_\varepsilon$ its compensated Poisson random measure. Denote $\hat{\overline{N}}_\varepsilon$ a compensated Poisson random measure with intensity $\overline{\chi}_\varepsilon \nu \times ds$, where $\overline{\chi}_\varepsilon = 1 - \chi_\varepsilon$.

The processes Z can then be represented in law as follows

$$\begin{aligned} Z_t &\stackrel{d}{=} \gamma_\varepsilon t + Z_t^\varepsilon + R_t^\varepsilon, \\ \gamma_\varepsilon &= \gamma - \int_{|y|\le 1} y\chi_\varepsilon \nu(dy) + \int_{|y|>1} y\overline{\chi}_\varepsilon \nu(dy), \\ Z_t^\varepsilon &= \int_0^t \int_{\mathbb{R}^d} yN_\varepsilon(dy, ds), \\ R_t^\varepsilon &= \int_0^t \int_{\mathbb{R}^d} y\hat{\overline{N}}_\varepsilon(dy, ds). \end{aligned}$$

We denote by $\lambda_\varepsilon = \int_{\mathbb{R}^d} \chi_\varepsilon(y)\nu(dy)$ the intensity of Z^ε, by $T_i^\varepsilon, i \in \mathbb{N}$ the ith jump time of Z^ε with $T_0^\varepsilon = 0$, and by $\Sigma_\varepsilon = \left(\int_{\mathbb{R}^d} y_i y_j \overline{\chi}_\varepsilon(y)\nu(dy)\right)_{1\le i,j\le d}$ the covariance matrix of R_1^ε. In the one-dimensional case, $d = 1$, we set $\sigma_\varepsilon^2 = (\Sigma_\varepsilon)_{11}$. Given $\varepsilon > 0$ and Lévy measure ν, one can compute $\lambda_\varepsilon, \Sigma_\varepsilon$ and generate the sequence (T_i^ε). The random variable R_1^ε will be approximated using a Gaussian random variable with mean zero and variance Σ_ε. This is the so-called Asmussen–Rosiński approximation.

2.2.1. Kohatsu–Tankov scheme in dimension one. The following one-dimensional scheme ($d = N = 1$) uses an explicit transformation between jump times in order to solve explicitly the ODE.

$$dX_t = h(X_t)dt, \quad X_0 = x.$$

Suppose that $1/h$ is locally integrable, this equation has a solution

$$X_t = \theta(t; x) = F^{-1}(t + F(x)),$$

where F is a primitive of $1/h$.

We define inductively $\hat{X}(0) = X_0$ and for $i \ge 0$,

$$\begin{aligned} \hat{X}(T_{i+1}^\varepsilon -) = \theta\Big(&\gamma_\varepsilon(T_{i+1}^\varepsilon - T_i^\varepsilon) + \sigma_\varepsilon(W(T_{i+1}^\varepsilon) - W(T_i^\varepsilon)) \\ &- \frac{1}{2}h'(\hat{X}(T_i^\varepsilon))\sigma_\varepsilon^2(T_{i+1}^\varepsilon - T_i^\varepsilon); \hat{X}(T_i^\varepsilon)\Big), \end{aligned} \tag{2.1}$$

$$\hat{X}(T_{i+1}^\varepsilon) = \hat{X}(T_{i+1}^\varepsilon -) + h(\hat{X}(T_{i+1}^\varepsilon -))\Delta Z(T_{i+1}^\varepsilon). \tag{2.2}$$

For an arbitrary point t, we define

$$\hat{X}(t) = \theta\Big(\gamma_\varepsilon(t - \eta_t) + \sigma_\varepsilon(W(t) - W(\eta_t)) - \frac{1}{2}h'(\hat{X}(\eta_t))\sigma_\varepsilon^2(t - \eta_t); \hat{X}(\eta_t)\Big), \tag{2.3}$$

where $\eta_t = \sup\{T_i^\varepsilon : T_i^\varepsilon \le t\}$.

The logic behind the above scheme should be clear. Between jumps we use a high-order approximation to the solution of the stochastic differential equation driven by the drift coefficient γ_ε and the Wiener process W which replaces the small jumps (i.e., Asmussen–Rosiński approximation). When a jump happens the corresponding jump is added to the system.

The rate of convergence of this scheme, under the condition $\int |y|^6\nu(dy) < \infty$ is given by

$$\left|\mathbb{E}(f(\hat{X}_1) - f(X_1))\right| \le C\Big(\frac{\sigma_\varepsilon^2}{\lambda_\varepsilon}(\sigma_\varepsilon^2 + |\gamma_\varepsilon|) + \int_{\mathbb{R}} |y|^3\bar{\chi}_\varepsilon\nu(dy)\Big),$$

for some constant $C > 0$ not depend on ε (see [8, Theorem 2]).

The above scheme can be applied when F can be computed explicitly. Otherwise, one has to resort to approximations for F and then the order of the approximation becomes an important issue. See [8] for more comments on this matter.

2.2.2. Kohatsu–Tankov scheme in higher dimension. This scheme uses instead a Taylor expansion between jumps as the respective stochastic differential equation between jumps can not be solved explicitly. We denote

$$\begin{aligned}
&\tilde{X}(t) = Y^0(t) + Y_1(t), \quad t > \eta_t,\\
&\tilde{X}(T_i^\varepsilon) = \tilde{X}(T_i^\varepsilon-) + h(\tilde{X}(T_i^\varepsilon-))\Delta Z(T_i^\varepsilon),\\
&Y^0(t) = \tilde{X}(\eta_t) + \int_{\eta_t}^t h(Y^0(t))\gamma_\varepsilon ds,\\
&Y_1(t) = \sum_{i=1}^N \int_{\eta_t}^t \frac{\partial h}{\partial x_i}(Y^0(s))Y_1^i(s)\gamma_\varepsilon ds + \int_{\eta_t}^t h(Y^0(s))dW^\varepsilon(s),
\end{aligned}$$

where

- W^ε is a d-dimensional Brownian motion with covariance matrix Σ_ε independent of Z;
- the random vector $Y_1(t)$ conditioned on $T_i^\varepsilon,\ i \in \mathbb{N},\ t \in (T_j^\varepsilon, T_{j+1}^\varepsilon)$ and $\tilde{X}(T_j^\varepsilon)$ is a Gaussian random vector with conditional covariance matrix $\Omega(t)$ which satisfies the (matrix) linear equation

$$\Omega(t) = \int_{\eta_t}^t (\Omega(s)M(s) + M^\perp(s)\Omega^\perp(s) + N(s))ds,$$

where $M^\perp$ denotes the transpose of the matrix M and

$$M_{ij}(t) = \frac{\partial h_{jk}(Y^0(t))}{\partial x_i}\gamma_\varepsilon^k \quad \text{and} \quad N(t) = h(Y^0(t))\Sigma_\varepsilon h^\perp(Y^0(t)).$$

The rate of convergence of the above scheme under the condition $\int |y|^6\nu(dy) < \infty$ is given by

$$\left|\mathbb{E}(f(\tilde{X}_1) - f(X_1))\right| \le C\Big(\frac{\|\Sigma_\varepsilon\|}{\lambda_\varepsilon}(\|\Sigma_\varepsilon\| + |\gamma_\varepsilon|) + \int_{\mathbb{R}} |y|^3\bar{\chi}_\varepsilon\nu(dy)\Big),$$

for some constant $C > 0$ which does not depend on ε (see Theorem 16 [8]).

2.3. Operator splitting schemes

We define $V_0 = \tilde{V}_0 - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{N} \frac{\partial V_i}{\partial x_j} V_i^{(j)}$. Then Equation (1.1) can be rewritten in the following Stratonovich form

$$X_t(x) = x + \sum_{i=0}^{d} \int_0^t V_i(X_{s-}(x)) \circ dB_s^i + \int_0^t h(X_{s-}(x)) dZ_s, \tag{2.4}$$

where $B_t^0 = t$. We define the semigroup P_t by

$$P_t f(x) = \mathbb{E}[f(X_t(x))],$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is a continuous smooth function with polynomial growth at infinity.

We will approximate $P_t f(x) = \mathbb{E}[f(X_t(x))]$ by using its Taylor expansion for small $t > 0$. We will first compute, using Itô's formula

$$\frac{P_h f(x) - f(x)}{h}.$$

For this, note that Itô's formula gives

$$\begin{aligned} f(X_h(x)) - f(x) = &\int_0^h \nabla f(X_s(x)) dX_s^c(x) \\ &+ \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{N} \int_0^h D_{ij}^2 f(X_s(x)) d\langle X^i(x), X^j(x)\rangle_s \\ &+ \sum_{s \le h} \{ f(X_{s-}(x) + h(X_{s-}(x))\Delta Z_s) \\ &\qquad - f(X_{s-}(x)) - \nabla f(X_{s-}(x)) h(X_{s-}(x)) \Delta Z_s \}. \end{aligned}$$

After taking expectations and limits we obtain:

$$\lim_{h \to 0} \frac{P_h f(x) - f(x)}{h} = Lf(x) = \sum_{k=0}^{d+1} L_i f(x), \tag{2.5}$$

where

$$\begin{aligned} L_0 f(x) &\equiv \tilde{V}_0 f(x) = \sum_{k=1}^{N} \frac{\partial f}{\partial x_k}(x) \tilde{V}_0^{(k)}(x), \\ L_i f(x) &= \frac{1}{2} V_i^2 f(x) = \frac{1}{2}\sum_{j,k=1}^{N} \frac{\partial^2 f(x)}{\partial x_j \partial x_k} V_i^{(j)} V_i^{(k)}(x), \quad i = 1, \dots, d, \\ L_{d+1} f(x) &= \nabla f(x) h(x)\gamma + \int (f(x + h(x)y) - f(x) - \nabla f(x) h(x) y)\nu(dy). \end{aligned} \tag{2.6}$$

From the above calculation one clearly understands that the operator L_0 is associated to the drift of Equation (2.4), L_i for $i = 1, \dots, d$ is associated to the ith

Brownian motion and L_{d+1} is associated to the Lévy process. Note also that

$$\frac{\partial P_t f(x)}{\partial t} = L P_t f(x)$$
$$\frac{\partial^k P_t f(x)}{\partial t^k} = L^k P_t f(x).$$

Operator L is called the generator of P and this fact is usually written as $P_t = e^{tL}$. Due to the semigroup property of P, say $P_{t_1+t_2} = P_{t_1} P_{t_2}$, one understands that in order to approximate X one needs only to approximate P_t for small values of t and then use the following composition property

IMPORTANT PROPERTY: Let Y^1 and Y^2 be two independent stochastic processes generating semigroups R^1 and R^2 and with generators K^1 and K^2 respectively, then

$$\mathbb{E}[f(Y_t^1(Y_t^2(x)))] = R_t^2 R_t^1 f(x) = e^{tK^2} e^{tK^1} f(x).$$

Note that the operators above are not in general commutative.

In fact, if we iterate the above arguments we have that for a smooth function f,

$$P_t f(x) = f(x) + tLf(x) + \frac{t^2}{2} L^2 f(x) + \cdots = e^{tL} f(x).$$

Example. In this example, we retake the case of the Euler scheme in Section 2.1 and analyze it in the light of the previous argument.

Now let Q be the "semigroup" associated to the Euler scheme. That is, define $Q_t f(x) = E[f(\bar{X}_t)]$ for $t \leq \frac{1}{n}$. Then one can obtain the following expansion

$$Q_t f(x) = f(x) + t\bar{L}_1 f(x) + \frac{t^2}{2} \bar{L}_2 f(x) + \cdots$$

In fact, let $h \leq 1/n$ then

$$\begin{aligned} f(X_h^n(x)) - f(x) = & \int_0^h \nabla f(X_s^n(x)) dX_s^{n,c}(x) \\ & + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^N \int_0^h D_{ij}^2 f(X_s^n(x)) d\langle X^{n,i}(x), X^{n,j}(x) \rangle_s \\ & + \sum_{s \leq h} \left\{ f(X_{s-}^n(x) + h(X_{s-}^n(x))\Delta Y_s) - f(X_{s-}^n(x)) \right. \\ & \left. \qquad - \nabla f(X_{s-}^n(x)) h(X_{s-}^n(x)) \Delta Y_s \right\}. \end{aligned}$$

After some calculation one obtains that $\bar{L}_1 = L$ and that $\bar{L}_2 \neq L_2$. Therefore one has that the local error $P_t f(x) - Q_t f(x) = O(t^2)$. The proof finishes by using the following telescoping decomposition

$$\begin{aligned} E[f(X_1(x))] - E[f(\bar{X}_1(x))] &= -\sum_{i=1}^{n} \left\{ Q_{1/n}^{i} P_{1-t_i} f(x) - Q_{1/n}^{i-1} P_{1-t_{i-1}} f(x) \right\} \\ &= \sum_{i=1}^{n} \left\{ Q_{1/n}^{i-1} Q_{1/n} P_{1-t_i} f(x) - Q_{1/n}^{i-1} P_{1/n} P_{1-t_i} f(x) \right\} \\ &= \sum_{i=1}^{n} \left\{ Q_{1/n}^{i-1} \left(Q_{1/n} - P_{1/n} \right) P_{1-t_i} f(x) \right\}. \end{aligned}$$

Remark 2.1. One also needs the "stability" property of the operator Q in order to finish the argument above. That is, we need two properties: (1) the different $Q_{1/n}f - P_{1/n}f$ is of order $O(n^{-2})$ under certain regularity conditions on f (e.g., $f \in C_p^3$); and (2) the iteration $Q_{1/n}^{i-1}$ preserves the error rate of $Q_{1/n} - P_{1/n}$ without demanding any further regularity of $(Q_{1/n} - P_{1/n})P_{1-t_i} f(x)$.

Next, we define the following stochastic processes $X_{i,t}(x)$, $i = 0, \ldots, d+1$, usually called coordinate processes, which will correspond to the operator decomposition in (2.5) and which are the unique solutions of

$$\begin{aligned} X_{0,t}(x) &= x + \int_0^t V_0(X_{0,s}(x))ds, \\ X_{i,t}(x) &= x + \int_0^t V_i(X_{i,s}(x)) \circ dB_s^i, \quad 1 \le i \le d, \\ X_{d+1,t}(x) &= x + \int_0^t h(X_{d+1,s-}(x))dZ_s. \end{aligned}$$

Then we define

$$Q_{i,t} f(x) = \mathbb{E}[f(X_{i,t}(x))],$$

for a continuous function $f : \mathbb{R}^N \to \mathbb{R}$ with polynomial growth at infinity.

$$P_t = e^{tL} = \sum_{k=0}^{m} \frac{t^k}{k!} L^k + \mathcal{O}(t^{m+1}).$$

Note that $L = \sum_{i=0}^{d+1} L_i$ and we also let

$$P_t^i = e^{tL_i} = \sum_{k=0}^{m} \frac{t^k}{k!} L_i^k + \mathcal{O}(t^{m+1}).$$

Our next goal is to approximate e^{tL}, through a combination of the "coordinate" semigroups e^{sL_i}'s such that

$$e^{tL} - \sum_{j=1}^{k} \xi_j e^{t_{1,j} A_{1,j}} \cdots e^{t_{\ell_j,j} A_{\ell_j,j}} = \mathcal{O}(t^{m+1}),$$

with $t_{i,j} = t_{i,j}(t) > 0$, $A_{i,j} \in \{L_0, L_1, \ldots, L_{d+1}\}$ and weights $\{\xi_j\} \subset [0,1]$ with $\sum_{j=1}^k \xi_j = 1$. In this case, one can define

$$Q_t = \sum_{j=1}^{k} \xi_j e^{t_{1,j}A_{1,j}} \cdots e^{t_{\ell_j,j}A_{\ell_j,j}}. \tag{2.7}$$

If needed one may further approximate each $e^{t_{1,j}A_{1,j}}$ (mth order scheme) and in that case the definition of Q has to be further modified.

For simplicity let $d+1=2$ then

$$\begin{aligned} e^{tL} &= I + tL + \frac{t^2}{2}L^2 + O(t^3), \\ e^{tL_1}e^{tL_2} &= (I + tL_1 + \frac{t^2}{2}L_1^2 + \cdots)(I + tL_2 + \frac{t^2}{2}L_2^2 + \cdots) \\ &= I + tL + \frac{t^2}{2}\left(L_2^2 + L_1^2 + 2L_1L_2\right) + O(t^3), \end{aligned}$$

then

$$e^{tL} - e^{tL_1}e^{tL_2} = O(t^2).$$

Therefore the composition of the semigroups in the above order will lead to an approximation with local error of order $O(t^2)$. This approximation can be improved by randomizing it as follows

$$e^{tL} - \frac{1}{2}e^{tL_1}e^{tL_2} - \frac{1}{2}e^{tL_2}e^{tL_1} = O(t^3),$$

since $L^2 = L_1^2 + L_2^2 + L_1L_2 + L_2L_1$. Finally one needs to obtain a stochastic representation for $\frac{1}{2}e^{tL_1}e^{tL_2} + \frac{1}{2}e^{tL_2}e^{tL_1}$ and possibly approximate each coordinate process. These approximation methods for semigroups can be generalized in higher dimension as follows:

Example. Examples of schemes of order $2 = O(t^3)$:

Ninomiya–Ninomiya (see [10]):

$$Q_t = \frac{1}{2}e^{\frac{t}{2}L_0}e^{tL_1}\cdots e^{tL_{d+1}}e^{\frac{t}{2}L_0} + \frac{1}{2}e^{\frac{t}{2}L_0}e^{tL_{d+1}}\cdots e^{tL_1}e^{\frac{t}{2}L_0}.$$

Ninomiya–Victoir (see [9]):

$$Q_t = \frac{1}{2}e^{tL_0}e^{tL_1}\cdots e^{tL_{d+1}} + \frac{1}{2}e^{tL_{d+1}}\cdots e^{tL_1}e^{tL_0}.$$

Splitting (Strang) method:

$$Q_t = e^{\frac{t}{2}L_0}\cdots e^{\frac{t}{2}L_d}e^{tL_{d+1}}e^{\frac{t}{2}L_d}\cdots e^{\frac{t}{2}L_0}. \tag{2.8}$$

It is easy to see that all the approximation operators Q mentioned above are special case of (2.7). For example, the operator Q in Splitting (Strang) method is deduced from (2.7) by putting $k=1; \xi=1; t_i = \frac{t}{2}$, for $1 \le i \le 2d+1, i \neq d+1$, $t_{d+1} = t$; $A_i = A_{2d+2-i} = L_{i-1}$, for $i = 1, \ldots, d+1$.

Splitting is a classical idea that is used in approximations for partial differential equations. The only new feature in the present situation is that we make use of stochastic representations in order to obtain the associated Monte Carlo method to (2.7). So the idea of this approximation method is to combine the above algebraic approach with its stochastic representation and if necessary the associated approximation of the stochastic representation in order to obtain the definition of Q.

The first approximation is obtained through the algebraic semigroup methods described above. The second approximation corresponds to an approximation to the corresponding semigroup $e^{t_{\ell_j,j}A_{\ell_j,j}}$ which is amenable to a stochastic representation and that can be easily simulated or easily approximated and then simulated. In the remainder of the paper, we will concentrate on this second aspect of the approximations.

2.4. Stochastic representations and their approximations

In this section we will show various cases where we approximate or simulate directly the stochastic representation of Q.

2.4.1. Diffusion process with a finite number of jumps per interval. In this section we will consider a full example by considering equation (2.4) in the particular case that Z is a compound Poisson process. First, we need to approximate the semigroup associated to the coordinate processes defined by

$$Q_{i,t}f(x) := E[f(X_{i,t}(x))].$$

In the case of $i = 1, \ldots, d$ we can approximate Q using the following result. Before that we need to introduce the exponential mapping. For given $\alpha : \mathbb{R}^N \to \mathbb{R}^N$, denote by $z_t(\alpha, x)$ the solution of

$$\frac{dz_s(\alpha, x)}{ds} = \alpha(z_s(\alpha, x)),\ z_0(\alpha, x) = x,\ s \in [0, 1].$$

Theorem 2.2. *Let $V_i : \mathbb{R}^N \to \mathbb{R}^N$ be a smooth function satisfying the linear growth condition: $|V_i(x)| \le C(1 + |x|)$. Let $z_s(B_t^i V_i, x)$, $s \in [0, 1]$ be the exponential map defined as above for fixed $t \in [0, 1]$.*

For $i = 0, 1, \ldots, d$, the sde

$$X_{i,t}(x) = x + \int_0^t V_i(X_{i,s}(x)) \circ dB_s^i$$

has a unique solution given by

$$X_{i,t}(x) = z_1(B_t^i V_i, x).$$

Idea of the proof: Differentiating, we obtain

$$\frac{dz_s(\alpha V_i, x)}{d\alpha} = \int_0^s V_i(z_u(\alpha V_i, x))du + \alpha \int_0^s \nabla V_i(z_u^i(\alpha, x)) \frac{dz_u(\alpha V_i, x)}{d\alpha} du.$$

This gives by Itô's formula that

$$dz_1(B_t^i V_i, x) = \frac{dz_1(B_t^i V_i, x)}{d\alpha} \circ dB_t^i.$$

Therefore the result follows if one proves that (exercise)

$$\frac{dz_1(\alpha V_i, x)}{d\alpha} = V_i(z_1(\alpha V_i, x)).$$

Now, the process X can be simulated as follows: First, we can solve for each time interval $(t_i, t_{i+1}]$ the coordinate processes equations

1. Solve (say exactly) the $d+1$ ODE's

$$X_{0,t/2}(x) = x + \int_0^{t/2} V_0(X_{0,s}(x))ds$$

$$\frac{dz_s(B_{t/2}^i V_i, x)}{ds} = B_{t/2}^i V_i(z_s(B_{t/2}^i V_i, x)),\ z_0(B_{t/2}^i V_i, x) = x,$$

   for $s \in [0,1]$ and $i = 1, \ldots, d$. Denote $X_{i,t}(x) = z_1(B_{t/2}^i V_i, x)$.
2. Solve (say exactly) the $d+1$ ODE's

$$\bar{X}_{0,t/2}(x) = x + \int_0^{t/2} V_0(\bar{X}_{0,s}(x))ds$$

$$\frac{dz_s(\bar{B}_{t/2}^i V_i, x)}{ds} = \bar{B}_{t/2}^i V_i(z_s(\bar{B}_{t/2}^i V_i, x)),\ z_0(\bar{B}_{t/2}^i V_i, x) = x,$$

   for $s \in [0,1]$, $i = 1, \ldots, d$ and $\bar{B}$ is an independent copy of B. Denote $\bar{X}_{i,t}(x) = z_1(\bar{B}_{t/2}^i V_i, x)$.
3. Solve (say exactly) the difference equation

$$X_{d+1,t}(x) = x + \int_0^t h(X_{d+1,s-}(x))dZ_s.$$

The global idea is that $X_{i,t}(x)$ represents the process that has as generator L^i. Hence, we use (say) the splitting (Strang) formula (2.8) to write

$$\hat{X}_t(x) = X_{0,t/2} \circ \cdots \circ X_{d,t/2} \circ X_{d+1,t} \circ \bar{X}_{d,t/2} \circ \cdots \circ \bar{X}_{1,t/2} \circ \bar{X}_{0,t/2}(x).$$

This gives a scheme of order 2.

On the other hand, if we approximate processes $X_{i,t}$ and $\bar{X}_{i,t}$ with a good high-order approximation, say Y_t^i and $\bar{Y}_t^i$, respectively, then we can obtain an approximate of X by

$$\tilde{X}_t(x) = Y_{t/2}^0 \circ \cdots \circ Y_{t/2}^d \circ Y_t^{d+1} \circ \bar{Y}_{t/2}^d \cdots \circ \bar{Y}_{t/2}^0(x).$$

The semigroup associated with the process X_t is Q_t in (2.7). Finally the Monte Carlo method is given by

$$\frac{1}{M} \sum_{j=1}^{M} f((\tilde{X}_{1/n} \circ \cdots \circ \tilde{X}_{1/n}(x))^{(j)}).$$

In the particular case that $X_{i,\cdot}$ can be solved exactly one can always take $Y^i_s = X_{i,s}$, $s = t/2,\ t$.

Remark 2.3.

1. If one can solve the above ODE and difference equations 1, 2 and 3 without much effort then the scheme can be implemented. But if so, there is no reason to use the splitting method of order 2. One can use a higher-order method that will lead to better accuracy just by using compositions.
2. It is very rarely the case when one can in fact solve explicitly 1, 2 and 3 above. Usually one has to approximate the solutions of the ODE's. Then the order of approximation has to match the order of the semigroup approximation method used. For example, in the above case if we use methods of order 2 to approach the ODE's then the order of the whole scheme will be order 2. In this case also the definition of Q has to be changed into the semigroup associated with the approximation process.
3. For the compounded Poisson case, we have that if λ is large, many jumps will appear in any interval and so the calculation time will be long. Later, we will see that we do not need to consider all the jumps in order to obtain an approximation of order 2. This can be intuitively understood because the probability of having two or more jumps in an interval of size t is $O(t^2)$.

2.4.2. On the two basic properties in order to prove the error of approximation. In order to find an approximation of order n, one needs to check the two conditions mentioned in Remark 2.1. That is,

1. $Q_t f$ preserves the regularity properties of the function f.
2. $(Q_t - P_t)(f) = O(t^{n+1})$.

According to the operator splitting scheme explained in the previous section, one may even verify these conditions for each of the operators used in the decomposition. This first property when written mathematically becomes:

(H1) For $f_p(x) := |x|^{2p}$ $(p \in \mathbf{N})$,

$$Q_t f_p(x) \le (1 + Kt) f_p(x) + K't$$

for $K = K(T,p)$, $K' = K'(T,p) > 0$.

This condition expresses the fact that Q does not alter the smoothness properties of the function f_p. The following condition expresses the fact that $P_t - Q_t = O(t^{n+1})$ and therefore the resulting scheme will be of order n. To be precise, we need to recall the definition of the functional space C^m_p for each $m \in \mathbb{N}$ and $p > 0$. For each function $f : \mathbb{R}^n \to \mathbb{R}$ in C^m, denote

$$\|f\|_{C^m_p} := \inf\{C \ge 0 : |\partial^\alpha_x f(x)| \le C(1 + |x|^p),\ 0 \le |\alpha| \le m, x \in \mathbb{R}^n\}.$$

Then, denote

$$C^m_p = \{f \in C^m : \|f\|_{C^m_p} < \infty\}.$$

Property 2 above when written mathematically becomes:

(H2) $$\left|E[f(\bar{X}_t)] - E[f(X_t)]\right| \le \|f\|_{C^{2n}_p}(1 + |x|^{p+n})t^{n+1}.$$

Or in a more generalized form for $q \equiv q(n,p)$ and $m \equiv m(n)$

$$|E[f(\bar{X}_t)] - E[f(X_t)]| \leq \|f\|_{C_p^m}(1+|x|^q)t^{n+1}.$$

In Section 2.4.5, we propose an scheme and verify that conditions (H1) and (H2) are valid in the case that $\int |y|\nu(dy) = \infty$.

2.4.3. The study of the jump-size adapted scheme using the operator splitting method. Here we only discuss the approximation of the $(d+1)$th coordinate which corresponds to the jump process. Define for $\varepsilon > 0$ the finite activity Lévy process (Z_t^ε) with the Lévy triple $(\gamma, 0, \nu^\varepsilon)$ where the Lévy measure ν^ε is defined by

$$\nu^\varepsilon(E) = \nu(E \cap \{y : |y| > \varepsilon\}), \; E \in \mathcal{B}(\mathbb{R}_0^d).$$

We consider the approximate SDE

$$Y_t^{d+1,\varepsilon}(x) = x + \int_0^t h(Y_{s-}^{d+1,\varepsilon}(x))(dZ_s^\varepsilon + \gamma_\varepsilon ds).$$

In this case it is clear that the order to approximation on the jumps components is given by

$$\begin{aligned}&E[f(X_{d+1,t}(x))] - E[f(Y_t^{d+1,\varepsilon}(x))]\\&\quad = t\int_{|y|\leq\varepsilon}(f(x+h(x)y) - f(x) - \nabla f(x)h(x)y)\nu(dy) + O(t^2).\end{aligned}$$

By a further Taylor expansion one obtains that

$$\begin{aligned}&\int_{|y|\leq\varepsilon}(f(x+h(x)y) - f(x) - \nabla f(x)h(x)y)\nu(dy)\\&\qquad \approx D^2 f(x)h(x)^{\otimes 2}\int_{|y|\leq\varepsilon}|y|^2\nu(dy) + R\int_{|y|\leq\varepsilon}|y|^3\nu(dy).\end{aligned}$$

Therefore one sees that if $\varepsilon > 0$ is chosen so that $\int_{|y|\leq\varepsilon}|y|^2\nu(dy) = Ct$ then (H2) is satisfied with $n = 1$. Furthermore, the Asmussen–Rosiński approach [2] corresponds to the first term in the above expansion.

The verification of (H1) in this case is done in [14].

2.4.4. Approximate small jumps scheme. In this section, we give an approximation scheme which uses a limited number of jumps per interval. We assume that $\int_{|y|<1}|y|\nu(dy) < \infty$. Then we further decompose the operator L_{d+1} defined in (2.6) as follows

$$\begin{aligned}L_{d+1} &= L_{d+1}^{1,\varepsilon} + L_{d+1}^{2,\varepsilon} + L_{d+1}^{3,\varepsilon},\\ L_{d+1}^{1,\varepsilon}f(x) &:= \nabla f(x)h(x)\left(\gamma - \int_{\varepsilon<|y|\leq 1} y\nu(dy)\right),\\ L_{d+1}^{2,\varepsilon}f(x) &:= \int_{|y|\leq\varepsilon}(f(x+h(x)y) - f(x) - \nabla f(x)h(x)y)\nu(dy),\end{aligned}$$

$$L^{3,\varepsilon}_{d+1} f(x) := \int_{\varepsilon<|y|} f(x+h(x)y) - f(x)\nu(dy).$$

The operator $L^{1,\varepsilon}_{d+1}$ can be exactly generated using

$$\bar{X}^{1,\varepsilon}_{d+1,t} = x + \left(\gamma - \int_{\varepsilon<|y|\le 1} y\nu(dy)\right) \int_0^t h\left(\bar{X}^{1,\varepsilon}_{d+1,s}\right) ds.$$

For $L^{2,\varepsilon}_{d+1}$ one can use Asmussen–Rosinski (see [14]). We discuss $L^{3,\varepsilon}_{d+1}$.

The approximation for $L^{3,\varepsilon}_{d+1}$ is defined as follows. Let $\lambda_\varepsilon = \int_{|y|>\varepsilon} \nu(dy)$, $G_\varepsilon(dy) = \lambda_\varepsilon^{-1} 1_{|y|>\varepsilon} \nu(dy)$, and let $Z^\varepsilon \sim G_\varepsilon$ and let S^ε be a Bernoulli random variable independent of Z^ε.

$$\bar{X}^{3,\varepsilon}_{d+1,t}(x) = \begin{cases} x & \text{if } S^\varepsilon = 0, \\ x + h(x)Z^\varepsilon & \text{if } S^\varepsilon = 1. \end{cases}$$

Lemma 2.4. *Assume that* $\left|\lambda_\varepsilon^{-1} \mathbb{P}\left[S^\varepsilon = 1\right] - t\right| \le Ct^2$ *then*

$$\left|\mathbb{E}\left[f(\bar{X}^{3,\varepsilon}_{d+1,t})\right] - f(x) - tL^{3,\varepsilon}_{d+1} f(x)\right| \le Ct^2 \left\|f\right\|_{C^1_p} (1+|x|^{p+1}) \int_{|y|>\varepsilon} |y|\nu(dy).$$

In [14] an approximation for $L^{3,\varepsilon}_{d+1}$ with importance sampling and restriction on the number of jumps is proposed.

Remark 2.5. This approximate small jumps scheme has some advantages in comparison with the Jump-size adapted discretization schemes presented in Section 2.2. The first advantage is that in the former scheme we can control the number of jumps needed to be simulated. This fact is important especially in the case that it takes time to generate jump sizes. The second advantage is that the former scheme can be applied for SDE driven by both Brownian motion and jump processes while the latter scheme can be applied only for SDE driven by pure jump processes.

An extension with at most two jumps per interval. Considering more jumps per interval will give higher-order approximations.

For $L^{3,\varepsilon}_{d+1}$ one can do the following: Let $G_\varepsilon(dy) = \lambda_\varepsilon^{-1} 1_{|y|>\varepsilon}\nu(dy)$, $\lambda_\varepsilon = \int_{|y|>\varepsilon} \nu(dy)$ and let Z_1^ε, $Z_2^\varepsilon \sim G_\varepsilon$ independent between themselves and let S_1^ε and S_2^ε be two independent Bernoulli random variables independent of Z_1^ε, Z_2^ε.

$$\bar{X}^{3,\varepsilon}_{d+1,t}(x) := \begin{cases} x & \text{if } S_1^\varepsilon = 0, \\ x + h(x)Z_1^\varepsilon & \text{if } S_1^\varepsilon = 1 \text{ and } S_2^\varepsilon = 0 \\ x + h(x)Z_1^\varepsilon + h(x+h(x)Z_1^\varepsilon)Z_2^\varepsilon & \text{if } S_1^\varepsilon = 1 \text{ and } S_2^\varepsilon = 1. \end{cases}$$

Denote

$$p_\varepsilon := \mathbb{P}\left[S_1^\varepsilon = 1\right]\left(1 + \mathbb{P}\left[S_2^\varepsilon = 1\right]\right),$$
$$q_\varepsilon := \mathbb{P}\left[S_1^\varepsilon = 1\right]\mathbb{P}\left[S_2^\varepsilon = 1\right].$$

Lemma 2.6. *Assume that* $\left|\lambda_\varepsilon^{-1} p_\varepsilon - t\right| \le Ct^3$ *and* $\left|2\lambda_\varepsilon^{-2} q_\varepsilon - t^2\right| \le Ct^3$ *then*

$$\begin{aligned}&\left|\mathbb{E}\left[f(\hat{X}^{3,\varepsilon}_{d+1,t})\right] - f(x) - tL^3_{d+1} f(x) - \frac{t^2}{2}\left(L^3_{d+1}\right)^2 f(x)\right| \\ &\quad \le Ct^3 \|f\|_{C^2_p} (1 + |x|^{p+2}) \left(1 + \left(\int_{|y|>\varepsilon} |y| \nu(dy)\right)^2\right).\end{aligned}$$

2.4.5. A case study. In some cases it is possible to introduce the limited number of jumps scheme even when $\int_{|y|\le 1} |y|\nu(dy) = \infty$. We suppose the one-dimensional case for simplicity. Let S be a subordinator (an increasing Lévy process on $\mathbb{R}$) with Lévy density ρ and drift γ_S. That is,

$$\begin{aligned} S_t &= \beta_0 t + \int_0^t \int_0^\infty z N_S(dz, dt), \\ \beta_0 &= \gamma_S - \int_0^\infty z\rho(z)dz, \end{aligned}$$

where N_S is a Poisson random measure on $[0,\infty) \times [0,\infty)$ with intensity $\rho(z)dz$ and

$$\beta_0 \ge 0, \qquad \int_0^\infty (1 \wedge z)\rho(z)dz < \infty. \tag{2.9}$$

Let $Z_t = \theta S_t + \sigma W_{S_t}$ where W is a standard Brownian motion independent of S. This is the setup in Section 2.2 in the particular case that the Lévy process Z is a subordinate to a Brownian motion with drift. It follows from Theorem 30.1 in [13] that Z is a Lévy process with the generating triplet (γ, A, ν) defined as follows

$$\begin{aligned} A &= \sigma\beta_0, \\ \nu(dy) &= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\Big(-\frac{(y-\theta t)^2}{2\sigma^2 t}\Big)\rho(t)dtdy, \qquad (2.10)\\ \gamma &= \theta\beta_0 + \int_0^\infty \int_{|y|\le 1} \frac{y}{\sqrt{2\pi\sigma^2 t}} \exp\Big(-\frac{(y-\theta t)^2}{2\sigma^2 t}\Big)\rho(t)dy\, dt. \end{aligned}$$

Let ϱ denote the Blumenthal–Getoor index of S. That is,

$$\varrho = \inf\{p > 0 : \int_0^1 z^p \rho(z)dz < \infty\}.$$

It follows from (2.9) that $\varrho \in [0, 1]$.

The Blumenthal–Getoor index plays an essential role in our approximation. The following result relates the Blumenthal–Getoor indices of S and Z.

Lemma 2.7. *The Blumenthal–Getoor index of S is ϱ if and only if the Blumenthal–Getoor index of Z is 2ϱ.*

Proof. Since the integral on $[-1,0]$ can be converted into an integral on $[0,1]$ by doing a change of variable $w=-y$, for any $\alpha\in(0,1)$, we have

$$\begin{aligned}&\int_{|y|\le 1}|y|^{2\alpha}\nu(dy)\\&\quad=\int_0^1 dy\int_0^\infty y^{2\alpha}\frac{1}{\sqrt{2\pi\sigma^2 t}}\Big[\exp\Big(-\frac{(y-\theta t)^2}{2\sigma^2 t}\Big)+\exp\Big(-\frac{(y+\theta t)^2}{2\sigma^2 t}\Big)\Big]\rho(t)dt\\&\quad=\int_0^1 dy\int_0^\infty \frac{y^{2\alpha}}{\sqrt{2\pi\sigma^2 t}}\exp\Big(-\frac{y^2+\theta^2t^2}{2\sigma^2 t}\Big)\Big[\exp\Big(\frac{\theta y}{\sigma^2}\Big)+\exp\Big(-\frac{\theta y}{\sigma^2}\Big)\Big]\rho(t)dt.\end{aligned}$$

Since $\exp\Big(\frac{\theta y}{\sigma^2}\Big)+\exp\Big(-\frac{\theta y}{\sigma^2}\Big)\ge 2$, we have

$$\int_{|y|\le 1}|y|^{2\alpha}\nu(dy)\ge 2\int_0^1 dy\int_0^\infty \frac{y^{2\alpha}}{\sqrt{2\pi\sigma^2 t}}\exp\Big(-\frac{y^2+\theta^2t^2}{2\sigma^2 t}\Big)\rho(t)dt.$$

Since the integrand is non-negative, by using the Fubini theorem, we have using the change of variables $z=\frac{y}{\sigma\sqrt{t}}$,

$$\int_{|y|\le 1}|y|^{2\alpha}\nu(dy)\ge C\sigma^{2\alpha}\int_0^1 dt\int_0^{1/(\sigma\sqrt{t})} z^{2\alpha}e^{-z^2/2}t^\alpha\exp\Big(-\frac{\theta^2 t}{2\sigma^2}\Big)\rho(t)dz.$$

For each $t\in(0,1)$, one has

$$\int_0^{1/(\sigma\sqrt{t})} z^{2\alpha}e^{-z^2/2}dz\ge\int_0^{1/\sigma} z^{2\alpha}e^{-z^2/2}dz>0,$$

and $\exp\big(-\frac{\theta^2 t}{2\sigma^2}\big)\ge\exp\big(-\frac{\theta^2}{2\sigma^2}\big)$. Hence,

$$\int_{|y|\le 1}|y|^{2\alpha}\nu(dy)\ge C\int_0^1 t^\alpha\rho(t)dt. \tag{2.11}$$

On the other hand, one has

$$\begin{aligned}&\int_{|y|\le 1}|y|^{2\alpha}\nu(dy)\\&\quad\le 2\int_0^1 dy\int_0^\infty \frac{y^{2\alpha}}{\sqrt{2\pi\sigma^2 t}}\exp\Big(-\frac{y^2+\theta^2t^2}{2\sigma^2 t}\Big)\exp\Big(\frac{\theta y}{\sigma^2}\Big)\rho(t)dt.\\&\quad=2\int_0^1 dy\Big(\int_0^1+\int_1^\infty\Big)\frac{y^{2\alpha}}{\sqrt{2\pi\sigma^2 t}}\exp\Big(-\frac{y^2+\theta^2t^2}{2\sigma^2 t}\Big)\exp\Big(\frac{\theta y}{\sigma^2}\Big)\rho(t)dt.\end{aligned}$$

The second term above is less than $\frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big(\frac{\theta}{\sigma^2}\Big)\int_1^\infty\rho(t)dt<\infty$ while the first term is bounded by (using again $z=\frac{y}{\sigma\sqrt{t}}$),

$$C\int_0^\infty z^{2\alpha}e^{-z^2/2}dz\int_0^1 t^\alpha\rho(t)dt\le C\int_0^1 t^\alpha\rho(t)dt.$$

Hence

$$\int_{|y|\le 1} |y|^{2\alpha}\nu(dy) \le C\big(1+\int_0^1 t^\alpha \rho(t)dt\big).$$

This fact together with (2.11) implies, for any $\alpha \in (0,1)$,

$$\int_{|y|\le 1} |y|^{2\alpha}\nu(dy) < \infty \Leftrightarrow \int_0^1 t^\alpha \rho(t)dt < \infty.$$

This yields the desired result. □

Remark 2.8. It follows from Lemma 2.7 that if the Blumenthal–Getoor index ρ of subordinator S is bigger than $1/2$, then

$$\int_{|y|\le 1} |y|\nu(dy) = \infty.$$

The following simple observation plays an important role in the next discussion.

Lemma 2.9. *Suppose that $\varrho \in (0,1)$ and $\int_1^\infty t\rho(t)dt < \infty$, then*

$$\Big|\int_{|y|\le 1} y\nu(dy)\Big| < \infty.$$

Proof. We have

$$\begin{aligned}
&\int_{|y|\le 1} y\nu(dy)\\
&= \int_0^1 dy \int_0^\infty \frac{y}{\sqrt{2\pi\sigma^2 t}} \exp\Big(-\frac{y^2+\theta^2t^2}{2\sigma^2 t}\Big)\Big[\exp\Big(\frac{\theta y}{\sigma^2}\Big) - \exp\Big(-\frac{\theta y}{\sigma^2}\Big)\Big]\rho(t)dt,
\end{aligned}$$

Hence as $e^x - e^{-x} \le 2xe^x$ for $x \in [0,1]$ and for each $\beta > 0$, $\sup_{x>0} x^\beta e^{-x} < \infty$, we obtain

$$\begin{aligned}
\Big|\int_{|y|\le 1} y\nu(dy)\Big| &\le 2\int_0^1 dy \int_0^\infty \frac{y}{\sqrt{2\pi\sigma^2 t}} \exp\Big(-\frac{y^2+\theta^2t^2}{2\sigma^2 t}\Big)\frac{|\theta|y}{\sigma^2}\exp\Big(\frac{|\theta|y}{\sigma^2}\Big)\rho(t)dt\\
&\le C\int_0^1 dy \int_0^\infty \Big(\frac{y^2}{2\sigma^2 t}\Big)^{1+\delta/2} \exp\Big(-\frac{y^2}{2\sigma^2 t}\Big) y^{-\delta} t^{(\delta+1)/2}\rho(t)dt\\
&\le C\int_0^1 y^{-\delta}dy \int_0^\infty t^{(\delta+1)/2}\rho(t)dt < \infty,
\end{aligned}$$

for some constant $\delta \in (2\varrho - 1, 1)$ where C is a positive constant that depends on σ^2. □

Throughout the rest of this section, we suppose that $\varrho < 1$. Then we can rewrite $\gamma = \theta\beta_0 + \int_{|x|\le 1} x\nu(dx)$. We decompose the operator L_{d+1} defined in (2.6) by $L_{d+1} = L^1_{d+1} + L^2_{d+1}$, where

$$\begin{aligned} L^1_{d+1}f(x) &:= \theta\beta_0 h(x)f'(x), \\ L^2_{d+1}f(x) &:= \int_{\mathbb{R}} \big(f(x+h(x)y) - f(x)\big)\nu(dy). \end{aligned} \tag{2.12}$$

The operator L^1_{d+1} can be exactly generated using

$$\overline{X}^1_{d+1,t}(x) = x + \theta\beta_0 \int_0^t h(\overline{X}^1_{d+1,s}(x))ds,$$

as before. The approximation for L^2_{d+1} is defined as follows: For some $\varepsilon \in (0,1)$ which will be specified later, let $H_\varepsilon(x) = C_\varepsilon^{-1} 1_{x>\varepsilon}\rho(x)$, $C_\varepsilon = \int_\varepsilon^\infty \rho(x)dx$ and $\zeta_\varepsilon = \int_0^\varepsilon x\rho(x)dx$. Furthermore let S^ε be a Bernoulli random variable with $p_i = \mathbb{P}[S^\varepsilon = i], i = 0, 1$. We define

$$\overline{X}^{2,\varepsilon}_t(x) = \begin{cases} x + h(x)\big(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t}Z\big) & \text{if } S^\varepsilon = 0, \\ x + h(x)\big(\zeta_\varepsilon t\theta + \theta U^\varepsilon + \sigma\sqrt{\zeta_\varepsilon t + U^\varepsilon}Z\big) & \text{if } S^\varepsilon = 1, \end{cases}$$

where Z is a standard normal random variable and U^ε is a random variable with density function H_ε. We suppose that Z, U^ε, S^ε and W are mutually independent. Throughout this section we assume without loss of generality that $t \le 1$.

We need the following auxiliary estimate.

Lemma 2.10. *For any $\varrho_0 \in (\varrho, 1)$, there exists a positive constant $C(\varrho_0)$ which does not depend on ε such that for any $\varepsilon \in (0,1)$,*

$$\zeta_\varepsilon \le C(\varrho_0)\varepsilon^{1-\varrho_0}, \tag{2.13}$$

$$C_\varepsilon = \int_\varepsilon^\infty \rho(z)dz \le C(\varrho_0)\varepsilon^{-\varrho_0}, \tag{2.14}$$

$$\int_\varepsilon^\infty \sqrt{z}\rho(z)dz \le C(\varrho_0)(1 + \varepsilon^{1/2-\rho_0}), \tag{2.15}$$

$$\int_0^\varepsilon z^{3/2}\rho(z)dz \le C(\varrho_0)\varepsilon^{3/2-\varrho_0}. \tag{2.16}$$

Proof. Because $\varrho_0 \in (\varrho, 1)$, there exists a positive constant $C_0 = C(\varrho_0)$ such that

$$\int_0^1 z^{\varrho_0}\rho(z)dz < C_0.$$

Hence one has the following estimate for ζ_ε,

$$\zeta_\varepsilon = \int_0^\varepsilon z\rho(z)dz = \int_0^\varepsilon z^{1-\varrho_0}z^{\varrho_0}\rho(z)dz \le \varepsilon^{1-\varrho_0}\int_0^\varepsilon z^{\varrho_0}\rho(z)ds \le C_0\varepsilon^{1-\varrho_0}.$$

Next one has,

$$\int_\varepsilon^\infty \rho(z)dz = \int_\varepsilon^1 \rho(z)dz + \int_1^\infty \rho(z)dz \le \int_\varepsilon^1 \left(\frac{z}{\varepsilon}\right)^{\varrho_0} \rho(z)dz + \int_1^\infty \rho(z)dz$$
$$\le C_0\varepsilon^{-\varrho_0} + \int_1^\infty \rho(z)dz \le C_1\varepsilon^{-\varrho_0},$$

where $C_1 = C_0 + \int_1^\infty \rho(z)dz < \infty$ since $\varepsilon < 1$ and $\int_1^\infty \rho(z)dz < \infty$. A similar calculation gives (2.15).

Finally, one has that as $\varrho_0 < \frac{3}{2}$,

$$\int_0^\varepsilon z^{3/2}\rho(z)dz = \int_0^\varepsilon z^{-\varrho_0+3/2} z^{\varrho_0}\rho(z)dz \le C_0\varepsilon^{-\varrho_0+3/2}. \qquad \square$$

The following lemma will be used to justify condition (H2).

Lemma 2.11. *Assume that $f \in C_p^4$ for some $p > 1$, $\mathbb{P}[S^\varepsilon = 1] = C_\varepsilon t < 1$ and $\int_1^\infty z^{p+2}\rho(z)dz < \infty$, then for each $\varrho_0 \in (\varrho, 1)$, there exists a positive constant $C(\varrho_0)$ which does not depend on ε and t such that*

$$\left|\mathbb{E}\left[f(\overline{X}_t^{2,\varepsilon}(x)\right] - f(x) - tL_{d+1}^2 f(x)\right| \tag{2.17}$$
$$\le C(\varrho_0)(1+|x|^{p+4})\|f\|_{C_p^4}\left(t^{3/2}\varepsilon^{3(1-\varrho_0)/2} + t^2\varepsilon^{3/2-2\varrho_0} + t^2\varepsilon^{1-\varrho_0} + t\varepsilon^{3/2-\varrho_0}\right).$$

Proof. Before we start the proof, we remind the reader the properties that will be used repeatedly without further mention. These are: 1. $\sup_{0<\varepsilon<1}\zeta_\varepsilon < \infty$ and $\zeta_\varepsilon \to 0$ as $\varepsilon \to 0$. 2. $|h(x)|^q \le C(1+|x|^q)$ and 3. $\sup_{0<\varepsilon<1} C_\varepsilon t \le 1$.

1) First we expand $\mathbb{E}\left[f(\overline{X}_t^{2,\varepsilon}(x))\right] - f(x)$. Set $p_i = \mathbb{P}[S^\varepsilon = i],\ i = 0, 1$. Using Taylor's expansion, one has

$$\mathbb{E}\left[f(\overline{X}_t^{2,\varepsilon}(x))\right] - f(x)$$
$$= p_0\int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} h(x)(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t}y)\int_0^1 f'(x + uh(x)(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t}y))du\,dy$$
$$+ C_\varepsilon^{-1}p_1\int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\int_\varepsilon^\infty h(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)\rho(z)$$
$$\times\int_0^1 f'(x + uh(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y))du\,dz\,dy = I_1 + I_2.$$

Using Taylor's expansion again, I_1 becomes

$$I_1 = p_0\int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} h(x)(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t}y)f'(x)dy$$
$$+ p_0\int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} h(x)^2(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t}y)^2\int_0^1 du$$
$$\times\int_0^1 uf''(x + uvh(x)(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t}y))dv\,dy$$

$$
\begin{aligned}
&= p_0 h(x) f'(x) \zeta_\varepsilon t\theta \\
&\quad + \frac{p_0}{2} h(x)^2 \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \zeta_\varepsilon t \sigma^2 y^2 f''(x)\, dy \\
&\quad + p_0 h(x)^3 \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \zeta_\varepsilon t \sigma^2 y^2 (\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t} y) \int_0^1 du \int_0^1 dv \\
&\qquad \int_0^1 u^2 v f'''(x + uvwh(x)(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t} y)) dw\, dy \\
&\quad + p_0 h(x)^2 \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} ((\zeta_\varepsilon t\theta)^2 + 2\sigma\theta y (\zeta_\varepsilon t)^{3/2}) \int_0^1 du \\
&\qquad \times \int_0^1 u f''(x + uvh(x)(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t} y)) dv\, dy \\
&= I_{11} + I_{12} + I_{13} + I_{14},
\end{aligned} \tag{2.18}
$$

where the second equation is obtained by writing $(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t} y)^2 = \sigma^2 y^2 \zeta_\varepsilon t + ((\zeta_\varepsilon t\theta)^2 + 2\sigma\theta y(\zeta_\varepsilon t)^{3/2})$ and using the fact that $\int_{\mathbb{R}} y e^{-y^2/2} dy = 0$.

The second term I_{12} can be rewritten as

$$
I_{12} = \frac{p_0}{2} \zeta_\varepsilon t \sigma^2 h(x)^2 f''(x). \tag{2.19}
$$

Since $f \in C_p^4$, one can show that

$$
\begin{aligned}
|I_{13}| &\le C(\zeta_\varepsilon t)^{3/2} \|f\|_{C_p^4} \int_{\mathbb{R}} e^{-y^2/2} y^2 (1 + |x|^3)(\sqrt{\zeta_\varepsilon t} + y) \int_0^1 du \int_0^1 dv \\
&\quad \times \int_0^1 u^2 v (1 + |x|^p) \Big(1 + u^p v^p \big((\zeta_\varepsilon t)^p + (\zeta_\varepsilon t)^{p/2} y^p\big)\Big) dw dy \\
&\le C(1 + |x|^{p+3}) \|f\|_{C_p^4} (\zeta_\varepsilon t)^{3/2}.
\end{aligned}
$$

It follows from (2.13) that

$$
|I_{13}| \le C(\varrho_0)(1 + |x|^{p+4}) \|f\|_{C_p^4} t^{3/2} \varepsilon^{3(1-\varrho_0)/2}.
$$

After using a similar argument for I_{14}, we finally get

$$
|I_{13}| + |I_{14}| \le C(\varrho_0)(1 + |x|^{p+4}) \|f\|_{C_p^4} t^{3/2} \varepsilon^{3(1-\varrho_0)/2}. \tag{2.20}
$$

Furthermore, one has

$$
\begin{aligned}
I_2 &= t h(x)^2 \int_{\mathbb{R}} dy \frac{e^{-y^2/2}}{\sqrt{2\pi}} \int_\varepsilon^\infty (\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z} y)^2 \rho(z) \int_0^1 du \int_0^1 u \\
&\quad \times f''(x + uvh(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z} y)) dv\, dz \\
&\quad + t h(x) f'(x) \int_{\mathbb{R}} dy \frac{e^{-y^2/2}}{\sqrt{2\pi}} \int_\varepsilon^\infty (\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z} y) \rho(z) dz.
\end{aligned}
$$

In the first integral, we decompose $(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z} y)^2 = (\zeta_\varepsilon t\theta)^2 + 2\zeta_\varepsilon t\theta(\theta z + \sigma\sqrt{\zeta_\varepsilon t + z} y) + (\theta z + \sigma\sqrt{\zeta_\varepsilon t + z} y)^2$ to define I_{21}, I_{22} and I_{23}; and in the second integral, θz together with $\zeta_\varepsilon t\theta$ define I_{24} and I_{25}, respectively. Note that, the last integral corresponding to $\sigma\sqrt{\zeta_\varepsilon t+z}$ is zero. In detail, we write

$$I_2=\sum_{i=1}^{5} I_{2i},$$

where

$$\begin{aligned}
I_{21}&=th(x)^2\int_{\mathbb{R}} dy\frac{e^{-y^2/2}}{\sqrt{2\pi}}\int_\varepsilon^\infty(\zeta_\varepsilon t\theta)^2\rho(z)\int_0^1 du\\
&\quad\times\int_0^1 uf''(x+uvh(x)(\zeta_\varepsilon t\theta+\theta z+\sigma\sqrt{\zeta_\varepsilon t+z}y))dv\,dz,\\
I_{22}&=th(x)^2\int_{\mathbb{R}} dy\frac{e^{-y^2/2}}{\sqrt{2\pi}}\int_\varepsilon^\infty 2\zeta_\varepsilon t\theta(\theta z+\sigma\sqrt{\zeta_\varepsilon t+z}y)\rho(z)\int_0^1 du\int_0^1 u\\
&\quad\times f''(x+uvh(x)(\zeta_\varepsilon t\theta+\theta z+\sigma\sqrt{\zeta_\varepsilon t+z}y))dv\,dz,\\
I_{23}&=th(x)^2\int_{\mathbb{R}} dy\frac{e^{-y^2/2}}{\sqrt{2\pi}}\int_\varepsilon^\infty(\theta z+\sigma\sqrt{\zeta_\varepsilon t+z}y)^2\rho(z)\int_0^1 du\int_0^1 u\\
&\quad\times f''(x+uvh(x)(\zeta_\varepsilon t\theta+\theta z+\sigma\sqrt{\zeta_\varepsilon t+z}y))dv\,dz,\\
I_{24}&=th(x)f'(x)\theta\int_\varepsilon^\infty z\rho(z)dz,
\end{aligned}$$

and

$$I_{25}=t^2\zeta_\varepsilon\theta h(x)f'(x)\int_\varepsilon^\infty\rho(z)dz=p_1t\zeta_\varepsilon\theta h(x)f'(x).$$

We have

$$I_{11}+I_{25}+I_{24}=th(x)f'(x)\theta\int_0^\infty z\rho(z)dz. \tag{2.21}$$

Since $f\in C_p^4$ and $\varrho<1$, one gets

$$\int_\varepsilon^\infty z^q\rho(z)dz\le\int_0^1 z\rho(z)dz+\int_1^\infty z^{p+2}\rho(z)dz<\infty,$$

for all $1\le q\le p+2$. Furthermore, one has

$$\begin{aligned}
|I_{21}|&\le Ct^3\zeta_\varepsilon^2(1+x^2)\|f\|_{C_p^4}\int_{\mathbb{R}} dye^{-y^2/2}\int_\varepsilon^\infty\rho(z)\int_0^1 du\\
&\quad\times\int_0^1 u\Big(1+|x|^p+u^pv^p(1+|x|^p)\big(t^p+z^p+(z^{p/2}+(\zeta_\varepsilon t)^{p/2})y^p\big)\Big)dv\,dz\\
&\le Ct^3\zeta_\varepsilon^2(1+|x|^{p+2})\|f\|_{C_p^4}\int_\varepsilon^\infty(1+z^p)\rho(z)dz.\\
&\le C(1+|x|^{p+2})\|f\|_{C_p^4}C_\varepsilon t^3\zeta_\varepsilon^2\\
&\le C(1+|x|^{p+2})\|f\|_{C_p^4}t^2\zeta_\varepsilon^2.
\end{aligned}$$

The last inequality follows from the fact that $p_1 = C_\varepsilon t \le 1$. It then follows from (2.13) that

$$|I_{21}| \le C(\varrho_0)(1+|x|^{p+4})\|f\|_{C_p^4} t^2 \varepsilon^{1-\varrho_0}.$$

By a similar argument, one has

$$\begin{aligned}
|I_{22}| &\le Ct^2\zeta_\varepsilon(1+|x|^{p+2})\|f\|_{C_p^4}\Big(\int_\varepsilon^\infty (\sqrt{z}+z^{p+1})\rho(z)dz + \sqrt{t\zeta_\varepsilon}\int_\varepsilon^\infty \rho(z)dz\Big)\\
&\le Ct^2\zeta_\varepsilon(1+|x|^{p+2})\|f\|_{C_p^4}\Big(1+\int_\varepsilon^\infty \sqrt{z}\rho(z)dz + C_\varepsilon\sqrt{t\zeta_\varepsilon}\Big)\\
&\le C(1+|x|^{p+2})\|f\|_{C_p^4}\Big(t^2\zeta_\varepsilon\int_\varepsilon^\infty \sqrt{z}\rho(z)dz + (t\zeta_\varepsilon)^{3/2} + t^2\zeta_\varepsilon\Big).
\end{aligned}$$

It follows from (2.13) and (2.15) that

$$|I_{22}| \le C(1+|x|^{p+2})\|f\|_{C_p^4}\big(t^2\varepsilon^{1-\varrho_0} + t^2\varepsilon^{3/2-2\varrho_0} + t^{3/2}\varepsilon^{3(1-\varrho_0)/2}\big). \tag{2.22}$$

Next, by applying Taylor's expansion for f'', one gets $I_{23} = I_{23a} + I_{23b}$, where

$$I_{23a} = th(x)^2 f''(x)\int_{\mathbb{R}} dy\frac{e^{-y^2/2}}{2\sqrt{2\pi}}\int_\varepsilon^\infty (\theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)^2\rho(z)dz, \tag{2.23}$$

and

$$\begin{aligned}
I_{23b} = th(x)^3 &\int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\int_\varepsilon^\infty (\theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)^2(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)\rho(z)\\
&\times \int_0^1 du\int_0^1 u^2 v\int_0^1 f'''(x + uvwh(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y))dw\,dv\,dz\,dy.
\end{aligned} \tag{2.24}$$

2) Next, we expand $L_{d+1}^2 f(x)$. It follows from (2.10) and (2.12) and Taylor's expansion for f that

$$\begin{aligned}
&L_{d+1}^2 f(x)\\
&= h(x)\int_{\mathbb{R}} y\int_0^1 f'(x+uh(x)y)du\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 z}}\exp\Big(-\frac{(y-\theta z)^2}{2\sigma^2 z}\Big)\rho(z)dz\,dy\\
&= h(x)\int_{\mathbb{R}}\int_0^\infty (\theta z + \sigma\sqrt{z}y)\int_0^1 f'(x+uh(x)(\theta z+\sigma\sqrt{z}y))du\frac{e^{-y^2/2}}{\sqrt{2\pi}}\rho(z)dzdy\\
&= h(x)^2\int_{\mathbb{R}} dy\frac{e^{-y^2/2}}{\sqrt{2\pi}}\Big(\int_0^\varepsilon + \int_\varepsilon^\infty\Big)(\theta z + \sigma\sqrt{z}y)^2\int_0^1 du\\
&\qquad \times\int_0^1 uf''(x+uvh(x)(\theta z+\sigma\sqrt{z}y))dv\rho(z)dz\\
&\quad + h(x)f'(x)\theta\int_0^\infty z\rho(z)dz\\
&= J_1 + J_2 + J_3,
\end{aligned}$$

where the second equation follows by using an appropriate change of variables. By applying Taylor's expansion for f'', one gets $J_2 = J_{21} + J_{22}$, where

$$\begin{aligned}J_{21} &= h(x)^2 \int_{\mathbb{R}} dy \frac{e^{-y^2/2}}{\sqrt{2\pi}} \int_{\varepsilon}^{\infty} (\theta z + \sigma\sqrt{z}y)^2 \int_0^1 du \int_0^1 uf''(x)dv\rho(z)dz \\ &= h(x)^2 f''(x) \int_{\mathbb{R}} dy \frac{e^{-y^2/2}}{2\sqrt{2\pi}} \int_{\varepsilon}^{\infty} (\theta z + \sigma\sqrt{z}y)^2 \rho(z)dz, \qquad (2.25)\end{aligned}$$

and

$$\begin{aligned}J_{22} = h(x)^3 \int_{\mathbb{R}} dy \frac{e^{-y^2/2}}{\sqrt{2\pi}} \int_{\varepsilon}^{\infty} (\theta z + \sigma\sqrt{z}y)^3 \int_0^1 du \int_0^1 u^2 v \\ \times \int_0^1 f'''(x + uvwh(x)(\theta z + \sigma\sqrt{z}y))dwdv\rho(z)dz. \qquad (2.26)\end{aligned}$$

Next, we write $(\theta z + \sigma\sqrt{z}y)^2 = \sigma^2 y^2 z + (\theta^2 z^2 + 2\sigma\theta y z^{3/2})$ and after applying Taylor's expansion for f'', we get

$$\begin{aligned}J_1 &= \sigma^2 h(x)^2 f''(x) \int_{\mathbb{R}} y^2 \frac{e^{-y^2/2}}{2\sqrt{2\pi}} dy \int_0^{\varepsilon} z\rho(z)dz \\ &\quad + \sigma^2 h(x)^3 \int_{\mathbb{R}} y^2 \frac{e^{-y^2/2}}{\sqrt{2\pi}} \int_0^{\varepsilon} \rho(z) \int_0^1 du \int_0^1 u^2 v(\theta z + \sigma\sqrt{z}y)^3 \\ &\qquad \times \int_0^1 f'''(x + uvwh(x)(\theta z + \sigma\sqrt{z}y))dw\, dv\, dz\, dy \\ &\quad + h(x)^2 \int_{\mathbb{R}} dy \int_0^{\varepsilon} dz(\theta^2 z^2 + 2\theta\sigma y z^{3/2})\rho(z) \frac{e^{-y^2/2}}{\sqrt{2\pi}} \int_0^1 du \\ &\qquad \times \int_0^1 uf''(x + uvh(x)(\theta z + \sigma\sqrt{z}y))dv \\ &= J_{11} + J_{12} + J_{13}.\end{aligned}$$

Using a similar argument as before, one has

$$|J_{12}| + |J_{13}| \le C(1 + |x|^{p+3})\|f\|_{C_p^4} \int_0^{\varepsilon} z^{3/2}\rho(z)dz.$$

Therefore, it follows from (2.16) that

$$|J_{12}| + |J_{13}| \le C(\varrho_0)(1 + |x|^{p+4})\|f\|_{C_p^4} \varepsilon^{3/2-\varrho_0}. \qquad (2.27)$$

3) Now we compare the factors of $\mathbb{E}\big[f(\overline{X}_t^{2,\varepsilon}(x))\big] - f(x)$ and $L_{d+1}^2 f(x)$. First, it follows from (2.21) that

$$I_{11} + I_{25} + I_{24} = tJ_3. \qquad (2.28)$$

And it follows from (2.19) that

$$tJ_{11} - I_{12} = \frac{1}{2}\sigma^2 p_1 t\zeta_{\varepsilon} h(x)^2 f''(x). \qquad (2.29)$$

Next, it follows from (2.23) and (2.25) that

$$I_{23a} - tJ_{21}$$

$$= th(x)^2 f''(x) \int_{\mathbb{R}} dy \frac{e^{-y^2/2}}{2\sqrt{2\pi}} \int_{\varepsilon}^{\infty} \Big(\sigma^2\zeta_\varepsilon t y^2 + 2\theta\sigma z(\sqrt{z+\zeta_\varepsilon t} - \sqrt{z})y\Big)\rho(z)dz,$$

hence as $p_1 = C_\varepsilon t$ it follows from the above and (2.29)

$$I_{23a} + I_{12} = t(J_{11} + J_{21}). \tag{2.30}$$

Next, we compare I_{23b} and J_{22}. It follows from (2.24) and (2.26) that $I_{23b} - tJ_{22} = K_1 + K_2$, where

$$K_1 = th(x)^3 \int_{\mathbb{R}} dy \int_{\varepsilon}^{\infty} \Big((\theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)^2(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)$$
$$- (\theta z + \sigma\sqrt{z}y)^3\Big)\rho(z) \int_0^1 du \int_0^1 u^2 v$$
$$\times \int_0^1 \frac{e^{-y^2/2}}{\sqrt{2\pi}} f'''(x + uvwh(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y))dw\,dv\,dz,$$

and

$$K_2 = th(x)^3 \int_{\mathbb{R}} dy \int_{\varepsilon}^{\infty} (\theta z + \sigma\sqrt{z}y)^3\rho(z) \int_0^1 du \int_0^1 dv \int_0^1 dw\, u^2 v \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$
$$\times \Big(f'''(x + uvwh(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)$$
$$- f'''(x + uvwh(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{z}y))\Big)dz.$$

We rewrite $K_1 = K_{11} + K_{12}$, where

$$K_{11} = t^2\zeta_\varepsilon\theta h(x)^3 \int_{\mathbb{R}} dy \int_{\varepsilon}^{\infty} (\theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)^2\rho(z) \int_0^1 du \int_0^1 u^2 v$$
$$\times \int_0^1 \frac{e^{-y^2/2}}{\sqrt{2\pi}} f'''(x + uvwh(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y))dw\,dv\,dz,$$

and

$$K_{12} = th(x)^3 \int_{\mathbb{R}} dy \int_{\varepsilon}^{\infty} \Big((\theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)^3 - (\theta z + \sigma\sqrt{z}y)^3\Big)\rho(z) \int_0^1 du$$
$$\times \int_0^1 u^2 v \int_0^1 \frac{e^{-y^2/2}}{\sqrt{2\pi}} f'''(x + uvwh(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y))dw\,dv\,dz.$$

By using a similar argument as before, we have

$$|K_{11}| \le C(1 + |x|^{p+3})\|f\|_{C_p^4}\zeta_\varepsilon t^2 \int_{\varepsilon}^{\infty} (\zeta_\varepsilon t + z + z^{p+2})\rho(z)dz$$
$$\le C(1 + |x|^{p+3})\|f\|_{C_p^4}\zeta_\varepsilon t^2(1 + C_\varepsilon\zeta_\varepsilon t)$$
$$\le C(\varrho_0)(1 + |x|^{p+4})\|f\|_{C_p^4} t^2\varepsilon^{1-\varrho_0},$$

where the last inequality follows from the fact that $C_\varepsilon t \leq 1$ and (2.13). Next, we will estimate K_{12}. First, we have

$$\begin{aligned}
&(\theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)^3 - (\theta z + \sigma\sqrt{z}y)^3 \\
&\quad = \sigma y(\sqrt{\zeta_\varepsilon t + z} - \sqrt{z})\big((\theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)^2 + (\theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)(\theta z + \sigma\sqrt{z}y) \\
&\qquad + (\theta z + \sigma\sqrt{z}y)^2\big) \\
&\quad \leq Cy(\sqrt{\zeta_\varepsilon t + z} - \sqrt{z})(z^2 + y^2 z + y^2\zeta_\varepsilon t).
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
&\int_\varepsilon^\infty (\sqrt{\zeta_\varepsilon t + z} - \sqrt{z}) z\rho(z)dz \leq \zeta_\varepsilon t \int_\varepsilon^\infty \frac{z\rho(z)}{\sqrt{\zeta_\varepsilon t + z} + \sqrt{z}} dz \leq \zeta_\varepsilon t \int_\varepsilon^\infty \sqrt{z}\rho(z)dz, \\
&\int_\varepsilon^\infty (\sqrt{\zeta_\varepsilon t + z} - \sqrt{z})\rho(z)dz \leq \sqrt{t\zeta_\varepsilon} \int_\varepsilon^\infty \rho(z)dz = C_\varepsilon \sqrt{t\zeta_\varepsilon},
\end{aligned}$$

and for all $q \geq 3/2$,

$$\int_\varepsilon^\infty (\sqrt{\zeta_\varepsilon t + z} - \sqrt{z}) z^q \rho(z)dz \leq \zeta_\varepsilon t \int_\varepsilon^\infty z^{q-1/2}\rho(z)dz \leq C\zeta_\varepsilon t,$$

we have

$$\begin{aligned}
|K_{12}| &\leq C(1 + |x|^{p+3})\|f\|_{C_p^4} t^2 \zeta_\varepsilon \Big(1 + C_\varepsilon(\zeta_\varepsilon t)^{1/2} + \int_\varepsilon^\infty \sqrt{z}\rho(z)dz\Big) \\
&\leq C(1 + |x|^{p+3})\|f\|_{C_p^4} \Big(t^2\zeta_\varepsilon + (\zeta_\varepsilon t)^{3/2} + t^2\zeta_\varepsilon \int_\varepsilon^\infty \sqrt{z}\rho(z)dz\Big) \\
&\leq C(1 + |x|^{p+4})\|f\|_{C_p^4} \big(t^2\varepsilon^{1-\varrho_0} + t^{3/2}\varepsilon^{3(1-\varrho_0)/2} + t^2\varepsilon^{3/2-2\varrho_0}\big), \qquad (2.31)
\end{aligned}$$

where the last inequality follows from (2.13)–(2.15). Now we estimate K_2. Since $f \in C_p^4$, for any $u, v \in \mathbb{R}$, one has $|f'''(u) - f'''(v)| \leq C\|f\|_{C_p^4}|u - v|(1 + |u|^p + |v|^p)$, hence

$$\begin{aligned}
|K_2| \leq\ & Ct|h(x)|^4 \|f\|_{C_p^4} \int_{\mathbb{R}} dy \int_\varepsilon^\infty |(\theta z + \sigma\sqrt{z}y)^3 y|(\sqrt{z + \zeta_\varepsilon t} - \sqrt{z})\rho(z) \\
&\times \int_0^1 du \int_0^1 dv \int_0^1 dw u^2 v \frac{e^{-y^2/2}}{\sqrt{2\pi}} uv \\
&\times \Big(1 + |x|^p + u^p v^p w^p (1 + |x|^p)\big(\zeta_\varepsilon^p t^p + z^p + (z^{p/2} + \zeta_\varepsilon^{p/2} t^{p/2}) y^p\big)\Big) dz \\
\leq\ & Ch(x)^4 \|f\|_{C_p^4} t^2 \zeta_\varepsilon \int_{\mathbb{R}} dy \int_\varepsilon^\infty |yz(\sqrt{z} + y)^3|\rho(z) \\
&\times \int_0^1 du \int_0^1 dv \int_0^1 dw\, u^2 v \frac{e^{-y^2/2}}{\sqrt{2\pi}} uv \\
&\times \Big(1 + |x|^p + u^p v^p w^p (1 + |x|^p)\big(\zeta_\varepsilon^p t^p + z^p + (z^{p/2} + \zeta_\varepsilon^{p/2} t^{p/2}) y^p\big)\Big) dz.
\end{aligned}$$

Therefore

$$|K_2| \le C(1+|x|^{p+4})\|f\|_{C_p^4} t^2 \zeta_\varepsilon \int_\varepsilon^\infty z\rho(z)dz$$
$$\le C(1+|x|^{p+4})\|f\|_{C_p^4} t^2 \varepsilon^{1-\varrho_0}, \tag{2.32}$$

since $\int_\varepsilon^\infty z\rho(z)dz < \infty$.

4) Finally, it follows from (2.20), (2.22), (2.27), (2.28) and (2.30)–(2.32) that for any $\varrho_0 \in (\varrho, 1)$, there exists a positive constant $C(\varrho_0)$ which does not depend on ε such that

$$\big|\mathbb{E}\big[f(\overline{X}_t^{2,\varepsilon}(x)\big] - f(x) - tL_{d+1}^2 f(x)\big|$$
$$\le C(\varrho_0)(1+|x|^{p+4})\|f\|_{C_p^4}\Big(t^{3/2}\varepsilon^{3(1-\varrho_0)/2} + t^2\varepsilon^{3/2-2\varrho_0} + t^2\varepsilon^{1-\varrho_0} + t\varepsilon^{3/2-\varrho_0}\Big),$$

this implies (2.17). □

Next, the parameter ε should be chosen in order to obtain the best bound in (2.17). After a simple calculation, we have the following result.

Lemma 2.12. *Assume that $f \in C_p^4$ and $\int_1^\infty z^{p+2}\rho(z)dz < \infty$. For each $\varrho_0 \in (\varrho, 1)$, if we choose $\varepsilon = O(t^{1/\varrho_0})$ and such that $C_\varepsilon t < 1$, then there exists a positive constant $C(\varrho_0)$ which does not depend on ε and t such that*

$$\big|\mathbb{E}\big[f(\overline{X}_t^{2,\varepsilon}(x)\big] - f(x) - tL_{d+1}^2 f(x)\big| \le C(\varrho_0)(1+|x|^{p+4})\|f\|_{C_p^4}(t^{1+1/\varrho_0} + t^{3/(2\rho_0)}).$$

This result shows that if $\varepsilon = O(t^{1/\rho_0})$ and $C_\varepsilon t < 1$ then the analog of (H2) will be satisfied ($n = 1$ if $\rho \ge 1/2$ and $n = 2$ if $\rho < 1/2$). We also remark that the fact that $\varepsilon = \kappa_0 t^{1/\rho_0}$ together with $C_\varepsilon t < 1$ results on a choice for κ_0.

The next lemma verifies condition (H1) which corresponds to the assumption $(\mathcal{M})$ in [14].

Lemma 2.13. *Assume that $\mathbb{P}[S^\varepsilon = 1] = C_\varepsilon t < 1$. Then for any $p \ge 2$ such that $\int_1^\infty z^p\rho(z)dz < \infty$, there exist constants K and K' satisfying*

$$\mathbb{E}\big[|\overline{X}_t^{2,\varepsilon}(x)|^p\big] \le (1+Kt)|x|^p + K't.$$

Proof. We first denote $f(x) = |x|^p$ and write $\mathbb{E}[f(\bar{X}_t^{2,\varepsilon}(x)] - f(x) = I_1 + I_2$ as in the first part of the proof of Lemma 2.11. We need to show that

$$|I_1| + |I_2| \le ct(1+|x|^p), \quad \forall x \in \mathbb{R}.$$

It follows from (2.18) that

$$|I_1| \le t\zeta_\varepsilon p\theta|h(x)||x|^{p-1}\int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy$$
$$+ t\zeta_\varepsilon p(p-1)h(x)^2 \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}(\sqrt{\zeta_\varepsilon t}\theta + \sigma y)^2 \int_0^1 du$$
$$\times \int_0^1 u|x + uvh(x)(\zeta_\varepsilon t\theta + \sigma\sqrt{\zeta_\varepsilon t}y)|^{p-2}dvdy.$$

Since $|h(x)| \le C(1+|x|)$, we get

$$\begin{aligned}|I_1| &\le Ct\zeta_\varepsilon(1+|x|^p)\\ &\le Ct(1+|x|^p),\end{aligned}$$

since ζ_ε is bounded. It remains to bound I_2. Since $C_\varepsilon^{-1}p_1 = t$, we have

$$\begin{aligned}I_2 \le tp \int_{\mathbb{R}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \int_\varepsilon^\infty & |h(x)|(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}|y|)\rho(z)\\ \times \int_0^1 & |x + uh(x)(\zeta_\varepsilon t\theta + \theta z + \sigma\sqrt{\zeta_\varepsilon t + z}y)|^{p-1} du dz dy.\end{aligned}$$

And it follows from (2.13) that

$$\begin{aligned}|I_2| &\le Ct(1+|x|^p)\Big(1 + \int_\varepsilon^\infty z^p\rho(z)dz\Big)\\ &\le Ct(1+|x|^p),\end{aligned}$$

since $C_\varepsilon t < 1$ and $\int_\varepsilon^\infty z^p\rho(z)dz \le \int_0^1 z\rho(z)dz + \int_1^\infty z^p\rho(z)dz < \infty$. □

Finally, the rate of convergence of our scheme can be established by following the guide in Section 2.4.2 under appropriate regularity conditions.

3. Numerical study

In our numerical study, we concentrate on examples from three points of view.

- Lévy measures with different values of the Blumenthal–Getoor index.
- Different types of expectations. That is, to consider the calculation of $\mathbb{E}[f(X_1)]$ for different functions f.
- Different types of SDE's (different types of coefficients). In particular, we consider oscillating type of coefficients $\sin(ax)$ for different values of a.

In some of the cases considered, the theory provided so far does not tell us the theoretical rate of convergence. Still, by doing the simulations one can learn that the proposed methods still apply and points to further possible theoretical extensions of these methods.

3.1. Example 1: Weak approximation for an SDE driven by a tempered stable Lévy process

Let Z be a tempered stable Lévy process with Lévy measure ν defined on $\mathbb{R}$ by

$$\nu(dy) = \frac{1}{|y|^{\alpha+1}}\big(c_+e^{-\lambda_+|y|}1_{y>0} + c_-e^{-\lambda_-|y|}1_{y<0}\big)dy.$$

3.1.1. We consider the following one-dimensional SDE

$$dX_t = \sin(aX_{t-})dZ_t, \quad X_0 = 1.$$

We will estimate $\mathbb{E}[f(X_1)]$ with $f(x) = 2 - 2\cos(x - X_0)$ using various schemes mentioned in the last section.

We choose the parameter values $a = 1.0$, $\lambda_+ = 1.0, c_+ = 1, \gamma = 1$ and $c_- = 0$. Figures 1 and 2 present the Monte Carlo estimators and the corresponding variances obtained by using various schemes with the jump index Blumenthal–Getoor $\alpha = 0.1$ and $\alpha = 0.9$, respectively. The symbols ES, JSAS, 1JS and 2JS stand for Euler scheme (Section 2.1), Jump size adapted scheme (Section 2.2), One jump scheme (Section 2.4.4) and Two jump scheme (Section 2.4.4), respectively.

In the case that these schemes use the Asmussen–Rosiński approximation for small jumps, we append in parentheses "AR" to their symbols.

In the following, we provide some detailed information about each scheme and compute their theoretical computational costs. To be precise, we fix the error of the estimate at a level ε_0 and compute the expected computational cost, which is needed asymptotically in order to reach this level of error in the weak sense, with respect to ε_0. As usual, these calculations are exact up to constants.

First, we remark that in any scheme, if needed, we will always use the RK4 method (Runge–Kutta scheme of order 4) to solve the ode $dx_t = \sin(ax_t)dt$ (see [4]) if needed.

1. *Euler scheme without AR-correction:* We use the scheme presented in page 187 [5] to generate Z. The method consists of simulating the big jumps and replacing the small ones with their expectation.

   Fixed a jump threshold level $\varepsilon > 0$, then we approximate Z by a compound Poisson process Z^ε which has finite Lévy measure with density

$$\nu^\varepsilon(y) = \frac{1}{|y|^{\alpha+1}}\big(c_+ e^{-\lambda_+|y|}1_{y>\varepsilon} + c_- e^{-\lambda_-|y|}1_{y<-\varepsilon}\big).$$

   Hence Z has intensity $\lambda_\varepsilon = \int_{\mathbb{R}} \nu^\varepsilon(y)dy$, and jump size distribution $p^\varepsilon(x) = \nu^\varepsilon(x)/\lambda_\varepsilon$. The jump size distribution can be simulated by using the acceptance-rejection method. It has been shown in [5] that the average number of loops needed to generate one random variable tends to 1 when $\varepsilon \to 0$. Hence the computational cost to generate Z^ε is proportional to $\lambda_\varepsilon = O(\varepsilon^{-\alpha})$. Therefore considering that there are t^{-1} time partition points, we obtain a total cost of $t^{-1} + \varepsilon^{-\alpha}$.

   On the other hand, the order of convergence of this scheme is

$$t + \int_{|y|\le\varepsilon} |y|^2 \nu(dy) \approx t + \varepsilon^{2-\alpha}.$$

   If we choose $t = \varepsilon^{2-\alpha} = \varepsilon_0$ then the computational cost to reach to error of level ε_0 is $(\frac{1}{\varepsilon_0})^{1\vee(\alpha/(2-\alpha))}$. One should remark that this computational cost blows up when $\alpha \to 2$ even if the error level ε_0 stays constant and sufficiently small.

2. *Euler scheme with AR-correction:* The increment of Z is generated as before with a modification: we replace the small jumps by a Brownian motion with the same local mean and variance as explained in Section 2.2.

   The order of convergence of this scheme is

$$t + \int_{|y|\le\varepsilon} |y|^3 \nu(dy) \approx t + \varepsilon^{3-\alpha}.$$

   As before, if we choose $t = \varepsilon^{3-\alpha} = \varepsilon_0$ then the computational cost to reach to error of level ε_0 is $(\frac{1}{\varepsilon})^{1\vee(\alpha/(3-\alpha))}$.
3. *JSAS without AR-correction:* The approximated solution $\hat{X}$ is defined inductively as follow: $\hat{X}(0) = X_0$ and for $i \ge 0$,

$$\hat{X}(T_{i+1}^{\varepsilon}-) = \theta\Big(\gamma_\varepsilon(T_{i+1}^{\varepsilon} - T_i^{\varepsilon}); \hat{X}(T_i^{\varepsilon})\Big),$$
$$\hat{X}(T_{i+1}^{\varepsilon}) = \hat{X}(T_{i+1}^{\varepsilon}-) + h(\hat{X}(T_{i+1}^{\varepsilon}-))\Delta Z(T_{i+1}^{\varepsilon}).$$

   For an arbitrary point t, we define

$$\hat{X}(t) = \theta\Big(\gamma_\varepsilon(t-\eta_t); \hat{X}(\eta_t)\Big),$$

   where $\eta_t = \sup\{T_i^{\varepsilon} : T_i^{\varepsilon} \le t\}$.

   Although this scheme was not directly studied in [8]. The same ideas give that the error is of the order $\varepsilon^{2-\alpha}$. Therefore the computational cost to reach to error of level ε_0 is $(\frac{1}{\varepsilon_0})^{\frac{\alpha}{2-\alpha}}$.

   Now we introduce the cost for the schemes with limited number of jumps. Recall that for these schemes, we only consider the case that $\alpha < 1$.
4. *JSAS with AR-correction:* The computational cost is proportional to $\lambda_\varepsilon = \int_{|y|>\varepsilon} \nu(dy) = O(\varepsilon^{-\alpha})$.

   The order of convergence of this scheme is

$$\frac{\sigma_\varepsilon^2}{\lambda_\varepsilon}(\sigma_\varepsilon^2 + |\gamma_\varepsilon|) + \int_{|y|\le\varepsilon} |y|^3 \bar{\chi}_\varepsilon \nu(dy) = \varepsilon^{2\wedge(3-\alpha)}.$$

   Hence, the computational cost to reach to error of level ε_0 is $\varepsilon_0^{-\alpha/(2\wedge(3-\alpha))}$.
5. 1*JS without AR-correction:* The weak error of this scheme is proportional to $t + \varepsilon^{2-\alpha}$. Therefore, the optimal choice of parameters is $t = \varepsilon^{2-\alpha} = \varepsilon_0$. The computational cost is proportional to t^{-1}. On the other hand, if we choose $t = \kappa\varepsilon^{2-\alpha}$, where the positive κ is small enough such that $\mathbb{P}[S^{\varepsilon} = 1] = \lambda_\varepsilon t < 1$, then the computational cost to reach an error of level ε_0 is ε_0^{-1}.
6. 1*JS with AR-correction:* In this case the weak error is of the order $t + \varepsilon^{3-\alpha}$. Then the calculation of cost follows as in the previous case, which gives a computational cost of ε_0^{-1}. The main difference with the previous scheme is that fitting the side condition $\mathbb{P}[S^{\varepsilon} = 1] = \lambda_\varepsilon t < 1$ becomes easier. In fact, we choose $\varepsilon = (\frac{t}{\kappa})^{1/\alpha}$, where the positive κ is small enough such that $\mathbb{P}[S^{\varepsilon} = 1] = \kappa\varepsilon^{\alpha}\lambda_\varepsilon = \kappa\varepsilon^{\alpha}\int_{|y|>\varepsilon}\nu(dy) < 1$, then the scheme is of order 1.

7. $2JS$ *without AR-correction:* The weak error is of order $t^2 + \varepsilon^{2-\alpha}$ if the side conditions stated in Lemma 2.6 are satisfied.

   In this case, one needs to consider a non-regular choice of parameters due to these side conditions. In fact, besides the condition that the weak error has to be of order ε_0 one also needs to have that $\lambda_\varepsilon t < 2$. This raises an optimization problem which one can solve easily. The solution is to take $t = \kappa\varepsilon^{\alpha\vee(1-\frac{\alpha}{2})}$, where the positive κ is small enough such that $\mathbb{P}[S_1^\varepsilon = 1] = \lambda_\varepsilon t - \frac{\lambda_\varepsilon^2 t^2}{2} < 1$ and $\mathbb{P}[S_2^\varepsilon = 1] = \frac{\lambda_\varepsilon t}{2-\lambda_\varepsilon t} < 1$.
   This choice gives a final computational cost of $(\frac{1}{\varepsilon_0})^{\frac{1}{2}\vee\frac{\alpha}{2-\alpha}}$.
8. $2JS$ *with AR-correction:* In this case the optimization problem that appeared in the previous case is simplified due to the higher weak rate of convergence and one may choose $t = \kappa\varepsilon^\alpha$, where the positive κ is small enough such that $\mathbb{P}[S_1^\varepsilon = 1] = \lambda_\varepsilon t - \frac{\lambda_\varepsilon^2 t^2}{2} < 1$ and $\mathbb{P}[S_2^\varepsilon = 1] = \frac{\lambda_\varepsilon t}{2-\lambda_\varepsilon t} < 1$, then the scheme is of order t^2. The computational cost to reach to error of level ε_0 is $\varepsilon_0^{-1/2}$.

   Putting this information on a table for the case $\alpha < 1$, we obtain

method	ES	ES(AR)	JSAS	JSAS(AR)
cost	ε_0^{-1}	ε_0^{-1}	$(\varepsilon_0)^{-\frac{\alpha}{2-\alpha}}$	$\varepsilon_0^{-\alpha/2}$
t	$\varepsilon^{2-\alpha}$	$\varepsilon^{3-\alpha}$	–	–
ε	$\varepsilon_0^{\frac{1}{2-\alpha}}$	$\varepsilon_0^{\frac{1}{3-\alpha}}$	$\varepsilon_0^{\frac{1}{2-\alpha}}$	$\varepsilon_0^{\frac{1}{2}}$
N	t^{-1}	t^{-1}	λ_ε	λ_ε
method	1JS	1JS(AR)	2JS	2JS(AR)
cost	ε_0^{-1}	$(\varepsilon_0)^{-\frac{1}{2}\vee\frac{\alpha}{2-\alpha}}$	ε_0^{-1}	$\varepsilon_0^{-1/2}$
t	$\varepsilon^{2-\alpha}$	ε^α	$\varepsilon^{\alpha\vee(1-\frac{\alpha}{2})}$	ε^α
ε	$\varepsilon_0^{\frac{1}{2-\alpha}}$	$\varepsilon_0^{\frac{1}{3-\alpha}}$	$\varepsilon_0^{\frac{1}{2-\alpha}}$	$\varepsilon_0^{\frac{1}{3-\alpha}}$
N	t^{-1}	t^{-1}	t^{-1}	t^{-1}

From this table one can deduce that the JSAS methods have the lowest theoretical expected computational cost while the Euler scheme methods perform the worst.

This table assumes the general situation where one does not have information of how to generate the increments of the Lévy process exactly.

Let us now procceed with the experimental results. The estimator and variance of each scheme is plotted as a function of $\log(N)$ where N is the numbers of discretization points n between 0 and 1. The parameter λ_ε appearing in the JSAS method is chosen equal to N in order to allow for comparison of computational cost.

We have decided to use this as it would seem the most natural measure of computational time. The only case where this will differ with theoretical com-

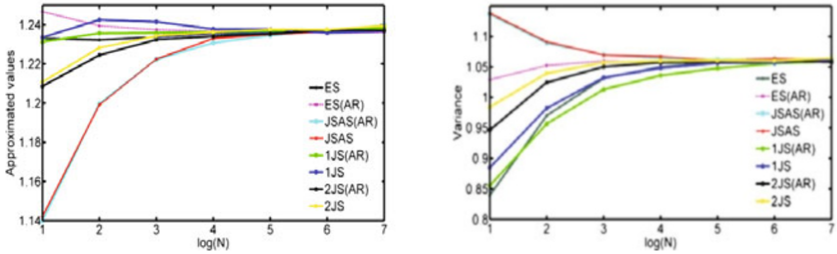

FIGURE 1. Numerical comparison of various schemes for estimating $\mathbb{E}[f(X_1)]$ with $\alpha = 0.1$. (Left: mean, right: variance)

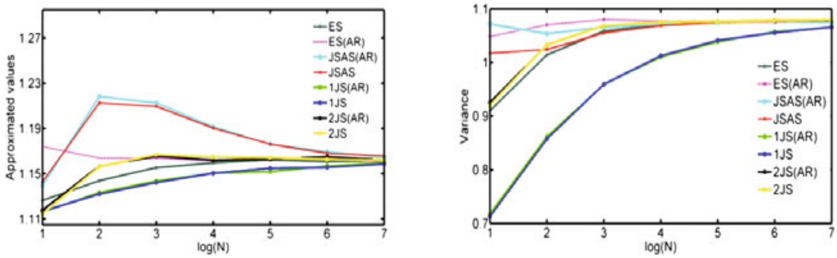

FIGURE 2. Numerical comparison of various schemes for estimating $\mathbb{E}[f(X_1)]$ with $\alpha = 0.9$. (Left: mean, right: variance)

putational time is in the case of the Euler scheme when all jumps have to be simulated.

For each point, we simulated 10^6 trajectories. In Figures 1 and 2, we see the convergence and the variance of each estimator. The computational times with respect to the case $\log(N) = 6$ are shown in Figure 3. This figure shows that the theoretical computational estimates are not necessarily accurate at this level and shows the difference of the constants in the asymptotic estimates. For example, the increase of computational time for the Euler scheme from $\alpha = 0.1$ to $\alpha = 0.9$ is due to the increase in the number of jumps. Even more, the fact that the JSAS schemes have a random number of partitions seems to play an important role in the computational time. In fact, asymptotically speaking, the number of calculations needed is a Poisson random variable with mean λ_ϵ which behaves like a Gaussian r.v. with variance proportional to λ_ϵ. From this figure, we can also see the increasing dependence of these constants with respect to the value of α.

Next, we perform the same simulation as above but with a different value of parameter a. More precisely, we choose $a = 5.0,\ \lambda_+ = 1.0,\ \alpha = 0.9, c_+ = 1, \gamma = 1$ and $c_- = 0$. The results are presented in Figures 4 and 5.

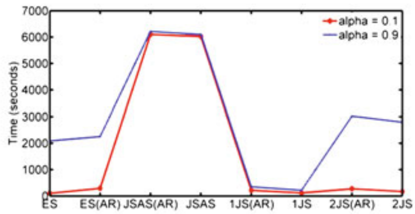

FIGURE 3. Computation time taken in the estimation of $\mathbb{E}[f(X_1)]$

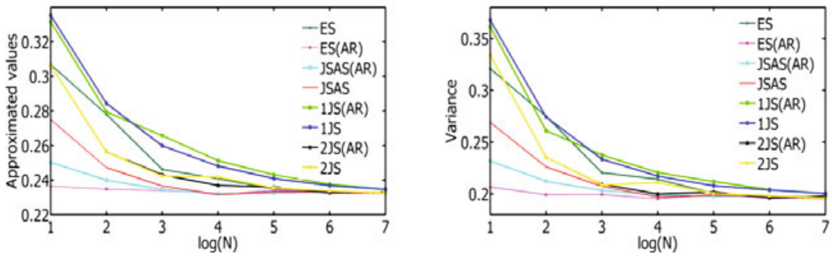

FIGURE 4. Numerical comparison of various schemes for estimating $\mathbb{E}[f(X_1)]$ with $a = 5.0$. (Left: mean, right: variance)

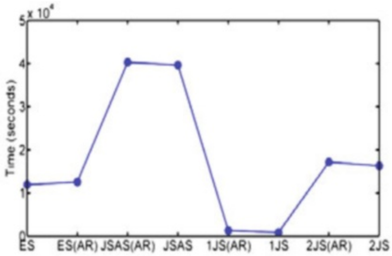

FIGURE 5. Computation time taken in the estimation of $\mathbb{E}[f(X_1)]$ with respect to $\log(N) = 6$

The conclusion is that in general the 1JS method is fast and gives good results for coefficients that do not oscillate too much. This contrasts with the theoretical results shown in the previous table. This seems to be caused by the size of the constants in the error expansions.

If the coefficients have rapidly growing derivatives then the method looses accuracy and one may better use the JSAS method which may be time consuming. In between these two methods one has the 2JS method. Therefore a practical issue

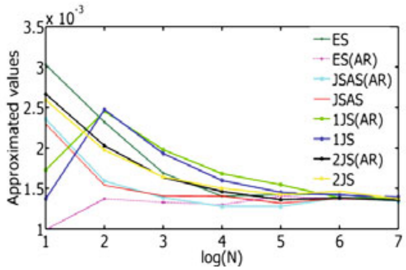

FIGURE 6. Numerical comparison of various schemes for estimating $\mathbb{P}[X_1 > 2]$

is how to determine before implementing the method which one should use and the range of applicability of each method.

3.1.2. The approximation schemes presented in Section 2 are only applicable for smooth functions f. However, in the next simulation, we will use this scheme to estimate the probability $\mathbb{P}(X_1 > x_0)$, or in other words, to estimate the expectation $\mathbb{E}(f(X_1))$ with $f(x) = 1_{x>x_0}$.

We choose $a = 5.0,\ \lambda_+ = 0.5,\ \alpha_+ = 0.9, c_+ = 1, \gamma = 1$ and $c_- = 0$. The results are presented in Figure 6. This study reveals that one may need to use an importance sampling method in order to improve the accuracy of the proposed method.

3.2. Example 2: Weak approximation for an SDE driven by a NIG Lévy process

Let Z be a normal inverse Gaussian Lévy process whose characteristic function is defined by

$$\phi_t(u) = \mathbb{E}(e^{iuZ_t}) = \exp\Big\{ -\delta t\Big(\sqrt{\alpha^2 - (\beta - iu)^2} - \sqrt{\alpha^2 - \beta^2}\Big)\Big\}$$

where $\alpha > 0$ and $\beta \in (-\alpha, \alpha)$ and $\delta > 0$ are parameters. The Lévy density is given by

$$\nu(x) = \frac{\delta\alpha}{\pi}\frac{e^{\beta x}K_1(\alpha|x|)}{|x|},$$

where K is a modified Bessel function of the second kind. The NIG process can be defined as

$$Z_t = \theta Y_t + \sigma W_{Y_t}, \tag{3.1}$$

where W is a standard Brownian motion and Y is a inverse Gaussian subordinator: a pure jump Lévy process with Lévy density $\rho(x) = \frac{2}{\sqrt{2\kappa\pi}}\frac{e^{-\frac{x}{2\kappa}}}{|x|^{3/2}}$ and therefore

$\rho_0 = 0.5$ in this case. The parameters (σ, θ, κ) and (α, β, δ) are related by

$$\begin{cases} \kappa = \frac{1}{\delta\sqrt{\alpha^2-\beta^2}} \\ \theta = \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}} \\ \sigma^2 = \frac{\delta}{\sqrt{\alpha^2-\beta^2}} \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{\sqrt{\theta^2+\sigma^2\kappa^{-1}}}{\sigma^2} \\ \beta = \theta\sigma^{-2} \\ \delta = \sigma\kappa^{-1/2}. \end{cases}$$

The representation (3.1) allows to simulate exact increments of NIG process in order to perform an Euler approximation scheme.

Let p_t be the density of $\theta t + \sigma W_t$, the density of the NIG process can be presented as $\nu(x) = \int_0^\infty p_t(x)\rho(t)dt$. We define the finite measure ν_ε via $\nu_\varepsilon(x) = \int_\varepsilon^\infty p_t(x)\rho(t)dt$, and then $\chi_\varepsilon(x) = \frac{\nu_\varepsilon(x)}{\nu(x)}$.

The constants $\lambda_\varepsilon, \gamma_\varepsilon, \sigma_\varepsilon$ are computed as follows

$$\lambda_\varepsilon = \sqrt{\frac{2}{\pi\kappa\varepsilon}} \exp\Big(-\frac{\varepsilon}{2\kappa}\Big) - \frac{2}{\kappa}N\Big(-\sqrt{\frac{\varepsilon}{\kappa}}\Big),$$

$$\gamma_\varepsilon = \theta - 2\theta N\Big(-\sqrt{\frac{\varepsilon}{\kappa}}\Big),$$

$$\sigma_\varepsilon^2 = (\sigma^2 + \kappa\theta^2)\Big(1 - 2N\Big(-\sqrt{\frac{\varepsilon}{\kappa}}\Big)\Big) - \sqrt{\frac{2\kappa\varepsilon}{\pi}} \exp\Big(-\frac{\varepsilon}{2\kappa}\Big)\theta^2,$$

where $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \exp(-\frac{x^2}{2})dx$.

We choose $\sigma = 0.5$; $\theta = 0.4$; $\kappa = 0.6$, and solve the one-dimensional SDE

$$dX_t = \sin(aX_t)d\tilde{Z}_t,$$

where $\tilde{Z}$ is the NIG process with drift adjusted to have $\mathbb{E}(\tilde{Z}_t) = 0$. In other words, $\tilde{Z}_t = Z_t - \theta t$. In this case, the values of λ_ε and σ_ε are the same as before but $\gamma_\varepsilon = -2\theta N\Big(-\sqrt{\frac{\varepsilon}{\kappa}}\Big)$.

We first use the representation (3.1) to define the Euler's scheme for X_1. Besides, we also use JSAS method and 1JS method to simulate X_1 as introduced in Section 2.4.5.

We now explain how to define the JSAS1 scheme (2.1)–(2.3) to simulate X_1:

1. $(T_i^\varepsilon)_{i\in\mathbb{N}}$ denotes jump times of a Poisson process whose intensity is λ_ε, $T_0^\varepsilon = 0$.
2. $(\Delta Z(T_i^\varepsilon))_{i\in\mathbb{N}}$ denotes a sequence of independent random variables whose density is $\frac{\nu_\varepsilon}{\lambda_\varepsilon}$.
3. The solution of the ODE $dX_t = \sin(aX_t)dt$ is approximated using the RK4 method.

A random variable with density $\frac{\nu_\varepsilon}{\lambda_\varepsilon}$ can be sampled using the following algorithm:

1. Sample a random variable Y with probability density $\frac{\rho(x)I_{x>\varepsilon}}{\lambda_\varepsilon}$ using the acceptance-rejection method (see [5], Example 6.9).

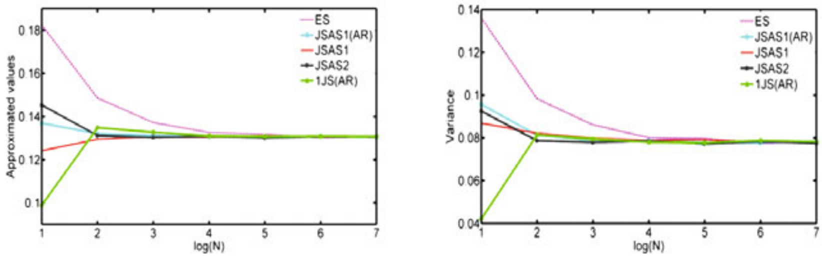

FIGURE 7. $a = 5,\ f(x) = 2 - 2\cos(x - X_0)$. Left: Mean. Right: Variance

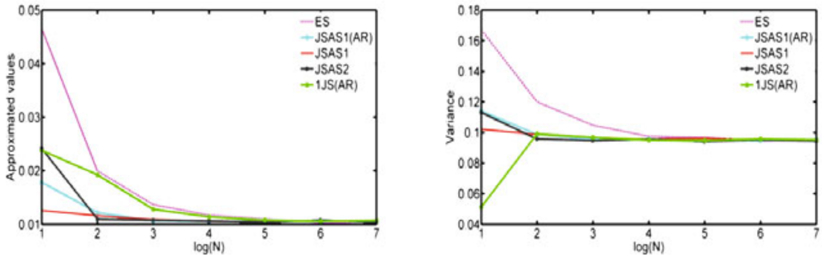

FIGURE 8. $a = 5,\ f(x) = 1_{\{x>3/2\}}$. Left: Mean. Right: Variance

2. Conditional on Y, sample X with law p_Y. It means that X is sampled by $X = \theta Y + \sigma\sqrt{Y}W'$, where W' is a standard normal distributed random variable.

Next, we use JSAS2 method introduced in Section 2.2.2 to simulate X_1. In this setting, $\Omega(t) = \sigma_\varepsilon^2 h^2(Y_t^0)(t - \eta_t)$.

Finally, we use 1JS(AR) method defined in Section 2.4.5 to simulate X_1. We remark that this method already incorporates the Asmussen–Rosiński approximation in its definition. One can also do a similar computational cost study for this case. We do not give details but only the following table.

method	ES	JSAS1	JSAS1(AR)	JSAS2	1JS(AR)
cost	ε_0^{-1}	ε_0^{-1}	$\varepsilon_0^{-1/2}$	$\varepsilon_0^{-1/2}$	$\varepsilon_0^{-1/2}$

Figures 7, 8 and 9 show the Monte Carlo estimation for $\mathbb{E}[f(X_1)]$ with $f(x) = 2-2\cos(x-X_0)$, $f(x) = 1_{\{x>3/2\}}$ and $f(x) = e^x$, and the corresponding variances, respectively, with $a = 5.0$.

Figures 10, 11 and 12 show the Monte Carlo estimation for $\mathbb{E}[f(X_1)]$ with $f(x) = 2 - 2\cos(x - X_0)$, $f(x) = 1_{\{x>3/2\}}$ and $f(x) = e^x$, and the corresponding

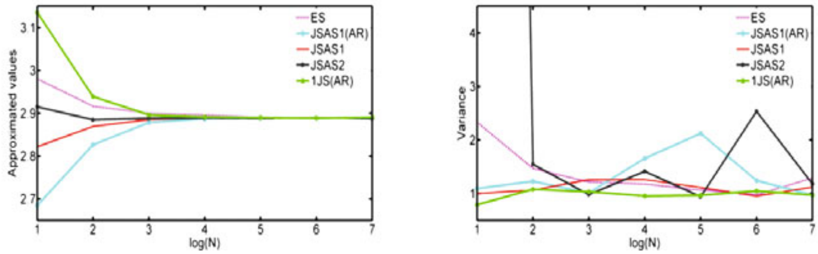

FIGURE 9. $a = 5,\ f(x) = e^x$. Left: Mean. Right: Variance

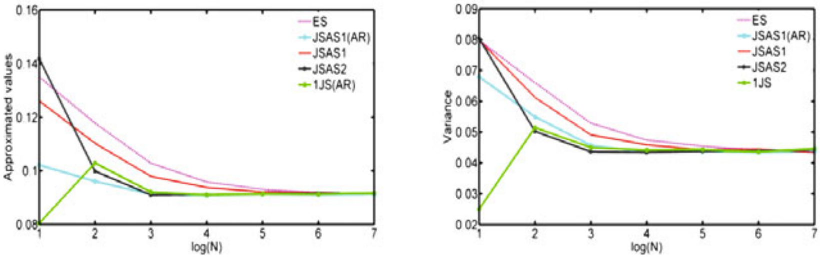

FIGURE 10. $a = 10,\ f(x) = 2 - 2\cos(x - X_0)$. Left: Mean. Right: Variance

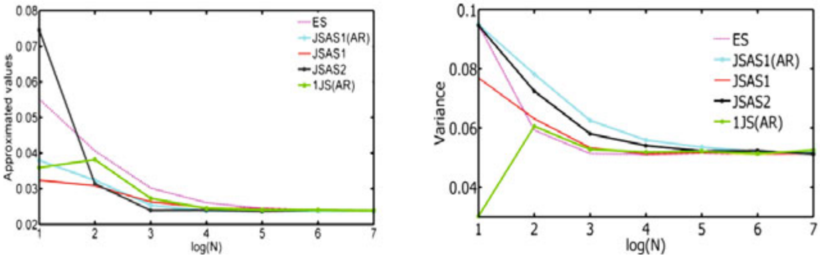

FIGURE 11. $a = 10,\ f(x) = 1_{\{x>3/2\}}$. Left: Mean. Right: Variance

variances, respectively, with $a = 10.0$. The computational time of each method with respect to the case $\log(N) = 7$ and $N_{MC} = 10^6$ are shown in Figure 13.

The conclusion is that, on one hand, JSAS methods have higher rates of convergence than the other methods. On the other hand, 1JS(AR) method defined in Section 2.4.5 is very fast and gives good results even when coefficients oscillate a lot.

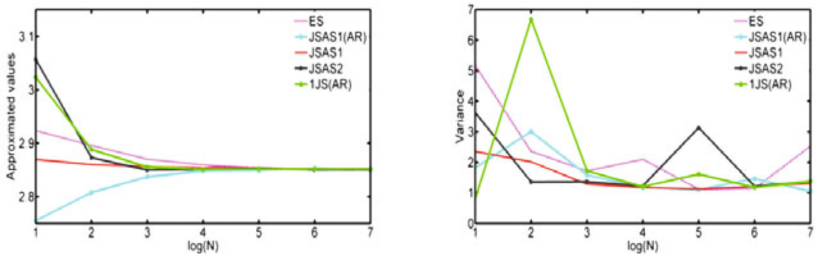

FIGURE 12. $a = 10$, $f(x) = e^x$. Left: Mean. Right: Variance

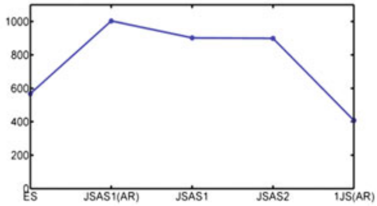

FIGURE 13. Computation time for estimating $\mathbb{E}f(X_1)$

3.3. Some conclusions

Throughout the experiments, we see that there is a big gap between theoretical asymptotic values and the actual computational results. So far, one can see that the 1JS is a fairly efficient scheme in most situations considering its accuracy and computational time. If high accuracy is required then the JSAS or 2JS can be used. Further studies are needed which may also incorporate new schemes. We have striven here for generality and therefore many faster schemes may be provided for particular situations.

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Itô's Formula for Banach-space-valued Jump Processes Driven by Poisson Random Measures

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Abstract. We prove Itô's formula for a general class of functions $H : \mathbb{R}_+ \times F \to G$ of class $C^{1,2}$, where F, G are separable Banach spaces, and jump processes driven by a compensated Poisson random measure.

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Keywords. Itô's formula, Poisson random measure, stochastic integral, Banach space of M-type 2.

1. Introduction

We prove Itô's formula for Banach-space-valued stochastic jump processes driven by a compensated Poisson random measure.

In this context, Itô's formula was originally given in [14]. However, there it was only proven for a smaller class of integrands, as stochastic integration for Banach-space-valued integrands was still not understood in the generality of the forthcoming papers [9, 10]. We remind here also the work of E. Hausenblas [6], where Itô's formula on Banach spaces was proven assuming additional conditions on the integrands. In a previous work of Graveraux and Pellaumail [5], where also additional conditions on the integrands are required, the Itô formula was not given in terms of the compensator. However, this is necessary in case the formula is used to find the generator of a Markov process. In this article, we provide an improvement of the work of [14]. Even if the methods are similar, the current article is presented in a clearer and direct way, due to integrands having general integrability conditions. The mark space $(E, \mathcal{E})$, associated to the compensated Poisson random measure, is also allowed to be more general than in [14], where E has been a separable Banach space. We refer Remark 2.1 and Remark 3.2, where this generalization is discussed. Moreover, an additional improvement is given, which is important for applications: The condition $H \in C_b^{1,2}(\mathbb{R}_+ \times F; G)$, i.e., that the function H and its partial derivatives are bounded, is not required any more. This means that Itô's formula can be applied to functionals such as $\| \cdot \|^2$, as discussed in Example 3.

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, and let $(F, \| \cdot \|)$ be a separable Banach space. Let $(E, \mathcal{E})$ be a measurable space which we assume to be a *Blackwell space* (see [1, 4]). Furthermore, let N be a time-homogeneous Poisson random measure on $\mathbb{R}_+ \times E$, see [7, Def. II.1.20]. Then its compensator is of the form $\nu(dt, dx) = dt\beta(dx)$, where β is a σ-finite measure on $(E, \mathcal{E})$. We call $q(dt, dx) = N(dt, dx) - \nu(dt, dx)$ the associated compensated Poisson random measure.

Remark 2.1. In previous works on this topic, see, e.g., [9, 10, 13, 14], the mark space $(E, \mathcal{E})$ is a separable Banach space equipped with its Borel σ-field, and N is the Poisson random measure associated to an E-valued Lévy process with Lévy measure β. The upcoming results from this Section, which we take from [9, 10, 13, 14], also hold true in our present, more general framework, and with analogous proofs (see, e.g., [2, 8] for the case where F is a separable Hilbert space). Indeed, the crucial point is whether the integral operator (2.3) can be extended to a continuous linear operator, and this property does not rely on the structure of E (see also Remark 2.4 below). The assumption that the measurable space $(E, \mathcal{E})$ is a Blackwell space is a standard assumption when dealing with random measures, see, e.g., [7, Chapter II.1]. It ensures the existence and uniqueness of the predictable compensator, see [7, Thm. II.1.8]. We remark that every Polish space with its Borel σ-field is a Blackwell space.

We fix an arbitrary $T \in \mathbb{R}_+$. Let us consider the set of progressively measurable functions on the time interval $[0, T]$, i.e.,

$$M^T(E/F) := \{f : \Omega \times [0,T] \times E \to F : f \text{ is } \mathcal{B}[0,T] \otimes \mathcal{E} \otimes \mathcal{F}_T\text{-measurable} \\ \text{and } f(t,x) \text{ is } \mathcal{F}_t\text{-measurable for all } t \in [0,T] \text{ and } x \in E\}.$$

Furthermore, we define

$$M_\nu^{T,2}(E/F) := \left\{ f \in M^T(E/F) : \int_0^T \int_E \mathbb{E}[\|f(t,x)\|^2] \nu(dt, dx) < \infty \right\},$$

where $\mathbb{E}[f]$ denotes the expectation with respect to the probability measure $\mathbb{P}$.

Definition 2.2. A function $f \in M^T(E/F)$ belongs to the set $\Sigma_T(E/F)$ *of simple functions*, if there exist $n, m \in \mathbb{N}$ such that

$$f(t,x) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} \mathbb{1}_{A_{k,l}}(x) \mathbb{1}_{F_{k,l}} \mathbb{1}_{(t_k, t_{k+1}]}(t) a_{k,l},$$

where $\beta(A_{k,l}) < \infty$, $t_k \in (0,T]$, $t_k < t_{k+1}$, $F_{k,l} \in \mathcal{F}_{t_k}$, $a_{k,l} \in F$, and for all $k \in 1,\dots,n-1$ we have $A_{k,l_1} \times F_{k,l_1} \cap A_{k,l_2} \times F_{k,l_2} = \emptyset$ if $l_1 \neq l_2$.

The set $\Sigma_T(E/F)$ of simple functions is dense in the Banach space $M_\nu^{T,2}(E/F)$ with norm

$$\|f\|_2 := \sqrt{\int_0^T \int_E \mathbb{E}[\|f(t,u)\|^2]\nu(dt,dx)}.$$

The proof only uses the fact that the simple functions are defined on the sets of a semiring which generates the σ-algebra $\mathcal{B}[0,T] \otimes \mathcal{E} \otimes \mathcal{F}_T$ and that the compensator is of the form $\nu(dt,dx) = dt\beta(dx)$, see [13, Theorem 4.2] (where this is proven for slightly more general compensated Poisson random measures having compensators $\alpha(dt)\beta(dx)$ with $\alpha(dt)$ being absolutely continuous w.r.t. Lebesgue measure). We remark that in [15, Chapter 2.4], which deals the case of real-valued integrands, a bigger set of simple functions is considered.

