Operator Theory Advances and Applications 227

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Operator Methods in Mathematical Physics

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This volume is dedicated to the memory of

Israel Gohberg

Israel Gohberg (1928–2009)

Contents

Introduction

This volume contains proceedings of the International Conference: Operator The-

ory, Analysis and Mathematical Physics – OTAMP 2010, held at the Mathematical
Research and Conference Center in Bedlewo, Poland, in August. The Conference
was the fifth one from the OTAMP series.
The current volume contain was the fifth one from the OTAMP series.
The current volume contains original results concerning among others the
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the Polish Academy of Sciences. We also thank the staff of the Bana We greatly appreciate financial support of the Institute of Mathematics
the Polish Academy of Sciences. We also thank the staff of the Banach Cent
in Bedlewo for essential help in transportation of the participants and smo Polish Academy of Sciences. We also thank the staff of the Banach Center edlewo for essential help in transportation of the participants and smooth eration.
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Inverse Scattering for Non-classical Impedance Schrödinger Operators

Sergio Albeverio, Rostyslav O. Hryniv, Yaroslav V. Mykytyuk and Peter A. Perry

Abstract. We review recent progress in the direct and inverse scattering theory for one-dimensional Schrödinger operators in impedance form. Two classes of non-smooth impedance functions are considered. Absolutely continuous impedances correspond to singular Miura potentials that are distributions from $W_{2,loc}^{-1}(\mathbb{R})$; nevertheless, most of the classic scattering theory for Schrödinger operators with Faddeev–Marchenko potentials is carried over to this singular setting, with some weak decay assumptions. The second class consists of discontinuous impedances and generates Schrödinger operators with unusual scattering properties. In the model case of piece-wise constant impedance functions with discontinuities on a periodic lattice the corresponding reflection coefficients are periodic. In both cases, a complete description of the scattering data is given and the explicit reconstruction method is derived.

Mathematics Subject Classification (2010). Primary: 34L25, Secondary: 34L40, 47L10, 81U40.

Keywords. Schrödinger operator, impedance function, inverse scattering problem.

Contents

1. Introduction

In this paper, we shall discuss inverse scattering problems for one-dimensional Schrödinger operators H in the impedance form.

$$
H := -\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}x}p^2\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{p},\tag{1.1}
$$

for non-smooth impedance functions p . Our aim is two-fold: firstly, we shall show that the classic inverse scattering theory for Schrödinger operators as discussed Schrödinger operators *H* in the impedance form,
 $H := -\frac{1}{p} \frac{d}{dx} p^2 \frac{d}{dx}$

for non-smooth impedance functions *p*. Our aim

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in, e.g., [27, 31, 40, 54, 60, 61, 63] := $-\frac{1}{p} \frac{1}{dx} p^2 \frac{1}{dx} p$, (1.1)
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Chands are then be then $\frac{1}{p}$ i h is contributed to the contribution of $\frac{1}{p}$ for non-smooth impedance functions p . Our aim is two-fold: firstly, we shall show
that the classic inverse scattering theory for Schrödinger operators as discussed
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in the physical literature), can successfully be extended to a much wider class of
operators, and shall give an account on a recent progre in the physical literature), can successfully be extended to a much wider class of operators, and shall give an account on a recent progress in this direction. Secondly, even though the general approach remains the same, $(1+|x|)dx$) (also called Jost–Bargmann potentials
successfully be extended to a much wider class of
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We note

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from those observed in the Faddeev-Marchenko case.
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are several We note that the above impedance Schrödinger
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form (1.1) . Firstly, Hamiltonians of many models of
in optics, e ten, at least for smooth enough p , in a more usual potential form; however, there are several reasons why our primary interest is in operators H in the impedance form (1.1) . Firstly, Hamiltonians of many models of m are several reasons why our primary interest is in operators H in the impedance
form (1.1). Firstly, Hamiltonians of many models of mathematical physics (e.g.,
in optics, electromagnetics etc.) take the form (1.1), or c in optics, electromagnetics etc.) take the form (1.1) , or can easily be transformed
to it; the corresponding example is given in Subsection 1.3. Secondly, (1.1) allows
a reduction to a first-order Dirac-type (or Zakha to it; the corresponding example is given in Subsection 1.3. Secondly, (1.1) allows a reduction to a first-order Dirac-type (or Zakharov–Shabat) system that is much

k with. Finally, a non-negative Schrödinger operator in potential form
be recast as (1.1) for a suitable impedance function p, cf. [48], while
oth p impedance Schrödinger operators may not possess a reasonable
m; in this can usually be recast as (1.1) for a suitable impedance function p, cf. [48], while
for non-smooth p impedance Schrödinger operators may not possess a reasonable
potential form; in this sense the class of operators (1.1) for non-smooth p impedance Schrödinger operators may not possess a reasonable potential form; in this sense the class of operators (1.1) is larger.

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potential form; in this sense the class of operators (1.1) is larger.
Indeed, set $u := (\log p)' = p'/p$. For absolutely continuous *p*, the operator Indeed, set $u := (\log p)' = p'/p$. For absolutely continuous p
can be written in the factorized form,
 $H = -\left(\frac{d}{dx} + u\right)\left(\frac{d}{dx} - u\right)$,
and becomes a Schrödinger operator in the potential form,

Indeed, set
$$
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$$
. For absolutely continuous p , the operator H can be written in the factorized form,
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H = -\left(\frac{d}{dx} + u\right)\left(\frac{d}{dx} - u\right),\tag{1.2}
$$
\nand becomes a Schrödinger operator in the potential form,
\n
$$
Hy = -y'' + qy,\tag{1.3}
$$

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$$

for the *Miura potential* $q := u' + u^2$. For this to be a regular potential, u must be = $\left(\frac{1}{dx} + u\right)\left(\frac{1}{dx} - u\right)$, (1.2)

perator in the potential form,
 $Hy = -y'' + qy$, (1.3)
 $y' + u^2$. For this to be a regular potential, u must be

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for the *Miura potential* $q := u' + u^2$. For this to be a regula
at least locally absolutely continuous and thus p a function
 $W_{1,loc}^2(\mathbb{R})$. We, however, will not require any continuity of
paper will discuss t = $-y'' + qy$, (1.3)
For this to be a regular potential, u must be
s and thus p a function from the Sobolev class
uire any continuity of u. The first part of the
s a function in $L_{2,loc}(\mathbb{R})$ with additional decay
ribution t for the *Miura potential* $q := u' + u^2$. For this to be a regular potential, u must be at least locally absolutely continuous and thus p a function from the Sobolev class $W_{1,loc}^2(\mathbb{R})$. We, however, will not require a at least locally absolutely continuous and thus p a function from the Sobolev class $W_{1,loc}^2(\mathbb{R})$. We, however, will not require any continuity of u . The first part of the paper will discuss the case where u is a $W^2_{1,\mathrm{loc}}$ (ℝ). We, however, will not require any continuity of u. The first part of the will discuss the case where u is a function in $L_{2,loc}(\mathbb{R})$ with additional decay rties and thus q will be a distribution that locally belong paper will discuss the case where *u* is a function in $L_{2,loc}(\mathbb{R})$ with additional decay properties and thus *q* will be a distribution that locally belongs to the Sobolev space $W_2^{-1}(\mathbb{R})$. In the second part, we properties and thus q will be a distribution that locally belongs to the Sobolev
space $W_2^{-1}(\mathbb{R})$. In the second part, we will treat the case where the impedance p
is discontinuous; then u contains the Dirac δ -func space W_2^{-1}
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is related t is discontinuous; then *u* contains the Dirac δ -functions and q – at least formally – involves their derivatives δ' . See the monographs [6] and [7] for detailed treatment of Schrödinger operators with singular pot

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involves their derivatives δ' . See the monographs [6] and [7] for detailed treatment
of Schrödinger operators with singular pot involves their derivatives δ' . See the monographs [6] and [7] for detailed treatment
of Schrödinger operators with singular potentials and extensive bibliography lists.
Yet another motivation for thinking of the Schröd Yet another motivation for thinking of the Schrödinger operator H in terms
of the function u rather than in terms of its potential q is given by Miura and
is related to completely integrable dispersive equations, cf is related to completely integrable dispersive equations, cf. [33]. Recall that the Miura map is defined as

$$
B: L_{2,loc}(\mathbb{R}) \to W_{2,loc}^{-1}(\mathbb{R}),
$$

\n
$$
u \mapsto u' + u^{2}.
$$

\n
$$
u \text{ that if } u(x,t) \text{ is a smooth solution of the mKdV}
$$

\n
$$
u'(x,t) = \frac{\partial u}{\partial x}(x,t) + u^{2}(x,t)
$$

\n
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Miura map is defined as
 $B: L_{2,loc}(\mathbb{R}) \to W_{2,loc}^{-1}(\mathbb{R}),$
 $u \mapsto u' + u^2.$

In 1968, Miura [62] observed that if $u(x,t)$ is a smooth solution of the mKdV equation, then iura $[62]$ obsen + u^2 .

is a smooth solution of the mKdV
 $+ u^2(x, t)$

or this reason, the Miura map has

nce and well-posedness questions for

$$
(Bu)(x,t) = \frac{\partial u}{\partial x}(x,t) + u^2(x,t)
$$

In 1968, Miura [62] observed that if $u(x, t)$ is a smooth solution of the mKdV equation, then
 $(Bu)(x, t) = \frac{\partial u}{\partial x}(x, t) + u^2(x, t)$

is a smooth solution of the KdV equation. For this reason, the Miura map has

played a fundam $(Bu)(x,t) = \frac{\partial u}{\partial x}(x,t) + u^2(x,t)$
is a smooth solution of the KdV equation. For this rea
played a fundamental role in the study of existence and we
these two equations. A suitable extension of the scatteri:
operators in the f played a fundamental role in the study of existence and well-posedness questions for
these two equations. A suitable extension of the scattering theory for Schrödinger
operators in the form (1.2) would allow to apply the i played these two equations. A suitable extension of the scattering theory for Schrödinger operators in the form (1.2) would allow to apply the inverse scattering transform method to study initial value problems for mKdV an operators in the form (1.2) would allow to apply the inverse scattering transform
method to study initial value problems for mKdV and other completely integrable
dispersive equations with highly singular initial data; c method to study initial value problems for mKdV and other completely integrable
dispersive equations with highly singular initial data; cf. [30] for a particular ex-
ample of the defocussing non-linear Schrödinger (NLS) e dispersive equations with highly singular initial data; cf. [30] for a particular example of the defocussing non-linear Schrödinger (NLS) equation with the Dirac delta-functions in potentials.
We shall concentrate on two

ample of the defocussing non-linear Schrodinger (NLS) equation with the Dirac
delta-functions in potentials.
We shall concentrate on two classes of operators H in (1.1) or (1.2), which
are in a sense "extremal" and for We shall concentrate on two classes of operators H in (1.1) or (1.2), which
are in a sense "extremal" and for which the direct and inverse scattering problems
have recently been quite thoroughly discussed. The first is have recently been quite thoroughly discussed. The first is the class of (locally) absolutely continuous impedance functions p , for which the corresponding $u = p'/p$ have certain integrability at infinity and which still absolutely continuous impedance functions p , for which the corresponding $u = p'/p$
have certain integrability at infinity and which still bears lots of properties found

for problems with Faddeev–Marchenko potentials; see [35, 36, 46 class is with piece-wise constant p [3, 4, 66, 67]; formally, the co is a discrete measure, i.e., the sum of the Dirac δ -functions. The operators (1.1) class is with piece-wise constant p [3, 4, 66, 67]; formally, the corresponding u is a discrete measure, i.e., the sum of the Dirac δ -functions. The Schrödinger operators (1.1) in this second class possess quite un is a discrete measure, i.e., the sum of the Dirac δ -functions. The Schrödinger
operators (1.1) in this second class possess quite unusual scattering properties [9,
10]; in a certain sense, the corresponding scattering % operators (1.1) in this second class possess quite unusual scattering properties [9, 10]; in a certain sense, the corresponding scattering theory might be viewed as a "discrete" analogue of the classical one.

Despite t

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Despite the formal similarities in the way the inverse scattering problems

are solved in both c "discrete" analogue of the classical one.

Despite the formal similarities in the way the inverse scattering problems

are solved in both cases, there are essential differences that do not allow to com-

bine the two meth Despite the formal similarities in are solved in both cases, there are essentine the two methods and to treat generic measures $d \log p$ without singula such a unified theory would be of muc electromagnetic scattering theory plyed in both cases, there are essential differences that do not allow to com-
the two methods and to treat generic piece-wise smooth impedances, i.e.,
ic measures dlog p without singular continuous components. Needless bine the two methods and to treat generic piece-wise smooth impedances, i.e., generic measures dlog p without singular continuous components. Needless to say, such a unified theory would be of much interest for many app

1.1. Basic definitions

is the electromagnetic scattering theory in stratified media.
 1.1. Basic definitions

Now we recall the main objects of the scattering theory for Schrödinger operators

in one dimension and describe in general terms th 1.1. Basic definitions
Now we recall the main objects of the scattering theory in one dimension and describe in general terms the
see, e.g., [19, 27, 31, 58, 59, 64, 67] for a detailed e
terms of the logarithmic derivativ in one dimension and describe in general terms the results we want to derive;
see, e.g., [19, 27, 31, 58, 59, 64, 67] for a detailed exposition. We shall work in
terms of the logarithmic derivative $u = p'/p$ of the impedanc see, e.g., [19, 27, 31, 58, 59, 64, 67] for a detailed exposition. We shall work in
terms of the logarithmic derivative $u = p'/p$ of the impedance function p ; the
precise assumptions on u will be stated in the next sectio terms of the logarithmic derivative $u = p'/p$ of the impedance function p ; the
precise assumptions on u will be stated in the next sections. In particular, in the
continuous case u will be in $L_2(\mathbb{R})$ with some decay a terms of the logarithmic derivative $u = p'/p$ of the impedance function p; the

the maximal domain in $L_{2,loc}(\mathbb{R})$. By definition, the Jost solutions $f_{\pm}(\cdot,\omega)$ for precise assumptions on *u* will be stated in the next sections. In particular, in the continuous case *u* will be in $L_2(\mathbb{R})$ with some decay at infinity, while for piece-wise constant *p* the function *u* is a measure continuous case *u* will be in $L_2(\mathbb{R})$ with some decay at infinity, while for piece-wise
constant *p* the function *u* is a measure.
Denote by I the differential expression generated by either (1.1) or (1.2) on
the ma constant *p* the function *u* is a measure.
Denote by I the differential exprection the maximal domain in $L_{2,loc}(\mathbb{R})$. By $\omega \in \mathbb{R}$ are solutions to the Schrödinger to $e^{\pm i\omega x}$ at $\pm \infty$, i.e., such that $f_+(x,\omega$ Denote by I the differential expression generated by either (1.1) or (1.2) on
naximal domain in $L_{2,loc}(\mathbb{R})$. By definition, the Jost solutions $f_{\pm}(\cdot,\omega)$ for
 \mathbb{R} are solutions to the Schrödinger equation $I(y) = \omega$ the maximal domain in $L_{2,loc}(\mathbb{R})$. By definition, the Jost solutions $f_{\pm}(\cdot, \omega)$ for $\omega \in \mathbb{R}$ are solutions to the Schrödinger equation $\mathfrak{l}(y) = \omega^2 y$ that are asymptotic to $e^{\pm i\omega x}$ at $\pm \infty$, i.e., such t $\omega \in \mathbb{R}$ are solutions to the Schrödinger equation $\mathfrak{l}(y) = \omega^2 y$ that are asymptotic
to $e^{\pm i\omega x}$ at $\pm \infty$, i.e., such that
 $f_+(x,\omega) = e^{i\omega x} (1 + o(1)), \quad x \to +\infty,$
 $f_-(x,\omega) = e^{-i\omega x} (1 + o(1)), \quad x \to -\infty.$
The Jost solutio

$$
f_{+}(x,\omega) = e^{i\omega x} (1 + o(1)), \qquad x \to +\infty,
$$

$$
f_{-}(x,\omega) = e^{-i\omega x} (1 + o(1)), \qquad x \to -\infty.
$$

to $e^{\pm i\omega x}$ at $\pm \infty$, i.e., such that
 $f_+(x,\omega) =$
 $f_-(x,\omega) =$

The Jost solutions exist for quinon-

For real nonzero ω , the solution $(x, \omega) = e^{i\omega x} (1 + o(1)),$ $x \to +\infty,$
 $(x, \omega) = e^{-i\omega x} (1 + o(1)),$ $x \to -\infty.$

t for quite a large class of impedance f

ne solutions $f_{-}(\cdot, \omega)$ and $f_{-}(\cdot, -\omega)$:
 $I(y) = \omega^2 y$ and thus there exist coeffic $(x, \omega) = e^{-i\omega x} (1 + o(1)),$ $x \to -\infty.$

t for quite a large class of impedance f

ne solutions $f_{-}(\cdot, \omega)$ and $f_{-}(\cdot, -\omega)$:
 $I(y) = \omega^2 y$ and thus there exist coeffic
 $(x, \omega) = a(\omega) f_{-}(x, -\omega) + b(\omega) f_{-}(x, \omega)$ The Jost solutions exist for quite a large class of impedance functions p (resp. u).
For real nonzero ω , the solutions $f_{-}(\cdot, \omega)$ and $f_{-}(\cdot, -\omega)$ form a fundamental
system of solutions of $I(y) = \omega^2 y$ and thus t system of solutions of $I(y) = \omega^2 y$ and thus there exist coefficients $a(\omega)$ and $b(\omega)$
such that
 $f_+(x, \omega) = a(\omega)f_-(x, -\omega) + b(\omega)f_-(x, \omega)$. (1.5)
As in the classic scattering theory for Schrödinger operators with potentials in

$$
f_{+}(x,\omega) = a(\omega)f_{-}(x,-\omega) + b(\omega)f_{-}(x,\omega). \tag{1.5}
$$

system of solutions of $\mathfrak{l}(y) = \omega^2 y$ and thus there exist coefficients $a(\omega)$ and $b(\omega)$
such that
 $f_+(x,\omega) = a(\omega)f_-(x,-\omega) + b(\omega)f_-(x,\omega)$. (1.5)
As in the classic scattering theory for Schrödinger operators with potentials i As in the
the Fadde
identities $(x, \omega) = a(\omega)f_{-}(x, -\omega) + b(\omega)f_{-}(x, \omega).$ (1.5)
tering theory for Schrödinger operators with potentials in
ko class, the coefficients *a* and *b* will be shown to verify the
 $(\omega)|^2 = 1$, $a(-\omega) = a(\omega)$, $b(-\omega) = b(\omega)$, and the relati the Faddeev–Marchenko class, the coefficients *a* and *b* will be shown to verify the
identities $|a(\omega)|^2 - |b(\omega)|^2 = 1$, $a(-\omega) = \overline{a(\omega)}$, $b(-\omega) = \overline{b(\omega)}$, and the relation
 $f_{-}(x, \omega) = a(\omega)f_{+}(x, -\omega) - b(-\omega)f_{+}(x, \omega)$. (1.6)
Th $2 - |b(\omega)|^2$

$$
f_{-}(x,\omega) = a(\omega)f_{+}(x,-\omega) - b(-\omega)f_{+}(x,\omega).
$$
 (1.6)

identities
$$
|a(\omega)|^2 - |b(\omega)|^2 = 1
$$
, $a(-\omega) = a(\omega)$, $b(-\omega) = b(\omega)$, and the relation
\n
$$
f_{-}(x, \omega) = a(\omega)f_{+}(x, -\omega) - b(-\omega)f_{+}(x, \omega).
$$
\n(1.6)
\nThe corresponding right r_{+} and left r_{-} reflection coefficients are introduced via
\n
$$
r_{+}(\omega) := -\frac{b(-\omega)}{a(\omega)}, \qquad r_{-}(\omega) := \frac{b(\omega)}{a(\omega)},
$$
\n(1.7)

Inverse Scattering for Impedance Schrödinger Operators 5

and $t := 1/a$ is the *transmission* coefficient. The motivation for these terms comes

from physics; indeed, the solution
 $t(\omega) f_+(x, \omega) = f_-(x, -\omega) + r_-(\omega) f_-(x, \omega)$

d

$$
t(\omega)f_+(x,\omega) = f_-(x,-\omega) + r_-(\omega)f_-(x,\omega)
$$

and $t := 1/a$ is the transmission coefficient. The motivation for these terms comes
from physics; indeed, the solution
 $t(\omega) f_+(x, \omega) = f_-(x, -\omega) + r_-(\omega) f_-(x, \omega)$
describes the plane monochromatic wave $e^{i\omega x}$ sent in from $-\$ $t(\omega) f_+(x, \omega) =$
describes the plane monochromatic
that after interaction with the ir
 $t(\omega) f_+(x; \omega)$ and partly gets reflections form the r $(\omega) f_+(x, \omega) = f_-(x, -\omega) + r_-(\omega) f_-(x, \omega)$
nonochromatic wave $e^{i\omega x}$ sent in from $-\infty$ (
n with the impedance p partly transmitically gets reflected back to $-\infty$ (the term
nts form the matrix
 $S(\omega) := \begin{pmatrix} t(\omega) & r_+(\omega) \\ r_-(\omega$ describes the plane monochromatic wave e^{i ωx} sent in from $-\infty$ (the term $f_-(x;-\omega)$)
that after interaction with the impedance p partly transmits to $+\infty$ (the term
 $t(\omega) f_+(x;\omega)$) and partly gets reflected back to $-\$ that after interaction with the impedance p partly transmits to $+\infty$ (the term $t(\omega) f_+(x;\omega)$) and partly gets reflected back to $-\infty$ (the term $r_-(\omega) f_-(x;\omega)$).
These coefficients form the matrix
 $S(\omega) := \begin{pmatrix} t(\omega) & r_+(\omega$ $\frac{1}{2}$

$$
S(\omega) := \begin{pmatrix} t(\omega) & r_+(\omega) \\ r_-(\omega) & t(\omega) \end{pmatrix}
$$

(ω) $f_+(x; \omega)$) and partly gets reflected back to $-\infty$ (the term $r_-(\omega) f_-(x; \omega)$).

These coefficients form the matrix
 $S(\omega) := \begin{pmatrix} t(\omega) & r_+(\omega) \\ r_-(\omega) & t(\omega) \end{pmatrix}$

called the *scattering matrix* for *H*. This matrix i $S(\omega) := \begin{pmatrix} t(\omega) & r_+(\omega) \\ r_-(\omega) & t(\omega) \end{pmatrix}$

d the *scattering matrix* for *H*. This matrix is une properties of the scattering coefficients *a* and

constructed from the right or left reflection coefficients

experience Sc called the *scattering matrix* for H . This matrix is unitary on the real line and the above properties of the scattering coefficients a and b imply that S can uniquely be reconstructed from the right or left refle above properties of the scattering coefficients *a* and *b* imply that *S* can uniquely
be reconstructed from the right or left reflection coefficient alone. We observe that
the impedance Schrödinger operator *H* is non-n the impedance Schrödinger operator H is non-negative and thus has no bound
states; therefore S comprises all the scattering information on H .
The direct scattering theory studies the properties of the scattering map

states; therefore *S* comprises all the scattering information on *H*.
The direct scattering theory studies the properties of the scatering via
 $S_+ : u \to r_+,$
 $S_- : u \to r_-$.
The inverse scattering problem is to reconstruct the

$$
S_+ : u \to r_+,
$$

$$
S_- : u \to r_-.
$$

The direct scattering theory studies the properties of the scattering maps S_{\pm}
ed via
 $S_{+}: u \to r_{+},$
 $S_{-}: u \to r_{-}.$
inverse scattering problem is to reconstruct the function u and thus the
idinger operator H from its sc The inverses Schrödinge The inverse scattering problem is to reconstruct the function u and thus the Schrödinger operator H from its scattering matrix (i.e., from its reflection coefficient r_+ or r_-). More exactly, for a given class of : $u \rightarrow r_$.

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plete solut
 $S_$ pm are on The inverse scattering problem is to reconstruct the function u and thus the Schrödinger operator H from its scattering matrix (i.e., from its reflection coefficient r_+ or r_-). More exactly, for a given class of Schrödinger operator *H* from its scattering matrix (i.e., from its reflection coefficient r_+ or r_-). More exactly, for a given class of Schrödinger operators, i.e., for a given class of functions u , a complete sol efficient r_+ or r_-). More exactly, for a given class of Schrödinger operators, i.e.,
for a given class of functions u , a complete solution of the inverse scattering prob-
lem consists in proving that the maps S_{\pm for a given class of functions u , a complete solution of the inverse scattering prob-
lem consists in proving that the maps S_{\pm} are one-to-one, finding their images, and
constructing the inverse maps S_{\pm}^{-1} .
Nex

lem consists in proving that the maps σ_{\pm} are one-to-one, finding their images, and
constructing the inverse maps S_{\pm}^{-1} .
Next we discuss in some more detail two classes of problems we will mostly
be interested i ±

1.2. Miura potentials: the case of absolutely continuous

constructing the inverse maps S_{\pm}^{-1} .

Next we discuss in some more detail two classes of problems we will mostly

be interested in.
 1.2. Miura potentials: the case of absolutely continuous p

To describe the fir $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ To describe the first class of problems to be studied, we start with the observation **1.2.** Miura pote
To describe the that if a Schröd
potential q admi
for $q \in W^{-1}_{2,\text{loc}}(0)$ that if a Schrödinger operator (1.3) has no bound states, then the corresponding
potential q admits a Riccati representation given by the Miura map (1.4). Namely,
for $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$ real-valued and $\varphi \in C_0^{\infty}(\$ potential q admits a Riccati representation given by the Miura map (1.4). Namely,
for $q \in W_{2,loc}^{-1}(\mathbb{R})$ real-valued and $\varphi \in C_0^{\infty}(\mathbb{R})$ define the Schrödinger form h
corresponding to (1.3) via
 $\mathfrak{h}(\varphi) := \int |\varphi'($ potential q admits a Riccati representation given by the Miura map (1.4). Namely,
for $q \in W_{2,loc}^{-1}(\mathbb{R})$ real-valued and $\varphi \in C_0^{\infty}(\mathbb{R})$ define the Schrödinger form h
corresponding to (1.3) via
 $\mathfrak{h}(\varphi) := \int |\varphi'($ $_{2,\text{loc}}$ (\mathbb{R}) it can value and $\varphi \in C_0$

$$
\mathfrak{h}(\varphi) := \int |\varphi'(x)|^2 dx + \langle q, |\varphi|^2 \rangle,
$$

for $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$ real-valued and $\varphi \in C_0^{\infty}(\mathbb{R})$ define the Schrödinger form \mathfrak{h}
corresponding to (1.3) via
 $\mathfrak{h}(\varphi) := \int |\varphi'(x)|^2 dx + \langle q, |\varphi|^2 \rangle$,
where $\langle \cdot, \cdot \rangle$ is the pairing between $W_{2,\text{loc}}^{-1$ $\mathfrak{h}(\varphi) := \int |\varphi'(x)|^2 dx + \langle q, |\varphi|$
where $\langle \cdot, \cdot \rangle$ is the pairing between $W_{2,\text{loc}}^{-1}(\mathbb{R})$ and W
in [48], if q is a real-valued distribution in $W_{2,\text{loc}}^{-1}(\mathbb{R})$
form \mathfrak{h} is nonnegative, then q may be p $\left.\begin{array}{c} (x) | \\ V_{2,\text{lo}}^{-1} \end{array}\right.$ ion
e pı
be µ a where $\langle \cdot, \cdot \rangle$ is the pairing between $W_{2,\text{loc}}^{-1}(\mathbb{R})$ and $W_{2,\text{comp}}^{1}$
in [48], if q is a real-valued distribution in $W_{2,\text{loc}}^{-1}(\mathbb{R})$ for w
form h is nonnegative, then q may be presented as $q = B$
 $L_{2,\text{loc}}(\mathbb{$ (ℝ). Then, as shown
hich the Schrödinger
u for a function $u \in$
ll *q* a *Miura potential*
tive for *q*. It is easily
a continuous function in [48], if q is a real-valued distribution in $W_{2,\text{loc}}^{-1}(\mathbb{R})$ for which the Schrödinger
form h is nonnegative, then q may be presented as $q = Bu$ for a function $u \in L_{2,\text{loc}}(\mathbb{R})$. Such a function u need not be unique form $\mathfrak h$ is nonnegative, then q may be presented as $q = Bu$ for a function $u \in L_{2,loc}(\mathbb{R})$. Such a function u need not be unique; we will call q a *Miura potential* and any function u satisfying $q = Bu$ a *Ricca* $L_{2,\mathrm{loc}}$ (ℝ). Such a function *u* need not be unique; we will call *q* a *Miura potential*
iny function *u* satisfying $q = Bu$ a *Riccati representative* for *q*. It is easily
chat two Riccati representatives for a given *q* differ and any function *u* satisfying $q = Bu$ a *Riccati representative* for *q*. It is easily seen that two Riccati representatives for a given *q* differ by a continuous function seen that two Riccati representatives for a given q differ by a continuous function

and that a Riccati representative u is the logarithmic derivative
distributional solution to the zero-energy Schrödinger equation $-y'$
One may not hope for extension of the scattering theory that
struct directly a singula and that a Riccati representative *u* is the logarithmic derivative of a positive distributional solution to the zero-energy Schrödinger equation $-y'' + qy = 0$.
One may not hope for extension of the scattering theory that wo distributional solution to the zero-energy Schrödinger equation $-y'' + qy = 0$.
One may not hope for extension of the scattering theory that would restruct directly a singular Miura potential $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$ of (1.3). t directly a singular Miura potential $q \in W_{2,\text{loc}}(\mathbb{K})$ of (1.3). However, the
sponding Riccati representative u is a regular function, and reconstruction
from the scattering data of H might be possible. Since, however, struct directly a singular Miura potential $q \in W_{2,\text{loc}}^{-1}(\mathbb{R})$ of (1.3). However, the
corresponding Riccati representative u is a regular function, and reconstruction
of u from the scattering data of H might be possibl corresponding Riccati representative *u* is a regular function, and reconstruction
of *u* from the scattering data of *H* might be possible. Since, however, a Miura
potential *q* possesses many different Riccati represent

of *u* from the scattering data of *H* might be possible. Since, however, a Miura
potential *q* possesses many different Riccati representatives, one has to single out
a distinguished *u* that should be recovered.
One suc potential q possesses many different Riccati representatives, one has to single out
a distinguished u that should be recovered.
One such possibility was suggested in [36], where the class Q_0 of Miura poten-
tials admit a distinguished u that should be recovered.
One such possibility was suggested in [index] admitting Riccati representatives $u \in \text{case}$, such a Riccati representative is unique of Miura potentials q and Schrödinger ope One such possibility was suggested in [36], where the class Q_0 of Miura poten-
admitting Riccati representatives $u \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ was considered. In this
such a Riccati representative is unique, thus giving a tials admitting Riccati representatives $u \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ was considered. In this case, such a Riccati representative is unique, thus giving a natural parametrization of Miura potentials q and Schrödinger operator

of Miura potentials q and Schrödinger operators (1.2) and (1.3). Below, we give
several examples from [36] of Miura potentials generated this way.
Example 1.1. Let u be an even function that for $x > 0$ equals $x^{-\alpha} \sin x^$ **Example 1.1.** Let u be an even function that for $x > 0$ equals $x^{-\alpha}$ that $\alpha > 1$ and $\beta > \alpha + 1$. Then u belongs to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and the Miura potential $q = u' + u^2$ is of the form
 $q(x) = \beta \operatorname{sign}(x)|x|^{\beta - \alpha - 1} \cos |$ **Example 1.1.** Let *u* be an even function that for $x > 0$ equals $x^{-\alpha} \sin x^{\beta}$. Assume
that $\alpha > 1$ and $\beta > \alpha + 1$. Then *u* belongs to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and the corresponding
Miura potential $q = u' + u^2$ is of the f

$$
q(x) = \beta \operatorname{sign}(x)|x|^{\beta - \alpha - 1} \cos |x|^{\beta} + \tilde{q}(x)
$$

that $\alpha > 1$ and $\beta > \alpha + 1$. Then *u* belongs to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and the corresponding
Miura potential $q = u' + u^2$ is of the form
 $q(x) = \beta \operatorname{sign}(x)|x|^{\beta - \alpha - 1} \cos |x|^{\beta} + \tilde{q}(x)$
for some bounded function \tilde{q} . Thus Miura potential $q = u' + u^2$ is of the form
 $q(x) = \beta \operatorname{sign}(x)|x|^{\beta - 1}$

for some bounded function \tilde{q} . Thus q is u

the corresponding Schrödinger operator por

trum filling out the positive semi-axis and

such potential $(x) = \beta \operatorname{sign}(x)|x|^{\beta - \alpha - 1} \cos |x|^{\beta} + \tilde{q}(x)$
ction \tilde{q} . Thus q is unbounded and osci
rödinger operator possesses only absolus
sitive semi-axis and the scattering and
defined. for some bounded function \tilde{q} . Thus q is unbounded and oscillatory; nevertheless,
the corresponding Schrödinger operator possesses only absolutely continuous spec-
trum filling out the positive semi-axis and the sc

trum filling out the positive semi-axis and the scattering and inverse scattering on
such potentials is well defined.
Example 1.2. Assume that $\phi \in C_0^{\infty}(\mathbb{R})$ is such that $\phi \equiv 1$ on $(-1, 1)$. Take
 $u(x) = \alpha \phi(x) \log |x$ **Example 1.2.** Assume that ϕ
 $u(x) = \alpha \phi(x) \log |x|$ with $\alpha >$
distributional derivative of log
Miura potential q is smooth or
gularity. See, e.g., [16, 34, 57] a **Example 1.2.** Assume that $\phi \in C_0^c$ Assume that $\phi \in C_0^{\infty}(\mathbb{R})$ is such that $\phi \equiv 1$ on $(-1, 1)$. Take
log |x| with $\alpha > 0$. Then $u \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$; moreover, since the
derivative of log |x| is the distribution P.v. 1/x, the corresponding
l q \boldsymbol{u} $(x) = \alpha \phi(x) \log |x|$ with $\alpha > 0$. Then $u \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$; moreover, since the istributional derivative of $\log |x|$ is the distribution P.v. 1/x, the corresponding fiura potential q is smooth outside the origin and has distributional derivative of log | x | is the distribution P.v. 1/ x , the corresponding
Miura potential q is smooth outside the origin and has there a Coulomb-type sin-
gularity. See, e.g., [16, 34, 57] and the referen

Miura potential q is smooth outside the origin and has there a Coulomb-type singularity. See, e.g., [16, 34, 57] and the references therein for discussion and rigorous treatment of Schrödinger operators with Coulomb poten Example 1.3 (Frayer [35]). The Riccati representative $u = \alpha \chi_{[-1,1]}$, with α and nonzero real constant and χ_{Δ} the indicator function of a set Δ , corresponds to the Miura potential
 $q = \alpha \delta(\cdot + 1) - \alpha \delta(\cdot - 1) + \alpha^$ **Example 1.3 (Frayer [35]).** The Riccati representative $u =$
nonzero real constant and χ_{Δ} the indicator function of a set Δ
Miura potential
 $q = \alpha \delta(\cdot + 1) - \alpha \delta(\cdot - 1) + \alpha^2 \chi_{[-1,1]},$ **Example 1.3 (Frayer [35]).** nonzero real constant and χ_{Δ} the indicator function of a set Δ , corresponds to the nonzero real constant and χ_{Δ} the indicator function of a set Δ , corresponds to the
Miura potential
 $q = \alpha \delta(\cdot + 1) - \alpha \delta(\cdot - 1) + \alpha^2 \chi_{[-1,1]},$
 δ being the Dirac delta-function centered at the origin.
The class \mathcal

$$
q = \alpha \delta(\cdot + 1) - \alpha \delta(\cdot - 1) + \alpha^2 \chi_{[-1,1]},
$$

 δ

 δ being the Dira
The class
all the correspon = $\alpha\delta(\cdot + 1) - \alpha\delta(\cdot - 1) + \alpha^2\chi_{[-1,1]},$
function centered at the origin.
iura potentials, however, is rather s
hrödinger operators (1.2) possess re
 δ : Section 2. In particular, Q_0 inclu
 δ with $\alpha > 0$ and δ bei The class Q_0 of Miura potentials, however, is ratled the corresponding Schrödinger operators (1.2) posses at the origin, see Section 2. In particular, Q_0 is ngular potential $q = \alpha \delta$, with $\alpha > 0$ and δ being the The class Q_0 of Miura potentials, however, is rather small in the sense that
all the corresponding Schrödinger operators (1.2) possess *resonance* (or *half-bound*
state) at the origin, see Section 2. In particular, Q state) at the origin, see Section 2. In particular, Q_0 includes heither the model
singular potential $q = \alpha \delta$, with $\alpha > 0$ and δ being the Dirac delta function centered
at the origin, nor generic (i.e., non-resonan singular potential $q = \alpha \delta$, with $\alpha > 0$ and δ being the Dirac delta function centered
at the origin, nor generic (i.e., non-resonant) Faddeev–Marchenko potentials. For
this reason a further extension of Q_0 is desi at the origin, and generic (i.e., nor resonant) Faddeev–Marchenthen products. For
this reason a further extension of Q_0 is desirable. this reason a further extension of \mathcal{Q}_0 is desirable.

wider class Q of Miura potentials q was suggested in [46]. Recall that
epresentative u gives rise to a strictly positive distributional solution y
nergy Schrödinger equation $-y'' + qy = 0$ via
 $u(x) = \exp\left(\int_x^x u(s) ds\right)$ Such a wider class Q of Miura potentials q was suggested in [46]. Recall that

diccati representative u gives rise to a strictly positive distributional solution y
 y zero-energy Schrödinger equation $-y'' + qy = 0$ via
 $y(x$ any Riccati representative *u* gives rise to a strictly positive distributional solution *y* of the zero-energy Schrödinger equation $-y'' + qy = 0$ via $y(x) = \exp\left(\int_0^x u(s) ds\right)$
and, conversely, any positive solution $y \in H_{\text{loc$

$$
y(x) = \exp\left(\int_0^x u(s) \mathrm{d}s\right)
$$

of the zero-energy Schrödinger equation $-y'' + qy = 0$ via
 $y(x) = \exp\left(\int_0^x u(s)ds\right)$

and, conversely, any positive solution $y \in H_{loc}^1(\mathbb{R})$ gives

sentative $u(x) = y'(x)/y(x)$. Thus, the set of Riccati rep

distribution potentia $(x) = \exp \left(\int_0^x u(s) \, ds \right)$
solution $y \in H^1_{\text{loc}}(\mathbb{R})$
Thus, the set of Ricc:
ameterized by positive
ized by $y(0) = 1$. An
nes y_{\pm} with the propen and, conversely, any positive solution $y \in H_{\text{loc}}^1(\mathbb{R})$ gives rise to a Riccati representative $u(x) = y'(x)/y(x)$. Thus, the set of Riccati representatives for a given distribution potential q is parameterized by positive sentative $u(x) = y'$
distribution potenti
Schrödinger equation
 $y'' = qy$ there are expansion distribution potential q is parameterized by positive solutions y to the zero-energy Schrödinger equation normalized by $y(0) = 1$. Among all positive solutions to y''

$$
\int_0^\infty \frac{\mathrm{d}s}{y_+^2(s)} = \int_{-\infty}^0 \frac{\mathrm{d}s}{y_-^2(s)} = +\infty,
$$

and then any positive solution y takes the form $y = \theta y_+ + (1 - \theta)y_-$ for some $\theta \in$ = qy there are extremal ones y_{\pm} with the properties that
 $\int_0^{\infty} \frac{ds}{y_+^2(s)} = \int_{-\infty}^0 \frac{ds}{y_-^2(s)} = +\infty,$

d then any positive solution y takes the form $y = \theta y_+ + (1)$. The corresponding extremal Riccati representa $\frac{ds}{dt}$
 $\frac{y}{dt}$ main
 $\frac{1}{2}$ $\overline{(s)} = \int_{-\infty}^{\infty} \overline{y^2_{-}(s)} dx = +\infty,$
 y' takes the form $y = \theta y_{+}$

and Riccati representative

addition that u_{\pm} are in

egrable at $-\infty$. The set of
 \overline{Q} , i.e., $\frac{ds}{ds}$
 $\frac{2}{s}$
 $\frac{1}{s}$ and then any positive solution y takes the form $y = \theta y_+ + (1 - \theta)y_-$ for some $\theta \in [0, 1]$. The corresponding extremal Riccati representatives $u_{\pm} = (\log y_{\pm})'$ belong to $L_{2,loc}(\mathbb{R})$; we will assume in addition that $u_{\$ [0, 1]. The corresponding extremal Riccati representatives $u_{\pm} = (\log y_{\pm})'$ belong
to $L_{2,loc}(\mathbb{R})$; we will assume in addition that u_{\pm} are in $L_2(\mathbb{R})$ and that u_{+} is
integrable at $+\infty$ and u_{-} is integra ′ integrable at $+\infty$ and u_{-} is integrable at $-\infty$. The set of all potentials with the
above properties is denoted by Q , i.e.,
 $Q := \{q = \overline{q} \in W_2^{-1}(\mathbb{R}) : \exists u_{\pm} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}^{\pm}) \text{ s.t. } q = u'_{+} + u_{+}^{2} = u'_{-}$

$$
\mathcal{Q} := \{ q = \overline{q} \in W_2^{-1}(\mathbb{R}) : \exists u_{\pm} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}^{\pm}) \text{ s.t. } q = u'_{+} + u_{+}^{2} = u'_{-} + u_{-}^{2} \}.
$$

integrable at +∞ and u_- is integrable at $-\infty$. The set of all potentials with the
above properties is denoted by Q , i.e.,
 $Q := \{q = \overline{q} \in W_2^{-1}(\mathbb{R}) : \exists u_{\pm} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}^{\pm}) \text{ s.t. } q = u'_+ + u_+^2 = u'_- + u_-^2\}.$
W $Q := \{q = \overline{q} \in W_2^{-1}(\mathbb{R}) : \exists u_{\pm} \in L_2(\mathbb{R})\}$
We observe that the Riccati representies are unique [44, Ch. IX.2 (ix)] a
 $u_{-} = u_{+}$, i.e., where the extremal so
This condition is very unstable under := { $q = \overline{q} \in W_2^{-1}(\mathbb{R})$: $\exists u_{\pm} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}^{\pm})$ s.t. $q = u'_{+} + u_{+}^2 = u'_{-} + u_{-}^2$
We observe that the Riccati representatives u_{+} and u_{-} with the above pi
es are unique [44, Ch. IX.2 (ix)] and We observe that the Riccati representatives u_+ and u_- with the above prop-

i are unique [44, Ch. IX.2 (ix)] and that Q_0 corresponds to the case where
 $= u_+, i.e.,$ where the extremal solutions y_- and y_+ are lin erties are unique [44, Ch. IX.2 (IX)] and that Q_0 corresponds to the case where $u_{-} = u_{+}$, i.e., where the extremal solutions y_{-} and y_{+} are linearly dependent.
This condition is very unstable under perturbati $u_-=u_+$, i.e., where the extremal solutions y_- and y_+ are linearly dependent.
This condition is very unstable under perturbation of q, whence the case $q \in \mathcal{Q}_0$ = u_+ , i.e., where the extremal solutions y_- and y_+ are linearly dependent.

s condition is very unstable under perturbation of q, whence the case $q \in \mathcal{Q}_0$

ht be considered "exceptional" and $q \in \mathcal{Q}_1 := \mathcal{Q}$ This condition is very unstable under perturbation of q, whence the case $q \in \mathcal{Q}_0$ might be considered "exceptional" and $q \in \mathcal{Q}_1 := \mathcal{Q} \setminus \mathcal{Q}_0$ "generic". A potential $q \in \mathcal{Q}$ is uniquely determined by the da might be considered "exceptional" and $q \in \mathcal{Q}_1 := \mathcal{Q} \setminus \mathcal{Q}_0$ "generic". A poten-
tial $q \in \mathcal{Q}$ is uniquely determined by the data $u_{-}|_{(-\infty,0)}, u_{+}|_{(0,\infty)}$, and the "jump"
 $(u_{-}-u_{+})(0)$. The set $\mathcal Q$ contains a tial $q \in \mathcal{Q}$ is uniquely determined by the data $u_{-}|_{(-\infty,0)}, u_{+}|_{(0,\infty)}$, and the "jump"
 $(u_{-}-u_{+})(0)$. The set $\mathcal Q$ contains all real-valued potentials of Faddeev–Marchenko

class generating non-negative Schröding $(-\infty,0)$, $\frac{u+1(0,\infty)}{v-1}$

tentials (e.g., with Dirac delta-functions and Coulomb-like singularities). For in-
stance, if $q = q_0 + q_1$ is such that $q_0 \in \mathcal{Q}$, $q_1 \in W_2^{-1}(\mathbb{R})$ is of compact support,
and the corresponding operator H of (1.3) i stance, if $q = q_0 + q_1$ is such that $q_0 \in \mathcal{Q}$, $q_1 \in W_2^{-1}(\mathbb{R})$ is of compact support,
and the corresponding operator H of (1.3) is non-negative, then $q \in \mathcal{Q}$; see [46].
In fact, a further extension is possible stance, if $q = q_0 + q_1$ is such that $q_0 \in \mathcal{Q}$, $q_1 \in W_2^{-1}$
and the corresponding operator H of (1.3) is non-nega
In fact, a further extension is possible; namely, on
from $\mathcal Q$ to any Faddeev-Marchenko potential; t and the corresponding operator H of (1.3) is non-negative, then $q \in \mathcal{Q}$; see [46].
In fact, a further extension is possible; namely, one can add a Miura potentia
from $\mathcal Q$ to any Faddeev-Marchenko potential; the r Q to any Faddeev–Marchenko potential; the resulting Schrödinger operator not be non-negative any longer but will in general have a finite number of tive eigenvalues. A comprehensive direct and inverse scattering theory from the angle of negative any longer but will in general have a finite number of negative eigenvalues. A comprehensive direct and inverse scattering theory for such operators can be developed; this will be discussed in [4

1.3. A physical example with discontinuous impedance

The second class of Schrödinger operators we shall treat are generated by (1.1) with operators can be developed; this will be discussed in [47].
 1.3. A physical example with discontinuous impedance

The second class of Schrödinger operators we shall treat are generated by (1.1) with

discontinuous impe **1.3.** A physical example with discontinuous impedance The second class of Schrödinger operators we shall treat are discontinuous impedances. There are many physically reto such operators $[3, 4, 66]$, and we present belo discontinuous impedances. There are many physically relevant problems leading to such operators $[3, 4, 66]$, and we present below one example. discontinuous impedances. There are many problems $\frac{1}{2}$ relevant problems leading to such operators $[3, 4, 66]$, and we present below one example.

Non-destructive testing of a layered isotropic medium is usu
probing by electromagnetic waves and leads to the Maxwell system
tric $E(\cdot, \omega)$ and magnetic $H(\cdot, \omega)$ components of the electromagn
For the planar probing wave ng by electromagnetic waves and leads to the Maxwell system for the elec-
 $E(\cdot, \omega)$ and magnetic $H(\cdot, \omega)$ components of the electromagnetic field [42].

the planar probing wave of frequency ω and normal incidence (al tric $E(\cdot, \omega)$ and magnetic $H(\cdot, \omega)$ components of the electromagnetic field [42].
For the planar probing wave of frequency ω and normal incidence (along the *x*-axis)
this system takes the form
 $\begin{cases} \frac{dE(x, \omega)}{dx} + i\$ For the planar probing wave of frequency ω and normal incidence (along the *x*-axis)
this system takes the form
 $\begin{cases} \frac{dE(x,\omega)}{dx} + i\omega\mu(x)H(x,\omega) = 0, \\ \frac{dH(x,\omega)}{dx} + i\omega\varepsilon(x)E(x,\omega) = 0, \end{cases}$

$$
\begin{cases}\n\frac{dE(x,\omega)}{dx} + i\omega\mu(x)H(x,\omega) = 0, \\
\frac{dH(x,\omega)}{dx} + i\omega\varepsilon(x)E(x,\omega) = 0, \\
\text{respectively the permeability and} \\
\text{lle transformation}\n\end{cases}
$$

, $\left\{\begin{array}{c} \end{array}\right.$ with
 μ and ε denoting res with μ and ε denoting respectively the permeability and permittivity of the me-
dium. Under the Liouville transformation
 $s(x) := \int_0^x \sqrt{\varepsilon(x')\mu(x')} dx', \qquad p(s) := \sqrt[4]{\frac{\varepsilon(s)}{\mu(s)}},$
the above Maxwell system assumes the form

$$
\left\{\n\begin{aligned}\n\frac{dH(x,\omega)}{dx} + i\omega\varepsilon(x)E(x,\omega) &= 0, \\
\text{with } \mu \text{ and } \varepsilon \text{ denoting respectively the permeability and permit} \\
\text{dium. Under the Liouville transformation} \\
s(x) &:= \int_0^x \sqrt{\varepsilon(x')\mu(x')} \, dx', \qquad p(s) := \sqrt[4]{\frac{\varepsilon(s)}{\mu(s)}}, \\
\text{the above Maxwell system assumes the form} \\
\left\{\n\begin{aligned}\n\frac{dE(s,\omega)}{ds} + \frac{i\omega}{p^2(s)}H(s,\omega) &= 0; \\
\frac{dH(s,\omega)}{ds} + i\omega p^2(s)E(s,\omega) &= 0,\n\end{aligned}\n\right.
$$

$$
s(x) := \int_0^{\infty} \sqrt{\varepsilon(x')}\mu(x') dx', \qquad p(s) := \int_0^s
$$

the above Maxwell system assumes the form

$$
\begin{cases} \frac{dE(s,\omega)}{ds} + \frac{\mathrm{i}\omega}{p^2(s)}H(s,\omega) = 0; \\ \frac{dH(s,\omega)}{ds} + \mathrm{i}\omega p^2(s)E(s,\omega) = 0, \end{cases}
$$
and yields the impedance Schrödinger equation

$$
-(p^2(s)E'(s,\omega))' = \omega^2 p^2(s)E(s,\omega)
$$
for the electric potential E. Clearly, the impedance p is d

$$
\left(\frac{u_{II}(s,\omega)}{ds} + i\omega p^{2}(s)E(s,\omega) = 0\right)
$$

ace Schrödinger equation

$$
-\left(p^{2}(s)E'(s,\omega)\right)' = \omega^{2}p^{2}(s)E(s,\omega)
$$
(1.8)
al *E*. Clearly, the impedance *p* is discontinuous at the in-
the layers.

 $-(p^2(s)E'(s,\omega))' = \omega^2 p$
for the electric potential E. Clearly, the imped
terface points between the layers.
The inverse scattering problem of interes
function n given the scattering data for the equ

terface points between the layers.
The inverse scattering problem of interest is to reconstruct the impedance
function p given the scattering data for the equation (1.8). Here, we consider the
model case where p is pi $(s, \omega)^{\prime} = \omega^2 p^2(s) E(s, \omega)$ (1.8)

rly, the impedance p is discontinuous at the in-

lem of interest is to reconstruct the impedance

ata for the equation (1.8). Here, we consider the

constant and has jumps at the points lattice $d\mathbb{Z}$, for some $d > 0$. Without loss of generality, we shall assume that $d = 1$, scaling appropriately the *s*-axis as necessary. Then in every interval $\Delta_i := (i, i+1)$ The inverse scattering prob
function p given the scattering da
model case where p is piece-wise c
lattice $d\mathbb{Z}$, for some $d > 0$. Witho
scaling appropriately the s-axis as
equation (1.8) takes the form $-E$ function p given the scattering data for the equation (1.8). Here, we consider the
model case where p is piece-wise constant and has jumps at the points of a regular
lattice $d\mathbb{Z}$, for some $d > 0$. Without loss of model case where *p* is piece-wise constant and has jumps at the points of a regular
lattice $d\mathbb{Z}$, for some $d > 0$. Without loss of generality, we shall assume that $d = 1$,
scaling appropriately the *s*-axis as neces lattice $d\mathbb{Z}$, for some $d > 0$. Without loss of generality, we shall assume that $d = 1$,
scaling appropriately the s-axis as necessary. Then in every interval $\Delta_j := (j, j+1)$
equation (1.8) takes the form $-E'' = \omega^2 E$, an scaling appropriately the *s*-axis as necessary. Then in every interval $\Delta_j := (j, j+1)$
equation (1.8) takes the form $-E'' = \omega^2 E$, and the impedance *p* only determines
the interface conditions at the lattice points $s \in \mathbb{$ equation (1.8) takes the form $-E'' = \omega^2 E$, and the impedance p only determines
the interface conditions at the lattice points $s \in \mathbb{Z}$.
We further set $y(s, \omega) = p(s)E(s, \omega)$ and find that y satisfies the equation
 $-\frac{1}{$

the interface conditions at the lattice points
$$
s \in \mathbb{Z}
$$
.
\nWe further set $y(s, \omega) = p(s)E(s, \omega)$ and find that y satisfies the equation
\n
$$
-\frac{1}{p(s)}\frac{d}{ds}p^2(s)\frac{d}{ds}\frac{y}{p(s)} = \omega^2 y,
$$
\n(1.9)
\nfor which the mathematical treatment of the corresponding direct and inverse scat-

tering problems is easier. Since the asymptotic behavior of the solutions $E(\cdot, \omega)$ and $y(\cdot, \omega)$ of equations (1.8) and (1.9) are the same up to the factor $p(s)$, the (3)
1933
1936) (e)
eat
ce
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) a
wo
or 1 (s) $\frac{d}{ds} \frac{y}{p(s)} = \omega^2 y$, (1.9)
of the corresponding direct and inverse scat-
symptotic behavior of the solutions $E(\cdot, \omega)$..9) are the same up to the factor $p(s)$, the
ions are equivalent. Clearly, (1.9) is just the
a tering problems is easier. Since the asymptotic behavior of the solutions $E(\cdot, \omega)$ and $y(\cdot, \omega)$ of equations (1.8) and (1.9) are the same up to the factor $p(s)$, the scattering problems for the two equations are equival and $y(\cdot, \omega)$ of equations (1.8) and (1.9) are the same up to the factor $p(s)$, the scattering problems for the two equations are equivalent. Clearly, (1.9) is just the spectral problem $Hy = \omega^2 y$ for the impedance Scrödin spectral problem $Hy = \omega^2 y$ for the impedance Scrödinger operator *H* of (1.1).

1.4. Some singular impedance Schrödinger operators not discussed

Inverse Scattering for Impedance Schrödinger Operators 9
1.4. Some singular impedance Schrödinger operators not discussed
There are some other classes of Schrödinger operators that can be reduced to
the impedance form. On the impedance form. One of the examples is the Schrödinger operator H_{κ} on the the impedance form. One of the examples is the Schrödinger operator H_{κ} on the
half-line generated by differential expressions
 $\ell_{\kappa}(y) := -y'' + \frac{\kappa(\kappa+1)}{x^2}y + qy$
with Bessel-type potentials $\kappa(\kappa+1)/x^2$, where $\$

$$
\ell_{\kappa}(y) := -y'' + \frac{\kappa(\kappa+1)}{x^2}y + qy
$$

 $\ell_{\kappa}(y) := -y'' + \frac{\kappa(\kappa)}{x}$
with Bessel-type potentials $\kappa(\kappa + 1)/x^2$, white
generations of κ such operators arise in the contract of the three dimensional Hamiltonian. $(y) := -y'' + \frac{\kappa(\kappa+1)}{x^2}y + qy$
 $\kappa(\kappa+1)/x^2$, where $\kappa \in [$

ators arise in the decomposition

and then κ is
 κ
 κ is
 κ
 κ
 κ
 κ
 with Bessel-type potentials $\kappa(\kappa + 1)/x^2$, where $\kappa \in [-\frac{1}{2}, \frac{1}{2}]$
integer values of κ such operators arise in the decomposition in
of the three-dimensional Hamiltonian $-\Delta$, and then κ is the
or *partial w* integer values of κ such operators arise in the decomposition in spherical harmonics
of the three-dimensional Hamiltonian $-\Delta$, and then κ is the *angular momentum*,
or *partial wave*, see [19, 26]. Operators of t of the three-dimensional Hamiltonian $-\Delta$, and then κ is the *angular momentum*, or *partial wave*, see [19, 26]. Operators of the form H_{κ} with non-integer values of κ arise in the study of scattering of wav of the three-dimensional Hamiltonian $-\Delta$, and then κ is the *angular momentum*,
or *partial wave*, see [19, 26]. Operators of the form H_{κ} with non-integer values
of κ arise in the study of scattering of wav or *partial wave*, see [19, 26]. Operators of the form H_{κ} with non-integer values
of κ arise in the study of scattering of waves and particles in conical domains (see,
e.g., [20]), as well as in the study of the of κ arise in the study of scattering of waves and particles in conical domains (see, e.g., [20]), as well as in the study of the Aharonov–Bohm effect [2]. See also the related paper [56], where inverse scattering is related paper [56], where inverse scattering is discussed for long-range oscillating
potentials leading to scattering functions with finite phase shifts.
We observe that for many q the differential expression ℓ_{κ} mi

potentials leading to scattering functions with finite phase shifts.
We observe that for many q the differential expression ℓ_{κ} might be written
in the factorized form (1.2)
 $-\left(\frac{d}{dx} - \frac{\kappa}{x} + v\right)\left(\frac{d}{dx} + \frac{\kappa}{x}$ We observe that for many q the differential expression ℓ_{κ} is the factorized form (1.2)
 $-\left(\frac{d}{dx} - \frac{\kappa}{x} + v\right)\left(\frac{d}{dx} + \frac{\kappa}{x} - v\right)$ We observe that for many q the differential expression ℓ_{κ} might be written
 $-\left(\frac{d}{dx} - \frac{\kappa}{x} + v\right)\left(\frac{d}{dx} + \frac{\kappa}{x} - v\right)$

suitable v, thus taking the impedance form (1.1) with $p = x^{-\kappa} \exp \int v$. Re-

$$
-\left(\frac{d}{dx} - \frac{\kappa}{x} + v\right)\left(\frac{d}{dx} + \frac{\kappa}{x} - v\right)
$$

with suitable v , thus taking
cently, direct and inverse s
in [8] It was demonstrated $+v$ eda
pro $\,$ ithe $\,$ isco $\,$ icul $\,$ rn
cor
us
th with suitable v, thus taking the impedance form (1.1) with $p = x^{-\kappa} \exp \int v$. Recently, direct and inverse scattering problem for H_{κ} on the half-line was studied in [8]. It was demonstrated there that the scattering f cently, direct and inverse scattering problem for H_{κ} on the half-line was studied
in [8]. It was demonstrated there that the scattering function F possesses some un-
usual properties, namely, that it is discontinu in [8]. It was demonstrated there that the scattering function F possesses some un-
usual properties, namely, that it is discontinuous at the origin and at infinity and
 $1 - F$ does not tend to zero. In particular, in the 1 − *F* does not tend to zero. In particular, in the model case $q \equiv 0$ the scattering function *F* was shown to take two different constant values for $ω < 0$ and $ω > 0$. As the scattering problems on half-line are form function *F* was shown to take two different constant values for $\omega < 0$ and $\omega > 0$.
As the scattering problems on half-line are formulated in somewhat different terms than on the whole line, we shall not discuss here th

than on the whole line, we shall not discuss here the inverse scattering problems
for the operator H_{κ} .
There are some works studying scattering problems for Hamiltonians on
the half-line with Coulombic reference pot for the operator H_{κ} .
There are some works studying scattering problems for Hamiltonians on
the half-line with Coulombic reference potential, see, e.g., [24, 25]. Although the
Coulomb-type singularity can be modeled alf-line with Coulombic reference potential, see, e.g., [24, 25]. Although the omb-type singularity can be modeled by Miura potentials treated here, see aple 1.2, taking $1/x$ as a reference potential again requires somewh Coulomb-type singularity can be modeled by Miura potentials treated here, see
Example 1.2, taking $1/x$ as a reference potential again requires somewhat different
techniques that could not be covered in this paper without Example 1.2, taking $1/x$ as a reference potential again requires somewhat different

techniques that could not be covered in this paper without significantly enlarging
its size.
The paper is organized as follows. In Section 2, we discuss the general ap-
proach to solve the inverse scattering problem for th proach to solve the inverse scattering problem for the impedance operators H for continuous and discontinuous impedance functions. The continuous case leading to Schrödinger operators with Miura potentials is studied in T
Proach
continu
Schrödi
of piece proach to solve the inverse scattering problem for the impedance operators H for continuous and discontinuous impedance functions. The continuous case leading to Schrödinger operators with Miura potentials is studied in Schrödinger operators with Miura potentials is studied in Section 3, and the case of piece-wise constant impedances in Section 4. Finally, the Appendix contains some basic properties of Wiener-type Banach algebras. of piece-wise constant impedances in Section 4. Finally, the Appendix contains some basic properties of Wiener-type Banach algebras. some basic properties of Wiener-type Banach algebras.

2. The general approach and main results

10 S. Albeverio, R. Hryniv, Ya. Mykytyuk and P. Perry

2. The general approach and main results

The extension of the scattering theory we are going to present here builds upon the

standard approach; however, we use exten standard approach; however, we use extensively the Banach algebra and Banach
space techniques and, whenever possible, avoid point evaluations of the functions
involved.
We define the Fourier transforms \mathcal{F}_+ and $\mathcal{$ space techniques and, whenever possible, avoid point evaluations of the functions

 $% \left\vert \psi _{n}\right\rangle _{n}$ where $% \left\vert \psi _{n}\right\rangle _{n}$

involved.
\nWe define the Fourier transforms
$$
\mathcal{F}_+
$$
 and \mathcal{F}_- via
\n
$$
(\mathcal{F}_+ f)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} f(s) ds,
$$
\n
$$
(\mathcal{F}_- f)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} f(s) ds
$$
\nfor summable f and extend them in the usual manner to distributions. It will be convenient to work with the functions\n
$$
F_+(x) := (F_+(x)) (x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} f(s) ds.
$$
\n(2.1)

$$
(\mathcal{F}_- f)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} f(s) ds
$$

for summable f and extend them in the usual manner to distributions. It will be
convenient to work with the functions

$$
F_+(x) := (\mathcal{F}_+ r_+)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} r_+(s) ds,
$$
(2.2)

$$
F_-(x) := (\mathcal{F}_- r_-)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} r_-(s) ds
$$
(2.3)
and to analyze the mappings $u \mapsto F_+$ and $u \mapsto F_-$ given by $\mathcal{F}_+ \circ \mathcal{S}_+$ and $\mathcal{F}_- \circ \mathcal{S}_-$.

$$
F_{-}(x) := (\mathcal{F}_{-}r_{-})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} r_{-}(s) ds
$$
 (2.3)

 $(x) := (\mathcal{F}_{+}r_{+})(x) = \frac{1}{2\pi} \int_{-\infty} e^{isx}r_{+}(s) ds,$ (2.2)
 $(x) := (\mathcal{F}_{-}r_{-})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx}r_{-}(s) ds$ (2.3)

appings $u \mapsto F_{+}$ and $u \mapsto F_{-}$ given by $\mathcal{F}_{+} \circ \mathcal{S}_{+}$ and $\mathcal{F}_{-} \circ \mathcal{S}_{-}$.

ings $\mathcal{S}_{$ $\frac{2\pi}{2\pi}$
 $\frac{1}{2\pi}$
 $\frac{u}{\sigma}$
 $\frac{1}{\sigma}$ $(x) := (\mathcal{F}_{-}r_{-})(x) = \frac{1}{2\pi} \int_{-\infty} e^{-isx}r_{-}(s) ds$ (2.3)
appings $u \mapsto F_{+}$ and $u \mapsto F_{-}$ given by $\mathcal{F}_{+} \circ \mathcal{S}_{+}$ and $\mathcal{F}_{-} \circ \mathcal{S}_{-}$.
ings \mathcal{S}_{\pm} are known to be close to the Fourier transforms in
ppings $\begin{array}{c} 2\pi\\ u\\ \text{o}\\ \text{F}_{-}\\ \text{tail}\\ \text{ase} \\ I \end{array}$ and to analyze the mappings $u \mapsto F_+$ and $u \mapsto F_-$ given by $\mathcal{F}_+ \circ \mathcal{S}_+$ and $\mathcal{F}_- \circ \mathcal{S}_-$.
The scattering mappings \mathcal{S}_\pm are known to be close to the Fourier transforms in
the sense that the mappings \mathcal

The scattering mappings O_{\pm} are known to be close to the Fourier transforms in
the sense that the mappings $\mathcal{F}_+ \circ \mathcal{S}_+$ and $\mathcal{F}_- \circ \mathcal{S}_-$ are close to the identity mapping
and, in particular, act continuous the sense that the mappings $\mathcal{F}_+ \circ \mathcal{O}_+$ and $\mathcal{F}_- \circ \mathcal{O}_-$ are close to the identity mapping
and, in particular, act continuously in certain spaces of interest, see [45, 46].
We shall concentrate mostly on the We shall concentrate mostly on the case of absolutely continuous impediate generating Miura potentials; then u is in $L_2(\mathbb{R})$. The differences in the discussed at the end of this section; then u is a discrete measure 2 We shall concentrate mostly on the case of absolutely continuous impedances
generating Miura potentials; then u is in $L_2(\mathbb{R})$. The differences in the discrete
case will be discussed at the end of this section; then u

2.1. The continuous case

generating Miura potentials; then u is in $L_2(\mathbb{R})$. The differences in the discrete
case will be discussed at the end of this section; then u is a discrete measure.
2.1. The continuous case, the Riccati representat **2.1. The continuous case**
 2.1. The continuous case

In the continuous case, the Riccati representative u is supposed to generate

Miura potential $q \in \mathcal{Q}$. A generic $q \in \mathcal{Q}$ possesses two different representa In the continuous case, the Riccati representative *u* is supposed to generate the Miura potential $q \in \mathcal{Q}$. A generic $q \in \mathcal{Q}$ possesses two different representatives u_{\pm} with the property that $u_{\pm} \in L_2(\mathbb{R$ Miura potential $q \in \mathcal{Q}$. A generic $q \in \mathcal{Q}$ possesses two different representatives u_{\pm} with the property that $u_{\pm} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}^{\pm})$ and is uniquely determined by the triple $\{u_{-\vert \mathbb{R}^{\pm}}, u_{+\vert \mathbb$ with the property that $u_{\pm} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}^{\pm})$ and is uniquely determined by the
triple $\{u_{-}\vert_{\mathbb{R}^{\pm}}, u_{+}\vert_{\mathbb{R}^{+}}, (u_{-}-u_{+})(0)\}$, and in the exceptional case $u_{-} = u_{+}$. Set
 $X := L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ with $X := L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ with the norm $||f||_X = ||f||_{L_1} + ||X||_{L_2}$; then Q might be

regarded as a subset of $X \oplus \mathbb{R}$, while \mathcal{Q}_0 is identified with X .
In what follows, we shall exploit the relation between the factorized Schrö-
dinger operators (1.2) and the AKNS-Dirac systems [1], for which t \mathbb{R}^- , $u_+|_{\mathbb{R}^+}$, $(u_- - u_+)(0)$ }, and in the exceptional case $u_- = u_+$. Set \mathbb{R}) \cap $L_2(\mathbb{R})$ with the norm $||f||_X = ||f||_{L_1} + ||X||_{L_2}$; then $\mathcal Q$ might be \mathbb{R} a subset of $X \oplus \mathbb{R}$, while $\mathcal Q_$:= $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ with the norm $||f||_X = ||f||_{L_1} + ||X||_{L_2}$; then Q might be arded as a subset of $X \oplus \mathbb{R}$, while Q_0 is identified with X .
In what follows, we shall exploit the relation between the facto regarded as a subset of $X \oplus \mathbb{R}$, while Q_0 is identified with X .
In what follows, we shall exploit the relation between t
dinger operators (1.2) and the AKNS–Dirac systems [1], fo
scattering theory is also well u dinger operators (1.2) and the AKNS–Dirac systems [1], for which the inverse
scattering theory is also well understood, at least for regular potentials [29, 38, 39,
43, 65, 68, 69]. Indeed, consider the reduced Dirac equa

43, 65, 68, 69]. Indeed, consider the reduced Dirac equation
\n
$$
LY := \left(B\frac{d}{dx} + u_{\pm}J\right)Y = \omega Y,
$$
\nwhere
\n
$$
B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$
\n(2.5)

where
\n
$$
LY := \left(B\frac{d}{dx} + u_{\pm}J\right)Y = \omega Y,
$$
\nwhere
\n
$$
B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$
\n(2.5)

Inverse Scattering for Impedance Schrödinger Operators 11
It is straightforward to verify that if $Y = (y_1, y_2)^T$ solves (2.4) with say u_+ , then y_1 and y_2 solve the respective Schrödinger equations $-y'' + q^{\pm}y = \omega^2$ It is straightforward to verify that if $Y = (y_1, y_2)^T$ solves (2.4) with say u_+ , then y_1 and y_2 solve the respective Schrödinger equations $-y'' + q^{\pm}y = \omega^2y$ with $q^{\pm} = \pm u'_+ + u_+^2$.
The appearance of the operato

$$
q^{\pm} = \pm u'_{+} + u^{2}_{+}.
$$

 y_1 and y_2 solve the respective Schrödinger equations $-y'' + q^{\pm}y = \omega^2 y$ with
 $q^{\pm} = \pm u'_{+} + u^{2}_{+}$.

The appearance of the operator L here should not be surprising since L is the

operator in the Lax pair for the m

 $= \pm u'_{+} + u^{2}_{+}$
ere should not
dified Korte
KNS Dirac
e can derive
nger operato The appearance of the operator *L* here should not be surprising since *L* is the first operator in the Lax pair for the modified Korteweg-de Vries equation.
The Jost solutions for the AKNS Dirac system (2.4) can be const The Jost solutions for the AKNS Dirac system (2.4) can be of
in an explicit manner, and thus we can derive the representations for
solutions f_- and f_+ for the Schrödinger operators H in the factorized
Then we shall explicit manner, and thus we can derive the representations for the Jost

ions f_{-} and f_{+} for the Schrödinger operators H in the factorized form (1.2).

we shall show that, for every fixed $x \in \mathbb{R}$, the Jost s solutions f_{-} and f_{+} for the Schrödinger operators H in the factorized form (1.2).
Then we shall show that, for every fixed $x \in \mathbb{R}$, the Jost solutions f_{-} and f_{+} are
such that the functions
 $e^{-i\omega x}f_{$ Then we shall show that, for every fixed $x \in \mathbb{R}$, the Jost solutions f_- and f_+ are
such that the functions
 $e^{-i\omega x} f_+(x,\omega) - 1$ and $e^{i\omega x} f_-(x,\omega) - 1$ (2.6)
are the Fourier transforms of some elements of X suppor

$$
e^{-i\omega x} f_{+}(x,\omega) - 1
$$
 and $e^{i\omega x} f_{-}(x,\omega) - 1$ (2.6)

line \mathbb{R}_+ . In particular, there is a kernel $K_+(x,t)$ such that, for every fixed $x \in \mathbb{R}$, e^{-ic}
are the Fourier transform
line \mathbb{R}_+ . In particular,
the function $K_+(x, \cdot)$
such that e^{$-i\omega x$} $f_{+}(x, \omega) - 1$ and $e^{i\omega x} f_{-}(x, \omega) - 1$ (2.6)
sforms of some elements of X supported on the positive half-
ar, there is a kernel $K_{+}(x, t)$ such that, for every fixed $x \in \mathbb{R}$,
·) is an element of X supported are the Fourier transforms of some elements of X supported on the positive half-
line \mathbb{R}_+ . In particular, there is a kernel $K_+(x,t)$ such that, for every fixed $x \in \mathbb{R}$,
the function $K_+(x, \cdot)$ is an element of X the function $K_+(x, \cdot)$ is an element of X supported on the half-line $t > x$ and

such that
 $f_+(x, \omega) = e^{i\omega x} + \int_x^{\infty} K_+(x, t)e^{i\omega t} dt.$ (2.7)

The kernel K_+ generates the corresponding transformation operator, see [59].

such that
\n
$$
f_{+}(x,\omega) = e^{i\omega x} + \int_{x}^{\infty} K_{+}(x,t)e^{i\omega t} dt.
$$
\n(2.7)
\nThe kernel K_{+} generates the corresponding transformation operator, see [59].

the corresponding Wiener-type algebra \hat{X} of continuous functions with the norm $f_+(x,\omega) = e^{i\omega x} + \int_x K_+(x,t)e^{i\omega t} dt.$ (2.7)

The kernel K_+ generates the corresponding *transformation operator*, see [59].

We regard the Fourier transforms $\hat{h} = \mathcal{F}_+ h$ of functions $h \in X$ as elements of

the corre The kernel K_+ generates the corresponding transformation operator, see [59].
We regard the Fourier transforms $\hat{h} = \mathcal{F}_+ h$ of functions $h \in X$ as element
the corresponding Wiener-type algebra \hat{X} of continuous We regard the Fourier transforms $h = \mathcal{F}_+ h$ of functions $h \in X$ as elements of orresponding Wiener-type algebra \hat{X} of continuous functions with the norm $:= \|h\|_X$. Adjoining the unity 1 to \hat{X} , we then prove th the corresponding Wiener-type algebra X of continuous functions with the norm $\|\hat{h}\|_{\hat{X}} := \|h\|_{X}$. Adjoining the unity 1 to \hat{X} , we then prove that the functions $\omega a(\omega)/(\omega + i)$ and $\omega b(\omega)/(\omega + i)$ are respectively el $||h||_{\hat{X}} := ||h||_{X}$. Adjoining the unity **1** to *X*, we then prove that the functions $\omega a(\omega)/(\omega + i)$ and $\omega b(\omega)/(\omega + i)$ are respectively elements of $\mathbf{1} + \hat{X}$ and \hat{X} , that a^{-1} belongs to $\mathbf{1} + \hat{X}$ and *a* a ωa $(\omega)/(\omega + i)$ and $\omega b(\omega)/(\omega + i)$ are respectively elements of $\mathbf{1} + X$ and X , that 1 belongs to $\mathbf{1} + \hat{X}$ and a admits an extension into the upper-half complex ne \mathbb{C}_+ as an analytic function that is continu a^{-1} belongs to **1** + *X* and *a* admits an extension into the upper-half complex \mathbb{C}_+ as an analytic function that is continuous and bounded on the closed r-half plane $\overline{\mathbb{C}}_+$ outside the disks $|\omega| \leq \varepsilon$, for ev plane \mathbb{C}_+ as an analytic function that is continuous and bounded on the closed
upper-half plane $\overline{\mathbb{C}_+}$ outside the disks $|\omega| \leq \varepsilon$, for every $\varepsilon > 0$. As a result, the
scattering coefficient r_+ belongs upper-half plane \mathbb{C}_+ outside the disks $|\omega| \leq \varepsilon$, for every $\varepsilon > 0$. As a result, the scattering coefficient r_+ belongs to \hat{X} ; in particular, the function $F_+ = \mathcal{F}_+ r_+$ belongs to X . Next, symmetry

scattering coefficient r_+ belongs to X ; in particular, the function $F_+ = \mathcal{F}_+ r_+$
belongs to X . Next, symmetry properties of a and b result in the same symmetry
for r_+ , viz., $r_+(-\omega) = \overline{r_+(\omega)}$ for real belongs to X. Next, sy<u>mmetry</u> properties of *a* and *b* result in the same symmetry
for r_+ , viz., $r_+(-\omega) = \overline{r_+(\omega)}$ for real ω ; also, $|r_+(\omega)| < 1$ for real nonzero ω .
Concerning the value $\omega = 0$, two cases are for r_+ , viz., $r_+(-\omega) = r_+(\omega)$ for real ω ; also, $|r_+(\omega)| < 1$ for real nonzero ω .
Concerning the value $\omega = 0$, two cases are possible. We say that the S
dinger operator H has a *resonance* (also called *half-bound* Concerning the value $\omega = 0$, two cases are possible. We say that the Schrö-
r operator *H* has a resonance (also called *half-bound state*) at $\omega = 0$, if the
energy Jost solutions $f_{-}(\cdot, 0)$ and $f_{+}(\cdot, 0)$ are linea dinger operator *H* has a *resonance* (also called *half-bound state*) at $\omega = 0$, if the zero-energy Jost solutions $f_-(\cdot, 0)$ and $f_+(\cdot, 0)$ are linearly dependent. In this case $q \in \mathcal{Q}_0$; moreover, *a* is then an e zero-energy Jost solutions $f_-(\cdot, 0)$ and $f_+(\cdot, 0)$ are linearly dependent. In this case $q \in \mathcal{Q}_0$; moreover, a is then an element of $\mathbf{1} + \hat{X}$ and, in particular, is bounded on the whole real line thus yielding $q \in \mathcal{Q}_0$; moreover, *a* is then an element of **1** + *X* and, in particular, is bounded
whole real line thus yielding the inequality $|r_+(0)| < 1$. Generically, $q \in \mathcal{Q}_1$
possesses no resonance at $\omega = 0$; then *a* is unbounded at on the whole real line thus yielding the inequality $|r_+(0)| < 1$. Generically, $q \in \mathcal{Q}_1$
and *H* possesses no resonance at $\omega = 0$; then *a* is unbounded at the origin and
 $r_+(0) = -1$. Moreover, the explicit way r_+ dep and *H* possesses no resonance at $\omega = 0$; then *a* is unbounded at the origin and $r_{+}(0) = -1$. Moreover, the explicit way r_{+} depends on *a* and *b* leads to the conclusion that the regularized coefficient $\tilde{r}_{+}(\omega$ $r+(0) = -1$. Moreover, the explicit way r_+ depends on *a* and *b* leads to the conclusion that the regularized coefficient
 $\widetilde{r}_+(\omega) := \frac{1 - |r_+(\omega)|^2}{\omega^2}$

belongs to \widehat{X} .

conclusion that the regularized coefficient ˜+() := 1 − ∣+()∣ 2 2

belongs to X .

Introduce now the set
\n
$$
\mathcal{R} := \{r \in \hat{X} \mid r(-\omega) = \overline{r(\omega)}, \ |r(\omega)| < 1 \text{ for } \omega \in \mathbb{R} \setminus \{0\} \}
$$
\nand its subsets\n
$$
\mathcal{R}_0 := \{r \in \mathcal{R} \mid |r(0)| < 1\}
$$
\nand\n
$$
\mathcal{R}_1 := \{r \in \mathcal{R} \mid r(0) = -1, \ \tilde{r} \in \hat{X}, \ \tilde{r}(0) \neq 0 \}.
$$

$$
\mathcal{R}_0 := \{ r \in \mathcal{R} \mid |r(0)| < 1 \} \tag{2.8}
$$

$$
:= \{r \in X \mid r(-\omega) = r(\omega), \ |r(\omega)| < 1 \quad \text{for} \quad \omega \in \mathbb{R} \setminus \{0\} \}
$$
\n
$$
\text{ks}
$$
\n
$$
\mathcal{R}_0 := \{r \in \mathcal{R} \mid \ |r(0)| < 1\} \tag{2.8}
$$
\n
$$
\mathcal{R}_1 := \{r \in \mathcal{R} \mid r(0) = -1, \ \tilde{r} \in \hat{X}, \ \tilde{r}(0) \neq 0\}, \tag{2.9}
$$
\n
$$
= (1 - |r(\omega)|^2) / \omega^2. \text{ The set } \mathcal{R}_0 \text{ is endowed with the topology of } \hat{X},
$$

 $\mathcal{R}_0 := \{ r \in \mathcal{R} \mid |r(0)| < 1 \}$ (2.8)

and $\mathcal{R}_1 := \{ r \in \mathcal{R} \mid r(0) = -1, \ \tilde{r} \in \hat{X}, \ \tilde{r}(0) \neq 0 \},$ (2.9)

where $\tilde{r}(\omega) := (1 - |r(\omega)|^2)/\omega^2$. The set \mathcal{R}_0 is endowed with the topology of \hat{X} ,

while the to $\mathcal{R}_1 := \{ r \in \mathcal{R} \mid r(0) = -1, \ \tilde{r} \in \hat{X}, \ \tilde{r}(0) \neq 0 \},$ (2.9)

where $\tilde{r}(\omega) := (1 - |r(\omega)|^2)/\omega^2$. The set \mathcal{R}_0 is endowed with the topology of \hat{X} ,

while the topology of \mathcal{R}_1 is determined by the dist

$$
d(r_1, r_2) = ||r_1 - r_2||_{\widehat{X}} + ||\widetilde{r}_1 - \widetilde{r}_2||_{\widehat{X}}.
$$

while the topology of \mathcal{R}_1 is determined by the distance
 $d(r_1, r_2) = ||r_1 - r_2||_{\tilde{X}} + ||\tilde{r}_1 - \tilde{r}_2||_{\tilde{X}}$.

One of our main results is that the sets \mathcal{R}_0 and \mathcal{R}_1 consist of the reflection

coefficien ²)/ ω^2 . The set \mathcal{R}_0 is endowed with the topology of X ,
is determined by the distance
1, r_2) = $||r_1 - r_2||_{\hat{X}} + ||\tilde{r}_1 - \tilde{r}_2||_{\hat{X}}$.
ults is that the sets \mathcal{R}_0 and \mathcal{R}_1 consist of the reflect coefficients for the class of Schrödinger operators under consideration in the resonant (i.e., exceptional $q \in \mathcal{Q}_0$) and non-resonant (i.e., generic $q \in \mathcal{Q}_1$) cases, respectively. More exactly, the typical result \mathcal{L}_1 is determined by the distance
 $(r_1, r_2) = ||r_1 - r_2||_{\hat{X}} + ||\tilde{r}_1 - \tilde{r}_2||_{\hat{X}}.$

esults is that the sets \mathcal{R}_0 and \mathcal{R}_1 c

is of Schrödinger operators under cc
 $q \in \mathcal{Q}_0$ and non-resonant (i.e., g One of our main results is that the sets κ_0 and κ_1 consist of the reflection
cients for the class of Schrödinger operators under consideration in the res-
t (i.e., exceptional $q \in \mathcal{Q}_0$) and non-resonant (i.e countant (i.e., exceptional $q \in \mathcal{Q}_0$) and non-resonant (i.e., generic $q \in \mathcal{Q}_1$) cases, respectively. More exactly, the typical result reads as follows. Denote by $X_{\mathbb{R}}$ the subset of real-valued functions in t respectively. More exactly, the typical result reads as follows. Denote by $X_{\mathbb{R}}$ the subset of real-valued functions in the space X ; $X_{\mathbb{R}}$ is a real subspace of X under the inherited topology. Recall that q subset of real-valued functions in the space X ; $X_{\mathbb{R}}$ is a real subspace of X under the inherited topology. Recall that $q \in \mathcal{Q}$ is parametrized by a pair $(w, \alpha) \in X \oplus \mathbb{R}^+$, in which $w|_{\mathbb{R}^{\pm}}$ are eq inherited topology. Recall that $q \in \mathcal{Q}$ is parametrized by a pair $(w, \alpha) \in X \oplus \mathbb{R}^+$,
in which $w|_{\mathbb{R}^\pm}$ are equal to $u_\pm|_{\mathbb{R}^\pm}$ and $\alpha = (u_- - u_+)(0)$; moreover, $\alpha = 0$ for
 $q \in \mathcal{Q}_0$ and $\alpha > 0$ for q in which $w|_{\mathbb{R}^{\pm}}$ are equal to $u_{\pm}|_{\mathbb{R}^{\pm}}$ and $\alpha = (u_{-} - u_{+})(0)$; moreover, $\alpha = 0$ for $q \in \mathcal{Q}_0$ and $\alpha > 0$ for $q \in \mathcal{Q}_1$.
 Theorem 2.1. The maps
 $\mathcal{S}_{\pm} : (w, \alpha) \mapsto r_{\pm}$

are homeomorphic betwe $\left.\left.\right|_{\mathbb{R}^{\pm}}\right|_{\mathbb{R}^{\pm}}$ are equal to $u_{\pm}\left.\right|_{\mathbb{R}^{\pm}}$ $q \in \mathcal{Q}_0$ and $\alpha > 0$ for $q \in \mathcal{Q}_1$.

Theorem 2.1. The maps

$$
\mathcal{S}_{\pm} : (w, \alpha) \mapsto r_{\pm}
$$

are homeomorphic between $X_{\mathbb{R}}$ and \mathcal{R}_0 and between $X_{\mathbb{R}} \oplus \mathbb{R}^+$ and \mathcal{R}_1 , respectively.

and $\alpha > 0$ for $q \in \mathcal{Q}_1$.

m 2.1. The maps

neomorphic between X_1

is gives a complete definition and S_{\pm} : $(w, \alpha) \mapsto r_{\pm}$

nd \mathcal{R}_0 and between

iption of the scatus

us solves the dire

It remains to fi tors under consideration, and thus solves the direct scattering problem and settles
existence in the inverse problem. It remains to find the procedure to actually de-
termine u given r_+ or r_- .
To this end we observe

$$
\frac{e^{i\omega x}f_{-}(x,\omega)}{a(\omega)} = e^{i\omega x}f_{+}(x,-\omega) + e^{i\omega x}r_{+}(\omega)f_{+}(x,\omega).
$$

termine *u* given r_+ or r_- .

To this end we observe that equation (1.6) yields the relation
 $\frac{e^{i\omega x}f_-(x,\omega)}{a(\omega)} = e^{i\omega x}f_+(x,-\omega) + e^{i\omega x}r_+(\omega)f_+(x,\omega)$.

The properties of *a* and of the function $e^{i\omega x}f_-(x,\omega)$ of $\frac{e^{i\omega x}f_{-}(x,\omega)}{a(\omega)} = e^{i\omega x}f_{+}(x,-\omega) + e^{i\omega x}r_{+}(\omega)f_{+}(x,\omega)$.
properties of a and of the function $e^{i\omega x}f_{-}(x,\omega)$ of (2.6) imply
side of the above equality is a function admitting a bound
on to the upper-half comple $\frac{e^{i\omega x}f_-(x,\omega)}{a(\omega)} = e^{i\omega x}f_+(x,-\omega) + e^{i\omega x}r_+(\omega)f_+(x,\omega)$.

of *a* and of the function $e^{i\omega x}f_-(x,\omega)$ of (2.6) implees above equality is a function admitting a bound

upper-half complex plane. Therefore the Fourier tr The properties of *a* and of the function e^{i ωx} $f_-(x,\omega)$ of (2.6) imply that the left-
hand side of the above equality is a function admitting a bounded analytic ex-
tension to the upper-half complex plane. Therefore tension to the upper-half complex plane. Therefore the Fourier transform of the right-hand side is supported on the positive half-line; working it out on account of the representation (2.7) leads to the well-known *Marche*

the representation (2.7) leads to the well-known *Marchenko equation* relating the Fourier transform
$$
F_+
$$
 of r_+ and the kernel K_+ of the transformation operator: $K_+(x,t) + F_+(x+t) + \int_x^\infty K_+(x,s)F_+(s+t) \, ds = 0, \qquad x < t.$ (2.10) For every fixed $x \in \mathbb{R}$, the above Marchenko equation is understood as an equality of X-valued functions of argument t for $t > x$. Given the reflection co-

Fourier transform F_+ of r_+ and the kernel K_+ of the transformation operator:
 $K_+(x,t) + F_+(x+t) + \int_x^{\infty} K_+(x,s)F_+(s+t) ds = 0, \qquad x < t.$ (2.1)

For every fixed $x \in \mathbb{R}$, the above Marchenko equation is understood as

eq $(x, t) + F_+(x + t) + \int_x^{\infty} K_+(x, s) F_+(s + t) ds = 0,$ $x < t.$ (2.10)
every fixed $x \in \mathbb{R}$, the above Marchenko equation is understood as an
of X-valued functions of argument t for $t > x$. Given the reflection co-
 r_+ and thus the For every fixed $x \in \mathbb{R}$, the above Marchenko equation is understood as an ity of X-valued functions of argument t for $t > x$. Given the reflection coent r_+ and thus the function F_+ , one can solve this equation fo equality of X-valued functions of argument t for $t > x$. Given the reflection co-
efficient r_+ and thus the function F_+ , one can solve this equation for $K_+(x, \cdot)$, efficient r_+ and thus the function F_+ , one can solve this equation for $K_+(x, \cdot)$, Inverse Scattering for Impedance Schrödinger Operators 13
for every fixed x. In the case of Faddeev–Marchenko potentials q the kernel K_+ is
absolutely continuous in the domain $x \le t$, and the equality for every fixed x. In the case of Faddeev–Marchenko potentials q the kernel K_+ is
absolutely continuous in the domain $x \le t$, and the equality
 $q(x) = -2\frac{d}{dx}K_+(x, x)$
holds almost everywhere.
For singular u, the kernel

$$
q(x) = -2\frac{d}{dx}K_+(x,x)
$$

absolutely continuous in the domain $x \le t$, and the equality
 $q(x) = -2\frac{d}{dx}K_+(x, x)$

holds almost everywhere.