The Itô integral of simple functions is defined as usual pathwise in a very natural way (see Chapter 3 in [13]): For $f \in \Sigma_T(E/F)$, the "natural stochastic integral" of f is given by

$$\int_0^T \int_E f(t,x)q(dt,dx) := \sum_{k=1}^{n-1} \sum_{l=1}^{m} a_{k,l} \mathbb{1}_{F_{k,l}} q((t_k,t_{k+1}] \cap (0,T] \times A_{k,l}).$$

Remark 2.3. Suppose the mark space $(E,\mathcal{E})$ is a separable Banach space equipped with its Borel σ-field. Then, for each $f \in \Sigma_T(E/F)$ we have

$$\int_0^T \int_E f(s,x)q(ds,dx) = \sum_{0<s\le T} f(s,\Delta X_s) - \mathbb{E}\Big[\sum_{0<s\le T} f(s,\Delta X_s)\Big], \tag{2.1}$$

where $(X_t)_{t\ge 0}$ is the Lévy process associated to the compensated Poisson random measure $q(ds,dx)$, and ΔX_s denotes the jump of X at time s, i.e., $\Delta X_s = X_s - \lim_{u\uparrow s} X_u$.

Let $\mathcal{M}_T^2(F)$ be the linear space of all F-valued square integrable martingales $M = (M_t)_{t\in[0,T]}$ with norm

$$\|M\|_{\mathcal{M}_T^2} = \big(\mathbb{E}[\|M_T\|^2]\big)^{1/2}.$$

The Itô integral for functions $f \in M_\nu^{T,2}(E/F)$ is well defined, if the linear operator

$$\Sigma_T(E/F) \to \mathcal{M}_T^2(F), \quad f \mapsto \bigg(\int_0^t \int_E f(s,x)q(ds,dx)\bigg)_{t\in[0,T]} \tag{2.2}$$

can uniquely be extended to a continuous linear operator

$$M_\nu^{T,2}(E/F) \to \mathcal{M}_T^2(F), \quad f \mapsto \bigg(\int_0^t \int_E f(s,x)q(ds,dx)\bigg)_{t\in[0,T]}. \tag{2.3}$$

In particular, if this is the case, for all $f \in M_\nu^{T,2}(E/F)$ there is a sequence $(f_n)_{n\in\mathbb{N}} \subset \Sigma_T(E/F)$ such that $\lim_{n\to\infty} \|f - f_n\|_2 = 0$ and

$$\lim_{n\to 0} \mathbb{E}\left[\left\| \int_0^T \int_E (f(s,x) - f_n(s,x))q(ds,dx)\right\|^2\right] = 0.$$

Remark 2.4. In [10] we have proven that the Itô Integral for simple functions in (2.2) can uniquely be extended to a continuous linear operator (2.3) if and only if there is a constant K_β, depending on the Lévy measure β, such that

$$\begin{aligned} &\mathbb{E}\left[\left\| \int_0^T \int_E f(s,x)q(ds,dx)\right\|^2\right] \\ &\quad \le K_\beta \mathbb{E}\left[\int_0^T \int_E \|f(s,x)\|^2 \beta(dx)ds\right] \quad \text{for all } f \in \Sigma_T(E/F). \end{aligned} \tag{2.4}$$

If the Banach space F is of M-type 2, then such a constant exists and does not depend on the Lévy measure β, see [9]. We also emphasize that – as indicated in Remark 2.1 – the proof of this result does not rely on the structure of E. Hence, the M-type 2 condition is a sufficient, but not necessary condition for the Itô integral with respect to compensated Poisson random measures to be well defined. Typical examples of M-type 2 Banach spaces are the Lebesgue spaces $L^p(G,\mathcal{G},\mu)$ with $2 \le p < \infty$ and $(G,\mathcal{G},\mu)$ being a measure space, see [11, 12]. On the other hand, if such a constant exists and does not depend on the Lévy measure β, i.e., $K_\beta = K$, then the Banach space F has to be of type 2, see [10]. We also remark that in case $F = H$ being a Hilbert space, the linear operator (2.3) is even an isometry, see [13].

There are separable Banach spaces which are not of M-type 2, as the following example shows:

Example. Let ℓ^1 be the space of all real-valued sequences $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ which are absolutely convergent, that is

$$\|x\|_{\ell^1} := \sum_{j=1}^\infty |x_j| < \infty.$$

Then $(\ell^1, \|\cdot\|_{\ell^1})$ is a separable Banach space which is not of M-type 2. Indeed, let $(e_j)_{j\in\mathbb{N}}$ be the standard unit sequences in ℓ^1, which are given by

$$e_1 = (1,0,\dots), \quad e_2 = (0,1,0,\dots), \quad \dots$$

Let $n \in \mathbb{N}$ be arbitrary. We denote by $(X_j^{(n)})_{j=1,\dots,n}$ independent random variables having a normal distribution $N(0,1/n)$, and we define the ℓ^1-valued process $M^{(n)} = (M_k^{(n)})_{k=0,\dots,n}$ as

$$M_0^{(n)} := 0 \qquad \text{and} \qquad M_k^{(n)} := \sum_{j=1}^k X_j^{(n)} e_j, \quad k = 1,\dots,n.$$

Then $M^{(n)}$ is a martingale with respect to the filtration $(\mathcal{F}_k^{(n)})_{k=0,\dots,n}$ given by

$$\mathcal{F}_0^{(n)} = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_k^{(n)} = \sigma(X_1^{(n)}, \dots, X_k^{(n)}), \quad k = 1, \dots, n.$$

Moreover, we have

$$\sum_{k=1}^n \mathbb{E}\big[\|M_k^{(n)} - M_{k-1}^{(n)}\|_{\ell^1}^2\big] = \sum_{k=1}^n \mathbb{E}[\|X_k^{(n)} e_k\|_{\ell^1}^2] = \sum_{k=1}^n \mathbb{E}\big[|X_k^{(n)}|^2\big] = 1$$

as well as

$$\begin{aligned}
\mathbb{E}\big[\|M_n^{(n)}\|_{\ell^1}^2\big] &= \mathbb{E}\left[\left\|\sum_{j=1}^n X_j^{(n)} e_j\right\|_{\ell^1}^2\right] = \mathbb{E}\left[\left(\sum_{j=1}^n |X_j^{(n)}|\right)^2\right] \\
&= \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}\big[|X_i^{(n)} X_j^{(n)}|\big] = \sum_{j=1}^n \mathbb{E}\big[|X_j^{(n)}|^2\big] + \sum_{i \neq j} \mathbb{E}\big[|X_i^{(n)}|\big]\mathbb{E}\big[|X_j^{(n)}|\big] \\
&= 1 + \sum_{i \neq j} \frac{2}{\pi n} = 1 + \frac{2n(n-1)}{\pi n} = 1 + \frac{2(n-1)}{\pi} \to \infty \quad \text{for } n \to \infty,
\end{aligned}$$

showing that ℓ^1 is not of M-type 2.

Remark 2.5. The space ℓ^1 has also been utilized in [17] in order to provide counterexamples for stochastic integration in Banach spaces with respect to a Wiener process. In [16], the stochastic integral with respect to a Wiener process has been defined on so-called UMD Banach spaces. This approach is based on a two-sided decoupling inequality for UMD spaces due to [3].

Together with Example 2, the next result shows that ℓ^1 is a separable Banach space, which is not of M-type 2, but where inequality (2.4) is satisfied for certain Poisson random measures.

Proposition 2.6. *We suppose that $\beta(E) < \infty$. Then inequality* (2.4) *is satisfied with $K_\beta = 4 + 6T\beta(E)$.*

Proof. Let $f \in \Sigma_T(E/F)$ be arbitrary. Then we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^T \int_E f(s,x) q(ds,dx)\right\|^2\right] \\
&\quad \leq 2\mathbb{E}\left[\left(\int_0^T \int_E \|f(s,x)\| N(ds,dx)\right)^2\right] + 2\mathbb{E}\left[\left(\int_0^T \int_E \|f(s,x)\| \beta(dx) ds\right)^2\right] \\
&\quad \leq 2\mathbb{E}\left[\left(\int_0^T \int_E \|f(s,x)\| q(ds,dx) + \int_0^T \int_E \|f(s,x)\| \beta(dx) ds\right)^2\right] \\
&\qquad + 2\mathbb{E}\left[\left(\int_0^T \int_E \|f(s,x)\| \beta(dx) ds\right)^2\right].
\end{aligned}$$

Thus, by the Itô isometry for real-valued integrands and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg\| \int_0^T \int_E f(s,x) q(ds,dx) \bigg\|^2\bigg] \\
&\leq 4\mathbb{E}\bigg[\bigg(\int_0^T \int_E \|f(s,x)\| q(ds,dx) \bigg)^2\bigg] + 6\mathbb{E}\bigg[\bigg(\int_0^T \int_E \|f(s,x)\| \beta(dx)ds \bigg)^2\bigg] \\
&\leq 4\mathbb{E}\bigg[\int_0^T \int_E \|f(s,x)\|^2 \beta(dx)ds \bigg] + 6T\beta(E)\mathbb{E}\bigg[\int_0^T \int_E \|f(s,x)\|^2 \beta(dx)ds \bigg].
\end{aligned}$$

Consequently, inequality (2.4) is satisfied with $K_\beta = 4 + 6T\beta(E)$. □

If the continuous linear operator (2.3) is well defined, then the definition of the Itô integral can be extended to all $f \in \mathcal{K}^2_{T,\beta}(E/F)$, where $\mathcal{K}^2_{T,\beta}(E/F)$ denotes the linear space of all progressively measurable $f \in M^T(E/F)$ such that

$$\mathbb{P}\bigg(\int_0^T \int_E \|f(s,x)\|^2 \beta(dx)ds < \infty \bigg) = 1.$$

For all $f \in \mathcal{K}^2_{T,\beta}(F)$ we define the sequence of stopping times

$$\tau_n := \inf\bigg\{ t \in [0,T] : \int_0^t \int_E \|f(s,x)\|^2 \beta(dx)ds \geq n \bigg\}, \quad n \in \mathbb{N}.$$

Note that $f\mathbb{1}_{[0,\tau_n]} \in M^{T,2}_\nu(E/F)$ for all $n \in \mathbb{N}$. Hence, we can define the Itô integral

$$\int_0^t \int_E f(s,x) q(ds,dx) := \lim_{n\to\infty} \int_0^t \int_E f(s,x)\mathbb{1}_{[0,\tau_n]} q(ds,dx), \quad t \in [0,T]$$

which is a local martingale.

In the sequel we will use Theorem 7.7 with Remark 7.8 from [13]. We recall the result here:

Theorem 2.7. *Let $f \in \mathcal{K}^2_{T,\beta}(E/F)$ be arbitrary and let $(f_n)_{n\in\mathbb{N}}$ be a sequence such that $f_n \in \mathcal{K}^2_{T,\beta}(E/F)$ for all $n \in \mathbb{N}$. Suppose that f_n converges $\nu \otimes \mathbb{P}$-almost surely to f on $\Omega \times [0,T] \times E$, when $n \to \infty$, and $\mathbb{P}$-almost surely*

$$\lim_{n\to\infty} \int_0^T \int_E \|f_n - f\|^2 d\nu = 0.$$

Assume there is $g \in \mathcal{K}^2_{T,\beta}(E/F)$ such that

$$\int_0^T \int_E \|f_n\|^2 d\nu \leq \int_0^T \int_E \|g\|^2 d\nu.$$

Then, we have

$$\int_0^t \int_E f(s,x) q(ds,dx) = \lim_{n\to\infty} \int_0^t \int_E f_n(s,x) q(ds,dx),$$

where the limit is in probability.

3. Itô's formula for Banach-space-valued functions

Let F be a separable Banach space and the integral operator (2.3) be a continuous linear operator for each $T \in \mathbb{R}_+$. By Remark 2.4 this is equivalent to state that there is a constant K_β such that (2.4) holds. As pointed out in Remark 2.4, this is in particular satisfied when the Banach space F is of M-type 2.

Remark 3.1. According to [7, Prop. II.1.14], there exist a sequence $(\tau_k)_{k\in\mathbb{N}}$ of finite stopping times with $[\![\tau_k]\!] \cap [\![\tau_l]\!] = \emptyset$ for $k \neq l$ and an E-valued optional process ξ such that for every optional process $f : \Omega \times \mathbb{R}_+ \times E \to H$ with

$$\mathbb{P}\bigg(\int_0^t \int_E \|f(s,x)\| N(ds,dx) < \infty\bigg) = 1 \quad \text{for all } t \geq 0$$

we have the identity

$$\int_0^t \int_E f(s,x) N(ds,dx) = \sum_{k\in\mathbb{N}} f(\tau_k, \xi_{\tau_k}) \mathbb{1}_{\{\tau_k \leq t\}}, \quad t \geq 0. \tag{3.1}$$

Remark 3.2. Suppose the mark space $(E, \mathcal{E})$ is a separable Banach space equipped with its Borel σ-field. From Remark 2.3 it follows that the stopping times τ_k in Remark 3.1 can be chosen to be the jump times of the corresponding Lévy process $(X_t)_{t\geq 0}$, with the random variables ξ_{τ_k} being the jumps of the process at time τ_k, that is $\xi = \Delta X$. An analogous statement for $E = \mathbb{R}^d$ and an adapted càdlàg process X can be found in [7, Prop.II.1.16]. The corresponding result, where E is a separable Banach space, is given by Theorem 5.1 in [13]. The result [7, Prop. II.1.14] used in Remark 3.1 allows us to use a more general mark space $(E, \mathcal{E})$ than in [13], i.e., a Blackwell space.

From now on, let G be another separable Banach space such that integral operator (2.3) with $F = G$ is a continuous linear operator for each $T \in \mathbb{R}_+$. Again by Remark 2.4, this ensures that all upcoming stochastic integrals are well defined, and that, for some constant $K_\beta > 0$, for each $T \in \mathbb{R}_+$ we have the estimates

$$\mathbb{E}\bigg[\bigg\|\int_0^T \int_E f(s,x) q(ds,dx)\bigg\|^2\bigg] \leq K_\beta E\bigg[\int_0^T \int_E \|f(s,x)\|^2 ds \beta(dx)\bigg]$$

for all $f \in M_\nu^{T,2}(E/F)$ and all $f \in M_\nu^{T,2}(E/G)$. We start with a version of Itô's formula, where the mark space is finite. Based on this result, we shall prove Theorem 3.6 later on.

Proposition 3.3. *We suppose that:*

- $H \in C^{1,2}(\mathbb{R}_+ \times F; G)$ *is a function.*
- $C \in \mathcal{E}$ *is a set with* $\beta(C) < \infty$.
- $f : \Omega \times \mathbb{R}_+ \times E \to F$ *is a progressively measurable process.*

- *$g : \Omega \times \mathbb{R}_+ \times E \to F$ is a progressively measurable process such that for all $t \in \mathbb{R}_+$ we have $\mathbb{P}$-almost surely*

$$\int_0^t \int_C \|g(s,x)\| \nu(ds,dx) < \infty. \tag{3.2}$$

- *Y is an Itô process of the form*

$$Y_t = Y_0 + \int_0^t \int_C f(s,x) N(ds,dx) + \int_0^t \int_C g(s,x) \nu(ds,dx), \quad t \geq 0.$$

Then, the following statements are true:

1. *For all $t \in \mathbb{R}_+$ we have $\mathbb{P}$-almost surely*

$$\int_0^t \|\partial_s H(s,Y_s)\| ds < \infty, \tag{3.3}$$

$$\int_0^t \int_C \|H(s,Y_{s-} + f(s,x)) - H(s,Y_{s-})\| N(ds,dx) < \infty, \tag{3.4}$$

$$\int_0^t \int_C \|\partial_y H(s,Y_s) g(s,x)\| \nu(ds,dx) < \infty. \tag{3.5}$$

2. *We have $\mathbb{P}$-almost surely*

$$\begin{aligned} H(t,Y_t) = H(0,Y_0) &+ \int_0^t \partial_s H(s,Y_s) ds \\ &+ \int_0^t \int_C \big(H(s,Y_{s-} + f(s,x)) - H(s,Y_{s-}) \big) N(ds,dx) \\ &+ \int_0^t \int_C \partial_y H(s,Y_s) g(s,x) \nu(ds,dx), \quad t \geq 0. \end{aligned}$$

Proof. Estimates (3.3), (3.5) hold true by (3.2) and the continuity of the partial derivatives $\partial_s H$, $\partial_y H$, and (3.4) is valid, because $\beta(C) < \infty$. We define the Itô processes

$$\begin{aligned} Y_t^N &:= Y_0 + \int_0^t \int_C f(s,x) N(ds,dx), \quad t \geq 0 \\ Z_t^\nu &:= \int_0^t \int_C g(s,x) \nu(ds,dx), \quad t \geq 0. \end{aligned}$$

Let $h \in C^{1,2,2}(\mathbb{R}_+ \times F \times F; G)$ be the function $h(t,y,z) := H(t,y+z)$. Furthermore, let $(\Pi_n)_{n \in \mathbb{N}}$ be a sequence of decompositions of $\mathbb{R}_+$ with $|\Pi_n| \to 0$. Let $t \in \mathbb{R}_+$ be arbitrary. Then, we have

$$
\begin{aligned}
H(t,Y_t)-H(0,Y_0) &= h(t,Y_t^N,Z_t^\nu)-h(0,Y_0^N,Z_0^\nu)\\
&= \lim_{n\to\infty}\sum_{t_i\in\Pi_n}\big(h(t_{i+1}\wedge t,Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})-h(t_i,Y^N_{t_i},Z^\nu_{t_i})\big)\\
&= \lim_{n\to\infty}\sum_{t_i\in\Pi_n}\big(h(t_{i+1}\wedge t,Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})-h(t_i,Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})\big)\\
&\quad+\lim_{n\to\infty}\sum_{t_i\in\Pi_n}\big(h(t_i,Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})-h(t_i,Y^N_{t_i},Z^\nu_{t_{i+1}\wedge t})\big)\\
&\quad+\lim_{n\to\infty}\sum_{t_i\in\Pi_n}\big(h(t_i,Y^N_{t_i},Z^\nu_{t_{i+1}\wedge t})-h(t_i,Y^N_{t_i},Z^\nu_{t_i})\big).
\end{aligned}
$$

By Taylor's theorem, the first term equals

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{t_i\in\Pi_n}\big(h(t_{i+1}\wedge t,Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})-h(t_i,Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})\big)\\
&\quad=\lim_{n\to\infty}\sum_{t_i\in\Pi_n}\int_0^1\partial_s h(t_i+\theta((t_{i+1}\wedge t)-t_i),Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})((t_{i+1}\wedge t)-t_i)d\theta\\
&\quad=\lim_{n\to\infty}\int_0^t\Bigg(\sum_{t_i\in\Pi_n}\int_0^1\partial_s h(t_i+\theta((t_{i+1}\wedge t)-t_i),Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})d\theta\\
&\qquad\qquad\mathbb{1}_{(t_i,t_{i+1}\wedge t]}(s)\Bigg)ds.
\end{aligned}
$$

Using Lebesgue's dominated convergence theorem, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{t_i\in\Pi_n}\big(h(t_{i+1}\wedge t,Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})-h(t_i,Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})\big)\\
&\quad=\int_0^t\partial_s h(s,Y^N_s,Z^\nu_s)ds=\int_0^t\partial_s H(s,Y_s)ds.
\end{aligned}
$$

By Remark 3.1 we have

$$
Y_t^N=Y_0+\sum_{k\in\mathbb{N}}f(\tau_k,\xi_{\tau_k})\mathbb{1}_{\{\tau_k\le t\}},\quad t\ge 0.
$$

Therefore, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{t_i\in\Pi_n}\big(h(t_i,Y^N_{t_{i+1}\wedge t},Z^\nu_{t_{i+1}\wedge t})-h(t_i,Y^N_{t_i},Z^\nu_{t_{i+1}\wedge t})\big)\\
&\quad=\lim_{n\to\infty}\sum_{t_i\in\Pi_n}\sum_{k\in\mathbb{N}}\big(h(t_i,Y^N_{\tau_k},Z^\nu_{t_{i+1}\wedge t})-h(t_i,Y^N_{\tau_k-},Z^\nu_{t_{i+1}\wedge t})\big)\mathbb{1}_{(t_i,t_{i+1}\wedge t]}(\tau_k)\\
&\quad=\lim_{n\to\infty}\sum_{t_i\in\Pi_n}\sum_{k\in\mathbb{N}}\big(h(t_i,Y^N_{\tau_k-}+f(\tau_k,\xi_{\tau_k}),Z^\nu_{t_{i+1}\wedge t})-h(t_i,Y^N_{\tau_k-},Z^\nu_{t_{i+1}\wedge t})\big)\\
&\qquad\qquad\mathbb{1}_{(t_i,t_{i+1}\wedge t]}(\tau_k)
\end{aligned}
$$

$$= \lim_{n\to\infty} \sum_{t_i\in\Pi_n} \int_{t_i}^{t_{i+1}\wedge t} \int_C \big(h(t_i, Y^N_{s-} + f(s,x), Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{s-}, Z^\nu_{t_{i+1}\wedge t})\big) N(ds,dx).$$

Consequently, by Lebesgue's dominated convergence theorem, the second term equals

$$\lim_{n\to\infty} \sum_{t_i\in\Pi_n} \big(h(t_i, Y^N_{t_{i+1}\wedge t}, Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{t_i}, Z^\nu_{t_{i+1}\wedge t})\big)$$
$$= \lim_{n\to\infty} \int_0^t \int_C \Bigg(\sum_{t_i\in\Pi_n} \big(h(t_i, Y^N_{s-} + f(s,x), Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{s-}, Z^\nu_{t_{i+1}\wedge t})\big) \mathbb{1}_{(t_i, t_{i+1}\wedge t]}(s)\Bigg) N(ds,dx)$$
$$= \int_0^t \int_C \big(h(s, Y^N_{s-} + f(s,x), Z^\nu_s) - h(s, Y^N_{s-}, Z^\nu_s)\big) N(ds,dx)$$
$$= \int_0^t \int_C \big(H(s, Y_{s-} + f(s,x)) - H(s, Y_{s-})\big) N(ds,dx).$$

By Taylor's theorem, for the third term we obtain

$$\lim_{n\to\infty} \sum_{t_i\in\Pi_n} \big(h(t_i, Y^N_{t_i}, Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{t_i}, Z^\nu_{t_i})\big)$$
$$= \lim_{n\to\infty} \sum_{t_i\in\Pi_n} \int_0^1 \partial_z h(t_i, Y^N_{t_i}, Z^\nu_{t_i} + \theta(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i}))(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i}) d\theta$$
$$= \lim_{n\to\infty} \sum_{t_i\in\Pi_n} \int_0^1 \partial_z h(t_i, Y^N_{t_i}, Z^\nu_{t_i} + \theta(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i})) \int_{t_i}^{t_{i+1}\wedge t} \int_C g(s,x)\nu(ds,dx) d\theta$$
$$= \lim_{n\to\infty} \sum_{t_i\in\Pi_n} \int_{t_i}^{t_{i+1}\wedge t} \int_C \int_0^1 \partial_z h(t_i, Y^N_{t_i}, Z^\nu_{t_i} + \theta(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i})) g(s,x) d\theta \nu(ds,dx).$$

Therefore, by Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty} \sum_{t_i\in\Pi_n} \big(h(t_i, Y^N_{t_i}, Z^\nu_{t_{i+1}\wedge t}) - h(t_i, Y^N_{t_i}, Z^\nu_{t_i})\big)$$
$$= \lim_{n\to\infty} \int_0^t \int_C \int_0^1 \Bigg(\sum_{t_i\in\Pi_n} \partial_z h(t_i, Y^N_{t_i}, Z^\nu_{t_i} + \theta(Z^\nu_{t_{i+1}\wedge t} - Z^\nu_{t_i})) g(s,x) d\theta \, \mathbb{1}_{(t_i, t_{i+1}\wedge t]}(s)\Bigg) \nu(ds,dx)$$
$$= \int_0^t \int_C \partial_z h(s, Y^N_s, Z^\nu_s) g(s,x)\nu(ds,dx) = \int_0^t \int_C \partial_y H(s, Y_s) g(s,x)\nu(ds,dx).$$

This completes the proof. □

Remark 3.4. In the proof of Proposition 3.3 we used the representation (3.1), whereas in [14] we have used the natural representation (2.1) of simple functions. Due to Remark 3.2, these methods are more or less equivalent, but the proof here is shorter and allows a more general mark space $(E, \mathcal{E})$.

Definition 3.5. We call a continuous, non-decreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ *quasi-sublinear* if there is a constant $C > 0$ such that

$$h(x+y) \le C\big(h(x) + h(y)\big), \quad x, y \in \mathbb{R}_+,$$
$$h(xy) \le Ch(x)h(y), \quad x, y \in \mathbb{R}_+.$$

Theorem 3.6. *We suppose that:*

- $H \in C^{1,2}(\mathbb{R}_+ \times F; G)$ *is a function such that*

$$\|\partial_y H(s,y)\| \le h_1(\|y\|), \quad (s,y) \in \mathbb{R}_+ \times F \tag{3.6}$$
$$\|\partial_{yy} H(s,y)\| \le h_2(\|y\|), \quad (s,y) \in \mathbb{R}_+ \times F \tag{3.7}$$

  for quasi-sublinear functions $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$.
- $B \in \mathcal{E}$ *is a set with* $\beta(B^c) < \infty$.
- $f : \Omega \times \mathbb{R}_+ \times E \to F$ *is a progressively measurable process such that for all* $t \in \mathbb{R}_+$ *we have* $\mathbb{P}$*-almost surely*

$$\begin{aligned}&\int_0^t \int_B \|f(s,x)\|^2 \nu(ds,dx) + \int_0^t \int_B h_1(\|f(s,x)\|)^2 \|f(s,x)\|^2 \nu(ds,dx) \\ &\quad + \int_0^t \int_B h_2(\|f(s,x)\|) \|f(s,x)\|^2 \nu(ds,dx) < \infty.\end{aligned} \tag{3.8}$$

- $g : \Omega \times \mathbb{R}_+ \times E \to F$ *is a progressively measurable process.*
- Y *is an Itô process of the form*

$$Y_t = Y_0 + \int_0^t \int_B f(s,x) q(ds,dx) + \int_0^t \int_{B^c} g(s,x) N(ds,dx), \quad t \ge 0.$$

  Then, the following statements are true:

1. *For all* $t \in \mathbb{R}_+$ *we have* $\mathbb{P}$*-almost surely*

$$\int_0^t \|\partial_s H(s,Y_s)\| ds < \infty, \tag{3.9}$$
$$\int_0^t \int_B \|H(s, Y_s + f(s,x)) - H(s,Y_s)\|^2 \nu(ds,dx) < \infty, \tag{3.10}$$
$$\int_0^t \int_B \|H(s, Y_s + f(s,x)) - H(s,Y_s) - \partial_y H(s,Y_s) f(s,x)\| \nu(ds,dx) < \infty, \tag{3.11}$$
$$\int_0^t \int_{B^c} \|H(s, Y_{s-} + g(s,x)) - H(s, Y_{s\_})\| N(ds,dx) < \infty. \tag{3.12}$$

2. *We have $\mathbb{P}$-almost surely*

$$
\begin{aligned}
H(t, Y_t) = H(0, Y_0) &+ \int_0^t \partial_s H(s, Y_s) ds \qquad (3.13) \\
&+ \int_0^t \int_B \big(H(s, Y_{s-} + f(s,x)) - H(s, Y_{s-}) \big) q(ds, dx) \\
&+ \int_0^t \int_B \big(H(s, Y_s + f(s,x)) - H(s, Y_s) - \partial_y H(s, Y_s) f(s,x) \big) \nu(ds, dx) \\
&+ \int_0^t \int_{B^c} \big(H(s, Y_{s-} + g(s,x)) - H(s, Y_{s-}) \big) N(ds, dx), \quad t \geq 0.
\end{aligned}
$$

Proof. Estimate (3.9) holds true by the continuity of the partial derivative $\partial_s H$, and (3.12) is valid, because $\beta(B^c) < \infty$. By Taylor's theorem, the Cauchy–Schwarz inequality and (3.6), we obtain $\mathbb{P}$-almost surely

$$
\begin{aligned}
&\int_0^t \int_B \| H(s, Y_s + f(s,x)) - H(s, Y_s) \|^2 \nu(ds, dx) \\
&\quad = \int_0^t \int_B \left\| \int_0^1 \partial_y H(s, Y_s + \theta f(s,x)) f(s,x) d\theta \right\|^2 \nu(ds, dx) \\
&\quad \leq \int_0^t \int_B \int_0^1 \| \partial_y H(s, Y_s + \theta f(s,x)) \|^2 \| f(s,x) \|^2 d\theta \nu(ds, dx) \\
&\quad \leq \int_0^t \int_B \int_0^1 h_1(\| Y_s + \theta f(s,x)) \|)^2 \| f(s,x) \|^2 d\theta \nu(ds, dx).
\end{aligned}
$$

Since h_1 is quasi-sublinear, for some constant $C > 0$ we get $\mathbb{P}$-almost surely

$$
\begin{aligned}
&\int_0^t \int_B \| H(s, Y_s + f(s,x)) - H(s, Y_s) \|^2 \nu(ds, dx) \\
&\quad \leq C^2 \int_0^t \int_B \int_0^1 \big(h_1(\|Y_s\|) + C h_1(\theta) h_1(\|f(s,x)\|) \big)^2 \| f(s,x) \|^2 d\theta \nu(ds, dx) \\
&\quad \leq 2C^2 \int_0^t \int_B h_1(\|Y_s\|)^2 \| f(s,x) \|^2 \nu(ds, dx) \\
&\qquad + 2C^4 h_1(1) \int_0^t \int_B h_1(\|f(s,x)\|)^2 \| f(s,x) \|^2 \nu(ds, dx) < \infty,
\end{aligned}
$$

showing (3.10). By Taylor's theorem and (3.7), we obtain $\mathbb{P}$-almost surely

$$
\begin{aligned}
&\int_0^t \int_B \| H(s, Y_s + f(s,x)) - H(s, Y_s) - \partial_y H(s, Y_s) f(s,x) \| \nu(ds, dx) \\
&\quad = \int_0^t \int_B \left\| \int_0^1 \partial_{yy} H(s, Y_s + \theta f(s,x)) (f(s,x), f(s,x)) d\theta \right\| \nu(ds, dx) \\
&\quad \leq \int_0^t \int_B \int_0^1 \| \partial_{yy} H(s, Y_s + \theta f(s,x)) \| \, \| f(s,x) \|^2 d\theta \nu(ds, dx)
\end{aligned}
$$

$$\leq \int_0^t \int_B \int_0^1 h_2(\|Y_s + \theta f(s,x)\|)\|f(s,x)\|^2 d\theta \nu(ds,dx).$$

Since h_2 is quasi-sublinear, for some constant $C > 0$ we get $\mathbb{P}$-almost surely

$$\begin{aligned}
&\int_0^t \int_B \|H(s, Y_s + f(s,x)) - H(s,Y_s) - \partial_y H(s,Y_s) f(s,x)\| \nu(ds,dx) \\
&\quad \leq C \int_0^t \int_B \int_0^1 \big(h_2(\|Y_s\|) + C h_2(\theta) h_2(\|f(s,x)\|)\big) \|f(s,x)\|^2 d\theta \nu(ds,dx) \\
&\quad \leq C \int_0^t \int_B h_2(\|Y_s\|)\|f(s,x)\|^2 \nu(ds,dx) \\
&\qquad + C^2 h_2(1) \int_0^t \int_B h_2(\|f(s,x)\|)\|f(s,x)\|^2 \nu(ds,dx) < \infty,
\end{aligned}$$

providing (3.11). Since the measure β is σ-finite, there exists a sequence $(C_n)_{n\in\mathbb{N}} \subset \mathcal{E}$ such that $C^n \uparrow E$ and $\beta(C_n) < \infty$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let Y^n be the Itô process

$$Y_t^n := Y_0 + \int_0^t \int_{B\cap C_n} f(s,x) q(ds,dx) + \int_0^t \int_{B^c \cap C_n} g(s,x) N(ds,dx), \quad t \geq 0.$$

Then, we can express Y^n as

$$\begin{aligned}
Y_t^n = Y_0 &+ \int_0^t \int_{C_n} \big(f(s,x)\mathbb{1}_B(x) + g(s,x)\mathbb{1}_{B^c}(x)\big) N(ds,dx) \\
&- \int_0^t \int_{B\cap C_n} f(s,x)\nu(ds,dx), \quad t \geq 0.
\end{aligned}$$

Note that, by the Cauchy–Schwarz inequality and (3.8), for each $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
&\int_0^t \int_{B\cap C_n} \|f(s,x)\|\nu(ds,dx) \\
&\quad \leq \big(t\beta(B\cap C_n)\big)^{1/2} \bigg(\int_0^t \int_{B\cap C_n} \|f(s,x)\|^2 \nu(ds,dx)\bigg)^{1/2} < \infty,
\end{aligned}$$

showing that condition (3.2) with $g = -f\mathbb{1}_B$ and $C = C_n$ is satisfied. Using Proposition 3.3, we obtain $\mathbb{P}$-almost surely

$$\begin{aligned}
H(Y_t^n) &= H(Y_0) + \int_0^t \partial_s H(s, Y_s^n) ds \\
&+ \int_0^t \int_{C_n} \big(H(s, Y_{s-}^n + f(s,x)\mathbb{1}_B(x) + g(s,x)\mathbb{1}_{B^c}(x)) - H(s, Y_{s-}^n)\big) N(ds,dx) \\
&- \int_0^t \int_{B\cap C_n} \partial_y H(s, Y_s^n) f(s,x) \nu(ds,dx), \quad t \geq 0.
\end{aligned}$$

We can rewrite this formula as

$$\begin{aligned}H(Y_t^n) = H(Y_0) &+ \int_0^t \partial_s H(s, Y_s^n) ds \\ &+ \int_0^t \int_{B \cap C_n} \big(H(s, Y_{s-}^n + f(s,x)) - H(s, Y_{s-}^n)\big) N(ds, dx) \\ &+ \int_0^t \int_{B^c \cap C_n} \big(H(s, Y_{s-}^n + g(s,x)) - H(s, Y_{s-}^n)\big) N(ds, dx) \\ &- \int_0^t \int_{B \cap C_n} \partial_y H(s, Y_s^n) f(s,x) \nu(ds, dx), \quad t \geq 0,\end{aligned}$$

and therefore, we obtain

$$\begin{aligned}H(Y_t^n) &= H(Y_0) + \int_0^t \partial_s H(s, Y_s^n) ds \\ &+ \int_0^t \int_{B \cap C_n} \big(H(s, Y_{s-}^n + f(s,x)) - H(s, Y_{s-}^n)\big) q(ds, dx) \\ &+ \int_0^t \int_{B \cap C_n} \big(H(s, Y_s^n + f(s,x)) - H(s, Y_s^n) - \partial_y H(s, Y_s^n) f(s,x)\big) \nu(ds, dx) \\ &+ \int_0^t \int_{B^c \cap C_n} \big(H(s, Y_{s-}^n + g(s,x)) - H(s, Y_{s-}^n)\big) N(ds, dx), \quad t \geq 0.\end{aligned}$$

Letting $n \to \infty$, by virtue of Theorem 2.7 we arrive at (3.13). □

Example. Suppose that $H \in C_b^{1,2}(\mathbb{R}_+ \times F; G)$ and

$$\int_0^t \int_B \|f(s,x)\|^2 \nu(ds, dx) < \infty \quad \text{for all } t \in \mathbb{R}_+.$$

Then Theorem 3.6 applies and yields the Itô formula (3.13), cf. [14].

Example. If $H \in L(F, G)$ is a continuous linear operator and

$$\int_0^t \int_B \|f(s,x)\|^2 \nu(ds, dx) < \infty \quad \text{for all } t \in \mathbb{R}_+,$$

then Theorem 3.6 applies and yields that $\mathbb{P}$-almost surely

$$\begin{aligned}H(Y_t) = H(Y_0) &+ \int_0^t \int_B H(f(s,x)) q(ds, dx) \\ &+ \int_0^t \int_{B^c} H(g(s,x)) N(ds, dx), \quad t \geq 0.\end{aligned}$$

Example. Suppose that F is a separable Hilbert space. Then $H(x) = \|x\|^2$ is of class $C^2(F; \mathbb{R})$ with

$$H_x(x)v = 2\langle x, v \rangle \quad \text{and} \quad H_{xx}(x)(v, w) = 2\langle v, w \rangle.$$

Therefore, we have

$$\|H_x(x)\| \leq 2\|x\| \quad \text{and} \quad \|H_{xx}(x)\| \leq 2.$$

Consequently, if

$$\int_0^t \int_B \|f(s,x)\|^2 \nu(ds,dx) + \int_0^t \int_B \|f(s,x)\|^4 \nu(ds,dx) < \infty \quad \text{for all } t \in \mathbb{R}_+,$$

then Theorem 3.6 applies and yields that $\mathbb{P}$-almost surely

$$\begin{aligned} \|Y_t\|^2 = \|Y_0\|^2 &+ \int_0^t \int_B \big(\|Y_{s-} + f(s,x)\|^2 - \|Y_{s-}\|^2\big) q(ds,dx) \\ &+ \int_0^t \int_B \big(\|Y_s + f(s,x)\|^2 - \|Y_s\|^2 - 2\langle Y_s, f(s,x)\rangle\big)\nu(ds,dx) \\ &+ \int_0^t \int_{B^c} \big(\|Y_{s-} + g(s,x)\|^2 - \|Y_{s-}\|^2\big) N(ds,dx), \quad t \geq 0. \end{aligned}$$

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Well-posedness for a Class of Dissipative Stochastic Evolution Equations with Wiener and Poisson Noise

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Abstract. We prove existence and uniqueness of mild and generalized solutions to a class of stochastic semilinear evolution equations driven by additive Wiener and Poisson noise. The non-linear drift term is supposed to be the evaluation operator associated to a continuous monotone function satisfying a polynomial growth condition. The results are extensions to the jump-diffusion case of the corresponding ones proved in [3] for equations driven by purely discontinuous noise.

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1. Introduction

Let D be a bounded domain of $\mathbb{R}^d$. The purpose of this note is to show that stochastic evolution equations of the type

$$du(t)+Au(t)\,dt+f(u(t))\,dt = B(t)\,dW(t)+\int_Z G(z,t)\,\bar{\mu}(dz,dt), \quad u(0)=u_0, \tag{1.1}$$

where A is a linear m-accretive operator on $L_2(D)$, $f:\mathbb{R}\to\mathbb{R}$ is a continuous increasing function of polynomial growth, W is a cylindrical Wiener noise on $L_2(D)$, and $\bar{\mu}$ is a compensated Poisson random measure, admit a unique mild solution. Precise assumptions on the notion of solution and on the data of the problem are given in the next section.

Global well-posedness of (1.1) in the case of purely discontinuous noise (i.e., with $B\equiv 0$) has been proved in [3] showing that solutions to regularized equations converge to a process which solves the original equation. This was achieved proving a priori estimates for the approximating processes by rewriting the regularized stochastic equations as deterministic evolution equations with random coefficients and using monotonicity arguments. These a priori estimates essentially rely, in turn, on a maximal inequality of Bichteler–Jacod type for stochastic convolutions on L_p spaces with respect to compensated Poisson random measures, that is also proved in [3].

The well-posedness results of [3] will be here extended to the more general class of equations like (1.1). We shall adapt the method used in [3], but instead of rewriting the regularized (stochastic) equations as deterministic equations with random coefficients, we shall rewrite them as stochastic equations driven just by Wiener noise (we might say that, in a sense, we "hide the jumps"), the solutions of which will be shown to satisfy suitable a priori estimates allowing to pass to the limit in the regularized equations.

The result might be interesting even in the case of equations driven only by a Wiener process (i.e., with $G \equiv 0$). In fact, the usual approach to establish well-posedness for such equations is to rewrite them as deterministic equations with random coefficients and to consider them on a Banach space of continuous functions. This approach requires the stochastic convolution to have paths in such a space of continuous functions. The latter condition (indeed a quite strong one) is not needed in our setting. Let us also remark that we do not assume that f is locally Lipschitz, hence all methods based on establishing first existence (and uniqueness) of a local solution and then continuation to a global solution cannot be applied. For recent results and a detailed bibliographic overview on dissipative parabolic stochastic PDEs driven by Wiener noise we refer to [1]. Some references to the much less extensive literature on equations with jumps are collected in [3].

Let us conclude this introductory section with some words about notation used throughout the paper: $a \lesssim b$ stands for $a \leq Nb$ for some constant N (if the constant N depends on parameters $p_1, \ldots, p_n$ we shall also write $N(p_1, \ldots, p_n)$ and $a \lesssim_{p_1,\ldots,p_n}$, respectively). For any $q \geq 0$, we set $q^* := q^2/2$. The duality mapping of a Banach space X with (algebraic and topological) dual X^* and duality form $\langle \cdot, \cdot \rangle$ is the (multi-valued) map $J : X \to 2^{X^*}$, $J : x \mapsto \{x^* \in X^* : \langle x^*, x\rangle = \|x\|_X^2\}$.

2. Main result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, with $T > 0$ fixed, be a filtered probability space satisfying the "usual conditions", and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. All stochastic elements will be defined on this stochastic basis, and any equality or inequality between random quantities will be meant to hold $\mathbb{P}$-almost surely. Let $(Z, \mathcal{Z}, m)$ be a measure space, $\bar{\mu}$ a Poisson measure on $[0, T] \times Z$ with compensator Leb $\otimes\, m$, where Leb stands for Lebesgue measure. Let D be an open bounded subset of $\mathbb{R}^d$ with smooth boundary. All Lebesgue spaces on D will be denoted without explicit mention of the domain, e.g., $L_p := L_p(D)$. Given $q \geq 1$ and a Banach space X, we shall denote the set of all X-valued random variables ξ such that $\mathbb{E}\|\xi\|_X^q < \infty$ by $\mathbb{L}_q(X)$. We call $\mathbb{H}_q(X)$ the set of all adapted X-valued processes such that

$$\|u\|_{\mathbb{H}_q(X)} := \big(\mathbb{E}\sup_{t\leq T}\|u(t)\|_X^q\big)^{1/q} < \infty.$$

For compactness of notation, we shall also write $\mathbb{L}_q$ in place of $\mathbb{L}_q(L_q)$. and $\mathbb{H}_q$ in place of $\mathbb{H}_q(L_q)$. We shall denote by W a cylindrical Wiener process on $L_2(D)$.

Let $p \geq 2$, and $f : \mathbb{R} \to \mathbb{R}$ be a continuous monotonically increasing function with $f(0) = 0$, such that $|f(x)| \lesssim 1 + |x|^{p/2}$ for all $x \in \mathbb{R}$. Moreover, let A be a linear (unbounded) m-accretive operator in the spaces L_2, L_p and L_{p^*}, and denote by S the strongly continuous semigroup generated by $-A$. Assuming that the realizations of A and S on the above spaces are consistent, we shall not distinguish them notationally. For notational convenience, we shall set throughout the paper $H := L_2$ and $E := L_p$.

Denoting by $\mathcal{L}(H \to E)$ and by $\gamma(H \to E)$ the space of bounded linear operators and γ-Radonifying operators from H to E, respectively, for any $q \geq 1$, the class of adapted processes $B : [0,T] \to \mathcal{L}(H \to E)$ such that

$$\|B\|^q_{\mathcal{L}^\gamma_q} := \mathbb{E}\int_0^T \|B(t)\|^q_{\gamma(H\to E)}\,dt < \infty$$

will be denoted by $\mathcal{L}^\gamma_q(H \to E)$. Similarly, denoting the predictable σ-algebra by $\mathcal{P}$ and the Borel σ-algebra of $\mathbb{R}^d$ by $\mathcal{B}(\mathbb{R}^d)$, the space of $\mathcal{P}\otimes\mathcal{Z}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable processes $g : [0,T] \times Z \times D \to \mathbb{R}$ such that

$$\|g\|^q_{\mathcal{L}^m_q} := \mathbb{E}\int_0^T\int_Z \|g(t,z)\|^q_{L_q}\,m(dz)\,dt + \mathbb{E}\int_0^T\Big(\int_Z \|g(t,z)\|^2_{L_q}\,m(dz)\Big)^{q/2}\,dt < \infty$$

will be denoted by $\mathcal{L}^m_q$. It was proved in [3] that, for any strongly continuous semigroup of positive contractions R on L_q, $q \in [2,\infty[$, one has the maximal inequality

$$\mathbb{E}\sup_{t\leq T}\Big\|\int_0^t\int_Z R(t-s)g(s,z)\,\bar{\mu}(ds,dz)\Big\|^q_{L_q} \lesssim \|g\|^q_{\mathcal{L}^m_q}. \tag{2.1}$$

Let us now define mild and generalized solutions of (1.1).

Definition 2.1. Let u_0 be an H-valued $\mathcal{F}_0$-measurable random variable. A (strongly) measurable adapted H-valued process u is a mild solution to (1.1) if, for all $t \in [0,T]$,

$$\begin{aligned} u(t) &+ \int_0^t S(t-s)f(u(s))\,ds \\ &= S(t)u_0 + \int_0^t S(t-s)B(s)\,dW(s) + \int_0^t\int_Z S(t-s)G(s,z)\,\bar{\mu}(dz,ds) \end{aligned}$$

and all integrals are well defined.

As is well known, the stochastic convolution with respect to W is well defined (as an H-valued random variable) if the operator Q_t is nuclear for all $t \in [0,T]$, where

$$Q_t := \int_0^t S(t-s)B(s)B^*(s)S^*(t-s)\,ds.$$

This condition is verified, for instance, if $B \in \mathcal{L}_2^\gamma$, i.e., if

$$\mathbb{E} \int_0^T \|B(s)\|^2_{\gamma(H \to H)} \, ds < \infty$$

(recall that $\gamma(H \to H)$ is just the space of Hilbert–Schmidt operators from H to itself). Similarly, the stochastic convolution with respect to $\bar{\mu}$ is well defined if $G \in \mathcal{L}_2^m$, i.e., if

$$\mathbb{E} \int_0^T \int_Z \|G(s,z)\|^2_{L_2} \, m(dz) \, ds < \infty.$$

The deterministic convolution term is well defined if $f(u) \in L_1([0,T] \to H)$, or if $u \in L_{p/2}([0,T] \to L_p)$.

Definition 2.2. A process $u \in \mathbb{H}_2$ is a generalized solution to (1.1) if there exist sequences $\{u_{0n}\}_n \subset \mathbb{L}_p$, $\{B_n\}_n \subset \mathcal{L}_p^\gamma$, $\{G_n\}_n \subset \mathcal{L}_{p^*}^m$, and $\{u_n\}_n \subset \mathbb{H}_2(T)$ such that $u_{0n} \to u_0$ in $\mathbb{L}_2$, $B_n \to B$ in $\mathcal{L}_2^\gamma$, $G_n \to G$ in $\mathcal{L}_2^m$ and $u_n \to u$ in $\mathbb{H}_2(T)$ as $n \to \infty$, where u_n is the (unique) mild solution of

$$du_n(t) + Au_n(t)\,dt + f(u_n(t))\,dt = B_n(s)\,dW(t) + \int_Z G_n(z)\,\bar{\mu}(dt,dz), \quad u_n(0) = u_{0n}.$$

Here are the results, which will be proved in the next sections.

Theorem 2.3. *Assume that $u_0 \in \mathbb{L}_p$, $B \in \mathcal{L}_p^\gamma$ and $G \in \mathcal{L}_{p^*}^m$. Then there exists a unique càdlàg mild solution $u \in \mathbb{H}_2$ to equation* (1.1) *such that $f(u) \in L^1[0,T] \to H)$.*

Theorem 2.4. *Assume that $u_0 \in \mathbb{L}_2$, $B \in \mathcal{L}_2^\gamma$, $G \in \mathcal{L}_2^m$. Then there exists a unique generalized solution to equation* (1.1).

Remark 2.5. By inspection of the corresponding proof in [3], it is clear that the same argument used there applies to Theorem 2.3 if one assumes $B \in \mathcal{L}_{p^*}^\gamma$, i.e.,

$$\mathbb{E} \sup_{t \le T} \|W_A(t)\|^{p^*}_{L_{p^*}} < \infty.$$

In the proof of Theorem 2.3 below we show that in fact $B \in \mathcal{L}_{p^*}^\gamma$ is too strong an assumption, and that $B \in \mathcal{L}_p^\gamma$ is enough. It is natural to conjecture that also $G \in \mathcal{L}_{p^*}^m$ is too strong, and that it should suffice to assume $G \in \mathcal{L}_p^m$. Unfortunately, thus far we have not been able to replace the exponent p^* by p in the hypotheses on G of Theorem 2.3.

Example. The above results apply to a large class of dissipative parabolic semilinear evolution equations perturbed by noise. For instance, if $-A$ is the generator of a sub-Markovian strongly continuous semigroup of contractions S on L_2, then it is well known that S induces a strongly continuous sub-Markovian contraction semigroup (consistent, hence denoted by the same symbol) on all L_p, $p \in [2,\infty[$. The operator $-A$ can then be, for example, the Dirichlet Laplacian, or a fractional power thereof. The assumptions on the diffusion coefficient B of Theorem 2.3 are automatically satisfied if, e.g., B is a deterministic time-independent operator belonging to $\mathcal{L}(L_2 \to L_\infty)$. It is well known that the latter condition implies $B \in \gamma(L_2 \to L_p)$ for all $p \ge 2$.

3. Proofs

3.1. Proof of Theorem 2.3

Let $f_\lambda := \lambda^{-1}(I-(I+\lambda f)^{-1})$, $\lambda > 0$, be the Yosida approximation of f, and recall that f_λ is Lipschitz continuous with Lipschitz constant bounded by $1/\lambda$. Let us consider the regularized equation

$$du_\lambda(t)+Au_\lambda(t)\,dt+f_\lambda(u_\lambda(t))\,dt = B(t)\,dW(t)+\int_Z G(z,t)\,\bar{\mu}(dz,dt), \quad u_\lambda(0)=u_0. \tag{3.1}$$

Assuming that $B \in \mathcal{L}_p^\gamma$ and $G \in \mathcal{L}_p^m$, one could prove by a fixed point argument that (3.1) admits a unique càdlàg mild E-solution (by which we mean, here and in the following, a mild solution with values in E).[1] However, we prefer to proceed in a less direct way, for reasons that will become apparent later. In particular, we "hide the jumps" in (3.1) writing an equation for the difference between u_λ and the stochastic convolution with respect to the Poisson random measure as follows: setting, for notational compactness,

$$W_A(t) := \int_0^t S(t-s)B(s)\,dW(s), \quad G_A(t) := \int_0^t\int_Z S(t-s)G(s,z)\,\bar{\mu}(ds,dz),$$

the integral form of (3.1) reads

$$u_\lambda(t) + \int_0^t S(t-s)f_\lambda(u_\lambda(s))\,ds = S(t)u_0 + W_A(t) + G_A(t), \tag{3.2}$$

which can be equivalently written as

$$u_\lambda(t) - G_A(t) + \int_0^t S(t-s)f_\lambda(u_\lambda(s) - G_A(s) + G_A(s))\,ds = S(t)u_0 + W_A(t),$$

hence also, setting $v_\lambda := u_\lambda - G_A$ and $\tilde{f}_\lambda(t,y) := f_\lambda(y + G_A(t))$, for $y \in \mathbb{R}$ and $t \geq 0$, as

$$v_\lambda(t) + \int_0^t S(t-s)\tilde{f}_\lambda(v_\lambda(s))\,ds = S(t)u_0 + W_A(t),$$

which is the mild form of

$$dv_\lambda(t) + Av_\lambda(t)\,dt + \tilde{f}_\lambda(t, v_\lambda(t))\,dt = B(t)\,dW(t), \qquad v_\lambda(0) = u_0. \tag{3.3}$$

It is clear that v_λ is a mild E-solution of (3.3) if and only if $v_\lambda + G_A$ is a mild E-solution of (3.1).

In the next Proposition we show that (3.3) admits a unique mild E-solution v_λ, hence identifying also the unique E-mild solution of (3.1).

Proposition 3.1. *If $u_0 \in \mathbb{L}_p$, $B \in \mathcal{L}_p^\gamma$ and $G \in \mathcal{L}_p^m$, then equation* (3.3) *admits a unique càdlàg mild E-solution $v_\lambda \in \mathbb{H}_p$. Therefore equation* (3.1) *admits a unique càdlàg mild E-solution $u_\lambda \in \mathbb{H}_p$, and $u_\lambda = v_\lambda + G_A$.*

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[1] Since $B \in \mathcal{L}_p^\gamma$ and $E = L_p$, $p \geq 2$, is a UMD Banach space (and S is strongly continuous on E by hypothesis), the stochastic convolution is well defined. One can thus lawfully talk about mild E-solutions. See, e.g., [4] for details.

Proof. We use a fixed point argument on the space $\mathbb{H}_p$. Let us consider the operator

$$\mathfrak{F} : \mathbb{H}_p \ni \phi \mapsto \Big(t \mapsto S(t)u_0 - \int_0^t S(t-s)\tilde{f}_\lambda(s,\phi(s))\,ds + W_A(t) \Big).$$

We shall prove that $\mathfrak{F}$ is a contraction on $\mathbb{H}_p$, if T is small enough. Since $u_0 \in \mathbb{L}_p$ and S is strongly continuous on L_p, it is clear that we can (and will) assume, without loss of generality, that $u_0 = 0$. Then

$$\big\|\mathfrak{F}(\phi)\big\|_{\mathbb{H}_p} \leq \big\|S * \tilde{f}_\lambda(\cdot,\phi)\big\|_{\mathbb{H}_p} + \big\|W_A\big\|_{\mathbb{H}_p}.$$

By a maximal inequality for stochastic convolutions we have

$$\|W_A\|^p_{\mathbb{H}_p} \lesssim \mathbb{E}\int_0^T \|B(t)\|^p_{\gamma(H\to E)}\,dt,$$

where the right-hand side is finite by assumption. Moreover, Jensen's inequality and strong continuity of S on L_p yield

$$\mathbb{E}\sup_{t\leq T}\Big\|\int_0^t S(t-s)\tilde{f}_\lambda(s,\phi(s))\,ds\Big\|^p_E \lesssim_T \mathbb{E}\sup_{t\leq T}\big\|\tilde{f}_\lambda(t,\phi(t))\big\|^p_E.$$

Since the Lipschitz constant of f_λ is not larger than $1/\lambda$, we have

$$|\tilde{f}_\lambda(t,x) - \tilde{f}_\lambda(t,y)| = |f_\lambda(x+G_A(t)) - f_\lambda(y+G_A(t))| \leq \frac{1}{\lambda}|x-y|,$$

hence

$$\begin{aligned}|\tilde{f}_\lambda(t,x)| &\leq |\tilde{f}_\lambda(t,x) - \tilde{f}_\lambda(t,0)| + |\tilde{f}_\lambda(t,0)|\\ &\leq \frac{1}{\lambda}|x| + |f_\lambda(G_A(t))| \leq \frac{1}{\lambda}|x| + \frac{1}{\lambda}|G_A(t)|,\end{aligned}$$

thus also

$$\mathbb{E}\sup_{t\leq T}\big\|\tilde{f}_\lambda(t,\phi(t))\big\|^p_E \lesssim_\lambda \mathbb{E}\sup_{t\leq T}\|\phi(t)\|^p_E + \mathbb{E}\sup_{t\leq T}\|G_A(t)\|^p_E,$$

where the right-hand side is finite because of (2.1) and because $G \in \mathcal{L}^m_p$ by hypothesis. We have thus proved that $\mathfrak{F}(\mathbb{H}_p) \subseteq \mathbb{H}_p$. Since $x \mapsto \tilde{f}_\lambda(t,x,\omega)$ is Lipschitz continuous, uniformly over $t \in [0,T]$ and $\omega \in \Omega$, analogous computations show that $\mathfrak{F}$ is Lipschitz on $\mathbb{H}_p$, with a Lipschitz constant that depends continuously on T. Choosing $T = T_0$, for a small enough T_0 such that $\mathfrak{F}$ is a contraction, and then covering the interval $[0,T]$ by intervals of lenght T_0, one obtains the desired existence and uniqueness of a fixed point of $\mathfrak{F}$ in a standard way. □

Remark 3.2.

(i) Note that we have assumed the more natural condition $G \in \mathcal{L}^m_p$ for the well-posedness of the regularized equation (3.1) rather than $G \in \mathcal{L}^m_{p^*}$. Let us show that the latter condition also ensures that $\|G_A\|_{\mathbb{H}_p}$ is finite: since D has finite Lebesgue measure and $p^* = p^2/2 \geq p$, Hölder's inequality implies

$$\mathbb{E}\sup_{t\leq T}\|G_A(t)\|^p_{L_p} \lesssim_D \mathbb{E}\sup_{t\leq T}\|G_A(t)\|^p_{L_{p^*}} \leq \Big(\mathbb{E}\sup_{t\leq T}\|G_A(t)\|^{p^*}_{L_{p^*}}\Big)^{2/p} < \infty.$$

(ii) The previous existence and uniqueness result also follows by an adaptation of [4, Thm. 6.2], which is a more general and more precise result about well-posedness for equations with Wiener noise and Lipschitz coefficients. In [4] the nonlinearity in the drift is Lipschitz continuous and satisfies a linear growth condition with a constant that does not depend on t and ω, hence it does not apply directly to our situation. A reasoning completely analogous to the above one permits however to circumvent this problem.

We shall need the following a priori estimate for the solution to the regularized equation (3.1).

Lemma 3.3. *Assume that $u_0 \in \mathbb{L}_p$, $B \in \mathcal{L}_p^\gamma$ and $G \in \mathcal{L}_{p^*}^m$. Then there exists a constant N, independent of λ, such that*

$$\mathbb{E} \sup_{t \le T} \|u_\lambda(t)\|_E^p \le N\big(1 + \mathbb{E}\|u_0\|_E^p\big).$$

Proof. Let v_λ be the mild E-solution to (3.3). For $\varepsilon > 0$, set

$$u_0^\varepsilon := (I + \varepsilon A)^{-1} u_0, \qquad B^\varepsilon(t) := (I + \varepsilon A)^{-1} B(t),$$
$$g_\lambda(t) := \tilde{f}_\lambda(t, v_\lambda), \qquad g_\lambda^\varepsilon(t) := (I + \varepsilon A)^{-1} g_\lambda(t),$$

and let w_λ^ε be the mild E-solution to

$$dw_\lambda^\varepsilon + Aw_\lambda\, dt + g_\lambda^\varepsilon\, dt = B^\varepsilon\, dW, \qquad w_\lambda^\varepsilon(0) = u_0^\varepsilon,$$

that is

$$w_\lambda^\varepsilon(t) = S(t)u_0^\varepsilon - \int_0^t S(t-s) g_\lambda^\varepsilon(s)\, ds + \int_0^t S(t-s) B^\varepsilon(s)\, ds$$

for all $t \in [0, T]$. It is easily seen that w_λ^ε is a strong solution, i.e., that one has

$$w_\lambda^\varepsilon(t) + \int_0^t \big(Aw_\lambda^\varepsilon(s) + g_\lambda^\varepsilon(s)\big)\, ds = u_0^\varepsilon + \int_0^t B(s)\, dW(s)$$

for all $t \in [0, T]$, and that $w_\lambda^\varepsilon = (I + \varepsilon A)^{-1} v_\lambda \to v_\lambda$ in $\mathbb{H}_p$ as $\varepsilon \to 0$. We are going to apply Itô's formula (in particular we shall use the version in [5, Thm. 3.1]) to obtain estimates for $\|w_\lambda^\varepsilon\|_E^p$. To this purpose, we have to check that

$$\mathbb{E}\Big(\int_0^T \|b(t)\|_E\, dt\Big)^p < \infty,$$

where $b := Aw_\lambda^\varepsilon + g_\lambda^\varepsilon$. One has

$$\mathbb{E}\Big(\int_0^T \|b(t)\|_E\, dt\Big)^p \lesssim \mathbb{E}\int_0^T \|b(t)\|_E^p\, dt \lesssim \mathbb{E}\int_0^T \|Aw_\lambda^\varepsilon\|_E^p\, dt + \mathbb{E}\int_0^T \|g_\lambda^\varepsilon\|_E^p\, dt,$$

where

$$\|Aw_\lambda^\varepsilon\|_E = \|A(I + \varepsilon A)^{-1} v_\lambda\|_E \lesssim_\varepsilon \|v_\lambda\|_E$$

and

$$\|g_\lambda^\varepsilon\|_E = \|(I + \varepsilon A)^{-1} f_\lambda(v_\lambda + G_A)\|_E \le \|f_\lambda(v_\lambda + G_A)\|_E \lesssim_\lambda \|v_\lambda\|_E + \|G_A\|_E,$$

hence

$$\mathbb{E}\Big(\int_0^T \|b(t)\|_E\,dt\Big)^p \lesssim_{\lambda,\varepsilon,T} \|v_\lambda\|^p_{\mathbb{H}_p} + \|G_A\|^p_{\mathbb{H}_p} < \infty,$$

which justifies applying Itô's formula. Setting $\psi(x) := \|x\|_E^p$, we have

$$\psi(w^\varepsilon_\lambda) + \int_0^t \langle Aw^\varepsilon_\lambda + g^\varepsilon_\lambda, \psi'(w^\varepsilon_\lambda)\rangle\,ds = \int_0^t \psi'(w^\varepsilon_\lambda)B^\varepsilon(s)\,dW(s) + R(t),$$

where R is a "remainder" term, the precise definition of which is given in [5]. Note that $\psi(u) = \big(\|u\|_E^2\big)^{p/2}$ and $\psi'(u) = p|u|^{p-2}u = p\|u\|_E^{p-2}J(u)$, where J is the duality mapping of E,

$$J : u \mapsto u|u|^{p-2}\|u\|_E^{2-p},$$

i.e., J is the Gâteaux (and Fréchet) derivative of $\|\cdot\|_E^2/2$. Since A is m-accretive on E, it holds

$$\langle Aw^\varepsilon_\lambda, \psi'(w^\varepsilon_\lambda)\rangle = p\|w^\varepsilon_\lambda\|_E^{p-2}\langle Aw^\varepsilon_\lambda, J(w^\varepsilon_\lambda)\rangle \geq 0.$$

Moreover, there exists $\delta > 0$ and $N = N(\delta) > 0$ such that (cf. [5])

$$\mathbb{E}\sup_{t\leq T}|R(t)| \leq \delta\mathbb{E}\sup_{t\leq T}\|w^\varepsilon_\lambda(t)\|^p_E + N\mathbb{E}\Big(\int_0^T \|B^\varepsilon(s)\|^2_{\gamma(H\to E)}\,ds\Big)^{p/2}$$

and, by some calculations based on Young's and Burkholder's inequalities,

$$\begin{aligned}&\mathbb{E}\sup_{t\leq T}\Big|\int_0^t \psi'(w^\varepsilon_\lambda(s))B^\varepsilon(s)\,dW(s)\Big|\\ &\qquad\lesssim \delta\mathbb{E}\sup_{t\leq T}\|w^\varepsilon_\lambda(t)\|^p_E + N\mathbb{E}\Big(\int_0^T \|B^\varepsilon(s)\|^2_{\gamma(H\to E)}\,ds\Big)^{p/2}.\end{aligned}$$

We thus arrive at the estimate

$$\mathbb{E}\sup_{t\leq T}\|w^\varepsilon_\lambda\|^p_E \lesssim \delta\mathbb{E}\sup_{t\leq T}\|w^\varepsilon_\lambda\|^p_E + \|B^\varepsilon\|^p_{\mathcal{L}^\gamma_p} + \mathbb{E}\sup_{t\leq T}\int_0^t \langle -g^\varepsilon_\lambda, w^\varepsilon_\lambda|w^\varepsilon_\lambda|^{p-2}\rangle\,ds.$$

Letting $\varepsilon \to 0$, we are left with

$$\mathbb{E}\sup_{t\leq T}\|v_\lambda\|^p_E \lesssim \delta\mathbb{E}\sup_{t\leq T}\|v_\lambda\|^p_E + \|B\|^p_{\mathcal{L}^\gamma_p} + \mathbb{E}\sup_{t\leq T}\int_0^t \big\langle -\tilde{f}_\lambda(s, v_\lambda(s)), \psi'(v_\lambda(s))\big\rangle\,ds.$$

Note that we have

$$\langle \tilde{f}_\lambda(t, v_\lambda), \psi'(v_\lambda)\rangle = p\|v_\lambda\|_E^{p-2}\langle \tilde{f}_\lambda(t, v_\lambda), J(v_\lambda)\rangle = p\|v_\lambda\|_E^{p-2}\langle f_\lambda(G_A + v_\lambda), J(v_\lambda)\rangle,$$

where, by accretivity of f_λ,

$$\begin{aligned}\langle f_\lambda(G_A + v_\lambda), J(v_\lambda)\rangle &= \langle f_\lambda(G_A + v_\lambda) - f(G_A), J(G_A + v_\lambda - G_A)\rangle\\ &\quad + \langle f_\lambda(G_A), J(v_\lambda)\rangle\\ &\geq \langle f_\lambda(G_A), J(v_\lambda)\rangle,\end{aligned}$$

hence, recalling that $\psi'(u) = pu|u|^{p-2}$,

$$\langle \tilde{f}_\lambda(t, v_\lambda), \psi'(v_\lambda)\rangle \geq p\langle f_\lambda(G_A), v_\lambda|v_\lambda|^{p-2}\rangle,$$

and, by Young's inequality with conjugate exponents p and $p' = p/(p-1)$,

$$\begin{aligned}\langle f_\lambda(G_A), v_\lambda |v_\lambda|^{p-2}\rangle &\lesssim N\|f_\lambda(G_A)\|_{L_p}^p + \delta\|v_\lambda|v_\lambda|^{p-2}\|_{L_{p'}}^{p'}\\ &= N\|f_\lambda(G_A)\|_{L_p}^p + \delta\|v_\lambda\|_{L_p}^p,\end{aligned}$$

so that

$$\begin{aligned}\mathbb{E}\sup_{t\le T}\left|\int_0^t \langle \tilde{f}_\lambda(t,v_\lambda), \psi'(v_\lambda)\rangle\,ds\right| &\lesssim \delta\mathbb{E}\sup_{t\le T}\|v_\lambda(t)\|_E^p + N\mathbb{E}\sup_{t\le T}\|f_\lambda(G_A(t))\|_E^p\\ &\lesssim 1 + \delta\mathbb{E}\sup_{t\le T}\|v_\lambda(t)\|_E^p + N\|G\|_{\mathcal{L}^m_{p^*}},\end{aligned}$$

where the last constant does not depend on λ.

Combining the above estimates and choosing δ small enough, we are left with

$$\mathbb{E}\sup_{t\le T}\|v_\lambda\|_E^p \lesssim 1 + \mathbb{E}\|u_0\|_E^p + \|G\|_{\mathcal{L}^m_{p^*}}^{p^*} + \|B\|_{\mathcal{L}^\gamma_p}^p,$$

with implicit constant independent of λ. □

Thanks to the a priori estimate just established, we are now going to show that $\{u_\lambda\}_\lambda$ is a Cauchy sequence in $\mathbb{H}_2$, hence that there exists $u \in \mathbb{H}_2$ such that $u_\lambda \to u$ in $\mathbb{H}_2$ as $\lambda \to 0$. In particular, we have

$$d(u_\lambda - u_\mu) + A(u_\lambda - u_\mu)\,dt + (f_\lambda(u_\lambda) - f_\mu(u_\mu))\,dt = 0,$$

from which we obtain, using the same argument as in [3, pp. 1539–1540],

$$\begin{aligned}\mathbb{E}\sup_{t\le T}\|u_\lambda - u_\mu\|_{L_2}^2 &\lesssim_T (\lambda+\mu)\big(\mathbb{E}\sup_{t\le T}\|f_\lambda(u_\lambda(t))\|_{L_2}^2 + \mathbb{E}\sup_{t\le T}\|f_\mu(u_\mu(t))\|_{L_2}^2\big)\\ &\lesssim (\lambda+\mu)\big(1 + \mathbb{E}\sup_{t\le T}\|u_\lambda(t)\|_{L_p}^p + \mathbb{E}\sup_{t\le T}\|u_\mu(t)\|_{L_p}^p\big).\end{aligned}$$

Since

$$\|u_\lambda\|_{\mathbb{H}_p} \le \|v_\lambda\|_{\mathbb{H}_p} + \|G_A\|_{\mathbb{H}_p}$$

and $\|G_A\|_{\mathbb{H}_p}$ is finite because $G \in \mathcal{L}^m_{p^*}$, we conclude that $\mathbb{E}\sup_{t\le T}\|u_\lambda(t)\|_E^p$ is bounded uniformly over λ, hence that there exists $u \in \mathbb{H}_2$ such that $u_\lambda \to u$ in $\mathbb{H}_2$ as $\lambda \to 0$.

As in [3], one can now pass to the limit as $\lambda \to 0$ in (3.2), concluding that u is indeed a mild solution of (1.1). Since $\mathbb{E}\sup_{t\le T}\|u\|_{L_p}^p < \infty$, one also gets that $f(u) \in L_1([0,T] \to H)$, hence, by the uniqueness results in [2], u is the unique càdlàg mild solution belonging to $\mathbb{H}_2$.

3.2. Proof of Theorem 2.4

We need the following lemma, whose proof is completely analogous to the proof of [3, Lemma 9], hence omitted.

Lemma 3.4. *Assume that u_{01}, $u_{02} \in \mathbb{L}_p$; B_1, $B_2 \in \mathcal{L}^\gamma_p$; G_1, $G_2 \in \mathcal{L}^m_{p^*}$, and denote the unique càdlàg mild solutions of*

$$du + Au\,dt + f(u)\,dt = B_1\,dW + \int_Z G_1\,d\bar{\mu}, \qquad u(0) = u_{01}$$

and

$$du + Au\,dt + f(u)\,dt = B_2\,dW + \int_Z G_2\,d\bar{\mu}, \qquad u(0) = u_{02}$$

by u_1 and u_2, respectively. Then one has

$$\mathbb{E}\sup_{t\le T}\|u_1(t) - u_2(t)\|_H^2 \lesssim_T \mathbb{E}\|u_{01} - u_{02}\|_H^2 + \mathbb{E}\int_0^T \|B_1(t) - B_2(t)\|^2_{\gamma(H\to H)}\,dt + \mathbb{E}\int_0^T\int_Z \|G_1(t,z) - G_2(t,z)\|_H^2\,m(dz)\,dt.$$

Let us consider sequences $\{u_{0n}\}_n \subset \mathbb{L}_p$, $\{B_n\}_n \subset \mathcal{L}_p^\gamma$ and $\{G_n\}_n \subset \mathcal{L}_{p^*}^m$ such that $u_{0n} \to u_0$ in $\mathbb{L}_2$, $B_n \to B$ in $\mathcal{L}_2^\gamma$ and $G_n \to G$ in $\mathcal{L}_2^m$ as $n \to \infty$. Denoting by u_n the unique mild solution in $\mathbb{H}_2$ of

$$du_n + Au_n\,dt + f(u_n)\,dt = B_n\,dW + \int_Z G_n\,d\bar{\mu}, \qquad u_n(0) = u_{0n},$$

the previous lemma yields

$$\mathbb{E}\sup_{t\le T}\|u_n(t) - u_m(t)\|_H^2 \lesssim_T \mathbb{E}\|u_{0n} - u_{0m}\|_H^2 + \mathbb{E}\int_0^T \|B_n(t) - B_m(t)\|^2_{\gamma(H\to H)}\,dt + \mathbb{E}\int_0^T\int_Z \|G_n(t,z) - G_m(t,z)\|_H^2\,m(dz)\,dt,$$

i.e., $\{u_n\}_n$ is a Cauchy sequence in $\mathbb{H}_2$. This implies that $u_n \to u$ in $\mathbb{H}_2$ as $n \to \infty$, and u is a generalized solution of (1.1). Since the limit does not depend on the choice of u_{0n}, B_n and G_n, the generalized solution is unique.

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Localization of Relative Entropy in Bose–Einstein Condensation of Trapped Interacting Bosons

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Abstract. We consider a system of interacting diffusions which is naturally associated to the ground state of the Hamiltonian of a system of N pair-interacting bosons and we give a detailed description of the phenomenon of the "localization of the relative entropy". The method is based on peculiar rescaling properties of the mean energy functional.

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1. Introduction

The new state of matter known as Bose–Einstein condensation was predicted from a theoretical point of view by Bose and Einstein in 1925 and it was confirmed in the experiment only in 1995. To be precise, it was observed that if a large number of identical pair interacting bosons is confined in a trap of macroscopic size at very low temperature, then almost all particles belong to a "condensed" cloud, where every particle is in the same macroscopic one-body quantum state, called the "wave function of the condensate". As far as the interacting case is concerned, this phenomenon was firstly investigated by Bogolubov [4] and later by Gross [5] and Pitaevskii [28]. In the Gross–Pitaevskii theory the wave function of the condensate satisfies a cubic nonlinear Schrödinger equation, in this context called Gross–Pitaevskii equation, where the effect of the interactions gives rise to the non linear term. This model has been widely confirmed by experimental results.

A completely rigorous derivation of the Bose–Einstein condensation, for the case of the ground state of a diluted pair-interacting Bose gas in a trap, was done quite recently by Lieb and Sieringer [22], exploiting a suitable scaling limit, consistent with the Gross–Pitaewskii theory, with the number of particles going to infinity. In particular they can prove that any finite-order reduced density matrix converges in the trace norm to the factorized one.

Stochastic tools have also been considered and in particular the interest in stochastic descriptions has increased during the last decade.

For example boson random point processes (fields or general Cox processes), have been exploited by many authors. For the ideal case we quote [6, 7, 11, 12, 13] and [8]. In particular in [8] the random point field describing the position distribution of the ideal boson gas in a state of Bose–Einstein condensation is obtained in the thermodynamic limit. Limit theorems for this field, including a large deviation principle, are established in [10].

For the interacting case interesting results were obtained in [14] and in [9].

We also quote the work [3], where the authors exploit a model of spatial random permutations, finding the occurrence of infinite cycles and [2], where large deviation principles are obtained for a model consisting of N mutually repellent Brownian Motions confined in a bounded region.

The possibility offered by Nelson processes, that can be rigorously associated to the quantum N-body Hamiltonians, was considered only very recently [15, 32]. In this approach the N-body system is described by a system of N interacting diffusions, the interaction being described by the structure of the Mean Energy Functional. Under the assumption of strict positivity and continous differentiability of the many-body ground state wave function, all one-particle diffusions have the same law. This allows to consider a generic one-particle process and to show that, in a proper scaling limit, such a process continuously remains outside a time dependent random *interaction set* with probability one and that its stopped version converges, in a relative entropy sense, toward a Markov diffusion whose drift is uniquely determined by wave function of the condensate [32].

In this paper we focus our attention on the scaling properties of the Mean Energy Functional which is associated to the system of the N interacting diffusions and we describe in detail the phenomenon of the concentration of relative entropy, which plays a fundamental role in understanding the peculiarities of the stochastic motion of a particle in the condensate.

2. Basics

We start by considering a single spinless quantum particle of mass m in a potential V. Denoting by ψ its wave function, we know that it is a solution of the Schrödinger equation

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\triangle + V\right)\psi.$$

We also know that, if V is of Rellich class and the initial kinetic energy is finite [15], then there exists a weak solution X to the three-dimensional stochastic differential equation

$$dX_t = \frac{\hbar}{m}\left(\mathrm{Re}\frac{\nabla\psi}{\psi} + \mathrm{Im}\frac{\nabla\psi}{\psi}\right)(X_t,t)dt + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} dW_t$$

where dW_t denotes the increment of a standard Brownian Motion.

Notably, the diffusion X satisfies the stochastic version of the second Newton's law

$$a_N(X_t,t) = -\frac{1}{m}\nabla V(X_t.t)$$

where a_N denotes the natural mean stochastic acceleration as introduced by Nelson in 1966 [27]. In addition, up to regularity assumptions, X is critical for the mean classical action functional [18] (see also [16] for a recent review).

The system we are considering consists of N pair interacting copies of such a particle, with Hamiltonian

$$H_N = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2m}\triangle_i + V(\mathbf{r_i})\right) + \sum_{1\le i<j\le N} v(\mathbf{r_i}-\mathbf{r_j}). \tag{2.1}$$

We adopt the following notations: bold letters denote vectors in $\mathbb{R}^3$, capital letters stochastic processes and $\hat{X} = (X_1,\ldots,X_N)$ arrays in $\mathbb{R}^{3N}$.

Under suitable assumptions on V and v one can prove the existence of the ground state Ψ_N of (2.1), which is unique up to a phase coefficient. We also assume that it is strictly positive and continuously differentiable (see [29], Thms. XIII.46 and XIII.47, for the regularity conditions on the potentials V and v implying the strictly positivity, and (XIII.11) for those implying the differentiability of the ground state wave function).

We denote by $\hat{X}$ the corresponding $3N$-dimensional Nelson's diffusion, whose generator is related to H_N by a Doob's transformation [17, 30] (see also [31] for extensions).

$\hat{X}$ is the N-body ground state process and it consists of a family of N three-dimensional one-particle interacting diffusions $(X_1,\ldots,X_N)$.

It satisfies the SDE, written in compact form,

$$d\hat{X}_t = \hat{b}(\hat{X}_t)dt + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} d\hat{W}_t \tag{2.2}$$

where $\hat{b} = (b_1,\ldots,b_N)$, $\hat{b}_i(\hat{X}_t) = \frac{\nabla_i\Psi_N}{\Psi_N}(\hat{X}_t)$ for $i = 1,\ldots,N$, and $\hat{W}$ is a $3N$-dimensional standard Brownian Motion.

If Bose–Einstein condensation occurs, the condensate is usually described by the order parameter $\phi_{GP} \in L^2(\mathbb{R}^3)$, also called the wave function of the condensate, which is the minimizer of the Gross–Pitaevskii functional

$$E^{GP}[\phi] = \int\left(\frac{\hbar^2}{2m}|\nabla\phi(r)|^2 + V(r)|\phi(r)|^2 + g|\phi(r)|^4\right)d\mathbf{r} \tag{2.3}$$

under the L^2-normalization condition

$$\int_{\mathbb{R}^3} |\phi^{GP}|^2 d\mathbf{r} = 1$$

and where $g > 0$ is a parameter depending on the interaction potential v (see also next assumption h2)). Therefore ϕ_{GP} solves the stationary cubic non-linear equation (in this context called Gross–Pitaevskii equation)

$$-\frac{\hbar^2}{2m}\triangle\phi + V\phi + 2g|\phi|^2\phi = \lambda\phi \tag{2.4}$$

λ denoting the chemical potential.

3. Mean energy and rescaling

The basic mathematical object which contains all elements necessary to prove, from first principles, the existence of BEC and its proper stochastic description, is the quantum mechanical energy of the N-body system in the ground state.

Its explicit expression, with $\hbar = m = 1$, is

$$\begin{aligned} E(\Psi_N) &= \langle \Psi_N, H_N \Psi_N \rangle \qquad (3.1)\\ &= \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} \frac{1}{2}|\nabla_i \Psi_N|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N + \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} V(\mathbf{r}_i)|\Psi_N|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N \\ &\quad + \sum_{1\le i<j\le N}^{N} \int v(\mathbf{r}_j - \mathbf{r}_i)|\Psi_N|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N. \end{aligned}$$

Exploiting the $3N$-dimensional ground state process $\hat{X}$, the kinetic quantum mechanical energy turns to be the sum of the expectation of the kinetic energies of the single particles at any time t and the quantum energy takes the more compact form

$$E(\Psi_N) = \mathbb{E}\left\{ \sum_{i=1}^{N} \left[\frac{1}{2} b_i^2(\hat{X}(t)) + V(X_i(t)) \right] + \sum_{1\le i<j\le N}^{N} [v(X_j(t) - X_i(t))] \right\}, \tag{3.2}$$

b_i being the drift of the interacting ith particle.

A possible rescaling which leads to the Gross–Pitaevskii description of the condensate is defined as follows [21]:

h1) V is locally bounded, positive and going to infinity when $|\mathbf{r_i}|$ goes to infinity. The interaction potential v is smooth, compactly supported, non-negative, spherically symmetric, with finite *scattering length* a ([23, Appendix C]).

h2) $N \longrightarrow \infty$ and the interaction potential v satisfies the Gross–Pitaevskii scaling [21], that is

$$v(r) = v_1\left(\frac{r}{a}\right)/a^2,$$
$$a = \frac{g}{4\pi N},$$

where v_1 has scattering length equal to 1. We notice that g is positive as a consequence of our assumptions on v (see h1)) and it is kept constant in the rescaling.

For given N and a we denote by $E_o(N,a)$ the ground state energy $E(\Psi_N)$ of the N-body system and by E_{GP} the minimum value of the Gross–Pitaevskii functional (2.3).

The following two theorems, proved in [21] and [22], clarify the two main properties of the rescaling procedure. The first is the important

Theorem 3.1 (Energy Theorem [21]). *If $N \uparrow +\infty$ with Na fixed, then*

$$\lim_{N\to\infty} \frac{E_0(N,a)}{N} = E_{GP}$$

and

$$\lim_{N\to\infty} \int |\Psi_N|^2 d\mathbf{r}_2 \cdots \mathbf{r}_N = |\phi^{GP}|^2.$$

Moreover there exists $s \in (0,1]$, depending on the interaction potential v through the solution of the zero-energy scattering equation, such that

$$\begin{aligned}\lim_{N\uparrow\infty} \int_{\mathbb{R}^{3N}} |\nabla_1\Psi_N(\mathbf{r}_1,\ldots,\mathbf{r}_N)|^2 d\mathbf{r}_1\cdots d\mathbf{r}_N \\ = \int_{\mathbb{R}^3} |\nabla\phi^{GP}(\mathbf{r})|^2 d\mathbf{r} + gs\int_{\mathbb{R}^3} (\phi^{GP})^4 d\mathbf{r},\end{aligned} \tag{3.3}$$

$$\lim_{N\uparrow\infty} \int_{\mathbb{R}^{3N}} V(\mathbf{r})|\Psi_N(\mathbf{r}_1,X)|^2 d\mathbf{r}_1\cdots d\mathbf{r}_N = \int V(\mathbf{r})|\phi_{GP}|^2 d\mathbf{r} \tag{3.4}$$

and

$$\lim_{N\uparrow\infty} \frac{1}{2}\sum_{j=2}^{N} \int_{\mathbb{R}^{3N}} v(|\mathbf{r}_1 - \mathbf{r}_j|)|\Psi_N(\mathbf{r}_1,\ldots,\mathbf{r}_N)|^2 d\mathbf{r}_1\cdots d\mathbf{r}_N = (1-s)g\int |\phi_{GP}|^4 d\mathbf{r}. \tag{3.5}$$

A second fundamental tool, originally called "Localization of energy" is the following.

Theorem 3.2 (Localization Theorem [22]). *Defining*

$$F^N(\mathbf{r}_2,\ldots,\mathbf{r}_N) := \left(\bigcup_{i=2}^{N} B^N(\mathbf{r}_i)\right)^c$$

where $B^N(\mathbf{r})$ *denotes the open ball centered in* $\mathbf{r}$ *with radius* $N^{-\frac{1}{3}-\delta}$ *where* $0 < \delta \leq \frac{4}{51}$,

$$\lim_{N\uparrow\infty} \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \dots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2,\dots,\mathbf{r}_N)} \left(\nabla_1 \frac{\Psi_N}{\phi_{GP}}\right)^2 (\phi_{GP})^2 d\mathbf{r}_1 = 0. \tag{3.6}$$

These "quantum mechanical" theorems allow to prove that, in the limit of N going to infinity, one has a complete condensation, in the sense that any finite-order reduced density matrix converges in the trace norm to the factorized one [22]. Moreover they can be seen as analytical tools which are crucial in understanding the scaling properties of the mean energy of the N-body interacting system represented by the system of interacting diffusions (2.2), the interaction being defined by the Mean Energy Functional $E(\Psi_N)$ by (3.2). They also allow to study the limit stochastic behavior of a single generic particle [32].

In particular the Localization Theorem is very interesting and gains a clear meaning in the stochastic framework.

For this reason we report in the appendix a synthesis of the main analytical steps which lead to its proof.

4. Localization of relative entropy and the BEC process

We now turn to the stochastic description and notice that the fixed time joint probability density of the N-body ground state process $\hat{X} = (X_1, \dots, X_N)$ is given by $\rho_N := |\Psi_N|^2$, which is invariant under spatial permutations. Moreover, as expected, if some smoothness conditions are assumed for Ψ_N, the processes $\{X_i\}_{i=1,\dots,N}$ are equal in law. To be more precise (see [32]), one can say that, if Ψ_N is the ground state of H_N and it is strictly positive and of class C^1, then the three-dimensional one-particle interacting diffusions $\{X_i\}_{i=1,\dots,N}$ are equal in law.

Motivated by the fact that the first part of the energy theorem claims that the one-particle marginal density of ρ_N converges to $|\phi_{GP}|^2$ in $L^1(\mathbb{R}^3)$, we consider a three-dimensional process X^{GP} with invariant density $\rho_{GP} := |\phi_{GP}|^2$ and we compare it with the generic interacting non-Markovian diffusion X_1.

We assume that X^{GP} is a solution of the SDE

$$dX_t^{GP} := u_{GP}(X_t^{GP})dt + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} dW_t$$

where,

$$u_{GP} := \frac{\nabla \phi_{GP}}{\phi_{GP}}.$$

Then, since ϕ_{GP} is a solution to the stationary Gross–Pitaevkii equation (2.4), a standard calculation in Stochastic Mechanics shows that Nelson acceleration of X^{GP} reads, quite reasonably,

$$a_N(X_t^{GP}) = -\frac{1}{m}\nabla \left\{ V(X_t^{GP}) + g\left|\phi_{GP}(X_t^{GP})\right|^2 \right\}.$$

(On the other side one could observe that now, by the non-linearity of (2.4), Doob's transformation is not expected to play any role.)

By the equality

$$| \phi_{GP} |^2 \left(\nabla \frac{\Psi_N}{\phi_{GP}}\right)^2 =| \Psi_N |^2 \left(\frac{\nabla \Psi_N}{\Psi_N} - \frac{\nabla \phi_{GP}}{\phi_{GP}}\right)^2$$

we see that the L^2 distance between the two drifts b_1^N and u_{GP}, is given by

$$\int_{\mathbb{R}^{3N}} \|b_1^N - u_{GP}\|^2 \rho_N d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int_{\mathbb{R}^{3N}} \left(\nabla_1 \frac{\Psi_N}{\phi_{GP}}\right)^2 | \phi_{GP} |^2 \, d\mathbf{r}_1 \cdots d\mathbf{r}_N. \quad (4.1)$$

This can be exploited to show that the localization theorem is related to the localization of the relative entropy between the generic one particle non-Markovian interacting diffusion and the process X^{GP}.

To do this we introduce a $3N$-dimensional process $\hat{X}^{GP}$ which satisfies a stochastic differential equation with the same diffusion coefficient as $\hat{X}$ and drift $\hat{u}_{GP}$, defined by

$$\hat{u}_{GP}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = (u_{GP}(\mathbf{r}_1), \ldots, u_{GP}(\mathbf{r}_N)).$$

We consider the measurable space $(\Omega^N, \mathcal{F}^N)$ where Ω^N is $C(\mathbb{R}^+ \to \mathbb{R}^{3N})$, and $\mathcal{F}^N$ is its Borel sigma-algebra. We denote by $\hat{Y} := (Y_1, \ldots, Y_N)$ the coordinate process and by $\mathcal{F}_t^N$ the natural filtration.

We denote by $\mathbb{P}_N$ and $\mathbb{P}_{GP}$ the measures corresponding to the weak solutions of the $3N$-dimensional stochastic differential equations

$$\begin{aligned} \hat{Y}_t - \hat{X}_0 &= \int_0^t \hat{b}^N(\hat{Y}_s) ds + \hat{W}_t, \\ \hat{Y}_t - \hat{X}_0 &= \int_0^t \hat{u}_{GP}(\hat{Y}_s) ds + \hat{W}'_t, \end{aligned} \quad (4.2)$$

where $\hat{X}_0$ is a random variable with probability density equal to $| \Psi_N |^2$ while $\hat{W}_t$ and $\hat{W}'_t$ are $3N$-dimensional $\mathbb{P}_N$ and $\mathbb{P}_{GP}$ standard Brownian Motions, respectively.

We will assume for simplicity that u_{GP} is bounded, that is in fact true if the trap is finite and ϕ_{GP} is smooth and strictly positive.

We recall that under our hypothesis on the potentials v and V, ϕ_{GP} is strictly positive and in $C^1(\mathbb{R}^3) \bigcap L^\infty(\mathbb{R}^3)$ and therefore $u_{GP} \in L^2(\mathbb{R}^3)$ (see [21, Thm. 2.1]). Then, since Ψ_N is the minimizer of $E^N[\Psi]$ and u_{GP} is bounded, the following finite energy conditions hold (with the shorthand notation $\hat{b}_s^N =: \hat{b}^N(\hat{Y}_s)$ and $\hat{u}_s^N =: \hat{u}_{GP}(\hat{Y}_s)$):

$$E_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N\|^2 ds < \infty, \quad (4.3)$$

$$E_{\mathbb{P}_N} \int_0^t \| \, \hat{u}_s^{GP} \|^2 ds < \infty. \quad (4.4)$$

Then, by Girsanov theorem, we have, for all $t > 0$,

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{GP}}|_{\mathcal{F}_t} = \exp\left\{-\int_0^t (\hat{b}_s^N - \hat{u}_s^{GP})\cdot d\hat{W}_s + \frac{1}{2}\int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds\right\}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^{3N}$. The relative entropy restricted to $\mathcal{F}_t$ reads

$$\mathcal{H}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_t} =: \mathbb{E}_{\mathbb{P}_N}[\log \frac{d\mathbb{P}_N}{d\mathbb{P}_{GP}}\ |_{\mathcal{F}_t}] = \frac{1}{2}E_{\mathbb{P}_N}\int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds.$$

Since under $\mathbb{P}_N$ the $3N$-dimensional process $\hat{Y}$ is a solution of (4.2) with invariant probability density $|\ \Psi_N\ |^2$, we can write, recalling also (4.3) and (4.4),

$$\begin{aligned}\frac{1}{2}E_{\mathbb{P}^N}\int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds &= \frac{1}{2}\int_0^t E_{\mathbb{P}_N}\|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds \\ &= \frac{1}{2}t\int_{\mathbb{R}^{3N}} \|\hat{b}^N(\mathbf{r}_1,\ldots,\mathbf{r}_N) - \hat{u}_{GP}(\mathbf{r}_1,\ldots,\mathbf{r}_N)\|^2 \rho_N d\mathbf{r}_1\ldots d\mathbf{r}_N\end{aligned}$$

so that, the symbol $\|\ .\ \|$ now denoting the euclidean norm in $\mathbb{R}^3$, we get

$$\begin{aligned}\mathcal{H}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_t} &= \frac{1}{2}t\int_{\mathbb{R}^{3N}}\sum_{i=1}^N \|b_i^N(\mathbf{r}_1,\ldots,\mathbf{r}_N) - u_{GP}(\mathbf{r}_i)\|^2 \rho_N d\mathbf{r}_1\ldots d\mathbf{r}_N \\ &= \frac{1}{2}Nt\int_{\mathbb{R}^{3N}} \|b_1^N(\mathbf{r}_1,\ldots,\mathbf{r}_N) - u_{GP}(\mathbf{r}_1)\|^2 \rho_N d\mathbf{r}_1\ldots d\mathbf{r}_N \\ &= \frac{1}{2}NE_{\mathbb{P}_N}\int_0^t \|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 ds\end{aligned}$$

where the symmetry of $\hat{b}^N$ and Ψ_N has been exploited.

Finally we get the sum of N identical *one-particle relative entropies*, each of them being defined by

$$\bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_t} =: \frac{1}{N}\mathcal{H}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_t} = \frac{1}{2}E_{\mathbb{P}_N}\int_0^t \|b_1^N(\hat{Y}_s) - u^{GP}(Y_1(s))\|^2 ds.$$

Recalling (3.3) in *Energy Theorem* and (4.1), we can write

$$\lim_{N\uparrow\infty}\int_{\mathbb{R}^{3N}} \|b_1^N - u_{GP}\|^2 \rho_N d\mathbf{r}_1\cdots d\mathbf{r}_N = gs\int_{\mathbb{R}^3}(\phi^{GP})^4 d\mathbf{r}.$$

As a consequence, for all $t > 0$, the one particle relative entropy is asymptotically finite but it does not go to zero in the scaling limit. This means that the process X^{GP} cannot be directly interpreted as the stochastic description of the generic particle in the condensate.

But the key point is that, for great N, the one particle process continously "lives" outside a properly defined "random interaction-set" $D_N(t)$.

We define it by the equality

$$D_N(t) := \bigcup_{i=2}^N B^N(X_i(t))$$

where $B^N(\mathbf{r})$ is again the ball with radius $N^{-1/3-\delta}$, $\quad 0<\delta\leq 4/51$ centered in $\mathbf{r}$. We also introduce the stopping time

$$\tau^N := \inf\{t \geq 0 : X_1(t) \in D_N(t)\}. \tag{4.5}$$

The following proposition claims that, in the scaling limit, a generic particle remains outside the *interaction-set*, for any finite time interval, with probability one.

Notice that the result is not obvious: even in dimension $d=3$, where the Lebesgue measure of $D_N(t)$ goes to zero for all t, it could happen that, asymptotically, such a set takes the form of a very complicated surface, dividing the physical three-dimensional space into smaller and smaller non connected regions.

Proposition 4.1 ([32]). *Let* h1) *and* h2) *hold and the ground state* Ψ_N *be of class* C^1. *Then in dimension* $d=3$, *for all* $t>0$, *we have*

$$\lim_{N\to\infty} \mathbb{P}(\tau^N > t \mid X_1(0) \notin D_N(0)) = 1$$

and τ^N *has an exponential distribution.*

This allows to apply the *Localization Theorem* to the stopped one-particle process:

Theorem 4.2. *Let* h1) *and* h2) *hold. Assume also that* Ψ_N *is of class* C^1 *and that* u_{GP} *is bounded. Then, with* τ^N *defined as in* (4.5), *we have*

$$\lim_{N\uparrow\infty} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP}) \mid_{\mathcal{F}_{t\wedge\tau^N}} = 0.$$

Proof, cf. [32]. Recalling (4.3) and (4.4) we can write

$$\begin{aligned}
\bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_{t\wedge\tau^N}} &= \frac{1}{2} E_{\mathbb{P}_N} \int_0^{t\wedge\tau^N} \|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 ds \\
&\leq \frac{1}{2}\int_0^t E_{\mathbb{P}_N}\{\|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 I_{\{Y_1\notin D_s^N\}}\}ds \\
&= \frac{1}{2} t E_{\mathbb{P}_N}\{\|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 I_{\{Y_1\notin D_s^N\}}\} \\
&= \frac{1}{2} t \int_{\mathbb{R}^{3N}} \|b_1^N(\mathbf{r}_1,\ldots,\mathbf{r}_N) - u_{GP}(\mathbf{r}_1)\|^2 I_{F^N(\mathbf{r}_2,\ldots,\mathbf{r}_N)}(\mathbf{r}_1)\rho_N^2 d\mathbf{r}_1\cdots d\mathbf{r}_N.
\end{aligned}$$

Thus, by (4.1) and the Localization Theorem, we finally get

$$\begin{aligned}
&\lim_{N\uparrow\infty} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_{t\wedge\tau^N}} \\
&= \frac{1}{2} t \lim_{N\uparrow\infty} \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \ldots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2,\ldots,\mathbf{r}_N)} \|b_1^N - u_{GP}\|^2 \rho_N^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = 0. \qquad \square
\end{aligned}$$

Concluding, for great N, $X_{t\wedge\tau^N}$ is close in the sense of relative entropy to the "BEC process" $X^{GP}_{t\wedge\tau^N}$, whenever at an arbitrary time-origine the particle is not in the random interacting set, while the probability that the particle hits such a set in a finite time becomes negligible.

Appendix

In this section we put $\frac{\hbar^2}{2m} = 1$.

The proof of the *Localization Theorem* is essentially devoted to establish a proper lower bound for the Energy Functional (3.1) and it is based on two results concerning the following interacting Hamiltonian for the homogeneous case

$$H_N^I = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \le i \le j \le N} v(|x_i - x_j|). \tag{A.1}$$

Lemma A.1 (Smoothing Lemma). *Let v be non-negative with finite range R_0 and let U be any non-negative function satisfying,*

$$\int U(r) r^2 dr \le 1 \qquad U(r) = 0 \quad r < R_0$$

then, a being the scattering length and $\epsilon \in (0,1)$,

$$H_N^I \ge \epsilon T_N + (1-\epsilon) a W_R$$

where

$$T_N = -\sum_i \Delta_i, \qquad W_R = \sum_i^N U(t_i)$$

with

$$t_i = t_i(x_1, x_2, \ldots, x_N) := \min_{j, j \ne i} |x_i - x_j|.$$

Moreover one can take

$$U(r) = 3(R^3 - R_0^3)^{-1} \qquad R_0 < r < R$$

and otherwise equal to zero, where R represents the range of the potential U.

We can observe that in the lower bound operator, only a part of the kinetic energy survives and the interaction potential is softer than v, but with a larger range.

The proof is based on a generalization of a Dyson's Lemma [25] due to Lieb and Yngvason [24].

Lemma A.2 (Lower Bound Theorem in a finite box [24]). *Let* (A.1) *the Hamiltonian for N interacting bosons in a cubic box Λ with side length L, where v is a spherically symmetric pair potential having finite scattering length a. Then there exists $\lambda > 0$ such that, denoting by $E_0(N,L)$ the ground state energy of H_N^I, with Neumann boundary conditions, one has*

$$\frac{E_0(N,L)}{N} \ge 4\pi\rho a (1 - CY^{1/17})$$

where $\rho = \frac{N}{L^3}$ is the particle density, $Y = 4\pi\rho\frac{a^3}{3}$ is the number of particles in the ball of radius a and L is such that $Y < \lambda$ and $\frac{L}{a} > C_1 Y^{-\frac{6}{17}}$.

Moreover C and C_1 are positive constants independent of N and L.

For the proof see [23, Thm. 2.4].

Proof of the Localization Theorem ([23], *Lemma* 7.3 *and* [22]). It is sufficient to show that, when $N \uparrow \infty$

$$\int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int_{F_N^c(\mathbf{r}_2,\ldots,\mathbf{r}_N)} \left(\nabla_1 \frac{\Psi_N}{\phi_{GP}}\right)^2 (\phi_{GP})^2 d\mathbf{r}_1$$
$$+ \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int |\Psi_N|^2 \left[\frac{1}{2}\sum_{k\geq 2} v(|\mathbf{r}-\mathbf{r}_k|) - 2g{\phi_{GP}}^2\right]$$
$$\geq -g \int |\phi_{GP}|^4 d\mathbf{r} - o(1). \tag{A.2}$$

This implies the thesis because (A.2) can be written as

$$\int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int \left(\nabla_1 \frac{\Psi_N}{\phi_{GP}}\right)^2 (\phi_{GP})^2 d\mathbf{r}_1$$
$$+ \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int |\Psi_N|^2 \left[\frac{1}{2}\sum_{k\geq 2} v(|\mathbf{r}-\mathbf{r}_k|) - 2g{\phi_{GP}}^2\right]$$
$$- \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2,\ldots,\mathbf{r}_N)} \left(\nabla_1 \frac{\Psi_N}{\phi_{GP}}\right)^2 (\phi_{GP})^2 d\mathbf{r}_1$$
$$\geq -g \int |\phi_{GP}|^4 d\mathbf{r} - o(1)$$

and then, by (3.3), (3.4) and (3.5) in the *Energy Theorem*, with the external potential V particularized to $2g|\phi_{GP}|^2$ in (3.4), one obtains the thesis (3.6).

To prove (A.2) one introduces a function F such that

$$\frac{\Psi_N}{\phi_{GP}(\mathbf{r}_1)} := \prod_{k\geq 2} \phi_{GP}(\mathbf{r}_k) F(\mathbf{r}_1,\ldots,\mathbf{r}_N).$$

Using the fact that F is *symmetric* in the particle coordinates, one can see that (A.2) is equivalent to

$$\frac{Q_\delta(F)}{N} \geq -g \int |\phi_{GP}|^4 d\mathbf{r} - o(1) \tag{A.3}$$

where Q_δ has the following definition

$$Q_\delta := \sum_{i=1}^{N} \int_{\Gamma_i^c} |\nabla_i F|^2 \prod_{k=1}^{N} |\phi_{GP}(\mathbf{r}_k)|^2 d\mathbf{r}_k$$
$$+ \sum_{1\leq i\leq j\leq N} \int v(|\mathbf{r}_i - \mathbf{r}_j|)|F|^2 \prod_{k=1}^{N} |\phi_{GP}(\mathbf{r}_k)|^2 d\mathbf{r}_k$$
$$- 2g \sum_{i=1}^{N} \int |\phi_{GP}(\mathbf{r}_i)|^2 |F|^2 \prod_{k=1}^{N} |\phi_{GP}(\mathbf{r}_k)|^2 d\mathbf{r}_k$$

with

$$\Gamma_i^c = \{(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{3N} | \min_{k \neq i} |\mathbf{r}_i - \mathbf{r}_k| \leq R'\}$$

where $R' = N^{-\frac{1}{3}-\delta}$.

To handle the expression of Q_δ one applies the "cell-method", considering the space as divided into cells of width L, and then minimizing over all possible distributions of the particles in the different cells. Since one is looking for a lower bound and v is positive, the interactions due to particles in different cells can be ignored. Finally one leaves the width of the cells going to zero.

Labeling cells with the index α, one has,

$$\inf_F Q_\delta(F) \geq \inf_{n_\alpha} \sum_\alpha \inf_{F_\alpha} Q_\delta^\alpha(F_\alpha)$$

where Q_δ^α is defined as Q_δ but with the integrations limited to the cell α. F_α is defined as F but with N replaced by n_α. The infimum is taken over all distributions such that $\sum_\alpha n_\alpha = N$.

One now fixes some $M > 0$ and considers only cells inside a cube Λ_M of side length M. For the cells belonging to Λ_M one can evaluate the maximum and minimum value of ρ_{GP}. For the cell α those are denoted by $\rho_{\alpha,max}$ and $\rho_{\alpha,min}$, respectively.

One then can observe that, if the range R of the smoothing potential U is sufficiently small, one can apply the *Smoothing Lemma* "in the cell α" and restrict all integrations to Γ_i^c.

This at the end leads to the inequality

$$Q_\delta^\alpha(F_\alpha) \geq \frac{\rho_{\alpha,\,\min}}{\rho_{\alpha,\,\max}} E_0^U(n_\alpha, L) - 8\pi a N \rho_{\alpha,\,\max} n_\alpha - \epsilon C_M n_\alpha \tag{A.4}$$

where $E_0^U(n_\alpha, L)$ is the ground state energy of

$$\sum_{i=1}^{n_\alpha} \left(-\frac{1}{2}\epsilon\triangle_i + (1-\epsilon) a U(t_i) \right)$$

with $C_M = \sup_{\mathbf{r}\in\Lambda_M} |\nabla\phi_{GP}(\mathbf{r})|^2$, independent of N.

To minimize (A.4) with respect to n_α one takes advantage of the *Lower Bound Theorem* and of Lemma 6.4 in [23], p. 55.

One finds after some manipulations that $\bar{n}_\alpha$ is at least of the order of NL^3.

If one takes $L \sim N^{-\frac{1}{10}}$, the range of smoothing potential U can be shown to be well estimated as $R \sim N^{-\frac{1}{17}}$ and the assumption on δ is sufficient to guarantee that R remains lower or equal to R', allowing the application of the *Smoothing Lemma* in constructing the lower bound for Q_δ^α.

Further standard manipulations then give rise, C_o denoting a positive constant independent of N, to the following inequality:

$$Q_\delta(F) \geq 4\pi a N^2 \int |\phi_{GP}|^4 [1 + C_o \cdot N^{-1/10}] - Y^{1/17} N C_M - 8\pi a N^2 \sup_{\mathbf{r}\notin\Lambda_M} |\phi_{GP}|^2(\mathbf{r}).$$

Dividing by N, taking $N \uparrow \infty$ and then $M \uparrow \infty$, one obtains the result. In fact, since ϕ_{GP} decreases more than exponentially at infinity ([21, Lemma A.5]), the last term is arbitrarily small for M large. This proves (A.3), which is equivalent to (A.2). □

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Multi-dimensional Semicircular Limits on the Free Wigner Chaos

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Abstract. We show that, for sequences of vectors of multiple Wigner integrals with respect to a free Brownian motion, componentwise convergence to semicircular law is equivalent to joint convergence. This result extends to the free probability setting some findings by Peccati and Tudor (2005), and represents a multi-dimensional counterpart of a limit theorem inside the free Wigner chaos established by Kemp, Nourdin, Peccati and Speicher (2011).

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1. Introduction

Let $W = \{W_t : t \geq 0\}$ be a one-dimensional standard Brownian motion (living on some probability space $(\Omega, \mathscr{F}, P)$). For every $n \geq 1$ and every real-valued, symmetric and square-integrable function $f \in L^2(\mathbb{R}_+^n)$, we denote by $I^W(f)$ the multiple Wiener–Itô integral of f, with respect to W. Random variables of this type compose the so-called nth *Wiener chaos* associated with f. In an infinite-dimensional setting, the concept of Wiener chaos plays the same role as that of the Hermite polynomials for the one-dimensional Gaussian distribution, and represents one of the staples of modern Gaussian analysis (see, e.g., [5, 10, 13, 15] for an introduction to these topics).

In recent years, many efforts have been made in order to characterize Central Limit Theorems (CLTs) – that is, limit theorems involving convergence in distribution to a Gaussian element – for random variables living inside a Wiener chaos. The following statement gathers the main findings of [14] (Part 1) and [16] (Part 2), and provides a complete characterization of (both one- and multi-dimensional) CLTs on the Wiener chaos.

Theorem 1.1 (See [14, 16]).

(A) *Let* $F_k = I^W(f_k)$, $k \geq 1$, *be a sequence of multiple integrals of order* $n \geq 2$, *such that* $E[F_k^2] \to 1$. *Then, the following two assertions are equivalent, as* $k \to \infty$*:*

- **(i)** F_k *converges in distribution to a standard Gaussian random variable* $N \sim \mathscr{N}(0,1)$*;*
- **(ii)** $E[F_k^4] \to 3 = E[N^4]$.

(B) *Let* $d \geq 2$ *and* $n_1, \ldots, n_d$ *be integers, and let* $(F_k^{(1)}, \ldots, F_k^{(d)})$, $k \geq 1$, *be a sequence of random vectors such that, for every* $i = 1, \ldots, d$, *the random variable* $F_k^{(i)}$ *lives in the* n_i*th Wiener chaos of* W. *Assume that, as* $k \to \infty$ *and for every* $i, j = 1, \ldots, d$, $E[F_k^{(i)} F_k^{(j)}] \to c(i,j)$, *where* $c = \{c(i,j) : i,j = 1, \ldots, d\}$ *is a positive definite symmetric matrix. Then, the following two assertions are equivalent, as* $k \to \infty$*:*

- **(i)** $(F_k^{(1)}, \ldots, F_k^{(d)})$ *converges in distribution to a centered* d*-dimensional Gaussian vector* $(N_1, \ldots, N_d)$ *with covariance* c*;*
- **(ii)** *for every* $i = 1, \ldots, d$, $F_k^{(i)}$ *converges in distribution to a centered Gaussian random variable with variance* $c(i,i)$.

Roughly speaking, Part (B) of the previous statement means that, for vectors of random variables living inside some fixed Wiener chaoses, *componentwise convergence to Gaussian always implies joint convergence.* The combination of Part (A) and Part (B) of Theorem 1.1 represents a powerful simplification of the so-called 'method of moments and cumulants' (see, e.g., [15, Chapter 11] for a discussion of this point), and has triggered a considerable number of applications, refinements and generalizations, ranging from Stein's method to analysis on homogenous spaces, random matrices and fractional processes – see the survey [9] as well as the monograph [10] for details and references.

Now, let $(\mathscr{A}, \varphi)$ be a non-commutative tracial W^*-probability space (in particular, $\mathscr{A}$ is a von Neumann algebra and φ is a trace – see Section 2.1 for details), and let $S = \{S_t : t \geq 0\}$ be a free Brownian motion defined on it. It is well known (see, e.g., [2]) that, for every $n \geq 1$ and every $f \in L^2(\mathbb{R}_+^n)$, one can define a free multiple stochastic integral with respect to f. Such an object is usually denoted by $I^S(f)$. Multiple integrals of order n with respect to S compose the so-called nth *Wigner chaos* associated with S. Wigner chaoses play a fundamental role in free stochastic analysis – see again [2].

The following theorem, which is the main result of [4], is the exact free analogous of Part (A) of Theorem 1.1. Note that the value 2 coincides with the fourth moment of the standard semicircular distribution $S(0,1)$.

Theorem 1.2 (See [4]). *Let* $n \geq 2$ *be an integer, and let* $(f_k)_{k\in\mathbb{N}}$ *be a sequence of mirror symmetric (see Section 2.2 for definitions) functions in* $L^2(\mathbb{R}_+^n)$, *each with* $\|f_k\|_{L^2(\mathbb{R}_+^n)} = 1$. *The following statements are equivalent.*

(1) *The fourth moments of the stochastic integrals* $I(f_k)$ *converge to* 2*, that is,*

$$\lim_{k\to\infty} \varphi(I^S(f_k)^4) = 2.$$

(2) *The random variables* $I^S(f_k)$ *converge in law to the standard semicircular distribution* $S(0,1)$ *as* $k \to \infty$*.*

The aim of this paper is to provide a complete proof of the following Theorem 1.3, which represents a free analogous of Part (B) of Theorem 1.1.

Theorem 1.3. *Let* $d \geq 2$ *and* $n_1, \ldots, n_d$ *be some fixed integers, and consider a positive definite symmetric matrix* $c = \{c(i,j) : i,j = 1,\ldots,d\}$*. Let* $(s_1,\ldots,s_d)$ *be a semicircular family with covariance* c *(see Definition* 2.10*). For each* $i = 1,\ldots,d$*, we consider a sequence* $(f_k^{(i)})_{k\in\mathbb{N}}$ *of mirror-symmetric functions in* $L^2(\mathbb{R}_+^{n_i})$ *such that, for all* $i,j = 1,\ldots,d$*,*

$$\lim_{k\to\infty} \varphi[I^S(f_k^{(i)})I^S(f_k^{(j)})] = c(i,j). \tag{1.1}$$

The following three statements are equivalent as $k \to \infty$*.*

(1) *The vector* $((I^S(f_k^{(1)}),\ldots,I^S(f_k^{(d)}))$ *converges in distribution to* $(s_1,\ldots,s_d)$*.*
(2) *For each* $i = 1,\ldots,d$*, the random variable* $I^S(f_k^{(i)})$ *converges in distribution to* s_i*.*
(3) *For each* $i = 1,\ldots,d$*,*

$$\lim_{k\to\infty} \varphi[I^S(f_k^{(i)})^4] = 2\,c(i,i)^2.$$

Remark 1.4. In the previous statement, the quantity $\varphi[I^S(f_k^{(i)})I^S(f_k^{(j)})]$ equals $\langle f_k^{(i)}, f_k^{(j)}\rangle_{L^2(\mathbb{R}_+^{n_i})}$ if $n_i = n_j$, and equals 0 if $n_i \neq n_j$. In particular, the limit covariance matrix c is necessarily such that $c(i,j) = 0$ whenever $n_i \neq n_j$.

Remark 1.5. Two additional references deal with non-semicircular limit theorems inside the free Wigner chaos. In [11], one can find necessary and sufficient conditions for the convergence towards the so-called Marčenko–Pastur distribution (mirroring analogous findings in the classical setting – see [8]). In [3], conditions are established for the convergence towards the so-called 'tetilla law' (or 'symmetric Poisson distribution' – see also [6]).

Combining the content of Theorem 1.3 with those in [4, 16], we can finally state the following Wiener–Wigner transfer principle, establishing an equivalence between multi-dimensional limit theorems on the classical and free chaoses.

Theorem 1.6. *Let* $d \geq 1$ *and* $n_1,\ldots,n_d$ *be some fixed integers, and consider a positive definite symmetric matrix* $c = \{c(i,j) : i,j = 1,\ldots,d\}$*. Let* $(N_1,\ldots,N_d)$ *be a* d*-dimensional Gaussian vector and* $(s_1,\ldots,s_d)$ *be a semicircular family, both with covariance* c*. For each* $i = 1,\ldots,d$*, we consider a sequence* $(f_k^{(i)})_{k\in\mathbb{N}}$ *of fully-symmetric functions (cf. Definition* 2.2*) in* $L^2(\mathbb{R}_+^{n_i})$*. Then:*

1. *For all* $i, j = 1, \dots, d$ *and as* $k \to \infty$, $\varphi[I^S(f_k^{(i)})I^S(f_k^{(j)})] \to c(i,j)$ *if and only if* $E[I^W(f_k^{(i)})I^W(f_k^{(j)})] \to \sqrt{(n_i)!(n_j)!}\, c(i,j)$.
2. *If the asymptotic relations in* (1) *are verified then, as* $k \to \infty$,

$$\big(I^S(f_k^{(1)}), \dots, I^S(f_k^{(d)})\big) \stackrel{\text{law}}{\to} (s_1, \dots, s_d)$$

if and only if

$$\big(I^W(f_k^{(1)}), \dots, I^W(f_k^{(d)})\big) \stackrel{\text{law}}{\to} \big(\sqrt{(n_1)!}N_1, \dots, \sqrt{(n_d)!}N_d\big).$$

The remainder of this paper is organized as follows. Section 2 gives concise background and notation for the free probability setting. Theorems 1.3 and 1.6 are then proved in Section 3.

2. Relevant definitions and notations

We recall some relevant notions and definitions from free stochastic analysis. For more details, we refer the reader to [2, 4, 7].

2.1. Free probability, free Brownian motion and stochastic integrals

In this note, we consider as given a so-called (*tracial*) W^* *probability space* $(\mathscr{A}, \varphi)$, where $\mathscr{A}$ is a von Neumann algebra (with involution $X \mapsto X^*$), and $\varphi : \to \mathbb{C}$ is a *tracial state* (or *trace*). In particular, φ is weakly continuous, positive (that is, $\varphi(Y) \geq 0$ whenever Y is a non-negative element of $\mathscr{A}$), faithful (that is, $\varphi(YY^*) = 0$ implies $Y = 0$, for every $Y \in \mathscr{A}$) and tracial (that is, $\varphi(XY) = \varphi(YX)$, for every $X, Y \in \mathscr{A}$). The self-adjoint elements of $\mathscr{A}$ are referred to as *random variables*. The *law* of a random variable X is the unique Borel measure on $\mathbb{R}$ having the same moments as X (see [7, Proposition 3.13]). For $1 \leq p \leq \infty$, one writes $L^p(\mathscr{A}, \varphi)$ to indicate the L^p space obtained as the completion of $\mathscr{A}$ with respect to the norm $\|a\|_p = \tau(|a|^p)^{1/p}$, where $|a| = \sqrt{a^*a}$, and $\|\cdot\|_\infty$ stands for the operator norm.

Definition 2.1. Let $\mathscr{A}_1, \dots, \mathscr{A}_n$ be unital subalgebras of $\mathscr{A}$. Let $X_1, \dots, X_m$ be elements chosen from among the $\mathscr{A}_i$'s such that, for $1 \leq j < m$, X_j and X_{j+1} do not come from the same $\mathscr{A}_i$, and such that $\varphi(X_j) = 0$ for each j. The subalgebras $\mathscr{A}_1, \dots, \mathscr{A}_n$ are said to be *free* or *freely independent* if, in this circumstance, $\varphi(X_1X_2 \cdots X_n) = 0$. Random variables are called freely independent if the unital algebras they generate are freely independent.

Definition 2.2. The (centered) *semicircular distribution* (or Wigner law) $S(0,t)$ is the probability distribution

$$S(0,t)(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2}\, dx, \quad |x| \leq 2\sqrt{t}.$$

Being symmetric around 0, the odd moments of this distribution are all 0. Simple calculations (see, e.g., [7, Lecture 2]) show that the even moments can be expressed in therms of the so-called *Catalan numbers*: for non-negative integers m,

$$\int_{-2\sqrt{t}}^{2\sqrt{t}} x^{2m} S(0,t)(dx) = C_m t^m,$$

where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the mth Catalan number. In particular, the second moment (and variance) is t while the fourth moment is $2t^2$.

Definition 2.3. A *free Brownian motion* S consists of: (i) a filtration $\{\mathscr{A}_t : t \geq 0\}$ of von Neumann sub-algebras of $\mathscr{A}$ (in particular, $\mathscr{A}_s \subset \mathscr{A}_t$, for $0 \leq s < t$), (ii) a collection $S = \{S_t : t \geq 0\}$ of self-adjoint operators in $\mathscr{A}$ such that: (a) $S_0 = 0$ and $S_t \in \mathscr{A}_t$ for every t, (b) for every t, S_t has a semicircular distribution with mean zero and variance t, and (c) for every $0 \leq u < t$, the increment $S_t - S_u$ is free with respect to $\mathscr{A}_u$, and has a semicircular distribution with mean zero and variance $t - u$.

For the rest of the paper, we consider that the W^*-probability space $(\mathscr{A}, \varphi)$ is endowed with a free Brownian motion S. For every integer $n \geq 1$, the collection of all operators having the form of a multiple integral $I^S(f)$, $f \in L^2(\mathbb{R}_+^n; \mathbb{C}) = L^2(\mathbb{R}_+^n)$, is defined according to [2, Section 5.3], namely: (a) first define $I^S(f) = (S_{b_1} - S_{a_1}) \cdots (S_{b_n} - S_{a_n})$ for every function f having the form

$$f(t_1, \ldots, t_n) = \mathbf{1}_{(a_1,b_1)}(t_1) \times \cdots \times \mathbf{1}_{(a_n,b_n)}(t_n), \tag{2.1}$$

where the intervals (a_i, b_i), $i = 1, \ldots, n$, are pairwise disjoint; (b) extend linearly the definition of $I^S(f)$ to 'simple functions vanishing on diagonals', that is, to functions f that are finite linear combinations of indicators of the type (2.1); (c) exploit the isometric relation

$$\langle I^S(f), I^S(g)\rangle_{L^2(\mathscr{A},\varphi)} = \int_{\mathbb{R}_+^n} f(t_1, \ldots, t_n)\overline{g(t_n, \ldots, t_1)}dt_1 \ldots dt_n, \tag{2.2}$$

where f, g are simple functions vanishing on diagonals, and use a density argument to define $I(f)$ for a general $f \in L^2(\mathbb{R}_+^n)$.

As recalled in the introduction, for $n \geq 1$, the collection of all random variables of the type $I^S(f)$, $f \in L^2(\mathbb{R}_+^n)$, is called the nth *Wigner chaos* associated with S. One customarily writes $I^S(a) = a$ for every complex number a, that is, the Wigner chaos of order 0 coincides with $\mathbb{C}$. Observe that (2.2) together with the above sketched construction imply that, for every $n, m \geq 0$, and every $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$,

$$\varphi[I^S(f)I^S(g)] = \mathbf{1}_{n=m} \times \int_{\mathbb{R}_+^n} f(t_1, \ldots, t_n)\overline{g(t_n, \ldots, t_1)}dt_1 \ldots dt_n, \tag{2.3}$$

where the right-hand side of the previous expression coincides by convention with the inner product in $L^2(\mathbb{R}_+^0) = \mathbb{C}$ whenever $m = n = 0$.

2.2. Mirror symmetric functions and contractions

Definition 2.4. Let n be a natural number, and let f be a function in $L^2(\mathbb{R}_+^n)$.

(1) The *adjoint* of f is the function $f^*(t_1,\dots,t_n) = \overline{f(t_n,\dots,t_1)}$.
(2) f is called *mirror symmetric* if $f = f^*$, i.e., if

$$f(t_1,\dots,t_n) = \overline{f(t_n,\dots,t_1)}$$

for almost all $t_1,\dots,t_n \geq 0$ with respect to the product Lebesgue measure
(3) f is called *fully symmetric* if it is real-valued and, for any permutation σ in the symmetric group Σ_n, $f(t_1,\dots,t_n) = f(t_{\sigma(1)},\dots,t_{\sigma(n)})$ for almost every $t_1,\dots,t_n \geq 0$ with respect to the product Lebesgue measure.

An operator of the type $I^S(f)$ is self-adjoint if and only if f is mirror symmetric.

Definition 2.5. Let n, m be natural numbers, and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$. Let $p \leq \min\{n,m\}$ be a natural number. The *pth contraction* $f \overset{p}{\frown} g$ of f and g is the $L^2(\mathbb{R}_+^{n+m-2p})$ function defined by nested integration of the middle p variables in $f \otimes g$:

$$\begin{aligned} &f \overset{p}{\frown} g\,(t_1,\dots,t_{n+m-2p}) \\ &\quad = \int_{\mathbb{R}_+^p} f(t_1,\dots,t_{n-p},s_1,\dots,s_p) g(s_p,\dots,s_1,t_{n-p+1},\dots,t_{n+m-2p})\,ds_1\cdots ds_p. \end{aligned}$$

Notice that when $p = 0$, there is no integration, just the products of f and g with disjoint arguments; in other words, $f \overset{0}{\frown} g = f \otimes g$.

2.3. Non-crossing partitions

A *partition* of $[n] = \{1,2,\dots,n\}$ is (as the name suggests) a collection of mutually disjoint nonempty subsets $B_1, \dots, B_r$ of $[n]$ such that $B_1 \sqcup \cdots \sqcup B_r = [n]$. The subsets are called the *blocks* of the partition. By convention we order the blocks by their least elements; i.e., $\min B_i < \min B_j$ iff $i < j$. If each block consists of two elements, then we call the partition a *pairing*. The set of all partitions on $[n]$ is denoted $\mathscr{P}(n)$, and the subset of all pairings is $\mathscr{P}_2(n)$.

Definition 2.6. Let $\pi \in \mathscr{P}(n)$ be a partition of $[n]$. We say π has a *crossing* if there are two distinct blocks B_1, B_2 in π with elements $x_1, y_1 \in B_1$ and $x_2, y_2 \in B_2$ such that $x_1 < x_2 < y_1 < y_2$.

If $\pi \in \mathscr{P}(n)$ has no crossings, it is said to be a *non-crossing partition*. The set of non-crossing partitions of $[n]$ is denoted $NC(n)$. The subset of non-crossing pairings is denoted $NC_2(n)$.

Definition 2.7. Let $n_1,\dots,n_r$ be positive integers with $n = n_1 + \cdots + n_r$. The set $[n]$ is then partitioned accordingly as $[n] = B_1 \sqcup \cdots \sqcup B_r$ where $B_1 = \{1,\dots,n_1\}$, $B_2 = \{n_1+1,\dots,n_1+n_2\}$, and so forth through $B_r = \{n_1+\cdots+n_{r-1}+1,\dots,n_1+\cdots+n_r\}$. Denote this partition as $n_1 \otimes \cdots \otimes n_r$.

We say that a pairing $\pi \in \mathscr{P}_2(n)$ *respects* $n_1 \otimes \cdots \otimes n_r$ if no block of π contains more than one element from any given block of $n_1 \otimes \cdots \otimes n_r$. The set of such respectful pairings is denoted $\mathscr{P}_2(n_1 \otimes \cdots \otimes n_r)$. The set of non-crossing pairings that respect $n_1 \otimes \cdots \otimes n_r$ is denoted $NC_2(n_1 \otimes \cdots \otimes n_r)$.

Definition 2.8. Let $n_1, \ldots, n_r$ be positive integers, and let $\pi \in \mathscr{P}_2(n_1 \otimes \cdots \otimes n_r)$. Let B_1, B_2 be two blocks in $n_1 \otimes \cdots \otimes n_r$. Say that π *links* B_1 *and* B_2 if there is a block $\{i, j\} \in \pi$ such that $i \in B_1$ and $j \in B_2$.

Define a graph C_π whose vertices are the blocks of $n_1 \otimes \cdots \otimes n_r$; C_π has an edge between B_1 and B_2 iff π links B_1 and B_2. Say that π *is connected* with respect to $n_1 \otimes \cdots \otimes n_r$ (or that π *connects the blocks of* $n_1 \otimes \cdots \otimes n_r$) if the graph C_π is connected. We shall denote by $NC_2^c(n_1 \otimes \cdots \otimes n_r)$ the set of all non-crossing pairings that both respect and connect $n_1 \otimes \cdots \otimes n_r$.

Definition 2.9. Let n be an even integer, and let $\pi \in \mathscr{P}_2(n)$. Let $f\colon \mathbb{R}_+^n \to \mathbb{C}$ be measurable. The *pairing integral* of f with respect to π, denoted $\int_\pi f$, is defined (when it exists) to be the constant

$$\int_\pi f = \int f(t_1, \ldots, t_n) \prod\nolimits_{\{i,j\} \in \pi} \delta(t_i - t_j)\, dt_1 \cdots dt_n.$$

We finally introduce the notion of a semicircular family (see, e.g., [7, Definition 8.15]).

Definition 2.10. Let $d \geq 2$ be an integer, and let $c = \{c(i, j) : i, j = 1, \ldots, d\}$ be a positive definite symmetric matrix. A d-dimensional vector $(s_1, \ldots, s_d)$ of random variables in $\mathscr{A}$ is said to be a *semicircular family with covariance* c if for every $n \geq 1$ and every $(i_1, \ldots, i_n) \in [d]^n$

$$\varphi(s_{i_1} s_{i_2} \cdots s_{i_n}) = \sum_{\pi \in NC_2(n)} \prod_{\{a,b\} \in \pi} c(i_a, i_b).$$

The previous relation implies in particular that, for every $i = 1, \ldots, d$, the random variable s_i has the $S(0, c(i, i))$ distribution – see Definition 2.2.

For instance, one can rephrase the defining property of the free Brownian motion $S = \{S_t : t \geq 0\}$ by saying that, for every $t_1 < t_2 < \cdots < t_d$, the vector $(S_{t_1}, S_{t_2} - S_{t_1}, \ldots, S_{t_d} - S_{t_{d-1}})$ is a semicircular family with a diagonal covariance matrix such that $c(i, i) = t_i - t_{i-1}$ (with $t_0 = 0$), $i = 1, \ldots, d$.

3. Proof of the main results

A crucial ingredient in the proof of Theorem 1.3 is the following statement, showing that contractions control all important pairing integrals. This is the generalization of Proposition 2.2. in [4] to our situation.

Proposition 3.1. *Let $d \geq 2$ and $n_1, \ldots, n_d$ be some fixed positive integers. Consider, for each $i = 1, \ldots, d$, sequences of mirror-symmetric functions $(f_k^{(i)})_{k \in \mathbb{N}}$ with $f_k^{(i)} \in L^2(\mathbb{R}_+^{n_i})$, satisfying:*

- *There is a constant $M > 0$ such that $\|f_k^{(i)}\|_{L^2(\mathbb{R}_+^{n_i})} \leq M$ for all $k \in \mathbb{N}$ and all $i = 1, \ldots, d$.*

- *For all $i = 1, \dots, d$ and all $p = 1, \dots, n_i - 1$,*

$$\lim_{k\to\infty} f_k^{(i)} \overset{p}{\frown} f_k^{(i)} = 0 \qquad \text{in} \quad L^2(\mathbb{R}_+^{2n_i-2p}).$$

Let $r \geq 3$, and let π be a connected non-crossing pairing that respects $n_{i_1} \otimes \cdots \otimes n_{i_r}$: $\pi \in NC_2^c(n_{i_1} \otimes \cdots \otimes n_{i_r})$. Then

$$\lim_{k\to\infty} \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} = 0.$$

Proof. In the same way as in [4] one sees that without restriction (i.e., up to a cyclic rotation and relabeling of the indices) one can assume that

$$\int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} = \int_{\pi'} (f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}) \otimes (f_k^{(i_3)} \otimes \cdots \otimes f_k^{(i_r)}),$$

where $0 < 2p < n_{i_1} + n_{i_2}$ and

$$\pi' \in NC_2^c\big((n_{i_1} + n_{i_2} - 2p) \otimes n_{i_3} \otimes \cdots \otimes n_{i_r}\big).$$

Note that $0 < 2p < n_{i_1} + n_{i_2}$ says that $f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}$ is not a trivial contraction (trivial means that either nothing or all arguments are contracted); of course, in the case $n_{i_1} \neq n_{i_2}$ it is allowed that $p = \min(n_{i_1}, n_{i_2})$.

By Lemma 2.1. of [4] we have then

$$\begin{aligned} &\left| \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} \right| \\ &\quad \leq \| f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)} \|_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} \cdot \| f_k^{(i_3)} \|_{L^2(\mathbb{R}_+^{n_{i_3}})} \cdots \| f_k^{(i_r)} \|_{L^2(\mathbb{R}_+^{n_{i_r}})} \\ &\quad \leq \| f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)} \|_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} \cdot M^{r-2}. \end{aligned}$$

Now we only have to observe that, by also using the mirror symmetry of $f_k^{(i_1)}$ and $f_k^{(i_2)}$, we have

$$\begin{aligned} \| f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)} \|^2_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} &= \left\langle f_k^{(i_1)} \overset{n_{i_1}-p}{\frown} f_k^{(i_1)}, f_k^{(i_2)} \overset{n_{i_2}-p}{\frown} f_k^{(i_2)} \right\rangle_{L^2(\mathbb{R}_+^{2p})} \\ \leq \| f_k^{(i_1)} \overset{n_{i_1}-p}{\frown} f_k^{(i_1)} \|_{L^2(\mathbb{R}_+^{2p})} &\cdot \| f_k^{(i_2)} \overset{n_{i_2}-p}{\frown} f_k^{(i_2)} \|_{L^2(\mathbb{R}_+^{2p})}. \end{aligned}$$

According to our assumption we have, for each $i = 1, \dots, d$ and each $q = 1, \dots, n_i - 1$, that

$$\lim_{k\to\infty} f_k^{(i)} \overset{q}{\frown} f_k^{(i)} = 0 \qquad \text{in} \qquad L^2(\mathbb{R}_+^{2n_i-2q}).$$

Since now at least one of the two contractions $\overset{n_{i_1}-p}{\frown}$ and $\overset{n_{i_2}-p}{\frown}$ is non-trivial, we can choose either $q = n_{i_1} - p$, $i = i_1$ or $q = n_{i_2} - p$, $i = i_2$ in the above, and this implies that

$$\lim_{k\to\infty} \| f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)} \|_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} = 0,$$

which gives our claim. □

We can now provide a complete proof of Theorem 1.3.

Proof of Theorem 1.3. The equivalence between (2) and (3) follows from [4]. Clearly, (1) implies (3), so we only have the prove the reverse implication. So let us assume (3). Note that, by Theorem 1.6 of [4], this is equivalent to the fact that all non-trivial contractions of $f_k^{(i)}$ converge to 0; i.e., for each $i = 1, \ldots, d$ and each $q = 1, \ldots, n_i - 1$ we have

$$\lim_{k\to\infty} f_k^{(i)} \overset{q}{\frown} f_k^{(i)} = 0 \qquad \text{in} \qquad L^2(\mathbb{R}_+^{2n_i-2q}). \tag{3.1}$$

We will use statement (3) in this form. In order to show (1), we have to show that any moment in the variables $I(f_k^{(1)}), \ldots, I(f_k^{(d)})$ converges, as $k \to \infty$, to the corresponding moment in the semicircular variables $s_1, \ldots, s_d$. So, for $r \in \mathbb{N}$ and positive integers $i_1, \ldots, i_r$, we consider the moments

$$\varphi\left[I^S(f_k^{(i_1)}) \cdots I^S(f_k^{(i_r)})\right].$$

We have to show that they converge, for $k \to \infty$, to the corresponding moment $\varphi(s_{i_1} \cdots s_{i_r})$. Note that our assumption (1.1) says that

$$\lim_{k\to\infty} \varphi[I^S(f_k^{(i)}) I^S(f_k^{(j)})] = c(i,j) = \varphi(s_i s_j).$$

By Proposition 1.38 in [4] we have

$$\varphi\left[I^S(f_k^{(i_1)}) \cdots I^S(f_k^{(i_r)})\right] = \sum_{\pi \in NC_2(n_{i_1} \otimes \cdots \otimes n_{i_r})} \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)}.$$

By Remark 1.33 in [4], any $\pi \in NC_2(n_{i_1} \otimes \cdots \otimes n_{i_r})$ can be uniquely decomposed into a disjoint union of connected pairings $\pi = \pi_1 \sqcup \cdots \sqcup \pi_m$ with $\pi_q \in NC_2^c(\bigotimes_{j\in I_q} n_{i_j})$, where $\{1, \ldots, r\} = I_1 \sqcup \cdots \sqcup I_m$ is a partition of the index set $\{1, \ldots, r\}$. The above integral with respect to π factors then accordingly into

$$\int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} = \prod_{q=1}^m \int_{\pi_q} \bigotimes_{j\in I_q} f_k^{(i_j)}.$$

Consider now one of those factors, corresponding to π_q. Since π_q must respect $\bigotimes_{j\in I_q} n_{i_j}$, the number $r_q := \# I_q$ must be strictly greater than 1. On the other hand, if $r_q \geq 3$, then, from (3.1) and Proposition 3.1, it follows that the corresponding pairing integral $\int_{\pi_q} \bigotimes_{j\in I_q} f_k^{(i_j)}$ converges to 0 in L^2. Thus, in the limit, only those π make a contribution, for which all r_q are equal to 2, i.e., where each of the π_q in the decomposition of π corresponds to a complete contraction between two of the appearing functions. Let $NC_2^2(n_{i_1} \otimes \cdots \otimes n_{i_r})$ denote the set of those pairings π. So we get

$$\lim_{k\to\infty} \varphi\left[I(f_k^{(i_1)}) \cdots I(f_k^{(i_r)})\right] = \sum_{\pi \in NC_2^2(n_{i_1} \otimes \cdots \otimes n_{i_r})} \lim_{k\to\infty} \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)},$$

We continue as in [4]: each $\pi \in NC_2^2(n_{i_1} \otimes \cdots \otimes n_{i_r})$ is in bijection with a non-crossing pairing $\sigma \in NC_2(r)$. The contribution of such a π is the product of the complete contractions for each pair of the corresponding $\sigma \in NC_2(r)$; but the complete contraction is just the L^2 inner product between the paired functions, i.e.,

$$\lim_{k\to\infty} \varphi\left[I^S(f_k^{(i_1)}) \cdots I^S(f_k^{(i_r)})\right] = \sum_{\sigma\in NC_2(r)} \prod_{\{s,t\}\in\sigma} c(i_s, i_t).$$

This is exactly the moment $\varphi(s_{i_1} \cdots s_{i_r})$ of a semicircular family $(s_1, \ldots, s_d)$ with covariance matrix c, and the proof is concluded. □

We conclude this paper with the proof of Theorem 1.6.

Proof of Theorem 1.6. Point (1) is a simple consequence of the Wigner isometry (2.3) (since each $f_k^{(i)}$ is fully symmetric, $f_k^{(i)}$ is in particular mirror-symmetric), together with the classical Wiener isometry which states that

$$E[I^W(f)I^W(g)] = \mathbf{1}_{n=m} \times n!\langle f, g\rangle_{L^2(\mathbb{R}_+^n)}$$

for every $n, m \geq 0$, and every $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$. For point (2), we observe first that the case $d = 1$ is already known, as it corresponds to [4, Theorem 1.8]. Consider now the case $d \geq 2$. Let us suppose that $\left(I^S(f_k^{(1)}), \ldots, I^S(f_k^{(d)})\right) \overset{\text{law}}{\to} (s_1, \ldots, s_d)$. In particular, $I^S(f_k^{(i)}) \overset{\text{law}}{\to} s_i$ for all $i = 1, \ldots, d$. By [4, Theorem 1.8] (case $d = 1$), this implies that $I^W(f_k^{(i)}) \overset{\text{law}}{\to} \sqrt{(n_i)!}N_i$. Since the asymptotic relations in (1) are verified, Theorem 1.1**(B)** leads then to $\left(I^W(f_k^{(1)}), \ldots, I^W(f_k^{(d)})\right) \overset{\text{law}}{\to} \left(\sqrt{(n_i)!}N_1, \ldots, \sqrt{(n_d)!}N_d\right)$, which is the desired conclusion. The converse implication follows exactly the same lines,and the proof is concluded. □

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Malliavin Calculus for Stochastic Point Vortex and Lagrangian Models

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Abstract. We explore the properties of solutions of two stochastic fluid models for viscous flow in two dimensions. We establish the absolute continuity of the law of the corresponding solution using Malliavin calculus.

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1. Introduction

The solution of the two-dimensional Cauchy problem for the Euler equation was first studied by Yudovich [21], Golovkin [8] and Kato [9]. In [9], the global existence and uniqueness of a solution was shown, and the quasi-Lipchitz condition of the velocity field was explored. Properties for flows with irregular velocity coefficient were also discussed in Le Bris and Lions [11], where the existence and uniqueness of the solution for a transport equation with partially Sobolev $W^{1,1}$ coefficient were proved. In Cao and He [3], the well-posedness for a stochastic differential equation (SDE) with quasi-Lipschitz coefficients was established. Fang and Zhang [7] proved the uniqueness of a strong solution of a SDE with non-Lipschitz coefficients relaxed by a logarithmic factor using the Gronwall's inequality. The continuous and homeomorphic flow properties of SDEs with non-Lipschitz coefficients were studied by Ren and Zhang [18] and Zhang [22].

On the other hand, the stochastic calculus of variation developed by Malliavin [12] is a useful tool for the study of absolute continuity of the law and smoothness of densities associated to solutions of SDEs. Most of the literature on this topic deals with continuously differentiable and global Lipschitz coefficients. In two-

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dimensional vortex flows, however, the velocity field only satisfies a quasi-Lipschitz condition. Recently, Kusuoka [10] discussed the existence of densities of solutions of SDEs with non-Lipschitz coefficients by Malliavin calculus. There are two major differences between [10] and our work. First, in Kusuoka [10], the SDE has Lipschitz drift coefficient and only uniformly bounded noise coefficient, our model instead contains standard noise coefficient but non-Lipschitz drift term as the flow velocity. Second, Kusuoka [10] proved the existence of densities of solutions in a class V_h which is larger than Sobolev spaces. The enlargement of solution spaces allowed the author to show the existence of densities without imposing ellipticity conditions on the noise coefficient.

The goal of this paper is to show the existence of a density function associated to the solution of a stochastic Lagrangian model. The idea is to use a stochastic chain rule for quasi-Lipschitz functions, which can be proved using a L_p estimate on the velocity gradient. This generalized stochastic chain rule extends the result in Nualart [16] to non-Lipschitz functions. The structure of this paper is as follows: in Section 2 we review some fluid dynamics background and introduce two important estimates on the velocity function. In Section 3, two different stochastic fluid models are introduced and their solvability is recalled. The main result on absolute continuity of the law for the solution of the stochastic Lagrangian model is proved in Section 4.

2. Fluid dynamics background

We recall the Navier–Stokes equations (NSEs) for incompressible viscous fluid in $\mathbb{R}^2$:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla p + \nu\nabla^2\mathbf{u}, \\ \nabla\cdot\mathbf{u} &= 0. \end{aligned} \tag{2.1}$$

Here $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t))$ is the velocity field, p is the pressure field and ν is the coefficient of kinematic viscosity. In 2-D, the vorticity equation is obtained by taking curl of (2.1).

$$\begin{aligned} \frac{\partial \omega}{\partial t} + (\mathbf{u}\cdot\nabla)\omega &= \nu\nabla^2\omega, \\ \mathbf{u}(x,t) &= (K*\omega)(x,t). \end{aligned} \tag{2.2}$$

Here ω denotes the vorticity field. Formula $\mathbf{u} = K*\omega$ is called the Biot–Savart law and can be written explicitly as:

$$\mathbf{u}(x,t) = K*\omega(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} (y_2 - x_2, x_1 - y_1)^T r^{-2}\omega(y,t)dy, \tag{2.3}$$

where $r = |x-y| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$.

Since $\nabla\cdot\mathbf{u} = 0$, (2.2) can be written as

$$\frac{\partial \omega}{\partial t} + \nabla\cdot(\mathbf{u}\omega) = \nu\nabla^2\omega. \tag{2.4}$$

We define A^* as an operator of the form

$$A^*\phi(y) = \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}(\sigma_{ij}\phi) - \sum_i \frac{\partial}{\partial y_i}(u_i\phi), \quad \phi \in C^2, \tag{2.5}$$

where

$$\sigma_{ij} = \begin{cases} \nu & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Equation (2.4) reduces to

$$\frac{d}{dt}\omega(x,t) = A^*\omega(x,t) \qquad \text{for all} \quad x \in \mathbb{R}^2. \tag{2.6}$$

A^* turns out to be the adjoint operator of A defined as

$$Af(y) = \sum_i \nu \frac{\partial^2 f}{\partial y_i^2} + \sum_i u_i(y,t)\frac{\partial f}{\partial y_i}, \qquad f \in C_0^2. \tag{2.7}$$

A is the infinitesimal generator of a diffusion process X_t of the form:

$$dX_t = \mathbf{u}(X_t,t)dt + \sigma_\nu dW_t, \tag{2.8}$$

where $\sigma_\nu = \begin{pmatrix} \sqrt{2\nu} & 0 \\ 0 & \sqrt{2\nu} \end{pmatrix}$ and W_t is the standard Brownian motion in $\mathbb{R}^2$. Marchioro and Pulvirenti [13] used this stochastic formulation to prove the convergence of a Navier–Stokes solution to the corresponding Euler solution in the limit $\nu \to 0$. In addition, equation (2.6) is the forward Kolmogorov equation with density ω associated to the law of solution X_t of (2.8). For studies of (2.8) and Navier–Stokes equations, readers should refer to the work of Chorin [5] and Arnaudon, Cruzeiro and Galamba [1].

Definition 2.1. We say a real-valued function $g(t,\cdot) : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the quasi-Lipschitz condition if

$$[g(t,\cdot)]_\varphi = \sup_{x,y\in\mathbb{R}^2, x\neq y} \frac{|g(t,y) - g(t,x)|}{\varphi(|y-x|)} < \infty \tag{2.9}$$

with

$$\varphi(r) = \begin{cases} 0 & , \quad r = 0, \\ -r\ln r + r & , 0 < r < 1, \\ r & , \quad r \geq 1. \end{cases} \tag{2.10}$$

We denote $C_b(\mathbb{R}^2)$ as the space of all uniformly bounded, continuous functions in $\mathbb{R}^2$, equipped with the norm

$$||g||_0 = \sup_{z\in\mathbb{R}^2} |g(z)|. \tag{2.11}$$

Lemma 2.2. *Given $T > 0$, the following inequalities hold*

1. *If $\omega(\cdot,t) \in C_b(\mathbb{R}^2)$, then*

$$[u_i]_\varphi = [K_i * \omega]_\varphi \leq c_1||\omega||_0. \quad \textit{for any} \quad t \in [0,T] \tag{2.12}$$

2. *If* $\omega(\cdot,t) \in \mathrm{L}^p(\mathbb{R}^2)$*,* $p > 1$*, then*

$$||\nabla u_i||_{\mathrm{L}^p} = ||\nabla(K_i * \omega)||_{\mathrm{L}^p} \leq c_p||\omega||_{\mathrm{L}^p}, \quad \textit{for any} \quad t \in [0,T] \tag{2.13}$$

where $i = 1, 2$*.* c_1 *and* c_p *are constants independent of* ω*.*

Proof. The proof of (2.12) can be found in Rautmann ([17], Proposition 2.1), where Kato's result [9] is extended to functions in the whole domain $\mathbb{R}^2$. Inequality (2.13) is proved by Chemin ([4], Theorem 3.1.1). □

3. Stochastic point vortex and lagrangian models

In this section, two stochastic fluid models originated from (2.8) are introduced. We first consider a model where the velocity is regularized according to the point vortex approximation. The second model describes the Lagrangian transport of a particle perturbed by the Gaussian noise, with drift presented by the flow velocity.

An approximation of the Biot–Savart law can be obtained by regularizing its singular kernel K. For $x = (x_1, x_2)$, denoting $d = |x| = \sqrt{x_1^2 + x_2^2}$ and considering a cutoff function

$$\zeta(x) = \bar{\zeta}(|x|) = \frac{1}{\pi}\exp(-r^2), \tag{3.1}$$

we can define

$$\zeta_\epsilon(x) = \epsilon^{-2}\zeta(\frac{x}{\epsilon}), \quad \text{for} \quad \epsilon > 0 \tag{3.2}$$

and compute

$$K_\epsilon(x) = K * \zeta_\epsilon = \frac{1}{2\pi r^2}(-x_2, x_1)\{1 - \exp(-r^2/\epsilon^2)\}. \tag{3.3}$$

For each fixed $\epsilon > 0$, it was shown that K_ϵ is bounded and globally Lipschitz (see Sritharan and Xu [19]).

Following Chorin [6], we can write the regularized velocity field in terms of its point vortex approximation as

$$\mathbf{u}_{\epsilon,t}(x) = \sum_{j=1}^{N} \alpha_j K_\epsilon(x - X^j(t)), \qquad \forall x \in \mathrm{R}^2. \tag{3.4}$$

Here N is the number of point vortices in the fluid, $X^j(t)$ represents the position of jth point at time t and α_j denotes the vorticity intensity for jth point. (3.4) is derived from the Biot–Savart law for vorticity field with the form

$$\omega(x,t) = \sum_{j=1}^{N} \alpha_j \delta_{X^j(t)}. \tag{3.5}$$

Given a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a σ-filtration $\{\mathcal{F}_t\}_{t\geq 0}$, we denote the initial position for ith point vortex as a random variable ξ^i, and introduce a stochastic point vortex model that is described by the following SDE:

$$dX^i(t) = \mathbf{u}_{\epsilon,t}(X^i(t))dt + \sigma(X^i(t))dW_t, \quad \text{for} \quad i = 1, \cdots, N. \tag{3.6}$$

Here W_t is the two-dimensional Brownian motion and $\sigma : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ is the multiplicative noise coefficient.

Before stating the theorem on the unique solvability of (3.6), we first define the diffusion matrix for the noise coefficient σ:

$$a(x,y) = \sigma(x)\sigma(y)^T, \tag{3.7}$$

and the associated seminorm

$$||a|| = \sum_{i=1}^{2} |a_i^i|. \tag{3.8}$$

Theorem 3.1. *Assume the diffusion matrix satisfies the following Lipschitz type and growth conditions:*

$$||a(y_1,y_1) - 2a(y_1,y_2) + a(y_2,y_2)|| \le K|y_1 - y_2|^2. \tag{3.9}$$

Then there exists a unique solution $X = (X(t), t \ge 0)$ to the SDE (3.6) with initial condition $X(0) = \xi$ and the process X is adapted.

Proof. See Sritharan and Xu [19]. □

Our second model describes the trajectory of one fluid particle perturbed by multiplicative Gaussian noise:

$$dX_i(t) = u_i(X(t),t)dt + \sum_{j=1}^{2} \sigma_i^j(X(t))dW_t^j \quad i = 1,2, \tag{3.10}$$

where X_i, $i = 1, 2$ denotes the coordinate of the fluid particle position. Equation (3.10) is a system of scalar-valued SDEs with the Navier–Stokes fluid velocity as the drift coefficient. We note here that this idea can be naturally applied to N particles [19] for numerical approximations.

Mikulevicius and Rozovskii [15] analyzed a similar model to (3.10) driven by the Stratonovich noise. The well-posedness of (3.10) was studied in Sritharan and Xu [20]. Here we include their result (Theorem 2.2, [20]) for completeness.

Theorem 3.2. *Suppose $\omega(\cdot,t) \in C_b(\mathbb{R}^2)$ for any $t \in [0,T]$ and the following Lipschitz-type and growth conditions on the diffusion coefficient hold:*

$$||a(y_1,y_1) - 2a(y_1,y_2) + a(y_2,y_2)|| \le K_1|y_1 - y_2|, \tag{3.11}$$

and

$$||a(y,y)|| \le K_2(1 + |y|). \tag{3.12}$$

Then there exists a unique solution $X = (X(t), 0 \le t \le T)$ to the SDE (3.10) with initial conditions satisfying $E(|X(0)|^2) < \infty$. Moreover, we have

$$E(|X(t)|^2) < \infty \quad \text{for any} \quad t \in [0,T]. \tag{3.13}$$

Proof. See Sritharan and Xu [20]. □

4. Malliavin calculus and absolute continuity of law

In this section, we briefly review some concepts and notations from the Malliavin calculus. Interested readers should refer to Nualart [16] for details.

Definition 4.1. Given H is a real separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_H$, we say a stochastic process $W = \{W(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$ is an isonormal Gaussian process if W is a centered Gaussian family of random variables, s.t.

$$E(W(h)W(g)) = \langle h, g \rangle_H \quad \text{for any} \quad h, g \in H. \tag{4.1}$$

We denote by $C_p^\infty(\mathbb{R}^m)$ the set of all infinitely continuously differentiable functions $f : \mathbb{R}^m \to \mathbb{R}$ such that f and all its partial derivatives have polynomial growth. Let $\mathcal{S}$ denote the class of smooth random variables such that a random variable $F \in \mathcal{S}$ has the form

$$F = f(W(h_1), \ldots, W(h_n)), \tag{4.2}$$

where f belongs to $C_p^\infty(\mathbb{R}^m)$, $h_1, \ldots, h_n$ are in H, and $n \geq 1$.

Definition 4.2. The Malliavin derivative of a smooth random variable F of the form (4.2) is the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), \ldots, W(h_n))h_i. \tag{4.3}$$

For any $p \geq 1$ we denote the domain of D in $\mathrm{L}^p(\Omega)$ by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$||F||_{1,p} = [E(|F|^p) + E(||DF||_H^p)]^{\frac{1}{p}}. \tag{4.4}$$

The closure property of the operator D from $D^{1,2}$ to L^2 is stated below.

Lemma 4.3. *Let $\{F_n, n \geq 1\}$ be a sequence of random variables in $\mathbb{D}^{1,2}$ that converges to F in $\mathrm{L}^2(\Omega)$ and such that*

$$\sup_n E(||DF_n||_H^2) < \infty. \tag{4.5}$$

Then F belongs to $\mathbb{D}^{1,2}$, and the sequence of derivatives $\{DF_n, n \geq 1\}$ converges to DF in the weak topology of $\mathrm{L}^2(\Omega; H)$.

Higher-order Malliavin derivatives can be defined similarly and adapted from lower-order derivatives.

Proposition 4.4. *Let F be a random variable in $\mathbb{D}^{k,\alpha}$ with $\alpha > 1$. Suppose that D^iF belongs to $\mathrm{L}^p(\Omega; H^{\otimes i})$ for $i = 0, 1, \ldots, k$ and for some $p > \alpha$. Then $F \in \mathbb{D}^{k,p}$, and there exists a sequence $G_n \in \mathcal{S}$ that converges to F in the norm $|| \cdot ||_{k,p}$.*

Let $\mathbb{L}^{1,2}$ be the class of processes $F \in \mathrm{L}^2(T \times \Omega)$ such that $F(t) \in \mathbb{D}^{1,2}$ for almost all t. It is a Hilbert space with the norm

$$||F||^2_{\mathbb{L}^2(T)} = ||F||^2_{\mathrm{L}^2(T\times\Omega)} + ||DF||^2_{\mathrm{L}^2(T\times\Omega)}. \tag{4.6}$$

Note that $\mathbb{L}^{1,2}$ is isomorphic to $\mathrm{L}^2(T;\mathbb{D}^{1,2})$.

Given the identification between $\mathrm{L}^2(\Omega;H)$ and $\mathrm{L}^2(T \times \Omega)$, we may denote $\{D_tF, t \in [0,T]\}$ as the stochastic process of the derivative of a random variable $F \in \mathbb{D}^{1,2}$. The following lemma shows that an Itô integral is differentiable if and only if its integrand is differentiable.

Lemma 4.5. *Let $W = \{W(t), t \in [0,1]\}$ be a one-dimensional Brownian motion. Consider a square integrable adapted process $F = \{F_t, t \in [0,1]\}$, and set $I_t = \int_0^t F_s dW_s$. Then the process F belongs to the space $\mathbb{L}^{1,2}$ if and only if I belongs to $\mathbb{D}^{1,2}$. In this case the process I belongs to $\mathbb{L}^{1,2}$, and we have*

$$D_rI_t = F_r + \int_r^t D_rF_s dW_s. \tag{4.7}$$

for all $r < t \in [0,1]$.

The stochastic chain rule for global Lipschitz functions is given below and its proof can be found in Nualart [16].

Proposition 4.6. *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F^1,\dots,F^m)$ is a random vector whose components belong to the space $\mathbb{D}^{1,p}$. Then $\varphi(F) \in \mathbb{D}^{1,p}$, $p \geq 1$, and*

$$D(\varphi(F)) = \sum_{i=1}^m \partial_i\varphi(F)DF^i. \tag{4.8}$$

Proposition 4.6 is crucial in obtaining the absolute continuity of the law for our models. Due to the point vortex approximation, the stochastic point vortex model (3.6) contains global Lipschitz coefficients and the following theorem can be applied directly. For the stochastic Lagrangian model (3.10), the drift coefficient is quasi-Lipschitz and thus Proposition 4.6 cannot be applied directly. We overcome this difficulty by applying (2.13) to derive an identity similar to (4.8) for the non-Lipschitz drift.

Theorem 4.7. *Let $F = (F^1,\dots,F^m)$ be a random vector satisfying the following conditions:*

1. *F^i belongs to the space $\mathbb{D}^{1,2}$, for all $i = 1,\dots,m$.*
2. *The matrix $\gamma_F = (\langle DF^i, DF^j\rangle)_{1\leq i,j\leq m}$ is invertible a.s.*

Then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$.

This criteria for absolute continuity of the law of a random variable is given in Bouleau and Hirsch [2]. Its proof relies on the properties of approximate derivatives.

For the stochastic Lagrangian model, we first show that the unique solution X_i of (3.10) satisfies the first condition of Theorem 4.7. In other words, we prove that the solution is first-order Malliavin differentiable and its Malliavin derivative satisfies a linear SDE. We extend results in Nualart [16] to SDEs with non-Lipschitz coefficients, in particular, to the stochastic Lagrangian equation (3.10) with quasi-Lipschitz drift coefficient $\mathbf{u}$.

Theorem 4.8. *Assume $\omega(\cdot,t) \in \mathrm{L}^p(\mathbb{R}^2)$ for any $p > 1$ and $t \in [0,T]$. Suppose that $X = \{X(t), t \in [0,T]\}$ is the unique solution to equation (3.10), where the diffusion coefficient is globally Lipschitz. Then $X_i(t)$ belongs to $\mathbb{D}^{1,2}$ for any $t \in [0,T]$ and $i = 1,2$. Moreover,*

$$\sup_{0\le r\le t} E(\sup_{r\le s\le T} |D_r X_i(s)|^2) < \infty, \tag{4.9}$$

and the derivative $D_r X_i(t)$ satisfies the following linear equation:

$$\begin{aligned} D_r X_i(t) = \sigma_i^j(X(r)) &+ \int_r^t \bar{\sigma}_{i,l}^j(X(s)) D_r(X_l(s)) dW_s^j \\ &+ \int_r^t \bar{u}_i^l(X(s)) D_r X_l(s) ds \end{aligned} \tag{4.10}$$

for $r \le t$ a.e., where $\bar{\sigma}_{i,l}^j(X(s))$ and $\bar{u}_i^l(X(s))$ are uniformly bounded and adapted one-dimensional processes for $l = 1,2$. Consequently, $D_r X_i$ is the unique adapted solution to equation (4.10).

Proof. Consider the Picard approximations given by

$$X^0(t) = x_0, \tag{4.11}$$

$$X^{n+1}(t) = x_0 + \int_0^t \sigma^j(X^n(s)) dW_s^j + \int_0^t \mathbf{u}(X^n(s), s) ds, \quad n \ge 0, \tag{4.12}$$

where $\sigma^j = (\sigma_1^j, \sigma_2^j)$. From Theorem 3.2 we know that

$$E(\sup_{s\le T} |X^n(s) - X(s)|^2) \to 0 \tag{4.13}$$

as n tends to infinity. We assume that for $i = 1,2$, $X_i^n \in \mathbb{D}^{1,2}$.

Let $\psi_n(x) \in C_o^\infty(\mathbb{R}^2)$ be a sequence of regularization kernels with compact support such that $\int_{\mathbb{R}^m} \psi_n(x) dx = 1$. Set $\mathbf{u}^m = \mathbf{u} * \psi_m$. The sequence $\{\mathbf{u}^m\}$ consists of functions with infinite differentiability, and it is obvious that $\lim_{m\to\infty} \mathbf{u}^m(x) = \mathbf{u}(x)$ for any $x \in \mathbb{R}^2$. Therefore u_i^m converges to u_i in $\mathrm{L}^2(\Omega)$ as m tends to infinity. In addition, by (2.13) in Lemma 2.2, we can show that $|\nabla u_i^m|$, $i = 1,2$ is bounded. In fact,

$$\begin{aligned} |\nabla u_i^m| &= |\nabla(u_i * \psi_m)| = |\nabla u_i * \psi_m| \\ &\le ||\nabla u_i||_{\mathrm{L}^p} ||\psi_m||_{\mathrm{L}^q} \le c||\omega||_{\mathrm{L}^p} < C. \end{aligned} \tag{4.14}$$

Proposition 4.6 tells us that for each n,

$$D(u_i^m(X^n(s), s)) = \sum_{j=1}^2 \partial_j u_i^m(X^n(s)) D X_j^n(s) \tag{4.15}$$

and $u_i^m(X^n(s),s) \in \mathbb{D}^{1,2}$. On the other hand, since $|\partial_i u_i^m(X^n(s),s)| < C$ and $DX_i^n(s) \in \mathrm{L}^2$ by the assumption, $\{D(u_i^m(X^n(s),s)), m \geq 1\}$ is bounded in $\mathrm{L}^2(\Omega;H)$. By Lemma 4.3, $u_i(X^n(s),s) \in \mathbb{D}^{1,2}$ and $\{D(u_i^m(X^n(s),s)), m \geq 1\}$ converges in the weak topology of $\mathrm{L}^2(\Omega;H)$ to $D(u_i(X^n(s),s))$.

(4.14) tells us that $\{\nabla u_i^m(X^n(s),s), m \geq 1\}$ is bounded by C. Hence, there exists some random vector $\bar{\mathbf{u}}_i(X^n(s)) = (\bar{u}_i^1(X^n(s)), \bar{u}_i^2(X^n(s)))$ as the weak limit of $\{\nabla u_i^m(X^n(s),s), m \geq 1\}$ such that, after taking the limit in (4.15), $u_i(X^n(s),s)$ belongs to $\mathbb{D}^{1,2}$ and

$$D_r u_i(X^n(s),s) = \bar{u}_i^l(X^n(s)) D(X_l^n(s)) \quad for \quad r \leq s. \tag{4.16}$$

A direct application of Proposition 4.6 implies that there exists bounded random variable $\bar{\sigma}_i^j(X^n(s))$, such that

$$D_r[\sigma_i^j(X^n(s))] = \bar{\sigma}_{i,l}^j(X^n(s)) D_r(X_l^n(s)), \quad for \quad r \leq s. \tag{4.17}$$

Since $\int_0^t u_i(s, X^n(s), s)ds$ and $\int_0^t \sigma_i^j(X^n(s))dW_s^j$ belong to $\mathbb{D}^{1,2}$,

$$D_r\left[\int_0^t u_i(X^n(s),s)ds\right] = \int_r^t D_r[u_i(X^n(s),s)]ds, \tag{4.18}$$

and by Lemma 4.5,

$$D_r\left[\int_0^t \sigma_i^j(X^n(s))dW_s^j\right] = \sigma_i^l(X^n(r)) + \int_r^t D_r[\sigma_i^j(X^n(s))\Big] dW_s^j. \tag{4.19}$$

for any $r \leq t$.

From equalities (4.18), (4.19) and (4.12), $X_i^{n+1}(t) \in \mathbb{D}^{1,2}$ for all $t \in [0,T]$, and we obtain

$$E(\sup_{r \leq s \leq t} |D_r X_i^{n+1}(s)|^2) \leq c[\gamma + TK^2 \int_r^t E(|D_r X_i^n(s)|^2)ds], \tag{4.20}$$

where

$$\gamma = \sup_{n,i} E\left(\sup_{0 \leq t \leq T} |X_i^n(t)|^2\right) < \infty. \tag{4.21}$$

Applying Gronwall's lemma to (4.20), we prove that derivatives of the sequence $X_i^n(t)$ are bounded in $\mathrm{L}^2(\Omega;H)$ uniformly in n by induction. Since (4.13) and Lemma 4.3, we could apply the Malliavin derivative D to equation (3.10) and derive the linear stochastic differential equation (4.10) for the derivative of $X_i(t)$. □

For SDEs with global Lipschitz coefficients of linear growth, Nualart ([16], Section 2.3.1) showed that if

$$\eta^T a(X_s)\eta \geq \lambda(s)|\eta|^2, \quad for\ any \quad \eta \in \mathbb{R}^2, \tag{4.22}$$

where $\lambda(s) > 0$, then the Malliavin matrix for the corresponding solution is invertible a.s. Under the same non-degenerate assumption (4.22), Theorem 4.7 and 4.8 imply the absolute continuity of law for the solution of (3.10).

Theorem 4.9. *Assume* $\omega(\cdot,t) \in C_b(\mathbb{R}^2) \cap \mathrm{L}^p(\mathbb{R}^2)$ *for any* $p > 1$ *and* $t \in [0,T]$. *Let* $\{X(t), t \in [0,T]\}$ *be the solution of the stochastic differential equation* (3.10). *Assume the diffusion coefficient is globally Lipschitz and the diffusion matrix* a^{ij} *satisfies* (4.22). *Then for any* $0 < t \leq T$ *the law of* $X(t)$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^2$, *thus there exists a density associated to the solution* $\{X(t), t \in [0,T]\}$.

Proof. See Theorem 2.3.1 in Nualart [16]. □

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Two Remarks on the Wasserstein Dirichlet Form

Wilhelm Stannat

Abstract. The Wasserstein diffusion is an Ornstein–Uhlenbeck type process on the set of all probability measures with the Wasserstein metric as intrinsic metric. Sturm and von Renesse constructed in [6] this process in the case of probability measures over the unit interval using Dirichlet form theory. An essential step in this construction is the closability of a certain gradient form, defined for smooth cylindrical test functions, in the space L^2 w.r.t. the entropic measure $\mathbb{Q}_\beta$. In this paper we will first give an alternative proof for this closability, avoiding the striking, but elaborate integration by parts formula for $\mathbb{Q}_\beta$ used in [6]. Second, we give explicit conditions under which certain finite-dimensional particle approximations introduced in the paper [1] by Andres and von Renesse do converge in the resolvent sense to the Wasserstein diffusion, a question that was left open in the above cited paper.

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1. The Wasserstein Dirichlet form

Let $I := [0,1]$ be the unit interval and $\mathcal{M}_1(I)$ the space of all probability measures on the Borel σ-algebra $\mathcal{B}(I)$ equipped with the weak topology. Recall that the quadratic Wasserstein distance

$$d_W(\mu,\nu) := \inf_{\gamma\in\mathcal{M}_1(\mu,\nu)} \left(\int_I\int_I |x-y|^2\,\gamma(dx,dy)\right)^{\frac{1}{2}}$$

where $\mathcal{M}_1(\mu,\nu)$ denotes the set of all couplings of μ and ν.

Let

$$\mathcal{G} = \{g : I \to I \mid g \text{ right-cont., non-decreas.}\}$$

be the set of monotone right-continuous transformations on I, considered as a subset of $L^2(I)$, and for $g \in \mathcal{G}$ let μ_g be the probability measure defined by

$$\int_I f\, d\mu_g := \int_I f(g(t))\, dt\,, f \in \mathcal{B}_+(I).$$

It is well known that the transformation

$$\Phi : \mathcal{G} \to \mathcal{M}_1(I)\,, g \mapsto \mu_g$$

defines an isometry between the spaces $\mathcal{G}$ and $\mathcal{M}_1(I)$.

The Wasserstein diffusion constructed by Sturm and von Renesse can be characterized as a diffusion process $(\mathbb{X}_t)_{t\geq 0}$ on $\mathcal{G}$ for which the following process

$$\begin{aligned} M_T^f := & \int_0^1 f(\mathbb{X}_T(t))\, dt - \frac{\beta}{2}\int_0^T \int_0^1 \Delta f(\mathbb{X}_s(t))\, dt\, ds \\ & - \frac{1}{2}\int_0^T \left(\sum_{I \in \text{gaps}\,(\mathbb{X}_s)} \left[\frac{\Delta f(I_-) + \Delta f(I_+)}{2} - \frac{f'(I_+) - f'(I_-)}{|I|} \right] \right. \\ & \left. - \frac{\Delta f(0) + \Delta f(1)}{2} \right) ds \end{aligned}$$

is a continuous martingale with quadratic variation

$$\langle M^f \rangle_T = \int_0^T \int_0^1 f'(\mathbb{X}_s(t))^2\, dt\, ds$$

for all $f \in C^2(I)$ with Neumann boundary conditions $f'(0) = f'(1) = 0$ (see [6]). Here $I = [I_-, I_+]$ is called a gap of the monotone map g if $I_- = g(t_-) < g(t_+) = I_+$ for some $t \in [0,1]$ and $\beta > 0$. The process therefore describes random fluctuations around the Neumann heat flow with intrinsic metric given by the Wasserstein distance.

An important feature of the process is that it is reversible w.r.t. the entropic measure $\mathbb{Q}_\beta$ which is charcterized as the unique probability measure on $\mathcal{G}$ having the finite-dimensional distributions

$$\begin{aligned} & \mathbb{Q}_\beta \left(g(t_1) \in dx_1, \dots, g(t_n) \in dx_n \right) \\ & = \frac{\Gamma(\beta)}{\prod_{k=1}^{n+1} \Gamma(\beta(t_k - t_{k-1}))} \prod_{k=1}^{n+1} (x_k - x_{k-1})^{\beta(t_k - t_{k-1}) - 1}\, dx \end{aligned}$$

for $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ on the set

$$\Sigma_n = \{(x_1, \dots, x_n) \in [0,1]^n \mid 0 = x_0 < x_1 < \cdots < x_{n+1} = 1\}\,.$$

The Dirichlet form associated with the Wasserstein diffusion can be obtained as the closure of the following pre-Dirichlet form

$$\mathcal{E}(f) := \int_{\mathcal{G}} \|F'(\cdot, g)\|^2_{L^2(I)}\, d\mathbb{Q}_\beta(g)$$

defined for all F contained in the space

$$\mathcal{F}C_b^1 := \{F(g) = \varphi(\langle f_1, g\rangle, \dots, \langle f_n, g\rangle) \mid n \geq 1, \\ \varphi \in C_b^1(\mathbb{R}^n), f_1, \dots, f_n \in L^2(I)\}$$

and

$$F'(t,g) := \sum_{k=1}^{n} (\partial_k \varphi)(\langle f_1, g\rangle, \dots, \langle f_n, g\rangle) f_k(t)$$

denotes the Fréchet derivative of F, considered as a function on $L^2(I)$. Here, $\langle f, g\rangle = \int_I f(t)g(t)\, dt$.

2. Closability of the Wasserstein Dirichlet form

The crucial step in the construction of the Wasserstein diffusion by von Renesse and Sturm is to prove that $\mathcal{E}$ is closable in $L^2(\mathbb{Q}_\beta)$ which has been achieved in the paper [6] using an explicit, however elaborate, integration by parts formula for the entropic measure $\mathbb{Q}_\beta$. Once closability is shown, it is not so difficult to see that its closure $(\mathcal{E}, D(\mathcal{E}))$ is a local regular Dirichlet form and application of Dirichlet form theory (see [3]) yields the existence of an associated diffusion process in $\mathcal{G}$.

We will use the following alternative representation of $\mathcal{E}$, obtained in Proposition 1.1 of [5], to give an independent proof for the closability of $\mathcal{E}$. Let

$$\Psi : \mathcal{M}_1(I) \to \mathcal{G}\,, \mu \mapsto g_\mu$$

be the usual identification of μ with its (right.-cont.) cumulative distribution function $g_\mu(t) := \mu([0,t])$, $t \in I$. Then $\mathbb{Q}_\beta = \Psi(\pi_{\beta\, dx})$, where $\pi_{\beta dx}$ denotes the Dirichlet measure on $\mathcal{M}_1(I)$ with intensity $\beta\, dx$ (see (2.2) below) and for given $F(\mu) := \varphi(\langle f_1, \mu\rangle, \dots, \langle f_n, \mu\rangle)$ with $f_i \in C_0^1(I)$ let

$$\tilde{F}(g) := \varphi(-\langle f_1', g\rangle, \dots, -\langle f_n', g\rangle).$$

Then

$$\mathcal{E}(\tilde{F}) = \tilde{\mathcal{E}}(F) := \int_{\mathcal{M}_1(I)} \left\| \frac{d}{dx} F'(\cdot, \mu) \right\|_{L^2(I)}^2 \pi_{\beta\, dx}(d\mu). \tag{2.1}$$

The closability of $\mathcal{E}$ in $L^2(\mathbb{Q}_\beta)$ is therefore equivalent to the closability of $\tilde{\mathcal{E}}$ in the space $L^2(\pi_{\beta dx})$.

We will formulate the closability result in a more general framework. To this end let S be a locally compact space, $\mathcal{M}_f(S)$ (resp. $(\mathcal{M}_1(S))$ be the space of all finite (resp. normalized) non-negative Borel measures on S equipped with the weak topology. For given $\nu \in \mathcal{M}_f(S)$ the Dirichlet measure π_ν with intensity ν is the unique probability measure on $\mathcal{M}_1(S)$ satisfying

$$\pi_\nu\left(\mu(A_1) \in dx_1, \dots, \mu(A_n) \in dx_n\right) = \pi_{(\nu(A_1), \dots, \nu(A_n))}(dx_1, \dots, dx_n), \tag{2.2}$$

for any measurable partition $A_1, \dots, A_n$ of S. Here,

$$\pi_q(dx_1, \dots, dx_n) \\ = \frac{\Gamma(q_{1:n})}{\Pi_{i=1}^n \Gamma(q_i)} x_1^{\nu(A_1)^{-1}} \cdot \dots \cdot x_n^{\nu(A_n)^{-1}} \cdot \delta_{(1-x_{1:n-1})}(dx_n) dx_{n-1} \dots dx_1$$

denotes the multivariate Dirichlet distribution on the $n-1$-dimensional simplex $\Delta_{n-1} \subseteq \mathbb{R}_+^n$ for $q \in \overset{\circ}{\mathbb{R}}{}_+^n$. Here, $q_{1:n} = q_1 + \dots + q_n$ and $x_{1:n-1} = x_1 + \dots + x_{n-1}$.

Let

$$\mathcal{D}_0 := \big\{ F(\mu) = \varphi(\mu(A_1), \dots, \mu(A_n)) \mid n \geq 1, \varphi \in C_b^1(\mathbb{R}^n), \\ \{A_1, \dots, A_n\} \text{ measurable partition of } S \big\}$$

be the space of continuously differentiable test functions based on finite measurable partitions of S. For $F \in \mathcal{D}_0$ that admits a representation of the type

$$F(\mu) = \varphi(\mu(A_1), \dots, \mu(A_n))$$

let

$$F'(\cdot, \mu) := \sum_{k=1}^n \varphi_{x_k}(\mu(A_1), \dots, \mu(A_n)) 1_{A_k}$$

be the differential of F at the point μ. Note that $F'(\cdot, \mu) \in L^2(\nu)$ and that

$$\|F'(\cdot, \mu)\|_{L^2(S,\nu)}^2 = \sum_{k=1}^n \varphi_{x_k}^2 (\mu(A_1), \dots, \mu(A_n)) \, \nu(A_k)$$

is bounded and measurable in μ, so that the integral

$$\mathcal{A}(F) := \int_{\mathcal{M}_1(S)} \|F'(\cdot, \mu)\|_{L^2(\nu)}^2 \, \pi_\nu(d\mu)$$

is well defined and finite. In the following we will consider $\mathcal{A}$ as a symmetric bilinear form in the space $L^2(\mathcal{M}_1(S), \pi_\nu)$ with domain $\mathcal{D}_0$. Our first main result is the following

Theorem 2.1. $(\mathcal{A}, \mathcal{D}_0)$ *is closable in* $L^2(\pi_\nu)$.

For the proof of the theorem let us fix an increasing sequence $\mathcal{A}^n = \{A_1^{(n)}, \dots, A_n^{(n)}\}$ of measurable partitions of S generating the Borel σ-algebra, i.e., $\mathcal{B}(S) = \sigma\left(\bigcup_n \mathcal{A}^n\right)$. Denote by

$$\mathcal{D}_0^{(n)} := \{F(\mu) = \varphi(\mu(A_1^{(n)}), \dots, \mu(A_n^{(n)})) : \varphi \in C_b^1(\mathbb{R}^n)\}$$

the set of continuously differentiable test functions based on the fixed partition $\mathcal{A}^n$ and note that $\mathcal{D}_0^{(n)} \subset \mathcal{D}_0^{(n+1)}$ is increasing with n. Let $\mathcal{A}^{(n)}$ be the restriction of $\mathcal{A}$ to $\mathcal{D}_0^{(n)}$. Then for $F(\mu) = \varphi(\mu(A_1^{(n)}), \ldots, \mu(A_n^{(n)})) \in \mathcal{D}_0^{(n)}$ we obtain that

$$\begin{aligned}\mathcal{A}^{(n)}(F) &= \int_{\mathcal{M}_1(S)} \|F'(\mu, \cdot)\|^2_{L^2(S,\nu)} \pi_\nu(d\mu) \\ &= \sum_{i=1}^n \nu(A_i^{(n)}) \int_{\Delta_{n-1}} (\partial_{x_i}\varphi)^2 d\nu^{(n)} =: \mathcal{E}^{(n)}(\varphi),\end{aligned}$$

where $\nu^{(n)} = \pi_{(\nu(A_1^{(n)}),\ldots,\nu(A_n^{(n)}))}$ denotes the Dirichlet distribution with parameters $\nu(A_1^{(n)}), \ldots, \nu^{(n)}(A_n^{(n)})$.

In the following denote by

$$T_n F := E\left(F \mid \mu(A_k^{(n)}), 1 \le k \le n\right), \quad F \in \mathcal{B}_b(\mathcal{M}_1(S))$$

the conditional expectation of F given $\sigma\{\mu(A_k^{(n)}) \mid 1 \le k \le n\}$.

Proposition 2.2. *Let $F \in \mathcal{D}_0^{(n)}$ and $m \le n$. Then $T_m F \in \mathcal{D}_0^{(m)}$ and*

$$\mathcal{A}^{(m)}(T_m F) \le \mathcal{A}^{(n)}(F) = \mathcal{A}(F).$$

The proof requires the following

Lemma 2.3. *Let $F \in \mathcal{D}_0^{(n+1)}$ admit the representation*

$$F(\mu) = \varphi\left(\mu\left(A_1^{(n+1)}\right), \ldots, \mu\left(A_{n+1}^{(n+1)}\right)\right)$$

and assume that

$$A_k^{(n)} = A_k^{(n+1)}, 1 \le k \le n-1, \quad A_n^{(n)} = A_n^{(n+1)} \cup A_{n+1}^{(n+1)}.$$

Then

$$\begin{aligned}&T_n\varphi(x_1, \ldots, x_n) \\ &\quad = \int_0^1 \varphi(x_1, \ldots, x_{n-1}, tx_n, (1-t)x_n)\pi_{\left(\nu\left(A_n^{(n+1)}\right), \nu\left(A_{n+1}^{(n+1)}\right)\right)}(dt)\end{aligned}$$

is a regular conditional expectation of F given $\sigma\{\mu(A_k^{(n)}) \mid 1 \le k \le n\}$, i.e., $T_n\varphi(\mu(A_1^{(n)}), \ldots, \mu(A_n^{(n)}))$ is a version of the conditional expectation T_nF. In particular, $T_nF \in \mathcal{D}_0^{(n)}$ and

$$\mathcal{A}^{(n)}(T_nF) \le \mathcal{A}^{(n+1)}(F) = \mathcal{A}(F).$$

Proof. To simplify the notation, let

$$\tilde{\nu} := \pi_{\left(\nu\left(A_n^{(n+1)}\right), \nu\left(A_{n+1}^{(n+1)}\right)\right)}.$$

Consider the transformation

$$T : \Delta_{n-1} \times [0,1] \to \Delta_n \,, (x,t) \mapsto (x_1, \ldots, x_{n-1}, tx_n, (1-t)x_n).$$

Then it is easy to see that

$$T\left(\nu^{(n)} \otimes \tilde{\nu}\right) = \nu^{(n+1)}.$$

Hence, for $G(\mu) = \psi(\mu(A_1^{(n)}), \ldots, \mu(A_n^{(n)}))$, it follows that

$$\begin{aligned}\int FG\, d\pi_\nu &= \int_{\Delta_n} \varphi(x_1, \ldots, x_{n+1})\psi(x_1, \ldots, x_n + x_{n+1})\nu^{(n+1)}(dx)\\ &= \int_{\Delta_{n-1}} \int_0^1 \varphi(x_1, \ldots, x_{n-1}, tx_n, (1-t)x_n)\tilde{\nu}(dt)\psi(x_1, \ldots, x_n)\nu^{(n)}(dx),\end{aligned}$$

which implies the first assertion.

For the proof of the second assertion first note that $\varphi \in C_b^1(\mathbb{R}^{n+1})$ implies $T_n\varphi \in C_b^1(\mathbb{R}^n)$ with $\partial_{x_i} T_n\varphi = T_n(\partial_i\varphi)$ for $1 \le i \le n-1$ and

$$\begin{aligned}\partial_{x_n} T_n\varphi(x_1, \ldots, x_n) = &\int_0^1 t(\partial_n\varphi)(x_1, \ldots, x_{n-1}, tx_n, (1-t)x_n)\tilde{\nu}(dt)\\ &+ \int_0^1 (1-t)(\partial_{n+1}\varphi)(x_1, \ldots, x_{n-1}, tx_n, (1-t)x_n)\tilde{\nu}(dt).\end{aligned}$$

Consequently,

$$\begin{aligned}&(\partial_{x_n} T_n\varphi)^2(x_1, \ldots, x_n)\\ &\quad\le \int_0^1 t^2\tilde{\nu}(dt)T_n(\partial_n\varphi)^2(x_1, \ldots, x_n) + \int_0^1 (1-t)^2\tilde{\nu}(dt)T_n(\partial_{n+1}\varphi)^2(x_1, \ldots, x_n)\\ &\quad\le \frac{\nu\left(A_n^{(n+1)}\right)}{\nu\left(A_n^{(n)}\right)} T_n(\partial_n\varphi)^2(x_1, \ldots, x_n) + \frac{\nu\left(A_{n+1}^{(n+1)}\right)}{\nu\left(A_n^{(n)}\right)} T_n(\partial_{n+1}\varphi)^2(x_1, \ldots, x_n).\end{aligned}$$

Using $(T_n\partial_i\varphi)^2 \le T_n(\partial_i\varphi)^2$ we can now conclude that

$$\begin{aligned}\mathcal{A}^{(n)}(T_nF) &= \sum_{i=1}^n \nu\left(A_i^{(n)}\right) \int_{\Delta_{n-1}} (\partial_{x_i} T_n\varphi)^2 d\nu^{(n)}\\ &\le \sum_{i=1}^{n-1} \nu\left(A_i^{(n+1)}\right) \int_{\Delta_{n-1}} T_n(\partial_i\varphi)^2 d\nu^{(n)} + \nu\left(A_n^{(n+1)}\right) \int_{\Delta_{n-1}} T_n(\partial_n\varphi)^2 d\nu^{(n)}\\ &\quad + \nu\left(A_{n+1}^{(n+1)}\right) \int_{\Delta_{n-1}} T_n(\partial_{n+1}\varphi)^2 d\nu^{(n)}\\ &= \sum_{i=1}^{n+1} \nu\left(A_i^{(n+1)}\right) \int_{\Delta_n} (\partial_i\varphi)^2 d\nu^{(n)} = \mathcal{A}^{(n+1)}(F).\end{aligned}$$

□

Iterating Lemma 2.3 we can now conclude that for $F \in \mathcal{D}(\mathcal{A}^{(n)})$ and $m \le n$ it follows that $T_mF \in \mathcal{D}(\mathcal{A}^{(m)})$ and

$$\mathcal{A}^{(m)}(T_mF) \le \mathcal{A}^{(m+1)}(T_{m+1}F) \le \cdots \le \mathcal{A}^{(n)}(F) = \mathcal{A}(F),$$

hence Proposition 2.2 is proven.

Proof of Theorem 2.1. It is sufficient to show that $(\mathcal{A}, \mathcal{D}_0^{(\infty)})$ is closable in $L^2(\pi_\nu)$, where $\mathcal{D}_0^{(\infty)} = \bigcup_{n\geq 1} \mathcal{D}_0^{(n)}$, since the domain of the closure of $(\mathcal{A}, \mathcal{D}_0^{(\infty)})$ will contain $\mathcal{D}_0$, using a simple approximation argument.

To this end let $(F_n)_{n\geq 1} \subset \mathcal{D}_0^{(\infty)}$ be an $\mathcal{A}$-Cauchy sequence with $\lim_{n\to\infty} F_n = 0$ in $L^2(\pi_\nu)$. Clearly, for any $m \geq 1$, it follows that

$$\lim_{n\to\infty} T_m F_n = 0 \quad \text{in } L^2(\pi_\nu),$$

and Proposition 2.2 implies that $(T_m F_n)_{n\geq 1}$ is $\mathcal{A}^{(m)}$-Cauchy. Since $\mathcal{A}^{(m)}$ is closable in $L^2(\pi_\nu)$ we conclude that

$$\lim_{n\to\infty} \mathcal{A}^{(m)}(T_m F_n) = 0.$$

Consequently,

$$\begin{aligned}\mathcal{A}(F_n) &= \lim_{m\to\infty} \mathcal{A}^{(m)}(T_m F_n) = \lim_{m\to\infty} \lim_{k\to\infty} \mathcal{A}^{(m)}(T_m F_n - T_m F_k) \\ &\leq \limsup_{k\to\infty} \mathcal{A}(F_n - F_k)\end{aligned}$$

and the right-hand side in the last inequality can be made arbitrarily small for large n. □

The Wasserstein Dirichlet form – extensions to the multivariate case

In this subsection we restrict our general setting to the case $\Omega \subset \mathbb{R}^d$ relatively compact and $d\nu = \beta\, dx$ with $\beta > 0$. Consider the bilinear form

$$\tilde{\mathcal{E}}(F) := \int_{\mathcal{M}_1(\Omega)} \| |\nabla_x F'(\cdot,\mu)| \|^2_{L^2(\Omega)} \pi_{\beta dx}(d\mu)$$

for

$$\begin{aligned}F \in \mathcal{D}_0(\tilde{\mathcal{E}}) := \{F(\mu) = \varphi(\langle f_1,\mu\rangle, \ldots, \langle f_n,\mu\rangle) : n \geq 1, \varphi \in C_b^1(\mathbb{R}^n), \\ f_1, \ldots, f_n \in C_0^2(\Omega)\}.\end{aligned}$$

Here, $\nabla_x \varphi = (\partial_{x_1}\varphi, \ldots, \partial_{x_d}\varphi)$ denotes the gradient of a function φ and $|\nabla_x \varphi|$ its euclidean norm.

Theorem 2.4. *Let Ω be such that the following Poincaré-inequality*

$$\int_\Omega f^2 dx \leq c_\Omega \int_\Omega |\nabla_x f|^2 dx \quad \forall f \in C_0^1(\Omega).$$

holds with some constant $c_\Omega > 0$. Then $(\tilde{\mathcal{E}}, \mathcal{D}_0(\tilde{\mathcal{E}}))$ is closable in $L^2(\pi_{\beta dx})$.

The proof relies on a comparison of $\tilde{\mathcal{E}}$ with the simpler Dirichlet form $\mathcal{A}$, which is closable in $L^2(\pi_{\beta dx})$ according to Theorem 2.1.

We will need the following

Proposition 2.5. *$\mathcal{D}_0(\tilde{\mathcal{E}}) \subset \mathcal{D}(\mathcal{A})$ and the following inequalities hold:*

(i)
$$\begin{aligned}\tilde{\mathcal{E}}(F,G) &= -\int_{\mathcal{M}_1(\Omega)} \langle F'(\cdot,\mu), \Delta_x G'(\cdot,\mu)\rangle_{L^2(\Omega)} \pi_{\beta dx}(d\mu) \\ &\le \frac{1}{2}\mathcal{A}(F) + \frac{1}{2}\int_{\mathcal{M}_1(\Omega)} \|\Delta_x G'(\cdot,\mu)\|^2_{L^2(\Omega)} \pi_{\beta dx}(d\mu)\end{aligned}$$

for $F, G \in \mathcal{D}_0(\tilde{\mathcal{E}})$.

(ii) $\mathcal{A}(F) \le c_\Omega \tilde{\mathcal{E}}(F)$ *for all* $F \in \mathcal{D}_0(\tilde{\mathcal{E}})$.

Proof. Clearly we can approximate any function $f \in C_0^2(\Omega)$ by a sequence of elementary functions g_n, $n \ge 1$, uniformly bounded, and converging to f pointwise. It follows that $\lim_{n\to\infty}\langle g_n, \mu\rangle = \langle f, \mu\rangle$ for all $\mu \in \mathcal{M}_1(\Omega)$. Now, fix $F \in \mathcal{D}_0(\tilde{\mathcal{E}})$ with the representation $F(\mu) = \varphi(\langle f_1,\mu\rangle, \ldots, \langle f_m,\mu\rangle)$, $\varphi \in C_b^1(\mathbb{R}^m)$, $f_1, \ldots, f_m \in C_0^2(\Omega)$. Let $g_n^{(i)}$, $n \ge 1$, be a sequence of uniformly bounded elementary functions converging to f_i pointwise. Then

$$F_n(\mu) = \varphi(\langle g_n^{(1)}, \mu\rangle, \ldots, \langle g_n^{(m)}, \mu\rangle) \to F(\mu)$$

for all μ and in $L^p(\pi_{\beta dx})$ for all finite p. Moreover,

$$\begin{aligned}\|F_n'(\cdot,\mu)\|^2_{L^2(\Omega)} &= \sum_{k,l=1}^m \varphi_{x_k x_l}(\langle g_n^{(1)},\mu\rangle, \ldots, \langle g_n^{(m)},\mu\rangle)\langle g_n^{(k)}, g_n^{(l)}\rangle_{L^2(\Omega)} \\ &\to \sum_{k,l=1}^m \varphi_{x_k x_l}(\langle f_1,\mu\rangle, \ldots, \langle f_m,\mu\rangle)\langle f_k, f_l\rangle_{L^2(\Omega)} \\ &= \|F'(\cdot,\mu)\|^2_{L^2(\Omega)}\end{aligned}$$

for all $\mu \in \mathcal{M}_1(\Omega)$ and in $L^p(\pi_\nu)$ for all finite p. It follows that $(F_n)_{n\ge1} \subset D(\mathcal{A})$ is an $\mathcal{A}$-Cauchy sequence, so that $F \in D(\mathcal{A})$ and

$$\mathcal{A}(F,F) = \lim_{n\to\infty} \mathcal{A}(F_n, F_n) = \sum_{k,l=1}^m \varphi_{x_k x_l}(\langle f_1,\mu\rangle, \ldots, \langle f_m\rangle)\langle f_k, f_l\rangle_{L^2(\Omega)}.$$

Note that for $F, G \in \mathcal{D}_0(\tilde{E})$ and $\mu \in \mathcal{M}_1(\Omega)$, a simple integration by parts yields

$$\begin{aligned}\langle \nabla_x F'(\cdot,\mu), \nabla_x G'(\cdot,\mu)\rangle_{L^2(\Omega)} &= -\int_\Omega F'(\cdot,\mu)\Delta_x G'(\cdot\mu)\,dx \\ &\le \frac{1}{2}\|F'(\cdot,\mu)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta_x G'(\cdot,\mu)\|^2_{L^2(\Omega)}\end{aligned}$$

and integration w.r.t. $\pi_{\beta dx}$ yields the first inequality (i).

For the proof of (ii) note that for given $F \in \mathcal{D}_0(\tilde{E})$ we have that $F'(\cdot,\mu) \in C_0^2(\Omega)$ and therefore

$$\|F'(\cdot,\mu)\|^2_{L^2(\Omega)} \le c_\Omega \|\nabla_x F'(\cdot,\mu)\|^2_{L^2(\Omega)}$$

and integrating the last inequality w.r.t. $\pi_{\beta dx}$ yields the assertion. □

Proof of Theorem 2.4. Let $(F_n)_{n\geq 1} \subset \mathcal{D}_0(\tilde{\mathcal{E}})$ be an $\tilde{\mathcal{E}}$-Cauchy sequence such that $\lim_{n\to\infty} F_n = 0$ in $L^2(\pi_{\beta dx})$. According to (ii) of Proposition 2.5 it follows that $(F_n)_{n\geq 1}$ is an $\mathcal{A}$-Cauchy sequence, hence

$$\lim_{n\to\infty} \mathcal{A}(F_n) = 0$$

since $\mathcal{A}$ is closed. For any $G \in \mathcal{D}_0(\tilde{\mathcal{E}})$ we now conclude from (i) of Proposition 2.5 that

$$\lim_{n\to\infty} \tilde{\mathcal{E}}(F_n, G) = 0.$$

Note that

$$\begin{aligned}\tilde{\mathcal{E}}(F_n) &= \tilde{\mathcal{E}}(F_n - F_m, F_n) + \tilde{\mathcal{E}}(F_m, F_n) \\ &\leq \frac{1}{2}\tilde{\mathcal{E}}(F_n - F_m) + \frac{1}{2}\tilde{\mathcal{E}}(F_n) + \tilde{\mathcal{E}}(F_m, F_n),\end{aligned}$$

hence

$$\tilde{\mathcal{E}}(F_n) \leq \tilde{\mathcal{E}}(F_n - F_m) + 2\tilde{\mathcal{E}}(F_m, F_n).$$

Consequently,

$$\begin{aligned}\tilde{\mathcal{E}}(F_n) &\leq \liminf_{m\to\infty} \tilde{\mathcal{E}}(F_n - F_m) + 2\tilde{\mathcal{E}}(F_m, F_n) \\ &= \liminf_{m\to\infty} \tilde{\mathcal{E}}(F_n - F_m)\end{aligned}$$

and the right-hand side of the last inequality can be made arbitrarily small for n large, since $(F_n)_{n\geq 1}$ is an $\tilde{\mathcal{E}}$-Cauchy sequence. It follows that $\lim_{n\to\infty} \tilde{\mathcal{E}}(F_n) = 0$, hence the assertion. □

Corollary 2.6. *The Wasserstein pre-Dirichlet form* $(\mathcal{E}, \mathcal{F}C_b^1)$ *is closable in* $L^2(\mathbb{Q}_\beta)$.

Proof. Using the identity (2.1) it suffices to show that

$$\tilde{\mathcal{E}}(F) := \int_{\mathcal{M}_1(I)} \|\frac{d}{dx}F'(\cdot, \mu)\|^2_{L^2(I)} \pi_{\beta dx}(d\mu)$$

with domain $\mathcal{D}_0(\tilde{\mathcal{E}})$ is closable in $L^2(\pi_{\beta dx})$, which is just Theorem 2.4. □

3. Convergence of finite-dimensional particle approximations

Let us introduce a class of finite-dimensional particle approximations for the Wasserstein diffusion. To this end consider a sequence $(a_N)_{N\geq 2} \subset]0,1]$, satisfying $Na_N < 1$ for all N, and $\lim_{N\to\infty} 1 - Na_N = 0$ and let $b_N := \frac{1-(N-2)a_N}{2}$. For fixed N we then consider the following differential operator

$$\begin{aligned}L^{(N)}f(x) = \Delta f(x) &+ (\beta a_N - 1) \sum_{k=2}^{N-1} \left(\frac{1}{x_k - x_{k-1}} - \frac{1}{x_{k+1} - x_k} \right) \partial_{x_k} f(x) \\ &+ \beta(b_N - 2a_N) \left(\frac{1}{x_1}\partial_{x_1} f(x) - \frac{1}{1 - x_{N-1}}\partial_{x_{N-1}} f(x) \right)\end{aligned}$$

on the space Σ_{N-1} with Neumann boundary conditions $\partial_{x_1} f(x) = 0$ if $x_1 = 0$, $\partial_{x_{N-1}} f(x) = 0$ if $x_{N-1} = 1$ and $\partial_{x_k} f(x) = \partial_{x_{k+1}} f(x)$ if $x_k = x_{k+1}$, $1 \le k \le N-2$. It is easy to see that $L^{(N)}$ has a symmetrizing measure

$$\nu^{(N)}(dx) = \frac{\Gamma(\beta(1-2a_N))}{\Gamma(\beta a_N)^{N-2}\,\Gamma(\beta(b_N - a_N))^2} \cdot x_1^{\beta(b_N-a_N)-1} \prod_{k=2}^{N-1} (x_k - x_{k-1})^{\beta a_N - 1} (1 - x_{N-1})^{\beta(b_N - a_N)-1} dx.$$

The associated symmetric (pre-)Dirichlet form is given by

$$\mathcal{E}^{(N)}(f) = \int_{\Sigma_{N-1}} |\nabla f|^2 \, d\nu^{(N)} \, , f \in C^1(\Sigma_{N-1})$$

in $L^2(\Sigma_{N-1}, \nu^{(N)})$. We denote the closure of $\mathcal{E}^{(N)}$ again with $(\mathcal{E}^{(N)}, D(\mathcal{E}^{(N)}))$, and the associated generator, which is a self-adjoint extension of $L^{(N)}$, again with $(L^{(N)}, D(L^{(N)}))$. It is easy to see that $\mathcal{E}^{(N)}$ is a local regular Dirichlet form and the associated diffusion process $((\mathbb{X}_t^{(N)})_{t\ge 0}, (\mathbb{P}_x)_{x\in\Sigma_{N-1}})$ is a solution of the martingale problem associated with $L^{(N)}$. The quantity a_N can be interpreted as the average distance between two neighboured particles in the approximation, and $b_N - a_N$ denotes the average distance from the left-most (resp. right-most) particle to the boundary.

To understand the limit $N \to \infty$ let us consider the projection

$$\Pi_N : \mathcal{G} \to \Sigma_{N-1} \, , g \mapsto (g(b_N), g(b_N + a_N), \ldots, g(b_N + (N-2)a_N)) \, ,$$

and let

$$a_k^{(N)} := b_N + (k-1)a_N, \quad g_k^{(N)} := g(a_k^{(N)}), \quad k = 1, \ldots, N.$$

For given $f \in C(I)$ and $s_N(f,x) := a_N \sum_{n=1}^{N-1} f(x_n)$ it follows that

$$\lim_{N\to\infty} s_N(f, \Pi_N g) = \lim_{N\to\infty} a_N \sum_{k=1}^{N-1} f\left(g_k^{(N)}\right) = \int_0^1 f(g(s))\, ds =: s(f,g)$$

$\mathbb{Q}_\beta$-a.s. and in $L^p(\mathbb{Q}_\beta)$ for all finite p. Consequently, if we consider the sequence

$$\psi_N(x) := \psi(s_N(f_1,x), \ldots, s_N(f_m,x)) \, , \psi \in C^1(\mathbb{R}^m) \, , f_k \in C(I) \tag{3.1}$$

it follows that

$$\lim_{N\to\infty} \psi_N(\Pi_N g) = \psi(s(f_1,g), \ldots, s(f_m,g)) =: F(g).$$

Similarly,

$$
\begin{aligned}
a_N^{-1}\mathcal{E}^{(N)}(\varphi_N) &= \sum_{k,l=1}^{m} a_N \sum_{n=1}^{N-1} \int_{\Sigma_{N-1}} (f_k' f_l')(x_n)(\partial_k\varphi\partial_l\varphi)(s_N(f_1,x),\dots)\,d\nu^{(N)} \\
&= \sum_{k,l=1}^{m} a_N \sum_{n=0}^{N-2} \int_{\mathcal{G}} (f_k' f_l')\left(g_n^{(N)}\right)(\partial_k\varphi\partial_l\varphi)\left(s_N(f_1,\Pi_N g),\dots\right)\,d\mathbb{Q}^\beta(g) \\
&\to \int_{\mathcal{G}} \left\| \sum_{k=1}^{m} \partial_k\varphi\left(s(f_1,g),\dots,s(f_m,g)\right) f_k'\circ g \right\|_{L^2(I)}^2 \, d\mathbb{Q}^\beta(g) \\
&= \mathcal{E}(F).
\end{aligned}
$$

Consequently, the quadratic forms associated with the finite-dimensional particle approximations converge on the set of finitely based test functions towards the Wasserstein Dirichlet integral. This does, however, not imply that the corresponding transition probabilities converge towards the transition probabilities of the Wasserstein diffusion, unless the Wasserstein Dirichlet form, restricted to the set of finitely based test functions $\mathcal{F}C_b^1$, would be Markov unique, i.e., its closure in $L^2(\mathbb{Q}_\beta)$ would be the only Dirichlet form on $L^2(\mathbb{Q}_\beta)$ extending $(\mathcal{E},\mathcal{F}C_b^1)$. This Markov-uniqueness however is open and also the question whether and in what sense the sequence of finite-dimensional diffusion approximations $(\mathbb{X}_t^{(N)})_{t\geq 0}$ converge to the Wasserstein diffusion was left open in the work [1] by Andres and von Renesse.

In the following we are going to prove the desired convergence of the transition semigroup under the assumption that

$$\lim_{N\to\infty} \frac{a_N}{b_N} = 0. \tag{3.2}$$

This means that the ratio of the average distance of two neighboured particles and the average distance of the right-most (resp. left-most) particle to the boundary converge to zero.

To state the convergence result precisely, let $R_\alpha^{(N)} := (\alpha - a_N^{-1}L^{(N)})^{-1}$, $\alpha > 0$, be the resolvent associated with $a_N^{-1}L^{(N)}$ and let $R_\alpha := (\alpha - L)^{-1}$ be the resolvent associated with the Wasserstein Dirichlet form.

Theorem 3.1. *Let* (a_N) *be such that* (3.2) *holds. Let* $\psi \in C^1(\mathbb{R}^m)$*,* ψ_N *be as in* (3.1) *and* $\psi_\infty(g) = \psi(s(f_1,g),\dots,s(f_m,g))$*. Then*

$$\lim_{N\to\infty} (R_\alpha^{(N)}\psi_N)\circ\Pi_N = R_\alpha\psi_\infty \quad \text{in } L^2(\mathcal{G},\mathbb{Q}_\beta).$$

Let us first sketch the main steps in the proof of Theorem 3.1. The main difficulty in the finite-dimensional approximation of the Wasserstein Dirichlet form is due to the fact that neither the evaluation map $g\mapsto g(t)$ nor the linear functionals $g\mapsto\langle f,g\rangle$ are contained in the domain $D(\mathcal{E})$, so that in particular $\varphi(\Pi_N g)\notin D(\mathcal{E})$.

We therefore introduce the following convolutions

$$\Phi^{(N)}(\varphi)(g) := \frac{1}{|I_N|}\int_{I_N} \varphi(g^{(N)}(s))\,ds\,, \varphi \in \mathcal{B}_b(\Sigma_{N-1}). \tag{3.3}$$

Here, $I_N = [0, a_N]$ and $|I_N| = a_N$ denotes its length. The properties of this convolution are quite similar to the properties of a Dirac-sequence on $\mathbb{R}$, like, e.g., $T_N f(x) = \sqrt{\frac{N}{2\pi}} \int f(x-y)\exp(-\frac{y^2}{2N})(dy)$. In particular, we have that

$$\lim_{N\to\infty} \Phi^{(N)}(\psi^{(N)}) = F \text{ in } L^p(\mathbb{Q}_\beta) \text{ for all finite } p.$$

A further difficulty now arises from the fact that we need this convergence uniformly w.r.t. the sup-norm $\|\psi^{(N)}\|_\infty$ in the sense that

$$\lim_{N\to\infty} \int_{\mathcal{G}} \Phi^{(N)}(f_N)\,d\mathbb{Q}_\beta - \int_{\Sigma_{N-1}} f_N d\tilde{\nu}_0^{(N)} = 0 \tag{3.4}$$

for $f_N \in C(\Sigma_{N-1})$ with $\sup_N \|f_N\|_\infty < \infty$ (cf. Lemma 3.6 below). Here, $\tilde{\nu}_s^{(N)}$, $s \in I_N$, is the distribution of

$$g^{(N)}(s) = \left(g_1^{(N)}(s), \ldots, g_{N-1}^{(N)}(s)\right)\,, \quad g_k^{(N)}(s) = g(b_N + (k-1)a_N + s)$$

under $\mathbb{Q}_\beta$, i.e.,

$$\tilde{\nu}_s^{(N)}(dx) = \frac{\Gamma(\beta)}{\Gamma(\beta(b_N+s))\Gamma(\beta a_N)^{N-2}\Gamma(\beta(b_N+a_N-s))} \cdot x_1^{\beta(b_N+s)-1} \prod_{k=2}^{N-1} (x_k - x_{k-1})^{\beta a_N - 1} (1-x_{N-1})^{\beta(b_N+a_N-s)-1}\,dx.$$

The desired convergence (3.4) reduces to show that

$$\int_{\mathcal{G}} \Phi^{(N)}(f_N)\,d\mathbb{Q}_\beta - \int_{\Sigma_{N-1}} f_N\,d\tilde{\nu}_0^{(N)} = \frac{1}{|I_N|}\int_{I_N}\left(\int_{\Sigma_{N-1}} f_N\,d\tilde{\nu}_s^{(N)} - \int_{\Sigma_{N-1}} f_N\,d\tilde{\nu}_0^{(N)}\right) ds \to 0$$

which requires some uniform control of the Radon–Nikodym derivatives $\frac{d\tilde{\nu}_s^{(N)}}{d\tilde{\nu}_0^{(N)}}$, $N \ge 1$. In order to ensure this we will show in Lemma 3.2 below that in fact

$$\lim_{N\to\infty} \int_{\Sigma_{N-1}} \left(\frac{d\tilde{\nu}_s^{(N)}}{d\tilde{\nu}_0^{(N)}} - 1\right)^2 d\tilde{\nu}_0^{(N)} = 0$$

if the condition (3.2) on the sequence a_N and b_N is satisfied. A similar control is required for the Radon–Nikodym derivatives $\frac{d\tilde{\nu}_s^{(N)}}{d\nu^{(N)}}$, $N \ge 1$ (see Lemma 3.3).

The crucial property of the convolutions $\Phi^{(N)}$ is that it maps C^1-functions φ into the domain $D(\mathcal{E})$ of the Wasserstein Dirichlet form with the additional fact that $\mathcal{E}(\Phi^{(N)}(\varphi)) \le c\tilde{\mathcal{E}}^{(N)}(\varphi)$ for some uniform constant c (cf. Lemma 3.4 below).

Applying this to the sequence $\Phi^{(N)}(\varphi_N)$, $\varphi_N = \alpha R_\alpha^{(N)} \psi_N$, $N \geq 1$, now yields a bounded sequence in $D(\mathcal{E})$, and any limit $F \in D(\mathcal{E})$ of some weakly convergent subsequence will satisfy the identity

$$\mathcal{E}_\alpha(F, \exp(s(h,\cdot)) = \int \psi_\infty \exp(s(h,\cdot))\, d\mathbb{Q}_\beta$$

which implies that $F = R_\alpha \psi_\infty$ and thus the assertion.

Lemma 3.2. *Let*

$$c_N(s) := \frac{\Gamma(\beta b_N)\Gamma(\beta(b_N + a_N))}{\Gamma(\beta(b_N + s))\Gamma(\beta(b_N + a_N - s))}, s \in I_N$$
$$d_N(s,x) := x_1^{\beta s}(1 - x_{N-1})^{-\beta s}, x \in \Sigma_{N-1}, s \in I_N.$$

so that

$$c_N(s) d_N(s,x) = \frac{d\tilde{\nu}_s^{(N)}}{d\tilde{\nu}_0^{(N)}}(x).$$

Then

$$\lim_{N\to\infty} \int_{\Sigma_N} \frac{1}{|I_N|} \int_{I_N} \left(1 - c_N(s) d_N(s,x)\right)^2 \, ds\, \tilde{\nu}_0^{(N)}(dx) = 0.$$

Proof. Note that

$$\begin{aligned}
&\int_{\Sigma_N} \frac{1}{|I_N|} \int_{I_N} \left(1 - c_N(s) d_N(s,x)\right)^2 ds\, \tilde{\nu}_0^{(N)}(dx) \\
&\quad = -1 + \frac{1}{|I_N|} \int_{I_N} c_N^2(s) \int_{\Sigma_{N-1}} d_N^2(s,x) \tilde{\nu}_0^{(N)}(dx)\, ds \\
&\quad = -1 + \frac{1}{|I_N|} \int_{I_N} c_N^2(s) \frac{\Gamma\left(\beta\left(b_N + 2s\right)\right)\Gamma\left(\beta\left(b_N + a_N - 2s\right)\right)}{\Gamma\left(\beta b_N\right)\Gamma\left(\beta(b_N + a_N)\right)} ds.
\end{aligned}$$

It is therefore sufficient to show that

$$\lim_{N\to\infty} \frac{1}{|I_N|} \int_{I_N} c_N^2(s) \frac{\Gamma\left(\beta\left(b_N + 2s\right)\right)\Gamma\left(\beta\left(b_N + a_N - 2s\right)\right)}{\Gamma\left(\beta b_N\right)\Gamma\left(\beta(b_N + a_N)\right)} ds = 1.$$

To this end note that we can write the integrand as

$$\begin{aligned}
&c_N^2(s) \frac{\Gamma\left(\beta\left(b_N + 2s\right)\right)\Gamma\left(\beta\left(b_N + a_N - 2s\right)\right)}{\Gamma\left(\beta b_N\right)\Gamma\left(\beta(b_N + a_N)\right)} \\
&\qquad = \frac{(b_N + s)^2 (b_N + a_N - s)^2}{b_N(b_N + a_N)(b_N + 2s)(b_N + a_N - 2s)} \cdot e_N(s)
\end{aligned}$$

with

$$e_N(s) = \frac{\Gamma(\beta(b_N + s) + 1)^2 \Gamma(\beta(b_N + a_N - s) + 1)^2}{\Gamma(\beta b_N + 1)\Gamma(\beta(b_N + a_N) + 1)\Gamma(\beta(b_N + 2s) + 1)\Gamma(\beta(b_N + a_N - 2s) + 1)}.$$

Note the two-sided estimate

$$\frac{1}{\Gamma(\beta(b_N + 2a_N) + 1)^4} \leq e_N(s) \leq \Gamma(\beta(b_N + a_N) + 1)^4$$

which implies that $e_N(s) \to 1$ for $N \to \infty$ uniformly in $s \in I_N$. Consequently

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{|I_N|} \int_{I_N} c_N^2(s) \frac{\Gamma\left(\beta\left(b_N+2s\right)\right)\Gamma\left(\beta\left(b_N+a_N-2s\right)\right)}{\Gamma\left(\beta b_N\right)\Gamma\left(\beta(b_N+a_N)\right)} ds \\
&\quad = \lim_{N\to\infty} \frac{1}{|I_N|} \int_{I_N} \frac{(b_N+s)^2(b_N+a_N-s)^2}{b_N(b_N+a_N)(b_N+2s)(b_N+a_N-2s)} ds \\
&\quad = \lim_{N\to\infty} \int_0^1 \frac{(b_N+a_N t)^2(b_N+a_N(1-t))^2}{b_N(b_N+a_N)(b_N+2a_N t)(b_N+a_N(1-2t))} dt \\
&\quad = \lim_{N\to\infty} \int_0^1 \frac{(1+\frac{a_N}{b_N}t)^2(1+\frac{a_N}{b_N}(1-t))^2}{(1+\frac{a_N}{b_N})(1+2\frac{a_N}{b_N}t)(1+\frac{a_N}{b_N}(1-2t))} dt = 1. \qquad \square
\end{aligned}$$

Lemma 3.3. *Let*

$$\begin{aligned}
c_N(s) &:= \frac{\Gamma(\beta)}{\Gamma(\beta(1-2a_N))} \frac{\Gamma(\beta(b_N-a_N))^2}{\Gamma(\beta(b_N+s))\Gamma(\beta(b_N+a_N-s))}\,, s \in I_N \\
d_N(s,x) &:= x_1^{\beta(a_N+s)}(1-x_{N-1})^{\beta(a_N-s)}\,, x \in \Sigma_{N-1}\,, s \in I_N.
\end{aligned}$$

so that

$$c_N(s) d_N(s,x) = \frac{d\tilde{\nu}_s^{(N)}}{d\nu^{(N)}}(x).$$

Then

$$\lim_{N\to\infty} \int_{\Sigma_N} \frac{1}{|I_N|} \int_{I_N} \left(1 - c_N(s) d_N(s,x)\right)^2 \, ds\, \nu^{(N)}(dx) = 0.$$

The proof is completely similar to the proof of the previous lemma, so that we can omit it.

For the next lemma recall from (3.3) the definition of the convolution $\Phi^{(N)}(\varphi)$.

Lemma 3.4. *Let $\varphi \in C^1(\Sigma_{N-1})$. Then $\Phi^{(N)}(\varphi) \in D(\mathcal{E})$ and*

$$\begin{aligned}
\mathcal{E}\left(\Phi^{(N)}(\varphi)\right) &\le \frac{1}{|I_N|} \sum_{k=1}^{N-1} \int \Phi^{(N)}\left((\partial_{x_k}\varphi)^2\right)(g)\, \mathbb{Q}_\beta(dg) \\
&= \frac{1}{|I_N|^2} \int_{I_N} \int_{\Sigma_{N-1}} |\nabla\varphi|^2 \, d\tilde{\nu}_s^{(N)}\, ds,
\end{aligned}$$

Moreover, there exists a uniform constant c such that

$$\mathcal{E}\left(\Phi^{(N)}(\varphi)\right) \le c\, \frac{1}{a_N} \tilde{\mathcal{E}}^{(N)}(\varphi).$$

Proof. It is easy to see by suitable approximations that $\Phi^{(N)}(\varphi)$ is in the domain of the closure of $\mathcal{E}$ and Fréchet differentiable with differential

$$\left(\Phi^{(N)}(\varphi)\right)'(s,g) = \frac{1}{|I_N|} \sum_{k=1}^{N-1} \partial_{x_k}\varphi\left(g^{(N)}\left(s - a_k^{(N)}\right)\right) 1_{\left[a_k^{(N)}, a_{k+1}^{(N)}\right]}(s).$$

Consequently,

$$\mathcal{E}\left(\Phi^{(N)}(\varphi)\right) = \frac{1}{|I_N|^2} \sum_{k=1}^{N-1} \int_{\mathcal{G}} \int_{I_N} (\partial_{x_k} \varphi)^2 \left(g^{(N)}(s)\right) ds\, \mathbb{Q}_\beta(dg)$$
$$= \frac{1}{|I_N|^2} \int_{I_N} \int_{\Sigma_{N-1}} |\nabla \varphi|^2 \, d\tilde{\nu}_s^{(N)} \, ds$$

which implies the first assertion. The second assertion follows from the fact that the Radon–Nikodym derivative $\frac{d\tilde{\nu}_s^{(N)}}{d\nu^{(N)}}$ can be uniformly bounded w.r.t. $x \in \Sigma_{N-1}$ and w.r.t. N since

$$\frac{d\tilde{\nu}_s^{(N)}}{d\nu^{(N)}}(x) \leq \frac{\Gamma(\beta)}{\Gamma(\beta(1-2a_N))} \frac{\Gamma(\beta(b_N - a_N))^2}{\Gamma(\beta(b_N + s))\Gamma(\beta(b_N + a_N - s))}$$
$$\leq \frac{\Gamma(\beta+1)}{\Gamma(\beta(1-2a_N)+1)} \frac{\Gamma(\beta(b_N - a_N)+1)^2}{\Gamma(\beta(b_N + s)+1)\Gamma(\beta(b_N + a_N - s)+1)}$$
$$\cdot \frac{(1-2a_N)(b_N+s)(b_N+a_N-2)}{(b_N-a_N)^2} \leq \Gamma(\beta+1)^3 \frac{(b_N+a_N)^2}{(b_N-a_N)^2}$$

is uniformly bounded w.r.t. N. □

We now consider in the limit $N \to \infty$ for given ψ as in (3.1) the sequence

$$\varphi_N = \alpha R_\alpha^{(N)} \psi_N \, , N \geq 1.$$

Lemma 3.5. *The sequence* $\left(\Phi^{(N)}(\varphi_N)\right)_{N\geq 1}$ *is bounded in* $\mathcal{D}(\mathcal{E})$.

Proof. It follows from Lemma 3.4 that $\Phi^{(N)}(\varphi_N) \in \mathcal{D}(\mathcal{E})$ and we can estimate

$$\mathcal{E}(\Phi^{(N)}(\varphi_N)) \leq c \frac{1}{a_N} \tilde{\mathcal{E}}^{(N)}(\varphi_N) = c \int_{\Sigma_{N-1}} (\psi_N - \alpha\varphi_N)\varphi_N \, d\tilde{\nu}^{(N)}$$
$$\leq c\|\psi\|_\infty^2.$$ □

Lemma 3.6. *Let* $f_N \in C(\Sigma_{N-1})$, $N \geq 1$, *be uniformly bounded, i.e.,*

$$M := \sup_{N\geq 1} \|f_N\|_{\infty,\Sigma_{N-1}} < \infty.$$

Then

(i) $$\lim_{N\to\infty} \int_{\mathcal{G}} \Phi^{(N)}(f_N)\, d\mathbb{Q}_\beta - \int_{\Sigma_{N-1}} f_N \, d\tilde{\nu}_0^{(N)} = 0,$$

(ii) $$\lim_{N\to\infty} \int_{\mathcal{G}} \Phi^{(N)}(f_N)\, d\mathbb{Q}_\beta - \int_{\Sigma_{N-1}} f_N \, d\nu^{(N)} = 0.$$

Proof. For the proof of (i) we use the notations of Lemma 3.2. We can then write

$$\begin{aligned}
&\left|\int_{\mathcal{G}} \Phi^{(N)}(f_N)(g)\mathbb{Q}_\beta(dg) - \int_{\Sigma_{N-1}} f_N\, d\tilde{\nu}_0^{(N)}\right| \\
&\quad = \Big|\frac{1}{|I_N|}\int_{I_N}\int_{\Sigma_{N-1}} f_N\, d\tilde{\nu}_s^{(N)}\, ds - \int_{\Sigma_{N-1}} f_N(x)\,\tilde{\nu}_0^{(N)}(dx)\Big| \\
&\quad = \Big|\frac{1}{|I_N|}\int_{I_N}\int_{\Sigma_{N-1}} f_N(x)(c_N(s)d_N(s,x)-1)\,\tilde{\nu}_0^{(N)}(dx)\,ds\Big| \\
&\quad \le \|f_N\|_{\infty,\Sigma_{N-1}}\left(\int_{\Sigma_{N-1}}\frac{1}{|I_N|}\int_{I_N}(c_N(s)d_N(s,x)-1)^2\,ds\,\tilde{\nu}_0^{(N)}(dx)\right)^{\frac{1}{2}}
\end{aligned}$$

which converges to zero according to Lemma 3.2 and the uniform boundedness of $\|f_N\|_\infty$.

The proof of (ii) is similar using the Lemma 3.3. □

Lemma 3.7. *Let $h \in C^1(I)$ and $p \in [1,\infty[$. Then*

$$\lim_{N\to\infty}\int_{\mathcal{G}}\frac{1}{|I_N|}\int_{I_N}\left|\exp\left(s(h,g)\right) - \exp\left(s_N(h,g^{(N)}(s))\right)\right|^p\, ds\mathbb{Q}_\beta(dg) = 0.$$

Proof. Note that for all $g \in \mathcal{G}$ and for all $s \in I_N$

$$\begin{aligned}
&\big|\exp\left(s(h,g)\right) - \exp\left(s_N(h,g^{(N)}(s))\right)\big|^p \\
&\quad \le \left|s(h,g) - s_N(h,g^{(N)}(s)\right|\exp(p\|h\|_\infty) \\
&\quad \le \Big(2b_N\|h\|_\infty \\
&\qquad + a_N\sum_{k=1}^{N-1}\int_{a_k^{(N)}}^{a_{k+1}^{(N)}}\left|h(g(r)) - h\left(g_k^{(N)}(s)\right)\right|\,dr\Big)\exp(p\|h\|_\infty) \\
&\quad \le \Big(2b_N\|h\|_\infty \\
&\qquad + a_N\|h'\|_\infty\sum_{k=1}^{N-1}\int_{a_k^{(N)}}^{a_{k+1}^{(N+1)}}\left|g(r) - g_k^{(N)}(s)\right|\,dr\Big)\exp(p\|h\|_\infty).
\end{aligned}$$

Using

$$\begin{aligned}
&\int_{\mathcal{G}}\frac{1}{|I_N|}\int_{I_N}\int_{a_k^{(N)}}^{a_{k+1}^{(N)}}\left|g(r) - g_k^{(N)}(s)\right|\,dr\,ds\,\mathbb{Q}_\beta(dg) \\
&\qquad = \frac{1}{|I_N|}\int_{I_N}\int_0^{a_N}|r-s|\,dr\,ds \le a_N
\end{aligned}$$

for $k = 1, \ldots, N-1$ and $s \in I_N$ and $Na_N \leq 1$ we obtain that

$$\int_{\mathcal{G}} \frac{1}{|I_N|} \int_{I_N} \left| \exp\left(s(h,g)\right) - \exp\left(s_N(h, g^{(N)}(s))\right)\right|^p \, ds \, \mathbb{Q}_\beta(dg)$$
$$\leq (2b_N + a_N)\left(\|h'\|_\infty + \|h\|_\infty\right) \exp(p\|h\|_\infty) \to 0\,, N \to \infty. \qquad \square$$

Proof of Theorem 3.1. Let $F \in \mathcal{D}(\mathcal{E})$ be the limit of some weakly convergent subsequence of $\Phi^{(N)}(\varphi_N)$, $N \geq 1$, again denoted by $\Phi^{(N)}(\varphi_N)$, $N \geq 1$. Fix $h \in C^1(I)$. Let $\mathcal{E}_\alpha(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \alpha(\cdot,\cdot)_{L^2(\mathbb{Q}_\beta)}$. Then

$$\mathcal{E}_\alpha(F, \exp(s(h,\cdot))) = \lim_{N\to\infty} \mathcal{E}_\alpha(\Phi^{(N)}(\varphi_N), \exp(s(h,\cdot))). \tag{3.5}$$

Note that

$$\lim_{N\to\infty} \int_{\mathcal{G}} \Phi^{(N)}(\varphi_N) \exp\left(s(h,\cdot)\right) - \Phi^{(N)}(\varphi_N \exp(s_N(h,\cdot)))\, d\mathbb{Q}_\beta = 0$$

by Lemma 3.7 and

$$\lim_{N\to\infty} \int_{\mathcal{G}} \Phi^{(N)}(\varphi_N \exp(s_N(h,\cdot)))\, d\mathbb{Q}_\beta - \int_{\Sigma_{N-1}} \varphi_N \exp(s_N(h,\cdot))\, d\nu^{(N)} = 0$$

by Lemma 3.6 so that

$$\lim_{N\to\infty} \int_{\mathcal{G}} \Phi^{(N)}(\varphi_N) \exp\left(s(h,\cdot)\right)\, d\mathbb{Q}_\beta - \int_{\Sigma_{N-1}} \varphi_N \exp(s_N(h,\cdot))\, d\nu^{(N)} = 0. \tag{3.6}$$

Next, observe that

$$\begin{aligned}
&\mathcal{E}\left(\Phi^{(N)}(\varphi_N), \exp(s(h,\cdot))\right) \\
&= \sum_{k=1}^{N-1} \int_{\mathcal{G}} \frac{1}{|I_N|} \int_{I_N} \partial_{x_k}\varphi_N\left(g^{(N)}(s)\right) h'\left(g_k^{(N)}(s)\right)\, ds\, \exp\left(s(h,g)\right)\, \mathbb{Q}_\beta(dg) \\
&= \sum_{k=1}^{N-1} \int_{\mathcal{G}} \Phi^{(N)}\left(\partial_{x_k}\varphi_N(g^{(N)}(s)) h'\left(g_k^{(N)}(s)\right)\right) \exp\left(s(h,g)\right)\, \mathbb{Q}_\beta(dg)
\end{aligned}$$

and thus

$$\begin{aligned}
&\lim_{N\to\infty} \mathcal{E}\left(\Phi^{(N)}(\varphi_N), \exp(s(h,\cdot))\right) - \sum_{k=1}^{N-1} \int_{\mathcal{G}} \Phi^{(N)}\left(\partial_{x_k}\varphi_N \partial_{x_k} \exp\left(s_N(h,\cdot)\right)\right)\, d\mathbb{Q}_\beta \\
&\quad \leq \lim_{N\to\infty} \|h'\|_\infty \left(\frac{1}{a_N} \sum_{k=1}^{N-1} \int_{\mathcal{G}} \Phi^{(N)}\left(\partial_{x_k}\varphi_N^2\right)\, d\mathbb{Q}_\beta\right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{\mathcal{G}} \int_{I_N} \left(\exp\left(s(h,g)\right) - \exp\left(s_N(h, g^{(N)}(s))\right)\right)^2\, ds\, \mathbb{Q}_\beta(dg)\right)^{\frac{1}{2}} = 0
\end{aligned}$$

again using Lemma 3.7 and 3.5. Using Lemma 3.2 and the assumption $Na_N \leq 1$ we have that

$$\sum_{k=1}^{N-1} \int_{\mathcal{G}} \Phi^{(N)} \left(\partial_{x_k} \varphi_N \partial_{x_k} \exp\left(s_N(h, \cdot)\right)\right) d\mathbb{Q}_\beta - \sum_{k=1}^{N-1} \int_{\Sigma_{N-1}} \partial_{x_k} \varphi_N \partial_{x_k} \exp\left(s_N(h, \cdot)\right) \, d\nu^{(N)}$$
$$\leq \|h'\|_\infty \exp(\|h\|_\infty) \left(a_N^{-1} \mathcal{E}^{(N)}(\varphi_N)\right)^{\frac{1}{2}} \left(\int_{\Sigma_{N-1}} \frac{1}{|I_N|} \int_{I_N} \left(\frac{d\tilde{\nu}_s^{(N)}}{d\nu^{(N)}} - 1\right)^2 ds \, d\nu^{(N)}\right)^{\frac{1}{2}} \longrightarrow 0$$

as $N \to \infty$ by Lemma 3.2 and the assumption $Na_N \leq 1$. Consequently,

$$\lim_{N\to\infty} \mathcal{E}\left(\Phi^{(N)}(\varphi_N), \exp(s(h, \cdot))\right) - \sum_{k=1}^{N-1} \int_{\Sigma_{N-1}} \partial_{x_k} \varphi_N \partial_{x_k} \exp\left(s_N(h, \cdot)\right) \, d\nu^{(N)} = 0. \tag{3.7}$$

Inserting (3.6) and (3.7) into (3.5) we obtain that

$$\begin{aligned}\lim_{N\to\infty} \mathcal{E}_\alpha \left(\Phi^{(N)}(\varphi_N), \exp\left(s(h, \cdot)\right)\right) &= \lim_{N\to\infty} \alpha \int_{\Sigma_{N-1}} \varphi_N \exp(s_N(h, \cdot)) \, d\nu^{(N)} \\ &\quad - \sum_{k=1}^{N-1} \int_{\Sigma_{N-1}} \partial_{x_k} \varphi_N \partial_{x_k} \exp\left(s_N(h, \cdot)\right) \, d\nu^{(N)} \\ &= \lim_{N\to\infty} \int_{\Sigma_{N-1}} \psi_N \exp\left(s_N(h, \cdot)\right) \, d\nu^{(N)} \\ &= \int_{\mathcal{G}} \psi_\infty \exp\left(s(h, \cdot)\right) \, d\mathbb{Q}_\beta.\end{aligned}$$

In the last equality we used the fact that

$$\lim_{N\to\infty} \int_{\Sigma_{N-1}} \psi_N \exp\left(s_N(h, \cdot)\right) \, d\nu^{(N)} - \int_{\mathcal{G}} \psi_N \circ \Pi_N \exp\left(s_N(h, \Pi_N)\right) \, d\mathbb{Q}_\beta = 0$$

by Lemma 3.6 and

$$\lim_{N\to\infty} \int_{\mathcal{G}} \psi_N \circ \Pi_N \exp\left(s_N(h, \Pi_N)\right) \, d\mathbb{Q}_\beta = \int_{\mathcal{G}} \psi_\infty \exp\left(s(h, \cdot)\right) \, d\mathbb{Q}_\beta.$$

Since span $\left\{\exp(s(h, \cdot)) \mid h \in C^1(I)\right\} \subset \mathcal{D}(\mathcal{E})$ dense, the last equality holds for all $G \in \mathcal{D}(\mathcal{E})$ which implies that $F = R_\alpha \psi_\infty$.