For singular u, the kernel K_+ allows no restriction ont

and the above formula becomes useless. Moreover, $(x) = -2\frac{d}{dx}K_{+}(x, x)$
 K_{+} allows no restrices useless. Moreover,

ot the potential q.

on between u and the For singular u, the kernel K_+ allows no restriction onto the diagonal $x = t$,
and the above formula becomes useless. Moreover, our aim is to determine the
Riccati representative u, and not the potential q.
To derive the

Riccati representative *u*, and not the potential *q*.

To derive the main relation between *u* and the transformation operators, we

assume for the time being that *q* is a Faddeev-Marchenko potential and consider

along To derive the main relation between *u* and the transformation operators, we
ne for the time being that *q* is a Faddeev-Marchenko potential and consider
; with (2.10) the equation
 $K_{-}(x,t) - F_{+}(x+t) - \int_{x}^{\infty} K_{-}(x,s)F_{+}(s+t$

$$
K_{-}(x,t) - F_{+}(x+t) - \int_{x}^{\infty} K_{-}(x,s)F_{+}(s+t) ds = 0, \qquad x < t,
$$

assume for the time being that q is a Faddeev–Marchenko potential and consider

along with (2.10) the equation
 $K_{-}(x,t) - F_{+}(x+t) - \int_{x}^{\infty} K_{-}(x,s)F_{+}(s+t) ds = 0, \qquad x < t,$

in which F_{+} is replaced with $-F_{+}$. In the reson $K_{-}(x,t) - F_{+}(x+t)$
in which F_{+} is replaced with $-r_{+}$ instead of r_{+} . Using the
resonant case, we conclude that $(x, t) - F_+(x + t) - \int_x^{\infty} K_-(x, s)F_+(s + t) ds = 0,$ $x < t$,
is replaced with $-F_+$. In the resonant case this corresponds to of r_+ . Using the fact that the scattering maps S_{\pm} are od
se, we conclude that $-r_+$ corresponds to in which F_+ is replaced with $-F_+$. In the resonant case this corresponds to taking $-r_+$ instead of r_+ . Using the fact that the scattering maps S_{\pm} are odd in the resonant case, we conclude that $-r_+$ correspon instead of r_+ . Using the fact that the scattering maps S_{\pm} are odd in the ant case, we conclude that $-r_+$ corresponds to the Riccati representative $-u$, at $-2\frac{d}{dx}K_-(x,x) = -u'(x) + u^2(x)$.
ducing the kernels resonant case, we conclude that $-r_+$ corresponds to the Riccati representative −u,
so that
 $-2\frac{d}{dx}K_-(x,x) = -u'(x) + u^2(x)$.
Introducing the kernels
 $M_+(x,t) := \frac{1}{\pi} \left[K_+(x,t) \pm K_-(x,t) \right],$ so that $$\rm Introducing\ the\ kernels$$

$$
-2\frac{d}{dx}K_{-}(x,x) = -u'(x) + u^{2}(x).
$$

$$
-2\frac{d}{dx}K_{-}(x,x) = -u'(x) + u^{2}(x).
$$

s

$$
M_{\pm}(x,t) := \frac{1}{2}\Big[K_{+}(x,t) \pm K_{-}(x,t)\Big],
$$

Internal $\frac{1}{2}$
we see that

$$
u(x) = -2M_{-}(x, x), \tag{2.11}
$$

 $(x, t) := \frac{1}{2} \Big[K_+(x, t) \pm K_-(x, t)$
 $u(x) = -2M_-(x, x),$

on. We note that the kernels Λ $\frac{1}{2}$ which is the equations

$$
u(x) = -2M_{-}(x, x),
$$
\n(2.11)

\nwhich is the desired relation. We note that the kernels M_{\pm} satisfy the system of equations

\n
$$
F_{+}(x+t) + M_{-}(x,t) + \int_{x}^{\infty} M_{+}(x,s)F_{+}(s+t) ds = 0,
$$
\n
$$
M_{+}(x,t) + \int_{x}^{\infty} M_{-}(x,s)F_{+}(s+t) ds = 0,
$$
\n
$$
(2.13)
$$

$$
M_{+}(x,t) + \int_{x}^{\infty} M_{-}(x,s)F_{+}(s+t) ds = 0,
$$
 (2.13)

 $\frac{1}{2}$
found in 1

found in the inverse theory for Zakharov–Shabat systems [72].
It turns out [36, 46] that the same relations take place also for the case of
Miura potentials in Q . Therefore, given the reflection coefficient r_{+} , one It turns out $[36, 46]$ that the same relations take place also for the case of $(x, t) + \int_x^{\infty} M_-(x, s) F_+(s+t) ds = 0,$ (2.13)

r Zakharov-Shabat systems [72].

t the same relations take place also for the case of

ore, given the reflection coefficient r_+ , one can form

system (2.12)-(2.13) with $F_+ = \math$ It turns out [36, 46] that the same relations take place
Miura potentials in Q. Therefore, given the reflection coefficies
the above Zakharov–Shabat system (2.12) – (2.13) with $F_+ = M_-$, and determine u from (2.11) .
I Miura potentials in Q . Therefore, given the reflection coefficient r_+ , one can form
the above Zakharov-Shabat system $(2.12)-(2.13)$ with $F_+ = \mathcal{F}_+r_+$, solve it for
 M_- , and determine u from (2.11) .
In fact, the $M_-,$ and determine u from (2.11) .

the above Zakharov–Shabat system (2.12) – (2.13) with $F_+ = \mathcal{F}_+ r_+$, solve it for M_- , and determine u from (2.11) .
In fact, the Riccati representative u determined via (2.11) will be integrable at $+\infty$ but nee , and determine u from (2.11).
In fact, the Riccati represent
 $\vdash \infty$ but need not be such at esentative u_+ of $q \in \mathcal{Q}$. To fin In fact, the Riccati representative *u* determined via (2.11) will be integrable ∞ but need not be such at $-\infty$; in other words, *u* is the extremal Riccati sentative *u*₊ of *q* \in *Q*. To find the other extrema at +∞ but need not be such at $-\infty$; in other words, *u* is the extremal Riccati
representative u_+ of $q \in \mathcal{Q}$. To find the other extremal Riccati representative u_-
 $u_$ representative u_+ of $q \in \mathcal{Q}$. To find the other extremal Riccati representative u_-
 \blacksquare 14 S. Albeverio, R. Hryniv, Ya. Mykytyuk and P. Perry
that is integrable at $-\infty$, we use the left reflection coefficient $r_$, take its Fourier
transform $F_- := \mathcal{F}_{-}r_-$ of (2.3), and form the "left" analogue

that is integrable at
$$
-\infty
$$
, we use the left reflection coefficient r_- , take its Fourier
transform $F_- := \mathcal{F}_- r_-$ of (2.3), and form the "left" analogue

$$
F_-(x+t) + M_-(x,t) + \int_{-\infty}^x M_+(x,s) F_-(s+t) ds = 0, \qquad (2.14)
$$

$$
M_+(x,t) + \int_{-\infty}^x M_-(x,s) F_-(s+t) ds = 0 \qquad (2.15)
$$
of the Marchenko-type system (2.12)–(2.13); then $u_- = 2M_-(x,x)$.
The final and the most difficult step will be to prove that $Bu_+ = Bu_- =: q$
in the sense of distributions and that the Schrödinger operator H with Mura

$$
M_{-}^{-}(x,t) + \int_{-\infty}^{x} M_{+}^{-}(x,s)F_{-}(s+t) ds = 0,
$$
\n(2.14)
\n
$$
M_{+}^{-}(x,t) + \int_{-\infty}^{x} M_{-}^{-}(x,s)F_{-}(s+t) ds = 0
$$
\n(2.15)
\nstem (2.12)–(2.13); then $u_{-} = 2M_{-}^{-}(x,x)$.
\nost difficult step will be to prove that $Bu_{+} = Bu_{-} =: q$
\nons and that the Schrödinger operator H with Mura

 $(x, t) + \int_{-\epsilon}^{x}$

a (2.12)-(2.

difficult ste

and that t

coefficient $(x, s)F_{-}(s+t) ds = 0$ (2.15)

hen $u_{-} = 2M_{-}^{-}(x, x)$.

be to prove that $Bu_{+} = Bu_{-} =: q$

chrödinger operator *H* with Miura

chave started with. The details are of the Marchenko-type system (2.12) – (2.13) ; then $u_- = 2M_-^-$
The final and the most difficult step will be to prove then
in the sense of distributions and that the Schrödinger oper
potential q has the reflection coeffi $\Gamma(x, x)$.
hat *Bu*.
erator *I* with. T The final and the most difficult step will be to prove that $Bu_+ = Bu_- =: q$
e sense of distributions and that the Schrödinger operator H with Miura
tial q has the reflection coefficient r_+ we have started with. The details in the sense of distributions and that the Schrödinger operator H with Miura
potential q has the reflection coefficient r_+ we have started with. The details are
discussed in Section 3.
2.2. The discrete case
In t

2.2. The discrete case

potential q has the reflection coefficient r_+ we have started with. The details are
discussed in Section 3.
2.2. The discrete case
In the case where p is piece-wise constant, no direct analogue of (2.11) exists.
Howe **2.2.** The discrete case
In the case where p is
However, there are me
discretized form, see S
lattice dZ ; without loss In the case where *p* is piece-wise constant, no direct analogue of (2.11) exists.
However, there are meaningful analogues of the transformation operators but in a
discretized form, see Section 4. Assume that *p* has jump discretized form, see Section 4. Assume that p has jumps at the sites of a regular lattice $d\mathbb{Z}$; without loss of generality, we may take $d = 1$. The function $u = (\log p)'$ is now the discrete measure, $u = -\sum u_k \delta(-k)$, wi lattice $d\mathbb{Z}$; without loss of generality, we may take $d = 1$. The function $u = (\log p)^t$
is now the discrete measure, $u = -\sum u_k \delta(\cdot - k)$, with
 $u_k := \log \frac{p(k+0)}{p(k-0)}$,
so that
 $p(x) = \exp\left\{\sum u_k\right\}$. is now the discrete measure, $u = -\sum u_k \delta(\cdot - k),$ with
 $u_k := \log \frac{p(k+0)}{p(k-0)},$ so that

$$
u_k := \log \frac{p(k+0)}{p(k-0)},
$$

$$
u_k := \log \frac{p(k+0)}{p(k-0)},
$$

$$
p(x) = \exp \Biggl\{ \sum_{k:k>x} u_k \Biggr\}.
$$

representatives u are :
 k) of finite total vari

 $p(x) = \exp\left\{\sum_{k:k>}\right\}$

In other words, Riccati representatives of measures $d\mu = \sum \mu_k \delta(\cdot - k)$ of finite tota $\mu := (\mu_k)$ belonging to $\ell_1(\mathbb{Z})$. Then the correspote elements in X is the Wiener algebra of 2π convergent In other words, Riccati representatives *u* are now elements of the space *X* easures $d\mu = \sum \mu_k \delta(\cdot - k)$ of finite total variation, i.e., with the sequences (μ_k) belonging to $\ell_1(\mathbb{Z})$. Then the corresponding set \hat of measures $d\mu = \sum \mu_k \delta(\cdot - k)$ of finite total variation, i.e., with the sequences $\mu := (\mu_k)$ belonging to $\ell_1(\mathbb{Z})$. Then the corresponding set \hat{X} of Fourier transforms of elements in X is the Wiener algebra of 2 μ := (μ_k) belonging to $\ell_1(\mathbb{Z})$. Then the corresponding set X of Fourier transforms
elements in X is the Wiener algebra of 2π -periodic functions with absolutely
nvergent Fourier series $\sum \mu_k e^{iks}$. In particular, t of elements in X is the Wiener algebra of 2π -periodic functions with absolutely
convergent Fourier series $\sum \mu_k e^{iks}$. In particular, the scattering coefficients a and
b and the reflection coefficients r_{\pm} are elem \boldsymbol{b}

convergent Fourier series $\sum \mu_k e^{iks}$. In particular, the scattering coefficients a and
 b and the reflection coefficients r_{\pm} are elements of \hat{X} , and $F_{\pm} = \mathcal{F}_{\pm} r_{\pm}$ are discrete

measures supported and the reflection coefficients r_{\pm} are elements of X , and $F_{\pm} = \mathcal{F}_{\pm}r_{\pm}$ are discrete
reasures supported by \mathbb{Z} .
The transformation operators exist and can be represented in the following
ay. Set $\Delta_j :=$ measures supported by \mathbb{Z} .

The transformation α

way. Set $\Delta_j := [j, j + 1]$

onto $L_2(\Delta_j)$, $P_j g(x) = g$

of Δ_j . It is convenient to
 $L_2(0,1) \otimes \ell_2(\mathbf{Z})$ and to rep
 $k \in \mathbb{Z}$, and $g(y, k) = g(y - k)$ way. Set $\Delta_j := [j, j + 1)$ and denote by P_j the orthogonal projector in $L_2(\mathbb{R})$
onto $L_2(\Delta_j)$, $P_j g(x) = g(x)\chi_{\Delta_j}(x)$, with χ_{Δ_j} being the characteristic function
of Δ_j . It is convenient to use the unitary equiva onto $L_2(\Delta_j)$, $P_j g(x) = g(x) \chi_{\Delta_j}(x)$, with χ_{Δ_j} being the characteristic function
of Δ_j . It is convenient to use the unitary equivalence of the spaces $L_2(\mathbb{R})$ and
 $L_2(0,1) \otimes \ell_2(\mathbf{Z})$ and to represent funct of Δ_j . It is convenient to use the unitary equivalence of the spaces $L_2(\mathbb{R})$ and $L_2(0,1) \otimes \ell_2(\mathbf{Z})$ and to represent functions $g(x)$ in $L_2(\mathbb{R})$ as $g(y, k)$, with $y \in [0, 1)$, $k \in \mathbb{Z}$, and $g(y, k) = g(y + k)$. $\scriptstyle{L_2}$ $k \in \mathbb{Z}$, and $g(y, k) = g(y + k)$. Under this identification, every bounded operator T in $L_2(\mathbb{R})$ can be written in the matrix form $(T_{m,n})_{m,n\in\mathbb{Z}}$, with $T_{m,n} := P_m T P_n$.

(0, 1)⊗ $\ell_2(\mathbf{Z})$ and to represent functions $g(x)$ in $L_2(\mathbb{R})$ as $g(y, k)$, with $y \in [0, 1)$, $\in \mathbb{Z}$, and $g(y, k) = g(y + k)$. Under this identification, every bounded operator in $L_2(\mathbb{R})$ can be written in the mat , and $g(y, k) = g(y + k)$. Under this identification, every bounded operator $L_2(\mathbb{R})$ can be written in the matrix form $(T_{m,n})_{m,n \in \mathbb{Z}}$, with $T_{m,n} := P_m T P_n$.

In these notations, the transformation operator K for the Schrö in $L_2(\mathbb{R})$ can be written in the matrix form $(T_{m,n})_{m,n\in\mathbb{Z}}$, with $T_{m,n} := P_m T P_n$.
In these notations, the transformation operator K for the Schrödinger oper-
or H_u can be constructed in an explicit way and its co In these notations, the transformation operator K for the Schrödinger oper-
 H_u can be constructed in an explicit way and its components $K_{m,n}$ can be

n to be of the form $K(m, n)T^{m+n}$, there $K(m, n)$ is a number and T ator H_u can be constructed in an explicit way and its components $K_{m,n}$ can be shown to be of the form $K(m, n)T^{m+n}$, there $K(m, n)$ is a number and T is the reflection operator in $L_2(\Delta_0)$, i.e., $(Tg)(y) = g(1 - y)$. Fur shown to be of the form $K(m, n) T^{m+n}$, there $K(m, n)$ is a number and T is the reflection operator in $L_2(\Delta_0)$, i.e., $(Tg)(y) = g(1 - y)$. Further, set $R(k) := \hat{r}_+(-k)$; flection operator in $L_2(\Delta_0)$, i.e., $(Tg)(y) = g(1 - y)$. Further, set $R(k) := \hat{r}_+(-k)$;
 $\hat{r}_+(-k) = \hat{r}_+(-k)$

e discrete analogue of the Marchenko equation reads

$$
K_{+}(m,n) + R(m+n) + \sum_{\xi=m+1}^{\infty} K_{+}(m,\xi)R(\xi+n) = 0, \qquad m < n,
$$

while the analogue of the Zakharov-Shabat system takes the form

$$
K_{+}(m, n) + R(m + n) + \sum_{\xi=m+1}^{\infty} K_{+}(m, \xi)R(\xi + n) = 0, \qquad m < n,
$$

while the analogue of the Zakharov-Shabat system takes the form

$$
M_{+}(m, n) + \sum_{\xi=m+1}^{\infty} M_{-}(m, \xi)R(\xi + n) = 0,
$$

$$
M_{-}(m, n) + R(m + n) + \sum_{\xi=m+1}^{\infty} M_{+}(m, \xi)R(\xi + n) = 0.
$$
As in the continuous case, the kernel M_{-} determines the function u ,
corresponding relation takes a somewhat different form, namely,

$$
\tanh u_{n} = M_{-}(n - 1, n + 1), \qquad n \in \mathbb{Z}.
$$

$$
\tanh u_n = M_-(n-1, n+1), \qquad n \in \mathbb{Z}.\tag{2.16}
$$

As in the continuous case, the kernel M_- determines the function u , but the corresponding relation takes a somewhat different form, namely,
 $\tanh u_n = M_-(n-1, n+1)$, $n \in \mathbb{Z}$. (2.16)

It is the difference between recove tanh $u_n = M_-(n-1, n+1)$, $n \in \mathbb{Z}$.
It is the difference between recovering u via (2.11) in the continuo
in the discrete case that does not allow to a unified approach
measures u with both continuous and discrete componen tanh $u_n = M_-(n-1, n+1)$, $n \in \mathbb{Z}$. (2.16)
tween recovering u via (2.11) in the continuous case and (2.16)
that does not allow to a unified approach to reconstructing
h continuous and discrete components.
olutions of the It is the difference between recovering u via (2.11) in the continuous case and (2.16) in the discrete case that does not allow to a unified approach to reconstructing measures u with both continuous and discrete comp in the discrete case that does not allow to a unified approach to reconstructing measures u with both continuous and discrete components.
The details of solutions of the inverse scattering problems in the continuous

and discrete cases respectively are given in Sections 3 and 4 below. details discrete cases respectively are given in Sections 3 and 4 below.
 Reverse scattering for Miura potentials
 Reverse scattering problem for Schrödinger

3. Inverse scattering for Miura potentials

3. Inverse scattering for Miura potentials
The standard method of solution of the inverse scattering problem f
operators is due to Marchenko and is based on the so-called transfe operators is due to Marchenko and is based on the so-called transformation operators. As mentioned above, for impedance Schrödinger equations a more efficient approach is that of Zakharov and Shabat for the related Dirac s ators. As mentioned above, for impedance Schrodinger equations a more efficient approach is that of Zakharov and Shabat for the related Dirac system (2.4). Below, we explain the basic steps in the solution of the inverse s we explain the basic steps in the solution of the inverse scattering problem for the
Schrödinger operator with Miura potentials using the reduced Dirac systems in
AKNS form.
3.1. Jost solutions
We shall start with the s Schrödinger operator with Miura potentials using the reduced Dirac systems in AKNS form.
3.1. Jost solutions
We shall start with the simpler case where $q \in \mathcal{Q}_0$, i.e., where the Riccati repre-
sentative u belongs to

3.1. Jost solutions

SCHRIM AKNS form.

S.1. Jost solutions

We shall start with the simpler case where $q \in \mathcal{Q}_0$, i.e., where the Riccati representative u belongs to $X := L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

An effective construction of the Jost solutio **3.1.** Jost sol
We shall sta
sentative u h
An effective

We shall start with the simpler case where $q \in \mathcal{Q}_0$, i.e., where the Riccati repre-
sentative u belongs to $X := L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.
An effective construction of the Jost solution for $q = u' + u^2$ with given $u \in X$
uses sentative *u* belongs to $X := L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.
An effective construction of the Jost so
uses the "conjugate" potentials $q^{-} = -i$
associated reduced Dirac equation (2.4). Na
solution matrix for the Dirac system (2.4), An effective construction of the Jost solution for $q = u'$
the "conjugate" potentials $q^{(-)} = -u' + u^2$ correspo
iated reduced Dirac equation (2.4). Namely, we first co
ion matrix for the Dirac system (2.4), i.e., a 2 × 2 mat + u^2 with given $u \in X$
onding to $-u$ and the
nstruct a fundamental
rix solution $U(\cdot, \omega)$ to uses the "conjugate" potentials $q^{(-)} = -u' + u^2$ corresponding to $-u$ and the associated reduced Dirac equation (2.4). Namely, we first construct a fundamental solution matrix for the Dirac system (2.4), i.e., a 2 × 2 matri solution matrix for the Dirac system (2.4), i.e., a 2 × 2 matrix solution $U(\cdot, \omega)$ to
 $\left(B\frac{d}{dx} + uJ\right)U = \omega U$

$$
\left(B\frac{d}{dx} + uJ\right)U = \omega U
$$

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that approaches $e^{-\omega x B}$ as x tends to $+\infty$ (recall that the matrices B and J were
defined in (2.5)). Noting that B has eigenvalues $\lambda_1 = -i$ and $\lambda_2 = +i$ with the
 defined in (2.5)). Noting that B has eigenvalues $\lambda_1 = -i$ and $\lambda_2 = +i$ with the

$$
v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix},
$$

defined in (2.5)). Noting that *B* has eigenvalues $\lambda_1 = -i$ and $\lambda_2 = +i$ with the corresponding eigenvectors
 $v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$,

we then see that the Jost solutions f_+ and $f_+^{(-)}$ for we then see that the Jost scheer relation we then see that the Jost solutions f_+ and $f_+^{(-)}$ for the potentials q and $q^{(-)}$ satisfy
the relation
 $\begin{pmatrix} f_+(x,\omega) \\ -if_+^{(-)}(x,\omega) \end{pmatrix} = U(x,\omega)v_1.$ (3.1)
Variation of constants formula leads to the integral equation

$$
v_1 = \begin{pmatrix} 0 \\ -i \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ i \end{pmatrix},
$$

olutions f_+ and $f_+^{(-)}$ for the potentials q and $q_-^{(-)}$ satisfy

$$
\begin{pmatrix} f_+(x,\omega) \\ -if_+^{(-)}(x,\omega) \end{pmatrix} = U(x,\omega)v_1.
$$

is formula leads to the integral equation

$$
e^{-\omega xB} - \int_x^{\infty} e^{-\omega(x-t)B} u(t)BJU(t,\omega) dt.
$$

eximation method, we set

Variat

$$
\left(-if_{+}^{(-)}(x,\omega)\right)^{-} = e(x,\omega)v_{1}.
$$
\n
$$
\text{Variation of constants formula leads to the integral equation}
$$
\n
$$
U(x,\omega) = e^{-\omega xB} - \int_{x}^{\infty} e^{-\omega(x-t)B} u(t)BJU(t,\omega) \,\mathrm{d}t.
$$
\n
$$
\text{where } U(x,\omega) = e^{-\omega xB} = \int_{x}^{\infty} e^{-\omega(x-t)B} u(t)BJU(t,\omega) \,\mathrm{d}t.
$$
\n
$$
U_{0}(x,\omega) = -\int_{-\infty}^{\infty} e^{-\omega(x-t)B} u(t)BJU_{n-1}(t,\omega) \,\mathrm{d}t.
$$

Using the successive approximation method, we set

$$
U_0(x,\omega) = \exp(-\omega x B)
$$

$$
U_0(x,\omega) = \exp(-\omega xB)
$$

and

$$
U_n(x,\omega) = -\int_x^{\infty} e^{-\omega(x-t)B} u(t)BJU_{n-1}(t,\omega) dt
$$

for $n \ge 1$; then U is formally given by the Volterra series $\sum_{n \ge 0} i$
In a straightforward recursive manner one derives the form

 $\frac{1}{2}$ for

$$
U_n(x,\omega) = -\int_x e^{-\omega(x-t)B} u(t) B J U_{n-1}(t,\omega) dt
$$

for $n \ge 1$; then U is formally given by the Volterra series $\sum_{n\ge 0} U_n$.
In a straightforward recursive manner one derives the formulas

$$
U_{2n}(x,\omega) = \int_{x < t_1 < \dots < t_{2n}} u(t_1) \cdots u(t_{2n}) e^{-\omega(x-2\sigma_{2n}(t))B} dt_1 \cdots dt_{2n}
$$

and

$$
U_{2n-1}(x,\omega)
$$

$$
U_{2n}(x,\omega) = \int_{x < t_1 < \dots < t_{2n}} u(t_1) \cdots u(t_{2n}) e^{-\omega(x - 2\sigma_{2n}(t))B} dt_1 \cdots dt_{2n}
$$

and

$$
U_{2n-1}(x,\omega)
$$

$$
= -\int_{x < t_1 < \dots < t_{2n-1}} u(t_1) \cdots u(t_{2n-1}) B J e^{\omega(x - 2\sigma_{2n-1}(t))B} dt_1 \cdots dt_{2n-1},
$$
where

$$
\sigma_n(t) := \sum_{j=1}^n (-1)^{j+1} t_j
$$

for $n \ge 1$ and $t := (t_1, t_2, \dots, t_n)$. The proof uses the anti-commutation relati

$$
\sigma_n(\mathbf{t}) := \sum_{j=1}^n (-1)^{j+1} t_j
$$

 $\sigma_n(t) := \sum_{j=1}^n (-1)^{j+1} t_j$

for $n \ge 1$ and $t := (t_1, t_2, \dots, t_n)$. The proof uses $BJ + BJ = 0$ (which implies that $J \exp(tB) = e$

identity
 $\sigma_{n+1}(t_1, \dots, t_{n+1}) = t_1 - \sigma_n(t_2,$ for $n \ge 1$ and $t := (t_1, t_2, ..., t_n)$. The proof uses the anti-commutation relation $BJ + BJ = 0$ (which implies that $J \exp(tB) = \exp(-tB)J$) together with the identity
 $\sigma_{n+1}(t_1, ..., t_{n+1}) = t_1 - \sigma_n(t_2, ..., t_{n+1})$.
Note that $||e^{\omega x B}|| = 1$ f BJ

$$
\sigma_{n+1}(t_1,\ldots,t_{n+1})=t_1-\sigma_n(t_2,\ldots,t_{n+1}).
$$

+ $BJ = 0$ (which implies that $J \exp(tB) = \exp(-tB)J$) together with the
tity
 $\sigma_{n+1}(t_1, \ldots, t_{n+1}) = t_1 - \sigma_n(t_2, \ldots, t_{n+1}).$
Note that $||e^{\omega x B}|| = 1$ for all real x and ω , with $|| \cdot ||$ being the operator
m on 2 × 2 matrices. T No
norm of
orm of $(t_1, \ldots, t_{n+1}) = t_1 - \sigma_n(t_2, \ldots, t_{n+1}).$
= 1 for all real x and ω , with $\|\cdot\|$
Therefore, with
 $\eta(x) := \int_x^{\infty} |u(s)| \, ds,$ Note that $||e^{\omega x B}|| = 1$ for all real x and ω , with $|| \cdot ||$ being the operator on 2 × 2 matrices. Therefore, with $\eta(x) := \int_x^{\infty} |u(s)| ds$, \mathbb{F} \mathbb{F} \mathbb{F}

$$
\eta(x) := \int_x^{\infty} |u(s)| \, ds,
$$

$$
||U_n(x,\omega)|| \le \frac{(\eta(x))^n}{n!},
$$

inverse
as uniformly in ω
or of B and $Jv_1 = -iv_2$

$$
u(t_1) \cdots u(t_{2n})
$$

ows that
 $||U_n(x,\omega)|| \leq \frac{(\eta(x))^n}{n!}$,

Therefore Schröder Operators 17 converges uniformly in $\omega \in \mathbb{R}$ to a bounded continuous so the Volterra series
function $U(\cdot, \omega)$ on $\mathbb F$
Since v_1 is an ei

$$
||U_n(x,\omega)|| \le \frac{\sqrt{N}}{n!},
$$

so the Volterra series for U converges uniformly in $\omega \in \mathbb{R}$ to a bounded continuous
function $U(\cdot,\omega)$ on \mathbb{R} .
Since v_1 is an eigenvector of B and $Jv_1 = -iv_2$, we compute that

$$
U_{2n}(x,\omega)v_1 = \left(\int_{x < t_1 < \dots < t_{2n}} u(t_1) \cdots u(t_{2n}) e^{i\omega(x - 2\sigma_{2n}(t))} dt\right) v_1
$$

and

$$
U_{2n-1}(x,\omega)v_1 = -\left(\int_{-\infty}^{\infty} u(t_1) \cdots u(t_{2n-1}) e^{-i\omega(x - 2\sigma_{2n-1}(t))} dt\right) v_2.
$$

$$
U_{2n}(x,\omega)v_1 = \left(\int_{x < t_1 < \dots < t_{2n}} u(t_1) \dots u(t_{2n}) e^{i\omega(x - 2\sigma_{2n}(t))} dt\right) v_1
$$

and

$$
U_{2n-1}(x,\omega)v_1 = -\left(\int_{x < t_1 < \dots < t_{2n-1}} u(t_1) \dots u(t_{2n-1}) e^{-i\omega(x - 2\sigma_{2n-1}(t))} dt\right) v_2.
$$

Relation (3.1) now yields the representation

$$
f_+(x,\omega) = a(x,\omega)e^{i\omega x} + b(x,\omega)e^{-i\omega x},
$$

for the Jost solution f_+ , with

$$
f_{+}(x,\omega) = a(x,\omega)e^{i\omega x} + b(x,\omega)e^{-i\omega x},
$$

for the Jost solution f_+ , with
 $a($: $\label{eq:1}$

$$
f_{+}(x,\omega) = a(x,\omega)e^{i\omega x} + b(x,\omega)e^{-i\omega x},
$$

for the Jost solution f_{+} , with

$$
a(x,\omega) = 1 + \sum_{n=1}^{\infty} a_{2n}(x,\omega),
$$

$$
b(x,\omega) = \sum_{n=1}^{\infty} b_{2n-1}(x,\omega),
$$

where

$$
a_{2n}(x,\omega) := \int_{x < t_{1} < \cdots < t_{2n}} u(t_{1}) \cdots u(t_{2n})e^{-2i\omega}
$$

and

where

$$
b(x,\omega) = \sum_{n=1} b_{2n-1}(x,\omega),
$$

\n
$$
a_{2n}(x,\omega) := \int_{x < t_1 < \dots < t_{2n}} u(t_1) \dots u(t_{2n}) e^{-2i\omega \sigma_{2n}(t)} dt
$$
 (3.2)
\n
$$
x,\omega) := - \int_{x < t_1 < \dots < t_{2n-1}} u(t_1) \dots u(t_{2n-1}) e^{2i\omega \sigma_{2n-1}(t)} dt.
$$
 (3.3)
\nobtain a more useful representation for $a(x,\omega)$ and $b(x,\omega)$ by recast-

and

$$
b_{2n-1}(x,\omega) := -\int_{x (3.3)
$$

ing
 σ_{2n} $(x, \omega) := -\int_{x < t_1 < \dots < t_{2n-1}} u(t_1) \cdots u(t_{2n-1}) e^{2i\omega \sigma_{2n-1}(t)} dt.$ (3.3)

obtain a more useful representation for $a(x, \omega)$ and $b(x, \omega)$ by recast-

i (3.3) as follows. In (3.2), let $s = -\sigma_{2n}(t)$ and $y_j = t_{j+1}$. Note that

i We can obtain a more useful representation for $a(x, \omega)$ and $b(x, \omega)$ by recast-
3.2) and (3.3) as follows. In (3.2), let $s = -\sigma_{2n}(t)$ and $y_j = t_{j+1}$. Note that
 $0 \le 0$ in the region of integration. We then have
 $a_{2n}(x,$ ing (3.2) and (3.3) as follows. In (3.2), let $s = -\sigma_{2n}(t)$ and $y_j = t_{j+1}$. Note that $\sigma_{2n}(t) \le 0$ in the region of integration. We then have
 $a_{2n}(x,\omega) = \int_0^\infty e^{2i\omega s} A_{2n}(x,s) ds$

where $\sigma_{2n}(t) \leq 0$ in the region of integration. We then have

$$
a_{2n}(x,\omega) = \int_0^\infty e^{2i\omega s} A_{2n}(x,s) \,ds
$$

$$
a_{2n}(x, \omega) = \int_0^\infty e^{2i\omega s} A_{2n}(x, s) ds
$$

where

$$
A_{2n}(x, s) = \int_{x < s-\sigma_{2n-1}(y) < y_1 < \dots < y_{2n-1}} u(s - \sigma_{2n-1}(y)) u(y_1) \dots u(y_{2n-1}) dy
$$

obeys the estimates

$$
||A_{2n}(x, \cdot)||_{L_1(\mathbb{R}^+)} \le \frac{\left(\eta(x)\right)^{2n}}{(2n)!}
$$

obeys the estimates

$$
||A_{2n}(x, \cdot)||_{L_1(\mathbb{R}^+)} \le \frac{(\eta(x))^{2n}}{(2n)!}
$$

for
$$
u \in L_1(\mathbb{R})
$$
 and
\n
$$
||A_{2n}(x, \cdot)||_{L_2(\mathbb{R}^+)} \le ||u||_2 \frac{(\eta(x))^{2n-1}}{(2n-1)!}
$$
\nfor $u \in X$. It follows that

for $u \in L_1(\mathbb{R})$ and
 $||A_2$
for $u \in X$. It follows that for $u \in X$. It follows that
 a

where

and

$$
A_{2n}(x, \cdot) \|_{L_2(\mathbb{R}^+)} \le \|u\|_2 \frac{(\eta(x))^{-\infty}}{(2n-1)!}
$$

at

$$
a(x,\omega) = 1 + \int_0^\infty e^{2i\omega s} A(x,s) ds
$$

$$
||A(x, \cdot)||_1 \le \cosh \eta(x) - 1
$$

$$
||A(x, \cdot)||_2 \le ||a||_2 \sinh \eta(x).
$$

eck that $\lim_{x \to \infty} A(x, \cdot) = 0$ and

$$
||A(x, \cdot)||_1 \le \cosh \eta(x) - 1
$$

 $||A(x, \cdot)||_2 \leq ||a||_2 \sinh \eta(x).$

and

and
 $||A(x, \cdot)||_1 \leq \cosh \eta(x) - 1$

Finally, it is easy to check that $\lim_{x\to\infty} A(x, \cdot) = 0$
 $\lim_{x\to-\infty} A(x, s)$ exists in $L_1(\mathbb{R}^+) \cap L_2(\mathbb{R}^+)$.

Next we consider $b(x, s)$. In (3.3), let $s = \sigma_{2n-1}$

the region of in

 $||A(x, \cdot)||_2 \le ||a||_2 \sinh \eta(x).$
Finally, it is easy to check that $\lim_{x\to\infty} A(x, \cdot) = 0$
 $\lim_{x\to-\infty} A(x, s)$ exists in $L_1(\mathbb{R}^+) \cap L_2(\mathbb{R}^+).$
Next we consider $b(x, s)$. In (3.3), let $s = \sigma_{2n-1}$
the region of integration. Th Finally, it is easy to check that $\lim_{x\to\infty} A(x, \cdot) = 0$ and that the limit $A(s) =$
 $\lim_{x\to-\infty} A(x, s)$ exists in $L_1(\mathbb{R}^+) \cap L_2(\mathbb{R}^+)$.

Next we consider $b(x, s)$. In (3.3), let $s = \sigma_{2n-1}(t)$ and note that $x < s$ in

th lim_{x→−∞} $A(x, s)$ exists in $L_1(\mathbb{R}^+) \cap L_2(\mathbb{R}^+)$.
Next we consider $b(x, s)$. In (3.3), let s
the region of integration. Thus with $y_j = t_{j+}$
 $b_{2n-1}(x, \omega) = -\int_x^{\infty} e^{2i\omega}$
where Next we consider $b(x, s)$. In (3.3), let $s = \sigma_{2n-1}(t)$ and note that $x < s$ in
egion of integration. Thus with $y_j = t_{j+1}$ for $1 \le j \le 2n-2$ we get
 $b_{2n-1}(x, \omega) = -\int_x^{\infty} e^{2i\omega s} B_{2n-1}(x, s) ds$,
e
 $u(s + \sigma_{2n-2}(y))u(y_1) \cdots u(y_{2$

$$
b_{2n-1}(x,\omega) = -\int_x^{\infty} e^{2i\omega s} B_{2n-1}(x,s) \,ds,
$$

the region of integration. Thus with
$$
y_j = t_{j+1}
$$
 for $1 \leq j \leq 2n - 2$ we get
\n
$$
b_{2n-1}(x,\omega) = -\int_x^{\infty} e^{2i\omega s} B_{2n-1}(x,s) ds,
$$
\nwhere
\n
$$
B_{2n-1}(x,s) = \int_{x < s+\sigma_{2n-2}(y) < y_1 < \dots < y_{2n-1}} u(s + \sigma_{2n-2}(y))u(y_1) \dots u(y_{2n-2}) dy
$$

\nobeys multi-linear estimates similar to those for A_{2n} . In particular,
\n
$$
b(x,\omega) = -\int_x^{\infty} e^{2i\omega s} B(x,s) ds
$$

\nwhere (taking X-norms on R and extending $B(x,s)$ to zero if $s < x$)

obeys multi-linear estimates similar to those for
$$
A_{2n}
$$
. In particular,
\n
$$
b(x,\omega) = -\int_x^{\infty} e^{2i\omega s} B(x,s) ds
$$
\nwhere (taking X-norms on R and extending $B(x, s)$ to zero if $s < x$
\n
$$
||B(x, \cdot)||_1 \le \sinh \eta(x)
$$
\nand\n
$$
||B(x, \cdot)||_2 \le ||u||_2 (\cosh \eta(x) - 1).
$$

where (taking X-norms on ℝ and extending $B(x, s)$ to zero if $s < x$)
 $||B(x, \cdot)||_1 \le \sinh \eta(x)$

and
 $||B(x, \cdot)||_2 \le ||u||_2 (\cosh \eta(x) - 1).$

We can show that the limit
 $B(s) = \lim_{x \to a} B(x, s)$

$$
||B(x, \cdot)||_1 \le \sinh \eta(x)
$$

$$
||B(x, \cdot)||_2 \le ||u||_2 (\cosh \eta(x) - 1).
$$

We can show that the limit
exists in X .

$$
|B(x, \cdot)|_1 \le \sinh \eta(x)
$$

.
$$
||B||_2 \le ||u||_2 (\cosh \eta(x))
$$

$$
B(s) = \lim_{x \to -\infty} B(x, s)
$$

 $(x, \cdot) \|_2 \le \|u\|_2 (\cosh \eta(x) - 1)$
it
 $B(s) = \lim_{x \to -\infty} B(x, s)$
ii-linear estimates for the dif-
notions constructed for u and a exists in X.
Finally, similar multi-
 $B_{2n+1} - \widetilde{B}_{2n+1}$ of such fund
and $u \mapsto B(x, \cdot)$ are continuous $(s) = \lim_{x \to -\infty} B(x, s)$
ar estimates for the s constructed for u
s mappings from X exists in X.
Finally
 $B_{2n+1} - \widetilde{B}_2$
and $u \mapsto B$
Introd
Introd Finally, similar multi-linear estimates for the differences $A_{2n} - A_{2n}$ and
 $1 - \widetilde{B}_{2n+1}$ of such functions constructed for u and \widetilde{u} show that $u \mapsto A(x, \cdot)$
 $u \mapsto B(x, \cdot)$ are continuous mappings from X to $C(\math$ $B_{2n+1} - B_{2n+1}$ of such functions constructed for u and \tilde{u} show that $u \mapsto A(x, \cdot)$

) are continuous mappings from X to $C(\mathbb{R}; X)$.

the matrix
 $V := \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$

e eigenvectors v_1 and v_2 of B and set $\Psi(x, \omega)$

$$
V := \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}
$$

and $u \mapsto B(x, \cdot)$ are continuous mappings from X to $C(\mathbb{R}; X)$.
Introduce the matrix
 $V := \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$
composed of the eigenvectors v_1 and v_2 of B and set $\Psi(x, \omega)$.
Then Ψ is a solution of the AKNS-ZS s $V := \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}$
osed of the eigenvectors v_1 and v_2 of B a
 Ψ is a solution of the AKNS-ZS system
 $\Psi' = \mathrm{i}\omega J_1 \Psi + uJ$ composed of the eigenvectors v_1 and v_2 of B and set $\Psi(x, \omega) := V^{-1}U(x, \omega)V$.
Then Ψ is a solution of the AKNS-ZS system
 $\Psi' = i\omega J_1 \Psi + uJ\Psi$

$$
\Psi' = \mathrm{i} \omega J_1 \Psi + u J \Psi
$$

Inverse Scattering for Impedance Schrödinger Operators
with $J_1 := \text{diag}\{1, -1\}$, and one can verify in a straightforward manner that

with
$$
J_1 := \text{diag}\{1, -1\}
$$
, and one can verify in a straightforward manner that
\n
$$
e^{-i\omega J_1} \Psi(x, \omega) = \begin{pmatrix} a(x, \omega) & \overline{b(x, \omega)} \\ b(x, \omega) & a(x, \omega) \end{pmatrix}
$$
\nis an element of the group $SU(1, 1)$; see [36]. In particular, we see that
\n
$$
|a(x, \omega)|^2 - |b(x, \omega)|^2 = 1.
$$
\nFrom the computations above it is clear that the limits

is an element of the group $SU(1, 1)$; see [36]. In particular, we see that
 $|a(x, \omega)|^2 - |b(x, \omega)|^2 = 1$.

From the computations above it is clear that the limits
 $a(\omega) = \lim_{x \to -\infty} a(x, \omega)$
 $b(\omega) = \lim_{x \to 0} b(x, \omega)$

$$
|a(x,\omega)|^2 - |b(x,\omega)|^2 = 1
$$

$$
|a(x,\omega)|^2 - |b(x,\omega)|^2 = 1.
$$

From the computations above it is clear that the limits

$$
a(\omega) = \lim_{x \to -\infty} a(x,\omega)
$$

$$
b(\omega) = \lim_{x \to -\infty} b(x,\omega)
$$

exist and define continuous functions of ω . Moreover,
$$
|a(\omega)|^2 = |b(\omega)|^2 + 1
$$

so that $|a(\omega)| \ge 1$, while the symmetry relations

$$
|a(\omega)|^2 = |b(\omega)|^2 + 1 \tag{3.4}
$$

exist and define continuous functions of
$$
\omega
$$
. Moreover,
\n
$$
|a(\omega)|^2 = |b(\omega)|^2 + 1
$$
\nso that $|a(\omega)| \ge 1$, while the symmetry relations
\n
$$
\overline{a(\omega)} = a(-\omega) \quad \text{and} \quad \overline{b(\omega)} = b(-\omega)
$$
\nare inherited from those for $a(x, \omega)$ and $b(x, \omega)$. More importantly:

so that $|a(\omega)| \ge 1$, while the symmetry relations
 $\overline{a(\omega)} = a(-\omega)$ and \overline{b}

are inherited from those for $a(x, \omega)$ and $b(x, \omega)$.
 Proposition 3.1. Suppose that $u \in L_1(\mathbb{R})$. Then t

integral representations $(\omega) = a(-\omega)$ and $b(\omega) = b(-\omega)$
se for $a(x, \omega)$ and $b(x, \omega)$. More import
ose that $u \in L_1(\mathbb{R})$. Then the functions
 $a(\omega) = 1 + \int_{-\infty}^{\infty} e^{2i\omega s} A(s) ds$ are inherited from those for $a(x, \omega)$ and $b(x, \omega)$. More importantly:
 Proposition 3.1. Suppose that $u \in L_1(\mathbb{R})$. Then the functions $a(\omega)$ integral representations
 $a(\omega) = 1 + \int_0^\infty e^{2i\omega s} A(s) ds$, **Proposition 3.1.** Suppose that $u \in L_1(\mathbb{R})$. Then the functions $a(\omega)$ and $b(\omega)$ admit
integral representations
 $a(\omega) = 1 + \int_0^{\infty} e^{2i\omega s} A(s) ds$,
 $b(\omega) = - \int_{-\infty}^{\infty} e^{2i\omega s} B(s) ds$, integral representations

$$
a(\omega) = 1 + \int_0^{\infty} e^{2i\omega s} A(s) ds,
$$

$$
b(\omega) = -\int_{-\infty}^{\infty} e^{2i\omega s} B(s) ds,
$$

$$
||A||_1 \le \cosh(||u||_1) - 1,
$$

where

$$
(\omega) = -\int_{-\infty} e^{2i\omega s} B(s) ds,
$$

$$
||A||_1 \le \cosh(||u||_1) - 1,
$$

$$
||B||_1 \le \sinh ||u||_1,
$$

$$
(\sinh \omega) = 1,
$$

$$
||B||_2 \le ||u||_2 \sinh ||u||_1,
$$

and the function $A \in L_1(\mathbb{R}^+)$ and $B \in L_1(\mathbb{R})$ depend continuously on u in $L_1(\mathbb{R})$.

If, also, $u \in L_2(\mathbb{R})$, then
 $||A||_2 \le ||u||_2 \sinh ||u||_1$,
 $||B||_2 \le ||u||_2 (\cosh ||u||_1 - 1)$,

and the maps $u \mapsto A$ and $u \mapsto B$ are c If, also, $u \in L_2(\mathbb{R})$, then

$$
||B||_1 \le \sinh ||u||_1,
$$

\n
$$
(\mathbb{R}^+)
$$
 and $B \in L_1(\mathbb{R})$ depend co
\n
$$
||A||_2 \le ||u||_2 \sinh ||u||_1,
$$

\n
$$
||B||_2 \le ||u||_2 (\cosh ||u||_1 - 1),
$$

\n
$$
||u \mapsto B \text{ are continuous as } n
$$

\n
$$
vely.
$$

\non, it is easy to derive the r

 (\mathbb{R}) , then
 $u \mapsto A$ as
 X respec cosh $||u||_1 - 1$
continuous as
to derive the
c) the functic and the maps $u \mapsto A$ and $u \mapsto B$ are continuous as maps from X to $L_1(\mathbb{R}^+) \cap L_2(\mathbb{R}^+)$ and to X respectively.
Using this proposition, it is easy to derive the representation of the Jost solution f_+ in the form (2 $L_2(\mathbb{R}^+)$ and to X respectively.

(ℝ⁺) and to *X* respectively.
Using this proposition, i
ution f_+ in the form (2.7), solution f_+ in the form (2.7), with $K(x, \cdot)$ the function in X for every $x \in \mathbb{R}$.

3.2. The scattering data

20 S. Albeverio, R. Hryniv, Ya. Mykytyuk and P. Perry

3.2. The scattering data

It follows from Proposition 3.1 that for $u \in X$ the scattering coefficients a and

b are elements of the Wiener algebras $1 \stackrel{\cdot}{+} \widehat{X}$ a It follows from Proposition 3.1 that for $u \in X$ the scattering coefficients *a* and
 b are elements of the Wiener algebras $\mathbf{1} + \hat{X}$ and \hat{X} , respectively. Since *a* is an

invertible element of $\mathbf{1} + \hat{X}$, \boldsymbol{b} invertible element of $\mathbf{1} + \hat{X}$, we see that $r_+(\omega) := -b(-\omega)/a(\omega)$ belongs to \hat{X} . The symmetry properties of a and b yield the relation $r(-\omega) = \overline{r(\omega)}$, while (3.4) implies that $|r(\omega)| < 1$ for all $\omega \in \mathbb{R}$. Thi invertible element of **1** + *X*, we see that $r_+(\omega) := -b(-\omega)/a(\omega)$ belongs to *X*. The symmetry properties of *a* and *b* yield the relation $r(-\omega) = \overline{r(\omega)}$, while (3.4) implies that $|r(\omega)| < 1$ for all $\omega \in \mathbb{R}$. This e symmetry properties of *a* and *b* yield the relation $r(-\omega) = r(\omega)$, while (3.4) implies
that $|r(\omega)| < 1$ for all $\omega \in \mathbb{R}$. This establishes the direct part of the scattering
problem, namely:
Proposition 3.2. For every

Proposition 3.2. For every real-valued $u \in X$, the reflection coefficient r_{+} of the

that $|r(\omega)| < 1$ for all $\omega \in \mathbb{R}$. This establishes the direct part of the scattering
problem, namely:
Proposition 3.2. For every real-valued $u \in X$, the reflection coefficient r_+ of the
corresponding Schrödinger o **Proposition 3.2.**
corresponding Scinary Contractor
 $u_{\pm} \in L_1(\mathbb{R}^{\pm}) \cap L_2$
for $a(x,\omega)$, $b(x,\omega)$ (2.8) .

for

e forn

mts re

There In the case where $q = u'_L$
 $L_1(\mathbb{R}^{\pm}) \cap L_2(\mathbb{R})$, we cannot
 (x, ω) , $b(x, \omega)$, $A(x, s)$, and i

as they only require half-line

at the following
 na 3.3. The representation j $+ u^2$
pass
 $\frac{\partial(x, s)}{\partial x}$
integ $\frac{2}{\epsilon} = u'_+ + u^2_+ \in \mathcal{Q}_1 := \mathcal{Q} \setminus \mathcal{Q}_0$ for some
to the limit $x \to -\infty$ in the above formulas
s). However, all the other arguments remain
grability of u, which holds for u_+ . Therefore,
ulas $+ + u_+ \in \mathcal{Q}_1$ $u_{\pm} \in L_1$ for $a(x, \omega)$, $b(x, \omega)$, $A(x, s)$, and $B(x, s)$. However, all the other arguments remain valid as they only require half-line integrability of u, which holds for u_+ . Therefore, we get the following valid as they only require half-line integrability of u, which holds for u_{+} . Therefore, valid as they only require half-line integrability of *u*, which holds for *u*₊. Therefore,
we get the following
Lemma 3.3. The representation formulas
 $f_{+}(x,\omega) = e^{i\omega x} + \int_{x}^{\infty} K_{+}(x,\zeta)e^{i\omega \zeta} d\zeta$,
 $f_{+}(x,\omega) = \int_{$

Lemma 3.3. The represent f :
 $f^{\left[1\atop{3}\right]}$ **Lemma 3.3.** The representation formulas

$$
f_{+}(x,\omega) = e^{i\omega x} + \int_{x}^{\infty} K_{+}(x,\zeta)e^{i\omega\zeta}d\zeta,
$$

\n
$$
f_{+}^{[1]}(x,\omega) = i\omega \left(e^{i\omega x} + \int_{x}^{\infty} K_{+,1}(x,\zeta)e^{i\omega\zeta}d\zeta\right)
$$

\n
$$
f' - u_{+}f \text{ and the kernels } K_{+}(x,\cdot) \text{ and } K_{+,1}
$$

\n
$$
x \in \mathbb{R}.
$$

\nwhy quasi-derivatives $f^{[1]} := f' - u_{+}f$ are used
\n f in the domain of the operator H_{u} , the deriv

hold, with $f_+^{[1]}$ X for every fixed $x \in \mathbb{R}$.

:= $f' - u_+ f$ and the kernels $K_+(x, \cdot)$ and $K_{+,1}(x, \cdot)$ belonging to
ced $x \in \mathbb{R}$.
n why quasi-derivatives $f^{[1]} := f' - u_+ f$ are used above comes from
or f in the domain of the operator H_u , the derivative f' need The reason why *quasi-derivatives* $f^{[1]} := f' - u_+ f$ are used above comes from
act that for f in the domain of the operator H_u , the derivative f' need not
ntinuous while $f^{[1]}$ is continuous. Denoting by $W_+\{f,g\} := f^{$ be continuous while $f^{[1]}$ is continuous. Denoting by $W_+\{f,g\} := f^{[1]}g - fg^{[1]}$ the
modified Wronskian, we get from the above representations the equality
 $W_+\{f_+(x,\omega), f_+(x,-\omega)\} = -2i\omega,$ (3.5)
for every nonzero $\omega \in \mathbb{R}$.

$$
W_{+} \{ f_{+}(x,\omega), f_{+}(x,-\omega) \} = -2i\omega, \tag{3.5}
$$

be continuous while $f^{[1]}$ is continuous. Denoting by $W_+\{f,g\} := f^{[1]}g - fg^{[1]}$ the
modified Wronskian, we get from the above representations the equality
 $W_+\{f_+(x,\omega), f_+(x,-\omega)\} = -2i\omega,$ (3.5)
for every nonzero $\omega \in \mathbb{R}$. $W_+ \{f_+(x,\omega), f_+(x,-\omega)\} = -2i\omega,$
for every nonzero $\omega \in \mathbb{R}$.
Clearly, there are analogous construction for the left Jost solutions
and their quasi-derivatives $f_-^{[1]} := f'_- - u_- f_-$ that use the Riccati re
tive u_- . We then $(x, \omega), f_{+}(x, -\omega)$ } = −2i ω , (3.5)

us construction for the left Jost solutions $f_{-}(x, \omega)$

:= $f'_{-} - u_{-}f_{-}$ that use the Riccati representa-

e coefficients *a* and *b* from the relations (1.5) and for every nonzero $\omega \in \mathbb{R}$.
Clearly, there are a
and their quasi-derivativ
tive u_{-} . We then detern
(3.5) as
 a Clearly, there are analogous construction for the left Jost solutions $f_-(x, \omega)$
their quasi-derivatives $f_-^{[1]} := f_-' - u_- f_-$ that use the Riccati representa-
 $u_-,$ We then determine the coefficients a and b from the relatio and their quasi-derivatives $f_{-}^{[1]} := f_{-}' - u_{-}f_{-}$
tive u_{-} . We then determine the coefficients a a
(3.5) as
 $a(\omega) = \frac{W_{+} \{f_{-}(x, \omega), f_{+}\}}{2i\omega}$
 $b(\omega) = -\frac{W_{+} \{f_{-}(x, -\omega)}{2i\omega}$ $\begin{align*}\n\frac{U_1}{\text{the coefficient}} &:= f_2' \\
&= \frac{W_+ \{f_-}{W_+ \}}{1} \\
&= -\frac{W_+ \{f_-}{W_+ \}}{1} \n\end{align*}$ tive *u*₋. We then determine the coefficients *a* and *b* from the relations (1.5) and

(3.5) as
 $a(\omega) = \frac{W_+ \{f_-(x,\omega), f_+(x,\omega)\}}{2i\omega}$, (3.6)
 $b(\omega) = -\frac{W_+ \{f_-(x,-\omega), f_+(x,\omega)\}}{2i\omega}$ (3.7)

Time the coefficients
$$
a
$$
 and b from the relations (1.5) and

\n
$$
a(\omega) = \frac{W_+ \{f_-(x,\omega), f_+(x,\omega)\}}{2i\omega},
$$
\n
$$
b(\omega) = -\frac{W_+ \{f_-(x,-\omega), f_+(x,\omega)\}}{2i\omega}
$$
\n(3.7)

(3.5) as
\n
$$
a(\omega) = \frac{W_+ \{f_-(x,\omega), f_+(x,\omega)\}}{2i\omega},
$$
\n
$$
b(\omega) = -\frac{W_+ \{f_-(x,-\omega), f_+(x,\omega)\}}{2i\omega}
$$
\n(3.6)
\nfor real nonzero ω . Using now the integral representations for the Jost solutions

 $(\omega) = \frac{W_+ \{f_-(x, \omega), f_+(x, \omega)\}}{2i\omega},$ (3.6)
 $(\omega) = -\frac{W_+ \{f_-(x, -\omega), f_+(x, \omega)\}}{2i\omega}$ (3.7)

ag now the integral representations for the Jost solutions

atives, and recalling that $v := u_- - u_+$ is a non-negative

arrive at th $(\omega) = -\frac{W_+ \{f_-(x, -\omega), f_+(x, \omega)\}}{2i\omega}$ (3.7)

ag now the integral representations for the Jost solutions

atives, and recalling that $v := u_- - u_+$ is a non-negative

arrive at the following conclusion. for real nonzero ω . Using now the integral representations for the Jost solutions f_{\pm} and their quasi-derivatives, and recalling that $v := u_{-} - u_{+}$ is a non-negative continuous function, we arrive at the following f_{\pm} and their quasi-derivatives, and recalling that $v := u_{-} - u_{+}$ is a non-negative continuous function, we arrive at the following conclusion.

Suppose that $q \in \mathcal{Q}$. Then the coefficients a and b admit the repre-
 $a(\omega) = 1 + \widehat{A}_1(\omega) - v(0) \left[\frac{1 + \widehat{A}_2(\omega)}{2i\omega} \right],$ **Lemma 3.4.** Suppose that $q \in \mathcal{Q}$. Then the coefficients a and b admit the representation

$$
a(\omega) = 1 + \hat{A}_1(\omega) - v(0) \left[\frac{1 + \hat{A}_2(\omega)}{2i\omega} \right],
$$

\n
$$
b(\omega) = \hat{B}_1(\omega) + v(0) \left[\frac{1 + \hat{B}_2(\omega)}{2i\omega} \right],
$$

\n
$$
j = 1, 2, \text{ are real-valued functions in }.
$$

\n
$$
A_2 = B_2 = 0 \text{ if } q \in \mathcal{Q}_0 \text{ and}
$$

\n
$$
+ \hat{A}_2(0) = 1 + \hat{B}_2(0) = f_+(0, 0) f_-(0, 0)
$$

\nThe maps $q \mapsto A_j$ and $q \mapsto B_j$ are con

 $2i\omega$
fun
 id
 $+$ $(0 \rightarrow I)$ in which A_j and B_j , j on

$$
1 + \widehat{A}_2(0) = 1 + \widehat{B}_2(0) = f_+(0,0) f_-(0,0)
$$

= 1, 2, are real-valued functions in X, with A_i supported
 $\hat{A}_2(0) = 1 + \hat{B}_2(0) = f_+(0,0)f_-(0,0)$

allelled maps $q \mapsto A_j$ and $q \mapsto B_j$ are continuous maps from Q

be generic case $q \in Q_1$ the coefficients $\omega a(\omega)/(\omega + i)$ [0, ∞). Moreover, $A_2 = B_2 = 0$ if $q \in Q_0$ and
 $1 + \widehat{A}_2(0) = 1 + \widehat{B}_2(0) = f_+$

nonzero if $q \in Q_1$. The maps $q \mapsto A_j$ and $q \mapsto$

o X.

It follows that in the generic case $q \in Q_1$ t
 ω // $(\omega + i)$ are elements of the is nonzero if $q \in \mathcal{Q}_1$. The maps $q \mapsto A_j$ and $q \mapsto B_j$ are continuous maps from \mathcal{Q} into X .

1 + $A_2(0) = 1 + B_2(0) = f_+(0, 0) f_-(0, 0)$.
The maps $q \mapsto A_j$ and $q \mapsto B_j$ are cont
in the generic case $q \in \mathcal{Q}_1$ the coefficient
ements of the Wiener algebras $1 + \hat{X}$ and
have singularity at the origin. The funct It follows that in the generic case $q \in \mathcal{Q}_1$ the coefficients $\omega a(\omega)/(\omega + i)$ and $/(\omega + i)$ are elements of the Wiener algebras $1 + \hat{X}$ and \hat{X} , respectively, but *l b* themselves have singularity at the origin. $\omega b(\omega)/(\omega+i)$ are elements of the Wiener algebras $1+\hat{X}$ and \hat{X} , respectively, but $(\omega)/(\omega + i)$ are elements of the Wiener algebras $1 + X$ and X , respectively, but
and b themselves have singularity at the origin. The function a never vanishes
the real line so that $1/a$ is well defined and belongs to $1 \dot$ \boldsymbol{a} on the real line so that $1/a$ is well defined and belongs to $\mathbf{1} + \hat{X}$; as a result, $r_{+}(\omega) := -b(-\omega)/a(\omega)$ and $r_{-}(\omega) = b(\omega)/a(\omega)$ are well defined and belong to \hat{X} .
In particular, r_{\pm} are continuous functions on the real line so that $1/a$ is well defined and belongs to $\mathbf{1} + X$; as a result, $r_+(\omega) := -b(-\omega)/a(\omega)$ and $r_-(\omega) = b(\omega)/a(\omega)$ are well defined and belong to \hat{X} .
In particular, r_{\pm} are continuous functions and r r_{+} In particular, r_{\pm} are continuous functions and $r_{\pm}(0) = -1$. More details are given in the following statement, where w_{\pm} stand for the restrictions of the Riccati representatives u_+ onto the respective half-line \mathbb{R}^{\pm} and the set \mathcal{R}_1 was introduced representatives u_{\pm} onto the respective half-line ℝ[±] and the set \mathcal{R}_1 was introduced
in (2.9).
Proposition 3.5. Suppose that $q \in \mathcal{Q}_1$. Then $r_{\pm} \in \mathcal{R}_1$, and the maps
 \mathcal{S}_{\pm} : $(w_+, w_-, v(0)) \mapsto r_{\pm}$

Proposit
and
are cont **Proposition 3.5.** Suppose that $q \in Q_1$. Then $r_{\pm} \in \mathcal{R}_1$, and the maps

$$
\mathcal{S}_{\pm} : (w_+, w_-, v(0)) \mapsto r_{\pm}
$$

$$
(w_+, w_-, v(0)) \mapsto \widetilde{r}_{\pm}
$$

transmission coefficient
truct $t = 1/a$ given either

and

$$
(w_+, w_-, v(0)) \mapsto \widetilde{r}_\pm
$$

are continuous.

3.3. Reconstruction of the transmission coefficient

Next, we show how to construct $t = 1/a$ given either one of the reflection coefficients. Again the case $q \in \mathcal{Q}_0$ is much simpler, and we shall concentrate on the non-resonant case $q \in \mathcal{Q}_1$.
We observe that formula

Next, we show how to construct $t = 1/a$ given either one of the reflection coefficients. Again the case $q \in \mathcal{Q}_0$ is much simpler, and we shall concentrate on the non-resonant case $q \in \mathcal{Q}_1$.
We observe that formula cients. Again the case $q \in \mathcal{Q}_0$ is much simpler, and we shall concentrate on the
non-resonant case $q \in \mathcal{Q}_1$.
We observe that formula (3.6) allows to extend a analytically in the open
upper-half plane \mathbb{C}^+ an non-resonant case $q \in \mathcal{Q}_1$.
We observe that form
upper-half plane \mathbb{C}^+ and larization
larization
of a extends to a bounded We observe that formula (3.6) allows to extend a analytically in the open
r-half plane \mathbb{C}^+ and this extension has no zeros in $\overline{\mathbb{C}^+}\setminus\{0\}$. Thus the regu-
tion
 $\widetilde{a}(\omega) := \frac{\omega}{\omega + i} a(\omega)$
extends to a boun

$$
\widetilde{a}(\omega) := \frac{\omega}{\omega + i} a(\omega)
$$

upper-half plane ℂ⁺ and this extension has no zeros in ℂ+ ∖ {0}. Thus the regularization
 $\tilde{a}(\omega) := \frac{\omega}{\omega + i} a(\omega)$

of a extends to a bounded holomorphic function in the upper half-plane with no

zeros in its closure. atization

of *a* extends to a bounded holomorphic fur-

zeros in its closure. Using the Schwarz form

from its real part $\text{Re} \log \tilde{a}(s) = \log |\tilde{a}(s)|$, w
 $\tilde{a}(z) = \exp \left(\frac{1}{\pi i} \int_{\mathbb{R}} \log |a(s)| \right)$ $\frac{1}{2}$ inc
we
og (ω)
on
a t
et
 $((s)$ of *a* extends to a bounded holomorphic function in the upper half-plane with no
zeros in its closure. Using the Schwarz formula to reconstruct the function $\log \tilde{a}$
from its real part $\text{Re}\log \tilde{a}(s) = \log |\tilde{a}(s)|$, we ge from its real part Re $\log \tilde{a}(s) = \log |\tilde{a}(s)|$, we get

zeros in its closure. Using the Schwarz formula to reconstruct the function
$$
\log \tilde{a}
$$

from its real part $\text{Re}\log \tilde{a}(s) = \log |\tilde{a}(s)|$, we get

$$
\tilde{a}(z) = \exp\left(\frac{1}{\pi i} \int_{\mathbb{R}} \log |\tilde{a}(s)| \frac{ds}{s - z}\right).
$$
(3.8)

It follows from (3.8) that
\n
$$
t(z) := 1/a(z) = \frac{z}{z+i} \exp\left\{\frac{1}{2\pi i} \int_{\mathbb{R}} \log\left[\left(1 - |r_{\pm}(s)|^2\right) \frac{s^2 + 1}{s^2}\right] \frac{ds}{s-z}\right\},\
$$
\nand t on the real line is given as a boundary value as $\text{Im } z \to 0$. In terms of
\nRiesz projector C_+ , we get the formula
\n
$$
t(\omega) = \frac{\omega}{\omega + i} \exp\left\{\left(C_+ \log\left[\left(1 - |r_{\pm}(s)|^2\right) \frac{s^2 + 1}{s^2}\right]\right)(\omega)\right\}.
$$

$$
t(z) := 1/a(z) = \frac{z}{z+i} \exp\left\{\frac{1}{2\pi i} \int_{\mathbb{R}} \log \left| \left(1 - |r_{\pm}(s)|^2\right) \frac{s^2 + 1}{s^2} \right| \frac{ds}{s - z} \right\},\
$$

and t on the real line is given as a boundary value as $\text{Im } z \to 0$. In terms of the
Riesz projector C_+ , we get the formula

$$
t(\omega) = \frac{\omega}{\omega + i} \exp\left\{\left(\mathcal{C}_+ \log \left[\left(1 - |r_{\pm}(s)|^2\right) \frac{s^2 + 1}{s^2}\right]\right)(\omega)\right\}.
$$
(3.9)
In particular, the number $\theta := \lim_{\omega \to 0} \left[2i\omega/t(\omega)\right]$ can be recovered from either
reflection coefficient.
We next show that formula (3.9) makes sense for every element $r \in \mathcal{R}_1$.
Indeed, the function

$$
\left(1 - |r(\omega)|^2\right) \frac{\omega^2 + 1}{\omega^2} = 1 - r(\omega)r(-\omega) + \tilde{r}(\omega)
$$
(3.10)

 $\frac{1}{3^2}$
De 1 (ω)
ver reflection coefficient.
We next show that formula (3.9) makes sense for every element $r \in \mathcal{R}_1$.
Indeed, the function

We next show that formula (3.9) makes sense for every element
$$
r \in \mathcal{R}_1
$$
.
Indeed, the function

$$
\left(1 - |r(\omega)|^2\right) \frac{\omega^2 + 1}{\omega^2} = 1 - r(\omega)r(-\omega) + \tilde{r}(\omega)
$$
(3.10)
belongs to the algebra $1 + \hat{X}$, does not vanish on the real line, and tends to 1 at
infinity. By the Wiener–Levi Lemma A.2, the function

 $\left(1\right)$
belongs to the algeb
infinity. By the Wie: $1 - |r(\omega)|^2 \int \frac{d\omega}{d\omega^2} = 1 - r(\omega)r(-\omega) + \tilde{r}(\omega)$ (3.10)

bra $1 + \hat{X}$, does not vanish on the real line, and tends to 1 at

ener-Levi Lemma A.2, the function
 $\log \left[\left(1 - |r(\omega)|^2 \right) \frac{\omega^2 + 1}{\omega^2} \right]$

$$
||^2 = 1 - r(\omega)r(-
$$

\n
$$
\widehat{X}
$$
, does not vanish on the
\ni Lemma A.2, the function
\n
$$
\log \left[\left(1 - |r(\omega)|^2 \right) \frac{\omega^2 + 1}{\omega^2} \right]
$$

belongs to the algebra $1 + X$, does not vanish on the real line, and tends to 1 at infinity. By the Wiener-Levi Lemma A.2, the function
 $\log \left[\left(1 - |r(\omega)|^2 \right) \frac{\omega^2 + 1}{\omega^2} \right]$

also belongs to $1 + \hat{X}$; in fact, since $\log \left[\left(1 - |r(\omega)|^2 \right) \frac{\omega^2 + 1}{\omega^2} \right]$
also belongs to $1 + \widehat{X}$; in fact, since it vanishes at infinit
the Riesz projector C_+ acts continuously in \widehat{X} , and exp
operation in $1 + \widehat{X}$ by the Wiener–Levi lemma. log $\left| \left(1 - |r(\omega)| \right) \right|$
t, since it vanis
continuously in
iener-Levi lem $\frac{1}{2}$
infinite definition also belongs to 1 + X; in fact, since it vanishes at infinity, it belongs to X. Finally,
the Riesz projector C_+ acts continuously in \hat{X} , and exponentiation is a continuous
operation in 1 + \hat{X} by the Wiener-Le the Riesz projector C_+ acts continuously in X , and exponentiation is a continuous
operation in $1 + \hat{X}$ by the Wiener-Levi lemma. We now define a function $\tilde{t} \in 1 + \hat{X}$
by (cf. (3.9))
 $\tilde{t} = \exp\left\{C_+ \log\left[\left(1 -$

$$
\widetilde{t} = \exp\left\{ \mathcal{C}_+ \log \left[\left(1 - |r(\omega)|^2 \right) \frac{\omega^2 + 1}{\omega^2} \right] \right\}.
$$
\n(3.11)

operation in 1+*X* by the Wiener–Levi lemma. We now define a function $t \in 1 + X$
by (cf. (3.9))
 $\tilde{t} = \exp \left\{ \mathcal{C}_+ \log \left[\left(1 - |r(\omega)|^2 \right) \frac{\omega^2 + 1}{\omega^2} \right] \right\}.$ (3.11)
Clearly, \tilde{t} is an invertible element of the Ban $\tilde{t} = \exp \left\{ \mathcal{C}_+ \log \left[\left(1 - |r(\omega)|^2 \right) \frac{\omega^2 + 1}{\omega^2} \right] \right\}.$ (3.11)
Clearly, \tilde{t} is an invertible element of the Banach algebra $1 + \hat{X}$. Moreover, the
following holds:
Lemma 3.6. The mappings
 $\mathcal{R}_1 \ni r \mapsto \$ $\frac{1}{2}$
 $\frac{1}{2}$
 \mapsto 1 Clearly, *t* is an invertible element of the Banach algebra $1 + X$. Moreover, the following holds:
 Lemma 3.6. The mappings
 $\mathcal{R}_1 \ni r \mapsto \tilde{t} \in 1 + \hat{X}$ and $\mathcal{R}_1 \ni r \mapsto 1/\tilde{t} \in 1 + \hat{X}$

are continuous.

Lemma 3.6. The mappings

$$
\mathcal{R}_1 \ni r \mapsto \tilde{t} \in 1 + \hat{X} \qquad \text{and} \qquad \mathcal{R}_1 \ni r \mapsto 1/\tilde{t} \in 1 + \hat{X}
$$

are continuous.

Lemma 3.6. Th
 R_1
are continuous.
We observ 1 + *X* and $\mathcal{R}_1 \ni r \mapsto 1/t \in 1 + X$
 \therefore the function in (3.10) is even and the R

odd ones, the function \tilde{t} enjoys the syn
 $t(\omega) = \omega \tilde{t}(\omega) / (\omega + i)$; then the above c maps even functions into odd ones, the function \tilde{t} enjoys the symmetry prop-
erty $\tilde{t}(-\omega) = \tilde{t}(\omega)$. We set $t(\omega) = \omega \tilde{t}(\omega) / (\omega + i)$; then the above considerations
show that
 $\frac{t(\omega)}{t(-\omega)} = \frac{\omega - i}{\omega + i} \frac{\tilde{t}(\$ erty $t(-\omega) = t(\omega)$. We set $t(\omega) = \omega t(\omega)/(\omega + i)$; then the above considerations
show that
 $\frac{t(\omega)}{t(-\omega)} = \frac{\omega - i}{\omega + i} \frac{\tilde{t}(\omega)}{\tilde{t}(-\omega)}$
also belongs to $1 + \tilde{X}$. The function
 $r^{\#}(\omega) = -\frac{t(\omega)}{t(-\omega)}r(-\omega)$

$$
\frac{t(\omega)}{t(-\omega)} = \frac{\omega - i}{\omega + i} \frac{t(\omega)}{\tilde{t}(-\omega)}
$$
\naction\n
$$
{}^{\#}(\omega) = -\frac{t(\omega)}{t(-\omega)} r(-\omega)
$$

also belongs to $1 + \widehat{X}$. The function also belongs to 1 + $X.$ The function $r^{\#}(\omega):$

$$
t(-\omega) \quad \omega + i t(-\omega)
$$

function

$$
r^{\#}(\omega) = -\frac{t(\omega)}{t(-\omega)}r(-\omega)
$$

thus belongs to
$$
\mathcal{R}_1
$$
 and, as $|t(\omega)/t(-\omega)| = 1$, we have
\n
$$
\frac{1 - |r^{\#}(\omega)|^2}{\omega^2} = \frac{1 - |r(\omega)|^2}{\omega^2} \in \widehat{X}.
$$
\nHence:
\n**Proposition 3.7.** For $r \in \mathcal{R}_1$, define \tilde{t} by (3.11) and set $t(\omega) = \omega \tilde{t}(\omega)/(\omega + i)$. Then
\nthe nonlinear map
\n
$$
\mathcal{I}: r \mapsto r^{\#}(\omega) := -\frac{t(\omega)}{(\omega + i)}r(-\omega)
$$

 $\frac{(\omega)}{\omega}$ Propose
the not
is a co **Proposition 3.7.** For $r \in \mathcal{R}_1$, define t by (3.11) and set $t(\omega) = \omega t(\omega)/(\omega + i)$. Then
 $:= -\frac{t(\omega)}{t(-\omega)}r(-\omega)$

ction coefficients for a Schrödinger operathe nonlinear map

$$
\mathcal{I}: r \mapsto r^{\#}(\omega) := -\frac{t(\omega)}{t(-\omega)}r(-\omega)
$$

on on \mathcal{R}_1 .
 r_{\pm} are the reflection coefficients
tial $q \in \mathcal{Q}$, then $\mathcal{I}r_{\pm} = r_{\mp}$.
n

is a continuous involution on \mathcal{R}_1 .

3.4. The inverse problem

tor *H* with Miura potential $q \in \mathcal{Q}$, then $\mathcal{I}r_{\pm} = r_{\mp}$.
 3.4. The inverse problem

In this subsection we solve the inverse scattering problem by proving the following

theorem. tor *H* with Miura potential $q \in \mathcal{Q}$, then $\mathcal{I}r_{\pm} = r_{\mp}$.
 3.4. The inverse problem

In this subsection we solve the inverse scattering pr

theorem.
 Theorem 3.8. Suppose that $r \in \mathcal{R}_j$, $j = 0, 1$. Then

Theorem 3.8. Suppose that $r \in \mathcal{R}_j$, $j = 0, 1$. Then there exists a unique $q \in \mathcal{Q}_j$
having r as its right reflection coefficient; moreover, the map $r \mapsto q$ is continuous.
The resonant case $r \in \mathcal{R}_0$ is much si **Theorem 3.8.** Suppose that $r \in \mathcal{R}_j$, j having r as its right reflection coefficient; moreover, the map $r \mapsto q$ is continuous.

Theorem
having r
The
procedur
class: we = 0, 1. Then there exists a unique $q \in \mathcal{Q}_j$
it; moreover, the map $r \mapsto q$ is continuous.
simpler and can be settled by the limiting
is for smooth potentials q in the Schwartz
case where $r \in \mathcal{R}_1$.
 $r \in \mathcal{R}_1$, pre

The resonant case $r \in \mathcal{R}_0$ is much simpler and can be settled by the limiting
dure from the well-known results for smooth potentials q in the Schwartz
we therefore concentrate on the case where $r \in \mathcal{R}_1$.
We thus s procedure from the well-known results for smooth potentials q in the Schwartz
class; we therefore concentrate on the case where $r \in \mathcal{R}_1$.
We thus suppose given a function $r \in \mathcal{R}_1$, presumed to be the right refle class; we therefore concentrate on the case where $r \in \mathcal{R}_1$.
We thus suppose given a function $r \in \mathcal{R}_1$, presumed
coefficient corresponding to a potential q_0 to be found.
construct $t(\omega)$ (and hence $a(\omega) := 1/t(\omega)$ We thus suppose given a function $r \in \mathcal{R}_1$, presumed to be the right reflection
cient corresponding to a potential q_0 to be found. From this data, we can
ruct $t(\omega)$ (and hence $a(\omega) := 1/t(\omega)$) using (3.9), and use th coefficient corresponding to a potential q_0 to be found. From this data, we can
construct $t(\omega)$ (and hence $a(\omega) := 1/t(\omega)$) using (3.9), and use the involution \mathcal{I}
to construct $r^{\#} = \mathcal{I}r$, a candidate for the

construct $t(\omega)$ (and hence $a(\omega) := 1/t(\omega)$) using (3.9), and use the involution \mathcal{I}
to construct $r^{\#} = \mathcal{I}r$, a candidate for the left reflection coefficient. Clearly, b is
defined as $r^{\#}a$.
We then form two Za to construct $r^{\#} = \mathcal{I}r$, a candidate for the left reflection coefficient. Clearly, *b* is
defined as $r^{\#}a$.
We then form two Zakharov-Shabat systems like $(2.12)-(2.13)$ but taking
the putative reflection coefficien defined as $r^{\#}a$.
We then the putative rest
these equations
determine cand
the Riccati dat. the putative reflection coefficients *r* and $r^{\#}$ instead of r_{+} and r_{-} and prove that these equations are uniquely soluble for the kernels M_{\pm} and $M_{\pm}^{\#}$. These kernels determine candidate right and le these equations are uniquely soluble for the kernels M_{\pm} and $M_{\pm}^{\#}$
determine candidate right and left Riccati representatives w and
the Riccati data
 $(w|_{\mathbb{R}^+}, w^{\#}|_{\mathbb{R}^-}, (w^{\#} - w)(0))$
of a distribution pote $\frac{1}{4} w^{\#}$, which give

(3.12)

continuity of the
 \mathbb{R}^+ , with $X^{\pm} :=$

$$
(w|_{\mathbb{R}^+}, w^\#|_{\mathbb{R}^-}, (w^\# - w)(0)) \tag{3.12}
$$

determine candidate right and left Riccati representatives w and $w^{\#}$, which give
the Riccati data
 $(w|_{\mathbb{R}^+}, w^{\#}|_{\mathbb{R}^-}, (w^{\#} - w)(0))$ (3.12)
of a distribution potential $q_0 \in \mathcal{Q}$. The construction exhibits c of a distribution
map from r to t
 $L_1(\mathbb{R}^{\pm}) \cap L_2(\mathbb{R}^{\pm})$
The most $(w^{\#} - w)(0)$ (3.12)
construction exhibits continuity of the
com \mathcal{R} to $X^+ \times X^- \times \mathbb{R}^+$, with X^{\pm} :=
lem is to justify the reconstruction by
 q_0 and that q_0 has reflection coefficients $L_1(\mathbb{R}^{\pm})\cap L_2(\mathbb{R}^{\pm}).$

of a distribution potential $q_0 \in \mathcal{Q}$. The construction exhibits continuity of the
map from r to the data (3.12) as maps from R to $X^+ \times X^- \times \mathbb{R}^+$, with $X^{\pm} :=$
 $L_1(\mathbb{R}^{\pm}) \cap L_2(\mathbb{R}^{\pm})$.
The most difficult map from *r* to the data (3.12) as maps from \mathcal{R} to $X^+ \times X^- \times \mathbb{R}^+$, with $X^{\pm} :=$
 $L_1(\mathbb{R}^{\pm}) \cap L_2(\mathbb{R}^{\pm})$.

The most difficult part of the problem is to justify the reconstruction by

showing that $w' + w^$ $(\mathbb{R}^{\pm}) \cap L_2(\mathbb{R}^{\pm})$.
The most divides with the most dividend r^* . It then for that for Levinsch showing that $w' + w^2 = (w^{\#})' + (w^{\#})^2 = q_0$ and that q_0 has reflection coefficients r and $r^{\#}$. It then follows from the uniqueness result (whose proof is a simple variant of that for Levinson's theorem given in [2 r and $r^{\#}$. It then follows from the uniqueness result (whose proof is a simple variant of that for Levinson's theorem given in [27]) that q_0 is the correct reconstruction. of that for Levinson's theorem given in [27]) that q_0 is the correct reconstruction.

Now we explain some details of the above algorithm. Taking the Fourier
transform F of the function r, we form the Marchenko-type system

$$
F(x+t) + M_-(x,t) + \int_x^{\infty} M_+(x,s)F(s+t) ds = 0,
$$
(3.13)

$$
M_+(x,t) + \int_x^{\infty} M_-(x,s)F(s+t) ds = 0.
$$
(3.14)
It is convenient to make change of variables $s = x + y$ in the above integrals and
to introduce the linear operator $T_F(x)$ on X^+ by

$$
M_{-}(x,t) + \int_{x}^{x} M_{+}(x,s)F(s+t) ds = 0,
$$
 (3.13)
\n
$$
M_{+}(x,t) + \int_{x}^{\infty} M_{-}(x,s)F(s+t) ds = 0.
$$
 (3.14)
\nchange of variables $s = x + y$ in the above integrals and
\nerator $T_{F}(x)$ on X^{+} by

 $(x, s)F(s + t) ds = 0.$ (3.14)
 $x = x + y$ in the above integrals and

by
 $(x + y + t) dt.$

$$
M_{+}(x,t) + \int_{x}^{\infty} M_{-}(x,s)F(s+t)
$$

ake change of variables $s = x + y$ in the
ar operator $T_{F}(x)$ on X^{+} by

$$
T_{F}(x)\psi(y) := \int_{0}^{\infty} \psi(t)F(x+y+t) dt
$$

It is convenient to make change of variables $s = x + y$ in the above integrals and
to introduce the linear operator $T_F(x)$ on X^+ by
 $T_F(x)\psi(y) := \int_0^\infty \psi(t)F(x + y + t) dt$.
Since $F \in X$, this operator is compact in X^+ . Substitut to introduce the linear operator $T_F(x)$ on X^+ by
 $T_F(x)\psi(y) := \int_0^\infty \psi(t)F(x \cdot \theta)$

Since $F \in X$, this operator is compact in X^+ . Su

equation (3.13) and using the notation $F_x(\cdot) := F$

equation for $\widetilde{M}_-(x, y) := M_-(x, x + y$ $(x)\psi(y) := \int_0^\infty \psi(t) F(x + y + t) dt.$
tor is compact in X^+ . Substituting t
g the notation $F_x(\cdot) := F(x + \cdot)$, we
= $M_-(x, x + y)$:
 $) + \widetilde{M}_-(x, y) - T_F^2(2x) \widetilde{M}_-(x, \cdot)(y) =$ Since $F \in X$, this operator is compact in X^+ . Substituting for M_+ from (3.14) in
equation (3.13) and using the notation $F_x(\cdot) := F(x + \cdot)$, we then get the following
equation for $\widetilde{M}_-(x, y) := M_-(x, x + y)$:
 $F_{2x}(y) + \widetilde{$ equation (3.13) and using the notation $F_x(\cdot) := F(x + \cdot)$, we then get the following
equation for $\widetilde{M}_-(x, y) := M_-(x, x + y)$:
 $F_{2x}(y) + \widetilde{M}_-(x, y) - T_F^2(2x)\widetilde{M}_-(x, \cdot)(y) = 0$,
or
 $(I - T_F^2(2x))\widetilde{M}_-(x, \cdot) = -F_{2x}$. equation for $M_-(x,y) := M_-(x,x+y)$:

$$
(x, y) := M_{-}(x, x + y):
$$

\n
$$
F_{2x}(y) + \widetilde{M}_{-}(x, y) - T_F^2(2x)\widetilde{M}_{-}(x, \cdot)(y) = 0,
$$

\n
$$
(I - T_F^2(2x))\widetilde{M}_{-}(x, \cdot) = -F_{2x}.
$$

\nove that the operator $I - T_F^2(2x)$ is boundedly

$$
(I - T_F^2(2x))\widetilde{M}_-(x, \cdot) = -F_{2x}.
$$

equation for M or $$\,{\rm We}\xspace$ now ${\rm p}\xspace$ $(y) + \overline{M}_{-}(x, y) - T_F^2(2x)\overline{M}_{-}(x, \cdot)(y) = 0,$
 $(I - T_F^2(2x))\overline{M}_{-}(x, \cdot) = -F_{2x}.$

and the operator $I - T_F^2(2x)$ is boundedly
 $|r(k)| < 1$ a.e. implies that $\ker_{L_2}(I \pm T_F($

f of Lemma 6.4.1 in [59]), and then comp In
fo:
in $(2x)$) $M_{-}(x, \cdot) = -F_{2x}$.

rator $I - T_F^2(2x)$ is bouse.

i.e. implies that ker_{L_2}(\ldots 6.4.1 in [59]), and the

m alternative gives the
 $= -(I - T_F^2(2x))^{-1}F_2$. We now prove that the operator $I - T_F^2(2x)$ is boundedly invertible in X^+ .

d, the inequality $|r(k)| < 1$ a.e. implies that $\ker_{L_2}(I \pm T_F(2x))$ is trivial (see,

cample, the proof of Lemma 6.4.1 in [59]), and then compactn Indeed, the inequality $|r(k)| < 1$ a.e. implies that ker $L_2(I \pm T_F(2x))$ is trivial (see, for example, the proof of Lemma 6.4.1 in [59]), and then compactness of $T_F(2x)$ in X^+ together with the Fredholm alternative gives t for example, the proof of Lemma 6.4.1 in [59]), and then compactness of $T_F(2x)$
in X^+ together with the Fredholm alternative gives the result. Therefore,
 $\widetilde{M}_-(x, \cdot) = -\left(I - T_F^2(2x)\right)^{-1} F_{2x}$
is the solution we are in X^+ together with the Fredholm alternative gives the result. Therefore,
 $\widetilde{M}_-(x, \cdot) = -\left(I - T_F^2(2x)\right)^{-1} F_{2x}$

is the solution we are looking for. Using the operator identity
 $(I - T)^{-1} = I + T(I - T)^{-1}$,

we can show th

$$
\widetilde{M}_{-}(x,\cdot) = -\big(I - T_F^2(2x)\big)^{-1} F_{2x}
$$

$$
I_{-}(x, \cdot) = -(I - T_{F}^{2}(2x))^{T} F_{\frac{1}{2}}
$$

ing for. Using the operator i

$$
(I - T)^{-1} = I + T(I - T)^{-1},
$$

$$
I_{-}(x, y) = -F(2x + y) + G(x,
$$

$$
(I - T)^{-1} = I + T(I - T)^{-1},
$$

\n
$$
\widetilde{M}_{-}(x, y) = -F(2x + y) + G(x, y)
$$

\nIt is jointly continuous in x and y
\n
$$
):= -2\widetilde{M}_{-}(x, 0) = 2F(2x) - 2G(
$$

 $(I-T)^{-1} = I + T(I-T)^{-1},$
we can show that
 $\widetilde{M}_{-}(x,y) = -F(2x + y) + G(x,y)$
for some function G that is jointly continuous in x and y. In

$$
w(x) := -2M_{-}(x,0) = 2F(2x) - 2G(x,0)
$$

 $\widetilde{M}_{-}(x, y) = -F(2x + y) + G(x, y)$
for some function G that is jointly continuous in x and y.
 $w(x) := -2\widetilde{M}_{-}(x, 0) = 2F(2x) - 2G(x)$
is well defined and can be shown to belong to the space L
 (c, ∞) . for some function G that is jointly continuous in x and y. In particular,
 $w(x) := -2\widetilde{M}_-(x,0) = 2F(2x) - 2G(x,0)$

is well defined and can be shown to belong to the space $L_1 \cap L_2$ on ever;
 (c, ∞) .