For the proof of the theorem it remains to show that

$$\lim_{N\to\infty} \Phi^{(N)}(\varphi_N) - \varphi_N \circ \Pi_N = 0$$

weakly in $L^2(\mathbb{Q}_\beta)$. But this follows from the fact that the sequence $\Phi^{(N)}(\varphi_N) - \varphi_N \circ \Pi_N$, $N \geq 1$, is bounded in $L^2(\mathbb{Q}_\beta)$ and for any $h \in C^1(I)$

$$\begin{aligned}
&\lim_{N\to\infty} \int_{\mathcal{G}} \left(\Phi^{(N)}(\varphi_N) - \varphi_N \circ \Pi_N\right) \exp(s(h,\cdot))\, d\mathbb{Q}_\beta \\
&= \lim_{N\to\infty} \int_{\mathcal{G}} \left(\Phi^{(N)}(\varphi_N) - \varphi_N \circ \Pi_N\right) \exp(s_N(h,\cdot)) \circ \Pi_N \, d\mathbb{Q}_\beta \\
&= \lim_{N\to\infty} \int_{\mathcal{G}} \Phi^{(N)}\left(\varphi_N \exp(s_N(h,\cdot))\right)\, d\mathbb{Q}_\beta \\
&\qquad - \int_{\Sigma_{N-1}} \varphi_N \exp(s_N(h,\cdot))\, d\tilde{\nu}_0^{(N)} = 0
\end{aligned}$$

using Lemma 3.3. This implies the assertion of the theorem. □

3.1. The particle approximation of the Wasserstein Dirichlet form over the unit circle

We can also consider the Wasserstein diffusion over the unit circle S^1 with state space

$$\mathcal{G}^1 = \{g : S^1 \to S^1 \mid g = g_\mu^{-1} \text{ for some } \mu \in \mathcal{M}_1(S^1)\}$$

where $g_\mu^{-1}(t) = \inf\{s \in [0,1] \mid \mu([0,s]) > t\}$ denotes the inverse cumulative distribution function of μ. The entropic measure $\mathbb{Q}_\beta^1$ is uniquely determined through its finite-dimensional distributions

$$\begin{aligned}
&\mathbb{Q}_\beta^1\left(g(t_1) \in dx_1, \ldots, g(t_n) \in dx_n\right) \\
&\quad = \frac{\Gamma(\beta)}{\prod_{k=1}^n \Gamma(\beta(t_k - t_{k-1}))} \prod_{k=1}^n (x_k - x_{k-1})^{\beta(t_k - t_{k-1}) - 1}\, dx
\end{aligned}$$

for any strictly increasing sequence $0 < t_1 < \cdots < t_n < 1$, identifying $t_0 = t_n$, on the set

$$\Sigma_n^1 = \left\{ (x_1, \ldots, x_n) \in (S^1)^n \mid \sum_{k=1}^n |[x_{k-1}, x_k]| = 1 \right\}$$

of n-partitions of S^1 identifying $x_0 = x_n$. $\mathbb{Q}_\beta^1$ is rotationally invariant in the sense that the laws of $g \mapsto g(\cdot + t)$ are independent of $t \in \mathbb{R}$, thereby identifying S^1 with $\mathbb{R}/\mathbb{Z}$.

The Wasserstein Dirichlet form in this case is the closure $(\mathcal{E}^1, D(\mathcal{E}^1))$ in $L^2(\mathbb{Q}_\beta^1)$ of the bilinear form

$$\int_{\mathcal{G}^1} \|F'(\cdot, g)\|_{L^2(S^1)}^2 \, d\mathbb{Q}_\beta^1(g),$$

defined on the space of all functions F that admit the representation $F(g) = \varphi(\langle f_1, g\rangle, \ldots, \langle f_m, g\rangle)$ for some $\varphi \in C_b^1(\mathbb{R}^m)$, $m \geq 1$, and $f_1, \ldots, f_m \in L^2(S^1)$.

We now consider the finite-dimensional approximations generated by

$$L^{1,(N)}f(x) = \Delta f(x) + (\beta\frac{1}{N-1} - 1)\sum_{k=1}^{N-1}\left(\frac{1}{x_k - x_{k-1}} - \frac{1}{x_{k+1} - x_k}\right)\partial_{x_k}f(x)$$

with Neumann boundary conditions $\partial_{x_k}f(x) = \partial_{x_{k_1}}f(x)$ if $x_k = x_{k-1}$, $k = 1,\ldots,N-1$. $L^{1,(N)}$ has symmetrizing measure

$$\nu^{1,(N)}(dx) = \frac{\Gamma(\beta)}{\Gamma\left(\beta\frac{1}{N-1}\right)^{N-1}}\prod_{k=1}^{N-1}(x_k - x_{k-1})^{\beta\frac{1}{N-1}-1}dx.$$

Similar to the case of the unit interval we denote the (closure) of the associated symmetric Dirichlet form

$$\mathcal{E}^{1,(N)}(f) = \int_{\Sigma^1_{N-1}} |\nabla f|^2\, d\nu^{1,(N)}\,, f \in C^1(\Sigma^1_{N-1})$$

in $L^2(\Sigma^1_{N-1}, \nu^{1,(N)})$ again with $(\mathcal{E}^{1,(N)}, D(\mathcal{E}^{1,(N)}))$.

The analogous convolutions in the periodic case are given by

$$\Phi^{1,(N)}(\varphi)(g) := \frac{1}{|I_N|}\int_{I_N}\varphi\left(g^{(N)}(s)\right)\, ds$$

for $\varphi \in \mathcal{B}_b(\Sigma^1_{N-1})$ where now $I_N = [0, \frac{1}{N-1}]$, and

$$g^{(N)}(s) = \left(g(s), \ldots, g\left(s + \frac{N-2}{N-1}\right)\right)\,, s \in I_N.$$

The convolution has analogous properties to the previous case, and due to the rotatonal invariance of the entropic measure in this case it follows that the (joint) distribution $\nu_s^{1,(N)}$ of $g^{(N)}(s)$ is independent of s and equal to $\nu^{1,(N)}$. In particular,

$$\int_{\mathcal{G}^1}\Phi^{1,(N)}(\varphi)\, d\mathbb{Q}^1_\beta = \int_{\Sigma^1_{N-1}}\varphi\, d\nu^{1,(N)}$$

for all N, so that as a consequence Lemmata 3.4, 3.5 and 3.6 hold without any further restriction. We can therefore state the following result:

Theorem 3.8. *Let $R^{1,(N)}_\alpha := (\alpha - (N-1)L^{1,(N)})^{-1}$, $\alpha > 0$, be the resolvent associated with $(N-1)L^{1,(N)}$ and let $R^1_\alpha := (\alpha - L^1)^{-1}$ be the resolvent associated with the Wasserstein Dirichlet form on the unit circle. Let*

$$\psi_N(x) = \psi\left(s_N(f_1, x), \ldots, s_N(f_m, x)\right)\,, \psi \in C^1(\mathbb{R}^m)\,, f_k \in C^1(S^1),$$
$$\psi_\infty(g) = \psi\left(s(f_1, g), \ldots, s(f_m, g)\right),$$

where $s_N f(x) = \frac{1}{N-1}\sum_{k=1}^{N-1} f(x_k)$ and $s(f,g) = \int_{S^1} f(g(s))\, ds$. Then

$$\lim_{N\to\infty}(R^{1,(N)}_\alpha\psi_N)\circ\Pi_N = R^1_\alpha\psi_\infty \quad \text{in } L^2(\mathcal{G}^1, \mathbb{Q}^1_\beta).$$

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Erratum

Arturo Kohatsu-Higa and José Manuel Corcuera

Abstract. We give some errata corresponding to the article "Statistical Inference and Malliavin Calculus" published in the Seminar on Stochastic Analysis, Random Fields and Applications VI, 2011.

Mathematics Subject Classification (2010). Primary 62Mxx, 60Hxx; Secondary 60Fxx, 62Fxx.

Keywords. Diffusion processes, Malliavin calculus, parametric estimation, Cramer-Rao lower bound, LAN property, LAMN property, jump-diffusion process, score function.

Acknowledgement of erratum

The authors of [CK] would like to point out that, due to a mistake that was observed by Eulalia Nualart in the comments of the subsection 4.1, the problem treated in that subsection remains as an open problem.

In fact, although the integration by parts formulae that appears in that section is correct, the asymptotic analysis of the conditional expectations is incorrect.

[CK] J.M. Corcuera and A. Kohatsu-Higa, *Statistical inference and Malliavin calculus.* Seminar on Stochastic Analysis, Random Fields and Applications VI, Birkäuser Verlag, 2011.

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Part II
Stochastic Methods in Financial Models

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Stochastic Modeling of Power Markets Using Stationary Processes

Fred Espen Benth and Heidar Eyjolfsson

Abstract. We make a survey over recent developments in stochastic modelling of power markets, with a particular focus on the application of stationary processes. We analyse the class of Lévy semistationary processes proposed by Barndorff-Nielsen, Benth and Veraart [1] for modelling electricity spot prices. We suggest and analyse different numerical methods for simulating the paths of these processes, a particulary important question for risk management studies in power markets. Finally, we discuss the aspect of pricing forward contracts based on a class of stationary models, and review some implications.

Mathematics Subject Classification (2010). Primary 60G10; Secondary 91B28, 65C05, 65C20.

Keywords. Energy markets, spot modeling, forward pricing, Lévy semistationary processes, numerical simulation, Fourier transform.

1. A Brief Survey of the Basics of Power Markets

Power markets world-wide have been liberalized over the last decades. The Nordic exchange NordPool starting in 1992 was one of the first market places where one could trade in power products. Nowadays, most developed countries have an organized market for power, where prices are determined by supply and demand.

The structure of these markets are reasonably similar, and we shall use NordPool and the German market EEX as the prime examples of such power markets. Typically, there is a so-called *spot market*, where one can buy or sell electricity. To operate on this market, one must be either a producer or a consumer of electricity, since any trade here involves a physical distribution of electricity from the seller to

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the buyer. The market place offers *forward and futures* contracts which are settled with respect to the power price over a pre-defined period. These periods are typically weeks, months, quarters and years at the EEX and NordPool markets. For example, one may buy forward contracts delivering electricity over the year 2012, 2013 and 2014. Since this delivery is *financial*, in the sense that a buyer of such a forward contract will receive the money-equivalent of the power delivered against the agreed forward price. Hence, one does not need to have physical installations to distribute power to act in this market segment. Finally, many of the developed power markets, like NordPool and EEX, organize European call and put options on the forward/futures. These are of course also financial contracts.

The spot market is highly different than other commodity spot markets in that prices are discrete, describing the price for the distribution of 1MW of power over an hour. The prices are settled on an auction *the day before*. This means that all players in the spot market hand in price and volume bids for all or some of the 24 hours next day. The market then collects all the buy and sell bids into *demand* and *supply* curves, one for each of the 24 hours, describing price as a function of volume implied by the bids. The intersection of these two curves defines the spot price for a particular hour. These 24 prices are revealed in the afternoon the day before. Any distribution of electricity at a given hour the next day will be at the spot price determined by the market for that particular hour. Prices in the weekend and Mondays are determined in the auction on Friday.

Noteworthy is that spot prices are truly time series with a fixed time resolution. In fact, one may view the spot market as a forward market, where the price is the (futures) price for delivery of 1MWh electricity at a specific hour.

There are many fundamental factors driving the spot prices, and the major ones include the fuels for power generation and the forces for demand. In the German EEX market coal and gas are used to a large extent for power production, and the varying cost of purchasing these commodities on the market will influence the power price significantly. Nuclear power is a major source of electricity, and relatively cheap in production compared to gas and coal fired plants. Gas and coal fired plants also must take into account carbon emissions, and must purchase allowance certificates on the market. The price of these allowances will also impact power prices. In the Nordic market, hydro power is dominating, which of course is a clean and renewable source of energy. Market integration via connecting cables indirectly transfer the cost of emission to such energy production. Another source of renewable power is wind, which is predicted to play a major role in the future power generation. Interestingly, wind power may lead to *negative* power prices, something one has in fact already observed at several instances in the EEX market. The reason being that wind power has priority in the grid, and the unexpected occurence of high production will lead to over-capacity. Someone will then get paid for increasing their consumption of power. On the demand side, temperatures play a major role. For example, in the Nordic market, cold winter periods are usually giving higher than normal power prices since demand goes up. In the NordPool market, one may get really high prices if such cold periods come along with low reservoir levels in the magazines for hydro power production, as could be observed in January 2011. Another interesting driver of power prices in the Nordic area is the snow situation in the spring, since this gives predictions for the coming water inflow into the reservoirs.

Sudden outages of power plants may lead to significant spikes in the power prices. Such spikes may also be a result of sudden changes in demand combined with outages in plants. For example, in the summer of 2009 Europe had a very hot period, where the use of air-conditioning significantly went up. On the other hand, power plants had to shut down their production due to lack of river water cold enough to cool their generators. This led to unusually high electricity prices.

A forward contract on power spot delivers financially the spot over an agreed period of time. For example, a forward contract with delivery in July, will give the buyer a stream of money in return for the agreed forward price. The stream of money will be the sum of the hourly spot prices over the month of July,

$$\sum_{i=1}^{31\times 24} S(i)$$

with $S(i)$ being the spot price for hour i in July. If the contract was entered at a time $t \leq \text{July}1$, with a forward price $F(t,T_1,T_2)$, T_1 being first hour of July 1, and T_2 being last hour of July 31, then the buyer of the contract pays in return for the money stream

$$(T_2 - T_1)F(t,T_1,T_2)\,.$$

From this we see that the forward price is stated in terms of Euro/MWh, which is the market convention. In essence, one is actually swapping a floating spot price $S(i)$ against a fixed spot price $F(t,T_1,T_2)$ with such an electricity forward. For this reason forward contracts written on electricity are sometimes called *swaps*.

At NordPool and EEX one can buy or sell European call and put options on a selection of forward contracts. In the OTC market, however, one finds an abundance of different types of exotic derivatives, written on forwards or on power spot prices directly. *Spread* options are contracts written on the price spread between different markets. A typical example is the spark spread, which is an option written on the difference between electricity and gas, gas being a fuel for electricity (we refer to Carmona and Durrleman [11] for an extensive discussion and analysis of spread options in energy markets). Other spread contracts are the so-called *Contracts-for-Difference* (CfD), which are traded on the NordPool market and settled on the difference between spot prices in various areas (for example, the spread between Norwegian and Swedish spot prices, which are often different due to congestion). Hence, CfD's are locational spread contracts. Another important class of options in power markets is *swing options*, which are options with multiple exercise rights and volume control. A simple example could be a contract where the holder has the right to buy electricity at a fixed price K at N different hours in a pre-defined period, being for example a month. Each time the holder decides to use her option, she can also decide the volume of electricity to be purchased, typically within some defined limits. The problem of the buyer is to choose the optimal N hours in the period, and the optimal volume each time to purchase. Hence, the analysis of swing options involves use of stochastic control theory (see Benth, Lempa and Nilssen [7] for the analysis of a flexible load contract, as an example)

2. Models for the spot price dynamics

As we have discussed above, the spot price of electricity is quoted for each hour of the day, and in this respect may be seen as a time series. A curious fact in these markets is that the spot moves at discrete times, whereas the forward and futures contracts are traded continuously. We want to discuss this in some more detail, and motivate a continuous-time modelling approach for the spot price dynamics of electricity.

We introduce a continuous-time stochastic process $S(t)$ on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, which we intepret as the *unobserved* instantaneous spot price of electricity, that is, the price of electricity at time t with delivery in the interval $[t, t+dt)$. Further, let the filtration $\mathcal{F}_t$ model the stream of information in the market. We can think of $S(t)$ as the price of electricity market participants know they would have to pay, *if* they could buy electricity at time t with delivery over the infinitesimal time interval $[t, t+dt]$. Integrating $S(t)$ over a specific hour would then yield a model for the actual spot price observed in the market. As is usual in the literature, we suppose that $S(t)$ is the electricity spot price, and that $S(t_d^i) = s_d^i$ where s_i^d is the observed spot price in the market at time t_d^i, that is, at hour i on day d.

We review some of the popular stochastic processes which have been suggested in the literature, leading up to stationary processes which will be our main focus next.

Lucia and Schwartz [15] proposed a two-factor Gaussian model for the time dynamics of spot prices observed in the NordPool market. They consider both an arithmetic and geometric model of the form

$$S(t) = \Lambda(t) + X(t) + Y(t),$$

where $S(t)$ is the spot price at time t in the arithmetic case, and the *logarithmic* spot price at time t for the geometric case. Here, $\Lambda(t)$ is the deterministic seasonality function, modelling the trend and mean variations of prices. Furthermore, X and Y are two stochastic processes defined as

$$dX(t) = -\alpha X(t)\,dt + \sigma\,dB(t),$$

and

$$dY(t) = \mu\,dt + \eta\,dW(t).$$

The parameters α, σ, μ and η are all constants and positive, except possibly μ. The processes B and W are correlated Brownian motions, with incremental correlation $\rho\,dt$, $\rho \in [-1, 1]$. The Lucia–Schwartz model explains the evolution of power spot prices through a short-term factor X being stationary, and a long-term non-stationary factor Y which is supposed to explain for example inflation, increasing demand and fuel costs.

By using Brownian motion driven factors, the Lucia–Schwartz model fails to account for the spikes frequently observed in power spot prices, being sudden huge price jumps followed by a strong reversion back to "normal levels". A spike in power prices can be of several magnitudes, and occurs as a result of sudden imbalances in supply and/or demand, like for example an unexpected shut-down of a major power plant. To account for the jumps in prices, Cartea and Figueroa [12] introduces a one-factor model for the spot price evolution in the UK power market as

$$S(t) = \Lambda(t)\exp(X(t)),$$

where

$$dX(t) = -\alpha X(t)\,dt + \sigma\,dB(t) + dL(t),$$

and $L(t)$ being a compound Poisson process. The unfortunate effect of such an approach is that the speed of mean reversion α is the same for both spikes (modelled through jumps of L) and the normal variations (modelled through B). We expect the mean reversion to be far stronger in case of spikes than in a market where prices fluctuate "normally", and this is indeed the case when doing estimation on data (see Benth, Kiesel and Nazarova [6]).

Of course, choosing a spot price model of an exponential form ensures positivity of prices. However, as we shall see later, such models are in general not suitable for (analytical) pricing of power forward contracts, since these are settled on the average of the spot over a delivery period. Arithmetic models constitute an attractive alternative.

Benth, Kallsen and Meyer-Brandis [5] suggested a general multi-factor model of the form

$$S(t) = \Lambda(t) + \sum_{i=1}^{n} X_i(t)$$

where

$$dX_i(t) = -\alpha_i X_i(t)\,dt + dL_i(t),$$

and $\alpha_i > 0$ are constants, for $i = 1,\ldots,n$. To ensure positivity of prices, the 'function $\Lambda(t)$ is interpreted as a seasonal "floor", and $L_i(t)$ are Lévy processes with only positive jumps, i.e., subordinators. With recent years' market developments, the condition of positivity of prices is no longer effective. As we indicated in the previous section, both EEX and NordPool have opened up for negative spot prices. Thus, one may suppose that one or more of the factors X_i can be driven by general Lévy processes. We refer to Benth, Šaltytė Benth and Koekebakker [8] for a general account on multi-factor models in power markets.

Garcia, Klüppelberg and Müller [14] have proposed a continuous-time autoregressive moving-average (CARMA) model for power spot prices. A CARMA model is defined via a special type of multi-dimensional Ornstein–Uhlenbeck process. Define the $p \times p$ matrix

$$A = \begin{bmatrix} \mathbf{0}_{p-1} & I_{p-1} \\ -\alpha_p \dots & \dots - \alpha_1 \end{bmatrix}$$

with α_i positive constants, $i = 1, \dots, p$, $\mathbf{0}_{p-1}$ the $p-1$-dimensional column vector of zero's, and I_{p-1} the $(p-1) \times (p-1)$-dimensional identity matrix. Define $\mathbf{X}(t)$ to be the p-dimensional Ornstein–Uhlenbeck process

$$d\mathbf{X}(t) = A\mathbf{X}(t)\, dt + \mathbf{e}_p dL(t),$$

with $\mathbf{e}_k$ the kth Euclidean unit vector and $L(t)$ a Lévy process. A process $Y(t) = \mathbf{b}'\mathbf{X}(t)$ is called a CARMA(p, q) process when the coefficients $b_0, b_1, \dots, b_{p-1}$ of $\mathbf{b}$ satisfy $b_q \neq 0$ and $b_j = 0$ for $q < j < p$.

The models discussed above consist in general of one or more stationary factors, along with a possibly non-stationary factor. Recently, a general class of stochastic processes called Lévy semistationary (LSS) processes have been proposed by Barndorff-Nielsen et al. [1], encompassing the models described above. Assume that the spot price is

$$S(t) = \Lambda(t) + \sum_{i=1}^{n} X_i(t), \tag{2.1}$$

with

$$X_i(t) = \int_{-\infty}^{t} g_i(t, s)\sigma_i(s)\, dL_i(s),$$

for deterministic "kernel functions" $g_i(t, s)$ and stochastic processes $\sigma_i(s)$, $i = 1, \dots, n$. We restrict to (two-sided) Lévy processses $L_i(t)$ being square integrable, and suppose that $g_i(t, s)$ and $\sigma_i(s)$ satisfy suitable integrability conditions to ensure well-defined processes $X_i(t)$ (see, e.g., Protter [16]). If $g_i(t, s) = g_i(t - s)$, we say that $X_i(t)$ is an LSS process. Typically, $\sigma_i(t)$ plays the role as stochastic volatility, and is conveniently assumed to be independent of $L_i(t)$ in the modelling. Furthermore, $\sigma_i^2(t)$ is frequently assumed to be an LSS process itself. If $\sigma_i^2(t)$ is stationary, then $X_i(t)$ also becomes stationary, whereas any g_i explicitly depending on t and s separately will lead to non-stationary models. One of the novelties of the LSS modelling framework is that by integrating from $-\infty$ rather than from zero, we model power prices *in stationarity*. As an example, let $\sigma_i = 1$ and

$$g(t - s) = \mathbf{b} \exp(A(t - s))\mathbf{e}_p$$

and we recover the stationary solution of a CARMA(p, q) model introduced above. Barndorff-Nielsen et al. [1] estimate an LSS model to EEX spot price data with L being a normal inverse Gaussian Lévy process and $g(t-s)$ a sum of two exponential functions.

3. A brief analysis of Lévy semistationary processes

LSS processes consitute an important general class of moving-average models for electricity spot prices. In this section we make an independent study of some properties of LSS processes interesting for modelling purposes.

Define the LSS process

$$X(t) = \int_{-\infty}^{t} g(t-s)\sigma(s)\,dL(s) \tag{3.1}$$

with L being a two-sided square integrable Lévy process with mean zero, that is, $\mathbb{E}[L(1)] = 0$. We restrict our attention to LSS processes with a stochastic volatility process $\sigma(t)$ where $\sigma^2(t)$ is modelled as a *stationary* process *independent* of L. Denote for the sequel the second moment of $\sigma(t)$ by $\kappa_2 := \mathbb{E}[\sigma^2(t)]$, which we assume to be finite. Furthermore, g is assumed to be a Borel measurable function on $\mathbb{R}_+$, the positive half-line including the origin, such that $\int_0^\infty g^2(u)\,du < \infty$. These conditions ensure that $X(t)$ is well defined, and square-integrable (see Protter [16]).

The characteristic function of X is easily computed by conditioning on the volatility process $\sigma(s)$:

$$\mathbb{E}[\exp(\mathrm{i}\theta X(t))] = \mathbb{E}[\exp(\int_{-\infty}^{t} \psi(\theta g(t-s)\sigma(s))\,ds)],$$

where $\psi(\theta)$ is the cumulant (i.e., the log-characteristic function) of $L(1)$. We observe that

$$\mathbb{E}[X(t)] = -\mathrm{i}\psi'(0)\int_{-\infty}^{t} g(t-s)\mathbb{E}[\sigma(s)]\,ds = 0$$

since $\mathbb{E}[L(1)] = \psi'(0) = 0$ by assumption. Furthermore, we find

$$\mathbb{E}[X^2(t)] = -\psi''(0)\kappa_2 \int_0^\infty g^2(s)\,ds. \tag{3.2}$$

Hence, $X(t)$ is second-order stationary. Note that if $\sigma(s) = 1$, then the cumulant function of $X(t)$ is

$$\ln \mathbb{E}[\exp(\mathrm{i}\theta X(t))] = \int_0^\infty \psi(\theta g(s))\,ds.$$

In the following proposition we show that LSS processes inherit Hölder continuity in variance from the kernel function g, which in turn may be applied to determine pathwise properties of the spot price model (recall (2.1)).

Proposition 3.1. *Suppose g is locally L^2-Hölder continuous of order $0 < p \le 1$, in the sense that there exists a $\delta > 0$ such that for all $0 < y \le \delta$ it holds*

$$\int_0^\infty (g(u+y) - g(u))^2\,du \le C|y|^{2p}. \tag{3.3}$$

Then, for $|t-s| \le \delta$,

$$\mathbb{E}\left[|X(t) - X(s)|^2\right] \le K|t-s|^p$$

for some positive constant K independent of t and s.

Proof. Suppose that $t > s$, then

$$X(t) - X(s) = \int_{-\infty}^{s} (g(t-u) - g(s-u))\sigma(s)\,dL(u) + \int_{s}^{t} g(t-u)\sigma(s)\,dL(u).$$

Now from (3.2) and the independent increment property of L

$$\begin{aligned}
&\mathbb{E}\left[|X(t) - X(s)|^2\right] \\
&\quad = (-\psi''(0))\kappa_2 \left\{ \int_{-\infty}^{s} (g((t-u) - g(s-u))^2\,du + \int_{s}^{t} g^2(t-u)\,du \right\} \\
&\quad = (-\psi''(0))\kappa_2 \left\{ \int_{0}^{\infty} (g(v + (t-s)) - g(v))^2\,dv + \int_{0}^{t-s} g^2(v)\,dv \right\}.
\end{aligned}$$

But, by a simple change of variables,

$$\int_{0}^{\infty} g^2(v + (t-s))\,dv = \int_{t-s}^{\infty} g^2(v)\,dv,$$

and therefore, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\mathbb{E}\left[|X(t) - X(s)|^2\right] &= 2(-\psi''(0))\kappa_2 \int_{0}^{\infty} g(v)(g(v) - g(v + (t-s)))\,dv \\
&\leq 2(-\psi''(0))\kappa_2 (\int_{0}^{\infty} g^2(v)\,dv)^{1/2} \\
&\qquad \times (\int_{0}^{\infty} (g(v + (t-s)) - g(v))^2\,dv)^{1/2}.
\end{aligned}$$

Hence, the result follows by the assumption on g. $\square$

Note that a sufficient condition for (3.3) on g to hold is if there exists a function h on $L^2(\mathbb{R}_+)$ such that

$$|g(u + y) - g(u)| \leq h(u)|y|^p \tag{3.4}$$

for a $\delta > 0$ such that $|y| < \delta$ and $0 < p \leq 1$.

The classical Ornstein–Uhlenbeck process with $g(x) = \exp(-\alpha x)$ satisfies the condition (3.3) in the lemma. However, in this case we can do a direct calculation to find

$$\mathbb{E}\left[|X(t) - X(s)|^2\right] = \frac{-\psi''(0)\kappa_2}{\alpha}(1 - \exp(-\alpha(t-s))).$$

Thus, for $t - s$ small, we have $1 - \exp(-\alpha(t-s)) \sim t - s$, and the process is locally Lipschitz continuous in variance. Another example which is relevant in electricity markets is to choose $g(x) = 1/(x + a)$, for $a > 0$ (see Barndorff-Nielsen et al. [1] and Bjerksund, Rasmussen and Stensland [9] for a motivation of this model). Then

$$|g(u + y) - g(u)| = \frac{|y|}{(u + a)(u + y + a)} \leq \frac{|y|}{(u + a)^2}.$$

Thus, (3.4) is satisfied with $p = 1$ and $h(u) = 1/(u + a)^2 \in L^2(\mathbb{R}_+)$.

In general an LSS process is not a semimartingale. Conditions that guarantee the semimartingale property for moving average processes driven by Lévy processes have been studied in, e.g., [4]. We give conditions for the semimartingale property of an LSS process:

Proposition 3.2. *Suppose that g is absolutely continuous with a derivative g' almost everywhere, and that $\int_0^\infty |g'(s)|^2\,ds < \infty$. If $|g(0)| < \infty$, then the process $X(t)$ for $t \geq 0$ is a semimartingale with representation*

$$dX(t) = \int_{-\infty}^{t} g'(t-s)\sigma(s)\,dL(s)\,dt + g(0)\sigma(t)\,dL(t).$$

Proof. First, decompose $X(t)$ for $t \geq 0$ as

$$X(t) = \int_{-\infty}^{0} g(t-s)\sigma(s)\,dL(s) + \int_0^t g(t-s)\sigma(s)\,dL(s).$$

By absolute continuity,

$$g(t-s) - g(-s) = \int_0^t g'(u-s)\,du$$

for $t \geq 0$, and therefore, using the stochastic Fubini theorem (see, e.g., Protter [16]),

$$\begin{aligned}\int_{-\infty}^{0} g(t-s)\sigma(s)\,dL(s) &= \int_{-\infty}^{0} (\int_0^t g'(u-s)\,du + g(-s))\sigma(s)\,dL(s)\\ &= \int_0^t \int_{-\infty}^{0} g'(u-s)\sigma(s)\,dL(s)\,du + \int_{-\infty}^{0} g(-s)\sigma(s)\,dL(s).\end{aligned}$$

Note that the last term is equal to $X(0)$. Next, using stochastic Fubini theorem again, we find

$$\begin{aligned}\int_0^t \int_0^s g'(s-u)\sigma(u)\,dL(u)\,ds &= \int_0^t \int_u^t g'(s-u)\,ds\sigma(u)\,dL(u)\\ &= \int_0^t \{g(t-u) - g(0)\}\,\sigma(u)\,dL(u)\\ &= \int_0^t g(t-u)\sigma(u)\,dL(u) - g(0)\int_0^t \sigma(u)\,dL(u).\end{aligned}$$

Recall that L is a square-integrable Lévy process with zero expectation, thus a martingale. The proof is complete. □

We remark that electricity spot is not storable, and thus not tradeable in the financial sense. There is therefore nothing wrong with modelling electricity prices by non-semimartingales, contrary to what is the usual situation in mathematical finance models.

The next lemma concerns continuity of LSS processes with respect to the kernel function g.

Lemma 3.3. *Consider two LSS processes* $X(t) = \int_{-\infty}^t g(t-s)\sigma(s)\,dL(s)$ *and* $Y(t) = \int_{-\infty}^t h(t-s)\sigma(s)\,dL(s)$*, where* $\int_0^\infty g^2(s)\,ds < \infty$ *and* $\int_0^\infty h^2(s)\,ds < \infty$*. Then,*

$$\mathbb{E}\left[|X(t) - Y(t)|^2\right] = (-\psi''(0))\kappa_2 |g - h|^2_{L^2(\mathbb{R}_+)}.$$

Proof. The proof goes by a straightforward calculation using (3.2):

$$\begin{aligned}\mathbb{E}\left[|X(t) - Y(t)|^2\right] &= \mathbb{E}\left[\left(\int_{-\infty}^t (g(t-s) - h(t-s))\sigma(s)\,dL(s)\right)^2\right] \\ &= (-\psi''(0))\kappa_2 \int_{-\infty}^t (g(t-s) - h(t-s))^2\,ds \\ &= (-\psi''(0))\kappa_2 \int_0^\infty (g(x) - h(x))^2\,dx.\end{aligned}$$

Hence, the lemma follows. □

In practice, for a given LSS spot price model, we would estimate the kernel function g from observed price data in the market. Such estimates are prone to statistical error, and hence we find g_ϵ rather than g itself, where ϵ is the error induced from statistical estimation, being a function of the number of data n at hand. The above result shows that the variance of $X(t)$ is robust towards this estimation error.

Let us consider an example of an approximation of a singular kernel g coming from applications to turbulence (see Barndorff-Nielsen and Schmiegel [3]). Suppose g is of the form

$$g(x) = x^{\nu-1}\mathrm{e}^{-\lambda x},$$

where $1/2 < \nu < 1$ and $\lambda > 0$. Note that g is singular at the origin, and $X(t)$ is thus, in general (unless L has bounded variaton), not a semimartingale process. By Proposition 3.2 and Lemma 3.3 we may approximate $X(t)$ with a semimartingale LSS process that has the non-singular kernel function

$$h_\epsilon(x) = \begin{cases} g(x) & \text{if } x \geq \epsilon \\ g(\epsilon) & \text{if } x \in [0, \epsilon]. \end{cases}$$

We easily find that

$$\begin{aligned}\int_0^\infty (g(x) - h_\epsilon(x))^2\,dx &\leq 2\int_0^\epsilon x^{2\nu-2}\mathrm{e}^{-2\lambda x}\,dx + 2\epsilon^{2\nu-1}\mathrm{e}^{-2\lambda\epsilon} \\ &\leq \frac{2\epsilon^{2\nu-1}}{2\nu-1} + 2\epsilon^{2\nu-1} = \frac{4\nu\epsilon^{2\nu-1}}{2\nu-1}.\end{aligned}$$

Thus we have the rate

$$|g - h_\epsilon|^2_{L^2(\mathbb{R}_+)} \leq \frac{4\nu\epsilon^{2\nu-1}}{2\nu-1},$$

from which we may observe that the closer ν is to $1/2$, the slower the rate is. If we want to simulate from $X(t)$, one would do numerical integration of $g(t-s)$ with respect to the paths of $L(s)$ and $\sigma(s)$ for $s \leq t$. To avoid problems around the singularity $s = t$, we can use h_ϵ rather than g in the numerical integration, with an error that we can control.

Another application of Lemma 3.3 is to view the LSS process $X(t)$ as a sliding window. To this end, fix $N > 0$, and consider

$$X_N(t) := \int_{t-N}^{t} g(t-s)\sigma(s)\,dL(s).$$

Since,

$$X_N(t) = \int_{-\infty}^{t} g(t-s)1(N \geq t-s)\sigma(s)\,dL(s)$$

we find from the lemma that

$$\mathrm{E}\left[|X(t) - X_N(t)|^2\right] = (-\psi''(0))\kappa_2 \int_N^\infty g^2(x)\,dx.$$

Since g is square integrable on $\mathbb{R}_+$, the integral on the right-hand side will tend to zero as N increases. This gives the interpretation of LSS processes as limits of moving average over a sliding window.

4. Simulation of Lévy semistationary processes

In this section we discuss methods to simulate an LSS process which is potentially more efficient than numerical brute-force integration. A "quick-and-dirty" Euler scheme is proposed, where one is iterating over some approximation of the kernel function g. The method introduces an error, and has restricted application. Alternatively, by applying the Fourier transform, we can do an Euler scheme in parallel with fast Fourier transform inversion in order to simulate a path of the LSS process. This scheme is iterative, and at the same time accounts for the memory in the process.

The problem to simulate the LSS process is as follows: let us say that we have $X(t)$, and want to know what is $X(t+\delta)$ for some increment in time $\delta > 0$. From the definition of $X(t)$, we want to simulate a sample from

$$X(t+\delta) = \int_{-\infty}^{t+\delta} g(t+\delta-s)\sigma(s)\,dL(s)\,.$$

A natural approach would be to integrate numerically $g(t+\delta-s)\sigma(s)$ from $-\infty$ to $t+\delta$ for a path of $L(s)\,, s \leq t+\delta$. If we have this path at time t, we simulate an independent increment $\Delta L(t) := L(t+\delta) - L(t)$, and use the simulated values of $L(s)\,, s \leq t$ along with this new outcome to numerically integrate $g(t+\delta-s)\sigma(s)$. We see that we cannot use information of $X(t)$, since this is defined as the integral of $g(t-s)\sigma(s)$. In order to obtain $X(t+\delta)$, we must perform a complete re-integration, using the whole path of $L(s)$.

To make matter slightly more simple, we suppose that the volatility $\sigma(t) = 1$ in the following. An approximative simulation method could be to use the following

Euler scheme idea: suppose that there exists a (positive) function $h(\delta)$ such that for all $u \geq 0$

$$|g(u+\delta) - h(\delta)g(u)| \leq c(u,\delta), \tag{4.1}$$

for some function $c(u,\delta)$. Note that naturally $h(\delta) \to 1$ when $\delta \to 0$, at least as long as g is continuous. Define the time series

$$\widetilde{X}(t+\delta) = h(\delta)\widetilde{X}(t) + \int_t^{t+\delta} g(t+\delta-s)\,dL(s), \tag{4.2}$$

for a given $\delta > 0$. We suppose that $\widetilde{X}(0) = X(0)$, and we note that the stochastic integral on the right-hand side becomes independent of $\widetilde{X}(t)$. Thus, we can easily simulate the time series at different time points given that we can sample from $X(0)$ and the independent random variables

$$\int_t^{t+\delta} g(t+\delta-s)\,dL(s).$$

Note that these increments have cumulant function given by

$$\ln \mathbb{E}[\exp(\mathrm{i}\theta \int_t^{t+\delta} g(t+\delta-s)\,dL(s))] = \int_0^{\delta} \psi(\theta g(s))\,ds.$$

Hence, their characteristics are independent of current time t. We easily derive the mean to be zero and variance to be $-\psi''(0)\int_0^{\delta} g^2(s)\,ds$.

We want to investigate to what extent $\widetilde{X}(t)$ resembles the path of $X(t)$. The following proposition on the closeness of one-step simulations holds (here, $\|\cdot\|_2$ denotes the L^2-norm):

Proposition 4.1. *Suppose g satisfies the condition* (4.1) *with $u \mapsto c(u,\delta)$ being square integrable for all $\delta \leq \epsilon$ for some $\epsilon > 0$. Then, for $\delta \leq \epsilon$, it holds*

$$\|\widetilde{X}(t+\delta) - X(t+\delta)\|_2 \leq h(\delta)\|\widetilde{X}(t) - X(t)\|_2 + \left((-\psi''(0))\int_0^{\infty} c^2(u,\delta)\,du\right)^{1/2}.$$

Proof. We have

$$\begin{aligned} X(t+\delta) &= \int_{-\infty}^{t+\delta} g(t+\delta-s)\,dL(s) \\ &= \int_{-\infty}^{t} g(t+\delta-s)\,dL(s) + \int_t^{t+\delta} g(t+\delta-s)\,dL(s) \\ &= h(\delta)X(t) + \int_{-\infty}^{t} \left(g(t+\delta-s) - h(\delta)g(t-s)\right)\,dL(s) \\ &\quad + \int_t^{t+\delta} g(t+\delta-s)\,dL(s). \end{aligned}$$

From the assumption on g in (4.1) and the triangle inequality, the proposition follows. □

We now apply this to analyse the variance of $\widetilde{X}(1)-X(1)$ for a given uniform sampling $\delta = 1/N$ of the unit interval. Assume further for notational simplicity that $-\psi''(0) = 1$. Suppose that $h(\delta) < 1$. Define

$$\Delta_k = \|\widetilde{X}(k/N) - X(k/N)\|_2$$

for $k = 0, 1, 2, \ldots, N$. We have that $\Delta_0 = 0$, and from the proposition we see that

$$\Delta_{k+1} \leq h(1/N)\Delta_k + \left(\int_0^\infty c^2(u, 1/N)\, du\right)^{1/2}.$$

Iterating this, we find that

$$\begin{aligned}\Delta_N &\leq \left(\int_0^\infty c^2(u, 1/N)\, du\right)^{1/2} \sum_{k=0}^{N-1} h(1/N)^k \\ &= \left(\int_0^\infty c^2(u, 1/N)\, du\right)^{1/2} \frac{1-h(1/N)^N}{1-h(1/N)} \\ &\leq \frac{\left(\int_0^\infty c^2(u, 1/N)\, du\right)^{1/2}}{1-h(1/N)}.\end{aligned}$$

In order for Δ_N to tend to zero as the partition gets finer, that is, as N tends to infinity, the fraction on the right-hand side must have a limit equal to zero. Note that if g is a continuous function (which is natural in most applications), $h(1/N)$ should tend to one and $c(u, 1/N)$ to zero as $N \to \infty$. By dominated convergence theorem, we get that the denominator also tends to zero as $N \to \infty$. So, we get a 0/0-expression, and this should have a zero limit in order to get convergence. However, it is more likely that the limit is bigger than zero, which then will be a maximal bound for the simulation error, as a result of simulating using an approximative kernel separation.

Let us look at some examples which satisfy the conditions on g. As a trivial example, let $g(u) = \exp(-\alpha u)$, yielding that $X(t)$ is an OU-process. Then $h(\delta) = \exp(-\alpha\delta) < 1$ and $c(u, \delta) = 0$. In this case we have an exact simulation of $X(t)$ for all choices of N, which is known. A less trivial example could be

$$g(u) = \frac{a}{u+b}\exp(-\alpha u). \tag{4.3}$$

Such a kernel function will be a blend of the choice suggested by Bjerksund et al. [9] and a standard OU-process, and thus constitute a potential kernel for applications in electricity. Choosing

$$h(\delta) = \frac{b}{b+\delta}\exp(-\alpha\delta),$$

we find

$$\begin{aligned}|g(u+\delta) - h(\delta)g(u)| &= \frac{a}{b}h(\delta)\mathrm{e}^{-\alpha u}\frac{\delta u}{(u+b)(u+b+\delta)} \\ &\leq \frac{a}{b}h(\delta)\mathrm{e}^{-\alpha u}\frac{\delta}{u+b}.\end{aligned}$$

Thus, let $c(u,\delta)=a\delta h(\delta)\exp(-\alpha u)/(b(u+b))$, and it holds that

$$\int_0^\infty c^2(u,\delta)\,du=\frac{a^2}{b^2}\left(b^{-1}-2\alpha\mathrm{e}^{2\alpha b}\mathrm{E}_1(2\alpha b)\right)\delta^2h^2(\delta)\le\frac{a^2}{b^3}\delta^2h^2(\delta),$$

where $\mathrm{E}_1(z)$ is the exponential integral. But, as δ tends to zero, $\delta h(\delta)/(1-h(\delta))$ will converge to $b/(\alpha b+1)$, and hence the error Δ_N will be bounded by a constant

$$\Delta_N\le\frac{a}{\sqrt{b}(1+\alpha b)},$$

as $N\to\infty$. If α is big, we can make this error small. A big α will correspond to a fast decay of the autocorrelation function for small lags, which is interpretable as a fast mean-reversion of the process X. In the context of electricity, this is a relevant case.

An alternative to the approximative Euler scheme is a more robust and accurate approach that makes use of the idea to convert integration with respect to g into a Fourier transform. We focus on LSS processes with stochastic volatility σ and a kernel function $g\in L^2(\mathbb{R}_+)$. Suppose that there exists a $\lambda>0$ such that the function

$$g_\lambda(x):=g(x)\mathrm{e}^{\lambda x}\in L^1(\mathbb{R}_+). \tag{4.4}$$

Let the Fourier transform of g_λ be (see Folland [13])

$$\widehat{g}_\lambda(y)=\int g_\lambda(x)\mathrm{e}^{-\mathrm{i}xy}\,dx.$$

Suppose now that $\widehat{g}_\lambda\in L^1(\mathbb{R})$. Then the inverse Fourier transform exists, and we have (see Folland [13])

$$g(x)=\frac{1}{2\pi}\int\widehat{g}_\lambda(y)\mathrm{e}^{(-\lambda+\mathrm{i}y)x}\,dy.$$

Insert this into the definition of the LSS process to get

$$X(t)=\frac{1}{2\pi}\int\widehat{g}_\lambda(y)\widehat{X}_\lambda(t,y)\,dy \tag{4.5}$$

where we have commuted integration using the stochastic Fubini theorem (see Protter [16]). Here,

$$\widehat{X}_\lambda(t,y)=\int_{-\infty}^t\mathrm{e}^{(\mathrm{i}y-\lambda)(t-s)}\sigma(s)\,dL(s).$$

Note that since $\lambda>0$, $\widehat{X}_\lambda(t,y)$ is a (complex-valued) LSS process for each $y\in\mathbb{R}$. We observe that for $\lambda=0$, the definition of $\widehat{X}_0(t,y)$ fails since the complex exponential has norm 1 (except under stronger conditions on σ than we have assumed here).

Fix $\delta > 0$, and we find

$$\begin{aligned}\widehat{X}_\lambda(t+\delta, y) &= \int_{-\infty}^{t+\delta} \mathrm{e}^{(\mathrm{i}y-\lambda)(t+\delta-s)}\sigma(s)\, dL(s) \\ &= \mathrm{e}^{(\mathrm{i}y-\lambda)\delta}\widehat{X}_\lambda(t,y) + \mathrm{e}^{(\mathrm{i}y-\lambda)\delta}\int_t^{t+\delta} \mathrm{e}^{(\mathrm{i}y-\lambda)(t-s)}\sigma(s)\, dL(s).\end{aligned}$$

Now, the residuals can be simulated by the approximation

$$\int_t^{t+\delta} \mathrm{e}^{(\mathrm{i}y-\lambda)(t-s)}\sigma(s)\, dL(s) \approx \sigma(t)\Delta L(t).$$

One can show that the variance of the error in this approximation is independent of y, and is of order δ.

Hence, to simulate $X(t+\delta)$, we do the following: Discretize the Fourier domain $y_i\, , i = 1, \ldots N$.

1. Simulate $\Delta L(t)$
2. For each $i = 1, \ldots, N$, simulate $\widehat{X}_\lambda(t+\delta, y_i)$ from $\widehat{X}(t, y_i)$ and $\Delta L(t)$
3. Compute numerically the inverse Fourier transform in (4.5).

Note the advantages here: We have the same residual term for every y_i, except from a deterministic scaling by a complex exponential. This means that to simulate $\widehat{X}_\lambda(t+\delta, y)$, we simulate the outcome of *one* random variable Z, and then compute

$$\widehat{X}_\lambda(t+\delta, y) = \exp((\mathrm{i}y-\lambda)\delta)\left\{\widehat{X}_\lambda(t,y) + Z\right\}.$$

Hence, in step 2 above, we just need to have stored the N previous values of $\widehat{X}_\lambda(t, y_i)$ along with the simulated Z, in order to compute the next iterative step. Notice also that the number of sampling points N depends on the damping properties of $\widehat{g}_\lambda$. The faster $\widehat{g}_\lambda(y)$ decays to zero for large values of y, the smaller interval of y_i's can be chosen. We can also easily change the kernel function g, without having to redo the whole simulation algorithm, since this is going on independently of g. This may prove advantageous in estimation studies, where one may want to simulate over parametric g's in order to find the optimal one. Finally, another advantage compared to direct numerical integration is that with the latter, the accuracy is linked to how many sample points we simulate the Lévy process in time, whereas with the Fourier technique this is converted into sampling an integral over space instead.

In principle, we could simulate $\widehat{X}_\lambda(t, y)$ exactly. For example, if $\sigma(s) = 1$, we have that the residual is an independent outcome of a random variable Z with cumulant.

$$\ln \mathbb{E}[\exp(i\theta \int_t^{t+\delta} \exp((\mathrm{i}y-\lambda)(t-s))\, dL(s))] = \int_0^{\delta} \psi(\theta \mathrm{e}^{(\mathrm{i}y-\lambda)u})\, du$$

Thus, error is from numerical integration in Fourier domain only, and not connected to the simulations which are in principle exact.

To illustrate the Fourier transform method let us consider the problem of simulating the modified Bjerksund model (4.3), with constant volatility $\sigma = 1$ and Brownian motion, B, as the driving Lévy process:

$$X(t) = \int_{-\infty}^{t} \frac{a}{t-s+b} \exp(-\alpha(t-s))\, dB(s).$$

In order for us be able to apply the Fourier transform method we need to verify that (4.4) holds and that $\widehat{g}_\lambda \in L^1(\mathbb{R})$ holds for some $\lambda > 0$. Clearly condition (4.4) is satisfied if and only if $\lambda < \alpha$, but the second condition is a bit more problematic. Indeed the condition $\widehat{g}_\lambda \in L^1(\mathbb{R})$ together with the Riemann–Lebesgue Lemma implies that g should be a continuous function on the entire real line, which fails at the origin ($x = 0$). However this problem may be amended by approximating g by some function g_ϵ, and applying Lemma 3.3 for assessing the goodness. Another path is to observe from Lemma 3.3 that we may choose any function h extending g into the negative real line, with $g = h$ on $\mathbb{R}_+$, without changing the LSS process. To that end, for a given $M > 0$ consider the function $h \in L^2(\mathbb{R})$ defined by

$$h(x) = \begin{cases} g(x) & \text{if } x \geq 0 \\ p(x) & \text{if } x \in [-M, 0] \\ 0 & \text{if } x \leq -M \end{cases} \tag{4.6}$$

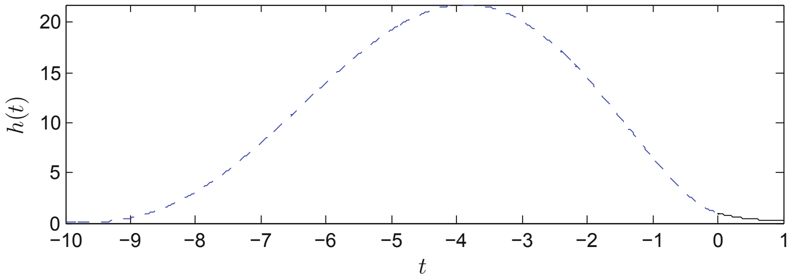
where $p(x) = \sum_{n=0}^{5} c_n x^n$ is a polynomial with coefficients determined by the conditions $p_\lambda(-M) = p'_\lambda(-M) = p''_\lambda(-M) = 0$, $p_\lambda(0) = g_\lambda(0)$, $p'_\lambda(0) = g'_\lambda(0)$, and $p''_\lambda(0) = g''_\lambda(0)$. Thus h_λ is a $C^2(\mathbb{R})$ function such that h agrees with g on $\mathbb{R}_+$ (see Figure 1). It is furthermore easy to check that $h'_\lambda, h''_\lambda \in L^1(\mathbb{R}_+)$ are continuous and vanish at infinity. Therefore (see Folland [13, Section 8.4]) $\widehat{h}_\lambda \in L^1(\mathbb{R})$ holds and we have a representation of the type (4.5) for the LSS process with the kernel function h, i.e.,

$$\begin{aligned} \int_{-\infty}^{t} g(t-s)\sigma(s)\, dL(s) &= \int_{-\infty}^{t} h(t-s)\sigma(s)\, dL(s) \\ &= \frac{1}{2\pi} \int \widehat{h}_\lambda(y) \widehat{X}_\lambda(t,y)\, dy. \end{aligned} \tag{4.7}$$

An advantage of this approach is that we are able to control the damping properties of $\widehat{h}_\lambda$ by means of calibrating the parameters $M > 0$ and $\lambda < \alpha$. In our case we get by integration by parts that

$$\begin{aligned} \widehat{h}_\lambda(y) &= \int_{-M}^{0} p(x) \mathrm{e}^{x(\lambda - \mathrm{i}y)}\, dx + \int_{0}^{\infty} \frac{a}{x+b} \mathrm{e}^{-x(\alpha-\lambda+\mathrm{i}y)}\, dx \\ &= \sum_{n=0}^{5} (-1)^n \frac{n! c_n - \mathrm{e}^{-M(\lambda-\mathrm{i}y)} p^{(n)}(-M)}{(\lambda - \mathrm{i}y)^{n+1}} + a \mathrm{e}^{b(\alpha-\lambda+\mathrm{i}y)}\, \mathrm{E}_1(b(\alpha - \lambda + \mathrm{i}y)), \end{aligned} \tag{4.8}$$

which is an integrable function with damping properties depending on the parameters M and λ (recall that $\mathrm{E}_1(z)$ is the exponential integral function). In fact one may choose the parameters such that the Fourier transform $\widehat{h}_\lambda$ will decrease



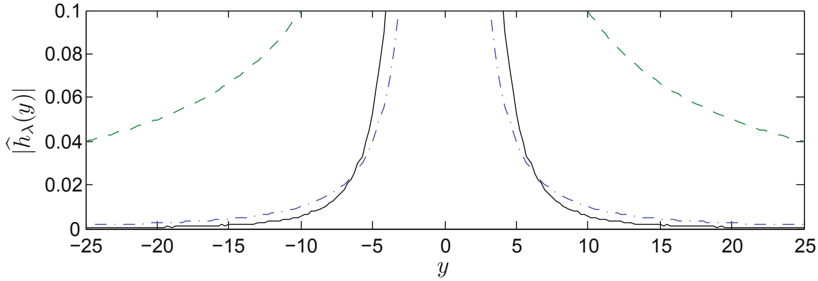


FIGURE 1. Above: h from (4.6) where $a = 1$, $b = 1$ and $\alpha = 1$. The dashed line is a polynomial which interploates the kernel function from the origin to $M = -10$. Below: The absolute value of the Fourier transform (4.8), where $\lambda = 0.95$ on the interval $[-25, 25]$. The dashed lines represent $y \mapsto 1/|y|$ and $y \mapsto 1/|y|^2$ respectively.

quite rapidly to zero as $|y| \to \infty$ (see Figure 1), and therefore one can numerically estimate the inverse Fourier transform (4.5) over a rather small domain.

Now consider comparing the Fourier transform and approximative Euler method (4.2) with numerical integration at each time step. We implemented all three methods in Matlab (R2011a) for the modified Bjerksund model on the unit interval $[0, 1]$ with parameters $a = 1$, $b = 1$ and $\alpha = 1$. A constant step size of $\delta = 0.01$ was used for increments in the time domain of all the methods. In the Fourier integration domain we estimated the integral (4.7) by means of cutting off its tails at ± 25, dividing the interval $[-25, 25]$ into 200 equal subintervals of length 0.25 and summing up the areas of the rectangles with length 1 and height equal to the distance from the left point of each subinterval to the integrand value at the same point. Note that in order to save time this numerical integration may be implemented by means of matrix multiplication. In our calculations we set $M = 10$ and $\lambda = 0.95$. In our experience given the extended kernel function h the selection of M and λ affects the shape of the Fourier transform $\widehat{h}_\lambda$ and thus the numerical integration in the Fourier domain. In the case of the modified Bjerksund model, we have observed that the Fourier transform $\widehat{h}_\lambda$ tends more rapidly to zero for high M. Whereas our numerical integration generally has lower error for λ close to one.

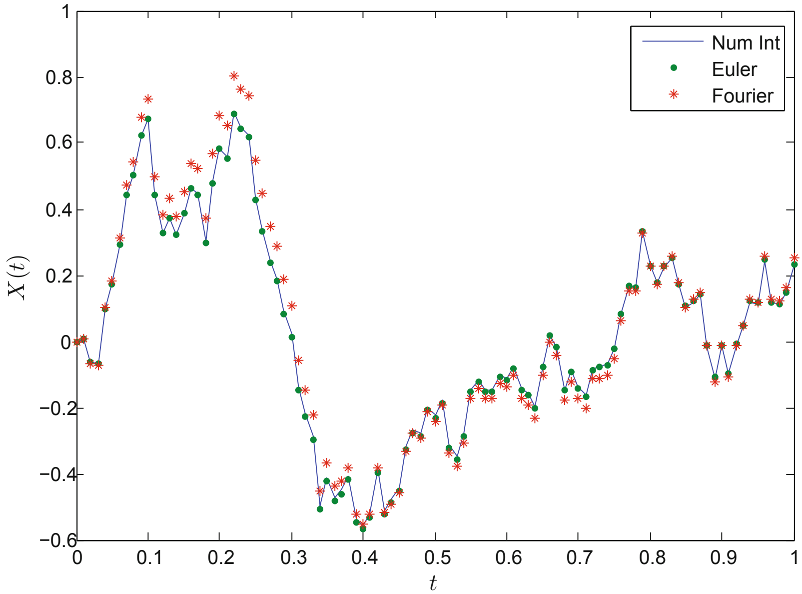


FIGURE 2. A comparison of simulation methods for the modified Bjerksund model with $a = 1$, $b = 1$, $\alpha = 1$ and $\lambda = 0.95$, using the Euler (4.2), numerical integration and Fourier methods on the interval $[0, 1]$ with $\delta = 0.01$ and the numerical integration in the Fourier domain on the interval $[-25, 25]$ with step size 0.25.

From this it is clear, that one should be careful when selecting the parameters of our Fourier method approach.

Using the tic,toc Matlab function we measured the effectivity of the respective methods in terms of speed. It turned out the Euler method was the fastest one, spending the time 0.00054sec, whereas the numerical integration method used 0.0023sec, and the Fourier method used 0.0021sec (the experiments were performed on a standard laptop computer). Note that the time given for the Fourier method excludes the computation of the Fourier transform, as repeated evaluation of the Fourier transform would not be required for simulation of multiple paths. From Figure 2, we see that the Fourier and approximative Euler both give very good replication of the "exact" path, here taken as the integral method. However it is evident that there is a rather big difference between the two methods in terms of how closely they follow the "exact" path. Indeed the path obtained by the approximative Euler scheme has a much lower error rate than the path obtained by the Fourier method. But that is hardly surprising given the extra source of error in the numerical integration of the inverse Fourier transform. We remark that the error in the Fourier method may potentially be lowered by more sophisticated methods of numerical integration, which in turn is likely to lead to increased computation time.

5. Deriving electricity forward prices from spot

Power markets trade in forward contracts with (financial) delivery of electricity over an agreed time period. Buying electricity forward[1] with delivery over the period $[T_1, T_2]$, will yield a profit/loss at time T_2 being

$$\int_{T_1}^{T_2} S(u)\, du - F(t, T_1, T_2)(T_2 - T_1),$$

where $F(t, T_1, T_2)$ is the *forward price* agreed with the seller of the contract at time $t \leq T_1$. As long as we assume a continuous-time spot price model, it is natural to suppose that delivery is the cumulative amount of electricity defined as the integral of $S(u)$ over the delivery period.

In case power would be a tradeable commodity, one could have used the buy-and-hold strategy to perfectly replicate the forward contract. In fact, we would have replicated delivery of $S(u)$ at each time instant $u \in [T_1, T_2]$. The result of such an exercise would be that

$$\begin{aligned} F(t, T_1, T_2) &= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(t) e^{-r(u-t)}\, du \\ &= \frac{S(t)}{r(T_2 - T_1)} \left\{ e^{-r(T_1 - t)} - e^{-r(T_2 - t)} \right\}, \end{aligned} \tag{5.1}$$

with r being the risk-free interest rate in the market. This forward price can be represented as the conditional expectation of the spot price, given a risk neutral probability Q. Hence,

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathbb{E}_Q[S(u) \,|\, \mathcal{F}_t]\, du, \tag{5.2}$$

where Q is an equivalent martingale measure for $S(t)$, that is, the discounted spot price is a martingale under Q.

In order to have the representation (5.2) of the forward price, we must have a spot price dynamics for which there exists (at least one) equivalent martingale measure Q. However, in the case of power, this condition turns out to be highly non-restrictive. As we recall, we cannot trade in the spot of electricity in the "mathematical finance" sense of the word due to non-storability. Hence, the buy-and-hold strategy fails for electricity, and we no longer have the no-arbitrage argument leading to the price (5.1), and in consequence (5.2). On the other hand, forwards are liquidly traded in power markets, and the no-arbitrage theory prescribes that their dynamics must be a martingale under some pricing measure Q. The standard approach (see Benth et al. [8]) in power markets is therefore to *assume* that the forward price is defined by (5.2) for *an equivalent measure* Q. We emphasise that Q does not need to be a martingale measure. With this definition of forward prices, the measure Q plays the role of modelling the risk premium in the

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[1] that is, going *long* a forward contract on electricity

spot-forward market, that is, the premium producers have to accept as discount in prices in order to hedge their production in forward contracts.

From the definition (5.2), we see that

$$F(t,T_1,T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t,u)\,du,$$

where $f(t,u)$ is the forward price at time $t \geq 0$ of a contract delivering at time $u \geq t$. Of course,

$$f(t,u) = \mathbb{E}_Q[S(u) \mid \mathcal{F}_t].$$

Thus, to derive the dynamics of a power forward contract $F(t,T_1,T_2)$, one can first compute the price of a fixed-maturity forward $f(t,u)$, and next find its average over the delivery period. In order to find $f(t,u)$, we must choose a measure Q.

We could use the tool-box of *incomplete markets theory* in mathematical finance to choose a probability Q. However, many of the approaches rest on deriving partial hedges of the derivative in question, and finding a Q based on the cost of (partial) hedging. As we have already argued, the spot is not suitable for trading in a portfolio, and thus such approaches fail. Indifference pricing is another alternative, based on hedging in correlated assets. For example, the shares of a major power utility company are dependent on the power prices, and one could exploit such a dependency in order to build partial hedges for power derivatives like the forward. We refer to Carmona [10] for an account of indifference pricing in incomplete markets, including power markets.

The *parametric approach* is the typical choice in the literature on pricing forwards in power markets (see Benth et al. [8]). The measure Q is chosen among a parametric family of equivalent probabilities, and the parameters are next estimated from historical forward prices. In this way, we model the risk premium, and estimate it from historical prices. We study this approach more closely for spot models being in the class of LSS processes. We will follow the analysis in Barndorff-Nielsen et al. [1] closely, and refer to this paper for further details and a more in-depth discussion.

Consider the LSS process $X(t)$ defined in (3.1). Introduce the measure Q by the Radon–Nikodym derivative process

$$Z(t) = \frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \exp\left(\theta L(t) - \psi(-\mathrm{i}\theta)\,t\right),$$

for $t \in [0,T^*]$, with T^* being some terminal time in the market covering all delivery periods of interest. In order for $Z(t)$ to be a martingale, we assume that the Lévy process L has exponential moments, that is, there exists a constant $c > 0$ such that for all $|\theta| \leq c$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\theta z}\,\ell(dz) < \infty,$$

where $\ell(dz)$ is the Lévy measure of L. Then $Z(t)$ is a martingale for all $|\theta| \leq c$. Hence, Q is an equivalent probability of P, and the measure change is called the

Esscher transform. One can see that the Lévy measure of L with respect to Q becomes

$$\ell_Q(dz) = \mathrm{e}^{\theta z}\ell(dz),$$

being an exponential tilting of the original measure. The measure change is restricted to times $t \geq 0$, treating $t = 0$ as the start of the economy. For simplicity, we do not make any measure change with respect to the volatility process $\sigma(t)$, although this could be done as well (see Barndorff-Nielsen et al. [1] for details on this).

Let us first suppose that the spot price $S(t)$ is the exponential of the LSS process $X(t)$,

$$S(t) = \Lambda(t)\exp(X(t)).$$

Recall that $\Lambda(t)$ is a deterministic seasonality function, naturally being positive and measurable. Implicitly, we suppose that $S(t)$ has a finite expectation in order to have well-defined forward prices. We find the following result in Barndorff-Nielsen et al. [1]:

Proposition 5.1. *Suppose that θ is such that*

$$\exp\left(\int_0^u \psi(-\mathrm{i}(\theta + g(u-s)\sigma(s))) - \psi(-\mathrm{i}\theta)\,ds\right)$$

has finite expectation for all $0 \leq u \leq T^*$. Then the forward price $f(t,u)$ at time $t \geq 0$ with delivery at time $u \geq t$ is

$$f(t,u) = \Lambda(u)\Psi(t,u;\theta)\exp\left(\int_{-\infty}^t g(u-s)\sigma(s)\,dL(s)\right),$$

where,

$$\Psi(t,u;\theta) = \mathbb{E}\left[\exp\left(\int_t^u \psi(-\mathrm{i}(\theta + g(u-s)\sigma(s)) - \psi(-\mathrm{i}\theta)\,ds\right) \mid \mathcal{F}_t\right]$$

Proof. See Barndorff-Nielsen et al. [1]. □

We observe that the forward price is a function of

$$\int_{-\infty}^t g(u-s)\sigma(s)\,dL(s),$$

which does not coincide with $X(t) = \ln S(t) - \ln\Lambda(t)$, unless $u = t$. Hence, in general, the forward price is not affine in the spot as is usual with most one-factor spot models used in electricity modelling. Furthermore, it is clear that in general it is impossible to integrate analytically $f(t,u)$ with respect to the delivery time u, and therefore we cannot get any closed-form expressions for power forward prices $F(t,T_1,T_2)$. The function $\Psi(t,u;\theta)$ measures implicitly the risk premium in the forward market. In the case of no volatility in the LSS process, $\sigma(s) = 1$, then $\Psi(t,u;\theta)$ in the forward price simplifies considerably to the deterministic expression

$$\Psi(t,u;\theta)=\exp\left(\int_0^{u-t}\psi(-\mathrm{i}(\theta+g(s)))-\psi(-\mathrm{i}\theta)\,ds\right).$$

This can be further computed given a specification of L, which gives ψ.

As a special case, consider now $L=B$, a Brownian motion (in which case the Esscher transform coincides with Girsanov transformation)

$$\frac{df(t,u)}{f(t,u)}=g(u-t)\sigma(t)\,dW(t),$$

with W being the Q Brownian motion obtained from the measure change. What is interesting here, is that the volatility of the forward is given by

$$\Sigma(t,u)=g(u-t)\sigma(t).$$

For the case of $X(t)$ being an OU-process, we find $g(u-t)=\exp(-\alpha(u-t))$ with $\alpha>0$ the speed of mean reversion. Then the volatility of the forward is lower than the spot volatility $\sigma(t)$ for all maturity times $u>t$, but is increasing towards the spot volatility as time to maturity converges to zero. This is known as the Samuelson effect (see Samuelson [17]). For LSS models, we find a similar effect, as

$$\lim_{u\downarrow t}\Sigma(t,u)=g(0)\sigma(t),$$

as long as $g(0)$ is well defined. Supposing in addition that g is differentiable with $\int_0^\infty g'(s)^2\,ds<\infty$, $X(t)$ will be in the class of semimartingale process by Proposition 3.2. We see from the dynamics of $X(t)$ in Proposition 3.2 that the volatility becomes $g(0)\sigma(t)$. Hence, for such LSS models, we have the Samuelson effect.

In general it seems very hard to integrate the forward price $f(t,u)$ over some delivery period in order to obtain an expression for a power forward, without resorting to numerical integration. It is desirable to have accessible a power forward price dynamics which can be computed efficiently in order to do risk management and derivatives pricing. This is an argument in favour of arithmetic models. To this end, consider

$$S(t)=\Lambda(t)+X(t)$$

where we suppose that X is an LSS process in $L^1(Q)$. We have the following from Barndorff-Nielsen et al. [1]:

Proposition 5.2. *The forward price is*

$$f(t,u)=\Lambda(u)+\Psi(t,u;\theta)+\int_{-\infty}^t g(u-s)\sigma(s)\,dL(s).$$

where

$$\Psi(t,u;\theta)=(-\mathrm{i}\psi'(-\mathrm{i}\theta))\int_t^u g(u-s)\mathbb{E}[\sigma(s)\,|\mathcal{F}_t]\,ds$$

Proof. See Barndorff-Nielsen et al. [1]. □

Note that we can integrate this $f(t, u)$ with respect to u by appealing to the stochastic Fubini theorem, and given a kernel function g we may potentially derive analytical expressions.

In the discussion above, we have restricted our attention to one-factor spot models, in the sense that the spot is driven by one LSS process. In practice, one may consider multi-factor models, which gives a richer structure for capturing a reasonable spot price evolution, but also to get more realistic power forward price dynamics. A further extension is to consider forward price models directly, with possibly infinitely many factors. Using ambit fields, see Barndorff-Nielsen et al. [2], one may provide a rich class of models for the forward dynamics.

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Evaluating Hybrid Products: The Interplay Between Financial and Insurance Markets

Francesca Biagini

Abstract. A current issue in the theory and practice of insurance and reinsurance markets is to find alternative ways of securitizing risks. Insurance companies have the possibility of investing in financial markets and therefore hedge against their risks with financial instruments. Furthermore they can sell part of their insurance risk by introducing insurance linked products on financial markets. Hence insurance and financial markets may no longer be considered as disjoint objects, but can be viewed as one arbitrage-free market. Here we provide an introduction to how mathematical methods for pricing and hedging financial claims such as the benchmark approach and local risk minimization can be applied to the valuation of hybrid financial insurance products, as well as to premium determination, risk mitigation and claim reserve management.

Mathematics Subject Classification (2010). Primary 97M30; Secondary 62P05, 60H30, 60G44.

Keywords. Benchmark approach, local risk minimization, actuarial premium determination, hybrid markets, martingale methods.

1. Introduction

A current issue in the theory and practice of insurance and reinsurance markets is to find alternative ways of securitizing risks. To this purpose, insurance companies have tried to take advantage of the vast potential of capital markets by introducing exchange-traded insurance-linked instruments such as *mortality derivatives* and *catastrophe insurance options*. At the same time, insurance products such as *unit-linked life insurance contracts*, where the insurance benefits depend on the price of some specific traded stocks, offer a combination of traditional life insurance and financial investment. Furthermore, new kinds of insurance instruments, which offer protection against risks connected to macro-economic factors such as unemployment, are recently offered on the market. Hence insurance and financial markets may no longer be viewed as disjoint objects, but can be considered as one arbitrage-free market. Here we provide an introduction to how mathematical methods for pricing and hedging financial claims can be applied to the valuation and hedging of the hybrid products mentioned above, as well as to premium determination, risk mitigation and claim reserve management. In this paper we propose to use the **benchmark approach** for pricing and the **(local) risk minimization** method for hedging purposes. We motivate these choices as follows.

We have already remarked that insurance markets and financial markets can be seen as one arbitrage-free market. However insurance claims are in general not replicable by other financial instruments, which implies that the hybrid market consisting of financial and insurance products is incomplete. As a consequence, there usually exist several equivalent (local) martingale measures, corresponding to the same numéraire, that guarantee the absence of arbitrage in the market. In incomplete markets a pricing and hedging criterion with a corresponding equivalent (local) martingale measure must then be selected. Rather natural and tractable are quadratic hedging criteria such as *mean-variance hedging* and *local risk minimization*, see [41] and [25] for extensive surveys on these methods. The local risk minimization approach provides for a given square-integrable contingent claim H a perfect hedge by using strategies that are not necessarily self-financing, with (discounted) portfolio value given by the gain of trade plus an instantaneous adjustment called the cost. The optimal strategy, when it exists, is determined by the property of having minimal risk, in the sense that the optimal cost is given by a square-integrable martingale strongly orthogonal to the martingale part of the asset price process. This implies that the optimal strategy is "self-financing on average", i.e. remains as close as possible to being self-financing. In this setting, one can hedge a contingent claim H by investing in the primary assets on the market and by compensating other sources of risks by using the cost. In particular (local) risk-minimization naturally appears as suitable hedging method when market incompleteness derives by the presence of an additional source of randomness external to the financial market (such as for example mortality risk, catastrophe risk, insurance risks), that is "orthogonal" to the asset price dynamics, but not necessarily independent of them and vice versa. This is the case of market models containing financial insurance-linked instruments, such as mortality derivatives (survival swaps, longevity bonds) recently introduced on the markets to hedge against systematic mortality risk in life insurance contracts, and unit-linked life insurance contracts, i.e., contracts that combine insurance benefits and financial investment. Some references on this topic are for example [1, 4, 5, 8, 9, 15, 16, 31, 32, 37] and [38].

Local risk minimization is mainly an hedging criterion and provides a no-arbitrage price only as a "by-product" of the method, but such a valuation is not its primary objective. Hence for what concerns the pricing issue, we consider here the benchmark approach, introduced in the literature by several authors ([20, 21, 28, 33, 34, 35]). The benchmark approach provides a pricing rule (*real-world pricing*) under the real-world probability measure $\mathbb{P}$ by using a particular discounting factor called *benchmark* or $\mathbb{P}$-*numéraire portfolio*, and does not require the existence of an equivalent (local) martingale measure (ELMM). The numéraire portfolio contains information on macro-economic influences and on risks generated by the complex of hybrid products on the market. Hence it can be seen as a general indicator of the market's financial and economic conditions (cost of capital, interest rates, expected investment returns, macro-economic influences, market dependence structure). Real-world pricing uses the numéraire as a measure of market performance and then results to be more natural than pricing by selecting a particular equivalent martingale measure. In this way we also benefit from the statistical advantages of working directly under the real-world probability measure.

Furthermore, for hedgeable claims, the real-world pricing formula gives their minimal price and for non-hedgeable claims this method is consistent with (asymptotic) utility indifference pricing as defined in [35] in a very general setting. Moreover there is an intrinsic relation between (local) risk minimization approach and real-world pricing, that justifies the use of the benchmark approach for pricing also in incomplete markets. To this extent we refer to the detailed discussion contained in Section 5.

For what concerns the application of the no-arbitrage pricing theory to premium determination for insurance contracts, this topic has been already discussed in the literature by several authors, see [17, 29, 40] and [42], as explained in Section 3.

Here we consider the benchmark approach also for actuarial application as more natural pricing method with respect to the martingale methods of the standard no-arbitrage pricing theory, since it keeps a close connection to the classical premium calculation principles, which also use the real-world probability measure $\mathbb{P}$.

Furthermore, in the case of real-world pricing of insurance contracts, we take directly in account the role of investment opportunities in assessing premiums and reserves, since the benchmark is a direct and intuitive global indicator of (hybrid) market performance. This is of course even more relevant for insurance structures depending heavily on the performance of financial markets and macro-economic factors, such as for example unemployment insurance products. On the contrary the choice of a particular martingale measure for actuarial applications appears quite artificial, since it is exclusively determined in relation to the primitive financial assets on the market. A detailed discussion on the relation between actuarial premium calculation principles and real-world pricing is contained in Section 5.

The structure of the paper is the following. First of all we introduce shortly the benchmark approach. Then in Section 3.1 we illustrate an application of real-world pricing to premium determination for unemployment insurance products, after having discussed no-arbitrage pricing of insurance claims in Section 3. Afterwards we consider local risk minimization for hybrid markets: in Section 4 we recall the main features of this hedging method and in Section 4.1 we apply it to dynamic hedging with longevity bonds. Finally a discussion on the relation between (local) risk minimization approach, real-world pricing and actuarial premium calculation principles concludes the paper in Section 5.

2. The benchmark approach

As stated in the introduction, we adopt the benchmark approach for our pricing issue. All fundamental results of this approach can be found in [35] for jump diffusion and Itô process driven markets and in [33] for a general semimartingale market.

Let $T > 0$ be a finite time horizon. We consider a frictionless financial market model in continuous time, which is set up on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t < T}$ that is assumed to satisfy $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \in [0, T]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, as well as the usual hypotheses, see [36].

On the market, we can find $d + 1$ non-negative, adapted tradable (primary) security account processes, represented by the $(d + 1)$-dimensional càdlàg semimartingale $S = (S_t)_{t \in [0,T]} = (S^0_t, S^1_t, \ldots, S^d_t)^{tr}_{t \in [0,T]}$. Here we interpret S^0_t as the value of the adapted, strictly positive savings account at time t, $t \in [0, T]$.

Let $L(S)$ denote the space of $\mathbb{R}^{d+1}$-valued, predictable strategies

$$\delta = (\delta_t)_{t \in [0,T]} = (\delta^0_t, \delta^1_t, \ldots, \delta^d_t)^{tr}_{t \in [0,T]} ,$$

for which the corresponding gain from trading in the assets, i.e., $\int_0^t \delta_s \cdot dS_s$, exists for all $t \in [0, T]$.

Here, δ^j_t represents the units of asset j held at time t by a market participant. The portfolio value S^δ_t at time $t \in [0, T]$ is then given by

$$S^\delta_t = \delta_t \cdot S_t = \sum_{j=0}^{d} \delta^j_t S^j_t .$$

A strategy $\delta \in L(S)$ is called *self-financing* if changes in the portfolio value are only due to changes in the assets and not due to in- or outflow of money, i.e., if

$$S^\delta_t = S^\delta_0 + \int_0^t \delta_s \cdot dS_s \ , \ t \in [0, T] \ ,$$

or equivalently

$$dS^\delta_t = \delta_t \cdot dS_t \ .$$

In the sequel we won't always request strategies to be self-financing. We write $\mathcal{V}^+_x$ $(\mathcal{V}_x)$ for the set of all strictly positive (non-negative), finite and self financing portfolios S^δ with initial capital $S^\delta_0 = x$. We now introduce the notion of the $\mathbb{P}$-numéraire portfolio.

Definition 2.1. A portfolio $S^{\delta_*} \in \mathcal{V}^+_1$ is called $\mathbb{P}$-*numéraire portfolio* if every non-negative portfolio $S^\delta \in \mathcal{V}_1$, discounted (or benchmarked) with S^{δ_*}, forms a $(\mathbb{F}, \mathbb{P})$-supermartingale. In particular, we have

$$\mathbb{E}\left[\frac{S^\delta_\sigma}{S^{\delta_*}_\sigma} \,\middle|\, \mathcal{F}_\tau \right] \le \frac{S^\delta_\tau}{S^{\delta_*}_\tau} \qquad \text{a.s.} \tag{2.1}$$

for all stopping times $0 \le \tau \le \sigma \le T$.

From now on, we choose the $\mathbb{P}$-numéraire portfolio as *benchmark*. We call any security, when expressed in units of the numéraire portfolio, a *benchmarked security* and refer to this procedure as *benchmarking*. The benchmarked value of a portfolio S^δ is given by

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}, \ t \in [0, T].$$

If a $\mathbb{P}$-numéraire portfolio exists, it is unique by the supermartingale property and Jensen's inequality, see [3].

To establish the further modeling framework, we make the following (rather weak) assumption, see [3], [28] or [35].

Assumption 2.2. The $\mathbb{P}$-numéraire portfolio $S^{\delta_*} \in \mathcal{V}_1^+$ exists in our market.

If it exists, the $\mathbb{P}$-numéraire portfolio is equal to the "growth optimal portfolio" (in short: GOP), which is defined as the portfolio with the maximal growth-rate in the market. It also satisfies several other optimality criteria, see [3, 26, 33] and [35], and can be approximated under fairly weak assumptions by a sequence of well-diversified portfolios (see Theorem 3.6 of [34]). The existence and uniqueness of the $\mathbb{P}$-numéraire portfolio can be shown in a sufficiently general setting, see [3], [28] or [35].

Definition 2.3. A benchmarked non-negative self-financing portfolio $\hat{S}^\delta$ is a *strong arbitrage* if it starts with zero initial capital, that is $\hat{S}_0^\delta = 0$, and generates some strictly positive wealth with strictly positive probability at a later time $t > 0$, that is $\mathbf{P}(\hat{S}_t^\delta > 0) > 0$.

With the existence of the $\mathbb{P}$-numéraire portfolio and the corresponding supermartingale property (2.1), strong arbitrage opportunities, as defined in Definition 2.3, are excluded, see [33]. There could still exist some weaker forms of arbitrage, which would require to allow for negative portfolios of total wealth, however. Because of the (often legally established) principle of limited liability, these portfolios should be excluded in a realistic market model: a market participant generally holds a non-negative portfolio of total wealth, otherwise he would have to declare bankruptcy. This holds in particular for insurance companies that must take care of several legal constraints for trading.

Let us now consider two portfolios $S^\delta \in \mathcal{V}_x$ and $S^{\delta'} \in \mathcal{V}_y$ with $\hat{S}_T^\delta = \hat{S}_T^{\delta'}$ $\mathbb{P}$-a.s. Let the benchmarked portfolio process $\hat{S}_t^\delta, t \in [0, T]$, be a martingale and the benchmarked portfolio process $\hat{S}_t^{\delta'}, t \in [0, T]$, be a supermartingale. Then

$$\hat{S}_t^\delta = \mathbb{E}\left[\hat{S}_T^\delta \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\hat{S}_T^{\delta'} \,\middle|\, \mathcal{F}_t\right] \le \hat{S}_t^{\delta'}, \quad \forall t \in [0, T]\,, \tag{2.2}$$

and in particular

$$x = \hat{S}_0^\delta \le \hat{S}_0^{\delta'} = y\,.$$

Then S^δ (if it exists) has minimal price among all benchmarked portfolios with the same terminal value. Hence, a rational (risk-averse) investor would always invest in a benchmarked martingale portfolio (if it exists). This justifies the following definition of "fair" wealth processes, see [33].

Definition 2.4. A portfolio process $S^\delta = (S_t^\delta)_{t\geq 0}$ is called *fair* if its benchmarked value process

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}, \quad t \in [0,T],$$

forms a $(\mathbb{F},\mathbb{P})$-martingale.

Definition 2.5. A *T-contingent claim* H is a $\mathcal{F}_T$-measurable random variable with $\mathbb{E}\left[\frac{|H|}{S_T^{\delta_*}}\right] < \infty$. We denote by

$$\widehat{H} := \frac{H}{S_T^{\delta_*}}$$

the *benchmarked payoff* of the T-contingent claim H.

According to Definition 2.4, it is natural to define the so-called *real-world pricing formula* for a T-contingent claim H as follows:

Definition 2.6. For a T-contingent claim H the *fair price* $P_t(H)$ of H at time $t \in [0,T]$ is given by

$$P_t(H) := S_t^{\delta_*}\mathbb{E}\left[\frac{H}{S_T^{\delta_*}}\middle|\mathcal{F}_t\right] = S_t^{\delta_*}\mathbb{E}\left[\hat{H}\,\middle|\,\mathcal{F}_t\right]. \tag{2.3}$$

Here (2.3) is addressed as *real-world pricing formula.*

Hence the corresponding benchmarked fair price process $(\hat{P}_t)_{t\in[0,T]} = \left(\frac{P_t}{S_t^{\delta_*}}\right)_{t\in[0,T]}$ forms a $(\mathbb{F},\mathbb{P})$-martingale.

Definition 2.7. We say that a non-negative benchmarked contingent claim $\hat{H} \in L^1(\mathcal{F}_T,\mathbb{P})$ is *hedgeable* if there exists a self-financing strategy $\delta^{\hat{H}} = (\delta_t^{\hat{H}})_{t\in[0,T]} = (\delta_t^{\hat{H},1}, \delta_t^{\hat{H},2}, \dots, \delta_t^{\hat{H},d})_{t\in[0,T]}^{tr}$ such that

$$\hat{H} = \hat{H}_0 + \int_0^T \delta_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u.$$

In the case of a hedgeable benchmarked payoff $\hat{H}$, the real-world pricing formula (2.3) provides the description for the fair portfolio of minimal price among all replicating self-financing portfolios for $\hat{H}$, since the benchmarked fair portfolio value forms by definition a $\mathbb{P}$-martingale. The benchmark approach allows other self-financing hedge portfolios to exist for $\hat{H}$, see [35]. However, these non-negative portfolios are not $\mathbb{P}$-martingales and, as supermartingales, therefore more expensive than the $\mathbb{P}$-martingale given by the benchmarked fair portfolio process obtained by (2.3), see (2.2).

Remark 2.8. If a T-contingent claim H and the value $S_T^{\delta_*}$ at time T of the $\mathbb{P}$-numéraire portfolio are independent, we get

$$\begin{aligned} P_t(H) &= S_t^{\delta_*}\mathbb{E}\Big[\frac{1}{S_T^{\delta_*}}|\mathcal{F}_t\Big]\,\mathbb{E}\left[H|\mathcal{F}_t\right] \\ &= P(t,T)\mathbb{E}\left[H|\mathcal{F}_t\right], \end{aligned} \tag{2.4}$$

where $P(t,T)$ is the fair price at time $t \leq T$ of a zero-coupon bond with nominal value one, paid at time T. This formula is often called the *actuarial pricing formula.*

3. No-arbitrage pricing of insurance claims

Pricing of random claims has ever been one of the core subjects in both actuarial and financial mathematics and there exist various approaches for calculating (fair) prices. The actuarial way of pricing usually considers the classical premium calculation principles that consist of *net premium* and *safety loading*: if H describes a random claim, which the insurance company has to pay (eventually) in the future at time τ, then a premium $P(H)$ to be charged for the claim is defined by

$$P(H) = \underbrace{\mathbb{E}\left[\frac{H}{D_\tau}\right]}_{\text{net premium}} + \underbrace{A\left(\frac{H}{D_\tau}\right)}_{\text{safety loading}}, \tag{3.1}$$

where D is a discounting factor chosen according to actuarial judgement (see also [29] for further remarks). Note that the net premium is the expected value of H with respect to the *real-world* (or *objective*) probability measure. Possible safety loadings could be $A(\frac{H}{D_\tau}) = 0$ (*net premium principle*), $A(\frac{H}{D_\tau}) = a \cdot \mathbb{E}\left[\frac{H}{D_\tau}\right]$ (*expected value principle*, where $a \geq 0$), $A(\frac{H}{D_\tau}) = a \cdot \mathbb{V}ar(\frac{H}{D_\tau})$ (*variance principle*, where $a > 0$) or $A(\frac{H}{D_\tau}) = a \cdot \sqrt{\mathbb{V}ar(\frac{H}{D_\tau})}$ (*standard deviation principle*, where $a > 0$), see, e.g., [39]. The existence of a safety loading is justified by ruin arguments and the risk-averseness of the insurance company.

Widely used pricing approaches in finance base on no-arbitrage assumptions (see, e.g., the famous papers of Black and Scholes [11] and Merton [30]). A financial market, consisting of several primary assets, is assumed to be in an economic equilibrium, in which riskless gains with positive probability (arbitrage) by trading in the assets are impossible. A fundamental result in this context is that absence of arbitrage is implied by the existence of an equivalent (local) martingale measure, i.e., a probability measure, which is equivalent to the real-world measure and according to which all assets, discounted with some numéraire, are (local) martingales. There are different versions of this result which is often called the fundamental theorem of asset pricing (in short: FTAP), see, e.g., [18], [19], [22], [24] or [27].

Based on the FTAP, it can then be shown that, at any time t, an arbitrage-free price $P_t(H)$ of a (contingent) claim H (paid at time $T \geq t$) can be defined by

$$P_t(H) := N_t \mathbb{E}_{\mathbb{Q}} \left[\frac{H}{N_T} | \mathcal{F}_t \right] , \tag{3.2}$$

where $\mathbb{Q}$ is an equivalent (local) martingale measure and $(N_t)_{t\in[0,T]}$ a discounting factor process.

From an economic point of view both the safety loading in equation (3.1) and the change to an equivalent (local) martingale measure in equation (3.2) express the risk-averseness of the insurance company. Hence, there have been several attempts to connect actuarial premium calculation principles with the financial no-arbitrage theory. The papers [17] and [42] both describe a competitive and liquid reinsurance market, in which insurance companies can "trade" their risks among each other. Since riskless profits shall be excluded also in this setting, the no-arbitrage theory applies and insurance premiums can be calculated by equation (3.2). Both papers actually show that under some assumptions there exist equivalent martingale measures, which explain premiums of the form (3.1), so that these principles provide arbitrage-free prices, too.

Therefore no-arbitrage pricing theory can be applied also to actuarial premium determination. To this purpose, in this paper we choose the benchmark approach, as we have already thoroughly explained in the Introduction. In Section 5 we comment extensively on the relation between actuarial premium calculation principles and real-world pricing. We now illustrate an application of real-world pricing to premium determination of unemployment insurance products.

3.1. Real-world pricing for unemployment insurance products

We first introduce the structure of the considered unemployment insurance products. The product's basic idea is that the insurance company compensates to some extend the financial deficiencies, which an unemployed insured person is exposed to. Here we only consider contracts with deterministic, a priori fixed claim payments c_i, which can be interpreted as an annuity during an unemployment period, and predefined payment dates T_i, $i = 1, \ldots, N$. An example for this kind of contracts is given by Payment Protection Insurance (in short: PPI) products against unemployment, which are linked to some payment obligation of an obligor to its creditor.

The following details of the insurance contract are important for the later model specifications:

- Regarding the method of premium payment, we have to differentiate between single rates, where the whole insurance premium is paid at the beginning of the contract, and periodical rates. For our modeling purpose, we want to focus on calculating single premiums. This is again motivated by PPI unemployment products, which are often sold as an add-on directly by the creditor. The insurance company then receives a single rate from the creditor, who in turn allocates this rate to the instalments.

- The obligor must have been employed at least for a certain period before the contract's conclusion. Hence we assume that she is employed at the beginning of the contract.

We also consider three time periods that belong to the exclusion clauses of the contracts and impact the insurance premium.

- The *waiting period* W starts with the beginning of the contract. If an insured person becomes unemployed at any time of this period, he is not entitled to receive any claim payments during the whole unemployment time.
- The *deferment period* D starts with the first day of unemployment. An insured person is not entitled to receive claim payments until the end of this period.
- The third period is comparable to the waiting period and is called the *requalification period* and denoted by R. The difference between waiting and requalification period is their beginning. The waiting period starts with the beginning of the contract and the requalification period with the end of any unemployment period that occurred during the contract's duration. If an insured person becomes (again) unemployed at any time of the requalification period, he is not entitled to receive any claim payment during the whole time of unemployment.

For existing unemployment insurance contracts, the waiting, deferment and requalification periods currently vary from three to twelve months.

According to the contract structure defined above, the random insurance claim H_i at the payment date T_i can be defined as

$$H_i(\omega) := c_i I_{\{W<\tau_1\le T_i-D,\tau_2>T_i\}\cup\bigcup_{j=2}^{\infty}\{\tau_{2j-1}-\tau_{2j-2}>R,W<\tau_{2j-1}\le T_i-D,\tau_{2j}>T_i\}}(\omega)\,,$$

where $(\tau_j)_{j\in\mathbb{N}}$ with $\tau_0 := 0$ are the random jump times of the employment-unemployment process $X := (X_t)_{t\in[0,T]}$ that describes at time t if the insured person is employed ($X_t = 0$) or not ($X_t = 1$).

Assumption 3.1. Every (random) insurance claim H_i of the unemployment insurance contract, paid at time T_i, is independent of the respective value $S_{T_i}^{\delta_*}$ of the $\mathbb{P}$-numéraire portfolio at time T_i.

Under this assumption we can apply the actuarial pricing formula (2.4), that requires in this case only the conditional joint distributions of the jump times $\tau_j, j \in \mathbb{N}$. However we note that this assumption may be too strong for a realistic model. The insurance claims obviously depend on macroeconomic unemployment factors, which in turn may have interdependencies with financial markets, represented by the $\mathbb{P}$-numéraire portfolio (or the GOP). For the study of dependence effects between the insurance claims and the $\mathbb{P}$-numéraire portfolio, we refer to [10].

Furthermore, we assume that there is the possibility of putting money on a bank account with constant interest rate $r > 0$, and that the employment-unemployment process X follows a time-homogeneous strong Markov chain with respect to $\mathbb{P}$ and

$$\mathcal{F}_t = \mathcal{F}_t^X = \sigma(X_u, u \leq t),\ t \in [0,T].$$

These assumptions may again be too strong for a realistic model. Actually, the probability of an insured person of getting unemployed or employed may depend on his past employment-unemployment development. An extension of this model can be found in [10].

Under these hypotheses the sojourn times $\tau_j - \tau_{j-1},\ j \geq 1$, given $X_0 = i_0,\ i_0 \in \{0,1\}$, are conditionally independent and exponentially distributed, with parameters given by the intensity matrix

$$\Lambda = \begin{pmatrix} \lambda_0 & -\lambda_0 \\ -\lambda_1 & \lambda_1 \end{pmatrix} \tag{3.3}$$

of X. In particular, we have

$$\mathbb{P}\left(\tau_1 - \tau_0 > t_1, \ldots, \tau_n - \tau_{n-1} > t_n \middle|\ X_0 = i_0\right) = e^{-\lambda_{i_0} t_1} \cdot \ldots \cdot e^{-\lambda_{i_{n-1}+1} t_n}\ ,$$

where $i_0, i_1, \ldots, i_{n-1} \in \{0,1\}$ with $i_k = 1 - i_{k-1},\ t_1, \ldots, t_n \in [0,\infty)$, and $\lambda_{i_0}, \ldots, \lambda_{i_{n-1}}$ are defined by (3.3), see [43].

For the sake of simplicity, we now assume that $t \in [T_{k-1}, W)$, that the insured person was employed at the actual beginning of the contract ($X_0 = 0$) and that the first jump to unemployment τ_1 has not occurred up to time t ($t < \tau_1$).

Proposition 3.2. *Under Assumption* 3.1 *we obtain the insurance premiums P_t for $X_0 = 0$ and $t \in [T_{k-1}, W)$ as follows:*

$$P_t = \sum_{i=k}^{N} S_t^{\delta_*} \mathbb{E}\left[\hat{H}_i \middle| \mathcal{F}_t\right] = \sum_{i=k}^{N} e^{-r(T_i - t)} \mathbb{E}\left[\hat{H}_i \middle| \mathcal{F}_t\right] \tag{3.4}$$

$$= \sum_{i=k}^{N} c_i e^{-(r+\lambda_1)(T_i - t)} \Big(\frac{\lambda_0}{\lambda_0 - \lambda_1} \big(e^{-(\lambda_0-\lambda_1)(W-t)} - e^{-(\lambda_0-\lambda_1)(T_i - D - t)} \big)$$

$$+ \lambda_0^2 \lambda_1 \int\limits_{\max\{W-t,R\}}^{T_i - D - t} \int\limits_{R}^{y} e^{-(\lambda_0-\lambda_1)x} \int\limits_{0}^{y-x} e^{-(\lambda_0-\lambda_1)u} I_0(2\sqrt{\lambda_0\lambda_1 u(y-x-u)}) du dx dy \Big), \tag{3.5}$$

where $I_0(x)$ is the modified first kind Bessel function of order 0*. In general, the modified first-order Bessel function $I_\alpha(x)$ of order $\alpha \in \mathbb{R}$ is given by*

$$I_\alpha(x) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}\ .$$

Proof. Pricing formula (3.4) derives by applying (2.4) and (3.5) by the assumptions on the employment-unemployment process X. For further details on the proof, we refer to [10]. □

Note that, due to the "loss of memory" property of X, it is sufficient to calculate the insurance premiums for $t \leq \tau_1$. Analogous computations deliver the price for all the other cases, see [10].

4. (Local) risk minimization for hybrid markets

We now turn to the hedging issue. To avoid technicalities, we focus on the case where the asset prices discounted with the savings account S^0 are given by local martingales under a given probability measure $\mathbb{P}$. We denote by $\bar{S}$ the vector of the $d+1$ discounted primary security accounts

$$\bar{S} := \left(\frac{S_t}{S_t^0}\right)_{t\in[0,T]} = (1, \bar{S}_t^1, \dots, \bar{S}_t^d)^{tr}_{t\in[0,T]}.$$

Remark 4.1. This assumption on the underlying asset price processes may appear quite restrictive. However if we choose as discounting factor the $\mathbb{P}$-numéraire portfolio, by Assumption 2.2 and Theorem 2.4 of [26] it follows that the vector process $\hat{S}$ of benchmarked primary security accounts is always a $\mathbb{P}$-local martingale, if S is given by a continuous semimartingale and also for a wide class of jump-diffusion models.

Under these assumptions on the discounted financial markets, we can apply the risk-minimization method as originally introduced in [23]. For further details, we also refer to [41].

Definition 4.2. An L^2-*admissible strategy* is any $\mathbb{R}^{d+1}$-valued predictable vector process $\delta = (\delta)_{t\in[0,T]} = (\delta_t^0, \delta_t^1, \dots, \delta_t^d)^{tr}_{t\in[0,T]}$ such that

(i) the associated discounted portfolio $\bar{S}^\delta$ is a square-integrable stochastic process whose left-limit is equal to $\bar{S}^\delta_{t-} = \delta_t \cdot \bar{S}_t$, $t \in [0,T]$,
(ii) the stochastic integral $\int \delta \cdot \mathrm{d}\bar{S}$ is such that

$$\mathbb{E}\left[\int_0^T \delta_u^{tr} \mathrm{d}[\bar{S}]_u \delta_u\right] < \infty. \tag{4.1}$$

Here $[\bar{S}] = ([\bar{S}^i, \bar{S}^j])_{i,j=1,\dots,d}$ denotes the matrix-valued optional covariance process of $\bar{S}$.

Since the market is not complete, we also admit strategies here that are not self-financing and may generate profits or losses over time as defined below.

Definition 4.3. For any L^2-admissible strategy δ, the *cost* process $\bar{C}^\delta$ is defined by

$$\bar{C}_t^\delta := \bar{S}_t^\delta - \int_0^t \delta_u \cdot \mathrm{d}\bar{S}_u - \bar{S}_0^\delta, \quad t \in [0,T]. \tag{4.2}$$

Here $\bar{C}_t^\delta$ describes the total costs incurred by δ over the interval $[0,t]$.

Definition 4.4. For an L^2-admissible strategy δ, the corresponding *risk* at time t is defined by

$$\bar{R}_t^\delta := \mathbb{E}\left[\left(\bar{C}_T^\delta - \bar{C}_t^\delta\right)^2 \middle| \mathcal{F}_t\right], \quad t \in [0,T],$$

where the cost process $\bar{C}^\delta$, given in (4.2), is assumed to be square-integrable.

We now wish to find an L^2-admissible strategy δ which minimizes the associated risk measured by the fluctuations of its cost process in a suitable sense.

Definition 4.5. Given a discounted contingent claim $\bar{H} \in L^2(\mathcal{F}_T, \mathbb{P})$, an L^2-admissible strategy δ is said to be *risk-minimizing* if the following conditions hold:

(i) $\bar{S}^{\delta}_T = \bar{H}$, $\mathbb{P}$-a.s.;
(ii) for any L^2-admissible strategy $\tilde{\delta}$ such that $\bar{S}^{\tilde{\delta}}_T = \bar{S}^{\delta}_T$ $\mathbb{P}$-a.s., we have

$$\bar{R}^{\delta}_t \leq \bar{R}^{\tilde{\delta}}_t \quad \mathbb{P}\text{-a.s. for every } t \in [0, T].$$

Lemma 4.6. *The cost process $\bar{C}^{\delta}$ associated to a risk-minimizing strategy δ is a $\mathbb{P}$-martingale for all $t \in [0, T]$.*

Proof. For the proof of Lemma 4.6, we refer to [41] and [6]. □

The martingale property of the cost process characterizes the mean-self-financing property of the strategy δ, i.e., L^2-admissible strategies that somehow are kept "self-financing on average".

The next result shows how to provide a risk-minimizing strategy for a given claim. Let $\mathcal{M}^2_0(\mathbb{P})$ be the space of all square-integrable martingales starting at null at the initial time.

Proposition 4.7. *Every discounted contingent claim $\bar{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ admits a unique risk-minimizing strategy δ with portfolio value $\bar{S}^{\delta}$ and cost process $\bar{C}^{\delta}$, given respectively by*

$$\delta = \delta^{\bar{H}}, \quad \bar{S}^{\delta}_t = \mathbb{E}\left[\bar{H} \middle| \mathcal{F}_t\right], \quad \bar{C}^{\delta}_t = L^{\bar{H}}_t,$$

for $t \in [0, T]$, where $\delta^{\bar{H}}$ and $L^{\bar{H}}$ are provided by the Galtchouk–Kunita–Watanabe (GKW) *decomposition of $\bar{H}$, i.e.,*

$$\bar{H} = \bar{H}_0 + \int_0^T \delta^{\bar{H}}_u \cdot \mathrm{d}\bar{S}_u + L^{\bar{H}}_T, \quad \mathbb{P}\text{-a.s.}, \tag{4.3}$$

where $\bar{H}_0 \in \mathbb{R}$, $\delta^{\bar{H}}$ is an $\mathbb{F}$-predictable vector process satisfying the integrability condition (4.1) *and $L^{\bar{H}} \in \mathcal{M}^2_0(\mathbb{P})$ is strongly orthogonal to each component of $\bar{S}$.*

Proof. The proof follows from Theorem 2.4 of [41] and Lemma 4.6. □

Thus, the problem of minimizing risk is reduced to finding the representation (4.3). Decomposition (4.3) is often addressed in the literature as the *Föllmer–Schweizer decomposition.*

We now illustrate an application of the local risk minimization approach to hedging of mortality derivatives.

4.1. Application to mortality risk: Dynamic hedging with longevity bonds

A large number of life insurance and pensions products have mortality and longevity as a primary source of risk. Life and pension insurance companies typically use deterministic mortality intensities when determining premiums and reserves. However empirical evidence (see [14] for a literature overview on this topic) shows that this assumption is not realistic, so companies are exposed also to changes in the mortality intensity, i.e., to *systematic mortality risk*. This risk cannot be diversified away by pooling (i.e., by using sufficiently large portfolios) as in the case of *unsystematic mortality risk*, i.e., the risk associated with the status of individual life, but on the contrary its impact increases for larger portfolios of insured persons. Here we use the terminology of (*systematic*) *mortality risk* to denote all forms of deviations in aggregate mortality rates from those anticipated. More precisely, it can be differentiated in *longevity risk*, i.e., the risk that aggregate survival rates for given cohorts are higher than anticipated, and *short-term, catastrophic mortality risk*, i.e., the risk, that over short period of time, mortality rates are very much higher than would be normally experienced (such as for example in the case of a pandemic influenza or a natural catastrophe). Although mortality and longevity risk can be re-insured, traditional reinsurance is becoming inadequate to offer sufficient protection against these risks. Furthermore the new regulatory regime Solvency II proposal, due to be adopted in 2012, will require insurance companies to hold significant additional capital to guarantee their annuity liabilities if longevity risk cannot be controlled effectively. Since existing markets provide no effective hedge for longevity and mortality risk, recent studies ([2, 12, 13] and [14]) have highlighted the need of encouraging the introduction of a life market in order to address the problem of an extremely fast ageing population and the risk of long retirement periods that cannot be afforded anymore by a shrinking (younger) labor force. Hence to this purpose, new forms of investment in mortality derivatives have been recently introduced as alternative or as a complement to traditional reinsurance. Some examples are the followings (for an exhaustive discussion on mortality products, we refer to [14]):

- *Longevity bonds*, where coupon payments are linked to the number of survivors in a given cohort. The first example of longevity bond in the history is represented by Tontine bonds issued by some European governments in the 17th and 18th centuries. The first modern longevity bonds were introduced in 2004 by the European Investment Bank and BNP Paribas.
- *Short-dated, mortality securities*: market traded securities, whose payments are linked to a mortality index. They allow the issuer to reduce its exposure to short-term catastrophic mortality risk. The first bond of this type was issued with great success by Swiss Re in 2004.
- *Survivor swaps*, where counterparties swap a fixed series of payments for a series of payments linked to the number of survivors in a given cohort. Until now a small number of survivor swaps have been traded only on an over-the-counter basis.

Other kinds of products such as *mortality options*, i.e., financial contracts with mortality rate as underlying, have been discussed only at a theoretical level in the literature. The AFPEN (the Association of French Pension Funds) has suggested to introduce also an *annuity futures* market. Some of these new investment products such as some longevity bonds have encountered the favor of the public. However the establishment of a life market is still at the beginning.

As a contribution to the ongoing discussion on the introduction of longevity markets, we now consider an application of risk minimization to dynamic hedging with longevity bonds. For further details on this issue, we refer to [9]. The mathematical setting is the following. The time of death $\tau > 0$ of a person is modeled as a random variable with $P(\tau > t) > 0$ for any $t \in [0, T]$, and we denote by $H_t = \mathbb{I}_{\{\tau \leq t\}}$ the counting process of death. Let $\mathbb{H} := (\mathcal{H}_t)_{t \in [0,T]}$ be the filtration, generated by H. We assume that the overall information is represented by the filtration $\mathbb{G} := \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} := (\mathcal{F}_t)_{t \in [0,T]}$ is the augmented natural filtration of some Brownian motion W. To avoid technical difficulties, we suppose that the hypothesis (H) holds, i.e., every $\mathbb{F}$-martingale remains a martingale in the larger filtration $\mathbb{G}$. In particular, W is a $\mathbb{G}$-martingale, and then by Lévy's characterization a $\mathbb{G}$-Brownian motion. The survival probability process G associated to τ is supposed to fulfill

$$G_t := \mathbb{P}\left(\tau > t \middle| \mathcal{F}_t\right) = \exp\left(-\int_0^t \mu_u \, du\right) =: \exp\left(-\Gamma_t\right), \ t \in [0, T],$$

where the *stochastic mortality intensity* μ is given by an $\mathbb{F}$-progressively measurable process driven by W. The counting process martingale M associated with the one-jump process H is given as

$$M_t = H_t - \int_0^t (1 - H_u) \, \mu_u du, \ t \in [0, T].$$

For simplicity we assume here to work with a fixed constant short rate r. We now suppose that it is possible to trade on the financial market in an instrument called a longevity bond which has present value

$$B_t = \int_0^t e^{-ru} G_u \, du, \ t \in [0, T].$$

The payment generated by this bond has the form of an annuity, where the declining rate is given by the survival probability for the age cohort of the insured person. The (discounted) value process associated with the longevity bond is thus given by the conditional expectation

$$V_t = \mathbb{E}\left[\int_0^T e^{-ru} G_u \, du \middle| \mathcal{G}_t\right], \ t \in [0, T] \,. \tag{4.4}$$

Remark 4.8. If we consider a benchmarked financial market $\hat{S}_t, \ t \in [0, T]$, then the pricing formula for the benchmarked value process of the longevity bond is given by

$$\hat{V}_t = \mathbb{E}\left[\int_0^T \frac{G_u}{S_u^{\delta^*}}\, du \middle| \mathcal{G}_t\right],\ t \in [0,T]\,. \tag{4.5}$$

In the case of dividends paying assets, the benchmark approach presents the disadvantage that we need to know the joint conditional distribution of (S^{δ^*}, G) to compute (4.5). Note however that when the interest rate is supposed to be constant and the discounted asset prices to be local martingales, then the pricing formulas (4.4) and (4.5) coincide, since the $\mathbb{P}$-numéraire portfolio is given in this case by the savings account S^0.

We assume the existence on the market of a *gratification annuity* with increasing, continuous rate payments equal to $1 - G_t$ as long as the insured person is alive, up to maturity T. As G_t can be inferred from the longevity index which itself bases on realized mortality of some representative group, such an instrument rewards longevity relative to the policyholder's own age cohort. The present value of a gratification annuity is given by

$$C^a = \int_0^T e^{-ru}\,(1 - H_u)\,(1 - G_u)\; du\,.$$

Our goal is now to hedge the risk exposure from having sold the gratification annuity by trading dynamically in the longevity bond with value process V. For this sake we need some technical assumptions. First we assume $e^{\Gamma_T} \in L^2(P)$, and introduce the spaces $L^2(W)$, $L^2(M)$ consisting of all predictable θ, ψ such that

$$\mathbb{E}\left[\int_0^T \theta_s^2\, ds\right] < \infty, \quad \mathbb{E}\left[\int_0^T \psi_s^2\, d\Gamma_s\right] < \infty.$$

The space Θ of admissible strategies consists of all predictable ϑ such that

$$\mathbb{E}\left[\int_0^T \vartheta_s^2\, d\left\langle V\right\rangle_s\right] < \infty.$$

In this setting the risk minimizing strategy for the gratification annuity can be found by first computing the GKW decompositions of V and $\mathbb{E}\left[C^a \middle| \mathcal{G}_t\right], t \in [0,T]$, with respect to the $\mathbb{G}$-martingales W and M. By comparing them, one can then deduce the Föllmer–Schweizer decomposition

$$E\left[C^a \middle| \mathcal{G}_t\right] = c + \int_0^t \vartheta_s^*\, dV_s + V_t^{\perp},$$

where $\vartheta \in \Theta$ and $V^{\perp}$ is a square integrable martingale strongly orthogonal to V (i.e., $VV^{\perp}$ is a local martingale). For further details we refer to [9].

Theorem 4.9. *Under the hypotheses above, by martingale representation, for each* $u \in [0,T]$ *there exists a constant* c_u *and a predictable process* $(\theta_{u,s})_{s\in[0,T]} \in L^2(W)$, *with* $\theta_{u,s} = 0$ *if* $s > u$, *such that*

$$\begin{aligned}\mathbb{E}\left[e^{-ru}\,(1-G_u)\,e^{-\Gamma_u} \middle| \mathcal{F}_t \right] &= c_u + \int_0^{t\wedge u} \theta_{u,s}\,dW_s \\ &= c_u + \int_0^t \theta_{u,s}\mathbb{I}_{[0,u]}(s)\,dW_s,\end{aligned}$$

for $t \in [0,T]$. *We set* $c := \int_0^T c_u\,du < \infty$. *Then the Föllmer–Schweizer decompositions of the gratification annuity* C^a *with respect to the longevity bond* V *is given by*

$$C^a = c + \int_0^T \eta_s\,dV_s + V_T^{\perp},$$

where $V_T^{\perp} = \int_{0+}^T \gamma_s^M\,dM_s$, *the predictable integrand* $\gamma^M \in L^2(M)$ *is equal to*

$$\gamma_s^M = -(1-H_{s-})e^{\Gamma_s}\int_s^T \left(c_u + \int_0^s \theta_{u,v}\,dW_v \right) du, \;\; s \in [0,T],$$

and $\eta \in \Theta$ *is uniquely determined by the equation*

$$\eta_s \xi_s = (1-H_{s-})e^{\Gamma_s}\int_s^T \theta_{u,s}\;du, \;\; s \in [0,T].$$

Here the predictable integrand $\xi \in L^2(W)$ *derives by the predictable martingale representation for the longevity bond*

$$V_t = E\left[\int_0^T e^{-ru}G_u \middle| \mathcal{F}_t \right] = V_0 + \int_0^t \xi_s\,dW_s, \;\; t \in [0,T].$$

Proof. For the proof we refer to [9]. □

5. Relation between (local) risk minimization approach and real-world pricing

We now discuss the relation between the (local) risk minimization approach and real-world pricing. For an exhaustive discussion of the connection between the risk minimization approach and real-world pricing, we also refer to [6]. For further details on the relation between the existence of the numéraire portfolio and the minimal martingale density, see [26].

For the sake of simplicity, we assume that the underlying financial market contains only continuous asset prices. Then by Theorem 2.4 of [26] it follows that the benchmarked asset price process $\hat{S}$ is given by a local martingale. Given a benchmarked contingent claim $\hat{H} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, by Proposition 4.7 there exists a unique risk-minimizing strategy $\xi^{\hat{H}}$, that can be obtained by the Galtchouk–Kunita–Watanabe decomposition of $\hat{H}$ with respect to $\hat{S}$ given by

$$\hat{H} = E[\hat{H}] + \int_0^T \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u + L_T^{\hat{H}}, \quad \mathbb{P}\text{-a.s.}, \tag{5.1}$$

where $\xi^{\hat{H}}$ is an $\mathbb{F}$-predictable vector process with $\mathbb{E}\left[\int_0^T {\xi_u^{\hat{H}}}^{tr} \mathrm{d}[\hat{S}]_u \xi_u^{\hat{H}}\right] < \infty$ and $L^{\hat{H}} = (L_t^{\hat{H}})_{t\in[0,T]}$ is a square-integrable martingale with $L_0^{\hat{H}} = 0$, strongly orthogonal to each component of $\hat{S}$. The benchmarked portfolio's value process associated to $\xi^{\hat{H}}$ is then $\mathbb{E}\left[\hat{H}\middle|\mathcal{F}_t\right]$, $t \in [0,T]$, with initial value $\mathbb{E}\left[\hat{H}\right]$ and benchmarked cost process $L^{\hat{H}}$. Hence the real-world pricing formula (2.3) coincides at any time $t \in [0,T]$ with the portfolio's value of the risk-minimizing strategy for $\hat{H}$ in incomplete markets where the benchmarked underlyings are local martingales. This is the case not only for continuous asset price models, but also for a large class of jump-diffusion models, see for example [35], Chapter 14, pages 513–549. Moreover we also remark that the risk-minimizing strategy is independent of the choice of the discounting factor in market models driven by continuous asset price processes or where the orthogonal martingale structure is generated by continuous martingales. For further details on this, we refer to [6] and [7].

Furthermore decomposition (5.1) allows us to decompose every square-integrable benchmarked contingent claim as the sum of its *hedgeable part* $\hat{H}^h$ and its *unhedgeable part* $\hat{H}^u$ such that we can write

$$\hat{H} = \hat{H}^h + \hat{H}^u,$$

where

$$\hat{H}^h := \hat{H}_0 + \int_0^T \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u$$

and

$$\hat{H}^u := L_T^{\hat{H}}.$$

Here the benchmarked hedgeable part $\hat{H}^h$ can be replicated perfectly, i.e.,

$$\hat{U}_{H^h}(t) = \mathbb{E}\left[\hat{H}^h\middle|\mathcal{F}_t\right] = \hat{H}_0 + \int_0^t \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u\,,\ t \in [0,T],$$

and $\xi^{\hat{H}}$ yields the fair strategy for the self-financing replication of the hedgeable part of $\hat{H}$. The remaining benchmarked unhedgeable part can be diversified and will be covered through the benchmarked cost process $L^{\hat{H}}$. In particular at $t = 0$ the initial value of the risk-minimizing strategy coincides with the real world price for the hedgeable part at $t = 0$, while the benchmarked unhedgeable part remains totally untouched. This is reasonable because any extra trading could only create unnecessary uncertainty and potential additional benchmarked profits or losses. However for $t > 0$ the cost $L^{\hat{H}}$ will be different from 0 and

$$\mathbb{E}\left[\hat{H}\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\hat{H}^h\middle|\mathcal{F}_t\right] + L_t^{\hat{H}},\ t \in [0,T],$$

can be interpreted as an actuarial valuation formula, with the difference that the expectation term involves only the hedgeable part of the claim. The safety loading is given here by the benchmarked cost process. For similar results on the relation between actuarial valuation principles and mean-variance hedging, we also refer to [40].

The connection between risk-minimization and real-world pricing is then an important insight which both gives a clear reasoning for pricing and hedging of contingent claims via real-world pricing also in incomplete markets, and contributes to justify the use of the benchmark approach also for actuarial applications.

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f-Divergence Minimal Equivalent Martingale Measures and Optimal Portfolios for Exponential Lévy Models with a Change-point

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Abstract. We study exponential Lévy models with change-point which is a random variable, independent from initial Lévy processes. On canonical space with initially enlarged filtration we describe all equivalent martingale measures for change-point model and we give the conditions for the existence of f-divergence minimal equivalent martingale measure. Using the connection between utility maximisation and f-divergence minimisation, we obtain a general formula for optimal strategy in change-point case for initially enlarged filtration and also for progressively enlarged filtration. We illustrate our results considering the Black–Scholes model with change-point.

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1. Introduction

The parameters of financial models are generally highly dependent on time: a number of events (for example the release of information in the press, changes in the price of raw materials or the first time a stock price hits some psychological level) can trigger a change in the behaviour of stock prices. This time-dependency of the parameters can often be described using a piece-wise constant function: we will call this case a change-point model. In this context, an important problem in financial mathematics will be option pricing and hedging. Of course, the time of

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change (change-point) for the parameters is not explicitly known, but it is often possible to make reasonable assumptions about its nature and use statistical tests for its detection.

Change-point problems have a long history, probably beginning with the papers of Page [47, 48] in an a-posteriori setting, and of Shiryaev [56] in a quickest detection setting. The problem was later considered in many papers, see for instance [3, 15, 22, 44, 50, 51, 61] and also the book [2] and references there. In the context of financial mathematics, the question was investigated in [9, 16, 23, 31, 32, 40, 59, 60, 62] and was often related to a quickest detection approach.

It should be noticed that not only quickest detection approach is interesting in financial mathematics, and this fact is related with pricing and hedging of so-called default models (see [1, 18] and references there). In mentioned papers a number of very important results was obtained but for the processes without jump part or with only one jump.

The models with jumps, like exponential Lévy models, in general, compromise the uniqueness of an equivalent martingale measure when such measure exists. So, one has to choose in some way an equivalent martingale measure to price. Many approaches have been developed and various criteria suggested for this choice of martingale measure, for example risk-minimization in an L^2-sense [20, 42, 54, 55], Hellinger integrals minimization [12, 13, 29], entropy minimization [19, 45, 21], f^q-martingale measures [35] or Esscher measures [33].

All these approaches can be considered in unified way using so-called f-divergences, introduced by Ciszar [14] and investigated in a number of papers and books (see for instance [43] and references there). It should also be noticed that a general characterisation of f-divergence minimal martingale measures with applications to exponential Lévy models was given first in [28].

We recall that for f a convex function on $\mathbb{R}^{+,*}$ and two measures Q and P such that $Q << P$, the f-divergence of Q with respect to P is defined as

$$f(Q|P) = \mathbb{E}_P \left[f\left(\frac{dQ}{dP}\right)\right]$$

where $\frac{dQ}{dP}$ is Radon–Nikodym density of Q with respect to P, and E_P is the expectation with respect to P. We recall that the utility maximisation is closely related to f-divergence minimisation via Fenchel–Legendre transform and this will be one of essential points to obtain an optimal strategy.

The aim of this paper is to study f-divergence minimal martingale measures and optimal portfolios from the point of view of utility maximization, for exponential Lévy model with change-point where the parameters of the model before and after the change are known and a change-point itself is a random variable, independent from initial Lévy processes. We remark that even complete models like Black–Scholes model, become to be incomplete in change-point setting. Moreover, in our case, the characteristics of price process depend on τ, and then, in general, simple conditioning with respect to τ and the use of the results on the processes with independent increments do not give us a right answer. For this reason we use dual approach.

We start by describing our model in more details. We assume the financial market consists of a non-risky asset B with interest rate $(r_t)_{t\geq 0}$, namely

$$B_t = B_0 \exp\left(\int_0^t r_s\, ds\right)$$

where

$$r_t = r\mathbb{I}_{\{\tau>t\}} + \tilde{r}\mathbb{I}_{\{\tau\leq t\}}, \tag{1.1}$$

with $r, \tilde{r}$ interest rates before and after change-point τ, and a d-dimensional risky asset $S = (S_t)_{t\geq 0}$,

$$S_t = {}^\top(S_0^{(1)} \exp(X_t^{(1)}), \ldots, S_0^{(d)} \exp(X_t^{(d)}))$$

where X is a stochastic process obtained by pasting in τ of two d-dimensional Lévy processes L and $\tilde{L}$ together:

$$X_t = L_t\mathbb{I}_{\{\tau>t\}} + (L_\tau + \tilde{L}_t - \tilde{L}_\tau)\mathbb{I}_{\{\tau\leq t\}}. \tag{1.2}$$

Here and further L and $\tilde{L}$ supposed to be independent Lévy processes with characteristics (b, c, ν) and $(\tilde{b}, \tilde{c}, \tilde{\nu})$, respectively which are independent from τ (for more details see [53]). Here $b, \tilde{b}$ stand for drift, $c, \tilde{c}$ denote the covariance matrix of Brownian part, and $\nu, \tilde{\nu}$ are Lévy measures on $\mathbb{R}^{d,*}$. To avoid unnecessary complications we assume up to now that the parameters r and $\tilde{r}$ in (1.1) are equal to zero, and that $S_0^{(i)} = 1$ for all $1 \leq i \leq d$.

To describe a probability space on which the process X is well defined, we consider $(D, \mathcal{G}, \mathbb{G})$ the canonical space of right-continuous functions with left-hand limits equipped with its natural filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ which satisfies standard conditions: it is right-continuous, $\mathcal{G}_0 = \{\emptyset, D\}$, $\bigvee_{t\geq 0} \mathcal{G}_t = \mathcal{G}$. On the product of such canonical spaces we define two independent Lévy processes $L = (L_t)_{t\geq 0}$ and $\tilde{L} = (\tilde{L}_t)_{t\geq 0}$ with characteristics (b, c, ν) and $(\tilde{b}, \tilde{c}, \tilde{\nu})$ respectively and denote by P and $\tilde{P}$ their respective laws which are assumed to be locally equivalent: $P \overset{\text{loc}}{\sim} \tilde{P}$. As we will consider the market on a fixed finite time interval, we are really only interested in $P|_{\mathcal{G}_T}$ and $\tilde{P}|_{\mathcal{G}_T}$ for a fixed $T \geq 0$ and the distinction between equivalence and local equivalence does not need to be made.

Our change-point will be represented by an independent random variable τ of law α taking values in $([0, T], \mathcal{B}([0, T])$. The set $\{\tau = T\}$ corresponds to the situation when the change-point does not take place, or at least not on the interval we are studying.

On the probability space $(D \times D \times [0, T], \mathcal{G} \times \mathcal{G} \times \mathcal{B}([0, T], P \times \tilde{P} \times \alpha)$ we define a measurable map X by (1.2) and we denote by $\mathbb{P}$ its law. In what follows we use $\mathbb{E}$ mainly for the expectation with respect to $\mathbb{P}$ but this notation will be also used for the expectation with respect to $P \times \tilde{P} \times \alpha$.

From point of view of observable processes we can have the following situations. If we observe only the process X then the natural probability space to work is $(D, \mathcal{G}, \mathbb{P})$ equipped with the right-continuous version of the natural filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ where $\mathcal{G}_t = \sigma\{X_s,\ s \leq t\}$ for $t \geq 0$. Now, if we observe not only the process X but also some complementary variables related with τ then we can take it into account by the enlargement of the filtration. First we consider the filtration $\mathbb{H}$ given by $\mathcal{H}_t = \sigma(\mathbb{I}_{\{\tau \leq s\}}, s \leq t)$ and note that $\mathcal{H}_T = \sigma(\tau)$. Then we introduce two filtrations: the initially enlarged filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$

$$\mathcal{F}_0 = \mathcal{G}_0 \vee \mathcal{H}_T,\ \ \mathcal{F}_t = \bigcap_{s>t} (\mathcal{G}_s \vee \mathcal{H}_T)$$

and the progressively enlarged filtration $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t\geq 0}$ which satisfies:

$$\hat{\mathcal{F}}_0 = \mathcal{G}_0 \vee \mathcal{H}_0,\ \ \hat{\mathcal{F}}_t = \bigcap_{s>t} (\mathcal{G}_s \vee \mathcal{H}_s).$$

In the case of additional information the most natural filtration from the point of view of observable events would be $\hat{\mathbb{F}}$. However, it is not so easy to obtain the explicit formulas of optimal strategies for progressively enlarged filtration. So, we start by investigation of optimal strategies for initially enlarged filtration. In special cases of exponential utility, logarithmic utility and power utility it gives us an optimal strategy for progressively enlarged filtration automatically. In order to obtain an optimal strategy in general case, we do projection (see Proposition 4.11 for the details).

The paper is organized in the following way. In Section 2 we start by recalling in unified way the facts about f-divergence minimal equivalent martingale measures for exponential Lévy models. This information will be used for investigation of change-point case.

In Section 3 we study the change-point case. On mentioned probability space and for initially enlarged filtration we describe first all equivalent martingale measures. Then, we introduce as hypotheses, such properties of f-divergence minimal equivalent martingale measures as a preservation of Lévy property and a scaling property. The question of preservation of Lévy property was considered in details in [6] and it was shown that the class of f-divergences preserving Lévy property is larger then common f-divergences, i.e., the functions such that $f''(x) = ax^\gamma$, $a > 0, \gamma \in \mathbb{R}$. We recall that these functions are those for which there exists $A > 0$ and real B, C such that $f(x) = Af_\gamma(x) + Bx + C$ where

$$f_\gamma(x) = \begin{cases} c_\gamma x^{\gamma+2} & \text{if } \gamma \neq -1, -2, \\ x \ln(x) & \text{if } \gamma = -1, \\ -\ln(x) & \text{if } \gamma = -2, \end{cases} \tag{1.3}$$

and $c_\gamma = \text{sign}[(\gamma + 1)\,(\gamma + 2)]$. The conditions for existence and the expression of Radon–Nikodym density $Z^*_T(\tau)$ of f-divergence minimal martingale measure for change-point model is given in Theorem 3.2. Then, in Corollaries 3.3 and 3.4 we give the corresponding results for common f-divergences and, finally, we apply the results to Black–Scholes change-point model.

In Section 4 we present first some facts about utility maximisation and the formulas for optimal strategies of single exponential Lévy model. Then we give a decomposition formula for $f'(Z^*_T(\tau))$ in initially enlarged filtration. These decompositions allow us via the result of [28] to identify optimal strategy (see Theorem 4.8). We illustrate these results by considering again the Black–Scholes model with a change-point.

2. f-divergence minimal EMM's for exponential Lévy model

We start by recalling in unified way the facts about f-divergence minimal martingale measures for exponential Lévy models. Namely, we will consider common f-divergences and we will discuss the preservation of Lévy property by f-divergence minimal locally equivalent martingale measures (EMM's). We will also mention the expressions for so-called Girsanov parameters when we change the initial measure P into f-divergence minimal EMM's and also the expression of Radon–Nikodym density of these measures in terms of Girsanov parameters.

Let now $L = (L_t)_{t\geq 0}$ be d-dimensional Lévy process with parameters (b, c, ν) where b is the drift parameter, c is a covariance matrix of Brownian part and ν is the Lévy measure, i.e., the measure on $\mathbb{R}^{d,*}$ which satisfies

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < +\infty.$$

We recall that the characteristic function of L_t for $t \in \mathbb{R}^+$ and $u \in \mathbb{R}$ is given then by:

$$\phi_t(u) = \mathbb{E}e^{i\langle u, L_t\rangle} = e^{\psi(u)t}$$

and in turn, the characteristic exponent

$$\psi(u) = i\,\langle u, b\rangle - \frac{1}{2}\,\langle cu, u\rangle + \int_{\mathbb{R}^d} (\exp(i\,\langle u, x\rangle) - 1 - i\,\langle u, h(x)\rangle)\nu(dx),$$

where from now on, h is the truncation function and $\langle\cdot,\cdot\rangle$ is a scalar product in $\mathbb{R}^d$. We set $S = (S_t)_{t\geq 0}$ with

$$S_t = {}^{\top}\left(S_0^{(1)}\exp(L_t^{(1)}), \ldots, S_0^{(d)}\exp(L_t^{(d)})\right)$$

for our risky asset and $B = (B_t)_{t\geq 0}$ for non-risky asset with constant interest rate r. We will suppose without loss of generality up to now that $S_0^{(i)} = 1$ for all $1 \leq i \leq d$ and $r = 0$.

Let T be a fixed horizon and $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be natural filtration. We recall that for a convex function f on $\mathbb{R}^{+,*}$, the f-divergence of the restriction Q_T of the measure Q with respect to the restriction P_T of the measure P to $\mathcal{G}_T$ is:

$$f(Q_T|P_T) = E_P\left[f\left(\frac{dQ_T}{dP_T}\right)\right].$$

Here by convention we set this integral equal to $+\infty$ if the corresponding function is not integrable. We recall that Q^*_T is an f-divergence minimal equivalent martingale measure if $f(Q^*_T|P_T) < +\infty$ and

$$f(Q^*_T|P_T) = \inf_{Q\in\mathcal{M}(P)} f(Q_T|P_T)$$

where $\mathcal{M}(P)$ is the set of locally equivalent martingale measures supposed to be non-empty. We also recall that an f-divergence minimal equivalent martingale measure Q^* is invariant under scaling if for all $x \in \mathbb{R}^{+,*}$

$$f(xQ^*_T|P_T) = \inf_{Q\in\mathcal{M}(P)} f(xQ_T|P_T).$$

It is called time-invariant if Q^* is the same for all $T > 0$. For a given exponential Lévy model $S = S_0\, e^L$, we say that an f-divergence minimal martingale measure Q^* preserves the Lévy property if L remains a Lévy process under Q^*.

We recall that the density Z of any equivalent to P measure Q can be written in the form $Z = \mathcal{E}(M)$ where $\mathcal{E}$ denotes the Doleans–Dade exponential and $M = (M_t)_{t\geq 0}$ is a local martingale. It follows from Girsanov theorem that there exist predictable functions β and Y verifying the following integrability conditions: for $t \geq 0$ (P-a.s.)

$$\int_0^t \langle c\beta_s, \beta_s\rangle\, ds < \infty,$$

$$\int_0^t \int_{\mathbb{R}^d} |\, h(y)\, (Y_s(y) - 1)\, |\nu^{X,P}(ds, dy) < \infty,$$

and such that

$$M_t = \int_0^t \langle \beta_s, dX^c_s\rangle + \int_0^t \int_{\mathbb{R}^d} (Y_s(y) - 1)(\mu^X - \nu^{X,P})(ds, dy)$$

where μ^X is a jump measure of the process X and $\nu^{X,P}$ is its compensator with respect to $(P, \mathbb{G})$, $\nu^{X,P}(ds, dy) = ds\, \nu(dy)$ (for more details see [34]). We will refer to (β, Y) as the Girsanov parameters of the change of measure from P into Q. It is known from Grigelionis result [30] that a semi-martingale is a process with independent increments under Q if and only if their semi-martingale characteristics are deterministic, i.e., the Girsanov parameters do not depend on ω, i.e., β depends only on time t and Y depends on (t, x) time and jump size. Since Lévy process is homogeneous process, it implies that X will remain a Lévy process under Q if and only if there exists $\beta \in \mathbb{R}^d$ and a positive measurable function Y such that $\beta_t(\omega) = \beta$ and $Y_t(\omega, y) = Y(y)$ ($P \times \lambda \times \nu$-a.s.).

We recall that if Lévy property is preserved, S will be a martingale under Q if and only if

$$b + \frac{1}{2}\,\mathrm{diag}(c) + c\beta + \int_{\mathbb{R}^d} [(e^y - 1)Y(y) - h(y)]\nu(dy) = 0$$

where $y = {}^{\top}(y_1, \ldots, y_d)$ and $e^y - 1$ is a vector with the components $e^{y_i} - 1$ for $1 \leq i \leq d$. This follows again from Girsanov theorem and reflects the fact that under Q the drift of S is equal to zero.

As it was mentioned, our aim in this section is to consider in more detail the class of minimal martingale measures for the functions which satisfy (1.3). In particular, the minimal measure for f will be the same as that for f_γ. Minimal measures for the different functions f_γ have been well studied (see [28] and further references). It has been shown in [19, 35, 37] that in all these cases, the minimal measure, when it exists, preserves the Lévy property.

Sufficient conditions for the existence of a minimal measure and an explicit expression of the associated Girsanov parameters have been given in the case of relative entropy in [21, 33] and for power functions in [35]. It was also shown in [33] that these conditions are in fact necessary in the case of relative entropy or for power functions. Our aim in this section is to give a unified expression of such conditions for all functions which satisfy $f''(x) = ax^\gamma$ and to show that, under some conditions, they are necessary and sufficient. We have already mentioned that f-divergence minimal martingale measures play an important role in the determination of utility maximising strategies. In this context, it is useful to have further invariance properties for the minimal measures such as scaling and time invariance properties. This is the case when $f''(x) = ax^\gamma$.

Theorem 2.1. *Consider a Lévy process X with characteristics (b, c, ν) and let f be a function such that $f''(x) = ax^\gamma$, where $a > 0$ and $\gamma \in \mathbb{R}$. Suppose that $c \neq 0$ or $\overset{\circ}{\text{supp}}\ (\nu) \neq \emptyset$. Then there exists an f-divergence minimal equivalent to P martingale measure Q preserving Lévy properties if and only if there exist constants $\alpha, \beta \in \mathbb{R}^d$ and measurable function $Y : \mathbb{R}^{d,*} \to \mathbb{R}^+$ such that*

$$Y(y) = (f')^{-1}(f'(1) + \alpha(e^y - 1)) \tag{2.1}$$

with, for $c \neq 0$, $\alpha = c_\gamma\,(\gamma+1)\,(\gamma+2)\,\beta$ if $\gamma \neq -1, -2$ and $\alpha = \beta$ if $\gamma = -1$ or $\gamma = -2$, and such that the following properties hold:

$$Y(y) > 0 \ \ \nu - a.e., \tag{2.2}$$

$$\int_{|y|\geq 1} |e^y - 1|\, Y(y)\nu(dy) < +\infty, \tag{2.3}$$

$$b + \frac{1}{2}\operatorname{diag}(c) + c\beta + \int_{\mathbb{R}^d} ((e^y - 1)Y(y) - h(y))\nu(dy) = 0. \tag{2.4}$$

If such a measure exists the Girsanov parameters associated with Q are: (β, Y) if $c \neq 0$, and $(0, Y)$ if $c = 0$. In addition, this measure is scale and time invariant.

We begin with some technical lemmas. For $Q \overset{\text{loc}}{\sim} P$ we denote by $(Z_t)_{t\geq 0}$ Radon–Nikodym density process of Q with respect to P.

Lemma 2.2. *Let Q be the measure preserving Lévy property. Then, $Q_T \sim P_T$ for all $T > 0$ iff*

$$Y(y) > 0 \ \ \nu - a.e.,$$

$$\int_{\mathbb{R}^d} (\sqrt{Y(y)} - 1)^2 \nu(dy) < +\infty. \tag{2.5}$$

Proof. See Theorem 2.1, p. 209 of [34]. □

Lemma 2.3. *Under $Q_T \sim P_T$, the condition $E_P|f(Z_T)| < \infty$ is equivalent to*

$$\int_{\mathbb{R}^d} [f(Y(y)) - f(1) - f'(1)(Y(y) - 1)]\nu(dy) < +\infty. \tag{2.6}$$

Proof. In our particular case, $E_P|f(Z_T)| < \infty$ is equivalent to the existence of $E_P f(Z_T)$. We use the Itô formula to express this integrability condition in predictable terms. Taking for $n \geq 1$ stopping times

$$s_n = \inf\{t \geq 0 : Z_t > n \,\text{or}\, Z_t < 1/n\}$$

where $\inf\{\emptyset\} = +\infty$, we get for $\gamma \neq -1, -2$ and $\alpha = \gamma + 2$ that P-a.s.

$$\begin{aligned} Z^\alpha_{T\wedge s_n} = 1 &+ \int_0^{T\wedge s_n} \alpha\, Z^\alpha_{s-}\, \langle \beta, dX^c_s\rangle + \frac{1}{2}\alpha\,(\alpha-1)\,\langle c\,\beta, \beta\rangle \int_0^{T\wedge s_n} Z^\alpha_{s-} ds \\ &+ \int_0^{T\wedge s_n} \int_{\mathbb{R}^d} Z^\alpha_{s-}(Y^\alpha(y) - 1)(\mu^X - \nu^{X,P})(ds, dy) \\ &+ \int_0^{T\wedge s_n} \int_{\mathbb{R}^d} Z^\alpha_{s-}[Y^\alpha(y) - 1 - \alpha(Y(y) - 1)]ds\ \nu(dy). \end{aligned}$$

Hence,

$$Z^\alpha_{T\wedge s_n} = \mathcal{E}(N^{(\alpha)} + A^{(\alpha)})_{T\wedge s_n}$$

where

$$N^{(\alpha)}_t = \int_0^t \alpha\ \langle \beta, dX^c_s\rangle + \int_0^t \int_{\mathbb{R}^d} (Y^\alpha(y) - 1)(\mu^X - \nu^{X,P})(ds, dy)$$

and

$$A^{(\alpha)}_t = \frac{t}{2}\alpha\,(\alpha-1)\,\langle c\beta, \beta\rangle + t\int_{\mathbb{R}^d} [Y^\alpha(y) - 1 - \alpha(Y(y) - 1)]\nu(dy).$$

Since $[N^{(\alpha)}, A^{(\alpha)}]_t = 0$ for each $t \geq 0$ we have

$$Z^\alpha_{T\wedge s_n} = \mathcal{E}(N^{(\alpha)})_{T\wedge s_n}\mathcal{E}(A^{(\alpha)})_{T\wedge s_n}.$$

In the case $\alpha > 1$ and $\alpha < 0$, and $E_P Z^\alpha_T < \infty$, we have by Jensen inequality

$$0 \leq Z^\alpha_{T\wedge s_n} \leq E_P(Z^\alpha_T \,|\, \mathcal{F}_{T\wedge s_n})$$

and since the right-hand side of this inequality form uniformly integrable sequence, $(Z^\alpha_{T\wedge s_n})_{n\geq 1}$ is also uniformly integrable. We remark that $A^{(\alpha)}_t \geq 0$ for all $t \geq 0$ and

$$\mathcal{E}(A^{(\alpha)})_{T\wedge s_n} = \exp(A^{(\alpha)}_{T\wedge s_n}) \geq 1.$$

It means that $(\mathcal{E}(N^{(\alpha)})_{T\wedge s_n})_{n\in\mathbb{N}^*}$ is uniformly integrable and

$$E_P(Z^\alpha_T) = \exp(A^{(\alpha)}_T).$$

Hence, (2.6) holds.

If (2.6) holds, then by the Fatou lemma and since $\mathcal{E}(N^{(\alpha)})$ is a positive local martingale we get

$$E_P(Z_T^\alpha) \le \underline{\lim}_{n\to\infty} E_P(Z_{T\wedge s_n}) \le \exp(A_T^{(\alpha)}) < \infty.$$

For $0 < \alpha < 1$, we have again

$$Z_{T\wedge s_n}^\alpha = \mathcal{E}(N^{(\alpha)})_{T\wedge s_n} \mathcal{E}(A^{(\alpha)})_{T\wedge s_n}$$

with uniformly integrable sequence $(Z_{T\wedge s_n}^\alpha)_{n\ge 1}$. Since

$$\mathcal{E}(A^{(\alpha)})_{T\wedge s_n} = \exp(A_{T\wedge s_n}^{(\alpha)}) \ge \exp(A_T^{(\alpha)}),$$

the sequence $(\mathcal{E}(N^{(\alpha)})_{T\wedge s_n})_{n\in\mathbb{N}^*}$ is uniformly integrable and

$$E_P(Z_T^\alpha) = \exp(A_T^{(\alpha)}).$$

For $\gamma = -2$ we have that $f(x) = x\ln(x)$ up to linear term and

$$\begin{aligned} Z_{T\wedge s_n}\ln(Z_{T\wedge s_n}) &= \int_0^{T\wedge s_n} (\ln(Z_{s-})+1) Z_{s-}\,\langle \beta, dX_s^c\rangle + \frac{1}{2}\langle c\,\beta,\beta\rangle \int_0^{T\wedge s_n} Z_{s-}ds \\ &+ \int_0^{T\wedge s_n}\int_{\mathbb{R}^d} Z_{s-}[\ln(Z_{s-})(Y(y)-1) + Y(y)\,\ln(Y(y)](\mu^X - \nu^{X,P})(ds,dy) \\ &+ \int_0^{T\wedge s_n}\int_{\mathbb{R}^d} Z_{s-}(\,Y(y)\ln(Y(y)) - Y(y) + 1\,)ds\,\nu(dy). \end{aligned}$$

Taking mathematical expectation we obtain:

$$\begin{aligned} &E_P[Z_{T\wedge s_n}\ln(Z_{T\wedge s_n})] \qquad (2.7)\\ &\quad = E_P\int_0^{T\wedge s_n} Z_{s-}\left[\frac{1}{2}\langle c\,\beta,\beta\rangle + \int_{\mathbb{R}^d}(\,Y(y)\ln(Y(y)) - Y(y)+1\,)\,\nu(dy)\right] ds. \end{aligned}$$

If $E_P[Z_T\ln(Z_T)] < \infty$, then the sequence $(Z_{T\wedge s_n}\ln(Z_{T\wedge s_n}))_{n\in\mathbb{N}^*}$ is uniformly integrable. In addition, $E_P(Z_{s-}) = 1$ and we obtain applying Lebesgue monotone convergence theorem that

$$E_P[Z_T\ln(Z_T)] = T\,[\frac{1}{2} < c\,\beta,\beta > + \int_{\mathbb{R}^d}(\,Y(y)\ln(Y(y)) - Y(y)+1\,)\nu(dy)]$$

and this implies (2.6). If (2.6), then by the Fatou lemma from (2.7) we deduce that $E_P[Z_T\ln(Z_T)] < \infty$.

For $\gamma = -1$, we have $f(x) = -\ln(x)$ and exchanging P and Q we get:

$$\begin{aligned} E_P[-\ln(Z_T)] &= E_Q[\tilde{Z}_T\ln(\tilde{Z}_T)] \\ &= T\left[\frac{1}{2}\langle c\,\beta,\beta\rangle + \int_{\mathbb{R}^d}(\,\tilde{Y}(y)\ln(\tilde{Y}(y)) - \tilde{Y}(y)+1\,)\nu^Q(dy)\right] \end{aligned}$$

where $\tilde{Z}_T = 1/Z_T$ and $\tilde{Y}(y) = 1/Y(y)$. But $\nu^Q(dy) = Y(y)\nu(dy)$ and, finally,

$$E_P[-\ln(Z_T)] = \frac{T}{2}\langle c\,\beta,\beta\rangle + T\int_{\mathbb{R}^d}(-\ln(Y(y)) + Y(y) - 1)\nu(dy)$$

which implies (2.6). Again from (2.6) we get that $E_P[-\ln(Z_T)]$ is finite. □

Lemma 2.4. *If the second Girsanov parameter Y has a particular form* (2.1) *then the condition*

$$\int_{|y|\geq 1} |e^y - 1|\, Y(y)\nu(dy) < +\infty$$

implies the conditions (2.5) *and* (2.6).

Proof. We can cut each integral in (2.5) and (2.6) on two parts and integrate on the sets $\{|y| \leq \delta\}$ and $\{|y| > \delta\}$ for some $\delta > 0$. Then we can use a particular form of Y and conclude easily writing Taylor expansion of order 2. □

Proof of Theorem 2.1. (Necessity) We suppose that there exists f-divergence minimal equivalent martingale measure Q preserving Lévy property of X. Then, since $Q_T \sim P_T$, the conditions (2.2), (2.5) follow from Theorem 2.1, p. 209 of [34]. From Theorem 3 of [6] we deduce that (2.1) holds. Then, the condition (2.3) follows from the fact that S is a martingale under Q. Finally, the condition (2.4) follows from Girsanov theorem since Q is a martingale measure and, hence, the drift of S under Q is zero.

(Sufficiency) We take β and Y verifying the conditions (2.2), (2.3), (2.4) and we construct

$$M_t = \int_0^t \langle \beta, dX_s^c\rangle + \int_0^t \int_{\mathbb{R}^d} (Y(y) - 1)(\mu^X - \nu^{X,P})(ds, dy).$$

As known from Theorem 1.33, p. 72–73, of [34], the last stochastic integral is well defined if

$$C(W) = T\int_{\mathbb{R}^d} (Y(y) - 1)^2 I_{\{|Y(y)-1|\leq 1\}}\nu(dy) < \infty,$$

$$C(W') = T\int_{\mathbb{R}^d} |Y(y) - 1| I_{\{|Y(y)-1|>1\}}\nu(dy) < \infty.$$

But the condition (2.3), the relation (2.1) and Lemma 2.4 implies (2.5). So, $(Y - 1) \in G_{\text{loc}}(\mu^X)$ and M is local martingale. Then we take

$$Z_T = \mathcal{E}(M)_T$$

and this defines the measure Q_T by its Radon–Nikodym density. Now, the conditions (2.2), (2.3) together with the relation (2.1) and Lemma 2.4 imply (2.5), and, hence, from Lemma 2.2 we deduce that $P_T \sim Q_T$.

Since $P_T \sim Q_T$, the Lemma 2.3 gives us the needed integrability condition: $E_P|f(Z_T)| < \infty$. Now, since (2.4) holds, Q is martingale measure, and it remains to show that Q is indeed f-divergence minimal. For that we take any equivalent martingale measure $\bar{Q}$ such that $E_P|f(\bar{Z}_T)| < \infty$ where $\bar{Z}_T = \dfrac{d\bar{Q}_T}{dP_T}$ and we show that

$$E_Q f'(Z_T) \leq E_{\bar{Q}} f'(Z_T). \tag{2.8}$$

If the mentioned inequality holds, the Theorem 2.2 of [28] implies that Q is an f-divergence minimal.

In the case $f_\gamma(x) = c_\gamma x^{\gamma+2}$ with $\gamma < -1$ we obtain from Theorem 4 in [6] and the particular form of Y that

$$Z_T^{\gamma+1} = E_Q(Z_T^{\gamma+1})\ \mathcal{E}(N^{(\gamma+1)})_T$$

where for $0 \le t \le T$

$$N_t^{(\gamma+1)} = \int_0^t (\gamma+1) E_Q(Z_{T-t})^{\gamma+1} \left\langle \beta, d\hat{X}_t \right\rangle$$

and $\hat{X}$ stands for the vector of stochastic logarithms of the components of S. So, $\mathcal{E}(N^{(\gamma+1)})$ is a positive local martingale with respect to $\bar{Q}$ and, hence, a supermartingale, and we get

$$E_{\bar{Q}} Z_T^{\gamma+1} \le E_Q Z_T^{\gamma+1}.$$

Multiplying by $c_\gamma(\gamma+2) < 0$ we obtain (2.8).

In the case $\gamma > -1$, $c_\gamma(\gamma+2) > 0$. We prove using again a particular form of f' and Y that

$$Z_T^{\gamma+1} = E_Q(Z_T^{\gamma+1}) + \int_0^T Z_{t-}^{\gamma+1}(\gamma+1) E_Q(Z_{T-t}^{\gamma+1}) \left\langle \beta, d\hat{X}_t \right\rangle.$$

We notice that the right-hand side is a positive local martingale with respect to $\bar{Q}$ Then, we take a localising sequence $(\tau_n)_{n\in\mathbb{N}}$ and the mathematical expectation with respect to $\bar{Q}$ in previous equality, and we show that the family $(\bar{Z}_{T\wedge\tau_n} Z_{T\wedge\tau_n}^{\gamma+1})_{n\in\mathbb{N}}$ is uniformly integrable with respect to P. In fact, using Cauchy–Schwarz inequality with $p = \frac{\gamma+2}{\gamma+1}$ and $q = \gamma+2$ we can show that this family has uniformly bounded expectation with respect to P and that it is uniformly continuous. As a consequence, we have (2.8).

In the case $\gamma = -1$ we prove again using Theorem 4 in [6] that

$$\ln(Z_T) = E_Q(\ln(Z_T)) + \left\langle \beta, \hat{X}_T \right\rangle.$$

The right-hand side is a local martingale with respect to $\bar{Q}$. Then, we take a localising sequence $(\tau_n)_{n\in\mathbb{N}}$ and we show that

$$E_Q \hat{X}_t = \lim_{n\to\infty} E_Q \hat{X}_{t\wedge\tau_n} = 0.$$

For $\gamma = -2$ we have:

$$-\frac{1}{Z_T} = E_Q\left(-\frac{1}{Z_T}\right) + \int_0^T \frac{1}{Z_s}\, d\hat{X}_s.$$

The integral in the right-hand side of this expression is a local martingale with respect to $\bar{Q}$. We show that its expectation with respect to $\bar{Q}$ is equal to zero and it proves that Q is minimal martingale measure.

Finally, note that the conditions which appear in Theorem 2.1 do not depend in any way on the time interval which is considered and, hence, the minimal measure always exists and its Girsanov parameters does not depend on T. So, the measure Q^* is time invariant. Furthermore, if Q^* is f-divergence minimal, the equality

$$f(cx) = Af(x) + Bx + C$$

with A, B, C constants, $A > 0$, gives

$$\begin{aligned} E_P\left[f\left(c\frac{d\bar{Q}}{dP}\right)\right] &= AE_P\left[f\left(\frac{d\bar{Q}}{dP}\right)\right] + B + C \\ &\geq AE_P\left[f\left(\frac{dQ}{dP}\right)\right] + B + C = E_P\left[f\left(c\frac{dQ}{dP}\right)\right] \end{aligned}$$

and Q is scale invariant. □

3. f-divergence minimal EMM's for change-point model

Here we describe all locally equivalent martingale measures (EMMs) for change point model leaving on our probability space equipped with initially enlarged filtration, and in particular in relation to the sets of EMMs of the two associated Lévy models L and $\tilde{L}$. For that we introduce the sets of equivalent martingale measures $\mathcal{M}(P, \mathbb{G})$ and $\mathcal{M}(P, \mathbb{F})$ related with L and the filtrations $\mathbb{G}$ and $\mathbb{F}$ respectively. We denote the relative sets for $\tilde{L}$ by $\mathcal{M}(\tilde{P}, \mathbb{G})$ and $\mathcal{M}(\tilde{P}, \mathbb{F})$.

3.1. EMMs for change-point model

We assume that the sets $\mathcal{M}(P, \mathbb{G})$ and $\mathcal{M}(\tilde{P}, \mathbb{G})$ are non-empty. Let $Q \in \mathcal{M}(P, \mathbb{F})$ and $\tilde{Q} \in \mathcal{M}(\tilde{P}, \mathbb{F})$. We introduce the Radon–Nikodym density processes $\zeta = (\zeta_t)_{t\geq 0}$ and $\tilde{\zeta} = (\tilde{\zeta}_t)_{t\geq 0}$ given by

$$\zeta_t = \frac{dQ_t}{dP_t}, \qquad \tilde{\zeta}_t = \frac{d\tilde{Q}_t}{d\tilde{P}_t}$$

where $Q_t, P_t, \tilde{Q}_t, \tilde{P}_t$ stand for the restrictions of the corresponding measures to the σ-algebra $\mathcal{F}_t$.

According to the result of Jeulin [36] (see also [4]), there exist the versions of density processes $\zeta, \tilde{\zeta}$ which can be written in the form $(\zeta_t(\tau))_{t\geq 0}$, $(\tilde{\zeta}_t(\tau))_{t\geq 0}$ in a way that for a.e. $u \in \text{supp}(\mathcal{L}(\tau))$, $\zeta(u) = (\zeta_t(u))_{t\geq 0}$ and $\tilde{\zeta}(u) = (\tilde{\zeta}_t(u))_{t\geq 0}$ are $(P, \mathbb{G})$ and respectively $(\tilde{P}, \mathbb{G})$ martingales. In what follows we take such versions of ζ and $\tilde{\zeta}$.

We also introduce for all $t > 0$

$$v_t = \frac{d\tilde{P}_t}{dP_t},$$

then

$$V_t = \mathbb{I}_{[\![0,\tau]\!]}(t) + \frac{v_t}{v_\tau}\mathbb{I}_{]\!]\tau,+\infty[\![}(t). \tag{3.1}$$

We remark that the measure $\mathbb{P}$ which is the law of X verify for $t \geq 0$:

$$\frac{d\mathbb{P}_t}{d\,P_t} = V_t.$$

To describe all EMMs leaving on our space we define the process $z = (z_t)_{t \geq 0}$ given by

$$z_t = \zeta_t \mathbb{I}_{[\![0,\tau]\!]}(t) + \zeta_\tau \frac{\tilde{\zeta}_t}{\tilde{\zeta}_\tau} \mathbb{I}_{]\!]\tau,+\infty[\![}(t). \tag{3.2}$$

Finally, we consider the measure $\mathbb{Q}$ such that

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = z_t. \tag{3.3}$$

Proposition 3.1. *A measure* $\mathbb{Q}$ *is an equivalent martingale measure for the exponential model* (1.2) *related to the process* X *iff its density process has the form* (3.3).

Proof. First we show that the process z is a density process with respect to $\mathbb{P}$ and that the process $S = (S_t)_{t \geq 0}$ such that

$$S_t = e^{L_t} \mathbb{I}_{[\![0,\tau]\!]}(t) + S_\tau e^{\tilde{L}_t - \tilde{L}_\tau} \mathbb{I}_{]\!]\tau,+\infty[\![}(t)$$

is a $(\mathbb{Q}, \mathbb{F})$-martingale.

We begin by noticing that if for all $u \in \text{supp}(\mathcal{L}(\tau))$, $M(u)$ and $\tilde{M}(u)$ are two strictly positive $(P, \mathbb{G})$ martingales , then $N(u) = (N_t(u))_{t \geq 0}$ such that

$$N_t(u) = \left[M_t(u) \mathbb{I}_{[\![0,u]\!]}(t) + M_u(u) \frac{\tilde{M}_t(u)}{\tilde{M}_u(u)} \mathbb{I}_{]\!]u,+\infty[\![}(t) \right]$$

is a $(P, \mathbb{G})$ martingale.

To show that z is a $(\mathbb{P}, \mathbb{F})$-martingale, we prove an equivalent fact that $(V_t\, z_t)_{t \geq 0}$ is a $(P, \mathbb{F})$ martingale. For that we use the relations (3.1), (3.2), we condition with respect to τ and we use the previous remark with $M_t(u) = \zeta_t(u)$ and $\tilde{M}_t(u) = \tilde{\zeta}_t(u) v_t$. Furthermore, we see that $\mathbb{E} z_t = 1$. To show that $S = (S_t)_{t \geq 0}$ is $(\mathbb{Q}, \mathbb{F})$-martingale we establish that $(V_t\, z_t\, S_t)_{t \geq 0}$ is a $(P, \mathbb{F})$-martingale. For this we use the same remark with $M_t(u) = e^{L_t} \zeta_t(u)$ and $\tilde{M}_t(u) = v_t \tilde{\zeta}_t(u) e^{\tilde{L}_t}$.

Conversely, z is the density of any equivalent martingale measure if and only if $(z_t\, S_t)_{t \geq 0}$ is a $(\mathbb{P}, \mathbb{F})$-martingale. But the last fact is equivalent to: for any bounded stopping time σ,

$$\mathbb{E}(z_\sigma\, S_\sigma) = 1.$$

Replacing σ by $\sigma \wedge \tau$ in previous expression we deduce that $(z_{t \wedge \tau})_{t \geq 0}$ is the density of a martingale measure for $(e^{L_{t \wedge \tau}})_{t \geq 0}$. In the same way, using the martingale properties of z we get for any bounded stopping time σ that

$$\mathbb{E} \left(\frac{z_\sigma\, S_\sigma}{z_{\sigma \wedge \tau}\, S_{\sigma \wedge \tau}} \right) = 1$$

and so $\left(\frac{z_t}{z_{t \wedge \tau}} \right)_{t \geq \tau}$ is the density of an equivalent martingale measure for the process

$$\left(e^{\tilde{L}_t - \tilde{L}_{t \wedge \tau}} \right)_{t \geq \tau}. \qquad \square$$

3.2. From EMM's to f-divergence minimal EMM's

In the following theorem we give an expression for the density of the f-divergence minimal EMM's $\mathbb{Q}^*_T$ with respect to $\mathbb{P}_T$ in our change-point framework. We denote by Q^* and $\tilde{Q}^*$ f-divergence minimal EMM's belonging to $\mathcal{M}(P,\mathbb{G})$ and $\mathcal{M}(\tilde{P},\mathbb{G})$ respectively.

We set for $t\in[0,T]$

$$z^*_T(t)=\zeta^*_t\,\frac{\tilde{\zeta}^*_T}{\tilde{\zeta}^*_t}$$

where ζ^* and $\tilde{\zeta}^*$ are the densities of Q^* and $\tilde{Q}^*$ with respect to P and $\tilde{P}$ respectively.

We introduce the following hypotheses:

($\mathcal{H}_1$): The f-divergence minimal equivalent martingale measures Q^* and $\tilde{Q}^*$ relative to L and $\tilde{L}$ exist.

($\mathcal{H}_2$): The f-divergence minimal equivalent martingale measures Q^* and $\tilde{Q}^*$ preserve the Lévy property and are scale and time invariant.

($\mathcal{H}_3$): For all $c>0$ and $t\in[0,T]$, we have

$$\sup_{0\le t\le T}\mathbb{E}|z^*_T(t)\,f'(c\,z^*_T(t))\,|<\infty$$

where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$.

Theorem 3.2. *Assume that f is a strictly convex function, $f\in C^1(\mathbb{R}^{+,*})$, and that ($\mathcal{H}_1$), ($\mathcal{H}_2$), ($\mathcal{H}_3$) hold. If the f-divergence EMM's $\mathbb{Q}^*$ for the change-point model (1.2) exists, then*

$$\frac{d\mathbb{Q}^*_T}{d\mathbb{P}_T}=c(\tau)\,z^*_T(\tau)$$

where $c(\cdot)$ is a measurable function $[0,T]\to\mathbb{R}^+$ such that $\mathbb{E}c(\tau)=1$.

For $c>0$, let

$$\lambda_t(c)=\mathbb{E}[\,z^*_T(t)\,f'(c\,z^*_T(t))\,]$$

and let $c_t(\lambda)$ be its right-continuous inverse.

If in addition there exists λ^* such that

$$\int_0^T c_t(\lambda^*)d\alpha(t)=1, \tag{3.4}$$

then the f-minimal equivalent martingale measure for a change-point model exists and the density $Z^*_T(\tau)$ of $\mathbb{Q}^*_T$ with respect to $\mathbb{P}_T$ is equal to

$$Z^*_T(\tau)=c^*(\tau)\,z^*_T(\tau)$$

where $c^*(t)=c_t(\lambda^*)$ for $t\in[0,T]$.

Corollary 3.3. *Assume that f is power function, $f(x)=c_\gamma x^{\gamma+2}$. Then under ($\mathcal{H}_1$) the f-divergence EMM for change-point model exist and $Z^*_T(\tau)=c^*(\tau)\,z^*_T(\tau)$ with*

$$c^*(t)=\frac{[\mathbb{E}(\,z^*_T(t)^{\gamma+2}\,)]^{-\frac{1}{\gamma+1}}}{\int_0^T[\mathbb{E}(\,z^*_T(t)^{\gamma+2}\,)]^{-\frac{1}{\gamma+1}}\,d\alpha(t)}.$$

Corollary 3.4. *Assume that* $f(x) = x\ln(x)$. *Then under* $(\mathcal{H}_1)$ *the* f*-divergence EMM for change-point model exist and* $Z_T^*(\tau) = c^*(\tau)\, z_T^*(\tau)$ *with*

$$c^*(t) = \frac{e^{-\mathbb{E}(\, z_T^*(t)\ln z_T^*(t)\,)}}{\int_0^T e^{-\mathbb{E}(\, z_T^*(t)\ln z_T^*(t)\,)}\, d\alpha(t)}.$$

Corollary 3.5. *Assume that* $f(x) = -\ln(x)$. *Then under* $(\mathcal{H}_1)$ *the* f*-divergence EMM for change-point model exist and* $Z_T^*(\tau) = c^*(\tau)\, z_T^*(\tau)$ *with* $c^*(t) = 1$.

Remark 3.6. We can also express the factor $c^*(t)$ in terms of f-divergences of the processes L and $\tilde{L}$. Namely, one can see easily that

$$\mathbb{E}(\, z_T^*(t)^{\gamma+2}\,) = \mathbb{E}(\, {\zeta_t^*}^{\gamma+2}\,)\, \mathbb{E}(\, {\tilde{\zeta}^*}_{T-t}^{\gamma+2}\,)$$

and that

$$\mathbb{E}(\, z_T^*(t)\ln z_T^*(t)\,) = \mathbb{E}(\, \zeta_t^*\ln\zeta_t^*\,) + \mathbb{E}(\, \tilde{\zeta}_{T-t}^*\ln\tilde{\zeta}_{T-t}^*\,),$$
$$\mathbb{E}(-\ln z_T^*(t)\,) = \mathbb{E}(\,-\ln\zeta_t^*\,) + \mathbb{E}(\,-\ln\tilde{\zeta}_{T-t}^*\,).$$

In turn, the last quantities can be easily expressed via the corresponding Girsanov parameters using the Itô formula as it was done in Lemma 2.3.

Proof of Theorem 3.2. For any equivalent martingale measure $\mathbb{Q}$ in change-point situation we have:

$$f(\mathbb{Q}_T\,|\,\mathbb{P}_T) = \mathbb{E}z_T = \mathbb{E}\left[f\left(\,\zeta_\tau(\tau)\,\frac{\tilde{\zeta}_T(\tau)}{\tilde{\zeta}_\tau(\tau)}\right)\right].$$

Since $L, \tilde{L}$ and τ are independent, we also have:

$$\mathbb{E}\left[f\left(\,\zeta_\tau(\tau)\,\frac{\tilde{\zeta}_T(\tau)}{\tilde{\zeta}_\tau(\tau)}\right)|\tau = u\right] = \mathbb{E}\left[f\left(\zeta_u(u)\,\frac{\tilde{\zeta}_T(u)}{\tilde{\zeta}_u(u)}\right)\right].$$

Using the fact that $\left(\dfrac{\zeta_t(u)}{\zeta_0(u)}\right)_{t\geq 0}$ is a density process of some equivalent martingale measure from $\mathcal{M}(P,\mathbb{G})$, and also time invariance and scaling properties we get that

$$\mathbb{E}\left[f\left(\zeta_u(u)\,\frac{\tilde{\zeta}_T(u)}{\tilde{\zeta}_u(u)}\right)|\,\sigma(\tilde{L})\right] \geq \mathbb{E}\left[f\left(\zeta_0(u)\zeta_u^*\,\frac{\tilde{\zeta}_T(u)}{\tilde{\zeta}_u(u)}\right)|\,\sigma(\tilde{L})\right].$$

In the same way and with similar arguments as before we deduce that

$$\mathbb{E}\left[f\left(\zeta_0(u)\zeta_u^*\,\frac{\tilde{\zeta}_T(u)}{\tilde{\zeta}_u(u)}\right)|\,\sigma(L)\right] \geq \mathbb{E}\left[f\left(\zeta_0(u)\zeta_u^*\,\frac{\tilde{\zeta}_T^*}{\tilde{\zeta}_u^*}\right)|\,\sigma(L)\right].$$

Finally,

$$\mathbb{E}\left[f\left(\,c(\tau)\,\zeta_\tau\,\frac{\tilde{\zeta}_T}{\tilde{\zeta}_\tau}\right)\right] \geq \mathbb{E}\left[f\left(\,c(\tau)\,\zeta_\tau^*\,\frac{\tilde{\zeta}_T^*}{\tilde{\zeta}_\tau^*}\right)\right]$$

where $c(\tau)$ is any positive random variable with expectation 1.

To find f-divergence minimal equivalent martingale measure we minimize the function

$$F(c)=\int_0^T \mathbb{E}[f(\,c(t)z^*_T(t)\,)]\,d\alpha(t)$$

over all Borelian bounded functions $c:[0,T]\to\mathbb{R}^{+,*}$ such that $\mathbb{E}c(\tau)=1$. For that we consider a linear space $\mathcal{L}$ of Borelian bounded functions $c:[0;T]\to\mathbb{R}$ with the norm $||c||=\sup_{t\in[0,T]}|c(t)|$ and also the cone of such positive functions.

We apply Kuhn–Tucker theorem (see [41]) to the function

$$F_\lambda(c)=F(c)-\lambda\int_0^T (c(t)-1)d\alpha(t)$$

with Lagrangian factor $\lambda>0$. We show that the Fréchet derivative $\frac{\partial F_\lambda}{\partial c}$ of $F_\lambda(c)$, defined by

$$\lim_{||\delta||\to 0}\frac{|F_\lambda(c+\delta)-F_\lambda(c)-\frac{\partial F_\lambda}{\partial c}\delta|}{||\delta||}=0 \tag{3.5}$$

is equal to:

$$\frac{\partial F_\lambda}{\partial c}(\delta)=\int_0^T (\mathbb{E}[f'(\,c(t)z^*_T(t)\,)z^*_T(t)]-\lambda)\,\delta(t)d\alpha(t). \tag{3.6}$$

In fact, by the Taylor formula, we have for $\delta\in\mathcal{L}$:

$$\begin{aligned}&F_\lambda(c+\delta)-F_\lambda(c)-\frac{\partial F_\lambda}{\partial c}\delta\\ &\quad=\int_0^T \mathbb{E}[(f'(\,(c(t)+\theta(t)\,)z^*_T(t))-f'(\,c(t)z^*_T(t)\,))z^*_T(t)]\delta(t)d\alpha(t)\end{aligned}$$

where $\theta(t)$ is a function which takes values in the interval $[0,\delta(t)]$. We remark that the modulus of the right-hand side in the previous equality is bounded from above by:

$$A_T=\sup_{t\in[0,T]}\mathbb{E}[\,|\,f'(\,(c(t)+\theta(t)\,)z^*_T(t))-f'(\,c(t)z^*_T(t)\,)\,|\,z^*_T(t)\,]\;||\delta||.$$

Since f' is continuous and increasing and the functions c and δ are bounded, hypothesis $(\mathcal{H}_3)$ implies that A_T is finite. We conclude by the Lebesgue dominated convergence theorem that (3.5) holds and then (3.6).

Then, in order to $\frac{\partial F_\lambda}{\partial c}\delta=0$ for all $\delta\in\mathcal{L}$, it is necessary and sufficient to take c such that

$$\mathbb{E}[\,z^*_T(t)\,f'(c(t)z^*_T(t))]-\lambda=0\quad\alpha\text{-a.s.}$$

Finally, for each $c>0$ and $t\in[0,T]$ we consider the function

$$\lambda_t(c)=\mathbb{E}[\;z^*_T(t)\,f'(cz^*_T(t))\,].$$

We see easily that it is increasing in c and that its right-continuous inverse $c_t(\lambda)$ satisfies:

$$\lambda=\mathbb{E}[\,z^*_T(t)\,f'(c_t(\lambda)z^*_T(t))\,].$$

Now, to obtain a minimizer c^*, it remains to find, if it exists, λ^* which satisfies (3.4). □

Proof of Corollaries 3.3*,* 3.4 *and* 3.5*.* First of all we remark that for common f-divergences, the hypothesis $(\mathcal{H}_1)$ implies $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Then, we obtain in power case $f(x) = c_\gamma x^{\gamma+2}$ that $\lambda_t(c) = (\gamma+2)\, c_\gamma\, c^{\gamma+1}\, \mathbb{E}[z_T^*(t)^{\gamma+2}]$. For $f(x) = x\ln(x)$ we get $\lambda_t(c) = \mathbb{E}[z_T^*(t) \ln z_T^*(t)] + \ln c + 1$. In the case $f(x) = -\ln x$ we get $\lambda_t(c) = -1/c$. Finally, we write down $c_t(\lambda)$ and we integrate with respect to α to find λ^* and the expression of $c^*(t)$. □

Example. A change-point Black–Scholes model. We apply the previous results when L and $\tilde{L}$ define Black–Scholes type models. Therefore, we assume that L and $\tilde{L}$ are continuous Lévy processes with characteristics $(b, c, 0)$ and $(\tilde{b}, c, 0)$ respectively, $c > 0$. As is well known, the initial models will be complete, with a unique equivalent martingale measure which defines a unique price for options. However, in our change-point model the martingale measure is not unique, and we have an infinite set of martingale measures of the form:

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}(X) = c(\tau)\, \exp\left(\int_0^T \beta_s dX_s^c - \frac{1}{2}\int_0^T \beta_s^2 c ds \right)$$

where $c(\cdot)$ is a measurable function $[0,T] \to \mathbb{R}^{+,*}$ such that $\mathbb{E}[c(\tau)] = 1$ and

$$\beta_s = -\frac{1}{c}\left[\left(b + \frac{c}{2}\right) \mathbb{I}_{[\![0,\tau]\!]}(s) + \left(\tilde{b} + \frac{c}{2}\right) \mathbb{I}_{]\!]\tau,+\infty[\![}(s) \right].$$

If for example $f(x) = c_\gamma x^{\gamma+2}$ with $\gamma \neq -1, -2$ then applying Theorem 3.2, we get

$$c^*(t) = \frac{e^{-\frac{\gamma+2}{2c}[(b+\frac{c}{2})^2 t + (\tilde{b}+\frac{c}{2})^2 (T-t)]}}{\int_0^T e^{-\frac{\gamma+2}{2c}[(b+\frac{c}{2})^2 t + (\tilde{b}+\frac{c}{2})^2 (T-t)]} d\alpha(t)}.$$

If $f(x) = x\ln(x)$, then

$$c^*(t) = \frac{e^{-\frac{1}{2c}[(b+\frac{c}{2})^2 t + (\tilde{b}+\frac{c}{2})^2 (T-t)]}}{\int_0^T e^{-\frac{1}{2c}[(b+\frac{c}{2})^2 t + (\tilde{b}+\frac{c}{2})^2 (T-t)]} d\alpha(t)}.$$

If $f(x) = -\ln(x)$, then $c^*(t) = 1$.

4. Optimal strategies for utility maximization

We start by recalling some useful basic facts about optimal strategies for utility maximization. Then some decomposition formulas will be given which permit us to find optimal strategies. We end up by giving the formulas for optimal strategies for utility maximization in change-point setting for both initially and progressively enlarged filtrations.

4.1. Some known facts

In this subsection, we are interested in finding optimal strategies for terminal wealth with respect to some utility functions. More precisely, we assume that our financial market consists of two assets: a non-risky asset B, with interest rate r, and a risky asset S, modelled using the change-point Lévy model defined in (1.2). We denote by $\vec{S} = (B, S)$ the price process and by $\vec{\Phi} = (\phi^0, \phi)$ the amount of money invested in each asset. According to usual terminology, a predictable $\vec{S}$-integrable process $\vec{\Phi}$ is said to be a self-financing admissible strategy if for every $t \in [0, T]$ and x initial capital

$$\left\langle \vec{\Phi}_t, \vec{S}_t \right\rangle = x + \int_0^t \left\langle \vec{\Phi}_u, d\vec{S}_u \right\rangle$$

where the stochastic integral in the right-hand side is bounded from below. We will denote by $\mathcal{A}$ the set of all self-financing admissible strategies. In order to avoid unnecessary complications, we will assume again that the interest rate r is 0, so that starting with an initial capital x, terminal wealth at time T is

$$V_T(\phi) = x + \int_0^T \langle \phi_s, dS_s \rangle .$$

Let u denote a strictly increasing, strictly concave, continuously differentiable function on $\mathrm{dom}(u) = \{x \in \mathbb{R} | u(x) > -\infty\}$ which satisfies

$$u'(+\infty) = \lim_{x \to +\infty} u'(x) = 0,$$
$$u'(\underline{x}) = \lim_{x \to \underline{x}} u'(x) = +\infty$$

where $\underline{x} = \inf\{u \in \mathrm{dom}(u)\}$.

We will say that ϕ^* defines an optimal strategy with respect to u if

$$\mathbb{E}_P \left[u \left(x + \int_0^T \langle \phi_s^*, dS_s \rangle \right) \right] = \sup_{\phi \in \mathcal{A}} \mathbb{E}_P \left[u \left(x + \int_0^T \langle \phi_s, dS_s \rangle \right) \right].$$

As in [37], we will say that ϕ^* is an asymptotically optimal strategy if there exists a sequence of admissible strategies $(\phi^{(n)})_{n \geq 1}$ such that

$$\lim_{n \to +\infty} E \left[u \left(x + \int_0^T \left\langle \phi_s^{(n)}, dS_s \right\rangle \right) \right] = \sup_{\phi \in \mathcal{A}} E \left[u \left(x + \int_0^T \langle \phi_s, dS_s \rangle \right) \right].$$

As known, there is a strong link between this optimization problem and the previous problem of finding f-divergence minimal martingale measures. Let f be the convex conjugate function of u:

$$f(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\} = u(I(y)) - yI(y)$$

where $I = (u')^{-1} = -f'$. We recall that in particular

$$\begin{aligned}
&\text{if } \; u(x) = \ln(x) && \text{then } \; f(x) = -\ln(x) - 1, \\
&\text{if } \; u(x) = \frac{x^p}{p}, p < 1 && \text{then } \; f(x) = -\frac{p-1}{p} x^{\frac{p}{p-1}}, \\
&\text{if } \; u(x) = 1 - e^{-x} && \text{then } \; f(x) = 1 - x + x\ln(x).
\end{aligned}$$

The following result gives us the relation between portfolio optimization and f-minimal martingale measures.

Theorem 4.1 (cf. [28]). *Let $x \in \mathbb{R}^+$ be fixed and $f \in C^1(\mathbb{R}^{+,*})$. Let Q^* be an equivalent martingale measure which satisfies*

$$\mathbb{E}_P \left| f\left(\lambda \frac{dQ_T^*}{dP_T}\right)\right| < \infty, \;\; \mathbb{E}_{Q^*} \left| f'\left(\lambda \frac{dQ_T^*}{dP_T}\right)\right| < \infty$$

for λ such that

$$-\mathbb{E}_{Q^*} f'\left(\lambda \frac{dQ_T^*}{dP_T}\right) = x.$$

Then, if Q^* is an f-divergence minimal martingale measure, there exists a predictable function ϕ^* such that $(\int_0^{\cdot} < \phi_u^*, dS_u >)$ is a Q^*-martingale and

$$-f'\left(\lambda \frac{dQ_T^*}{dP_T}\right) = x + \int_0^T \langle \phi_u^*, dS_u \rangle \,.$$

If the last relation holds, then $\vec{\Phi} = (\phi^0, \phi^*)$ with $\phi_t^0 = x + \int_0^t < \phi_u^*, dS_u > - < \phi_t^*, S_t >$ is an asymptotically optimal portfolio strategy. Moreover, if $\underline{x} > -\infty$, this strategy is optimal.

Proof. The first part of the theorem is a slight adaptation of [37]. We do however recall the proof for the reader's ease. We denote $Z_T = \dfrac{dQ_T^*}{dP_T}$.

As f' is strictly increasing, continuous and due to imposed integrability conditions, the function $\lambda \mapsto E_{Q^*}[f'(\lambda Z_T)]$ is also increasing and continuous. Furthermore, since $f' = -(u')^{-1}$, we have $\lim_{\lambda \to 0} E_{Q^*}[f'(\lambda Z_T)] = -\infty$ and $\lim_{\lambda \to +\infty} E_{Q^*}[f'(\lambda Z_T)] = -\underline{x}$. Hence, for all $x > \underline{x}$, there exists $\lambda > 0$ such that $E_{Q^*}[f'(\lambda Z_T)] = -x$. As Q^* is minimal for the function $x \mapsto f(\lambda x)$, it follows from Theorem 3.1 of [28], that there exists a predictable process ϕ^* such that

$$-f'(\lambda Z_T) = x + \int_0^T \langle \phi^*, dS_s \rangle$$

and furthermore $\int_0^{\cdot} < \phi_s^*, dS_s >$ defines a Q^*-martingale. Then, from the definition of the convex conjugate, we have

$$u\left(x + \int_0^T \langle \phi^*, dS_s \rangle \right) = f(\lambda Z_T) - \lambda Z_T \, f'(\lambda Z_T) \tag{4.1}$$

and, hence,

$$E_P\left[\left|u\left(x+\int_0^T \langle \phi^*, dS_s\rangle\right)\right|\right] \leq E_P|f(\lambda Z_T)| + \lambda E_P[Z_T|f'(\lambda Z_T)|] < \infty.$$

If now ϕ denotes any admissible strategy, we have from $u(x) \leq xy + f(y)$ for all $x, y \in \mathbb{R}^{+,*}$ and (4.1) that

$$\begin{aligned} u\left(x+\int_0^T \langle \phi, dS_s\rangle\right) &\leq \left(x+\int_0^T \langle \phi, dS_s\rangle\right)\lambda Z_T + f(\lambda Z_T) \\ &\leq \left(x+\int_0^T \langle \phi, dS_s\rangle\right)\lambda Z_T + u\left(x+\int_0^T \langle \phi^*, dS_s\rangle\right) + \lambda Z_T f'(\lambda Z_T). \end{aligned}$$

Taking expectation, we obtain since $E_P(Z_T\, f'(\lambda Z_T)) = -x$, that

$$E_P\left[u\left(x+\int_0^T \langle \phi, dS_s\rangle\right)\right] \leq E_P\left[u\left(x+\int_0^T \langle \phi^*, dS_s\rangle\right)\right] + \lambda E_{Q^*}\left[\int_0^T \langle \phi, dS_s\rangle\right].$$

Now, under Q^*, $\int_0^{\cdot} \langle \phi, dS_s\rangle$ is a local martingale and if it is bounded from below, then $E_{Q^*}\left[\int_0^T \langle \phi, dS_s\rangle\right] \leq 0$. Therefore,

$$E_P\left[u\left(x+\int_0^T \langle \phi, dS_s\rangle\right)\right] \leq E_P\left[u\left(x+\int_0^T \langle \phi^*, dS_s\rangle\right)\right].$$

Furthermore, if $\underline{x} > -\infty$, we note that $\int_0^T \langle \phi_s^*, dS_s\rangle \ \geq \underline{x} - x$, so that ϕ^* defines an admissible strategy, and hence is a u-optimal strategy.

When $\underline{x} = -\infty$, we can construct using the definition of $\mathcal{A}$ a sequence of admissible strategies $\phi^{(n)}$ such that for all $0 \leq t \leq T$, $(\phi^{(n)} \cdot S)_t \geq -n$ and such that

$$\lim_{n\to+\infty} E\left[u\left(x+\int_0^T \left\langle \phi^{(n)}, dS_s\right\rangle\right)\right] = \sup_{\phi\in\mathcal{A}} E\left[u\left(x+\int_0^T \langle \phi, dS_s\rangle\right)\right].$$

Finally, ϕ^* is asymptotically u-optimal. □

In the following theorem proved in [7] we give a unified expression of u-optimal strategy for exponential Lévy model. We denote by (β^*, Y^*) the Girsanov parameters for changing of the measure P into Q^*. We put also

$$\xi_t(x) = E_{Q^*}[f''(xZ_{T-t})Z_{T-t}].$$

Theorem 4.2. *Let u be a $\mathcal{C}^3(]\underline{x}, +\infty[)$ utility function and f its convex conjugate. Assume there exists an f-minimal martingale measure Q^* which preserves the Lévy property and such that the integrability conditions are satisfied: for all $\lambda > 0$ and all compact set $K \subseteq \mathbb{R}^+$*

$$E_P|f(\lambda Z_T)| < +\infty, \quad E_Q|f'(\lambda\, Z_T)| < +\infty, \quad \sup_{t\leq T}\sup_{\lambda\in K} E_Q[f''(\lambda Z_t)Z_t] < +\infty.$$

Then for any fixed initial capital $x > \underline{x}$, there exists an asymptotically u-optimal strategy ϕ^*. In addition, ϕ^* defines an u-optimal strategy as soon as $\underline{x} > -\infty$.

Furthermore, if $c \neq 0$, we have for $1 \leq i \leq d$ that

$$\phi_s^{*,(i)} = -\frac{\lambda \beta^{*,(i)} Z_{s-}}{S_{s-}^{(i)}} \xi_s(\lambda Z_{s-})$$

where λ is a unique solution to the equation $E_{Q^*}(-f'(\lambda Z_T)) = x$.

If $c = 0$, $\overset{\circ}{\text{supp}}\,(\nu) \neq \emptyset$ and it contains zero, and Y^* is not identically 1, then $f''(x) = ax^\gamma$ with $a > 0$ and $\gamma \in \mathbb{R}$, and for $1 \leq i \leq d$

$$\phi_s^{*,(i)} = -\frac{\lambda \alpha^{*,(i)} Z_{s-}}{S_{s-}} \xi_s(\lambda Z_{s-})$$

where again λ is a unique solution to the equation $E_{Q^*}(-f'(\lambda Z_T)) = x$ and the constant α^* is related with the second Girsanov parameter Y^* by the formula:

$$\alpha^{*,(i)} = \exp(-y_0)\, Y(y_0)^\gamma \, \frac{\partial Y}{\partial y_i}(y_0)$$

where y_0 is chosen arbitrarily in $\overset{\circ}{\text{supp}}\,(\nu)$.

In the case of classical utilities we obtain the following result.

Proposition 4.3. *Consider a Lévy process X with characteristics (b, c, ν) and let f be a function such that $f''(x) = ax^\gamma$, where $a > 0$ and $\gamma \in \mathbb{R}$. Let u_f be its concave conjugate. Assume there exist $\alpha, \beta \in \mathbb{R}^d$ and a measurable function $Y : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^+$ such that*

$$Y(y) = (f')^{-1}(f'(1) + \alpha(e^y - 1))$$

with, for $c \neq 0$, $\alpha = c_\gamma\,(\gamma+1)\,(\gamma+2)\,\beta$ if $\gamma \neq -1, -2$ and $\alpha = \beta$ if $\gamma = -1$ or $\gamma = -2$, and such that the following properties hold:

$$Y(y) > 0 \;\; \nu - a.e.,$$

$$\int_{y \geq 1} (e^y - 1) Y(y) \nu(dy) < \infty.$$

$$b + \frac{1}{2}\operatorname{diag}(c) + c\beta + \int_{\mathbb{R}^d} ((e^y - 1)Y(y) - h(y))\nu(dy) = 0.$$

Then if $c \neq 0$, there exists an asymptotically optimal strategy ϕ^* given by

$$\phi_s^{*,(i)} = \alpha_\gamma(x) \frac{\beta^{(i)}\; Z_{s-}^{\gamma+1}}{E_{Q^*}[Z_s^{\gamma+1}]\, S_{s-}^{(i)}},$$

where x is initial capital and

$$\alpha_\gamma(x) = (\gamma+1)(x + f'(1)) - a. \tag{4.2}$$

If $c = 0$ and $\overset{\circ}{\text{supp}}\,(\nu) \neq \emptyset$, then

$$\phi_s^{*,(i)} = \alpha_\gamma(x) \frac{\alpha^{(i)}\; Z_{s-}^{\gamma+1}}{E_{Q^*}[Z_s^{\gamma+1}]\, S_{s-}^{(i)}},$$

In addition, ϕ^* is optimal as soon as $\gamma \neq -1$.

4.2. A decomposition formula for initially enlarged filtration

We use the structure of $\mathbb{Q}^*$ presented in Theorem 3.2 to write down a decomposition formula mentioned in Theorem 4.1 for $f'(\lambda Z_T^*(\tau))$. First of all we give the expressions for Girsanov parameters when changing the measure $\mathbb{P}$ into $\mathbb{Q}^*$.

Lemma 4.4. *Let Girsanov parameters of the f-divergence minimal equivalent martingale measures Q^* and $\tilde{Q}^*$ are (β^*, Y^*) and $(\tilde{\beta}^*, \tilde{Y}^*)$ respectively. Then the Girsanov parameters when changing from $\mathbb{P}$ to $\mathbb{Q}^*$ are:*

$$\beta_t^*(\tau) = \beta^* \mathbb{I}_{[\![0,\tau]\!]}(t) + \tilde{\beta}^* \, \mathbb{I}_{]\!]\tau,+\infty[\![}(t)$$
$$Y_t^*(\tau) = Y^* \mathbb{I}_{[\![0,\tau]\!]}(t) + \tilde{Y}^* \, \mathbb{I}_{]\!]\tau,+\infty[\![}(t).$$

It should be noticed that Girsanov parameters $\beta^*(\tau)$, $Y^*(\tau)$ do not define entirely the measure $\mathbb{Q}^*$. In fact, $\mathcal{F}_0 = \sigma(\tau)$ is not trivial and $Z_0^*(\tau) = c(\tau)$.

To write a decomposition formula of $(\mathbb{Q}^*, \mathbb{F})$ martingale as a stochastic integral with respect to $(S_t)_{t\geq 0}$ of some $\mathbb{F}$-predictable function $(\phi_t(\tau)))_{t\geq 0}$ it is sufficient to proceed by conditioning with respect to τ. In fact, from Proposition 2.1 of [4] we know that $N(\tau)$ is $(\mathbb{Q}^*, \mathbb{F})$ martingale iff for a.e. $u \in \mathrm{supp}(\mathcal{L}(\tau))$, $N(u)$ is $(\mathbb{Q}_u^*, \mathbb{G})$ martingale, where $\mathbb{Q}_u^*$ is conditional probability $\mathbb{Q}^*$ given $\tau = u$. Moreover, $(\phi_t(\tau)))_{t\geq 0}$ is $\mathbb{F}$-predictable iff for a.e. $u \in \mathrm{supp}(\mathcal{L}(\tau))$, $(\phi_t(u)))_{t\geq 0}$ is $\mathcal{P}(\mathbb{G}\otimes\mathbb{B}(\mathbb{R}^+))$ measurable.

We introduce for fixed $u \in \mathrm{supp}(\mathcal{L}(\tau))$, $x \geq 0$ and $t \in [0, T]$ the quantities

$$\rho^{(u)}(t, x) = \mathbb{E}_{\mathbb{Q}^*}(f'(Z_T^*(\tau)) \,|\, \tau = u, \, Z_t^*(u) = x)$$

and we remark that

$$\rho^{(u)}(t, x) = \mathbb{E}_{\mathbb{Q}_u^*}(f'(Z_T^*(u)) \,|\, Z_t^*(u) = x)$$

where $\mathbb{Q}_u^*$ is conditional probability $\mathbb{Q}^*$ given $\tau = u$. We notice that for regular conditional probabilities and for right-continuous versions of conditional expectations we have: $\mathbb{P}$-a.s. for all $t \in [0, T]$

$$\mathbb{E}_{\mathbb{Q}^*}(\, f'(Z_T^*(\tau)) \,|\, \mathcal{F}_t) = \rho^{(\tau)}(t, Z_t^*(\tau)).$$

To simplify the notation we introduce $\eta_{T-t}(u)$ such that

$$\eta_{T-t}(u) = \frac{z_T^*(u)}{z_t^*(u)}.$$

As a consequence of previous formulas, we have

$$\rho^{(u)}(t, x) = \mathbb{E}_{\mathbb{Q}_u^*}[f'(x\eta_{T-t}(u))].$$

Now, we would like to use the Itô formula for $\rho^{(u)}(t, Z_t^*(u))$. But the mentioned function is not sufficiently smooth and we will proceed by approximations. For that we construct a sequence of functions $(\phi_n)_{n\geq 1}$.

Lemma 4.5. *Let f be convex function belonging to $C^3(\mathbb{R}^{+,*})$. There exists a sequence of bounded functions $(\phi_n)_{n\geq 1}$, which are of class $\mathcal{C}^2$ on $\mathbb{R}^{+,*}$, increasing,* *such that for all $n \geq 1$, ϕ_n coincides with f' on the compact set $[\frac{1}{n}, n]$ and such that for sufficiently big n the following inequalities hold for all $x, y > 0$:*

$$|\phi_n(x)| \leq 4|f'(x)|+\alpha \ , \ |\phi_n'(x)| \leq 3f''(x) \ , \ |\phi_n(x)-\phi_n(y)| \leq 5|f'(x)-f'(y)| \quad (4.3)$$

where α is a real positive constant.

Proof. We set, for $n \geq 1$,

$$A_n(x) = f'(\frac{1}{n}) - \int_{x \vee \frac{1}{2n}}^{\frac{1}{n}} f''(y)(2ny-1)^2(5-4ny)dy$$

$$B_n(x) = f'(n) + \int_n^{x\wedge(n+1)} f''(y)(n+1-y)^2(1+2y-2n)dy$$

and finally

$$\phi_n(x) = \begin{cases} A_n(x) & \text{if } \ 0 \leq x < \frac{1}{n}, \\ f'(x) & \text{if } \ \frac{1}{n} \leq x \leq n, \\ B_n(x) & \text{if } \ x > n. \end{cases}$$

□

Proof. We can verify easily that ϕ_n coincide with f' on $[\frac{1}{n}, n]$ and that the properties (4.3) hold. □

Now we replace f' by ϕ_n in previous formulas and we introduce

$$\rho_n^{(u)}(t,x) = \mathbb{E}_{\mathbb{Q}^*}(\phi_n(Z_T^*(\tau))\,|\,\tau = u,\, Z_t^*(u) = x).$$

It is not difficult to see that

$$\rho_n^{(u)}(t, Z_T^*(u)) = \mathbb{E}_{\mathbb{Q}_u^*}[\phi_n(x\eta_{T-t}(u))].$$

In the next lemma we give a decomposition formula for $\rho_n^{(u)}$. For that we put

$$\xi_t^{(n,u)}(x) = \mathbb{E}_{\mathbb{Q}_u^*}[\eta_{T-t}(u)\phi_n'(x\eta_{T-t}(u))]$$

and

$$H_t^{(n,u)}(x,y) = \mathbb{E}_{\mathbb{Q}_u^*}([\phi_n(x\eta_{T-t}(u)Y_t^*(y)) - \phi_n(x\eta_{T-t}(u))]).$$

Lemma 4.6. *We have $\mathbb{Q}_u^*$-a.s., for all $t \leq T$,*

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_u^*}[\phi_n(Z_T^*(u))\,|\,\mathcal{G}_t] \\ &\quad = \mathbb{E}_{\mathbb{Q}_u^*}[\phi_n(Z_T^*(u))] + \int_0^t Z_{s-}^*(u)\,\xi_s^{(n,u)}(Z_{s-}^*(u))\left\langle \beta_s^*(u), dX_s^{(c),\mathbb{Q}_u^*}\right\rangle \\ &\qquad\qquad + \int_0^t\int_{\mathbb{R}^d} H_s^{(n,u)}(Z_{s-}^*(u),y)\,(\mu^X - \nu^{X,\mathbb{Q}_u^*})(ds,dy) \end{aligned}$$

where $\nu^{X,\mathbb{Q}_u^*}$ is a compensator of the jump measure μ^X with respect to $(\mathbb{G}, \mathbb{Q}_u^*)$.

Proof. In order to apply the Itô formula to $\rho_n^{(u)}$, we show that ρ_n is twice continuously differentiable with respect to x and once with respect to t on the set $x \geq \epsilon$, $\epsilon > 0$ and $t \in [0, T]$ and that the corresponding derivatives are bounded on the mentioned set. Then we apply the Itô formula to $\rho_n^{(u)}$ but stopped at stopping times

$$s_m = \inf\left\{t \geq 0 \,|\, Z_t^*(u) \leq \frac{1}{m}\right\},$$

with $m \geq 1$ and $\inf\{\emptyset\} = \infty$.

From strong Markov property of Lévy processes we have:

$$\rho_n^{(u)}(t \wedge s_m, Z_{t\wedge s_m}^*(u)) = E_{\mathbb{Q}_u^*}(\phi_n(Z_T^*(u)) \,|\, \mathcal{G}_{t\wedge s_m})$$

and we remark that $(E_{\mathbb{Q}_u^*}(\phi_n(Z_T^*(u)) \,|\, \mathcal{G}_{t\wedge s_m}))_{t\geq 0}$ is a $\mathbb{Q}_u^*$-martingale. By the Itô formula we obtain that:

$$\begin{aligned}
\rho_n^{(u)}(t\wedge s_m, Z_{t\wedge s_m}^*(u)) &= \rho_n^{(u)}(0, Z_0^*(u)) + \int_0^{t\wedge s_m} \frac{\partial \rho_n^{(u)}}{\partial s}(s, Z_{s-}^*(u))ds \\
&+ \int_0^{t\wedge s_m} \frac{\partial \rho_n^{(u)}}{\partial x}(s, Z_{s-}^*(u))dZ_{s-}^*(u) + \frac{1}{2}\int_0^{t\wedge s_m} \frac{\partial^2 \rho_n^{(u)}}{\partial x^2}(s, Z_{s-}^*(u))d\langle Z^{*,c}(u)\rangle_s \\
&+ \int_0^{t\wedge s_m}\int_{\mathbb{R}}\left[\rho_n^{(u)}(s, Z_{s-}^*(u)+y) - \rho_n^{(u)}(s, Z_{s-}^*(u)) - \frac{\partial \rho_n^{(u)}}{\partial x}(s, Z_{s-}^*(u))y\right]\mu^{Z^*}(ds, dy).
\end{aligned}$$

Then we can write that

$$\rho_n^{(u)}(t \wedge s_m, Z_{t\wedge s_m}^*(u)) = A_{t\wedge s_m} + M_{t\wedge s_m}$$

where for $0 \leq t \leq T$

$$\begin{aligned}
A_t &= \int_0^t \frac{\partial \rho_n^{(u)}}{\partial s}(s, Z_{s-}^*(u))ds + \frac{1}{2}\int_0^t \frac{\partial^2 \rho_n^{(u)}}{\partial x^2}(s, Z_{s-}^*(u))d\langle Z^{*,c}(u)\rangle_s \\
&+ \int_0^t\int_{\mathbb{R}}\left[\rho_n^{(u)}(s, Z_{s-}^*(u)+y) - \rho_n^{(u)}(s, Z_{s-}^*(u)) - \frac{\partial \rho_n^{(u)}}{\partial x}(s, Z_{s-}^*(u))y\right]\nu^{Z^*,\mathbb{Q}_u^*}(ds, dy)
\end{aligned}$$

and

$$\begin{aligned}
M_t &= \int_0^t \frac{\partial \rho_n^{(u)}}{\partial x}(s, Z_{s-}^*(u))dZ_{s-}^*(u) \\
&+ \int_0^t\int_{\mathbb{R}}\left[\rho_n^{(u)}(s, Z_{s-}^*(u)+y) - \rho_n^{(u)}(s, Z_{s-}^*(u))\right](\mu^{Z^*}(ds, dy) - \nu^{Z^*,\mathbb{Q}_u^*}(ds, dy)).
\end{aligned}$$

But since A is predictable process and $(E_{\mathbb{Q}_u^*}(\phi_n(Z_T^*(u)) \,|\, \mathcal{G}_{t\wedge s_m}))_{t\geq 0}$ is a $\mathbb{Q}_u^*$-martingale, we obtain that $\mathbb{Q}_u^*$-a.s., $A_t = 0$ for all $0 \leq t \leq T$.

From [52], Corollary 2.4, p. 59, we get since $\sigma(\cup_{m=1}^{\infty}\mathcal{G}_{t\wedge s_m}) = \mathcal{G}_t$ that

$$\lim_{m\to\infty} \rho_n^{(u)}(t \wedge s_m, Z_{t\wedge s_m}^*(u)) = E_{\mathbb{Q}^*}(\phi_n(Z_T^*(u)) \,|\, \mathcal{G}_t).$$

Moreover, we remark that for all $x \in \mathbb{R}$ and $s \in [0,T]$

$$\frac{\partial \rho_n^{(u)}}{\partial x}(s,x) = \xi_s^{(n,u)}(x)$$

and all $x, y \in \mathbb{R}$ and $s \in [0,T]$

$$H_s^{(n,u)}(x,y) = \rho_n^{(u)}(s, x\,Y_s^*(y)) - \rho_n^{(u)}(s,x).$$

Using the definition of local martingales we conclude that the decomposition of lemma holds. □

The next step consists to pass to the limit in previous decomposition. For that let us denote for $0 \le t \le T$

$$\xi_t^{(u)}(x) = \mathbb{E}_{\mathbb{Q}_u^*}[\eta_{T-t}(u) f''(x\eta_{T-t}(u))]$$

and

$$H_t^{(u)}(x,y) = \mathbb{E}_{\mathbb{Q}_u^*}([f'(x\eta_{T-t}(u)Y_t^*(y)) - f'(x\eta_{T-t}(u))]).$$

Lemma 4.7. *We have $\mathbb{Q}_u^*$-a.s., for all $t \le T$,*

$$\mathbb{E}_{\mathbb{Q}_u^*}(f'(Z_T^*(u))|\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}_u^*}[f'(Z_T^*(u))] + \int_0^t Z_{s-}^*(u)\xi_s^{(u)}(Z_{s-}(u))\left\langle \beta_s^*(u), dX_s^{(c),\mathbb{Q}_u^*}\right\rangle + \int_0^t \int_{\mathbb{R}^d} H_s^{(u)}(Z_{s-}(u),y)(\mu^X - \nu^{X,\mathbb{Q}_u^*})(ds,dy)$$

where $\nu^{X,\mathbb{Q}_u^*}$ is a compensator of the jump measure μ^X with respect to $(\mathbb{G}, \mathbb{Q}_u^*)$.

Proof. The proof consists to show the convergence in probability of stochastic integrals and conditional expectations using the properties of ϕ_n cited in Lemma 4.5 and can be performed in the same way as in [7]. □

4.3. Optimal strategies in a change-point situation for initially enlarged filtration

Let u be a utility function belonging to $C^3(]\underline{x}, +\infty[)$ and f its convex conjugate, $f \in C^3(\mathbb{R}^{+,*})$. We suppose that $\mathcal{M}(P) \neq \emptyset$ and $\mathcal{M}(\tilde{P}) \neq \emptyset$ and we introduce the following hypothesis

$(\mathcal{H}_4)$: For each compact set K of $\mathbb{R}^{+,*}$ we have:

$$\sup_{\lambda \in K} \sup_{t \in [0,T]} E_{Q^*}[\zeta_t^* \, f''(\lambda\, \zeta_t^*)] < \infty, \quad \sup_{\lambda \in K} \sup_{t \in [0,T]} E_{\tilde{Q}^*}[\tilde{\zeta}_t^* \, f''(\lambda\, \tilde{\zeta}_t^*)] < \infty$$

where ζ^* and $\tilde{\zeta}^*$ are the densities of the f-divergence EMM's Q^* and $\tilde{Q}^*$ with respect to P and $\tilde{P}$ respectively.

Theorem 4.8. *Let u be a strictly concave function belonging to $C^3(]\underline{x}, +\infty[)$ and $x > \underline{x}$. Suppose that a convex conjugate f of u satisfy $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and* (3.4)*. Then for change-point model* (1.2) *there exists an $\mathbb{F}$-optimal strategy ϕ^*. If*

$c \neq 0$, then for $1 \leq i \leq d$

$$\phi_t^{*,(i)} = -\lambda \frac{{\beta_t}^{*,(i)}(\tau)\, Z^*_{t-}(\tau)}{S^{(i)}_{t-}}\, \xi_t^{(\tau)}(\lambda\, Z^*_{t-}(\tau))$$

with $\beta^*(\tau)$ defined in Lemma 4.4 and λ such that $\mathbb{E}_{\mathbb{Q}^*}(-f'(\lambda\, Z^*_T(\tau))) = x$.

If $c = 0$ and $\overset{\circ}{\text{supp}}\ (\nu) \neq \emptyset$, $\overset{\circ}{\text{supp}}\ (\tilde{\nu}) \neq \emptyset$ both supports containing 0,and Y, $\tilde{Y}$ are not identically 1, then $f''(x) = ax^\gamma$ with $a > 0$ and $\gamma \in \mathbb{R}$, and the optimal strategies are defined by the same formula but with the replacement of $\beta^*_t(\tau)$ by $\alpha^*_t(\tau)$ such that

$$\alpha_t^{*,(i)}(\tau) = e^{-y_{0,i}} Y^*(y_0)^\gamma\ \frac{\partial Y^*}{\partial y_i}(y_0)\mathbb{I}_{\{\tau > t\}} + e^{-y_{1,i}} \tilde{Y}^*(y_1)^\gamma\ \frac{\partial \tilde{Y}^*}{\partial y_i}(y_1)\mathbb{I}_{\{\tau \leq t\}}$$

with any $y_0 \in \overset{\circ}{\text{supp}}\ (\nu)$ and $y_1 \in \overset{\circ}{\text{supp}}\ (\tilde{\nu})$.

Proof. From Theorem 3.2 and the hypotheses $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and (3.4) it follows that there exists an f- divergence minimal martingale measure $\mathbb{Q}^*$. Since the processes X and S are $\mathbb{F}$-adapted, applying Theorem 3.1 in [28], we have the existence of an $\mathbb{F}$-adapted optimal strategy ϕ^* such that

$$-f'(\lambda Z^*_T(\tau)) = x + \int_0^T < \phi^*_u, dS_u >$$

and such that $\int_0^{\cdot} < \phi^*_u, dS_u >$ defines a martingale with respect to $(\mathbb{Q}, \mathbb{F})$. Moreover, for all $x > \underline{x}$, there exists $\lambda > 0$ such that $E_{Q^*}[f'(\lambda Z_T)] = -x$. Then, to get our formulas for $c \neq 0$, we compare the decomposition of Lemma 4.7 and the decomposition of Theorem 4.1. For $c = 0$ we use first the Theorem 3 of [6] to prove that $f''(x) = ax^\gamma$. Then, again we compare the decomposition of Lemma 4.7 and the decomposition of Theorem 4.1 and we obtain the result. □

Corollary 4.9. *Let u be common utility function and let f be its convex conjugate, $f''(x) = ax^\gamma$, where $a > 0$ and $\gamma \in \mathbb{R}$. Then for $x > \underline{x}$ there exists an u-asymptotically optimal strategy if and only if the conditions of Theorem 2.1 are verified for both processes L and $\tilde{L}$. Furthermore, if $c \neq 0$ then $1 \leq i \leq d$*

$$\phi_t^{*,(i)} = -A_t(\tau)\, \frac{\beta_t^{*,(i)}(\tau)\, (Z^*_{t-}(\tau))^{\gamma+1}}{S^{(i)}_{t-}}$$

with

$$A_t(\tau) = \alpha_\gamma(x)\, \frac{\mathbb{E}(\, [z^*_T(\tau)/z^*_t(\tau)]^{\gamma+2} \mid \tau\,)}{\mathbb{E}(\, [Z^*_T(\tau)]^{\gamma+2}\,)}$$

where $\alpha_\gamma(x)$ is defined by (4.2). If $c = 0$, then we have the same formula for ϕ^*_t with replacement of $\beta^*_t(\tau)$ by $\alpha^*_t(\tau)$ given in Theorem 4.8. In addition, ϕ^* is optimal as soon as $\gamma \neq -1$.

Proof. According to Theorem 2.1, under the assumptions (2.2), (2.3) and (2.4), the Lévy model associated with L has an f-divergence minimal equivalent martingale measure which preserves the Lévy property and is scale invariant. The same is true for the Lévy model associated with $\tilde{L}$. Then, the existence of f-divergence EMM's for change-point model follows from Theorem 3.2 and the formulas for strategies follow directly from Theorem 4.8. □

Remark 4.10. From Corollary 4.9 we can see the following. Let ψ^* and $\tilde{\psi}^*$ be u-optimal strategies for the exponential Lévy models L and $\tilde{L}$ respectively. Then the u-optimal strategy for corresponding change-point model can be written as

$$\phi_t^* = B_t(\tau)\psi_t^* \, \mathbb{I}_{\{\tau\geq t\}} + \tilde{B}_t(\tau)\tilde{\psi}_t^* \, \mathbb{I}_{\{\tau<t\}}$$

where

$$B_t(\tau) = (c^*(\tau))^{\gamma+1} \frac{\mathbb{E}(\,[\zeta^*_{\tau-t}]^{\gamma+2}\,|\,\tau)\,\mathbb{E}(\,[\tilde{\zeta}^*_{T-\tau}]^{\gamma+2}\,|\,\tau\,)}{\mathbb{E}(\,[Z^*_T(\tau)]^{\gamma+2}\,)}\,\mathbb{E}(\zeta^*_t)^{\gamma+2},$$

$$\tilde{B}_t(\tau) = (c^*(\tau))^{\gamma+1} (\frac{\zeta_\tau}{\tilde{\zeta}_\tau})^{\gamma+1} \frac{\mathbb{E}(\,[\tilde{\zeta}^*_{T-t}]^{\gamma+2}\,)}{\mathbb{E}(\,[Z^*_T(\tau)]^{\gamma+2}\,)}\,\mathbb{E}(\tilde{\zeta}^*_t)^{\gamma+2}.$$

When u is exponential utility, and, hence, $\gamma = -1$, we see that $B_t(\tau) = \tilde{B_t}(\tau) = 1$ and the optimal strategy ϕ^* can be obtained by pasting together at τ two optimal strategies ψ^* and $\tilde{\psi}^*$:

$$\phi_t^* = \psi_t^* \, \mathbb{I}_{\{\tau\geq t\}} + \tilde{\psi}_t^* \, \mathbb{I}_{\{\tau<t\}}.$$

For logarithmic utility ($\gamma = -2$) since $c^*(\tau) = 1$ we get:

$$\phi_t^* = \psi_t^* \, \mathbb{I}_{\{\tau\geq t\}} + \frac{\tilde{\zeta}_\tau}{\zeta_\tau}\tilde{\psi}_t^* \, \mathbb{I}_{\{\tau<t\}}.$$

For power utility we use the expression for $c^*(\tau)$ to simplify the expression of $B_t(\tau)$ which becomes to be equal to

$$B_t = \left(\mathbb{E}\left([Z^*_T(\tau)]^{\gamma+2}\right)\right)^{-1} \left(\int_0^T [\mathbb{E}(\,z^*_T(t)^{\gamma+2}\,)]^{-\frac{1}{\gamma+1}}\,d\alpha(t)\right)^{-(\gamma+1)}.$$

Then,

$$\phi_t^* = B_t\,\psi_t^* \, \mathbb{I}_{\{\tau\geq t\}} + \tilde{B}_t(\tau)\tilde{\psi}_t^* \, \mathbb{I}_{\{\tau<t\}}.$$

In mentioned cases, ϕ^* is already adapted with respect to progressively enlarged filtration.

Example. Optimal strategy for Black–Scholes model with change point and exponential utility. As before, we now want to apply the results when L and $\tilde{L}$ define Black–Scholes type models. Therefore, we assume that L and $\tilde{L}$ are continuous Lévy processes with characteristics $(b, c, 0)$ and $(\tilde{b}, c, 0)$ respectively. Let τ be a random variable bounded by T which is independent from L and $\tilde{L}$. Then the asymptotically optimal strategy from the point of view of maximization of exponential utility $u(x) = 1 - \exp(-x)$ will be:

$$\phi_t^{*,(i)} = -\frac{\beta_t^{(i)}}{S_{t-}^{(i)}} = \frac{(b^{(i)} + c_{i,i}/2)}{cS_t^{(i)}}\,\mathbb{I}_{[\![0,\tau]\!]}(t) + \frac{(\tilde{b}^{(i)} + c_{i,i}/2)}{cS_t^{(i)}}\,\mathbb{I}_{]\!]\tau,+\infty[\![}(t).$$

4.4. Optimal strategies in progressively enlarged filtration

We denote as before by $\mathcal{M}(\mathbb{P},\mathbb{F})$ and $\mathcal{M}(\mathbb{P},\hat{\mathbb{F}})$ the sets of equivalent martingale measures related with initially enlarged filtration $\mathbb{F}$ and progressively enlarged filtration $\hat{\mathbb{F}}$. Since $\hat{\mathcal{F}}_T = \mathcal{F}_T$, for any f convex function

$$\inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{P},\mathbb{F})} \mathbb{E}f\left(\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\right) = \inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{P},\hat{\mathbb{F}})} \mathbb{E}f\left(\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\right).$$

From previous part and under the conditions of Theorem 2.1 for exponential Lévy models related with L and $\tilde{L}$, we have:

$$\inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{P},\mathbb{F})} \mathbb{E}f\left(\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\right) = \mathbb{E}f\left(\frac{d\mathbb{Q}^*_T}{d\mathbb{P}_T}\right) = \mathbb{E}f(Z^*_T(\tau)).$$

Then, since $Z^*_T(\tau)$ is $\hat{\mathcal{F}}_T$-measurable, $\mathbb{Q}^*_T$ is the restriction of the minimal equivalent martingale measure on $\hat{\mathcal{F}}_T$.

First of all we remark that when the minimal equivalent measure $\mathbb{Q}^*$ exists, there exists a $(\mathbb{Q}^*,\mathbb{F})$-predictable process ϕ^* such that (cf. [28])

$$-f'(\lambda Z^*_T(\tau)) = x_0 + \int_0^T \langle \phi^*_s, dS_s\rangle \tag{4.4}$$

where $(\int_0^{\cdot} \langle \phi^*_s, dS_s\rangle)$ is $(\mathbb{Q}^*,\mathbb{F})$-martingale and λ is a constant such that

$$-E_{\mathbb{Q}^*} f'(\lambda Z^*_T(\tau)) = x_0.$$

At the same time, there exists a $(\mathbb{Q}^*,\hat{\mathbb{F}})$-predictable process ϕ^* such that

$$-f'(\lambda Z^*_T(\tau)) = x_0 + \int_0^T \langle \hat{\phi}^*_s, dS_s\rangle \tag{4.5}$$

where $(\int_0^{\cdot} \langle \hat{\phi}^*_s, dS_s\rangle)$ is $(\mathbb{Q}^*,\hat{\mathbb{F}})$-martingale and λ is again a constant such that

$$-E_{\mathbb{Q}^*} f'(\lambda Z^*_T(\tau)) = x_0.$$

In the following proposition we give the result of shrinking, which gives us $\hat{\phi}^*$ in general case.

Proposition 4.11. *Let the conditions of Theorem* 2.1 *are satisfied for exponential Lévy models related with L and $\tilde{L}$. Suppose that c is not degenerated or $\nu \neq 0$ and $\tilde{\nu} \neq 0$. Then $\mathbb{Q}^* \times \lambda$-a.s.*

$$\hat{\phi}^*_s = E_{\mathbb{Q}^*}(\phi^*_s \,|\, \hat{\mathcal{F}}_s).$$

Proof. We denote by $\mathcal{P}(\mathbb{Q}^*,\hat{\mathbb{F}})$ and $\mathcal{P}(\mathbb{Q}^*,\mathbb{F})$ the sets of predictable functions related with $\mathbb{Q}^*$, $\hat{\mathbb{F}}$ and $\mathbb{F}$ respectively. Since

$$\mathcal{P}(\mathbb{Q}^*,\hat{\mathbb{F}}) \subseteq \mathcal{P}(\mathbb{Q}^*,\mathbb{F})$$

and S is a $(\mathbb{Q}^*,\mathbb{F})$-martingale, the process $(\int_0^{\cdot} \langle \hat{\phi}^*_s, dS_s\rangle)$ is a local $(\mathbb{Q}^*,\mathbb{F})$-martingale. But since it is uniformly integrable, it is a $(\mathbb{Q}^*,\mathbb{F})$-martingale.

Let us denote $N = (N_t)_{0 \le t \le T}$ with

$$N_t = \int_0^t \langle (\hat{\phi}_s^* - \phi_s^*), dS_s \rangle.$$

From (4.4) and (4.5) we deduce that $N_t = 0$ for all $t \in [0, T]$. Then for any truncation level $k > 0$ for the jumps, the truncated process $N^{(k)}$ is equal to zero, too. Then for all $t \in [0, T]$,

$$E_{\mathbb{Q}^*}[N^{(k)}, N^{(k)}]_t = E_{\mathbb{Q}^*} \langle N^{(k)}, N^{(k)} \rangle_t = 0 \tag{4.6}$$

where $\langle \cdot, \cdot \rangle$ stands for predictable quadratic variation of $N^{(k)}$. To write the expression of $\langle N^{(k)}, N^{(k)} \rangle$ we denote for $t \in [0, T]$

$$c_t = c\mathbb{I}_{\{\tau > t\}} + \tilde{c}\mathbb{I}_{\{\tau \le t\}}, \qquad \hat{Y}_t = \hat{Y}^*(y)\mathbb{I}_{\{\tau > t\}} + \hat{\tilde{Y}}^*(y)\mathbb{I}_{\{\tau \le t\}},$$

where $\hat{Y}$ and $\hat{\tilde{Y}}$ are second Girsanov parameters corresponding to the stochastic exponential of $\exp(L)$ and $\exp(\tilde{L})$ when we change P and $\tilde{P}$ into Q^* and $\tilde{Q}^*$ respectively, and

$$\nu_t(dy) = \nu(dy)\mathbb{I}_{\{\tau > t\}} + \tilde{\nu}(dy)\mathbb{I}_{\{\tau \le t\}}.$$

Then, for $1 \le i \le d$ the diagonal element i of $\langle N^{(k)}, N^{(k)} \rangle_t$ is equal to

$$\int_0^t \frac{(\hat{\phi}_s^{*,(i)} - \phi_s^{*,(i)})^2}{S_{s-}^{(i)}} c_s^{(i,i)} ds + \int_0^t \int_{\mathbb{R}^d} \frac{(\hat{\phi}_s^{*,(i)} - \phi_s^{*,(i)})^2}{S_{s-}^{(i)}} y^2 \mathbb{I}_{\{|y| \le k\}} \hat{Y}_s(y) \nu_s(dy) ds$$

and from (4.6) it equal to zero for all $t \in [0, T]$. Since $P \overset{\text{loc}}{\ll} \tilde{P}$, we have $c = \tilde{c}$. If $c_{i,i} \neq 0$ or $\nu \neq 0$ and $\tilde{\nu} \neq 0$ then we get that $\mathbb{Q}^* \times \lambda$-a.s.

$$\frac{(\hat{\phi}_s^{*,(i)} - \phi_s^{*,(i)})^2}{S_{s-}^{(i)}} = 0.$$

It implies that $\mathbb{Q}^* \times \lambda$-a.s.

$$\hat{\phi}_s^{*,(i)} - \phi_s^{*,(i)} = 0$$

and it proves the result of our proposition. □

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Optimal Investment-consumption for Partially Observed Jump-diffusions

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Abstract. We deal with an optimal consumption-investment problem under restricted information in a financial market where the risky asset price follows a non-Markovian geometric jump-diffusion process. We assume that agents acting in the market have access only to the information flow generated by the stock price and that their individual preferences are modeled through a power utility. We solve the problem with a two steps procedure. First, by using filtering results we reduce the partial information problem to a full information one involving only observable processes. Next, by using dynamic programming, we characterize the value process and the optimal-consumption strategy in terms of solution to a backward stochastic differential equation.

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Keywords. Utility maximization, optimal stochastic control, partial information, backward stochastic differential equations, jump-diffusion processes.

1. Introduction

In this paper we study an extension of the classical Merton optimal investment-consumption problem to a partially observable financial market in which asset prices follow geometric jump-diffusions. A single agent manages his portfolio by investing in a bond and in the stock asset S_t and chooses a portfolio-consumption strategy in order to maximize on a finite horizon his total expected utility from consumption and terminal wealth. The agent's information is described by the natural filtration of the stock price process,$\{\underline{\mathrm{F}}_t^S\}_{t\in[0,T]}$, hence his decisions must be adapted to $\{\underline{\mathrm{F}}_t^S\}_{t\in[0,T]}$ and this leads to a utility maximization problem under restricted information.

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Utility maximization problems in a full information setting have been largely studied in the literature by using different approaches, such as convex duality methods, stochastic control techniques based on the Hamilton–Jacobi–Bellman equation or backward stochastic differential equations (see for example [2, 8, 11, 14, 17, 20, 21, 25] and references therein). Portfolio selection problems with partial information have been studied among others in [16, 23, 24] in a continuous setting, in [1, 18] for jump-diffusions and in [6] in the case where the risky asset follows a Markov pure jump process. In [1] it is assumed that investors are only able to observe the stock price process and not the Markov chain which drives the jump intensity. In [18] a default model is studied where investors only observe asset prices and default times, while the drift of the asset price dynamics and the default intensities are not directly observable.

The contribution of this note consists in solving the utility maximization problem with intermediate consumption under partial information in a general jump-diffusion setting. More precisely, we do not assume Markovianity of the asset price dynamics and we work with a jump component described by a general integer-valued measure.

The outline of the paper is as follows. In Section 2, we describe the market model and the optimal investment-consumption problem under restricted information. In Section 3, by projection on the information flow we reduce the partial observable problem to a full information one and we give a representation theorem for $\underline{F}_t^S$-martingales. In Section 4, we formulate the full information problem (with respect to the filtration $\{\underline{F}_t^S\}_{t\in[0,T]}$) as a stochastic control problem. The special form of the power utility leads to a factorization of the associated value process into a part depending on the current wealth and the so-called opportunity process J_t ([21, 22]) around which our analysis is built. In Section 5, by using dynamic programming we show that J_t solves a backward stochastic differential equation and we provide a feedback formula for the optimal consumption in terms of J_t. We discuss the particular case of bounded investment strategies and finally we characterize the opportunity process in the case of non constrained strategies via a sequence of solutions of Lipschitz BSDEs. We conclude the section providing a verification result and giving as application a simplified model where the risky asset dynamics is driven by two independent point processes whose intensities are not directly observed by investors.

2. The market model and problem formulation

In this paper, we consider a complete filtered probability space $(\Omega, \{\underline{F}_t\}_{t\in[0,T]}, P)$ endowed with a Brownian motion W_t with values in $\mathbb{R}$ and a Poisson random measure $N(dt, d\zeta)$ independent of W_t. Here T is a fixed final time. The financial market consists of a nonrisky asset, with price process normalized to unity, and one risky asset with logreturn process Y_t given by the following jump-diffusion process

$$dY_t = b_t dt + \sigma_t dW_t + \int_Z K(t;\zeta)N(dt,d\zeta), \quad Y_0 = 0. \tag{2.1}$$

The mean measure of $N(dt, d\zeta)$ is denoted by $\nu(d\zeta)\ dt$ with $\nu(d\zeta)$ a σ-finite measure on a measurable space $(Z, \underline{Z})$. The coefficients b_t and σ_t are progressive $\underline{F}_t$-adapted processes with $\sigma_t > 0$ P-a.s. $\forall t \in [0,T]$, and $K(t;\zeta)$ is an $\mathbb{R}$-valued $(P, \underline{F}_t)$-predictable process joint measurable w.r.t. $(t,\zeta) \in [0,T] \times Z$. We also assume some requirements for (2.1) to be well defined

$$\mathbb{E}\int_0^T |b_t|dt < \infty \quad \mathbb{E}\int_0^T \sigma_t^2 dt < \infty \quad \mathbb{E}\int_0^T \int_Z |K(t;\zeta)|\nu(d\zeta)dt < \infty \tag{2.2}$$

and which entail that Y_t has finite first moment. The price S_t of the risky asset follows a geometric jump-diffusion process given by

$$S_t = S_0 e^{Y_t} \quad S_0 \in \mathbb{R}^+.$$

From Itô's formula we get that S_t solves the following differential equation

$$dS_t = S_{t^-}\left\{\mu_t dt + \sigma_t dW_t + \int_Z \widetilde{K}(t;\zeta)N(dt,d\zeta)\right\}$$

where

$$\mu_t = b_t + \frac{1}{2}\sigma_t^2, \quad \widetilde{K}(t;\zeta) = e^{K(t;\zeta)} - 1.$$

We are interested in solving an optimal portfolio problem for an agent who has access only to the observable flow generated by asset prices

$$\underline{F}_t^S = \sigma\{S_s; s \le t\} = \underline{F}_t^Y = \sigma\{Y_s; s \le t\} \subseteq \underline{F}_t.$$

We shall call this situation the case of partial information to distinguish it from the case of full information where investors observe the whole filtration $\{\underline{F}_t\}_{t\in[0,T]}$. We assume that $\{\underline{F}_t^S\}_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and completeness.

The investor starts with initial capital $z_0 > 0$, invests at any time $t \in [0,T]$ the fraction θ_t of the wealth Z_t in stock S_t and also consumes at the rate $C_t Z_t$. We consider both cases of utility from terminal wealth only and with intermediate consumption. As in [21] and [22], to unify the notations we introduce the measure $\mu(dt)$ on $[0,T]$ by $\mu(dt) = 0$ in the case without consumption and $\mu(dt) = dt$ in the case with consumption and assume the convention $C_T = 1$ (which means that all the remaining wealth is consumed at time T).

Because the agent's information is described by the filtration $\{\underline{F}_t^S\}_{t\in[0,T]}$ the decisions (θ_t, C_t) must be adapted to $\underline{F}_t^S$. By considering $\underline{F}_t^S$-predictable, self-financing trading strategies, the dynamics of the wealth process controlled by the investment-consumption process $(\theta_t,\ C_t)$ evolves according with

$$dZ_t = Z_{t^-}\left(\theta_t \frac{dS_t}{S_{t^-}} - C_t\mu(dt)\right), \quad Z_0 = z_0. \tag{2.3}$$

The solution process Z_t to (2.3) of course depends on the chosen strategy (θ, C). To be precise we should therefore denote the process Z_t by $Z_t^{\theta,C}$ but sometimes we will suppress θ, C.

For an agent with power utility

$$U(x) = \frac{x^\alpha}{\alpha} \quad 0 < \alpha < 1$$

the objective is to maximize over a suitable class of strategies $\underline{\mathrm{A}}$ either the expected utility from terminal wealth

$$\sup_{(\theta,C)\in\underline{\mathrm{A}}} \mathbb{E}\Big[U(Z_T^{\theta,C})\Big]$$

and with intermediate consumption

$$\sup_{(\theta,C)\in\underline{\mathrm{A}}} \mathbb{E}\Big[\int_0^T U(C_t Z_t^{\theta,C})\ dt + U(Z_T^{\theta,C})\Big].$$

Defining $\mu^0(dt) = \mu(dt) + \delta_{\{T\}}(dt)$, where δ_a denotes the Dirac measure at the point a, both the cases can be written as

$$\sup_{(\theta,C)\in\underline{\mathrm{A}}} \mathbb{E}\Big[\int_0^T U(C_t Z_t^{\theta,C})\mu^0(dt)\Big]. \tag{2.4}$$

Let us come back to the market model. We introduce the discrete random measure ([4],[13]) associated to the jump component of Y_t

$$m(dt, dx) = \sum_{s:\Delta Y_s \neq 0} \delta_{\{s,\Delta Y_s\}}(dt, dx) \tag{2.5}$$

and observe that for any real-valued function $f(x)$ the following equality holds

$$\int_0^t \int_Z f(K(s;\zeta))1\!\!1_{\{K(s;\zeta)\neq 0\}}(s,\zeta)N(ds, d\zeta) = \int_0^t \int_{\mathbb{R}} f(x)m(ds, dx). \tag{2.6}$$

We recall Proposition 2.2 in [5] which provides the $(P, \underline{F}_t)$-local characteristics of $m(dt, dx)$ in terms of the measure $\nu(d\zeta)$.

Proposition 2.1. *Let* $\forall t \in [0, T]$, $\forall A \in \underline{B}(\mathbb{R})$ *(where* $\underline{B}(\mathbb{R})$ *denotes the family of Borel sets of* $\mathbb{R}$*)*

$$D_t^A(\omega) = \{\zeta \in Z : K(t, \omega; \zeta) \in A \setminus \{0\}\} \subseteq D_t(\omega) = \{\zeta \in Z : K(t, \omega; \zeta) \neq 0\}.$$

Under the assumption

$$\mathbb{E}\int_0^T \nu(D_s)\ ds < \infty \tag{2.7}$$

the $(P, \underline{F}_t)$*-predictable projection of* m *is given by*

$$m^p(dt, dx) = \lambda_t \Phi_t(dx)dt$$

where λ_t *is a non-negative* $\underline{F}_t$*-predictable process and* $\Phi_t(dx)$ *is an* $\underline{F}_t$*-predictable process taking values in the space of probability measures over* $(\mathbb{R}, \underline{B}(\mathbb{R}))$ *and they* *satisfy* $\forall A \in \underline{B}(\mathbb{R})$

$$m^p(dt, A) = \lambda_t \Phi_t(A)dt = \nu(D_t^A)dt. \tag{2.8}$$

In particular $\lambda_t = \nu(D_t)$ *provides the* $(P, \underline{F}_t)$*-predictable intensity of the point process* $N_t = m((0,t], \mathbb{R})$ *which counts the total number of jumps of* Y *until time* t*.*

Remark 2.2. Equation (2.8) can be also written as

$$m^p(dt, dx) = \lambda_t \Phi_t(dx)dt = \int_{D_t} \delta_{K(t;\zeta)}(dx)\nu(d\zeta)dt.$$

Let us observe that the local characteristics $(\lambda_t, \Phi_t(dx))$ of $m(dt, dx)$ are not observable by investors since the process $K(t;\zeta)$ is not $\underline{F}_t^S$-adapted.

The $(P, \underline{F}_t)$-semimartingale structure of the risky asset S_t is described in the following proposition.

Proposition 2.3. *Under* (2.2), (2.7) *and in addition*

$$\mathbb{E}\int_0^T \int_Z |\widetilde{K}(t;\zeta)|\nu(d\zeta) < \infty \tag{2.9}$$

S_t *is a* $(P, \underline{F}_t)$*-semimartingale with the decomposition*

$$S_t = S_0 + M_t^S + A_t^S$$

where

$$A_t^S = \int_0^t S_r \mu_r dr + \int_0^t \int_{\mathbb{R}} S_{r^-}(e^x - 1)\lambda_r \Phi_r(dx)dr$$

is a process with finite variation paths, and

$$M_t^S = \int_0^t S_r \sigma_r dW_r + \int_0^t \int_{\mathbb{R}} S_{r^-}(e^x - 1)(m(dr, dx) - \lambda_r \Phi_r(dx)dr)$$

is a $(P, \underline{F}_t)$*-local martingale.*

Proof. Under (2.2), (2.7) and (2.9), the process

$$\int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_Z \widetilde{K}(s;\zeta)N(ds, d\zeta)$$
$$= \int_0^t \left\{ \mu_s + \int_Z \widetilde{K}(s;\zeta)\nu(d\zeta) \right\} ds + \int_0^t \sigma_s dW_s + \int_0^t \int_Z \widetilde{K}(s;\zeta)(N(ds, d\zeta) - \nu(d\zeta)ds)$$

is a $(P, \underline{F}_t)$-semimartingale, hence S_t is a semimartingale being the Doléans-Dade exponential of a semimartingale. The expressions of the processes A_t^S and M_t^S follow by Equation (2.6). □

3. Reduction to an optimization problem with complete information

To solve the utility maximization problem under partial information we first reduce it to a full information one involving only $\underline{\mathrm{F}}_t^S$-adapted processes. To this aim we need to compute the $(P, \underline{\mathrm{F}}_t^S)$-predictable projection of the integer-valued measure $m(dt, dx)$.

From now on we will denote by $\widehat{R}_t$ the $(P, \underline{\mathrm{F}}_t^S)$-optional projection of a generic process R_t, satisfying $\mathbb{E}|R_t| < \infty\ \forall t \in [0, T]$, defined as the unique optional process (in a P-indistinguishable sense) such that for each $\underline{\mathrm{F}}_t^S$-stopping time τ, $\widehat{R}_\tau = \mathbb{E}[R_\tau | \underline{\mathrm{F}}_\tau^S]$ P-a.s. on $\{\tau < \infty\}$.

Remark 3.1. We recall two well-known facts: for every $(P, \underline{\mathrm{F}}_t)$-martingale m_t, the projection $\widehat{m}_t$ is a $(P, \underline{\mathrm{F}}_t^S)$-martingale and that for any progressively measurable process Ψ_t with $\mathbb{E}\int_0^T |\Psi_t| dt < \infty$

$$\widehat{\int_0^t \Psi_s ds} - \int_0^t \widehat{\Psi}_s ds$$

is a $(P, \underline{\mathrm{F}}_t^S)$-martingale. Note that this implies that $E\int_0^T \Psi_t dt = \mathbb{E}\int_0^T \widehat{\Psi}_t dt$.