"Left" analogues of $(x) := -2M_{-}(x, 0) = 2F(2x) - 2G(x, 0)$

h be shown to belong to the space $L_1 \cap L_2$

s of the above objects constructed for r^{\dagger}

and belongs to $L_1 \cap L_2$ on every half-line
 $(x) = 2\widetilde{M}^{\#}(x, 0) = -2F^{\#}(2x) + 2G^{\#}(x, 0$ is well defined and can be shown to belong to the space $L_1 \cap L_2$ on every half-line (c, ∞) .

"Left" analogues of the above objects constructed for $r^{\#}$ instead of r_{-} produce a function $w^{\#}$ that belongs to

 (c, ∞) .
 $\overset{\text{(i)}}{=}$

duce a

for son

suffices

$$
w^{\#}(x) = 2\widetilde{M}_{-}^{\#}(x,0) = -2F^{\#}(2x) + 2G^{\#}(x,0)
$$

"Left" analogues of the above objects constructed for $r^{\#}$ instead of r_{-} pro-
a function $w^{\#}$ that belongs to $L_1 \cap L_2$ on every half-line $(-\infty, c)$ and equals
 $w^{\#}(x) = 2\widetilde{M}_{-}^{\#}(x, 0) = -2F^{\#}(2x) + 2G^{\#}(x,$ duce a function $w^{\#}$ that belongs to $L_1 \cap L_2$ on every half-line $(-\infty, c)$ and equals $w^{\#}(x) = 2\widetilde{M}_{-}^{\#}(x, 0) = -2F^{\#}(2x) + 2G^{\#}(x, 0)$
for some continuous function $G^{\#}$. The difference $w^{\#} - w$ is continuou $(x) = 2\overline{M}_{-}^{\#}(x,0) = -2F^{\#}(2x) + 2G^{\#}(x,0)$
function $G^{\#}$. The difference $w^{\#} - w$ is con
 $F + F^{\#}$ is a continuous function. Recalli for some continuous function $G^{\#}$. The difference $w^{\#} - w$ is continuous; indeed, it suffices to show that $F + F^{\#}$ is a continuous function. Recalling that $F = \mathcal{F}_{+}r$ suffices to show that $F + F^{\#}$ is a continuous function. Recalling that $F = \mathcal{F}_{+}r$ and $F^{\#} = \mathcal{F}_{-}r^{\#}$ (cf. (2.2)–(2.3)), we get that

$$
-r^{\#} (\text{cf. } (2.2) - (2.3)), \text{ we get that}
$$

\n
$$
F(x) + F^{\#}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (r(\omega) + r^{\#}(-\omega)) e^{i\omega x} d\omega,
$$

\n
$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 - \frac{t(-\omega)}{t(\omega)}\right) r(\omega) e^{i\omega x} d\omega.
$$

\n
$$
\text{erve that } t(-\omega)/t(\omega) \text{ belongs to the Wiener algebra } 1 + \hat{X} \text{ and tends}
$$

\nity, so that $1 - t(-\omega)/t(\omega)$ is in \hat{X} and thus in $L_2(\mathbb{R})$; as a result,
\nthe Fourier transform of the integrable function $(1 - t(-\omega)/t(\omega))r(\omega)$

 $=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(1-\frac{t(-\omega)}{t(\omega)}\right)r(\omega)e^{i\omega x} d\omega.$
 $t(\omega)$ belongs to the Wiener algebra **1**
 $t(-\omega)/t(\omega)$ is in \hat{X} and thus in $L_2(\mathbb{I})$

porm of the integrable function $(1-t(\omega))$ $\begin{align} 2\pi \ \cdot \omega \ \text{and} \ \$ (ω)
 $\sum_{n=1}^{\infty}$ and $\sum_{n=1}^{\infty}$
 $\sum_{n=1}^{\infty}$ to 1 at infinity, so that $1 - t(-\omega)/t(\omega)$ is in \hat{X} and thus in $L_2(\mathbb{R})$; as a result, $F + F^{\#}$ is the Fourier transform of the integrable function $(1 - t(-\omega)/t(\omega))r(\omega)$ and thus is continuous.
The next step in the rec F

to 1 at infinity, so that $1 - t(-\omega)/t(\omega)$ is in X and thus in $L_2(\mathbb{R})$; as a result,
 $F + F^{\#}$ is the Fourier transform of the integrable function $(1 - t(-\omega)/t(\omega))r(\omega)$

and thus is continuous.

The next step in the reco are the right and left extremal Riccati representatives of the potential $q_0 \in \mathcal{Q}$ potential $q := w' + w^2$, i.e., that The next step in the reconstruction algorithm is to show that w and $w^{\#}$
he right and left extremal Riccati representatives of the potential $q_0 \in \mathcal{Q}$
sponding to the data (3.12). We do this by first showing that is the kernel of the transformation operator for the Schrödinger operator H_w with corresponding to the data (3.12). We do this by first showing that $K := M_+ + M_-$
is the kernel of the transformation operator for the Schrödinger operator H_w with
potential $q := w' + w^2$, i.e., that
 $f(x, \omega) := e^{i\omega x} + \int_x^{\infty} K(x$ is the kernel of the transformation operator for the Schrödinger operator H_w with
potential $q := w' + w^2$, i.e., that
 $f(x, \omega) := e^{i\omega x} + \int_x^{\infty} K(x, s)e^{i\omega s} ds$
is the right Jost solution for H_w . Similarly, the kernel $K^{\#} :=$

$$
f(x,\omega) := e^{i\omega x} + \int_x^{\infty} K(x,s)e^{i\omega s} ds
$$

potential $q := w' + w^2$, i.e., that
 $f(x, \omega) :=$

is the right Jost solution for H_w

from the kernels $M_{\pm}^{\#}$ solving the $(x, \omega) := e^{i\omega x} + \int_x K(x, s)e^{i\omega s} ds$

a for H_w . Similarly, the kernel $K^{\#}$

lving the "left" Marchenko-type sy
 $\#$, is the kernel of the transform
 $w^{\#}$ with potential $q^{\#} := (w^{\#})' + (w^{\#})'$ is the right Jost solution for H_w . Similarly, the kernel $K^{\#} := M^{\#}_+ + M^{\#}_-$ formed
from the kernels $M^{\#}_\pm$ solving the "left" Marchenko-type system $(2.14)-(2.15)$ but
with F_- replaced by $F^{\#}$, is the kernel of $+$ $-$ ± with F_{-} replaced by $F^{\#}$, is the kernel of the transformation operator for the
Schrödinger operator $H_{w^{\#}}$ with potential $q^{\#} := (w^{\#})' + (w^{\#})^2$, i.e.,
 $f^{\#}(x,\omega) := e^{-i\omega x} + \int_{-\infty}^{x} K^{\#}(x,s)e^{-i\omega s} ds$
is the left with F_{-} replaced by $F^{\#}$, is the kernel of the transformation operator for the
Schrödinger operator $H_{w^{\#}}$ with potential $q^{\#} := (w^{\#})' + (w^{\#})^2$, i.e.,
 $f^{\#}(x,\omega) := e^{-i\omega x} + \int_{-\infty}^{x} K^{\#}(x,s)e^{-i\omega s} ds$
is the left

Schrödinger operator
$$
H_{w^{\#}}
$$
 with potential $q^{\#} := (w^{\#})' + (w^{\#})^2$, i.e.,
\n
$$
f^{\#}(x,\omega) := e^{-i\omega x} + \int_{-\infty}^{x} K^{\#}(x,s)e^{-i\omega s} ds
$$
\nis the left Jost solution for $H_{w^{\#}}$.
\nIt turns out that the functions f and $f^{\#}$ are related as follows
\n**Lemma 3.9.** The following holds:

is the left Jost solution for $H_{w^{\#}}$.
It turns out that the function
Lemma 3.9. The following holds.
 $f^{\#}(x,\omega) = a(x)$
 $f(x,\omega) = a(x)$ **Lemma 3.9.** The following holds:

$$
f^{\#}(x,\omega) = a(\omega)f(x,-\omega) - b(-\omega)f(x,\omega), \qquad (3.15)
$$

It turns out that the functions
$$
f
$$
 and $f^{\#}$ are related as follows.
\n**na 3.9.** The following holds:
\n
$$
f^{\#}(x,\omega) = a(\omega)f(x,-\omega) - b(-\omega)f(x,\omega), \qquad (3.15)
$$
\n
$$
f(x,\omega) = a(\omega)f^{\#}(x,-\omega) + b(\omega)f^{\#}(x,\omega), \qquad (3.16)
$$

where a and b are constructed from r as explained at the beginning of this subsection.

As a result, we conclude that $f(\cdot, \omega)$ solves the equations $-y'' + qy = \omega^2 y$ and $(x, \omega) = a(\omega) f^{#}(x, -\omega) + b(\omega) f^{#}(x, \omega),$ (3.16)
structed from r as explained at the beginning of this subsec-
notice that $f(\cdot, \omega)$ solves the equations $-y'' + qy = \omega^2 y$ and
ich implies that $q = q^{\#} = q_0$ as distributions in W^{- As a result, we conclude that $f(\cdot, \omega)$ solves the equations $-y'' + qy = \omega^2 y$ and $+ q^{\#}y = \omega^2 y$, which implies that $q = q^{\#} = q_0$ as distributions in $W_{2,loc}^{-1}(\mathbb{R})$.
lows that f and $f^{\#}$ are respectively the right $-y'' + q^{\#}y = \omega^2 y$, which implies that $q = q^{\#} = q_0$ as distributions in $W_{2,\text{loc}}^{-1}$
It follows that f and $f^{\#}$ are respectively the right and left Jost solutions of
Schrödinger operator with Miura potential $q_0 \in \$ It follows that f and $f^{\#}$ are respectively the right and left Jost solutions of the Schrödinger operator with Miura potential $q_0 \in \mathcal{Q}_1$, and now (3.15) and (3.16)
show that r and $r^{\#}$ are respectively its right and left reflection coefficients. This
completes the reconstruction procedure and th Schrödinger operator with Miura potential $q_0 \in \mathcal{Q}_1$, and now (3.15) and (3.16)
show that r and $r^{\#}$ are respectively its right and left reflection coefficients. This
completes the reconstruction procedure and the show that r and $r^{\#}$ are respectively its right and left reflection coefficients. This completes the reconstruction procedure and the proof of Theorem 3.8.
3.5. Sobolev properties of the scattering mappings

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3.5. Sobolev properties of the scattering mappings

One of the motivations for extending the inverse scattering theory is the possibility

to study solvability of som to study solvability of some completely integrable nonlinear partial differential equations with irregular initial data [14, 22, 33, 37]. For instance, the Cauchy
problem for the modified Korteweg-de Vries (mKdV) equation on the line is
 $u_t + u_{xxx} + 6u^2 u_x = 0;$ (3.17)
The existence of global weak solution

$$
u_t + u_{xxx} + 6u^2 u_x = 0;
$$

\n
$$
u(x; 0) = u_0(x).
$$
\n(3.17)

problem for the modified Korteweg-de Vries (mKdV) equation on the line is
 $u_t + u_{xxx} + 6u^2 u_x = 0;$ (3.17)
 $u(x;0) = u_0(x).$

The existence of global weak solutions for initial data in $L_2(\mathbb{R})$ was proven by

Kato [51] and, $u_t + u_{xxx} + 6u^2u_x = 0;$
 $u(x; 0) = u_0(x).$

The existence of global weak solutions for initial data in $L_2(\mathbb{R})$ was prove

Kato [51] and, independently, by Kruzhkov and Faminskii [55]; see also [17

[41]. Kato's result give $(x; 0) = u_0(x)$.

for initial data in $L_2(\mathbb{R})$ was proven by

kov and Faminskii [55]; see also [17] and

uniform continuity of the solution in the

Ponce, and Vega [52] showed local well-

r space $H^s(R)$ if $s \geq \frac{1}{4}$ The existence of global weak solutions for initial data in $L_2(\mathbb{R})$ was proven by Kato [51] and, independently, by Kruzhkov and Faminskii [55]; see also [17] and [41]. Kato's result gives existence but no uniform conti [41]. Kato's result gives existence but no uniform continuity of the solution in the initial data. On the other hand, Kenig, Ponce, and Vega [52] showed local well-posedness for initial data in the Sobolev space $H^s(R)$ i initial data. On the other hand, Kenig, Ponce, and Vega [52] showed local well-
posedness for initial data in the Sobolev space $H^s(R)$ if $s \geq \frac{1}{4}$, while Colliander,
Keel, Staffilani, Takaoka, and Tao [23] proved gl

posedness for initial data in the Sobolev space $H^{\circ}(R)$ if $s \geq \frac{1}{4}$, while Colliander,
Keel, Staffilani, Takaoka, and Tao [23] proved global well-posedness for initial data
in $H^{s}(R)$ for $s > \frac{1}{4}$.
The classica posedness for initial data in the Sobolev space $H^s(R)$ if $s \geq \frac{1}{4}$
Keel, Staffilani, Takaoka, and Tao [23] proved global well-posed
in $H^s(R)$ for $s > \frac{1}{4}$.
The classical inverse scattering method for mKdV on the in $H^s(R)$ for $s > \frac{1}{4}$.
The classical inverse scattering method for mKdV on the line (see Wadati [71]
and Tanaka [70], Beals and Coifman [11–13] and in a more general setting the
monograph of Beals, Deift and Tomei [1 in $H^s(R)$ for $s > \frac{1}{4}$
The classical i
and Tanaka [70], I
monograph of Bea
the inverse scatter
The crucial observa
R. Miura [37] in 19 r 3 ks i: ")" Tanaka [70], Beals and Coifman [11–13] and in a more general setting the graph of Beals, Deift and Tomei [14]) constructs a classical solution using nverse scattering transform for initial data u_0 in the Schwartz class monograph of Beals, Deift and Tomei [14]) constructs a classical solution using
the inverse scattering transform for initial data u_0 in the Schwartz class $S(\mathbb{R})$.
The crucial observation (made originally by C. Gardn the inverse scattering transform for initial data u_0 in the Schwartz class $S(\mathbb{R})$. The crucial observation (made originally by C. Gardner, J. Green, M. Kruscal and R. Miura [37] in 1974 for the KdV equation is that if the function $u(x, t)$ solves

$$
H(t) := -\left(\frac{\mathrm{d}}{\mathrm{d}x} + u(x,t)\right) \left(\frac{\mathrm{d}}{\mathrm{d}x} - u(x,t)\right),\,
$$

then the reflection coefficient $r(t)$ for $H(t)$ evolves in a straightforward manner the mKdV equation (3.17), and $H(t)$ is the family of Schrödinger operators in the
factorized form (1.2) with Riccati representatives $u(x,t)$,
 $H(t) := -\left(\frac{d}{dx} + u(x,t)\right) \left(\frac{d}{dx} - u(x,t)\right)$,
then the reflection coefficient $r(t)$ f factorized form (1.2) with Riccati representatives $u(x, t)$,
 $H(t) := -\left(\frac{d}{dx} + u(x, t)\right)\left(\frac{d}{dx} - u(x, t)\right)$

then the reflection coefficient $r(t)$ for $H(t)$ evolves in a

in time, namely, $r(t, \omega) = e^{8it\omega^3} r(0, \omega)$. Thus given t $(t) := -\left(\frac{d}{dx} + u(x, t)\right)\left(\frac{d}{dx} - u(x, t)\right)$
efficient $r(t)$ for $H(t)$ evolves in a st
 $= e^{8it\omega^3} r(0, \omega)$. Thus given the initiatering data for the operator $H(0)$
cconstructs the potential $u(x, t)$ of t
ation.
e to apply this $\frac{1}{d}$
 $\frac{1}{d}$
 $\frac{1}{d}$ $\frac{\mathrm{d}x}{r(0)}$
for $\frac{\mathrm{d}x}{\mathrm{d}x}$
for $\frac{\mathrm{d}x}{\mathrm{d}x}$ $\frac{1}{16}$
 $\frac{1}{16}$ ve $\frac{1}{16}$ then the reflection coefficient $r(t)$ for $H(t)$ evolves in a straightforward manner
in time, namely, $r(t, \omega) = e^{8it\omega^3} r(0, \omega)$. Thus given the initial condition $u(x, 0) =$
 $u_0(x)$, one finds the scattering data for the ope in time, namely, $r(t, \omega) = e^{8it\omega^3}r$
 $u_0(x)$, one finds the scattering d

evolution, and then reconstructs

solving the mKdV equation.

The only obstacle to apply

that the mKdV flow does not p

to find smaller classes o $(0, \omega)$. Thus given the initial condition $u(x, 0)$ = ata for the operator $H(0)$, calculates their time ithe potential $u(x, t)$ of the operator $H(t)$ thus this method to initial data of low regularity is reserve the inclu u_0

(*x*), one finds the scattering data for the operator $H(0)$, calculates their time
olution, and then reconstructs the potential $u(x,t)$ of the operator $H(t)$ thus
lving the mKdV equation.
The only obstacle to apply this m evolution, and then reconstructs the potential $u(x, t)$ of the operator $H(t)$ thus
solving the mKdV equation.
The only obstacle to apply this method to initial data of low regularity is
that the mKdV flow does not preserve The only obstacle to a
that the mKdV flow does n
to find smaller classes of in
reflection coefficients remain
nection with using the inver
the defocussing nonlinear Sc the mKdV flow does not preserve the inclusion $r \in \mathcal{R}$. Therefore we need
id smaller classes of initial data u_0 , for which the corresponding classes of
tion coefficients remain invariant. Similarly, the same questio to find smaller classes of initial data u_0 , for which the corresponding classes of reflection coefficients remain invariant. Similarly, the same questions arise in connection with using the inverse scattering for the Z to find smaller classes of initial data u_0 , for which the corresponding classes of reflection coefficients remain invariant. Similarly, the same questions arise in connection with using the inverse scattering for the Z

nection with using the inverse scattering for the ZS-AKNS systems [36] to solve
the defocussing nonlinear Schrödinger equation. In other words, the problem is to
study the properties of the scattering maps for various spa study the properties of the scattering maps for various spaces of u.
Fourier-type properties of the map $q \mapsto r$ have been studied by many authors,
including Cohen [21], Deift and Trubowitz [27], Faddeev [32], and Zhou [73 Fourier-type properties of the map $q \mapsto r$ have been studied by many authors,
ding Cohen [21], Deift and Trubowitz [27], Faddeev [32], and Zhou [73]. These
prs imposed weighted L_1 assumptions on q and obtained regula authors imposed weighted L_1 assumptions on q and obtained regularity results for r in terms of L_{∞} -norms of r and its derivatives. Kappeler and Trubowitz [49, 50] studied Sobolev space mapping properties of r in terms of L_{∞} -norms of r and its derivatives. Kappeler and Trubowitz [49, 50] in terms of L_{∞} -norms of r and its derivatives. Kappeler and Trubowitz [49, 50]
udied Sobolev space mapping properties of the scattering map and observed that,
milarly to the Fourier transform, whenever q is inte similarly to the Fourier transform, whenever q is integrable with the weight $\langle x \rangle^k :=$
similarly to the Fourier transform, whenever q is integrable with the weight $\langle x \rangle^k :=$

 $k \geq 3$, then the reflection coefficient r is $k-1$ times differentiable,
hness of q is reflected in the weight integrability of r . They extend
to potentials with finitely many bound states in [50] and also prove
n while smoothness of q is reflected in the weight integrability of r . They extend
their results to potentials with finitely many bound states in [50] and also prove
analyticity and investigate the differential of the s ²)^{$k/2$}, $k \ge 3$, then the reflection coefficient *r* is $k - 1$ times differentiable, moothness of *q* is reflected in the weight integrability of *r*. They extend sults to potentials with finitely many bound states i their results to potentials with finitely many bound states in [50] and also prove analyticity and investigate the differential of the scattering map. Deift and Zhou

 $S: u \mapsto r$

is locally invertible bi-Lipschitz map between real space $L_2(\mathbb{R}, \langle x \rangle^s dx)$ and special equation with the initial data in weighted Sobolev spaces.

In our paper [45] we showed that, for any $s > \frac{1}{2}$, the mapping
 $S: u \mapsto r$

is locally invertible bi-Lipschitz map between real space $L_2(\mathbb{R}, \langle x \rangle^s dx)$ and In our paper [45] we showed that, for any $s > \frac{1}{2}$, the
 $S: u \mapsto r$

is locally invertible bi-Lipschitz map between real space L_2

subspace of $W_2^s(\mathbb{R})$. However, the space $W_2^s(\mathbb{R})$ is not inva

induced flow In our paper [45] we showed that, for any $s > \frac{1}{2}$
 $S: u \mapsto r$

ally invertible bi-Lipschitz map between real spa

pace of $W_2^s(\mathbb{R})$. However, the space $W_2^s(\mathbb{R})$ is not

eed flow of r if $s > \frac{1}{2}$, whence thi $\begin{align} \text{ce } L_2(\mathbb{R}, \langle x \rangle^s \text{a} \text{is invariant and} \ \text{eredy applicable} \ \text{in} \ \text{in}$: $u \mapsto r$
etween
ee $W_2^s(\mathbb{R}^n)$
result is
, to studing space is locally invertible bi-Lipschitz map between real space $L_2(\mathbb{R}, \langle x \rangle^s dx)$ and special
subspace of $W_2^s(\mathbb{R})$. However, the space $W_2^s(\mathbb{R})$ is not invariant under the mKdV-
induced flow of r if $s > \frac{1}{2}$, whe subspace of $W_2^s(\mathbb{R})$. However, the space W_2^s
induced flow of r if $s > \frac{1}{2}$, whence this result
mKdV. It might be possible, however, to st
 $W_2^{2s}(\mathbb{R}, \langle x \rangle^s dx)$ since the corresponding sp
coefficients is pre (ℝ) is not invariant under the mKdV-
is not directly applicable to solving the
cudy the mKdV equation in the space
pace $W_2^s(\mathbb{R}, \langle x \rangle^{2s} dx)$ of the reflection
low. This and other related questions induced flow of r if $s > \frac{1}{2}$
mKdV. It might be pos
 $W_2^{2s}(\mathbb{R}, \langle x \rangle^s dx)$ since the
coefficients is preserved
will be discussed elsewh sible, however, to study the mKdV equation in the space
are corresponding space $W_2^s(\mathbb{R}, \langle x \rangle^{2s} dx)$ of the reflection
under the mKdV flow. This and other related questions
are [18]. $W_2^{2s}(\mathbb{R}, \langle x \rangle^s dx)$ since the corresponding space $W_2^s(\mathbb{R}, \langle x \rangle^{2s} dx)$ of the reflection coefficients is preserved under the mKdV flow. This and other related questions will be discussed elsewhere [18].
4. The ca coefficients is preserved under the mKdV flow. This and other related questions
will be discussed elsewhere [18].
4. The case of discontinuous impedance function
In this section, we show that the case of discontinuous im

4. The case of discontinuous impedance function

4. The case of discontinuo
In this section, we show that the
to Schrödinger operators whose to Schrödinger operators whose scattering matrices possess completely different
properties than those observed in the previous section. The corresponding func-
tions $u = (\log p)'$ now contain Dirac δ -functions and hence are properties than those observed in the previous section. The corresponding func-
tions $u = (\log p)'$ now contain Dirac δ -functions and hence are not summable.
Moreover, a common sense suggests that the scattering transforms tions $u = (\log p)'$ now contain Dirac δ -functions and hence are not summable.
Moreover, a common sense suggests that the scattering transforms S_{\pm} act approx-
imately as the Fourier transforms and thus the reflection co Moreover, a common sense suggests that the scattering transforms σ_{\pm} act approximately as the Fourier transforms and thus the reflection coefficients r_{\pm} should have properties typical to those of the Fourier tra have properties typical to those of the Fourier transform of u . We shall show that, indeed, r_{\pm} contain (almost-) periodic components and, as a result, do not tend to zero at infinity, contrary to what was observed

indeed, r_{\pm} contain (almost-) periodic components and, as a result, do not tend to zero at infinity, contrary to what was observed for singular Miura potentials in the previous section.
Inverse scattering for Schrödin previous section.
Inverse scattering for Schrödinger operators with discontinuous impedances
were considered before (see, e.g., [3–5, 19, 66, 67], but only piece-wise smooth
impedances with a finite number of discontinuit Inverse scat
were considered
impedances with
called layer-strip
the problem was
dard methods, w considered before (see, e.g., [3–5, 19, 66, 67], but only piece-wise smooth
dances with a finite number of discontinuities were allowed. Also, the so-
I layer-stripping method was used, i.e., on every interval of continui impedances with a finite number of discontinuities were allowed. Also, the so-
called layer-stripping method was used, i.e., on every interval of continuity of p ,
the problem was transformed to the potential form and th called layer-stripping method was used, i.e., on every interval of continuity of p , the problem was transformed to the potential form and then solved by the stan-
dard methods, while the discontinuity in the impedance w dard methods, while the discontinuity in the impedance was determined form the asymptotics of the scattering data. One then had to recalculate the scattering data for the next interval of continuity, and then repeat the p for the next interval of continuity, and then repeat the process until all discontinuities have been treated. Unfortunately, no method has been suggested that would automatically determine p along with all its jumps in

The shave been treated. Unfortunately, no method has been suggested that would
automatically determine p along with all its jumps in a generic situation.
To better expose the main effects encountered in this problem, we centrate on the model example of piece-wise constant impedance function p having
jumps at points of a regular lattice, taken to be $\mathbb Z$ without loss of generality. We
shall comment in Subsection 4.8 on possible extens jumps at points of a regular lattice, taken to be $\mathbb Z$ without loss of generality. We shall comment in Subsection 4.8 on possible extensions.

4.1. The operators

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 4.1. The operators

Throughout this section, the function p is assumed constant on the intervals $\Delta_j := (j, j + 1)$ and hence takes the form

$$
p(s) = \exp\left\{\sum_{j\;:\;j>s} u_j\right\}
$$

Throughout this section, the function p is assumed constant on the intervals $\Delta_j :=$
 $(j, j + 1)$ and hence takes the form
 $p(s) = \exp\left\{\sum_{j : j > s} u_j\right\}$

for some real numbers u_j . Since in applications only bounded and uni $(j, j + 1)$ and hence takes the form
 $p(s)$
for some real numbers u_j . Since in
tive impedances are of interest, it is
belongs to $\ell_1(\mathbb{Z})$. Equation (1.9) is $(s) = \exp\Big{\sum_{j \,:\, j > s} \,:\, \begin{equation} \text{in application} \text{at } s \text{ is natural to} \end{equation}}$
it is natural to
 $H = H_u \text{ gener.}$ for some real numbers u_j . Since in applications only bounded and uniformly positive impedances are of interest, it is natural to assume that the sequence $u := (u_j)$ belongs to $\ell_1(\mathbb{Z})$. Equation (1.9) is the spectral tive impedances are of interest, it is natural to assume that the sequence $\mathbf{u} := (u_j)$
belongs to $\ell_1(\mathbb{Z})$. Equation (1.9) is the spectral problem $Hy = \omega^2 y$ for the corre-
sponding Schrödinger operator $H = H_u$ genera

$$
I := -\frac{1}{p(s)} \frac{d}{ds} p^2(s) \frac{d}{ds} \frac{1}{p(s)}.
$$

belongs to $\ell_1(\mathbb{Z})$. Equation (1.9) is the spectral problem $Hy = \omega^2 y$ for the corre-
sponding Schrödinger operator $H = H_u$ generated by the differential expression
 $\mathfrak{l} := -\frac{1}{p(s)} \frac{d}{ds} p^2(s) \frac{d}{ds} \frac{1}{p(s)}$.
The dif sponding Schrödinger operator $H = H_u$ generated by the differential expression
 $\mathfrak{l} := -\frac{1}{p(s)} \frac{d}{ds} p^2(s) \frac{d}{ds} \frac{1}{p(s)}$.

The differential expression \mathfrak{l} acts as $\mathfrak{l} y := -y''$ on its domain consisting of functio $:= -\frac{1}{p(s)} \frac{d}{ds} p^2(s) \frac{d}{ds}$

acts as $[y := -y'$ on

' are locally absolut

tions
 $e^{u_j} y(j+) = y(j)$ $\mathfrak{r}(s)$
 \mathfrak{r}_y oral
 \mathfrak{r}_y or y' ($\frac{1}{3}$
 $\frac{1}{3}$
 $\frac{1}{3}$
 $\frac{1}{3}$ $\begin{align} \n\begin{cases}\ns & \text{if } s \leq 0 \\
y & \text{if } s \leq 0 \\
y & \text{if } s \leq 0\n\end{cases} \n\end{align}$ The differential expression *l* acts as $\lceil y := -y'' \rceil$ on its domain consisting of functions *y* such that both y/p and py' are locally absolutely continuous, i.e., of functions *y* satisfying the interface conditions
 \overline{y} such that both y/p and py' are locally absolutely continuous, i.e., of functions $e^{u_j}y(j+) = y(j-),$
 $e^{-u_j}y'(j+) = y'(j-)$ (IF_j)

every lattice point $s = j$. The operator H is the realization of $\mathfrak h$ in $L_2(\mathbb{R})$, i.e.,

$$
e^{u_j} y(j+) = y(j-),
$$

\n
$$
e^{-u_j} y'(j+) = y'(j-).
$$
\n(IF_j)

 $e^{u_j}y(j+) = y(j-),$
 $e^{-u_j}y'(j+) = y'(j-)$

at every lattice point $s = j$. The operator *H* is the
 $Hy = -y''$ on the domain

dom $H = \{y \in W_2^2(\mathbb{R} \setminus \mathbb{Z}) \mid \forall j \in \mathbb{Z},\}$ $e^{-u_j} y'(j+) = y'(j-)$ (IF j)

he operator H is the realization of h in $L_2(\mathbb{R})$, i.e.,
 $W_2^2(\mathbb{R} \setminus \mathbb{Z}) \mid \forall j \in \mathbb{Z}$, (IF j) holds}.

is operator H is symmetric; in fact [53], it is also $Hy = -y''$ on the domain

dom
$$
H = \{ y \in W_2^2(\mathbb{R} \setminus \mathbb{Z}) \mid \forall j \in \mathbb{Z}, \quad (\text{IF}_j) \text{ holds} \}.
$$

at every lattice point $s = j$. The operator H is the realization of \mathfrak{h} in $L_2(\mathbb{R})$, i.e., $Hy = -y''$ on the domain
dom $H = \{y \in W_2^2(\mathbb{R} \setminus \mathbb{Z}) \mid \forall j \in \mathbb{Z}, \text{ (IF}_j\}$ holds}.
It is immediate to see that the op $=-y''$ on the domain
dom $H = \{$
i mmediate to see th
adjoint on the above dom $H = \{y \in W_2^2(\mathbb{R} \setminus \mathbb{Z}) \mid \forall j \in \mathbb{Z}, \text{ (IF}_j) \text{ holds} \}.$

e to see that the operator H is symmetric; in fact

the above domain.
 ions and the scattering data

solution $f_+(\cdot, \omega)$, if exists, must be of the form It is immediate to see that the operator *H* is symmetric; in fact [53], it is also
self-adjoint on the above domain.
4.2. Jost solutions and the scattering data
The right Jost solution $f_+(\cdot,\omega)$, if exists, must be of

4.2. Jost solutions and the scattering data

4.2. Jost solutions and the scatter-
The right Jost solution $f_+(\cdot,\omega)$,
 $f_+(s,\omega)$
on each interval Δ_i . The interface

$$
f_{+}(s,\omega) = a_{j}e^{i\omega s} + b_{j}e^{-i\omega s}
$$

The right Jost solution
$$
f_{+}(\cdot, \omega)
$$
, if exists, must be of the form
\n
$$
f_{+}(s, \omega) = a_{j}e^{i\omega s} + b_{j}e^{-i\omega s}
$$
\non each interval Δ_{j} . The interface conditions at the point $x = j$ force the relation\n
$$
\begin{pmatrix} a_{j-1} \\ b_{j-1} \end{pmatrix} = M(j, \omega) \begin{pmatrix} a_{j} \\ b_{j} \end{pmatrix}
$$
\nwith\n
$$
M(j, \omega) := \begin{pmatrix} \cosh u_{j} & e^{-2i\omega j} \sinh u_{j} \\ e^{2i\omega j} \sinh u_{j} & \cosh u_{j} \end{pmatrix}.
$$
\nWe observe that the matrix $M(j, \omega)$ for all $\omega \in \mathbb{R}$ belongs to the group $SU(1, 1)$.
\nDenoting by |A| the norm of a matrix A and by $I_{2} := \text{diag}(1, 1)$ the unity matrix

$$
M(j,\omega) := \begin{pmatrix} \cosh u_j & e^{-2\mathrm{i}\omega j}\sinh u_j \\ e^{2\mathrm{i}\omega j}\sinh u_j & \cosh u_j \end{pmatrix}.
$$

 $M(j, \omega) := \begin{pmatrix} \cosh u_j & e^{-2i\omega j} \sinh u_j \e^{\cosh u_j} & \cosh u_j \end{pmatrix}$

We observe that the matrix $M(j, \omega)$ for all $\omega \in \mathbb{R}$ belongs

Denoting by |A| the norm of a matrix A and by $I_2 := \text{diag}$

in \mathbb{C}^2 , we see that the inclusi We observe that the matrix $M(j, \omega)$ for all $\omega \in \mathbb{R}$ belongs to the group $SU(1, 1)$.
Denoting by |A| the norm of a matrix A and by $I_2 := diag(1, 1)$ the unity matrix
in \mathbb{C}^2 , we see that the inclusion $(\mu_j) \in \ell_1(\mathbb$ Denoting by |A| the norm of a matrix A and by $I_2 := diag(1, 1)$ the unity matrix
in \mathbb{C}^2 , we see that the inclusion $(\mu_j) \in \ell_1(\mathbb{Z})$ yields the inequality
 $\sum_{j \in \mathbb{Z}} |M(j, \omega) - I_2| < \infty$, in \mathbb{C}^2 , we see that the inclusion $(\mu_i) \in \ell_1(\mathbb{Z})$ yields the inequality

$$
\sum_{j\in\mathbb{Z}}|M(j,\omega)-I_2|<\infty,
$$

whence for every
$$
k \in \mathbb{Z}
$$
 the product
\n
$$
M_k(\omega) := \prod_{j>k}^{\longrightarrow} M(j, \omega) := \lim_{m \to \infty} M(k+1, \omega) \cdots M(m, \omega)
$$
\nconverges to an element of $SU(1, 1)$. Also, by the same reason, the limit\n
$$
M(\omega) := \lim_{n \to -\infty} M_n(\omega)
$$
\nexists and belongs to $SU(1, 1)$.

$$
M(\omega) := \lim_{n \to -\infty} M_n(\omega)
$$

converges to an element of
$$
SU(1,1)
$$
. Also, by the same reason, the limit
\n
$$
M(\omega) := \lim_{n \to -\infty} M_n(\omega)
$$
\nexists and belongs to $SU(1,1)$.
\nSet now\n
$$
\begin{pmatrix} a_n(\omega) \\ b_n(\omega) \end{pmatrix} := M_n(\omega) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} a(\omega) \\ b(\omega) \end{pmatrix} := M(\omega) \begin{pmatrix} 1 \\ 0 \end{pmatrix};
$$
\nthen for every fixed $\omega \in \mathbb{R}$, we have that $a_n(\omega) \to a(\omega)$ and $b_n(\omega)$.

exists and belongs to $SU(1, 1)$.

Set now
 $\begin{pmatrix} a_n(\omega) \\ b_n(\omega) \end{pmatrix} := M_n$

then, for every fixed $\omega \in \mathbb{R}$, v
 $n \to -\infty$. Also, $M_n(\omega) \to I_2$ as for ever
 $-\infty$. Also
thus the (ω)
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an (ω)
ed (ω)
 $\frac{1}{2}$
al al
ats $:= M_n(\omega) \begin{pmatrix} 0 \end{pmatrix}, \qquad \begin{pmatrix} \overline{u}(1) \\ b(\omega) \end{pmatrix} := M(\omega)$
 $\in \mathbb{R}$, we have that $a_n(\omega) \to a(\omega)$ and $b_n(\omega)$ indeed determine the right Jo
 $b_n(\omega)$ indeed determine the right Jo $\begin{bmatrix} 0 \\ v \\ e \\ 0 \end{bmatrix}$ $\begin{array}{c} \n\text{0} \ \text{0} \ \$ (ω)
 (ω)
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 ω
 $(\omega$
 ω (ω)
 (ω)
 $n(\omega)$

is us $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $(\epsilon$
e r ; then, for every fixed $\omega \in \mathbb{R}$, we have that $a_n(\omega) \to a(\omega)$ and $b_n(\omega) \to b(\omega)$ as $n \to -\infty$. Also, $M_n(\omega) \to I_2$ as $n \to \infty$ yields $a_n(\omega) \to 1$ and $b_n(\omega) \to 0$ as $n \to \infty$ and thus the $a_n(\omega)$ and $b_n(\omega)$ indeed determin $n \to -\infty$

. Also, $M_n(\omega) \to I_2$ as $n \to \infty$ yields $a_n(\omega) \to 1$ and $b_n(\omega) \to 0$ as $n \to \infty$
the $a_n(\omega)$ and $b_n(\omega)$ indeed determine the right Jost solution $f_+(\cdot, \omega)$
coefficients a_n and b_n possess some useful properties, which and thus the $a_n(\omega)$ and $b_n(\omega)$ indeed determine the right Jost solution $f_+(\cdot, \omega)$
via (1.5).
The coefficients a_n and b_n possess some useful properties, which we now
discuss. Assume first that a sequence $\mathbf{u} = (u_n$ The coefficients a_n and b_n possess some useful properties, which we now
discuss. Assume first that a sequence $\mathbf{u} = (u_n)$ is such that, for some $m \in \mathbb{N}$, we
have $u_n = 0$ if $|n| > m$. Then one can prove that, for discuss. Assume first that a sequence $u = (u_n)$ is such that, for some $m \in \mathbb{N}$, we
have $u_n = 0$ if $|n| > m$. Then one can prove that, for every $n \in \mathbb{Z}$, the functions
 a_n and $\omega \mapsto e^{-2i\omega(n+1)}b_n(\omega)$ belong to the Ha have $u_n = 0$ if $|n| > m$. Then one can prove that, for every $n \in \mathbb{Z}$, the functions a_n and $\omega \mapsto e^{-2i\omega(n+1)}b_n(\omega)$ belong to the Hardy space H^+ of functions that are bounded and analytic in the upper-half complex p a_n are bounded and analytic in the upper-half complex plane \mathbb{C}_+ . Moreover, for the Fourier series expansions of a_n and b_n ,

are bounded and analytic in the upper-half complex plane
$$
\mathbb{C}_+
$$
. Moreover, for the Fourier series expansions of a_n and b_n ,
\n
$$
a_n(\omega) = \sum_{m\geq 0} \hat{a}_n(m) e^{i\omega m}, \qquad b_n(\omega) = \sum_{m\geq 2(n+1)} \hat{b}_n(m) e^{i\omega m} \qquad (4.2)
$$
\nthe interface condition (4.1) yields the relations
\n
$$
\hat{a}_n(0) = \cosh \mu_{n+1} \hat{a}_{n+1}(0), \qquad \hat{b}_n(2n+2) = \sinh \mu_{n+1} \hat{a}_{n+1}(0). \qquad (4.3)
$$
\nPassing to the limit as $m \to \infty$ shows that the above properties hold for all $\mathbf{u} \in \ell_1$.
\nIt follows that for every fixed $\mathbf{u} \in \ell_2$ and $s = n + u$ with $n \in \mathbb{Z}$ and $u \in (0, 1)$, the

$$
\hat{a}_n(0) = \cosh \mu_{n+1} \hat{a}_{n+1}(0), \qquad \hat{b}_n(2n+2) = \sinh \mu_{n+1} \hat{a}_{n+1}(0). \tag{4.3}
$$

 $\hat{a}_n(0) = \cosh \mu_{n+1} \hat{a}_{n+1}(0), \qquad \hat{b}_n(2n+2)$

Passing to the limit as $m \to \infty$ shows that the all

It follows that, for every fixed $u \in \ell_1$ and $s = n$

function of ω
 $e^{-i\omega s} f_+(s,\omega) = a_n(\omega) + \epsilon$ (0) = cosh $\mu_{n+1}\hat{a}_{n+1}(0)$, $b_n(2n + 2) = \sinh \mu_{n+1}\hat{a}_{n+1}(0)$. (4.3)

b the limit as $m \to \infty$ shows that the above properties hold for all $u \in \ell_1$,

that, for every fixed $u \in \ell_1$ and $s = n + y$ with $n \in \mathbb{Z}$ and y Passing to the limit as $m \to \infty$ shows that the above properties hold for all $u \in \ell_1$.
It follows that, for every fixed $u \in \ell_1$ and $s = n + y$ with $n \in \mathbb{Z}$ and $y \in (0, 1)$, the function of ω
 $e^{-i\omega s} f_+(s, \omega) = a_n(\omega)$ It follows that, for every fixed $u \in \ell_1$ and $s = n + y$ with $n \in \mathbb{Z}$ and $y \in (0, 1)$, the
function of ω
 $e^{-i\omega s} f_+(s, \omega) = a_n(\omega) + e^{-2i\omega s} b_n(\omega)$
belongs to the Hardy space H^+ . In a similar manner, one concludes that

$$
e^{-i\omega s} f_{+}(s,\omega) = a_{n}(\omega) + e^{-2i\omega s} b_{n}(\omega)
$$

$$
e^{i\omega s}f_{-}(s,\omega)\in H^{+}
$$
\n(4.4)

function of ω
belongs to th
for every $x \in$
half plane \mathbb{C}^2 $e^{-i\omega s} f_{+}(s, \omega) = a_{n}(\omega) + e^{-2i\omega s} b_{n}(\omega)$
space H^{+} . In a similar manner, one $e^{i\omega s} f_{-}(s, \omega) \in H^{+}$
ve also that none of a and a_{n} has zerwise the corresponding impedance
ve eigenvalues, which is impossible. belongs to the Hardy space H^+ . In a similar manner, one concludes that
 $e^{i\omega s} f_-(s,\omega) \in H^+$

for every $x \in \mathbb{R}$. Observe also that none of a and a_n has zeros in the ope

half plane \mathbb{C}^+ as otherwise the co for every $x \in \mathbb{R}$. Observe also that none of a and a_n has zeros in the open upper-
half plane \mathbb{C}^+ as otherwise the corresponding impedance Schrödinger operators
would have non-positive eigenvalues, which is for every $x \in \mathbb{R}$. Observe also that none of a and a_n has zeros in the open upper-
half plane \mathbb{C}^+ as otherwise the corresponding impedance Schrödinger operators
would have non-positive eigenvalues, which is

half plane \mathbb{C}^+ as otherwise the corresponding impedance Schrödinger operators
would have non-positive eigenvalues, which is impossible.
We next recall the following
Definition 4.1. The Wiener algebra W is a compl We next recall the following
Definition 4.1. The Wiener algebra W is a complex Bandic continuous functions h with absolutely convergent Founder the point-wise multiplication and the norm $||h||_W$: **ition 4.1.** The *Wiener algeb* notinuous functions h with a r the point-wise multiplication **Definition 4.1.** The *Wiener algebra W* is a complex Banach algebra of 2π -perio-The *Wiener algebra W* is a complex Banach algebra of 2π -perio-
functions *h* with absolutely convergent Fourier series $\sum_{n\in\mathbb{Z}}h_n e^{inx}$ -wise multiplication and the norm $||h||_W := \sum_{n\in\mathbb{Z}}|h_n|$. dic continuous functions *h* with absolutely convergent Fourier series \sum
under the point-wise multiplication and the norm $||h||_W := \sum_{n \in \mathbb{Z}} |h_n|$. $n\in\mathbb{Z}$ h_n e^{inx} under the point-wise multiplication and the norm $\|h\|_W:=\sum_{n\in\mathbb{Z}}|h_n|.$

30 S. The above considerations might be summarized as follows:
 Proposition 4.2. The functions a_n , b_n , a , and b are π -periodic e

Wiener algebra W, depend therein continuously on $u \in \ell_{1,\mathbb{R}}$ and s

mates **osition 4.2.** The functions a_n , b_n , a , and b are π -perioding eralgebra W , depend therein continuously on $u \in \ell_{1,\mathbb{R}}$ and s
 $\|a_n\|_W + \|b_n\|_W, \|a\|_W + \|b\|_W \le \exp{\{\|u\|\}}.$ **Proposition 4.2.** The functions a_n , b_n , a , and b are π -periodic elements of the Wiener algebra W, depend therein continuously on $u \in \ell_{1,\mathbb{R}}$ and satisfy the estimates

$$
||a_n||_W + ||b_n||_W, ||a||_W + ||b||_W \le \exp{\{||u||\}}.
$$

The Wiener algebra W has a nice property that the spectrum of an element The Wiener algebra *W* has a nice property that the spectrum of an element W is the range Ran $h := \{h(x) \mid x \in \mathbb{R}\}$; as a result, h is invertible in W ever h does not vanish on \mathbb{R} . Now, the inclusion $M(\omega) \in$ $h \in W$ whenever h does not vanish on \mathbb{R} . Now, the inclusion $M(\omega) \in SU(1, 1)$ yields the
relation
 $|a(\omega)|^2 - |b(\omega)|^2 = 1$
for real ω ; therefore a does not vanish on \mathbb{R} and thus is invertible in W. It follows

$$
|a(\omega)|^2 - |b(\omega)|^2 = 1
$$

whenever *h* does not vanish on ℝ. Now, the inclusion $M(\omega) \in SU(1, 1)$ yields the relation
 $|a(\omega)|^2 - |b(\omega)|^2 = 1$
for real ω ; therefore *a* does not vanish on ℝ and thus is invertible in *W*. It follows
that the reflectio for real
that the
belong t $(\omega)|^2 - |b(\omega)|^2 = 1$

ranish on ℝ and thi

r₋(ω) = $b(\omega)/a(\omega)$

() = r(x), r(-ω) for real ω ; therefore *a* does not vanish on ℝ and thus is invertible in *W*. It follows
that the reflection coefficients $r_{-}(\omega) = b(\omega)/a(\omega)$ and $r_{+}(\omega) = -b(-\omega)/a(\omega)$
belong to *W* as well. Set
 $\mathcal{R} := \{r \in W \mid r(x + \pi) = r(x$ that the reflection coefficients $r_-(\omega) = b(\omega)/a(\omega)$ and $r_+(\omega) = -b(-\omega)/a(\omega)$
belong to W as well. Set
 $\mathscr{R} := \{r \in W \mid r(x + \pi) = r(x), \quad r(-\omega) = \overline{r(\omega)}, \quad ||r||_{\infty} < 1\};$ (4.5)
then the above reasoning establish the direct part of the

$$
\mathcal{R} := \{ r \in W \mid r(x + \pi) = r(x), \quad r(-\omega) = \overline{r(\omega)}, \quad ||r||_{\infty} < 1 \};\tag{4.5}
$$

belong to W as well. Set
 $\mathscr{R} := \{r \in W \mid$

then the above reasoning
 Corollary 4.3. For every $:= \{ r \in W \mid r(x + π) = r(x), \quad r(-ω) = r(ω), \quad ||r||_{∞} < 1 \};$ (4.5)

ove reasoning establish the direct part of the inverse problem, namely:

3. For every real-valued sequence $u \in \ell_1$ the reflection coefficients r_

ee corresponding S **Corollary 4.3.** For every real-valued sequence $u \in \ell_1$ the reflection coefficients $r_$ and r_+ of the corresponding Schrödinger operator H_u belong to $\mathscr R$.

The next step is to show that, firstly, the reflection coefficients determine uniquely the corresponding scattering coefficients a and b , and, in fact, that every uniquely the corresponding scattering coefficients a and b, and, in fact, that every $r \in \mathcal{R}$ generates some a and b that have properties the genuine scattering coefficients do. Secondly, a continuous involution $\mathcal{I$ $r \in \mathcal{R}$ generates some a and b that have properties the genuine scattering coefgenerates some *a* and *b* that have properties the genuine scattering coef-
do. Secondly, a continuous involution *I* on $\mathscr R$ exists such that the right
is reflection coefficients for every Schrödinger operator H_u wi and left reflection coefficients for every Schrödinger operator H_u with $u \in \ell_{1,\mathbb{R}}$ are
related via $r_- = Tr_+$.
Lemma 4.4.
(i) Every $r \in \mathcal{R}$ admits a unique representation in the form $r = b/a$ with $a, b \in$

Lemma 4.4.

- and left reflection coefficients for every Schrödinger operator H_u with $u \in \ell_{1,\mathbb{R}}$ are

related via $r_- = Tr_+$.

Lemma 4.4.

(i) Every $r \in \mathcal{R}$ admits a unique representation in the form $r = b/a$ with $a, b \in$
 W s related via $r_- = \mathcal{I}r_+$.
 Lemma 4.4.

(i) Every $r \in \mathcal{R}$ adt
 W such that a is
 $|a|^2 = (1 - |r|^2)^-$

(ii) For $r \in \mathcal{R}$, take (i) Every $r \in \mathcal{R}$ admits a unique representation in the form $r = b/a$ with $a, b \in W$ such that a is invertible in W , zero-free in \mathbb{C}^+ , and satisfies $\hat{a}(0) > 0$ and $|a|^2 = (1 - |r|^2)^{-1}$.

(ii) For $r \in \mathcal{R}$, ta $\alpha(0) > 0$ and
 $\alpha(\omega)$. Then
	- $= (1 |r|^2)^{-1}$.
 $r \in \mathcal{R}$, take a

	function $r^{\#}$ als

	continuous incontinuous income that $\omega \in \mathcal{L}$ (ii) For $r \in \mathcal{R}$, take a and b as in part (i) and set $r^{\#}(\omega) := -b(-\omega)/a(\omega)$. Then
the function $r^{\#}$ also belongs to \mathcal{R} and the mapping
 $\mathcal{I}: r \mapsto r^{\#}$
is a continuous involution on \mathcal{R} .
(iii) Assume th the function $r^{\#}$ also belongs to $\mathscr R$ and the mapping

$$
\mathcal{I}:r\mapsto r^\#
$$

is a continuous involution on R.

- : $r \mapsto r^{\#}$

..
 $\downarrow t$ r_{\pm} are
 $\downarrow t$ $r_{\pm} = r_{\mp}$
 \downarrow , 1] $\ni z$ -(iii) Assume that $\mathbf{u} \in \ell_{1,\mathbb{R}}$ and that r_{\pm} are the reflection coefficients for the Schrödinger operator $H_{\mathbf{u}}$; then $Tr_{\pm} = r_{\mp}$.
(iv) For every $r \in \mathcal{R}$, the mapping $[0,1] \ni z \rightarrow (zr)^{\#} \in \mathcal{R}$ is r Schrödinger operator $H_{\mathbf{u}}$; then $Tr_{\pm} = r_{\mp}$.
For every $r \in \mathcal{R}$, the mapping $[0,1] \ni z \rightarrow$
- (iv) For every $r \in \mathcal{R}$, the mapping $[0, 1] \ni z \rightarrow (zr)^{\#} \in \mathcal{R}$ is real analytic.

4.3. Transformation operators

Interaction operators
Ition $f_{+}(\cdot, \omega)$ can also be represented via the transformation operator.
Ition a representation only for the values of f_{+} at the lattice points
the Fourier series expansions (4.2) of a_s an The Jost solution $f_+(\cdot, \omega)$ can also be represented via the transformation operator.
We shall need such a representation only for the values of f_+ at the lattice points $s \in \mathbb{Z}$. Using the Fourier series expansions

We shall need such a representation only for the values of
$$
f_+
$$
 at the lattice points
\n $s \in \mathbb{Z}$. Using the Fourier series expansions (4.2) of a_s and b_s in (1.5), we see that
\n
$$
f_+(s+0,\omega) = a_s(\omega)e^{i\omega s} + b_s(\omega)e^{-i\omega s}
$$
\n
$$
= \sum_{t \ge s} e^{i\omega t} \left[\hat{a}_s(t-s) + \hat{b}_s(t+s) \right].
$$
\nFor $s, t \in \mathbb{Z}$, we set
\n
$$
M_+(s,t) := \frac{\hat{a}_s(t-s)}{\hat{a}_s(s)} - \delta(s,t), \qquad M_-(s,t) := \frac{\hat{b}_s(t+s)}{\hat{a}_s(s,t)},
$$
\n(4.6)

$$
= \sum_{t \ge s} e^{i\omega t} \left[\hat{a}_s(t-s) + \hat{b}_s(t+s) \right].
$$

For $s, t \in \mathbb{Z}$, we set

$$
M_+(s,t) := \frac{\hat{a}_s(t-s)}{\hat{a}_s(0)} - \delta(s,t), \qquad M_-(s,t) := \frac{\hat{b}_s(t+s)}{\hat{a}_s(0)}, \qquad (4.6)
$$

$$
K_{\pm}(s,t) := M_+(s,t) \pm M_-(s,t), \qquad (4.7)
$$

where $\delta(s,t)$ is the Kronecker delta. It then follows that, for all $\omega \in \mathbb{R}$ and $s \in \mathbb{Z}$,

 K_{\pm}

$$
M_{+}(s,t) := \frac{\hat{a}_{s}(t-s)}{\hat{a}_{s}(0)} - \delta(s,t), \qquad M_{-}(s,t) := \frac{b_{s}(t+s)}{\hat{a}_{s}(0)}, \tag{4.6}
$$

\n
$$
K_{\pm}(s,t) := M_{+}(s,t) \pm M_{-}(s,t), \qquad (4.7)
$$

\nwhere $\delta(s,t)$ is the Kronecker delta. It then follows that, for all $\omega \in \mathbb{R}$ and $s \in \mathbb{Z}$,
\n
$$
f_{+}(s+0,\omega) = \hat{a}_{s}(0) \left(e^{i\omega s} + \sum_{t=s+1}^{\infty} K_{+}(s,t)e^{i\omega t}\right).
$$

\nEqualities (4.3) yield the crucial relation
\n
$$
M_{-}(s-1,s+1) = \tanh u_{s}, \qquad s \in \mathbb{Z},
$$

\nthat allows one to uniquely reconstruct the numbers u_{s} from the kernel M_{-} . Ob-
\ngroup that analogous procedure in the continuous case, since the value of ω is the

$$
M_{-}(s-1, s+1) = \tanh u_s, \qquad s \in \mathbb{Z},
$$

 $M_{-}(s-1, s+1) = \tanh$
allows one to uniquely reconstruct the numeral that analogous procedure in the continuation of M_{-} on the diagonal, cf. (2.11). $(s-1, s+1) = \tanh u_s,$ $s \in \mathbb{Z}$,
ely reconstruct the numbers u_s fron
cedure in the continuous case gives
diagonal, cf. (2.11).
ne" transformation operator for *H*
ng the unitary equivalence of $L_2(\mathbb{R})$

serve that analogous procedure in the continuous case gives the value of u as the restriction of M_- on the diagonal, cf. (2.11).
 Remark 4.5. The "genuine" transformation operator for H_u can be constructed in a simi serve that analogous procedure in the continuous case gives the value of u as the restriction of M_- on the diagonal, cf. (2.11).
 Remark 4.5. The "genuine" transformation operator for H_u can be constructed in a si restriction of M_- on the diagonal, cf. (2.11).
 Remark 4.5. The "genuine" transformation

in a similar manner. Using the unitary equiv

we can represent every function $g \in L_2(\mathbb{R})$

restrictions onto Δ_n , with $g(n$ *Remark* 4.5. The "genuine" transformation operator for H_u can be constructed 4.5. The "genuine" transformation operator for H_u can be constructed
lar manner. Using the unitary equivalence of $L_2(\mathbb{R})$ and $\ell_2(\mathbb{Z}) \otimes L_2(0, 1)$,
epresent every function $g \in L_2(\mathbb{R})$ via the sequence $(g(n, y))_{$ in a similar manner. Using the unitary equivalence of $L_2(\mathbb{R})$ and $\ell_2(\mathbb{Z}) \otimes L_2(0, 1)$,
we can represent every function $g \in L_2(\mathbb{R})$ via the sequence $(g(n, y))_{n \in \mathbb{Z}}$ of its
restrictions onto Δ_n , with $g(n, y)$ we can represent every function $g \in L_2(\mathbb{R})$ via the sequence $(g(n, y))_{n \in \mathbb{Z}}$ of its
restrictions onto Δ_n , with $g(n, y) := g(n + y)$ for $n \in \mathbb{Z}$ and $y \in (0, 1)$. For ease of
notation, we shall write $g(n, \cdot)$ as g_n .

Substituting the Fourier series expansions for a_n and b_n in the expression for

restrictions onto
$$
\Delta_n
$$
, with $g(n, y) := g(n + y)$ for $n \in \mathbb{Z}$ and $y \in (0, 1)$. For ease of notation, we shall write $g(n, \cdot)$ as g_n .
\nSubstituting the Fourier series expansions for a_n and b_n in the expression for f at the point $s = n + y \in \Delta_n$, we get\n
$$
f_n(y) = \sum_{m \ge n} \hat{a}_n(m - n) e^{i\omega(m + y)} + \sum_{m \ge n+2} \hat{b}_n(m + n) e^{i\omega(m - y)}
$$
\n
$$
=:\hat{a}_n(0) \left[e^{i\omega(n+\cdot)} + \sum_{m > n} [A(n, m) + B(n, m)] e^{i\omega(m+\cdot)} \right](y),
$$
\nwhere $A(n, m)$ and $B(n, m)$ are operators in $L_2(0, 1)$ given by\n
$$
A(n, m) = \frac{\hat{a}_n(m - n)}{\hat{a}_n(0)} I, \qquad B(n, m) = \frac{\hat{b}_n(m + n + 1)}{\hat{a}_n(0)} T
$$

$$
=:\hat{a}_n(0)\Big[e^{i\omega(n+\cdot)} + \sum_{m>n} [A(n,m) + B(n,m)]e^{i\omega(m+\cdot)}\Big]\Big(\text{where } A(n,m) \text{ and } B(n,m) \text{ are operators in } L_2(0,1) \text{ given by}
$$
\n
$$
A(n,m) = \frac{\hat{a}_n(m-n)}{\hat{a}_n(0)}I, \qquad B(n,m) = \frac{\hat{b}_n(m+n+1)}{\hat{a}_n(0)}T
$$
\nand T is the reflection operator in $L_2(0,1)$ defined via $Tf(y) = f(1 -$

and *T* is the reflection operator in $L_2(0, 1)$ defined via $Tf(y) = f(1 - y)$.

It turns out that the operator
$$
K
$$
 in $L_2(\mathbb{R})$ given by
\n
$$
(\mathcal{K}g)_n = \widehat{a}_n(0) \Big[g_n + \sum_{m>n} (A(n,m) + B(n,m))g_m \Big]
$$
\nis the transformation operator for H_u . In other words, for every $g \in W_2^2$

It turns out that the operator $\mathcal K$ in $L_2(\mathbb{R})$ given by
 $(\mathcal K g)_n = \hat{a}_n(0) \Big[g_n + \sum_{m>n} (A(n,m) + B(n)) \Big]$

e transformation operator for H_u . In other words

unction $\mathcal K g$ belongs to the domain of H_u and H_u .

that $(\mathcal{K}g)_n = \hat{a}_n(0) \Big[g_n + \sum_{m>n} (A(n,m) + B(n,m)) g_m$

aation operator for H_u . In other words, for evertherm belongs to the domain of H_u and $H_u \mathcal{K}g = -$

anction $\mathcal{K}g$ belongs to W_2^2 outside the integer portions can is the transformation operator for H_u . In other words, for every $g \in W_2^2(\mathbb{R})$,
the function $\mathcal{K}g$ belongs to the domain of H_u and $H_u\mathcal{K}g = -\mathcal{K}g''$. Indeed, the
fact that the function $\mathcal{K}g$ belongs to the function $\mathcal{K}g$ belongs to the domain of H_u and $H_u\mathcal{K}g = -\mathcal{K}g''$. Indeed, the fact that the function $\mathcal{K}g$ belongs to W_2^2 outside the integer points and satisfies the interface conditions can be veri fact that the function Kg belongs to W_2^2 outside the integer points and satisfies
the interface conditions can be verified in a straightforward manner first for g of
support contained in $(j - \frac{1}{2}, j + \frac{1}{2})$, for so 2 the interface conditions can be verified in a straightforward manner first for g of support contained in $(j - \frac{1}{2}, j + \frac{1}{2})$, for some $j \in \mathbb{Z}$ and then using the linearity.
Finally, one can show that the operator support contained in $(j - \frac{1}{2}, j + \frac{1}{2})$
Finally, one can show that the op
similarity of the operators H_u and
4.4. Derivation of the Marchenko
The Marchenko equation relating
the Fourier transform of the reflect $+$ $\frac{1}{2}$
e op
and
and
nko
ing
flect

4.4. Derivation of the Marchenko equation

Finally, one can show that the operator K is boundedly invertible and performs
similarity of the operators H_u and H_0 . We shall not need this fact in what follows.
4.4. Derivation of the Marchenko equation
The Mar similarity of the operators H_u and H_0 . We shall not need this fact in what follows.
4.4. Derivation of the Marchenko equation
The Marchenko equation relating the kernel of the transformation operator and
the Fourie similarity of the operators H_u and H_0 . We shall not need this fact in what follows.
4.4. Derivation of the Marchenko equation
The Marchenko equation relating the kernel of the transformation operator and
the Fourie the Fourier transform of the reflection coefficient can now be derived in a standard manner. The only difference is that, because r_+ is a periodic function and the kernel K is in a sense piece-wise constant, the discre

manner. The only difference is that, because r_+ is a periodic function and the kernel K is in a sense piece-wise constant, the discrete Fourier transform should be used.
Assume that f_{\pm} are the Jost solution for th kernel *K* is in a sense piece-wise constant, the discrete Fourier transform should
be used.
Assume that f_{\pm} are the Jost solution for the operator $H_{\mathbf{u}}$, with some $\mathbf{u} \in$
 $\ell_{1,\mathbb{R}}$, and that *a*, *b*, and Assume that f_{\pm} are the Jost solution for the operator H_{u} , with some $u \in$
 $\ell_{1,\mathbb{R}}$, and that a , b , and $r = r_{+}$ are the corresponding scattering and reflection

coefficients. Since a never vanishes on $\ell_{1,\mathbb{R}}$, and that a, b, and $r = r_+$ are the corresponding scattering and reflection
ficients. Since a never vanishes on the real line, the relations (1.6) and (1.7)
ly that
 $\frac{e^{i\omega x}f_-(x,\omega)}{a(\omega)} = e^{i\omega x}f_+(x,-\omega) + r(\omega)e^{i\omega x}f_+(x,\$

$$
\frac{e^{i\omega x}f_{-}(x,\omega)}{a(\omega)} = e^{i\omega x}f_{+}(x,-\omega) + r(\omega)e^{i\omega x}f_{+}(x,\omega).
$$

coefficients. Since *a* never vanishes on the real line, the relations (1.6) and (1.7)

imply that
 $\frac{e^{i\omega x}f_{-}(x,\omega)}{a(\omega)} = e^{i\omega x}f_{+}(x,-\omega) + r(\omega)e^{i\omega x}f_{+}(x,\omega).$

By (4.4), for every $x \in \mathbb{R}$ the function $\omega \mapsto e^{i\omega x}$ $\frac{e^{i\omega x}f_{-}(x,\omega)}{a(\omega)} = e^{i\omega x}f_{+}(x,-\omega) + r(\omega)e^{i\omega x}f_{+}(x,\omega).$

By (4.4), for every $x \in \mathbb{R}$ the function $\omega \mapsto e^{i\omega x}f_{-}(x,\omega)$ belon

bra $W^{+} := W \cap H^{+}$, and thus the same is true of the function
 $g(x,\omega) := e^{i\omega x}f_{+}(x,-$ By (4.4), for every $x \in \mathbb{R}$ the function $\omega \mapsto e^{i\omega x} f_-(x, \omega)$ belongs to the algebra $W^+ := W \cap H^+$, and thus the same is true of the function
 $g(x, \omega) := e^{i\omega x} f_+(x, -\omega) + r(\omega) e^{i\omega x} f_+(x, \omega)$.

Set
 $R(s) := \hat{r}(-s), \qquad s \in \$

bra
$$
W^+ := W \cap H^+
$$
, and thus the same is true of the function
\n
$$
g(x,\omega) := e^{i\omega x} f_+(x,-\omega) + r(\omega)e^{i\omega x} f_+(x,\omega).
$$
\nSet

$$
R(s) := \hat{r}(-s), \qquad s \in \mathbb{Z}; \tag{4.9}
$$

the then equality (4.8) for each $s \in \mathbb{Z}$ yields

$$
g(x,\omega) := e^{i\omega x} f_{+}(x,-\omega) + r(\omega)e^{i\omega x} f_{+}(x,\omega).
$$

\n
$$
R(s) := \hat{r}(-s), \qquad s \in \mathbb{Z}; \tag{4.9}
$$

\n
$$
\text{ity (4.8) for each } s \in \mathbb{Z} \text{ yields}
$$

\n
$$
\frac{g(s+0,\omega)}{\hat{a}_s(0)} = 1 + \sum_{t=s+1}^{\infty} K_{+}(s,t)e^{-i\omega(t-s)} + \sum_{n \in \mathbb{Z}} \hat{r}(n)e^{i\omega(n+2s)}
$$

\n
$$
+ \sum_{\xi=s+1}^{\infty} \sum_{n \in \mathbb{Z}} \hat{r}(n)K_{+}(s,\xi)e^{i\omega(\xi+s+n)}
$$

\n
$$
= 1 + \sum_{t \in \mathbb{Z}} L_{+}(s,t)e^{-i\omega(t-s)},
$$

\n
$$
L_{+}(s,t) := K_{+}(s,t) + R(s+t) + \sum_{n \in \mathbb{Z}} K_{+}(s,\xi)R(\xi+t).
$$

where

$$
= 1 + \sum_{t \in \mathbb{Z}} L_{+}(s, t) e^{-i\omega(t - s)},
$$

$$
L_{+}(s, t) := K_{+}(s, t) + R(s + t) + \sum_{\xi = s + 1}^{\infty} K_{+}(s, \xi) R(\xi + t).
$$

Recalling that
$$
g(s + 0, \cdot) \in W^+
$$
, we derive the following discrete analogue of the
\nMarchenko equation for all $s, t \in \mathbb{Z}$ with $s < t$:
\n
$$
K_+(s,t) + R(s+t) + \sum_{\xi=s+1}^{\infty} K_+(s,\xi)R(\xi+t) = 0.
$$
\n(4.10)
\n4.5. Derivation of the Zakharov-Shabat system
\nIt follows from the analysis of Subsection 4.2 that the sign change $u \mapsto -u$ does not
\neffect the functions a , which changes to b is a result, we have $x_i = -x_i$.

4.5. Derivation of the Zakharov–Shabat system

 $(s, t) + R(s + t) + \sum_{\xi=s+1}^{\infty} K_{+}(s, \xi)R(\xi + t) = 0.$ (4.10)

he Zakharov-Shabat system

malysis of Subsection 4.2 that the sign change $u \mapsto -u$ does not
 a_n , while b_n change to $-b_n$; as a result, we have $r_{\pm,-u} = -r_{\pm,u}$.
 affect the functions a_n , while b_n change to $-b_n$; as a result, we have $r_{\pm,-\mathbf{u}} = -r_{\pm,\mathbf{u}}$.
Therefore the counterpart of (4.10) for K_{-} reads

after the functions
$$
a_n
$$
, while b_n change to $-b_n$; as a result, we have $r_{\pm,-u} = -r_{\pm,u}$. Therefore the counterpart of (4.10) for K_- reads

\n
$$
K_-(s,t) - R(s+t) - \sum_{\xi=s+1}^{\infty} K_-(s,\xi)R(\xi+t) = 0. \tag{4.11}
$$
\nRecalling the definition of the kernels M_+ and M_- in (4.6), one can recast the system (4.10)–(4.11) as

\n
$$
M_+(s,t) + \sum_{\xi=s+1}^{\infty} M_-(s,\xi)R(\xi+t) = 0, \tag{4.12}
$$

$$
K_{-}(s,t) - R(s+t) - \sum_{\xi=s+1} K_{-}(s,\xi)R(\xi+t) = 0.
$$
 (4.11)
Recalling the definition of the kernels M_{+} and M_{-} in (4.6), one can recast the
system (4.10)–(4.11) as

$$
M_{+}(s,t) + \sum_{\xi=s+1}^{\infty} M_{-}(s,\xi)R(\xi+t) = 0,
$$
 (4.12)

$$
M_{-}(s,t) + R(s+t) + \sum_{\xi=s+1}^{\infty} M_{+}(s,\xi)R(\xi+t) = 0
$$
 (4.13)
for $s, t \in \mathbb{Z}$ and $s < t$, which is a discrete analogue of the Zakharov-Shabat system.
4.6 Solutions of the Zakharov Shabat system.

$$
M_{+}(s,t) + \sum_{\xi=s+1}^{\infty} M_{-}(s,\xi)R(\xi+t) = 0,
$$
(4.12)

$$
M_{-}(s,t) + R(s+t) + \sum_{\xi=s+1}^{\infty} M_{+}(s,\xi)R(\xi+t) = 0
$$
(4.13)
for $s, t \in \mathbb{Z}$ and $s < t$, which is a discrete analogue of the Zakharov–Shabat system.
4.6. Solution of the Zakharov–Shabat system
Take now an arbitrary element r of \mathcal{R} ; our aim is to show that it is the right
reflection coefficient for a Schrödinger operator H_{u} with some real-valued sequence

4.6. Solution of the Zakharov–Shabat system

for $s, t \in \mathbb{Z}$ and $s < t$, which is a discrete analogue of the Zakharov–Shabat system.
4.6. Solution of the Zakharov–Shabat system
Take now an arbitrary element r of \mathcal{R} ; our aim is to show that it is the right Take now an arbitrary element r of $\mathscr R$; our aim is to show that it is the right
reflection coefficient for a Schrödinger operator H_u with some real-valued sequence
 $u \in \ell_1$. We shall do this by first solving the Z reflection coefficient for a Schrödinger operator H_u with some real-valued sequence $u \in \ell_1$. We shall do this by first solving the Zakharov-Shabat system with the sequence R defined via (4.9) and then determining u $u \in \ell_1$

sequence *R* defined via (4.9) and then determining **u** via (2.16).
\nThe sequence
$$
R(n)
$$
 generates a continuous operator \mathcal{R} in $\ell_2 := \ell_2(\mathbb{Z})$ via
\n
$$
(\mathcal{R}x)(s) := \sum_{t \in \mathbb{Z}} R(s+t)x(t).
$$
\n(4.14)
\nTaking the (inverse) Fourier transform of (4.14), we see that
\n
$$
\|\mathcal{R}\| = \sup_{\omega \in \mathbb{R}} |r(\omega)| \le \|R\|_1,
$$

$$
\|\mathcal{R}\| = \sup_{\omega \in \mathbb{R}} |r(\omega)| \le \|R\|_1,
$$

Taking the (inverse) Fourier transform of (4.14), we see that
 $\|\mathcal{R}\| = \sup_{\omega \in \mathbb{R}} |r(\omega)| \leq \|R\|_1$,

where $\|R\|_1$ denotes the ℓ_1 -norm of the sequence R. In particular, $\|\mathcal{R}\| < 1$; this

inequality is used cru where $||R||_1$ denotes the ℓ_1 -norm of the sequence R. In particular, $||R|| < 1$; this $||\mathcal{R}|| = \sup_{\omega \in \mathbb{R}} |r(\omega)| \le ||R||_1,$
where $||R||_1$ denotes the ℓ_1 -norm of the sequence R. In part
inequality is used crucially to establish the unique solvabili
Shabat system. For $s \in \mathbb{Z}$, we denote by \mathcal{P}_s = sup $|r(\omega)| \leq ||R||_1$,

i of the sequence R.

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stablish the unique s

note by P_s an orthop
 $:=\begin{cases} x(t) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$ where $||R||_1$ denotes the ℓ_1 -norm of the sequence R . In particular, $||R|| < 1$; this
inequality is used crucially to establish the unique solvability of the Zakharov-
Shabat system. For $s \in \mathbb{Z}$, we denote by $\mathcal{$

$$
(\mathcal{P}_s x)(t) := \begin{cases} x(t) & \text{if } t > s, \\ 0 & \text{otherwise,} \end{cases}
$$
\nAlso, \mathcal{X} shall stand for the s

\n
$$
\Omega := \{(s, t) \in \mathbb{Z}^2 \mid s < t\}
$$

Shabat system. For $s \in \mathbb{Z}$, we denote by \mathcal{P}_s an orthoprojector in ℓ_2 given by
 $(\mathcal{P}_s x)(t) := \begin{cases} x(t) & \text{if } t > s, \\ 0 & \text{otherwise,} \end{cases}$

and set $\mathcal{R}_s := \mathcal{P}_s \mathcal{RP}_s$. Also, \mathcal{X} shall stand for the set of all Shabat system. For $s \in \mathbb{Z}$, we denote by \mathcal{P}_s an orthoprojector in ℓ_2 given by
 $(\mathcal{P}_s x)(t) := \begin{cases} x(t) & \text{if } t > s, \\ 0 & \text{otherwise,} \end{cases}$

and set $\mathcal{R}_s := \mathcal{P}_s \mathcal{R} \mathcal{P}_s$. Also, \mathcal{X} shall stand for the set 0 otherwise,

ull stand for the s
 $t \in \mathbb{Z}^2 \mid s < t$ } and set $\mathcal{R}_s := \mathcal{P}_s \mathcal{RP}_s$. Also, \mathcal{X} shall stand for the set of all complex-valued
functions on
 $\Omega := \{(s, t) \in \mathbb{Z}^2 \mid s < t\}$

$$
\Omega := \{ (s, t) \in \mathbb{Z}^2 \mid s < t \}
$$

$$
||X||_{\mathcal{X}}^2 := \sup_{s \in \mathbb{Z}} \sum_{t=s+1} |X(s,t)|^2
$$

with
 $||X||_{\mathcal{X}}^2 := \sup_{s \in \mathbb{Z}} \sum_{t=s+1} |X(s,t)|^2$

finite; the set $\mathcal X$ is a Banach space with respect to the norm $|| \cdot$

stands for the subspace of $\mathcal X$ defined by $\frac{$ finite; the set X is a Banach space with respect to the norm $|| \cdot ||_{\mathcal{X}}$. Also, \mathcal{X}_0
stands for the subspace of X defined by
 $\mathcal{X}_0 := \{ X \in \mathcal{X} \mid \lim_{s \to +\infty} \sum_{t>s} |X(s,t)|^2 = 0 \}.$
Then the following holds.
Theorem 4

$$
\|X\|_{\mathscr{X}}^2 := \sup_{s \in \mathbb{Z}} \sum_{t=s+1} |X(s,t)|^2
$$

is a Banach space with respect to the norm
pace of \mathscr{X} defined by

$$
\mathscr{X}_0 := \{ X \in \mathscr{X} \mid \lim_{s \to +\infty} \sum_{t>s} |X(s,t)|^2 = 0 \}.
$$

stands for the subspace of \mathscr{X} defined by
 $\mathscr{X}_0 := \{ X \in \mathscr{X} \mid \lim_{s \to +} \$

Then the following holds.
 Theorem 4.6. For every $r \in \mathscr{R}$, define

Zakharov-Shabat sustem (4.12) - (4.13) h := { $X \in \mathcal{X} \mid \lim_{s \to +\infty}$

ds.
 $y r \in \mathcal{R}$, define R
 $n (4.12)$ -(4.13) has

This solution is give $(s, t)|^2 = 0$ }.

(i) via $R(n)$:

(i) wie solution (**Theorem 4.6.** For every
Zakharov-Shabat system
to the space $\mathcal{X}_0 \times \mathcal{X}$. Th
M. **Theorem 4.6.** For every $r \in \mathcal{R}$, define $R \in \ell_1(\mathbb{Z})$ via $R(n) := \hat{r}(-n)$. Then the Zakharov-Shabat system (4.12)-(4.13) has a unique solution (M_+, M_-) belonging to the space $\mathcal{X}_0 \times \mathcal{X}$. This solution is give Zakharov–Shabat system (4.12) – (4.13) has a unique solution (M_+, M_-) belonging to the space $\mathscr{X}_0 \times \mathscr{X}$. This solution is given by

$$
m (4.12)–(4.13) has a unique solution (M_+, M_-) belongingThis solution is given by\n
$$
M_+(s,t) = \langle (I - \mathcal{R}_s^2)^{-1} \mathcal{R}_s \mathcal{R} e_s, e_t \rangle,
$$
\n
$$
M_-(s,t) = -\langle (I - \mathcal{R}_s^2)^{-1} \mathcal{P}_s \mathcal{R} e_s, e_t \rangle,
$$
\n
$$
solar product in \ell_2 and (e_s) the standard orthonormal\nands continuously in \mathcal{X}_0 \times \mathcal{X} on r \in \mathcal{R}.
$$
\n
$$
a \text{ related uniqueness result stating that no two different } r \in \text{solutions of the corresponding Zakharov–Shabat systems}
$$
$$

) しょころ r with $\langle \cdot, \cdot \rangle$ denoting the scalar product in ℓ_2 and basis therein, and depends continuously in $\mathscr{X}_0 \times \mathscr{X}$ on $r \in \mathscr{R}$.

 $(s, t) = -\langle (I - \mathcal{R}_s^2) \rangle$
calar product in ℓ :
continuously in \mathcal{R}
elated uniqueness nutions of the corre-
inverse scattering $\begin{array}{c} \gamma \ 2 \zeta \in \mathbb{R} \ \mathbb{R} \end{array}$ ${normal}$
nt $r \in$
stems. (*e_s*) the standard orthonormal
 \therefore on $r \in \mathcal{R}$.

stating that no two different $r \in$

ing Zakharov–Shabat systems.

Em of the next subsection.

the solutions M_{\pm} of the related We shall also need a related uniqueness result stating that no two different $r \in$ n share the same solutions of the corresponding Zakharov–Shabat systems.
will be crucial in the inverse scattering problem of the next subs \mathcal{R} can share the same solutions of the corresponding Zakharov–Shabat systems.

is will be crucial in the inverse scattering problem of the next subsection.
 eorem 4.7. Assume that $r_1, r_2 \in \mathcal{R}$ are such that the solutions M_{\pm} of the related
 kharov–Shabat systems (4.12)–(4.13) *coincide.* **Theorem 4.7.** Assume that $r_1, r_2 \in \mathcal{R}$ are such that the solutions M_{\pm} of the related \mathcal{R} Zakharov–Shabat systems (4.12) – (4.13) coincide. Then $r_1 = r_2$.

Theorem 4.7. Assume that $r_1, r_2 \in \mathcal{R}$ are such that the solutions M_{\pm} of the reductions of r_1 and R_2 the corresponding ℓ_1 -sequences formed from Fourier coefficients of r_1 and r_2 (namely, $R_j(n) := \$ (4.12)–(4.13) coincide. Then $r_1 = r_2$.

I R_2 the corresponding ℓ_1 -sequences

nd r_2 (namely, $R_j(n) := \hat{r}_j(-n)$) and

Set $R := R_1 - R_2$; then R satisfies $t + t$) + $\sum_{n=1}^{\infty} K_{+}(s, \xi)R(\xi + t) = 0$ *Proof.* Denote by R_1 and R_2 the corresponding ℓ_1 -sequences formed from the Denote by R_1 and R_2 the corresponding ℓ_1 -sequences formed from the
coefficients of r_1 and r_2 (namely, $R_j(n) := \hat{r}_j(-n)$) and recall that K_+ is
y $K_+ = M_+ + M_-$. Set $R := R_1 - R_2$; then R satisfies the rela

Fourier coefficients of
$$
r_1
$$
 and r_2 (namely, $R_j(n) := \hat{r}_j(-n)$) and recall that K_+ is
given by $K_+ = M_+ + M_-$. Set $R := R_1 - R_2$; then R satisfies the relations

$$
R(s + t) + \sum_{\xi=s+1}^{\infty} K_+(s,\xi)R(\xi + t) = 0
$$
(4.16)
for all $s \in \mathbb{Z}$ and all $t > s$. Using properties of the kernel K_+ and (4.16), we shall
show that
(i) there is $N \in \mathbb{Z}$ such that $R(n) = 0$ for all $n \ge N$;
(ii) the N above can be taken arbitrary;

for all $s \in \mathbb{Z}$ and all $t > s$. Using properties of the kernel K_+ and (4.16), we shall
show that
(i) there is $N \in \mathbb{Z}$ such that $R(n) = 0$ for all $n \ge N$;
(ii) the N above can be taken arbitrary;
which results i

(ii) the *N* above can be taken arbitrary;

which results in $R \equiv 0$.

For (i), it suffices to show existence $\sum_{\xi>s} |K_{+}(s)|$

andeed, (4.16) yields the inequality For (i), it suffices to show existence of $s \in \mathbb{Z}$ such that

(i) there is
$$
N \in \mathbb{Z}
$$
 such that $R(n) = 0$ for all $n \ge N$;
\n(ii) the N above can be taken arbitrary;
\nwhich results in $R \equiv 0$.
\nFor (i), it suffices to show existence of $s \in \mathbb{Z}$ such that
\n
$$
\sum_{\xi>s} |K_{+}(s,\xi)| < 1;
$$
\nindeed, (4.16) yields the inequality
\n
$$
\max_{n>2s} |R(n)| \le \max_{n>2s} |R(n)| \cdot \sum_{\xi>s} |K_{+}(s,\xi)|,
$$
\n(4.17)

which results in $R \equiv 0$.
For (i), it suffices to show existently $\sum_{\xi>s}$
indeed, (4.16) yields the inequality

$$
\max_{n>2s} |R(n)| \leq \max_{n>2s} |R(n)| \cdot \sum_{\xi>s} |K_{+}(s,\xi)|,
$$

17) forces that $R(n) = 0$ for all $n > 2s$. This is proved by using the ulae (4.15) for solutions M_{\pm} and the fact that $\sum_{t>s} |R_j(t)|$ tends to $+\infty$. Part (ii) is then established by induction. which by (4.17) forces that $R(n) = 0$ for all $n > 2s$. This is proved by using the explicit formulae (4.15) for solutions M_{\pm} and the fact that $\sum_{t>s} |R_j(t)|$ tends to zero as $s \to +\infty$. Part (ii) is then established by $t>s|R_j$

4.7. The inverse scattering problem

explicit formulae (4.15) for solutions M_{\pm} and the fact that $\sum_{t>s} |R_j(t)|$ tends to
zero as $s \to +\infty$. Part (ii) is then established by induction. \Box
4.7. The inverse scattering problem
Finally we show that every zero as $s \to +\infty$. Part (ii) is then established by induction. □
 4.7. The inverse scattering problem

Finally we show that every $r \in \mathcal{R}$ is a right reflection coefficient for a Schrödinger

operator H_u correspond

$$
\tanh u_j = M_-(j-1, j+1), \qquad j \in \mathbb{Z},
$$

operator H_u corresponding to a unique sequence $\mathbf{u} = (u_j)_{j \in \mathbb{Z}} \in \ell_{1,\mathbb{R}}$. The considerations of the previous sections imply that such a \mathbf{u} must verify the relation $\tanh u_j = M_-(j-1,j+1)$, $j \in \mathbb{Z}$,
where t where the kernel M_{-} satisfies the Zakharov–Shabat system (4.12) – (4.13) , in which erations of the previous sections imply that such a u must verify the relation
 $\tanh u_j = M_-(j-1, j+1), \quad j \in \mathbb{Z},$

where the kernel M_- satisfies the Zakharov-Shabat system (4.12)-(4.13), in w
 $R(j) := \hat{r}(-j).$

This suggest tanh $u_j = M_-(j-1, j+1)$, $j \in \mathbb{Z}$,
satisfies the Zakharov-Shabat system (
e following reconstruction algorithm:
m the sequence $R = (R(n)) \in \ell_{1,\mathbb{R}}$ with
rrov-Shabat system $(4.12)-(4.13)$ with $R(j) := \widehat{r}(-j).$

- $R(n)$) $\in \ell_{1,\mathbb{R}}$
- where the kernel M_{-} satisfies the Zakharov–Shabat system (4.12)–(4.13), in which $R(j) := \hat{r}(-j)$.

This suggests the following reconstruction algorithm:

(1) given $r \in \mathcal{R}$, form the sequence $R = (R(n)) \in \ell_{1,\mathbb{R}}$ wi (*j*) := $\hat{r}(-j)$.
This sugg
(1) given $r \in$
(2) solve the
(M_+, M_-
 $M_-(j-1)$
3) define u_i given $r \in \mathcal{R}$, form the sequence $R = (R(n)) \in \ell_{1,\mathbb{R}}$ with $R(n) := \hat{r}(-n)$;
solve the Zakharov-Shabat system $(4.12)-(4.13)$ with the given R to
 $(M_+, M_-) \in \mathcal{X}_0 \times \mathcal{X}$ and form the sequence $\mathbf{v} := (v(j))$ with $v(j)$
 M (1) given $r \in \mathcal{R}$, form the sequence $R = (2)$ solve the Zakharov-Shabat system ($(M_+, M_-) \in \mathcal{X}_0 \times \mathcal{X}$ and form the $M_-(j-1, j+1)$ for $j \in \mathbb{Z}$;
(3) define $u_j \in \mathbb{R}$ via tanh $u_j := v(j)$ and $3y$ Theorem 4.6 step $\begin{align} &\frac{1}{2} \log \left(\frac{n}{2} \right) \ &\log \left(\frac{n}{2} \right) \$ $M_{-}(j-1,j+1)$ for $j \in \mathbb{Z}$;
-

(2) solve the Zakharov–Shabat system $(4.12)-(4.13)$ with the given R to get $(M_+, M_-) \in \mathcal{X}_0 \times \mathcal{X}$ and form the sequence $\mathbf{v} := (v(j))$ with $v(j) := M_-(j-1, j+1)$ for $j \in \mathbb{Z}$;
(3) define $u_j \in \mathbb{R}$ via tanh $u_j := v(j)$ an $(M_+, M_-) \in \mathcal{X}_0 \times \mathcal{X}$ and form the sequence **v** := $(v(j))$ with $v(j) := M_-(j-1, j+1)$ for $j \in \mathbb{Z}$;
define $u_j \in \mathbb{R}$ via tanh $u_j := v(j)$ and form $u := (u_j)_{j \in \mathbb{Z}}$.
theorem 4.6 step (2) can always be performed, but $(j - 1, j + 1)$ for $j \in \mathbb{Z}$;
ne $u_j \in \mathbb{R}$ via tanh u_j :
rem 4.6 step (2) can al
1. We shall prove this is
ne final step is to show
s r we have started with
first start with r of sm (3) define $u_j \in \mathbb{R}$ via tanh $u_j := v(j)$ and form $\mathbf{u} := (u_j)_{j \in \mathbb{Z}}$.
By Theorem 4.6 step (2) can always be performed, but \mathbf{u} can $\mathbf{v} \parallel_{\infty} < 1$. We shall prove this is the case and, moreover, that $s \in \ell_$ By Theorem 4.6 step (2) can always be performed, but u can only be defined if $||\mathbf{v}||_{\infty} < 1$. We shall prove this is the case and, moreover, that so-defined u belongs to ℓ_1 . The final step is to show that the $\|\mathbf{v}\|_{\infty}$ < 1. We shall prove this is the case and, moreover, that so-defined *u* belongs to ℓ_1 . The final step is to show that the Schrödinger operator corresponding to 1. We shall prove this is the case and, moreover, that so-defined u belongs
ne final step is to show that the Schrödinger operator corresponding to
is r we have started with as its right reflection coefficient.
first st

to ℓ_1 . The final step is to show that the Schrödinger operator corresponding to
this **u** has *r* we have started with as its right reflection coefficient.
We first start with *r* of small norm.
Lemma 4.8. Assume tha this **u** has *r* we have started with as its right reflection coefficient.
We first start with *r* of small norm.
Lemma 4.8. Assume that $r \in \mathcal{R}$ is such that $||r||_W \leq \frac{1}{2}$; then the sequence **v** belongs to ℓ_1 **Lemma 4.8.** Assume that $r \in \mathcal{R}$ is such that $||r||_W \leq \frac{1}{2}$; then the corresponding sequence **v** belongs to ℓ_1 and $\|\mathbf{v}\|_1 \leq \frac{2}{3}$.