Let us denote by $\underline{\mathrm{P}}(\underline{\mathrm{F}}_t^S)$ the $\underline{\mathrm{F}}_t^S$-predictable σ-field on $(0, T] \times \Omega$.

Proposition 3.2. *Let us assume* (2.7). *The* $(P, \underline{F}_t^S)$*-predictable projection,* $\nu^p(dt, dx)$*, of* $m(dt, dx)$ *is given by* $\nu^p(dt, dx) = \nu_t^p(dx)dt$*, where* $\nu_t^p(dx)$ *is a measure-valued* $\underline{F}_t^S$*-predictable process satisfying* $\nu_t^p(dx) = \widehat{(\lambda_t \Phi_t)}(dx)$*,* $dP \times dt$*-a.e. More precisely, for each* $H(t, x)$*,* $\underline{P}(\underline{F}_t^S)$*-measurable*

$$\begin{aligned}\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} H(t, x)\ \nu_t^p(dx)dt\right] &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} H(t, x)\widehat{(\lambda_t \Phi_t)}(dx)dt\right] \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} H(t, x)\ m(dt, dx)\right].\end{aligned}$$

Proof. By definition of $(P, \underline{\mathrm{F}}_t)$-predictable projection of the integer-valued measure $m(dt, dx)$ it follows that, for each $H(t, x)$ $(P, \underline{\mathrm{F}}_t)$-predictable process jointly measurable w.r.t. $(t, x) \in [0, T] \times \mathbb{R}$, verifying the condition

$$\mathbb{E}\int_0^T \int_{\mathbb{R}} |H(r, x)| \lambda_r \Phi_r(dx) dr < \infty,$$

the process

$$m_t = \int_0^t \int_{\mathbb{R}} H(r, x)(m(dr, dx) - \lambda_r \Phi_r(dx)dr) \tag{3.1}$$

is a $(P, \underline{F}_t)$-martingale. Let us now consider in (3.1) a process $H(t,x)$ which is $(P, \underline{F}_t^S)$-predictable. By Remark 3.1 we get that

$$\int_0^t \int_{\mathbb{R}} H(r,x)m(dr,dx) - \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} H(r,x)\lambda_r \Phi_r(dx)dr | \underline{F}_t^S\right]$$

is a $(P, \underline{F}_t^S)$-martingale, and

$$\int_0^t \int_{\mathbb{R}} H(r,x)m(dr,dx) - \int_0^t \int_{\mathbb{R}} H(r,x)\widehat{\lambda_r \Phi_r}(dx)dr$$

is a $(P, \underline{F}_t^S)$-martingale. In particular, for any $A \in \underline{B}(\mathbb{R})$

$$m((0,t],A) - \int_0^t \widehat{\nu(D_s^A)}ds = m((0,t],A) - \int_0^t \int_A \widehat{\lambda_s \Phi_s}(dx)ds$$

is a $(P, \underline{F}_t^S)$-martingale. Hence, since $\widehat{\nu(D_t^A)}$ is a progressively measurable process, it provides the $(P, \underline{F}_t^S)$-intensity of $N_t(A) = m((0,t],A)$ and as in [4, Theorem T13] one can find a $(P, \underline{F}_t^S)$-intensity, λ_t^A, that is predictable. It suffices define λ_t^A, for any $A \in \underline{B}(\mathbb{R})$, as the Radon–Nikodym derivatives of $P(d\omega)\widehat{\nu(D_t^A)}(\omega)dt$ w.r.t. $P(d\omega)dt$ on $\underline{P}(\underline{F}_t^S)$. □

Throughout the paper we denote by $m^S(dt,dx)$ the $(P, \underline{F}_t^S)$-compensated martingale random measure

$$m^S(dt,dx) = m(dt,dx) - \nu_t^p(dx)dt$$

and we recall that, for any $H(t,x)$, jointly measurable process, $\underline{F}_t^S$-predictable such that

$$\mathbb{E}\int_0^T \int_Z |H(t,x)|\nu_t^p(dx)dt < \infty \quad \left(\text{resp.} \int_0^T \int_Z |H(t,x)|\nu_t^p(dx)dt < \infty \quad P\text{-a.s.}\right)$$

the process $\int_0^T \int_Z H(t,x)m^S(dt,dx)$ is a $(P, \underline{F}_t^S)$-martingale (resp. local-martingale).

Next, assuming

$$\mathbb{E}\int_0^T \frac{|b_t|}{\sigma_t}dt < \infty, \tag{3.2}$$

and the volatility σ_t to be $\underline{F}_t^S$-adapted, we introduce the innovation process

$$I_t := W_t + \int_0^t \frac{1}{\sigma_s}(b_s - \widehat{b}_s)ds.$$

By extending classical results in filtering theory ([19]) to our frame we have the following

Proposition 3.3. *Let σ_t to be $\underline{F}_t^S$-adapted. The random process $\{I_t\}_{t\in[0,T]}$ is a $(P, \underline{F}_t^S)$-Wiener process.*

Proof. By Equation (2.6) we get that $\int_Z K(t;\zeta)N(dt,d\zeta) = \int_{\mathbb{R}} x\, m(dt,dx)$. Hence, taking into account Equation (2.1), we have

$$dI_t = \frac{1}{\sigma_t}\left\{dY_t - \widehat{b}_t - \int_{\mathbb{R}} x\, m(dt,dx)\right\},$$

which implies that I_t is an $\underline{\mathrm{F}}^S_t$-adapted process. We now compute the following conditional expectation, $\forall s \le t$

$$\mathbb{E}\left[I_t - I_s \mid \underline{\mathrm{F}}^S_s\right] = \mathbb{E}\left[\int_s^t \left\{\frac{b_u}{\sigma_u} - \frac{\widehat{b}_u}{\sigma_u}\right\} du \mid \underline{\mathrm{F}}^S_s\right] + E[W_t - W_s \mid \underline{\mathrm{F}}^S_s].$$

Since, the first term of the right-hand side vanishes because of the properties of the conditional expectation and the second one vanishes because W_t is an $\underline{\mathrm{F}}_t$-Brownian motion and $\underline{\mathrm{F}}^S_t \subseteq \underline{\mathrm{F}}_t$ we get that I_t is a $(P, \underline{\mathrm{F}}^S_t)$-martingale. Finally, the thesis follows by the Lévy Theorem. □

Taking into account (2.6), Propositions 3.2 and 3.3, we are able to give the $(P, \underline{\mathrm{F}}^S_t)$-decompositions of the semimartingales Y_t and S_t

$$Y_t = Y_0 + \int_0^t \left\{\widehat{b}_s + \int_{\mathbb{R}} x\nu^p(dx)\right\} ds + \int_0^t \sigma_s dI_s + \int_0^t \int_{\mathbb{R}} x\, m^S(ds,dx) \tag{3.3}$$

$$\begin{aligned} S_t = S_0 &+ \int_0^t S_s\left\{\widehat{\mu}_s + \int_{\mathbb{R}} (e^x - 1)\nu^p_s(dx)\right\} ds \\ &+ \int_0^t S_s\sigma_s dI_s + \int_0^t \int_{\mathbb{R}} S_{s^-}(e^x - 1) m^S(ds,dx). \end{aligned} \tag{3.4}$$

Remark 3.4. Let us observe that by Proposition 3.2 and assumptions (2.9) we get that

$$\begin{aligned} \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |e^x - 1| \nu^p_t(dx) ds\right] &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |e^x - 1| \widehat{(\lambda_t \Phi_t)}(dx) dt\right] \\ = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |e^x - 1| \lambda_t \Phi_t(dx) dt\right] &= \mathbb{E} \int_0^T \int_Z |\widetilde{K}(t;\zeta)| \nu(d\zeta) < \infty. \end{aligned}$$

By virtue of (3.4) the wealth process Z_t induced by the investment-consumption strategy (θ_t, C_t), satisfies

$$dZ_t = Z_{t^-}\left(\theta_t \widehat{\mu}_t dt - C_t \mu(dt) + \theta_t \sigma_t dI_t + \theta_t \int_{\mathbb{R}} (e^x - 1) m(dt,dx)\right).$$

Then the utility maximization problem defined in (2.4) can be now treated as a full information problem since all the processes involved are adapted to the observable flow $\{\underline{\mathrm{F}}^S_t\}_{t\in[0,T]}$.

The last part of this section is devoted to derive a martingale representation theorem for $(P, \underline{\mathrm{F}}^S_t)$-martingales. Let us observe that from Proposition 3.3 it follows that

$$\underline{\mathrm{F}}^I_t \vee \underline{\mathrm{F}}^m_t \subseteq \underline{\mathrm{F}}^S_t$$

where $\underline{\mathrm{F}}_t^m = \sigma\{m((0,s],A); s \leq t, A \in \underline{\mathrm{B}}(\mathbb{R})\}$, and in general this inclusion holds in a strict sense. From now on we will assume a stronger condition than (3.2), that is

$$\mathbb{E}\int_0^T \left(\frac{b_s}{\sigma_s}\right)^2 ds < \infty \quad P\text{-a.s.} \tag{3.5}$$

and we consider the positive local martingale defined as the Doléans-Dade exponential of the $(P, \underline{\mathrm{F}}_t)$-martingale $-\int_0^t \frac{b_s}{\sigma_s} dW_s$,

$$L_t = \mathrm{Exp}\left(-\int_0^t \frac{b_s}{\sigma_s} dW_s\right) = \exp\left\{-\int_0^t \frac{b_s}{\sigma_s} dW_s - \frac{1}{2}\int_0^t \left(\frac{b_s}{\sigma_s}\right)^2 ds\right\}.$$

We shall make the usual standing assumption

Assumption A: L_t is a $(P, \underline{\mathrm{F}}_t)$-martingale, that is $\mathbb{E}[L_T] = 1$.

Under this last assumption we can define on $\underline{\mathrm{F}}_T$ a probability measure Q equivalent to P such that

$$\frac{dQ}{dP}|_{\underline{\mathrm{F}}_T} = L_T. \tag{3.6}$$

By Girsanov theorem the process

$$\widetilde{W}_t := W_t + \int_0^t \frac{b_s}{\sigma_s} ds$$

is a $(Q, \underline{\mathrm{F}}_t)$-Wiener process, moreover since by the definition of I_t the following equality is fulfilled

$$\widetilde{W}_t = I_t + \int_0^t \frac{\widehat{b}_s}{\sigma_s} ds \tag{3.7}$$

it turns out that the process $\widetilde{W}_t$ is $\underline{\mathrm{F}}_t^S$-adapted, and as a consequence

$$\widehat{L}_t = \mathbb{E}[L_t | \underline{\mathrm{F}}_t^S] = \frac{dQ}{dP}|_{\underline{\mathrm{F}}_t^S} = \mathrm{Exp}\left(-\int_0^t \frac{\widehat{b}_s}{\sigma_s} dI_s\right). \tag{3.8}$$

Let us notice that, by Jensen's inequality and (3.5)

$$\mathbb{E}\int_0^T \frac{(\widehat{b}_t)^2}{\sigma_t^2} dt \leq \mathbb{E}\int_0^T \frac{\widehat{b_t^2}}{\sigma_t^2} dt = \mathbb{E}\int_0^T \left(\frac{b_t}{\sigma_t}\right)^2 dt < \infty.$$

In order to derive a representation theorem for $(P, \underline{\mathrm{F}}_t^S)$-martingales we need an additional assumption on σ_t. Since σ_t is $\underline{\mathrm{F}}_t^S$-adapted and $\underline{\mathrm{F}}_t^S = \underline{\mathrm{F}}_t^Y$ there exists for each $t \in [0,T]$ a Borel measurable $H_t : D_{\mathbb{R}}[0,T] \to (0, +\infty)$ such that $\sigma_t = H_t(Y_{\cdot\wedge t})$ P-a.s. Here $D_{\mathbb{R}}[0,T]$ denotes the space of càdlàg $\mathbb{R}$-valued paths endowed with the Skorokhod metric, and we assume that H_t satisfies a global Lipschitz condition on $D_{\mathbb{R}}[0,T]$.

We summarize below all the conditions introduced in this section that we shall use from now on

Assumptions B: Assumption A, (2.2), (2.7), (2.9), (3.5) and assume σ_t to be $\underline{\mathrm{F}}_t^S$-adapted and such that H_t satisfies a global Lipschitz condition on $D_{\mathbb{R}}[0,T]$.

Lemma 3.5. *Under Assumptions B, the filtration $\underline{F}_t^S$ coincides with the filtration generated by $\widetilde{W}_t$ and the jump measure $m(dt,dx)$, that is*

$$\underline{F}_t^S = \underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m.$$

Proof. Since $\widetilde{W}_t$ and $m(dt,dx)$ are $\underline{F}_t^S$-adapted we have that $\underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m \subseteq \underline{F}_t^S$. To prove the converse, let us observe that, taking into account (3.3) and (3.7), the process Y_t solves under the probability Q, defined by (3.6), the following equation driven by $\widetilde{W}_t$ and $m(dt,dx)$

$$dY_t = \sigma_t d\widetilde{W}_t + \int_{\mathbb{R}} x\ m(dt,dx). \tag{3.9}$$

Finally, since $\sigma_t = H_t(Y_{.\wedge t})$ P-a.s. and $H_t : D_{\mathbb{R}}[0,T] \to (0,+\infty)$ satisfies a global Lipschitz condition on $D_{\mathbb{R}}[0,T]$, the stochastic functional differential equation (3.9) has a unique strong solution $\underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m$-adapted, hence $\underline{F}_t^S = \underline{F}_t^Y \subseteq \underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m$, and this concludes the proof. □

Finally we are able to prove the announced martingale representation theorem, which extend to a non-Markovian case Proposition 2.6 in [7].

Proposition 3.6. *Under Assumptions B, every $(P,\underline{F}_t^S)$-local-martingale M_t admits the decomposition*

$$M_t = M_0 + \int_0^t \int_{\mathbb{R}} \eta(t,x) m^S(ds,dx) + \int_0^t \psi_s dI_s$$

where $\eta(t,x)$ is a $\underline{F}_t^S$-predictable process and ψ_t a $\underline{F}_t^S$-adapted process such that

$$\int_0^T \int_{\mathbb{R}} |\eta(t,x)| \nu_t^p(dx)dt < \infty, \quad \int_0^T \psi_t^2 dt < \infty \quad P\text{-a.s.}$$

Proof. Let Q be the probability measure defined on $\underline{F}_T$ by (3.6). Notice that $\int_0^T \nu_t^p(\mathbb{R})dt < \infty$ P-a.s. since $\int_0^T \nu_t^p(\mathbb{R})dt = \int_0^T \widehat{\lambda}_t dt$ P-a.s. and, by (2.7), $\mathbb{E}\int_0^T \widehat{\lambda}_t dt = \mathbb{E}\int_0^T \lambda_t dt < \infty$. Hence, recalling that $\underline{F}_t^S = \underline{F}_t^{\widetilde{W}} \vee \underline{F}_t^m$ we can apply Remark 3.2 in [3] which states that for any $\widetilde{M}_t$, $(Q,\underline{F}_t^S)$- local-martingale, there exist two $\underline{F}_t^S$-adapted processes $\widetilde{\phi}(t,x)$ predictable and $\widetilde{\psi}_t$ such that

$$\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \int_{\mathbb{R}} \widetilde{\eta}(s,x) m^S(ds,dx) + \int_0^t \widetilde{\psi}_s d\widetilde{W}_s$$

with

$$\int_0^T \int_{\mathbb{R}} |\widetilde{\eta}(t,x)| \nu_t^p(dx) < \infty, \quad \int_0^T \widetilde{\psi}_t^2 dt < \infty \quad Q\text{-a.s.}$$

Let M_t be a $(P,\underline{F}_t^S)$-local martingale, by Kallianpur–Striebel formula $\widetilde{M}_t = M_t \widehat{L}_t^{-1}$ is a $(Q,\underline{F}_t^S)$-local martingale, where $\widehat{L}_t$ is defined in (3.8). We can write $M_t = \widetilde{M}_t \widehat{L}_t$

and by the product rule we deduce

$$\begin{aligned}dM_t &= \widetilde{M}_{t-}d\widehat{L}_t + \widehat{L}_{t-}d\widetilde{M}_t + d\langle \widetilde{M}^c, \widehat{L^c}\rangle_t + d\Big(\sum_{s\le t}\Delta\widetilde{M}_s\Delta\widehat{L}_s\Big)\\ &= \widehat{L}_t\Big(\widetilde{\psi}_t - \frac{\widehat{b}_t}{\sigma_t}\widetilde{M}_t\Big)dI_t + \int_{\mathbb{R}}\widehat{L}_{t-}\widetilde{\eta}(t,x)m^S(dt,dx)\end{aligned}$$

which gives the martingale representation for M_t with $\psi_t = \widehat{L}_t\widetilde{\psi}_t - \frac{\widehat{b}_t}{\sigma_t}M_t$ and $\eta(t,x) = \widehat{L}_{t-}\widetilde{\eta}(t,x)$. □

4. The optimal investment-consumption problem

In this section we focus on formulating the $\underline{\mathrm{F}}_t^S$-optimal investment-consumption problem as a stochastic control problem. We begin by recalling that the wealth process Z_t satisfies

$$\begin{aligned}dZ_t &= Z_{t-}\Big(\theta_t\frac{dS_t}{S_{t-}} - C_t\mu(dt)\Big) &(4.1)\\ &= Z_{t-}\Big\{\theta_t\widehat{\mu}_t dt - C_t\mu(dt) + \theta_t\sigma_t dI_t + \theta_t\int_{\mathbb{R}}(e^x-1)m(dt,dx)\Big\}.\end{aligned}$$

The set of admissible strategies $\underline{\mathrm{A}}$ consists of all the pairs (θ_t, C_t), where θ_t is an $\mathbb{R}$-valued, $\underline{\mathrm{F}}_t^S$-predictable process and C_t a non-negative $\underline{\mathrm{F}}_t^S$-adapted process such that $C_T = 1$ and

$$\int_0^T\Big\{|\theta_t\widehat{\mu}_t - C_t| + \theta_t^2\sigma_t^2 + |\theta_t|\int_{\mathbb{R}}|e^x-1|\nu_t^p(dx)\Big\}dt < \infty \quad P\text{-a.s.} \qquad (4.2)$$

$$\forall x\in\mathbb{R}\quad 1+\theta_t(e^x-1) > 0 \quad dP\times dt\text{-a.e.} \qquad (4.3)$$

Proposition 4.1. *Let $\{\theta_t, C_t\}_{t\in[0,T]}$ be an admissible strategy. Then the wealth equation has a unique positive solution $Z_t^{\theta,C}$ given by*

$$Z_t^{\theta,C} = z_0 e^{\int_0^t\int_{\mathbb{R}}\log(1+\theta_s(e^x-1))m(ds,dx)+\int_0^t\theta_s\sigma_s dI_s+\int_0^t(\theta_s\widehat{\mu}_s-\frac{1}{2}\theta_s^2\sigma_s^2)ds-\int_0^t C_s\mu(ds)}. \qquad (4.4)$$

Proof. Equation (4.1) can be written as $dZ_t = Z_{t-}dM_t^{\theta,C}$, where from (4.2)

$$\begin{aligned}M_t^{\theta,C} := &\int_0^t\{\theta_s\widehat{\mu}_s + \theta_s\int_{\mathbb{R}}(e^x-1)\nu_s^p(dx)\}ds - \int_0^t C_s\mu(ds)\\ &+\int_0^t\theta_s\sigma_s dI_s + \int_0^t\theta_s\int_{\mathbb{R}}(e^x-1)m^S(ds,dx)\end{aligned}$$

is a $(P,\underline{\mathrm{F}}_t^S)$-semimartingale. By the Doléans-Dade Theorem we get that there exists a unique semimartingale $Z_t^{\theta,C}$ given by

$$Z_t^{\theta,C} = z_0\, e^{M_t^{\theta,C}-\frac{1}{2}<(M^{\theta,C})^c>_t}\Pi_{s\le t}(1+\Delta M_s^{\theta,C})e^{-\Delta M_s^{\theta,C}}.$$

Moreover, $Z_t^{\theta,C} > 0$ if and only if $1 + \Delta M_s^{\theta,C} = 1 + \int_{\mathbb{R}} \theta_s(e^x - 1)m(\{s\}, dx) > 0$ $\forall s \leq t$, and this condition is implied by (4.3). Finally, by standard computation we derive expression (4.4). □

Remark 4.2. Let us observe that the pair $(\theta_t, C_t) = (0, 0)$, $\forall t \in [0, T)$, is an admissible strategy whose associated wealth is given by $Z_t^{0,0} = z_0$.

Remark 4.3. For any $(\theta, C) \in \underline{\mathrm{A}}$, the following inequality is fulfilled

$$\int_{\mathbb{R}} |(1 + \theta_t(e^x - 1))^\alpha - 1| \nu_t^p(dx) \leq \int_{\mathbb{R}} |\theta_t||e^x - 1| \nu_t^p(dx) < \infty \quad P\text{-a.s.} \tag{4.5}$$

As a consequence

$$\begin{aligned} M_t(\alpha) := & \int_0^t \int_{\mathbb{R}} \{[1 + \theta_s(e^x - 1)]^\alpha - 1\} m^S(ds, dx) + \int_0^t \alpha \theta_s \sigma_s dI_s \\ & + \int_0^t \alpha(\theta_s \widehat{\mu}_s ds - C_s \mu(ds)) + \int_0^t \int_{\mathbb{R}} \{[1 + \theta_s(e^x - 1)]^\alpha - 1\} \nu_s^p(dx) ds \end{aligned}$$

is a $(P, \underline{\mathrm{F}}_t^S)$-semimartingale and by (4.4), using standard computations, we have

$$Z_t^\alpha = z_0^\alpha e^{\frac{1}{2}\alpha(\alpha-1)\int_0^t \theta_s^2 \sigma_s^2 ds} \operatorname{Exp}(M_t(\alpha)) \tag{4.6}$$

where we recall Exp denotes the Doléans-Dade exponential.

From now on we shall furthermore assume that

$$\sup_{(\theta,C) \in \underline{\mathrm{A}}} \mathbb{E}\left[\int_0^T (C_t Z_t)^\alpha \mu^0(dt) \right] < \infty.$$

As usual in stochastic control frame we introduce the associated value process which gives a dynamic extension of the optimization problem (2.4) to each initial time $t \in [0, T]$. For any $t \in [0, T]$, $(\bar{\theta}, \bar{C}) \in \underline{\mathrm{A}}$, let us consider the set of strategies coinciding with $(\bar{\theta}, \bar{C})$ until time t

$$\underline{\mathrm{A}}_t(\bar{\theta}, \bar{C}) := \{(\theta, C) \in \underline{\mathrm{A}} : (\theta_s, C_s) = (\bar{\theta}_s, \bar{C}_s), s \leq t\}$$

and define the value process as

$$V_t(\bar{\theta}, \bar{C}) = \operatorname{ess} \sup_{(\theta,C) \in \underline{\mathrm{A}}_t(\bar{\theta}, \bar{C})} \mathbb{E}\left[\int_t^T \frac{(C_s Z_s)^\alpha}{\alpha} \mu^0(ds) \mid \underline{\mathrm{F}}_t^S \right].$$

From the dynamic programming principle ([10]) $\forall (\bar{\theta}, \bar{C}) \in \underline{\mathrm{A}}$

$$V_t(\bar{\theta}, \bar{C}) + \int_0^t \frac{(\bar{C}_s Z_s^{\bar{\theta}, \bar{C}})^\alpha}{\alpha} \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale and $(\theta^*, C^*) \in \underline{\mathrm{A}}$ is optimal for problem (2.4) if and only if

$$V_t(\theta^*, C^*) + \int_0^t \frac{(C_s^* Z_s^{\theta^*, C^*})^\alpha}{\alpha} \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-martingale. By Equation (4.4) we get that, for any $(\bar{\theta}, \bar{C}) \in \underline{\mathrm{A}}$

$$V_t(\bar{\theta}, \bar{C}) = \frac{(Z_t^{\bar{\theta}, \bar{C}})^\alpha}{\alpha} J_t$$

where the càdlàg process J_t does not depend on $(\bar{\theta}, \bar{C})$ and is defined as

$$J_t = \operatorname{ess} \sup_{(\theta,C)\in \underline{\mathrm{A}}_t} \mathbb{E}\left[\int_t^T \frac{(C_s Z_s)^\alpha}{Z_t^\alpha} \mu^0(ds) \mid \underline{\mathrm{F}}_t^S \right], \tag{4.7}$$

here $\underline{\mathrm{A}}_t$ denotes the set of admissible strategies over $[t, T]$. The process J_t is the so-called opportunity process and it is a suitable tool to derive results about the optimal investment-consumption strategy. In particular, the Bellman optimality principle can be stated as follows.

Proposition 4.4. *The following properties hold true:*

(i) $\{J_t\}_{t\in[0,T]}$ *is the smallest càdlàg* $\underline{F}_t^S$*-adapted process s.t.* $J_T = 1$ *and* $\forall(\theta, C) \in \underline{A}$, $(Z_t^{\theta,C})^\alpha J_t + \int_0^t (C_s Z_s^{\theta,C})^\alpha \mu(ds)$ *is a* $(P, \underline{F}_t^S)$*-supermartingale.*

(ii) $(\theta^*, C^*) \in \underline{A}$ *is an optimal investment-consumption strategy if and only if* $(Z_t^{\theta^*,C^*})^\alpha J_t + \int_0^t (C_s^* Z_s^{\theta^*,C^*})^\alpha \mu(ds)$ *is a* $(P, \underline{F}_t^S)$*-martingale.*

We give now some other properties of the process J_t.

Proposition 4.5. $\forall t \in [0, T]$, $J_t \geq 1$, *P-a.s. and* $\sup_{t\in[0,T]} \mathbb{E}[J_t] \leq J_0 < \infty$.

Proof. Since $(\theta_t, C_t) = (0, 0)$ $\forall t \in [0, T)$ is an admissible strategy, by (4.7) we get that $J_t \geq 1$ and, from Proposition 4.4, J_t is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale. Then $\mathbb{E}(J_t) \leq J_0$, where $J_0 = \frac{\alpha}{z_0^\alpha} \sup_{(\theta,C)\in\underline{\mathrm{A}}} \mathbb{E}\left[\int_0^T U(C_t Z_t)\mu^0(dt) \right] < \infty$. □

5. A BSDE approach

In this section, we address the problem of characterizing dynamically the opportunity process J_t. In all this section we make the class of hypotheses summarized in **Assumptions B**. First, let us fix some notations

- $\underline{\mathrm{S}}^p, 1 \leq p \leq +\infty$, denotes the space of $\mathbb{R}$-valued $\underline{\mathrm{F}}_t^S$-adapted stochastic processes $\{H_t\}_{t\in[0,T]}$ with $\|H\|_{\underline{\mathrm{S}}^p} = \| \sup_{t\in[0,T]} \mid H_t \mid \|_{L^p} < \infty$.
- $\underline{\mathrm{L}}^2_{\nu^p}$ ($\underline{\mathrm{L}}^1_{\nu^p,\mathrm{loc}}$) denotes the space of $\mathbb{R}$-valued $\underline{\mathrm{F}}_t^S$-predictable processes $\{U(t,x)\}_{t\in[0,T]}$ indexed by x with

$$\mathbb{E}\int_0^T \int_{\mathbb{R}} \mid U(t,x) \mid^2 \nu_t^p(dx)dt < \infty$$

$$\left(\text{resp. } \int_0^T \int_{\mathbb{R}} \mid U(t,x) \mid^2 \nu_t^p(dx)dt < \infty, P\text{-a.s.} \right).$$

- $\underline{\mathrm{L}}^2$ ($\underline{\mathrm{L}}^2_{\mathrm{loc}}$) denotes the space of $\mathbb{R}$-valued $\underline{\mathrm{F}}^S_t$-adapted processes $\{R_t\}_{t\in[0,T]}$ with

$$\mathbb{E}\int_0^T |\, R_t\,|^2\, dt < \infty \quad \left(\text{resp.} \int_0^T |\, R_t\,|^2\, dt < \infty \quad P\text{-a.s.}\right).$$

From Proposition 4.4, since $(\theta_t, C_t) = (0,0) \in \underline{\mathrm{A}}$, the process $\{J_t\}_{t\in[0,T]}$, is a $(P, \underline{\mathrm{F}}^S_t)$-supermartingale and it admits a unique Doob–Meyer decomposition

$$J_t = m^J_t - A_t$$

with m^J_t a $(P, \underline{\mathrm{F}}^S_t)$-local martingale and A_t a nondecreasing $(P, \underline{\mathrm{F}}^S_t)$-predictable process with $A_0 = 0$. By the martingale representation result (Proposition 3.6) there exist $\Gamma(t,x) \in \underline{\mathrm{L}}^1_{\nu^p,\mathrm{loc}}$ and $R_t \in \underline{\mathrm{L}}^2_{\mathrm{loc}}$ such that

$$m^J_t = \int_0^t \int_{\mathbb{R}} \Gamma(s,x) m^S(ds,dx) + \int_0^t R_s dI_s. \tag{5.1}$$

Theorem 5.1. *If there exists an optimal strategy* $(\theta^*, C^*) \in \underline{A}$ *for the utility maximization problem* (2.4)*, the process* $\{J_t, \Gamma(t,x), R_t\}_{t\in[0,T]}$ *solves the following BSDE*

$$\begin{aligned} J_t = 1 &- \int_t^T \int_{\mathbb{R}} \Gamma(s,x) m^S(ds,dx) - \int_t^T R_s dI_s \\ &+ \int_t^T \operatorname*{ess\ sup}_{(\theta,C)\in\underline{A}} \Big\{ f(s,J,\Gamma,R,\theta)ds + (C^\alpha_s - \alpha C_s J_s)\mu(ds)\Big\} \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} f(t,y,u,r,\theta) = &\int_{\mathbb{R}} \big(y + u(t,x)\big)\big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big]\nu^p_t(dx) \\ &+ \alpha\theta_t\sigma_t r + \Big\{\alpha\theta_t\widehat{\mu}_t + \frac{\alpha(\alpha-1)}{2}\sigma^2_t\theta^2_t\Big\} y. \end{aligned} \tag{5.3}$$

Moreover, the optimal strategy (θ^*, C^*) *realizes the essential supremum in* (5.2) *and* $C^*_t = (J_t)^{\frac{1}{\alpha-1}}$, *P-a.s..*

Proof. For any $(\theta, C) \in \underline{\mathrm{A}}$ we apply the product rule to compute $(Z^{\theta,C}_t)^\alpha J_t$

$$\begin{aligned} (Z^{\theta,C}_t)^\alpha J_t = z^\alpha_0 J_0 &+ \int_0^t J_{s-} d(Z^{\theta,C}_s)^\alpha + \int_0^t (Z^{\theta,C}_{s-})^\alpha dJ_s \\ &+ \sum_{s\le t} \Delta(Z^{\theta,C}_s)^\alpha \Delta J_s + d\left\langle Z^{\theta,C}, J\right\rangle_t. \end{aligned} \tag{5.4}$$

Since by (5.1) and (4.6)

$$\Delta J_s = \int_{\mathbb{R}} \Gamma(s,x) m(\{s\},dx),$$

$$\Delta(Z^{\theta,C}_s)^\alpha = (Z^{\theta,C}_{s-})^\alpha \int_{\mathbb{R}} [\{1+\theta_s(e^x-1)\}^\alpha - 1] m(\{s\},dx),$$

we get that (5.4) becomes

$$\begin{aligned} d\big((Z_t^{\theta,C})^\alpha J_t\big) &= (Z_{t^-}^{\theta,C})^\alpha dm_t^J \\ &\quad + (Z_{t^-}^{\theta,C})^\alpha J_{t^-}\left\{\frac{\alpha(\alpha-1)}{2}\sigma_t^2\theta_t^2 dt + dM_t(\alpha)\right\} - (Z_{t^-}^{\theta,C})^\alpha dA_t \\ &\quad + \int_{\mathbb{R}} \big(J_{t^-} + \Gamma(t,x)\big)(Z_{t^-}^{\theta,C})^\alpha \big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big] m(dt,dx) \\ &\quad + \alpha\theta_t\sigma_t R_t (Z_t^{\theta,C})^\alpha dt. \end{aligned}$$

Then, taking into account Equation (4.6)

$$\begin{aligned} &d\big((Z_t^{\theta,C})^\alpha J_t\big) \\ &= dM_t^J - (Z_{t^-}^{\theta,C})^\alpha \Bigg[dA_t - \int_{\mathbb{R}}\big(J_t + \Gamma(t,x)\big)[\{1+\theta_t(e^x-1)\}^\alpha - 1]\nu_t^p(dx)dt \\ &\qquad - \frac{\alpha(\alpha-1)}{2}\sigma_t^2\theta_t^2 J_t dt - \alpha J_t(\theta_t\widehat{\mu}_t dt - C_t\mu(dt)) - \alpha\theta_t\sigma_t R_t dt\Bigg] \end{aligned}$$

with

$$\begin{aligned} M_t^J = M_0^J &+ \int_0^t \int_{\mathbb{R}} (Z_{s^-}^{\theta,C})^\alpha \Gamma(s,x)\{1+\theta_s(e^x-1)\}^\alpha m^S(ds,dx) \\ &+ \int_0^t \int_{\mathbb{R}} (Z_{s^-}^{\theta,C})^\alpha J_{s^-}\big[\{1+\theta_s(e^x-1)\}^\alpha - 1\big] m^S(ds,dx) \\ &+ \alpha \int_0^t \theta_s\sigma_s (Z_s^{\theta,C})^\alpha J_s dI_s. \end{aligned} \tag{5.5}$$

Based on the above derivations, we obtain

$$\begin{aligned} &d\big((Z_t^{\theta,C})^\alpha J_t\big) + C_t^\alpha (Z_t^{\theta,C})^\alpha \mu(dt) \\ &\quad = dM_t^J - (Z_{t^-}^{\theta,C})^\alpha [dA_t - f(t,J,\Gamma,R,\theta)dt - (C_t^\alpha - \alpha C_t J_t)\mu(dt)] \end{aligned} \tag{5.6}$$

with $f(t,y,u,r,\theta)$ given by (5.3). Since, by the Bellman optimality principle (Proposition 4.4), $\forall(\theta,C)\in\underline{\mathrm{A}}$

$$(Z_t^{\theta,C})^\alpha J_t + \int_0^t C_r^\alpha (Z_r^{\theta,C})^\alpha \mu(dr) \tag{5.7}$$

is a $(P,\underline{\mathrm{F}}_t^S)$-supermartingale it follows that (5.5) is a $(P,\underline{\mathrm{F}}_t^S)$- local martingale and $dA_t - f(t,J,\Gamma,R,\theta)dt - (C_t^\alpha - \alpha C_t J_t)\mu(dt) \geq 0$, which in turn implies

$$dA_t \geq \operatorname*{ess\ sup}_{(\theta,C)\in\underline{\mathrm{A}}} [f(t,J,\Gamma,R,\theta)dt + (C_t^\alpha - \alpha C_t J_t)\mu(dt)].$$

On the other hand, again by the Bellman optimality principle, $(\theta^*,C^*)\in\underline{\mathrm{A}}$ is an optimal strategy if and only if the associated process given in (5.7) by replacing

(θ, C) by (θ^*, C^*) is a $(P, \underline{\mathrm{F}}_t^S)$-martingale. Thus if and only if

$$\begin{aligned} dA_t &= f(t, J, \Gamma, R, \theta^*)dt + \{(C_t^*)^\alpha - \alpha C_t^* J_t\}\mu(dt) \\ &= \operatorname{ess} \sup_{(\theta,C)\in \underline{\mathrm{A}}} [f(t, J, \Gamma, R, \theta) + (C_t^\alpha - \alpha C_t J_t)\mu(dt)]. \end{aligned}$$

To conclude the proof, let us notice that since the essential supremum of $C_t^\alpha - \alpha C_t J_t$ is attained in $(J_t)^{\frac{1}{\alpha-1}}$ this implies that $C_t^* = (J_t)^{\frac{1}{\alpha-1}}$, P-a.s. □

Remark 5.2. Conditions for existence of optimal strategies can be found in [14] for the case of terminal wealth and [15] for the case with consumption.

Remark 5.3. By Proposition 4.5, $\forall t \in [0, T]$ $J_t \geq 1$ P-a.s., thus if (θ^*, C^*) is an optimal investment-consumption strategy then $C_t^* = (J_t)^{\frac{1}{\alpha-1}}$, which in turn implies that $\forall t \in [0, T]$, $0 \leq C_t^* \leq 1$, P-a.s.

We now study the utility maximization problem defined in (2.4) over the subset $\underline{\mathrm{A}}^k \subset \underline{\mathrm{A}}$ of admissible strategies, $(\theta, C) \in \underline{\mathrm{A}}$, such that θ is uniformly bounded by k, with $k \geq 1$. In this frame the process J_t is replaced by

$$J_t^k = \operatorname{ess} \sup_{(\theta,C)\in \underline{\mathrm{A}}_t^k} \mathbb{E}\bigg[\int_t^T \frac{(C_s Z_s)^\alpha}{Z_t^\alpha} ds + \frac{Z_T^\alpha}{Z_t^\alpha} \mid \underline{\mathrm{F}}_t^S\bigg], \tag{5.8}$$

here $\underline{\mathrm{A}}_t^k$ denotes the set of admissible strategies $\underline{\mathrm{A}}^k$ over $[t, T]$. We introduce for any $(\theta, C) \in \underline{\mathrm{A}}$ the process

$$\xi_t^{\theta,C} := \mathbb{E}\bigg[\int_t^T \frac{(C_s Z_s)^\alpha}{Z_t^\alpha} ds + \frac{Z_T^\alpha}{Z_t^\alpha} \mid \underline{\mathrm{F}}_t^S\bigg].$$

Proposition 5.4. *Let us assume* $\forall t \in [0, T]$

$$|\widetilde{K}(t; \zeta)| \leq c, \quad \lambda_t = \nu(D_t) \leq c, \quad |b_t| \leq c, \quad \sigma_t \leq c \quad P\text{-a.s.} \tag{5.9}$$

with c positive constant. Then, for any $(\theta, C) \in \underline{\mathrm{A}}^k$, $\xi_t^{\theta,C}$ *is uniformly bounded on t by a constant independent of* (θ, C).

Proof. Firstly, we observe that assumptions (5.9) imply

$$\widehat{\lambda_t \Phi_t}(\mathbb{R}) = \mathbb{E}[\lambda_t | \underline{\mathrm{F}}_t^S] \leq c \quad P\text{-a.s.}$$

and since $\nu_t^p(dx) = \widehat{\lambda_t \Phi_t}(dx)$, $dP \times dt$-a.e.

$$\int_{\mathbb{R}} |e^x - 1| \nu_t^p(dx) = \int_{\mathbb{R}} |e^x - 1| \widehat{\lambda_t \Phi_t}(dx) = \mathbb{E}\bigg[\int_Z \widetilde{K}(t; \zeta)\nu(d\zeta) | \underline{\mathrm{F}}_t^S\bigg] \leq c^2 \quad P\text{-a.s.} \tag{5.10}$$

$\forall (\theta, C) \in \underline{\mathrm{A}}^k$ let us consider the probability measure $P^{\theta,\alpha}$ defined on $\underline{\mathrm{F}}_T^S$ as

$$\frac{dP^{\theta,\alpha}}{dP}\Big|_{\underline{\mathrm{F}}_T^S} = L_T^\theta = \operatorname{Exp}(M^{\theta,\alpha})_T$$

with
$$M_t^{\theta,\alpha} := \int_0^t \alpha\theta_s\sigma_s dI_s + \int_0^t \int_{\mathbb{R}} [(1+\theta_s(e^x-1))^\alpha - 1] m^S(ds,dx)$$
By the Doléans-Dade exponential formula for all $t \le s \le T$
$$\frac{Z_s^\alpha}{Z_t^\alpha} = \frac{L_s^\theta}{L_t^\theta} \exp\left\{\alpha \int_t^s \left[\left(\theta_r\ \widehat{\mu}_r + \frac{\alpha-1}{2}\theta_r^2\sigma_r^2\right)dr - C_r\mu(dr)\right]\right.$$
$$\left. + \alpha \int_t^s \int_Z [(1+\theta_r(e^x-1))^\alpha - 1]\nu_r^p(dx)dr\right\}$$
and, taking into account (4.5), we get
$$\mathbb{E}\left[\frac{Z_s^\alpha}{Z_t^\alpha}|\underline{\mathrm{F}}_t^S\right] \le \mathbb{E}^{\theta,\alpha}\left[\exp\left\{\alpha \int_t^s |\theta_r\widehat{\mu}_r|dr + \alpha \int_t^s \int_Z |\theta_r||e^x-1|\nu_r^p(dx)dr\right\}|\underline{\mathrm{F}}_t^S\right]$$
where $\mathbb{E}^{\theta,\alpha}$ denotes the expectation w.r.t. $P^{\theta,\alpha}$. Finally, by (5.9) and (5.10),
$$\mathbb{E}\left[\frac{Z_s^\alpha}{Z_t^\alpha}|\underline{\mathrm{F}}_t^S\right] \le e^{c(k)(s-t)} \quad P\text{-a.s.}$$
with $c(k)$ a suitable positive constant independent of (θ, C), which in turn implies that $\forall t \in [0,T]$
$$\xi_t^{\theta,C} = \mathbb{E}\left[\int_t^T \frac{(C_sZ_s)^\alpha}{Z_t^\alpha}\mu^0(ds) \mid \underline{\mathrm{F}}_t^S\right] \le (k+1)e^{c(k)T} \quad P\text{-a.s.} \qquad \square$$

Lemma 5.5. *Under* (5.9), $\forall(\theta,C) \in \underline{A}^k$, *the process* $\{\xi_t^{\theta,C}, \Gamma^{\theta,C}(t,x),\ R_t^{\theta,C}\}_{t\in[0,T]}$ *is the unique solution in* $\underline{S}^2 \times \underline{L}^2_{\nu^p} \times \underline{L}^2$ *to the BSDE*
$$\xi_t^{\theta,C} = 1 - \int_t^T \int_{\mathbb{R}} \Gamma^{\theta,C}(s,x)m^S(ds,dx) - \int_t^T R_s^{\theta,C}dI_s \tag{5.11}$$
$$+ \int_t^T \left[f(s,\xi^{\theta,C},\Gamma^{\theta,C},R^{\theta,C},\theta)ds + (C_s^\alpha - \alpha C_s\xi_s^{\theta,C})\mu(ds)\right]$$
with $f(s,y,u,r,\theta)$ *given in* (5.3).

Proof. As in [2] we consider the space $L(\mathbb{R},\nu^p)$ of measurable functions $u(x)$ with the topology of convergence in measure and define for $u, \widetilde{u} \in L(\mathbb{R},\nu^p)$,
$$\|u-\widetilde{u}\|_t = \left(\int_{\mathbb{R}} |u(x)-\widetilde{u}(x)|^2\nu_t^p(dx)\right)^{\frac{1}{2}}. \tag{5.12}$$
By (5.9), $\forall(\theta,C) \in \underline{A}^k$, $u(x) \in L(\mathbb{R},\nu^p)$ and $y \in \mathbb{R}$ there exists a positive constant $d(k)$, independent of (θ,C), such that
$$\int_{\mathbb{R}} \big(y+u(x)\big)\big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big]\nu_t^p(dx) \tag{5.13}$$
$$\le |\theta_t| \int_{\mathbb{R}} \big\{|y|+|u(x)|\big\}|e^x-1|\nu_t^p(dx) \le d(k)\{|y|+\|u\|_t\} \quad P\text{-a.s.}$$

Observing that the generator of BSDE (5.11) is given by

$$g(t,y,u,r,\theta,C) = \int_{\mathbb{R}} \big(y+u(x)\big)\big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big]\nu_t^p(dx) \qquad (5.14)$$
$$+ \alpha\theta_t\sigma_t r + C_t^\alpha + \left\{\alpha(\theta_t\widehat{\mu}_t - C_t) + \frac{\alpha(\alpha-1)}{2}\sigma_t^2\theta_t^2\right\}y$$

in the case with intermediate consumption (by (5.14) without the part in C_t if there is no intermediate consumption), it follows that it is uniformly Lipschitz in (y,u,r). By classical results (see for instance Proposition 3.2 in [2]) there exists a unique solution $(\widetilde{\xi},\widetilde{\Gamma},\widetilde{\mathbb{R}}) \in \underline{\mathrm{S}}^2 \times \underline{\mathrm{L}}^2_{\nu^p} \times \underline{\mathrm{L}}^2$ to BSDE (5.11) and following the same computations as in the proof of Theorem 5.1 we get that

$$d\big((Z_t^{\theta,C})^\alpha\widetilde{\xi}_t\big) + C_t^\alpha (Z_t^{\theta,C})^\alpha\mu(dt) = dM_t^{\widetilde{\xi}}$$

where

$$dM_t^{\widetilde{\xi}} = \int_{\mathbb{R}} (Z_{t^-}^{\theta,C})^\alpha\widetilde{\Gamma}(t,x)\{1+\theta_t(e^x-1)\}^\alpha m^S(dt,dx)$$
$$+ \int_{\mathbb{R}} (Z_{t^-}^{\theta,C})^\alpha\widetilde{\xi}_{t^-}\big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big]m^S(dt,dx) + \alpha\theta_t\sigma_t(Z_t^{\theta,C})^\alpha\widetilde{\xi}_t dI_t.$$

Equation (4.6) and conditions (5.9) imply that $\forall(\theta,C) \in \underline{\mathrm{A}}^k$

$$\sup_{t\in[0,T]} (Z_t^{\theta,C})^\alpha \le e^{d(N_T+|I_T|+T)} \quad P\text{-a.s.}$$

where $N_t = m((0,t],\mathbb{R})$ and d is a suitable positive constant. Now, the intensity λ_t of the point process N_t is bounded by c, hence for any constant b, $\mathbb{E}[e^{bN_T}] \le e^{(e^b-1)c}$. This entails that $\forall(\theta,C) \in \underline{\mathrm{A}}^k$, $(Z_t^{\theta,C})^\alpha$ belongs to $\underline{\mathrm{S}}^p$, for any $p \ge 1$. Therefore $M_t^{\widetilde{\xi}}$ is a $(P,\underline{\mathrm{F}}_t^S)$-uniformly integrable martingale, whose t-time value is the $\underline{\mathrm{F}}_t^S$-conditional expectation of its terminal value, which implies that $\widetilde{\xi}_t = \xi_t^{\theta,C}$. □

Now we are in a position to solve the investment-consumption problem in the case of bounded strategies.

Proposition 5.6. *Under* (5.9), *the following hold:*

- $(J_t^k, \Gamma^k(t,x), R_t^k) \in \underline{S}^2 \times \underline{L}^2_{\nu^p} \times \underline{L}^2$ *is the unique solution to BSDE*

$$J_t^k = 1 - \int_t^T\int_{\mathbb{R}} \Gamma^k(s,x)m^S(ds,dx) - \int_t^T R_s^k dI_s \qquad (5.15)$$
$$+ \int_t^T \operatorname{ess}\sup_{(\theta,C)\in\underline{A}^k} [f(s,J^k,\Gamma^k,R^k,\theta)ds + (C_s^\alpha - \alpha C_s J_s^k)\mu(ds)]$$

  with $f(s,y,u,r,\theta)$ *given in* (5.3).
- *There exists an optimal strategy* $(\theta^k,C^k) \in \underline{A}^k$ *for* (5.8).
- *A strategy* $(\theta^k,C^k) \in \underline{A}^k$ *is optimal if and only if it attains the essential supremum in* (5.15).

Proof. To prove that J^k is a solution to BSDE (5.15) we follow the same lines of the proof of Theorem 5.1. From Proposition 4.4, since $(\theta_t, C_t) = (0,0) \in \underline{\mathrm{A}}^k$, the process $\{J_t^k\}_{t\in[0,T]}$ is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale and it admits a unique Doob–Meyer decomposition

$$J_t^k = m_t^{J^k} - A_t^{J^k}$$

with $m_t^{J^k}$ a $(P, \underline{\mathrm{F}}_t^S)$-local martingale and $A_t^{J^k}$ a nondecreasing $(P, \underline{\mathrm{F}}_t^S)$-predictable process with $A_0^{J^k} = 0$. By the martingale representation result there exist $\Gamma^k(t,x) \in \underline{\mathrm{L}}^1_{\nu^p,\mathrm{loc}}$ and $R_t^k \in \underline{\mathrm{L}}^2_{\mathrm{loc}}$ such that

$$m_t^{J^k} = \int_0^t \int_{\mathbb{R}} \Gamma^k(s,x) m^S(ds,dx) + \int_0^t R_s^k dI_s.$$

Again by the Bellman optimality principle (Proposition 4.4)

$$\forall (\theta, C) \in \underline{\mathrm{A}}^k, \quad (Z_t^{\theta,C})^\alpha J_t^k + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds) \tag{5.16}$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale. By applying the product rule and following the same computations as in the proof of Theorem 5.1 (see Equation (5.6)), we get that $\forall (\theta, C) \in \underline{\mathrm{A}}^k$

$$d\big((Z_t^{\theta,C})^\alpha J_t^k\big) + C_t^\alpha (Z_t^{\theta,C})^\alpha \mu(dt) = dM_t^{J^k} - (Z_{t^-}^{\theta,C})^\alpha \left\{ dA_t^{J^k} - dF(t, J, \Gamma, R, \theta, C) \right\}$$

where $dF(t,y,u,r,\theta,C) = f(t,y,u,r,\theta)dt + (C^\alpha - \alpha C y)\mu(dt)$ and M^{J^k} is a $(P, \underline{\mathrm{F}}_t^S)$-local martingale. As a consequence

$$dA_t^{J^k} \geq \operatorname*{ess\,sup}_{(\widetilde{\theta}, \widetilde{C}) \in \underline{\mathrm{A}}^k} dF(t, J^k, \Gamma^k, R^k, \widetilde{\theta}, \widetilde{C})$$

and $(\theta^k, C^k) \in \underline{\mathrm{A}}^k$ is an optimal strategy for the problem (5.8) if and only if

$$dA_t^{J^k} \geq \operatorname*{ess\,sup}_{(\widetilde{\theta}, \widetilde{C}) \in \underline{\mathrm{A}}^k} dF(t, J^k, \Gamma^k, R^k, \widetilde{\theta}, \widetilde{C}) = dF(t, J^k, \Gamma^k, R^k, \theta^k, C^k).$$

Notice that for any fixed $(t, \omega, J^k, \Gamma^k, R^k)$, $F(t, J^k, \Gamma^k, R^k, w_1, w_2)$ is continuous with respect to the pair $(w_1, w_2) \in [-k,k] \times [0,1]$, since the following inequality holds

$$|\Gamma^k(t,x)||\{1 + w_1(e^x - 1)\}^\alpha - 1| \leq |\Gamma^k(t,x)||w_1||e^x - 1|$$

and, taking into account that $\Gamma^k(t,x)$ and $|e^x - 1| \in \underline{\mathrm{L}}^2_{\nu^p}$, we can apply Lebesgue's Theorem on dominated convergence. Therefore, by a predictable selection theorem we have that there exists $(\theta^k, C^k) \in \underline{\mathrm{A}}^k$ which realizes the essential supremum of $F(t, J^k, \Gamma^k, R^k, \theta, C)$ over $\underline{\mathrm{A}}^k$. Hence $(\theta^k, C^k) \in \underline{\mathrm{A}}^k$ is an optimal strategy for the problem (5.8) and (J^k, Γ^k, R^k) solves BSDE (5.15).

It remains to prove uniqueness of the solutions to BSDE (5.15). It is sufficient to consider the case with intermediate consumption. Notice that the generator of

BSDE (5.15) in such a case can be written as

$$\widetilde{g}(t,y,u,r) = \text{ess} \sup_{(\theta,C)\in\underline{\mathrm{A}}^k} g(t,y,u,r,\theta,C),$$

where $g(t,y,u,r,\theta,C)$ is given in (5.14).

Since we have, $\forall (y,u,r), (\widetilde{y},\widetilde{u},\widetilde{r}) \in \mathbb{R}\times L(\mathbb{R},\nu^p)\times\mathbb{R}$

$$\widetilde{g}(t,y,u,r) \le \text{ess} \sup_{(\theta,C)\in\underline{\mathrm{A}}^k} |g(t,y,u,r,\theta,C) - g(t,\widetilde{y},\widetilde{u},\widetilde{r},\theta,C)| + g(t,\widetilde{y},\widetilde{u},\widetilde{r})$$

by (5.9) and (5.13) we obtain

$$\widetilde{g}(t,y,u,r) - \widetilde{g}(t,\widetilde{y},\widetilde{u},\widetilde{r}) \le L\big(|y-\widetilde{y}| + \|u-\widetilde{u}\|_t + |r-\widetilde{r}|\big)$$

(see (5.12) for the definition of $\|u-\widetilde{u}\|_t$) and by symmetry $g(t,y,u,r)$ is uniformly Lipschitz in (y,u,r).

Applying classical results it follows that $(J^k,\Gamma^k,R^k)\in \underline{\mathrm{S}}^2\times\underline{\mathrm{L}}^2_{\nu^p}\times\underline{\mathrm{L}}^2$ is the unique solution to BSDE (5.15). □

We now come back to the non constrained case and we give a characterization of the value process J_t as the limit of the sequence $\{J^k_t\}_{k\ge1}$. Let us observe that this result does not require the existence of an optimal investment-strategy for the investment-consumption problem (2.4).

Proposition 5.7. *For any $t\in[0,T]$, we have that*

$$J_t = \lim_{k\to\infty} J^k_t \quad P\text{-}a.s.$$

Proof. We follow the same lines of the proof of Theorem 4.1 in [17]. Fix $t\in[0,T]$, since $\underline{\mathrm{A}}^k_t\subset\underline{\mathrm{A}}^{k+1}_t$ $\forall k$, we have that $\{J^k_t\}_{k\ge1}$ is an increasing sequence and we define the random variable

$$J'(t) = \lim_{k\to\infty} J^k_t \quad P\text{-a.s.}$$

Now observing that $\underline{\mathrm{A}}^k_t\subset\underline{\mathrm{A}}_t$ $\forall k$, we get that $J^k_t\le J_t$ and therefore $J'(t)\le J_t$ P-a.s.

Before proving the opposite inequality we first observe that by monotone convergence theorem for conditional expectation, since J^k_t are $\underline{\mathrm{F}}^S_t$-supermartingales $\forall k$, $J'(t)$ is a $\underline{\mathrm{F}}^S_t$-supermartingale, and we can consider its càdlàg version which we denote by J'_t. By the Doob–Meyer decomposition we can write

$$dJ'_t = \int_{\mathbb{R}} \Gamma'(t,x) m^S(dt,dx) + R'_t dI_t - dA'_t$$

with $\Gamma'(t,x)\in L^1_{\nu^p,\mathrm{loc}}$, $R'_t\in L^2_{\mathrm{loc}}$ and A'_t a nondecreasing $(P,\underline{\mathrm{F}}^S_t)$-predictable process. Following the same computations as in Theorem 5.1 (see Equation (5.6)) the product rules gives, $\forall(\theta,C)\in\underline{\mathrm{A}}$

$$d\big((Z^{\theta,C}_t)^\alpha J'_t\big) + C^\alpha_t (Z^{\theta,C}_t)^\alpha \mu(dt) \tag{5.17}$$
$$= dM^{J'}_t - (Z^{\theta,C}_{t^-})^\alpha \left[dA'_t - f(t,J',\Gamma',R',\theta,C)dt - (C^\alpha - \alpha C J'_t)\mu(dt)\right]$$

where $M_t^{J'}$ is a $(P, \underline{\mathrm{F}}_t^S)$-local martingale defined as in (5.5). We now want to prove that $\forall(\theta, C) \in \underline{\mathrm{A}}$

$$(Z_t^{\theta,C})^\alpha J_t' + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingales. Let $\widetilde{\underline{\mathrm{A}}}$ be the set of uniformly bounded admissible strategies. Since $\forall(\theta, C) \in \widetilde{\underline{\mathrm{A}}}$ there exists $n \geq 1$ such that $(\theta, C) \in \underline{\mathrm{A}}^n$, we have that $(\theta, C) \in \underline{\mathrm{A}}^k$ $\forall k \geq n$, and taking into account Equation (5.16), that

$$(Z_t^{\theta,C})^\alpha J_t^k + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale. By monotone convergence theorem we derive that

$$\forall(\theta, C) \in \widetilde{\underline{\mathrm{A}}}, \quad (Z_t^{\theta,C})^\alpha J_t' + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds)$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale and by Equation (5.17) we have

$$\forall(\theta, C) \in \widetilde{\underline{\mathrm{A}}}, \quad dA_t' - [f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)] \geq 0.$$

Thus

$$dA_t' \geq \operatorname*{ess\,sup}_{(\theta,C)\in\widetilde{\underline{\mathrm{A}}}} [f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)].$$

Now, since $\forall(\theta, C) \in \underline{\mathrm{A}}$, $\theta_t = \lim_k \theta_t^k$ with $\theta_t^k = \theta_t 1\!\!1_{|\theta_t| \leq k} \in \widetilde{\underline{\mathrm{A}}}$, we get

$$\begin{aligned} &\operatorname*{ess\,sup}_{(\theta,C)\in\widetilde{\underline{\mathrm{A}}}} [f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)] \\ &\quad = \operatorname*{ess\,sup}_{(\theta,C)\in\underline{\mathrm{A}}} [f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)] \end{aligned}$$

hence $dA_t' \geq \operatorname{ess\,sup}_{(\theta,C)\in\underline{\mathrm{A}}}[f(t, J', \Gamma', R', \theta, C)dt + (C^\alpha - \alpha C J_t')\mu(dt)]$. Again by (5.17)

$$\forall(\theta, C) \in \underline{\mathrm{A}} \quad M_t^{J'} \geq (Z_t^{\theta,C})^\alpha J_t' + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds) \geq 0$$

is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale, since it is a non-negative local martingale. This implies that $(Z_t^{\theta,C})^\alpha J_t' + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha \mu(ds)$ is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale $\forall(\theta, C) \in \underline{\mathrm{A}}$. Finally, by Bellman principle $J_t' \geq J_t$ P-a.s. $\forall t \in [0, T]$ and this concludes the proof. □

We conclude this section by giving a verification result for the general case and providing an example which can be solved using this result.

Proposition 5.8. *Under the assumptions:*

(i) *there exists a solution* $(\widetilde{J}_t, \widetilde{\Gamma}(t, x), \widetilde{R}_t)$ *to BSDE* (5.2) *such that* $M_t^{\widetilde{J}}$ *defined in* (5.5) *is a* $(P, \underline{F}_t^S)$*-local martingale*

(ii) *there exists* $(\theta^*, C^*) \in \underline{A}$ *which attains the essential supremum in Equation* (5.2) *with* $(J_t, \Gamma(t, x), R_t)$ *replaced by* $(\widetilde{J}_t, \widetilde{\Gamma}(t, x), \widetilde{R}_t)$

(iii) $\xi_t^{\theta^*,C^*}$ *is the unique solution to BSDE* (5.11) *associated with* (θ^*, C^*).

Then $\widetilde{J}_t = J_t$ *P-a.s. for any* $t \in [0,T]$, *and* (θ^*, C^*) *is an optimal strategy.*

Proof. Let $(\widetilde{J}_t, \widetilde{\Gamma}(t,x), \widetilde{R}_t)$ be a solution to BSDE (5.2), by applying the product rule and following the same computations as in the proof of Theorem 5.1 (see Equation (5.6)), we get that $\forall (\theta, C) \in \underline{\mathrm{A}}$

$$\begin{aligned} &d\big((Z_t^{\theta,C})^\alpha \widetilde{J}_t\big) + C_t^\alpha (Z_t^{\theta,C})^\alpha \mu(dt) \\ &\quad = dM_t^{\widetilde{J}} - (Z_{t^-}^{\theta,C})^\alpha \left\{ \operatorname*{ess\ sup}_{(\widetilde{\theta},\widetilde{C})\in\underline{\mathrm{A}}} dF(t,\widetilde{J},\widetilde{\Gamma},\widetilde{R},\widetilde{\theta},\widetilde{C}) - dF(t,\widetilde{J},\widetilde{\Gamma},\widetilde{R},\theta,C) \right\} \end{aligned}$$

where $dF(t,y,u,r,\theta,C) = f(t,y,u,r,\theta)dt + (C^\alpha - \alpha C y)\mu(dt)$ and $M^{\widetilde{J}}$ is a $(P, \underline{\mathrm{F}}_t^S)$-local martingale such that $M_0^{\widetilde{J}} = z_0^\alpha J_0$. Notice now that

$$M_t^{\widetilde{J}} \geq (Z_t^{\theta,C})^\alpha \widetilde{J}_t + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha ds \geq 0$$

and since every non-negative local martingale is a supermartingale the process $M^{\widetilde{J}}$ is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale.

Thus $\forall (\theta, C) \in \underline{\mathrm{A}}$, $(Z_t^{\theta,C})^\alpha \widetilde{J}_t + \int_0^t C_s^\alpha (Z_s^{\theta,C})^\alpha ds$ is a $(P, \underline{\mathrm{F}}_t^S)$-supermartingale, and from Bellman principle it yields that $\widetilde{J}_t \geq J_t$ P-a.s. for any $t \in [0,T]$.

To prove the opposite inequality, let us observe that by (ii), $\widetilde{J}_t$ solves BSDE Equation (5.11) associated to $(\theta^*, C^*) \in \underline{\mathrm{A}}$, and by (iii), $\widetilde{J}_t = \xi_t^{\theta^*,C^*} \leq \operatorname{ess\,sup}_{(\widetilde{\theta},\widetilde{C})\in\underline{\mathrm{A}}} \xi_t^{\theta,C} = J_t$, P-a.s. for any $t \in [0,T]$. Hence $\widetilde{J}_t = J_t$, P-a.s. and (θ^*, C^*) is an optimal strategy. □

Example. We now present a particular model where the risky asset follows a geometric jump-diffusion driven by two independent point processes whose intensities are not directly observed by investors. Let us assume

$$K(t;\zeta) = \sum_{j=1}^{2} K_j(t) \mathbb{1}_{D_j(t)}(\zeta)$$

with $K_1(t) > 0, K_2(t) < 0$ $(P, \underline{\mathrm{F}}_t^S)$-predictable processes and $D_j(t)$, $j = 1,2$, $(P, \underline{\mathrm{F}}_t^S)$-predictable processes taking values in $\underline{\mathrm{Z}}$. In this particular case the logreturn process solves

$$dY_t = b_t dt + \sigma_t dW_t + \sum_{j=1}^{2} K_j(t) N_t^j$$

with $N_t^j = N((0,t), D_j(t))$, $j = 1,2$, independent counting processes with $(P, \underline{\mathrm{F}}_t)$-predictable intensities given by $\lambda_t^j = \nu(D_j(t))$. In this model the agent can observe the processes $K_j(t)$ but not the intensities λ_t^j. As in the general case we assume σ_t

a strictly positive $\underline{\mathrm{F}}^S_t$-adapted process. The integer-valued random measure defined in (2.5) and its $(P, \underline{\mathrm{F}}^S_t)$-predictable dual projection are given by

$$m(dt, dx) = \sum_{j=1}^{2} \delta_{K_j(t)}(dx) N_t^j, \quad \nu^p(dt, dx) = \sum_{j=1}^{2} \delta_{K_j(t)}(dx) \widetilde{\lambda}_t^j dt$$

respectively, where $\widetilde{\lambda}_t^j$, $j = 1, 2$, denote the $(P, \underline{\mathrm{F}}^S_t)$-predictable intensities of N_t^j. From now on we assume $\forall t \in [0, T]$, P-a.s.

$$|b_t| \leq A_2,\ |\sigma_t| \leq A_2,\ A_1 \leq \lambda_t^j \leq A_2,\ A_1 \leq K_j(t) \leq A_2,\ j = 1, 2 \tag{5.18}$$

with A_i, $i = 1, 2$, positive constants. We consider the case with intermediate consumption. The BSDE (5.2) adapted to this particular model is given by

$$\begin{aligned} J_t = 1 - \sum_{j=1}^{2} \int_t^T \Gamma(s, j)(N_t^j - \widetilde{\lambda}_t^j dt) - \int_t^T R_s dI_s \\ + \int_t^T \operatorname*{ess\ sup}_{(\theta, C) \in \underline{\mathrm{A}}} h(s, J, \Gamma(1), \Gamma(2), R, \theta, C) ds \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} h(t, y, u_1, u_2, r, \theta, C) = \sum_{j=1}^{2} \left(y + u_j\right) \left[\{1 + \theta_t (e^{K_j(t)} - 1)\}^\alpha - 1\right] \widetilde{\lambda}_t^j \\ + \alpha \theta_t \sigma_t r + C_t^\alpha + \left\{\alpha(\theta_t \widehat{\mu}_t - C_t) + \frac{\alpha(\alpha - 1)}{2} \sigma_t^2 \theta_t^2\right\} y. \end{aligned}$$

We begin by observing that by (4.3) any admissible trading strategy θ_t necessarily satisfies $\theta_t \in \left(-\frac{1}{e^{K_1(t)}-1}, \frac{1}{e^{K_2(t)}-1}\right)$ for a.e. t and assumption (5.18) yields that admissible investment strategies take values in a compact space. Following similar computations as those performed in the proofs of Lemma 5.5 and Proposition 5.6 we obtain that the generator of the BSDE (5.19) is uniformly Lipschitz in (y, u_1, u_2, r).

From classical results there exists a unique solution, $(\widetilde{J}_t, \widetilde{\Gamma}(t, 1), \widetilde{\Gamma}(t, 2), \widetilde{R}_t) \in \underline{\mathrm{S}}^2 \times \underline{\mathrm{L}}_1^2 \times \underline{\mathrm{L}}_2^2 \times \underline{\mathrm{L}}^2$, to the BSDE (5.19). Here $\underline{\mathrm{L}}_i^2$ denotes the space of $\mathbb{R}$-valued $\underline{\mathrm{F}}^S_t$-predictable processes $\{U(t)\}_{t \in [0,T]}$ such that $\mathbb{E} \int_0^T |U(t)|^2 \widetilde{\lambda}_t^i dt < \infty$.

Finally, we have that for any fixed (t, y, u_1, u_2, r) the essential supremum of $h(t, y, u_1, u_2, r, \theta, C)$ is achieved at $\left(\theta^*(t, y, u_1, u_2, r), C^* = y^{\frac{1}{\alpha - 1}}\right)$ where $\theta^*(t, y, u_1, u_2, r)$ is such that $\frac{\partial h}{\partial \theta}|_{\theta = \theta^*} = 0$. Indeed, it is sufficient to observe that $\frac{\partial^2 h}{\partial^2 \theta} < 0$ P-a.s. and that

$$\lim_{\theta \to \frac{-1}{e^{K_1(t)} - 1}} \frac{\partial h}{\partial \theta} = +\infty, \quad \lim_{\theta \to \frac{1}{e^{K_2(t)} - 1}} \frac{\partial h}{\partial \theta} = -\infty \quad P\text{-a.s.}$$

Proposition 5.8 implies that $\widetilde{J}_t$ coincides with the opportunity process and the unique optimal investment-consumption strategy is given by

$$(\theta_t^*, C_t^*) = \left(\theta^*(t, \widetilde{J}_t, \widetilde{\Gamma}(t,1), \widetilde{\Gamma}(t,2), \widetilde{R}_t), (\widetilde{J}_t)^{\frac{1}{\alpha-1}}\right)$$

with $(\widetilde{J}_t, \widetilde{\Gamma}(t,1), \widetilde{\Gamma}(t,2), \widetilde{R}_t)$ unique solution of BSDE (5.19).

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Stochastic Control and Pricing Under Swap Measures

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Abstract. This paper relates to an approach described in [6] which, for the pricing of bonds and bond derivatives, is alternative to the classical approach that involves martingale measures and is based on the solution of a stochastic control problem, thereby avoiding a change of measure. It turns out that this approach can be extended to various situations where traditionally a change of measure is involved via a change of numeraire. In the present paper we study this extension for the case of Swap measures that are relevant in the classical approach to the pricing of Swaps and Swaptions.

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1. Introduction

In a recent paper [6], a new approach has been proposed for the derivation of bonds and bond derivatives prices in a diffusion-type multivariate factor model for the term structure of interest rates which, while yielding the same arbitrage free prices, is alternative to the classical derivation. It is based on the solution of a stochastic control problem and its key feature can be described as follows. In the classical approach a fundamental tool are martingale measures that can be obtained by a Girsanov-type measure transformation. The latter implies a change of drift in the dynamics of the factors which however preserves the trajectories. Now, the drift of a diffusion-type factor process can also be changed by a feedback control as it is done in stochastic control. With the latter approach the trajectories are changed, but the measure remains unchanged.

An immediate implication of this key feature is the novel insight that it becomes equivalent to compute prices either on the basis of a traditional measure change or by solving an optimal stochastic control problem. In fact, since the values that one ultimately observes are the prices, it is irrelevant whether these values are generated by considering the same trajectories of the factors under a different measure or by considering different trajectories (which one does not even observe) under the same measure. What is relevant is that in both ways one generates the same prices. The major novelty of our approach can thus be seen in the linking of stochastic optimal control theory with the classical martingale approach thereby providing an alternative representation of the prices of bonds and interest rate derivatives under a multifactor term structure. The use of system theoretic tools also allows for much simpler formulae for computing bond derivatives prices.

In [6] the approach via a stochastic control interpretation is worked out in detail for prices and forward prices of bonds and then generalized also to forward measures in view of the pricing of more general derivative products. This generalization to forward measures hints at the possibility to extend the approach also to different situations. One such possible extensions concerns Swap measures that are relevant in interest rate derivative pricing, in particular for Swaps and Swaptions. The major purpose of the present paper is now to work out this latter extension.

In order to describe the approach for Swap measures, it is unavoidable to summarize the main steps of the approach in [6]. This is done in the first five sections of the paper, where we also specify the model and the assumptions and recall some basic facts from arbitrage pricing and stochastic control. Furthermore, in addition to recalling in these five sections the approach in [6], in Subsection 5.1 we also present a control interpretation of expectations under forward measures, which will then be useful also in the case of Swap measures. Finally, Sections 6 and 7 contain the novel part with respect to [6], namely the stochastic control interpretation of Swap rates and of expectations under Swap measures respectively.

2. Arbitrage-free term structure

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ be a given filtered probability space. Consider the l-dimensional Markovian factor process $x(\cdot)$, evolving under $\mathbb{Q}$, according to the dynamics

$$dx(t) = \mathbf{f}(t, x(t))dt + \mathbf{g}(t, x(t))dw_t, \qquad t \in [0, T], \qquad x(0) = 0, \tag{2.1}$$

where $T > 0$, $\mathbf{f}$ is an l-dimensional vector function and $\mathbf{g}$ is a matrix function of dimensions $l \times k$, w is a k-dimensional $(\mathbb{Q}, \mathcal{F}_t)$ Wiener process.

For the bond prices we consider a notation of the form $p(t, T, x(t))$, where t is the time variable, T is the date of maturity and $x(t)$ is the value of the factor process $x(\cdot)$ at time t. Analogously, the forward rate corresponding to $p(t, T, x)$ will be denoted by $f(t, T, x) := -\frac{\partial}{\partial T} \ln p(t, T, x)$ and the short rate by $r(t, x) := f(t, t, x)$.

We shall make the following

Assumption 2.1. There exists some constant $M > 0$ such that, uniformly in $t \in [0, T]$:

- $||\mathbf{f}(t, x)|| \leq M(1 + ||x||), \qquad ||\mathbf{g}(t, x)|| \leq M$
- $|r(t, x)| \leq M(1 + ||x||^2)$.

We have the well-known Term Structure Equation (see, e.g., [1, 2])

Theorem 2.2. *In an arbitrage-free bond market, with the filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$, *the function* $p(t, T, x)$ *is the unique solution* (*see Remark* 2.3 *below*) *of the PDE*

$$\begin{cases} \frac{\partial}{\partial t} p(t,T,x) + \mathbf{f}'(t,x)\nabla_x p(t,T,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}(\mathbf{g}'(t,x)\nabla_{xx} p(t,T,x)\mathbf{g}(t,x)) \\ \qquad\qquad - r(t,x)p(t,T,x) = 0 \\ p(T,T,x) = 1. \end{cases} \tag{2.2}$$

Remark 2.3. It is possible to prove (see [3, Ch. 6, Section 4]) that, under Assumption 2.1, the stochastic differential equation (2.1) has a unique strong solution and that the solution to (2.2), if it exists, is unique within the class of functions satisfying the growth condition $|p(t, T, x)| \le Ce^{C||x||^2}$ for all $t \le T$ and all $x \in \mathbb{R}^l$, where C is a positive constant possibly depending on T.

3. Stochastic control in interest rate derivative pricing

As mentioned in the Introduction, the traditional pricing techniques are based on measure changes, where the trajectories of the stochastic processes involved are preserved, but a modification in their drift term is implicit in the measure transformation.

Our approach, instead, makes this drift modification for the factor process explicit, while maintaining the original martingale measure $\mathbb{Q}$. Moreover, we obtain drift changes by introducing a control process and choosing a suitable objective function. We shall show that the prices arising from a suitable stochastic control formulation are the same as those calculated by the usual methods.

These control problems are obtained in the following three steps, illustrated here for the case of bond prices:

- apply a *logarithmic transform* to the bond price;
- use the available pricing equations to obtain a PDE for the transformed price;
- identify an HJB equation, and the corresponding stochastic control problem, associated to such a PDE.

As a first instance of our argument, in this section we investigate the connection between bond prices and stochastic optimal control, following the approach in [4]. Here we assume that the factor process evolves, under the standard martingale measure $\mathbb{Q}$, according to the general dynamics (2.1).

Now we put

$$W(t, T, x) := -\ln p(t, T, x). \tag{3.1}$$

Remembering (2.2), we obtain that the function $W(t,T,x)$ in (3.1) satisfies

$$\begin{cases} \frac{\partial}{\partial t}W(t,T,x) + \mathbf{f}'(t,x)\nabla_x W(t,T,x) \\ \qquad\qquad - \frac{1}{2}\nabla_x W'(t,T,x)\mathbf{gg}'(t,x)\nabla_x W(t,T,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}W(t,T,x)\mathbf{g}(t,x)\right) + r(t,x) = 0 \\ W(T,T,x) = 0. \end{cases} \tag{3.2}$$

Remark 3.1. As usual, it is easy to check that the bond price $p(t,T,x)$ is a solution to (2.2) and, in view of Remark 2.3, it is unique. Notice then that (3.2) is the equation satisfied by a one-to-one transformation of $p(t,T,x)$, and thus it also has a unique solution.

Consider next the following stochastic control problem:

$$\begin{cases} dx(t) = \left[\mathbf{f}(t,x(t)) + \mathbf{g}(t,x(t))u(t)\right]dt + \mathbf{g}(t,x(t))dw_t \\ W(t,T,x) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^T \left(\frac{1}{2}u'(s)u(s) + r(s,x(s))\right)ds\right\}, \end{cases} \tag{3.3}$$

where $\mathcal{U}$ denotes the class of admissible control laws, namely the control processes for which the first equation in (3.3) has a unique solution in probability law and the expected cost, namely $J(t,T,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\{\int_t^T \left(\frac{1}{2}u'(s)u(s) + r(s,x(s))\right)ds\}$, has finite value.

It is possible to prove (see [4]) the following

Proposition 3.2. *The bond price $p(t,T,x)$ can be expressed as*

$$p(t,T,x) = \exp\left[-W(t,T,x)\right],$$

where $W(t,T,x)$ is the optimal value function of the stochastic control problem (3.3)*.*

In the field of stochastic control, we have the following (see [5])

Sufficient Condition for Admissibility: *Given a process $u(\cdot)$, suppose that there exist some constants M and K such that:*

- $||u(t,x)|| \le M(1+||x||)$ *for all* $(t,x) \in [0,T]\times\mathbb{R}^l$*;*
- *for any bounded $B \subset \mathbb{R}^l$ and any T_0 in $(0,T)$,*

$$||u(t,x) - u(t,y)|| \le K||x-y||$$

for all $x,y \in B$ and $0 \le t \le T_0$.

Then $u(\cdot)$ is an admissible control law. Notice that K may depend on B and T_0, while both M and K may depend on $u(\cdot)$.

Thus, a possible choice in order to have admissibility for the optimal control law in (3.3) is to make the following

Assumption 3.3. The gradient of $W(t,T,x)$, solution of (3.2), satisfies a linear growth condition, i.e.,

$$||\nabla_x W(t,T,x)|| \leq M(1+||x||) \quad \text{for all } x \in \mathbb{R}^l,$$

for some constant $M > 0$, uniformly in $t \in [0,T]$.

Remark 3.4. Notice that such a hypothesis is not void: it is satisfied, for example, in the case of linear dynamics as discussed in [6].

Remark 3.5. Assumption 3.3 will turn out to be sufficient in order to assure that the optimal control law of problem (3.3) is an admissible control law, in the sense of the Sufficient Condition for Admissibility. However, such a hypothesis is not strictly needed: it can be substituted by another one implying just admissibility for the optimal control law.

4. Forward prices

Since now we have a complete control interpretation for the bond prices maturing at a given T, we consider the more complex problem of pricing derivatives on these bonds that have a maturity τ, with $t \leq \tau \leq T$. For this purpose, in this section we first consider computing the expected value at time t of the T-bond price at time τ. We refer to such a quantity as the *forward price* of the T-bond. More precisely, using the forward measure Q^τ, the one with $p(t,\tau,x(t))$ as numeraire, we want to calculate

$$\mathbb{E}^{\mathbb{Q}^\tau}_{t,x}\left\{p(\tau,T,x(\tau))\right\}. \tag{4.1}$$

Our purpose in this section is to obtain a control description for the forward price (4.1). For this purpose we define the process $x^\tau(t)$, with dynamics

$$\begin{aligned} dx^\tau(t) &= \left[\mathbf{f}(t,x^\tau(t)) - \mathbf{gg}'(t,x^\tau(t))\nabla_x W(t,\tau,x^\tau(t))\right]dt + \mathbf{g}(t,x^\tau(t))dw_t,\\ x^\tau(0) &= 0, \end{aligned} \tag{4.2}$$

where the function $W(t,\tau,x)$ is the unique solution of the PDE in (3.2), with $T = \tau$.

Moreover, let us put

$$p^\tau(t,T,x) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{p(\tau,T,x^\tau(\tau))\right\} = \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\exp\left[-W(\tau,T,x^\tau(\tau))\right]\right\}, \tag{4.3}$$

where the second equality comes from (3.1). The Kolmogorov backward equation associated to (4.3) is

$$\begin{cases} \frac{\partial}{\partial t}p^\tau(t,T,x) + \left[\mathbf{f}'(t,x) - (\nabla_x W)'(t,\tau,x)\mathbf{gg}'(t,x)\right]\nabla_x p^\tau(t,T,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}p^\tau(t,T,x)\mathbf{g}(t,x)\right) = 0 \\ p^\tau(\tau,T,x) = \exp[-W(\tau,T,x)]. \end{cases} \tag{4.4}$$

Remark 4.1. Differently from what concerns equation (2.2), Assumption 2.1 is not sufficient in order to guarantee uniqueness of the solution to (4.4). However, since it is sufficient to require $\nabla_x W(t,\tau,x)$ to have at most linear growth in x, under Assumption 3.3, we indeed have uniqueness.

Putting

$$W^\tau(t,T,x) := -\ln p^\tau(t,T,x),$$

analogously to what has been made in the previous section, the PDE (4.4) becomes

$$\begin{cases} \frac{\partial}{\partial t} W^\tau(t,T,x) + \left[\mathbf{f}'(t,x) - (\nabla_x W)'(t,\tau,x)\mathbf{g}\mathbf{g}'(t,x)\right] \nabla_x W^\tau(t,T,x) \\ \qquad\qquad - \frac{1}{2}\left(\nabla_x W^\tau\right)'(t,T,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x W^\tau(t,T,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx} W^\tau(t,T,x)\mathbf{g}(t,x)\right) = 0 \\ W^\tau(\tau,T,x) = W(\tau,T,x). \end{cases} \tag{4.5}$$

For reasons of admissibility of the optimal control law (4.7) below, as explained in Remark 3.5, we make the following

Assumption 4.2. The gradient of $W^\tau(t,T,x)$, solution of (4.5), has at most linear growth, i.e.,

$$||\nabla_x W^\tau(t,T,x)|| \leq M(1+||x||) \quad \text{for all } x \in \mathbb{R}^l,$$

for some constant $M > 0$, uniformly in $t \in [0,T]$.

Since it has a similar structure to (3.2), also the PDE (4.5) can be seen as resulting from a HJB equation, namely (dropping the arguments of the functions)

$$\begin{cases} \frac{\partial}{\partial t} W^\tau + \inf_{u\in\mathbb{R}^k}\left\{\left[\mathbf{f}' - (\nabla_x W)'\mathbf{g}\mathbf{g}' + u'\mathbf{g}'\right]\nabla_x W^\tau \right. \\ \qquad \left. + \frac{1}{2}\mathrm{tr}(\mathbf{g}'\nabla_{xx}W^\tau\mathbf{g}) + \frac{1}{2}u'u\right\} = 0, \\ W^\tau(\tau,T,x) = W(\tau,T,x) \end{cases} \tag{4.6}$$

with the usual solution

$$u^*(t,x;W^\tau) = -\mathbf{g}'(t,x)\nabla_x W^\tau(t,T,x). \tag{4.7}$$

Thus, equation (4.6) is the HJB equation originating from the following stochastic control problem

$$\begin{cases} dx^\tau(t) = \Big[\mathbf{f}(t,x^\tau(t)) - \mathbf{g}\mathbf{g}'(t,x^\tau(t))\nabla_x W(t,\tau,x^\tau(t)) \\ \qquad\qquad + \mathbf{g}(t,x^\tau(t))u(t)\Big]dt + \mathbf{g}(t,x^\tau(t))dw_t \\ W^\tau(t,T,x) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^\tau \frac{1}{2}u'(s)u(s)ds + W(\tau,T,x^\tau(\tau))\right\}. \end{cases} \tag{4.8}$$

The symbol $\mathcal{U}$ denotes the class of the admissible control laws, for which the first equation in (4.8) has a unique solution in probability law and the expected cost $J(t,\tau,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^\tau \frac{1}{2}u'(s)u(s)ds + W(\tau,T,x^\tau(\tau))\right\}$ has finite value.

Remark 4.3. For the well-posedness of a stochastic control problem, also a condition on the terminal cost is needed. More precisely, we have to require it to have at most polynomial growth (again, see [5]). Notice that, thanks to Assumption 3.3, the stochastic control problem (4.8) satisfies this requirement. If we substitute Assumption 3.3 with another one implying just admissibility in (3.3), we need to require additionally that $W(t,T,x)$ has at most polynomial growth in x.

Now we are ready to give the control interpretation promised above for the forward prices, based on problem (4.8). Indeed, it is possible to prove the following (see [6])

Proposition 4.4. *For $t \leq \tau$, it holds*

$$\mathbb{E}^{\mathbb{Q}^\tau}_{t,x}\left\{p(\tau, T, x(\tau))\right\} = p^\tau(t, T, x) = \exp\left[-W^\tau(t, T, x)\right].$$

5. Forward measures and a general pricing formula

In this section we show the existence of a close connection between the factor process $x^\tau(\cdot)$ defined in (4.2) and the forward measure $\mathbb{Q}^\tau$, for each $\tau > 0$. More precisely, for a given expectation taken with respect to the forward measure $\mathbb{Q}^\tau$, we are interested in expressing such an expected value by using the standard martingale measure $\mathbb{Q}$, by means of a suitable modification to the original factor process $x(\cdot)$, evolving according to (2.1). This is specified in the following two propositions.

Proposition 5.1. *Given $\tau > 0$, let t be a fixed time-instant, with $0 \leq t \leq \tau$, and let x be a fixed vector in $\mathbb{R}^l$. Let $x(s)$, $s \in [t, \tau]$, be the process satisfying* (2.1) *with $x(t) = x$, and let $x^\tau(s)$, $s \in [t, \tau]$, be the process satisfying the dynamics in* (4.8) *with $x^\tau(t) = x$. Then the random variable $x(\tau)$ has the same distribution under the forward measure $\mathbb{Q}^\tau$* (*the one with numeraire $p(t, \tau, x(t))$*) *as the random variable $x^\tau(\tau)$ under the standard martingale measure $\mathbb{Q}$* (*the one with numeraire $B(t)$*).

This proposition can be proved analogously to Proposition 4.1 in [6].

The following proposition can now be obtained (see always [6]).

Proposition 5.2. *Given a date of maturity τ and a τ-claim $F(x(\tau))$, its arbitrage-free price at time t, with $t \leq \tau$, is*

$$\begin{aligned}\pi(t) &= \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\exp\left[-\int_t^\tau r(s, x(s))ds\right] \cdot F(x(\tau))\right\} \\ &= p(t, \tau, x) \cdot \mathbb{E}^{\mathbb{Q}^\tau}_{t,x}\left\{F(x(\tau))\right\}.\end{aligned}$$

Then, if $W(t, \tau, x)$ is the unique solution of (3.2) *with $T = \tau$, we have the following representation for $\pi(t)$:*

$$\pi(t) = \exp\left[-W(t, \tau, x)\right] \cdot \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{F\left(x^\tau(\tau)\right)\right\}.$$

5.1. Control interpretation for expectations under forward measures

Proposition 5.1 allows one to obtain a control description for expected values with respect to forward measures. It will be achieved by introducing a further control problem, obtained by adding a control term to the dynamics (4.2), i.e., by adding a second control term to the original factor process dynamics (2.1) (indeed, dynamics (4.2) originate from problem (3.3)). The result in this section somehow generalizes what has been made in Section 4 for forward prices. More precisely, we have the following: considering expectations under a forward measure, let $Y(x)$ be a real-valued positive function and define

$$d(t,\tau,x) := \mathbb{E}^{\mathbb{Q}^\tau}_{t,x}\left\{Y(x(\tau))\right\}.$$

Thanks to Proposition 5.1, we get an alternative representation for $d(t,\tau,x)$, namely

$$d(t,\tau,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{Y(x^\tau(\tau))\right\}. \tag{5.1}$$

The Kolmogorov backward equation associated to (5.1) is

$$\begin{cases} \frac{\partial}{\partial t}d(t,\tau,x) + \left[\mathbf{f}'(t,x) - (\nabla_x W)'(t,\tau,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x d(t,\tau,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}d(t,\tau,x)\mathbf{g}(t,x)\right) = 0 \\ d(\tau,\tau,x) = Y(x). \end{cases}$$

As usual, we apply the logarithmic transform to $d(t,\tau,x)$, and so we define

$$W^Y(t,\tau,x) := -\ln d(t,\tau,x).$$

We have the following PDE for $W^Y(t,\tau,x)$

$$\begin{cases} \frac{\partial}{\partial t}W^Y(t,\tau,x) + \left[\mathbf{f}'(t,x) - (\nabla_x W)'(t,\tau,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x W^Y(t,\tau,x) \\ \qquad\qquad - \frac{1}{2}(\nabla_x W^Y)'(t,\tau,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x W^Y(t,\tau,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}W^Y(t,\tau,x)\mathbf{g}(t,x)\right) = 0 \\ W^Y(\tau,\tau,x) = -\ln Y(x). \end{cases}$$

As in the previous sections, also this PDE results from a HJB equation, in particular the one originating from the stochastic control problem

$$\begin{cases} dx^\tau(t) = \left[\mathbf{f}(t,x^\tau(t)) - \mathbf{g}\mathbf{g}'(t,x^\tau(t))\nabla_x W(t,\tau,x^\tau(t)) \right. \\ \qquad\qquad\qquad \left. + \mathbf{g}(t,x^\tau(t))u(t)\right]dt + \mathbf{g}(t,x^\tau(t))dw_t \\ W^Y(t,\tau,x) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^\tau \frac{1}{2}u'(s)u(s)ds - \ln Y\left(x^\tau(\tau)\right)\right\} \end{cases} \tag{5.2}$$

where $\mathcal{U}$ denotes the class of the control processes for which the first equation in (5.2) has a unique solution in probability law and the expected cost $J(t,\tau,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^\tau \frac{1}{2}u'(s)u(s)ds - \ln Y\left(x^\tau(\tau)\right)\right\}$ has finite value.

Thus

$$d(t,\tau,x) = \exp\left[-W^Y(t,\tau,x)\right],$$

with $W^Y(t,\tau,x)$ the optimal value function in (5.2).

Remark 5.3. Notice that we need to assume that $Y(x)$ is regular enough in order for $-\ln Y(x)$ to have at most polynomial growth and for $\nabla_x W^Y(t,\tau,x)$ to have at most linear growth (for reasons of well-posedness of the stochastic control problem (5.2) and admissibility of the resulting optimal control law, see Remarks 3.5 and 4.3).

6. Swap rates

The main purpose of this paper is to apply the previous control approach to swap measures. We use notations taken from [1] (see also [2]). Given a set of increasing dates $T_0, T_1, \dots, T_N$ and choosing $n \in \{0, \dots, N-1\}$, a fundamental quantity arising in this context is the so-called *swap rate* $R^{n,N}(t, x(t))$, given by

$$R^{n,N}(t, x(t)) := \frac{p(t, T_n, x(t)) - p(t, T_N, x(t))}{C^{n,N}(t, x(t))},$$

where

$$C^{n,N}(t, x(t)) := \sum\nolimits_{i=n+1}^{N} \alpha_i \cdot p(t, T_i, x(t)),$$

with $\alpha_i := T_i - T_{i-1}$. Let $\mathbb{Q}^{n,N}$ be the swap measure, namely a probability measure, equivalent to $\mathbb{Q}$, under which $R^{n,N}(t, x(t))$ is a martingale. We thus have

$$\mathbb{E}_{t,x}^{\mathbb{Q}^{n,N}} \left\{ R^{n,N}(T_n, x(T_n)) \right\} = \frac{p(t, T_n, x) - p(t, T_N, x)}{C^{n,N}(t, x)}. \tag{6.1}$$

Remark 6.1. In what follows we shall assume that

$$p(t, T_n, x) > p(t, T_N, x) \quad \textit{for all } (t, x) \in [0, T_n] \times \mathbb{R}^l.$$

This hypothesis is not restrictive at all, since (remember that $p(t, T_n, x) \geq p(t, T_N, x)$ if $r(t, x) \geq 0$) we have $p(t, T_n, x) = p(t, T_N, x)$ if and only if $r(s, x(s)) = 0$ a.e. in $[T_n, T_N]$. In other words, the above assumption is always satisfied in the real world.

We are interested in obtaining a control description for (6.1). First of all, we prove the following

Lemma 6.2. *Let $T_0, T_1, \dots, T_N$ be a set of increasing maturities and let $p(t, T_i, x)$ be the price of the zero-coupon T_i-bond at time t, for $i = 0, 1, \dots, N$. Let $k \in \{0, 1, \dots, N-1\}$ and let $P(t, x)$ be an arbitrary linear combination of bonds evaluated at time t for $x(t) = x$, i.e.,*

$$P(t, x) = \sum\nolimits_{i=k}^{N} \beta_i \cdot p(t, T_i, x),$$

for some $\beta_i \in \mathbb{R}$, $i = k, \dots, N$. Then $P(t, x)$ is the unique solution of the PDE

$$\begin{cases} \frac{\partial}{\partial t} P(t, x) + \mathbf{f}'(t, x) \nabla_x P(t, x) + \frac{1}{2} \mathrm{tr} \left(\mathbf{g}'(t, x) \nabla_{xx} P(t, x) \mathbf{g}(t, x) \right) \\ \qquad\qquad - r(t, x) P(t, x) = 0 \\ P(T_k, x) = \sum_{i=k}^{N} \beta_i \cdot p(T_k, T_i, x). \end{cases}$$

Proof. From Theorem 2.2, each function $p(t, T_i, x)$ satisfies

$$\frac{\partial}{\partial t} p(t, T_i, x) + \mathbf{f}'(t, x) \nabla_x p(t, T_i, x) + \frac{1}{2} \mathrm{tr} \left(\mathbf{g}'(t, x) \nabla_{xx} p(t, T_i, x) \mathbf{g}(t, x) \right)$$
$$- r(t, x) p(t, T_i, x) = 0.$$

Since $P(t, x)$ is a linear combination of them, it is solution of the same PDE. The choice of the boundary condition is obvious, while the uniqueness of the solution comes from Assumption 2.1. □

Using Lemma 6.2, $C^{n,N}(t,x)$ satisfies (from now on, we shall omit the superscript n, N in all the PDEs)

$$\begin{cases} \frac{\partial}{\partial t}C(t,x) + \mathbf{f}'(t,x)\nabla_x C(t,x) + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}C(t,x)\mathbf{g}(t,x)\right) \\ \qquad\qquad - r(t,x)C(t,x) = 0 \\ C(T_n,x) = \sum_{i=n+1}^{N} \alpha_i \cdot p(T_n,T_i,x). \end{cases} \tag{6.2}$$

Let us put (making explicit the dependence on the instant T_n)

$$Z^{n,N}(t,T_n,x) := -\ln C^{n,N}(t,x). \tag{6.3}$$

From (6.2), $Z^{n,N}(t,T_n,x)$ is the unique (for the same reasons as in Remark 2.3) solution of

$$\begin{cases} \frac{\partial}{\partial t}Z(t,T_n,x) + \mathbf{f}'(t,x)\nabla_x Z(t,T_n,x) \\ \qquad - \frac{1}{2}\left(\nabla_x Z\right)'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x Z(t,T_n,x) \\ \qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}Z(t,T_n,x)\mathbf{g}(t,x)\right) + r(t,x) = 0 \\ Z(T_n,T_n,x) = -\ln \sum_{i=n+1}^{N} \alpha_i \cdot p(T_n,T_i,x). \end{cases} \tag{6.4}$$

It is easy to recognize that this PDE has the same form as (3.2), except for the terminal condition. Thus one can carry out the same arguments as in the proof of Proposition 3.2 (given in [6]), observing that (6.4) can be seen as originating from the HJB equation (dropping the arguments of the functions)

$$\frac{\partial}{\partial t}Z + \inf_{u\in\mathbb{R}^k}\left\{\left[\mathbf{f}' + u'\mathbf{g}'\right]\nabla_x Z + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'\ \nabla_{xx}Z\ \mathbf{g}\right) + \frac{1}{2}u'u + r\right\} = 0,$$

whose optimal control is

$$u^*(t,x;Z^{n,N}) = -\mathbf{g}'(t,x)\nabla_x Z^{n,N}(t,T_n,x). \tag{6.5}$$

In order to guarantee admissibility for such a control law, in the sense of the Sufficient Condition for Admissibility in Section 3, we make the following

Assumption 6.3. The gradient of $Z^{n,N}(t,T_n,x)$, solution of (6.4), satisfies a linear growth condition, i.e.,

$$||\nabla_x Z^{n,N}(t,T_n,x)|| \le M(1+||x||) \quad \textit{for all } x \in \mathbb{R}^l,$$

for some constant $M > 0$, uniformly in $t \in [0,T]$.

Such a hypothesis is not too restrictive. Indeed, we have the following

Lemma 6.4. *Under Assumption* 3.3*, and supposing that* $r(t,x) \ge 0$*,* $Z^{n,N}(t,T_n,x)$ *has at most quadratic growth.*

Proof. Given two dates of maturity T_1, T_2, with $T_1 < T_2$, since the spot rate is non-negative, we have $p(t,T_1,x) \ge p(t,T_2,x)$. Using this fact, we get

$$p(t,T_N,x)\sum_{i=n+1}^{N}\alpha_i \le C^{n,N}(t,x) = \sum_{i=n+1}^{N}\alpha_i\cdot p(t,T_i,x) \le p(t,T_{n+1},x)\sum_{i=n+1}^{N}\alpha_i.$$

So, remembering (3.1), we obtain

$$-\ln \sum_{i=n+1}^{N} \alpha_i + W(t, T_{n+1}, x) \leq Z^{n,N}(t, T_n, x) \leq -\ln \sum_{i=n+1}^{N} \alpha_i + W(t, T_N, x).$$

Since Assumption 3.3 implies that $W(t, T_{n+1}, x)$ and $W(t, T_N, x)$ have at most quadratic growth, the proof is concluded. □

The function $Z^{n,N}(t, T_n, x)$ in (6.3) is now the optimal value function of the stochastic control problem

$$\begin{cases} dx(t) = [\mathbf{f}(t, x(t)) + \mathbf{g}(t, x(t))u(t)]\, dt + \mathbf{g}(t, x(t))dw_t \\ Z^{n,N}(t, T_n, x) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\{\int_t^{T_n} \left(\frac{1}{2}u'(s)u(s) + r(s, x(s))\right) ds \\ \qquad - \ln \sum_{i=n+1}^{N} \alpha_i \cdot p(T_n, T_i, x(T_n))\}, \end{cases} \tag{6.6}$$

where $\mathcal{U}$ is the class of the admissible control laws, for which the first equation in (6.6) has a unique solution in probability law and the expected cost

$$J(t, T_n, x, u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^{T_n} \left(\frac{1}{2}u'(s)u(s) + r(s, x(s))\right) ds - \ln \sum_{i=n+1}^{N} \alpha_i \cdot p(T_n, T_i, x(T_n))\right\}$$

has finite value.

Substituting (6.5) into the dynamics in (6.6), we obtain

$$\begin{aligned} dx^{n,N}(t) = &\left[\mathbf{f}(t, x^{n,N}(t)) - \mathbf{g}\mathbf{g}'(t, x^{n,N}(t))\nabla_x Z^{n,N}(t, T_n, x^{n,N}(t))\right] dt \\ &+ \mathbf{g}(t, x^{n,N}(t))dw_t. \end{aligned} \tag{6.7}$$

In order to give an idea of the significance of the dynamics (6.7), we compute the Girsanov kernel $L^{n,N}$ of the measure transformation from $\mathbb{Q}$ to the forward measure $\mathbb{Q}^{n,N}$. By using (6.4), we first have

$$\begin{aligned} dZ(t, T_n, x) &= \frac{\partial}{\partial t} Z(t, T_n, x)\ dt + \mathbf{f}'(t, x)\nabla_x Z(t, T_n, x)\ dt \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t, x)\nabla_{xx} Z(t, T_n, x)\mathbf{g}(t, x)\right) dt \\ &\quad + (\nabla_x Z)'(t, T_n, x)\mathbf{g}(t, x)\ dw_t \\ &= \left[\frac{1}{2}(\nabla_x Z)'(t, T_n, x)\mathbf{g}\mathbf{g}'(t, x)\nabla_x Z(t, T_n, x) - r(t, x)\right] dt \\ &\quad + (\nabla_x Z)'(t, T_n, x)\mathbf{g}(t, x)\ dw_t. \end{aligned}$$

Thus

$$\begin{aligned} dC(t, x) &= de^{-Z(t, T_n, x)} \\ &= -C(t, x)\ dZ(t, T_n, x) \\ &\quad + \frac{1}{2}\ C(t, x)(\nabla_x Z)'(t, T_n, x)\mathbf{g}\mathbf{g}'(t, x)\nabla_x Z(t, T_n, x)\ dt \end{aligned}$$

$$
\begin{aligned}
= & -C(t,x)\left[\frac{1}{2}(\nabla_x Z)'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x Z(t,T_n,x)-r(t,x)\right]dt \\
& -C(t,x)(\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\,dw_t \\
& +\frac{1}{2}\,C(t,x)(\nabla_x Z)'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x Z(t,T_n,x)\,dt \\
= & \,C(t,x)r(t,x)\,dt-C(t,x)(\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\,dw_t,
\end{aligned}
$$

and so, recalling (see [1]) that $L^{n,N}(t)$ is given by $L^{n,N}(t)=\frac{C(t,x)}{B(t)\,C(0,0)}$, namely by the normalized ratio of the numéraires,

$$
\begin{aligned}
dL^{n,N}(t) &= d\left(\frac{C(t,x)}{B(t)C(0,0)}\right) \\
&= \frac{dC(t,x)}{B(t)C(0,0)}+\frac{C(t,x)}{C(0,0)}\,d\left(\frac{1}{B(t)}\right) \qquad (6.8)\\
&= \frac{C(t,x)}{B(t)C(0,0)}r(t,x)\,dt-\frac{C(t,x)}{B(t)C(0,0)}(\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\,dw_t \\
&\quad -\frac{C(t,x)}{B(t)C(0,0)}r(t,x)\,dt \\
&= -L^{n,N}(t)(\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\,dw_t.
\end{aligned}
$$

Then the Girsanov kernel is exactly $-\mathbf{g}'(t,x(t))\nabla_x Z^{n,N}(t,T_n,x(t))$, i.e., the minimizer (6.5). It follows that (6.7) represents the factor process dynamics under the forward measure $\mathbb{Q}^{n,N}$. This leads us to claim that the expected value

$$
\mathbb{E}_{t,x}^{\mathbb{Q}^{n,N}}\left\{R^{n,N}(T_n,x(T_n))\right\}
$$

can be computed as an expectation with respect to the standard martingale measure $\mathbb{Q}$, assuming that the factor process evolves according to (6.7), instead of (2.1), $x(\cdot)$ and $x^{n,N}(\cdot)$ having the same initial condition x and $Z^{n,N}(t,T_n,x)$ being the solution of (6.4).

Indeed, defining the quantity

$$
\mathcal{R}^{n,N}(t,x):=\mathbb{E}_{t,x}^{\mathbb{Q}}\left\{R^{n,N}(T_n,x^{n,N}(T_n))\right\},
$$

we can prove the following

Proposition 6.5. *For $t\le T_n$, it holds*

$$
\mathbb{E}_{t,x}^{\mathbb{Q}^{n,N}}\left\{R^{n,N}(T_n,x(T_n))\right\}=\mathcal{R}^{n,N}(t,x).
$$

Proof. From (6.1), we have to show that

$$
\mathcal{R}^{n,N}(t,x)=\frac{p(t,T_n,x)-p(t,T_N,x)}{C^{n,N}(t,x)}. \qquad (6.9)
$$

Inspired by the proof of Proposition 4.4 (given in [6]), let $M(t,x) := p(t,T_n,x) - p(t,T_N,x)$. From Lemma 6.2, $M(t,x)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}M(t,x) + \mathbf{f}'(t,x)\nabla_x M(t,x) + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}M(t,x)\mathbf{g}(t,x)\right) \\ \qquad\qquad - r(t,x)M(t,x) = 0 \\ M(T_n,x) = p(T_n,T_n,x) - p(T_n,T_N,x) = 1 - p(T_n,T_N,x). \end{cases}$$

Defining (notice that, according to Remark 6.1, we may assume $M(t,x) > 0$)

$$D(t,x) := -\ln M(t,x),$$

this $D(t,x)$ is the solution of

$$\begin{cases} \frac{\partial}{\partial t}D(t,x) + \mathbf{f}'(t,x)\nabla_x D(t,x) - \frac{1}{2}\left(\nabla_x D\right)'(t,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x D(t,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}D(t,x)\mathbf{g}(t,x)\right) + r(t,x) = 0 \\ D(T_n,x) = -\ln\left(1 - p(T_n,T_N,x)\right). \end{cases} \tag{6.10}$$

Moreover, the Kolmogorov backward equation associated to $\mathcal{R}^{n,N}(t,x)$ is (we write only $\mathcal{R}$ instead of $\mathcal{R}^{n,N}$)

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{R}(t,x) + \left[\mathbf{f}'(t,x) - (\nabla_x Z)'(t,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x\mathcal{R}(t,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}\mathcal{R}(t,x)\mathbf{g}(t,x)\right) = 0 \\ \mathcal{R}(T_n,x) = \frac{1-p(T_n,T_N,x)}{\sum_{i=n+1}^{N}\alpha_i\cdot p(T_n,T_i,x)}. \end{cases}$$

Again, applying a logarithmic transform to $\mathcal{R}^{n,N}(t,x)$ and making explicit the dependence on the time-instant T_n, we put

$$W^{\mathcal{R}}(t,T_n,x) := -\ln\mathcal{R}^{n,N}(t,x).$$

The function $W^{\mathcal{R}}(t,T_n,x)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}W^{\mathcal{R}}(t,T_n,x) + \left[\mathbf{f}'(t,x) - (\nabla_x Z)'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x W^{\mathcal{R}}(t,T_n,x) \\ \qquad\qquad - \frac{1}{2}(\nabla_x W^{\mathcal{R}})'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x W^{\mathcal{R}}(t,T_n,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}W^{\mathcal{R}}(t,T_n,x)\mathbf{g}(t,x)\right) = 0 \\ W^{\mathcal{R}}(T_n,T_n,x) = -\ln\frac{1-p(T_n,T_N,x)}{\sum_{i=n+1}^{N}\alpha_i\cdot p(T_n,T_i,x)}. \end{cases} \tag{6.11}$$

In order to prove (6.9), it suffices to show that

$$\exp\left[-W^{\mathcal{R}}(t,T_n,x)\right] = \frac{\exp\left[-D(t,x)\right]}{\exp\left[-Z^{n,N}(t,T_n,x)\right]},$$

i.e., that

$$W^{\mathcal{R}}(t,T_n,x) + Z^{n,N}(t,T_n,x) = D(t,x).$$

Let

$$\tilde{W}(t,x) := W^{\mathcal{R}}(t,T_n,x) + Z^{n,N}(t,T_n,x).$$

From equations (6.11) for $W^{\mathcal{R}}(t,T_n,x)$ and (6.4) for $Z^{n,N}(t,T_n,x)$, we have

$$\begin{aligned}
-\frac{\partial}{\partial t}\tilde{W} = & -\frac{\partial}{\partial t}W^{\mathcal{R}} - \frac{\partial}{\partial t}Z = \mathbf{f}'\nabla_x W^{\mathcal{R}} - (\nabla_x Z)'\,\mathbf{g}\mathbf{g}'\nabla_x W^{\mathcal{R}} \\
& - \frac{1}{2}\left(\nabla_x W^{\mathcal{R}}\right)'\mathbf{g}\mathbf{g}'\nabla_x W^{\mathcal{R}} + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'\nabla_{xx}W^{\mathcal{R}}\mathbf{g}\right) \\
& + \mathbf{f}'\nabla_x Z - \frac{1}{2}(\nabla_x Z)'\,\mathbf{g}\mathbf{g}'\nabla_x Z + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'\nabla_{xx}Z\mathbf{g}\right) + r \\
= & \,\mathbf{f}'\left[\nabla_x W^{\mathcal{R}} + \nabla_x Z\right] + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'\left[\nabla_{xx}W^{\mathcal{R}} + \nabla_{xx}Z\right]\mathbf{g}\right) \\
& - \frac{1}{2}\left[\nabla_x W^{\mathcal{R}} + \nabla_x Z\right]'\mathbf{g}\mathbf{g}'\left[\nabla_x W^{\mathcal{R}} + \nabla_x Z\right] + r \\
= & \,\mathbf{f}'\nabla_x\tilde{W} + \frac{1}{2}\mathrm{tr}(\mathbf{g}'\nabla_{xx}\tilde{W}\mathbf{g}) - \frac{1}{2}(\nabla_x\tilde{W})'\mathbf{g}\mathbf{g}'\nabla_x\tilde{W} + r.
\end{aligned}$$

The boundary condition is

$$\begin{aligned}
\tilde{W}(T_n,x) &= W^{\mathcal{R}}(T_n,T_n,x) + Z^{n,N}(T_n,T_n,x) \\
&= -\ln\frac{1-p(T_n,T_N,x)}{\sum_{i=n+1}^{N}\alpha_i\cdot p(T_n,T_i,x)} - \ln\sum_{i=n+1}^{N}\alpha_i\cdot p(T_n,T_i,x) \\
&= -\ln\left(1-p(T_n,T_N,x)\right) \\
&= D(T_n,x).
\end{aligned}$$

Thus, $\tilde{W}(t,x)$ satisfies the same PDE as $D(t,x)$, namely the PDE in (6.10), and has the same terminal value. Since the equation in (6.10) has unique solution (for the same reasons as in Remark 2.3) and is satisfied by both $\tilde{W}(t,x)$ and $D(t,x)$, we get

$$D(t,x) = \tilde{W}(t,x) = W^{\mathcal{R}}(t,T_n,x) + Z^{n,N}(t,T_n,x) \qquad \text{for } t \le T_n. \qquad \square$$

Proposition 6.5 leads to a control interpretation for expectations on swap rates as in (6.1). As in Sections 3 and 4, in order to have also for (6.6) an admissible control problem we make the following

Assumption 6.6. The gradient of $W^{\mathcal{R}}(t,T_n,x)$, solution of (6.11), has at most linear growth, i.e.,

$$||\nabla_x W^{\mathcal{R}}(t,T_n,x)|| \le M(1+||x||) \quad \text{for all } x\in\mathbb{R}^l,$$

for some constant $M>0$, uniformly in $t\in[0,T]$.

The PDE in (6.11) can be seen as resulting from the HJB equation

$$\frac{\partial}{\partial t}W^{\mathcal{R}} + \inf_{u\in\mathbb{R}^k}\left\{\left[\mathbf{f}' - (\nabla_x Z)'\mathbf{g}\mathbf{g}' + u'\mathbf{g}'\right]\nabla_x W^{\mathcal{R}} + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'\nabla_{xx}W^{\mathcal{R}}\mathbf{g}\right) + \frac{1}{2}u'u\right\} = 0.$$

Thus, we obtain

$$\mathbb{E}_{t,x}^{\mathbb{Q}^{n,N}}\left\{R^{n,N}(T_n,x(T_n))\right\} = \exp\left[-W^{\mathcal{R}}(t,T_n,x)\right],$$

where $W^{\mathcal{R}}(t,T_n,x)$ is the optimal value function of the stochastic control problem

$$\begin{cases} dx^{n,N}(t) = [\mathbf{f}(t,x^{n,N}(t)) - \mathbf{gg}'(t,x^{n,N}(t))\nabla_x Z^{n,N}(t,T_n,x^{n,N}(t)) \\ \qquad\qquad + \mathbf{g}(t,x^{n,N}(t))u(t)]dt + \mathbf{g}(t,x^{n,N}(t))dw_t \\ W^{\mathcal{R}}(t,T_n,x) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^{T_n} \frac{1}{2}u'(s)u(s)ds - \ln \frac{1-p(T_n,T_N,x^{n,N}(T_n))}{\sum_{i=n+1}^N \alpha_i \cdot p(T_n,T_i,x^{n,N}(T_n))}\right\} \end{cases}$$

with $\mathcal{U}$ denoting again the class of the admissible control laws, for which the first equation in the control problem above has a unique solution in probability law and the expected cost

$$J(t,T_n,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^{T_n} \frac{1}{2}u'(s)u(s)ds - \ln \frac{1-p(T_n,T_N,x^{n,N}(T_n))}{\sum_{i=n+1}^N \alpha_i \cdot p(T_n,T_i,x^{n,N}(T_n))}\right\}$$

has finite value.

7. Swap measures and a general pricing formula

Analogously to what has been made in Section 5, it is possible to establish a connection between the factor process $x^{n,N}(\cdot)$ and the swap measure $\mathbb{Q}^{n,N}$; more precisely, such a process is the key element in order to calculate expectations under $\mathbb{Q}^{n,N}$ by using the standard martingale measure $\mathbb{Q}$. Indeed, we have the following

Proposition 7.1. *Let $T_0, T_1, \ldots, T_N$ be a set of increasing maturities. Fix a vector x in $\mathbb{R}^l$, $n \in \{0,1,\ldots,N-1\}$ and a time-instant t, with $0 \le t \le T_n$. Let $x(s)$, $s \in [t,T_n]$, be the process satisfying* (2.1) *with $x(t) = x$, and let $x^{n,N}(s)$, $s \in [t,T_n]$, be the process satisfying* (6.7) *with $x^{n,N}(t) = x$. Then the random variable $x(T_n)$ has the same distribution under the swap measure $\mathbb{Q}^{n,N}$ (the one with numeraire $C^{n,N}(t,x(t))$) as the random variable $x^{n,N}(T_n)$ under the standard martingale measure $\mathbb{Q}$ (the one with numeraire $B(t)$).*

Proof. The proof is analogous to the one of Proposition 4.1 in [6] and we outline here its main steps. We recall that (see (6.8))

$$dL^{n,N}(s) = -L^{n,N}(s)(\nabla_x Z)'(s,T_n,x)\mathbf{g}(s,x)\, dw_s,$$

i.e., the Girsanov kernel of the measure transformation from $\mathbb{Q}$ to $\mathbb{Q}^{n,N}$ is

$$\sigma(s) = -\mathbf{g}'(s,x(s))\nabla_x Z^{n,N}(s,T_n,x(s)).$$

Thus, the process $w_s^{Q^{n,N}}$, defined by

$$dw_s^{Q^{n,N}} = dw_s^Q + \mathbf{g}'(s,x(s))\nabla_x Z^{n,N}(s,T_n,x(s))ds, \tag{7.1}$$

is a Wiener process under $\mathbb{Q}^{n,N}$. Consider the factor process $x(\cdot)$, satisfying (2.1) under $\mathbb{Q}$. Substituting (7.1) into (2.1), we get (6.7) under $\mathbb{Q}^{n,N}$. Since $x(t) = x^{n,N}(t) = x$, the distribution is the same. □

We also deduce a pricing equation in the following analog of Proposition 5.2.