We first start with r of small norm.
 na 4.8. Assume that $r \in \mathcal{R}$ is such

nce **v** belongs to ℓ_1 and $||\mathbf{v}||_1 \leq \frac{2}{3}$.

This can be derived from the explicials

lakharov-Shabat system. Therefore

sponding This can be derived from the explicit formula (4.15) for the solution M_- of
lakharov-Shabat system. Therefore u is well defined; we then consider the
sponding Schrödinger operator H_u and determine its right reflecti the Zakharov–Shabat system. Therefore u is well defined; we then consider the corresponding Schrödinger operator H_u and determine its right reflection coefficient $r_{u,+}$ and the corresponding kernels $M_{u,\pm}$. In fac ficient $r_{u,+}$ and the corresponding kernels $M_{u,\pm}$. In fact, both M_{\pm} and $M_{u,\pm}$
satisfy the discrete hyperbolic system with the same initial conditions determined
by **v** and therefore coincide [10]. It follows ficient $r_{u,+}$ and the corresponding kernels $M_{u,\pm}$. In fact, both M_{\pm} and $M_{u,\pm}$
satisfy the discrete hyperbolic system with the same initial conditions determined
by **v** and therefore coincide [10]. It follows by **v** and therefore coincide [10]. It follows that the sequence $R_u(n) := \hat{r}_{u,+}(-n)$
along with $R(n)$ verifies the Zakharov-Shabat system with the same M_{\pm} ; by The-
orem 4.7, $r_{u,+} = r$. This completes the solution of t by **v** and therefore coincide [10]. It follows that the sequence $R_u(n) := \hat{r}_{u,+}(-n)$
along with $R(n)$ verifies the Zakharov–Shabat system with the same M_{\pm} ; by The-
orem 4.7, $r_{u,+} = r$. This completes the solution of t in the case when $||r||_W < \frac{1}{2}$.
In a generic case, we first show that the sequence **v** belongs to ℓ_1 at $+\infty$.

along with $R(n)$ verifies the Zakharov–Shabat system with the same M_{\pm} ; by Theorem 4.7, $r_{u,+} = r$. This completes the solution of the inverse scattering problem in the case when $||r||_W < \frac{1}{2}$.
In a generic case, we orem 4.7, $r_{\mathbf{u},+} = r$. This completes the solution of the inverse scattering problem
in the case when $||r||_W < \frac{1}{2}$.
In a generic case, we first show that the sequence **v** belongs to ℓ_1 at $+\infty$.
To this end we o in the case when $||r||_W < \frac{1}{2}$
In a generic case, we
To this end we observe tha
on $R(n)$ with $n > N$; choos
reasonings yields the result
At the second step we
 $x \mapsto -x$ to show that $\mathbf{v} \in \ell_1$ f t s .
. In a generic case, we first show that the sequence **v** belongs to ℓ_1 at $+\infty$.
is end we observe that, by (4.15), the values of $v(n)$ for $n > N$ only depend
 (n) with $n > N$; choosing N so that $\sum_{n>N} |R(n)| < \frac{1}{2}$ an

To this end we observe that, by (4.15), the values of $v(n)$ for $n > N$ only depend
on $R(n)$ with $n > N$; choosing N so that $\sum_{n>N} |R(n)| < \frac{1}{2}$ and using the above
reasonings yields the result.
At the second step we expl on $R(n)$ with $n > N$; choosing N so that $\sum_{n>N} |R(n)| < \frac{1}{2}$
reasonings yields the result.
At the second step we exploit the way u and r_{\pm} behav
 $x \mapsto -x$ to show that $\mathbf{v} \in \ell_1$. Namely, set $r^{\#} := \mathcal{I}r$, w At the second step we exploit the way u and r_{\pm} behave under the reflection $x \mapsto -x$ to show that $\mathbf{v} \in \ell_1$. Namely, set $r^{\#} := \mathcal{I}r$, with the involution \mathcal{I} defined in Lemma 4.4. Regarding r as a pu At the second step we exploit the way u and r_{\pm} behave under the reflection $x \mapsto -x$ to show that $\mathbf{v} \in \ell_1$. Namely, set $r^{\#} := \mathcal{I}r$, with the involution \mathcal{I} defined in Lemma 4.4. Regarding r as a pu $x \mapsto -x$ to show that $\mathbf{v} \in \ell_1$. Namely, set $r^{\#} := \mathcal{I}r$, with the involution \mathcal{I} defined in Lemma 4.4. Regarding r as a putative right reflection coefficient for the Schrödinger operator to be found, we c Lemma 4.4. Regarding r as a putative right reflection coefficient for the Schrödinger
operator to be found, we conclude that $r^{\#}$ will then be its left reflection coefficient.
If r were a genuine right reflection c operator to be found, we conclude that $r^{\#}$ will then be its left reflection coefficient.
If r were a genuine right reflection coefficient for some Schrödinger operator $H_{\boldsymbol{u}}$
 $H_{\boldsymbol{u}}$ If r were a genuine right reflection coefficient for some Schrödinger operator H_u 36 S. Albeverio, R. Hryniv, Ya. Mykytyuk and P. Perry

and $u^{\#} = u(n, r^{\#})$ is the sequence constructed for $r^{\#}$ instead of r, then we would

have the relation
 $u(n, r^{\#}) = -u(-n, r)$

for all $n \in \mathbb{Z}$. In particular, t

$$
u(n,r^\#) = -u(-n,r)
$$

sufficiently small. Using now the analytic dependence on a small parameter ε of for all $n \in \mathbb{Z}$. I
sufficiently small
the sequences \boldsymbol{u} a
an arbitrary $r \in$ $(n, r^{\#}) = -u(-n, r)$
this relation holds v
ne analytic depender
ted from εr , we can ϵ
is that the sequence of for all $n \in \mathbb{Z}$. In particular, this relation holds whenever the W-norm of r is sufficiently small. Using now the analytic dependence on a small parameter ε of the sequences \boldsymbol{u} and \boldsymbol{v} constructed fro sufficiently small. Using now the analytic dependence on a small parameter ε of the sequences \boldsymbol{u} and \boldsymbol{v} constructed from εr , we can establish the above equality for an arbitrary $r \in \mathcal{R}$. This shows the sequences **u** and **v** constructed from εr , we can establish the above equality for
an arbitrary $r \in \mathcal{R}$. This shows that the sequence **v** constructed for every $r \in \mathcal{R}$
belongs to ℓ_1 . The final step is

an arbitrary $r \in \mathcal{R}$. This shows that the sequence **v** constructed for every $r \in \mathcal{R}$ belongs to ℓ_1 . The final step is to use the same analyticity to prove that $||\mathbf{v}||_{\infty} < 1$; see [10] for details.
Thus gi belongs to ℓ_1 . The final step is to use the same analyticity to prove that $||\mathbf{v}||_{\infty} < 1$;
see [10] for details.
Thus given any $r \in \mathcal{R}$, we can successfully perform the steps (i) to (iii) in the
above reconstr Thus given an
above reconstructif
The fact that the coefficient equal to
ness result of Thee
problem for the cla Thus given any $r \in \mathcal{R}$, we can successfully perform the steps (i) to (iii) in the e reconstruction algorithm and to determine a real-valued sequence u in ℓ_1 .
fact that the corresponding Schrödinger operator H_u above reconstruction algorithm and to determine a real-valued sequence u in ℓ_1 .
The fact that the corresponding Schrödinger operator H_u has the right reflection
coefficient equal to the r we have started from is a The fact that the corresponding Schrödinger operator H_u has the right reflection coefficient equal to the r we have started from is again justified using the uniqueness result of Theorem 4.7. This completes the solution coefficient equal to the r we have started from is again justified using the unique-
ness result of Theorem 4.7. This completes the solution of the inverse scattering
problem for the class of impedance Schrödinger opera

4.8. Some generalizations

problem for the class of impedance Schrödinger operators under consideration.
 4.8. Some generalizations

Most of the above considerations can be generalized for the situation where the

discontinuity points x_k of the **4.8. Some generalizations**
Most of the above considerations can be generalized for the situation where the
discontinuity points x_k of the impedance function p do not form a periodic latti
Assume that the set $\{x_k\}$ d discontinuity points x_k of the impedance function p do not form a periodic lattice.
Assume that the set $\{x_k\}$ does not have finite accumulation points, that x_k are labelled in increasing order, and determine the Assume that the set $\{x_k\}$ does not have finite accumulation points, that x_k are labelled in increasing order, and determine the sequence u_k from the relation Assume that the set $\{x_k\}$ does not have finite accumulation points, that x_k are labelled in increasing order, and determine the sequence u_k from the relation $p(x) = \exp\left\{\sum_{k:x_k>x} u_k\right\}$.
Then under the assumption th

$$
p(x) = \exp\Big{\sum_{k:x_k>x} u_k\Big}.
$$

labelled in increasing order, and determine the sequence u_k from the relation
 $p(x) = \exp\left\{\sum_{k:x_k>x} u_k\right\}.$

Then under the assumption that the sequence (u_k) belongs to ℓ_1 we can de

the scattering coefficients $a(\omega)$ $(x) = \exp\left\{\sum_{k:x_k>}\right\}$
hat the sequence
) and $b(\omega)$ as in
ra W; however,
urier series [15] Then under the assumption that the sequence (u_k) belongs to ℓ_1 we can define
the scattering coefficients $a(\omega)$ and $b(\omega)$ as in Subsection 4.2. They are no longer
elements of the Wiener algebra W ; however, they ar the scattering coefficients $a(\omega)$ and $b(\omega)$ as in Subsection 4.2. They are no longer
elements of the Wiener algebra W; however, they are almost-periodic functions
with absolutely summable Fourier series [15] and thus el elements of the Wiener algebra W ; however, they are almost-periodic functions
with absolutely summable Fourier series [15] and thus elements of a generalized
Wiener-type algebra W_{ap} . The spectrum of an element in thi

Wiener-type algebra W_{ap} . The spectrum of an element in this algebra is the closure
of its range over \mathbb{R} ; thus we again get that a is invertible in W_{ap} and $r \in W_{ap}$.
Next, the function a belongs to the Hardy s Next, the function *a* belongs to the Hardy space H^+ , and this opens the door to deriving an analogue of the Marchenko equation. It takes the form (4.10), where the variables *s* and *t* belong to the additive group $\$ Next, the function *a* belongs to the Hardy space H^+ , and this opens the
to deriving an analogue of the Marchenko equation. It takes the form (4.10),
e the variables *s* and *t* belong to the additive group Γ spanne

where the variables *s* and *t* belong to the additive group Γ spanned by $\{x_j\}$ and
the summation in ξ is over Γ as well.
Due to the condition $||r||_{\infty} < 1$, the corresponding Zakharov–Shabat system
might be shown t the summation in ξ is over Γ as well.
Due to the condition $||r||_{\infty} < 1$,
might be shown to possess a unique
inverse scattering problem is to find
thus actually determine the sequence
Schrödinger operator $H_{\mathbf{u}}$ Due to the condition $||r||_{\infty} < 1$, the corresponding Zakharov–Shabat system
t be shown to possess a unique solution in $\ell_2(\Gamma)$. The difficult part of the
se scattering problem is to find a replacement for the relation might be shown to possess a unique solution in $\ell_2(\Gamma)$. The difficult part of the inverse scattering problem is to find a replacement for the relation (2.16) and thus actually determine the sequence (u_k) and then justi thus actually determine the sequence (u_k) and then justify that the corresponding Schrödinger operator H_u has the reflection coefficient r have started with. This will be discussed in a future work that is currently in Schrödinger operator H_u has the reflection coefficient r have started with. This will be discussed in a future work that is currently in progress.

Appendix

Example 1 and $\widehat{L}^1(\mathbb{R})$ the Wiener algebra of Fourier transforms (2.1) of functions
th norm $\|\widehat{f}\|_{\widehat{L}^1} := \|f\|_{L^1}$, and by \widehat{X} the Banach algebra that is the
 $= L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ under the Fourier t Let us denote by $L^1(\mathbb{R})$ the Wiener algebra of Fourier transforms (2.1) of functions
in $L^1(\mathbb{R})$ with norm $\|\hat{f}\|_{\widehat{L}^1} := \|f\|_{L^1}$, and by \widehat{X} the Banach algebra that is the
image of $X = L^1(\mathbb{R}) \cap L^2(\mathbb$ $\widehat{L}^1(\mathbb{R})$ the in $L^1(\mathbb{R})$ with norm $||\hat{f}||_{\widehat{L}^1} := ||f||_{L^1}$, and by X the Banach algebra that is the
image of $X = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ under the Fourier transform, equipped with the norm
 $||\hat{f}||_{\widehat{X}} := ||f||_X$. We also denote in $L^1(\mathbb{R})$ with norm $\|\hat{f}\|_{\widehat{L^1}} := \|f\|_{L^1}$, and by \widehat{X} the Banach algebra that is the image of $X = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ under the Fourier transform, equipped with the norm image of $X = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ under the Fourier transform, equipped with the norm $||\hat{f}||_{\hat{X}} := ||f||_X$. We also denote by $\mathbf{1} + \hat{X}$ the unital extension of \hat{X} obtained by adding the constant functions and adding the constant functions and norming $1 + \hat{X}$ with the norm

$$
||c + \hat{f}||_{1+\hat{X}} = |c| + ||\hat{f}||_{\hat{X}}
$$

adding the constant functions and norming $1 + X$ with the norm
 $||c + \hat{f}||_{1+\hat{X}} = |c| + ||\hat{f}||_{\hat{X}};$

we similarly define $1 + \widehat{L}^1(\mathbb{R})$. The Fourier transform extends to 1

the constant 1 into the convolution identit and norming $1 + X$ with
 $+ \hat{f} \|_{1+\hat{X}} = |c| + \| \hat{f} \|_{\hat{X}};$

The Fourier transform extion identity δ .

g results.
 $\alpha + \hat{h} \in 1 + \hat{X}$. Then j

is non-vanishing on \mathbb{R} L^1

we similarly define $1 + L^1(\mathbb{R})$. The Fourier transform extends to $1 + X$ by mapping
the constant 1 into the convolution identity δ .
We will need the following results.
Lemma A.1. Suppose that $f = \alpha + \hat{h} \in 1 + \hat{X}$. the constant 1 into the convolution identity δ .
 Lemma A.1. Suppose that $f = \alpha + \hat{h} \in 1 + \hat{X}$.

algebra $1 + \hat{X}$ if and only if f is non-vanishin
 Proof. If f is invertible in $1 + \hat{X}$ it is also invertions. **Lemma A.1.** Suppose that $f = \alpha + h \in$ algebra $1 + \hat{X}$ if and only if f is non-vanishing on $\mathbb R$ and $\alpha \neq 0$.

na A.1. Suppose that $f = \alpha + \hat{h} \in$
ra $1 + \hat{X}$ if and only if f is non-volt.
If f is invertible in $1 + \hat{X}$ it is als
results by the Wiener theorem. If,
 $d \alpha \neq 0$ then f is invertible in $1 + \hat{h}$ = $\alpha + h \in 1 + X$. Then *f* is invertible in the Banach
f is non-vanishing on ℝ and $\alpha \neq 0$.
 $\vdash \hat{X}$ it is also invertible in $1 + \widehat{L}^1(\mathbb{R})$, so the condition
heorem. If, on the other hand, *f* does not vanish on 1 + *X if and only if f is non-vanishing on* ℝ *and* $\alpha \neq 0$.

if *f* is invertible in 1 + \hat{X} it is also invertible in 1 + $\widehat{L}^1(\mathbb{R})$

sary by the Wiener theorem. If, on the other hand, *f* dα
 $\alpha \neq 0$, *Proof.* If f is invertible in $1 + X$ it is also invertible in $1 + L^1$ If *f* is invertible in 1 + *X* it is also invertible in 1 + *L*
ssary by the Wiener theorem. If, on the other hand,
 $\alpha \neq 0$, then *f* is invertible in $1 + \widehat{L}^1(\mathbb{R})$ with $f^{-1} = c$
d to check that $g \in L^2(\mathbb{R})$. Wi (ℝ), so the condition
 f does not vanish on
 $-1 + \hat{g}$ for $g \in L^1(\mathbb{R})$.
 \Box

1 and compute that
 \Box

for $1 + \hat{X}$. is necessary by the Wiener theorem. If, on the other hand, f does not vanish on \mathbb{R} and $\alpha \neq 0$, then f is invertible in $1 + \widehat{L}^1(\mathbb{R})$ with $f^{-1} = \alpha^{-1} + \widehat{g}$ for $g \in L^1(\mathbb{R})$.
We need to check that $g \$ \mathbb{R} and $\alpha \neq 0$, then *f* is invertible in 1 + *L*¹ R and $\alpha \neq 0$, then *f* is invertible in $1 + L^{1}(\mathbb{R})$ with $f^{-1} = \alpha^{-1} + \widehat{g}$ for $g \in L^{1}(\mathbb{R})$.
We need to check that $g \in L^{2}(\mathbb{R})$. Without loss we take $\alpha = 1$ and compute that $\widehat{g} = -(1 + \widehat{h})^{-1}\widehat{h}$, wh $\hat{g} = -(1+h)^{-1}h$

We need to check that $g \in L^2(\mathbb{R})$. Without loss we take $\alpha = 1$ and compute that $\hat{g} = -(1 + \hat{h})^{-1}\hat{h}$, which shows that $g \in L^2(\mathbb{R})$ as required. \Box
We now have an analogue of the Wiener-Levi theorem for $1 \dot{+}$ = $-(1 + h)^{-1}h$, which shows that $g \in L^2(\mathbb{R})$ as required. \Box
We now have an analogue of the Wiener-Levi theorem for $1 \dot{+} \hat{X}$.
 emma A.2. Assume that $f \in 1 + \hat{X}$ and that ϕ is a function that is analytic in $)^{-1}h$

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ne m We now have an analogue of the Wiener–Levi theorem for $1 + X$.
 na A.2. Assume that $f \in 1 + \hat{X}$ and that ϕ is a function that is

ben neighborhood Ω of the closure of the range of f. Then $\phi \circ f \in \Omega$

over, the m $1 \dotplus X$ and that ϕ is a function that is analytic in
closure of the range of f. Then $\phi \circ f \in 1 \dotplus \hat{X}$ and,
 $f \mapsto \phi \circ f$
into itself when restricted to functions with range
bset of Ω . an open neighborhood moreover, the map

 $f \mapsto \phi \circ f$

 Ω of the closure of the range of f. Then $\phi \circ f \in 1 + X$ and,
 $f \mapsto \phi \circ f$
 $m \ 1 + \widehat{X}$ into itself when restricted to functions with range
 m pact subset of Ω .

the that, according to the above, the spectrum of f is an analytic map from contained in a fixed compact subset of Ω .

1 $\dot{+}$ *X* into itself when restricted to functions with range
pact subset of Ω .
that, according to the above, the spectrum of *f* in 1 $\dot{+}$ \hat{X}
e of its range. Then the standard functional calculus for
thus Ω.
ug
th Proof. % coincides with the closure of its range. Then the standard functional calculus for Banach algebras applies, thus yielding the result.
 \Box
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Acknowledgement

Banach algebras applies, thus yielding the result.
 \Box
 \Box
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Finite and Infinite Gap Jacobi Matrices

Jacob S. Christiansen

Abstract. The present paper reviews the theory of bounded Jacobi matrices whose essential spectrum is a finite gap set, and it explains how the theory can be extended to also cover a large number of infinite gap sets. Two of the central results are generalizations of Denisov–Rakhmanov's theorem and Szegő's theorem, including asymptotics of the associated orthogonal polynomials. When the essential spectrum is an interval, the natural limiting object J_0 has constant Jacobi parameters. As soon as gaps occur, ℓ say, the complexity increases and the role of J_0 is taken over by an ℓ -dimensional isospectral torus of periodic or almost periodic Jacobi matrices.

Mathematics Subject Classification (2010). Primary 47B36; Secondary 42C05. **Keywords.** Orthogonal polynomials, Szegő's theorem, Isospectral torus, Parreau–Widom sets.

1. Introduction

$$
\int_{\mathbb{R}} |x|^n d\mu(x) < \infty \quad \text{for all } n \ge 0. \tag{1.1}
$$

Let $d\mu$ be a probability measure on ℝ with moments of all orders, that is,
 $\int_{\mathbb{R}} |x|^n d\mu(x) < \infty$ for all $n \ge 0$.

When $d\mu$ is nontrivial (i.e., supp($d\mu$) is infinite), we can apply the Gram-Sprocess to 1, $x, x^$ When $d\mu$ is nontrivial (i.e., supp $(d\mu)$ is infinite), we can apply the Gram–Schmidt
process to $1, x, x^2, ...$ and obtain a sequence $\{P_n\}_{n\geq 0}$ of orthonormal polynomials
 $\langle P_n, P_m \rangle := \int_{\mathbb{R}} P_n(x) P_m(x) d\mu(x) = \delta_{nm},$ (1.2)

$$
\int_{\mathbb{R}} |x|^n d\mu(x) < \infty \text{ for all } n \ge 0. \tag{1.1}
$$
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$$
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$$
\npositive leading coefficient and is of degree n. It is a basic fact

\nals satisfy a three-term recurrence relation of the form

\n
$$
= a_{n+1} P_{n+1}(x) + b_{n+1} P_n(x) + a_n P_{n-1}(x), \quad n \ge 0 \tag{1.3}
$$
\nrted by a Stepo Besearch Grant (00.064947) from the Danish Besearch

process to 1, $x, x^2, ...$ and obtain a sequence $\{P_n\}_{n\geq 0}$ of orthonormal polynomials
 $\langle P_n, P_m \rangle := \int_{\mathbb{R}} P_n(x) P_m(x) d\mu(x) = \delta_{nm},$ (1.2)

where each P_n has positive leading coefficient and is of degree *n*. It is a basi where each P_n has positive leading coefficient and is of degree *n*. It is a basic fact
that such polynomials satisfy a three-term recurrence relation of the form
 $xP_n(x) = a_{n+1}P_{n+1}(x) + b_{n+1}P_n(x) + a_nP_{n-1}(x)$, $n \ge 0$ (1

$$
xP_n(x) = a_{n+1}P_{n+1}(x) + b_{n+1}P_n(x) + a_nP_{n-1}(x), \quad n \ge 0
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 (1.3)

 $xP_n(x) = a_{n+1}P_{n+1}(x) + b_{n+1}P_n(x) + a_nP_{n-1}(x), \quad n \ge 0$
The author was supported by a Steno Research Grant (09-064947) from the Danish
Council for Nature and Universe. $(x) = a_{n+1}P_{n+1}(x) + b_{n+1}P_n(x) + a_nP_{n-1}(x), \quad n \ge 0$ (1.3)
upported by a Steno Research Grant (09-064947) from the Danish Research
e and Universe. The author was supported by a Steno Research Grant (09-064947) from the Danish Research Council for Nature and Universe.

with $a_n = \langle P_{n-1}, xP_n \rangle > 0$ and $b_n = \langle P_{n-1}, xP_{n-1} \rangle \in \mathbb{R}$ for $n \ge 1$ (by convention, $P_{-1}(x) \equiv 0$). To see this, simply expand xP_n in terms of $P_0, P_1, \ldots, P_{n+1}$ and use the orthogonality relation (1.2). Note al with $a_n = \langle P_{n-1}, xP_n \rangle > 0$ and $b_n = \langle P_{n-1}, xP_{n-1} \rangle \in \mathbb{R}$ for $n \ge 1$ (by convention,
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the orthogonality relation (1.2). Note P_{-1}

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(x) \equiv 0
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. To see this, simply expand xP_n in terms of $P_0, P_1, \ldots, P_{n+1}$ and use orthogonality relation (1.2). Note also that\n
$$
P_n(x) = \frac{1}{a_1 \cdots a_n} \left(x^n - (b_1 + \cdots + b_n)x^{n-1} + \cdots \right) \text{ for } n \ge 1.
$$
\n(1.4)

\nThe spectral theorem for orthonormal polynomials (also known as Favard's term) states that for any pair of sequences $\{a_n, b_n\}^{\infty}$, $\in (0, \infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, there are a specific property of sequences of a_n to a_n and b_n are the same as follows:

 $P_n(x) = \frac{1}{a_1 \cdots a_n} \Big(x^n - (b_1 + \cdots + b_n)$
The spectral theorem for orthonormal po
theorem) states that for any pair of sequences
exists a probability measure du on $\mathbb R$ such that $f(x) = \frac{1}{a_1 \cdots a_n} \Big(x^n - (b_1 + \cdots + b_n) x^{n-1} + \cdots \Big)$ for $n \ge 1$. (1.4)
eetral theorem for orthonormal polynomials (also known as Favard's
tes that *for any pair of sequences* $\{a_n, b_n\}_{n=1}^{\infty} \in (0, \infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb$ em) states that *for any pair of sequences* $\{a_n, b_n\}_{n=1}^{\infty} \in (0, \infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, *there a probability measure dµ on* \mathbb{R} *such that the polynomials generated by* (1.3), $P_0(x) = 1$, *satisfy the ortho* theorem) states that *for any pair of sequences* $\{a_n, b_n\}_{n=}^{\infty}$
exists a probability measure dµ on \mathbb{R} *such that the polyno*
with $P_0(x) = 1$, *satisfy the orthogonality relation* (1.2).
of orthogonality nee $\sum_{n=1}^{\infty} \in (0, \infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, there
iomials generated by (1.3),
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rence with $P_0(x) = 1$, satisfy the orthogonality relation (1.2). In general, this measure $(x) = 1$, satisfy the orthogonality relation (1.2). In general, this measure
ogonality need not be unique. But when the recurrence coefficients are
d, say a_n , $|b_n| \le C$, then $d\mu$ is indeed unique and supp $(d\mu)$ is cont bounded, say a_n , $|b_n| \le C$, then $d\mu$ is indeed unique and supp $(d\mu)$ is contained in $[-3C, 3C]$. Conversely, if $d\mu$ has compact support, then the associated recurrence coefficients are bounded by $\max_{x \in \text{supp}(d\mu)} |x$ [-3C, 3C]. Conversely, if $d\mu$ has compact support, then the associated recurrence
coefficients are bounded by
 $\max_{x \in \text{supp}(d\mu)} |x| < \infty$
and the polynomials are dense in $L^2(d\mu)$. We shall henceforth assume that supp(

$$
\max_{x\in \text{supp}(d\mu)} |x| < \infty
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[−3C, 3C]. Conversely, if $d\mu$ has compact support, then the associated recurrence
coefficients are bounded by
 $\max_{x \in \text{supp}(d\mu)} |x| < \infty$
and the polynomials are dense in $L^2(d\mu)$. We shall henceforth assume that supp $(d\$ and the polynomials are density compact. $x \in supp(d\mu)$
 $L^2(d\mu)$. V

relation (

al matrice and the polynomials are dense in $L^2(d\mu)$. We shall henceforth assume that supp($d\mu$)
is compact.
The three-term recurrence relation (1.3) links orthogonal polynomials to Ja-
cobi matrices, that is, tridiagonal matrices

cobi matrices, that is, tridiagonal matrices of the form

$$
J = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 & a_2 \\ & a_2 & b_3 & a_3 \\ & & \ddots & \ddots \end{pmatrix}
$$
 (1.5)

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"it Fix J in $(1.5$
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 J is viewed
and we shall (or of
spect), its
ectral with $a_n > 0$ and $b_n \in \mathbb{R}$. In fact, the matrix J in (1.5) represents the operator of multiplication by the identity function x in the Hilbert space $L^2(d\mu)$ with respect to the orthonormal basis $\{P_n\}_{n\geq 0}$. multiplication by the identity function *x* in the Hilbert space $L^2(d\mu)$ with respect
to the orthonormal basis $\{P_n\}_{n\geq 0}$. When *J* is viewed as an operator on $\ell^2(\mathbb{N})$, its
spectrum $\sigma(J)$ coincides with supp

to the orthonormal basis $\{P_n\}_{n\geq 0}$. When J is viewed as an operator on $\ell^2(\mathbb{N})$, its
spectrum $\sigma(J)$ coincides with supp $(d\mu)$ and we shall refer to $d\mu$ as the spectral
measure of J .
In spectral theory for spectrum $\sigma(J)$ coincides with supp($d\mu$) and we shall refer to $d\mu$ as the spectral
measure of J.
In spectral theory for orthogonal polynomials, one studies the relation be-
tween nontrivial probability measures $d\mu$ measure of *J*.
In spect
tween nontriv
sequences $\{a_n\}$
paper is to gi
and the recur
sequences. As tween nontrivial probability measures $d\mu$ satisfying (1.1) on one hand and pairs of sequences $\{a_n, b_n\}_{n=1}^{\infty} \in (0, \infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ on the other hand. The aim of the present paper is to give a general view paper is to give a general view of the situation where $d\mu$ is compactly supported and the recurrence coefficients (also known as Jacobi parameters) are bounded sequences. As already mentioned, there is a one-one corresp $\sum_{n=1}^{\infty} \in (0, \infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ on the other hand. The aim of the present general view of the situation where $d\mu$ is compactly supported e coefficients (also known as Jacobi parameters) are bounded ady ment paper is to give a general view of the situation where $d\mu$ is compactly supported
and the recurrence coefficients (also known as Jacobi parameters) are bounded
sequences. As already mentioned, there is a one-one corresp sequences. As already mentioned, there is a one-one correspondence between these two classes of objects and we shall focus on results that explain how qualitative features of the Jacobi parameters, say, are reflected in t two classes of objects and we shall focus on results that explain how qualitative fea-
tures of the Jacobi parameters, say, are reflected in the measure of orthogonality,
and vice versa.
Throughout, we shall write the pro tures of the Jacobi parameters, say, are reflected in the measure of orthogonality,
and vice versa.
Throughout, we shall write the probability measure $d\mu$ as
 $d\mu = f(x)dx + d\mu_s,$ (1.6) and vice versa.
Throughout, we shall write the probability measure $d\mu$ as
 $d\mu = f(x)dx + d\mu_s,$ (1.6)

 $\label{prop:1}$ Through
o $\,$ Throughout, we shall write the probability measure $d\mu$ as

$$
d\mu = f(x)dx + d\mu_{\rm s},\tag{1.6}
$$

Finite and Infinite Gap Jacobi Matrices 45
with $d\mu_s$ singular to dx. Rather than $\sigma(J)$, many of the results are more suitably
formulated in terms of $\sigma_{\rm ess}(J)$, the essential spectrum of J. By definition,

$$
\sigma_{\rm ess}(J) := \{ x \in \sigma(J) \mid x \text{ is not an isolated eigenvalue of } J \}. \tag{1.7}
$$

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As reg formulated in terms of $\sigma_{\text{ess}}(J)$, the essential spectrum of J . By definition,
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As regards proofs, in particular, a key role is played by the *m*-function (or transform of d

$$
(J) := \{x \in \sigma(J) \mid x \text{ is not an isolated eigenvalue of } J\}. \tag{1.7}
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\nofs, in particular, a key role is played by the *m*-function (or Stieltjes μ) defined by

\n
$$
m(z) := m_{\mu}(z) = \int \frac{d\mu(x)}{x - z}, \quad z \in \mathbb{C} \setminus \text{supp}(d\mu). \tag{1.8}
$$
\nunction is known to be a Nevanlinna–Pick function (i.e., Im $m(z) \geq 0$ and we have

\n
$$
m(z) = -1/z + \mathcal{O}(z^{-2}) \tag{1.9}
$$
\net, one can write down the Laurent expansion of m_{μ} around ∞ in the case of d_{μ} . Now, in the fact that the behavior of m_{μ} is a constant, we can use the following equations:

As regards proofs, in particular, a key role is played by the *m*-function (or Stieltjes
transform of $d\mu$) defined by
 $m(z) := m_{\mu}(z) = \int \frac{d\mu(x)}{x - z}$, $z \in \mathbb{C} \setminus \text{supp}(d\mu)$. (1.8)
This analytic function is known to be transform of $d\mu$) defined by
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This analytic function is know

for Im $z \ge 0$) and we have

$$
m(z) = -1/z + \mathcal{O}(z^{-2})
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\n(1.9)

This analytic function is known to be a Nevanlinna–Pick function (i.e., Im $m(z) \ge 0$
for Im $z \ge 0$) and we have
 $m(z) = -1/z + \mathcal{O}(z^{-2})$ (1.9)
near ∞ . In fact, one can write down the Laurent expansion of m_{μ} around for Im $z \ge 0$) and we have
near ∞ . In fact, one can
terms of the moments of a
 $\lim_{\varepsilon \downarrow 0} m(x + i\varepsilon)$ exist for (z) = −1/z + $\mathcal{O}(z^{-2})$ (1.9)
down the Laurent expansion of m_{μ} around ∞ in
ore importantly, the boundary values $m(x + i0) :=$
 $\in \mathbb{R}$ and
 $(x + i\varepsilon) dt \xrightarrow{w} d\mu$ as $\varepsilon \downarrow 0$. (1.10) near ∞. In fact, one can write down the Laurent expansion of m_{μ} around ∞ in
terms of the moments of $d\mu$. More importantly, the boundary values $m(x + i0) :=$
 $\lim_{\varepsilon \downarrow 0} m(x + i\varepsilon)$ exist for a.e. $x \in \mathbb{R}$ and
 $\frac{1}{$ terms of the moments of $d\mu$. More importantly, the boundary values $m(x + i0) :=$
 $\lim_{\varepsilon \downarrow 0} m(x + i\varepsilon)$ exist for a.e. $x \in \mathbb{R}$ and
 $\frac{1}{\pi} \operatorname{Im} m_{\mu}(x + i\varepsilon) dt \xrightarrow{w} d\mu$ as $\varepsilon \downarrow 0$. (1.10)

To be even more specifi

$$
\frac{1}{\pi} \operatorname{Im} m_{\mu}(x + i\varepsilon) dt \xrightarrow{w} d\mu \text{ as } \varepsilon \downarrow 0. \tag{1.10}
$$

ic,

$$
f(x) = \frac{1}{\pi} \operatorname{Im} m_{\mu}(x + i0) \text{ a.e. on } \mathbb{R} \tag{1.11}
$$

$$
\lim_{\varepsilon \downarrow 0} m(x + i\varepsilon) \text{ exist for a.e. } x \in \mathbb{R} \text{ and}
$$
\n
$$
\frac{1}{\pi} \operatorname{Im} m_{\mu}(x + i\varepsilon) dt \xrightarrow{w} d\mu \text{ as } \varepsilon \downarrow 0. \tag{1.10}
$$
\nTo be even more specific,
\n
$$
f(x) = \frac{1}{\pi} \operatorname{Im} m_{\mu}(x + i0) \text{ a.e. on } \mathbb{R} \tag{1.11}
$$
\nand
\n
$$
\mu_{\text{s}}(\{x\}) = \lim_{\varepsilon \to 0} \varepsilon \operatorname{Im} m_{\mu}(x + i\varepsilon) \text{ for all } x \in \mathbb{R}. \tag{1.12}
$$

$$
f(x) = \frac{1}{\pi} \operatorname{Im} m_{\mu}(x + i0) \text{ a.e. on } \mathbb{R}
$$
\n(1.11)

\nand

\n
$$
\mu_{\mathbf{s}}(\{x\}) = \lim_{\varepsilon \to 0} \varepsilon \operatorname{Im} m_{\mu}(x + i\varepsilon) \text{ for all } x \in \mathbb{R}. \tag{1.12}
$$
\nSo isolated mass points of $d\mu$ (or isolated eigenvalues of J) are poles of m .

\nThe simplest compact subsets of \mathbb{R} that have positive measure are intervals of the form $[\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$. In Section 2, we shall consider the

So i
of t
situ = $\lim_{\varepsilon \to 0} \varepsilon \text{Im} m_\mu(x + i\varepsilon)$ for all $x \in \mathbb{R}$. (1.12)
 $d\mu$ (or isolated eigenvalues of *J*) are poles of *m*.

t subsets of \mathbb{R} that have positive measure are intervals
 $\infty < \alpha < \beta < \infty$. In Section 2, we So isolated mass points of $d\mu$ (or isolated eigenvalues of J) are poles of m .
The simplest compact subsets of \mathbb{R} that have positive measure are in
of the form $[\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$. In Section 2, we sh The simplest compact subsets of ℝ that have positive measure are intervals
e form $[\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$. In Section 2, we shall consider the
tion when $\sigma_{\text{ess}}(J)$ has this form and without loss of generality we situation when $\sigma_{\text{ess}}(J)$ has this form and without loss of generality we may assume
that $-\alpha = \beta = 2$. The associated Jacobi parameters are often – but not always
– close to 1 and 0 as $n \to \infty$. Orthogonal polynomials o situation when $\sigma_{\text{ess}}(J)$ has this form and without loss of generality we may assume
that $-\alpha = \beta = 2$. The associated Jacobi parameters are often – but not always
– close to 1 and 0 as $n \to \infty$. Orthogonal polynomials o that $-\alpha = \beta = 2$. The associated Jacobi parameters are often – but not always – close to 1 and 0 as $n \to \infty$. Orthogonal polynomials on a compact interval are intimately related to Jacobi parameters that are asymptotically

– close to 1 and 0 as $n \to \infty$. Orthogonal polynomials on a compact interval are intimately related to Jacobi parameters that are asymptotically constant. As we shall see, the theory is well developed and many precise r shall see, the theory is well developed and many precise results are available.
In Section 3, we generalize our studies to finite gap sets ϵ , that is, finite
unions of closed intervals. When ϵ is the union of two or In Section 3, we generalize our studies to finite gap sets \mathbf{e} , that is, finite unions of closed intervals. When \mathbf{e} is the union of two or more disjoint intervals, the complement $\overline{\mathbb{C}} \setminus \mathbf{e}$ is no long unions of closed intervals. When \mathfrak{e} is the union of two or more disjoint intervals, the complement $\overline{\mathbb{C}} \setminus \mathfrak{e}$ is no longer simply connected. This is to be overcome by using the universal covering map. Pe the complement $\mathbb{C} \setminus \mathfrak{e}$ is no longer simply connected. This is to be overcome by
using the universal covering map. Perhaps more seriously, the structure of the
Jacobi parameters changes. They are no longer asymp Jacobi parameters changes. They are no longer asymptotically constant but rather asymptotically periodic or almost periodic. The natural limit point (viz., the free Jacobi matrix) also has to be replaced by an ℓ -dimens asymptotically periodic or almost periodic. The natural limit point (viz., the free
Jacobi matrix) also has to be replaced by an ℓ -dimensional torus, where ℓ counts
the number of gaps in ϵ .
Finally, in Section 4

Jacobi matrix) also has to be replaced by an ℓ -dimensional torus, where ℓ counts the number of gaps in **e**.
Finally, in Section 4 we consider infinite gap sets of Parreau–Widom type.
This notion of regular compact the number of gaps in ε .
Finally, in Section
This notion of regular
among others. The thec
nite gap sets can be exte notion of regular compact sets includes Cantor sets of positive measure, \log others. The theory is less developed, but many results that hold for fi-
gap sets can be extended to the infinite gap setting. among others. The theory is less developed, but many results that hold for finite gap sets can be extended to the infinite gap setting. mute gap sets can be extended to the infinite gap setting.
 $\label{eq:1}$

2. Perturbations of the free Jacobi matrix
The most natural choice of Jacobi parameters is
 $a_n \equiv 1$ and $b_n \equiv 0$ **2. Perturbations of the free Jacobi matrix**

$$
a_n \equiv 1 \text{ and } b_n \equiv 0. \tag{2.1}
$$

The most natural choice of Jacobi parameters is
 $a_n \equiv 1$ and $b_n \equiv 0$. (2.1)

As is well known, the associated orthogonal polynomials are Chebyshev of the 2nd

kind

$$
U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)}, \quad x = 2\cos(\theta).
$$

They are orthogonal on the interval $[-2,2]$ with respect to the semicircle law kind
 $U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}$, $x = 2\cos \theta$.

They are orthogonal on the interval $[-2, 2]$ with respect to the semicircle law
 $f_0(x) = \sqrt{4 - x^2}/2\pi$. We shall follow the standard terminology and refer to $U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad x = 2\cos\theta.$
They are orthogonal on the interval $[-2, 2]$ with respection $f_0(x) = \sqrt{4 - x^2}/2\pi$. We shall follow the standard termino $J_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

They are orthogonal on the interval
$$
[-2, 2]
$$
 with respect to the semicircle law
\n
$$
f_0(x) = \sqrt{4 - x^2}/2\pi.
$$
 We shall follow the standard terminology and refer to\n
$$
J_0 = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}
$$
\nas the free Jacobi matrix.
\nIf $a_n \to 1$ and $b_n \to 0$, then $J = \{a_n, b_n\}_{n=1}^{\infty}$ is a compact perturbation
\nof J_0 and hence $\sigma_{\text{ess}}(J) = [-2, 2]$ by Weyl's theorem. There may be points in

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ure $= \{a_n,$
Weyl's
lated ma ${a_n, b_n}_{n=1}^{\infty}$
eyl's theo:
d mass po
tes that t $\frac{1}{2}$ ation
ts in If $a_n \to 1$ and $b_n \to 0$, then $J = \{a_n, b_n\}_{n=1}^{\infty}$ is a compact perturbation
of J_0 and hence $\sigma_{\text{ess}}(J) = [-2, 2]$ by Weyl's theorem. There may be points in
supp $(d\mu) \setminus [-2, 2]$, but these are all isolated mass points $n=1$ supp $(d\mu) \setminus [-2, 2]$, but these are all isolated mass points that can only accumulate
at ± 2 . Moreover, a result of Nevai [14] states that the ratio $P_{n+1}(x)/P_n(x)$ has a
limit for $x \notin \sigma(J)$.
The condition $\sigma_{\rm ess}(J) = [-2,$

supp($d\mu$) \[-2, 2], but these are all isolated mass points that can only accumulate
at ± 2 . Moreover, a result of Nevai [14] states that the ratio $P_{n+1}(x)/P_n(x)$ has a
limit for $x \notin \sigma(J)$.
The condition $\sigma_{\rm ess}(J) = [-2$ at ± 2 . Moreover, a result of Nevai [14] states that the ratio $P_{n+1}(x)/P_n(x)$ has a
limit for $x \notin \sigma(J)$.
The condition $\sigma_{\text{ess}}(J) = [-2, 2]$, on the other hand, is by itself not strong
enough to imply $a_n \to 1$ and $b_n \to$ limit for $x \notin \sigma(J)$.
The conditio
enough to imply ϵ
example). An ext
Rakhmanov theor
[-2, 2], then $a_n \to$
trix (i.e., the matr The condition $\sigma_{\text{ess}}(J) = [-2, 2]$, on the other hand, is by itself not strong
gh to imply $a_n \to 1$ and $b_n \to 0$ (see, e.g., [21, Section 1.4] for a counter-
ple). An extra condition is needed and for $d\mu$ as in (1.6), t enough to imply $a_n \to 1$ and $b_n \to 0$ (see, e.g., [21, Section 1.4] for a counter-
example). An extra condition is needed and for $d\mu$ as in (1.6), the Denisov-
Rakhmanov theorem [9] states that if $\sigma_{\rm ess}(J) = [-2, 2]$ and example). An extra condition is needed and for $d\mu$ as in (1.6), the Denisov-
Rakhmanov theorem [9] states that if $\sigma_{\rm ess}(J) = [-2, 2]$ and $f(x) > 0$ a.e. on
 $[-2, 2]$, then $a_n \to 1$ and $b_n \to 0$. Denoting by J_n the n time $[-2, 2]$, then $a_n \to 1$ and $b_n \to 0$. Denoting by J_n the n times stripped Jacobi matrix (i.e., the matrix obtained from J by removing the first n rows and columns), the above conclusion can also be formulates as $J_n \to$

[−2, 2], then $a_n \to 1$ and $b_n \to 0$. Denoting by J_n the *n* times stripped Jacobi matrix (i.e., the matrix obtained from J by removing the first n rows and columns), the above conclusion can also be formulates as J trix (i.e., the matrix obtained from J by removing the first n rows and columns),
the above conclusion can also be formulates as $J_n \to J_0$ strongly.
The more detailed spectral analysis involves the rate of convergence the above conclusion can also be formulates as $J_n \to J_0$ strongly.

The more detailed spectral analysis involves the rate of co

Jacobi parameters. Of particular interest are the cases of Hilb

trace-class perturbations o bi parameters. Of particular interest are the cases of Hilbert–Schmidt and class perturbations of J_0 . A deep result of Killip and Simon [12] classifies the ral measures of all Jacobi matrices $J = \{a_n, b_n\}_{n=1}^{\infty}$ for trace-class perturbations of J_0 . A deep result of Killip and Simon [12] classifies the
spectral measures of all Jacobi matrices $J = \{a_n, b_n\}_{n=1}^{\infty}$ for which
 $\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty$. (2.3)
They all have

spectral measures of all Jacobi matrices
$$
J = \{a_n, b_n\}_{n=1}^{\infty}
$$
 for which
\n
$$
\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty.
$$
\n(2.3)
\nThey all have
\n
$$
\text{supp}(d\mu) = [-2, 2] \cup \{x_k\},
$$
\nwhere $\{x_k\}$ is a countable set of isolated mass points, possibly empty, and are
\nprecisely those probability measures of the form (1.6) that satisfy

$$
supp(d\mu) = [-2,2] \cup \{x_k\},\
$$

supp $(d\mu) = [-2, 2] \cup \{x_k\}$,
where $\{x_k\}$ is a countable set of isolated mass point
precisely those probability measures of the form (1.6)
 $\int_{-2}^{2} \log f(x) \sqrt{4 - x^2} dx > -$

where
$$
\{x_k\}
$$
 is a countable set of isolated mass points, possibly empty, and are
precisely those probability measures of the form (1.6) that satisfy

$$
\int_{-2}^{2} \log f(x) \sqrt{4 - x^2} dx > -\infty
$$
(2.4)

and

$$
\sum_{k} (|x_k| - 2)^{3/2} < \infty.
$$
\n(2.5)

\nillip-Simon's theorem relies on sum rules, obtained from a function. More precisely, one shows that

factorization of the m -function. More precisely, one shows that 2 om a
 (2.6)
 $\text{ere } B$

$$
M(z) := -m(z + 1/z), \quad |z| < 1 \tag{2.6}
$$

factorization of the *m*-function. More precisely, one shows that
 $M(z) := -m(z + 1/z), \quad |z| < 1$ (2.6)

is a meromorphic Herglotz function and hence of the form $M = B \cdot O$, where B

is an alternating Blaschke product and O an oute (z) := $-m(z + 1/z)$, $|z| < 1$ (2.6)
function and hence of the form $M = B \cdot O$, where B
product and O an outer function (see [18] for details).
from computing the Taylor coefficients of $\log(M(z)/z)$ is a meromorphic Herglotz function and hence of the form $M = B \cdot O$, where B
is an alternating Blaschke product and O an outer function (see [18] for details).
The sum rules now result from computing the Taylor coefficients is an alternating Blaschke product and O an outer function (see [18] for details).
The sum rules now result from computing the Taylor coefficients of $\log(M(z)/z)$
in two different ways.
Note that
 $\phi(z) := z + 1/z$ (2.7)
is the un

$$
\phi(z) := z + 1/z \tag{2.7}
$$

The sum rules now result from computing the Taylor coefficients of $\log(M(z)/z)$
in two different ways.
Note that
 $\phi(z) := z + 1/z$ (2.7)
is the unique conformal mapping of the unit disk \mathbb{D} onto $\overline{\mathbb{C}}\setminus[-2,2]$ for which Note that
is the unique conform:
 ∞ and $\lim_{z\to 0} z\phi(z)$
goes back at least to
Compared to (2) $\phi(z) := z + 1/z$ (2.7)

unique conformal mapping of the unit disk \mathbb{D} onto $\overline{\mathbb{C}} \setminus [-2, 2]$ for which $\phi(0) =$

od $\lim_{z\to 0} z\phi(z) = 1$. The use of ϕ in the theory of orthogonal polynomials

back at least to Szegő. is the unique conformal mapping of the unit disk \mathbb{D} onto $\mathbb{C}\setminus[-2, 2]$ for which $\phi(0) = \infty$ and $\lim_{z\to 0} z\phi(z) = 1$. The use of ϕ in the theory of orthogonal polynomials goes back at least to Szegő.
Compared ∞

and
$$
\lim_{z\to 0} z\phi(z) = 1
$$
. The use of ϕ in the theory of orthogonal polynomials
is back at least to Szegő.
Compared to (2.3), the a priori stronger condition

$$
\sum_{n=1}^{\infty} |a_n - 1| + |b_n| < \infty
$$
(2.8)
is conjectured by Nevai [13] and later proven by Killing and Simon [12] to imply

Compared to (2.3) , the
was conjectured by Nevai [1]
the Szeg \tilde{c} condition, that is $\sum_{n=1}^{\infty} |a_n - 1| + |b_n| < \infty$
conjectured by Nevai [13] and later proven by Killig
zegő condition, that is, the Szegő condition, that is,
 \int_{-2}^{2} .
In turn, (2.9) is closely related to

$$
\sum_{n=1} |a_n - 1| + |b_n| < \infty
$$
 (2.8)
was conjectured by Nevai [13] and later proven by Killing and Simon [12] to imply
the Szegő condition, that is,

$$
\int_{-2}^{2} \frac{\log f(x)}{\sqrt{4 - x^2}} dx > -\infty.
$$
 (2.9)
In turn, (2.9) is closely related to

$$
a_1 \cdots a_n \neq 0
$$
 (2.10)
and

$$
\sum_{n=1}^{\infty} \frac{(-1 - 2)^{1/2}}{2}
$$
 (2.11)

$$
a_1 \cdots a_n \nrightarrow 0 \tag{2.10}
$$

$$
J_{-2} \sqrt{4 - x^2}
$$

\nd to
\n
$$
a_1 \cdots a_n \nrightarrow 0
$$
\n
$$
\sum_k (|x_k| - 2)^{1/2} < \infty.
$$
\n(2.11)

and \sum_{k} what is known as Szegő's theorem $\langle \infty,$ (2.11)
 if (2.11) *holds*, *then* (2.9) *is equiva-*

2.11) so as formulated by Simon and Wh
lent
Zlat
(2.9 t
1
h
h $quiva$ -
on and
when
d, and What is known as Szegő's theorem states that if (2.11) holds, then (2.9) is equiva-
lent to (2.10). Moreover, (2.9)–(2.10) implies (2.11) so as formulated by Simon and
Zlatoš [22], any two imply the third. In the setting lent to Zlatoš [22], any two imply the third. In the setting of Szegő's theorem (i.e., when

(2.9)–(2.11) hold), the product in (2.10) has a positive limit, (2.3) is satisfied, and

both of the series
 $\sum_{n=1}^{\infty} (a_n - 1)$, $\sum_{$

$$
\sum_{n=1}^{\infty} (a_n - 1), \sum_{n=1}^{\infty} b_n
$$
 (2.12)
Furthermore, a result of Peherstorfer and Yuditskii

$$
P_n(z + 1/z) \to \frac{B(z)D(z)}{1 - z^2}
$$
 (2.13)

both of the series
 $\sum_{n=1}^{\infty} (a_n - 1), \sum_{n=1}^{\infty} b_n$ (2.12)

are conditionally convergent. Furthermore, a result of Peherstorfer and Yuditskii

[15] states that are conditionally
 $[15] \text{ states that}$ \mathbf{r}_{1}

are conditionally convergent. Furthermore, a result of Peherstorfer and Yuditskii
[15] states that

$$
z^n P_n(z+1/z) \rightarrow \frac{B(z)D(z)}{1-z^2}
$$
(2.13)

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uniformly on compact subsets of D, where B is the Blaschke product

$$
B(z) = \prod_{k} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - z_k z}, \quad z_k = \frac{1}{2} \left(x_k - \sqrt{x_k^2 - 4} \right)
$$
and D the outer function

$$
D(z) = \exp \left\{ \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \left(\frac{|\sin \theta|}{\pi f(2 \cos \theta)} \right) \frac{d\theta}{4\pi} \right\}.
$$

$$
B(z) = \prod_{k} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - z_k z}, \quad z_k = \frac{1}{2} \Big(x_k - \sqrt{x_k^2 - 4} \Big)
$$

function

$$
D(z) = \exp \Big\{ \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \Big(\frac{|\sin \theta|}{\pi f(2 \cos \theta)} \Big) \frac{d\theta}{4\pi} \Big\}.
$$

ver asymptotic behavior is known as Szegő asymp

$$
U_n(z + 1/z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \sim \frac{z^{-n}}{1 - z^2},
$$

$$
z^n
$$
 by $1/U_n(z + 1/z)$ on the left-hand side in (2.

This type of power asymptotic behavior is known as Szegő asymptotics. Note that and *D* the outer function
 $D(z) = e$

This type of power asymprosince
 $U_n(z)$ $\frac{4\pi}{3}$
syl $,$
 $\frac{1}{2}$

$$
U_n(z+1/z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \sim \frac{z^{-n}}{1 - z^2},
$$

since
 $U_n(z+1/z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \sim \frac{z^{-n}}{1 - z^2}$,

we can replace z^n by $1/U_n(z+1/z)$ on the left-hand side in (2.13) if the factor
 $1-z^2$ on the right-hand side is removed too.

While the Szegő condition implies we c:
 $1-z$
it is

 $(z + 1/z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \sim \frac{z^{-n}}{1 - z^2}$,
 $/U_n(z + 1/z)$ on the left-hand side is

d side is removed too.

ndition implies Szegő asymptotics, as

ndition. Examples for which (2.11) f

s a limit are given by Damanik a we can replace z^n by $1/U_n(z + 1/z)$ on the left-hand side in (2.13) if the factor $1 - z^2$ on the right-hand side is removed too.
While the Szegő condition implies Szegő asymptotics, as has long been known, it is not a nece 1 − z^2 on the right-hand side is removed too.
While the Szegő condition implies Szegő
it is not a necessary condition. Examples for
hand side of (2.13) has a limit are given by
importantly, [8] proves that $z^n P_n(z + 1/z)$ not a necessary condition. Examples for which (2.11) fails and yet the left-
side of (2.13) has a limit are given by Damanik and Simon in [8]. More
rtantly, [8] proves that $z^n P_n(z + 1/z)$ has a limit for all $z \in \mathbb{D}$ if hand side of (2.13) has a limit are given by Damanik and Simon in [8]. More
importantly, [8] proves that $z^n P_n(z + 1/z)$ has a limit for all $z \in \mathbb{D}$ if and only if
(2.3) holds and the series in (2.12) are conditionally c

3. Finite gap Jacobi matrices

side of (2.13), however, is only correct when (2.9) holds.
 3. Finite gap Jacobi matrices

In this section, we shall consider Jacobi matrices $J = \{a_n, b_n\}_{n=1}^{\infty}$ for which $\sigma_{\text{ess}}(J)$ **3. Finite gap Jacobi matrices**
In this section, we shall consider Jacobi matrices $J = \{a_n\}$ is a finite gap set, that is, a set of the form

In this section, we shall consider Jacobi matrices
$$
J = \{a_n, b_n\}_{n=1}^{\infty}
$$
 for which $\sigma_{\text{ess}}(J)$ is a finite gap set, that is, a set of the form\n
$$
\mathfrak{e} = \bigcup_{j=1}^{\ell+1} [\alpha_j, \beta_j], \quad \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_{\ell+1}.\tag{3.1}
$$
\nApart from a single interval, such a finite union of closed intervals is the simplest time of compact sets in \mathbb{R} with positive measure (and no isolated points). Note that the set \mathfrak{e} is the sum of the set of \mathbb{R}^n .

 $\mathfrak{e} = \bigcup_{j=1}^{\ell+1} [\alpha_j, \beta_j], \quad \alpha_1 < \beta$

Apart from a single interval, such a finite u

type of compact sets in R with positive m

that ℓ counts the number of sons in a and if $\frac{1}{15}$
 $\frac{1}{15}$
 $\frac{1}{15}$ (1)
plest
Note
d for type of compact sets in $\mathbb R$ with positive measure (and no isolated points). Note that ℓ counts the number of gaps in $\mathfrak e$ and when $\ell \geq 1$, two questions arise:

• Is there a natural choice of J that can serv

- type of compact sets in ℝ with positive measure (and no isolated points). Note
that ℓ counts the number of gaps in \mathfrak{e} and when $\ell \geq 1$, two questions arise:
• Is there a natural choice of *J* that can serve as ∙
	- What replaces the conformal mapping ϕ in (2.7) when $\overline{\mathbb{C}} \setminus \mathfrak{e}$ is no longer

The answer to the first question is negative. There is no single J that will take Is there a natural choice of J that can serve as a limit point, like J_0 did for
the interval $[-2, 2]$?
What replaces the conformal mapping ϕ in (2.7) when $\overline{\mathbb{C}} \setminus \mathfrak{e}$ is no longer
simply connected?
answe the interval $[-2, 2]$?
What replaces the
simply connected?
answer to the first α
the role of J_0 . Even
everal sequences of p
 $b_{n+2} = b_n$ for all n) What replaces the conformal mapping ϕ in (2.7) when $\mathbb{C} \setminus \mathfrak{e}$ is no longer
simply connected?
answer to the first question is negative. There is no single *J* that will take
the role of J_0 . Even when $\mathfrak{e$ The answer to the first question is negative. There is no single *J* that will take
over the role of J_0 . Even when **c** only has one gap, say $\mathbf{c} = [-2, -1] \cup [1, 2]$, there
are several sequences of periodic Jacobi param over the role of J_0 . Even when \mathfrak{e} only has one gap, say $\mathfrak{e} = [-2, -1] \cup [1, 2]$, there are several sequences of periodic Jacobi parameters with period 2 (i.e., $a_{n+2} = a_n$ and $b_{n+2} = b_n$ for all *n*) leading t are several sequences of periodic Jacobi parameters with period 2 (i.e., $a_{n+2} = a_n$ and $b_{n+2} = b_n$ for all n) leading to the right spectrum, namely **c**. And it seems impossible to pick out one that should be more natura and $b_{n+2} = b_n$ for all *n*) leading to the right spectrum, namely **c**. And it seems
impossible to pick out one that should be more natural than all the others. In
fact, the Denisov-Rakhmanov theorem is known to fail when fact, the Denisov–Rakhmanov theorem is known to fail when $[-2, 2]$ is replaced by a finite gap set with at least one gap. The Jacobi parameters need not approach a finite gap set with at least one gap. The Jacobi paramete

Finite and Infinite Gap Jacobi Matrices 49
a single point. Rather, they approach a set which is topologically a circle (or a
1-dimensional torus) when $\ell = 1$.
For a general finite gap set \mathfrak{e} as in (3.1), Simon [19 1-dimensional torus) when $\ell = 1$.
For a general finite gap set \mathfrak{e} as in (3.1), Simon [19,20] suggested to introduce
the so-called isospectral torus $\mathcal{T}_{\mathfrak{e}}$ of dimension ℓ . The structure of this limiting
 For a general finite gap set \mathfrak{e} as in (3.1), Simon [19,20] suggested to introduce o-called isospectral torus $\mathcal{T}_{\mathfrak{e}}$ of dimension ℓ . The structure of this limiting t is carefully described in [4]. It cons the so-called isospectral torus $\mathcal{T}_{\mathfrak{e}}$ of dimension ℓ . The structure of this limiting
object is carefully described in [4]. It consists of all Jacobi matrices whose *m*-
function is a minimal Herglotz function object is carefully described in [4]. It consists of all Jacobi matrices whose m-
function is a minimal Herglotz function on the two-sheeted Riemann surface S
associated with \mathfrak{e} . Loosely speaking, one can think o function is a minimal Herglotz function on the two-sheeted Riemann surface σ associated with ϵ . Loosely speaking, one can think of S as two copies of $\mathbb{C} \setminus \epsilon$ glued together suitably. Alternatively, \mathcal{T}_{ϵ} $n=-\infty$

glued together suitably. Alternatively, r_{ϵ} is the collection of all two-sided Jacobi
matrices $J = \{a_n, b_n\}_{n=-\infty}^{\infty}$ that have spectrum ϵ and are reflectionless on ϵ (see,
e.g., [17,23] for more details).
The matrices $J = \{a_n, b_n\}_{n=-\infty}^{\infty}$ that have spectrum \mathfrak{e} and are reflectionless on \mathfrak{e} (see, e.g., [17,23] for more details).
The isospectral torus is invariant under coefficient stripping, a very useful fact. The isospectral torus is
fact. If J' is a point on \mathcal{T}_{ϵ} , tl
or almost periodic sequences
rational harmonic measure (i
the equilibrium measure of ϵ).
harmonic measure. The spect If J' is a point on \mathcal{T}_{ϵ} , then the Jacobi parameters $\{a'_n, b'_n\}_{n=1}^{\infty}$ are periodic
most periodic sequences, depending on whether the intervals in ϵ all have
nal harmonic measure (i.e., whether $\mu_{\epsilon}([\alpha_j,$ fact. If J' is a point on \mathcal{T}_{ϵ} , then the Jacobi parameters $\{a'_n, b'_n\}_{n=1}^{\infty}$ are periodic
or almost periodic sequences, depending on whether the intervals in ϵ all have
rational harmonic measure (i.e., whe or almost periodic sequences, depending on whether the intervals in \mathbf{c} all have
rational harmonic measure (i.e., whether $\mu_{\mathbf{c}}([\alpha_j, \beta_j]) \in \mathbb{Q}$ for all j, where $d\mu_{\mathbf{c}}$ is
the equilibrium measure of $\mathbf{c$ rational harmonic measure (i.e., whether $\mu_{\mathfrak{e}}([\alpha_j, \beta_j]) \in \mathbb{Q}$ for all j, where $d\mu_{\mathfrak{e}}$ is
the equilibrium measure of \mathfrak{e}). We say that \mathfrak{e} is periodic if all $[\alpha_j, \beta_j]$ have rational
harmonic mea $([\alpha_j,\beta_j]) \in \mathbb{Q}$ the equilibrium measure of \mathbf{e}). We say that \mathbf{e} is periodic if all $[\alpha_j, \beta_j]$ have rational
harmonic measure. The spectral measure of J' is also very regular. It is purely
absolutely continuous on \mathbf{e} wi harmonic measure. The spectral measure of J' is also very regular. It is purely
absolutely continuous on \mathfrak{e} with a density that satisfies the Szegő condition (see
(3.3) below). Besides, it has at most one mass po absolutely continuous on \mathfrak{e} with a density that satisfies the Szegő condition (see
(3.3) below). Besides, it has at most one mass point in each of the ℓ gaps in \mathfrak{e} and
no other singular part. For later us

(3.3) below). Besides, it has at most one mass point in each of the ℓ gaps in \mathfrak{e} and no other singular part. For later use, we pick J^{\sharp} to be a suitable reference point on \mathcal{T}_{ϵ} , namely a Jacobi matrix no other singular part. For later use, we pick J^{\sharp} to be a suitable reference point
on \mathcal{T}_{ϵ} , namely a Jacobi matrix whose spectral measure has no singular part at all.
A remarkable result of Remling [17] general on r_{ϵ} , namely a Jacobi matrix whose spectral measure has no singular part at all.
A remarkable result of Remling [17] generalizes the Denisov-Rakhmanov theorem to finite gap sets. It states that if $\sigma_{\rm ess}(J) = \mathfrak{e}$ to finite gap sets. It states that if $\sigma_{\text{ess}}(J) = \mathfrak{e}$ and $f(x) > 0$ a.e. on \mathfrak{e} , then
rbit of J under coefficient stripping approaches the isospectral torus $\mathcal{T}_{\mathfrak{e}}$. The
mee of J_n 's need not have a lim orem to finite gap sets. It states that if $\sigma_{\rm ess}(J) = \mathfrak{e}$ and $f(x) > 0$ a.e. on \mathfrak{e} , then
the orbit of J under coefficient stripping approaches the isospectral torus $\mathcal{T}_{\mathfrak{e}}$. The
sequence of J_n 's need n the orbit of J under coefficient stripping approaches the isospectral torus \mathcal{T}_{ϵ} sequence of J_n 's need not have a limit, but any of its accumulation points (essentially right limits) lie on \mathcal{T}_{ϵ} . In order to ensure convergence to some point on the isospectral torus and not only the torus as a tially right limits) lie on r_{ϵ} . In order to ensure convergence to some point on the isospectral torus and not only the torus as a set, stronger assumptions on J are needed.
We say that a Jacobi matrix $J = \{a_n, b_n\}_{n$

isospectral torus and not only the torus as a set, stronger assumptions on J are
needed.
We say that a Jacobi matrix $J = \{a_n, b_n\}_{n=1}^{\infty}$ with spectral measure $d\mu$ of
the form (1.6) belongs to the Szegő class for $\$ W
the form
where {
dition

$$
supp(d\mu) = \mathfrak{e} \cup \{x_k\},
$$

We say that a Jacobi matrix $J = \{a_n, b_n\}_{n=1}^{\infty}$ with spectral measure $d\mu$ of
orm (1.6) belongs to the Szegő class for **e** if
 $\text{supp}(d\mu) = \mathfrak{e} \cup \{x_k\},$
 $\in \{x_k\}$ is a countable set of isolated mass points satisfyi the form (1.6) belongs to the Szegő class for **c** if

supp $(d\mu) = \mathbf{c} \cup \{x\}$

where $\{x_k\}$ is a countable set of isolated mass p

dition
 $\sum \text{dist}(x_k, \mathbf{c})^{1/2}$ where ${x_k}$ is a countable set of isolated mass points satisfying the Blaschke con-
dition
 $\sum_{k} \text{dist}(x_k, \mathfrak{e})^{1/2} < \infty$ (3.2)
and f obeys the Szegő condition
 $\int \frac{\log f(x)}{x}$ (3.3) dition \sum_k and f obeys the Szegő condition

$$
\text{supp}(d\mu) = \mathfrak{e} \cup \{x_k\},
$$

of isolated mass points satisfying the Blaschke con-

$$
\sum_{k} \text{dist}(x_k, \mathfrak{e})^{1/2} < \infty
$$
 (3.2)
ion

$$
\sum_{k} \text{dist}(x_k, \mathfrak{e})^{1/2} < \infty \tag{3.2}
$$
\naddition

\n
$$
\int_{\mathfrak{e}} \frac{\log f(x)}{\text{dist}(x, \mathbb{R} \setminus \mathfrak{e})^{1/2}} dx > -\infty. \tag{3.3}
$$
\nen (3.2) holds, (3.3) is equivalent to

\n
$$
\frac{a_1 \cdots a_n}{\text{Cap}(\mathfrak{e})^n} \nrightarrow 0. \tag{3.4}
$$

and f obeys the Szegő condition
 $\int_{\mathfrak{e}} \frac{1}{\text{dist}}$
It is proven in [5] that when (3.2)

It is proven in [5] that when (3.2) holds, (3.3) is equivalent to
\n
$$
\frac{a_1 \cdots a_n}{\text{Cap}(\epsilon)^n} \nrightarrow 0.
$$
\n(3.4)

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In fact, just as for Szegő's theorem on $[-2, 2]$, any two of (3.2) – (3.4) imply the
third. While the sequence in (3.4) no longer has a limit, it turns out to be asymp-
totically periodic/almost p *third.* While the sequence in (3.4) no longer has a limit, it turns out to be asymp-

Another result of Christiansen, Simon, and Zinchenko [5] states that if J belongs to the Szegő class for ε , there is a unique point $J' \in \mathcal{T}_{\varepsilon}$ so that

$$
|a_n - a'_n| + |b_n - b'_n| \to 0. \tag{3.5}
$$

totically periodic/almost periodic.

Another result of Christiansen, Simon, and Zinchenko [5] states that if J

belongs to the Szegő class for **e**, there is a unique point $J' \in \mathcal{T}_{\epsilon}$ so that
 $|a_n - a'_n| + |b_n - b'_n| \to 0.$ + $|b_n - b'_n| \to 0.$ (3.5)
 $J'_n \to 0$ strongly (i.e., the orbit of J under

rbit of J' on \mathcal{T}_{ϵ}). To explain which point on

ement about the asymptotics of P_n , we first

ver by the universal covering map of $\mathbb{D$ Equivalently, this means that $J_n - J'_n \to 0$ strongly (i.e., the orbit of J under
coefficient stripping approaches the orbit of J' on \mathcal{T}_{ϵ}). To explain which point on
the torus to pick and to make a statement abou $\frac{n}{\cdot}$.

the torus to pick and to make a statement about the asymptotics of P_n , we first
need to answer the second question.
In short, the role of ϕ is taken over by the universal covering map of $\mathbb D$ onto
 $\Omega := \overline{\mathbb C} \setminus \math$ the torus to pick and to make a statement about the asymptotics of P_n , we first
need to answer the second question.
In short, the role of ϕ is taken over by the universal covering map of $\mathbb D$ onto
 $\Omega := \overline{\mathbb C} \setminus \math$ In short, the role of ϕ is taken over by the universal covering map of $\mathbb D$ onto $\Omega := \overline{\mathbb C} \setminus \mathfrak{e}$. This is the standard tool for 'lifting' functions on multiply connected domains to the unit disk. The universa domains to the unit disk. The universal covering map $\psi : \mathbb{D} \to \Omega$ is only locally
one-to-one and each point in Ω has infinitely many preimages in \mathbb{D} . These are
related to one another through a Fuchsian group domains to the unit disk. The universal covering map $\psi : \mathbb{D} \to \Omega$ is only locally
one-to-one and each point in Ω has infinitely many preimages in \mathbb{D} . These are
related to one another through a Fuchsian group

$$
\psi(z) = \psi(w) \iff \exists \gamma \in \Gamma : z = \gamma(w).
$$

one-to-one and each point in Ω has infinitely many preimages in \mathbb{D} . These are
related to one another through a Fuchsian group Γ of Möbius transformations,
 $\psi(z) = \psi(w) \iff \exists \gamma \in \Gamma : z = \gamma(w)$.
We fix ψ uniquely by $\psi(z) = \psi(w) \iff \exists \gamma \in \Gamma : z = \gamma(w).$
We fix ψ uniquely by also requiring that $\psi(0) = \infty$ and $\lim_{z\to 0} z\psi(z) > 0$. Γ isomorphic to the fundamental group $\pi_1(\Omega)$ and hence a free group on ℓ generato say $\gamma_1, \ldots, \gamma_\ell$. $(z) = \psi(w) \iff \exists \gamma \in \Gamma : z = \gamma(w).$
also requiring that $\psi(0) = \infty$ and line mental group $\pi_1(\Omega)$ and hence a free giorities of Γ , we introduce the open set $z := \{z \in \mathbb{D} : |\gamma'(z)| < 1 \text{ for all } \gamma \neq \text{id} \}$ We fix ψ uniquely by also requiring that $\psi(0) = \infty$ and $\lim_{z\to 0} z\psi(z) > 0$. Γ is
isomorphic to the fundamental group $\pi_1(\Omega)$ and hence a free group on ℓ generators,
say $\gamma_1, \dots, \gamma_\ell$.
To get a better picture

$$
\mathbb{F} := \{ z \in \mathbb{D} : |\gamma'(z)| < 1 \text{ for all } \gamma \neq \text{id} \}. \tag{3.6}
$$

isomorphic to the fundamental group $\pi_1(\Omega)$ and hence a free group on ℓ generators,
say $\gamma_1, \ldots, \gamma_\ell$.
To get a better picture of Γ , we introduce the open set
 $\mathbb{F} := \{ z \in \mathbb{D} : |\gamma'(z)| < 1 \text{ for all } \gamma \neq \text{id} \}.$ (3.6) say $\gamma_1, \ldots, \gamma_\ell$.
To get a
This is a function of $\overline{\mathbb{I}}$ and $\overline{\mathbb{I}}$
symmetric in $\mathbb{F} := \{z \in \mathbb{D} : |\gamma'(z)| < 1 \text{ for all } \gamma \neq \text{id} \}$
is a fundamental domain for Γ , that is, no two point
r Γ and $\overline{\mathbb{F}}$ contains at least one point from each Γ -orbi
netric in the real line and consists of the uni := { $z \in \mathbb{D}: |\gamma'(z)| < 1$ for all $\gamma \neq id$ }. (3.6)
domain for Γ , that is, no two points of \mathbb{F} are equivalent
is at least one point from each Γ -orbit. Geometrically, \mathbb{F} is
line and consists of the unit dis : |γ′
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pon This is a fundamental domain for Γ, that is, no two points of \mathbb{F} are equivalent
under Γ and $\overline{\mathbb{F}}$ contains at least one point from each Γ-orbit. Geometrically, \mathbb{F} is
symmetric in the real line and consis under Γ and $\mathbb F$ contains at least one point from each Γ -orbit. Geometrically, $\mathbb F$ is symmetric in the real line and consists of the unit disk with 2ℓ orthocircles (and their interior) removed. The circular symmetric in the real line and consists of the unit disk with 2ℓ orthocircles (and
their interior) removed. The circular arcs in the upper (or lower) half-disk, say
 C_1, \ldots, C_ℓ , are in one-one correspondence with the ga C_1,\ldots,C_ℓ , are in one-one correspondence with the gaps in \mathfrak{e} under the covering

 C_1, \ldots, C_ℓ , are in one-one correspondence with the gaps in \mathfrak{e} under the covering
map ψ . In fact, one can take the generator γ_j to be reflection in C_j following
complex conjugation.
The multiplicative gro map ψ . In fact, one can take the generator γ_j to be reflection in C_j following
complex conjugation.
The multiplicative group of characters on Γ , denoted Γ^* , turns out to play
an important role. Since an ele The multiplicative group of characters on Γ, denoted Γ^{*}, turns out to play
an important role. Since an element in Γ^{*} is determined from its values on the
generators of Γ, we can think of Γ^{*} as an ℓ -dimensional to an important role. Since an element in Γ^* is determined from its values on the
generators of Γ , we can think of Γ^* as an ℓ -dimensional torus. The point is that
 \mathcal{T}_{ϵ} and Γ^* are homeomorphic. To get generators of Γ, we can think of Γ^* as an ℓ -dimensional torus. The point is that $\mathcal{T}_{\mathfrak{e}}$ and Γ^* are homeomorphic. To get hold of a homeomorphism between these two ℓ -dimensional tori, we first introduc \mathcal{T}_{ϵ} and Γ^* are homeomorphic. To get hold of a homeomorphism between these two and Γ^* are homeomorphic. To get hold of a homeomorphism between these two
imensional tori, we first introduce the Jost function of an element in the Szegő
ss. Let $d\mu^{\sharp} = f^{\sharp}(x)dx$ be the spectral measure of J^{\sharp} , ℓ class. Let $d\mu^{\sharp} = f^{\sharp}(x)dx$ be the spectral measure of J^{\sharp} , our reference point on \mathcal{T}_{ϵ} .
For J in the Szegő class of ϵ , we define its Jost function by

class. Let
$$
d\mu^{\sharp} = f^{\sharp}(x)dx
$$
 be the spectral measure of J^{\sharp} , our reference point on \mathcal{T}_{ϵ} .
For J in the Szegő class of ϵ , we define its Jost function by

$$
u(z; J) = \prod_{k} B(z, z_{k}) \exp\left\{\int_{0}^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\left(\frac{f^{\sharp}(\psi(e^{i\theta}))}{f(\psi(e^{i\theta}))}\right) \frac{d\theta}{4\pi}\right\}, \quad z \in \mathbb{D} \quad (3.7)
$$
where $\{z_{k}\}$ are the unique points in $\overline{\mathbb{F}}$ with Im $z_{k} \ge 0$ and $\psi(z_{k}) = x_{k}$. This analytic function turns out to be character automorphic, that is, there exists $\chi_{J} \in \Gamma^{*}$ such

 $\frac{4\pi}{k}$
:e e where $\{z_k\}$ are the unique points in $\mathbb F$ with $\text{Im } z_k \ge 0$ and $\psi(z_k) = x_k$. This analytic
function turns out to be character automorphic, that is, there exists $\chi_j \in \Gamma^*$ such
there is $\chi_j \in \Gamma^*$ such function turns out to be character automorphic, that is, there exists $\chi_j \in \Gamma^*$ such
that is, there exists $\chi_j \in \Gamma^*$ such that

$$
u(\gamma(\cdot); J) = \chi_J(\gamma)u(\cdot; J) \text{ for all } \gamma \in \Gamma.
$$
\n
$$
\text{the map}
$$
\n
$$
\mathcal{T}_e \ni J \longrightarrow \chi, \in \Gamma^*, \tag{3.9}
$$

$$
\mathcal{T}_{\epsilon} \ni J \longrightarrow \chi_J \in \Gamma^*,\tag{3.9}
$$

Most importantly, the map
 $\mathcal{T}_{\epsilon} \ni J \longrightarrow \chi_J \in \Gamma^*,$

essentially the Abel map, is a homeomorphism (see, e.g., [4] for details).

We are now able to explain which point J' on \mathcal{T}_{ϵ} is the right one for (3.5) to Example 1 and $\frac{1}{N}$ is the map of $\frac{1}{N}$ and $\frac{1}{N}$ are map $\frac{1}{N}$ and $\frac{1}{N}$ are map $\frac{1}{N}$ and $\frac{1}{N}$ are $\frac{1}{N}$ and $\frac{1}{N}$ and $\frac{1}{N}$ are $\frac{1$ Γ^{*}, (3.9)
(see, e.g., [4] for details).
' on \mathcal{T}_{ϵ} is the right one for (3.5) to
This fact is proven in [5] by use of
and a technical lemma stating that
blies convergence of the associated
by takes a few lines We are now able to explain which point J' on \mathcal{T}_{ϵ} is the right one for (3.5) to hold: Take the unique point for which $\chi_{J'} = \chi_{J}$. This fact is proven in [5] by use of Remling's theorem, the homeomorphism (3.9), hold: *Take the unique point for which* $\chi_{J'} = \chi_{J}$. This fact is proven in [5] by use of Remling's theorem, the homeomorphism (3.9), and a technical lemma stating that strong convergence to a point on the torus implies strong convergence to a point on the torus implies convergence of the associated
characters. We repeat the proof here as it merely takes a few lines.
For contradiction, suppose that
 $|a_n - a'_n| + |b_n - b'_n| \nrightarrow 0$.
Then there is

$$
|a_n - a'_n| + |b_n - b'_n| \nrightarrow 0
$$

characters. We repeat the proof here as it merely takes a few lines.
For contradiction, suppose that
 $|a_n - a'_n| + |b_n - b'_n| \nrightarrow 0$.
Then there is a subsequence $\{n_k\}$ so that J and J' have different right limits, say
 $K \neq K'$ For contradiction, suppose that
 $|a_n - a'_n| + |b_n - b'_n| \neq 0$.
Then there is a subsequence $\{n_k\}$ so that J and J' have different r:
 $K \neq K'$. Due to Remling's theorem, both K and K' lie on \mathcal{T}_{ϵ} , and there is a subsequence $\{n_k\}$ so K' . Due to Remling's theorem,
 $\chi_{J_{n_k}} \longrightarrow \chi_K$ + | $b_n - b'_n$
that *J* an
both *K* as
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$$
\chi_{J_{n_k}} \longrightarrow \chi_K
$$
 and $\chi_{J'_{n_k}} \longrightarrow \chi_{K'}$

Then there is a subsequence $\{n_k\}$ so that J and J' have different right limits, say $K \neq K'$. Due to Remling's theorem, both K and K' lie on \mathcal{T}_{ϵ} , and we have
 $\chi_{J_{n_k}} \longrightarrow \chi_K$ and $\chi_{J'_{n_k}} \longrightarrow \chi_{K'}$
since $\neq K'$. Due to Remling's theorem, both K and K' lie on \mathcal{T}_{ϵ} , and we have
 $\chi_{J_{n_k}} \longrightarrow \chi_K$ and $\chi_{J'_{n_k}} \longrightarrow \chi_{K'}$

ce $J_{n_k} \rightarrow K$ and $J'_{n_k} \rightarrow K'$ strongly. As $\chi_J = \chi_{J'}$, we also have $\chi_{J_n} = \chi_{K'}$

at $\chi_K = \chi_{K'}$ that $\chi_K = \chi_{K'}$. This contradicts the fact that $K \neq K'$.
The Jost function also enters the picture in connection with the asymptotic $\chi'_{n_k} \to K'$ strongly. As $\chi_j = \chi_{j'}$, we also have $\chi_{j_n} = \chi_{j'_n}$

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ave strongly. As $\chi_J = \chi_{J'}$, we also have $\chi_{J_n} = \chi_{J'_n}$

s the fact that $K \neq K'$.

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thonormal polynomials associated with J' (no

(i), we have
 $\chi_n(\psi(z)) \longrightarrow \frac{u(z; J)}{u(z; J')}$ (tic
to
0) that $\chi_{\kappa} = \chi_{\kappa'}$. This contradicts the fact that $K \neq K'$
The Jost function also enters the picture in conn
behavior of P_n . With P'_n the orthonormal polynomials
be confused with the derivative), we have
 $\frac{P_n(\psi$ $\frac{1}{3}$ vior of P_n . With P'_n the orthonormal polynomials associated with J' (not to infused with the derivative), we have
 $\frac{P_n(\psi(z))}{P'_n(\psi(z))} \longrightarrow \frac{u(z; J)}{u(z; J')}$ (3.10)
rmly on compact subsets of \mathbb{F} , the fundamental doma

$$
\frac{P_n(\psi(z))}{P'_n(\psi(z))} \longrightarrow \frac{u(z;J)}{u(z;J')}
$$
\n(3.10)

be confused with the derivative), we have
 $\frac{P_n(\psi(z))}{P'_n(\psi(z))} \longrightarrow \frac{u(z;J)}{u(z;J')}$

uniformly on compact subsets of \mathbb{F} , the fundamental domain

should be compared with (2.13) and the fact that $u(z;J_0) = 1$. t_n'' the orthonormal polynomials associated with J' (not to
erivative), we have
 $\frac{P_n(\psi(z))}{P'_n(\psi(z))} \longrightarrow \frac{u(z; J)}{u(z; J')}$ (3.10)
subsets of \mathbb{F} , the fundamental domain for Γ . This result
th (2.13) and the fact that uniformly on compact subsets of \mathbb{F} , the fundamental domain for Γ . This result

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damental domain for Γ . This result
that $u(z; J_0) = 1$.
on, and Zinchenko [6] set out to find
n that still imply Szegő asymptotics
) has a limit). At first sight, it may uniformly on compact subsets of \mathbb{F} , the fundamental domain for Γ . This result
should be compared with (2.13) and the fact that $u(z; J_0) = 1$.
Along the lines of [8], Christiansen, Simon, and Zinchenko [6] set out should be compared with (2.13) and the fact that $u(z; J_0) = 1$.

Along the lines of [8], Christiansen, Simon, and Zinchenk

weaker assumptions than the Szegő condition that still imply

(in the sense that the left-hand sid er assumptions than the Szegő condition that still imply Szegő asymptotics
ne sense that the left-hand side of (3.10) has a limit). At first sight, it may
like
 $\sum_{n=1}^{\infty} (a_n - a'_n)^2 + (b_n - b'_n)^2 < \infty$ (3.11) (in the sense that the left-hand side of (3.10) has a limit). At first sight, it may look like

look like\n
$$
\sum_{n=1}^{\infty} (a_n - a'_n)^2 + (b_n - b'_n)^2 < \infty \tag{3.11}
$$
\nand conditional convergence of\n
$$
\sum_{n=1}^{\infty} (a_n - a'_n)^2 + (b_n - b'_n)^2 < \infty \tag{3.12}
$$

and conditional convergence of

$$
\sum_{n=1}^{\infty} (a_n - a'_n)^2 + (b_n - b'_n)^2 < \infty \tag{3.11}
$$
\nace of

\n
$$
\sum_{n=1}^{\infty} (a_n - a'_n), \quad \sum_{n=1}^{\infty} (b_n - b'_n) \tag{3.12}
$$
\nmore careful analysis shows that the periodicity/almost

will be sufficient. But a more
periodicity has to be taken into convergence with a more involved $(a_n - a'_n), \sum_{n=1}^{\infty} (b_n - b'_n)$
careful analysis shows
to account and one need
lved set of assumptions is
e reader is referred to [6] that the periodicity/almost
s to replace the conditional
avolving the harmonic mea-
for more details. periodicity has to be taken into account and one needs to replace the conditional convergence with a more involved set of assumptions involving the harmonic measures $\mu_{\mathfrak{e}}([\alpha_j, \beta_j])$ for all *j*. The reader is referred to [6] for more details. sures $\mu_{\mathfrak{e}}([\alpha_j, \beta_j])$ for all *j*. The reader is referred to [6] for more details.

J.S. Christiansen

The generalized Nevai conjecture has recent
setting by Frank and Simon [10]. They answer
 $\{a_n, b_n\}_{n=1}^{\infty}$ is a Jacobi matrix with spectral meas setting by Frank and Simon [10]. They answer in the affirmative that if $J =$
 $\{a_n, b_n\}_{n=1}^{\infty}$ is a Jacobi matrix with spectral measure dµ of the form (1.6) and
 $\sum_{n=1}^{\infty} |a_n - a'_n| + |b_n - b'_n| < \infty$ (3.13)

for some po ${a_n, b_n}_{n=1}^\infty$ is a Jacobi matrix with spectral measure $d\mu$ of the form

matrix with spectral measure
$$
d\mu
$$
 of the form (1.6) and

\n
$$
\sum_{n=1}^{\infty} |a_n - a'_n| + |b_n - b'_n| < \infty
$$
\n(3.13)

\n, then the Szegő condition (3.3) holds. Hence there is convergence to \mathcal{T}_{ϵ} . Among other things, [10] relies on an

+ $|b_n - b'_n| < \infty$ (3.13)
 egő condition (3.3) *holds*. Hence there is
 $\circ \mathcal{T}_{\epsilon}$. Among other things, [10] relies on an

the gaps of ϵ .
 $\circ \mathcal{T}_{\epsilon}$, on the other hand, is much less un-

imon theorem can be proved for some point J' on \mathcal{T}_{ϵ} , then the Szegő condition (3.3) holds. Hence there is

(3.3) *holds*. Hence there is
ther things, [10] relies on an
ther hand, is much less un-
can be proved for all finite
 ι ϵ is periodic has proven by
of [7] is to handle the periimproved Birman–Schwinger bound in the gaps of **ε**.
The situation of l^2 -convergence to \mathcal{T}_{ϵ} , on the other hand, is much less understood. Whether or not the Killip–Simon theorem can be proved for all finite gap s improved Birman–Schwinger bound in the gaps of ε .
The situation of ℓ^2 -convergence to $\mathcal{T}_{\varepsilon}$, on the
derstood. Whether or not the Killip–Simon theoren
gap sets is still an open question. That it is true whe
D The situation of ℓ^2 -convergence to \mathcal{T}_{ϵ} , on the other hand, is much less un-
ood. Whether or not the Killip-Simon theorem can be proved for all finite
ets is still an open question. That it is true when \mathfrak{e} gap sets is still an open question. That it is true when \mathfrak{e} is periodic has proven by
Damanik, Killip, and Simon [7]. The ingenious idea of [7] is to handle the peri-
odic case by use of matrix orthogonal polynomia odic case by use of matrix orthogonal polynomials. But this method only applies
to periodic ϵ . The proof of Killip-Simon's theorem for $[-2, 2]$ relies among other
things on the explicit form of ϕ . The universal cove things on the explicit form of ϕ . The universal covering map, in turn, is much more complicated. Even if one succeeds in finding ψ explicitly, the expression at hand will still be too difficult to work with. New ins will still be too difficult to work with. New insight is needed to really understand the concept of ℓ^2 -convergence to the isospectral torus.
 4. Infinite gap Jacobi matrices

Every compact set $\mathbf{E} \subset \mathbb{R}$ can be written in the form the concept of ℓ^2 -convergence to the isospectral torus.
 4. Infinite gap Jacobi matrices

Every compact set $E \subset \mathbb{R}$ can be written in the form

4. Infinite gap Jacobi matrices

$$
\mathbf{E} = [\alpha, \beta] \setminus \bigcup_{j} (\alpha_j, \beta_j), \tag{4.1}
$$

Every compact set $\mathbf{E} \subset \mathbb{R}$ can be written in the form
 $\mathbf{E} = [\alpha, \beta] \setminus \bigcup_j (\alpha_j, \beta_j),$

where \cup_j is a countable union of disjoint open subint

to (α_j, β_j) as a 'gap' in E and now mainly focus on th

gaps. In o = $[\alpha, \beta] \setminus \bigcup_j (\alpha_j, \beta_j)$, (4.1)
of disjoint open subintervals of $[\alpha, \beta]$. We shall refer
ow mainly focus on the situation of infinitely many
cheory, a few restrictions have to be put on E. But
e room for Cantor sets of where \cup_j is a countable union of disjoint open subintervals of $[\alpha, \beta]$. We shall refer
to (α_j, β_j) as a 'gap' in **E** and now mainly focus on the situation of infinitely many
gaps. In order to develop the theory, a f to (α_j, β_j) as a 'gap' in **E** and now mainly focus on the situation of infinitely many gaps. In order to develop the theory, a few restrictions have to be put on **E**. But among others, there will still be room for Canto

among others, there will still be room for Cantor sets of positive measure.
First of all, we shall always assume that $|E| > 0$ to allow for an absolutely
continuous part of $d\mu$. This in particular implies that the logar First of all, we shall always assume that $|E| > 0$ to allow for an absolutely continuous part of $d\mu$. This in particular implies that the logarithmic capacity of **F**, denoted Cap(**E**), is positive so that the domain Ω continuous part of $d\mu$. This in particular implies that the logarithmic capacity of **F**, denoted Cap(**E**), is positive so that the domain $\Omega = \overline{C} \setminus E$ has a Green's function.
We denote by g the Green's function for We denote by g the Green's function for Ω with pole at ∞ . This function is known to be positive and harmonic on Ω , and We denote by g the Green's function for Ω with pole at ∞ . This function is known
to be positive and harmonic on Ω , and
 $g(z) = \log |z| + \gamma(\mathbf{E}) + o(1)$
near ∞ , where $e^{-\gamma(\mathbf{E})} = \text{Cap}(\mathbf{E})$.
To avoid dealing with

$$
g(z) = \log|z| + \gamma(\mathbf{E}) + o(1)
$$

 $g(z) = \log |z|$
near ∞ , where $e^{-\gamma(E)} = \text{Cap}(E)$.
To avoid dealing with isolated pothat E is regular, that is, $(z) = \log |z| + \gamma(E) + o(1)$
(E).
isolated points in the esse
 $\lim_{z \to x} g(z) = 0$ for all $x \in E$ near ∞, where $e^{-\gamma(E)} = Cap(E)$.
To avoid dealing with isolated that E is regular, that is,
 $\lim_{\Omega \ni z \to a}$ $\frac{1}{\sqrt{2}}$

E is regular, that is,

$$
\lim_{\Omega \ni z \to x} g(z) = 0 \text{ for all } x \in \mathbb{E}.
$$
 (4.2)

Finite and Infinite Gap Jacobi Matrices 53
Hence g has precisely one critical point in each gap of E. Denoting by c_j the critical
point in (α_j, β_j) , we impose the so-called Parreau–Widom condition,
 $\sum_j g(c_j) < \infty$. (4.3

$$
\sum_{j} g(c_j) < \infty. \tag{4.3}
$$

Hence g has precisely one critical point in each gap of **E**. Denoting by c_j the critical point in (α_j, β_j) , we impose the so-called Parreau–Widom condition,
 $\sum_j g(c_j) < \infty$. (4.3)
While Widom was interested in Riemann s point in (α_j, β_j) , we impose the so-called Parreau–Widom condition,
 $\sum_j g(c_j) < \infty$.

While Widom was interested in Riemann surfaces with sufficiently m

functions, the notion becomes useful to us as the equilibrium meas (c_j) < ∞. (4.3)
nn surfaces with sufficiently many analytic
io us as the equilibrium measure $d\mu$ _E of E
see, e.g., [2] for a detailed proof). Moreover,
on E is of bounded characteristic when lifted
2) is larger to be functions, the notion becomes useful to us as the equilibrium measure $d\mu_{\mathbf{E}}$ of **E** turns out to be absolutely continuous (see, e.g., [2] for a detailed proof). Moreover, the *m*-function for measures supported on functions, the notion becomes useful to us as the equilibrium measure $d\mu_{\mathbf{E}}$ of **E** turns out to be absolutely continuous (see, e.g., [2] for a detailed proof). Moreover, the *m*-function for measures supported on

the *m*-function for measures supported on **E** is of bounded characteristic when lifted
to \mathbb{D} .
The Parreau–Widom condition (4.3) is known to be satisfied for compact sets
that are homogeneous in the sense of Carleso to D.
that
is an
Carle are homogeneous in the sense of Carleson [1]. By definition, this means there $\varepsilon > 0$ such that
 $\frac{|(x-\delta, x+\delta) \cap \mathbf{E}|}{\delta} \ge \varepsilon$ for all $x \in \mathbf{E}$ and all $\delta <$ diam(\mathbf{E}). (4.4)

son introduced this geometric con

$$
\frac{|(x-\delta, x+\delta)\cap \mathbf{E}|}{\delta} \ge \varepsilon \text{ for all } x \in \mathbf{E} \text{ and all } \delta < \text{diam}(\mathbf{E}).\tag{4.4}
$$

is an $\varepsilon > 0$ such that
 $\frac{|(x - \delta, x + \delta) \cap \mathbf{E}|}{\delta} \ge \varepsilon$ for all $x \in \mathbf{E}$ and all $\delta <$ diam(\mathbf{E}). (4.4)

Carleson introduced this geometric condition to avoid the possibility of certain

parts of \mathbf{E} to $\frac{(x-\delta,x+\delta)\cap \mathbf{E}}{\delta} \geq \varepsilon$ for all $x \in \mathbf{E}$ and all $\delta <$ diam(\mathbf{E}). (4.4)

coduced this geometric condition to avoid the possibility of certain

be very thin, compared to Lebesgue measure. To get an explicit
 parts of **E** to be very thin, compared to Lebesgue measure. To get an explicit example of an infinite gap set which is homogeneous, remove the middle $1/4$ from the interval [0, 1] and continue removing subintervals of le example of an infinite gap set which is homogeneous, remove the middle $1/4$ from the interval [0, 1] and continue removing subintervals of length $1/4^n$ from the middle of each of the 2^{n-1} remaining intervals. The se the interval [0, 1] and continue removing subintervals of length $1/4^n$ from the middle of each of the 2^{n-1} remaining intervals. The set **E** of what is left in [0, 1] is a Cantor set of length $1/2$, and the reader ma

middle of each of the 2^{n-1} remaining intervals. The set **E** of what is left in [0, 1] is
a Cantor set of length 1/2, and the reader may check that $|(x - \delta, x + \delta) \cap \mathbf{E}| \ge \delta/4$
for all $x \in \mathbf{E}$ and all $\delta < 1$.
Just a Cantor set of length 1/2, and the reader may check that $|(x - \delta, x + \delta) \cap \mathbf{E}| \ge \delta/4$
for all $x \in \mathbf{E}$ and all $\delta < 1$.
Just as in the finite gap setting, we can make use of the covering space
formalism. In fact, the s for all $x \in E$ and all $\delta < 1$.
Just as in the finite
formalism. In fact, the sem
gap sets of Parreau–Wido
 $\sigma_{\rm ess}(J) = E$ and spectral m
mass points of $d\mu$ outside as in the fact, the seminal paper [23] of Sodin and Yuditskii deals with infinite
tets of Parreau–Widom type. Let $J = \{a_n, b_n\}_{n=1}^{\infty}$ be a Jacobi matrix with
 J) = **E** and spectral measure $d\mu$ of the form (1.6). Den gap sets of Parreau–Widom type. Let $J = \{a_n, b_n\}_{n=1}^{\infty}$ be a Jacobi matrix with $\sigma_{\text{ess}}(J) = \mathbf{F}$ and spectral measure $d\mu$ of the form (1.6). Denote by $\{x_k\}$ the possible mass points of $d\mu$ outside **E**. We sa gap sets of Parreau–Widom type. Let $J = \{a_n, b_n\}_{n=1}^{\infty}$ be a Jacobi matrix with $\sigma_{\text{ess}}(J) = \mathbf{E}$ and spectral measure $d\mu$ of the form (1.6). Denote by $\{x_k\}$ the possible mass points of $d\mu$ outside \mathbf{E} . $n=1$

$$
\int_{\mathbf{E}} \log f(x) \, d\mu_{\mathbf{E}}(x) > -\infty. \tag{4.5}
$$

mass points of $d\mu$ outside **E**. We say that $d\mu$ (or J) satisfies the Szegő condition if
 $\int_{\mathbf{E}} \log f(x) d\mu_{\mathbf{E}}(x) > -\infty.$ (4.5)

As follows at once when recalling the explicit form of $d\mu_{\mathbf{c}}$ (see, e.g., [21, mass points of $d\mu$ outside **E**. We say that $d\mu$ (or *J*) satisfies the Szegő condition if
 $\int_{\mathbf{E}} \log f(x) d\mu_{\mathbf{E}}(x) > -\infty.$ (4.5)

As follows at once when recalling the explicit form of $d\mu_{\mathbf{E}}$ (see, e.g., [21 log $f(x) d\mu_{\mathbf{E}}(x) > -\infty$. (4.5)

ng the explicit form of $d\mu_{\mathbf{E}}$ (see, e.g., [21, Chap. 5]),

ralizing (3.3). On condition that
 $\sum_{k} g(x_k) < \infty$, (4.6)

d that $M := m \circ \psi$ is of bounded characteristic on As follows at once when recalling the explicit form of $d\mu_{\mathfrak{e}}$ (see, e.g., [21, Chap. 5]),
this is the natural way of generalizing (3.3). On condition that
 $\sum_{k} g(x_k) < \infty$, (4.6)
Sodin and Yuditskii [23] showed that

$$
\sum_{k} g(x_k) < \infty,\tag{4.6}
$$

Sodin and Yuditskii [23] showed that $M := m \circ \psi$ is of bounded characteristic on $\mathbb D$ and without a singular inner part. Hence it admits a factorization of the form

$$
\sum_{k} g(x_{k}) < \infty, \tag{4.6}
$$

Sodin and Yuditskii [23] showed that $M := m \circ \psi$ is of bounded characteristic on
 \mathbb{D} and without a singular inner part. Hence it admits a factorization of the form

$$
M(z) = B_{\infty}(z) \exp\left\{\int_{0}^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|M(e^{i\theta})| \frac{d\theta}{2\pi}\right\} \tag{4.7}
$$

with B_{∞} the Blaschke product of zeros and poles, and this paves the way for step-
by-step sum rules. Comparing the constant terms in (4.7) and iterating *n* times

 $M(z) = B_{\infty}(z) \exp\left\{ \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|M(e^{i\theta})| \frac{d\theta}{2\pi} \right\}$ (4.7)
th B_{∞} the Blaschke product of zeros and poles, and this paves the way for step-step sum rules. Comparing the constant terms in (4.7) $(z) = B_{\infty}(z) \exp \left\{ \int_0^{2\pi}$
the product of zeros and
Comparing the constants $\sum_{k=1}^{n} (g(x_k) - g(x_k))$ $\frac{+z}{-z}$
oles
tern
 $+\frac{1}{z}$ $\log |\cos \theta|$
 $\log \sin \theta$ $(e^{i\theta}) \Big| \frac{d\theta}{2\pi} \Big\}$ (4.7)
his paves the way for step-
4.7) and iterating *n* times
 $\left(\frac{f(t)}{f_n(t)}\right) d\mu_{\mathbf{E}}(t)$, (4.8) $\begin{pmatrix} 2\pi \ \text{ive} \end{pmatrix}$ with B_{∞} the Blaschke product of zeros and poles, and this paves the way for step-
by-step sum rules. Comparing the constant terms in (4.7) and iterating *n* times
lead to
 $\log\left(\frac{a_1 \cdots a_n}{\text{Cap}(E)^n}\right) = \sum_k \left(g(x_k) - g(x_{n,k$

by-step sum rules. Comparing the constant terms in (4.7) and iterating *n* times lead to\n
$$
\log\left(\frac{a_1\cdots a_n}{\text{Cap}(E)^n}\right) = \sum_k \left(g(x_k) - g(x_{n,k})\right) + \frac{1}{2} \int_E \log\left(\frac{f(t)}{f_n(t)}\right) d\mu_E(t),\tag{4.8}
$$

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where $\{x_{n,k}\}$ are the eigenvalues of J_n outside **E** and f_n is the absolutely continuous

part of its spectral measure. Interpreting the integral on the right-hand side in

terms of relative en part of its spectral measure. Interpreting the integral on the right-hand side in
terms of relative entropies, one can show that the $Szeg\tilde{\phi}$ condition is equivalent to
 $\frac{a_1 \cdots a_n}{\text{Cap}(E)^n} \nrightarrow 0$ (4.9)
provided that

$$
\frac{a_1 \cdots a_n}{\text{Cap}(\mathbf{E})^n} \nrightarrow 0 \n\tag{4.9}
$$

terms of relative entropies, one can show that the $Szeg\tilde{\sigma}$ condition is equivalent to
 $\frac{a_1 \cdots a_n}{\text{Cap}(E)^n} \neq 0$ (4.9)

provided that (4.6) holds. The details are given in [2] and the proof also shows

that the seq Cap(E)^{*n*} ⁷⁷^o (4.9)
tails are given in [2] and the proof also shows
nded above and below. While one direction is
ther involves some cutting and pasting in the
3).
sets, the isospectral torus \mathcal{T}_E will be infin provided that (4.6) holds. The details are given in [2] and the proof also shows
ence in (4.9) is bounded above and below. While one direction is
d using (4.8), the other involves some cutting and pasting in the
before applying (4.8).

straightforward using (4.8), the other involves some cutting and pasting in the Jacobi matrix before applying (4.8).
For general Parreau–Widom sets, the isospectral torus \mathcal{T}_E will be infinite dimensional and we equip Jacobi matrix before applying (4.8).
For general Parreau–Widom sets, the isospectral torus \mathcal{T}_E will be infinite
dimensional and we equip it with the product topology. It is known that Remling's
theorem generalizes an For general Parreau–Widom s
dimensional and we equip it with the
theorem generalizes and one can ask
a point on T_E and not only the isosp
remains a homeomorphism, the same
hold, an extra condition on E turns of For general Parreau–Widom sets, the isospectral torus $T_{\rm E}$ will be infinite insignal and we equip it with the product topology. It is known that Remling's em generalizes and one can ask if elements in the Szegő class theorem generalizes and one can ask if elements in the Szegő class still approach
a point on \mathcal{T}_E and not only the isospectral torus as a set. Provided the Abel map
remains a homeomorphism, the same proof as in Sectio a point on \mathcal{T}_E and not only the isospectral torus as a set. Provided the Abel map
remains a homeomorphism, the same proof as in Section 3 should work. For this to
hold, an extra condition on E turns out to be needed. a point on $f_{\mathbf{E}}$ and not only the isospectral torus as a set. I rovided the Abel map
remains a homeomorphism, the same proof as in Section 3 should work. For this to
hold, an extra condition on **E** turns out to be ne hold, an extra condition on E turns out to be needed. The so-called direct Cauchy theorem has to be valid (see $[24]$, $[11]$). These and related issues are treated in the upcoming paper $[3]$. A recent article of Yudit theorem has to be valid (see [24], [11]). These and related issues are treated in the upcoming paper [3]. A recent article of Yuditskii [25] points out that Parreau-Widom sets for which the direct Cauchy theorem holds are the upcoming paper [3]. A recent article of Yuditskii [25] points out that Parreau-Widom sets for which the direct Cauchy theorem holds are still more general than homogeneous sets. Asymptotics of orthogonal polynomials on Widom sets for which the direct Cauchy theorem holds are still more general than
homogeneous sets. Asymptotics of orthogonal polynomials on homogeneous sets
were treated by Peherstorfer and Yudiskii in [16]. homogeneous sets. Asymptotics of orthogonal polynomials on homogeneous sets

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- were treated by Peherstorfer and Yudiskii in [16].
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Derivation of Ginzburg-Landau Theory for a One-dimensional System with Contact Interaction

Rupert L. Frank, Christian Hainzl, Robert Seiringer and Jan Philip Solovej

Abstract. In a recent paper [7] we give the first rigorous derivation of the celebrated Ginzburg-Landau (GL) theory, starting from the microscopic Bardeen-Cooper-Schrieffer (BCS) model. Here we present our results in the simplified case of a one-dimensional system of particles interacting via a δ -potential.

Mathematics Subject Classification (2010). Primary 82D50; Secondary 81Q20, 35Q56.

Keywords. Ginzburg-Landau theory, BCS theory, semiclassical analysis, superfluidity, superconductivity.

1. Introduction and main results

1.1. Introduction

description of the phenomenon of superconductivity. Their model examined the *macroscopic* properties of a superconductor in a phenomenological way, without explaining its microscopic mechanism. In the GL theory the super *macroscopic* properties of a superconductor in a phenomenological way, without explaining its microscopic mechanism. In the GL theory the superconducting state is represented by a complex order parameter $\psi(x)$, which is *macroscopic* properties of a superconductor in a phenomenological way, without is microscopic mechanism. In the GL theory the superconducting state
d by a complex order parameter $\psi(x)$, which is zero in the normal
n-zero in the superconducting state. The order parameter $\psi(x)$ can
l as a macroscopi is represented by a complex order parameter $\psi(x)$, which is zero in the normal state and non-zero in the superconducting state. The order parameter $\psi(x)$ can be considered as a macroscopic wave-function whose square $|\psi$ is represented by a complex order parameter $\psi(x)$, which is zero in the normal
state and non-zero in the superconducting state. The order parameter $\psi(x)$ can
be considered as a macroscopic wave-function whose square $|\psi$ 2

state and non-zero in the superconducting state. The order parameter $\psi(x)$ can
be considered as a macroscopic wave-function whose square $|\psi(x)|^2$ is proportional
to the density of superconducting particles.
In 1957 Barde be considered as a macroscopic wave-function whose square $|\psi(x)|^2$ is proportional
to the density of superconducting particles.
In 1957 Bardeen, Cooper and Schrieffer [2] formulated the first *microscopic*
explanation of In 1957 Bardeen, Cooper and Schrieffer [2] formulated the first *microscopic* explanation of superconductivity starting from a first principle Hamiltonian. In a major breakthrough they realized that this phenomenon can be major breakthrough they realized that this phenomenon can be described by the *pairing-mechanism*. The superconducting state forms due to an instability of the normal state in the presence of an attraction between the part pairing-mechanism. The superconducting state forms due to an instability of the normal state in the presence of an attraction between the particles. In the case of a metal the attraction is made possible by an interaction pairing-mechanism. The superconducting state forms due to an instability of the be presence of an attraction between the particles. In the case of tion is made possible by an interaction through the lattice. For a metal the attraction is made possible by an interaction through the lattice. For 58 R.L. Frank, C. Hainzl, R. Seiringer and J.P. Solovej
other systems, like superfluid cold gases, the interaction is of local type. In the BCS
theory the superconducting state, which is made up by pairs of particles of o

to the critical temperature, the order parameter $\psi(x)$ and the pair-wavefunction spin, the *Cooper-pairs*, is described by a two-particle wave-function $\alpha(x, y)$.
A connection between the two approaches, the phenomenological GL theory
and the microscopic BCS theory, was made by Gorkov [3] who showed th he microscopic BCS theory, was made by Gorkov [3] who showed that, close
e critical temperature, the order parameter $\psi(x)$ and the pair-wavefunction
y) are proportional. A simpler argument was later given by de Gennes [4 $\alpha(x, y)$ are proportional. A simpler argument was later given by de Gennes [4].

to the critical temperature, the order parameter $\psi(x)$ and the pair-wavefunction $\alpha(x, y)$ are proportional. A simpler argument was later given by de Gennes [4].
Recently we presented in [7] a mathematical proof of the eq (*x, y*) are proportional. A simpler argument was later given by de Gennes [4].
Recently we presented in [7] a mathematical proof of the equivalence of two models, GL and BCS, in the limit when the temperature *T* is clos two models, GL and BCS, in the limit when the temperature T is close to the critical
ical temperature T_c , i.e., when $h = [(T_c - T)/T_c]^{1/2} \ll 1$, where T_c is the critical
temperature for the translation-invariant BCS equat ical temperature T_c , i.e., when $h = [(T_c - T)/T_c]^{1/2} \ll 1$, where T_c is the critical
temperature for the translation-invariant BCS equation. The mathematical aspects
of this equation where studied in detail in [8, 6, 9, 1 $\frac{1/2}{\cdot}$ of this equation where studied in detail in [8, 6, 9, 10, 11]. In the present paper
we present this result in the simplified case a of one-dimensional system where the
particles interact via an attractive contact interact we present this result in the simplified case a of one-dimensional system where the

$$
V(x - y) = -a\delta(x - y) \quad \text{with } a > 0. \tag{1.1}
$$

particles interact via an attractive contact interaction potential of the form
 $V(x - y) = -a\delta(x - y)$ with $a > 0$. (1.1)

We assume that the system is subject to a weak external potential W, which

varies on a large scale $1/h$ $V(x - y) = -a\delta(x - y)$ with $a > 0$.
We assume that the system is subject to a weak external potential W, varies on a large scale $1/h$ compared to the microscopic scale of order 1.
variations of the system on the macroscopic sca $(x - y) = -a\delta(x - y)$ with $a > 0$. (1.1)
e system is subject to a weak external potential W, which
 $1/h$ compared to the microscopic scale of order 1. Since
on the macroscopic scale cause a change in energy of the
at the externa We assume that the system is subject to a weak external potential W , which
s on a large scale $1/h$ compared to the microscopic scale of order 1. Since
tions of the system on the macroscopic scale cause a change in energ varies on a large scale $1/h$ compared to the microscopic scale of order 1. Since
variations of the system on the macroscopic scale cause a change in energy of the
order h^2 , we assume that the external potential W is al order h^2 , we assume that the external potential W is also of the order h^2 . Hence
we write it as $h^2W(hx)$, with x being the microscopic variable. The parameter h
will play the role of a semiclassical parameter.
We w order h^2 , we assume that the external potential *W* is also of the order h^2 . Hence
we write it as $h^2W(hx)$, with *x* being the microscopic variable. The parameter *h*
will play the role of a semiclassical parameter we write it as $h^2W(hx)$, with *x* being the microscopic variable. The parameter *h*
will play the role of a semiclassical parameter.
We will prove that, to leading order in *h*, the Cooper pair wave function
 $\alpha(x, y)$ and

 $\alpha(x, y)$ and the GL function $\psi(x)$ are related by

$$
\alpha(x,y) = \psi\left(h\frac{x+y}{2}\right)\alpha^0(x-y) \tag{1.2}
$$

We will prove that, to leading order in *h*, the Cooper pair wave function
 $\alpha(x, y)$ and the GL function $\psi(x)$ are related by
 $\alpha(x, y) = \psi\left(h\frac{x+y}{2}\right)\alpha^0(x-y)$ (1.2)

where α^0 is the translation invariant minimizer of t (x, y) and the GL function $\psi(x)$ are related by
 $\alpha(x, y) = \psi\left(h\frac{x+y}{2}\right)c$

there α^0 is the translation invariant minimizer

lar, the argument \bar{x} of the order parameter ψ

action of the BCS state, which varies o $(x, y) = \psi \left(h \frac{y}{2}\right) \alpha^{0}(x - y)$ (1.2)

n invariant minimizer of the BCS functional. In partic-

the order parameter $\psi(\bar{x})$ describes the *center-of-mass*

which varies on the macroscopic scale. To be precise, we
 $\frac{1}{2$ $\frac{+y}{2}$
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tl
tl)
tt where α^0 is the translation invariant minimizer of the BCS functional. In particular, the argument \bar{x} of the order parameter $\psi(\bar{x})$ describes the *center-of-mass* motion of the BCS state, which varies on the mac ular, the argument \bar{x} of the order parameter $\psi(\bar{x})$ describes the *center-of-mass* motion of the BCS state, which varies on the macroscopic scale. To be precise, we shall prove that $\alpha(x, y) = \frac{1}{2}(\psi(hx) + \psi(hy))\alpha^0(x - y$

shall prove that $\alpha(x, y) = \frac{1}{2}(\psi(hx) + \psi(hy))\alpha^{0}(x - y)$ to leading order in *h*, which agrees with (1.2) to this order.
For simplicity we restrict our attention to contact potentials of the form (1.1), but our method can be For simplicity we restric (1.1), but our method can be g
details. The proof presented h
applies to any dimension $d \leq 3$.
magnetic field in one dimension
invariant problem is particular but our method can be generalized to other kinds of interactions; see [7] for ls. The proof presented here is simpler than the general proof in [7] which es to any dimension $d \leq 3$. There are several reasons for this. F details. The proof presented here is simpler than the general proof in [7] which
applies to any dimension $d \leq 3$. There are several reasons for this. First, there is no
magnetic field in one dimension. Second, for a con applies to any dimension $d \leq 3$. There are several reasons for this. First, there is no magnetic field in one dimension. Second, for a contact interaction the translation invariant problem is particularly simple and the magnetic field in our simulation. Simple in the corresponding gap equation has
an explicit solution. Finally, several estimates are simpler in one dimension due
the boundedness of the Green's function for the Laplacian.
1. in an explicit solution. Finally, several estimates are simpler in one dimension due
the boundedness of the Green's function for the Laplacian.
1.2. The BCS functional
We consider a macroscopic sample of a fermionic syste

1.2. The BCS functional

and the boundedness of the Green's function for the Laplacian.
 1.2. The BCS functional

We consider a macroscopic sample of a fermionic system, in one spatial dimension.

Let $\mu \in \mathbb{R}$ denote the chemical potential **1.2. The BCS functional**
We consider a macroscopic sample of a fermionic system, in
Let $\mu \in \mathbb{R}$ denote the chemical potential and $T > 0$ the temp Let $\mu \in \mathbb{R}$ denote the chemical potential and $T > 0$ the temperature of the sample.
Let $\mu \in \mathbb{R}$ denote the chemical potential and $T > 0$ the temperature of the sample. Derivation of GL Theory for a 1D System 59
The fermions interact through the attractive two-body potential given in (1.1). In
addition, they are subject to an external force, represented by a potential $W(x)$.
In BCS theor addition, they are subject to an external force, represented by a potential $W(x)$.

$$
\Gamma = \left(\begin{array}{cc} \gamma & \alpha \\ \bar{\alpha} & 1 - \bar{\gamma} \end{array}\right)
$$

satisfying $0 \leq \Gamma \leq 1$ as an operator on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. The bar denotes complex $\Gamma = \begin{pmatrix} \gamma & \alpha \\ \bar{\alpha} & 1 - \bar{\gamma} \end{pmatrix}$

ying $0 \le \Gamma \le 1$ as an operator on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. The bar denotes complex gation, i.e., $\bar{\alpha}$ has the integral kernel $\alpha(x, y)$. In particular, Γ is assumed to of a 2×2 operator-valued matrix
 Γ :

satisfying $0 \le \Gamma \le 1$ as an operat

conjugation, i.e., $\bar{\alpha}$ has the integr

be hermitian, which implies that $\Gamma = \begin{pmatrix} \gamma & \alpha \\ \bar{\alpha} & 1 - \bar{\gamma} \end{pmatrix}$

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egral kernel $\alpha(x, \alpha)$

t γ is hermitian

There are no s

nnction is the pr satisfying $0 \leq \Gamma \leq 1$ as an operator on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. The bar denotes complex
conjugation, i.e., $\bar{\alpha}$ has the integral kernel $\alpha(x, y)$. In particular, Γ is assumed to
be hermitian, which implies that conjugation, i.e., $\bar{\alpha}$ has the integral kernel $\alpha(x, y)$. In particular, Γ is assumed to
be hermitian, which implies that γ is hermitian and α is symmetric (i.e., $\gamma(x, y) =$
 $\overline{\gamma(y, x)}$ and $\alpha(x, y) = \alpha(y, x)$.) Th be hermitian, which implies that γ is hermitian and α is symmetric (i.e., $\gamma(x, y) = \overline{\gamma(y, x)}$ and $\alpha(x, y) = \alpha(y, x)$.) There are no spin variables in Γ . The full, spin dependent Cooper pair wave function is the produ γ

dependent Cooper pair wave function is the product of α with an antisymmetric spin singlet.
We are interested in the effect of weak and slowly varying external fields, described by a potential $h^2W(hx)$. In order to av dependent Cooper pair wave function is the product of α with an antisymmetric spin singlet.
We are interested in the effect of weak and slowly varying external fields, described by a potential $h^2W(hx)$. In order to av We are
described by
conditions,
In particula.
The aim the
We fine ibed by a potential $h^2W(hx)$. In order to avoid having to introduce boundary
tions, we assume that the system is infinite and periodic with period h^{-1} .
rticular, W should be periodic. We also assume that the state Γ conditions, we assume that the system is infinite and periodic with period h^{-1} .
In particular, W should be periodic. We also assume that the state Γ is periodic.
The aim then is to calculate the free energy per unit conditions, we assume that the system is infinite and periodic with period h^{-1} .
In particular, W should be periodic. We also assume that the state Γ is periodic.
The aim then is to calculate the free energy per unit

In particular, W should be periodic. We also assume that the state
$$
\Gamma
$$
 is periodic.
The aim then is to calculate the free energy per unit volume.
We find it convenient to do a rescaling and use macroscopic variables instead
of the microscopic ones. In macroscopic variables, the BCS functional has the form

$$
\mathcal{F}^{BCS}(\Gamma) := \text{Tr} \left(-h^2 \nabla^2 - \mu + h^2 W(x) \right) \gamma - T S(\Gamma) - ah \int_C |\alpha(x, x)|^2 dx \quad (1.3)
$$

where C denotes the unit interval [0, 1]. The entropy equals $S(\Gamma) = -\text{Tr} \Gamma \ln \Gamma$.
The BCS state of the system is a minimizer of this functional over all admissible Γ .

where C denotes the unit interval [0, 1]. The entropy equals $S(\Gamma) = -\text{Tr} \Gamma \ln \Gamma$.
The BCS state of the system is a minimizer of this functional over all admissible Γ .
The symbol Tr in (1.3) stands for the trace per uni (Γ) := Tr $\left(-h^2\nabla^2 - \mu + h^2W(x)\right)\gamma - TS(\Gamma) - ah \int_C |\alpha(x, x)|^2 dx$ (1.3)
denotes the unit interval [0, 1]. The entropy equals $S(\Gamma) = -\text{Tr}\Gamma \ln \Gamma$.
S state of the system is a minimizer of this functional over all admissible Γ .
e sy where *C* denotes the unit interval [0, 1]. The entropy equals $S(\Gamma) = -\text{Tr} \Gamma \ln \Gamma$.
The BCS state of the system is a minimizer of this functional over all admissible Γ .
The symbol Tr in (1.3) stands for the trace per u The symbol Tr in (1.3) stands for the trace per unit volume. More precisely, if B is a periodic operator (meaning that it commutes with translation by 1), then Tr B equals, by definition, the (usual) trace of χB , with if *B* is a periodic operator (meaning that it commutes with translation by 1), then Tr *B* equals, by definition, the (usual) trace of χB , with χ the characteristic function of *C*. The location of the interval is function of C . The location of the interval is obviously of no importance. It is not difficult to see that the trace per unit volume has the usual properties like cyclicity, and standard inequalities like Hölder's inequ and standard inequalities like Hölder's inequality hold. This is discussed in more and standard inequalities like Hölder's inequality hold. This is discussed in more detail in [7].
 Assumption 1.1. We assume that W is a bounded, periodic function with period 1 and $\int_C W(x) dx = 0$.
 1.2.1. The translati

Assumption 1.1. We assume that W is a bounded, periodic function with period 1 and $\int_{\mathcal{C}} W$

detail in [7].
 Assumption 1.1. We assume that W is a bounded, periodic function with period 1 and $\int_C W(x) dx = 0$.
 1.2.1. The translation-invariant case. In the translation-invariant case $W = 0$ one can restrict $\mathcal{F$ **Assumption**
and $\int_{\mathcal{C}} W(x)$
1.2.1. The t
can restrict
invariant sta 1 $(x) dx = 0.$

e translation

ct \mathcal{F}^{BCS} to state in for In the translation-invariant case $W = 0$ one
iant states. We write a general translation-
atrix
 $\frac{\tilde{\alpha}(-ih\nabla)}{1 - \tilde{\gamma}(-ih\nabla)}$, (1.4)
ntegral kernels can restrict \mathcal{F}^{BCS} to translation-invariant states. We write a general translation-
invariant state in form of the 2 × 2 matrix
 $\Gamma = \begin{pmatrix} \frac{\tilde{\gamma}(-ih\nabla)}{\tilde{\alpha}(-ih\nabla)} & \tilde{\alpha}(-ih\nabla) \\ \frac{\tilde{\alpha}(-ih\nabla)}{\tilde{\alpha}(-ih\nabla)} & 1 - \tilde$

$$
\Gamma = \begin{pmatrix} \frac{\tilde{\gamma}(-ih\nabla)}{\tilde{\alpha}(-ih\nabla)} & \tilde{\alpha}(-ih\nabla) \\ \frac{\tilde{\gamma}(-ih\nabla)}{\tilde{\alpha}(-ih\nabla)} & 1 - \tilde{\gamma}(-ih\nabla) \end{pmatrix},
$$
(1.4)

$$
\alpha = [\Gamma]_{12}
$$
 have integral terms

$$
\alpha(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{\alpha}(hp) e^{ip(x-y)} dp.
$$

invariant state in form of the
$$
z \times z
$$
 matrix

\n
$$
\Gamma = \left(\frac{\tilde{\gamma}(-ih\nabla)}{\tilde{\alpha}(-ih\nabla)} \quad 1 - \tilde{\gamma}(-ih\nabla) \right),
$$
\nthat is, $\gamma = [\Gamma]_{11}$ and $\alpha = [\Gamma]_{12}$ have integral terms

\n
$$
\gamma(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{\gamma}(hp) e^{ip(x-y)} dp \quad \text{and} \quad \alpha(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{\alpha}(hp) e^{ip(x-y)} dp.
$$

The fact that Γ is admissible means that $\tilde{\alpha}(p) = \overline{\tilde{\alpha}(-p)}$, that $0 \leq |\tilde{\alpha}(p)|^2 \leq \tilde{\gamma}(p)(1-\tilde{\gamma}(-p))$ for any $p \in \mathbb{R}$. For states of this form the isocomes
 $\mathcal{F}^{\text{BCS}}(\Gamma) = \int (h^2p^2 - \mu)\tilde{\gamma}(hp) \frac{dp}{dt} - T \int S$ The fact that Γ is admissible means that $\tilde{\alpha}(p) = \tilde{\alpha}(-p)$, that $0 \le \tilde{\gamma}(p) \le 1$ and $|\tilde{\alpha}(p)|^2 \le \tilde{\gamma}(p)(1-\tilde{\gamma}(-p))$ for any $p \in \mathbb{R}$. For states of this form the BCS functional becomes
 $\mathcal{F}^{\text{BCS}}(\Gamma) = \int_{\$ $|\tilde{\alpha}(p)|^2 \leq \tilde{\gamma}$

$$
|\tilde{\alpha}(p)|^2 \leq \tilde{\gamma}(p)(1-\tilde{\gamma}(-p)) \text{ for any } p \in \mathbb{R}. \text{ For states of this form the BCS functional becomes}
$$
\n
$$
\mathcal{F}^{\text{BCS}}(\Gamma) = \int_{\mathbb{R}} (h^2 p^2 - \mu) \tilde{\gamma}(hp) \frac{dp}{2\pi} - T \int_{\mathbb{R}} S(\tilde{\Gamma}(hp)) \frac{dp}{2\pi} - ah \left| \int_{\mathbb{R}} \tilde{\alpha}(hp) \frac{dp}{2\pi} \right|^2, (1.5)
$$
\nwith $S(\tilde{\Gamma}(p)) = -\text{Tr}_{\mathbb{C}^2} \tilde{\Gamma}(p) \ln \tilde{\Gamma}(p)$ and $\tilde{\Gamma}(p)$ the 2×2 matrix obtained by replacing $-i\nabla$ by p in (1.4).
\nIn the following, we are going to summarize some well-known facts about

 $-i\nabla$ by p in (1.4).

 $(\Gamma) = \int_{\mathbb{R}} (h^2 p^2 - \mu) \tilde{\gamma}(hp) \frac{dp}{2\pi} - T \int_{\mathbb{R}} S(\tilde{\Gamma}(hp)) \frac{dp}{2\pi} - ah \left| \int_{\mathbb{R}} \tilde{\alpha}(hp) \frac{dp}{2\pi} \right|^2$, (1.5)
 $(\tilde{\Gamma}(p)) = -\text{Tr}_{\mathbb{C}^2} \tilde{\Gamma}(p) \ln \tilde{\Gamma}(p)$ and $\tilde{\Gamma}(p)$ the 2×2 matrix obtained by replacing
 $y \$ 2π
by
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fl: with $S(\Gamma(p)) = -\text{Tr}_{\mathbb{C}^2} \Gamma(p) \ln \Gamma(p)$ and $\Gamma(p)$ the 2×2 matrix obtained by replacing $-i\nabla$ by p in (1.4).

In the following, we are going to summarize some well-known facts about

the translation-invariant functional (In the following, we are going to summarize some well-known facts about
the translation-invariant functional (1.5). For given $a > 0$, we define the critical
temperature $T_c > 0$ by the equation ranslation-invariant functional (1.5). For given $a > 0$, we define the critical
erature $T_c > 0$ by the equation
 $\frac{1}{a} = \int_{\mathbb{R}} \frac{\tanh\left(\frac{p^2 - \mu}{2T_c}\right)}{p^2 - \mu} \frac{dp}{2\pi}$. (1.6)
is the form in which the gap-equation is u

$$
\frac{1}{a} = \int_{\mathbb{R}} \frac{\tanh\left(\frac{p^2 - \mu}{2T_c}\right)}{p^2 - \mu} \frac{dp}{2\pi}.
$$
\n(1.6)

This is the form in which the gap-equation is usually presented in the physics literature, see, e.g., $[13, 4]$. The fact that there is a unique solution to this equation temperature $T_c > 0$ by the equation
 $\frac{1}{a} = \int_{\mathbb{R}}$
This is the form in which the gap-ecenture, see, e.g., [13, 4]. The fact the follows from the strict monotonicity $\frac{1}{a}$
 $\frac{1}{b}$
 $\frac{1}{b}$
 $\frac{1}{b}$ a
ac
d
s a $\frac{\tanh\left(\frac{p^2-\mu}{2T_c}\right)}{p^2-\mu}$ uation is u
at there is
at there is
 $\text{of } t/\tanh t$
 $p) = (1+\text{ex}$ (1.6)
 $\frac{2\pi}{2\pi}$. (1.6)

tally presented in the physics lit-

unique solution to this equation
 $r t > 0$. If $T \geq T_c$, then the mini-
 $((h^2p^2-\mu)/T))^{-1}$. If $0 < T < T_c$,
 $\Delta_0 > 0$ of the *BCS gap equation* erature, see, e.g., [13, 4]. The fact that there is a unique solution to this equation follows from the strict monotonicity of $t/\tanh t$ for $t > 0$. If $T \geq T_c$, then the minimizer of (1.5) satisfies $\tilde{\alpha} \equiv 0$ and $\tilde{\gamma$ follows from the strict monotonicity of $t/$ tanh t for $t > 0$. If $T \geq T_c$, then the mini-
mizer of (1.5) satisfies $\tilde{\alpha} \equiv 0$ and $\tilde{\gamma}(hp) = (1 + \exp((h^2p^2 - \mu)/T))^{-1}$. If $0 < T < T_c$,
on the other hand, then there is a uni mizer of (1.5) satisfies $\tilde{\alpha} \equiv 0$ and $\tilde{\gamma}(hp) = (1 + \exp((h^2 p^2 - \mu)/T))^{-1}$. If $0 < T < T_c$,

on the other hand, then there is a unique solution $\Delta_0 > 0$ of the *BCS gap equation*
 $\frac{1}{a} = \int_{\mathbb{R}} \frac{1}{K_T^0(p)} \frac{dp}{2\pi},$ (1.

on the other hand, then there is a unique solution
$$
\Delta_0 > 0
$$
 of the *BCS gap equation*
\n
$$
\frac{1}{a} = \int_{\mathbb{R}} \frac{1}{K_T^0(p)} \frac{dp}{2\pi},
$$
\nwhere (1.7)

$$
\frac{1}{a} = \int_{\mathbb{R}} \overline{K_T^0(p)} \frac{1}{2\pi},
$$
\n(1.7)\n
$$
K_T^0(p) = \frac{\sqrt{(p^2 - \mu)^2 + \Delta_0^2}}{\tanh\left(\frac{1}{2T}\sqrt{(p^2 - \mu)^2 + \Delta_0^2}\right)}.
$$
\n(1.8)\n
$$
\text{izer of } (1.5) \text{ is given by}
$$
\n
$$
\tilde{\Gamma}^0(hp) = \left(1 + \exp\left(\frac{1}{T}H_{\Delta_0}^0(hp)\right)\right)^{-1}
$$
\n(1.9)

Moreover, the minimizer of (1.5) is given by

$$
\tanh\left(\frac{1}{2T}\sqrt{(p^2-\mu)^2+\Delta_0^2}\right)
$$
\n
\n
$$
\text{or of } (1.5) \text{ is given by}
$$
\n
$$
\tilde{\Gamma}^0(hp) = \left(1 + \exp\left(\frac{1}{T}H_{\Delta_0}^0(hp)\right)\right)^{-1}
$$
\n
$$
H_{\Delta_0}^0(p) = \begin{pmatrix} p^2 - \mu & -\Delta_0 \\ -\Delta_0 & -p^2 + \mu \end{pmatrix}.
$$
\n
$$
p)]_{12} \text{ one easily deduces from } (1.9) \text{ that}
$$
\n
$$
\Delta_0 \tag{1.12}
$$

with
\n
$$
\tilde{\Gamma}^{0}(hp) = \left(1 + \exp\left(\frac{1}{T}H_{\Delta_{0}}^{0}(hp)\right)\right)^{-1}
$$
\nwith
\n
$$
H_{\Delta_{0}}^{0}(p) = \begin{pmatrix} p^{2} - \mu & -\Delta_{0} \\ -\Delta_{0} & -p^{2} + \mu \end{pmatrix}.
$$
\nWriting $\tilde{\alpha}^{0}(hp) = [\tilde{\Gamma}^{0}(hp)]_{12}$ one easily deduces from (1.9)
\n
$$
\tilde{\alpha}^{0}(p) = \frac{\Delta_{0}}{2K_{T}^{0}(p)}.
$$
\nTo summarize, in the case $W \equiv 0$ the functional \mathcal{F}^{1} for $0 < T < T$ whose off-diagonal element does not vanish.

Writ

$$
\tilde{\alpha}^0(p) = \frac{\Delta_0}{2K_T^0(p)}.
$$
\n(1.10)

Writing $\tilde{\alpha}^0(hp) = [\tilde{\Gamma}^0(hp)]_{12}$ one easily deduces from (1.9) that
 $\tilde{\alpha}^0(p) = \frac{\Delta_0}{2K_T^0(p)}$.

To summarize, in the case $W \equiv 0$ the functional \mathcal{F}^{BCS} h

for $0 < T < T_c$ whose off-diagonal element does not $\tilde{\alpha}^0(p) = \frac{\Delta_0}{2K_T^0(p)}$. (1.10)
 $W \equiv 0$ the functional \mathcal{F}^{BCS} has a minimizer Γ^0

nal element does not vanish and has the integral
 $\beta = \frac{\Delta_0}{2} \int_{\mathbb{R}} \frac{1}{K_T^0(hp)} e^{ip(x-y)} \frac{dp}{2\pi}$. (1.11) $\lim_{r\to 0}$ To summarize, in the case $W \equiv 0$ the functional \mathcal{F}^{BCS} has a minimizer Γ^0
 $\langle T \rangle \langle T \rangle \langle T \rangle = \frac{\Delta_0}{2} \int_{\mathbb{R}} \frac{1}{K_T^0(hp)} e^{ip(x-y)} \frac{dp}{2\pi}$. (1.11)

mphasize that the function α^0 depends on *T*. For *T* c

$$
\alpha^{0}((x-y)/h) = \frac{\Delta_{0}}{2} \int_{\mathbb{R}} \frac{1}{K_{T}^{0}(hp)} e^{ip(x-y)} \frac{dp}{2\pi}.
$$
 (1.11)

for $0 < T < T_c$ whose off-diagonal element does not vanish and has the integral
kernel
 $\alpha^0((x - y)/h) = \frac{\Delta_0}{2} \int_{\mathbb{R}} \frac{1}{K_T^0(hp)} e^{ip(x-y)} \frac{dp}{2\pi}$. (1.11)
We emphasize that the function α^0 depends on *T*. For *T* close t $\alpha^0((x - y)/h) = \frac{\Delta_0}{2} \int_{\mathbb{R}} \frac{1}{K_T^0(hp)}$
We emphasize that the function α^0 depends on *T*.
case of interest, we have $\Delta_0 \sim \text{const}(1 - T/T_c)^{1/2}$. $\frac{1}{2}$ (n)
(n $rac{\Delta P}{2\pi}$. (1.11)
lose to T_c , which is the We emphasize that the function α^0 depends on *T*. For *T* close to *T_c*, which is the case of interest, we have $\Delta_0 \sim \text{const}(1 - T/T_c)^{1/2}$. case of interest, we have $\Delta_0 \sim \text{const}(1 - T/T_c)^{1/2}$.

1.3. The GL functional

Definition of $H_{\text{loc}}^1(\mathbb{R})$. For numbers $b_1, b_3 > 0$ and $b_2 \in \mathbb{R}$ the GL) functional is given by
 $|w'(x)|^2 + b_2W(x)|\psi(x)|^2 + b_2(1 - |\psi(x)|^2)^2 dx$. (1.12)

Let
$$
\psi
$$
 be a periodic function in $H_{loc}^1(\mathbb{R})$. For numbers $b_1, b_3 > 0$ and $b_2 \in \mathbb{R}$ the
Ginzburg-Landau (GL) functional is given by
\n
$$
\mathcal{E}(\psi) = \int_C \left(b_1 |\psi'(x)|^2 + b_2 W(x) |\psi(x)|^2 + b_3 (1 - |\psi(x)|^2)^2 \right) dx.
$$
\n(1.12)
\nWe denote its ground state energy by
\n
$$
E^{GL} = \inf \{ \mathcal{E}(\psi) | \psi \in H_{per}^1 \}.
$$
\nUnder our assumptions on W it is not difficult to show that there is a corresponding
\nminimizer, which satisfies a second-order differential equation known as the GL

$$
E^{\rm GL} = \inf \{ \mathcal{E}(\psi) \, | \, \psi \in H^1_{\rm per} \} \, .
$$

 $(x)|$
state
on $E^{\text{GL}} = \inf \{$
Under our assumptions on W it is not ominimizer, which satisfies a second-o-
equation. = inf $\{\mathcal{E}(\psi) | \psi \in H^1_{\text{p}}\}$
is not difficult to show
cond-order differentiants
Similarly FBCS Under our assumptions on *W* it is not difficult to show that there is a corresponding
minimizer, which satisfies a second-order differential equation known as the GL
equation.
1.4. Main results
Recall the definition of

1.4. Main results

1.4. Main results
1.4. Main results
Recall the definition of the BCS functional \mathcal{F}^{BCS} in (1.3). We define the energy
 $F^{\text{BCS}}(T,\mu)$ as the difference between the infimum of \mathcal{F}^{BCS} over all admissib 1.4. Main Recall the $F^{\text{BCS}}(T)$, and the f Recall the definition of the BCS functional \mathcal{F}^{BCS} in (1.3). We define the energy $F^{\text{BCS}}(T,\mu)$ as the difference between the infimum of \mathcal{F}^{BCS} over all admissible Γ and the free energy of the norma

$$
\Gamma_0 := \left(\begin{array}{cc} \gamma_0 & 0\\ 0 & 1 - \bar{\gamma}_0 \end{array}\right) \tag{1.13}
$$

$$
F^{\text{BCS}}(T,\mu) \text{ as the difference between the infimum of } \mathcal{F}^{\text{BCS}} \text{ over all admissible } \Gamma
$$

and the free energy of the normal state

$$
\Gamma_0 := \begin{pmatrix} \gamma_0 & 0 \\ 0 & 1 - \bar{\gamma}_0 \end{pmatrix} \tag{1.13}
$$

with $\gamma_0 = (1 + e^{(-h^2 \nabla^2 + h^2 W(x) - \mu)/T})^{-1}$. That is,

$$
F^{\text{BCS}}(T,\mu) = \inf_{\Gamma} \mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma_0). \tag{1.14}
$$

Note that

$$
\mathcal{F}^{\text{BCS}}(\Gamma_0) = -T \text{ Tr } \ln \left(1 + \exp \left(-\left(-h^2 \nabla^2 - \mu + h^2 W(x)\right)\right) / T\right). \tag{1.15}
$$

For small *h* this behaves like an (explicit) constant times h^{-1} . Under further regu-

$$
\mathcal{F}^{\rm BCS}(\Gamma_0) = -T \, \text{Tr} \ln \left(1 + \exp \left(- \left(-h^2 \nabla^2 - \mu + h^2 W(x) \right) \right) / T \right) \,. \tag{1.15}
$$

 $(T, \mu) = \inf_{\Gamma} \mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma_0).$ (1.14)
 $\inf_{\Gamma} (1 + \exp(-(-h^2 \nabla^2 - \mu + h^2 W(x))) / T).$ (1.15)

ke an (explicit) constant times h^{-1} . Under further regu-

(1.15) can be expanded in powers of h. We do not need

t F^{BC}
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Since (Γ_0) = −*T* Tr ln (1 + exp (- (- $h^2\nabla^2$ – μ + $h^2W(x)$)

this behaves like an (explicit) constant times h^{-1} .

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For small *h* this behaves like an (explicit) constant times h^{-1} . Under further regularity assumptions on W , (1.15) can be expanded in powers of *h*. We do not need this, however, since we are only interested in the larity assumptions on W , (1.15) can be expanded in powers of h . We do not need
this, however, since we are only interested in the difference $F^{\text{BCS}}(T,\mu)$.
Since Γ_0 is an admissible state, one always has $F^{\text{BC$ this, however, since we are only interested in the difference $F^{\text{BCS}}(T,\mu)$.

Since Γ_0 is an admissible state, one always has $F^{\text{BCS}}(T,\mu) \leq 0$. If

inequality $F^{\text{BCS}}(T,\mu) < 0$ holds, then the system is said to

Since Γ_0 is an admissible state, one always has $F^{\text{BCS}}(T,\mu) \leq 0$. If the strict
ality $F^{\text{BCS}}(T,\mu) < 0$ holds, then the system is said to be in a superconducting
uperfluid, depending on the physical interpretatio inequality $F^{\text{BCS}}(T,\mu) < 0$ holds, then the system is said to be in a superconducting
(or superfluid, depending on the physical interpretation) state.
Theorem 1.2. Let Assumption 1.1 be satisfied, and let $T_c > 0$ be t **Theorem 1.2.** Let Assumption 1.1 be satisfied, and let $T_c > 0$
perature in the translation invariant case, defined in (1.6). Let
are coefficients b_1 , b_2 and b_3 , given explicitly in (1.20)–(1.22) l
 $F^{\text{BCS}}(T_c(1$ **Theorem 1.2.** Let Assumption 1.1 be satisfied, and let $T_c > 0$ be the critical tem-1.1 be satisfied, and let $T_c > 0$ be the critical tem-
riant case, defined in (1.6). Let $D > 0$. Then there
iven explicitly in (1.20)-(1.22) below, such that
 Dh^2), μ) = $h^3 (E^{GL} - b_3) + o(h^3)$ (1.16)
rror term $o(h^3)$ perature in the translation invariant case, defined in (1.6). Let $D > 0$. Then there
are coefficients b_1 , b_2 and b_3 , given explicitly in (1.20)-(1.22) below, such that
 $F^{\text{BCS}}(T_c(1 - Dh^2), \mu) = h^3 (E^{\text{GL}} - b_3) + o(h^3$ are coefficients b_1 , b_2 and b_3 , given explicitly in (1.20) - (1.22) below, such that

\n- (,
$$
b_2
$$
 and b_3 , given explicitly in (1.20) – (1.22) below, such that $F^{\rm BCS}(T_c(1-Dh^2), \mu) = h^3(E^{\rm GL} - b_3) + o(h^3)$ (1.16) *precisely, the error term* $o(h^3)$ *satisfies* $-\text{const } h^{3+\frac{1}{3}} \leq o(h^3) \leq \text{const } h^5$.
\n- (f) is an approximate minimizer of $\mathcal{F}^{\rm BCS}$ at $T = T_c(1-h^2D)$, in
\n

as $h \to 0$. More precisely, the error term $o(h^3)$ satisfies

$$
-\operatorname{const} h^{3+\frac{1}{3}} \le o(h^3) \le \operatorname{const} h^5
$$

 $(T_c(1 - Dh^2), \mu) = h^3 (E^{GL} - b_3) + o(h^3)$ (1.16)

ely, the error term $o(h^3)$ satisfies
 $-\text{const } h^{3+\frac{1}{3}} \le o(h^3) \le \text{const } h^5$.

an approximate minimizer of \mathcal{F}^{BCS} at $T = T_c(1 - h^2 D)$, in 0. More precisely, the error term $o(h^3)$ satisfies
 $-\text{const } h^{3+\frac{1}{3}} \leq o(h^3) \leq \text{const } h$

preover, if Γ is an approximate minimizer of \mathcal{F}^{B}

is $\mathcal{F}^{\text{BCS}}(\Gamma) \leq \mathcal{F}^{\text{BCS}}(\Gamma_0) + h^3 (E^{\text{GL}}$ const $h^{3+\frac{1}{3}} \leq o(h^3) \leq \text{const } h^5$.

approximate minimizer of \mathcal{F}^{BCS}
 $(\Gamma) \leq \mathcal{F}^{\text{BCS}}(\Gamma_0) + h^3 (E^{\text{GL}} - b_3)$ Moreover, if Γ is an approximate minimizer of \mathcal{F}^{BCS} at $T = T_c(1 - h^2 D)$, in
ense that
 $\mathcal{F}^{BCS}(\Gamma) \leq \mathcal{F}^{BCS}(\Gamma_0) + h^3 (E^{GL} - b_3 + \epsilon)$ (1.17) the sense that

$$
\mathcal{F}^{\text{BCS}}(\Gamma) \le \mathcal{F}^{\text{BCS}}(\Gamma_0) + h^3 \left(E^{\text{GL}} - b_3 + \epsilon \right)
$$
 (1.17)

for some small $\epsilon >$

for some small
$$
\epsilon > 0
$$
, then the corresponding α can be decomposed as
\n
$$
\alpha(x, y) = \frac{1}{2} (\psi(x) + \psi(y)) \alpha^{0} (h^{-1}(x - y)) + \sigma(x, y)
$$
\n
$$
\text{with } \mathcal{E}^{\text{GL}}(\psi) \le E^{\text{GL}} + \epsilon + \text{const } h^{\frac{1}{3}}, \alpha^{0} \text{ defined in (1.11), and}
$$
\n
$$
\int |\sigma(x, y)|^{2} dx dy \le \text{const } h^{3 + \frac{1}{3}}.
$$
\n(1.19)

with $\mathcal{E}^{\text{GL}}(\psi) \leq E^{\text{GL}} + \epsilon + \text{const} \, h^{\frac{1}{3}}, \, \alpha^0 \text{ defined in}$

$$
(x, y) = \frac{1}{2} (\psi(x) + \psi(y)) \alpha^{0} (h^{-1}(x - y)) + \sigma(x, y)
$$
(1.18)
\n
$$
{}^{G\text{L}} + \epsilon + \text{const } h^{\frac{1}{3}}, \alpha^{0} \text{ defined in (1.11), and}
$$

\n
$$
\int_{\mathcal{C} \times \mathbb{R}} |\sigma(x, y)|^{2} dx dy \le \text{const } h^{3 + \frac{1}{3}}.
$$
(1.19)
\nents in the GL functional
\nexplicit expressions for the coefficients in the GL functional we
\nactions
\n
$$
\frac{(z/2)}{z}, \qquad q_{1}(z) = \frac{e^{2z} - 2ze^{z} - 1}{z}, \qquad q_{2}(z) = \frac{2e^{z}(e^{z} - 1)}{z}.
$$

1.5. The coefficients in the GL functional

 $(\psi) \leq E^{\text{GL}} + \epsilon + \text{const } h^{\frac{1}{3}}, \ \alpha^0 \text{ defined in (1.11)}, \text{ and}$
 $\int_{\mathcal{C} \times \mathbb{R}} |\sigma(x, y)|^2 dx dy \leq \text{const } h^{3 + \frac{1}{3}}.$
 coefficients in the GL functional

to give explicit expressions for the coefficients in the functions

introduce the functions
\n
$$
g_0(z) = \frac{\tanh(z/2)}{z}, \qquad g_1(z) = \frac{e^{2z} - 2ze^z - 1}{z^2(1 + e^z)^2}, \qquad g_2(z) = \frac{2e^z(e^z - 1)}{z(e^z + 1)^3}.
$$
\nSetting, as usual, $\beta_c = T_c^{-1}$ we define
\n
$$
c = \frac{2\int_{\mathbb{R}} \left[g_0(\beta_c(q^2 - \mu)) - \beta_c(q^2 - \mu)g_1(\beta_c(q^2 - \mu)) \right] dq}{\beta_c \int_{\mathbb{R}} \frac{g_1(\beta_c(q^2 - \mu))}{q^2 - \mu} dq}.
$$
\nThe three coefficients of the GL functional turn out to be as follows,

$$
g_0(z) = \frac{\tanh(z/2)}{z}, \qquad g_1(z) = \frac{e^{2z} - 2ze^z - 1}{z^2(1 + e^z)^2}, \qquad g_2(z) = \frac{2e^z}{z(\epsilon)}
$$

Setting, as usual, $\beta_c = T_c^{-1}$ we define

$$
c = \frac{2\int_{\mathbb{R}} \left[g_0(\beta_c(q^2 - \mu)) - \beta_c(q^2 - \mu)g_1(\beta_c(q^2 - \mu)) \right] dq}{\beta_c \int_{\mathbb{R}} \frac{g_1(\beta_c(q^2 - \mu))}{q^2 - \mu} dq}.
$$
The three coefficients of the GL functional turn out to be as follows,

$$
b_1 = cD \frac{\beta_c^2}{16} \int_{\mathbb{R}} \left(g_1(\beta_c(q^2 - \mu)) + 2\beta_c q^2 g_2(\beta_c(q^2 - \mu)) \right) \frac{dq}{2\pi},
$$

$$
b_1 = cD \frac{\beta_c^2}{16} \int_{\mathbb{R}} \left(g_1(\beta_c(q^2 - \mu)) + 2\beta_c q^2 g_2(\beta_c(q^2 - \mu)) \right) \frac{dq}{2\pi}, \tag{1.20}
$$

$$
b_1 = cD \frac{\beta_c^2}{16} \int_{\mathbb{R}} \left(g_1(\beta_c(q^2 - \mu)) + 2\beta_c q^2 g_2(\beta_c(q^2 - \mu)) \right) \frac{dq}{2\pi},
$$
(1.20)
\n
$$
b_2 = cD \frac{\beta_c^2}{4} \int_{\mathbb{R}} g_1(\beta_c(q^2 - \mu)) \frac{dq}{2\pi}
$$
(1.21)
\nand
\n
$$
b_3 = (cD)^2 \frac{\beta_c^2}{16} \int_{\mathbb{R}} \frac{g_1(\beta_c(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi}.
$$
(1.22)

$$
b_1 = cD \frac{\beta_c^2}{16} \int_{\mathbb{R}} \left(g_1(\beta_c(q^2 - \mu)) + 2\beta_c q^2 g_2(\beta_c(q^2 - \mu)) \right) \frac{dq}{2\pi},
$$
(1.20)
\n
$$
b_2 = cD \frac{\beta_c^2}{4} \int_{\mathbb{R}} g_1(\beta_c(q^2 - \mu)) \frac{dq}{2\pi}
$$
(1.21)
\nand
\n
$$
b_3 = (cD)^2 \frac{\beta_c^2}{16} \int_{\mathbb{R}} \frac{g_1(\beta_c(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi}.
$$
(1.22)
\nWe shall now discuss the signs of these coefficients. First note that $g_0(z) - zg_1(z) = (zg_0(z))' > 0$ and $g_1(z)/z > 0$, which implies that $c > 0$. Using $g_1(z)/z >$

 $\frac{zg_1}{a}$ = $(cD)^2 \frac{\beta_c^2}{16} \int_{\mathbb{R}} \frac{g_1(\beta_c(q^2 - \mu))}{q^2 - \mu}$

aall now discuss the signs of t
 $(g_0(z))' > 0$ and $g_1(z)/z > 0$, v

e see that $b_3 > 0$. In contrast,

on the value of $\beta_c \mu$ (which de

sive, as the following computa $\frac{\beta_c^2}{16}$ *J*
discu
0 an
 b_3 }
alue
ne fo $\frac{1}{2\pi}$. (1.22)

ese coefficients. First note that $g_0(z)$ –

hich implies that $c > 0$. Using $g_1(z)/z >$

the coefficient b_2 may have either sign,

bends on a and μ). The coefficient b_1 is

ion shows: using th We shall now discuss the signs of these coefficients. First note that $g_0(z)$ –
 $(y_0(z))' > 0$ and $g_1(z)/z > 0$, which implies that $c > 0$. Using $g_1(z)/z >$
 \sin , we see that $b_3 > 0$. In contrast, the coefficient b_2 may $(z) = (z g_0(z))' > 0$ and $g_1(z)/z > 0$, which implies that $c > 0$. Using $g_1(z)/z >$
gain, we see that $b_3 > 0$. In contrast, the coefficient b_2 may have either sign,
bending on the value of $\beta_c \mu$ (which depends on a and μ 0 again, we see that $b_3 > 0$. In contrast, the coefficient b_2 may have either sign,
depending on the value of $\beta_c \mu$ (which depends on a and μ). The coefficient b_1 is
again positive, as the following computatio

depending on the value of
$$
\beta_c \mu
$$
 (which depends on a and μ). The coefficient b_1 is
again positive, as the following computation shows: using the fact that $g_2(z) = g'_1(z) + (2/z)g_1(z)$ we find

$$
b_1 = cD \frac{\beta_c^2}{16} \int_{\mathbb{R}} \left(g_1(\beta_c(q^2 - \mu)) + 2\beta_c q^2 \left(g'_1(\beta_c(q^2 - \mu)) + \frac{2g_1(\beta_c(q^2 - \mu))}{\beta_c(q^2 - \mu)} \right) \right) \frac{dq}{2\pi}
$$

$$
= cD \frac{\beta_c^2}{16} \int_{\mathbb{R}} \left(g_1(\beta_c(q^2 - \mu)) + q \frac{d}{dq} \left(g_1(\beta_c(q^2 - \mu)) \right) + 4q^2 \frac{g_1(\beta_c(q^2 - \mu))}{q^2 - \mu} \right) \frac{dq}{2\pi}
$$

$$
= cD \frac{\beta_c^2}{4} \int_{\mathbb{R}} q^2 \frac{g_1(\beta_c(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi}.
$$
The claimed positivity is now again a consequence of $g_1(z)/z > 0$.

 46 The claimed positivity is now again a consequence of $g_1(z)/z > 0$.
 $\frac{1}{z}$
2. Sketch of the proof

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In the following we will consider temperatures $T = T_c(1 - Dh^2)$. It is not difficult

to see that the solution Δ_0 of the BCS gap equation (1.7) is of order $\Delta_0 = O(h)$.
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to see that the solution Δ_0 of the BCS gap equation (1.7) is of order $\Delta_0 = O(h)$.
It is useful to rewrite the BCS functional in a more

$$
\Delta = \Delta(x) = -\psi(x)\Delta_0,
$$

to see that the solution Δ_0 of the BCS gap equation (1.7) is of order $\Delta_0 = O(h)$.

It is useful to rewrite the BCS functional in a more convenient way. Define Δ to be the multiplication operator
 $\Delta = \Delta(x) = -\psi(x)\Delta_0$ be the multiplication operator
 $\Delta = \Delta(x) = -\psi(x)\Delta_0$,
 Δ_0 is the solution of the BCS equation (1.7) for temperature T, and ψ a

dic function in $H_{\text{loc}}^2(\mathbb{R})$. Define further
 $H = \begin{pmatrix} -h^2\nabla^2 - \mu + h^2W(x) & \Delta \end{pmatrix}$ $\Delta = \Delta$
where Δ_0 is the solution of the BC
periodic function in $H_{\text{loc}}^2(\mathbb{R})$. Define
 $H_{\Delta} = \begin{pmatrix} -h^2 \nabla^2 - \mu + h \ \Delta \end{pmatrix}$

$$
\Delta = \Delta(x) = -\psi(x)\Delta_0,
$$

where Δ_0 is the solution of the BCS equation (1.7) for temperature *T*, and ψ a
periodic function in $H_{\text{loc}}^2(\mathbb{R})$. Define further

$$
H_{\Delta} = \begin{pmatrix} -h^2 \nabla^2 - \mu + h^2 W(x) & \Delta \\ \overline{\Delta} & h^2 \nabla^2 + \mu - h^2 W(x) \end{pmatrix}.
$$
(2.1)
Formally, we can write the BCS functional as

Formally, we can write the BCS functional as

periodic function in
$$
H_{loc}^2(\mathbb{R})
$$
. Define further
\n
$$
H_{\Delta} = \begin{pmatrix} -h^2 \nabla^2 - \mu + h^2 W(x) & \Delta \\ \nabla & h^2 \nabla^2 + \mu - h^2 W(x) \end{pmatrix}.
$$
\n(2.1)
\nFormally, we can write the BCS functional as
\n
$$
\mathcal{F}^{BCS}(\Gamma) = -\operatorname{Tr}(-h^2 \nabla^2 - \mu + h^2 W) + \frac{1}{2} \operatorname{Tr} H_{\Delta} \Gamma - TS(\Gamma)
$$
\n
$$
+ \frac{1}{4ha} \Delta_0^2 \int_C |\psi(x)|^2 dx - ha \int_C \left| \frac{\Delta_0 \psi(x)}{2ha} - \alpha(x, x) \right|^2 dx. \quad (2.2)
$$
\nThe first two terms on the right are infinite, of course, only their sum is well defined. For an upper bound, we can drop the very last term. The terms on the first line are minimized for $\Gamma_{\Delta} = 1/(1 + e^{\frac{1}{T}H_{\Delta}})$, which we choose as a trial state.

The first two terms on the right are infinite, of course, only their sum is well $\frac{1}{T}$ m $^{\rm s}$
 $^{\rm se}$
 $^{\rm is}$
 $^{\rm s}$ h_1
 \rightarrow \rightarrow $\frac{1}{4ha}\Delta_0^2 \int_C |\psi(x)|^2 dx - ha \int_C \left| \frac{\Delta_0 \psi(x)}{2ha} - \alpha(x, x) \right|^2 dx$. (2.2)

on the right are infinite, of course, only their sum is well

bound, we can drop the very last term. The terms on the

d for $\Gamma_{\Delta} = 1/(1 + e^{\frac{1}{T}H_{\Delta}})$ (*x*)|
are
can
 $1/(1$
(Γ₀) defined. For an upper bound, we can drop the very last term. The terms on the
first line are minimized for $\Gamma_{\Delta} = 1/(1 + e^{\frac{1}{T}H_{\Delta}})$, which we choose as a trial state.
Then
 $F^{\text{BCS}}(T,\mu) \leq \mathcal{F}^{\text{BCS}}(\Gamma_{\Delta}) - \mathcal{F}^$

first line are minimized for
$$
\Gamma_{\Delta} = 1/(1 + e^{\frac{1}{T}H_{\Delta}})
$$
, which we choose as a trial state.
\nThen
\n $F^{\text{BCS}}(T,\mu) \leq \mathcal{F}^{\text{BCS}}(\Gamma_{\Delta}) - \mathcal{F}^{\text{BCS}}(\Gamma_0)$ (2.3)
\n $\leq -\frac{T}{2} \text{Tr} \left[\ln(1 + e^{-\frac{1}{T}H_{\Delta}}) - \ln(1 + e^{-\frac{1}{T}H_0}) \right] + \frac{1}{4ha} \Delta_0^2 \int_C |\psi(x)|^2 dx$.
\nTo complete the upper bound, we have to evaluate $\text{Tr}[\ln(1 + e^{-H_{\Delta}/T}) - \ln(1 + e^{-H_0/T})]$. This is done via a contour integral representation and semiclassical types of estimates.

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vi $\begin{bmatrix} \text{Tr} \ \text{oper} \ \text{me} \ \text{v} \end{bmatrix}$ $\ln(1 + e^{-\frac{1}{T}H_{\Delta}}) - \ln(1 + e^{-\frac{1}{T}H_{0}}) + \frac{1}{4ha}\Delta_{0}^{2}$
bound, we have to evaluate Tr[ln(1 + e^{-I}
ria a contour integral representation and sem
is divided into several steps. We first aim
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u∈ $\ln(\ln n)$
 $\frac{1}{2}$ $\begin{array}{c} \hline h \ + \ {\rm d} \ \hline \end{array}$ $|(x)|$
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n a To complete the upper bound, we have to evaluate Tr[ln(1 + $e^{-H_0/T}$) – ln(1 + $e^{-H_0/T}$)]. This is done via a contour integral representation and semiclassical types of estimates.
The lower bound is divided into several $e^{-H_0/T}$

% of estimates. The lower bound is divided into several steps. We first aim at an a priori bound on α for a general state Γ , which has lower energy than the translation-invariant state. With The love
bound on α
invariant sta bound on α for a general state Γ , which has lower energy than the translation-
invariant state. With
 $H_{\Delta_0}^0 = \begin{pmatrix} -h^2 \nabla^2 - \mu & \Delta_0 \\ \Delta_0 & -h^2 \nabla^2 + \mu \end{pmatrix}$,
we can rewrite the BCS functional in the form
 $\mathcal{$

$$
H_{\Delta_0}^0 = \begin{pmatrix} -h^2 \nabla^2 - \mu & \Delta_0 \\ \Delta_0 & -h^2 \nabla^2 + \mu \end{pmatrix},
$$

we can rewrite the BCS functional in the form

$$
H_{\Delta_0}^0 = \begin{pmatrix} -h & \sqrt{-\mu} & \Delta_0 \\ \Delta_0 & -h^2 \nabla^2 + \mu \end{pmatrix},
$$

we can rewrite the BCS functional in the form

$$
\mathcal{F}^{BCS}(\Gamma) = -\operatorname{Tr}(-h^2 \nabla^2 - \mu + h^2 W) + \frac{1}{2} \operatorname{tr} H_{\Delta_0}^0 \Gamma - TS(\Gamma)
$$

$$
+ h^2 \operatorname{Tr} W\gamma + \frac{1}{4ha} \Delta_0^2 - ha \int_c \left| \frac{\Delta_0}{2ha} - \alpha(x, x) \right|^2 dx. \quad (2.4)
$$

From the BCS equation and the definition of α^0 in (1.11) we conclude that

$$
\alpha^0(0) = \frac{1}{2\pi h} \int_{\mathbb{R}} \frac{\Delta_0}{2K_T^0(p)} dp = \frac{\Delta_0}{2ah}, \quad (2.5)
$$

$$
+ h^{2} \operatorname{Tr} W\gamma + \frac{1}{4ha} \Delta_{0}^{2} - ha \int_{\mathcal{C}} \left| \frac{\Delta_{0}}{2ha} - \alpha(x, x) \right| dx. \quad (2.4)
$$

From the BCS equation and the definition of α^{0} in (1.11) we conclude that

$$
\alpha^{0}(0) = \frac{1}{2\pi h} \int_{\mathbb{R}} \frac{\Delta_{0}}{2K_{T}^{0}(p)} dp = \frac{\Delta_{0}}{2ah}, \qquad (2.5)
$$

and hence
\n
$$
\mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma^0) \ge \frac{T}{2} \mathcal{H}(\Gamma, \Gamma^0) + h^2 \operatorname{Tr} W(\gamma - \gamma^0) - ah \int_{\mathcal{C}} |\alpha(x, x) - \alpha^0(0)|^2 dx,
$$
\n(2.6)

where H denotes the relative entropy

$$
{}^3(\Gamma) - \mathcal{F}^{BCS}(\Gamma^0) \ge \frac{T}{2} \mathcal{H}(\Gamma, \Gamma^0) + h^2 \operatorname{Tr} W(\gamma - \gamma^0) - ah \int_C |\alpha(x, x) - \alpha^0(0)|^2 dx,
$$

(2.6)
e \mathcal{H} denotes the relative entropy

$$
\mathcal{H}(\Gamma, \Gamma^0) = \frac{2}{T} \left(\frac{1}{2} \operatorname{tr} H_{\Delta_0}^0 \Gamma - TS(\Gamma) + \frac{1}{2} \operatorname{tr} H_{\Delta_0}^0 \Gamma^0 - TS(\Gamma^0) \right)
$$

$$
= \operatorname{Tr} \left[\Gamma \left(\ln \Gamma - \ln \Gamma^0 \right) + (1 - \Gamma) \left(\ln(1 - \Gamma) - \ln(1 - \Gamma^0) \right) \right]. \qquad (2.7)
$$
that the left side of (2.6) is necessarily non-positive for a minimizing state Γ .
One of the essential steps in our proof, which is used on several occasions,
mma 5.1. This lemma presents a lower bound on the relative entropy of the

Note that the left side of (2.6) is necessarily non-positive for a minimizing state Γ .

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en Γ
bin
r = Tr $[\Gamma (\ln \Gamma - \ln \Gamma^0) + (1 - \Gamma) (\ln(1 - \Gamma) - \ln(1 - \Gamma^0))]$. (2.7)
left side of (2.6) is necessarily non-positive for a minimizing state Γ .
he essential steps in our proof, which is used on several occasions,
This lemma presents a ι
16
Γε One of the essential steps in our proof, which is used on several occasions,
is Lemma 5.1. This lemma presents a lower bound on the relative entropy of the
form
 $\mathcal{H}(\Gamma,\Gamma^0) \geq \text{Tr}\left[H^0(\Gamma-\Gamma^0)^2\right]$
 $\left(\text{Tr}\Gamma(1-\Gamma) - \text{Tr}\Gamma$ is Lemma 5.1. This lemma presents a lower bound on the relative entropy of the

form
\n
$$
\mathcal{H}(\Gamma,\Gamma^0) \ge \text{Tr}\left[H^0\left(\Gamma-\Gamma^0\right)^2\right] + \frac{1}{3}\frac{\left(\text{Tr}\,\Gamma(1-\Gamma)-\text{Tr}\,\Gamma_0(1-\Gamma^0)\right)^2}{|\text{Tr}\,\Gamma(1-\Gamma)-\text{Tr}\,\Gamma^0(1-\Gamma^0)|+\text{Tr}\,\Gamma^0(1-\Gamma^0)},\qquad(2.8)
$$
\nwhere $H^0 = (1-2\Gamma_0)^{-1}\ln((1-\Gamma^0)/\Gamma^0)$. In our case here, it equals $K_T^0(-ih\nabla)/T$, with K_T^0 defined in (1.8). From (2.6) we deduce that for a minimizer Γ
\n $0 \ge \mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma^0) \ge \text{Tr}\, K_T^0(\gamma-\gamma^0)^2 + h^2 \text{Tr}\, W(\gamma-\gamma^0)$ (2.9)

wher $\frac{1}{3}$
 $\frac{1}{6}$

$$
3 |\text{Tr } \Gamma(1 - \Gamma) - \text{Tr } \Gamma^{0}(1 - \Gamma^{0})| + \text{Tr } \Gamma^{0}(1 - \Gamma^{0})
$$
\nwhere $H^{0} = (1 - 2\Gamma_{0})^{-1} \ln((1 - \Gamma^{0})/\Gamma^{0})$. In our case here, it equals $K_{T}^{0}(-ih\nabla)/T$,
\nwith K_{T}^{0} defined in (1.8). From (2.6) we deduce that for a minimizer Γ
\n
$$
0 \geq \mathcal{F}^{BCS}(\Gamma) - \mathcal{F}^{BCS}(\Gamma^{0}) \geq \text{Tr } K_{T}^{0}(\gamma - \gamma^{0})^{2} + h^{2} \text{Tr } W(\gamma - \gamma^{0})
$$
\n
$$
+ \int_{\mathcal{C}} \langle \alpha(\cdot, y) - \alpha^{0}(\frac{-y}{h}) | K_{T}^{0}(-ih\nabla) - a\delta(\frac{-y}{h}) | \alpha(\cdot, y) - \alpha^{0}(\frac{-y}{h}) \rangle dy
$$
\n
$$
+ \frac{1}{3} \frac{T (\text{Tr } [\gamma(1 - \gamma) - \gamma^{0}(1 - \gamma^{0}) - |\alpha|^{2} + |\alpha^{0}|^{2}])^{2}}{|\text{Tr } [\gamma(1 - \gamma) - \gamma^{0}(1 - \gamma^{0}) - |\alpha|^{2} + |\alpha^{0}|^{2}]| + \text{Tr } [\gamma^{0}(1 - \gamma^{0}) - |\alpha^{0}|^{2}]},
$$
\nwhere $\langle \cdot | \cdot \rangle$ denotes the inner product in $L^{2}(\mathbb{R})$. Observe that the term in the second line is a convenient way to write $\text{Tr } K_{T}^{0}(\alpha - \alpha^{0})^{*}(\alpha - \alpha^{0}) - ah \int_{\mathcal{C}} |\alpha(x, x) - \alpha^{0}(0)|^{2} dx$.

 $+$ code to the set of $\frac{1}{e}$ $\frac{1}{3}$ (n fi K comentications of the set of the se $\frac{1}{\gamma}$ are posentially proper power and the set of $\frac{1}{\gamma}$ are posentially proper posentially considered to $\frac{1}{\gamma}$ $(1 - \gamma) - \gamma^0 (1 - \gamma^0) - |\alpha|^2 + |\alpha^0|^2$
 $(1 - \gamma^0) - |\alpha|^2 + |\alpha^0|^2] + \text{Tr}[\gamma^0 (1$

coduct in $L^2(\mathbb{R})$. Observe that the

e Tr $K_T^0 (\alpha - \alpha^0)^* (\alpha - \alpha^0) - ah \int_C |\alpha|$

t side and the Schwarz inequality

otain first that $\text{Tr } K_T^0 (\gamma - \gamma^0)^$ Tr $[\gamma(1-\gamma) - \gamma^0(1-\gamma^0) - |\alpha|^2 + |\alpha^0|^2]| + \text{Tr} [\gamma^0(1-\gamma^0) - |\alpha^0|$
enotes the inner product in $L^2(\mathbb{R})$. Observe that the term in the s
enient way to write Tr $K_T^0(\alpha-\alpha^0)^*(\alpha-\alpha^0) - ah \int_C |\alpha(x, x) - \alpha^0(0)$
st line on the right sid] where $\langle \cdot | \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R})$. Observe that the term in the second
line is a convenient way to write Tr $K_T^0(\alpha - \alpha^0)^*(\alpha - \alpha^0) - ah \int_C |\alpha(x, x) - \alpha^0(0)|^2 dx$.
From the first line on the right side and line is a convenient way to write Tr $K_T^0(\alpha - \alpha^0)^*(\alpha - \alpha^0) - ah$ \bar{F}
From the first line on the right side and the Schwarz inequa
fact that $K_T^0 - a\delta \geq 0$, we obtain first that Tr $K_T^0(\gamma - \gamma^0)$
with the last line this $\int_T^0 (\alpha - \alpha^0)^* (\alpha - \alpha^0) - ah \int_C |\alpha(x, x) - \alpha^0(0)|^2 dx$ $(x, x) - \alpha^{0}(0)$ |
together with
{ $O(h^3)$. Toge
 $O(h)$.
ground state, γ m (2.9) that er
n $\frac{0}{T} - a\delta \geq 0$, we obtain first that $\text{Tr } K_T^0$ $2^2 \leq O$

fact that $K_T^0 - a\delta \geq 0$, we obtain first that $\text{Tr } K_T^0 (\gamma - \gamma^0)^2 \leq O(h^3)$. Together
with the last line this further gives the a priori bound $||\alpha||_2^2 \leq O(h)$.
Next, we use that $K_T^0 - a\delta$ has α^0 as unique zero ener fact that $K_T^0 - a\delta \ge 0$, we obtain first that Tr K_T^0
with the last line this further gives the a priori bou
Next, we use that $K_T^0 - a\delta$ has α^0 as unique z
a gap of order one above zero, and we can further
necessa with the last line this further gives the a priori bound $||\alpha||_2^2$
Next, we use that $K_T^0 - a\delta$ has α^0 as unique zero energy
a gap of order one above zero, and we can further conclude
necessarily of the form
 $\alpha(x, y) =$ $\begin{align} \mathbf{r}(\mathbf{h}). \ \mathbf{r}(\mathbf{h}) = \mathbf{r}(\mathbf{h}). \end{align}$ Next, we use that $K_T^0 - a\delta$ has α^0 as unique zero energy ground state, with

i of order one above zero, and we can further conclude from (2.9) that α is

sarily of the form
 $\alpha(x, y) = \frac{1}{2}(\psi(x) + \psi(y))\alpha^0((x - y)/h) + \beta(x$ Next, we use that $K_T^0 - a\delta$ has α^0 as unique zero energy ground state, with

$$
\alpha(x, y) = \frac{1}{2}(\psi(x) + \psi(y))\alpha^{0}((x - y)/h) + \beta(x, y),
$$

a gap of order one above zero, and we can further conclude from (2.9) that α is
necessarily of the form
 $\alpha(x, y) = \frac{1}{2}(\psi(x) + \psi(y))\alpha^{0}((x - y)/h) + \beta(x, y)$,
with $\|\beta\|_{2}^{2} \leq O(h^{3})$. This information about the decomposition $\alpha(x, y)$
with $\|\beta\|_2^2 \le O(h^3)$. The form of the form of the form of the semi-
 $\mathcal{F}^{\rm BCS}(\Gamma) - \mathcal{F}^{\rm BCS}(\Gamma_\Delta)$
the computation we all $(x, y) = \frac{1}{2}(\psi(x) + \psi(y))\alpha^{0}((x - y)/h) + \beta(x, y),$

(i). This information about the decomposition of by means of a lower bound of the type (2.8), then (Γ_{Δ}) is very small compared to h^{3} . This reduces we already did in the with $||\beta||_2^2 \le O(h^3)$. This information about the decomposition of α then allows us to deduce, again by means of a lower bound of the type (2.8), that the difference $\mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma_{\Delta})$ is very smal with $\|\beta\|_2^2 \le O(h^3)$. This information about the decomposition of α then allows us $\mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma_{\Delta})$ is very small compared to h^3 . This reduces the problem to the computation we already did in the upper bound.

3. Semiclassics

Derivation of GL Theory for a 1D System 65

3. Semiclassics

One of the key ingredients in both the proof of the upper and the lower bound are

detailed semiclassical asymptotics for operators of the form

$$
H_{\Delta} = \begin{pmatrix} -h^2 \nabla^2 - \mu + h^2 W(x) & \Delta(x) \\ \overline{\Delta(x)} & h^2 \nabla^2 + \mu - h^2 W(x) \end{pmatrix} . \tag{3.1}
$$

detailed semiclassical asymptotics for operators of the form
 $H_{\Delta} = \begin{pmatrix} -h^2 \nabla^2 - \mu + h^2 W(x) & \Delta(x) \\ \Delta(x) & h^2 \nabla^2 + \mu - h^2 W(x) \end{pmatrix}$ (3.1)

Here $\Delta(x) = -h\psi(x)$ with a periodic function ψ , which is of order one as $h \to 0$ $H_{\Delta} = \begin{pmatrix} -h^2 \nabla^2 - \mu + h^2 W(x) & \Delta(x) \\ \Delta(x) & h^2 \nabla^2 + \mu - h^2 \end{pmatrix}$
Here $\Delta(x) = -h\psi(x)$ with a periodic function ψ , which is (but might nevertheless depend on h). We are interested In contrast to traditional semiclassic $\hbar v$
er
tra
un
 ψ $\Delta(x)$ $h^2 \nabla^2 + \mu - h^2 W(x)$ \int
periodic function ψ , which is of order one as $h \to 0$
md on h). We are interested in the regime $h \to 0$.
miclassical results [12, 16] we work under minimal
 ψ and W. To be precise, w (but might nevertheless depend on h). We are interested in the regime $h \to 0$.
In contrast to traditional semiclassical results [12, 16] we work under minimal
smoothness assumptions on ψ and W. To be precise, we assume In contrast to traditional semiclassical results $[12, 16]$ we work under minimal smoothness assumptions on ψ and W. To be precise, we assume Assumption 1.1 smoothness assumptions on ψ and W . To be precise, we assume Assumption 1.1
for W and that ψ is a periodic function in $H_{\text{loc}}^2(\mathbb{R})$.
Our first result concerns the free energy.
Theorem 3.1. Let
 $f(z) = -\ln(1 + e$ for *W* and that ψ is a periodic function in $H_{\text{loc}}^2(\mathbb{R})$.