Proposition 7.2. *Given a date of maturity T_n, a T_n-claim $F(x(T_n))$, whose arbitrage-free price at time t, with $t \leq T_n$, is*

$$\begin{aligned}\pi(t) &= \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\exp\left[-\int_t^{T_n} r(s,x(s))ds\right]\cdot F(x(T_n))\right\}\\ &= C^{n,N}(t,x)\cdot \mathbb{E}^{\mathbb{Q}^{n,N}}_{t,x}\left\{\frac{F(x(T_n))}{C^{n,N}(T_n,x(T_n))}\right\},\end{aligned}$$

this price $\pi(t)$ admits a representation of the form

$$\pi(t) = \exp\left[-Z^{n,N}(t,T_n,x)\right]\cdot \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\frac{F(x^{n,N}(T_n))}{C^{n,N}(T_n,x^{n,N}(T_n))}\right\},$$

where $Z^{n,N}(t,T_n,x)$ is the unique solution of (6.4).

7.1. Control interpretation for expectations under swap measures

By proceeding exactly as in Subsection 5.1, it is possible to obtain a control description for expectations with respect to swap measures. Given a positive function $V(x)$, we put

$$\mathcal{E}(t,T_n,x) := \mathbb{E}^{\mathbb{Q}^{n,N}}_{t,x}\left\{V(x(T_n))\right\}$$

and making use of Proposition 7.1, we get

$$\mathcal{E}(t,T_n,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{V(x^{n,N}(T_n))\right\}. \tag{7.2}$$

Following the standard procedure, we write the Kolmogorov backward equation associated to (7.2), and then put

$$W^V(t,T_n,x) := -\ln\mathcal{E}(t,T_n,x).$$

The function W^V satisfies

$$\begin{cases}\frac{\partial}{\partial t}W^V(t,T_n,x) + \left[\mathbf{f}'(t,x) - (\nabla_x Z^{n,N})'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x W^V(t,T_n,x)\\ \qquad -\frac{1}{2}(\nabla_x W^V)'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x W^V(t,T_n,x)\\ \qquad +\frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}W^V(t,T_n,x)\mathbf{g}(t,x)\right) = 0\\ W^V(T_n,T_n,x) = -\ln V(x).\end{cases}$$

Our control interpretation is given by the fact that $W^V(t,T_n,x)$ can be seen also as the optimal value function of the stochastic control problem

$$\begin{cases}dx^{n,N}(t) = [\mathbf{f}(t,x^{n,N}(t)) - \mathbf{g}\mathbf{g}'(t,x^{n,N}(t))\nabla_x Z^{n,N}(t,T_n,x^{n,N}(t))\\ \qquad + \mathbf{g}(t,x^{n,N}(t))u(t)]dt + \mathbf{g}(t,x^{n,N}(t))dw_t\\ W^V(t,T_n,x) = \inf_{u(\cdot)\in\mathcal{U}}\mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^{T_n}\frac{1}{2}u'(s)u(s)ds - \ln V\left(x^{n,N}(T_n)\right)\right\}\end{cases} \tag{7.3}$$

where $\mathcal{U}$ denotes the class of the control processes for which the first equation in (7.3) has a unique solution in probability law and the expected cost $J(t,T_n,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^{T_n}\frac{1}{2}u'(s)u(s)ds - \ln V\left(x^{n,N}(T_n)\right)\right\}$ has finite value. As in Subsection 5.1, we need to assume that $V(x)$ is regular enough in order to have a well-posed stochastic control problem and an admissible optimal control law, i.e., in order for $-\ln V(x)$ to have at most polynomial growth and for the gradient of $W^V(t, T_n, x)$ to have at most linear growth.

Remark 7.3. It is easy to see that the entire argument in Sections 6 and 7 works exactly as for the forward measures. Indeed, in both cases:

- the logarithm of the numeraire is the optimal value function of a stochastic control problem (see (3.3) for forward measures and (6.6) for swap measures);
- the optimal control law $u^*(\cdot)$ coincides with the Girsanov kernel of the measure transformation from $\mathbb{Q}$ to the new martingale measure, when discussing forward prices and swap rates respectively;
- the optimally controlled factor process is distributed under $\mathbb{Q}$ as the original factor process under the equivalent martingale measure (compare Propositions 5.1 and 7.1);
- a second control problem (see (5.2) and (7.3)), obtained by further controlling the factor process, permits to give a control interpretation to all expectations.

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Affine Variance Swap Curve Models

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Abstract. This paper provides a brief overview of the stochastic modeling of variance swap curves. Focus is on affine factor models. We propose a novel drift parametrization which assures that the components of the state process can be matched with any pre-specified points on the variance swap curve. This should facilitate the empirical estimation for such stochastic models. Moreover, sufficient and yet flexible conditions that guarantee positivity of the rates are readily available. We finally discuss the relation and differences to affine yield-factor models introduced by Duffie and Kan [8]. It turns out that, in contrast to variance swap models, their yield factor representation requires imposing constraints on systems of nonlinear equations that are often not solvable in closed form.

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1. Variance swaps

Variance swap rates are becoming increasingly available over-the-counter at many different maturities. It becomes vital to design and estimate stochastic term structure models for the variance swap rates, see, e.g., Carr and Wu [3]. We first give a brief overview of the stochastic modeling of variance swap curves.

Let S denote the price process of an underlying stock index modeled on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$. We assume that $\mathbb{Q}$ is a risk-neutral pricing measure, and that S is a semimartingale of the form

$$\frac{dS_t}{S_{t-}} = r_t\,dt + \sigma_t\,dW_t + \int_{\mathbb{R}} (\mathrm{e}^x - 1)\,(\mu(dt,dx) - \nu_t(dx)\,dt)\,.$$

Here W is a standard Brownian motion, and $\mu(dt, dx)$ denotes the jump measure associated to $\log S$. That is, $\Delta \log S_t = \int_{\mathbb{R}} x\mu(dt, dx)$. The $\mathbb{Q}$-compensator $\nu_t(dx)\,dt$

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of $\mu(dt, dx)$ is assumed absolutely continuous. Finally, we have some non-negative predictable risk-free short rate r and volatility process σ, and we assume that

$$\int_0^T r_t\,dt < \infty \quad \text{a.s.,} \quad \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \sigma_t^2\,dt + \int_0^T \int_{\{|x|\geq 1\}} \mathrm{e}^x \nu_t(dx)\,dt\right] < \infty$$

for all finite T. See Jacod and Shiryaev [13] for the relevant background on semimartingales.

The annualized realized variance of S over a given time horizon $[t, t+\tau]$ is defined as the annualized quadratic variation of the log returns, which equals

$$\begin{aligned}\mathrm{RV}(t,\tau) &= \frac{1}{\tau}\left([\log S]_{t+\tau} - [\log S]_t\right)\\ &= \frac{1}{\tau}\left(\int_t^{t+\tau} \sigma_s^2\,ds + \int_t^{t+\tau}\int_{\mathbb{R}} x^2\,\mu(ds, dx)\right).\end{aligned}$$

A variance swap on S, initiated at t and with time to maturity τ, pays the difference

$$\mathrm{RV}(t,\tau) - \mathrm{VS}(t,\tau)$$

between the annualized realized variance and the variance swap rate, $\mathrm{VS}(t,\tau)$, fixed at t. By convention the variance swap rate is such that the variance swap contract has value zero at inception. Assuming further that

$$r \text{ and } [\log S] \text{ are } \mathcal{F}_t\text{-conditionally independent under } \mathbb{Q},$$

risk-neutral pricing then implies that

$$\mathrm{VS}(t,\tau) = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{RV}(t,\tau) \mid \mathcal{F}_t\right] = \frac{1}{\tau}\int_t^{t+\tau} f(t,s)\,ds$$

where we define the T-forward variance $f(t,T)$ prevailing at t as

$$f(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\sigma_T^2 + \int_{\mathbb{R}} x^2\,\nu_T(dx) \mid \mathcal{F}_t\right].$$

For $t = T$ we obtain the spot variance

$$v_t \equiv f(t,t) = \sigma_t^2 + \int_{\mathbb{R}} x^2\,\nu_t(dx). \tag{1.1}$$

Note the analogy – and difference – to the forward curve of interest rates, see, e.g., [12]. The T-forward variance $f(t,T)$ is a martingale for $t \in [0,T]$ under the risk-neutral measure $\mathbb{Q}$. In contrast, the T-forward interest rate is a martingale under the respective T-forward measure $\mathbb{Q}^T$.

The aim is now to model the stochastic evolution of the forward variance curve $f(t,T)$, or equivalently, the spot variance v_t. Any semimartingale price process S whose characteristics satisfy the consistency condition (1.1) is then compatible to this variance swap model. Related literature where this program has been carried out includes Buehler [2], Egloff et al. [9], Cont and Kokholm [6], and Filipović et al. [10].

Throughout the paper, we denote by $\mathbf{1} = (1, \ldots, 1)^\top$ the vector with all entries equal to 1, and we shall follow the convention

$$\frac{e^s - 1}{s} = \frac{s}{e^s - 1} = 1 \quad \text{for } s = 0. \tag{1.2}$$

2. The VIX formula

Before we proceed with modeling the variance swap curve, we derive in this auxiliary section an alternative valuation formula for variance swap rates based on a replication argument. This formula underlies the computation of the VIX (Chicago Board of Options Exchange Volatility Index), which is defined as variance swap on the S&P 500 index with a time to maturity of 30 days.

We start with an elementary identity.

Lemma 2.1. *For reals $a, b > 0$ we have*

$$\int_0^a \frac{1}{K^2}(K - b)^+ \, dK + \int_a^\infty \frac{1}{K^2}(b - K)^+ \, dK = \log(a) - \log(b) + \frac{1}{a}(b - a). \tag{2.1}$$

We now assume that the futures contract on S with maturity T is traded and continuously marked to market. Moreover, in this section we assume that

interest rates r_t are deterministic.

We denote by $P(t, T) = e^{-\int_t^T r_s ds}$ the discount bonds, and the futures price process by $F_t = S_t / P(t, T)$. Then

$$d \log F_t = \frac{dF_t}{F_{t-}} - \frac{1}{2}\sigma_t^2 \, dt - \int_{\mathbb{R}} (e^x - 1 - x) \, \mu(dt, dx).$$

Hence the realized variance over the time interval $[t, T]$, with time to maturity $\tau = T - t$, can be written as

$$\begin{aligned}
\tau \mathrm{RV}(t, \tau) &= \int_t^T \sigma_s^2 \, ds + \int_t^T \int_{\mathbb{R}} x^2 \, \mu(ds, dx) \\
&= 2(\log(F_t) - \log(F_T)) + 2\int_t^T \frac{dF_s}{F_{s-}} - 2\int_t^T\!\!\int_{\mathbb{R}} \left(e^x - 1 - x - \frac{x^2}{2}\right) \mu(ds, dx) \\
&= 2\left(\int_0^{F_t} \frac{1}{K^2}(K - S_T)^+ \, dK + \int_{F_t}^\infty \frac{1}{K^2}(S_T - K)^+ \, dK\right) \\
&\quad + 2\int_t^T \left(\frac{1}{F_{s-}} - \frac{1}{F_t}\right) dF_s - 2\int_t^T \int_{\mathbb{R}} \left(e^x - 1 - x - \frac{x^2}{2}\right) \mu(ds, dx).
\end{aligned}$$

In the last equality we used (2.1) and the fact that $F_T = S_T$. We thus obtained a replication strategy for the realized variance modulo some error term due to the jumps of S: holding static a basket of European options and trading dynamically in the futures contract by holding $2\left(\frac{1}{F_{s-}}-\frac{1}{F_t}\right)$ futures at time s. Taking $\mathbb{Q}$-expectation gives for the variance swap rate

$$\mathrm{VS}(t,\tau)=\mathrm{VIX}(t)+\epsilon$$

with the VIX rate

$$\mathrm{VIX}(t)=\frac{2}{\tau}\int_0^\infty \frac{\Theta_t(K,t+\tau)}{P(t,t+\tau)K^2}\,dK$$

and error term

$$\epsilon=-\frac{2}{\tau}\mathbb{E}_{\mathbb{Q}}\left[\int_t^{t+\tau}\int_{\mathbb{R}}\left(\mathrm{e}^x-1-x-\frac{x^2}{2}\right)\nu(dx)ds \mid \mathcal{F}_t\right]$$

where $\Theta_t(K,t+\tau)$ denotes the price of an out-of-the-money European option with strike price K and maturity $t+\tau$ (a call option when $K>F_t$ and a put option when $K\le F_t$). The error term ϵ only appears when S has jumps. It is typically non-negative, since log-returns are negatively skewed. Hence the VIX is lower biased since it neglects jump risk premium. This model-free valuation has been derived in increasing order of generality by Britten–Jones and Neuberger [1], Jiang and Tian [14], Carr and Wu [3].

3. Variance swap rate factor models

We now introduce a – possibly non-Markovian – multi-factor model for the variance swap term structure. We let X be a semimartingale state process with values in $\mathbb{R}^m$, solving the stochastic differential equation

$$dX_t=\kappa\,(\theta-X_t)\,dt+dM_t \tag{3.1}$$

for some parameters $\kappa\in\mathbb{R}^{m\times m}$, $\theta\in\mathbb{R}^m$, and where M is a multivariate martingale, which can possibly also depend on other "unspanned stochastic volatility" factors.[1] We further assume that the spot variance is an affine function of the state variable:

$$v_t=\phi_0+\psi_0^\top X_t \tag{3.2}$$

for some parameters $\phi_0\in\mathbb{R}$ and $\psi_0\in\mathbb{R}^m$. Under these assumptions, it follows that the term structure of annualized variance swap rates becomes affine in X_t:

$$\begin{aligned}\mathrm{VS}(t,\tau)&=\frac{1}{\tau}\int_0^\tau \mathbb{E}_{\mathbb{Q}}\left[v_{t+s}\mid\mathcal{F}_t\right]ds=\frac{1}{\tau}\int_0^\tau\left(\phi(s)+\psi(s)^\top X_t\right)ds\\&=\frac{\Phi(\tau)}{\tau}+\frac{\Psi(\tau)^\top}{\tau}X_t\end{aligned} \tag{3.3}$$

[1]The reader is referred to Collin–Dufresne and Goldstein [5] for the definition of the concept of unspanned stochastic volatility.

where ϕ and ψ are given by

$$\phi(\tau) = \phi_0 + \int_0^\tau \theta^\top \kappa^\top \psi(s)\, ds = \phi_0 + \theta^\top \psi_0 - \theta^\top \psi(\tau)$$
$$\psi(\tau) = \mathrm{e}^{-\kappa^\top \tau}\psi_0,$$

and we denote their integrals by

$$\begin{aligned}\Phi(\tau) &= \int_0^\tau \phi(s)\, ds = \phi_0\tau + \theta^\top \psi_0 \tau - \theta^\top \Psi(\tau) \\ \Psi(\tau) &= \int_0^\tau \psi(s)\, ds.\end{aligned} \tag{3.4}$$

Indeed, for any $T > 0$ it is easily seen that $N_t = \phi(T-t) + \psi(T-t)^\top X_t$ is a martingale with $N_T = v_T$. Hence $\phi(T-t) + \psi(T-t)^\top X_t = \mathbb{E}_\mathbb{Q}[v_T \mid \mathcal{F}_t]$, and (3.3) follows for $T = t + s$, as desired.

Note that the constituents of the term structure, i.e., the functions ϕ and ψ, only depend on the drift parameters κ and θ, and the spot variance parameters ϕ_0 and ψ_0. The aim is to find a specification so that the factors X_t match pre-specified points on the variance swap curve:

$$X_{it} = \mathrm{VS}(t, \tau_i), \quad i = 1, \dots, m \tag{3.5}$$

for some fixed maturities $0 \le \tau_1 < \cdots < \tau_m$. In view of (3.3) it follows that property (3.5) holds if and only if[2]

$$\frac{\Phi(\tau_i)}{\tau_i} = 0 \quad \text{and} \quad \frac{\Psi(\tau_i)}{\tau_i} = \mathbf{e_i}, \quad i = 1, \dots, m, \tag{3.6}$$

where $\mathbf{e_i}$ denotes the ith standard basis vector in $\mathbb{R}^m$, and where for $\tau = 0$ we set $\Phi(\tau)/\tau = \phi_0$ and $\Psi(\tau)/\tau = \psi_0$. Following Duffie and Kan's [8] terminology for yield curve models, we call a factor model satisfying (3.3) and (3.5) an affine *variance swap rate factor model.*

We shall now derive conditions on κ, θ, ϕ_0, and ψ_0 for (3.6) to hold under the standing assumption that

$$\kappa \text{ is diagonalizable.} \tag{3.7}$$

That is, there exists an invertible real $m \times m$-matrix S whose columns are linearly independent eigenvectors of κ, and such that

$$\kappa = SLS^{-1} \text{ for the diagonal matrix } L = \mathrm{diag}(\lambda_1, \dots, \lambda_m) \tag{3.8}$$

consisting of the real eigenvalues λ_i of κ. Here is our main result.

[2]Here we make the mild assumption that the support of the random variable X_t contains an open set in $\mathbb{R}^m$.

Theorem 3.1. *The matching condition* (3.5) *holds if and only if the eigenvalues* λ_i *of* κ *are mutually different and there exists some* $\ell \in \mathbb{R}$ *such that*

$$\theta = \ell \mathbf{1} \tag{3.9}$$

$$\phi_0 = \ell \left(1 - \mathbf{1}^\top \psi_0\right) \tag{3.10}$$

$$\psi_0 = \left(S^{-1}\right)^\top \mathbf{1} \tag{3.11}$$

$$S = \left(\mathbf{w_1} \mid \cdots \mid \mathbf{w_m}\right)^\top \tag{3.12}$$

with $\mathbf{w_i}$ *given by*

$$\mathbf{w_i} = (L\tau_i)^{-1} \left(I - \mathrm{e}^{-L\tau_i}\right) \mathbf{1}, \quad i = 1, \ldots, m. \tag{3.13}$$

In particular, for the boundary case $\tau_1 = 0$ *where* $X_{1t} = \mathrm{VS}(t, 0) = v_t$ *is the spot variance, we have* $\phi_0 = 0$ *and* $\psi_0 = \mathbf{e_1}$.

Moreover, the above parameters κ, θ, ϕ_0, ψ_0, *and thus* Φ *and* Ψ, *are invariant with respect to a permutation of the eigenvalues* λ_i.

It is worth noting that (3.9) implies that all components of X_t, whence all benchmark variance swap rates $\mathrm{VS}(t, \tau_i)$, mean-revert to the same level ℓ. While this may seem to be a stringent statistical restriction on the model, we must keep in mind that this property holds under the risk-neutral pricing measure $\mathbb{Q}$, which is different from the objective probability measure $\mathbb{P}$ in general. The mean-reversion levels of the variance swap rates $\mathrm{VS}(t, \tau_i)$ under $\mathbb{P}$ can thus still be mutually different.

Before proving Theorem 3.1, we state an immediate corollary which is most useful from a modeler point of view.

Corollary 3.2. *For any choice of real parameters* ℓ *and* $\lambda_1 < \cdots < \lambda_m$ *there exists a unique set of parameters* $\phi_0 \in \mathbb{R}$, $\psi_0 \in \mathbb{R}^m$, $\theta \in \mathbb{R}^m$, *and* $\kappa \in \mathbb{R}^{m \times m}$ *in the class of* (3.7) *such that the matching condition* (3.5) *holds. More specifically, they are explicitly given by* (3.8)–(3.13).

Moreover, this parametrization is exhaustive in the sense that it is in a one-to-one relation with every drift and spot variance specification in (3.1)–(3.3) *in the class of* (3.7) *such that the matching condition* (3.5) *holds.*

Note that the order of the eigenvalues λ_i matters if the martingale part M of X depends on them, see Remark 4.1 below. It is reasonable to restrict to non-negative eigenvalues λ_i since then, and only then, the variance swap curve (3.3) is bounded as a function of $\tau \in [0, \infty)$.

Proof of Theorem 3.1. To simplify notation, we shall write $\kappa^\top = \mathcal{S} L \mathcal{S}^{-1}$ with $\mathcal{S} = \left(S^{-1}\right)^\top$ in what follows. Using the convention (1.2), we then obtain the explicit expressions

$$\psi(\tau) = \mathcal{S}\, \mathrm{e}^{-L\tau}\, \mathcal{S}^{-1} \psi_0$$

$$\Psi(\tau) = \mathcal{S} L^{-1} \left(I - \mathrm{e}^{-L\tau}\right) \mathcal{S}^{-1} \psi_0.$$

Combining this with (3.4), we infer that property (3.6) is equivalent to the following conditions: for all $i = 1, \dots, m$,

$$\phi_0 + \theta^\top \psi_0 - \theta^\top \mathbf{e_i} = 0 \tag{3.14}$$

$$\mathcal{S}(L\tau_i)^{-1} \left(I - \mathrm{e}^{-L\tau_i}\right) \mathcal{S}^{-1}\psi_0 = \mathbf{e_i}. \tag{3.15}$$

We first note that (3.14) holds if and only if all components $\theta_i = \theta^\top \mathbf{e_i} = \phi_0 + \theta^\top \psi_0$ of θ are identical. Hence (3.14) holds if and only if $\theta = \ell\mathbf{1}$ for some $\ell \in \mathbb{R}$ such that $\ell\left(1 - \mathbf{1}^\top \psi_0\right) = \phi_0$, which proves (3.9) and (3.10).

Next we denote the linearly independent column vectors of $\mathcal{S}^{-1}$ by $\mathbf{w_i} = \mathcal{S}^{-1}\mathbf{e_i}$. Then, using convention (1.2) again, condition (3.15) can be rewritten as

$$\left(I - \mathrm{e}^{-L\tau_i}\right)^{-1} (L\tau_i)\, \mathbf{w_i} \equiv \mathcal{S}^{-1}\psi_0 = \sum_{j=1}^{m} \psi_{0j} \mathbf{w_j}, \quad i = 1, \dots, m. \tag{3.16}$$

We claim that all components of the vector $\mathcal{S}^{-1}\psi_0$ must be nonzero. Indeed, suppose the kth component of $\mathcal{S}^{-1}\psi_0$ were zero. Then (3.16) implies that $\mathbf{w_i}_k = 0$ for all i. But then $\mathcal{S}^{-1}$ cannot be invertible, which is absurd. Now suppose there are m linearly independent vectors $\mathbf{w_i}$ which satisfy (3.16), and let D be any invertible diagonal matrix. It then follows by inspection that (3.16) also holds for $\tilde{\mathbf{w}}_\mathbf{i} = D\mathbf{w_i}$ in lieu of $\mathbf{w_i}$. Consequently, after some appropriate transformation if necessary, we can assume that $\mathcal{S}^{-1}\psi_0 = \mathbf{1}$ without loss of generality. We thus have shown that (3.15) holds if and only if $\mathbf{w_i}$ is of the form (3.13) and $\psi_0 = \mathcal{S}\,\mathbf{1}$, where $\mathcal{S}^{-1} = (\mathbf{w_1} \mid \cdots \mid \mathbf{w_m})$, which is (3.11) and (3.12).

Finally, we note that the vectors $\mathbf{w_i}$ defined by (3.13) are linearly independent if and only if the eigenvalues λ_i of κ are mutually different. We also note that $\tau_1 = 0$ implies $\mathbf{w_1} = \mathbf{1}$ and thus $\psi_0 = \mathbf{e_1}$ by linear independence of the vectors $\mathbf{w_i}$, see (3.16). The invariance of κ, θ, ϕ_0, and ψ_0 with respect to a permutation of the eigenvalues λ_i follows by inspection. Thus the theorem is proved. □

4. Non-negative variance swap rates

Theorem 3.1 does not assert nonnegativity of the implied variance swap curve (3.3). Negative variance swap rates are clearly non-desirable. In this section we give sufficient conditions on the specification (3.8)–(3.13) to produce non-negative swap rates. We first observe that the variance swap rates $\mathrm{VS}(t,\tau)$ remain non-negative for all t and τ if and only if the spot variance v_t is non-negative for all t. Moreover, from (3.10) and (3.11) it follows that

$$v_t = \phi_0 + \psi_0^\top X_t = \mathbf{1}^\top S^{-1} \left(X_t - \ell\mathbf{1}\right) + \ell. \tag{4.1}$$

On the other hand, in view of (3.8) and (3.9), the stochastic differential equation for X reads

$$dX_t = SLS^{-1} \left(\ell\mathbf{1} - X_t\right) dt + dM_t. \tag{4.2}$$

This suggests to look at the transformed state process

$$Z_t = S^{-1}(X_t - \ell\mathbf{1}) + \ell\pi,$$

for some fixed $\pi \in [0,1]^m$ such that $\mathbf{1}^\top \pi = 1$. It satisfies the stochastic differential equation

$$dZ_t = S^{-1} dX_t = L(\ell\pi - Z_t)\,dt + S^{-1} dM_t.$$

Note that the drift of Z_t is fully decoupled. Sufficient conditions on the parameters ℓ and L, and the martingale part M_t such that Z_t takes values in the positive orthant $\mathbb{R}^m_+$ are well known, see, e.g., [7, Theorem A.5] or [12, Section 10.7.1]. This in turn implies a non-negative spot variance, which expressed in terms of Z_t reads

$$v_t = \mathbf{1}^\top Z_t.$$

The recipe for constructing non-negative variance swap rate factor models now reads as follows:

(i) Fix some $\ell \geq 0$ and m mutually distinct eigenvalues $\lambda_1, \ldots, \lambda_m \geq 0$. Define L and S as in (3.8) and Theorem 3.1.

(ii) Let Z_t be a jump-diffusion process with state space $\mathbb{R}^m_+$ of the form

$$dZ_t = L(\ell\pi - Z_t)\,dt + dN_t$$

for some suitably specified martingale N_t. For example, for the diffusion case we would set

$$dN_t = \Sigma(Z_t)\,dW_t$$

where W_t is an m-dimensional Brownian motion, and $\Sigma(z)$ is a $\mathbb{R}^{m\times m}$-valued dispersion function such that for any boundary point $z \in \partial\mathbb{R}^m_+$ the orthogonal diffusion components vanish:

$$\Sigma(z)^\top \mathbf{e_i} = 0 \quad \text{if } z_i = 0. \tag{4.3}$$

Indeed, such a specification asserts that the diffusion Z_t is $\mathbb{R}^m_+$-valued, see [12, Lemma 10.11].

(iii) Then $v_t = \mathbf{1}^\top Z_t$ defines a non-negative variance swap process with matched points on the curve

$$X_{it} = \mathrm{VS}(t, \tau_i)$$

for the transformed state process $X_t = S(Z_t - \ell\pi) + \ell\mathbf{1}$. The jump-diffusion X_t a fortiori takes values in $\mathbb{R}^m_+$ and is characterized by its stochastic differential equation (4.2) with martingale part $dM_t = S dN_t$. The spot variance can be expressed in terms of X_t by (4.1). For the above diffusion example we would have

$$dM_t = S\Sigma\left(S^{-1}(X_t - \ell\mathbf{1}) + \ell\pi\right) dW_t \tag{4.4}$$

where Σ is any dispersion function satisfying the invariance condition (4.3).

Remark 4.1. Note that, while the spot variance, ϕ_0, ψ_0, and drift specification, κ, θ, of X_t is invariant with respect to a permutation of the eigenvalues λ_i, the martingale specification (4.4) is not. In other words, the parameters λ_i also influence the martingale part of X_t, in order to assert the nonnegativity of the variance swap rates. Hence the order of the eigenvalues λ_i matters for the full specification of X_t.

5. Numerical example

We shall now consider a simple numerical example. A more detailed analysis and empirical study of variance swap rate factor models is given in [11].

We let $m = 2$. Following up the above recipe, we let Z_t be an $\mathbb{R}^2_+$-valued affine diffusion of the form

$$dZ_t = L\,(\ell\pi - Z_t)\,dt + \Sigma(Z_t)\,dW_t$$

with $\Sigma(z) = \operatorname{diag}\left(\sigma_1\sqrt{z_1}, \sigma_2\sqrt{z_2}\right)$, for some positive parameters σ_1, σ_2. The invariance condition (4.3) is obviously satisfied. Indeed, Z_t is well defined and $\mathbb{R}^2_+$-valued for any choice of $\ell \geq 0$ and mutually distinct eigenvalues $\lambda_1, \lambda_2 \geq 0$, see, e.g., [12, Theorem 10.8]. For illustration, we now fix the following parameter values

$$\pi = (1/2, 1/2)^\top, \quad \ell = 0.2, \quad \lambda_1 = 1, \quad \lambda_2 = 3.$$

Moreover, we fix the maturity dates $\tau_1 = 1/4$ and $\tau_2 = 1$, where time is measured in years. Figure 1 shows the resulting basis functions $\Phi(\tau)/\tau$, $\Psi_1(\tau)/\tau$, and $\Psi_2(\tau)/\tau$ in the term structure equation (3.3). The matching conditions (3.6) can be verified by inspection.

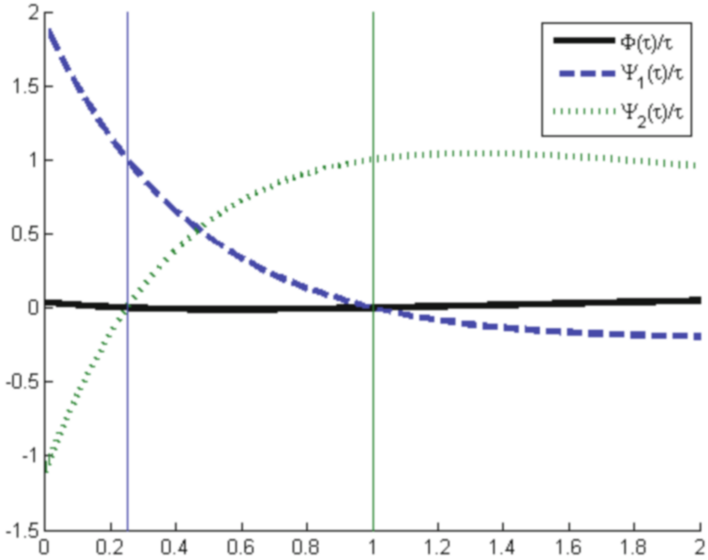

Figure 1. The term structure basis functions $\Phi(\tau)/\tau$, $\Psi_1(\tau)/\tau$, and $\Psi_2(\tau)/\tau$. The vertical lines indicate $\tau_1 = 1/4$ and $\tau_2 = 1$. Inspection shows that the matching conditions (3.6) hold.

In line with (4.2) and (4.4), the corresponding state process $X_t = S(Z_t - \ell\pi) + \ell\mathbf{1}$ is characterized by the stochastic differential equation

$$dX_t = \begin{pmatrix} 6.4 & -7.6 \\ 2.4 & -2.4 \end{pmatrix} \left(\begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix} - X_t \right) dt + \Sigma_X(X_t)\, dW_t$$

with dispersion function $\Sigma_X(x)$ given by

$$\Sigma_X(x) = \begin{pmatrix} 0.9\sigma_1\sqrt{-1.9x_1 + 4.3x_2 - 0.4} & 0.7\sigma_2\sqrt{3.8x_1 - 5.4x_2 + 0.4} \\ 0.6\sigma_1\sqrt{-1.9x_1 + 4.3x_2 - 0.4} & 0.3\sigma_2\sqrt{3.8x_1 - 5.4x_2 + 0.4} \end{pmatrix}.$$

For the following illustration we set $\sigma_1 = \sigma_2 = 0.2$, and $Z_0 = \ell\pi$. Figures 2 and 3 show a common $\mathbb{Q}$-trajectory of the vector-valued processes Z_t and X_t, respectively. While Figure 2 gives evidence of the independence of the two components of Z_t, Figure 3 illustrates that the components of X_t, that is, the variance swap rates $\mathrm{VS}(t,\tau_1)$ and $\mathrm{VS}(t,\tau_2)$, are correlated.

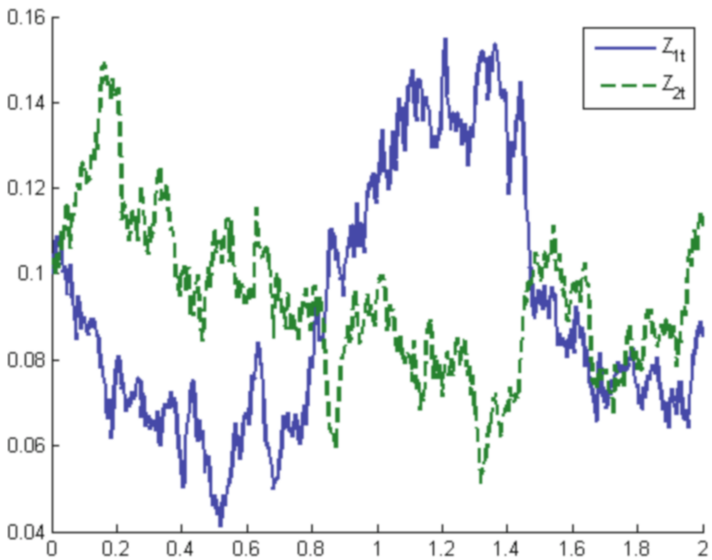

FIGURE 2. A trajectory of the process Z_t, whose components are independent by definition.

6. Comparison to affine yield-factor models

In their seminal paper, Duffie and Kan [8] introduce the class of affine factor models for the term structure of interest rates. Generically, such an affine term structure model is first written in terms of some latent diffusion state vector X_t.

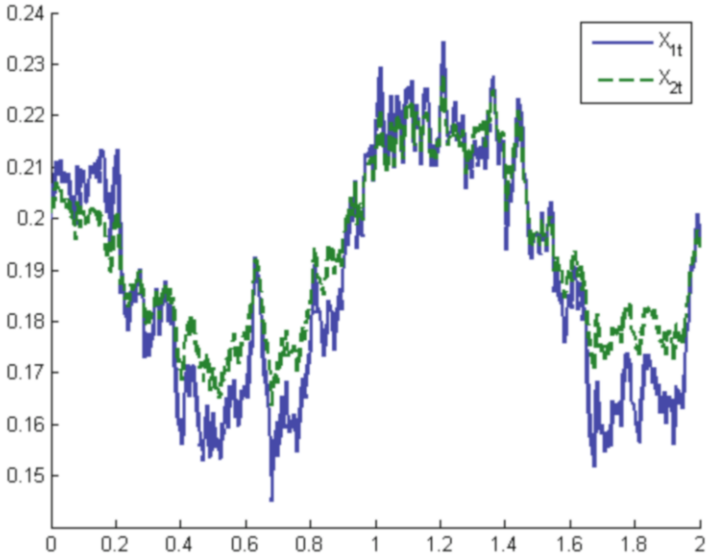

FIGURE 3. The corresponding trajectory of the state process X_t, whose components are the variance swap rates $\mathrm{VS}(t, \tau_1)$ and $\mathrm{VS}(t, \tau_2)$, which are obviously correlated.

That is, the yield curve is an affine function of X_t,

$$y(t, \tau) = \frac{A(\tau)}{\tau} + \frac{B(\tau)^\top}{\tau} X_t, \tag{6.1}$$

for some deterministic functions $A(\tau)$ and $B(\tau)$, which are given as solutions to some system of non-linear (Riccati) ODEs determined by the characteristics of X. The diffusion X in turn is necessarily an affine process, meaning that its drift and diffusion matrix are affine functions in the state X_t. Duffie and Kan [8] then also impose matching conditions similar to (3.5), such that

$$y(t, \tau_i) = X_{it}, \quad i = 1, \dots, m \tag{6.2}$$

for some fixed maturities $0 \le \tau_1 < \cdots < \tau_m$. As above, this is equivalent[3] to

$$\frac{A(\tau_i)}{\tau_i} = 0 \quad \text{and} \quad \frac{B(\tau_i)}{\tau_i} = \mathbf{e_i}, \quad i = 1, \dots, m.$$

Since here A and B solve non-linear ODEs, it is much more difficult to find an a priori parametrization, such as in Theorem 3.1 for the variance swap curves, that are consistent with the matching condition (6.2). Indeed, Duffie and Kan [8] state that they "do not have theoretical results describing how certain coefficients can be fixed in advance so as to achieve consistency with [the matching condition (6.2)]."

[3]Under the mild assumption that the support of the random variable X_t contains an open set in $\mathbb{R}^m$.

Instead, they propose the practical but indirect solution of first specifying an arbitrary affine factor model (6.1) with latent state vector X_t. In a second step they change the variables via the affine transformation

$$Y_{it} = \frac{A(\tau_i)}{\tau_i} + \frac{B(\tau_i)^\top}{\tau_i} X_t.$$

Provided the $m \times m$-matrix K with ith row vector given by $B(\tau_i)^\top/\tau_i$ is non-singular, this change of variables is possible. The process Y is an affine diffusion and the yield curve becomes affine in Y_t of the form

$$y(t,\tau) = \frac{A^*(\tau)}{\tau} + \frac{B^*(\tau)^\top}{\tau} Y_t,$$

for $A^*(\tau) = A(\tau) + B(\tau)^\top K^{-1} k$ and $B^*(\tau)^\top = B(\tau)^\top K^{-1}$, where $k_i = A(\tau_i)/\tau_i$. By construction, the matching condition (6.2) holds in these new coordinates: $y(t,\tau_i) = Y_{it}$. However, the characteristics of Y are now given in terms of K, which again depends via a solution of the non-linear Riccati equations on the original parameters of X. Apart from simple two-factor models, this approach is often difficult to implement and therefore has not been widely used, see also Collin-Dufresne et al. [4].

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Efficient Second-order Weak Scheme for Stochastic Volatility Models

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Abstract. Stochastic volatility models can be seen as a particular family of two-dimensional stochastic differential equations (SDE) in which the volatility process follows an autonomous one-dimensional SDE. We take advantage of this structure to propose an efficient discretization scheme with order two of weak convergence. We prove that the order two holds for the asset price and not only for the log-asset as usually found in the literature. Numerical experiments confirm our theoretical result and we show the superiority of our scheme compared to the Euler scheme, with or without Romberg extrapolation.

Mathematics Subject Classification (2010). 60H35, 65C30, 91G60.

Keywords. Discretization schemes, weak convergence, Lamperti transform, stochastic volatility models.

1. Introduction

The limitations of the Black & Scholes framework are widely accepted, especially the constant volatility assumption. The use of stochastic volatility models instead is now standard market practice but, unfortunately, there are only few cases where exact option pricing formulae exist for the pricing of simple European options. One has to resort to semi-analytical formulae or to discretization schemes combined with Monte Carlo approximations of expectations to price options. The aim of this article is to propose a simple and yet competitive discretization scheme of stochastic volatility models which has a second-order weak convergence property, much more efficient than the famous Euler scheme with order one of weak convergence commonly used by practitioners.

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In our study, we will consider the following specification which nests almost all the known stochastic volatility models:

$$\begin{cases} dS_t = rS_t dt + f(Y_t)S_t\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right); \quad S_0 = s_0 > 0 \\ dY_t = b(Y_t)dt + \sigma(Y_t)dW_t; \quad Y_0 = y_0 \end{cases} \tag{1.1}$$

where $(S_t)_{t\in[0,T]}$ is the asset price, r the instantaneous interest rate, $(B_t)_{t\in[0,T]}$ and $(W_t)_{t\in[0,T]}$ are independent standard one-dimensional Brownian motions, $\rho \in [-1,1]$ is the correlation between the Brownian motions respectively driving the asset price and the process $(Y_t)_{t\in[0,T]}$ which solves a one-dimensional autonomous stochastic differential equation. The volatility process is $(f(Y_t))_{t\in[0,T]}$ where the transformation function f is usually taken positive and strictly monotonic in order to ensure that the effective correlation between the stock price and the volatility keeps the same sign (the function σ usually takes non-negative values).

There exists an extensive literature on numerical integration of stochastic differential equations by the Euler scheme (see for example [1, 2] and [3]). More recently, many discretization schemes of higher order of weak convergence have appeared in the literature. Among others, we cite the work of Kusuoka [4, 5], the Ninomiya and Victoir [6] scheme which we will use hereafter, the Ninomiya and Ninomiya [7] scheme and the cubature on Wiener space by Lyons and Victoir [8].

But, with the exception of the Heston model ($f(y) = \sqrt{y}$, $b(y) = \kappa(\theta - y)$ and $\sigma(y) = \nu\sqrt{y}$) where both f and σ are singular, the development of specific discretization schemes for stochastic volatility models has only received little attention. In the present paper, we assume that the functions f, σ and b are smooth which means that we do not deal with the Heston model.

In [16], we take advantage of the structure of (1.1) to construct and analyze simple and robust ad'hoc discretization schemes which have nice convergence properties. We introduce a scheme devoted to the pricing of path-dependent options and prove that the Wasserstein distance between its law on the discretization grid and the law of the solution to (1.1) on the grid converges with order one, i.e., like a constant multiplied by the time-step. We also propose a scheme devoted to the pricing of vanilla options with potential order two of weak convergence and confirm this presumed order by numerical experiments. The aim of the present paper is to actually prove this order two property.

To exhibit this scheme, we perform a logarithmic change of variables for the asset: the two-dimensional process $(X_t := \log(S_t), Y_t)_{t\in[0,T]}$ solves the following SDE

$$\begin{cases} dX_t = \left(r - \frac{1}{2}f^2(Y_t)\right)dt + f(Y_t)\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right); \, X_0 = \log(s_0). \\ dY_t = b(Y_t)dt + \sigma(Y_t)dW_t; \, Y_0 = y_0 \end{cases} \tag{1.2}$$

Our main idea is to get rid in the first equality of the stochastic integral involving the common Brownian motion $(W_t)_{t\in[0,T]}$.

In all what follows, we assume that

($\mathcal{A}$) f and σ are $\mathcal{C}^1$ functions and $\sigma > 0$.

One can then define the primitive $F(y) = \int_0^y \frac{f}{\sigma}(z)dz$ and apply Itô's formula to get

$$dF(Y_t) = \frac{f}{\sigma}(Y_t)dY_t + \frac{1}{2}(\sigma f' - f\sigma')(Y_t)dt.$$

Therefore $(X_t, Y_t)_{t\in[0,T]}$ solves

$$\begin{cases} dX_t = \rho dF(Y_t) + h(Y_t)dt + \sqrt{1-\rho^2} f(Y_t)dB_t \\ dY_t = b(Y_t)dt + \sigma(Y_t)dW_t \end{cases}$$

where $h : y \mapsto r - \frac{1}{2}f^2(y) - \rho(\frac{b}{\sigma}f + \frac{1}{2}(\sigma f' - f\sigma'))(y)$.

Integrating the stochastic differential equation (1.2) gives

$$X_t = \log(s_0) + \rho(F(Y_t) - F(y_0)) + \int_0^t h(Y_s)ds + \sqrt{1-\rho^2}\int_0^t f(Y_s)dB_s.$$

We are only left with an integral with respect to time which can be handled by the use of a trapezoidal scheme and a stochastic integral where the integrand is independent of the Brownian motion. Hence, conditionally on $(Y_t)_{t\in[0,T]}$,

$$X_T \sim \mathcal{N}\left(\log(s_0) + \rho(F(Y_T) - F(y_0)) + m_T \,,\, (1-\rho^2)v_T\right)$$

where $m_T = \int_0^T h(Y_s)ds$ and $v_T = \int_0^T f^2(Y_s)ds$. This suggests that, in order to properly approximate the law of X_T, one should accurately approximate the law of Y_T and carefully handle integrals with respect to time of functions of the process $(Y_t)_{t\in[0,T]}$. We thus define our weak scheme as follows

Weak_2 scheme

$$\overline{X}_T^N = \log(s_0) + \rho(F(\overline{Y}_T^N) - F(y_0)) + \overline{m}_T^N + \sqrt{(1-\rho^2)\overline{v}_T^N}\,G$$

where G is a normal random variable and $(\overline{m}_T^N, \overline{v}_T^N)$ is computed independently using the Ninomiya–Victoir scheme $(\overline{Y}_{t_k}^N)_{0\leq k\leq N}$ applied to $(Y_t)_{t\in[0,T]}$ on the grid $\left(t_k = \frac{kT}{N}\right)_{0\leq k\leq N}$:

$$\overline{m}_T^N = \frac{T}{N}\sum_{k=0}^{N-1}\frac{h(\overline{Y}_{t_k}^N) + h(\overline{Y}_{t_{k+1}}^N)}{2} \quad \text{and} \quad \overline{v}_T^N = \frac{T}{N}\sum_{k=0}^{N-1}\frac{f^2(\overline{Y}_{t_k}^N) + f^2(\overline{Y}_{t_{k+1}}^N)}{2}.$$

Note that, conditionally on $(\overline{Y}_{t_k}^N)_{0\leq k\leq N}$, $\overline{X}_t^N$ is also a Gaussian random variable with mean $\log(s_0) + \rho(F(\overline{Y}_T^N) - F(y_0)) + \overline{m}_T^N$ and variance $(1-\rho^2)\overline{v}_T^N$.

It is well known that the Ninomiya and Victoir [6] scheme is of weak order two. For the sake of completeness, we give its definition in our setting:

$$\begin{cases} \overline{Y}_0^N = y_0 \\ \forall 0 \le k \le N-1,\ \overline{Y}_{t_{k+1}}^N = e^{\frac{T}{2N}V_0} e^{(W_{t_{k+1}}-W_{t_k})V} e^{\frac{T}{2N}V_0}\left(\overline{Y}_{t_k}^N\right) \end{cases}$$

where

$$V_0 : x \mapsto b(x) - \frac{1}{2}\sigma\sigma'(x) \quad \text{and} \quad V : x \mapsto \sigma(x). \tag{1.3}$$

The notation $e^{tV}(x)$ stands for the solution, at time t and starting from x, of the ODE $\eta'(t) = V(\eta(t))$. What is nice with our setting is that we need to integrate only one-dimensional ODEs which can be solved explicitly. Indeed, if ζ is a primitive of $\frac{1}{V}$: $\zeta(t) = \int_0^t \frac{1}{V(s)} ds$, then the solution writes as $\eta(t) = \zeta^{-1}(t + \zeta(x))$.

Note that our scheme can be seen as a splitting scheme for the SDE satisfied by $(Z_t = X_t - \rho F(Y_t), Y_t)$:

$$\begin{cases} dZ_t = h(Y_t)dt + \sqrt{1-\rho^2} f(Y_t) dB_t \\ dY_t = b(Y_t)dt + \sigma(Y_t)dW_t\,. \end{cases} \tag{1.4}$$

The differential operator associated to (1.4) writes as

$$\begin{aligned} \mathcal{L}v(z,y) &= h(y)\frac{\partial v}{\partial z} + b(y)\frac{\partial v}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 v}{\partial y^2} + \frac{(1-\rho^2)}{2} f^2(y)\frac{\partial^2 v}{\partial z^2} \\ &= \mathcal{L}_Y v(z,y) + \mathcal{L}_Z v(z,y) \end{aligned}$$

where

$$\mathcal{L}_Y v(z,y) = b(y)\frac{\partial v}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 v}{\partial y^2}$$

and

$$\mathcal{L}_Z v(z,y) = h(y)\frac{\partial v}{\partial z} + \frac{(1-\rho^2)}{2} f^2(y)\frac{\partial^2 v}{\partial z^2}.$$

One can check that our scheme amounts to first integrate exactly $\mathcal{L}_Z$ over a half time step then apply the Ninomiya–Victoir scheme to $\mathcal{L}_Y$ over a time step and finally integrate exactly $\mathcal{L}_Z$ over a half time step. According to results on splitting (see Alfonsi [9] or Tanaka and Kohatsu-Higa [10] for example) one expects this scheme to exhibit second-order weak convergence. Actually, according to Theorem 1.17 in Alfonsi [9], our scheme has potential second-order of weak convergence. To deduce formally the order two of weak convergence, one needs to check regularity of the solution of the backward Kolmogorov equation associated with the model.

We will use a slightly different point of view to prove our convergence result stated in the second section. Indeed we need to apply test functions with exponential growth to X_T to be able to analyze weak convergence of the stock price and not only of its logarithm. We are going to take advantage of the Gaussian conditional distributions of both X_T and $\overline{X}_T^N$ given $(W_t)_{t\le T}$ to integrate such functions. Then we will analyse the weak convergence of the triplet $(\overline{Y}_T^N, \overline{m}_T^N, \overline{v}_T^N)$

to (Y_T, m_T, v_T) using the backward Kolmogorov equation associated with the degenerate SDE (2.2).

The third section is devoted to numerical experiments which confirm the theoretical rates of convergence.

Notations. We will consider, for a number of time steps $N \geq 1$, the uniform subdivision $\prod_N = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of $[0, T]$ with the discretization step $\delta_N = \frac{T}{N}$.

We denote by $\underline{f^2}$ the greatest lower bound of the function $\psi : y \mapsto f^2(y)$ and by $\overline{f^2}$ its lowest upper bound.

2. Analysis of the order of weak convergence

Under regularity assumptions on the coefficients of (1.1), we prove order two of weak convergence for the log-asset price:

Theorem 2.1. *Suppose that $\rho \in (-1, 1)$. If the following assumptions hold*

($\mathcal{H}1$) *b, σ, h and f are $\mathcal{C}^6$ functions with bounded derivatives and their sixth-order derivatives are globally Hölder continuous. Moreover, h and f are bounded.*

($\mathcal{H}2$) *F is $\mathcal{C}^6$ and bounded together with all its derivatives.*

($\mathcal{H}3$) $\underline{f^2} > 0$

then, for any measurable function g verifying $\exists c \geq 0, \mu \in [0, 2)$ such that $\forall x \in \mathbb{R}, |g(x)| \leq ce^{|x|^\mu}$, there exists $C > 0$ such that

$$\left|\mathbb{E}\Big(g(X_T)\Big) - \mathbb{E}\left(g(\overline{X}_T^N)\right)\right| \leq \frac{C}{N^2}.$$

In terms of the asset price, we easily deduce the following corollary where the payoff function α may exhibit polynomial growth at infinity:

Corollary 2.2. *Under the assumptions of Theorem 2.1, for any measurable function α verifying $\exists c \geq 0, \mu \in [0, 2)$ such that $\forall y > 0, |\alpha(y)| \leq ce^{|\log(y)|^\mu}$, there exists $C > 0$ such that*

$$\left|\mathbb{E}\left(\alpha(S_T)\right) - \mathbb{E}\left(\alpha(e^{\overline{X}_T^N})\right)\right| \leq \frac{C}{N^2}$$

Proof of the theorem. The idea of the proof consists in conditioning by the Brownian motion which drives the volatility process and then applying the weak error analysis of Talay and Tubaro [1].

As stated above, conditionally on $(W_t)_{t\in[0,T]}$, both X_T and $\overline{X}^N_T$ are Gaussian random variables and one can easily show that

$$\begin{aligned}\epsilon &:= \left|\mathbb{E}\left[g(X_T) - g(\overline{X}^N_T)\right]\right| \\ &= \left|\int_{\mathbb{R}} g(x)\mathbb{E}\left[\frac{\exp\left(-\frac{(x-\log(s_0)+\rho F(y_0)-\rho F(Y_T)-m_T)^2}{2(1-\rho^2)v_T}\right)}{\sqrt{2\pi(1-\rho^2)v_T}}\right.\right. \\ &\qquad \left.\left. - \frac{\exp\left(-\frac{(x-\log(s_0)+\rho F(y_0)-\rho F(\overline{Y}^N_T)-\overline{m}^N_T)^2}{2(1-\rho^2)\overline{v}^N_T}\right)}{\sqrt{2\pi(1-\rho^2)\overline{v}^N_T}}\right] dx\right| .\end{aligned}$$

For $x \in \mathbb{R}$, denote by γ_x the function

$$\begin{aligned}\gamma_x : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^*_+ &\to \mathbb{R} \\ (y,m,v) &\mapsto \frac{\exp\left(-\frac{(x-\log(s_0)+\rho F(y_0)-\rho F(y)-\varphi(m))^2}{2(1-\rho^2)\psi(v)}\right)}{\sqrt{2\pi(1-\rho^2)\psi(v)}}\end{aligned}$$

where $\varphi : \mathbb{R} \to [-2T\sup_{z\in\mathbb{R}}|h(z)|, 2T\sup_{z\in\mathbb{R}}|h(z)|]$ and $\psi : \mathbb{R} \to [\frac{T}{2}\underline{f^2}, 2T\overline{f^2}]$ are smooth functions with bounded derivatives such that $\varphi(m) = m$ for $m \in [-T\sup_{z\in\mathbb{R}}|h(z)|, T\sup_{z\in\mathbb{R}}|h(z)|]$ and $\psi(v) = v$ for $v \in [T\underline{f^2}, T\overline{f^2}]$.

Since m_T and $\overline{m}^N_T$ (resp. v_T and $\overline{v}^N_T$) belong to $[-T\sup_{z\in\mathbb{R}}|h(z)|, T\sup_{z\in\mathbb{R}}|h(z)|]$ (resp. $[T\underline{f^2}, T\overline{f^2}]$), $\epsilon \le \int_{\mathbb{R}} g(x)e(x)dx$ where

$$e(x) = \left|\mathbb{E}\left[\gamma_x(Y_T, m_T, v_T) - \gamma_x(\overline{Y}^N_T, \overline{m}^N_T, \overline{v}^N_T)\right]\right|.$$

Consequently, it is enough to show the following intermediate result: $\exists C, K > 0$ such that $\forall x \in \mathbb{R}$,

$$e(x) \le \frac{C}{N^2}e^{-Kx^2}. \tag{2.1}$$

We naturally consider the following three-dimensional degenerate SDE:

$$\begin{cases} dY_t = \sigma(Y_t)dW_t + b(Y_t)dt; & Y_0 = y_0 \\ dm_t = h(Y_t)dt; & m_0 = 0 \\ dv_t = f^2(Y_t)dt; & v_0 = 0. \end{cases} \tag{2.2}$$

Notice that in the Ninomiya–Victoir scheme applied to this three-dimensional SDE, the approximation of the solution at time t_{k+1} is deduced from the one at time t_k by applying $e^{\frac{\delta_N}{2}\tilde{V}_0}e^{(W_{t_{k+1}}-W_{t_k})\tilde{V}}e^{\frac{\delta_N}{2}\tilde{V}_0}$ where $\tilde{V}_0(y) = (b(y) - \frac{1}{2}\sigma\sigma'(y), h(y), f^2(y))$ and $\tilde{V}(y) = (\sigma(y), 0, 0)$. The resulting approximation of (Y_T, m_T, v_T) is very close to $(\overline{Y}^N_T, \overline{m}^N_T, \overline{v}^N_T)$ but with equality only for the first coordinate: indeed in the integration of the ODE corresponding to $\tilde{V}_0$, this first coordinate evolves thus modifying the coefficients $h(y)$ and $f^2(y)$ which, in contrast, remain constant in our scheme. In order to prove (2.1), we need to analyze the dependence of the error on x and not only on N. That is why we resume the error analysis of Ninomiya and Victoir [6] in a more detailed fashion.

For $x \in \mathbb{R}$, let us define the function $u_x : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}$ by

$$u_x(t,y,m,v) = \mathbb{E}\left[\gamma_x\left((Y_{T-t}, m_{T-t}, v_{T-t})^{(y,m,v)}\right)\right]$$

where we denote by $(Y_{T-t}, m_{T-t}, v_{T-t})^{(y,m,v)}$ the solution at time $T-t$ of (2.2) starting from (y,m,v).

The remainder of the proof leans on the following lemmas, the proofs of which are postponed to the appendix. We will use the standard notation for partial derivatives: for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, d being a positive integer, we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_d$ its length and by ∂_α the differential operator $\partial^{|\alpha|}/\partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}$.

Lemma 2.3. *Under assumptions* $(\mathcal{H}1)$, $(\mathcal{H}2)$ *and* $(\mathcal{H}3)$, *we have that*

i) u_x *is* $\mathcal{C}^3$ *with respect to the time variable and* $\mathcal{C}^6$ *with respect to the space variables. Moreover, it solves the following PDE*

$$\begin{cases} \partial_t u_x + \mathcal{L}u_x = 0 \\ u_x(T,y,m,v) = \gamma_x(y,m,v) \end{cases} \tag{2.3}$$

where $\mathcal{L}$ *is the differential operator associated to* (2.2)*:*

$$\mathcal{L}u(y,m,v) = \frac{\sigma^2(y)}{2}\frac{\partial^2 u}{\partial y^2} + b(y)\frac{\partial u}{\partial y} + h(y)\frac{\partial u}{\partial m} + f^2(y)\frac{\partial u}{\partial v}.$$

ii) *For any multi-index* $\alpha \in \mathbb{N}^3$ *and integer* l *such that* $2l + |\alpha| \le 6$, *there exists* $C_{l,\alpha}, K_{l,\alpha} > 0$ *and* $p_{l,\alpha} \in \mathbb{N}$ *such that* $\forall t \in [0,T]$, $\forall (y,m,v) \in \mathbb{R}^3$,

$$\left|\partial_t^l \partial_\alpha u_x(t,y,m,v)\right| \le C_{l,\alpha} e^{-K_{l,\alpha}x^2}\left(1 + |y|^{p_{l,\alpha}}\right).$$

Lemma 2.4. *Under assumption* $(\mathcal{H}1)$,

$$\forall q \in \mathbb{N}, \sup_{0 \le k \le N} \mathbb{E}\left(\left|\overline{Y}_{t_k}^N\right|^q\right) < \infty.$$

Now, following the error analysis of Talay and Tubaro [1], we write that

$$\left|\mathbb{E}\left[\gamma_x(Y_T, m_T, v_T) - \gamma_x(\overline{Y}_T^N, \overline{m}_T^N, \overline{v}_T^N)\right]\right| \le \sum_{k=0}^{N-1} \eta_k(x)$$

where $\eta_k(x) = \left|\mathbb{E}\left[u_x(t_{k+1}, \overline{Y}_{t_{k+1}}^N, \overline{m}_{t_{k+1}}^N, \overline{v}_{t_{k+1}}^N) - u_x(t_k, \overline{Y}_{t_k}^N, \overline{m}_{t_k}^N, \overline{v}_{t_k}^N)\right]\right|$ and $\forall 0 \le k \le N$,

$$\overline{m}_{t_k}^N = \delta_N \sum_{j=0}^{k-1} \frac{h(\overline{Y}_{t_j}^N) + h(\overline{Y}_{t_{j+1}}^N)}{2} \quad \text{and} \quad \overline{v}_{t_k}^N = \delta_N \sum_{j=0}^{k-1} \frac{f^2(\overline{Y}_{t_j}^N) + f^2(\overline{Y}_{t_{j+1}}^N)}{2}.$$

Using the Markov property for the first term in the expectation and Taylor's formula together with PDE (2.3) for the second, we get

$$\eta_k(x) = \Bigg|\mathbb{E}\Bigg[\phi_x(t_{k+1},\overline{Y}^N_{t_k},\overline{m}^N_{t_k},\overline{v}^N_{t_k}) - u_x(t_{k+1},\overline{Y}^N_{t_k},\overline{m}^N_{t_k},\overline{v}^N_{t_k}) + \delta_N \mathcal{L}u_x(t_{k+1},\overline{Y}^N_{t_k},\overline{m}^N_{t_k},\overline{v}^N_{t_k}) - \frac{\delta_N^2}{2}\mathcal{L}^2 u_x(t_{k+1},\overline{Y}^N_{t_k},\overline{m}^N_{t_k},\overline{v}^N_{t_k}) + \frac{1}{2}\int_{t_k}^{t_{k+1}} \frac{\partial^3 u_x}{\partial t^3}(t,\overline{Y}^N_{t_k},\overline{m}^N_{t_k},\overline{v}^N_{t_k})(t-t_k)^2 dt\Bigg]\Bigg|$$

with $\phi_x(t_{k+1},y,m,v) = \mathbb{E}\left[\Gamma_y(\overline{Y}^{N,y}_{t_1})\right]$ where

$$\Gamma_y(z) = u_x\left(t_{k+1}, z, m+\delta_N\frac{h(z)+h(y)}{2}, v+\delta_N\frac{f^2(z)+f^2(y)}{2}\right).$$

Using Taylor's formula, we can show that $\forall z \in \mathbb{R}$,

$$\Gamma_y(z) = \Gamma_{y,1}(z) + \delta_N \Gamma_{y,2}(z) + \frac{\delta_N^2}{2}\Gamma_{y,3}(z) + R_0(z)$$

where

$$\Gamma_{y,1}(z) = u_x(t_{k+1},z,m,v)$$

$$\Gamma_{y,2}(z) = \frac{h(z)+h(y)}{2}\frac{\partial u_x}{\partial m}(t_{k+1},z,m,v) + \frac{f^2(z)+f^2(y)}{2}\frac{\partial u_x}{\partial v}(t_{k+1},z,m,v)$$

$$\Gamma_{y,3}(z) = \left(\left(\frac{h(z)+h(y)}{2}\right)^2 \frac{\partial^2 u_x}{\partial m^2}(\cdot) + \left(\frac{f^2(z)+f^2(y)}{2}\right)^2\frac{\partial^2 u_x}{\partial v^2}(\cdot)\right)(t_{k+1},z,m,v) + 2\frac{h(z)+h(y)}{2}\,\frac{f^2(z)+f^2(y)}{2}\,\frac{\partial^2 u_x}{\partial m \partial v}(t_{k+1},z,m,v)$$

and

$$R_0(z) = \int_0^{\delta_N}\frac{(\delta_N-t)^2}{2}dt\Bigg(\left(\frac{h(z)+h(y)}{2}\right)^3\frac{\partial^3 u_x}{\partial m^3}(\cdot) + \left(\frac{f^2(z)+f^2(y)}{2}\right)^3\frac{\partial^3 u_x}{\partial v^3}(\cdot) + 3\left(\frac{f^2(z)+f^2(y)}{2}\right)^2\left(\frac{h(z)+h(y)}{2}\right)\frac{\partial^3 u_x}{\partial m \partial v^2}(\cdot) + 3\left(\frac{h(z)+h(y)}{2}\right)^2\left(\frac{f^2(z)+f^2(y)}{2}\right)\frac{\partial^3 u_x}{\partial m^2 \partial v}(\cdot)\Bigg)\left(\overline{t}, z, \overline{m}, \overline{v}\right) \tag{2.4}$$

where $\overline{t} = t_{k+1}, \overline{m} = m + t\frac{h(z)+h(y)}{2}$ and $\overline{v} = v + t\frac{f^2(z)+f^2(y)}{2}$. So

$$\phi_x(t_{k+1},y,m,v) = \mathbb{E}\left[\Gamma_{y,1}\left(\overline{Y}^{N,y}_{t_1}\right) + \delta_N\Gamma_{y,2}\left(\overline{Y}^{N,y}_{t_1}\right) + \frac{\delta_N^2}{2}\Gamma_{y,3}\left(\overline{Y}^{N,y}_{t_1}\right) + R_0\left(\overline{Y}^{N,y}_{t_1}\right)\right].$$

With a slight abuse of notations, we define the first-order differential operators V_0 and V acting on $\mathcal{C}^1$ functions by $V_0\xi(x) = V_0(x)\xi'(x)$ and $V\xi(x) = V(x)\xi'(x)$ for $\xi \in \mathcal{C}^1(\mathbb{R})$ where the functions V_0 and V are defined in (1.3). We make the same expansions as in Ninomiya and Victoir [6] but with making the remainder terms explicit in order to check if they have the good behavior with respect to x. We can show after tedious but simple computations that

$$\begin{aligned}
\mathbb{E}\left[\Gamma_{y,1}(\overline{Y}_{t_1}^{N,y})\right] &= \Gamma_{y,1}(y) + \frac{\delta_N}{2}\left(V^2\Gamma_{y,1}(y) + 2V_0\Gamma_{y,1}(y)\right) \\
&\quad + \frac{\delta_N^2}{8}\Big(4{V_0}^2\Gamma_{y,1}(y) + 2V_0V^2\Gamma_{y,1}(y) \\
&\quad + 2V^2V_0\Gamma_{y,1}(y) + V^4\Gamma_{y,1}(y)\Big) + \mathbb{E}\left(R_1(y)\right)
\end{aligned}$$

$$\mathbb{E}\left[\Gamma_{y,2}(\overline{Y}_{t_1}^{N,y})\right] = \Gamma_{y,2}(y) + \frac{\delta_N}{2}\left(V^2\Gamma_{y,2}(y) + 2V_0\Gamma_{y,2}(y)\right) + \mathbb{E}\left(R_2(y)\right)$$

$$\mathbb{E}\left[\Gamma_{y,3}(\overline{Y}_{t_1}^{N,y})\right] = \Gamma_{y,3}(y) + \mathbb{E}\left(R_3(y)\right)$$

where

$$\begin{aligned}
R_1(y) = &\int_0^{\frac{\delta_N}{2}}\int_0^{s_1}\int_0^{s_2} {V_0}^3\Gamma_{y,1}(e^{s_3V_0}e^{W_{\delta_N}V}e^{\frac{\delta_N}{2}V_0}(y))ds_3ds_2ds_1 \\
&+ \int_0^{W_{\delta_N}}\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\int_0^{s_4}\int_0^{s_5} V^6\Gamma_{y,1}(e^{s_6V}e^{\frac{\delta_N}{2}V_0}(y))ds_6\cdots ds_1 \\
&+ \frac{\delta_N}{2}\int_0^{W_{\delta_N}}\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} V^4V_0\Gamma_{y,1}(e^{s_4V}e^{\frac{\delta_N}{2}V_0}(y))ds_4ds_3ds_2ds_1 \\
&+ \frac{\delta_N^2}{8}\int_0^{W_{\delta_N}}\int_0^{s_1} V^2{V_0}^2\Gamma_{y,1}(e^{s_2V}e^{\frac{\delta_N}{2}V_0}(y))ds_2ds_1 \\
&+ \int_0^{\frac{\delta_N}{2}}\int_0^{s_1}\int_0^{s_2} {V_0}^3\Gamma_{y,1}(e^{s_3V_0}(y))ds_3ds_2ds_1 \\
&+ \frac{\delta_N}{2}\int_0^{\frac{\delta_N}{2}}\int_0^{s_1} {V_0}^2V^2\Gamma_{y,1}(e^{s_2V_0}(y))ds_2ds_1 \qquad (2.5)\\
&+ \frac{\delta_N}{2}\int_0^{\frac{\delta_N}{2}}\int_0^{s_1} {V_0}^3\Gamma_{y,1}(e^{s_2V_0}(y))ds_2ds_1 \\
&+ \frac{\delta_N^2}{8}\int_0^{\frac{\delta_N}{2}} V_0V^4\Gamma_{y,1}(e^{s_1V_0}(y))ds_1 \\
&+ \frac{\delta_N^2}{8}\int_0^{\frac{\delta_N}{2}} {V_0}^3\Gamma_{y,1}(e^{s_1V_0}(y))ds_1 \\
&+ \frac{\delta_N^2}{4}\int_0^{\frac{\delta_N}{2}} V_0V^2V_0\Gamma_{y,1}(e^{s_1V_0}(y))ds_1,
\end{aligned}$$

$$
\begin{aligned}
R_2(y) = & \int_0^{\frac{\delta_N}{2}} \int_0^{s_1} {V_0}^2 \Gamma_{y,2}(e^{s_2 V_0} e^{W_{\delta_N} V} e^{\frac{\delta_N}{2} V_0}(y)) ds_2 ds_1 \\
& + \int_0^{W_{\delta_N}} \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} V^4 \Gamma_{y,2}(e^{s_4 V} e^{\frac{\delta_N}{2} V_0}(y)) ds_4 ds_3 ds_2 ds_1 \\
& + \frac{\delta_N}{2} \int_0^{W_{\delta_N}} \int_0^{s_1} V^2 V_0 \Gamma_{y,2}(e^{s_2 V} e^{\frac{\delta_N}{2} V_0}(y)) ds_2 ds_1 \\
& + \int_0^{\frac{\delta_N}{2}} \int_0^{s_1} {V_0}^2 \Gamma_{y,2}(e^{s_2 V_0}(y)) ds_2 ds_1 \qquad (2.6) \\
& + \frac{\delta_N}{2} \int_0^{\frac{\delta_N}{2}} V_0 V^2 \Gamma_{y,2}(e^{s_1 V_0}(y)) ds_1 + \frac{\delta_N}{2} \int_0^{\frac{\delta_N}{2}} {V_0}^2 \Gamma_{y,2}(e^{s_1 V_0}(y)) ds_1,
\end{aligned}
$$

$$
\begin{aligned}
R_3(y) = & \int_0^{\frac{\delta_N}{2}} V_0 \Gamma_{y,3}(e^{s_1 V_0} e^{W_{\delta_N} V} e^{\frac{\delta_N}{2} V_0}(y)) ds_1 \qquad (2.7) \\
& + \int_0^{W_{\delta_N}} \int_0^{s_1} V^2 \Gamma_{y,3}(e^{s_2 V} e^{\frac{\delta_N}{2} V_0}(y)) ds_2 ds_1 + \int_0^{\frac{\delta_N}{2}} V_0 \Gamma_{y,3}(e^{s_1 V_0}(y)) ds_1.
\end{aligned}
$$

Putting all the terms together, one can check that

$$
\begin{aligned}
\phi_x(t_{k+1}, y, m, v) = & \, u_x(t_{k+1}, y, m, v) + \delta_N \mathcal{L} u_x(t_{k+1}, y, m, v) \\
& + \frac{\delta_N^2}{2} \mathcal{L}^2 u_x(t_{k+1}, y, m, v) + R(y)
\end{aligned}
$$

where $R(y) = \mathbb{E}\left[R_0(\overline{Y}_{t_1}^{N,y}) + R_1(y) + \delta_N R_2(y) + \frac{\delta_N^2}{2} R_3(y)\right]$. Finally,

$$
e(x) \le \sum_{k=0}^{N-1} \mathbb{E}\left[\left|\frac{1}{2} \int_{t_k}^{t_{k+1}} \frac{\partial^3 u_x}{\partial t^3}\left(t, \overline{Y}_{t_k}^N, \overline{m}_{t_k}^N, \overline{v}_{t_k}^N\right)(t - t_k)^2 dt\right| + \left|R\left(\overline{Y}_{t_k}^N\right)\right|\right]. \qquad (2.8)
$$

From Lemmas 2.3 and 2.4, we deduce that there exists $C_1, K_1 > 0$ such that

$$
\sum_{k=0}^{N-1} \mathbb{E}\left[\left|\frac{1}{2} \int_{t_k}^{t_{k+1}} \frac{\partial^3 u_x}{\partial t^3}\left(t, \overline{Y}_{t_k}^N, \overline{m}_{t_k}^N, \overline{v}_{t_k}^N\right)(t - t_k)^2 dt\right|\right] \le \frac{1}{N^2} C_1 e^{-K_1 x^2}. \qquad (2.9)
$$

On the other hand, a close look to (2.4), (2.5), (2.6) and (2.7) convinces us that the term $\mathbb{E}\left[\left|R\left(\overline{Y}_{t_k}^N\right)\right|\right]$ is of order $\frac{1}{N^3}$ and that it involves only derivatives of u_x and of the coefficients of the SDE (2.2). So, thanks to Lemmas 2.3 and 2.4, there exists $C_2, K_2 > 0$ such that

$$
\sum_{k=0}^{N-1} \mathbb{E}\left[\left|R\left(\overline{Y}_{t_k}^N\right)\right|\right] \le \frac{1}{N^2} C_2 e^{-K_2 x^2}. \qquad (2.10)
$$

Putting (2.9) and (2.10) in (2.8), we conclude that (2.1) holds. □

Remark 2.5.

- As for plain vanilla options pricing, observe that, by the Romano and Touzi [11] formula,

$$\mathbb{E}\left(e^{-rT}\alpha(S_T)|(Y_t)_{t\in[0,T]}\right) = BS_{\alpha,T}\left(\overline{s},\overline{v}\right)$$

where $\overline{s} = s_0 e^{\rho(F(Y_T)-F(y_0))+m_T+(\frac{(1-\rho^2)v_T}{2T}-r)T}$, $\overline{v} = \frac{(1-\rho^2)v_T}{T}$ and $BS_{\alpha,T}(s,v)$ stands for the price of a European option with pay-off α and maturity T in the Black–Scholes model with initial stock price s, volatility $\sqrt{v}$ and constant interest rate r. When, like for a call or a put option, $BS_{\alpha,T}$ is available in a closed form, one should approximate $\mathbb{E}\left(e^{-rT}\alpha(S_T)\right)$ by

$$\frac{1}{M}\sum_{i=1}^{M} BS_{\alpha,T}\left(s_0 e^{\rho(F(\overline{Y}_T^{N,i})-F(y_0))+\overline{m}_T^{N,i}+(\frac{(1-\rho^2)\overline{v}_T^{N,i}}{2T}-r)T}, \frac{(1-\rho^2)\overline{v}_T^{N,i}}{T}\right)$$

where M is the total number of Monte Carlo samples and the index i refers to independent draws. Indeed, the conditioning provides a variance reduction.
- In the special case of an Ornstein–Uhlenbeck process driving the volatility, $(Y_t)_{t\in[0,T]}$ is solution of the following SDE

$$dY_t = \nu dW_t + \kappa(\theta - Y_t)dt,\ Y_0 = y_0$$

with $\nu > 0$ and $\kappa,\theta \in \mathbb{R}$. One should then replace the Ninomiya–Victoir scheme by the true solution $Y_t = y_0 e^{-\kappa t} + \theta(1-e^{-\kappa t}) + \nu\int_0^t e^{-\kappa(t-s)}dW_s$. The order two of weak convergence is then preserved.

3. Numerical illustration of weak convergence

For the following numerical computations, we are going to consider Scott's model (see [12]):

$$\begin{cases} dS_t = rS_t dt + \sigma_0 e^{Y_t} S_t\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right) \\ dY_t = \kappa(\theta - Y_t)dt + \nu dW_t \end{cases}$$

$$\Rightarrow f(y) = \sigma_0 e^y,\ b(y) = \kappa(\theta - y) \text{ and } \sigma(y) = \nu$$

with the following set of parameters: $S_0 = 100, r = 0.05, T = 1, \sigma_0 = 0.25, y_0 = 0, \kappa = 1, \theta = 0, \nu = \frac{7\sqrt{2}}{20}, \rho = -0.2$ and $f : y \mapsto \sigma_0 e^y$. Then Y is an Ornstein–Uhlenbeck process which can be simulated exactly on the time-grid $\prod_N$.

We compute the price of a call option with strike $K = 100$ and maturity $T = 1$. For our scheme, using both the conditioning variance reduction technique presented in Remark 2.5 and the opportunity to simulate exactly the process Y, we approximate the price of the option by

$$P_{\text{Weak2}}^{N,M} = \frac{1}{M}\sum_{i=1}^{M} BS_{\alpha,T}\left(s_0 e^{\rho(F(Y_T^i)-F(y_0))+\tilde{m}_T^{N,i}+(\frac{(1-\rho^2)\tilde{v}_T^{N,i}}{2T}-r)T}, \frac{(1-\rho^2)\tilde{v}_T^{N,i}}{T}\right)$$

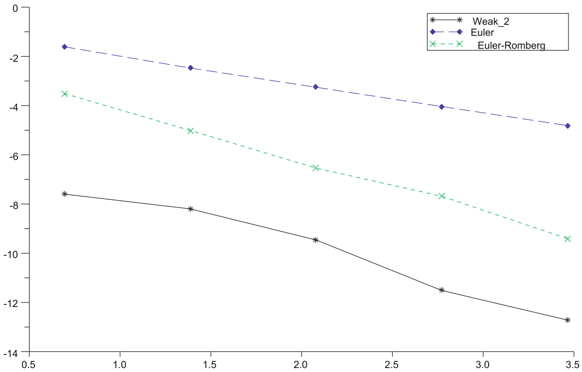

FIGURE 1. Illustration of the convergence rate for the call option

where

$$(\tilde{m}_T^{N,i}, \tilde{v}_T^{N,i}) = \delta_N \sum_{k=0}^{N-1} \left(\frac{h(Y_{t_k}^i) + h(Y_{t_{k+1}}^i)}{2}, \frac{f^2(Y_{t_k}^i) + f^2(Y_{t_{k+1}}^i)}{2} \right)$$

and the index i refers to independent draws. For the Euler schemes, we approximate the price of the option by

$$P_{\text{Euler}}^{N,M} = \frac{1}{M} \sum_{i=1}^{M} BS_{\alpha,T} \left(s_0 e^{\rho \sum_{k=0}^{N-1} f(Y_{t_k}^i)(W_{t_{k+1}}^i - W_{t_k}^i) - \rho^2 \tilde{v}_T^{N,i} T}, \frac{(1-\rho^2)\tilde{v}_T^{N,i}}{T} \right).$$

In Figure 1 we draw the logarithm $\log\left(\left|P_{\text{exact}} - P_{\text{scheme}}^{N,M}\right|\right)$ of the pricing error where the reference call price $P_{\text{exact}} \approx 12.82603$ is obtained by a multilevel Monte Carlo with an accuracy of $5bp$ (see [15] for the multilevel Monte Carlo method), as a function of the logarithm of the number N of time steps. In order to avoid statistical noise, we make $M = 10^7$ simulations.

We see that, as expected, the Weak_2 scheme and Euler scheme with Romberg extrapolation exhibit a weak convergence of order two and converge much faster than the Euler scheme. For the same number of time steps, the precision obtained by our scheme is better than the one obtained by the Euler scheme with Romberg extrapolation.

Finally, note that the weak scheme does not require the simulation of additional terms when compared to the Euler scheme, with and without Romberg extrapolation. Combined with its second-order weak convergence, this makes the Weak_2 scheme very competitive for the pricing of plain vanilla European options. In Figure 2, we give the relative error of each scheme as a function of the computation time needed when we fix the number of simulation to $M = 100\,000$. The vertical bars in the figure represent the confidence interval. We see that our scheme reaches a high level of precision in less than a second. It takes at least five seconds for the Euler to reach the same level of precision. Certainly, the use of Romberg

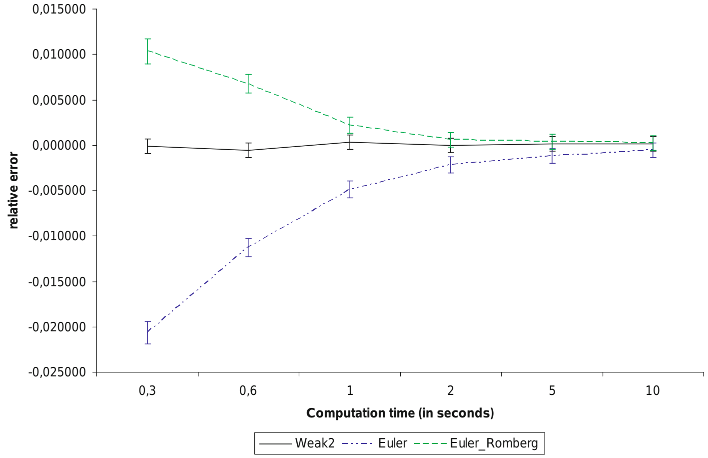

FIGURE 2. Convergence of the call price with respect to time

extrapolation improves the ratio precision/computation time but it is still much less efficient compared to our scheme.

4. Conclusion

In this article, we have capitalized on the particular structure of stochastic volatility models to propose and analyze an ad'hoc discretization scheme with order two of weak convergence. Our numerical experiments confirm this theoretical rate of convergence. The Euler scheme with Romberg extrapolation also achieves second-order weak convergence but we show that our scheme is much more competitive.

Appendix. Proofs

A.1. Proof of Lemma 2.3

We refer to [13, Theorem 7.1, p. 295] for the existence of a classical solution to the PDE (2.3), C^1 with respect to the time variable and C^2 with respect to (y, m, v). Then the Feynmac Kac representation formula ensures that this function is equal to u_x.

Let us now check the additional regularity properties. To do so, we first derive the following estimation of the derivatives of γ_x:

$$\begin{aligned}&\forall \beta \in \mathbb{N}^3 \text{ such that } \beta_1 \leq 6, \exists\, C_\beta, K_\beta > 0 \text{ such that}\\&\forall (y, m, v) \in \mathbb{R}^3, \quad |\partial_\beta \gamma_x(y, m, v)| \leq C_\beta e^{-K_\beta x^2}.\end{aligned} \tag{A.1}$$

Indeed, using Leibniz's formula, one can show that $\partial_\beta \gamma_x(y,m,v)$ can be written as a weighted sum of terms of the form

$$\zeta_k = \frac{(x - \log(s_0) + \rho F(y_0) - \rho F(y) - \varphi(m))^{k_2}}{(\psi(v))^{k_1 + \frac{1}{2}}} \times \exp\left(-\frac{(x - \log(s_0) + \rho F(y_0) - \rho F(y) - \varphi(m))^2}{2(1-\rho^2)\psi(v)}\right) \times \prod_{i=1}^{k_3} \left(F^{(i)}(y)\right)^{a_i} \left(\varphi^{(i)}(m)\right)^{b_i} \left(\psi^{(i)}(v)\right)^{c_i}$$

where $k = (k_1, k_2, k_3)$ belongs to a finit set $I_\beta \subset \mathbb{N}^3$ and $(a_i, b_i, c_i)_{0 \le i \le k_3}$ are constants taking value in $\mathbb{N}^3$. Using assumptions $(\mathcal{H}2)$ and $(\mathcal{H}3)$, Young's inequality and the boundedness of φ and ψ together with their derivatives, we show that $\exists C_k, K_k > 0$ such that $|\zeta_k| \le C_k e^{-K_k x^2}$ which yields the desired result.

By inverting expectation and differentiations, we see that for any multi-index $\beta \in \mathbb{N}$ such that $\beta_1 \le 6$, the derivative $\partial_\beta u_x(t,y,m,v)$ exists and is a continuous function equal to the expectation of a product between derivatives of the flow $(y,m,v) \to (Y_{T-t}, m_{T-t}, v_{T-t})^{(y,m,v)}$ and derivatives of the function γ_x evaluated at $(Y_{T-t}, m_{T-t}, v_{T-t})^{(y,m,v)}$. Using result (A.1) and the fact that, under assumptions $(\mathcal{H}1)$ and $(\mathcal{H}2)$, the derivatives of the flow satisfy a system of SDEs with Lipschitz continuous coefficients (see for example Kunita [13]) we obtain that for $\beta \in \mathbb{N}$ such that $\beta_1 \le 6$,

$$\exists\, C_\beta, K_\beta > 0,\ \forall (y,m,v) \in \mathbb{R}^3,\ |\partial_{\beta_1} u_x(t,y,m,v)| \le C_\beta e^{-K_\beta x^2}. \tag{A.2}$$

Now, let us fix $\alpha \in \mathbb{N}^3, l \in \mathbb{N}$ such that $2l + |\alpha| \le 6$ and $(t,y,m,v) \in [0,T] \times \mathbb{R}^3$. Thanks to the PDE (2.2), the derivative $\partial_t^l \partial_\alpha u_x(t,y,m,v)$ exists and is a continuous function equal to $(-1)^l \partial_\alpha \mathcal{L}^l u_x(t,y,m,v)$. One can check that the right-hand side is equal to a weighted sum of terms of the form $\partial_\beta u_x(t,y,m,v) \times \pi_\gamma(b,\sigma,f,h)$ where $\beta \in \mathbb{N}^3$ is multi-index belonging to a finite set $I^1_{\alpha,l} \subset \{0,1,\ldots,6\} \times \mathbb{N}^2$, γ is a suffix belonging to a finite set $I^2_{\alpha,l}$ and $\pi_\gamma(b,\sigma,f,h)$ is a product of terms involving the functions b, σ, f, h and their derivatives up to order 4.

Assumptions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ yield that $\exists c^2_{l,\alpha} \ge 0$ and $q_{l,\alpha} \in \mathbb{N}$ such that

$$\forall \gamma \in I^2_{\alpha,l}, |\pi_\gamma(b,\sigma,f,h)| \le c^2_{l,\alpha}(1 + |y|^{q_{l,\alpha}}). \tag{A.3}$$

Gathering (A.3) and (A.2) enables us to conclude.

A.2. Proof of Lemma 2.4

Making the link between ODEs and SDEs (see Doss [14]), one can check that $(\overline{Y}^N_{t_1}, \ldots, \overline{Y}^N_{t_N})$ has the same distribution law as $(\overline{\overline{Y}}_{2t_1}, \ldots, \overline{\overline{Y}}_{2t_N})$ where $(\overline{\overline{Y}}_t)_{t \in [0,2T]}$ is solution of the following inhomogeneous SDE

$$\overline{\overline{Y}}_t = y_0 + \int_0^t \overline{b}(s, \overline{\overline{Y}}_s) ds + \int_0^t \overline{\sigma}(s, \overline{\overline{Y}}_s) dW_s$$

with, $\forall (s,y) \in [0,2T] \times \mathbb{R}$,

$$\overline{b}(s,y) = \begin{cases} \frac{1}{2}\sigma\sigma'(y) & \text{if } s \in \bigcup_{k=0}^{N-1} \left[\frac{(4k+1)T}{2N}, \frac{(4k+3)T}{2N}\right] \\ b(y) - \frac{1}{2}\sigma\sigma'(y) & \text{otherwise} \end{cases}$$

and

$$\overline{\sigma}(s,y) = \begin{cases} \sigma(y) & \text{if } s \in \bigcup_{k=0}^{N-1} \left[\frac{(4k+1)T}{2N}, \frac{(4k+3)T}{2N}\right] \\ 0 & \text{otherwise} \end{cases}.$$

Since these coefficients have a uniform in time linear growth in the spatial variable, one easily concludes.

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Bid-Ask Spread Modelling, a Perturbation Approach

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Abstract. Our objective is to study liquidity risk, in particular the so-called "bid-ask spread", as a by-product of market uncertainties. "Bid-ask spread", and more generally "limit order books" describe the existence of different sell and buy prices, which we explain by using different risk aversions of market participants. The risky asset follows a diffusion process governed by a Brownian motion which is uncertain. We use the error theory with Dirichlet forms to formalize the notion of uncertainty on the Brownian motion. This uncertainty generates noises on the trajectories of the underlying asset and we use these noises to expound the presence of bid-ask spreads. In addition, we prove that these noises also have direct impacts on the mid-price of the risky asset. We further enrich our studies with the resolution of an optimal liquidation problem under these liquidity uncertainties and market impacts. To complete our analysis, some numerical results will be provided.

Mathematics Subject Classification (2010). 60G15, 60H10, 60G45, 91G80.

Keywords. Error theory using Dirichlet Forms, order books, bid-ask spread, optimal control problem, liquidity.

Introduction

A stock index is defined as an aggregate value produced by combining several stocks or other investment vehicles together and expressing their total values against a base value from a specific date. Stock Market indexes, for instance, are intended to represent an entire stock market and thus track the market's changes over time. They are generally used as benchmark to measure the relative performance of any given investment portfolio. But they are not directly tradable assets. An increasing role of the asset management industry is to provide investors with

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investment tools capable of replicating a wide range of indices. These investment tools are often known as trackers or Exchange Traded Funds (ETF). To put it into perspective, investments in ETF have significantly increased over the last decade.

Let us now consider a process X representing a stock market index, i.e., a non-tradable asset, which is governed by a given SDE (stochastic differential equation). We also consider the process S representing an ETF on the non-tradable index X.

Our main objective in this paper is to address two important questions arising from the trading of the ETF. The first question concerns the price value and its trajectory behavior compared to that of the index. Beyond this first question, we will, in particular, address the microstructure problem of the transaction market and attempt to model the financial and economic rationale behind the existence of the bid-ask spread. The second question is linked to the practical study of optimal liquidation strategy in the context of non-liquidity.

Managing a tracker fund is not as simplistic as it first looks. Indeed, the impossibility to trade continuously, the lack of liquidity in the transaction of the index's components, liquidity and transaction costs, etc, imply that the values S of the index tracker fund and the index X may be highly correlated but not identical, i.e., this phenomenon is called tracking errors.

By value of the tracker fund, we mean the "fundamental value" obtained by summing parts of the tracker fund. Due to tracking errors, which are non-observable, the "theoretical" value of the tracker fund is itself a non-observable process, i.e., at each time, it is a random variable. Furthermore, should we take into account the liquidity problem, the "value" of the tracker fund should be the liquidation value of the assets owned in the fund. The liquidation value is very different from the last-transaction based price or the mid-price. However, from the investors' point of view, the price value of the tracker fund is also and especially its quoted prices. Indeed, the ETF is itself a trading asset. As such, to invest in (disinvest from) the tracker fund, one has to go through the market and make appropriate buy (sell) order transaction. Its price therefore depends on the dynamics of the overall buy and sell orders.

To sum up, the dynamics of the price value S of the ETF has two driving factors: a fundamental factor and a trading factor. The fundamental factor considers the tracker fund as the aggregate value produced by combining its components while the trading factor deals with the microstructure of market transactions, i.e., when the liquidity risk and cost are factored in.

In addressing our first question on the dynamics of the price value S and its trajectory, we will in particular study how the non-observability of the fundamental values of the tracker fund explains the existence of the limit order book and more specifically the bid-ask spread. Indeed, at any fixed time, the price value S is not observable but it is a random variable where its law or at least its mean and variance may be characterized. In the context of this randomness and non-observability, the existence of different risk aversions of market participants directly explains the presence of different types of orders, i.e., limit and market orders and in particular the presence of many buy and sell limit order prices forming the limit order book. In order to focus on the modelling of the bid-ask spread, we consider a "representative" price-setter market participant who places all limit orders, in particular the best buy and best sell limit orders, i.e., the prices and the number of shares he is willing to buy and sell. Prior to setting the buy and sell orders, the representative agent obtains the distribution of possible asset values from the market information but has no possibility to observe the assets' realized values. A rational decision is to send a buy (sell) limit order with a lower (higher) price with respect to the assets mean value such that their difference justifies the risk taken.

In the mathematical finance literature, there are several approaches modelling liquidity risk. One approach is to explain liquidity risk by the presence of an insider [2, 17]. Another approach is the market manipulation literature where prices are assumed to depend directly on the trading strategies, see [3] and [10]. A third approach is to consider liquidity risk in terms of the difference between the bid and ask prices, i.e., the existence of a bid-ask spread. Transaction costs [16] are one way to model the bid-ask spread. In [18], liquidity risk is expressed by the presence of transaction costs and market manipulation. Another way is to directly model the bid-ask spread or the order books and explain its presence as being intrinsic to the financial markets, driven by trades between different market participants.

Bid-ask spread and order books play a crucial role in many financial problems such as unwinding large block of shares for large investors and hedging strategy of options for traders. These problems were investigated by the likes of [1] and [20]. Due to the complexity of the study, state process representing the underlying price is assumed to have simple dynamics such as Bachelier's dynamics [20], or is assumed to be a martingale process [1].

However, in the above models, liquidity risk is considered a posteriori. In other words, assuming the existence of liquidity risk, different approaches are used to replicate its effects. To our knowledge, few studies in the fields of mathematical finance have attempted to model the financial and economic rationales behind the existence of the bid-ask spread. This is precisely the objective in this paper: study and explain, in the context of the ETF trading, the existence of the bid-ask spread, and more generally limit order book, as a by-product of market uncertainties.

The mathematical formulation of such problems relies on the specification of a coherent framework to describe the remaining randomness on prices. In our study, the asset value depends on two random sources: the first one describes the evolution of the asset mean value while the second delineates the shape of asset (sell-buy) prices at a given fixed time. The coupling of the two probability spaces, with its respective filtration, requires complex tools and represents the principal drawback of this kind of approach. Therefore, we choose a different strategy based on error theory using Dirichlet forms formalism developed in [8] and [9]. The advantages of this approach are inherent to its elasticity and powerful tools.

Such an approach provides us with a perfect knowledge of the bid-ask spread component of the order book, i.e., the best/highest bid price and the best/lowest ask price of the order book placed by the "representative" price-setter market participant. Once our bid-ask spread model obtained, as in [20] or [1], we investigate an optimal liquidation problem for a large portfolio.

The article is organized as follows. In Section 1, we introduce the economic model for bid-ask spread and we present the analysis of prices variance and bias. In Section 2, we study an optimal liquidation problem associated with the bid-ask spread model developed in the previous section. And finally, in Section 3, we provide some numerical results.

1. The model

In this section, we aim at modelling the dynamics of the ask and bid prices of an ETF fund. Our objective is to define an asset price model that considers the bid-ask spread as an inherent part of asset price evolution.

We consider a process X representing an observable but non-tradable benchmark index, e.g., an industrial sector index, which is governed by a given stochastic differential equation (SDE). We also consider a process S representing an ETF on this index. The first question arising from the trading of the asset S concerns its price value. In particular, we address the way the "representative agent" sets his best bid and ask price and the resulting dynamics of those two prices. It is clear that the price of the asset S is generally very closed to the tracked index X.

1.1. Theoretical analysis of path sensitivity and approximation

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the process X is governing by the following SDE.

$$dX_t \quad = \quad r\, X_t\, dt + X_t\, \sigma(t,\, X_t)\, dW_t, \tag{1.1}$$

where r is the drift and σ a function on $\mathbb{R}^+ \times \mathbb{R}$ that verifies the following assumptions.

Assumption 1.1 (Underlying diffusion).

1. SDE (1.1) admits a unique strong solution, denoted X, such that X is square-integrable and does not explode in finite time with probability 1.
2. The solution X of SDE (1.1) is always positive.
3. $f(x) = x\,\sigma(t,\, x)$ is twice derivable function in x and the derivatives are Lipschitz and bounded.

These assumptions cover a large class of stochastic models in finance. In particular, Assumption 1.1 is satisfied by log-normal diffusion and a large part of local volatility models. For Constant Elasticity of Variance model, we may refer to [11].

To simplify our notations, we denote the first and second derivatives of $x\,\sigma(t,\, x)$:

$$\zeta(t,\, x) = \sigma(t,\, x) + x\frac{\partial \sigma}{\partial x}(t,\, x), \qquad \eta(t,\, x) = 2\frac{\partial \sigma}{\partial x}(t,\, x) + x\frac{\partial^2 \sigma}{\partial x^2}(t,\, x).$$

We assume that the price process S of the ETF follows the same SDE as the one of the index X, but its Brownian motion is perturbed by the problem of replication. The process S is therefore assumed to follow the following SDE

$$dS_t \quad = \quad r\, S_t\, dt + \sigma(t,\, S_t)\, S_t\, dB_t,$$

where B is a Brownian motion, which is almost explained by W but characterized by a small uncertainty. We assume that

$$B_t \quad = \quad \sqrt{e^{-\epsilon}}\, W_t + \sqrt{1 - e^{-\epsilon}}\, \widehat{W}_t, \tag{1.2}$$

where ϵ is a small parameter and $\widehat{W}$ is a Brownian motion, independent w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ where $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}$, that resume all hedging errors.

The two Brownian motions, W and $\widehat{W}$, play different roles. W describes the market information, that is progressively known through the index value. Therefore, at time t the information $\mathcal{F}_t$ is known, whereas the information $\mathcal{G}_t = \sigma\{\widehat{W}_s, 0 \leq s \leq t\}$ is unknown or unobservable. To deal with this problem, we choose to follow error theory approach using Dirichlet forms developed on [8] and [9]. We fix an error structure $(\Omega,\, \mathcal{F},\, \mathbb{P},\, \mathbb{D},\, \Gamma)$, where $(\Omega,\, \mathcal{F},\, \mathbb{P})$ is the Wiener space in which the Brownian motion B lives, while Γ is an Ornstein–Uhlenbeck carré du champ operator with constant weight θ (see Section 3 in [9]). Using this theory, formula (1.2), known as Mehler formula, is automatically justified (see Section VI.2. in [8]).

Error theory enables us to find a limited expansion of the law of the price of illiquid asset due to the noise on Brownian motion. In particular, we have the following results.

Theorem 1.2 (Law of illiquid asset price). *Under Assumption* 1.1 *and a Gaussian approximation, the uncertainty on Brownian motion is transmitted to the process S, which represents the illiquid asset price. Then, any realization $\overline{\omega}$ of process X, at time t, fixes a ($\mathcal{F}_t$-conditional) random variable $S_t(\overline{\omega})$ described by*

$$S_t(\overline{\omega},\, \widehat{\omega}) \quad = \quad X_t(\overline{\omega}) + \epsilon\, \mathcal{A}[S]_t(\overline{\omega}) + \sqrt{\epsilon\, \Gamma[S]_t(\overline{\omega})}\, \widetilde{\mathcal{N}}(\widehat{\omega}),$$

where $\widetilde{\mathcal{N}}$ is a centered reduced Gaussian random variable independent w.r.t. $\mathcal{F}_t$, while $\Gamma[S]_t(\overline{\omega})$ and $\mathcal{A}[S]_t(\overline{\omega})$ are given by

$$\begin{cases} \Gamma[S]_t = \theta\, M_t^2 \displaystyle\int_0^t \frac{X_s^2\, \sigma^2(s,\, X_s)}{M_s^2}\, ds + \Gamma[S]_0\, M_t^2\ , \\[2ex] \mathcal{A}[S]_t = M_t \displaystyle\int_0^t \frac{\eta(s,\, X_s)\, \Gamma[S]_s - \theta\, X_s\, \sigma(s,\, X_s)}{2\, M_s}\, \big[dW_s - \zeta(s,\, X_s)\, ds\big]\ , \\[2ex] M_t = \mathcal{E}\left\{ \displaystyle\int_0 \zeta(s,\, X_s)\, dW_s + \int_0 r\, ds \right\}_t\ , \end{cases} \tag{1.3}$$

where $\mathcal{E}$ denotes the Doleans–Dade exponential.

The proof of this theorem is mainly based on the truncated expansion in error theory using Dirichlet forms, see [8] and [9]. The two following lemmas form the main backbone of the proof. Indeed, they give the expression of the variance $\Gamma[S]$ and the bias $\mathcal{A}[S]$.

Lemma 1.3 (Variance due to Brownian motion). *Let X be the solution of SDE* (1.1) *and assume that Assumption* 1.1 *holds. Then, the uncertainty effect on process S satisfies the following SDE*

$$\begin{aligned} d\Gamma[S]_t &= 2\,\zeta(t,\,X_t)\,\Gamma[S]_t\,dW_t \\ &\quad + \left[2\,r + \zeta^2(t,\,X_t)\right]\Gamma[S]_t\,dt + \theta\,\sigma^2(t,\,X_t)\,X_t^2\,dt. \end{aligned} \tag{1.4}$$

Moreover, $\Gamma[S]$ has the following closed form solution

$$\Gamma[S]_t = \theta\,M_t^2 \int_0^t \frac{X_s^2\,\sigma^2(s,\,X_s)}{M_s^2}\,ds + \Gamma[S]_0\,M_t^2.$$

Lemma 1.4 (Bias due to Brownian motion). *Let X be the solution of SDE* (1.1) *and assume that Assumption* 1.1 *holds. Then, the bias effect on process S satisfies the following SDE*

$$d\mathcal{A}[S]_t = r\mathcal{A}[S]_t\,dt + \left[\zeta(t,\,X_t)\mathcal{A}[S]_t + \frac{1}{2}\eta(t,\,X_t)\,\Gamma[S]_t - \frac{\theta}{2}\sigma(t,\,X_t)\,X_t\right]dW_t. \tag{1.5}$$

Moreover, $\mathcal{A}[S]$ has the following closed form solution

$$\mathcal{A}[S]_t = M_t \int_0^t \frac{\eta(s,\,X_s)\,\Gamma[S]_s - \theta\,X_s\,\sigma(s,\,X_s)}{2\,M_s}\left[dW_s - \zeta(s,\,X_s)\,ds\right].$$

The proofs of these lemmas are postponed to the Appendix.

Remark 1.5 (Closed forms). Equations (1.3) show an interesting property of processes $\Gamma[S]$ and $\mathcal{A}[S]$, it is easy to check that the law of $(\Gamma[S]_t,\,\mathcal{A}[S]_t)_{t\geq 0}$ is completely explicit given the law of $(X_t,\,W_t)_{t\geq 0}$. Therefore, Equations (1.3) are closed forms in the sense of involving only algebraic operations and stochastic integrals.

Remark 1.6 (Black–Scholes case). In the particular case of σ constant, i.e., in the Black–Scholes model, Equations (1.3) are simplified with $\Gamma[S]_t$ proportional to the square of X_t and $\mathcal{A}[S]_t$ proportional to X_t.

Moreover, we have the following corollary:

Corollary 1.7 (Equilibrium price). *The equilibrium price, i.e., the mean of the price distribution, is given by*

$$S_t^M \;=\; \mathbb{E}\left[S_t \,\middle|\, \mathcal{F}_t\right] = X_t + \epsilon\,\mathcal{A}[S]_t.$$

The equilibrium price is therefore different from X_t. In particular, this shift exists in Black–Scholes framework. However, in this case, this shift is proportional to X_t, so it is possible to include it into the starting point S_0^M. This shift can explain tracking errors usually observed on ETF-markets, for instance see [14].

Finally as a corollary of the two previous lemmas, we have the following Markov property.

Corollary 1.8 (Markov property). *The triplet* $\widetilde{X} = (X, \Gamma[S], \mathcal{A}[S])$ *is a Markovian process if and only if* X *is Markovian.*

This assertion is a direct consequence of the fact that $\Gamma[S]$ verifies SDE (1.4) which only depends on the process X, and finally $\mathcal{A}[S]$ follows SDE (1.5) which depends on both X and $\Gamma[S]$.

1.2. Bid-ask model

Theorem 1.2 gives us the law of the illiquid asset price given the value of the benchmark/index. In this subsection, we explain how this approach can be used to define bid and ask prices and suggest a model that reproduces it.

We consider the presence of many agents on the market, all informed about the economic evolution of the benchmark price X, but without money-market intelligence about the residual information drawn by the perturbation, i.e., the independent Brownian motion $\widehat{W}$. We assume that all agents are risk adverse and can estimate the distribution of the uncertainty of the illiquid asset price, at any fixed time t, given by Theorem 1.2. We now consider uniquely price-setter agents or liquidity providers who place limit orders as opposed to market orders placed by price-taker agents, liquidity takers. Indeed, their aggregated limit orders constitute an order book and therefore the bid-ask spread. It stands to reason that, at any given time t, there exists a price-setter agent with minimal risk aversion with respect to other agents. This agent accepts to buy the asset at a price S_t^B higher than the prices proposed by the other agents. Thus, the price proposed by this agent is the bid price and it is denoted by S_t^B. This price is completely defined by the law of the illiquid asset and the risk aversion of this agent. A symmetric analysis generates the ask price S_t^A.

Let us assume, for the sake of simplicity, that there exists a representative price-setter agent who always submits the best buy and sell prices, which we respectively define as best bid price S_t^B and best ask price S_t^A. Indeed, we assume that he accepts to buy the illiquid asset at a price S_t^B such that the risk of overvaluing of this asset is equal to a supportable risk probability χ_B. Therefore he takes the risk against the expected earnings, see Figure 1. In conclusion, S_t^B is the χ_B-quantile of the illiquid asset price distribution given by the uncertainty on the Brownian motion, see Theorem 1.2.

The definition of ask price is symmetric, i.e., S_t^A is the $(1 - \chi_A)$-quantile of the illiquid asset price distribution given by the uncertainty on the Brownian motion. It is clear that $\chi_A + \chi_B < 1$, since the representative agent is risk-adverse.

Definition 1.9 (Static bid and ask prices). Let χ_B and χ_A with $\chi_A + \chi_B < 1$ be risks taken by the "representative price setter" in respectively overvaluing and undervaluing the illiquid asset at a given time t. The corresponding bid S_t^B and

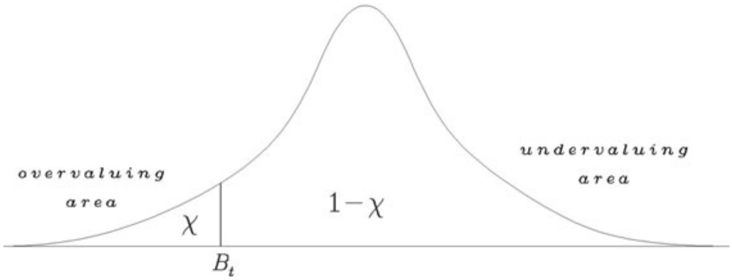

FIGURE 1. Bid price definition, defined by a risk probability χ.

ask S_t^A prices are defined as follow

$$S_t^B = X_t + \epsilon\,\mathcal{A}[S]_t + \sqrt{\epsilon\,\Gamma[S]_t}\,\widetilde{\mathcal{N}}^{-1}(\chi_B),$$
$$S_t^A = X_t + \epsilon\,\mathcal{A}[S]_t + \sqrt{\epsilon\,\Gamma[S]_t}\,\widetilde{\mathcal{N}}^{-1}(1-\chi_A).$$

Since the law of residual uncertainty is always Gaussian, the definition of the bearable risk is equivalent to the definition of the trader utility function. For sake of simplicity, we fix the same supportable risk for sell S_t^B and buy S_t^A prices, i.e., $\chi_B = \chi_A = \chi$. In Figure 2, we study an example and consider the CEV model. Given a trajectory of X, we compute the evolution of the bid price S^B, the mid-price S^M and the ask price S^A.

Remark 1.10. The trajectories of X and S^M are different. This is due to the fact that X is not linear w.r.t. W in CEV diffusion, see Section 2.3, hence the error introduces a bias, see Corollary 1.7.

In order to define a bid-ask model, we have to choose a dynamic for this risk aversion, since a static risk aversion is very restrictive. The dynamic risk aversion is not only justified by the very nature of the “representative price-setter agent” but also by market order flow from price-taker agents.

We now turn to the choice of a dynamics of the bid-ask spread. In the economic literature, bid-ask spread depends mainly on two factors: the value of the stock and the trading volume, see [22]. In particular, the bid-ask spread converges to zero (resp. infinity) when the asset price goes to zero (resp. infinity). However the relative spread[1] converges to a strictly positive constant (resp. zero) when the asset price goes to zero (resp. infinity). This effect can be explained endogenously with the evolution of the variance $\Gamma[S]$, see Section 2.3 for an analysis in CEV case.

The trading volume equally plays a leading role. If we analyze two assets with almost the same price but with different trading volumes, we notice that the lower the trading volume, i.e., the more illiquid the asset is, the larger bid-ask spread, see for instance [24]. An economic explanation is that the traders accept higher

[1]The relative spread is defined as the ratio between bid-ask spread and the asset.

FIGURE 2. An example with CEV model: given a single path of the process X, in black, we can compute explicitly the mid-price, in light grey, and the bid and the ask prices using a standard deviation, in dark grey.

risks if they can easily close their positions, which is possible with the presence of many counterparts. Historical data shows that average trading volumes are mean-reverting on medium term, the bid-ask spread shows the same behavior.

In order to satisfy the above behavior, we use a continuous time finite state markov chain process L representing market liquidity, all state values of L are positive. We assume that the markov chain is irreducible, so that we can say that L is a mean reverting process. Finally, we suppose that $(L_t)_{t\geq 0}$ is independent with respect to $(\mathcal{F}_t)_{t\geq 0}$. We consider the following model of the bid-ask spread.

Definition 1.11 (Bid and ask model). At any time t, given the value of the benchmark $X_t(\overline{\omega})$, the bid and ask prices are given by

$$\begin{cases} S_t^A = & X_t + \epsilon\,\mathcal{A}[S]_t + \sqrt{\epsilon\,\Gamma[S]_t}\;L_t, \\ S_t^B = & X_t + \epsilon\,\mathcal{A}[S]_t - \sqrt{\epsilon\,\Gamma[S]_t}\;L_t. \end{cases} \tag{1.6}$$

Remark 1.12. The choice of this model is justified by the following properties.

- *Positivity*: The ask price is always higher than the bid price.
- *Closed forms*: In our model, all terms, excepted the underlying X, have an explicit form. The law of X is the unique law that we have to estimate

numerically. This computation can be easily performed using a Monte Carlo method.

- *Error tracking*: The mid-price S^M is different to the benchmark one, since a systematic bias exists. The two prices are relatively closed given a small parameter ϵ.
- *Separation*: In our model, the bid-ask spread is explained by two independent factors. The first factor concerns the sensitivity of the benchmark/index level path w.r.t. the Brownian motion W. In an economic point of view, it corresponds to the sensitivity w.r.t. "market" information. The second one is risk aversion of market participants which mainly depends on trading volumes.
- *Mean reverting*: If the underlying value is relatively stable, the bid-ask spread shows a mean reverting behavior.
- *Bid-ask spread tails*: Given the evolution of the benchmark, the law of the bid-ask spread is lognormal, so extremely wide or small spreads are possible but with a very low probability.

2. Optimal liquidation portfolio problem

2.1. The Economic Motivations and the Objective Functions

The above bid-ask spread model, as defined in Definition 1.11, highlights the market imperfections due to liquidity risk. Thus, a natural but challenging problem for both professionals and academics in finance to solve is the optimal portfolio liquidation problem. Let us consider a price-taker investor who decides to close his position over a finite horizon, he has to define a trading strategy which maximizes his terminal portfolio value. Since the attempt to sell the whole block of shares causes, generally, a lack of balance between supply and demand, it results in an average selling price well below the best pre-order bid price. In practice, large orders are generally split into a number of consecutive small orders to reduce the overall price impact.

Let us therefore investigate a problem of an investor seeking to liquidate N shares of stock over a finite time horizon T. To solve this problem, we consider a discrete framework by assuming that trading occurs only at discrete times $t_0 < t_1 < \cdots < t_n = T$. A strategy decision π for the investor is a sequence $(\pi_i)_{0\leq i\leq n}$ valued in $[0, N]$ where π_i is $\mathcal{F}_{t_i}$-measurable and represents the number of shares to be sold at time t_i. We define an admissible strategy as being a strategy π such that $\sum_{i=0}^{n} \pi_i = N$. As such we define the set of admissible strategies $\mathcal{A}(t_i, p)$ as

$$\mathcal{A}(t_i, p) = \left\{\pi = \{\pi_i, \dots, \pi_n\},\, \pi_j \geq 0 \quad \forall j \in \{i, \dots, n\} \ \text{ and } \ \sum_{j=i}^{n} \pi_j = p\right\}.$$

Price impact. In addition to the existence of the bid-ask spread as evidenced in the previous section, we equally take into account a lack of market depth by assuming that marginal selling prices are non-increasing. Indeed, there is no infinite liquidity at either the best bid price nor at the best ask price. For that purpose, we introduce an impact function g which indicates the average price obtained at a sell market order. More precisely, when an investor submits a sell order of x number of shares through a single sell order at time t, the obtained average price $\overline{S}_t^B(x)$ is assumed

$$\overline{S}_t^B(x) \quad = \quad S_t^B\, g(x), \tag{2.1}$$

where the function g verifies the following assumption.

Assumption 2.1 (Trading impact function).

1. g is a continuous positive deterministic function independent to S.
2. g is non-increasing.
3. $h(x) = x\, g(x)$ is strictly non-decreasing and concave.

Remark 2.2.

1. We assume that trading impact is temporary when trading occurs. Only the price takers who place market price orders (at best selling prices) pay the liquidity costs. After the trades, the order book is refilled with limit orders from other market participants [1].
2. $g(x)$ corresponds to the ratio between the average stock price received following the sale of x shares at market price order and the best bid price. This average price obviously decreases with the number of traded shares.
3. The marginal price $[(x+\delta x)g(x+\delta x)-xg(x)]S^B$ should be non-negative and non-increasing. Therefore, the function $h(x) = x\, g(x)$ must be non-decreasing and concave. The concavity comes from the shape of the order book, which displays a maximum around the best bid price, see [22].

Objective function. The objective of the investors is to maximize their net present value (wealth) from the sales of the stock shares in holding. To fully describe our state process, we should take into account not only the processes X and L but also $\Gamma[S]$ and $\mathcal{A}[S]$. As such, the state process to consider is $Z = (\tilde{X}, L)$, where $\tilde{X}$ is defined as in Corollary 1.8. At any initial time t_i and any state value $(z,\, p)$ of the variables $(Z_{t_i},\, P_{t_i})$, with P_{t_i} the number of stock shares that we initially have at time t_i, we define our reward function for any strategy $\pi \in \mathcal{A}(t_i, p)$ by

$$J(i,\, z,\, p,\, \pi) = \mathbb{E}\left[\sum_{j=i}^{n} e^{-\rho(t_j - t_i)} \pi_j\, S_{t_j}^B\, g(\pi_j) \Big| \mathcal{F}_{t_i}\right],$$

where ρ represents the interest rate.

The objective of the investor is to maximize this reward function over all admissible strategies. We therefore introduce the following value function

$$v(i,\, z,\, p) = \sup_{\pi \in \mathcal{A}(t_i, p)} (J(i,\, z,\, p, \pi)). \tag{2.2}$$

For an initial state $(i,\, z,\, p)$, $\widehat{\pi} \in \mathcal{A}(t_i, p)$ is called an optimal strategy if

$$v(i,\, z,\, p) = J(i,\, z,\, p,\, \widehat{\pi}).$$

In the sequel, we restrict the set of admissible strategies $\mathcal{A}(t_i,p)$ to Markov strategies subset of $\mathcal{A}(t_i,p)$, which is denoted $\bar{\mathcal{A}}(t_i,p)$ (that is possible from Proposition 8.1 of [5]).