Our first result concerns the free energy.
 Theorem 3.1. Let
 $f(z) = -\ln(1 + e^{-z})$

and define
 $g_0(z) = \frac{f'(-z) - f'(z)}{z} = \frac{\tan z}{z}$

Theorem 3.1. Let

$$
f(z) = -\ln(1 + e^{-z}), \qquad (3.2)
$$

and define

rem 3.1. Let
\n
$$
f(z) = -\ln(1 + e^{-z}) ,
$$
\n
$$
g_0(z) = \frac{f'(-z) - f'(z)}{z} = \frac{\tanh(\frac{1}{2}z)}{z} ,
$$
\n(3.2)

$$
f(z) = -\ln(1 + e^{-z}) ,
$$
\n(3.2)
\n
$$
g_0(z) = \frac{f'(-z) - f'(z)}{z} = \frac{\tanh(\frac{1}{2}z)}{z} ,
$$
\n(3.3)
\n
$$
g_1(z) = -g'_0(z) = \frac{f'(-z) - f'(z)}{z^2} + \frac{f''(-z) + f''(z)}{z} = \frac{e^{2z} - 2ze^z - 1}{z^2(1 + e^z)^2} \quad (3.4)
$$
\n
$$
g_2(z) = g'_1(z) + \frac{2}{z}g_1(z) = \frac{f'''(z) - f'''(-z)}{z} = \frac{2e^z(e^z - 1)}{z(e^z + 1)^3} .
$$
\n(3.5)
\n
$$
g_1(z) = \ln(1 + e^{-z})
$$
\n
$$
g_2(z) = g'_1(z) + \frac{2}{z}g_1(z) = \frac{f'''(z) - f'''(-z)}{z} = \frac{2e^z(e^z - 1)}{z(e^z + 1)^3} .
$$

and

$$
-g_0'(z) = \frac{f'(-z) - f'(z)}{z^2} + \frac{f''(-z) + f''(z)}{z} = \frac{e^{2z} - 2ze^z - 1}{z^2(1 + e^z)^2}
$$
(3.4)
\n
$$
g_2(z) = g_1'(z) + \frac{2}{z}g_1(z) = \frac{f'''(z) - f'''(-z)}{z} = \frac{2e^z(e^z - 1)}{z(e^z + 1)^3}.
$$
(3.5)
\n
$$
g_1(z) = g_2'(z) + \frac{2}{z}g_1(z) = \frac{f'''(z) - f'''(-z)}{z(e^z + 1)^3} = \frac{2e^z(e^z - 1)}{z(e^z + 1)^3}.
$$
(3.6)
\n
$$
g_1(z) = g_1'(z) + \frac{2}{z}g_1(z) = \frac{f'''(z) - f'''(-z)}{z}
$$

Then, for any $\beta >$

$$
g_2(z) = g_1'(z) + \frac{2}{z} g_1(z) = \frac{f'''(z) - f'''(-z)}{z} = \frac{2e^z (e^z - 1)}{z (e^z + 1)^3}.
$$
\n
$$
(3.5)
$$
\n
$$
\frac{h}{\beta} \text{Tr} \left[f(\beta H_\Delta) - f(\beta H_0) \right] = h^2 E_1 + h^4 E_2 + O(h^6) \left(\|\psi\|_{H^1(C)}^6 + \|\psi\|_{H^2(C)}^2 \right),
$$
\n
$$
(3.6)
$$
\n
$$
e^{-\beta} \text{er}
$$
\n
$$
E_1 = -\frac{\beta}{2} \|\psi\|_2^2 \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi}
$$
\n
$$
E_2 = \frac{\beta^2}{2} \|\psi'\|_2^2 \int \left(g_1(\beta(q^2 - \mu)) + 2\beta q^2 g_2(\beta(q^2 - \mu)) \right) \frac{dq}{2\pi}
$$

where

$$
E_1 = -\frac{\beta}{2} ||\psi||_2^2 \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi}
$$

and

$$
E_1 = -\frac{\beta}{2} ||\psi||_2^2 \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi}
$$

\n
$$
E_2 = \frac{\beta^2}{8} ||\psi'||_2^2 \int_{\mathbb{R}} (g_1(\beta(q^2 - \mu)) + 2\beta q^2 g_2(\beta(q^2 - \mu))) \frac{dq}{2\pi}
$$

\n
$$
+ \frac{\beta^2}{2} \langle \psi | W | \psi \rangle \int_{\mathbb{R}} g_1(\beta(q^2 - \mu)) \frac{dq}{2\pi}
$$

\n
$$
+ \frac{\beta^2}{8} ||\psi||_4^4 \int_{\mathbb{R}} \frac{g_1(\beta(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi}.
$$

\nrecisely, we claim that the diagonal entries of the 2 × 2 r
\n
$$
H_{\Delta}) - f(\beta H_0)
$$
 are locally trace class and that the sum of
\nme is given by (3.6). We sketch the proof of Theorem 3.1

More precisely, we claim that the diagonal entries of the 2×2 matrix-valued $+\frac{\beta^2}{8} \|\psi\|_4^4 \int_{\mathbb{R}} \frac{g_1(\beta(q^2-\mu))}{q^2-\mu}$
ly, we claim that the diagon
 $-f(\beta H_0)$ are locally trace c
given by (3.6). We sketch th
r to [7] for some technicalit $rac{a}{2\pi}$.
al en ass a
e proces. More precisely, we claim that the diagonal entries of the 2×2 matrix-valued
tor $f(\beta H_{\Delta}) - f(\beta H_0)$ are locally trace class and that the sum of their traces
nit volume is given by (3.6). We sketch the proof of Theorem per unit volume is given by (3.6) . We sketch the proof of Theorem 3.1 in Subsection 6.2 below and refer to [7] for some technicalities. $p = 0.2$ below and refer to [7] for some technicalities.

Our second semiclassical result concerns the behavior of $(1 + \exp(\beta H_L))$
the limit $h \to 0$. More precisely, we are interested in $[(1 + \exp(\beta H_L))$
 $[\cdot]_{12}$ denotes the upper off-diagonal entry of an operator-valued 2 :
this purp Our second semiclassical result concerns the behavior of $(1 + \exp(\beta H_{\Delta}))^{-1}$ in
mit $h \to 0$. More precisely, we are interested in $[(1 + \exp(\beta H_{\Delta}))^{-1}]_{12}$, where
denotes the upper off-diagonal entry of an operator-valued 2 × 2 [·]₁₂ denotes the upper off-diagonal entry of an operator-valued 2 × 2 matrix. For
this purpose, we define the H^1 norm of a periodic operator η by
 $||\eta||_{H^1}^2 = \text{Tr} [\eta^*(1 - h^2 \nabla^2)\eta]$. (3.7)
In Subsection 6.3 we s [\cdot]12 denotes the upper on-diagonal entry of an operator-valued 2 × 2 matrix. For
this purpose, we define the H^1 norm of a periodic operator η by
 $\|\eta\|_{H^1}^2 = \text{Tr} [\eta^*(1 - h^2 \nabla^2)\eta]$. (3.7)
In Subsection 6.3 we

$$
\|\eta\|_{H^1}^2 = \text{Tr}\left[\eta^*(1 - h^2 \nabla^2)\eta\right].\tag{3.7}
$$

1 prove

$$
\rho(z) = (1 + e^z)^{-1}
$$

Then

$$
(3.8)
$$

Theorem 3.2. Let

e
\n
$$
\rho(z) = (1 + e^{z})^{-1}
$$
\n(3.8)

Theorem 3.2. Let
\n
$$
\rho(z) = (1 + e^{z})^{-1}
$$
\n(3.8)
\nand let g_0 be as in (3.3). Then
\n
$$
[\rho(\beta H_{\Delta})]_{12} = \frac{\beta h}{4} (\psi(x) g_0(\beta(-h^2 \nabla^2 - \mu)) + g_0(\beta(-h^2 \nabla^2 - \mu)) \psi(x)) + \eta_1 + \eta_2
$$
\nwhere
\n
$$
\|\eta_1\|_{H^1}^2 \le C h^5 \|\psi\|_{H^2(\mathcal{C})}^2
$$
\n(3.9)
\nand
\n
$$
\|\eta_2\|_{H^1}^2 \le C h^5 \left(\|\psi\|_{H^1(\mathcal{C})}^2 + \|\psi\|_{H^1(\mathcal{C})}^6 \right).
$$
\n(3.10)

where

$$
\|\eta_1\|_{H^1}^2 \le Ch^5 \|\psi\|_{H^2(\mathcal{C})}^2 \tag{3.9}
$$

and

$$
\|\eta_2\|_{H^1}^2 \le Ch^5 \left(\|\psi\|_{H^1(\mathcal{C})}^2 + \|\psi\|_{H^1(\mathcal{C})}^6 \right). \tag{3.10}
$$

4. Upper bound

+ $\|\psi\|_{H^1(\mathcal{C})}^6$). (3.10)

(3.10)
 κ ed *D* > 0 and denote by Δ_0 the

e following we write, as usual, $\beta =$ We assume that $T = T_c(1 - Dh^2)$ with a fixed $D > 0$ and denote by Δ_0 the solution of the BCS gap equation (1.7). In the following we write, as usual, $\beta = T^{-1} = \beta_c(1 - Dh^2)^{-1}$ with $\beta_c = T_c^{-1}$. It is well known that the Gi solution of the BCS gap equation (1.7). In the following we write, as usual, $\beta = T^{-1} = \beta_c (1 - Dh^2)^{-1}$ with $\beta_c = T_c^{-1}$. It is well known that the Ginzburg-Landau functional has a minimizer ψ , which is a periodic $H_{\text{loc$ $T^{-1} = \beta_c (1 - Dh^2)^{-1}$ with $\beta_c = T_c^{-1}$. It is well known that the Ginzburg-Landau
functional has a minimizer ψ , which is a periodic $H_{\text{loc}}^2(\mathbb{R})$ function. We put
 $\Delta(x) = -\Delta_0 \psi(x)$,
and define H_{Δ} by (2.1).
To ob functional has a minimizer ψ , which is a periodic $H_{\text{loc}}^2(\mathbb{R})$ function. We put
 $\Delta(x) = -\Delta_0 \psi(x)$,

and define H_{Δ} by (2.1).

To obtain an upper bound for the energy we use the trial state
 $\Gamma_{\Delta} = (1 + e^{\beta H_{\Delta$

$$
\Delta(x) = -\Delta_0 \psi(x),
$$

To obtain an upper bound for the energy we use the trial state

$$
\Gamma_{\Delta} = \left(1 + e^{\beta H_{\Delta}}\right)^{-1}
$$

and define
$$
H_{\Delta}
$$
 by (2.1).
\nTo obtain an upper bound for the energy we use the trial state
\n
$$
\Gamma_{\Delta} = (1 + e^{\beta H_{\Delta}})^{-1}.
$$
\nDenoting its off-diagonal element by $\alpha_{\Delta} = [\Gamma_{\Delta}]_{12}$, we have the upper bound
\n
$$
\mathcal{F}^{BCS}(\Gamma_{\Delta}) - \mathcal{F}^{BCS}(\Gamma_{0}) = -\frac{1}{2\beta} \text{Tr} [\ln(1 + e^{-\beta H_{\Delta}}) - \ln(1 + e^{-\beta H_{0}})] + \frac{\Delta_{0}^{2}}{4ha} ||\psi||_{2}^{2} - ha \int_{C} \left| \frac{\Delta_{0}\psi(x)}{2ha} - \alpha_{\Delta}(x, x) \right|^{2} dx \qquad (4.1)
$$
\n
$$
\leq -\frac{1}{2\beta} \text{Tr} [\ln(1 + e^{-\beta H_{\Delta}}) - \ln(1 + e^{-\beta H_{0}})] + \frac{\Delta_{0}^{2}}{4ha} ||\psi||_{2}^{2}.
$$
\nThe first term on the right side was evaluated in Theorem 3.1.

Applying this theorem with
$$
\psi
$$
 replaced by $(\Delta_0/h)\psi$ we obtain that
\n
$$
\mathcal{F}^{BCS}(\Gamma_{\Delta}) - \mathcal{F}^{BCS}(\Gamma_0)
$$
\n
$$
\leq -\frac{h\beta}{4} \frac{\Delta_0^2}{h^2} \|\psi\|_2^2 \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi} + \frac{h^3}{2} \left[\frac{\beta^2}{8} \frac{\Delta_0^2}{h^2} \|\psi'\|_2^2 \int_{\mathbb{R}} (g_1(\beta(q^2 - \mu)) + 2\beta q^2 g_2(\beta(q^2 - \mu))) \frac{dq}{2\pi} + \frac{\beta^2}{2} \frac{\Delta_0^2}{h^2} \langle \psi|W|\psi \rangle \int_{\mathbb{R}} g_1(\beta(q^2 - \mu)) \frac{dq}{2\pi} + \frac{\beta^2}{8} \frac{\Delta_0^4}{h^4} \|\psi\|_4^4 \int_{\mathbb{R}} \frac{g_1(\beta(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi} \right] + \frac{\Delta_0^2}{4ha} \|\psi\|_2^2 + O(h^5).
$$
\n(4.2)
\nIn the estimate of the remainder we used that ψ is H^2 and that $\Delta_0 \leq Ch$.
\nNext, we use that by definition (1.7) of Δ_0 the first and the last term on the right side of (4.2) cancel to leading order and that one has
\n $h\beta \Delta_0^2 \frac{\Delta_0^2}{\|\psi\|^2} \int_{\mathbb{R}} \frac{\beta(q^2 - \mu)}{q^2 - \mu} \frac{dq}{2\pi} \frac{\Delta_0^2}{\|\psi\|^2}$

 $\frac{1}{2}$ the $\frac{1}{2}$ $\frac{\Delta_0}{4ha} \|\psi\|_2^2$ e estimat
Next, we side of ($-\frac{h\beta}{4}$ + $O(h^5)$. (4.2)

e of the remainder we used that ψ is H^2 and that $\Delta_0 \le Ch$.

use that by definition (1.7) of Δ_0 the first and the last term on the

..2) cancel to leading order and that one has
 $\frac{\Delta_0^2}{\Delta_0$

In the estimate of the remainder we used that
$$
\psi
$$
 is H^2 and that $\Delta_0 \leq Ch$.
\nNext, we use that by definition (1.7) of Δ_0 the first and the last term on the
\nright side of (4.2) cancel to leading order and that one has
\n
$$
-\frac{h\beta}{4} \frac{\Delta_0^2}{h^2} ||\psi||_2^2 \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi} + \frac{\Delta_0^2}{4ha} ||\psi||_2^2
$$
\n
$$
= \frac{h\beta}{4} \frac{\Delta_0^2}{h^2} ||\psi||_2^2 \int_{\mathbb{R}} \left(g_0(\beta\sqrt{(q^2 - \mu)^2 + \Delta_0^2}) - g_0(\beta(q^2 - \mu)) \right) \frac{dq}{2\pi}
$$
\n
$$
= -\frac{h^3\beta^2}{8} \frac{\Delta_0^4}{h^4} ||\psi||_2^2 \int_{\mathbb{R}} \frac{g_1(\beta(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi} + O(h^5).
$$
\nWe conclude that
\n $\mathcal{F}^{\text{BCS}}(\Gamma_{\Delta}) - \mathcal{F}^{\text{BCS}}(\Gamma_0)$

We conclude that

$$
= -\frac{h^3 \beta^2}{8} \frac{\Delta_0^4}{h^4} ||\psi||_2^2 \int_{\mathbb{R}} \frac{g_1(\beta(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi} + O(h^5).
$$

We conclude that

$$
\mathcal{F}^{BCS}(\Gamma_{\Delta}) - \mathcal{F}^{BCS}(\Gamma_0)
$$

$$
\leq \frac{h^3}{2} \left[\frac{\beta^2}{8} \frac{\Delta_0^2}{h^2} ||\psi'||_2^2 \int_{\mathbb{R}} (g_1(\beta(q^2 - \mu)) + 2\beta q^2 g_2(\beta(q^2 - \mu))) \frac{dq}{2\pi} + \frac{\beta^2}{2} \frac{\Delta_0^2}{h^2} \langle \psi | W | \psi \rangle \int_{\mathbb{R}} g_1(\beta(q^2 - \mu)) \frac{dq}{2\pi} + \frac{\beta^2}{8} \frac{\Delta_0^4}{h^4} ||\psi||_4^4 \int_{\mathbb{R}} \frac{g_1(\beta(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi} - \frac{\beta^2}{4} \frac{\Delta_0^4}{h^4} ||\psi||_2^2 \int_{\mathbb{R}} \frac{g_1(\beta(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi} + O(h^5).
$$
 (4.3)
Up to an error of the order $O(h^5)$ we can replace $\beta = \beta_c (1 - Dh^2)^{-1}$ by β_c on the right side. Our last task is then to compute the asymptotics of Δ_0/h . To do so, we rewrite the BCS gap equation (1.7) as

ro
1
|
|- $\frac{\Delta_0^4}{h^4}$
r of
r l;
e B $\left(\frac{\beta(q^2-\mu)}{q^2-\mu}\right)\frac{dq}{2\pi}\right] + O(h^5).$ (4.3)
 $O(h^5)$ we can replace $\beta = \beta_c(1-Dh^2)^{-1}$ by β_c on the

then to compute the asymptotics of Δ_0/h . To do so,

aation (1.7) as
 $\frac{dq}{2\pi} = \frac{1}{2} = \beta \int q_0 \left(\beta \sqrt{(q^2-\mu)^2 + \Delta$ 2π
n r
pu
as
3
} Up to an error of the order $O(h^5)$ we can replace $\beta = \beta_c (1 - Dh^2)^{-1}$ by β_c on the right side. Our last task is then to compute the asymptotics of Δ_0/h . To do so, we rewrite the BCS gap equation (1.7) as
 $\beta_c \int_{\mathbb$ right side. Our last task is then to compute the asymptotics of Δ_0/h . To do so, we rewrite the BCS gap equation (1.7) as

$$
\beta_c \int_{\mathbb{R}} g_0(\beta_c(q^2 - \mu)) \frac{dq}{2\pi} = \frac{1}{a} = \beta \int_{\mathbb{R}} g_0\left(\beta \sqrt{(q^2 - \mu)^2 + \Delta_0^2}\right) \frac{dq}{2\pi}.
$$

A simple computation shows that

$$
\Delta_0^2 = Dh^2 \frac{\int_{\mathbb{R}} \left[g_0(\beta_c(q^2 - \mu)) - \beta_c(q^2 - \mu)g_1(\beta_c(q^2 - \mu))\right] dq}{\int_{\mathbb{R}} [g_0(\beta_c(q^2 - \mu)) - \beta_c(q^2 - \mu)g_1(\beta_c(q^2 - \mu))] dq}
$$
(1+O(l))

$$
\beta_c \int_{\mathbb{R}} g_0(\beta_c(q^2 - \mu)) \frac{dq}{2\pi} = \frac{1}{a} = \beta \int_{\mathbb{R}} g_0\left(\beta \sqrt{(q^2 - \mu)^2 + \Delta_0^2}\right) \frac{dq}{2\pi}.
$$

A simple computation shows that

$$
\Delta_0^2 = Dh^2 \frac{\int_{\mathbb{R}} \left[g_0(\beta_c(q^2 - \mu)) - \beta_c(q^2 - \mu)g_1(\beta_c(q^2 - \mu))\right] dq}{\beta_c \int_{\mathbb{R}} \frac{g_1(\beta_c(q^2 - \mu))}{2(q^2 - \mu)} dq} \left(1 + O(h^2)\right).
$$

Inserting this into (4.3) and using the fact that $\mathcal{E}(\psi) = E^{\text{GL}}$ we arribound claimed in Theorem 1.2. Inserting this into (4.3) and using the fact that $\mathcal{E}(\psi) = E^{\text{GL}}$ we arrive at the upper
bound claimed in Theorem 1.2.
5. Lower bound
5.1. The relative entropy

5. Lower bound

5. Lower bound
 5.1. The relative entropy
 Δz a proliminant to our pract. **5.1. The relative entropy**

for the relative entropy. In this subsection H^0 and $0 \leq \Gamma \leq 1$ are arbitrary self-
adjoint operators in a Hilbert space, not necessarily coming from BCS theory. Let
 $\Gamma^0 := (1 + \exp(\beta H^0))^{-1}$. It is well known that
 $\mathcal{$ $\Gamma^0 := (1 + \exp(\beta H^0))^{-1}$. It is well known that

$$
\mathcal{H}(\Gamma, \Gamma^0) = \text{Tr} \left(\beta H^0 \Gamma + \Gamma \ln \Gamma + (1 - \Gamma) \ln(1 - \Gamma) + \ln \left(1 + \exp(-\beta H_0) \right) \right)
$$

 $\Gamma^0 := (1 + \exp(\beta H^0))^{-1}$. It is well known that
 $\mathcal{H}(\Gamma, \Gamma^0) = \text{Tr} (\beta H^0 \Gamma + \Gamma \ln \Gamma + (1 - \Gamma) \ln(1 - \Gamma) + \ln (1 + \exp(-\beta H_0)))$

is non-negative and equals to zero if and only if $\Gamma = \Gamma^0$. Solving this equation

for H^0 , i.e., $H^0 = \$ $(\Gamma, \Gamma^0) = \text{Tr} (\beta H^0 \Gamma + \Gamma \ln \Gamma + (1 - \Gamma) \ln(1 - \Gamma) + \ln (1 + \exp(-\beta H_0)))$

-negative and equals to zero if and only if $\Gamma = \Gamma^0$. Solving this equals, i.e., $H^0 = \beta^{-1}(\ln(1 - \Gamma^0) - \ln \Gamma^0)$, we can rewrite $\mathcal{H}(\Gamma, \Gamma^0)$ as a rely,
 $\$ is non-negative and equals to zero if and only if $\Gamma = \Gamma^0$. Solving this equation
for H^0 , i.e., $H^0 = \beta^{-1}(\ln(1 - \Gamma^0) - \ln \Gamma^0)$, we can rewrite $\mathcal{H}(\Gamma, \Gamma^0)$ as a relative
entropy,
 $\mathcal{H}(\Gamma, \Gamma^0) = \text{Tr} [\Gamma(\ln \Gamma - \ln \Gamma^0)$ for H^0 , i.e., $H^0 = \beta^{-1}(\ln(1 - \Gamma^0) - \ln \Gamma^0)$, we can rewrite $\mathcal{H}(\Gamma, \Gamma^0)$ as a relative
entropy,
 $\mathcal{H}(\Gamma, \Gamma^0) = \text{Tr} [\Gamma (\ln \Gamma - \ln \Gamma^0) + (1 - \Gamma) (\ln(1 - \Gamma) - \ln(1 - \Gamma^0))]$. (5.1)
The following lemma quantifies the positivity of

$$
\mathcal{H}(\Gamma,\Gamma^0) = \text{Tr}\left[\Gamma\left(\ln\Gamma - \ln\Gamma^0\right) + (1-\Gamma)\left(\ln(1-\Gamma) - \ln(1-\Gamma^0)\right)\right].\tag{5.1}
$$

 $\mathcal{H}(1)$
The foll
from $[5]$
Lemma (Γ, Γ⁰) = Tr [Γ (ln Γ – ln Γ⁰) + (1 – Γ) (ln(1 – Γ) – ln(1 – Γ⁰)

llowing lemma quantifies the positivity of *H* and improves an

5].
 a 5.1. For any $0 \le \Gamma \le 1$ and any Γ₀ of the form $\Gamma^0 = (1 + e^{\beta})$
 $\Gamma^0 = (1 +$ α
γ $\frac{1}{2}$ esult

from [5].

Lemma 5.1. For any
$$
0 \le \Gamma \le 1
$$
 and any Γ_0 of the form $\Gamma^0 = (1 + e^{\beta H^0})^{-1}$,
\n**Lemma 5.1.** For any $0 \le \Gamma \le 1$ and any Γ_0 of the form $\Gamma^0 = (1 + e^{\beta H^0})^{-1}$,
\n
$$
\mathcal{H}(\Gamma, \Gamma^0) \ge \text{Tr} \left[\frac{\beta H^0}{\tanh(\beta H^0/2)} \left(\Gamma - \Gamma^0 \right)^2 \right] + \frac{1}{3} \frac{\left(\text{Tr} \Gamma(1 - \Gamma) - \text{Tr} \Gamma^0(1 - \Gamma^0) \right)^2}{\left[\text{Tr} \Gamma(1 - \Gamma) - \text{Tr} \Gamma^0(1 - \Gamma^0) \right] + \text{Tr} \Gamma^0(1 - \Gamma^0)}
$$
\n*Proof.* It is tedious, but elementary, to show that for real numbers $0 < x, y < 1$,
\n
$$
\frac{x}{\sqrt{1 + \frac{1 - y}{\sqrt{1 - \frac{y}{\sqrt{1 - \frac{y
$$

$$
+\frac{1}{3} \frac{\left(\text{Tr }\Gamma(1-\Gamma)-\text{Tr }\Gamma^{0}(1-\Gamma^{0})\right)^{2}}{|\text{Tr }\Gamma(1-\Gamma)-\text{Tr }\Gamma^{0}(1-\Gamma^{0})|+\text{Tr }\Gamma^{0}(1-\Gamma^{0})}.
$$

Proof. It is tedious, but elementary, to show that for real numbers $0 < x, y < 1$,

$$
x \ln \frac{x}{y} + (1-x) \ln \frac{1-x}{1-y} \ge \frac{\ln \frac{1-y}{y}}{1-2y}(x-y)^{2} + \frac{1}{3} \frac{(x(1-x)-y(1-y))^{2}}{|x(1-x)-y(1-y)|+y(1-y)}.
$$
Using joint convexity we see that

$$
\frac{(x(1-x)-y(1-y))^{2}}{|x(1-x)-y(1-y)|+y(1-y)}
$$

$$
x \ln \frac{x}{y} + (1-x) \ln \frac{1-x}{1-y} \ge \frac{\ln \frac{1-y}{y}}{1-2y} (x-y)^2 + \frac{1}{3} \frac{(x(1-x)-y(1-y))^2}{|x(1-x)-y(1-y)|+y(1-y)}
$$

Using joint convexity we see that

$$
\frac{(x(1-x)-y(1-y))^2}{|x(1-x)-y(1-y)|+y(1-y)}
$$

$$
= 4 \sup_{0 < b < 1} [b(1-b) |x(1-x)-y(1-y)| - b^2 y(1-y)].
$$

Let us replace on the right side the modulus |a| by max{a, -a}, and then u
Klein's inequality [15, Section 2.1.4] for either of the expressions. This implies t

 $(1 - x) - y(1 - y)| + y(1 - y)|$
= 4 sup
 $0 < b < 1$
us replace on the right side
n's inequality [15, Section 2.1.
t. = 4 sup $\left[b(1-b)|x(1-x) - y(1-y)| - b^2y(1-y)\right]$
ght side the modulus |a| by max{a, -a}, and then
ction 2.1.4] for either of the expressions. This implies Let us replace on the right side the modulus |a| by max{ $a, -a$ }, and then use
Klein's inequality [15, Section 2.1.4] for either of the expressions. This implies the
result. K is inequality [15, Section 2.1.4] for either of the expressions. The implies the expressions. \Box result. \Box

5.2. A priori estimates on α

the solution of α
reviewing some facts about the translation-invariant case $W \equiv$
ion 1.2.1. Recall that Γ^0 denotes the minimizer of $\mathcal F$ in the
t case. It can be written as $\Gamma^0 = (1 + e^{\beta H_{\Delta_0}^0})^{-1}$ with We begin by briefly reviewing some facts about the translation-invariant case $W \equiv 0$; see also Subsection 1.2.1. Recall that Γ^0 denotes the minimizer of \mathcal{F} in the translation-invariant case. It can be written a 0; see also Subsection 1.2.1. Recall that Γ^0 denotes the minimizer of \mathcal{F} in the
translation-invariant case. It can be written as $\Gamma^0 = (1 + e^{\beta H_{\Delta_0}^0})^{-1}$ with
 $H_{\Delta_0}^0 = \begin{pmatrix} -h^2 \nabla^2 - \mu & -\Delta_0 \\ -\Delta_0 & h^2 \nab$ translation-invariant case. It can be written as $\Gamma^0 = (1 + e^{\beta H_{\Delta_0}^0})^{-1}$ with

$$
H_{\Delta_0}^0 = \begin{pmatrix} -h^2 \nabla^2 - \mu & -\Delta_0 \\ -\Delta_0 & h^2 \nabla^2 + \mu \end{pmatrix}.
$$

translation-invariant case. It can be written as $\Gamma^0 = (1 + e^{\beta H_{\Delta_0}^0})^{-1}$ with
 $H_{\Delta_0}^0 = \begin{pmatrix} -h^2 \nabla^2 - \mu & -\Delta_0 \\ -\Delta_0 & h^2 \nabla^2 + \mu \end{pmatrix}$.

Here Δ_0 is the solution of the BCS gap-equation (1.7) and $\beta^{-1} = T = T_c$ ne
ree
na
no
lio $\frac{2}{2}$ + $\frac{1}{2}$ + $\frac{1}{2}$ vas
us $\frac{1}{2}$ + $\frac{1}{2}$ vas
us $\frac{1}{2}$ + $\frac{1$ Δ_0 $h^2 \nabla^2 + \mu$

b-equation (1.7) and Γ_0 which was de

ial W and has no

ernel of the off-dia

n in (1.11). From Notice the distinction between Γ^0 and Γ_0 which was defined in (1.13). The latter one, Γ_0 , contains the external potential W and has no off-diagonal term.
Recall also that we denote the kernel of the off-diagon

Notice the distinction between Γ^0 and Γ_0 which was defined in (1.13). The latter
one, Γ_0 , contains the external potential W and has no off-diagonal term.
Recall also that we denote the kernel of the off-diagon one, Γ_0 , contains the external potential *W* and has no off-diagonal term.

Recall also that we denote the kernel of the off-diagonal entry $\alpha^0 =$
 $\alpha^0((x-y)/h)$, which is explicitly given in (1.11). From this explici Recall also that we denote the kernel of the off-diagonal entry $\alpha^0 = [\Gamma^0]_{12}$ by $(x - y)/h$, which is explicitly given in (1.11). From this explicit representation the fact that $\Delta_0 \le Ch$ we conclude, in particular, that and the fact that $\Delta_0 \leq Ch$ we conclude, in particular, that

$$
\alpha^{0}((x-y)/h), \text{ which is explicitly given in (1.11). From this explicit representation and the fact that } \Delta_{0} \le Ch \text{ we conclude, in particular, that}
$$
\n
$$
\|\alpha^{0}\|_{2}^{2} = \int_{0}^{1} dy \int_{\mathbb{R}} dx \, |\alpha^{0}((x-y)/h)|^{2} = h \int_{\mathbb{R}} |\alpha^{0}(x)|^{2} dx \le Ch. \qquad (5.2)
$$
\nMoreover, the BCS gap-equation (1.7) is equivalent to\n
$$
(K_{T}^{0}(-ih\nabla) - ah\delta(x))\alpha^{0}(x/h) = 0. \qquad (5.3)
$$
\nThis implies, in particular, that

Moreover, the BCS gap-equation (1.7) is equivalent to
\n
$$
(K_T^0(-ih\nabla) - ah\delta(x))\alpha^0(x/h) = 0.
$$
\nThis implies, in particular, that

$$
(K_T^0(-ih\nabla) - ah\delta(x))\alpha^0(x/h) = 0.
$$
 (5.3)
This implies, in particular, that

$$
\text{Tr}\,K_T^0(-ih\nabla)\alpha^0\overline{\alpha^0} - ah|\alpha^0(0)|^2 = 0.
$$
 (5.4)
Now we turn to the case of general W. Our goal in this subsection is to prove

 $(K_T^0(-ih\nabla) - ah\delta(x))\alpha^0(x/h) = 0.$ (5.3)

ular, that
 Γ r $K_T^0(-ih\nabla)\alpha^0\overline{\alpha^0} - ah|\alpha^0(0)|^2 = 0.$ (5.4)

ne case of general W. Our goal in this subsection is to prove

energy state satisfies bounds similar to (5.2) and (5.4). Tr $K_T^0(-i$
Now we turn to the case of
that the α of *any* low-energy st.
Proposition 5.2. Any admissible Tr $K_T^0(-ih\nabla)\alpha^0\overline{\alpha^0} - ah|\alpha^0(0)|$

he case of general W. Our goal

energy state satisfies bounds si

admissible Γ with $\mathcal{F}^{\text{BCS}}(\Gamma) \leq \mathcal{J}$
 $||\alpha||^2 = \int_0^1 dx \int_0^1 du |\alpha(x, y)|^2$ = 0. (5.4)

1 this subsection is to prove

iilar to (5.2) and (5.4).

^{3CS}(Γ ⁰) *satisfies*

(5.5)

Now we turn to the case of general W. Our goal in this subsection is to prove
that the
$$
\alpha
$$
 of *any* low-energy state satisfies bounds similar to (5.2) and (5.4).
Proposition 5.2. Any admissible Γ with $\mathcal{F}^{\text{BCS}}(\Gamma) \leq \mathcal{F}^{\text{BCS}}(\Gamma^0)$ satisfies

$$
\|\alpha\|_2^2 = \int_0^1 dx \int_{\mathbb{R}} dy |\alpha(x, y)|^2 \leq Ch \qquad (5.5)
$$

and

$$
0 \leq \text{Tr } K_T^0(-ih\nabla)\alpha \overline{\alpha} - ah \int_C |\alpha(x, x)|^2 dx \leq Ch^3,
$$
 (5.6)

and

$$
\|\alpha\|_2^2 = \int_0^{\infty} dx \int_{\mathbb{R}} dy |\alpha(x, y)|^2 \le Ch \tag{5.5}
$$

$$
0 \le \text{Tr } K_T^0(-ih\nabla)\alpha\overline{\alpha} - ah \int_C |\alpha(x, x)|^2 dx \le Ch^3,
$$

i.e. the proof into two steps.
string point is the representation

Proof.

Step

where
$$
\alpha = [\Gamma]_{12}
$$
.
\n*Proof.* We divide the proof into two steps.
\n*Step 1.* Our starting point is the representation
\n
$$
\mathcal{F}^{BCS}(\Gamma) - \mathcal{F}^{BCS}(\Gamma^0) = \frac{1}{2\beta} \mathcal{H}(\Gamma, \Gamma^0) + h^2 \operatorname{Tr} \gamma W - ah \int_{\mathcal{C}} |\alpha(x, x) - \alpha^0(0)|^2 dx
$$
\n(5.7)
\nfor any admissible Γ , with the relative entropy $\mathcal{H}(\Gamma, \Gamma^0)$ defined in (5.1). We note
\nthat Γ^0 is of the form $(1 + e^{\beta H^0})^{-1}$ with $H^0 = H^0_{\Delta_0}$. We use Lemma 5.1 to bound

 \int note for any admissible Γ, with the relative entropy $\mathcal{H}(\Gamma,\Gamma^0)$ defined in (5.1). We note
that Γ^0 is of the form $(1 + e^{\beta H^0})^{-1}$ with $H^0 = H^0_{\Delta_0}$. We use Lemma 5.1 to bound that Γ^0 is of the form $(1+e^{\beta H^0})^{-1}$ with $H^0=H^0_{\Delta_0}$

 $\mathcal{H}(\Gamma, \Gamma^0)$ from below. Since $x \mapsto x/\tanh x$ is even, we can replare
absolute value $E(-ih\nabla) = \sqrt{(-h^2\nabla^2 - \mu)^2 + \Delta_0^2}$, and thus
Tr $\left[\frac{H_{\Delta_0}^0}{\tanh \beta H^0} (\Gamma - \Gamma_0)^2\right] = \text{Tr} [K_T^0(-ih\nabla)(\Gamma - \Gamma_0)^2]$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$
\mathcal{H}(\Gamma, \Gamma^0) \text{ from below. Since } x \mapsto x/\tanh x \text{ is even, we can replace } H_{\Delta_0}^0 \text{ is absolute value } E(-ih\nabla) = \sqrt{(-h^2\nabla^2 - \mu)^2 + \Delta_0^2}, \text{ and thus}
$$
\n
$$
\text{Tr}\left[\frac{H_{\Delta_0}^0}{\tanh\frac{\beta}{2}H_{\Delta_0}^0}(\Gamma - \Gamma_0)^2\right] = \text{Tr}\left[K_T^0(-ih\nabla)(\Gamma - \Gamma_0)^2\right]
$$
\n
$$
= 2 \text{Tr}\, K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 + 2 \text{Tr}\, K_T^0(-ih\nabla)(\alpha - \alpha^0)(\overline{\alpha - \alpha^0})
$$
\nwith\n
$$
K_T^0(hp) = \frac{E(hp)}{\tanh\frac{\beta E(hp)}{2}}
$$

$$
K_T^0(hp) = \frac{E(hp)}{\tanh\frac{\beta E(hp)}{2}}
$$

= $2 \text{Tr} K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 + 2 \text{Tr} K_T^0(-ih\nabla)(\alpha - \alpha^0)(\overline{\alpha - \alpha^0})$
 $K_T^0(hp) = \frac{E(hp)}{\tanh \frac{\beta E(hp)}{2}}$

th the aid of Lemma 5.1 and the assumption $\mathcal{F}^{\text{BCS}}(\Gamma) \leq \mathcal{F}^{\text{BCS}}$
 $\Gamma(5,7)$ the basic inequality

$$
K_T^0(hp) = \frac{E(hp)}{\tanh \frac{\beta E(hp)}{2}}
$$

from (1.8). With the aid of Lemma 5.1 and the assumption $\mathcal{F}^{BCS}(\Gamma) \leq \mathcal{F}^{BCS}(\Gamma^0)$
we obtain from (5.7) the basic inequality

$$
0 \geq \text{Tr } K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 + h^2 \text{Tr } \gamma W
$$
(5.8)
$$
+ \text{Tr } K_T^0(-ih\nabla)(\alpha - \alpha^0)(\overline{\alpha - \alpha^0}) - ah \int_C |\alpha(x, x) - \alpha^0(0)|^2 dx
$$

$$
+ \frac{T}{3} \frac{\left(\text{Tr } [\gamma(1 - \gamma) - \gamma^0(1 - \gamma^0) - \alpha\overline{\alpha} + \alpha^0\overline{\alpha^0}]\right)^2}{\left|\text{Tr } [\gamma(1 - \gamma) - \gamma^0(1 - \gamma^0) - \alpha\overline{\alpha} + \alpha^0\overline{\alpha^0}]\right| + \text{Tr } [\gamma^0(1 - \gamma^0) - \alpha^0\overline{\alpha^0}]}
$$
.
In the following step we shall derive the claimed a priori estimates on α from this inequality.
Step 2. We begin by discussing the first line on the right side of (5.8). Using the fact that $\text{Tr } W\gamma^0 = 0$ (since *W* has mean value zero) and the Schwarz inequality

 $\frac{1}{x}$
 $\frac{1}{x}$ $\begin{array}{c} \mbox{Tr} \Big[\vspace{1.5mm} \begin{array}{c} \mbox{wing} \\\mbox{begin} \end{array} \begin{array}{c} \mbox{begin} \mbox{reg} \\\mbox{array} \end{array} \end{array} \end{array}$ $(1 - \gamma) - \gamma^0 (1 - \gamma^0) - \alpha \overline{\alpha} + \alpha^0 \overline{\alpha^0})$ + Tr $\left[\gamma^0 (1 - \gamma^0) - \alpha^0 \overline{\alpha^0} \right]$
step we shall derive the claimed a priori estimates on α from
the by discussing the first line on the right side of (5.8). Using
 $= 0$ In the following step we shall derive the claimed a priori estimates on α from this inequality.
 Step 2. We begin by discussing the first line on the right side of (5.8). Using the fact that Tr $W\gamma^0 = 0$ (since *W* Step 2. We
fact that $\frac{1}{x}$ we obtain
 $\frac{1}{x}$ $\frac{1}{x}$ =

Step 2. We begin by discussing the first line on the right side of (5.8). Using the fact that Tr
$$
W\gamma^0 = 0
$$
 (since W has mean value zero) and the Schwarz inequality we obtain the lower bound\n\n
$$
\text{Tr } K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 + h^2 \text{Tr } W\gamma
$$
\n
$$
= \frac{1}{2} \text{Tr } K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 + \frac{1}{2} \text{Tr } K_T^0(\gamma - \gamma^0)^2 + h^2 \text{Tr } W(\gamma - \gamma^0)
$$
\n
$$
\geq \frac{1}{2} \text{Tr } K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 - h^4 \frac{1}{2} \text{Tr } W (K_T^0(-ih\nabla))^{-1} W
$$
\n
$$
\geq \frac{1}{2} \text{Tr } K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 - Ch^3.
$$
\n\n(5.9) The last step used that Tr $W (K_T^0(-ih\nabla))^{-1} W \leq ||W||_{\infty}^2 \text{Tr } K_T^0(-ih\nabla)^{-1} \leq Ch^{-1}.$ \nNext, we treat the second line on the right side of (5.8). Recall that the BCS can equation in the form (5.3) says that the operator $K_T^0(-ih\nabla) - ah\delta(x)$ has an

 $\left(K_T^0(-ih\nabla)\right)^{-1}W \leq \|W\|_{\infty}^2 \operatorname{Tr} K_T^0$ Ch^{-1} .

Tr $K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 - Ch^3$. (5.9)

(6.9)

(7.9)

(7 The last step used that Tr $W(K_T^0(-ih\nabla))^{-1} W \le ||W||_{\infty}^2 \text{Tr } K_T^0(-ih\nabla)^{-1} \le Ch^{-1}.$

Next, we treat the second line on the right side of (5.8). Recall that the BCS

gap equation in the form (5.3) says that the operator K_T^0 N
C.C.
V eigenvalue zero with eigenfunction $\alpha^{0}(x/h)$. Hence

gap equation in the form (5.3) says that the operator
$$
K_T^0(-ih\nabla) - ah\delta(x)
$$
 has an eigenvalue zero with eigenfunction $\alpha^0(x/h)$. Hence
\n
$$
\text{Tr } K_T^0(-ih\nabla)(\alpha - \alpha^0)(\overline{\alpha - \alpha^0}) - ah \int_C |\alpha(x, x) - \alpha^0(0)|^2 dx
$$
\n
$$
= \text{Tr } K_T^0(-ih\nabla)\alpha \overline{\alpha} - ah \int_C |\alpha(x, x)|^2 dx.
$$

Derivation of GL Theory for a 1D System 71
Since a delta potential creates at most one bound state, zero must be the ground

state energy of
$$
K_T^0(-ih\nabla) - ah\delta(x)
$$
, and we deduce that
\n
$$
\text{Tr } K_T^0(-ih\nabla)\alpha \overline{\alpha} - ah \int_{\mathcal{C}} |\alpha(x, x)|^2 dx \ge 0.
$$
\nThis information, together with (5.9) and (5.8), yields
\n
$$
\text{Tr}(1 - h^2 \nabla^2)(\gamma - \gamma^0)^2 \le C \text{Tr } K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 \le Ch^3,
$$
\n
$$
\text{Tr } K_T^0(-ih\nabla)\alpha \overline{\alpha} - ah \int_{\mathcal{C}} |\alpha(x, x)|^2 dx \le Ch^3
$$
\n(5.11)

state energy of
$$
K_T^0(-ih\nabla) - ah\delta(x)
$$
, and we deduce that
\n
$$
\text{Tr } K_T^0(-ih\nabla)\alpha \overline{\alpha} - ah \int_{\mathcal{C}} |\alpha(x, x)|^2 dx \ge 0.
$$
\nThis information, together with (5.9) and (5.8), yields
\n
$$
\text{Tr}(1 - h^2 \nabla^2)(\gamma - \gamma^0)^2 \le C \text{Tr } K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 \le Ch^3,
$$
\n
$$
\text{Tr } K_T^0(-ih\nabla)\alpha \overline{\alpha} - ah \int_{\mathcal{C}} |\alpha(x, x)|^2 dx \le Ch^3
$$
\n(5.11)
\nand
\n
$$
(\text{Tr } [\gamma(1 - \gamma) - \gamma^0(1 - \gamma^0) - \alpha \overline{\alpha} + \alpha^0 \overline{\alpha^0})]^2
$$

Tr
$$
K_T^0(-ih\nabla)\alpha\overline{\alpha} - ah \int_C |\alpha(x, x)|^2 dx \ge 0
$$
.
\ntogether with (5.9) and (5.8), yields
\n ${}^2\nabla^2/(\gamma - \gamma^0)^2 \le C \operatorname{Tr} K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 \le Ch^3$, (5.10)
\nTr $K_T^0(-ih\nabla)\alpha\overline{\alpha} - ah \int_C |\alpha(x, x)|^2 dx \le Ch^3$ (5.11)

Tr
$$
(1 - h^2 \nabla^2)(\gamma - \gamma^0)^2 \le C \operatorname{Tr} K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 \le Ch^3,
$$
 (5.10)
\nTr $K_T^0(-ih\nabla)\alpha\overline{\alpha} - ah \int_C |\alpha(x, x)|^2 dx \le Ch^3$ (5.11)
\nd
\n
$$
\left(\operatorname{Tr} \left[\gamma(1 - \gamma) - \gamma^0(1 - \gamma^0) - \alpha\overline{\alpha} + \alpha^0\overline{\alpha^0}\right]\right)^2
$$
\n
$$
\left|\operatorname{Tr} \left[\gamma(1 - \gamma) - \gamma^0(1 - \gamma^0) - \alpha\overline{\alpha} + \alpha^0\overline{\alpha^0}\right]\right| + \operatorname{Tr} \left[\gamma^0(1 - \gamma^0) - \alpha^0\overline{\alpha^0}\right] \le Ch^3.
$$
\n(5.12)
\n: know that Tr $\left[\gamma^0(1 - \gamma^0) - \alpha^0\overline{\alpha^0}\right] \le Ch^{-1}$ from the explicit solution in the
\nnslation invariant case, and therefore (5.12) yields

 $\sqrt{\frac{1}{1+\frac$ $\left[\frac{1}{2}\right]$ knc $(1 - \gamma) - \gamma^0 (1 - \gamma^0) - \alpha \overline{\alpha} + \alpha^0 \overline{\alpha^0}$ + Tr $\left[\gamma^0 (1 - \gamma^0) - \alpha^0 \overline{\alpha^0} \right]$
 σ that Tr $\left[\gamma^0 (1 - \gamma^0) - \alpha^0 \overline{\alpha^0} \right] \leq Ch^{-1}$ from the explicit solon invariant case, and therefore (5.12) yields
 $\left| \text{Tr} \left[\gamma$ We know that Tr $[\gamma^0(1 - \gamma^0) - \alpha^0 \overline{\alpha^0}] \leq Ch^{-1}$ from the explicit solution in the translation invariant case, and therefore (5.12) yields
 $\left| \text{Tr} \left[\gamma(1 - \gamma) - \gamma^0(1 - \gamma^0) - \alpha \overline{\alpha} + \alpha^0 \overline{\alpha^0} \right] \right| \leq Ch.$ (5.13)

In

$$
\int_{\tan^{-1} \cos \theta}^{\pi} \cos \theta \, d\theta
$$
\n
$$
\left| \text{Tr} \left[\gamma (1 - \gamma) - \gamma^0 (1 - \gamma^0) - \alpha \overline{\alpha} + \alpha^0 \overline{\alpha^0} \right] \right| \leq Ch. \tag{5.13}
$$
\nor, the result is an a priori estimate on α we use (5.10) and the Schwarz and

\n
$$
\int_{\tan^{-1} \cos \theta}^{\pi} \cos \theta \, d\theta \, d\theta
$$

 $\left|\text{Tr}\left[\gamma(1-\gamma)-\gamma^0(1-\gamma^0)-\alpha\overline{\alpha}+\alpha^0\right]\right|$
In order to derive from this an a priori estimate on α wielding to bound
 $\left|\text{Tr}(\gamma-\gamma^0)\right| \leq h^{-2}\text{Tr }K_T^0(-ih\nabla)(\gamma-\gamma^0)^2 + h^2$.

$$
\left| \text{Tr} \left[\gamma (1 - \gamma) - \gamma^0 (1 - \gamma^0) - \alpha \overline{\alpha} + \alpha^0 \overline{\alpha^0} \right] \right| \le Ch. \tag{5.13}
$$

order to derive from this an a priori estimate on α we use (5.10) and the Schwarz
uality to bound

$$
\left| \text{Tr}(\gamma - \gamma^0) \right| \le h^{-2} \text{Tr} K_T^0 (-ih\nabla)(\gamma - \gamma^0)^2 + h^2 \text{Tr} \left(K_T^0 (-ih\nabla) \right)^{-1} \le Ch
$$

In order to derive from this an a priori estimate on
$$
\alpha
$$
 we use (5.10) and the Schwarz
inequality to bound
 $|\text{Tr}(\gamma - \gamma^0)| \le h^{-2} \text{Tr } K_T^0(-ih\nabla)(\gamma - \gamma^0)^2 + h^2 \text{Tr } (K_T^0(-ih\nabla))^{-1} \le Ch$
and
 $|\text{Tr}(\gamma^2 - (\gamma^0)^2)| = |\text{Tr}(\gamma - \gamma^0)(\gamma + \gamma^0)| \le h^{-2} \text{Tr}(\gamma - \gamma^0)^2 + h^2 \text{Tr}(\gamma + \gamma^0)^2 \le Ch$.
Finally, since $\text{Tr }\alpha^0\overline{\alpha^0} \le Ch$ (see (5.2)), we conclude from (5.13) that $\text{Tr }\alpha\overline{\alpha} \le Ch$,
as claimed.

5.3. Decomposition of α
Here we quantify in which sense $\alpha(x, y)$ is close to $\frac{1}{2} (\psi(x) + \psi(y)) \alpha^0(h^{-1}(x - y))$.
There is one technical point that we would like to discuss before stating the result.

as claimed. Γ
si \mathbf{e}

5.3. Decomposition of α

Finally, since Tr $\alpha^0 \overline{\alpha^0} \le Ch$ (see (5.2)), we conclude from (5.13) that Tr $\alpha \overline{\alpha} \le Ch$,
as claimed.
5.3. Decomposition of α
Here we quantify in which sense $\alpha(x, y)$ is close to $\frac{1}{2}(\psi(x) + \psi(y)) \alpha^0(h^{-1}(x - y))$.
T **5.3. Decomposition of** α
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There is one technical point that we would like to discuss before stating t Here we quantify in which sense $\alpha(x, y)$ is close to $\frac{1}{2} (\psi(x) + \psi(y)) \alpha^{0} (h^{-1}(x - y))$.
There is one technical point that we would like to discuss before stating the result.
The asymptotic form $\frac{1}{2} (\psi(x) + \psi(y)) \alpha^{0} (h^{-1}(x$ 2 The asymptotic form $\frac{1}{2}(\psi(x) + \psi(y)) \alpha^{0}(h^{-1}(x - y))$ will allow us in the next
subsection to use the semiclassical results in a similar way as in the proof of the
upper bound. Our semiclassics, however, require ψ to be The asymptotic form $\frac{1}{2}(\psi(x) + \psi(y)) \alpha^0(h^{-1}(x - y))$ will allow us in the next
subsection to use the semiclassical results in a similar way as in the proof of the
upper bound. Our semiclassics, however, require ψ to be i 2 upper bound. Our semiclassics, however, require ψ to be in H^2 . While we naturally
get an H^1 condition, the H^2 condition is achieved by introducing an additional
parameter $\epsilon > 0$, which will later chosen to go

get an H^1 condition, the H^2 condition is achieved by introducing an additional
parameter $\epsilon > 0$, which will later chosen to go to zero as $h \to 0$.
Proposition 5.3. Let Γ be admissible with $\mathcal{F}^{BCS}(\Gamma) \leq \mathcal$ parameter $\epsilon > 0$, which will later chosen to go to zero as $h \to 0$.
 Proposition 5.3. Let Γ be admissible with $\mathcal{F}^{BCS}(\Gamma) \leq \mathcal{F}^{BCS}(\Gamma^0)$

sufficiently small $\epsilon \geq h > 0$, the operator $\alpha = [\Gamma]_{12}$ can be deco **Proposition 5.3.** Let Γ be admissible with $\mathcal{F}^{\text{BCS}}(\Gamma) \leq \mathcal{F}^{\text{BCS}}(\Gamma^0)$. Then for every sufficiently small $\epsilon \geq h > 0$, the operator $\alpha = [\Gamma]_{12}$ can be decomposed as $\alpha(x, y) = \frac{1}{2} (\psi(x) + \psi(y)) \alpha^0(h^{-1}(x - y)) + \sigma$

sufficiently small
$$
\epsilon \ge h > 0
$$
, the operator $\alpha = [\Gamma]_{12}$ can be decomposed as
\n
$$
\alpha(x, y) = \frac{1}{2} (\psi(x) + \psi(y)) \alpha^{0} (h^{-1}(x - y)) + \sigma(x, y)
$$
\n
$$
\text{with a periodic function } \psi \in H^{2}(\mathcal{C}) \text{ satisfying}
$$
\n
$$
\|\psi\|_{H^{1}} \le C, \qquad \|\psi\|_{H^{2}} \le C\epsilon h^{-1}
$$
\n(5.15)

with a periodic function $\psi \in H^2(\mathcal{C})$ satisfying
 $\|\psi\|_{H^1} \leq C$, $\|\psi\|$

$$
\|\psi\|_{H^1} \le C \,, \qquad \|\psi\|_{H^2} \le C\epsilon h^{-1} \tag{5.15}
$$

and with

and with
\n
$$
\|\sigma\|_{H^1}^2 \le C\epsilon^{-2}h^3.
$$
\n(5.16)
\nMore precisely, one has $\sigma = \sigma_1 + \sigma_2$ with
\n
$$
\|\sigma_1\|_{H^1}^2 \le Ch^3
$$
\n(5.17)

More precisely, one has $\sigma = \sigma_1 + \sigma_2$ with

$$
\overline{\sigma_2} \text{ with}
$$
\n
$$
\|\sigma_1\|_{H^1}^2 \le Ch^3
$$
\n
$$
\tilde{\psi}(x) + \tilde{\psi}(y) \alpha^0 (h^{-1}(x - y)),
$$
\n(5.17)

and with σ_2 of the form

$$
e \text{ has } \sigma = \sigma_1 + \sigma_2 \text{ with}
$$

$$
\|\sigma_1\|_{H^1}^2 \le Ch^3
$$

$$
form
$$

$$
\sigma_2(x, y) = \frac{1}{2} \left(\tilde{\psi}(x) + \tilde{\psi}(y)\right) \alpha^0 (h^{-1}(x - y)),
$$

$$
ransform \text{ of } \tilde{\psi} \text{ supported in } \{|p| \ge \epsilon h^{-1}\}. \text{ The}
$$

 $% \[5mm] % \begin{equation} \includegraphics[width=0.5\columnwidth]{figures/fig_0}}% \caption{The figure shows the function of the function ρ in the 1 and $1/3$ for $n=1/3$ and $1/3$ for $n=1/3$ and $1/3$ for $n=1/3$ and $1/3$ for $n=1/3$ for $n=1/3$ and $1/3$ for $n=1/3$ for $n=1/3$ and $1/3$ for $n=1/3$ for $n=$ $(x, y) = \frac{1}{2} (\tilde{\psi}(x) + \tilde{\psi}(y)) \alpha^{0} (h^{-1}(x - y)),$

nsform of $\tilde{\psi}$ supported in $\{|p| \ge \epsilon h^{-1}\}$. The
 $\{|p| < \epsilon h^{-1}\}$.

he H^{1} norm of an operator was introduce

n write (5.6) as where the Fourier transform of ψ supported in $\{|p| \ge \epsilon h^{-1}\}$. The Fourier transform
of ψ is supported in $\{|p| < \epsilon h^{-1}\}$.
We recall that the H^1 norm of an operator was introduced in (3.7).
Proof. Step 1. We can w of ψ is supported in $\{|p| < \epsilon h^{-1}\}.$

Proof. Step 1. We can write (5.6) as

We recall that the
$$
H^1
$$
 norm of an operator was introduced in (3.7).
\nf. Step 1. We can write (5.6) as
\n
$$
0 \le \int_C \langle \alpha(\cdot, y) | K_T^0(-ih\nabla) - ah \delta(\cdot - y) | \alpha(\cdot, y) \rangle dy \le Ch^3.
$$
\n
$$
(5.18)
$$
\nthe operator $K_T^0(-ih\nabla)$ acts on the *x* variable of $\alpha(x, y)$, and $\langle \cdot | \cdot \rangle$ denotes
\ntandard inner product on $L^2(\mathbb{R})$.

 $\boldsymbol{\mathcal{I}}$

the standard inner product on $L^2(\mathbb{R})$.
Now we recall that the operator $K_T^0 - ah\delta(\cdot - y)$ on $L^2(\mathbb{R})$ has a unique ground
state, proportional to $\alpha^0(h^{-1}(\cdot - y))$, with ground state energy zero. There are no 0 ≤
e op
daro
w wo
opo
eige! $(\cdot, y)|K_T^0(-ih\nabla) - ah\delta(\cdot - y)|\alpha(\cdot, y)\rangle dy \le Ch^3.$ (5.18)
 $\text{tr } K_T^0(-ih\nabla)$ acts on the *x* variable of $\alpha(x, y)$, and $\langle \cdot | \cdot \rangle$ denotes
 tr product on $L^2(\mathbb{R})$.

Il that the operator $K_T^0-ah\delta(\cdot - y)$ on $L^2(\mathbb{R})$ has a Here, the operator $K_T^0(-ih\nabla)$ acts on the *x* variable of $\alpha(x, y)$, and $\langle \cdot | \cdot \rangle$ denotes
the standard inner product on $L^2(\mathbb{R})$.
Now we recall that the operator $K_T^0-ah\delta(\cdot-y)$ on $L^2(\mathbb{R})$ has a unique ground
st the standard inner product on $L^2(\mathbb{R})$.

Now we recall that the operator *I*

state, proportional to $\alpha^0(h^{-1}(\cdot - y))$,

further eigenvalues and the bottom o

2*T*. In particular, there is a lower bou
 $\psi_0(y) = \left(h \int |\alpha^0(x$ Now we recall that the operator $K_T^0 - ah\delta(\cdot - y)$ on $L^2(\mathbb{R})$ has a unique ground, proportional to $\alpha^0(h^{-1}(\cdot - y))$, with ground state energy zero. There are now er eigenvalues and the bottom of its essential spectrum is Now we recall that the operator $K_T^0 - ah\delta(\cdot - y)$ on $L^2(\mathbb{R})$ has a unique ground state, proportional to $\alpha^0(h^{-1}(\cdot - y))$, with ground state energy zero. There are no
further eigenvalues and the bottom of its essential spectrum is $\Delta_0/\tanh\left[\frac{\Delta_0}{2T}\right] \ge$
2T. In particular, there is a lower bound, inde

further eigenvalues and the bottom of its essential spectrum is
$$
\Delta_0 / \tanh\left[\frac{\Delta_0}{2T}\right] \ge 2T
$$
. In particular, there is a lower bound, independent of *h*, on the gap. We write
\n
$$
\psi_0(y) = \left(h \int_{\mathbb{R}^d} |\alpha^0(x)|^2 dx\right)^{-1} \int_{\mathbb{R}^d} \alpha^0(h^{-1}(x-y)) \alpha(x,y) dx \qquad (5.19)
$$
\nand decompose
\n
$$
\alpha(x,y) = \psi_0(y) \alpha^0(h^{-1}(x-y)) + \sigma_0(x,y).
$$
\nThen (5.18) together with the uniform lower bound on the gap of $K_T^0 - ah\delta(\cdot - y)$ yields the bound $\|\sigma_0\|_2^2 \le Ch^3$. We can also symmetric and write

$$
\alpha(x, y) = \psi_0(y)\alpha^0(h^{-1}(x - y)) + \sigma_0(x, y)
$$

and $\alpha(x, y) = \psi_0(y)\alpha^0(h^{-1}(x - y)) + \sigma_0(x, y)$.

Then (5.18) together with the uniform lower bound on the gap

yields the bound $\|\sigma_0\|_2^2 \le Ch^3$. We can also symmetrize and $\sigma_1(x, y) = \sigma_0(x, y) + \frac{1}{2}(\psi(x) - \psi(y))\alpha^0(h^{-1}(x - y))$

The Then (5.18) together with the uniform lower bound on the gap of K_q^0
yields the bound $\|\sigma_0\|_2^2 \le Ch^3$. We can also symmetrize and write
 $\sigma_1(x, y) = \sigma_0(x, y) + \frac{1}{2} (\psi(x) - \psi(y)) \alpha^0(h^{-1}(x - y))$.
Then
 $\alpha(x, y) = \frac{1}{2} (\psi_0(x) + \psi_$ $\frac{f_0}{f} - ah\delta(\cdot - y)$

(5.20)

(5.21) yields the bound $\|\sigma_0\|_2^2 \le Ch^3$
 $\sigma_1(x, y) = \sigma_0(x,$

Then
 $\alpha(x, y) = \frac{1}{2} (\psi_0(x, y))$

again with

$$
\sigma_1(x, y) = \sigma_0(x, y) + \frac{1}{2} (\psi(x) - \psi(y)) \alpha^0(h^{-1}(x - y)). \tag{5.20}
$$

$$
\sigma_1(x, y) = \sigma_0(x, y) + \frac{1}{2} (\psi(x) - \psi(y)) \alpha^0(h^{-1}(x - y)).
$$
\n(5.20)
\n
$$
\alpha(x, y) = \frac{1}{2} (\psi_0(x) + \psi_0(y)) \alpha^0(h^{-1}(x - y)) + \sigma_1(x, y)
$$
\n(5.21)
\n
$$
\|\sigma_1\|_2^2 \leq Ch^3.
$$
\n(5.22)
\n10.667. (5.17). Before proving the second half in Step 4 below

$$
\|\sigma_1\|_2^2 \le Ch^3. \tag{5.22}
$$

 $\alpha(x, y) = \frac{1}{2} (\psi_0(x) + \psi_0(y)) \alpha^0(h^{-1}(x - y)) + \sigma_1(x, y)$ (5.21)

again with
 $\|\sigma_1\|_2^2 \leq Ch^3$. (5.22)

This proves the first half of (5.17). Before proving the second half in Step 4 below

we need to study ψ .
 Step 2. W This prove
we need to
 $Step 2$. We we need to study ψ .
Step 2. We claim th
and

Step 2.

This proves the first half of (5.17). Before proving the second half in Step 4 below
we need to study
$$
\psi
$$
.
Step 2. We claim that

$$
\int_{\mathcal{C}} |\psi_0(x)|^2 dx \leq C
$$
(5.23)
and

$$
\int_{\mathcal{C}} |\psi'(x)|^2 dx \leq C
$$
(5.24)

$$
\int_{\mathcal{C}} |\psi_0(x)|^2 dx \le C \tag{5.23}
$$
\n
$$
\int_{\mathcal{C}} |\psi_0'(x)|^2 dx \le C \tag{5.24}
$$

follows by Schwarz's inequality
\n
$$
\int_{\mathcal{C}} |\psi_0(x)|^2 dx \le \frac{\|\alpha\|_2^2}{h \int_{\mathbb{R}} |\alpha^0(x)|^2 dx}
$$
\n5) and (5.2). In order to prove (5.24) we use again Schwarz's

and our bounds (5.5) and (5.2) . In order to prove (5.24) we use again Schwarz's inequality,

$$
\int_{\mathcal{C}} |\psi_0(x)|^2 dx \leq \frac{e^{-\frac{|x|^2}{2}}}{h \int_{\mathbb{R}} |\alpha^0(x)|^2 dx}
$$

and our bounds (5.5) and (5.2). In order to prove (5.24) we use again Schwarz's inequality,

$$
\int_{\mathcal{C}} |\psi_0'(x)|^2 dx \leq \frac{\int_{\mathbb{R} \times \mathcal{C}} |(\nabla_x + \nabla_y) \alpha(x, y)|^2 dx dy}{h \int |\alpha^0(x)|^2 dx}.
$$
(5.25)
Lemma 5.4 below bounds the numerator by a constant times

$$
h^{-2} \int_{\mathcal{C}} \langle \alpha(\cdot, y) | K_T^0(-ih\nabla) - ah \delta(\cdot - y) | \alpha(\cdot, y) \rangle dy,
$$

and therefore (5.24) is a consequence of (5.18) and (5.2).

Lemma 5.4 below bounds the numerator by a constant times

$$
\int_{\mathcal{C}} |\psi_0'(x)|^2 dx \le \frac{\sin 2\pi x}{h \int |\alpha^0(x)|^2 dx}.
$$

ow bounds the numerator by a constant times

$$
h^{-2} \int_{\mathcal{C}} \langle \alpha(\cdot, y) | K_T^0(-ih\nabla) - ah \delta(\cdot - y) | \alpha(\cdot, y) \rangle dy,
$$

5.24) is a consequence of (5.18) and (5.2).

 $h^{-2} \int_C \langle \alpha(\cdot, y) | K_T^0(-ih\nabla) - ah \delta(\cdot - y) | \alpha(\cdot, y)$
and therefore (5.24) is a consequence of (5.18) and (5.2).
Step 3. Next, we establish the remaining bound $\|\nabla \sigma_1\|_2^2 \leq C$
formula (5.20) for σ_1 . First of all, usi $(\cdot, y)|K_T^0(-ih\nabla) - ah\delta(\cdot - y)|\alpha(\cdot, y)\rangle dy,$
 α , consequence of (5.18) and (5.2).
 \sinh the remaining bound $\|\nabla\sigma_1\|_2^2 \leq Ch$ in $\arcsin \delta$ all, using the fact that
 $K(-ih\nabla) \geq c(1-h^2\nabla^2)$ Step 3. Next, we establish the remaining bound $\|\nabla \sigma_1\|_2^2 \leq Ch$

$$
K(-ih\nabla) \ge c(1 - h^2 \nabla^2)
$$

Step 3. Next, we establish the remaining bound $\|\nabla \sigma_1\|_2^2 \leq Ch$ in (5.17). We use
formula (5.20) for σ_1 . First of all, using the fact that
 $K(-ih\nabla) \geq c(1-h^2\nabla^2)$
one easily deduces from (5.18) that $\|\nabla \sigma_0\|$ one easily deduces from (5.18) that $\|\nabla \sigma_0\|_2^2 \le Ch$

(5.24)
 $\int_{\mathcal{C}} |\psi_0'(x)|^2 \int_{\mathbb{R}} |\alpha^0(h^{-1}(x-y))|$

Finally,
 $h^{-2} \int_{\mathbb{R}} |\psi_0(x) - \psi_0(y)|^2 |(\alpha^0)'(h^{-1}(x))|$ $2 dx dy \leq Ch$.
 $\frac{1}{(x-y)})^2 dx dy$ (5.24)
Finally,

$$
K(-ih\nabla) \ge c(1 - h^2 \nabla^2)
$$

from (5.18) that $\|\nabla \sigma_0\|_2^2 \le Ch$. Moreover, b

$$
\int_{\mathcal{C}} |\psi_0'(x)|^2 \int_{\mathbb{R}} |\alpha^0(h^{-1}(x - y))|^2 dx dy \le Ch.
$$

 $\frac{1}{2}$

$$
\int_{\mathcal{C}} |\psi_0'(x)|^2 \int_{\mathbb{R}} |\alpha^0(h^{-1}(x-y))|^2 dx dy \le Ch.
$$

$$
h^{-2} \int_{\mathcal{C} \times \mathbb{R}} |\psi_0(x) - \psi_0(y)|^2 |(\alpha^0)'(h^{-1}(x-y))|^2 dx dy
$$

$$
= h^{-1} 4 \sum_{p \in 2\pi \mathbb{Z}} |\hat{\psi}_0(p)|^2 \int_{\mathbb{R}} |(\alpha^0)'(x)|^2 \sin^2(\frac{1}{2} h p x) dx
$$

$$
\le h \sum_{p \in 2\pi \mathbb{Z}} |p|^2 |\hat{\psi}_0(p)|^2 \int_{\mathbb{R}} |(\alpha^0)'(x)|^2 x^2 dx \le Ch,
$$

1(5.24) and the fact that $\int_{\mathbb{R}} |(\alpha^0)'(x)|^2 x^2 dx$ is finite. T
of the fact that the Fourier transform of α^0 is given h

 $|\hat{\psi}_0(p)|^2 \int_{\mathbb{R}} |(\alpha^0)|^2$
ct that $\int_{\mathbb{R}} |(\alpha^0)|^2$
e Fourier trans
completes the p (x)
 (x)
 $\frac{1}{x}$
 $\frac{1}{x}$ $x^2 dx$ $\frac{\Delta_0}{2(2\pi)^{1/2}}\left(K_T^0\right)^{-1}$

where we used (5.24) and the fact that $\int_{\mathbb{R}} |(\alpha^0)'(x)|$
consequence of the fact that the Fourier transform
function $\frac{\Delta_0}{2(2\pi)^{1/2}} (K_T^0)^{-1}$. This completes the proof
Step 4. Finally, for each $\epsilon \geq h$ we decom (x)
forn
prode
 $\frac{1}{2}$
.15) is given by the smooth
17).
+ $\tilde{\psi}$, where the Fourier
 $p \ge \epsilon h^{-1}$ }, respectively. consequence of the fact that the Fourier transform of α^0 is given by the smooth
function $\frac{\Delta_0}{2(2\pi)^{1/2}} (K_T^0)^{-1}$. This completes the proof of (5.17).
Step 4. Finally, for each $\epsilon \geq h$ we decompose $\psi_0 = \psi + \til$ function $\frac{\Delta_0}{2(2\pi)^{1/2}} (K_T^0)^{-1}$. This completes the proof of (5.17).
 Step 4. Finally, for each $\epsilon \geq h$ we decompose $\psi_0 = \psi + \hat{\psi}$

transforms of ψ and $\tilde{\psi}$ are supported in $\{|p| < \epsilon h^{-1}\}$ and $\{|p| \geq$ Step transforms of ψ and $\tilde{\psi}$ are supported in $\{|p| < \epsilon h^{-1}\}$ and $\{|p| \ge \epsilon h^{-1}\}$, respectively.
Clearly, the bounds (5.23) and (5.24) imply (5.15).
Moreover, $\|\tilde{\psi}\|_2 \le C\epsilon^{-1}h$ and $\|\tilde{\psi}'\|_2 \le C$, and hence
 $\sigma_2(x$ transforms of ψ and $\bar{\psi}$ are supported in { $|p| < \epsilon h^{-1}$ } and { $|p| \ge \epsilon h^{-1}$ }, respectively.
Clearly, the bounds (5.23) and (5.24) imply (5.15).
Moreover, $\|\tilde{\psi}\|_2 \le C\epsilon^{-1}h$ and $\|\tilde{\psi}'\|_2 \le C$, and hence
 $\sigma_2(x,$

Moreover, $\|\tilde{\psi}\|_2 \leq C\epsilon^{-1}h$ and $\|\tilde{\psi}'\|_2 \leq C$, and hence

Moreover,
$$
\|\tilde{\psi}\|_2 \leq C\epsilon^{-1}h
$$
 and $\|\tilde{\psi}'\|_2 \leq C$, and hence
\n
$$
\sigma_2(x, y) = \frac{1}{2} \left(\tilde{\psi}(x) + \tilde{\psi}(y) \right) \alpha^0(h^{-1}(x - y))
$$
\nsatisfies $\|\sigma_2\|_{H^1}^2 \leq h^3 \epsilon^{-2}$. This completes the proof of the prop

satisfies $||\sigma_2||_{H^1}^2 \leq h^3 \epsilon^{-2}$. This completes the proof of the proposition. \Box

Lemma 5.4. For some constant $C > 0$.

In the previous proof we made use of the following
\n**Lemma 5.4.** For some constant
$$
C > 0
$$
,
\n
$$
h^2 \int_{\mathbb{R} \times \mathcal{C}} |(\nabla_x + \nabla_y) \alpha(x, y)|^2 dx dy
$$
\n
$$
\leq C \int_{\mathcal{C}} \langle \alpha(\cdot, y) | K_T^0(-ih\nabla) - ah \delta(\cdot - y) | \alpha(\cdot, y) \rangle dy \qquad (5.26)
$$
\nfor all periodic and symmetric α (i.e., $\alpha(x, y) = \alpha(y, x)$).
\nProof. By expanding $\alpha(x, y)$ in a Fourier series
\n
$$
\alpha(x, y) = \sum_{x \in 2\pi} e^{ip(x+y)/2} \alpha_p(x - y) \qquad (5.27)
$$

for all periodic and symmetric α

Proof. By expanding $\alpha(x, y)$ in a Fourier series

periodic and symmetric
$$
\alpha
$$
 (i.e., $\alpha(x, y) = \alpha(y, x)$).
\nBy expanding $\alpha(x, y)$ in a Fourier series
\n
$$
\alpha(x, y) = \sum_{p \in 2\pi\mathbb{Z}} e^{ip(x+y)/2} \alpha_p(x - y)
$$
\n(5.27)
\nng that $\alpha_p(x) = \alpha_p(-x)$ for all $p \in 2\pi\mathbb{Z}$ we see that (5.26) is equivalent to
\n
$$
K_T^0(-ih\nabla + hp/2) + K_T^0(-ih\nabla - hp/2) - 2a\,\delta(x/h) \geq \frac{2}{C}h^2p^2
$$
\n(5.28)
\n $p \in 2\pi\mathbb{Z}$. This inequality holds for all $p \in \mathbb{R}$, in fact, for an appropriate

$$
K_T^0(-ih\nabla + hp/2) + K_T^0(-ih\nabla - hp/2) - 2a\,\delta(x/h) \ge \frac{2}{C}h^2p^2\tag{5.28}
$$

and using that $\alpha_p(x) = \alpha_p(-x)$ for all $p \in 2\pi\mathbb{Z}$ we see that (5.26) is equivalent to
 $K_T^0(-ih\nabla + hp/2) + K_T^0(-ih\nabla - hp/2) - 2a \delta(x/h) \ge \frac{2}{C}h^2p^2$ (5.28)

for all $p \in 2\pi\mathbb{Z}$. This inequality holds for all $p \in \mathbb{R}$

 $(-ih\nabla + hp/2) + K_T^0(-ih\nabla - hp/2) - 2a\,\delta(x/h) \ge \frac{2}{C}h^2p^2$ (5.28)
 $\ge 2\pi\mathbb{Z}$. This inequality holds for all $p \in \mathbb{R}$, in fact, for an appropriate
 $C > 0$, as we shall now show.
 $\ge K_T^0 \ge \text{const}(1 + h^2(-i\nabla + p/2)^2)$, it suff \overline{C} f si u \overline{C} for all $p \in 2\pi\mathbb{Z}$. This inequality holds for all $p \in \mathbb{R}$, in fact, for an appropriate
choice of $C > 0$, as we shall now show.
Since $K_T^0 \ge \text{const}(1 + h^2(-i\nabla + p/2)^2)$, it suffices to consider the case of hp
small. choice of $C > 0$, as we shall now show.

Since $K_T^0 \ge \text{const}(1 + h^2(-i\nabla + p))$

small. If $\kappa = \Delta_0 / \tanh\left[\frac{\Delta_0}{2T}\right] \ge 2T$ denoto

ah above zero, and $h^{-1/2}\phi_0(x/h)$ its
 $\alpha^0(x/h)$,
 $K_T^0(-ih\nabla + hp/2) + K_T^0(-ih\nabla - p)$ Since $K_T^0 \ge \text{const}(1 + h^2(-i\nabla + p/2)^2)$, it suffices to consider the case of hp

. If $\kappa = \Delta_0 / \tanh\left[\frac{\Delta_0}{2T}\right] \ge 2T$ denotes the gap in the spectrum of $K_T^0(-ih\nabla)$ -

above zero, and $h^{-1/2}\phi_0(x/h)$ its normalized ground $T \leq$ $\alpha^{0}(x/h)$,

small. If
$$
\kappa = \Delta_0 / \tanh\left[\frac{\Delta_0}{2T}\right] \geq 2T
$$
 denotes the gap in the spectrum of $K_T^0(-ih\nabla) - ah\delta$ above zero, and $h^{-1/2}\phi_0(x/h)$ its normalized ground state, proportional to $\alpha^0(x/h)$,
\n $K_T^0(-ih\nabla + hp/2) + K_T^0(-ih\nabla - hp/2) - 2ah\delta$
\n $\geq \kappa \left[e^{ihxp/2}(1 - |\phi_0\rangle\langle\phi_0|)e^{-ihxp/2} + e^{-ihxp/2}(1 - |\phi_0\rangle\langle\phi_0|)e^{ihxp/2}\right]$
\n $\geq \kappa \left[1 - \left|\int |\phi_0(x)|^2 e^{-ihxp} dx\right|\right].$ (5.29)
\nIn order to see the last inequality, simply rewrite the term as $\kappa(2 - |f\rangle\langle f| - |g\rangle\langle g|)$,
\nwhere $|\langle f|g\rangle|^2 = | \int |\phi_0(x)|^2 e^{-ihxp} dx|$, and compute the smallest eigenvalue of
\nthe corresponding 2 × 2 matrix. Since ϕ_0 is reflection symmetric, normalized and

 $\begin{align} 1 - \ \frac{1}{2} & = \ \frac{1}{2} \ \$ (x) |
nequ
(x)|
ma
∞ (s $g\rangle\langle g|$),
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ed and
ave In order to see the last inequality, simply rewrite the term as $\kappa(2-|f\rangle\langle f|-|g\rangle\langle g|)$,
where $|\langle f|g\rangle|^2 = | \int |\phi_0(x)|^2 e^{-ihx p} dx |$, and compute the smallest eigenvalue of
the corresponding 2×2 matrix. Since ϕ_0 is re where $|\langle f|g \rangle|^2 = | \int |\phi_0(x)|^2 e^{-ihx p} dx|$
the corresponding 2×2 matrix. Since
satisfies $\int x^2 |\phi_0|^2 dx < \infty$ (see Step 4 i
 $1 - \left| \int |\phi_0(x)|^2 e^{-ihx p} dx \right| = \int$
This completes the proof of (5.28). $\begin{array}{ccc} \begin{bmatrix} 1 & \vert \varphi_0(u) \vert & \vert & \vert & \vert & \vert & \vert \end{bmatrix} & \begin{bmatrix} 1 & \vert \varphi_0(u) \vert & \vert & \vert \vert & \vert \vert & \vert \vert & \vert \vert \end{bmatrix} & \begin{bmatrix} 1 & \vert \vert & \vert \vert & \vert \vert & \vert \vert \vert & \vert \vert \vert & \vert \vert \vert & \vert \vert \vert \end{bmatrix} & \begin{bmatrix} 1 & \vert \vert & \vert \vert & \vert \vert \vert & \vert \vert \vert & \vert \vert \vert \vert & \vert \$

where
$$
|\langle f|g\rangle|^2 = | \int |\phi_0(x)|^2 e^{-ihxp} dx|
$$
, and compute the smallest eigenvalue of
the corresponding 2×2 matrix. Since ϕ_0 is reflection symmetric, normalized and
satisfies $\int x^2 |\phi_0|^2 dx < \infty$ (see Step 4 in the proof of Proposition 5.3), we have

$$
1 - \left| \int |\phi_0(x)|^2 e^{-ihxp} dx \right| = \int |\phi_0(x)|^2 (1 - \cos(hpx)) dx \ge ch^2p^2.
$$
This completes the proof of (5.28).
5.4. The lower bound
Pick a Γ with $\mathcal{F}^{\text{BCS}}(\Gamma) \le \mathcal{F}^{\text{BCS}}(\Gamma^0)$ and let ψ be as in Proposition 5.3 (depending
on some parameter $\epsilon > h$ to be chosen later). As before we let $\Delta(x) = -\psi(x)\Delta_0$

5.4. The lower bound

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e wi
e pa $\binom{1}{0}$. This completes the proof of (5.28).
 5.4. The lower bound

Pick a Γ with $\mathcal{F}^{\text{BCS}}(\Gamma) \leq \mathcal{F}^{\text{BCS}}(\Gamma^0)$ and let ψ be as in Proposition 5.3 (depending

on some parameter $\epsilon \geq h$ to be chosen later). As be on some parameter $\epsilon \geq h$ to be chosen later). As before we let $\Delta(x) = -\psi(x)\Delta_0$ on some parameter $\epsilon \geq h$ to be chosen later). As before we let $\Delta(x) = -\psi(x)\Delta_0$
and define H_{Δ} by (2.1). We also put $\Gamma_{\Delta} = (1 + \exp(\beta H_{\Delta}))^{-1}$. and define H_{Δ} by (2.1). We also put $\Gamma_{\Delta} = (1 + \exp(\beta H_{\Delta}))^{-1}$.

$$
\sqcup
$$

Our starting point is the representation
\n
$$
\mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma_0) = -\frac{T}{2} \text{Tr} \left[\ln(1 + e^{-\beta H_{\Delta}}) - \ln(1 + e^{-\beta H_{0}}) \right]
$$
\n
$$
+ \frac{T}{2} \mathcal{H}(\Gamma, \Gamma_{\Delta}) + \Delta_0 \text{Re} \int_C \overline{\psi(x)} \alpha(x, x) dx - ha \int_C |\alpha(x, x)|^2 dx. \quad (5.30)
$$
\nCompare with (5.7).) According to the decomposition (5.14) which, in view of
\nne BCS gap equation (1.7), reads on the diagonal
\n
$$
\alpha(x, x) = \psi(x)\alpha^0(0) + \sigma(x, x) = \frac{\Delta_0 \psi(x)}{2ah} + \sigma(x, x),
$$
\ne can obtain the lower bound

vit
p

the BCS gap equation (1.7), reads on the diagonal
\n
$$
\alpha(x, x) = \psi(x)\alpha^{0}(0) + \sigma(x, x) = \frac{\Delta_{0}\psi(x)}{2ah} + \sigma(x, x),
$$
\nwe can obtain the lower bound
\n
$$
\tau^{BCS}(\Gamma) = \tau^{BCS}(\Gamma) \geq \frac{T}{\Gamma_{\Gamma}}\left[\ln(1 + e^{-\beta H_{\Delta}}) - \ln(1 + e^{-\beta H_{\Delta}})\right]
$$

$$
\alpha(x, x) = \psi(x)\alpha^{0}(0) + \sigma(x, x) = \frac{\Delta_{0}\psi(x)}{2ah} + \sigma(x, x),
$$

we can obtain the lower bound

$$
\mathcal{F}^{BCS}(\Gamma) - \mathcal{F}^{BCS}(\Gamma_{0}) \ge -\frac{T}{2} \text{Tr} [\ln(1 + e^{-\beta H_{\Delta}}) - \ln(1 + e^{-\beta H_{0}})]
$$

$$
+ \frac{T}{2} \mathcal{H}(\Gamma, \Gamma_{\Delta}) - ah \int_{\mathcal{C}} |\sigma(x, x)|^{2} dx.
$$

For the first two terms on the right side we apply the semiclassics from 3.1. Arguing as in the proof of the upper bound and taking into acc
bounds on ψ from Proposition 5.3 we obtain

 $\frac{1}{2}$ at $\frac{1}{N}$ $\begin{matrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix}$ $\frac{T}{2}$ H(Γ, Γ_Δ) – ah $\int_{\mathcal{C}} |\sigma(x,x)|$
at side we apply the semicl
the upper bound and takin
we obtain
 $\mathcal{E}^{\text{GL}}(\psi) - b_3$) – $C\epsilon^2 h^3$

rem 3.1. Arguing as in the proof of the upper bound and taking into account the bounds on
$$
\psi
$$
 from Proposition 5.3 we obtain
\n
$$
\mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma_0) \geq h^3 \left(\mathcal{E}^{\text{GL}}(\psi) - b_3 \right) - C \epsilon^2 h^3
$$
\n
$$
+ \frac{T}{2} \mathcal{H}(\Gamma, \Gamma_{\Delta}) - a h \int_{\mathcal{C}} |\sigma(x, x)|^2 dx. \tag{5.31}
$$
\nOur final task is to bound the last two terms from below. In the remainder of this subsection we shall show that

Our final task is to bound the last two terms from below. In the remainder of this

$$
+\frac{T}{2}\mathcal{H}(\Gamma,\Gamma_{\Delta}) - ah \int_{\mathcal{C}} |\sigma(x,x)|^2 dx.
$$
 (5.31)
Our final task is to bound the last two terms from below. In the remainder of this
subsection we shall show that

$$
\frac{T}{2}\mathcal{H}(\Gamma,\Gamma_{\Delta}) - ah \int_{\mathcal{C}} |\sigma(x,x)|^2 dx \geq -C \left(\epsilon h^3 + \epsilon^{-2} h^4\right).
$$
 (5.32)
The choice $\epsilon = h^{1/3}$ will then lead to
 $\mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma_0) \geq h^3 \left(E^{\text{GL}} - b_3\right) - Ch^{3+1/3}$,
which is the claimed lower bound.
In order to prove (5.32) we again use the lower bound on the relative entropy

$$
\mathcal{F}^{\rm BCS}(\Gamma) - \mathcal{F}^{\rm BCS}(\Gamma_0) \ge h^3 \left(E^{\rm GL} - b_3 \right) - C h^{3+1/3},
$$

The choice $\epsilon = h^{1/3}$ will then lead to
 $\mathcal{F}^{\text{BCS}}(\Gamma) - \mathcal{F}^{\text{BCS}}(\Gamma_0)$

which is the claimed lower bound.

In order to prove (5.32) we again

from Lemma 5.1 to estimate which is the claimed lower bound.
In order to prove (5.32) we again use the lower bound on the relative entropy
from Lemma 5.1 to estimate In order to prove (5.32) we agree from Lemma 5.1 to estimate
 $T \mathcal{H}(\Gamma, \Gamma_{\Delta}) \geq 7$.
The next lemma will allow us to r

Lemma 5.1 to estimate
\n
$$
T \mathcal{H}(\Gamma, \Gamma_{\Delta}) \ge \text{Tr} \left[\frac{H_{\Delta}}{\tanh \frac{1}{2T} H_{\Delta}} (\Gamma - \Gamma_{\Delta})^2 \right]. \tag{5.33}
$$
\nnext lemma will allow us to replace the operator H_{Δ} in this bound by H_0 .
\n**na 5.5.** There is a constant $c > 0$ such that for all sufficiently small $h > 0$
\n
$$
\frac{H_{\Delta}}{\tanh \frac{1}{2T} H_{\Delta}} \ge (1 - ch) K_T^0(-ih \nabla) \otimes \mathbb{I}_{\mathbb{C}^2}. \tag{5.34}
$$

 $T \mathcal{H}(\Gamma, \Gamma)$
The next lemma will allow u
Lemma 5.5. *There is a cons* The next lemma will allow us to replace the operator H_{Δ} in this bound by H_0 .
 Lemma 5.5. There is a constant $c > 0$ such that for all sufficiently small $h > 0$
 $\frac{H_{\Delta}}{\tanh \frac{1}{2T} H_{\Delta}} \ge (1 - ch) K_T^0(-ih\nabla) \otimes \mathbb{I$

The next lemma will allow us to replace the operator
$$
H_{\Delta}
$$
 in this bound by H_0 .
\n**Lemma 5.5.** There is a constant $c > 0$ such that for all sufficiently small $h > 0$
\n
$$
\frac{H_{\Delta}}{\tanh \frac{1}{2T} H_{\Delta}} \ge (1 - ch) K_T^0(-ih\nabla) \otimes \mathbb{I}_{\mathbb{C}^2}.
$$
\n(5.34)

Proof.

Proof. An application of Schwarz's inequality yields that for every
$$
0 < \eta < 1
$$
 $H^2_{\Delta} \geq (1 - \eta) \left(H^0_{\Delta_0} \right)^2 - \eta^{-1} (\Delta_0^2 ||\psi - 1||_{\infty}^2 + h^2 ||W||_{\infty}^2).$ The expansion formula $[14, (4.3.91)]$ \n
$$
\frac{x}{\tanh(x/2)} = 2 + \sum_{k=1}^{\infty} \left(2 - \frac{2k^2 \pi^2}{x^2/4 + k^2 \pi^2} \right)
$$
\nshows that $x \mapsto \sqrt{x}/\tanh\sqrt{x}$ is an operator monotone function. This one

$$
(1 - \eta) \left(H_{\Delta_0}^0\right)^2 - \eta^{-1} (\Delta_0^2 \|\psi - 1\|_{\infty}^2 + h^2 \|W\|_{\infty}^2
$$

$$
\text{ula } [14, (4.3.91)]
$$

$$
\frac{x}{\tanh(x/2)} = 2 + \sum_{k=1}^{\infty} \left(2 - \frac{2k^2 \pi^2}{x^2/4 + k^2 \pi^2}\right)
$$

$$
\overline{x}/\tanh\sqrt{x} \text{ is an operator monotone function}
$$

as that

$$
\frac{(1 - \eta)^{-1/2}\sqrt{H_{\Delta}^2 + \eta^{-1}(\Delta_0^2 \|\psi - 1\|_{\infty}^2 + h}}{\tanh\frac{1}{2T}(1 - \eta)^{-1/2}\sqrt{H_{\Delta}^2 + \eta^{-1}(\Delta_0^2 \|\psi - 1\|_{\infty}^2}}
$$

 $\frac{x}{\tanh(x/2)} = 2 +$ shows that $x \mapsto \sqrt{x}/\tanh\sqrt{x}$ is an monotonicity implies that

shows that
$$
x \mapsto \sqrt{x}/\tanh \sqrt{x}
$$
 is an operator monotone function. This operator
monotonicity implies that

$$
K_T^0(-ih\nabla) \otimes \mathbb{I}_{\mathbb{C}^2} \leq \frac{(1-\eta)^{-1/2}\sqrt{H_{\Delta}^2 + \eta^{-1}(\Delta_0^2 \|\psi - 1\|_{\infty}^2 + h^2\|W\|_{\infty}^2)}}{\tanh \frac{1}{2T}(1-\eta)^{-1/2}\sqrt{H_{\Delta}^2 + \eta^{-1}(\Delta_0^2 \|\psi - 1\|_{\infty}^2 + h^2\|W\|_{\infty}^2)}}
$$

$$
\leq (1-\eta)^{-1/2} \frac{\sqrt{H_{\Delta}^2 + \eta^{-1}(\Delta_0^2 \|\psi - 1\|_{\infty}^2 + h^2\|W\|_{\infty}^2)}}{\tanh \frac{1}{2T}\sqrt{H_{\Delta}^2 + \eta^{-1}(\Delta_0^2 \|\psi - 1\|_{\infty}^2 + h^2\|W\|_{\infty}^2)}}
$$

$$
\leq (1-\eta)^{-1/2} \left(1 + \frac{1}{4T^2\eta}(\Delta_0^2 \|\psi - 1\|_{\infty}^2 + h^2\|W\|_{\infty}^2)\right) \frac{H_{\Delta}}{\tanh \frac{1}{2T}H_{\Delta}}
$$
for $0 < \eta < 1$. The Sobolev inequality and (5.15) show that $\|\psi\|_{\infty} \leq C \|\psi\|_{H^1} \leq C$,
and hence the lemma follows by choosing $\eta = h$.
To proceed, we denote $\alpha_{\Delta} = [\Gamma_{\Delta}]_{12}$ and recall from Theorem 3.2 that

$$
\frac{\Delta_0}{\det \mathcal{L}^0} (\mathcal{L}^*\nabla)^{-1} + \mathcal{L}^0 (\mathcal{L}^*\nabla)^{-1} (\mathcal{L}^*\nabla)^{-1} (\mathcal{L}^*\nabla)^{-1} (\mathcal{L}^*\nabla)^{-1} (\mathcal{L}^*\nabla)^{-1} (\mathcal{L}^*\nabla)^{-1} (\mathcal{L}^*\nabla)^{-1} (\mathcal{L}^*\nabla)^{-1} (\mathcal{L}^*\nabla)^{-1} (\mathcal{L}^
$$

ノー・・・・ こうしょう for $0 < \eta < 1$. The Sobolev inequality and (5.15) show that $\|\psi\|_{\infty} \le C \|\psi\|_{H^1} \le C$,
and hence the lemma follows by choosing $\eta = h$.
To proceed, we denote $\alpha_{\Delta} = [\Gamma_{\Delta}]_{12}$ and recall from Theorem 3.2 that
 $\alpha_{\Delta} = \$ and hence the lemma follows by choosing $\eta = h$. \Box
To proceed, we denote $\alpha_{\Delta} = [\Gamma_{\Delta}]_{12}$ and recall from Theorem 3.2 that
 $\alpha_{\Delta} = \frac{\Delta_0}{4} \left(\psi K_T^0(-ih\nabla)^{-1} + K_T^0(-ih\nabla)^{-1} \psi \right) + \eta_1 + \eta_2$

To proceed, we denote
$$
\alpha_{\Delta} = [\Gamma_{\Delta}]_{12}
$$
 and recall from Theorem 3.2 that
\n
$$
\alpha_{\Delta} = \frac{\Delta_0}{4} \left(\psi K_T^0 (-ih\nabla)^{-1} + K_T^0 (-ih\nabla)^{-1} \psi \right) + \eta_1 + \eta_2
$$
\n
$$
= \frac{1}{2} \left(\psi \alpha^0 + \alpha^0 \psi \right) + \eta_1 + \eta_2
$$
\n
$$
\eta_1 \text{ and } \eta_2 \text{ satisfying the bounds (3.9) and (3.10). The second equality is the explicit form (1.10) of α^0 . Comparing this with (5.14) we infer the\n
$$
\alpha = \alpha_{\Delta} + \sigma - \eta_1 - \eta_2.
$$
$$

= $\frac{1}{2} (\psi \alpha^0 + \alpha^0 \psi) + \eta_1 + \eta_2$
tisfying the bounds (3.9) an
form (1.10) of α^0 . Compar:
 $\alpha = \alpha_{\Delta} + \sigma - \alpha$
(5.34) imply that f:
3 with η_1 and η_2 satisfying the bounds (3.9) and (3.10). The second equality follows
from the explicit form (1.10) of α^0 . Comparing this with (5.14) we infer that
 $\alpha = \alpha_{\Delta} + \sigma - \eta_1 - \eta_2$. (5.35)
Then (5.33) and

$$
\alpha = \alpha_{\Delta} + \sigma - \eta_1 - \eta_2. \tag{5.35}
$$

from the explicit form (1.10) of
$$
\alpha^0
$$
. Comparing this with (5.14) we infer that
\n
$$
\alpha = \alpha_{\Delta} + \sigma - \eta_1 - \eta_2.
$$
\n(5.35)
\nThen (5.33) and (5.34) imply that
\n
$$
\frac{T}{2} \mathcal{H}(\Gamma, \Gamma_{\Delta}) - ah \int_{\mathcal{C}} |\sigma(x, x)|^2 dx
$$
\n
$$
\geq (1 - ch) \operatorname{Tr} K_T^0(-ih\nabla)(\alpha - \alpha_{\Delta})(\overline{\alpha - \alpha_{\Delta}}) - ah \int_{\mathcal{C}} |\sigma(x, x)|^2 dx
$$
\n
$$
\geq (1 - ch) \operatorname{Tr} K_T^0(-ih\nabla)\sigma \overline{\sigma} - ah \int_{\mathcal{C}} |\sigma(x, x)|^2 dx
$$
\n
$$
-(1 - ch) 2 \operatorname{Re} \operatorname{Tr} K_T^0(-ih\nabla)\sigma(\overline{\eta_1 + \eta_2}).
$$
\n(5.36)
\nIn order to bound the first term on the right side from below we are going to choose a parameter $\rho \geq 0$ such that $ch + \rho \leq 1/2$. Here c is the constant from

(1 − *ch*)2 Re Tr $K_T^0(-ih\nabla)\sigma(\overline{\eta_1 + \eta_2})$. (5.36)

I the first term on the right side from below we are going to

er $\rho \ge 0$ such that $ch + \rho \le 1/2$. Here *c* is the constant from choose a parameter $\rho \ge 0$ such that $ch + \rho \le 1/2$. Here *c* is the constant from α .

(5.34). (Eventually, we will pick either
$$
\rho = 0
$$
 or $\rho = 1/4$, say.) Note that
\n
$$
(1 - ch) \operatorname{Tr} K_T^0(-ih\nabla) - ah\delta = \rho K_T^0(-ih\nabla) + (1 - 2ch - 2\rho)(K_T^0(-ih\nabla) - ah\delta) + (ch + \rho)(K_T^0(-ih\nabla) - 2ah\delta).
$$
\nWe recall that the operator $K_T^0(-ih\nabla) - ah\delta$ is non-negative and that the operator $K_T^0(-ih\nabla) - 2ah\delta$ has a negative eigenvalue of order one (by the form boundedness of δ with respect to $K^0(-i\nabla)$.) Hence $K^0(-ih\nabla) - 2ah\delta \geq -C$, with a constant

 $(1 - ch) \text{Tr } K_T^0(-ih\nabla) - ah\delta = \rho K_T^0(-ih\nabla) + (1 - 2ch - 2\rho)(K_T^0(-ih\nabla) - ah\delta)$
 $+ (ch + \rho)(K_T^0(-ih\nabla) - 2ah\delta).$

We recall that the operator $K_T^0(-ih\nabla) - ah\delta$ is non-negative and that the operato
 $K_T^0(-ih\nabla) - 2ah\delta$ has a negative eigenva + $(ch + \rho)(K_T^0(-ih\nabla) - 2ah\delta)$.

is non-negative and that the ope

f order one (by the form bounded
 $ih\nabla) - 2ah\delta \geq -C_1$ with a con-

oe somewhat important to keep

here a numbering.) Moreover, We recall that the operator $K_T^0(-ih\nabla) - ah\delta$ is non-negative and that the operator $K_T^0(-ih\nabla) - 2ah\delta$ has a negative eigenvalue of order one (by the form boundedness of δ with respect to $K_T^0(-i\nabla)$). Hence $K_T^0(-ih\nabla$ We recall that the operator $K_T^0(-ih\nabla) - ah\delta$ is non-negative and that the operator K^0_T $(-ih\nabla)-2ah\delta$ has a negative eigenvalue of order one (by the form boundedness

i with respect to $K_T^0(-i\nabla)$). Hence $K_T^0(-ih\nabla) - 2ah\delta \geq -C_1$ with a constant

independent of h. (In the following it will be somewhat impor of δ with respect to $K_T^0(-i\nabla)$). Hence K_T^0
 C_1 independent of h . (In the following it w.

of various constants, therefore we introdu

the fact that $K_T^0(-ih\nabla) \ge c_1(1-h^2\nabla^2)$ v
 $(1-ch) \text{Tr } K_T^0(-ih\nabla) - ah\delta \ge$ $T^{(v)}$ by μ . Hence \mathbf{r}_T $(-ih\nabla) - 2ah\delta \geq -C_1$ with a constant
Il be somewhat important to keep track
ce here a numbering.) Moreover, using
ve arrive at the lower bound
 $c_1\rho(1-h^2\nabla^2) - C_1(ch + \rho)$,
t side of (5.36) that C_1 of various constants, therefore we introduce here a numbering.) Moreover, using

$$
(1 - ch) \operatorname{Tr} K_T^0(-ih\nabla) - ah \delta \ge c_1 \rho (1 - h^2 \nabla^2) - C_1(ch + \rho),
$$

which means for the first term on the right side of (5.36) that

the fact that
$$
K_T^0(-ih\nabla) \ge c_1(1 - h^2\nabla^2)
$$
 we arrive at the lower bound
\n
$$
(1 - ch) \operatorname{Tr} K_T^0(-ih\nabla) - ah \delta \ge c_1 \rho (1 - h^2\nabla^2) - C_1(ch + \rho),
$$
\nwhich means for the first term on the right side of (5.36) that
\n
$$
(1-ch) \operatorname{Tr} K_T^0(-ih\nabla) \sigma \overline{\sigma} - ah \int_{\mathcal{C}} |\sigma(x, x)|^2 dx \ge c_1 \rho ||\sigma||_{H^1}^2 - C_1(ch + \rho) ||\sigma||_2^2.
$$
 (5.37)

We now turn to the second term on the right side of (5.36) . Theorem 3.2, to- $(1-ch) \text{Tr } K_T^0(-ih\nabla)\sigma\overline{\sigma}-ah \int_{\mathcal{C}} |\sigma(x,x)|^2 dx \geq c_1 \rho \|\sigma\|_{H^1}^2 - C_1$
We now turn to the second term on the right side of (5.36)
gether with the bounds (5.15) on ψ , implies that $\|\eta_1 + \eta_2\|_{H^1}^2 \leq$
combined with $(1-ch) \text{Tr } K_T^0(-ih\nabla)\sigma\overline{\sigma}-ah \int_C |\sigma(x,x)|^2 dx \geq c_1 \rho \|\sigma\|_{H^1}^2 - C_1(ch+\rho)\|\sigma\|_2^2$. (5.37)
We now turn to the second term on the right side of (5.36). Theorem 3.2, to-
ether with the bounds (5.15) on ψ , implies that $\|\eta_1 + \$ gether with the bounds (5.15) on ψ , implies that $||\eta_1 + \eta_2||_{H^1}^2 \leq C\epsilon^2 h^3$. This bound,
combined with $||\sigma||_{H^1}^2 \leq C\epsilon^{-2}h^3$ from Lemma 5.3, however, is not good enough.
(It leads to an error of order h^3 .) gether with the bounds (5.15) on ψ , implies that $\|\eta_1 + \eta_2\|_{H^1}^2 \leq C\epsilon^2 h^3$. This bound,

$$
\text{Tr}\, K_T^0(-ih\nabla)\sigma_2\overline{\eta_1} = 0\,.
$$

(It leads to an error of order h^3 .) Instead, we shall make use of the observation
that in the decompositions $\sigma = \sigma_1 + \sigma_2$ and $\eta_1 + \eta_2$ one has
 $Tr K_T^0(-ih\nabla)\sigma_2\overline{\eta_1} = 0$.
This can be seen by writing out the trace that in the decompositions $\sigma = \sigma_1 + \sigma_2$ and $\eta_1 + \eta_2$ one has
 $\text{Tr } K_T^0(-ih\nabla)\sigma_2\overline{\eta_1} = 0$.

This can be seen by writing out the trace in momentum space

the Fourier transform of the ψ involved in σ_2 has su Tr $K_T^0(-ih\nabla)\sigma_2\overline{\eta_1} = 0$.

out the trace in momen
 ψ involved in σ_2 has sup
 η_1 has support in $\{|p| <$

(7) and (3.10) on σ_1 and
 $\overline{\rho_2}$
 $\geq |\text{Tr } K_T^0(-ih\nabla)\sigma_1\overline{\eta}$

Using the estimates (5.17) and (3.10) on σ_1 and η_2 we conclude that

the Fourier transform of the
$$
\psi
$$
 involved in σ_2 has support in $\{|p| \ge \epsilon h^{-1}\}$, whereas
the one of the ψ involved in η_1 has support in $\{|p| < \epsilon h^{-1}\}$ (see also (6.22)).
Using the estimates (5.17) and (3.10) on σ_1 and η_2 we conclude that
 $|\text{Tr } K_T^0(-ih\nabla)\sigma(\overline{\eta_1 + \eta_2})| \le |\text{Tr } K_T^0(-ih\nabla)\sigma_1\overline{\eta_1}| + |\text{Tr } K_T^0(-ih\nabla)\sigma\overline{\eta_2}|$
 $\le C_2 \left(\epsilon h^3 + h^{5/2} ||\sigma||_{H^1}\right)$. (5.38)
Combining (5.36), (5.37) and (5.38) we find that
 $\frac{T}{2} \mathcal{H}(\Gamma, \Gamma_{\Delta}) - ah \int_C |\sigma(x, x)|^2 dx$

$$
\leq C_2 \left(\epsilon h^3 + h^{5/2} \|\sigma\|_{H^1} \right).
$$
\n(5.38)
\nCombining (5.36), (5.37) and (5.38) we find that
\n
$$
\frac{T}{2} \mathcal{H}(\Gamma, \Gamma_{\Delta}) - ah \int_{\mathcal{C}} |\sigma(x, x)|^2 dx
$$
\n
$$
\geq c_1 \rho \|\sigma\|_{H^1}^2 - C_1 (ch + \rho) \|\sigma\|_2^2 - 2C_2 \left(\epsilon h^3 + h^{5/2} \|\sigma\|_{H^1} \right).
$$
\n(5.39)
\nNext, we are going to distinguish two cases, according to whether $4C_1 \|\sigma\|_2^2 \leq c_1 \|\sigma\|_{H^1}^2$ or not. In the first case, we choose $\rho = 1/4$ and h so small that $ch + \rho \leq 1/2$. In this way we can bound the previous expression from below by

 $(ch + \rho) \|\sigma\|_2^2 - 2C_2 \left(\epsilon h^3 + h^{5/2} \|\sigma\|_{H^1}\right)$. (5.39)
aguish two cases, according to whether $4C_1 \|\sigma\|_2^2 \le$
case, we choose $\rho = 1/4$ and h so small that $ch + \rho \le$
ad the previous expression from below by
 $\int_0^{3} h^$ $2C_2$
s, a
 $\rho =$
 \exp
 $) \ge$ Next, we are going to distinguish two cases, according to whether $4C_1 \|\sigma\|_2^2$
 $c_1 \|\sigma\|_{H^1}^2$ or not. In the first case, we choose $\rho = 1/4$ and h so small that $ch + \rho$
 $1/2$. In this way we can bound the previous $c_1 \|\sigma\|_{H^1}^2$ or not. In the first case, we choose $\rho = 1/4$ and h so small that $ch + \rho \le 1/2$. In this way we can bound the previous expression from below by
 $\frac{1}{8}c_1 \|\sigma\|_{H^1}^2 - 2C_2 \left(\epsilon h^3 + h^{5/2} \|\sigma\|_{H^1} \right) \$

1/2. In this way we can bound the previous expression from below by
\n
$$
\frac{1}{8}c_1 \|\sigma\|_{H^1}^2 - 2C_2 \left(\epsilon h^3 + h^{5/2} \|\sigma\|_{H^1}\right) \ge -8c_1^{-1}C_2^2 h^5 - 2C_2 \epsilon h^3.
$$
\nThis proves the claimed (indeed, a better) bound (5.32) in this case.

Now assume, conversely, that $4C_1 \|\sigma\|_2^2 > c_1 \|\sigma\|_{H^1}^2$. Then we
and bound (5.39) from below by
 $-C_1 ch \|\sigma\|_2^2 - 2C_2 \left(\epsilon h^3 + h^{5/2} \|\sigma\|_{H^1}\right)$

Now assume, conversely, that
$$
4C_1 \|\sigma\|_2^2 > c_1 \|\sigma\|_{H^1}^2
$$
. Then we choose $\rho = 0$ and bound (5.39) from below by\n
$$
-C_1 ch \|\sigma\|_2^2 - 2C_2 \left(\epsilon h^3 + h^{5/2} \|\sigma\|_{H^1}\right)
$$
\n
$$
\geq -C_1 ch \|\sigma\|_2^2 - 2C_2 \left(\epsilon h^3 + h^{5/2} \left(4C_1/c_1\right)^{1/2} \|\sigma\|_2\right).
$$
\nThe bound (5.16) on $\|\sigma\|_2$ now leads again to the claimed lower bound (5.32). This concludes the proof of the lower bound to the free energy in Theorem 1.2. Concerning the statement about approximate minimizers we note that $\mathcal{F}^{\rm BCS}(\Gamma^0)$ – $\mathcal{F}^{\rm BCS}(\Gamma, \cdot) = O(h^{3})$ and that our a priori bounds on α in Proposition 5.3 remain

Concerning the statement about approximate minimizers we note that $\mathcal{F}^{\text{BCS}}(\Gamma^0)$ $2C_2$
eads
:he l
app:
: a-p + $h^{5/2}$ (4 C_1/c_1)
n to the claimed
bound to the fre
ate minimizers v
bounds on α in
 $\text{CS}(\Gamma) \leq \mathcal{F}^{\text{BCS}}(\Gamma)$ The bound (5.16) on $||\sigma||_2$ now leads again to the claimed lower bound (5.32).
This concludes the proof of the lower bound to the free energy in Theorem Concerning the statement about approximate minimizers we note that Concerning the statement about approximate minimizers we note that $\mathcal{F}^{\text{BCS}}(\Gamma^0) \mathcal{F}^{\text{BCS}}(\Gamma_0) = O(h^3)$ and that our a-priori bounds on α in Proposition 5.3 remain
true under the weaker condition that $\mathcal{F$ $\mathcal{F}^{\text{BCS}}(\Gamma_0) = O(h^3)$ and that our a-priori bounds on α in Proposition 5.3 remain $(\Gamma_0) = O(h^3)$ and that our a-priori bounds on α in Proposition 5.3 remain
under the weaker condition that $\mathcal{F}^{BCS}(\Gamma) \leq \mathcal{F}^{BCS}(\Gamma^0) + Ch^3$. We leave the
s to the reader.
roof of semiclassical asymptotics true under the weaker condition that $\mathcal{F}^{BCS}(\Gamma) \leq \mathcal{F}^{BCS}(\Gamma^0) + Ch^3$. We leave the
details to the reader.
6. Proof of semiclassical asymptotics
In this section we shall sketch the proofs of Theorems 3.1 and 3.2 co

6. Proof of semiclassical asymptotics

6. Proof of semic
In this section we show
semiclassical asympton % semiclassical asymptotics. We shall skip some technical details and refer to [7] for a thorough discussion.
 6.1. Preliminaries

It will be convenient to use the following abbreviations **6.1. Preliminaries**

It will be convenient to use the following abbreviations
 $k = -h^2 \nabla^2 - \mu + h^2 W(x), \qquad k_0 = -h^2 \nabla^2 - \mu.$ (6.1)

6.1. Preliminaries

$$
k = -h^2 \nabla^2 - \mu + h^2 W(x), \qquad k_0 = -h^2 \nabla^2 - \mu. \tag{6.1}
$$

6.1. Preliminaries
It will be convenient to $k = -1$
We will frequently have $k = -h^2 \nabla^2 - \mu + h^2 W(x)$, $k_0 = -$
We will frequently have to bound various norms of the
in the contour Γ defined by $\text{Im } z = \pm \pi/(2\beta)$ for $\beta > 0$
bounds separately.
For $p \ge 1$, we define the *p*-norm of a periodic op = $-h^2\nabla^2 - \mu + h^2W(x)$, $k_0 = -h^2\nabla^2 - \mu$. (6.1)

t have to bound various norms of the resolvents $(z - k)^{-1}$ for z

defined by Im $z = \pm \pi/(2\beta)$ for $\beta > 0$. We state these auxiliary

c define the p-norm of a periodic oper We will frequently have to bound various norms of the resolvents $(z - k)^{-1}$ for z
in the contour Γ defined by $\text{Im } z = \pm \pi/(2\beta)$ for $\beta > 0$. We state these auxiliary
bounds separately.
For $p \ge 1$, we define the *p*-no

$$
||A||_p = (\text{Tr}|A|^p)^{1/p} \tag{6.2}
$$

in the contour Γ defined by Im $z = \pm \pi/(2\beta)$ for $\beta > 0$. We state these auxiliary
bounds separately.
For $p \ge 1$, we define the *p*-norm of a periodic operator A by
 $||A||_p = (\text{Tr }|A|^p)^{1/p}$ (6.2)
where Tr stands again for For $p \ge 1$, we define the *p*-norm of a periodic operator *A* by
 $||A||_p = (\text{Tr } |A|^p)^{1/p}$

where Tr stands again for the trace per unit volume. We note the

multiplier $A(-ih\nabla)$, these norms are given as
 $||A(-ih\nabla)||_p = h^{-1/p$

$$
||A||_p = (\text{Tr } |A|^p)^{1/p} \tag{6.2}
$$

where Tr stands again for the trace per unit volume. We note that for a Fourier
multiplier $A(-ih\nabla)$, these norms are given as

$$
||A(-ih\nabla)||_p = h^{-1/p} \left(\int_{\mathbb{R}} |A(q)|^p \frac{dq}{2\pi}\right)^{1/p}.
$$

$$
(6.3)
$$
The usual operator norm will be denoted by $||A||_{\infty}$.
Lemma 6.1. For $z = t \pm i\pi/(2\beta)$ and all sufficiently small h one has

$$
||(z - k)^{-1}||_p \le C h^{-1/p} \times \left\{\begin{array}{ll} t^{-1/(2p)} & \text{for } t \gg 1 \\ |t|^{-1+1/(2p)} & \text{for } t \ll -1 \end{array} \right. \quad \text{if } 1 \le p \le \infty, \quad (6.4)
$$

$$
||A(-ih\nabla)||_p = h^{-1/p} \left(\int_{\mathbb{R}} |A(q)|^p \frac{d\mathbf{x}}{2\pi}\right) \tag{6.3}
$$

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Lemma 6.1. For $z = t \pm i\pi/(2\beta)$ and all sufficiently small h one has

$$
||(z - k)^{-1}||_p \le C h^{-1/p} \times \left\{\begin{array}{ll} t^{-1/(2p)} & \text{for } t \gg 1 \\ |t|^{-1+1/(2p)} & \text{for } t \ll -1 \end{array} \quad \text{if } 1 \le p \le \infty, \quad (6.4)
$$

as well as

$$
||(z - k)^{-1}||_{\infty} \le C \times \left\{\begin{array}{ll} 1 & \text{for } t \gg 1 \\ |t|^{-1} & \text{for } t \ll -1 \end{array} \right. \tag{6.5}
$$

as well as

$$
C h^{-1/p} \times \left\{ \begin{array}{ll} t^{(1-p)/p} & \text{for } t \ll -1 \\ |t|^{-1+1/(2p)} & \text{for } t \ll -1 \end{array} \right. \quad \text{if } 1 \le p \le \infty, \quad (6.4)
$$

$$
\left\| (z-k)^{-1} \right\|_{\infty} \le C \times \left\{ \begin{array}{ll} 1 & \text{for } t \gg 1 \\ |t|^{-1} & \text{for } t \ll -1 \end{array} \right. \quad (6.5)
$$

es are easily derived with k_0 instead of k by evaluating the ral. Since the spectra of k and k_0 agree up to $O(h^2)$ the same *Proof.* The estimates are easily derived with k_0 instead of k by evaluating the The estimates are easily derived with k_0 instead of k by evaluating the onding integral. Since the spectra of k and k_0 agree up to $O(h^2)$ the same hold for k .
 Oof of Theorem 3.1 and and and and and and and corresponding integral. Since the spectra of k and k_0 agree up to $O(h^2)$ the same
bounds hold for k .
6.2. Proof of Theorem 3.1
The function f in (3.2) is analytic in the strip $|\text{Im } z| < \pi$, and we can write bounds hold for k. \Box
 6.2. Proof of Theorem 3.1

The function f in (3.2) is analytic in the strip $|\text{Im } z| < \pi$, and we can write

6.2. Proof of Theorem 3.1

$$
f(\beta H_{\Delta}) - f(\beta H_0) = \frac{1}{2\pi i} \int_{\Gamma} f(\beta z) \left[\frac{1}{z - H_{\Delta}} - \frac{1}{z - H_0} \right] dz,
$$

The function f in (3.2) is analytic in the strip $|\text{Im } z| < \pi$, and we can write
 $f(\beta H_{\Delta}) - f(\beta H_0) = \frac{1}{2\pi i} \int_{\Gamma} f(\beta z) \left[\frac{1}{z - H_{\Delta}} - \frac{1}{z - H_0} \right] dz$,

where Γ is the contour $z = r \pm i \frac{\pi}{2\beta}$, $r \in \mathbb{R}$. We empha $(\beta H_{\Delta}) - f(\beta H_0) = \frac{1}{2\pi i} \int_{\Gamma} f(\beta z)$

he contour $z = r \pm i \frac{\pi}{2\beta}$, $r \in \mathbb{R}$. N

s not true for the operators $f(\beta H_0)$

ation from infinity), but only for

im that $\frac{1}{1}$
 $\frac{1}{1}$ —
h:
) where Γ is the contour $z = r \pm i \frac{\pi}{2\beta}$
resentation is not true for the operation of a contribution from infinity), buy
We claim that
 $[f(\beta H_{\Delta})]_{11} =$ resentation is not true for the operators $f(\beta H_{\Delta})$ and $f(\beta H_0)$ separately (because of a contribution from infinity), but only for their difference. resentation is not true for the operators $f(\beta H_{\Delta})$ and $f(\beta H_0)$ separately (because
of a contribution from infinity), but only for their difference.
We claim that
 $[f(\beta H_{\Delta})]_{11} = \overline{[f(\beta H_{\Delta})]_{22}} - \beta \overline{[H_{\Delta}]_{22}}$. (6

$$
[f(\beta H_{\Delta})]_{11} = \overline{[f(\beta H_{\Delta})]_{22}} - \beta \overline{[H_{\Delta}]_{22}}.
$$
\n(6.6)

We claim that
 $[f(\beta H_{\Delta})]_{11} = \overline{[f(\beta H_{\Delta})]_{22}} - \beta \overline{[H_{\Delta}]_{22}}$.

Recall that $[\cdot]_{ij}$ denotes the *ij* element of an operator-valued

(6.6), we introduce the unitary matrix Il that $[\cdot]_{ij}$ derived that the introduce $[f(\beta H_{\Delta})]_{11} = [f(\beta H_{\Delta})]_{22} - \beta [H_{\Delta}]_{22}.$ (6.6)

ces the *ij* element of an operator-valued 2 × 2 matrix. To see

e unitary matrix
 $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Recall that $[\cdot]_{ij}$ denotes the *ij* element of an operator-valued 2 × 2 matrix. To see (6.6), we introduce the unitary matrix
 $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and note that

$$
U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

and note that

$$
[f(\beta H_{\Delta})]_{11} = -[Uf(\beta H_{\Delta})]
$$
On the other hand, $UH_{\Delta}U = -\overline{H_{\Delta}}$, which implies

$$
\langle -1 \quad 0 \rangle
$$

$$
[f(\beta H_{\Delta})]_{11} = -[Uf(\beta H_{\Delta})U]_{22}.
$$

$$
\Delta U = -\overline{H_{\Delta}},
$$
 which implies that

On the other

$$
[f(\beta H_{\Delta})]_{11} = -[Uf(\beta H_{\Delta})U]_{22}.
$$

\n
$$
H_{\Delta}U = -\overline{H_{\Delta}},
$$
 which implies that
\n
$$
Uf(\beta H_{\Delta})U = f(-\beta \overline{H_{\Delta}}) = \overline{f(-\beta H_{\Delta})}.
$$

\nfollows from the fact that $f(-z) = f(z)$
\n8) and the corresponding formula for

On the other hand, $UH_{\Delta}U = -H_{\Delta}$, which implies that
 $Uf(\beta H_{\Delta})U = f(-\beta \overline{H_{\Delta}}) = \overline{f(-\beta H_{\Delta})}$

The claim (6.6) now follows from the fact that $f(-z)$ =

Subtracting (6.6) and the corresponding formula
 H_{Δ} and H $(\beta H_{\Delta})U = f(-\beta H_{\Delta}) = f(-\beta H_{\Delta}).$
ows from the fact that $f(-z) = f(z)$
and the corresponding formula for
a the diagonal we find that the two
omplex conjugates of each other. Si The claim (6.6) now follows from the fact that $f(-z) = f(z) - z$.

Subtracting (6.6) and the corresponding formula for H_0 and H_0 coincide on the diagonal we find that the two dia $f(\beta H_{\Delta}) - f(\beta H_0)$ are complex conjugates Subtracting (6.6) and the corresponding formula for H_0 and noting that
and H_0 coincide on the diagonal we find that the two diagonal entries of
 I_{Δ}) - $f(\beta H_0)$ are complex conjugates of each other. Since their $_{H_\Delta}$ and H_0 coincide on the diagonal we find that the two diagonal entries of H_{Δ}) - $f(\beta H_0)$ are complex conjugates of each other. Since their trace is real onclude that

Tr $[f(\beta H_{\Delta}) - f(\beta H_0)] = \frac{1}{\pi i} \int_{\Gamma} f(\beta z) \operatorname{Tr$ f

$$
(\beta H_{\Delta}) - f(\beta H_0)
$$
 are complex conjugates of each other. Since their trace is real
re conclude that

$$
\text{Tr}\left[f(\beta H_{\Delta}) - f(\beta H_0)\right] = \frac{1}{\pi i} \int_{\Gamma} f(\beta z) \text{ Tr}\left[\frac{1}{z - H_{\Delta}} - \frac{1}{z - H_0}\right]_{11} dz.
$$

For technical details concerning the interchange of the trace and the integral we
refer to [7].)
The resolvent identity and the fact that

Tr $[f(\beta H_{\Delta})]$
(For technical derefer to [7].) $\frac{1}{1}$

 $r_{\rm F1}$.

refer to [7].)
The resolvent identity and the fact that

$$
\delta := H_{\Delta} - H_0 = -h \left(\begin{array}{cc} 0 & \psi(x) \\ \overline{\psi(x)} & 0 \end{array} \right)
$$
(6.7)

80 R.L. Frank, C. Hainzl, R. Seiringer and J.P. Solovej
is off-diagonal (as an operator-valued 2×2 matrix) implies that

$$
\begin{aligned}\n\text{Tr}\left[\frac{1}{z - H_{\Delta}} - \frac{1}{z - H_0}\right]_{11} &= \text{Tr}\left[\frac{1}{z - H_0} \left(\delta \frac{1}{z - H_0}\right)^2\right]_{11} \\
&+ \text{Tr}\left[\frac{1}{z - H_0} \left(\delta \frac{1}{z - H_0}\right)^4\right]_{11} + \text{Tr}\left[\frac{1}{z - H_{\Delta}} \left(\delta \frac{1}{z - H_0}\right)^6\right]_{11} \\
&=: I_1 + I_2 + I_3. \\
\text{In the following we shall prove that} \\
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_1 dz &= -\frac{h\beta^2}{2} ||\psi||_2^2 \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi}\n\end{aligned}
$$

$$
=: I_1 + I_2 + I_3.
$$

In the following we shall prove that

$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_1 dz = -\frac{h\beta^2}{2} ||\psi||_2^2 \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi} + \frac{h^3 \beta^3}{8} ||\psi'||_2^2 \int_{\mathbb{R}} (g_1(\beta(q^2 - \mu)) + 2\beta q^2 g_2(\beta(q^2 - \mu))) \frac{dq}{2\pi} + \frac{h^3 \beta^3}{2} \langle \psi | W | \psi \rangle \int_{\mathbb{R}} g_1(\beta(q^2 - \mu)) \frac{dq}{2\pi} + O(h^5) ||\psi||_{H^2}^2,
$$
(6.8)

$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_2 dz = \frac{h^3 \beta^3}{8} ||\psi||_4^4 \int_{\mathbb{R}} \frac{g_1(\beta(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi} + O(h^5) ||\psi||_{H^1}^3 ||\psi||_{H^2}
$$
(6.9)
and

$$
+\frac{1}{2}\langle\psi|W|\psi\rangle\int_{\mathbb{R}}g_{1}(\beta(q^{2}-\mu))\frac{1}{2\pi} +O(h^{5})\|\psi\|_{H^{2}}^{2},
$$
\n
$$
+O(h^{5})\|\psi\|_{H^{2}}^{2},
$$
\n(6.8)\n
$$
\frac{1}{\pi i}\int_{\Gamma}f(\beta z)I_{2} dz = \frac{h^{3}\beta^{3}}{8}\|\psi\|_{4}^{4}\int_{\mathbb{R}}\frac{g_{1}(\beta(q^{2}-\mu))}{q^{2}-\mu}\frac{dq}{2\pi} +O(h^{5})\|\psi\|_{H^{1}}^{3}\|\psi\|_{H^{2}} \quad (6.9)
$$
\nand\n
$$
\frac{1}{\pi i}\int_{\Gamma}f(\beta z)I_{3} dz = O(h^{5})\|\psi\|_{H^{1}}^{6}.
$$
\nThis will clearly prove (3.6). We will treat the three terms I_{3} , I_{2} and I_{1} (in this order) separately.

$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_3 dz = O(h^5) ||\psi||_{H^1}^6.
$$
\n(6.10)

 $\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_3 dz = O(h^5) ||\psi||_{H^1}^6$. (6.10)

This will clearly prove (3.6). We will treat the three terms I_3 , I_2 and I_1 (in this

order) separately.
 I₃: With the notation k introduced in (6.1) at the $\frac{1}{\sqrt{2}}$
3.

 I_3 : With the
we have
 $I_3 = \text{Tr} \left[\frac{1}{z} \right]$
Using Hölder's in I_3 : With the notation k introduced in (6.1) at the beginning of this section,

This will clearly prove (3.6). We will treat the three terms
$$
I_3
$$
, I_2 and I_1 (in this order) separately.
\n***I*₃**: With the notation *k* introduced in (6.1) at the beginning of this section,
\nwe have
\n
$$
I_3 = \text{Tr}\left[\frac{1}{z - H_{\Delta}}\right]_{11} \Delta \frac{1}{z + k} \Delta^{\dagger} \frac{1}{z - k} \Delta \frac{1}{z + k} \Delta^{\dagger} \frac{1}{z - k} \Delta \frac{1}{z + k} \Delta^{\dagger} \frac{1}{z - k}.
$$
\nUsing Hölder's inequality for the trace per unit volume (see [7]) and the fact that
\n $|z - H_{\Delta}| \ge \pi/(2\beta)$, we get

 $I_3 = \text{Tr} \left[\frac{1}{z - 1} \right]$
Using Hölder's inection $|z - H_{\Delta}| \ge \pi/(2\beta)$ $\Delta \frac{1}{z+k} \Delta^{\dagger} \frac{1}{z-k} \Delta \frac{1}{z+k} \Delta^{\dagger} \frac{1}{z-k} \Delta \frac{1}{z+k} \Delta^{\dagger} \frac{1}{z-k}$
for the trace per unit volume (see [7]) and the
t
 $\frac{\beta}{\Gamma} h^6 ||\psi||^6_{\infty} ||(z-k)^{-1}||^3_6 ||(z+k)^{-1}||^3_6$. $+k$
|) a:
 \cdot $|z-H_{\Delta}| \ge \pi/(2\beta)$, we get
 $|I_3| \le \frac{2\beta}{\pi} h^6 ||\psi||_{\infty}^6 ||(z-k)^{-1}||_6^3 ||(z+k)^{-1}||_6^3$.
Together with (6.4), this yields
 Ch^5 $|z - H_{\Delta}| \ge \pi/$

$$
-H_{\Delta}|_{11} - z + k - z - k - z + k - z - k - z +
$$

equality for the trace per unit volume (see [7])
), we get

$$
|I_3| \leq \frac{2\beta}{\pi} h^6 \|\psi\|_{\infty}^6 \|(z - k)^{-1}\|_6^3 \|(z + k)^{-1}\|_6^3.
$$

4), this yields

$$
|I_3| \leq \frac{Ch^5}{1 + |z|^3} \|\psi\|_{\infty}^6.
$$

stant to get a decay faster than $|z|^{-2}$, since we

Together with (6.4) , this yields

$$
|I_3| \le \frac{Ch^5}{1+|z|^3} \|\psi\|_{\infty}^6.
$$

There it was important to get a
 I_3 against the function f which y Sobolev inequalities we have $1 + |z|$
faster
aves line Here it was important to get a decay faster than $|z|^{-2}$, since we need to integrate I_3 against the function f which behaves linearly at $-\infty$. Since $||\psi||_{\infty} \leq C ||\psi||_{H^1}$ by Sobolev inequalities we have complete $^{-2}$

 I_2 : We continue with

time with
\n
$$
I_2 = \text{Tr} \frac{1}{z - k} \Delta \frac{1}{z + k} \Delta^{\dagger} \frac{1}{z - k} \Delta \frac{1}{z + k} \Delta^{\dagger} \frac{1}{z - k}.
$$
\nidentity we have

By the resolvent identity we have

$$
= \text{Tr} \frac{1}{z - k} \Delta \frac{1}{z + k} \Delta^{\dagger} \frac{1}{z - k} \Delta \frac{1}{z + k} \Delta^{\dagger} \frac{1}{z - k}.
$$
\nentity we have

\n
$$
\frac{1}{z - k} = \frac{1}{z - k_0} + \frac{1}{z - k_0} h^2 W \frac{1}{z - k}.
$$
\n(6.11)

\nove, we can bound

\n
$$
\begin{array}{ccc}\n1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1\n\end{array}
$$

$$
\frac{1}{z-k} = \frac{1}{z-k_0} + \frac{1}{z-k_0} h^2 W \frac{1}{z-k}.
$$
\nUsing Hölder as above, we can bound

\n
$$
\left| \text{Tr} \left(\frac{1}{z-k} - \frac{1}{z-k_0} \right) \Delta \frac{1}{z+k} \Delta^{\dagger} \frac{1}{z-k} \Delta \frac{1}{z+k} \Delta^{\dagger} \frac{1}{z-k} \right|
$$
\n
$$
\leq h^6 \|W\|_{\infty} \left\| (z-k_0)^{-1} \right\|_{\infty} \| \psi \|_{\infty}^4 \| (z-k)^{-1} \|_{3}^3 \| (z+k)^{-1} \|_{\infty}^2.
$$
\n(6.12)

\nBy (6.4) and (6.5) this is bounded by $Ch^5 \| \psi \|_{H^1(C)}^4 (1 + |z|^{5/2})^{-1}$. What we effectively have achieved for this error is, therefore, to replace one factor of $(z-k_0)^{-1}$ in I_2 by a factor of $(z-k_0)^{-1}$.

\nIn particular, the corresponding factor $(z-k_0)^{-1}$.

 $+k$
 \parallel_{∞}
ed b
r is
pro + k)⁻¹
(c)⁽c) re $(z - k_0)^{-1} \|\infty \|\psi\|_{\infty}^{\ast} \|(z - k)^{-1}\|_{3}^{3} \|(z + k)^{-1}\|_{\infty}^2$. (6.12)

is is bounded by $Ch^5 \|\psi\|_{H^1(\mathcal{C})}^4 (1 + |z|^{5/2})^{-1}$. What we effector

or this error is, therefore, to replace one factor of $(z - k)^{-1}$
 $-k_0)^{-1}$
 By (6.4) and (6.5) this is bounded by $Ch^5 \|\psi\|_{H^1(\mathcal{C})}^4 (1+|z|^{5/2})$
tively have achieved for this error is, therefore, to replace one
in I_2 by a factor of $(z - k_0)^{-1}$
In exactly the same way we proceed with the re $(1+|z|^{5/2})^{-1}$. What we effec-
eplace one factor of $(z-k)^{-1}$
emaining factors $(z-k)^{-1}$ and
now be replaced by k_0 in the
fect the bounds.
quals

tively have achieved for this error is, therefore, to replace one factor of $(z - k)^{-1}$
in I_2 by a factor of $(z - k_0)^{-1}$
In exactly the same way we proceed with the remaining factors $(z - k)^{-1}$ and
 $(z + k)^{-1}$ in I_2 . The in I_2 by a factor of $(z - k_0)^{-1}$
In exactly the same way $(z + k)^{-1}$ in I_2 . The only different terms we have already treated
The final result is that ($\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac$ In exactly the same way we proceed with the remaining factors $(z-k)^{-1}$ and
 $(k)^{-1}$ in I_2 . The only difference is that k might now be replaced by k_0 in the

s we have already treated, but this does not effect the bo

$$
(z+k)^{-1}
$$
 in I_2 . The only difference is that k might now be replaced by k_0 in the
terms we have already treated, but this does not effect the bounds.
The final result is that $(\pi i)^{-1} \int_{\Gamma} f(\beta z) I_2 dz$ equals

$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) \operatorname{Tr} \left[\frac{1}{z - k_0} \Delta \frac{1}{z + k_0} \Delta^{\dagger} \frac{1}{z - k_0} \Delta \frac{1}{z + k_0} \Delta^{\dagger} \frac{1}{z - k_0} \right] dz + O(h^5) ||\psi||_{H^1}^4,
$$
and it remains to compute the asymptotics of the integral.
Let us indicate how to perform the trace per unit volume Tr[...]. In terms
of integrals the trace can be written as

$$
\frac{h^4}{(2\pi)^4} \int_0^1 dx_1 \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_3 \int_{\mathbb{R}} dx_4 \int_{\mathbb{R}} dp_1 dp_2 dp_3 dp_4 \overline{\psi(x_1)} \psi(x_2) \overline{\psi(x_3)} \psi(x_4)
$$

Let us indicate how to perform the trace per unit volume Tr[...]. In terms
of integrals the trace can be written as

$$
\frac{h^4}{(2\pi)^4} \int_0^1 dx_1 \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_3 \int_{\mathbb{R}} dx_4 \int_{\mathbb{R}^4} dp_1 dp_2 dp_3 dp_4 \overline{\psi(x_1)} \psi(x_2) \overline{\psi(x_3)} \psi(x_4)
$$

$$
\times \frac{e^{ip_1(x_1-x_2)}}{(z-(h^2p_1^2-\mu))^2} \frac{e^{ip_2(x_2-x_3)}}{z+(h^2p_2^2-\mu)} \frac{e^{ip_3(x_3-x_4)}}{z-(h^2p_3^2-\mu)} \frac{e^{ip_4(x_4-x_1)}}{z+(h^2p_4^2-\mu)}.
$$
(6.13)
Since ψ is periodic with period one we have

$$
\psi(x_j) = \sum_{l_j \in 2\pi\mathbb{Z}} \hat{\psi}(l_j) e^{ix_j l_j}.
$$
We insert this into the above integral and perform the integrals over x_2, x_3, x_4 .
This leads to δ -distributions such that we can subsequently perform the integrals

$$
\psi(x_j) = \sum_{l_j \in 2\pi \mathbb{Z}} \hat{\psi}(l_j) e^{ix_j l_j}.
$$

ove integral and perform to
s such that we can subsec

 $\frac{1}{2}$ $\frac{1}{2}$ Since ψ is periodic with period one we have
 $\psi(x_j) = \sum_{l_j \in 2\pi \mathbb{Z}} \hat{\psi}$

We insert this into the above integral and

This leads to δ -distributions such that we cover p_2, p_3, p_4 , as well as the integral over x $(x_j) = \sum_{l_j \in 2\pi \mathbb{Z}} \hat{\psi}(l_j) e^{ix_j l_j}.$

e integral and perform t

such that we can subseque integral over x_1 . In this $\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1$ over p_2, p_3, p_4 , as well as the integral over x_1 . In this way we obtain

We insert this into the above integral and perform the integrals over
$$
x_2, x_3, x_4
$$
.
This leads to δ -distributions such that we can subsequently perform the integrals
over p_2, p_3, p_4 , as well as the integral over x_1 . In this way we obtain

$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) \text{Tr} \left[\frac{1}{z - k_0} \Delta \frac{1}{z + k_0} \Delta^{\dagger} \frac{1}{z - k_0} \Delta \frac{1}{z + k_0} \Delta^{\dagger} \frac{1}{z - k_0} \right] dz
$$

$$
= h^3 \sum_{p_1, p_2, p_3 \in 2\pi \mathbb{Z}} \widehat{\psi}(p_1) \widehat{\psi^*}(p_2) \widehat{\psi}(p_3) \widehat{\psi^*}(-p_1 - p_2 - p_3) F(hp_1, hp_2, hp_3)
$$

with
\n
$$
F(p_1, p_2, p_3) = \frac{\beta^4}{\pi i} \int_{\Gamma} dz f(\beta z) \int_{\mathbb{R}} \frac{dq}{2\pi} \frac{1}{(z - \beta((q + p_1 + p_2 + p_3)^2 + \mu))^2}
$$
\n
$$
\times \frac{1}{z + \beta((q + p_1 + p_2)^2 - \mu)} \frac{1}{z - \beta((q + p_1)^2 - \mu)} \frac{1}{z + \beta(q^2 - \mu)}.
$$
\nThe leading behavior is given by
\n
$$
F(0, 0, 0) \sum_{\nu=0}^{\infty} \widehat{\psi}(p_1) \widehat{\psi^*}(p_2) \widehat{\psi}(p_3) \widehat{\psi^*}(-p_1 - p_2 - p_3) = F(0, 0, 0) \|\psi\|_4^4.
$$

$$
\times \frac{1}{z + \beta((q + p_1 + p_2)^2 - \mu)} \overline{z - \beta((q + p_1)^2 - \mu)} \overline{z + \beta(q^2 - \mu)}
$$

The leading behavior is given by

$$
F(0,0,0) \sum_{p_1, p_2, p_3 \in 2\pi\mathbb{Z}} \widehat{\psi}(p_1) \widehat{\psi^*}(p_2) \widehat{\psi}(p_3) \widehat{\psi^*}(-p_1 - p_2 - p_3) = F(0,0,0) \|\psi\|_4^4.
$$

The integral $F(0,0,0)$ can be calculated explicitly and we obtain

$$
F(0,0,0) = \frac{\beta^3}{8} \int_{\mathbb{R}} \frac{g_1(\beta(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi}
$$

The integral
$$
F(0,0,0)
$$
 can be calculated explicitly and we obtain
\n
$$
F(0,0,0) = \frac{\beta^3}{8} \int_{\mathbb{R}} \frac{g_1(\beta(q^2 - \mu))}{q^2 - \mu} \frac{dq}{2\pi}
$$
\nwith g_1 from (3.4). In order to estimate the remainder we use the
\n
$$
|F(p_1, p_2, p_3) - F(0,0,0)| \le \text{const} (p_1^2 + p_2^2 + p_3^2)
$$
\nUsing Schwarz and Hölder we can bound

4). In order to estimate the remainder we use the
\n
$$
|F(p_1, p_2, p_3) - F(0, 0, 0)| \le \text{const} (p_1^2 + p_2^2 + p_3^2)
$$
.
\nand Hölder we can bound

with
$$
g_1
$$
 from (3.4). In order to estimate the remainder we use the fact that [7]
\n
$$
|F(p_1, p_2, p_3) - F(0, 0, 0)| \le \text{const} (p_1^2 + p_2^2 + p_3^2).
$$
\nUsing Schwarz and Hölder we can bound
\n
$$
\sum_{p_1, p_2, p_3 \in 2\pi \mathbb{Z}} p_1^2 \left| \hat{\psi}^*(p_1) \hat{\psi}^*(p_2) \hat{\psi}(p_3) \hat{\psi}(-p_1 - p_2 - p_3) \right| \le \text{const} \|\psi\|_{H^2} \|\psi\|_{H^1}^3
$$
\nand equally with p_1^2 replaced by p_2^2 and p_3^2 . Hence we conclude that

$$
\sum_{p_1, p_2, p_3 \in 2\pi \mathbb{Z}} p_1^2 |\psi^*(p_1)\psi^*(p_2)\psi(p_3)\psi(-p_1 - p_2 - p_3)| \le \text{const } ||\psi||_{H^2} ||\psi||_{H^1}^3
$$
\nand equally with p_1^2 replaced by p_2^2 and p_3^2 . Hence we conclude that\n
$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_2 dz
$$
\n
$$
= h^3 \sum_{p_1, p_2, p_3 \in 2\pi \mathbb{Z}} \widehat{\psi}(p_1)\widehat{\psi^*(p_2)}\widehat{\psi}(p_3)\widehat{\psi^*(p_1 - p_2 - p_3)}F(hp_1, hp_2, hp_3) + O(h^5)||\psi||_{H^1}^4
$$
\n
$$
= h^3F(0, 0, 0)||\psi||_4^4 + O(h^5)||\psi||_{H^1}^3 ||\psi||_{H^2}.
$$
\nThis is what we claimed in (6.9).\n
$$
I_1: \text{ Finally, we examine the contribution of}
$$
\n
$$
I_1 = \text{Tr}\left[\frac{1}{z - k}\Delta \frac{1}{z + k}\Delta^{\dagger} \frac{1}{z - k}\right].
$$

 \boldsymbol{I}_1 : Finally, we examine the
 $I_1 = \text{Tr} \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$.
Using the resolvent identity (6.1). I_1

$$
I_1 = \text{Tr}\left[\frac{1}{z - k} \Delta \frac{1}{z + k} \Delta^{\dagger} \frac{1}{z - k}\right].
$$

the resolvent identity (6.11) we can write $I_1 = I_1^a +$

 $\frac{1}{2}$, where $\frac{1}{2}$

$$
I_1 = \text{Tr}\left[\frac{1}{z-k}\Delta\frac{1}{z+k}\Delta^{\dagger}\frac{1}{z-k}\right].
$$

Using the resolvent identity (6.11) we can write $I_1 = I_1^a + I_1^b + I_1^c$

$$
I_1^a = \text{Tr}\left[\frac{1}{z-k_0}\Delta\frac{1}{z+k_0}\Delta^{\dagger}\frac{1}{z-k_0}\right]
$$

and

$$
I_1^b = \text{Tr}\left[\frac{1}{z - k_0}(k - k_0)\frac{1}{z - k_0}\Delta \frac{1}{z + k_0}\Delta^{\dagger} \frac{1}{z - k_0} + \frac{1}{z - k_0}\Delta \frac{1}{z + k_0}(k_0 - k)\frac{1}{z + k_0}\Delta^{\dagger} \frac{1}{z - k_0} + \frac{1}{z - k_0}\Delta \frac{1}{z + k_0}\Delta^{\dagger} \frac{1}{z - k_0}(k - k_0)\frac{1}{z - k_0}\right].
$$

consists of the rest. We claim that

$$
z) I_1^a dz = -\frac{h\beta^2}{2} ||\psi||_2^2 \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi}
$$

$$
+\frac{1}{z-k_0}\Delta \frac{1}{z+k_0}\Delta^{\dagger} \frac{1}{z-k_0}(k-k_0)\frac{1}{z-k_0}
$$
\nThe part I_1^c consists of the rest. We claim that\n
$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_1^a dz = -\frac{h\beta^2}{2} ||\psi||_2^2 \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi} \\
+ \frac{h^3\beta^3}{8} ||\psi'||_2^2 \int_{\mathbb{R}} (g_1(\beta(q^2 - \mu)) + 2\beta q^2 g_2(\beta(q^2 - \mu))) \frac{dq}{2\pi} \\
+ O(h^5) ||\psi||_{H^2}^2, \qquad (6.14)
$$
\n
$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_1^b dz = \frac{h^3\beta^3}{2} \langle \psi | W | \psi \rangle \int_{\mathbb{R}} g_1(\beta(q^2 - \mu)) \frac{dq}{2\pi} + O(h^5) ||\psi||_{H^2} ||\psi||_{H^1}
$$
\n(6.15)

$$
+\frac{1}{8}\|\psi'\|_{2}^{2}\int_{\mathbb{R}}\left(g_{1}(\beta(q^{2}-\mu))+2\beta q^{2}g_{2}(\beta(q^{2}-\mu))\right)\frac{1}{2\pi} +O(h^{5})\|\psi\|_{H^{2}}^{2},\tag{6.14}
$$
\n
$$
\frac{1}{\pi i}\int_{\Gamma}f(\beta z) I_{1}^{b}dz = \frac{h^{3}\beta^{3}}{2}\langle\psi|W|\psi\rangle\int_{\mathbb{R}}g_{1}(\beta(q^{2}-\mu))\frac{dq}{2\pi}+O(h^{5})\|\psi\|_{H^{2}}\|\psi\|_{H^{1}}\tag{6.15}
$$
\nd\n
$$
\frac{1}{\pi i}\int_{\Gamma}f(\beta z) I_{1}^{c}dz = O(h^{5})\|\psi\|_{H^{1}}^{2}.\tag{6.16}
$$
\nearly, this will imply (6.8). We begin with I_{1}^{c} . These terms contain at least five resolvents, where at least

$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_1^c dz = O(h^5) \|\psi\|_{H^1}^2.
$$
\n(6.16)
\n(6.8).
\nThese terms contain at least five resolvents, where at least
\n $\ln (z - k_{\#})^{-1}$ and at least one term of the form $(z + k_{\#})^{-1}$.

Clearly, this will imply (6.8) .

 $\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_1^c dz = O(h^5) ||\psi||_{H^1}^2.$ (6.16)
Clearly, this will imply (6.8).
We begin with I_1^c . These terms contain at least five resolvents, where at least
two terms are of the form $(z - k_{\#})^{-1}$ and at least $\frac{1}{\pi}$ 6 Maple We begin with I_1^c . These terms contain at least five resolvents, where at least
two terms are of the form $(z - k_{\#})^{-1}$ and at least one term of the form $(z + k_{\#})^{-1}$.
(Here $k_{\#}$ stands for any of the operators k or (Here $k_{\#}$ stands for any of the operators k or k_0 .) Moreover, they contain at least two factors of $k - k_0$. The terms are either of the type

$$
A = \text{Tr}\,\frac{1}{z - k_0}(k - k_0)\frac{1}{z - k}\Delta\frac{1}{z + k_0}(k_0 - k)\frac{1}{z + k}\Delta^{\dagger}\frac{1}{z - k_0}
$$
(6.17)

$$
A = \text{Tr}\,\frac{1}{z - k_0}(k - k_0)\frac{1}{z - k}\Delta\frac{1}{z + k_0}(k_0 - k)\frac{1}{z + k}\Delta^{\dagger}\frac{1}{z - k_0}
$$
(6.17)
(at least three minus signs) or of the type

$$
B = \text{Tr}\,\frac{1}{z - k_0}\Delta\frac{1}{z + k_0}(k_0 - k)\frac{1}{z + k_0}(k_0 - k)\frac{1}{z + k}\Delta^{\dagger}\frac{1}{z - k}
$$
(6.18)
(only two minus signs). Terms of the first type we bound by

$$
|A| \leq Ch^6 \|W\|_{\infty}^2 \|\psi\|_{\infty}^2 \|(z - k_0)^{-1}\|_{\infty}^2 \|(z + k_0)^{-1}\|_3 \|(z + k)^{-1}\|_3 \|(z - k)^{-1}\|_3.
$$
By (6.4) and (6.5) this can be estimated by $Ch^5 |z|^{-2+1/6}$ if Re $z \geq 1$ and by

$$
|A| \le Ch^6 \|W\|_{\infty}^2 \|\psi\|_{\infty}^2 \|(z - k_0)^{-1}\|_{\infty}^2 \|(z + k_0)^{-1}\|_3 \|(z + k)^{-1}\|_3 \|(z - k)^{-1}\|_3.
$$

= Tr $\frac{1}{z-k_0} \Delta \frac{1}{z+k_0} (k_0 - k) \frac{1}{z+k_0} (k_0 - k) \frac{1}{z+k} \Delta^{\dagger} \frac{1}{z-k}$ (6.18)

minus signs). Terms of the first type we bound by
 $h^6 \|W\|_{\infty}^2 \|\psi\|_{\infty}^2 \|(z - k_0)^{-1}\|_{\infty}^2 \|(z + k_0)^{-1}\|_3 \|(z + k)^{-1}\|_3 \|(z - k)^{-1}\|_3$ + k_0 ^{(co} $z + k_0$)
erms of the first typ
 $(z - k_0)^{-1} ||^2_{\infty} ||(z +$
an be estimated b
l. This bound is fin
be bounded simila $+k$

ad b
 $z +$
 $z+1$

ateg

acin $-2+1/6$ $Ch^{5}|z|^{-2-1/6}$

 $|A| \leq Ch^6 ||W||_{\infty}^2 ||\psi||_{\infty}^2 ||(z - k_0)^{-1}||_{\infty}^2 ||(z + k_0)^{-1}||_3 ||(z + k_0)^2$
By (6.4) and (6.5) this can be estimated by $Ch^5 |z|^{-2+1/6}$
 $Ch^5 |z|^{-2-1/6}$ if Re $z \leq -1$. This bound is finite when integrificant terms of type $(z - k_0)^{-1} \|{}_{\infty}^2 \| (z + k_0)^{-1} \|_3 \| (z + k)^{-1} \|_3 \| (z - k)$

can be estimated by $Ch^5 |z|^{-2+1/6}$ if Re $z \ge 1$

1. This bound is finite when integrated against f

i be bounded similarly by replacing z by $-z$. Ind
 $= 0$, we By (6.4) and (6.5) this can be estimated by $Ch^5|z|^{-2+1/6}$ if Re $z \ge 1$ and by $Ch^5|z|^{-2-1/6}$ if Re $z \le -1$. This bound is finite when integrated against $f(\beta z)$.
Terms of type B can be bounded similarly by replacing z if Re $z \le -1$. This bound is finite when integrated against $f(\beta z)$.

type B can be bounded similarly by replacing z by $-z$. Indeed, $e \int_{\Gamma} z B dz = 0$, we can replace $f(\beta z)$ by $f(-\beta z) = f(\beta z) - \beta z$ without changing the value Terms of type *B* can be bounded similarly by replacing *z* by $-z$. Indeed, we that since $\int_{\Gamma} z B dz = 0$, we can replace $f(\beta z)$ by $f(-\beta z) = f(\beta z) - \beta z$ in the express of the value of the integral. I.e., we can integrate *B* note that since $\int_{\Gamma} z B dz$
the integrand without ch
against a function that c
for positive t, instead of
estimate (6.16). = 0, we can replace $f(\beta z)$ by $f(-\beta z) = f(\beta z) - \beta z$ in
anging the value of the integral. I.e., we can integrate *B*
ecays exponentially for negative *t* and increases linearly
the other way around. These considerations lead the integrand without changing the value of the integral. I.e., we can integrate B against a function that decays exponentially for negative t and increases linearly for positive t , instead of the other way around. against a function that decays exponentially for negative t and increases linearly for positive t , instead of the other way around. These considerations lead to the estimate (6.16). for positive t , instead of the other way around. These considerations lead to the estimate (6.16).

Next, we discuss the term I_1^a . After doing the contour integratives
 $(\pi i)^{-1} \int_{\Gamma} f(\beta z) I_1^a dz = h \sum_{p \in 2\pi \mathbb{Z}} |\hat{\psi}(p)|^2 G(hp)$ Next, we discuss the term I_1^a . After doing the contour integral the term I_1^a
 $(\pi i)^{-1} \int_{\Gamma} f(\beta z) I_1^a dz = h \sum_{p \in 2\pi \mathbb{Z}} |\hat{\psi}(p)|^2 G(hp)$
 $G(p) = -\frac{\beta}{\Gamma} \int \frac{\tanh(\frac{1}{2}\beta((q+p)^2 - \mu)) + \tanh(\frac{1}{2}\beta(q^2 - \mu))}{\tanh(\frac{1}{2}\beta(q^2 - \mu))$ gives

$$
(\pi i)^{-1} \int_{\Gamma} f(\beta z) I_1^a dz = h \sum_{p \in 2\pi \mathbb{Z}} |\hat{\psi}(p)|^2 G(hp)
$$

$$
(\pi i)^{-1} \int_{\Gamma} f(\beta z) I_1^a dz = h \sum_{p \in 2\pi\mathbb{Z}} |\hat{\psi}(p)|^2 G(hp)
$$

with

$$
G(p) = -\frac{\beta}{2} \int_{\mathbb{R}} \frac{\tanh\left(\frac{1}{2}\beta((q+p)^2 - \mu)\right) + \tanh\left(\frac{1}{2}\beta(q^2 - \mu)\right)}{(p+q)^2 + q^2 - 2\mu} \frac{dq}{2\pi}.
$$

By definition (3.3) we have

$$
G(0) = -\frac{\beta^2}{2} \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi}.
$$

Integrating by parts we can write

 $\ddot{ }$

$$
(p+q)^2 + q^2 - 2\mu
$$

have

$$
G(0) = -\frac{\beta^2}{2} \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi}
$$

can write

$$
G(0) = -\frac{\beta^2}{2} \int_{\mathbb{R}} g_0(\beta(q^2 - \mu)) \frac{dq}{2\pi}.
$$

Integrating by parts we can write

$$
G''(0) = \frac{\beta^3}{4} \int_{\mathbb{R}} \left(g_1(\beta(q^2 - \mu)) + 2\beta q^2 g_2(\beta(q^2 - \mu)) \right) \frac{dq}{2\pi}
$$
with g_1 and g_2 from (3.4) and (3.5). Moreover, one can show that [7]

$$
|G(p) - G(0) - \frac{1}{2}p^2 G''(0)| \leq Cp^4.
$$

From this we conclude that

$$
\frac{1}{2} \int_{\mathbb{R}} f(\beta z) I_1^a dz = h \sum |\hat{\psi}(p)|^2 (G(0) + \frac{1}{2}G''(0)h^2 p^2) + O(h^3)
$$

$$
|G(p) - G(0) - \frac{1}{2}p^2 G''(0)| \leq C p^4.
$$

with
$$
g_1
$$
 and g_2 from (3.4) and (3.5). Moreover, one can show that [7]
\n
$$
\left|G(p) - G(0) - \frac{1}{2}p^2G''(0)\right| \le Cp^4.
$$
\nFrom this we conclude that
\n
$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_1^a dz = h \sum_{p \in 2\pi \mathbb{Z}} |\hat{\psi}(p)|^2 (G(0) + \frac{1}{2}G''(0)h^2p^2) + O(h^5) \|\psi\|_{H^2}^2
$$
\n
$$
= hG(0) \|\psi\|_2^2 + \frac{1}{2}G''(0)h^3 \|\psi'\|_2^2 + O(h^5) \|\psi\|_{H^2}^2,
$$
\nwhich is what we claimed in (6.14).
\nFinally, we proceed to I_1^b . After the contour integration we find

$$
= hG(0) \|\psi\|_2^2 + \frac{1}{2}G''(0)h^3 \|\psi'\|_2^2 + O(h^5) \|\psi\|_{H^2}^2,
$$

which is what we claimed in (6.14).
Finally, we proceed to I_1^b . After the contour integration we find

$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_1^b dz = h^3 \sum_{p,q \in 2\pi \mathbb{Z}} \widehat{\psi}^*(p)\widehat{\psi}(q)\widehat{W}(-p-q)L(hp, hq),
$$

where

$$
L(p,q) = \beta^3 \int_{\mathbb{R}} L(p,q,k) \frac{dk}{2\pi}
$$

with

where $% \alpha$ with

$$
a = h3 \sum_{p,q \in 2\pi \mathbb{Z}} \widehat{\psi}^*(p)\widehat{\psi}(q)\widehat{W}(-1)
$$

$$
L(p,q) = \beta^3 \int_{\mathbb{R}} L(p,q,k) \frac{dk}{2\pi}
$$

$$
L(p,q) = \beta^3 \int_{\mathbb{R}} L(p,q,k) \frac{dk}{2\pi}
$$

\n
$$
L(p,q,k) = \frac{1}{\pi i} \int_{\Gamma} \ln(2 + e^{-\beta z} + e^{\beta z}) \frac{1}{z + k^2 - \mu} \frac{1}{z - p^2 + \mu} \frac{1}{z - q^2 + \mu}
$$

\n
$$
\times \left(\frac{1}{z - p^2 + \mu} + \frac{1}{z - q^2 + \mu} + \frac{1}{z + k^2 - \mu}\right) dz.
$$

\n
$$
L(0,0) = \frac{\beta^3}{2} \int_{\mathbb{R}} g_1(\beta(k^2 - \mu)) \frac{dk}{2\pi}
$$

\n
$$
\text{see [7] for details}
$$

\n
$$
|L(p,q) - L(0,0)| \le C (p^2 + q^2).
$$

We have

$$
\times \left(\frac{}{z - p^2 + \mu} + \frac{}{z - q^2 + \mu} + \frac{}{z} \right)
$$

$$
L(0,0) = \frac{\beta^3}{2} \int_{\mathbb{R}} g_1(\beta(k^2 - \mu)) \frac{dk}{2\pi}
$$

$$
|L(p,q) - L(0,0)| \le C (p^2 + q^2)
$$

and (see $[7]$ for details) α (see [7] for details)

$$
2 \int_{\mathbb{R}} 31(\nu(\nu - \nu)) \cdot 2\pi
$$

$$
|L(p,q) - L(0,0)| \le C (p^2 + q^2).
$$

By the Schwarz inequality we can bound
\n
$$
\sum_{p,q\in 2\pi\mathbb{Z}} \left| \hat{\psi}^*(p)\hat{\psi}(q)\widehat{W}(-p-q)(p^2+q^2) \right| \leq C \|W\|_2 \|\psi\|_{H^2} \|\psi\|_{H^1},
$$
\nand obtain
\n
$$
\frac{1}{m!} \int f(\beta z) I_1^b dz = h^3 L(0,0) \sum \hat{\psi}^*(-p)\hat{\psi}(q)\widehat{W}(p-q) + O(h^5) \|\psi\|_{H^2} \|\psi\|_{H^1}
$$

$$
\sum_{p,q\in 2\pi\mathbb{Z}} \left| \hat{\psi}^*(p)\hat{\psi}(q)\overline{W}(-p-q)(p^2+q^2) \right| \le C \|W\|_2 \|\psi\|_{H^2} \|\psi\|_{H^1},
$$
\nand obtain\n
$$
\frac{1}{\pi i} \int_{\Gamma} f(\beta z) I_1^b dz = h^3 L(0,0) \sum_{p,q\in 2\pi\mathbb{Z}} \hat{\psi}^*(-p)\hat{\psi}(q)\overline{W}(p-q) + O(h^5) \|\psi\|_{H^2} \|\psi\|_{H^1}
$$
\n
$$
= \frac{h^3 \beta^3}{2} \langle \psi|W|\psi \rangle \int_{\mathbb{R}} g_1(\beta(k^2-\mu)) \frac{dk}{2\pi} + O(h^5) \|\psi\|_{H^2} \|\psi\|_{H^1}.
$$
\nThis concludes the proof of Theorem 3.1.\n6.3. Proof of Theorem 3.2\nSince the function ρ in (3.8) is analytic in the strip $|\text{Im } z| < \pi$, we can write $[\rho(\beta H_\Delta)]_{12}$ with the aid of a contour integral representation as

6.3. Proof of Theorem 3.2

 $\frac{1}{2} \langle \psi | W | \psi \rangle$

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3.2

1 (3.8) is an

1 of a conton 6.3. Proof of Theorem 3.2.
Since the function ρ in (3.8) is analytic $[\rho(\beta H_{\Delta})]_{12}$ with the aid of a contour inte

Since the function
$$
\rho
$$
 in (3.8) is analytic in the strip $|Im z| < \pi$, we can write
 $[\rho(\beta H_{\Delta})]_{12}$ with the aid of a contour integral representation as

$$
[\rho(\beta H_{\Delta})]_{12} = \frac{1}{2\pi i} \int_{\Gamma} \rho(\beta z) \left[\frac{1}{z - H_{\Delta}}\right]_{12} dz,
$$
(6.19)
where Γ is again the contour $Im z = \pm \pi/(2\beta)$. We expand $(z - H_{\Delta})^{-1}$ using the
resolvent identity and note that, since $H_{\Delta} = H_0 + \delta$ with a H_0 diagonal and δ

 $[\rho(\beta H_{\Delta})]_{12}$ with the aid of a contour integral representation as
 $[\rho(\beta H_{\Delta})]_{12} = \frac{1}{2\pi i} \int_{\Gamma} \rho(\beta z) \left[\frac{1}{z - H_{\Delta}} \right]_{12} dz$,

where Γ is again the contour $\text{Im } z = \pm \pi/(2\beta)$. We expand $(z$ -

resolvent identit $[\rho(\beta H_{\Delta})]_{12} = \frac{1}{2\pi i} \int_{\Gamma} \rho(\beta z)$
he contour Im $z = \pm \pi/(2\beta)$
and note that, since $H_{\Delta} =$
the terms containing an oo
 $(\Delta)^{-1}$. In this way arrive at $[\rho(\beta H_{\Delta})]_{12} = \eta_0 + \eta_1$ $\frac{1}{1}$ + $\frac{1}{16}$ \log the and δ to the where Γ is again the contour Im $z = \pm \pi/(2\beta)$. We expand $(z - H_{\Delta})^{-1}$ using the resolvent identity and note that, since $H_{\Delta} = H_0 + \delta$ with a H_0 diagonal and δ off-diagonal, only the terms containing an odd numbe resolvent identity and note that, since $H_{\Delta} = H_0 + \delta$ with a H_0 diagonal and δ
off-diagonal, only the terms containing an odd number of δ 's contribute to the
12-entry of $(z - H_{\Delta})^{-1}$. In this way arrive at the d off-diagonall, only the terms containing an odd number of δ 's contribute to the
 $(z - H_{\Delta})^{-1}$. In this way arrive at the decomposition
 $[\rho(\beta H_{\Delta})]_{12} = \eta_0 + \eta_1 + \eta_2^a + \eta_2^b$, (6.20)
 $\frac{\partial}{\partial \rho} \left(\frac{\partial}{\partial \rho} \right) \left(\psi \frac{1}{\rho_0^a +$

$$
[\rho(\beta H_{\Delta})]_{12} = \eta_0 + \eta_1 + \eta_2^a + \eta_2^b, \qquad (6.20)
$$

12-entry of
$$
(z - H_{\Delta})^{-1}
$$
. In this way arrive at the decomposition
\n
$$
[\rho(\beta H_{\Delta})]_{12} = \eta_0 + \eta_1 + \eta_2^a + \eta_2^b,
$$
\n(6.20)
\nwhere
\n
$$
\eta_0 = -\frac{h}{4\pi i} \int_{\Gamma} \rho(\beta z) \left(\psi \frac{1}{z^2 - k_0^2} + \frac{1}{z^2 - k_0^2} \psi \right) dz,
$$
\n(6.21)
\n
$$
\eta_1 = \frac{h}{4\pi i} \int_{\Gamma} \rho(\beta z) \left(\frac{1}{z - k_0} [\psi, k_0] \frac{1}{z^2 - k_0^2} + \frac{1}{z^2 - k_0^2} [\psi, k_0] \frac{1}{z + k_0} \right) dz,
$$
\n(6.22)
\n
$$
\int_{\Gamma} \rho(\beta z) \left(\frac{1}{z - k_0} [\psi, k_0] \frac{1}{z^2 - k_0^2} + \frac{1}{z^2 - k_0^2} [\psi, k_0] \frac{1}{z + k_0} \right) dz,
$$
\n(6.23)

$$
\eta_0 = -\frac{1}{4\pi i} \int_{\Gamma} \rho(\beta z) \left(\psi \frac{1}{z^2 - k_0^2} + \frac{1}{z^2 - k_0^2} \psi \right) dz, \qquad (6.21)
$$
\n
$$
\eta_1 = \frac{h}{4\pi i} \int_{\Gamma} \rho(\beta z) \left(\frac{1}{z - k_0} \left[\psi, k_0 \right] \frac{1}{z^2 - k_0^2} + \frac{1}{z^2 - k_0^2} \left[\psi, k_0 \right] \frac{1}{z + k_0} \right) dz, \quad (6.22)
$$
\n
$$
\eta_2^a = -\frac{h^3}{2\pi i} \int_{\Gamma} \rho(\beta z) \frac{1}{z - k_0} \left(W \frac{1}{z - k} \psi + \psi \frac{1}{z + k_0} W \right) \frac{1}{z + k} dz \qquad (6.23)
$$

$$
\eta_0 = -\frac{h}{4\pi i} \int_{\Gamma} \rho(\beta z) \left(\psi \frac{1}{z^2 - k_0^2} + \frac{1}{z^2 - k_0^2} \psi \right) dz, \qquad (6.21)
$$

\n
$$
\eta_1 = \frac{h}{4\pi i} \int_{\Gamma} \rho(\beta z) \left(\frac{1}{z - k_0} [\psi, k_0] \frac{1}{z^2 - k_0^2} + \frac{1}{z^2 - k_0^2} [\psi, k_0] \frac{1}{z + k_0} \right) dz, \quad (6.22)
$$

\n
$$
\eta_2^a = -\frac{h^3}{2\pi i} \int_{\Gamma} \rho(\beta z) \frac{1}{z - k_0} \left(W \frac{1}{z - k} \psi + \psi \frac{1}{z + k_0} W \right) \frac{1}{z + k} dz \qquad (6.23)
$$

\nd
\n
$$
\eta_2^b = -\frac{h^3}{2\pi i} \int_{\Gamma} \rho(\beta z) \frac{1}{z - k} \psi \frac{1}{z + k} \overline{\psi} \frac{1}{z - k} \psi \left[\frac{1}{z - H_{\Delta}} \right]_{22} dz. \qquad (6.24)
$$

and

$$
\eta_{1} = \frac{h}{4\pi i} \int_{\Gamma} \rho(\beta z) \left(\frac{1}{z - k_{0}} \left[\psi, k_{0} \right] \frac{1}{z^{2} - k_{0}^{2}} + \frac{1}{z^{2} - k_{0}^{2}} \left[\psi, k_{0} \right] \frac{1}{z + k_{0}} \right) dz , \quad (6.22)
$$
\n
$$
\eta_{2}^{a} = -\frac{h^{3}}{2\pi i} \int_{\Gamma} \rho(\beta z) \frac{1}{z - k_{0}} \left(W \frac{1}{z - k} \psi + \psi \frac{1}{z + k_{0}} W \right) \frac{1}{z + k} dz \quad (6.23)
$$
\nand\n
$$
\eta_{2}^{b} = -\frac{h^{3}}{2\pi i} \int_{\Gamma} \rho(\beta z) \frac{1}{z - k} \psi \frac{1}{z + k} \overline{\psi} \frac{1}{z - k} \psi \left[\frac{1}{z - H_{\Delta}} \right]_{22} dz . \quad (6.24)
$$
\nA simple residue computation yields\n
$$
\eta_{0} = -\frac{h}{4} \left(\psi \frac{\rho(\beta k_{0}) - \rho(-\beta k_{0})}{k_{0}} + \frac{\rho(\beta k_{0}) - \rho(-\beta k_{0})}{k_{0}} \psi \right)
$$
\n
$$
- \frac{h\beta}{2\pi i} \left(\psi \frac{\rho(\beta k_{0}) + \rho(\beta k_{0}) \psi}{k_{0}} \right)
$$

$$
= -\frac{h^3}{2\pi i} \int_{\Gamma} \rho(\beta z) \frac{1}{z - k} \psi \frac{1}{z + k} \overline{\psi} \frac{1}{z - k} \psi \left[\frac{1}{z - H_{\Delta}} \right]_{22} dz.
$$

A simple residue computation yields

$$
\eta_0 = -\frac{h}{4} \left(\psi \frac{\rho(\beta k_0) - \rho(-\beta k_0)}{k_0} + \frac{\rho(\beta k_0) - \rho(-\beta k_0)}{k_0} \psi \right)
$$

$$
= \frac{h\beta}{4} (\psi g_0(\beta k_0) + g_0(\beta k_0) \psi),
$$

86 R.L. Frank, C. Hainzl, R. Seiringer and J.P. Solovej
which is the main term claimed in the theorem. In the following we shall prove
that that $\|\eta_1\|_{H^1}^2 \leq Ch^5 \|\psi\|_{H^2}^2$, (6.25)
 $\|\eta_2^a\|_{H^1}^2 \leq Ch^5 \|\psi\|_{H^1}^2$, (6.26)

and $\|\eta_2^b\|_{H^1}^2 < Ch^5 \|\psi\|_{H^1}^6$. (6.27)

$$
\|\eta_1\|_{H^1}^2 \le Ch^5 \|\psi\|_{H^2}^2\,,\tag{6.25}
$$

$$
\|\eta_2^a\|_{H^1}^2 \le Ch^5 \|\psi\|_{H^1}^2\,,\tag{6.26}
$$

 $\frac{1}{\sqrt{2}}$

$$
\|\eta_2^n\|_{H^1}^2 \le Ch^5 \|\psi\|_{H^1}^2, \tag{6.26}
$$

$$
\|\eta_2^n\|_{H^1}^2 \le Ch^5 \|\psi\|_{H^1}^6.
$$

3.2.
¹ norm of η_1 is given by

$$
\|p_{H^1}^2 = h \sum_{p \in 2\pi \mathbb{Z}} |\hat{\psi}(p)|^2 J(hp)
$$

This clearly implies Theorem 3.2.

$$
\|\eta_1\|_{H^1}^2 = h \sum_{p \in 2\pi\mathbb{Z}} |\hat{\psi}(p)|^2 J(hp)
$$

$$
\eta_1
$$
: The square of the H^1 norm of η_1 is given by
\n
$$
\|\eta_1\|_{H^1}^2 = h \sum_{p \in 2\pi \mathbb{Z}} |\hat{\psi}(p)|^2 J(hp)
$$
\nwith
\n
$$
J(p) = \frac{\beta^4}{4} \int_{\mathbb{R}} ((q+p)^2 - q^2)^2 (1+q^2) |F(q+p,q) - F(q,q+p)|^2 \frac{dq}{2\pi}
$$
\nand $F(p,q)$ equals

and $F(p,q)$ equals

$$
J(p) = \frac{\beta^4}{4} \int_{\mathbb{R}} \left((q+p)^2 - q^2 \right)^2 (1+q^2) |F(q+p,q) - F(q,q+p)|^2 \frac{dq}{2\pi}
$$

and $F(p,q)$ equals

$$
\frac{1}{p^2 - \mu} \frac{1}{1 + e^{\beta(p^2 - \mu)}} \frac{1}{1 + e^{\beta(q^2 - \mu)}} \left(\frac{e^{\beta(p^2 - \mu)} - e^{\beta(q^2 - \mu)}}{p^2 - q^2} + \frac{e^{\beta(p^2 + q^2 - 2\mu)} - 1}{p^2 + q^2 - 2\mu} \right).
$$
One can show that $0 \le J(p) \le Cp^4$ [7], which yields the desired bound (6.25).
 η_2^2 : This term is a sum of two terms and we begin by bounding the first or

 $\frac{1}{\pi}$
 $\frac{1}{\pi}$
 $\frac{1}{\pi}$
a $\frac{1}{30}$
e: $1 + e^{\beta(p^2 - \mu)} 1 + e^{\beta(q^2 - \mu)}$ $p^2 - q^2$ $p^2 + q^2 - 2\mu$

show that $0 \le J(p) \le Cp^4$ [7], which yields the desired bound (6

f This term is a sum of two terms and we begin by bounding the f
 $h^3(2\pi i)^{-1} \int \rho(\beta z)(z - k_0)^{-1} W(z - k$ $\begin{array}{c} \hline 3 \ 0 \\ \hline 1 \\ 1 \end{array}$ $\frac{d}{dx}$
e $\frac{2}{16}$ One can show that $0 \leq J(p) \leq Cp^4$ [7], which yields the desired bound (6.25).
 η_2^2 : This term is a sum of two terms and we begin by bounding the first c

that is, $-h^3(2\pi i)^{-1} \int \rho(\beta z)(z - k_0)^{-1}W(z - k)^{-1}\psi(z + k)^{-1} dz$. Usi $\eta_2^{\mathbf{a}}$ that is, $-h^3(2\pi i)^{-1} \int \rho(\beta z)(z - k_0)^{-1}W(z - k)^{-1}\psi(z + k)^{-1} dz$. Using Hölder's
inequality for the trace per unit volume we find that the square of the H^1 norm
of the integrand can be bounded by
 $\text{Tr}\left[\frac{1-h^2\nabla^2}{|z-k_0|^2}W$ inequality for the trace per unit volume we find that the square of the H^1 norm of the integrand can be bounded by

Tr
$$
\left[\frac{1-h^2\nabla^2}{|z-k_0|^2}W\frac{1}{z-k}\psi\frac{1}{|z+k|^2}\overline{\psi}\frac{1}{\overline{z}-k}W\right]
$$

\n $\leq \left\|\frac{1-h^2\nabla^2}{|z-k_0|^2}\right\|_{\infty} \|W\|_{\infty}^2 \|\psi\|_{\infty}^2 \|(z-k)^{-1}\|_{\infty}^2 \|(z+k)^{-1}\|_{2}^2.$
\n*b* bound this we use (6.4) and (6.5), as well as the fact to k_0 |⁻²||_∞ is bounded by $C|z|^{-1}$ if Re $z \leq -1$ and by $C|z|$ if ws similarly as (6.5).) In particular, we conclude that for 1

In order to bound this we use (6.4) and (6.5), as well as the fact that $\|$ (1 – $\frac{1-h^2\nabla^2}{|z-k_0|^2}$ $||W||_{\infty}^2 ||\psi||_{\infty}^2 ||(z-k)^{-1}||_{\infty}^2 ||(z+k)$

d this we use (6.4) and (6.5), as well as the
 ∞ is bounded by $C|z|^{-1}$ if Re $z \le -1$ and by ∞

ilarly as (6.5).) In particular, we conclude t In order to bound this we use (6.4) and (6.5), as well as the fact that $||(1 - h^2 \nabla^2)|z - k_0|^{-2}||\infty$ is bounded by $C|z|^{-1}$ if Re $z \le -1$ and by $C|z|$ if Re $z \ge 1$.
(This follows similarly as (6.5).) In particular, we $|h^2\nabla^2||z-k_0|^{-2}||_{\infty}$ is bounded by $C|z|^{-1}$ (This follows similarly as (6.5).) In particular, we conclude that for Re $z \le -1$
the previous quantity is bounded by $Ch^{-1} \|\psi\|_{\infty}^2 |z|^{-7/2}$. The square root of this is
integrable against $\rho(\beta z)$ and we arrive at the the previous quantity is bounded by $Ch^{-1} \|\psi\|_{\infty}^2 |z|^{-7/2}$. The square root of this is
integrable against $\rho(\beta z)$ and we arrive at the bound $Ch^{5/2} \|\psi\|_{\infty}$ for the H^1 norm.
For the positive z direction, we not $\int_{-\infty}^{2}|z|^{-7/2}$

For the positive z direction, we notice that $\rho(\beta z)$ decays exponentially leading to a finite result after z integration.
For the second term in η_2^{α} we proceed similarly. It is important to first notice that $\rho(z)$ For the positive *z* direction, we notice that $\rho(\beta z)$ decays exponentially leading to
a finite result after *z* integration.
For the second term in η_2^a we proceed similarly. It is important to first notice
that ρ a finite result after z integration.
For the second term in η_2^a w
that $\rho(z) = 1 - \rho(-z)$, however,
integrates to zero. Proceeding as
 η_2^b : Finally, we consider η_2^b .
volume and bounding $[(z - H_{\Delta})]$ For the second term in η_2^a we proceed similarly. It is important to first notice $\rho(z) = 1 - \rho(-z)$, however, and that the 1 does not contribute anything but rates to zero. Proceeding as above we arrive at (6.26).
 η_2 For the second term in η_2^a we proceed similarly. It is important to first notice

that $\rho(z) = 1 - \rho(-z)$, however, and that the 1 does not contribute anything but
integrates to zero. Proceeding as above we arrive at (6.26).
 η_2^b : Finally, we consider η_2^b . Using Hölder's inequality for the trace $\eta_2^{\mathbf{b}}$: Finally, we consider $\eta_2^{\mathbf{b}}$. Using Hölder's inequality for the trace per unit volume and bounding $[(z - H_{\Delta})^{-1}]_{22}$ by $2\beta/\pi$ for $z \in \Gamma$ we find that the square $\eta_2^{\mathbf{b}}$: Finally, we consider η_2^b

In of the integrand is bounded by
\n
$$
\frac{4\beta^2}{\pi^2} \|\psi\|_{\infty}^6 \left\| \frac{1 - h^2 \nabla^2}{|z - k_0|^2} \right\|_{\infty} \|(z - k)^{-1}\|_{\infty}^2 \|(z + k)^{-1}\|_2^2.
$$
\nthe bound for η_2^a one can show that for Re $z \le -1$ this is bounded $|z|^{-7/2}$. This leads to (6.27).

of the H^1 norm of the integrand is bounded by
 $\frac{4\beta^2}{\pi^2} ||\psi||_{\infty}^6 \left\| \frac{1 - h^2 \nabla^2}{|z - k_0|^2} \right\|_{\infty} ||(z - k)|$

Similarly as in the bound for η_2^a one can show t

by $Ch^{-1} ||\psi||_{\infty}^6 |z|^{-7/2}$. This leads to $(z - k)^{-1} \|_{\infty}^{2} \| (z + k)$
show that for Re $z \le$
7).
Erwin Schrödinger
the outhors are great

Acknowledgment

Similarly as in the bound for η_2^a
by $Ch^{-1} \|\psi\|_{\infty}^6 |z|^{-7/2}$. This lead
Acknowledgment
Part of this work was carried o
matical Physics in Vienna, Aus
and hospitality during their vis $\frac{a_2}{2}$ one can show that for Re $z \le -1$ this is bounded
ds to (6.27).
out at the Erwin Schrödinger Institute for Mathe-
stria, and the authors are grateful for the support
isit. Financial support via U.S. NSF grants by $Ch^{-1} \|\psi\|_{\infty}^{6} |z|^{-7/2}$
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Eigenfunction Expansions Associated with the One-dimensional Schrödinger Operator

Daphne J. Gilbert

Abstract. We consider the form of eigenfunction expansions associated with the time-independent Schrödinger operator on the line, under the assumption that the limit point case holds at both of the infinite endpoints. It is well known that in this situation the multiplicity of the operator may be one or two, depending on properties of the potential function. Moreover, for values of the spectral parameter in the upper half complex plane, there exist Weyl solutions associated with the restrictions of the operator to the negative and positive half-lines respectively, together with corresponding Titchmarsh-Weyl functions.