2.2. Theoretical solution of the optimization problem

We now prove the existence of a solution to our optimization problem (2.2) and its uniqueness. Given an initial N stock shares of the risky assets, our objective is to prove that an optimal strategy in liquidating our portfolio exists in $\mathcal{A}(t_0,N)$ and this one is unique. For notation convenience, we shall denote Z_i (resp. S_i, P_i) for Z_{t_i} (resp. S_{t_i}, P_{t_i}). Using the dynamic programming principle, we have:

Theorem 2.3 (Existence). *Under Assumptions* 1.1 *and* 2.1*, there exists an optimal policy* $\widehat{\pi} = (\widehat{\pi}_0, \dots, \widehat{\pi}_n)$ *to the optimization problem, such that* $\widehat{\pi} \in \bar{\mathcal{A}}(t_0,N)$*. This optimal strategy is given by the argmax in the following programming equation*

$$\begin{cases} v(i,\, z,\, p) = \operatorname*{ess\,sup}_{0 \le \pi_i \le p} \Big\{ \pi_i\, s_i^B\, g(\pi_i) \\ \qquad\qquad\qquad + \mathbb{E}\Big[e^{-\rho(t_{i+1}-t_i)} v(i+1,\, Z_{i+1}^{i,z},\, p-\pi_i) \Big| \mathcal{F}_{t_i} \Big] \Big\}, \\ v(n,\, z,\, p) = p\, s_n^B\, g(p), \end{cases} \tag{2.3}$$

where s_i^B *is defined by the components of the variable* z_i *(see the definition of* S^B *in* (1.6)*).*

Proof. This is an immediate application of Proposition 8.5 of [5]. From (1.6), we have

$$\mathbb{E}\left[S_t^B\right] = \mathbb{E}\left[X_t\right] + \epsilon\, \mathbb{E}\left[\mathcal{A}[S]_t\right] - \mathbb{E}\left[\sqrt{\epsilon\, \Gamma[S]_t}\right]\, \mathbb{E}\left[L_t\right].$$

From Assumption 1.1, X does not explode in finite time and $\sigma(t,\, x)$, $\zeta(t,\, x)$ and $\eta(t,\, x)$ are Lipschitz and bounded, thus $\mathbb{E}\big[\mathcal{A}[S]_t\big] < \infty$ and $\mathbb{E}\big[\sqrt{\epsilon\, \Gamma[S]_t}\big] < \infty$. Since the process L is a finite state markov chain, it is clear that $\mathbb{E}\left[L_t\right] < \infty$ and as the process X is square integrable, we also have that $\mathbb{E}\left[X_t\right] < \infty$. Therefore $\mathbb{E}\left[S_t^B\right] < \infty$ and it enables us to check assumptions (F^+) and (F^-) in Proposition 8.5 of [5]. Then, it remains to prove that the supremum in relation (2.3) is attained. This immediately follows from the continuity of $v(i+1,\, z,\, p)$ with respect to p, which is the case thanks to Assumption 2.1. □

We now turn to the uniqueness property of the optimal strategy.

Theorem 2.4 (Uniqueness). *Under Assumptions* 1.1 *and* 2.1*, there is at most one solution to optimization problem* (2.2)*.*

Proof. We first introduce the following function ϑ defined for any $x \le y$ as

$$\vartheta(i,\, z,\, x,\, y) = x\, s_i^B\, g(x) + \mathbb{E}\Big[e^{-\rho \Delta_i} v(i+1,\, Z_{i+1}^{i,z},\, y-x) \Big| \mathcal{F}_{t_i} \Big], \qquad i \in \{1, \dots, n-1\},$$

with $\Delta_i := t_{i+1} - t_i$.

We now prove by recurrence that ϑ is concave w.r.t. (x, y) and the value function v is strictly concave w.r.t. p for all $i \in \{0, \dots, n\}$.

We first note that for $i = n$, $v(n, z, p)$ is strictly concave w.r.t. p, according to Assumption 2.1. We can easily verify that $\vartheta(n, z, x, y)$ is concave w.r.t. (x, y). Assuming that for $i+1$, $v(i+1, z, p)$ and $\vartheta(i+1, z, x, y)$ are respectively strictly concave w.r.t. p and to (x, y), let prove that it is equally the case for i.

Let $0 \le \lambda \le 1$, (x_1, y_1) and (x_2, y_2), with $0 \le x_i \le y_i \le N$, we have

$$\begin{aligned}
&\vartheta(i, z, \lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2) \\
&\quad = (\lambda x_1 + (1-\lambda) x_2)\, s_i^B\, g(\lambda x_1 + (1-\lambda) x_2)) \\
&\qquad + \mathbb{E}\Big[e^{-\rho \Delta_i} v(i+1, Z_{i+1}^{i,z}, \lambda (y_1 - x_1) + (1-\lambda)(y_2 - x_2))\Big| \mathcal{F}_{t_i}\Big],
\end{aligned}$$

since the first term is strictly concave, we obtain

$$\begin{aligned}
&(\lambda x_1 + (1-\lambda) x_2)\, s_i^B\, g(\lambda x_1 + (1-\lambda) x_2) \\
&\quad > \lambda x_1\, s_i^B\, g(x_1) + (1-\lambda) x_2\, s_i^B\, g(x_2),
\end{aligned}$$

and by recurrence the second term is strictly concave, we obtain

$$\begin{aligned}
&v(i+1, Z_{i+1}^{i,z}, \lambda (y_1 - x_1) + (1-\lambda)(y_2 - x_2)) \\
&\quad > \lambda\, v(i+1, Z_{i+1}^{i,z}, y_1 - x_1) + (1-\lambda)\, v(i+1, Z_{i+1}^{i,z}, y_2 - x_2).
\end{aligned}$$

Taking the expectation, we get

$$\begin{aligned}
&\mathbb{E}\Big[e^{-\rho \Delta_i} v(i+1, Z_{i+1}^{i,z}, \lambda (y_1 - x_1) + (1-\lambda)(y_2 - x_2))\Big| \mathcal{F}_{t_i}\Big] \\
&\quad \ge \lambda\, \mathbb{E}\Big[e^{-\rho \Delta_i} v(i+1, Z_{i+1}^{i,z}, y_1 - x_1)\Big| \mathcal{F}_{t_i}\Big] \\
&\qquad + (1-\lambda)\, \mathbb{E}\Big[e^{-\rho \Delta_i} v(i+1, Z_{i+1}^{i,z}, y_2 - x_2)\Big| \mathcal{F}_{t_i}\Big].
\end{aligned}$$

Thus $\vartheta(i, z, x, y)$ is strictly concave w.r.t. (x, y) as the sum of two strictly concave functions.

We now prove that $v(i, z, y)$ is strictly concave w.r.t. y. For any $0 \le x_1 \le y_1$ and $0 \le x_2 \le y_2$, from the expression of v in (2.3), we have for all $0 \le \lambda \le 1$

$$v(i, z, \lambda y_1 + (1-\lambda) y_2) \ge \vartheta(i, z, \lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2).$$

Since $\vartheta(i, z, x, y)$ is strictly concave with respect to (x, y), we get

$$v(i, z, \lambda y_1 + (1-\lambda)\, y_2) > \lambda\, \vartheta(i, z, x_1, y_1) + (1-\lambda)\, \vartheta(i, z, x_2, y_2).$$

The latter equality holds for any positive $x_1 \le y_1$ and $x_2 \le y_2$. In particular since the supremum is attained (from Theorem 2.3), we can take x_1^* and x_2^* such that

$$\begin{aligned}
v(i, z, y_1) &= \vartheta(i, z, x_1^*, y_1) = \sup_{0 \le x \le y_1} \vartheta(i, z, x, y_1) \\
v(i, z, y_2) &= \vartheta(i, z, x_2^*, y_2) = \sup_{0 \le x \le y_2} \vartheta(i, z, x, y_2)
\end{aligned}$$

thus

$$v(i, z, \lambda y_1 + (1-\lambda) y_2) > \lambda\, v(i, z, y_1) + (1-\lambda)\, v(i, z, y_2).$$

Hence, $v(i,\, z,\, p)$ is strictly concave w.r.t. p. We have therefore proved the strict concavity of both functions. Using relation (2.3) and the above concavity property, we may obtain by iteration at most one solution to the optimization problem. □

2.3. Log-normal and constant elasticity of variance diffusions

For numerical and implementation purpose, we now consider our two particular diffusion models, with the first being the log-normal diffusion, i.e.,

$$dX_t = r\, X_t\, dt + \sigma\, X_t\, dW_t.$$

It is clear that this diffusion verifies Assumption 1.1. In this case, we remark that the bias $\mathcal{A}[S]_t$ and the variance $\Gamma[S]_t$ become proportional respectively to X_t and X_t^2. As a result,

$$S_t^B = X_t\, [1 - \epsilon\, \mathbf{a} + \sqrt{\epsilon\, \gamma}\, L_t]\,,$$

where $\mathbf{a}$ and γ are constants, and the average price at which we sell a quantity π_i at time t_i, given by formula (2.1), is simplified and we have the following average price

$$\overline{S}_{t_i}^B(\pi_i) = X_{t_i}\, g(\pi_i)\ \ [1 - \epsilon\, \mathbf{a} + \sqrt{\epsilon\, \gamma}\, L_{t_i}]\,.$$

As such, we consider an extension of the Black–Scholes model, which is the CEV model, see [11]. This model takes into account the heteroscedasticity of the asset returns and explains the down-slopping behavior of the implied volatility, see for instance [19].

Assumption 2.5 (CEV diffusion). The volatility function $\sigma(t,\, X_t)$ is equal to $\sigma\, X_t^{\alpha}$, where σ is a positive constant and α is constant and belongs to $(-1,\, 1)$. We also assume that $X_t \geq \xi > 0$ for all $t \in [0,\, T]$. That is SDE (1.1) is replaced by the following SDE

$$dX_t \quad = \quad r\, X_t\, dt + \sigma\, X_t^{\alpha+1}\, dW_t. \tag{2.4}$$

The CEV diffusion, unfortunately, does not verify Assumption 1.1. However, all previous results still hold and their proofs remain substantially the same with the main difference coming from some properties of CEV diffusion that can be found in [15].

Under Assumption 2.5, we have also the following rewriting of Theorem 1.2.

Corollary 2.6 (Constant elasticity of variance model). *Under Assumption* 2.5*, the result of Theorem* 1.2 *remains true and Equations* (1.3) *are replaced by*

$$\begin{cases} \Gamma[S]_t = \theta\, M_t^2 \displaystyle\int_0^t \frac{\sigma^2\, X_s^{2\alpha+2}}{M_s^2}\, ds + \Gamma[S]_0\, M_t^2, \\ \mathcal{A}[S]_t = M_t \displaystyle\int_0^t \frac{\alpha\,(\alpha+1)\, X_s^{\alpha-1}\, \Gamma[S]_s - \theta\, \sigma\, X_s^{\alpha+1}}{2\, M_s} \times \\ \qquad\qquad \big[dW_s - \sigma\,(\alpha+1)\, X_s^{\alpha}\, ds\big], \\ M_t = \mathcal{E}\left\{\sigma\,(\alpha+1) \displaystyle\int_0^t X_s^{\alpha}\, dW_s + r\, t\right\}. \end{cases}$$

Moreover, the martingale parts of the Doob decomposition of $\sqrt{\Gamma[S]_t}$ *and* $\frac{\sqrt{\Gamma[S]_t}}{X_t}$ *are respectively*

$$\begin{cases} \sigma\,(\alpha+1)\,X_t^{\alpha}\,\sqrt{\Gamma[S]_t}\,dW_t, \\ \sigma\,\alpha\,X_t^{\alpha-1}\,\sqrt{\Gamma[S]_t}\,dW_t. \end{cases}$$

Proof. The proof of the first part is just a simplification of relation (1.3) in the case of CEV diffusion. The second part is an easy application of the Itô formula. □

Remark 2.7. In particular, we may notice that in CEV case:

- The fluctuations of the absolute spread, which are proportional to $\sqrt{\Gamma[S]}$, are always positively correlated with the underlying X.
- The fluctuations of the relative spread, which are proportional to the ratio $\sqrt{\Gamma[S]}$ over X, are negatively (resp. positively) correlated with the underlying X if the CEV-exponent α is negative (resp. positive). The case usually treated in literature is when the CEV-exponent is smaller than one, for instance see [6]. Therefore, the relative bid-ask spread grows when the asset price falls, whereas the absolute bid-ask spread falls with the benchmark.

This remark is important, since it is well known on financial markets that the bid-ask spread converges to zero (resp. infinity) when the asset price goes to zero (resp. infinity). Instead, the relative spread grows when the asset price goes to zero and converges to zero when the asset price goes to infinity. The previous remark says that our model can explain endogenously this effect with the evolution of the variance $\Gamma[S]$ if we suppose that the parameter α is negative, i.e., the diffusion is sub-linear. This case is usually presented in literature as a way to explain why the BS model overprices in-the-money call options and underprices out-of-the-money ones, see for instance [11] and [19].

3. Numerical results

In this section, we provide some numerical results of the optimal strategy of the liquidation problem. For this purpose, we use two models.

1. The Black–Scholes model with the drift equal to zero. This model is used as a basic reference, since we have closed form expressions.
2. A CEV model with the drift equal to zero and a negative CEV-exponential. This model is used to evaluate the impact of sensitivity w.r.t. the Brownian motion.

We consider the following trading impact function g

$$g(x) = \exp(-\lambda x)\ , \tag{3.1}$$

where the constant $\lambda < \frac{1}{N}$, with N the total number of shares to liquidate. It is rather clear that function g verifies Assumption 2.1 on the interval $[0, N]$. We restrict our function to the set $[0, N]$ given the fact that the optimal liquidation strategy abstains to buy shares at any time.

The optimal liquidation strategy is determined by using the dynamic programming equation (2.3).

The classical approach to solve this kind of problem is to discretize all processes and to reduce the computation on a finite probability space, see [4]. Thus, we discretize our processes using Monte Carlo simulations for CEV one and closed forms for log-normal one.

3.1. Black–Scholes case

We first consider the case when the underlying X follows SDE (2.4) with $\alpha = 0$, i.e., the Black–Scholes model. The dynamic programming principle (2.3) gives us the optimal strategy to liquidate our portfolio. The strategy depends on three factors:

– the level of the underlying X,
– the value of the liquidity process L,
– the residual quantity of stocks that we still have to sell.

In Figures 3, 4, and 5, we present our numerical results in the Black–Scholes case, in particular, the dependencies of the optimal selling strategy on the three factors mentioned above.

Remark 3.1 (Black–Scholes case).

- We find that the optimal strategy is completely independent with respect to underlying X_t (see Figure 3). This result is coherent with the literature, see [1], and [20], given the fact that the spread is proportional to the underlying price in this particular case.
- The optimal strategy is almost linear with respect to the number of remaining stocks (see Figure 4). A slight concavity is equally worth noticing. This effect is explained by the presence of an exponential cost (3.1) on optimization problem (2.2), which breaks the linearity of the problem and prevents very large orders.
- Finally, the main result is that the optimal strategy decreases when the bid-ask spread increases and the dependence is almost linear till the spread is lower than its long term average (see Figure 5). When the spread is bigger than its mean, the optimal strategy is to keep all remaining stocks. This result is very interesting since it says that the optimal strategy depends mainly on the bid-ask spread and its equilibrium law. The optimal strategy can be resumed by we have to sell when the spread is small and to wait a better time when it is wide.

3.2. CEV Case

We consider the case when the underlying follows SDE (2.4) with $\alpha = -0.7$. The numerical simulations show that the results found in the Black–Scholes case remain

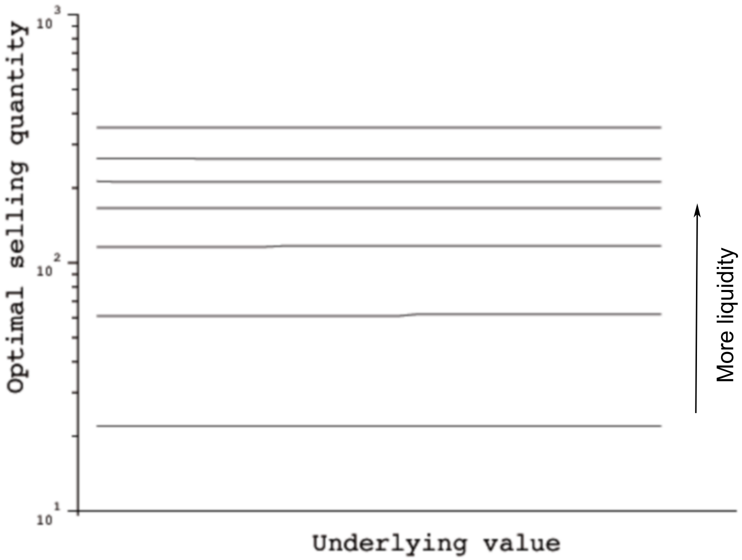

FIGURE 3. BS case: optimal selling quantities as function of the underlying value for different values of liquidity process L_t.

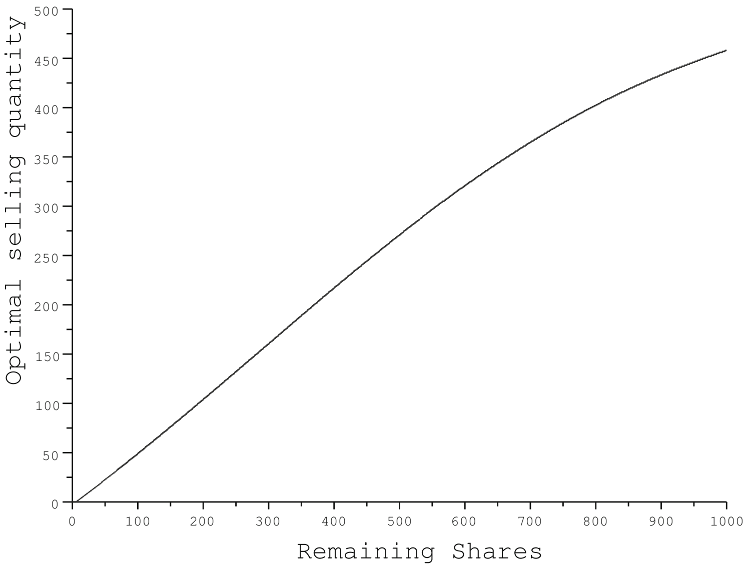

FIGURE 4. BS case: optimal selling quantities as function of the remaining shares owned by the investor.

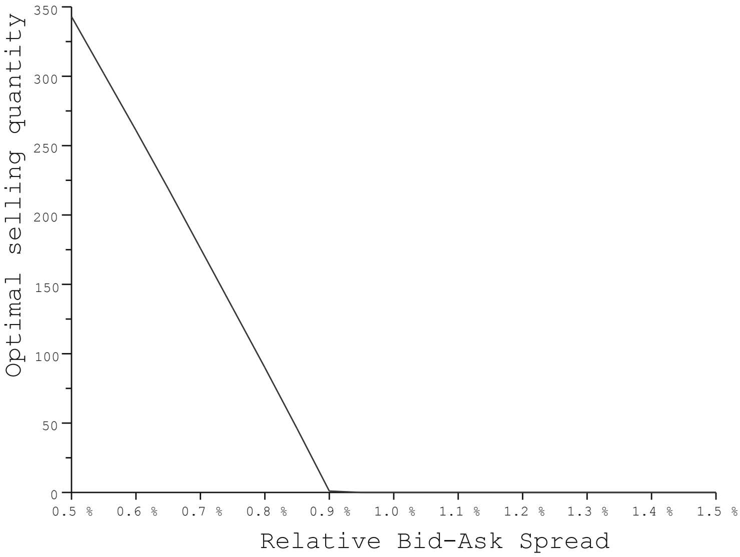

FIGURE 5. BS case: optimal selling quantities as function of the relative bid-ask spread.

true, except for the dependencies of the optimal selling strategies on underlying value (see Figure 6).

Remark 3.2. (CEV Case)

1. The optimal selling strategy is completely unaffected by underlying price when the stock is highly liquid. However, when it is illiquid, the optimal selling strategy is positively correlated with the price of underlying asset.
2. We also have analyzed the impact of a change on the CEV exponential α. When this parameter increases to zero, the dependency on the price of underlying asset is lessened.

Economic interpretation/explanation. *We may explain the effect mentioned in the first point of Remark* 3.2 *by the non-linear dependency of the bid-ask spread on the underlying value* (*see Corollary* 2.6)*. Indeed, when the price of underlying asset falls, the relative bid-ask spread* (*in percentage of the asset price*) *increases, which in turn incites the investor to delay the selling or sell a smaller number of shares.*

The below Figure 7 shows the shape of the selling region (above the curve) and the non-selling region (below the curve) at a given time and a given level of stock price.

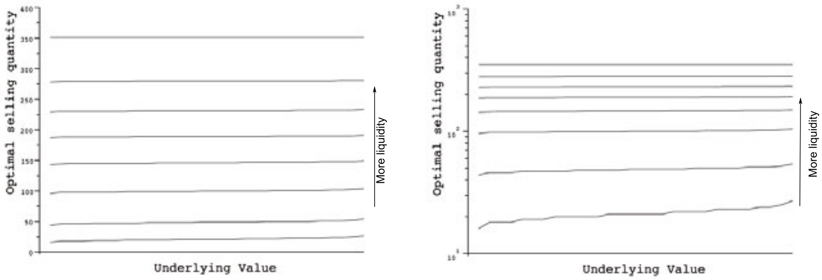

FIGURE 6. CEV case: optimal selling quantities as function of the underlying value with linear scale (left) and logarithmic scale (right).

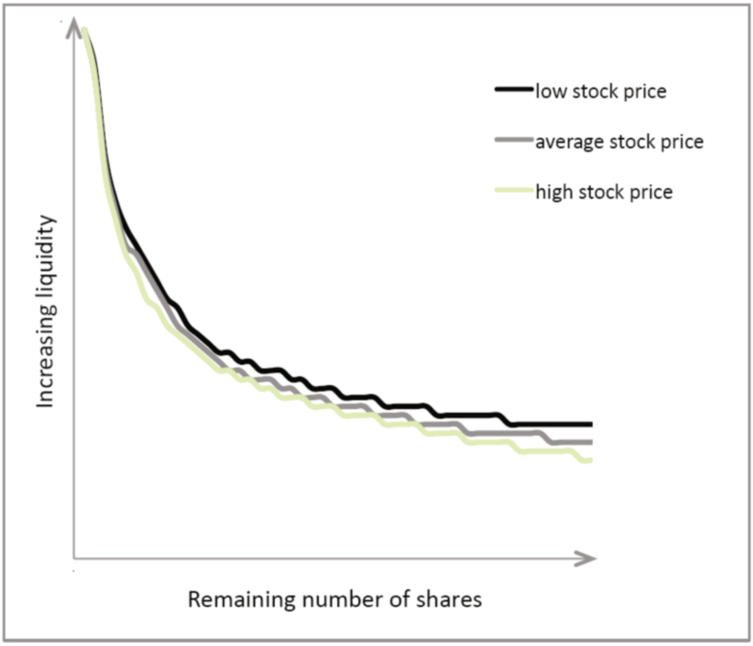

FIGURE 7. CEV case: an example of the selling regions as function of the remaining number of shares.

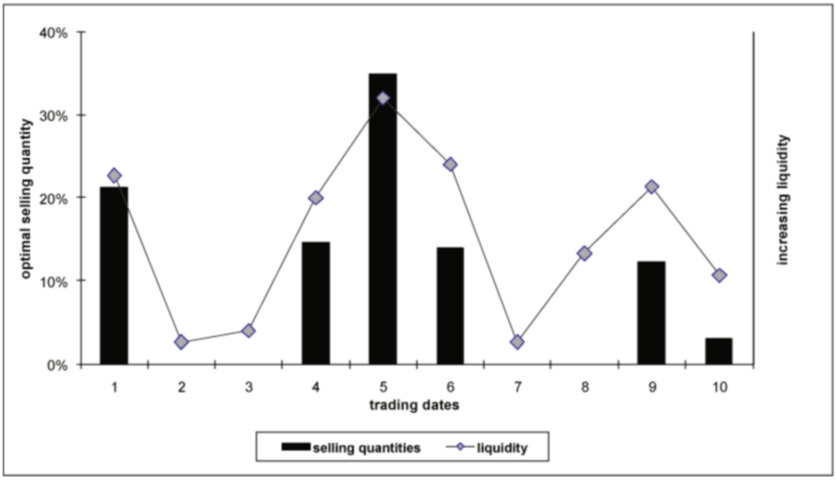

FIGURE 8. CEV case: an example of the optimal selling strategy given a liquidity trajectory.

Appendix. Proofs of Lemmas 1.3 and 1.4

We start with the proof of Lemma 1.3.

Proof. The proof is split into three steps.

Step 1: We compute the SDE satisfied by the sharp of S, see Section V.2 in [8] for the definition and more details:

$$dS_t^{\#} = r\, S_t^{\#}\, dt + \zeta(t,\, X_t)\, S_t^{\#}\, dW_t + \sqrt{\theta}\, \sigma(t,\, X_t)\, X_t\, d\widetilde{W}_t,$$

where $\widetilde{W}$ is an independent Brownian motion defined in a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ copy of the original probability space. Assumption 1.1 insures that the previous SDE admits a square integrable solution.

Step 2: We apply the Itô formula to $(S^{\#})^2$ and take the expectation under the probability $\widetilde{\mathbb{P}}$, since one of the properties of the sharp operator is that $\Gamma[S]_t = \widetilde{\mathbb{E}}\left[(S_t^{\#})^2\right]$, see Sections VI.2 and VII.4 in [8]. Therefore we find SDE (1.4).

Step 3: Finally, we prove that SDE (1.4) admits the closed form solution (1.3). Using the methods developed in Section 5 in [9] and noticing that the SDE verified by the sharp is linear, we may apply a variation of constant method, see for instance V.9 in [23], and obtain a closed form for $S^{\#}$. Then, we compute easily the expectation under $\widetilde{\mathbb{P}}$ of the square of $S^{\#}$. Another possibility is to check that the first equation in (1.3) is solution to SDE (1.4). □

We now turn to the proof of Lemma 1.4.

Proof. The proof is based on an L^2-convergence argument, see for instance [13], by using the fact that operators $\Gamma[\cdot]$ and $\mathcal{A}[\cdot]$ are closed, see [8].

We define a partition $\{\tau_i\}_{i=0,\dots,n}$ of the interval $[0, T]$, where T is a sufficient large time. We approximate X with the following process

$$Z_t = \sum_{i=0}^{n-1} \sigma(\tau_i, Z_{\tau_i})\, Z_{\tau_i}\left(W_{\tau_{i+1}\wedge t} - W_{\tau_i\wedge t}\right) + r\sum_{i=0}^{n-1} Z_{\tau_i}\left(\tau_{i+1}\wedge t - \tau_i\wedge t\right).$$

It is clear that Z converges to X when the partition step goes to zero, thanks to hypotheses 1 and 3 of Assumption 1.1. Then, we apply the bias operator on Z (see Section 6 in [9]), and we find

$$\begin{aligned}
\mathcal{A}[Z]_t = &\sum_{i=0}^{n-1}\Big\{\zeta(\tau_i, Z_{\tau_i})\,\mathcal{A}\,[Z]_{\tau_i}\,\left(W_{\tau_{i+1}\wedge t} - W_{\tau_i\wedge t}\right)\\
&\qquad + r\mathcal{A}\,[Z]_{\tau_i}\,\left(\tau_{i+1}\wedge t - \tau_i\wedge t\right)\Big\}\\
&+\sum_{i=0}^{n-1}\Big\{\sigma(\tau_i, Z_{\tau_i})Z_{\tau_i}\,\mathcal{A}\left[W_{\tau_{i+1}\wedge t} - W_{\tau_i\wedge t}\right]\\
&\qquad + \zeta(\tau_i, Z_{\tau_i})\,\widetilde{\mathbb{E}}\left[Z^{\#}_{\tau_i}\left(W^{\#}_{\tau_{i+1}\wedge t} - W^{\#}_{\tau_i\wedge t}\right)\right]\Big\}\\
&+\frac{1}{2}\sum_{i=0}^{n-1}\eta(\tau_i, Z_{\tau_i})\,Z_{\tau_i}\,\Gamma\left[Z_{\tau_i}\right]\,\left(W_{\tau_{i+1}\wedge t} - W_{\tau_i\wedge t}\right).
\end{aligned} \tag{A.1}$$

Then, we take the limit in SDE (A.1), when the step of the partition goes to zero and we have to prove that equation (A.1) converges to the integral form of equation (1.5). For sake of simplicity on notation, we compute all equations at the final time T. We start with the remark that Z converges to X in L^2-norm when the step partition goes to zero. It is also clear that $\Gamma[Z]$ converges to $\Gamma[S]$ due to the fact that Γ is a closed operator, see for instance [8], and the result of Lemma 1.3, this convergence is in L^1-norm but it is easy to check that it is true in L^2-norm too. Then, under Assumption 1.1, we will prove that

$$\sum_{i=0}^{n-1}\eta(\tau_i, Z_{\tau_i})\,\Gamma[Z]_{\tau_i}\left(W_{\tau_{i+1}\wedge t} - W_{\tau_i\wedge t}\right) \quad \overset{L^2}{\to} \quad \int_0^T \eta(t, X_t\,\Gamma[S]_t\,dW_t. \tag{A.2}$$

We separate the last integral using the partition $(\tau_i)_{i=0,\dots,n}$ and we evaluate the difference in L^2-norm, so we find

$$\begin{aligned}
&\mathbb{E}\Big[\Big\{\,\eta(\tau_i, Z_{\tau_i})\,\Gamma[Z]_{\tau_i}\left(W_{\tau_{i+1}} - W_{\tau_i}\right) - \int_{\tau_i}^{\tau_{i+1}}\eta(t, X_t)\,\Gamma[S]_t\,dW_t\,\Big\}^2\Big]\\
&\quad = \mathbb{E}\Big[\Big\{\eta(\tau_i, Z_{\tau_i})\,\Gamma[Z]_{\tau_i}(W_{\tau_{i+1}} - W_{\tau_i}) - \eta(\tau_i, X_{\tau_i})\,\Gamma[Z]_{\tau_i}(W_{\tau_{i+1}} - W_{\tau_i})\\
&\qquad + \eta(\tau_i, X_{\tau_i})\,\Gamma[Z]_{\tau_i}\left(W_{\tau_{i+1}} - W_{\tau_i}\right) - \eta(\tau_i, X_{\tau_i})\,\Gamma[S]_{\tau_i}\left(W_{\tau_{i+1}} - W_{\tau_i}\right)\\
&\qquad + \eta(\tau_i, X_{\tau_i})\,\Gamma[S]_{\tau_i}\,\left(W_{\tau_{i+1}} - W_{\tau_i}\right) - \int_{\tau_i}^{\tau_{i+1}}\eta(t, X_t)\,\Gamma[S]_t\,dW_t\Big\}^2\Big]
\end{aligned}$$

$$
\begin{aligned}
< \mathbb{E}\Big[\big\{[\eta(\tau_i,\, Z_{\tau_i}) - \eta(\tau_i,\, X_{\tau_i})]\; \Gamma[Z]_{\tau_i}\; \big(W_{\tau_{i+1}} - W_{\tau_i}\big)\big\}^2\Big] \\
+\, \mathbb{E}\Big[\big\{\eta(\tau_i,\, X_{\tau_i})\; [\Gamma[Z]_{\tau_i} - \Gamma[S]_{\tau_i}]\; \big(W_{\tau_{i+1}} - W_{\tau_i}\big)\big\}^2\Big] \\
+\, \mathbb{E}\Big[\Big\{\eta(\tau_i,\, X_{\tau_i})\, \Gamma[S]_{\tau_i}\; \big(W_{\tau_i} - W_{\tau_{i-1}}\big) - \int_{\tau_i}^{\tau_{i+1}} \eta(t,\, X_t)\, \Gamma[S]_t dW_t\Big\}^2\Big].
\end{aligned}
$$

The first expectation converges to zero thanks to the Lipschitz hypothesis on $\eta(t,\, x)$ w.r.t. x. The second expectation goes to zero using the fact that $\Gamma[Z]$ converges to $\Gamma[S]$ in L^2-norm. The last expectation converges to zero when the partition step goes to zero in accordance with the definition of the stochastic integral. Then the limit (A.2) is proved.

Using the same arguments and Gronwall lemma, see for instance Chapter V in [23], we have

$$
\sum_{i=0}^{n-1} \zeta(\tau_i,\, Z_{\tau_i})\, \mathcal{A}\,[Z]_{\tau_i}\; \big(W_{\tau_{i+1}\wedge t} - W_{\tau_i\wedge t}\big) \quad \overset{L^2}{\to} \quad \int_0^T \zeta(t,\, X_t)\, \mathcal{A}\,[S]_t\; dW_t
$$

and

$$
\sum_{i=0}^{n-1} \mathcal{A}\,[Z]_{\tau_i}\; (\tau_{i+1} - \tau_i) \quad \overset{L^2}{\to} \quad \int_0^T \mathcal{A}\,[S]_t\; dt.
$$

We study the third term in equation (A.1). We find

$$
\sum_{i=0}^{n-1} \sigma(\tau_i,\, Z_{\tau_i})\, Z_{\tau_i}\, \mathcal{A}\left[W_{\tau_{i+1}\wedge t} - W_{\tau_i\wedge t}\right] = -\frac{\theta}{2} \sum_{i=0}^{n-1} \sigma(\tau_i,\, Z_{\tau_i})\, Z_{\tau_i}\; \left[W_{\tau_{i+1}} - W_{\tau_i}\right],
$$

thanks to the chain rule of semigroup $\mathcal{A}$, see Section 3 in [9]. We also remark that $\sigma(\tau_i,\, Z_{\tau_i})\, Z_{\tau_i}$ converges to $\sigma(\tau_i,\, X_{\tau_i})\, X_{\tau_i}$, thanks to Assumption 1.1. Using always the same arguments used to prove limit (A.2), we have therefore that

$$
\sum_{i=0}^{n-1} \sigma(\tau_i,\, Z_{\tau_i})\, Z_{\tau_i}\, \mathcal{A}\left[W_{\tau_{i+1}\wedge t} - W_{\tau_i\wedge t}\right] \quad \overset{L^2}{\to} \quad -\frac{\theta}{2}\int_0^T \sigma(t,\, X_t)\, X_t\, dW_t.
$$

Finally, we analyze the term

$$
\sum_{i=0}^{n-1} \zeta(\tau_i,\, Z_{\tau_i})\, \widetilde{\mathbb{E}}\left[\left(Z^{\#}_{\tau_i}\right)\, \left(W^{\#}_{\tau_{i+1}\wedge t} - W^{\#}_{\tau_i\wedge t}\right)\right].
$$

We introduce a conditional expectation with respect to the σ-algebra $\widetilde{\mathcal{F}}_{\tau_i} = \sigma\{(W^{\#}_s,\, W_u)\,|\, u,\, s < \tau_i\}$

$$
\begin{aligned}
\sum_{i=0}^{n-1} \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[\left(Z^{\#}_{\tau_i}\right)\left(W^{\#}_{\tau_{i+1}\wedge t} - W^{\#}_{\tau_i\wedge t}\right)\middle|\, \widetilde{\mathcal{F}}_{\tau_i}\right]\right] \\
= \sum_{i=0}^{n-1} \widetilde{\mathbb{E}}\left[\left(Z^{\#}_{\tau_i}\right) \widetilde{\mathbb{E}}\left[W^{\#}_{\tau_{i+1}} - W^{\#}_{\tau_i}\middle|\, \widetilde{\mathcal{F}}_{\tau_i}\right]\right] = 0,
\end{aligned}
$$

using the fact that $Z^{\#}$ is adapted to the filtration $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ and $W^{\#}$ remains a Brownian motion w.r.t. filtration $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$, since $W^{\#}$ and W are independent. As a consequence, the fourth term in equation (A.1) is always equal to zero and we have proved the convergence of equation (A.1) to the integral form of equation (1.5). Now it is easy to check that the second equation in (1.3) solves SDE (1.5) □

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Optimal Portfolio in a Regime-switching Model

Adrian Roy L. Valdez and Tiziano Vargiolu

Abstract. In this paper we derive the solution of the classical Merton problem, i.e., maximizing the utility of the terminal wealth, in the case when the risky assets follow a diffusion model with switching coefficients. We show that the optimal portfolio is a generalisation of the corresponding one in the classical Merton case, with portfolio proportions which depend on the market regime. We perform our analysis via the classical approach with the Hamilton–Jacobi–Bellman equation. First we extend the mutual fund theorem as presented in [5] to our framework. Then we show explicit solutions for the optimal strategies in the particular cases of exponential, logarithm and power utility functions.

Mathematics Subject Classification (2010). 35K40, 60H30, 91G10, 91G80, 93E20.

Keywords. Utility maximization, regime-switching models, HJB equation, mutual fund theorem, exponential utility, logarithmic utility, power utility.

1. Introduction

The standard model for a financial market is the following. Let the bond price B_t satisfy

$$dB_t = rB_t dt,$$

with $r > 0$ deterministic, and the stock prices $S_t = (S_t^1, \dots, S_t^d)$ satisfy

$$dS_t = \text{diag }(S_t)(\mu(t, S_t)\ dt + \Sigma(t, S_t)\ dW_t),$$

where $\mu : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\Sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are suitable functions and W is a d-dimensional Brownian motion defined on a suitable probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Assume also that Σ has full rank. In this situation the classical problems (utility maximization, pricing and hedging of derivatives) are solved.

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However, these kinds of model fail in incorporating sudden changes in the dynamics of the assets, occurring for example during a financial crisis. Among the various models which are more suitable for this, we will analyse the so-called regime-switching models, introduced for the first time in [12]: while these models are now broadly used in various fields of financial mathematics (see, e.g., [1, 2, 3, 4, 6, 7, 10, 14, 17, 22] and references therein), it seems that the classical problem of utility maximisation of terminal wealth has not yet been addressed in this framework.

To fix the ideas, let us consider the classical Merton problem: assume that $\mu_t \equiv \mu$ and $\Sigma_t \equiv \Sigma$ are deterministic and known, and an agent wants to maximise his(her) expected utility from terminal wealth

$$\mathbb{E}[U(X_T)]$$

where the self-financing portfolio X has dynamics

$$dX_t = X_t \left[(h_t \cdot \mu + (1 - h_t \cdot \mathbf{1})r)\, dt \ + \ h_t \Sigma dW_t \right]$$

with h_t^i the proportion of wealth invested in the ith risky asset, $i = 1, \dots, d$, and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$. It is well known (see for example [5]) that, if for example $U(x) = x^\gamma / \gamma$, then the optimal portfolio allocation is given by the constant proportions

$$\hat{h}_t \equiv h := \frac{1}{1-\gamma} (\Sigma \Sigma^T)^{-1} (\mu - r\mathbf{1}).$$

Thus, no matter what happens in the world, an investor would always try to keep these portfolio proportions. A natural question arises: is this world too simple?

The idea of regime-switching models is that the “economy” can assume $m \geq 2$ different states, and when there is a change of state, prices change their dynamics. In this simple example, coefficients μ and Σ depend on this state. As reported in [15], the typical intuitive situation for this is the following: the economy switches between $m = 2$ states, i.e., “business as usual” (BAU, state 1) and “crisis” (state 2), with the typical following stylized facts:

- (much) higher variances in Σ_2 than in Σ_1;
- significantly larger correlations from Σ_2 than from Σ_1, reflecting contagion effects;
- predominantly negative μ_2 $(-r\mathbf{1})$, reflecting down market effects.

The way this could impact the classical Merton problem is seen in the following numerical example.

Example. Assume $d = 2$ and that in the BAU state both assets have yield equal $0.01 + r$ and volatility 0.2, with correlation 0.1; this can be represented as $\mu_1 - r\mathbf{1} = (0.01, 0.01)$ and

$$\Sigma_1 \Sigma_1^T = \begin{pmatrix} 0.04 & 0.004 \\ 0.004 & 0.04 \end{pmatrix}.$$

For an agent with risk-aversion coefficient $\gamma = 0.1$, this gives the optimal portfolio

$$h = \frac{1}{1-0.1}(\Sigma_1\Sigma_1^T)^{-1}(\mu_1 - r\mathbf{1}) \simeq (0.2525, 0.2525)$$

so this agent will invest the 25.25% of its wealth in each of the two risky assets.

Assume now that in the "crisis" state assets do not change their yield, but the two volatilities increase to 0.3 and the correlation to 0.6. Thus, now we still have $\mu_2 - r\mathbf{1} = (0.01, 0.01)$ and

$$\Sigma_2\Sigma_2^T = \begin{pmatrix} 0.09 & 0.054 \\ 0.054 & 0.09 \end{pmatrix}.$$

For the same agent as above, this gives the optimal portfolio

$$h = \frac{1}{1-0.1}(\Sigma_2\Sigma_2^T)^{-1}(\mu_2 - r\mathbf{1}) \simeq (0.0771, 0.0771)$$

so, under the "crisis" state, the agent will diminish his(her) investment down to 7.71% in each of the two risky assets.

The aim of this paper is to present a way to make the above argument (and optimal portfolios) rigorous. In particular, in Section 2 we will define the regime-switching model, frame the utility maximization problem in this context and present a classical way to solve it, namely the dynamic programming approach with Hamilton–Jacobi–Bellman (HJB) equation, which in this framework becomes a system of stochastic differential equations: this allows to derive a mutual fund theorem, which generalizes the classical one holding in diffusion models (see, e.g., [5]). In Sections 3, 4 and 5, respectively, we analyze the classical cases of exponential, logarithm and power utility functions in greater detail, generalizing classical results.

2. The model

As in [4], we begin by assuming that the bond price B_t satisfies

$$dB_t = rB_t dt,$$

for some $r > 0$, and that there are $m \in \mathbb{N}$ states of the world and d (non-defaultable) risky assets, with values in $D := (0, \infty)^d$. Let the stock prices $S_t = (S_t^1, \ldots, S_t^d)$ satisfy

$$\begin{cases} dS_t = \text{diag}\,(S_t)(\mu_{\eta_{t-}}(t, S_t)\; dt + \Sigma_{\eta_{t-}}(t, S_t)\; dW_t), \\ d\eta_t = \sum_{k,j=1}^m (j-k)\mathbb{I}_{\{k\}}(\eta_{t-})dN_t^{kj}, \end{cases} \tag{2.1}$$

where, for each $i = 1, \ldots, m$, $\mu_i : [0,T] \times D \to \mathbb{R}^d$, $\Sigma_i : [0,T] \times D \to \mathbb{R}^{d\times d}$ are functions such that $(t, s) \to \text{diag}\,(s)\mu_i(t, s)$ and $(t, s) \to \text{diag}\,(s)\Sigma_i(t, s)$ are C^1 on $[0,T] \times D$, W is a Brownian motion and $N = (N^{kj})_{1\le k,j\le m}$ is a multivariate $\mathbb{F}$-adapted point process such that

$$(N_t^{kj})_t \text{ has } (P, \mathbb{F})\text{-intensity } \lambda^{kj}(t, S_t)$$

with bounded C^1 functions $\lambda^{kj} : [0,T] \times D \longrightarrow [0,\infty)$; W and N are independent and are both defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. This implies (see [4]) that for all $(t, s, k) \in [0,T] \times D \times \{1, \dots, m\}$ there is a unique strong solution (S, η) to Equation (2.1) starting from $(S_t, \eta_t) = (s, k)$, up to a possibly finite random explosion time. Thus, we also assume that

$$\mathbb{P}\{S_u \in D \text{ for all } u \in [t,T]\} = 1 \qquad \text{for all } (t,s,k) \in [0,T] \times D \times \{1, \dots, m\}.$$

Remark 2.1. As in [14] (and implicitly in [2, 3, 4, 7, 8]), we assume that the process η is observable: in fact, if we assume that the Σs are distinct, then the local quadratic variation-covariation of the risky assets S in any small interval to the left of t will yield $\Sigma_{\eta_{t-}}$ exactly. Hence, even if S is not Markovian, (S, η) is jointly so.

We now build a self-financing portfolio with initial capital $X_0 > 0$, with the following dynamics:

$$dX_t = \big(\pi_t \cdot \mu_{\eta_{t-}}(t, S_t) + (X_t - \pi_t \cdot \mathbf{1})r\big)\ dt + \pi_t \Sigma_{\eta_{t-}}(t, S_t) dW_t \tag{2.2}$$

where

$$\pi_t^i = S_t^i \theta_t^i$$

is the wealth invested in the ith risky asset, with θ_t^i being the number of ith stocks in hand at time t. An alternative definition of X, more usual when we have the additional constraint $X_t > 0$ (for example when dealing with logarithmic or power utility function) is the following:

$$dX_t = X_t \left[\big(h_t \cdot \mu_{\eta_{t-}}(t, S_t) + (1 - h_t \cdot \mathbf{1})r\big)\ dt + h_t \Sigma_{\eta_{t-}}(t, S_t) dW_t\right]$$

where

$$h_t^i = \frac{S_t^i \theta_t^i}{X_t} = \frac{\pi_t^i}{X_t}$$

is, as in the Introduction, the proportion of wealth invested in the ith risky asset, with θ_t^i being the number of ith stocks in hand at time t.

We define

$$J(t, x, s, \eta; \pi) := \mathbb{E}[U(X_T^{t,x,s,\eta;\pi})]$$

and the *value function*

$$V(t, x, s, \eta) = \sup_{\pi \in \Theta} J(t, x, s, \eta; \pi) \tag{2.3}$$

where $(X^{t,x,s,\eta;h}, S^{t,x,s,\eta;h}, \eta^{t,x,s,\eta;h})$ is the three-dimensional controlled Markov process starting from (x, s, η) at time t with the dynamics defined by Equations (2.1) and (2.2) with the control $\pi \in \Theta[t,T]$, where $\Theta[t,T]$ is the set of *admissible* controls, i.e., predictable processes on $[t,T]$ such that Equation (2.2) has a unique strong solution $X^{t,x,s,\eta;\pi}$ for each initial condition (x, s, η) at time t such that $\mathbb{E}[U(X_T^{t,x,s,\eta;\pi})] \in \mathbb{R}$, $(e^{-ru} X_u^\pi)_u \in M^2([t,T])$ and $(\pi_u U(X_u^\pi))_u \in M^2([t,T])$.

As already noted, the three state variables (X, S, η) in (2.1) and (2.2) form a Markov process, with infinitesimal generator given by

$$A^{\pi}V(t,x,s,k) := (L^{\pi}_{x} + L^{\pi}_{xs})V(t,x,s,k) + \sum_{j=1}^{m}\lambda^{kj}\left[V(t,x,s,j) - V(t,x,s,k)\right] \tag{2.4}$$

for all $\pi \in \mathbb{R}^d$, $t \in [0,T]$, $x \in \mathbb{R}$, $s \in D$, $k = 1, \dots, m$, where

$$L^{\pi}_{x}V := rxV_x + (\mu_k - r\mathbf{1}) \cdot \pi V_x + \frac{1}{2}\|\pi\Sigma_k\|^2 V_{xx},$$
$$L^{\pi}_{xs}V := \mu_k \bar{s} V_s + \frac{1}{2}\mathrm{tr}\,(\bar{s}\Sigma_k\Sigma_k^T\bar{s}V_{ss}) + \pi\Sigma_k(\bar{s}\Sigma_k)^T V_{xs}$$

with $\bar{s} :=$ diag s for all $s \in D$, $g_k(t,s) = g(t,s,k)$ for $g = \mu$, σ, V_t, V_x and V_{xx} are the scalar derivatives with respect to the variables t and x, while V_s is the gradient with respect to the vector $s = (s^1, \dots, s^d)$, V_{ss} is the Hessian matrix and V_{sx} is the gradient of V_x with respect to s. The operator A^{π} is linked to the process (X, S, η) via the so-called *Dynkyn formula*

$$\mathbb{E}[f(T, X^{\pi}_T, S_T, \eta_T)] - \mathbb{E}[f(t, X^{\pi}_t, S_t, \eta_t)] = \mathbb{E}\left[\int_t^T A^{\pi_u} f(u,, X^{\pi}_u, S_u, \eta_u)\, du\right] \tag{2.5}$$

for sufficiently regular f.

We can now write the "integro-differential" Hamilton–Jacobi–Bellman (HJB) equation related to the utility maximization problem (2.3) above in a similar manner following the arguments of [19], and obtain

$$V^k_t + \sup_{\pi} A^{\pi}V = 0 \tag{2.6}$$

where $V^i(t,x,s) := V(t,x,s,i)$, $V(t,x,s) := (V(t,x,s,1), \dots, V(t,x,s,m))$ and the "integrals" are on the space $\{1, \dots, m\}$, and final condition

$$V^k(T,x,s) = U(x), \qquad (x,s,k) \in \mathbb{R} \times D \times \{1, \dots, m\}. \tag{2.7}$$

Solving for $\hat{\pi}^k$ which maximizes the Hamiltonian, we obtain

$$\hat{\pi}^k = -\frac{(\Sigma_k\Sigma_k^T)^{-1}(\mu_k - r\mathbf{1})V^k_x + \bar{s}V^k_{xs}}{V^k_{xx}}. \tag{2.8}$$

Notice that if V is independent of s, then the HJB equation simplifies to

$$V^k_t + \sup_h L^{\pi}_x V^k + \sum_{j=1}^{m}\lambda^{kj}\left[V^j - V^k\right] = 0 \tag{2.9}$$

and the optimal portfolio reduces to one of Merton type

$$\hat{\pi}^k = -\frac{V^k_x}{V^k_{xx}}(\Sigma_k\Sigma_k^T)^{-1}(\mu_k - r\mathbf{1}). \tag{2.10}$$

The rigorous link between the utility maximization problem (2.3) and the HJB equation (2.6) is given, as usual, by the following verification theorem. In order to present it, we follow the results in [11] and [13]. First define

$$\mathcal{D} := \left\{ \begin{array}{l} f \quad = (f_1, \ldots, f_m) \in C^{1,2}([0,T] \times \mathbb{R} \times D; \mathbb{R}^m) \text{ such that} \\ \forall t \in [0,T], \text{ the Dynkyn formula (2.5) holds } \forall \pi \in \Theta[t,T] \end{array} \right\}.$$

The usual choice for possibly discontinuous Markov processes is $\mathcal{D} := C_0^2([0,T] \times \mathbb{R})$, the C^2 functions vanishing at infinity: in fact for this space it is always possible to prove that the Dynkyn formula holds. However, this space is too small for our purposes, as typical utility functions are unbounded, so we have to define $\mathcal{D}$ more generally, as done also in [20].

We can now state the following verification theorem, which is a particular case of [13, Theorem III.8.1].

Theorem 2.2 (Verification Theorem). *Let $K \in \mathcal{D}$ be a classical solution to* (2.6) *with final condition* (2.7)*, and assume that there exists an admissible control $\pi^* \in \Theta[t,T]$ such that*

$$\pi_u^* \in \arg\max_{\pi} A^\pi K(u, X_u^\pi, S_u, \eta_u) \qquad \mathbb{P}\text{-a.s. for all } u \in [t,T].$$

Then $K(t,x,s,k) = J^{\pi^*}(t,x,s,k) = V^k(t,x,s)$.

Thus, the utility maximisation problem boils down to finding a regular solution of the HJB equation. The usual procedure for this is to guess a particular solution for a given utility function U and to see whether this particular candidate satisfies the Verification Theorem above. Since the most challenging task in doing this is usually to check whether this candidate solution belongs to $\mathcal{D}$, here we present a technical lemma which will be used in the following sections.

Lemma 2.3. *If the Σ_k are bounded in (t,s), then $V \in \mathcal{D}$ if, for all $\bar{t} \in [0,T]$ and $\pi \in \Theta[\bar{t},T]$, we have that for all $k, j = 1, \ldots, m$,*

$$\mathbb{E}\left[\int_{\bar{t}}^{T} \|V_x(t,X_t,S_t,\eta_t)\pi_u\|^2 + \|V_s(t,X_t,S_t,\eta_t) \cdot S_u\|^2 \, dt\right] < +\infty, \tag{2.11}$$

$$\mathbb{E}\left[\int_{\bar{t}}^{T} |V(t,X_t,S_t,j) - V(t,X_t,S_t,k)|\lambda^{kj}(t,S_t) \, dt\right] < +\infty. \tag{2.12}$$

Proof. For $\bar{t} \in [0,T]$ and $\pi \in \Theta[\bar{t},T]$ we have

$$dV(t, X_t^\pi, S_t, \eta_t) = A^\pi V(t, X_t^\pi, S_t, \eta_{t-}) \, dt + dM_t$$

where the process M is defined by $M_{\bar{t}} := 0$ and the dynamics

$$\begin{aligned} dM_t &:= V_x \pi_t \Sigma_t \, dW_t + V_s \text{diag } S_t \Sigma_t \, dW_t \\ &\quad + \sum_{j,k=1}^{m} [V(t,X_t,S_t,j) - V(t,X_t,S_t,k)] \mathbf{1}_{\{k\}}(\eta_{t-})(dN_t^{kj} - \lambda^{kj}(t,S_t) \, dt). \end{aligned}$$

The lemma follows from the fact that the Dynkyn formula holds if M is a martingale. Sufficient conditions for this are Equation (2.12) and

$$\mathbb{E}\left[\int_{\bar{t}}^{T} \|V_x(t,X_t,S_t,\eta_t)\pi_u\Sigma(u,X_u,S_u,\eta_u)\|^2 + \|V_s(u,X_u,S_u,\eta_u)\text{diag } S_u\Sigma(u,X_u,S_u,\eta_u)\|^2\, du\right] < +\infty.$$

If the Σ_k are bounded in (t,s), then this is implied by Equation (2.11). □

Remark 2.4. The requirement that the Σ_k be bounded in (t,s) is quite natural: in fact, since the diffusion coefficient (conditioned to $\eta_{t-}=k$) in Equation (2.1) is diag $(S_t)\Sigma_k(t,S_t)$, a classical sufficient condition to have a unique solution is to require it (and the drift) to be Lipschitz and with sublinear growth with respect to S_t: this is morally equivalent to the fact that the Σ_k are bounded.

A first consequence of Theorem 2.2 is a generalization of the classical mutual fund theorem (see, e.g., [5]). This result is obtained in the case when μ_k, Σ_k do not depend on S, and is valid with any utility function U such that Equation (2.9) has a smooth solution.

Corollary 2.5 (Mutual fund theorem). *If μ_k, Σ_k and λ^{kj} do not depend on S and Equation* (2.9) *has a smooth solution $V=(V^k)_k$, then the optimal portfolio strategy is given by the feedback control $\hat{\pi}_t := \hat{\pi}^k(t,X_t,S_t)|_{k=\eta_{t-}}$, where the functions $\hat{\pi}^k$, $k=1,\dots,m$ are defined as in Equation* (2.10).

Proof. Since the μ_k, Σ_k and λ^{kj} do not depend on S, as well as the final condition $U(x)$, we can search for a solution of the form $V^k(t,x)$, thus $V^k_s=V^k_{ss}=V^k_{xs}=0$ and the optimal strategy in Equation (2.8) becomes Equation (2.10). Substituting this in Equation (2.6), we obtain Equation (2.9): if this equation has a smooth solution, then the Verification Theorem 2.2 applies, and the theorem follows. □

Remark 2.6. Roughly speaking, results known as "mutual fund theorems" (for a much more general treatment see [21]) say that the optimal portfolio consists of a possibly dynamic allocation between two fixed mutual funds: in this particular situation, the first fund consists only of the risk-free asset B, while the second fund is given by the fixed vector

$$(\Sigma_{\eta_{t-}}(t)\Sigma^T_{\eta_{t-}}(t))^{-1}(\mu_{\eta_{t-}}(t)-r\mathbf{1})$$

for all $t\in[0,T]$, which *depends neither on the particular utility function used nor on the individual prices of the risky assets*, but still depends on time t and on the state η_{t-}. The amount of this fund to be taken is given by the scalar

$$-\frac{V_x^{\eta_{t-}}(t,X_t)}{V_{xx}^{\eta_{t-}}(t,X_t)}$$

which is typically positive as the functions V^k are typically increasing and concave.

Of course, the major assumption of Theorem 2.2 and Corollary 2.5 is to have a smooth solution V. This is satisfied in the following three particular cases, which are quite standard, namely the exponential, logarithmic and power utility functions: what really happens is that in some of these cases one obtains results also with more general assumptions.

3. Exponential utility

We now analyze the particular case when $U(x) = -\alpha e^{-\alpha x}$, with $\alpha > 0$.

Lemma 3.1. *Assume that $U(x) = -\alpha e^{-\alpha x}$, with $\alpha > 0$, and that for all $k = 1, \dots, m$ the functions μ_k, $\Sigma_k \Sigma_k^T$ are locally Lipschitz and bounded, Σ_k is nonsingular for all (t,s), $\Sigma_k^{-1}\mu_k$ is bounded and $\lambda^{kj} \in C_b^1([0,T] \times D)$ for all $k, j = 1, \dots, m$. Then:*

1. *There exists a unique classical solution $C := (C^k)_{k=1,\dots,m} \in C_b^{1,2}([0,T] \times D; \mathbb{R}^m)$ for the following system of PDEs:*

$$\begin{cases} C_t^k + rs \cdot C_s^k + \frac{1}{2} \mathrm{tr}\,(\bar{s}\Sigma_k \Sigma_k^T \bar{s} C_{ss}^k) \\ \qquad\qquad - \dfrac{e^{-r(T-t)}}{\alpha} \left[\displaystyle\sum_{j=1}^m \left(e^{-\alpha\phi(C^k - C^j)} - 1 \right) \lambda^{kj} + \frac{1}{2} z_k^2 \right] = rC^k, \\ C^k(T) = 0 \end{cases} \tag{3.1}$$

where the functions ϕ and z_k^2 are defined as

$$\phi(t) := e^{r(T-t)}, \;\; z_k^2(t,s) := (\mu_k(t,s) - r\mathbf{1})^T (\Sigma_k(t,s)\Sigma_k^T(t,s))^{-1} (\mu_k(t,s) - r\mathbf{1}). \tag{3.2}$$

2. *The function*

$$V^k(t,x,s) = -\alpha e^{-\alpha\phi(t)(x - C^k(t,s))}$$

with $\phi(t) := e^{r(T-t)}$, belongs to $\mathcal{D}$ and satisfies the HJB Equation (2.9).

3. *The optimal portfolio strategy $\hat{\pi}_t^k$ is given by*

$$\hat{\pi}_t := \left. \frac{(\Sigma_k(t,S_t)\Sigma_k(t,S_t)^T)^{-1}(\mu_k(t,S_t) - r\mathbf{1}) + \alpha\phi(t)\mathrm{diag}\,(S_t) C_s^k(t,S_t)}{\alpha\phi(t)} \right|_{k=\eta_{t-}}.$$

Proof. Point 1 follows from [4, Theorem 2.4]. For point 2, the partial derivatives of V^k are given by

$$\begin{cases} V_t^k = \left[C_t^k + rx - rC^k\right] \alpha\phi V^k \\ V_x^k = -\alpha\phi V^k \\ V_{xx}^k = (\alpha\phi)^2 V^k \\ V_s^k = C_s^k \alpha\phi V^k \\ V_{ss}^k = [\alpha\phi C_s^k \otimes C_s^k + C_{ss}^k]\alpha\phi V^k \\ V_{xs}^k = -C_s^k (\alpha\phi)^2 V^k. \end{cases}$$

In order to prove $V \in \mathcal{D}$, we check Equations (2.11)–(2.12): first,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\bar{t}}^{T} \|V_x(t,X_t,S_t,\eta_t)\pi_u\|^2 + \|V_s(t,X_t,S_t,\eta_t)\cdot S_u\|^2\ dt\right] \\
&\quad = \mathbb{E}\left[\int_{\bar{t}}^{T} \|\alpha\phi(t)U(X_t)e^{\alpha\phi(t)C(t,S_t)}\pi_t\|^2\ dt\right] \\
&\quad\quad + \mathbb{E}\left[\int_{\bar{t}}^{T} \|\alpha\phi(t)V(t,X_t,S_t,\eta_t)C_s(t,X_t,S_t,\eta_t)\cdot S_t\|^2\ dt\right].
\end{aligned}$$

Since C and ϕ are bounded, the first addend is finite since $\pi \in \Theta[\bar{t},T]$, and by the same argument the second addend reduces to

$$M\mathbb{E}\left[\int_{\bar{t}}^{T} \|e^{-\alpha\phi(t)X_t^\pi}S_t\|^2\ dt\right] \le M(T-\bar{t})\mathbb{E}\left[\int_{\bar{t}}^{T} e^{-2\alpha\phi(t)X_t^\pi}\ dt\right]^{\frac{1}{2}}\mathbb{E}\left[\int_{\bar{t}}^{T}\|S_t\|^2\ dt\right]^{\frac{1}{2}}$$

for a suitable M: the final product is finite by standard SDE estimates (see for example [13, Appendix D], so Equation (2.11) is satisfied. As concerns Equation (2.12), it reduces to

$$\begin{aligned}
&\mathbb{E}\left[\int_{\bar{t}}^{T} e^{-\alpha\phi(t)X_t^\pi}|e^{\alpha\phi(t)C^j(t,S_t)} - e^{\alpha\phi(t)C^k(t,S_t)}|\lambda^{kj}(t,S_t)\ dt\right] \\
&\quad \le M\mathbb{E}\left[\int_{\bar{t}}^{T} e^{-\alpha\phi(t)X_t^\pi}\ dt\right]
\end{aligned}$$

for a suitable M: as before, this quantity is finite, so also Equation (2.12) is satisfied. Thus, $V \in \mathcal{D}$ by Lemma 2.3. Substituting its derivatives in the operators L_x^π and L_{xs}^π appearing in Equation (2.6), we get

$$\begin{aligned}
L_x^\pi V^k &:= \alpha\phi V^k\left[-rx - (\mu_k - r\mathbf{1})\cdot\pi + \frac{1}{2}\alpha\phi\|\pi\Sigma_k\|^2\right], \\
L_{xs}^\pi V^k &:= \alpha\phi V^k\left[\mu_k\bar{s}C_s^k + \frac{1}{2}\mathrm{tr}\ (\bar{s}\Sigma_k\Sigma_k^T\bar{s}[\alpha\phi C_s^k\otimes C_s^k + C_{ss}^k]) - \alpha\phi\pi\Sigma_k(\bar{s}\Sigma_k)^T C_s^k\right] \\
&:= \alpha\phi V^k\left[\mu_k\bar{s}C_s^k + \frac{1}{2}\alpha\phi\|\bar{s}C_s^k\Sigma_k\|^2 + \frac{1}{2}\mathrm{tr}\ (\bar{s}\Sigma_k\Sigma_k^T\bar{s}C_{ss}^k) - \alpha\phi\pi\Sigma_k(\bar{s}\Sigma_k)^T C_s^k\right].
\end{aligned}$$

The maximizer in Equation (2.8) becomes

$$\hat{\pi}^k = \frac{(\Sigma_k\Sigma_k^T)^{-1}(\mu_k - r\mathbf{1}) + \alpha\phi\bar{s}C_s^k}{\alpha\phi}.$$

Substituting all this into Equation (2.6) and dividing by $\alpha\phi V^k$, we obtain Equation (3.1), which is satisfied by C. Finally, point 3 follows from the $\hat{\pi}^k$ just obtained. □

Remark 3.2. Point 2 is not a surprise here: in fact, in the exponential case one can check directly from the definition of V in Equation (2.3) and from the dynamics of X in Equation (2.2) that

$$V(t, x, s, \eta) = e^{-\alpha\phi(t)x} V(t, 0, s, \eta)$$

for all (t, x, s, η), so that the value function V is (exp-)affine in x.

As a particular case, we can see that the Mutual Fund Theorem 2.5 holds true in this situation.

Corollary 3.3. *If λ^{kj} and z_k^2 do not depend on s for all $k, j = 1, \ldots, m$, then the discounted wealth invested in the risky assets is given by*

$$e^{r(T-t)}\hat{\pi}_t = \left.\frac{(\Sigma_k(t, S_t)\Sigma_k(t, S_t)^T)^{-1}(\mu_k(t, S_t) - r\mathbf{1})}{\alpha}\right|_{k=\eta_{t-}}$$

Moreover, if also μ_k and Σ_k do not depend on s for all $k = 1, \ldots, m$, then the above optimal discounted wealth invested in the risky assets only depends on (t, η_{t-}).

Proof. In this case, the solution of the system of ODEs

$$\begin{cases} C_t^k - \frac{e^{-r(T-t)}}{\alpha}\left[\sum_{j=1}^m \left(e^{-\alpha\phi(C^k - C^j)} - 1\right)\lambda^{kj} + \frac{1}{2}z_k^2\right] = rC^k, \\ C^k(T) = 0 \end{cases}$$

is also solution of Equation (3.1), so that $C_s \equiv 0$. Thus,

$$\hat{\pi}_t = \left.\frac{(\Sigma_k(t, S_t)\Sigma_k(t, S_t)^T)^{-1}(\mu_k(t, S_t) - r\mathbf{1})}{\alpha\phi(t)}\right|_{k=\eta_{t-}}$$

and by multiplying for $\phi(t)$ we obtain the desired result. □

4. Logarithmic utility

In this and the following section, as we will have the constraint $X_t > 0$ for all $t \in [0, T]$, as definition of strategy we adopt h (the proportion of wealth in the risky assets) instead of π. This means that in the infinitesimal generator Equation (2.4) we must substitute $L_x^{xh} + L_{xs}^{xh}$ for $L_x^\pi + L_{xs}^\pi$.

In the case of a logarithmic utility function, the optimal portfolio is of the general form in Equation (2.8) even when μ_k, Σ_k and λ^{kj} depend on S.

Proposition 4.1. *Assume that $U(x) = \log x$, and that for all $k = 1, \ldots, m$ the functions μ_k, $\Sigma_k\Sigma_k^T$ are locally Lipschitz and bounded, Σ_k is nonsingular for all (t, s), $\Sigma_k^{-1}\mu_k$ is bounded and $\lambda^{kj} \in C_b^1([0, T] \times D)$ for all $k, j = 1, \ldots, m$. Then:*

1. *There exists a unique classical solution $C := (C^k)_{k=1,\ldots,m} \in C_b^{1,2}([0, T] \times D; \mathbb{R}^m)$ for the following system of PDEs:*

$$\begin{cases} C_t^k + r + \mu_k \bar{s} C_s^k + \frac{1}{2}\mathrm{tr}\,(\bar{s}\Sigma_k\Sigma_k^T\bar{s}C_{ss}^k) + \frac{1}{2}z_k^2 + \sum_{j=1}^m \left(C^j - C^k\right)\lambda^{kj} = 0, \\ C^k(T) = 0, \end{cases} \tag{4.1}$$

where the functions ϕ and z_k^2 are defined as in Equation (3.2).

2. *The function*
$$V^k(t,x,s) = \log x + C^k(t,s)$$
belongs to $\mathcal{D}$ and satisfies the HJB Equation (2.9)*.*
3. *The optimal portfolio proportion $\hat{h}_t^k$ is given by*
$$\hat{h}_t := (\Sigma_k(t,S_t)\Sigma_k(t,S_t)^T)^{-1}(\mu_k(t,S_t) - r\mathbf{1})\big|_{k=\eta_{t-}}.$$

Proof. Just as above, point 1 follows from [4, Theorem 2.4]. For point 2, the partial derivatives of V_k are now given by
$$V_t^k = C_t^k, \quad V_x^k = \frac{1}{x}, \quad V_{xx}^k = -\frac{1}{x^2}, \quad V_s^k = C_s^k, \quad V_{ss}^k = C_{ss}^k, \quad V_{sx}^k = 0.$$
In order to prove $V \in \mathcal{D}$, we check Equations (2.11)–(2.12): first, Equation (2.11) reduces to
$$\mathbb{E}\left[\int_{\bar{t}}^T \|h_t\|^2 \, dt\right] + \mathbb{E}\left[\int_{\bar{t}}^T \|C_s(t,S_t,\eta_t)\cdot S_t\|^2 \, dt\right].$$
Since C is bounded, the second term is finite, and the first term is finite by definition of $\Theta[\bar{t},T]$. Equation (2.12) reduces to
$$\mathbb{E}\left[\int_{\bar{t}}^T |C(t,S_t,j) - C(t,S_t,k)|\lambda^{kj}(t,S_t) \, dt\right] < +\infty$$
which is true, due to the boundedness of C and of the λ^{kj}. Thus, $V \in \mathcal{D}$. Substituting its derivatives in the operators L_x^{xh} and L_{xs}^{xh} appearing in Equation (2.6), we get
$$L_x^{xh}V^k := r + (\mu_k - r\mathbf{1})\cdot h - \frac{1}{2}\|h\Sigma_k\|^2,$$
$$L_{xs}^{xh}V^k := \mu_k \bar{s} C_s^k + \frac{1}{2}\mathrm{tr}\,(\bar{s}\Sigma_k\Sigma_k^T\bar{s}C_{ss}^k).$$
The maximizer in Equation (2.8) becomes
$$\hat{h}^k = (\Sigma_k\Sigma_k^T)^{-1}(\mu_k - r\mathbf{1}).$$
Substituting all this into Equation (2.6), we obtain Equation (4.1), which is satisfied by C. Finally, point 3 follows from the $\hat{h}^k$ just obtained. □

This result in some sense extends the result of Merton [18], in that the optimal portfolio has the form of the Merton optimal portfolio, even if the coefficients μ_k and Σ_k can depend on the price of the risky assets S in general. If they do not depend on t and S, however, we retrieve the usual "constant proportions" result, which follows.

Corollary 4.2. *If μ_k and Σ_k do not depend on (t,s) for all $k = 1,\ldots,m$, then the optimal portfolio proportions in the risky assets are given by*
$$\hat{h}_t := (\Sigma_k\Sigma_k^T)^{-1}(\mu_k - r\mathbf{1})\big|_{k=\eta_{t-}}$$
which only depends on η_{t-}.

Proof. The proof is straightforward from point 3 of Proposition 4.1. □

5. Power utility

In the case of a power utility function, we do not get general results as in the two previous cases, unless we assume that μ_k, Σ_k and λ^{kj} do not depend on s.

Proposition 5.1. *Assume that $U(x) = x^\gamma/\gamma$, with $\gamma < 1$, $\gamma \neq 0$, and that for all $k, j = 1, \dots, m$ the functions μ_k, Σ_k and λ^{kj} do not depend on s; besides, μ_k, $\Sigma_k \Sigma_k^T$ are locally Lipschitz and bounded, Σ_k is nonsingular for all t, $\Sigma_k^{-1}\mu_k$ is bounded and $\lambda^{kj} \in C_b^1([0,T])$. Then:*

1. *There exists a unique classical solution $C := (C^k)_{k=1,\dots,m} \in C_b^{1,2}([0,T];\mathbb{R}^m)$ for the following system of ODEs:*

$$\begin{cases} C_t^k + r + \dfrac{1}{2}\dfrac{1}{(1-\gamma)} z_k^2 + \dfrac{1}{\gamma}\sum_{j=1}^m \left(e^{\gamma(C^j - C^k)} - 1\right)\lambda^{kj} = 0, \\ \hfill C^k(T) = 0. \end{cases} \tag{5.1}$$

   where the functions ϕ and z_k^2 are defined as in Equation (3.2).

2. *The function*

$$V^k(t,x,s) = \frac{\left(xe^{C^k(t)}\right)^\gamma}{\gamma}$$

   belongs to $\mathcal{D}$ and satisfies the HJB Equation (2.9).

3. *The optimal portfolio proportion $\hat{h}_t^k$ is given by*

$$\hat{h}_t := \frac{1}{1-\gamma}(\Sigma_k(t)\Sigma_k(t)^T)^{-1}(\mu_k(t) - r\mathbf{1})\Big|_{k=\eta_{t-}}. \tag{5.2}$$

Proof. As above, point 1 follows from [4, Theorem 2.4]. For point 2, the partial derivatives of V_k are now given by

$$V_t^k = \gamma C_t^k V^k, \quad V_x^k = \frac{\gamma}{x}V^k, \quad V_{xx}^k = -\frac{\gamma(1-\gamma)}{x^2}V^k, \quad V_s^k = V_{sx}^k = V_{ss}^k = 0.$$

In order to prove $V \in \mathcal{D}$, we check Equations (2.11)–(2.12): firstly, Equation (2.11) reduces to

$$\mathbb{E}\left[\int_{\bar{t}}^T \|\gamma V(t, X_t, S_t, \eta_t) h_t\|^2 \, dt\right]$$

which is finite by definition of $\Theta[\bar{t}, T]$. Equation (2.12) reduces to

$$\mathbb{E}\left[\int_{\bar{t}}^T \frac{1}{\gamma} X_t^\gamma |e^{\gamma C^j(t)} - e^{\gamma C^k(t)}| \lambda^{kj}(t) \, dt\right] \leq M\mathbb{E}\left[\int_{\bar{t}}^T X_t^\gamma \, dt\right]$$

for a suitable M, since C and the λ^{kj} are bounded. This quantity is finite by definition of $\Theta[\bar{t}, T]$. Thus, $V \in \mathcal{D}$. Substituting its derivatives in the operators L_x^{xh} and L_{xs}^{xh} appearing in Equation (2.6), we get

$$\begin{aligned} L_x^{xh} V^k &:= \gamma V^k \left(r + (\mu_k - r\mathbf{1}) \cdot h - \frac{1}{2}(1-\gamma)\|h\Sigma_k\|^2 \right), \\ L_{xs}^{xh} V^k &:= 0. \end{aligned}$$

The maximizer in Equation (2.8) becomes

$$\hat{h}^k = \frac{(\Sigma_k \Sigma_k^T)^{-1}(\mu_k - r\mathbf{1})}{(1-\gamma)}.$$

Substituting all this into Equation (2.9) and dividing by γV^k, we obtain Equation (5.1), which is satisfied by C. Finally, point 3 follows from the $\hat{h}^k$ just obtained. □

Also this result in some sense extends the result of Merton [18], in that the optimal portfolio has the form of the Merton optimal portfolio. If μ_k and Σ_k do not depend on t, we again retrieve the usual "constant proportions" result, which follows.

Corollary 5.2. *If μ_k and Σ_k do not depend on (t, s) for all $k = 1, \ldots, m$, then the optimal portfolio proportions in the risky assets are given by*

$$\hat{h}_t := \frac{1}{1-\gamma}(\Sigma_k \Sigma_k^T)^{-1}(\mu_k - r\mathbf{1})\Big|_{k=\eta_{t-}}$$

which only depends on η_{t-}.

Proof. The proof is straightforward from point 3 of Proposition 4.1. □

Remark 5.3. If μ_k, Σ_k or λ^{kj} also depend on s, then Equation (5.1) must be modified as follows:

$$\begin{cases} C_t^k + r + \dfrac{1}{2}\dfrac{1}{(1-\gamma)} z_k^2 + \dfrac{1}{\gamma}\displaystyle\sum_{j=1}^{m}\left(e^{\gamma(C^j - C^k)} - 1\right)\lambda^{kj} \\ \qquad + \left(\mu_k - \dfrac{1}{1-\gamma}(\mu_k - r\mathbf{1})\right)\bar{s}C_s + \dfrac{1}{2}\mathrm{tr}\,(\bar{s}\Sigma_k\Sigma_k^T\bar{s}C_{ss}) \\ \qquad + \dfrac{1}{2}\dfrac{\gamma}{1-\gamma}\|\bar{s}C_s\Sigma_k\|^2 = 0, \\ C^k(T) = 0, \end{cases}$$

i.e., the additional terms in the second and third lines must be included, one of which is a nonlinear function of the gradient C_s. This is a quasilinear system of PDEs, to which the results of [4] do not apply, and that in general needs a theory which is more complex and beyond the scope of this paper.

We are now able to reconsider the initial Example 1.1 and to make it rigorous.

Example (Example 1.1 continued). Assume $d = 2$ and let $\mu_k - r\mathbf{1} \equiv (0.01, 0.01)$ for $k = 1, 2$ and

$$\Sigma_k := \begin{cases} \begin{pmatrix} 0.04 & 0.004 \\ 0.004 & 0.04 \end{pmatrix} & \text{for } k = 1, \\ \\ \begin{pmatrix} 0.09 & 0.054 \\ 0.054 & 0.09 \end{pmatrix} & \text{for } k = 2. \end{cases}$$

Then the optimal portfolio strategy for an investor with $U(x) = x^\gamma/\gamma$ with $\gamma = 0.1$, by Equation (5.2) is given by

$$\begin{aligned}\hat{h}_t &:= \frac{1}{1-\gamma}(\Sigma_k \Sigma_k^T)^{-1}(\mu_k - r\mathbf{1})\Big|_{k=\eta_{t-}} \\ &= (0.2525, 0.2525)\mathbf{1}_{\{\eta_{t-}=1\}} + (0.0771, 0.0771)\mathbf{1}_{\{\eta_{t-}=2\}}.\end{aligned}$$

Thus, our optimal investor always switches between holding 25.25% of its wealth in each of the two risky assets in the "normal" state, and 7.71% in each of the two risky assets during a "crisis" state.

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Part III
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Can there Be Excessive Mathematization of the World?

Nicolas Bouleau

Abstract. If we consider the evolution of ideas regarding chance since Antiquity (Aristotle, Cicero), the appearance of the calculations during the 17th century (Pascal, Fermat), to the theory of hedging on financial markets, we see: a) an extraordinary development of mathematics to manipulate randomness b) the increasing use in this language in economics in the 20th century c) a gradual eviction in the backyard of all that concerns the interpretation of phenomena. The shift to a collective work involving interpretation is an urgent need in the contemporary controversies: financial crises, long-term, biodiversity, but it faces a passive resistance due to the comfort of the agreement on mathematics.

The question on which we focus here is on what philosophical bases and under what circumstances can there be excessive mathematization of the world? This question is asked repeatedly about the economy. To elucidate this difficult problem we address it in a broader scope than just the economy, for knowledge in general. We discuss when and how to diagnose excessive mathematization and what it means. This leads us to ask: why normal science and revolutions in jolts? Why orthodox economics and crises?

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1. The contribution of mathematics to knowledge: Some history and preliminary remarks

Since the beginnings of civilization mathematics has been associated with most forms of knowledge. Early examples are Archimedes's work in engineering and, from the same era, The *Nine Chapters* about land measures and economy in China. Few areas have not been influenced in some way by mathematics. From this long and multi-faceted history we extract some key features.

1.1. The Baconian program served by mathematics

It is in *Il Saggiatore* (The Assayer) in 1623 that Galileo posits that the universe is written in the language of mathematics. This $\alpha\pi o\varphi\tau\varepsilon\gamma\mu\alpha$ as it is called, became the foundation of all Western science. This clarifies Francis Bacon's program, which asserts that man has a Promethean perspective, because he is subject to God choosing to share his power. He can conquer, dominate and transform nature. Galileo tells us how he can know and understand it. In fact later in his work – as Alexander Koyre has clearly shown [15] – Galileo proceeds essentially by thought experiments following mathematical reasoning, not by experiments providing data for subsequent modeling.

He believed that mathematics was a sufficient sign of the essence of God in nature that nature would reveal its secrets purely by geometric and algebraic deductions. Over a century later, Kant built his philosophy around the explicit idea that mathematics, although not based in sensory experience (*a priori* judgments), nevertheless teaches about the world (synthetic judgments). Subsequently mathematics has gradually yielded the philosophical throne of *synthetic a priori judgments*, but without ever losing the prestige of a natural fertility. In the early 19th century there was a separation with mathematics on one side, taking a modern and rigorous turn in the writings of Gauss, Cauchy and Bolzano, and philosophy on the other side, which, with Hegel's *Logic*, had no mathematical element. But then the emergence of non-Euclidean geometries and crises in the foundations of mathematics gave rise to a plurality of views about mathematics and its role in the development of scientific knowledge. At the end of the 19th and 20th centuries, with the development of physics that became the focus of epistemology, mathematics is, with variations depending on the authors, mainly considered as a servant of the natural sciences (cf. [2, 21]); we refer to this as its ancillary role.

1.2. The appearance of mathematics in economics

Sociology, as introduced by Auguste Comte, takes a non-mathematical road, except through the use of statistics, particularly by Durkheim. Subsequently it acquired its own methodological bases with Max Weber in the early 20th century. Economics, on the other hand, was mathematized as early as the mid 19th century with Jules Dupuit and Augustin Cournot, without really using statistics. Prior to this, economics presented itself as a kind of philosophy of accounting operations. After Dupuit and Cournot economics was full of talk of derivatives, equations and integrals. How did math come to be accepted into the very heart of this social science?

To answer this we follow the path of Jules Dupuit (1804–1865). A civil engineer, he realized that one can do better than simply fixing a single price for the tolls on a bridge since, whatever the price, some users will find it too expensive, while others would happily pay an even higher toll. He is the inventor of what today is called market segmentation. Having a good mathematical training he had the intuition that with a single price one cannot recover all of the integral of the curve that quantifies the willingness to pay; one can only recover that of a truncated curve. This idea of an integral is quite clear in his articles.

Yet we must note that this "willingness to pay" is a poorly defined concept. It depends on many factors, the weather, time of day, seasons, and a thousand social and economic causes. It seems impossible to measure. A collection of experiments measuring traffic against toll level would not provide a curve but *a cloud of points*. It also depends on the tolls levied on other crossings, and on whether users collude and sell their rights of crossing etc.

In the early 19th century, this concept was debated under the name '*utility*'. Dupuit pursued the belief that the mathematical phenomenon that he had discovered *would help to clarify the concept*. He postulated the existence of this quantity as a property of the commodity being exchanged and its price, which is shared according to the benefits of the seller/manufacturer and the consumer. "*Political economics*, he wrote [as opposed to social economics], *should measure the utility of an object by the sacrifice that each consumer is prepared to make in order to acquire it*" and he took the still famous example of a bridge [13]: "[the utility of a toll bridge] *can be separated into two main parts:*

1) *the lost utility, which corresponds to those crossings that would have occurred if the toll were abolished but which do not take place with the current charge, and*
2) *the utility produced, which corresponds to the crossings which do take place. This latter splits into two further parts:*
   a) *utility for the producer, i.e., the money raised by the toll, and*
   b) *utility for the consumer, i.e., the excess value of the service over the price it costs."*

Dupuit explains [12]: "[In a shop we see] *the fine, the very fine, the super fine, the extra fine, which, though from the same barrel and showing no difference other than the superlative of the label, are sold at very different prices*" and this changes the optimization of public taxes: "*So when the bridge is built and the State establishes a tariff, it stops caring about production costs. It charges less for a heavy cart which wears out the bridge more, than for a carriage with good suspension. Why two different prices for the same service? Because the poor do not value the crossing as highly as the rich, and raising the tariff would only prevent them from using the bridge.*" He explains: "*The goal is always the same: to charge for the service rendered, not what it costs, but what the buyer thinks its value to be*" [12].

Dupuit fully realizes that, being defined by thought experiments, this notion of utility is difficult to measure. He acknowledges that it is abstract. "*It may be objected that the calculation for which we have given the formula is based on data that no statistics can provide, thus we will never be able to express precisely the utility provided by a machine, by a road, by any work ...*" But he advances the famous argument, which has been repeated endlessly by neo-classicists ever since, that economic science is only an approximation. It is this argument that led to all the ambiguities in the passage from descriptive to normative and to the performativity of discourse, and which opened economics up to all the mathematical refinements imaginable.

Dupuit starts from a mathematical property and uses it to account for the psychological, and it is interesting to compare his approach with that of Condorcet, who, at the end of the preceding century, proposed a different kind of mathematization of the social.

Condorcet, a great mathematician, aimed to use the calculus of probabilities to understand the propagation and sharing of a "reason to believe" (what we call today the degree of certainty), a concept somewhat similar to that of utility but based on the truth or fallacy of judgments. He pursued this program at length, making, along the way, the great discovery of the "paradox of the vote of an assembly" (about 3 options A, B, C, the majority may be obtained on the preference $A < B$, a majority on $B < C$, and also a majority on $C < A$). But he did not think that it would be possible to go so far as to calculate peoples' behavior.

"*On the use of language of geometry, the amount of universal commodity, that of a particular commodity, these can be approximated by numbers, but the urge to buy and sell cannot be calculated. Yet the changes in price depend on this moral quantity which, in turn, depends on opinions and passions. It is a beautiful idea to try to calculate everything, but look at the greatest mathematicians of Europe, the likes of d'Alembert and Lagrange. They seek to understand the motion of three attracting bodies: they assume that these bodies are point masses, or are very nearly spherical, and yet this issue, despite being limited by a hundred conditions that make calculation easier, has occupied them for twenty years without an answer. The effect of the forces acting on the head of the dullest shopkeeper is much more difficult to calculate.*" [11]

Condorcet's approach starts from the psychological, the reason to believe, and attempts a mathematization of sociality by the calculus of probabilities. His epistemology is an extension of that of Laplace: we cannot determine everything – principles, laws of forces and their way of acting – only the calculation of probability is relevant. It is an approach with an a priori limitation of science. Condorcet had to spell out all his assumptions – independence or correlation of opinions etc. – before doing calculations.

Dupuit, on the other hand, can immediately perform calculations, and does so in his articles, he constructs concepts which interpret price curves (assumed to be obtained). His concepts require very strong assumptions of independence, but he leaves the details of these hypotheses to be spelt out and improved later.

These features – the independence of agents presented as approximation, the progression from prices and quantities to concepts and then, during the 19th century, production function, and problem-solving by local differentiation – these will be the backbone of the neo-classical theory with Stanley Jevons, Carl Menger, Léon Walras (general equilibrium), von Böhm-Bawerk, Vilfredo Pareto (theory of optimum), Irving Fisher, etc. creating an evocative and highly flexible language that is still in use today.

1.3. Advanced mathematization of finance

This is a very recent and well-known phenomenon, whose history I have recounted elsewhere [4]. I will simply explain how an apparently very clever mathematization of risk, helped lead financiers away from safe practices and facilitated the emergence of the subprime crisis. The crisis has occured in an era when finance is thoroughly mathematized, as a result of the "Black–Scholes revolution". A rediscovery of the work of Bachelier and the use of Brownian motion in modeling, and developments of stochastic calculus after the Second World War, particularly the work of K. Itô (1915–2008), provided a mathematical language (that of semimartingales) in which the non-arbitrage principle could be expressed under broad assumptions that were suitable for operational cases. Methods for pricing and hedging options were thus provided by partial differential equations. The simplest case is when volatility is constant, but it is clear to everyone that these methods are largely *perfectible*, a point which is epistemologically essential.

This led to three historical phenomena: the development of derivatives markets in the U.S. first, then Japan and Europe, a transformation of professional profiles in banks and a call for new mathematical skills, and an enhanced political role for finance which was felt during the construction of the European Union and then in the globalization movement.

From the hedging of (European or American) options on stocks and currencies, the mathematical formalization then spread to more delicate issues: rate models. In particular, the bond market and the term structure of interest rates. The Cox–Ingersoll–Ross and Heath–Jarrow–Morton models allow the non-arbitrage principle to be applied here. Furthermore the theory can make use of infinite-dimensional models that must be simplified and calibrated to the current data. These model the behavior of agents over five, ten or twenty years and are therefore highly uncertain, this uncertainty being expressed in the language of probability theory.

But the most ambitious level of mathematization goes even further and deals with securitization of debts and risk assessments. Putting risks on the market is *a priori* a good idea, in the sense that it is better not to put all your eggs in one basket. But this assumes that the players (banks, insurance companies) can assess the risks.

This gave rise to a mathematical innovation worth mentioning here. It was noted that to estimate the risk of a portfolio of contingent claims, the classical method known as 'value at risk', based on a criterion of the form (level of losses, probability of this level), entailed some logical difficulties. It has been shown that any criterion satisfying the desired consistency was of a particular mathematical form called a '*coherent risk measure*' cf. [5]. We emphasize that these tools allow calculations for complex portfolios assuming known probability of rare events, i.e., the tails of probability distributions which have great influence on the results. These methods, in other words, yield a quantification based on unknowns.

In the credit-risk market financial institutions have mathematical tools to estimate risks on reassembled portfolios for the purpose of exchanging them and improving the situation of each individual with respect to their own utility function and their aversion to risk. It has often been stressed in the commentaries on the crisis that the new tools of these markets especially CDO and CDS (credit default swaps) did not encourage operators to exercise caution. That is correct. The changes in the way agents dealt with risk when protected by insurance, termed 'moral hazard' by the Anglo-Saxons, surely had a role in making the 'soufflé' of the crisis rise. But equally important is the fact that it was wrong to think that the risk was 'in the portfolio'. The risk is interpretative in nature and just as "the beauty of the Parthenon is not found in the dust of the Parthenon", so these mathematical tools do not see the global economic interpretations related to the decline in U.S. household savings etc.

1.4. The quantification of uncertainty is a removal of meaning

From an epistemological point of view this fundamental fact needs to be stressed. It is the *significance* of the event that creates the risk. The probabilistic representation of risk is classically a pair of mathematical quantities: 1) a probability law that governs the states that can arise, 2) a random variable, i.e., a function that maps each state to the damage, that is to say the cost (counted algebraically if there are also benefits). This representation by a pair of quantities is a mathematical model both too simple and too ideal for thinking about risk. It is too ideal because we are almost never in a situation where this model is well informed. We do not know the tails of probability distributions because they concern rare events for which there is insufficient data. We do not know what correlations occur to assess the damage and we do not have a full description of what can happen. Moreover the model is too simplistic because it removes the reasons that make us interested in the events as if their translation into costs could be done automatically and objectively.

The true purpose of risk analysis is to move forward with a little foresight in organizing facts and social practices. It may be the risk that a child be knocked down while crossing the street, the risk that the air of Paris be toxic, that the failure of one business will cause that of others, etc. The intellectual operation of probabilizing a situation is fundamentally one of removing meaning. It is largely problematic for all matters concerning human behavior. Risk analysis necessarily involves *understanding* interpretations.

It is the meaning of the event that creates the risk. As an example, suppose a particular type of cancer is found in a certain proportion of the Swiss population. This proportion is then used to estimate the risk. If it is subsequently found that most of the people with this cancer had consumed cannabis twenty years ago, say, then all cannabis users become potential patients. The risk is much higher; the meaning of the event has changed. Reducing risk to a probability distribution of sums of money amounts to trusting mathematization as an approximation, as if it were describing a physical reality, whereas it is actually a question of meaning whose subjectivity permeates every interaction between the agents. This epistemological point is extremely important. They are interpretations, and hence meanings, that are replaced by numbers.

Recently there have been significant improvements in financial analysis, especially with the so-called coherent risk measures. *All these methods for making decisions in the face of uncertainty have the innate defect of assuming the interpretative process to be closed.* Yet, on the contrary, new interpretations are constantly emerging. Once a new reading is made, new risks are created, but perceived only by those who understand it. If in 2006, nobody had seen the growth of house prices and the decline of household savings in the United States as a phenomenon open to several interpretations, the corresponding risk would not have been perceived. Mathematization of risk conceals these difficulties behind assumptions about the tails of probability distributions. It is not enough to say that those are poorly known. They are by nature provisional and changeable according to the interpretative knowledge that agents bring from their understanding of economic phenomena.

1.5. In liberal economics, every quantification opens a possible extension to the market

There are numerous examples. The most recent is the quantification of research work. Up until the end of the last century, the quality of researchers was seen in terms of idiosyncratic talents that could only be truly appreciated by researchers themselves experienced in the same type of activities. Putting in place all the machinery of publication indices and journal citations has profoundly disrupted the working relations in the profession. I will say no more. The result has been the emergence of an international market for students, teachers and researchers, with Universities being faced with a new logic where their financial budgets determine what league of intellectual athletes they can afford.

Another example, one which is more serious in its long-term consequences, is biodiversity. Mathematization here is based on separating species into two categories. On the one hand are the '*remarkable*' species, those officially considered as threatened. For these species we calculate the cost of conservation much as for historical monuments. On the other hand for the '*ordinary*' species we calculate the *ecological service* they provide, from prokaryotes (bacteria) to eukaryotes (higher species) by standard methods of cost-benefit analysis. One can then buy and sell any part of nature or exchange it against goods or services already quantified by the economy.

2. When and how is there excessive mathematization?

We now examine the particular type of inefficiency and problem that suggests a diagnosis of excessive mathematization.

2.1. We only realize after the fact

The recent financial crisis is quite illustrative in this regard. While the crisis had not yet occurred – except in the eyes of some non-orthodox observers as there always are – every agent and every financial institution believed that they should estimate the risk of their portfolios (comprised of complex products such as credit derivatives) by the methods best suited to the very mathematical nature of these products. Coherent risk measures make assumptions on the tails of laws but enable one to handle multiple scenarios. The weak point is that they omit scenarios based on global interpretations where the value of each portfolio cannot be calculated by considering the others as *ne varietur*.

Once the crisis had started, and after the resultant upheavals, what happened was the result of political forces: on one hand a strong current of opinion emerged urging the adoption of regulatory measures in order to avoid future crises or at least limit their damage, on the other hand most financial workers felt that all that was needed was to take into account the interpretation that had been neglected, to improve, in other words, the global readings of risky situations by strengthening the role of rating agencies in particular. The latter have now been warned, and have learnt to keep in mind the previously neglected facts (resistance to "stress" of the various institutions, etc.). For public opinion we are back where we started, with the same tools with the same defects.

2.2. Calculations conceal ignorance

This is obvious for financial risks. Because we do not know how to quantify counterparty risks, or those related to market liquidity, and much less those which are due to human error or to changes in the law, very precise calculations are mixed with crude estimates hoping that they will have no appreciable impact on the outcome. Applying sophisticated calculations, such as coherent risk measures, to complex portfolios supposes that the risks are expressed perfectly in the ontology of the objects considered at the outset. In other words it adds a second level: one ignores one's ignorance. This affects the market (organized or OTC) in credits and their derivatives. By the market, portfolios acquire a value where everyone trusts everybody else's calculations though they are no better. This leads to an instability that may be called "methodological moral hazard" which is the belief that mathematics is able to capture new interpretations if the calculations are done by everyone. This kind of instability is worse than in conventional markets in assets and their options because the timescales are much longer (tens of years instead of tens of months) and the punishment of economic reality comes much more slowly.

2.3. The ancillary role of mathematics as servant is confused with that of the subjects being served

The previous idea can be generalized to all situations of mathematized knowledge. Let us take the case of physics. It is obviously helpful to physics when the mathematics used by physicists is improved. There is a real fertility there which has been particularly emphasized by Gaston Bachelard. But it works with the same interpretations as the served science. We are in the syntactic part of normal science in Kuhn's sense. Although Bachelard, with his usual talent, shows that mathematics can suggest questions for physicists, it is impossible to get genuinely new interpretations of phenomena occuring in the domain of the master discipline in this way. Mathematization is an essential component in the phenomenon of scientific crisis as described by Thomas Kuhn.

2.4. That a theoretical representation be perfectible does not mean it is the only way to deal with reality and does not guarantee that it is capable of taking into account every aspect of the situation in question

By theoretical representation I mean a semi-artificial language using mathematics, as in physics or modeling. The fundamental point is that perfectibility gives the illusion of completeness. Ptolemy's geocentric planetary system provides a good example: the excess of mathematization lies in cycles and hypocycles that can be added at will. The original system was improved by Tycho Brahe and is infinitely perfectible, and the excess only became apparent after the new interpretation given by Copernicus. The only flaw in Ptolemy's system is that it has no place for this new interpretation. Yet the new interpretation was much less precise, at least initially, when Copernicus was proposing heliocentric circles. But this is astronomy not planar geometry, and the new reading acquires legitimacy from the fact that it too could be a starting point for improvements; it also has room for possible enhancements. Galileo cannot depart from this new interpretation because he recognized in Jupiter and its satellites a Copernican system. Nevertheless, having, at that pre-Newtonian time, only a *kinematic* description of phenomena, he has no compelling argument against the geocentric system. He was accused during his trial of basing his position on "beliefs" that are not in the sacred texts. It is a case of one interpretation against another, a situation cleverly analysed by Augustin Cournot cf. [6]. The position of Cardinal Robert Bellarmine is that faith has a monopoly of beliefs and that science must remain a means of describing what is allowed in God's creation.

2.5. There is confusion between creativity of the representation and creativity of the world

Within a system of thought, especially one that is perfectible, one cannot see a reason to escape the system. This is related to Quine's remarks on ontological commitment and on the near impossibility of talking about things we either don't know about or deny the existence of. Quine emphasizes our strong tendency to "*talk and think about objects*" [19] both in ordinary language and in physical or economic theories where agents and objects are subject to certain relationships. "*It is hard to say how else there is to talk, not because our objectifying pattern is an invariable trait of human nature, but because we are bound to adapt any alien pattern to our own in the very process of understanding or translating the alien sentences.*" Quine also takes into account the ontological conflicts in order to clarify them. The novelty of the famous article "On What There Is" [20] is the proposal of a definition of ontological commitment which in principle applies quite generally. In fact these fine arguments inspired by mathematical logic are based on the use of logical quantifiers and are quite abstract, and they do not focus on the emergence of new objects.

A more concrete historical example is very illuminating: the abandonment of the natural scale in music. The octave, fifth and other basic musical intervals correspond initially to the division of a vibrating string into simple fractions, one-half for an octave, two-thirds for the fifth, three-fourths for the fourth, etc. This is a strict mathematization of the harmony that is actually perceived by the ear through sound frequencies. If we move from fifth to fifth by iterating the operation of taking two-thirds of the length, then translating these divisions back onto the original octave yields the intervals of the so-called Pythagorean scale. It is very close to the mathematics of vibrating strings, which is the natural (and scientific) basis of sound. It took more than twenty centuries before the natural scale and its improvements were abandoned and the so-called "even-tempered" scale, which gives exactly the same role to all intervals, was adopted. The instruments built on the even-tempered scale do not give preference to a particular key, but they do not respect fully the laws of vibrating strings. The creativity of the musicians has won over that of mathematics in music. The victory is in fact not total, because of some harmonics that are heard as dissonance, etc. But the point to emphasize here is that the idealized world of mathematics has been put to one side in favour of a world based on social practice.

3. Why normal science and jolts of revolutions? Why orthodox economics and crises?

Things seem to move like tectonic plates, in jolts. Why is this? How can we implement a production of knowledge that goes beyond the Kuhnian epistemology?

3.1. As Kuhn thought, normal science is very close to the Popperian vision

Only the modalities of its functioning are seen with a more social emphasis on paradigms as shared understandings of scientific communities. The real difference with Popper is that the disorder that precedes a crisis is more complex than simply encountering a decisive experiment that could refute the theory: there are also attempts to negotiate with the forms of interpretations. Usually the plasticity of the paradigms allows the acceptance of new facts or events in the theory. Kuhn takes the example of a child learning to distinguish ducks, swans and geese in a zoo, with his father playing the role of experimental verdict. He stresses the importance of slightly fuzzy categories whose vagueness is not mathematically quantified [16]. But in certain historical situations, the various ways of arranging things lead to choices that are too artificial (properties of the ether, for example), which gives rise to the search for and the legitimization of more radical interpretative changes.

3.2. But most mathematization situations are not Popperian

Economic theories are not likely to be refuted by any observations of facts. The social environment is constantly changing and is never the same twice. Specialized models with predictive aims are probabilistic and cannot be falsified by a single event. More generally, mathematizations useful for studying changes in the environment (pollution, climate change) are always open to several competing models, each based on a different perspective (extrapolation from ice cores or CO_2 emissions), each perfectible as new data become available. The simplest generic example is that of modeling the flow of a river for flood forecasting. Families of models based on Gaussian ARMA factoring in 1) the water depth, 2) the flow rate, 3) the logarithm of the depth and 4) the logarithm of flow, are each infinitely perfectible if new measured data are available yet they do not give the same probabilities of reaching a certain level. This does not mean that these models are useless, far from it. It just shows that it is not because reality is plural that it is not scientific. In fact, for one type of phenomenon, the data are always finite in number and a finite number of points can be matched either by polynomials or by combinations of real exponentials or trigonometric functions etc. If you think about the immense range of subjects opened up by modeling, then you quickly become convinced that it is the Popperian cases that are the exception. For a theory to be a credible candidate to be Popperian, in addition to agree with past experiments, it must have a fixed number of parameters, each fixed numerically. It is hard to think of any apart from gravitation and some physical theories. Probabilistic theories never fall into this category because an infinite number of events is needed to determine a probability distribution. Now for theories with an inifinite number of parameters, or theories belonging to a family of perfectible representations (as in our example of ARMA times series) the question whether they are popperian or not is not relevant because an negative experiment refutes only numerical values of the parameters (cf. [3]).

This remark also applies equally well to normal science in the sense of Kuhn. It is an extremely restrictive view of knowledge. Let us be more precise.

3.3. It is the monism required at each step that causes the jolts

Where does the new interpretation that is characteristic of a scientific revolution come from? It can only come from differences in the subject community. In other words, the jolts come from the absolute will that the community accept only one truth. Yet this is one particular vision of knowledge and social organization of science. If we accept instead that 'reality' is also, and indeed primarily, people, groups, with their abilities, their habits, their psychology, and their means of interacting with their environment, we see that the only way to capture, or at least to take some account of, the innovation in the world is to make space for the instances where new representations are constructed: users' associations, professional groups, consulting experts, victims of unforeseen circumstances, etc. As Funtowicz and Ravetz have thoroughly analyzed, this route leads to a *better quality* of knowledge, more reliable and in which we can have more confidence [14].

It is a pluralistic knowledge, but that is not to say that it is relativistic. This distinction is crucial. Specifically, as soon as one demands a certain level of rigor and consistency, one is limited to a *small number* of different approaches, just as the major political ideas concern a limited number of parties in multiparty parliamentary systems. To say that departing from the monism of unique truth leads one into relativism is the coarse argument of dominant representations, which the jolts of scientific crises regularly refute.

Nevertheless, if the implementation of such pluralistic knowledge is progressing well in some areas such as climate change or the protection of sensitive areas (despite clashes with political power, which are nothing new), it presents particular difficulties for economics. With globalization, knowledge about economic exchanges has a strong tendency to monism. One would think, however, that the growing environmental problems should lead us to greater tolerance in the implementation of specific economic experiments and their running as a condition of better support for natural equilibriums.

4. Interpretative pluralism is not destructive of knowledge; it is a better type of knowledge

We now propose to examine more thoroughly the features of that better quality and what role mathematics can play. This will necessitate a step back from science as it is currently most often understood and practiced. Beyond the concept of "confined research" introduced by Michel Callon [10], it appears that what is at stake is the *conquering* character of the Baconian program and the *masculine virtues* connected with them.

For convenience we shall use the term challenge-science to describe the view, held until recently by most scientists, that sees knowledge as *a challenge to nature*. It challenges nature to a duel. The honor in the game is to respect the assumptions that govern the rules for experiments. This includes Popperian science and Kuhn's normal science. In fact it is very old; the induction principle advocated by many philosophers and scientists to account for knowledge is similar in nature. Put simply, Popper proposes an induction articulated on a theory. Instead of accepting the thesis that knowledge is essentially philosophical in its ability to spot a pattern and extrapolate it – an idea championed simultaneously (in 1843) by John Stuart Mill and by Augustin Cournot who finely analysed it – thus drawing from a large number of results, or a large number of circumstances, a prospective law that is to be evaluated, Popper strengthens the criterion by requiring that we move from observed facts to a representation with the dress of a theory, that is to say, based on a mathematical syntax like mechanics as formulated by Lagrange or Hamilton. Historically, it is indisputable that during the whole period where industrialization had not yet complexified technology too much, science was practiced with little experimentation and as many challenges were presented to colleagues as to nature. The discoveries at the time of Pascal, Fermat and Father Mersenne were often announced as puzzles, whose answer was known only to the finder, to challenge the wit of contemporaries.

In these early years of the 21st century, a new awareness, unique in the history of man, is happening. Endless continual growth is impossible, and even if the limit is not yet reached, the current pace is so destructive that it must be drastically curbed [8]. It is becoming less and less clear that using challenge-science vis-à-vis the environment, with new technical devices and a progressive mathematization to calculate the economic optimum by cost-benefit analyses in the context of democracy and liberal economy, can overcome the global challenges: arable land, species, climate change, pollution of soil and water, etc. New options for production and consumption (e.g., use oriented product service systems, etc.) and for democratic structures (new bicameralism [9, 17]) are probably essential. But, more fundamentally, we must also consider the question of what kind of knowledge. The epistemological question of how knowledge is produced also arises.

4.1. What logical status can the new knowledge have?

Is there "room" for anything else? What are the characteristics of forms of knowledge that are not falsifiable theories – are there any? They would eventually be forgotten but they are innumerable. Included in this field are all *useful discoveries* that form the logical category complementary to that of refutable hypotheses. The vast majority of knowledge about animal, mineral and vegetable, and a great deal of technical expertise, is of this type.

In this class we find most of the chemistry that has long been viewed as pre-scientific when compared with physics. The great chemist Henry Le Chatelier in the early twentieth century says: "*These two sciences have a similar purpose, they both study phenomena that result in transformations of energy, i.e., mechanical, calorific, electrical or chemical power. In teaching physics one refers only to the laws of natural phenomena: the laws of Mariotte, Gay-Lussac, Ohm, Joule, Descartes, Carnot, etc.* [...] *In chemistry, on the other hand, there is an endless list of small particular facts* [...] *the material thus accumulated will be very useful for the subsequent establishment of science but they do not yet constitute it in any way*" [18]. Why such a disgrace? Is it justified in terms of services rendered?

This class also contains most medical and environmental knowledge. Long before Popper, Claude Bernard wrote the following about medicine: "*in science you can make two kinds of discoveries. Some are predicted by theory; these suppose two conditions: a very advanced science, e.g., physics, and simplicity of the phenomena. The other kind are unexpected: they appear unexpectedly in the experiment, not as corollaries of the theory and devoted to confirm it, but always outside of it and therefore contrary to it*" [1].

More generally, outside the challenge-science category lies all the knowledge about how the world is, what features make it the way we find it, and not another that follows the same laws. This is not inconsistent with general knowledge in Aristotle style, but these innumerable and fortuitous data, that reflect what life and history have made, are essential for nature and the society. Besides, without them challenge-science is nothing. Computers can help us to store them but they do not reduce to dimensions or coordinates. They are interpretative like the new paradigms that Kuhnian revolutions bring. We must therefore accept that some are complementary – plural answers to the same question, differing accounts written in different styles and emphasizing different points.

4.2. A knowledge whose social function is not prediction but caution and care

We have to make a place for stories, testimonies, for what makes our current understanding of the world in all its diversity. They are the basis for the uses and values that give meaning to representations, even scientific ones.

With regard to mathematics, there is no reason to exclude it, we need it here too. But symbols may be used more freely than in axiomatized theories. It is perfectly legitimate to reveal a phenomenon, to represent a trend or a natural evolution using existing scientific languages from the established sciences or from engineering which are semi-artificial languages with partial mathematization. For managing natural equilibriums of life and for working on collective decisions of social groups, it is necessary to allow various representations and even different rationalities to coexist. The use of mathematics as thought patterns, for the linguistic value of symbols and combinations thereof, is useful and desirable. They are not reserved for expressing the truths of challenge-science.

4.3. The main tool of a better quality science is critical and contradictory modeling

The models are able first to take into account the distinctive features of situations and to apply proven knowledge to them and secondly to translate, by the ordinary language which forms the internal cement and the external context, an interpretation of the complexity into what we are interested in.

If they are not to be seen as low level or amateur challenge-science, it is essential that models be always viewed as a facet of a plurality. Firstly, they must be validated by data with the same rigor as usually required by scientists. This validation is not a test of truth, but simply a process of eliminating the unlikely. Secondly they must be recognized as a social expression, i.e., a form of communication from an agent (be that a group, association, company, territorial entity, etc.) to an audience in order to contribute to a decision and therefore subject to criticism by other models. Knowledge is no longer formed exclusively by a struggle between theory and nature but by a contest between models. This process obviously requires a specific organizational context, just as challenge-science requires cautious experimental protocols. The "rules" are not currently codified, but the experiments are underway at international level for the IPCC and in the public debates, citizen juries etc., in a kind of applied living epistemology still under development.

To critique a model is difficult. The quantitative arguments are linked together, everything is connected. It is a huge task to draw out all the implicit assumptions of a model. Even though we know that every model is arbitrary in some aspects, we do not see this arbitrariness explicitly. When discussing one model, our thinking remains stuck in a rut. The best way is to build another model from scratch – the options are much clearer then.

To construct another model, the dualities introduced by the philosophy of science are relevant – they facilitate a dialectic setting for the occurrence of what may be called co-truths. Let us consider a few examples.

Discrete / continuous. Much of the economic theory can be developed without individualizing agents or goods. Some scholars find it illuminating to derive global laws from a micro-economic individual rationality. When studying traffic, we may use flow models or we may model each vehicle individually. Sometimes it is thought that discretization, spatial or temporal, simplifies the problems, with the recurrence rules being more elementary than differential equations and finite element algorithms reducing partial differential equations to simple algebra. But often the opposite happens: the discrete probabilities are sometimes intractable and some algorithms (such as Kalman), are best understood in continuous time.

Descriptive / explanatory. In 1970, two American authors, G.E.P. Box and G.M. Jenkins took methods invented by Wiener for signal processing and applied them to economic predictions. Treating annual series without any regard to their economic meaning, they sometimes obtained better predictions. This is the fundamental duality which we began with in this article. In the history of science, it often occurs in successive periods. The purely descriptive approach can be an advance when it frees us from certain loaded interpretations. On the other hand, explanations allow a reading to shed light on situations other than those already considered.

Quantitative / qualitative. The philosophical work of René Thom has brilliantly illustrated that mathematics provides representation tools that go far beyond the quantitative. A huge field of natural phenomena can be addressed qualitatively through a language adapted to the evolution of forms.

Deterministic / random. A huge number of modeling situations involve risks. The instinctive tendency of modelers is to probabilize the uncertainties – we have already discussed this tendency. This provides a very efficient syntax thanks to the stochastic calculus developed in the 20th century. But this, especially in the tails of laws, conceals ignorance. Uncertainty is sometimes better illustrated by some typical or extreme trajectories obtained from different scenarios.

Image / symbol. Let us take the example of dance. Dozens of notation systems have been developed by the choreographers to record ballets, either based on a limited vocabulary of successive steps (Feuillet system 1700) or more elaborate, noting the dancer's energy in each movement (Laban system 1927). The problem is one of modeling, with the usual constraints of relevance for the choreographer and dancers. *But is this not a false problem since film and video can provide us with an almost perfect image of the ballet ?* The image reproduces, it can provide the perfect illusion of reality, but it does not, by itself, allow choreographic creation. The notation systems have the immense superiority of enabling one to record a ballet that has never been danced.

Critiques of models cannot come from recipes or an *a priori* classification, especially since, as we have emphasized, their relevance depends on the social group that proposes them. *The quality* of the plural knowledge thus produced comes particularly from the things that it can draw out of reality but which challenge-science fails to see. Applied in good conditions of open democracy, it is likely to show hidden effects, unnoticed risks, possibly unsuspected solutions. Challenge-science instead, with the successive stages of its rockets, heads only in one direction.

5. Conclusion: The problem is not that there is too much mathematics, but that it is used exclusively as a framework for theories that claim to univocal truth

The propensity to mathematize more and more can occur in the development of a classical theoretical line of thought as much as one based on modeling, especially if one assigns a value of absolute truth to the interpretative framework we work in, so that syntactic developments will be seen as revealing reality. This occurs in modeling because the modelers tend to think that their models are reality. But faced with other models they are forced to acknowledge *the scope* of their approach. In contrast, in a Popperian conception, mathematization can be pursued without any restraint, until a crisis occurs. Our analysis of mathematization is an Ariadne's thread that opens up the philosophy of knowledge to a new and immense field of thought. It turns away from the jousts, catapults and knights-in-armor of the conquering knowledge, it takes a step back, whereupon challenge-science starts to look like a very particular way of understanding the world.

It is ultimately a choice between what is important and what is not. A river basin for example, may remain for centuries. But we are faced here with contradictory logics, politicians who want to develop jobs, farmers who want to irrigate, associations that want to respect the landscape, companies that want to build dams for electricity, etc. Often neither the economic interest nor the democratic vote, can overcome the basic dominance of selfishness. Maintaining the scenes of natural life involves intermediate languages between native speech and falsifiable science, languages which oppose but do not destroy each other, which, by their plurality, are open to *the interpretation of data* and *the imagination of eventualities.*

About mathematics itself, there is no need worry. Real mathematicians know what drives them: the pleasure of an intellectual game [7]. Maths does not need to be the framework for a grand and unique building of knowledge. On the contrary, freedom from applications and doctrines has always been maintained: non-Euclidean geometries, non-standard analysis, etc. Explorations off the beaten track are rewarded with the surprise of the treasures discovered there.

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