In this paper, we establish some alternative forms of the eigenfunction expansion which exhibit the underlying structure of the spectrum and the asymptotic behaviour of the corresponding eigenfunctions. We focus in particular on cases where some or all of the spectrum is simple and absolutely continuous. It will be shown that in this situation, the form of the relevant part of the expansion is similar to that of the singular half-line case, in which the origin is a regular endpoint and the limit point case holds at infinity. Our results demonstrate the key role of real solutions of the differential equation which are pointwise limits of the Weyl solutions on one of the half-lines, while all solutions are of comparable asymptotic size at infinity on the other half-line.

Mathematics Subject Classification (2010). Primary 34L10, 47E05, 47B25. **Keywords.** Eigenfunction expansions, Sturm-Liouville problems, unbounded selfadjoint operators.

1. Introduction

The study of eigenfunction expansions associated with singular differential operators of the Sturm-Liouville type was initiated by Weyl [6], [19], and subsequently
generalised by Stone, Kodaira, Titchmarsh, and others [16], [14], [17], [18]. Partic-
ularly well known are the Fourier, Hermite and Legend ularly well known are the Fourier, Hermite and Legendre expansions, the Laguerre
 ϵ

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polynomials and the Fourier-Bessel series, all of which have widespread applica-
tions in engineering and the physical sciences. It therefore seems worthwhile to
explore the general behaviour of the eigenfu tions in engineering and the physical sciences. It therefore seems worthwhile to explore the general behaviour of the eigenfunctions which contribute to such expansions, to investigate the relationship between the eigenfun

pansions, to investigate the relationship between the eigenfunctions and spectral properties, and to consider under what circumstances the formal structure of the standard expansions can be improved.
In the case of singul standard expansions can be improved.
In the case of singular Sturm-Liouville operators, the eigenfunctions are them-
selves solutions of the differential equations. If the multiplicity of the spectrum is
two then every so In the case of singular Sturm-Liou
selves solutions of the differential equa
two then every solution of $Lu = \lambda u$
since the dimension of the solution spa
linearly independent solutions to featu
similar way, it seems reasona In solutions of the differential equations. If the multiplicity of the spectrum is
then every solution of $Lu = \lambda u$ may be regarded as an eigenfunction, but
the dimension of the solution space is also two, we expect not mor two then every solution of $Lu = \lambda u$ may be regarded as an eigenfunction, but
since the dimension of the solution space is also two, we expect not more than two
linearly independent solutions to feature as integral kernels linearly independent solutions to feature as integral kernels in the expansions. In a similar way, it seems reasonable to expect that when the spectrum, or a part of the spectrum, is simple then for each relevant value of similar way, it seems reasonable to expect that when the spectrum, or a part of the spectrum, is simple then for each relevant value of λ , precisely one linearly independent solution should feature in the expansion. Th spectrum, is simple then for each relevant value of λ , precisely one linearly independent solution should feature in the expansion. This expectation has recently been confirmed for some specific classes of singular Stu

confirmed for some specific classes of singular Sturm-Liouville operators, which include cases where both singular endpoints are limit point (see, e.g., [7], [8]). It is the purpose of this paper to demonstrate that, in t include cases where both singular endpoints are limit point (see, e.g., [7], [8]).
It is the purpose of this paper to demonstrate that, in the case where part
or all of the absolutely continuous spectrum is simple, the co It is the purpose of this paper to demonstrate that, in the case where α or all of the absolutely continuous spectrum is simple, the corresponding parthe expansion can always be reformulated in such a way that the inte It is the absolutely continuous spectrum is simple, the corresponding part of expansion can always be reformulated in such a way that the integral kernels up to scalar multiples, the unique eigenfunctions themselves. To a the expansion can always be reformulated in such a way that the integral kernels
are, up to scalar multiples, the unique eigenfunctions themselves. To achieve this
result we start from the Weyl-Kodaira expansion formula [are, up to scalar multiples, the unique eigenfunctions themselves. To achieve this result we start from the Weyl-Kodaira expansion formula [3], [14], and following the method of Kac [12], [13], diagonalise the spectral de result we start from the Weyl-Kodaira expansion formula [3], [14], and following
the method of Kac [12], [13], diagonalise the spectral density matrix in order to
identify the eigenfunctions and simplify the expansion. It the method of Kac [12], [13], diagonalise the spectral density matrix in order to
identify the eigenfunctions and simplify the expansion. It turns out that in the
process of reformulating the part of the expansion where t identify the eigenfunctions and simplify the expansion. It turns out that in the
process of reformulating the part of the expansion where the spectral multiplicity
is one, the contribution of the half-line operators $H_{-\in$ process of reformulating the part of the expansion where the spectral multiplicity
is one, the contribution of the half-line operators $H_{-\infty}$ and H_{∞} is reflected through
their respective Titchmarsh-Weyl m-function is one, the contribution of the half-line operators $H_{-\infty}$ and H_{∞} is reflected through
their respective Titchmarsh-Weyl m-functions and corresponding spectral densi-
ties. This information enables the asymptotic b their respective Titchmarsh-Weyl *m*-functions and corresponding spectral densities. This information enables the asymptotic behaviour of the eigenfunctions at $\pm \infty$ to be determined in terms of the theory of subordinac their respective Titchmarsh-Weyl *m*-functions and corresponding spectral densi-
ties. This information enables the asymptotic behaviour of the eigenfunctions at
 $\pm \infty$ to be determined in terms of the theory of subordin ±∞

 $\pm \infty$ to be determined in terms of the theory of subordinacy [10], [11], and details
of the process will be demonstrated through worked examples.
Note. Throughout the paper the use of the term "eigenfunctions" is not e process will be demonstrated through worked examples.

. Throughout the paper the use of the term "eigenfunctions" is not restricted

ecific real solutions of the differential equation which are in $L_2(\mathbf{R})$, but also **Note.** Throughout the paper the use of the term "eigenfunction
to specific real solutions of the differential equation which are
includes "eigendifferentials" in the terminology of Weyl [4], [1
relevant solutions associa **Note.** Throughout the paper the use of the term "eigenfunctions" is not restricted to specific real solutions of the differential equation which are in $L_2(\mathbf{R})$, but also
includes "eigendifferentials" in the terminology of Weyl [4], [19], as well as other
relevant solutions associated with the essent relevant solutions associated with the essential spectrum. Where appropriate we will distinguish between singular eigenfunctions and absolutely continuous eigenfunctions depending on whether the corresponding λ -values will distinguish between singular eigenfunctions and absolutely continuous eigenfunctions depending on whether the corresponding λ -values are in the minimal supports of the singular, respectively absolutely continuous supports of the singular, respectively absolutely continuous parts of the spectral measure, and the term generalised eigenfunction will be used to refer to an eigen-
function of either type. functions depending on whether the corresponding λ -values are in the minimal
supports of the singular, respectively absolutely continuous parts of the spectral
measure, and the term generalised eigenfunction will be us measure, and the term generalised eigenfunction will be used to refer to an eigenmeasure of either type.
 $\label{eq:1}$

2. Mathematical background

Eigenfunction Expansions for Schrödinger Operators 91

2. Mathematical background

In this section we briefly summarize the relevant underlying theory from which

the main results of this paper are obtained.

Consider the

$$
Lu := -u'' + q(r)u = \lambda u, \quad -\infty < r < \infty,
$$

the main results of this paper are obtained.

Consider the differential operator H on $L_2(-\infty, \infty)$ associated with
 $Lu := -u'' + q(r)u = \lambda u, \quad -\infty < r < \infty$,

where $q(r) : \mathbf{R} \to \mathbf{R}$ is locally integrable, $\lambda \in \mathbf{R}$ is the Consider the differential operator H on $L_2(-\infty, \infty)$ associated with
 $Lu := -u'' + q(r)u = \lambda u, \quad -\infty < r < \infty,$

where $q(r) : \mathbf{R} \to \mathbf{R}$ is locally integrable, $\lambda \in \mathbf{R}$ is the spectral paramethe differential expression L := −u'' + q(r)u = λu , -∞ < r < ∞,
s locally integrable, $\lambda \in \mathbf{R}$ is the spect
ion L is in Weyl's limit point case at ±
rator H is defined by
 $Hf = Lf$, $f \in \mathcal{D}(H)$, where $q(r) : \mathbf{R} \to \mathbf{R}$ is locally integrable, $\lambda \in \mathbf{R}$ is the spectral parameter, and
the differential expression L is in Weyl's limit point case at $\pm \infty$. In this case the
unique self-adjoint operator H is defi the differential expression *L* is in Weyl's limit point case at $\pm \infty$. In this case the
unique self-adjoint operator *H* is defined by
 $Hf = Lf$, $f \in \mathcal{D}(H)$,
where $f \in \mathcal{D}(H)$ if
(i) $f, Lf \in L_2(\mathbf{R})$,
(ii) f, f' a

$$
Hf = Lf, \quad f \in \mathcal{D}(H),
$$

unique self-adjoint operator H is defined by
 $Hf = Lf$, f

where $f \in \mathcal{D}(H)$ if

(i) $f, Lf \in L_2(\mathbf{R})$,

(ii) f, f' are locally absolutely continuous

We refer to H as the Schrödinger operator on the line.
It is convenient in this context to use the so-called splitting method to analyse where $f \in \mathcal{D}(H)$ if

(i) $f, Lf \in L_2(\mathbf{R})$

(ii) f, f' are local

We refer to H as t

It is convenient

the spectrum of H

function sum (i) $f, Lf \in L_2(\mathbf{R})$,

(ii) f, f' are locally
 Je refer to H as the

It is convenient

the spectrum of H

inction expansion.
 H to $L_2([0, \infty))$ (ii) f, f' are locally absolutely continuous on **R**.
We refer to *H* as the Schrödinger operator on the
It is convenient in this context to use the so-c
he spectrum of *H* and derive appropriate form
unction expansion. W We refer to *H* as the Schrödinger operator on the line.
It is convenient in this context to use the so-called the spectrum of *H* and derive appropriate formulatio function expansion. We first define the half-line operat the spectrum of *H* and derive appropriate formulations of the associated eigen-
function expansion. We first define the half-line operator H_{∞} to be the restriction
of *H* to $L_2([0, \infty))$ with a Dirichlet boundary c function expansion. We first define the half-line operator H_{∞} to be the restriction
of H to $L_2([0, \infty))$ with a Dirichlet boundary condition at $r = 0$, and choose a
fundamental set of solutions $\{u_1(r, z), u_2(r, z)\}$ o

$$
u_1(0, z) = u'_2(0, z) = 0, \qquad u_2(0, z) = u'_1(0, z) = 1,\tag{2.1}
$$

of *H* to $L_2([0, \infty))$ with a Dirichlet boundary condition at $r = 0$, and choose a fundamental set of solutions $\{u_1(r, z), u_2(r, z)\}$ of $Lu = zu$, $z \in \mathbf{C}$, to satisfy $u_1(0, z) = u'_2(0, z) = 0$, $u_2(0, z) = u'_1(0, z) = 1$, (2.1) w fundamental set of solutions $\{u_1(r, z), u_2(r, z)\}$ of $Lu = zu$, $z \in \mathbf{C}$, to satisfy
 $u_1(0, z) = u'_2(0, z) = 0$, $u_2(0, z) = u'_1(0, z) = 1$,

where for $i = 1, 2$, $u'_i(0, z)$ denotes the value at $r = 0$ of the derivati
 $u_i(r, z)$ wi $(u, z) = u'_2(0, z) = 0$, $u_2(0, z) = u'_1(0, z) = 1$, (2.1)

2, $u'_i(0, z)$ denotes the value at $r = 0$ of the derivative of

ct to r. Associated with H_{∞} , there exists a Herglotz function
 u^+ known as the Titchmarsh-Weyl fu where for $i = 1, 2, u'_i(0, z)$ denotes the value at $r = 0$ of the derivative of $u_i(r, z)$ with respect to r . Associated with H_{∞} , there exists a Herglotz function $m_{\infty}(z) : \mathbf{C}^+ \to \mathbf{C}^+$ known as the Titchmarshwhere for $i = 1, 2, u'_i(0, z)$ denotes the value at $r = 0$ of the derivative of $u_i(r, z)$ with respect to r. Associated with H_{∞} , there exists a Herglotz function (r, z) with respect to r. Associated with H_{∞} , there exists a Herglotz function $_{\infty}(z) : \mathbf{C}^+ \to \mathbf{C}^+$ known as the Titchmarsh-Weyl function, such that the sotion $u_2(r, z) + m_{\infty}(z) u_1(r, z)$ of $Lu = zu$ is in $L_2([0$ $m_{\infty}(z) : \mathbf{C}^+ \to \mathbf{C}^+$ known as the Titchmarsh-Weyl function, such that the so-
lution $u_2(r, z) + m_{\infty}(z) u_1(r, z)$ of $Lu = zu$ is in $L_2([0, \infty))$ for all $z \in \mathbf{C}^+$. The
related spectral function $\rho_{\infty}(\lambda) : \mathbf{R} \to$ lution $u_2(r, z) + m_\infty(z) u_1(r, z)$ of $Lu = zu$ is in $L_2([0, \infty))$ for all $z \in \mathbb{C}^+$. The related spectral function $\rho_\infty(\lambda) : \mathbb{R} \to \mathbb{R}$ is non-decreasing, continuous on the right and generates a non-negative Borel-Stieltjes measure μ_{∞} on **R**. Its derivative, the

$$
\rho'_{\infty}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi} \operatorname{Im} m_{\infty}(z), \quad z = \lambda + i\epsilon,
$$
\n(2.2)

and generates a non-negative Borel-Stieltjes measure μ_{∞} on **R**. Its derivative, the
spectral density function $\rho'_{\infty}(\lambda)$, exists and satisfies
 $\rho'_{\infty}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi} \operatorname{Im} m_{\infty}(z)$, $z = \lambda + i\epsilon$, (2.2)
fo spectral density function $\rho'_{\infty}(\lambda)$, exists and satisfies
 $\rho'_{\infty}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi} \operatorname{Im} m_{\infty}(z)$, z

for Lebesgue and μ_{∞} -almost all $\lambda \in \mathbf{R}$, and the spe

by
 $\sigma(H^-) := \mathbf{R} \setminus \{\lambda \in \mathbf{R} : a \text{ is constant in a negative}\$ $(\lambda) = \lim_{z \downarrow \lambda}$
most all λ
 $\in \mathbb{R} : \rho_{\infty}$:
bsed set co $\begin{array}{c} \pi \in \mathbb{R}^n \\ 0 \in \mathbb{R}^n \end{array}$ Im $m_{\infty}(z)$, $z = \lambda + i\epsilon$, (2.2)
 R, and the spectrum $\sigma(H_{\infty})$ may be defined

onstant in a neighbourhood $N(\lambda)$ of λ ,

sining the points of increase of $\rho_{\infty}(\lambda)$.

 σ

for Lebesgue and μ_{∞} -almost all $\lambda \in \mathbf{R}$, and the spectrum $\sigma(H_{\infty})$ may be defined
by
 $\sigma(H_{\infty}) := \mathbf{R} \setminus \{\lambda \in \mathbf{R} : \rho_{\infty} \text{ is constant in a neighbourhood } N(\lambda) \text{ of } \lambda\},$
which is the smallest closed set containing the points of inc v
by
pr
im $(H_{\infty}) := \mathbf{R} \setminus \{ \lambda \in \mathbf{R} : \rho_{\infty} \text{ is constant in a neighbourhood } N(\lambda) \text{ of } \lambda \},$
is the smallest closed set containing the points of increase of $\rho_{\infty}(\lambda)$.
The half-line operator $H_{-\infty}$ on $L_2((-\infty, 0])$ is defined in a similar way
aal d which is the smallest closed set containing the points of increase of $\rho_{\infty}(\lambda)$.

The half-line operator $H_{-\infty}$ on $L_2((-\infty, 0])$ is defined in a similar w

principal difference being that the Titchmarsh-Weyl function The half-line operator $H_{-\infty}$ on $L_2((-\infty, 0])$ is defined in a similar way, the ipal difference being that the Titchmarsh-Weyl function $m_{-\infty}(z)$ has negative inary part on \mathbb{C}^+ , so that the spectral density ρ' principal difference being that the Titchmarsh-Weyl function $m_{-\infty}(z)$ has negative
imaginary part on \mathbb{C}^+ , so that the spectral density $\rho'_{-\infty}(\lambda)$ satisfies
 $\rho'_{-\infty}(\lambda) = -\lim_{z \uparrow \lambda} \frac{1}{\pi} \operatorname{Im} m_{-\infty}(z), \quad z = \lambda +$

imaginary part on
$$
\mathbb{C}^+
$$
, so that the spectral density $\rho'_{-\infty}(\lambda)$ satisfies

$$
\rho'_{-\infty}(\lambda) = -\lim_{z \uparrow \lambda} \frac{1}{\pi} \operatorname{Im} m_{-\infty}(z), \quad z = \lambda + i\epsilon,
$$
(2.3)

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for Lebesgue and $\mu_{-\infty}$ -almost all $\lambda \in \mathbf{R}$. We note that the spectrum of both
half-line operators is simple, that is to say, both H_{∞} and $H_{-\infty}$ have spectral
multiplicity one. half-line operators is simple, that is to say, both H_{∞} and $H_{-\infty}$ have spectral

The standard form of the eigenfunction expansion associated with the selfadjoint operator H_{∞} is as follows for $f(r) \in L_2([0,\infty))$:

adjoint operator
$$
H_{\infty}
$$
 is as follows for $f(r) \in L_2([0, \infty))$:
\n
$$
f(r) = \lim_{\omega \to \infty} \int_{-\omega}^{\omega} u_1(r, \lambda) G(\lambda) d\rho_{\infty}(\lambda),
$$
\nwhere $u_1(r, \lambda), 0 \le r < \infty$, satisfies the boundary condition in (2.1) with $z = \lambda \in$
\n**R**, and
\n
$$
G(\lambda) = \lim_{\eta \to \infty} \int_{0}^{\eta} u_1(r, \lambda) f(r) dr,
$$

 $(r, \lambda) G(\lambda) d\rho_{\infty}(\lambda),$ (2.4)

oundary condition in (2.1) with $z = \lambda \in$
 $u_1(r, \lambda) f(r) dr,$ **R**

$$
G(\lambda) = \lim_{\eta \to \infty} \int_0^{\eta} u_1(r, \lambda) f(r) dr,
$$

where $u_1(r, \lambda)$, $0 \le r < \infty$, satisfies the boundary condition in (2.1) with $z = \lambda \in \mathbf{R}$, and
 $G(\lambda) = \lim_{\eta \to \infty} \int_0^{\eta} u_1(r, \lambda) f(r) dr$,

where convergence in the mean is in $L_2([0, \infty))$ and $L_2(\mathbf{R}; d\rho_{\infty}(\lambda))$ respec here \vert . No e poi $(\lambda) = \frac{\lim_{\eta \to \infty}}{\lambda} \int_0^{\eta} u_1(r, \lambda) f(r) dr$,
mean is in $L_2([0, \infty))$ and L_2
contributes to the expansion
 ρ_{∞} , or more precisely, belong with W.
with we see that the integral k
solution of $Lu = \lambda u$ satisfying where convergence in the mean is in $L_2([0, \infty))$ and $L_2(\mathbf{R}; d\rho_{\infty}(\lambda))$ respectively [2]. Note that $G(\lambda)$ only contributes to the expansion (2.4) for those λ which are points of increase of ρ_{∞} , or more prec [2]. Note that $G(\lambda)$ only contributes to the expansion (2.4) for those λ which
are points of increase of ρ_{∞} , or more precisely, belong to a minimal support of
 μ_{∞} (see Definition 2 below). We see that the are points of increase of ρ_{∞} , or more precisely, belong to a minimal support of μ_{∞} (see Definition 2 below). We see that the integral kernel of both transform and inverse transform is a solution of $Lu = \lambda u$ s μ_{∞} and inverse transform is a solution of $Lu = \lambda u$ satisfying the Dirichlet boundary condition at $r = 0$, namely $u_1(r, \lambda)$, and it follows that $u_1(r, \lambda)$ is an eigenfunction of H_{∞} for those λ contributing to the sp condition at $r = 0$, namely $u_1(r, \lambda)$, and it follows that $u_1(r, \lambda)$ is an eigenfunction
of H_{∞} for those λ contributing to the spectrum of the operator. Note that the
spectral function $\rho_{\infty}(\lambda)$ is constant of H_{∞} for those λ contributing to the spectrum of the operator. Note that the spectral function $\rho_{\infty}(\lambda)$ is constant on each open interval of the resolvent set, so that there is no contribution to the integra spectral function $\rho_{\infty}(\lambda)$ is constant on each open interval of the resolvent set, so
that there is no contribution to the integral in (2.4) when λ is in the resolvent set.
The expansion associated with the operato

that there is no contribution to the integral in (2.4) when λ is in the resolvent set.
The expansion associated with the operator $H_{-\infty}$ has a similar form, with obvious
adjustments.
In the case of the full line oper The expansion associated with the operator $H_{-\infty}$ has a similar form, with obvious
adjustments.
In the case of the full line operator H , the analogue of the Titchmarsh-Weyl
m-function is a 2 × 2 M-matrix, which is def In the case of the full line operator *H*, the analogue of the Titchmarsh-Weyl
 m-function is a 2 × 2 *M*-matrix, which is defined in terms of the scalar *m*-functions

associated with $H_{-\infty}$ and H_{∞} by
 $M(z) := \frac{$

m-function is a 2 × 2 *M*-matrix, which is defined in terms of the scalar *m*-functions associated with
$$
H_{-\infty}
$$
 and H_{∞} by\n
$$
M(z) := \frac{1}{m_{-\infty} - m_{\infty}} \begin{pmatrix} m_{-\infty} m_{\infty} & \frac{1}{2} (m_{-\infty} + m_{\infty}) \\ \frac{1}{2} (m_{-\infty} + m_{\infty}) & 1 \end{pmatrix}
$$
\nfor $z \in \mathbb{C}^+$. The associated matrix spectral function for $\lambda \in \mathbb{R}$,\n
$$
(\rho_{ij}(\lambda)) := \begin{pmatrix} \rho_{11}(\lambda) & \rho_{12}(\lambda) \\ \rho_{21}(\lambda) & \rho_{22}(\lambda) \end{pmatrix},
$$
\nis continuous on the right, has bounded variation on compact subintervals of the

$$
(\rho_{ij}(\lambda)) := \begin{pmatrix} \rho_{11}(\lambda) & \rho_{12}(\lambda) \\ \rho_{21}(\lambda) & \rho_{22}(\lambda) \end{pmatrix},
$$

is continuous on the right, has bounded variation on compact subintervals of the for $z \in \mathbb{C}^+$. The associated matrix spectral function for $\lambda \in \mathbb{R}$,
 $(\rho_{ij}(\lambda)) := \begin{pmatrix} \rho_{11}(\lambda) & \rho_{12}(\lambda) \\ \rho_{21}(\lambda) & \rho_{22}(\lambda) \end{pmatrix}$,

is continuous on the right, has bounded variation on compact

real line, and g $(\rho_{ij}(\lambda)) := \begin{pmatrix} \rho_{11}(\lambda) & \rho_{12}(\lambda) \\ \rho_{21}(\lambda) & \rho_{22}(\lambda) \end{pmatrix}$
t, has bounded variation on
a Borel-Stieltjes measure (
nents of the corresponding de
 $(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi}$ Im $M_{ij}(z)$, $z =$ real line, and generates a Borel-Stieltjes measure (μ_{ij}) which is positive semi-
definite [14]. The components of the corresponding density matrix $(\rho'_{ij}(\lambda))$ satisfy
 $\rho'_{ij}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi} \text{Im } M_{ij}(z), \quad z = \lambda + i\epsilon,$

() satisfy ′

definite [14]. The components of the corresponding density matrix $(\rho'_{ij}(\lambda))$
 $\rho'_{ij}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi} \operatorname{Im} M_{ij}(z), \quad z = \lambda + i\epsilon,$

Lebesgue and μ_{ij} -almost everywhere on **R** for each $i, j = 1, 2$, and the sp

of *H* $(\lambda) = \lim_{z \downarrow \lambda}$
t everywh
R : (ρ_{ij}) i $\frac{1}{\pi}$ e: Im $M_{ij}(z)$, $z = \lambda + i\epsilon$, (2.6)

a on **R** for each $i, j = 1, 2$, and the spectrum

onstant in a neighbourhood $N(\lambda)$ of λ . Lebesgue and μ_{ij} -almost everywhere on **R** for each $i, j = 1, 2$, and the spectrum
of *H* is given by
 $\sigma(H) := \mathbf{R} \setminus \{ \lambda \in \mathbf{R} : (\rho_{ij}) \text{ is constant in a neighbourhood } N(\lambda) \text{ of } \lambda \}.$ of H is given by
 $\sigma(H) := \mathbf{R}$

 $\sigma(H) := \mathbf{R} \setminus \{ \lambda \in \mathbf{R} : (\rho_{ij}) \text{ is constant in a neighbourhood } N(\lambda) \text{ of } \lambda \}.$

Eigenfunction Expansions for Schrödinger Operators 93

As noted above, both the half-line operators, $H_{-\infty}$ and H_{∞} , have spectral

multiplicity one; however, in the case of the full line operator H where both end-As noted above, both the half-line operators, $H_{-\infty}$ and H_{∞} , have spectral
plicity one; however, in the case of the full line operator H where both end-
s are limit point, some or all of the spectrum may have mu multiplicity one; however, in the case of the full line operator *H* where both end-
points are limit point, some or all of the spectrum may have multiplicity two. Here
the standard formulation of the expansion in its mos

the standard formulation of the expansion in its most general form is given by the
\nWeyl-Kodaira formula [3], [14], from which we have for
$$
f \in L_2(\mathbf{R})
$$
,
\n
$$
f(r) = \lim_{\omega \to \infty} \int_{-\omega}^{\omega} \sum_{i=1}^{2} \sum_{j=1}^{2} u_i(r, \lambda) F_j(\lambda) d\rho_{ij}(\lambda),
$$
\n(2.7)
\nwhere
\n
$$
F(\lambda) = \{F_1(\lambda), F_2(\lambda)\} = \lim_{\eta \to \infty} \left\{ \int_{-\eta}^{\eta} u_1(r, \lambda) f(r) dr, \int_{-\eta}^{\eta} u_2(r, \lambda) f(r) dr \right\},
$$
\nand $u_1(r, \lambda), u_2(r, \lambda)$, satisfy (2.1) at $r = 0$, with $z = \lambda \in \mathbf{R}$, convergence of the

$$
F(\lambda) = \{F_1(\lambda), F_2(\lambda)\} = \frac{\lim_{\eta \to \infty}}{\eta} \left\{ \int_{-\eta}^{\eta} u_1(r,\lambda) f(r) dr, \int_{-\eta}^{\eta} u_2(r,\lambda) f(r) dr \right\},\,
$$

and $u_1(r, \lambda), u_2(r, \lambda)$, satisfy (2.1) at $r = 0$, with $z = \lambda \in \mathbf{R}$, convergence of the $F(\lambda) = \{F_1(\lambda), F_2(\lambda)\} = \lim_{\eta \to \infty} \left\{ \int_{-\eta}^{\eta} u_1(r, \lambda) f(r) dr, \int_{-\eta}^{\eta} u_2(r, \lambda) f(r) dr \right\}$
and $u_1(r, \lambda), u_2(r, \lambda)$, satisfy (2.1) at $r = 0$, with $z = \lambda \in \mathbb{R}$, convergence of integrals being in $L_2(\mathbb{R})$ and $L_2(\mathbb{R}; d\rho$ and $u_1(r, \lambda)$, $u_2(r, \lambda)$, satisfy (2.1) at $r = 0$, with $z = \lambda \in \mathbf{R}$, convergence of the
integrals being in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_{ij}(\lambda))$, respectively. An advantage of this
form of the eigenfunction expansion integrals being in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_{ij}(\lambda))$, respectively. An advantage of this
form of the eigenfunction expansion is that it has very general application, so that
with suitable adjustments it may also be app

- the endpoints $-\infty$, ∞ , are replaced by a, b, respectively, and the decomposi-
- with suitable adjustments it may also be applied to cases where

 the endpoints $-\infty$, ∞ , are replaced by a, b , respectively, and the decomposition point 0 by c , where $-\infty \le a < c < b \le \infty$,

 the boundary condition • the endpoints $-\infty$, ∞ , are replaced by *a*, *b*, respectively, ar
tion point 0 by *c*, where $-\infty \le a < c < b \le \infty$,
• the boundary condition at the decomposition point $c \in \mathbb{R}$
 $\sin(\alpha)u'(c,\lambda) = 0$ for some $\alpha \in [0, \pi)$ the endpoints $-\infty$, ∞ , are replaced by a, b , respectively, and the decomposition point 0 by c , where $-\infty \le a < c < b \le \infty$,
the boundary condition at the decomposition point $c \in \mathbf{R}$ is $\cos(\alpha)u(c, \lambda) + \sin(\alpha)u'(c, \lambda) = 0$ tion point 0 by c, where $-\infty \le a < c < b \le \infty$,
the boundary condition at the decomposition p
sin(α)u'(c, λ) = 0 for some $\alpha \in [0, \pi)$,
one or both of the endpoints is in the limit circ
urther details, see [10].
However ∙
	- one or both of the endpoints is in the limit circle case.

 $\sin(\alpha)u'(c, \lambda) = 0$ for some $\alpha \in [0, \pi)$,

• one or both of the endpoints is in the limit circle case.

For further details, see [10].

However, a significant drawback of the Weyl-Kodaira formula is that in the

case of s $\sin(\alpha)u'(c,\lambda) = 0$ for some $\alpha \in [0, \pi)$,
one or both of the endpoints is in the
urther details, see [10].
However, a significant drawback of the
of simple spectrum those solutions of t
ions of the operator H cannot be ider urther details, see [10].
However, a significant drawback of the Weyl-Kodaira
of simple spectrum those solutions of the differential equ
ions of the operator H cannot be identified directly fra
s. We note that the formu However, a significant
case of simple spectrum thos
functions of the operator H
stands. We note that the fo
tiplicity of the spectrum, as
two. It will be shown that of simple spectrum those solutions of the differential equation which are eigen-
ions of the operator H cannot be identified directly from the expansion as it
ls. We note that the formulation (2.7) contains four terms functions of the operator H cannot be identified directly from the expansion as it stands. We note that the formulation (2.7) contains four terms, whereas the multiplicity of the spectrum, and hence the dimension of t tiplicity of the spectrum, and hence the dimension of the eigenspaces, is at most
two. It will be shown that whenever the spectrum of H has multiplicity one, the
expansion can be reduced to a much simpler form which repli features of (2.4) in the half-line case. In this situation the eigenfunctions are the integral kernels in the simplified expansion and can be completely characterised in terms of their subordinacy properties.
To clarify t

integral kernels in the simplified expansion and can be completely characterised
in terms of their subordinacy properties.
To clarify the relevance of the theory of subordinancy in this context, we first
briefly introduce in terms of their subordinacy properties.

To clarify the relevance of the theory of subordinancy in this context, we first

briefly introduce some key features. A subordinate solution of $Lu = \lambda u$, $r \ge 0$,

when L is regu To clarify the relevance of the theor
briefly introduce some key features. A s
when L is regular at 0 and in the limit p
Definition 1. A solution $u_s(r, \lambda)$ of $Lu =$
be *subordinate* at infinity if briefly introduce some key features. A subordinate solution of $Lu = \lambda u$, $r \ge 0$,
when L is regular at 0 and in the limit point case at infinity, is defined as follows:
Definition 1. A solution $u_s(r, \lambda)$ of $Lu = \lambda u$, $-\infty$

when *L* is regular at 0 and in the limit point case at infinity, is defined as follows:
 Definition 1. A solution $u_s(r, \lambda)$ of $Lu = \lambda u$, $-\infty < \lambda < \infty$, $0 \le r < \infty$, is said to

be *subordinate* at infinity if
 $\lim_{N \to \infty}$ **Definition 1.**

$$
\lim_{N \to \infty} \frac{\|u_s(r, \lambda)\|_N}{\|u(r, \lambda)\|_N} = 0,
$$
\n
$$
\|u(r, \lambda)\|_N
$$
\n
$$
\|u(r, \lambda)\|_{\infty}
$$
\n
$$
\text{from } u_s(r, \lambda).
$$

A solution $u_s(r, \lambda)$ of $Lu = \lambda u$, $-\infty < \lambda < \infty$, $0 \le r < \infty$, is said to
 ie at infinity if
 $\lim_{N \to \infty} \frac{\|u_s(r, \lambda)\|_N}{\|u(r, \lambda)\|_N} = 0$,

denotes $(\int_0^N | \cdot |^2 dr)^{1/2}$, and $u(r, \lambda)$ denotes any solution of $Lu = \lambda u$

arly ind be subordinate at infinity if
 $\text{where } \|\;.\;\|_N \text{ denotes }(\int_0^N\mid$ which is linearly independentle $\int dr$
from $\frac{\langle \langle \cdot, \cdot \rangle \rangle_{\parallel N}}{\langle r, \lambda \rangle \parallel_{N}} = 0,$

and $u(r, \lambda)$ den

fr, λ). where $|| \cdot ||_N$ denotes $(\int_0^N | \cdot |^2 dr)^{1/2}$, and $u(r, \lambda)$ denotes any solution of $Lu = \lambda u$
which is linearly independent from $u_s(r, \lambda)$. which is linearly independent from $u_s(r, \lambda)$.

Thus a subordinate solution is unique up to m
of λ , and is asymptotically smaller than any li
same equation, in the sense of limiting ratios of
to the eigenfunction expansion associated with
must either of λ , and is asymptotically smaller than any linearly independent solution of the
same equation, in the sense of limiting ratios of Hilbert space norms. To contribute
to the eigenfunction expansion associated with $H_{\$ to the eigenfunction expansion associated with H_{∞} , a solution $u(r, \lambda)$ of $Lu = \lambda u$
must either
(a) satisfy the boundary condition at $r = 0$ in the case where no solution is
subordinate at infinity, or
(b) satisfy the

- to the eigenfunction expansion associated with H_{∞} , a solution $u(r, \lambda)$ of $Lu = \lambda u$
must either
(a) satisfy the boundary condition at $r = 0$ in the case where no solution is
subordinate at infinity, or
(b) satisfy the
-

(a) satisfy the boundary condition at $r = 0$ in the case where no solution is
subordinate at infinity, or
(b) satisfy the boundary condition at $r = 0$ and be subordinate at infinity,
since the set of all $\lambda \in \mathbf{R}$ for (b) satisfy the boundary condition at $r = 0$ and be subordinate at infinity,
ince the set of all $\lambda \in \mathbf{R}$ for which no solution satisfies (a) or (b) has
neasure zero [11]. In fact the solutions of $Lu = \lambda u$, $r \ge 0$ whi since the set of all $\lambda \in \mathbf{R}$ for which no solution satisfies (a) or (b) has μ_{∞} -
measure zero [11]. In fact the solutions of $Lu = \lambda u$, $r \ge 0$ which satisfy (a) are
absolutely continuous eigenfunctions of H_{∞ measure zero [11]. In fact the solutions of $Lu = \lambda u$, $r \ge 0$ which satisfy (a) are
absolutely continuous eigenfunctions of H_{∞} , and solutions which satisfy (b) are
singular eigenfunctions of H_{∞} . The definition absolutely continuous eigenfunctions of H_{∞} , and solutions which satisfy (b) are
singular eigenfunctions of H_{∞} . The definition of subordinate solutions and the
distinguishing properties of the eigenfunctions ar

distinguishing properties of the eigenfunctions are entirely analogous in the case
of $H_{-\infty}$.
The relationship between the eigenfunctions and the corresponding parts of
the spectrum of H can be made more precise using of $H_{-\infty}$.
The relationship between the eigenfunctions and the corresponding parts of
the spectrum of H can be made more precise using the concept of a minimal
support of a Borel-Stieltjes measure.
Definition 2. A sub pectrum of H can be made more precise using the concept of a minimal
ort of a Borel-Stieltjes measure.
ition 2. A subset S of **R** is said to be a minimal support of a measure ν on
the following conditions hold:
 $\nu(\math$

the spectrum of *H* can be made more precise using the concept of a minimal
support of a Borel-Stieltjes measure.
Definition 2. A subset *S* of **R** is said to be a *minimal support* of a measure ν on
R if the follo **Definition 2.** A subset S of **R** is said **R** if the following conditions hold:
(i) $\nu(\mathbf{R} \setminus S) = 0$,
(ii) if S_0 is a subset of S such that where $|\cdot|$ denotes Lebesgue measure **Definition 2. R** if the following conditions hold:

(i) $\nu(\mathbf{R} \setminus S) = 0$,

(ii) if S_0 is a subset of S such that $\nu(S_0) = 0$, then $|S_0| = 0$,

where $| \cdot |$ denotes Lebesgue measure.

Note that a minimal support of a measure ν is unique up to sets of ν - and Lebe (i) $\nu(\mathbf{R} \setminus S) = 0$,

(ii) if S_0 is a subset of S such the

here | . | denotes Lebesgue measure

ote that a minimal support of a m

easure zero and, in the case of

here the spectrum is concentrate where the spectrum is concentrated (for further details, see [11]). Corresponding to the decomposition of a Borel-Stieltjes measure ν into absolutely continuous where $|\cdot|$ denotes Lebesgue measure.
Note that a minimal support of a measure
measure zero and, in the case of a s
where the spectrum is concentrated (
to the decomposition of a Borel-Stie
and singular parts, $\nu_{a.c.}$ a Note that a minimal support of a measure ν is unique up to sets of ν - and Lebesgue
measure zero and, in the case of a spectral measure, provides an indication of
where the spectrum is concentrated (for further detai where the spectrum is concentrated (for further details, see [11]). Corresponding
to the decomposition of a Borel-Stieltjes measure ν into absolutely continuous
and singular parts, $\nu_{a.c.}$ and ν_s , there exist mini to the decomposition of a Borel-Stieltjes measure ν into absolutely continuous
and singular parts, $\nu_{a.c.}$ and ν_s , there exist minimal supports, $S(\nu_{a.c.})$ and $S(\nu_s)$
respectively such that $S(\nu_{a.c.}) \cup S(\nu_s) = S(\nu)$ and singular parts, $\nu_{a.c.}$ and ν_s , there exist minimal supports, $S(\nu_{a.c.})$ and $S(\nu_{s.})$
respectively such that $S(\nu_{a.c.}) \cup S(\nu_s) = S(\nu)$ and $S(\nu_{a.c.}) \cap S(\nu_s) = \emptyset$, where $S(\nu)$
is a minimal support of ν . For H_{\in respectively such that $S(\nu_{a.c.}) \cup S(\nu_s) = S(\nu)$ and $S(\nu_{a.c.}) \cap S(\nu_s) = \emptyset$, where $S(\nu)$
is a minimal support of ν . For H_{∞} , as shown in [11], minimal supports $\mathcal{M}_{a.c.}(H_{\infty})$,
 $\mathcal{M}_{s.}(H_{\infty})$ of the absolutely $\frac{\mathcal{W}_{s}}{c}$

is a minimal support of ν . For H_{∞} , as shown in [11], minimal supports $\mathcal{M}_{a.c.}(H_{\infty})$,
 $\mathcal{M}_{s.}(H_{\infty})$ of the absolutely continuous and singular parts respectively of μ_{∞} are

as follows:
 $\mathcal{M}_{a.c}(H$ (H_{∞}) of the absolutely continuous and singular parts respectively of μ_{∞} are
ollows:
 $A_{a.c}(H_{\infty}) = \{\lambda \in \mathbf{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at } \infty\},$
 $M_{s.}(H_{\infty}) = \{\lambda \in \mathbf{R} : \text{there exists a solution of } Lu = \lambda u \text{ which satisfies the
Dirichlet boundary condition at 0 and is subordinate at } \infty\}$
imal s $M_{a.c}(H$
 $M_s(H)$
Minimal s $\mathcal{M}_{a.c}(H_{\infty}) = \{\lambda \in \mathbf{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at } \infty \},\$ \mathcal{M}_s .

 $(H_{\infty}) = \{\lambda \in \mathbf{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at } \infty\},$
 $(H_{\infty}) = \{\lambda \in \mathbf{R} : \text{there exists a solution of } Lu = \lambda u \text{ which satisfy}\n\begin{aligned}\n\text{Dirichlet boundary condition at 0 and is subordinate at}\n\text{1 supports, } \mathcal{M}_{a.c}(H_{-\infty}) \text{ and } \mathcal{M}_{s.}(H_{-\infty}) \text{, of the absolutely contr}\n\text{parts of } \mu_{-\infty} \text{ are obtained by replacing } \infty \text{ with } -\infty \text{ in the ab}\n\end{aligned}$ $(H_{\infty}) = {\lambda \in \mathbf{R}}$: there exists a solution of $Lu = \lambda u$ which satisfies the
Dirichlet boundary condition at 0 and is subordinate at ∞ }
I supports, $\mathcal{M}_{a,c}(H_{-\infty})$ and $\mathcal{M}_s(H_{-\infty})$, of the absolutely continuous a
 $\mathcal{M}_{a.c}(H_{-\infty})$ and $\mathcal{M}_{s.}(H_{-\infty})$, of the absolutely continuous $\mu_{-\infty}$ are obtained by replacing ∞ with $-\infty$ in the above ϵ the Lebesgue measure of a minimal support of a spectral measure of the correspon Minimal supports, $\mathcal{M}_{a.c}(H_{-\infty})$ and $\mathcal{M}_s(H_{-\infty})$, of the absolutely continuous and
singular parts of $\mu_{-\infty}$ are obtained by replacing ∞ with $-\infty$ in the above equa-
tions. Note that the Lebesgue measure of singular parts of $\mu_{-\infty}$ are obtained by replacing ∞ with $-\infty$ in the above equations. Note that the Lebesgue measure of a minimal support of a spectral measure and the Lebesgue measure of the corresponding spectr and the Lebesgue measure of the corresponding spectrum are not in general equal.
For example, in the case where there is dense singular spectrum on a real inter-
val $[c, d]$, the singular spectrum, being a closed set, will For example, in the case where there is dense singular spectrum on a real inter-
val $[c, d]$, the singular spectrum, being a closed set, will contain $[c, d]$, so that the
Lebesgue measure of the interval is $d - c$; however, val $[c, d]$, the singular spectrum, being a closed set, will contain $[c, d]$, so that the Lebesgue measure of the interval is $d - c$; however, the minimal support of the Lebesgue measure of the interval is $d - c$; however, the minimal support of the of a spectral measure will always have Lebesgue measure zero. A
on can also arise in relation to the absolutely continuous spectrum
mple 6.5 in [10]).
se of a full-line operator H on $L_2(\mathbf{R})$ with two limit point end-p

similar situation can also arise in relation to the absolutely continuous spectrum
(see, e.g., Example 6.5 in [10]).
In the case of a full-line operator H on $L_2(\mathbf{R})$ with two limit point end-points
and spectral measur (see, e.g., Example 6.5 in [10]).

In the case of a full-line operator H on $L_2(\mathbf{R})$ with two limit point end-points

and spectral measure μ , a minimal support of μ is given by $\mathcal{M} = \mathcal{M}_{a.c.}(H) \cup \mathcal{M}_{s.}(H)$, $\mathcal{M}_s(H)$, where

In the case of a full-line operator
$$
H
$$
 on $L_2(\mathbf{R})$ with two limit point end-points
and spectral measure μ , a minimal support of μ is given by $\mathcal{M} = \mathcal{M}_{a.c.}(H) \cup$
 $\mathcal{M}_{s.}(H)$, where
 $\mathcal{M}_{a.c.}(H) = \{ \lambda \in \mathbf{R} : \text{either no solution of } Lu = \lambda u \text{ exists which issubordinate at $-\infty$, or no solution of $Lu = \lambda u \text{ exists}$
which is subordinate at $+\infty$, or both}
 $\mathcal{M}_{s.}(H) = \{ \lambda \in \mathbf{R} : \text{a solution of } Lu = \lambda u \text{ exists which issubordinate at both $\pm \infty \}$
(see [9]). The nature of the eigenfunctions when H has simple spectrum will be
considered in Section 4.$$

subordinate at both $\pm \infty$
re of the eigenfunctions with 4.
the spectral density is (see in Section 4.
 3. Diagonalising the spectral density matrix

In order to simplify the eigenfunction expansion in the case where both endpoints

3. Diagonalising the spectral density matrix

3. Diagonalising the
In order to simplify the
are limit point, we follow are limit point, we follow the method of I.S. Kac $[12]$, $[13]$, and begin by introducing a spectral density matrix. Let $M_{\tau}(z)$ denote the trace of the M-matrix in (2.5),

a spectral density matrix. Let
$$
M_{\tau}(z)
$$
 denote the trace of the M-matrix in (2.5),
so that for $z \in \mathbb{C}^{+}$

$$
M_{\tau}(z) := M_{11}(z) + M_{22}(z)
$$

$$
= \frac{m_{-\infty}(z) m_{\infty}(z) + 1}{m_{-\infty}(z) - m_{\infty}(z)}.
$$
(3.1)
Since $m_{-\infty}$ and m_{∞} are anti-Herglotz and Herglotz functions respectively, it is
straightforward to check that $M_{\tau}(z)$ is Herglotz, so that Im $M_{\tau}(z) > 0$ for $z \in \mathbb{C}^{+}$.
It follows that a non-decreasing function $\rho_{\tau}(\lambda) : \mathbb{R} \to \mathbb{R}$ exists such that

 $=\frac{m_{-\infty}(z) m_{\infty}(z)+1}{m_{-\infty}(z)-m_{\infty}(z)}$
erglotz and Herglotz fi
(z) is Herglotz, so that
function $\rho_{\tau}(\lambda) : \mathbf{R} \to \mathbf{R}$
 $\frac{1}{\pi} \text{Im } M_{\tau}(z), \quad z = \lambda$ Since $m_{-\infty}$ and m_{∞} are anti-Herglotz and Herglotz functions respectively, it is Since $m_{-\infty}$ and m_{∞} are anti-Herglotz and Herglotz functions respectively, it is
straightforward to check that $M_{\tau}(z)$ is Herglotz, so that $\text{Im } M_{\tau}(z) > 0$ for $z \in \mathbb{C}^+$.
It follows that a non-decreasing f

$$
\rho'_{\tau}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi} \operatorname{Im} M_{\tau}(z), \quad z = \lambda + i\epsilon,
$$
\n(3.2)

straightforward to check that $M_{\tau}(z)$ is Herglotz, so that $\text{Im } M_{\tau}(z) > 0$ for $z \in \mathbb{C}^+$.

It follows that a non-decreasing function $\rho_{\tau}(\lambda) : \mathbb{R} \to \mathbb{R}$ exists such that
 $\rho'_{\tau}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi} \text{Im }$ It follows that a non-decreasing function $\rho_{\tau}(\lambda) : \mathbf{R} \to \mathbf{R}$ exists such that
 $\rho'_{\tau}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi} \operatorname{Im} M_{\tau}(z), \quad z = \lambda + i\epsilon,$

exists and is satisfied for Lebesgue and μ_{τ} -almost all $\lambda \in \mathbf{R}$, wh (λ) = $\lim_{z \downarrow \lambda} \frac{1}{\pi} \text{Im } M_{\tau}(z)$, $z = \lambda + i\epsilon$, (3.2)

or Lebesgue and μ_{τ} -almost all $\lambda \in \mathbf{R}$, where μ_{τ} is the

ties measure generated by $\rho_{\tau}(\lambda)$. Moreover, since $(\rho_{ij}(\lambda))$,
 μ_{ij} is absolut $\frac{1}{\pi}$ 16 roof exists and is satisfied for Lebesgue and μ_{τ} -almost all $\lambda \in \mathbf{R}$, where μ_{τ} is the
non-negative Borel-Stieltjes measure generated by $\rho_{\tau}(\lambda)$. Moreover, since $(\rho_{ij}(\lambda))$
is positive semi-definite, μ_{ij} non-negative Borel-Stieltjes measure generated by $\rho_{\tau}(\lambda)$. Moreover, since $(\rho_{ij}(\lambda))$ is positive semi-definite, μ_{ij} is absolutely continuous with respect to μ_{τ} for each

is positive semi-definite,
$$
\mu_{ij}
$$
 is absolutely continuous with respect to μ_{τ} for each
 $i, j = 1, 2$, from which it may be inferred that for $\lim_{z \downarrow \lambda} \text{Im } M_{\tau}(z) \neq 0$,

$$
d\rho_{ij}(\lambda) = \frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda) d\rho_{\tau}(\lambda) = \lim_{z \downarrow \lambda} \frac{\text{Im} M_{ij}(z)}{\text{Im} M_{\tau}(z)} d\rho_{\tau}(\lambda), \quad z = \lambda + i\epsilon,
$$
(3.3)
Lebesgue and μ_{τ} -almost everywhere on **R** (see [10], Lemma 5.3). We refer to

$$
\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = \left(\begin{array}{cc} \frac{d\rho_{11}}{d\rho_{\tau}}(\lambda) & \frac{d\rho_{12}}{d\rho_{\tau}}(\lambda) \\ \frac{d\rho_{13}}{d\rho_{\tau}}(\lambda) & 0 \end{array}\right)
$$
(3.4)

$$
d\rho_{ij}(\lambda) = \frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)d\rho_{\tau}(\lambda) = \lim_{z \downarrow \lambda} \frac{\text{Im}M_{ij}(z)}{\text{Im}M_{\tau}(z)} d\rho_{\tau}(\lambda), \quad z = \lambda + i\epsilon,
$$
 (3.3)
Lebesgue and μ_{τ} -almost everywhere on **R** (see [10], Lemma 5.3). We refer to\n
$$
\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = \begin{pmatrix} \frac{d\rho_{11}}{d\rho_{\tau}}(\lambda) & \frac{d\rho_{12}}{d\rho_{\tau}}(\lambda) \\ \frac{d\rho_{21}}{d\rho_{\tau}}(\lambda) & \frac{d\rho_{22}}{d\rho_{\tau}}(\lambda) \end{pmatrix}
$$
 (3.4)

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as the spectral density matrix for *H*. Since the set

$$
S_0 := \{ \lambda \in \mathbf{R} : \lim_{z \downarrow \lambda} \text{Im } M_\tau(z) = 0 \}
$$
has μ_τ -measure zero, we may take

as the spectral density matrix for *H*. Since the set
\n
$$
S_0 := \{ \lambda \in \mathbf{R} : \lim_{z \downarrow \lambda} \text{Im } M_\tau(z) = 0 \}
$$
\nhas μ_τ -measure zero, we may take\n
$$
\begin{pmatrix} \frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \\ \frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all } \lambda \in S_0.
$$
\n(3.5)
\nAlso, noting that the limits as $z \downarrow \lambda$ in (3.3) exist for Lebesgue and μ_τ -almost all\n
$$
\lambda \in \mathbf{R} \setminus S_0, \text{ we have that}
$$
\n
$$
\frac{d\rho_{11}}{d\rho_\tau}(\lambda) = \frac{\text{Im } m_{-\infty}(\lambda) \mid m_{\infty}(\lambda) \mid^2 - \text{Im } m_{\infty}(\lambda) \mid m_{-\infty}(\lambda) \mid^2}{D(\lambda)}, \qquad (3.6)
$$
\n
$$
\frac{d\rho_{12}}{d\rho_1}(\lambda) = \frac{d\rho_{21}}{D(\lambda)}
$$

has μ_{τ} -measure zero, we may take
 $\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right)$ =

Also, noting that the limits as $z \downarrow$
 $\lambda \in \mathbf{R} \setminus S_0$, we have that
 $d\rho_{11}$ (λ) = Im $m_{-\infty}(\lambda)$ $\lambda \in \mathbf{R} \setminus S_0$, we have that

$$
\left(\frac{\overline{H_{ij}}(\lambda)}{d\rho_{\tau}}(\lambda)\right) = \begin{pmatrix} 0 & 0 \ 0 & 0 \end{pmatrix} \quad \text{for all } \lambda \in S_0.
$$
 (3.5)
Also, noting that the limits as $z \downarrow \lambda$ in (3.3) exist for Lebesgue and μ_{τ} -almost all
 $\lambda \in \mathbf{R} \setminus S_0$, we have that

$$
\frac{d\rho_{11}}{d\rho_{\tau}}(\lambda) = \frac{\text{Im } m_{-\infty}(\lambda) |m_{\infty}(\lambda)|^2 - \text{Im } m_{\infty}(\lambda) |m_{-\infty}(\lambda)|^2}{D(\lambda)},
$$
 (3.6)

$$
\frac{d\rho_{12}}{d\rho_{\tau}}(\lambda) = \frac{d\rho_{21}}{d\rho_{\tau}}(\lambda)
$$

$$
\frac{d\rho_{11}}{d\rho_{\tau}}(\lambda) = \frac{\text{Im } m_{-\infty}(\lambda) \mid m_{\infty}(\lambda) \mid^{2} - \text{Im } m_{\infty}(\lambda) \mid m_{-\infty}(\lambda) \mid^{2}}{D(\lambda)}, \qquad (3.6)
$$

$$
\frac{d\rho_{12}}{d\rho_{\tau}}(\lambda) = \frac{d\rho_{21}}{d\rho_{\tau}}(\lambda)
$$

$$
= \frac{\text{Im } m_{-\infty}(\lambda) \text{ Re } m_{\infty}(\lambda) - \text{Im } m_{\infty}(\lambda) \text{ Re } m_{-\infty}(\lambda)}{D(\lambda)}, \qquad (3.7)
$$

$$
\frac{d\rho_{22}}{d\rho_{\tau}}(\lambda) = \frac{\text{Im } m_{-\infty}(\lambda) - \text{Im } m_{\infty}(\lambda)}{D(\lambda)}, \qquad (3.8)
$$

ervwhere on **R** $\setminus S_0$, where

$$
= \frac{\operatorname{Im} m_{-\infty}(\lambda) \operatorname{Re} m_{\infty}(\lambda) - \operatorname{Im} m_{\infty}(\lambda) \operatorname{Re} m_{-\infty}(\lambda)}{D(\lambda)},
$$
(3.7)

$$
\frac{d\rho_{22}}{d\rho_{\tau}}(\lambda) = \frac{\operatorname{Im} m_{-\infty}(\lambda) - \operatorname{Im} m_{\infty}(\lambda)}{D(\lambda)},
$$
(3.8)
erywhere on $\mathbb{R} \setminus S_0$, where

$$
\lambda) = \operatorname{Im} m_{-\infty}(\lambda) (1 + |m_{\infty}(\lambda)|^2) - \operatorname{Im} m_{\infty}(\lambda) (1 + |m_{-\infty}(\lambda)|^2),
$$

$$
(\lambda), m_{\infty}(\lambda),
$$
 denote the normal limits of $m_{-\infty}(z)$, $m_{\infty}(z)$, respectively
 $\mathbb{R}.$
following theorem establishes a rigorous correlation between the rank of

$$
D(\lambda) = \text{Im } m_{-\infty}(\lambda) \left(1 + |m_{\infty}(\lambda)|^2\right) - \text{Im } m_{\infty}(\lambda) \left(1 + |m_{-\infty}(\lambda)|^2\right),
$$

almost everywhere on **R** \ S_0 , where
 $D(\lambda) = \text{Im } m_{-\infty}(\lambda) (1 + | m_{\infty})$

and $m_{-\infty}(\lambda)$, $m_{\infty}(\lambda)$, denote the no

as $z \downarrow \lambda \in \mathbf{R}$.

The following theorem establish

the spectral density matrix and the $(\lambda) = \text{Im } m_{-\infty}(\lambda) (1 + | m_{\infty}(\lambda) |^2) - \text{Im } m_{\infty}(\lambda) (1 + | m_{-\infty}(\lambda) |^2)$
 $\infty(\lambda)$, $m_{\infty}(\lambda)$, denote the normal limits of $m_{-\infty}(z)$, $m_{\infty}(z)$, res
 $\in \mathbb{R}$.

e following theorem establishes a rigorous correlation be

as $z \downarrow \lambda \in \mathbf{R}$.
The following theorem establishes a rigorous correlation between the rank of the spectral density matrix and the multiplicity of the spectrum of *H*.

and $m_{-\infty}(\lambda)$, $m_{\infty}(\lambda)$, denote the normal limits of $m_{-\infty}(z)$, $m_{\infty}(z)$, respectively
as $z \downarrow \lambda \in \mathbf{R}$.
The following theorem establishes a rigorous correlation between the rank of
the spectral density matrix the spectral density matrix and the multiplicity of the spectrum of H .
 Theorem 1 (Kac). The spectral multiplicity of H is two if and only if the μ_{τ} -

measure of the set
 $\mathcal{M}_2 := \left\{ \lambda \in E : \text{rank} \left(\frac{d\rho_{ij}}{d$ **Theorem 1 (Kac).** The spectral multiplicity of H is two if and only if the μ_{τ} measure of the set

$$
\mathcal{M}_2 := \left\{ \lambda \in E : \text{rank}\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda) \right) = 2 \right\}
$$

is strictly positive, where

$$
A_2 := \left\{ \lambda \in E : \text{rank}\left(\frac{\Delta_{P,j}}{d\rho_{\tau}}(\lambda)\right) = 2 \right\}
$$

re

$$
E := \left\{ \lambda \in \mathcal{M} : \left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) \text{ exists} \right\}
$$

ort \mathcal{M} of μ_{τ} . The set \mathcal{M}_2 is a max.
 $l \lambda \in (\mathbf{R} \setminus \mathcal{M}_2), H$ has multiplicity

$$
\text{rank}\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = 1.
$$

 $M_2 := \left\{ \lambda \in E : \text{rank} \left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda) \right) = 2 \right\}$

is strictly positive, where
 $E := \left\{ \lambda \in \mathcal{M} : \left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda) \right) \text{ exists} \right\}$

for some minimal support M of μ_{τ} . The set M_2 is a maximal set of mult 2, and for μ_{τ} -almost all $\lambda \in (\mathbf{R} \setminus M_2)$, *H* has multiplicity one with

rank $\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = 1$.

An important consequence of the theorem, as recognized by Kac, is
 Corollary 1. Let

$$
\lambda \in \mathcal{M} : \left(\frac{2\pi ij}{d\rho_{\tau}}(\lambda)\right)
$$

of μ_{τ} . The set \mathcal{M}_2 is
 $\mathbf{R} \setminus \mathcal{M}_2$, H has mul
rank $\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = 1$.
the theorem, as recog
 $\lim m_{-\infty}(\lambda) < 0$, Im
 $\varepsilon \Delta$ denotes the symm

Corollary 1. Let

$$
S_d := \{ \lambda \in \mathbf{R} : \text{Im } m_{-\infty}(\lambda) < 0, \text{ Im } m_{\infty}(\lambda) > 0 \}.
$$
\n
$$
S_d) = 0, \text{ where } \triangle \text{ denotes the symmetric different}
$$

Corollary 1. Let $S_d := \{ \lambda \in \mathbf{R} : \text{Im } m_{-\infty}(\lambda) < 0, \text{ Im } m_{\infty}(\lambda) > 0 \}.$
Then $\mu_{\tau}(\mathcal{M}_2 \triangle S_d) = 0$, where \triangle denotes the symmetric difference. Then $\mu_{\tau}(\mathcal{M}_2 \triangle S_d) = 0$, where \triangle denotes the symmetric difference.
 $\mu_{\tau}(\mathcal{M}_2 \triangle S_d) = 0$, where \triangle denotes the symmetric difference.

implies that only the absolutely continuous part of the spectrum
iplicity two, or contain a non-trivial subset with multiplicity two. It
at the degenerate spectrum is supported on
 $\mathbf{R} :$ no solution of $Lu = \lambda u$ is subor can have multiplicity two, or contain a non-trivial subset with multiplicity two. It
also implies that the degenerate spectrum is supported on
 $S' := \{ \lambda \in \mathbf{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at either } -\infty \text{ or at } \infty \},\$
where $\mu_{\tau}(S_d \triangle S') = 0$

 S^{\prime}

also implies that the degenerate spectrum is supported on
 $S' := \{ \lambda \in \mathbf{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at either } -\infty \text{ or at } \infty \},$

where $\mu_{\tau}(S_d \triangle S') = 0$ (see [10] for further details).

We now use the spectral density matrix to rearran $S' := \{ \lambda \in \mathbf{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at } \theta \}$
where $\mu_{\tau}(S_d \triangle S') = 0$ (see [10] for further details).
We now use the spectral density matrix to rearrange
mulation (2.7) in such a way that the generalised eigenfunc := { $\lambda \in \mathbf{R}$: no solution of $Lu = \lambda u$ is subordinate at either $-\infty$ or at ∞ },

re $\mu_{\tau}(S_d \triangle S') = 0$ (see [10] for further details).

We now use the spectral density matrix to rearrange the Weyl-Kodaira for

titi where $\mu_{\tau}(S_d \triangle S')$
We now use
mulation (2.7) in :
explicitly in the e
integral on the rig where $\mu_{\tau}(S_d \Delta S') = 0$ (see [10] for further details).
We now use the spectral density matrix to rearrange the Weyl-Kodaira for-
mulation (2.7) in such a way that the generalised eigenfunctions of *H* are exhibited
expli

mulation (2.7) in such a way that the generalised eigenfunctions of *H* are exhibited
explicitly in the expansion. Using (3.3)–(3.5), it is straightforward to see that the
integral on the right-hand side of (2.7) may be expressed as follows:

$$
\int_{-\omega}^{\omega} \sum_{i=1}^{2} \sum_{j=1}^{2} u_i(r,\lambda) F_j(\lambda) d\rho_{ij}(\lambda) = \int_{-\omega}^{\omega} (U(r,\lambda))^T \left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) F(\lambda) d\rho_{\tau}(\lambda)
$$
(3.9)
where the superfix *T* denotes the transpose, and

$$
U(r,\lambda) = \begin{pmatrix} u_1(r,\lambda) \\ u_2(r,\lambda) \end{pmatrix}, \quad F(\lambda) = \begin{pmatrix} F_1(\lambda) \\ F_2(\lambda) \end{pmatrix},
$$

$$
(r, \lambda)F_j(\lambda)d\rho_{ij}(\lambda) = \int_{-\omega} (U(r, \lambda))^T \left(\frac{\omega_{Pij}}{d\rho_{\tau}}(\lambda)\right) F
$$

ix *T* denotes the transpose, and

$$
U(r, \lambda) = \begin{pmatrix} u_1(r, \lambda) \\ u_2(r, \lambda) \end{pmatrix}, \qquad F(\lambda) = \begin{pmatrix} F_1(\lambda) \\ F_2(\lambda) \end{pmatrix},
$$

$$
F_2(\lambda) \text{ as in (2.7). Since the spectral density mawe may decompose it in such a way that
$$
\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = P^T D P,
$$
$$

where the superfix T denotes the transpose, and
 $U(r, \lambda) = \begin{pmatrix} u_1(r, \lambda) \\ u_2(r, \lambda) \end{pmatrix}$, $F(\lambda)$

with $F_1(\lambda)$ and $F_2(\lambda)$ as in (2.7). Since the spe

and symmetric, we may decompose it in such a with $F_1(\lambda)$ and $F_2(\lambda)$ as in (2.7). Since the spectral density matrix (3.4) is real
and symmetric, we may decompose it in such a way that
 $\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = P^T D P,$ (3.10)
where

with
$$
F_1(\lambda)
$$
 and $F_2(\lambda)$ as in (2.7). Since the spectral density matrix (3.4) is real
and symmetric, we may decompose it in such a way that

$$
\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = P^T D P,
$$
(3.10)
where

$$
D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$
according as the rank of the spectral density matrix is 1 or 2 respectively, and P

$$
\left(\frac{-P_{ij}}{d\rho_{\tau}}(\lambda)\right) = P^I D P,
$$
\n
$$
D = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \text{ or } \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right),
$$
\nif the spectral density matrix is 1.

is a 2×2 matrix given by according as the rank of the spectral density matrix is 1 or 2 respectively, and P

$$
D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

according as the rank of the spectral density matrix is 1 or 2 respectively, and P
is a 2 × 2 matrix given by

$$
P = \begin{pmatrix} \sigma_{11}^{1/2} & \sigma_{11}^{-1/2} \sigma_{12} \\ 0 & \sigma_{11}^{-1/2} (\sigma_{11}\sigma_{22} - \sigma_{12}^{-2})^{1/2} \end{pmatrix}
$$
(3.11)
for $\sigma_{11} \neq 0$, where for each $i, j = 1, 2, \sigma_{ij}$ denotes $(d\rho_{ij}/d\rho_{\tau})(\lambda)$. Note that $P^T P =$

is a 2 [×] 2 matrix given by ead
as
at: 0 $\sigma_{11}^{-1/2} (\sigma_{11}\sigma_{22} - \sigma_{12})$
 $i = 1, 2, \sigma_{ij}$ denotes $(d\rho_{ij})$

cank if and only if the d

strictly positive. For nor $\int_0^{1/2} d\rho_\tau$
tern
triv $T P =$
 σ_{12}^2 , of
 $1 = 0$, for $\sigma_{11} \neq 0$, where for each $i, j = 1, 2, \sigma_{ij}$ denotes $(d\rho_{ij}/d\rho_{\tau})(\lambda)$. Note that $P^T P = (\sigma_{ij}(\lambda))$ and that P has full rank if and only if the determinant, $\sigma_{11}\sigma_{22} - \sigma_{12}^2$, of the spectral density matrix i $(\sigma_{ij}(\lambda))$ and that P has full rank if and only if the determinant, $\sigma_{11}\sigma_{22} - \sigma_{12}^2$
the spectral density matrix is strictly positive. For non-trivial cases where σ_{11} =
we have
 $D = P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

strictly positive. For non-trivial cases where
$$
\sigma_{11} = 0
$$
,
\n
$$
D = P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
$$
\n(3.12)
\nwe semi-definite property of the spectral density ma-
\n $|\leq 1$ for $i, j = 1, 2$ (see [10]).

the spectral density matrix is strictly positive. For non-trivial cases where $\sigma_{11} = 0$,
we have $D = P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, (3.12)
taking into account the positive semi-definite property of the spectral density ma-
 taking in
trix and $= P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
semi-definite proximation is a form $i, j = 1$, $\frac{1}{2}$ matrix and the fact that $0 \leq |\sigma_{ij}| \leq 1$ for $i, j = 1, 2$ (see [10]). trix and the fact that 0 ≤∣ ∣≤ 1 for , = 1, 2 (see [10]).

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Let M_1 denote the set of all $\lambda \in E$ such that $((d\rho_{ij}/d\rho_{\tau}))$
one. From (2.7), (3.9) and (3.10), we have for $f(r) \in L_2(\mathbf{R})$, Let \mathcal{M}_1 denote the set of all $\lambda \in E$ such that $((d\rho_{ij}/d\rho_{\tau})(\lambda))$ has multiplicity

one. From (2.7), (3.9) and (3.10), we have for
$$
f(r) \in L_2(\mathbf{R})
$$
,
\n
$$
f(r) = \lim_{\omega \to \infty} \int_{-\omega}^{\omega} (U(r, \lambda))^T P^T D P F(\lambda) d\rho_\tau(\lambda)
$$
\n
$$
= \lim_{\omega \to \infty} \int_{-\omega}^{\omega} (V(r, \lambda))^T D G(\lambda) d\rho_\tau(\lambda), \qquad (3.13)
$$
\nwhere
\n
$$
P U(r, \lambda) = V(r, \lambda) = \begin{pmatrix} v_1(r, \lambda) \\ v_2(r, \lambda) \end{pmatrix}, \quad P F(\lambda) = G(\lambda) = \begin{pmatrix} G_1(\lambda) \\ G_2(\lambda) \end{pmatrix}, \quad (3.14)
$$
\nand convergence is in $L_2(\mathbf{R})$. Using (3.11)–(3.14), this leads to

$$
= \omega \to \infty \int_{-\omega} (V(r,\lambda))^T D G(\lambda) d\rho_{\tau}(\lambda),
$$
(3.13)
where

$$
PU(r,\lambda) = V(r,\lambda) = \begin{pmatrix} v_1(r,\lambda) \\ v_2(r,\lambda) \end{pmatrix}, \quad PF(\lambda) = G(\lambda) = \begin{pmatrix} G_1(\lambda) \\ G_2(\lambda) \end{pmatrix},
$$
(3.14)
and convergence is in $L_2(\mathbf{R})$. Using (3.11)–(3.14), this leads to

$$
f(r) = \omega \to \infty \begin{cases} \int_{(-\omega,\omega)\cap\mathcal{M}_1} v(r,\lambda) G(\lambda) d\rho_{\tau}(\lambda) \end{cases}
$$
(3.17)

$$
PU(r,\lambda) = V(r,\lambda) = \begin{pmatrix} v_1(r,\lambda) \\ v_2(r,\lambda) \end{pmatrix}, \quad PF(\lambda) = G(\lambda) = \begin{pmatrix} G_1(\lambda) \\ G_2(\lambda) \end{pmatrix}, \quad (3.14)
$$

and convergence is in $L_2(\mathbf{R})$. Using (3.11)–(3.14), this leads to

$$
f(r) = \stackrel{\text{l.i.m.}}{\omega \to \infty} \left\{ \int_{(-\omega,\omega) \cap \mathcal{M}_1} v(r,\lambda) G(\lambda) d\rho_\tau(\lambda) + \sum_{i=1}^2 \int_{(-\omega,\omega) \cap \mathcal{M}_2} v_i(r,\lambda) G_i(\lambda) d\rho_\tau(\lambda) \right\}, \quad (3.15)
$$

where

$$
G(\lambda) = \frac{1 \text{.i.m.}}{\eta \to \infty} \int_{-\eta}^{\eta} v(r,\lambda) f(r) dr,
$$

with

$$
G(\lambda) = \lim_{\eta \to \infty} \int_{-\eta}^{\eta} v(r, \lambda) f(r) dr,
$$

 $with$

$$
G(\lambda) = \frac{\lim_{\eta \to \infty}}{\eta} \int_{-\eta}^{\eta} v(r, \lambda) f(r) dr,
$$

\n
$$
v(r, \lambda) = \begin{cases} \sigma_{11}^{1/2} u_1(r, \lambda) + \sigma_{11}^{-1/2} \sigma_{12} u_2(r, \lambda) & \sigma_{11} \neq 0 \\ u_2(r, \lambda) & \sigma_{11} = 0, \end{cases}
$$

\n
$$
i = 1, 2,
$$

\n
$$
G_i(\lambda) = \frac{\lim_{\eta \to \infty}}{\eta} \int_{-\eta}^{\eta} v_i(r, \lambda) f(r) dr,
$$

\n
$$
v_r(n, \lambda) = \sigma_{11}^{1/2} v_r(n, \lambda) + \sigma_{11}^{-1/2} \sigma_{12} v_r(n, \lambda)
$$
 (3.16)

and and for $i = 1, 2$,
with

$$
G_i(\lambda) = \lim_{\eta \to \infty} \int_{-\eta}^{\eta} v_i(r, \lambda) f(r) dr,
$$

$$
G_i(\lambda) = \lim_{\eta \to \infty} \int_{-\eta}^{\eta} v_i(r, \lambda) f(r) dr,
$$

$$
v_1(r, \lambda) = \sigma_{11}^{1/2} u_1(r, \lambda) + \sigma_{11}^{-1/2} \sigma_{12} u_2(r, \lambda),
$$

$$
v_2(r, \lambda) = \sigma_{11}^{-1/2} (\sigma_{11} \sigma_{22} - \sigma_{12}^2)^{1/2} u_2(r, \lambda).
$$

$$
\mathcal{M}_k := \left\{ \lambda \in \mathbf{R} : \text{rank} \left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda) \right) = k \right\},
$$

with
 $\label{eq:11} \text{Also for } k=1,\,2,$

$$
a_2(r,\lambda) = \sigma_{11}^{-1/2} (\sigma_{11} \sigma_{22} - \sigma_{12}^2)^{1/2} u_2(r,\lambda).
$$

$$
\mathcal{M}_k := \left\{ \lambda \in \mathbf{R} : \text{rank}\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = k \right\},
$$
the integrals is in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_{\tau}(\lambda))$

Also for $k = 1, 2$,
and convergence
From (3.15)
 $\lambda \in \mathcal{M}_2$, a basis f := $\left\{ \lambda \in \mathbf{R} : \text{rank} \left(\frac{d\rho_{ij}}{d\rho_{\tau}} \right) \right\}$
tegrals is in $L_2(\mathbf{R})$ and
is an eigenfunction of .
genspace is $\{v_1(r,\lambda), v_2\}$
pectrum of *H* is simple
has a similar form to t (L_2)
 L_2
 H for (r, λ)
so t $= k$
 $k; d\rho$
 $=$ acl
 \therefore No
 \downarrow the the and convergence of the integrals is in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_{\tau}(\lambda))$, respectively.
From (3.15), $v(r, \lambda)$ is an eigenfunction of H for each $\lambda \in \mathcal{M}_1$, and for $\lambda \in \mathcal{M}_2$, a basis for the eigenspace is $\{v_$ From (3.15), $v(r, \lambda)$ is an eigenfunction of H for each $\lambda \in M_1$, and for each M_2 , a basis for the eigenspace is $\{v_1(r, \lambda), v_2(r, \lambda)\}\)$. Note from Kac's theorem, if $\mu_{\tau}(\mathcal{M}_2) = 0$ the spectrum of H is simple $\lambda \in \mathcal{M}_2$ that if $\mu_{\tau}(\mathcal{M}_2) = 0$ the spectrum of H is simple so that the second term in (3.15). is null and the expansion has a similar form to that of the half-line case in (2.4) .
4. The case of simple spectrum

of simple spectrum
of simple spectrum
we focus on the particular case where the spectrum of *H* is simple
solutely continuous. In this situation $\mu_{\tau}(\mathcal{M}_2) = 0$, and hence as
on 3, the expansion in (3.15) reduces to a In this section, we focus on the particular case where the spectrum of *H* is simple
and purely absolutely continuous. In this situation $\mu_{\tau}(\mathcal{M}_2) = 0$, and hence as
shown in Section 3, the expansion in (3.15) reduce shown in Section 3, the expansion in (3.15) reduces to a simpler form, so that for $f(r) \in L_2(\mathbf{R})$

$$
f(r) \in L_2(\mathbf{R})
$$

\n
$$
f(r) = \lim_{\omega \to \infty} \int_{-\omega}^{\omega} v(r, \lambda) G(\lambda) d\rho_{\tau}(\lambda)
$$
(4.1)
\nwhere $v(r, \lambda)$ is as in (3.16), and
\n
$$
G(\lambda) = \lim_{\eta \to \infty} \int_{-\eta}^{\eta} v(r, \lambda) f(r) dr,
$$

\nwith convergence in the mean in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_{\tau}(\lambda))$ respectively.

where $v(r, \lambda)$ is as in (3.16), and

$$
G(\lambda) = \frac{\text{l.i.m.}}{\eta \to \infty} \int_{-\eta}^{\eta} v(r, \lambda) f(r) dr,
$$

where $v(r, \lambda)$ is as in (3.16), and
 $G(\lambda) =$

with convergence in the mean in

We now investigate the str

relationship to the Titchmarsh-V $(\lambda) = \frac{\text{l.i.m.}}{\eta \to \infty} \int_{-\eta}^{\eta} v(r, \lambda) f(r) dr,$
ean in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_{\tau})$
he structure of the eigenfunctiansh-Weyl functions, $m_{-\infty}$, m_c
with $H_{-\infty}$ and H_{∞} , respectiv
 $v(r, \lambda)$ as $r \to \pm \infty$ to be with convergence in the mean in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_{\tau}(\lambda))$ respectively.
We now investigate the structure of the eigenfunction $v(r, \lambda)$ in te
relationship to the Titchmarsh-Weyl functions, $m_{-\infty}, m_{\infty}$, and t We now investigate the structure of the eigenfunction $v(r, \lambda)$ in terms of its
onship to the Titchmarsh-Weyl functions, $m_{-\infty}$, m_{∞} , and the corresponding
solutions associated with $H_{-\infty}$ and H_{∞} , respective relationship to the Titchmarsh-Weyl functions, $m_{-\infty}$, m_{∞} , and the corresponding
Weyl solutions associated with $H_{-\infty}$ and H_{∞} , respectively. This will enable the
asymptotic behaviour of $v(r, \lambda)$ as $r \to \pm \$ Weyl solutions associated with $H_{-\infty}$ and H_{∞} , respectively. This will enable the asymptotic behaviour of $v(r, \lambda)$ as $r \to \pm \infty$ to be ascertained in terms of the theory of subordinacy.
From (3.11) and (3.14), we h asymptotic behaviour of $v(r, \lambda)$ as $r \to \pm \infty$ to be ascertained in terms of the
theory of subordinacy.
From (3.11) and (3.14), we have for $\sigma_{11} \neq 0$, $\sigma_{22} \neq 0$,
 $v(r, \lambda) = \left(\sigma_{11}^{1/2} \ \sigma_{11}^{-1/2} \sigma_{12}\right) \left(\begin{array}{c} u$

$$
v(r,\lambda) = \begin{pmatrix} \sigma_{11}^{1/2} & \sigma_{11}^{-1/2} & \sigma_{12} \end{pmatrix} \begin{pmatrix} u_1(r,\lambda) \\ u_2(r,\lambda) \end{pmatrix},
$$
 (4.2)

From (3.11) and (3.14), we have for $\sigma_{11} \neq 0$, $\sigma_{22} \neq 0$,
 $v(r, \lambda) = \left(\sigma_{11}^{1/2} \sigma_{11}^{-1/2} \sigma_{12}\right) \left(\begin{array}{c} u_1(r, \lambda) \\ u_2(r, \lambda) \end{array}\right)$

since by Kac's theorem the assumption that *H* has simple

det(σ_{ij}) = 0, a $(r, \lambda) = \left(\sigma_{11}^{1/2} \quad \sigma_{11}^{-1/2} \quad \sigma_{12}\right) \left(\begin{array}{c} u_1(r, \lambda) \\ u_2(r, \lambda)\end{array}\right)$
em the assumption that *H* has simple
nce that the second row of *P* in (3.11)
it follows from (3.12) and (3.11) th
spectively.
 \cdot , λ) is a (r, λ) \prime

imple spectrum implies that

(3.11) is zero. In cases where

1) that $v(r, \lambda) = u_2(r, \lambda)$ or
 $u_1(r, \lambda)$ and $u_2(r, \lambda)$, whose

ure of the spectrum at λ . In since by Kac's theorem the assumption that *H* has simple spectrum implies that $det(\sigma_{ij}) = 0$, and hence that the second row of *P* in (3.11) is zero. In cases where $\sigma_{11} = 0$ or $\sigma_{22} = 0$, it follows from (3.12) and (3 det(σ_{ij}) = 0, and hence that the second row of *P* in (3.11) is zero. In cases where $\sigma_{11} = 0$ or $\sigma_{22} = 0$, it follows from (3.12) and (3.11) that $v(r, \lambda) = u_2(r, \lambda)$ or $v(r, \lambda) = u_1(r, \lambda)$, respectively.
From (4.2), $v(r, \lambda) = u_1(r, \lambda)$, respectively.

 $\sigma_{11} = 0$ or $\sigma_{22} = 0$, it follows from (3.12) and (3.11) that $v(r, \lambda) = u_2(r, \lambda)$ or $v(r, \lambda) = u_1(r, \lambda)$, respectively.
From (4.2), $v(r, \lambda)$ is a linear combination of $u_1(r, \lambda)$ and $u_2(r, \lambda)$, whose coefficients are fu $(r, \lambda) = u_1(r, \lambda)$, respectively.
From (4.2), $v(r, \lambda)$ is a loefficients are functions of λ
rder to determine the coeffici
 $S_1 := {\lambda \in \mathbf{R} : n \atop S_2 := {\lambda \in \mathbf{R} : n \atop \dots}}$ From (4.2), $v(r, \lambda)$ is a linear combination of $u_1(r, \lambda)$ and $u_2(r, \lambda)$, whose
cients are functions of λ which reflect the nature of the spectrum at λ . In
to determine the coefficients, we first introduce the foll

coefficients are functions of
$$
\lambda
$$
 which reflect the nature of the spectrum at λ . In
order to determine the coefficients, we first introduce the following disjoint sets:
 $S_1 := \{ \lambda \in \mathbf{R} : m_{-\infty}(\lambda) \in \mathbf{C}^-, m_{\infty}(\lambda) \in \mathbf{R} \cup \{ \infty \} \}$
 $S_2 := \{ \lambda \in \mathbf{R} : m_{-\infty}(\lambda) \in \mathbf{R} \cup \{ \infty \}, m_{\infty}(\lambda) \in \mathbf{C}^+ \}$
 $S_3 := \{ \lambda \in \mathbf{R} : m_{-\infty}(\lambda) = m_{\infty}(\lambda) \in \mathbf{R} \cup \{ \infty \} \}$
 $S_4 := \{ \lambda \in \mathbf{R} : m_{-\infty}(\lambda) \in \mathbf{C}^-, m_{\infty}(\lambda) \in \mathbf{C}^+ \}$
Note that $S_1 \cup S_2$ is a minimal support of the absolutely continuous part of the
spectral measure μ_{τ} in \mathcal{M}_1 . The set S_3 is a minimal support of the singular part
of μ_{τ} and always has Lebesgue measure zero. Altogether the set $\{ S_i : i = 1, ..., 3 \}$

Note that $S_1 \cup S_2$ is a minimal support of the absolutely continuous part of the spectral measure μ_{τ} in \mathcal{M}_1 . The set S_3 is a minimal support of the singular part := { $\lambda \in \mathbf{R} : m_{-\infty}(\lambda) = m_{\infty}(\lambda) \in \mathbf{R} \cup \{\infty\}$ }

:= { $\lambda \in \mathbf{R} : m_{-\infty}(\lambda) \in \mathbf{C}^-$, $m_{\infty}(\lambda) \in \mathbf{C}^+$ }

2 is a minimal support of the absolutely co:
 μ_{τ} in \mathcal{M}_1 . The set S_3 is a minimal support
 := { $\lambda \in \mathbf{R} : m_{-\infty}(\lambda) \in \mathbf{C}^-$, $m_{\infty}(\lambda) \in \mathbf{C}^+$ }

2 is a minimal support of the absolutely co
 μ_{τ} in \mathcal{M}_1 . The set S_3 is a minimal suppor

has Lebesgue measure zero. Altogether the

mal support Note that $S_1 \cup S_2$ is a minimal support of the absolutely continuous part of the spectral measure μ_{τ} in \mathcal{M}_1 . The set S_3 is a minimal support of the singular part of μ_{τ} and always has Lebesgue measur spectral measure μ_{τ} in \mathcal{M}_1 . The set S_3 is a minimal support of the singular part
of μ_{τ} and always has Lebesgue measure zero. Altogether the set $\{S_i : i = 1, ..., 3\}$
constitutes a minimal support of the p of μ_{τ} and always has Lebesgue measure zero. Altogether the set $\{S_i : i = 1, ..., 3\}$ constitutes a minimal support of the part of the spectral measure μ_{τ} corresponding to the simple part of the spectrum, so is equ constitutes a minimal support of the part of the spectral measure μ_{τ} corresponding
to the simple part of the spectrum, so is equal to \mathcal{M}_1 up to sets of μ_{τ} - and Lebesgue
measure zero. Also from Corollary to the simple part of the spectrum, so is equal to M_1 up to sets of μ_{τ} - and Lebesgue
measure zero. Also from Corollary 1, S_4 and M_2 differ at most by μ_{τ} - and Lebesgue
null sets, so that the degenerate measure zero. Also from Corollary 1, S_4 and M_2 differ at most by μ_{τ} - and Lebesgue
null sets, so that the degenerate part of the spectrum is supported on those values
of λ on which both $m_{-\infty}(\lambda)$ and m_{\in of λ on which both $m_{-\infty}(\lambda)$ and $m_{\infty}(\lambda)$ are strictly complex (see also [10]). Note that the resolvent set is the largest open set in $\mathbb{R}\setminus\{S_i : i = 1, ..., 4\}$. that the resolvent set is the largest open set in $\mathbf{R}\backslash\{S_i:i=1,\ldots,4\}.$

100 D.J. Gilbert

In this section we concentrate particularly on S_1 and S_2 which are associated

with the simple part of the absolutely continuous spectrum. In fact it is sufficient to

consider S_1 in detail, sin In this section we concentrate particularly on S_1 and S_2 which are associated
the simple part of the absolutely continuous spectrum. In fact it is sufficient to
der S_1 in detail, since the derivations and results

consider S_1 in detail, since the derivations and results are almost entirely analogous
for S_2 .
For simplicity of exposition, suppose in the first instance that $S_1 = \mathbf{R}$. If
 $\lambda \in S_1$, so that $m_{\infty}(\lambda) \in \mathbf{R} \$ for S_2 .
F
 $\lambda \in S_1$
and (3)
and he For simplicity of exposition, suppose in the first instance that $S_1 = \mathbf{R}$. If S_1 , so that $m_{\infty}(\lambda) \in \mathbf{R} \cup \{\infty\}$, then since $m_{-\infty}(\lambda) \in \mathbf{C}^-$, we have from (3.6)
3.7),
 $\sigma_{11} = \frac{(m_{\infty}(\lambda))^2}{1 + (m_{\infty}(\lambda))^2}$ $\lambda \in S_1$ and (3.7), σ_{11} = σ_{12} and hence from (4.2),

, so that
$$
m_{\infty}(\lambda) \in \mathbf{R} \cup \{\infty\}
$$
, then since $m_{-\infty}(\lambda) \in \mathbf{C}^-$, we have from (3.6)
\n.7),
\n
$$
\sigma_{11} = \frac{(m_{\infty}(\lambda))^2}{1 + (m_{\infty}(\lambda))^2}, \qquad \sigma_{12} = \frac{m_{\infty}(\lambda)}{1 + (m_{\infty}(\lambda))^2}
$$
(4.3)
\n
$$
v(r, \lambda) = \frac{u_2(r, \lambda) + m_{\infty}(\lambda) u_1(r, \lambda)}{(1 + (m_{\infty}(\lambda))^2)^{1/2}},
$$
(4.4)
\nis a real solution of the differential equation and a scalar multiple of the
\nis a limit of the Weral solution for U , with $u_r(\lambda)$ in (a) to (∞) and (∞) are the

$$
\frac{(m_{\infty}(\lambda))^2}{1 + (m_{\infty}(\lambda))^2}, \qquad \sigma_{12} = \frac{m_{\infty}(\lambda)}{1 + (m_{\infty}(\lambda))^2}
$$
(4.3)

$$
v(r, \lambda) = \frac{u_2(r, \lambda) + m_{\infty}(\lambda) u_1(r, \lambda)}{(1 + (m_{\infty}(\lambda))^2)^{1/2}}, \qquad (4.4)
$$

n of the differential equation and a scalar multiple of the
Weyl solution for H_{∞} , viz. $u_2(r, z) + m_{\infty}(z) u_1(r, z)$, as $z \downarrow \lambda$.

which is a real soluti
pointwise limit of the
Note also that if m_{∞} $(r, \lambda) = \frac{u_2(r, \lambda) + m_{\infty}(\lambda) u_1(r, \lambda)}{(1 + (m_{\infty}(\lambda))^2)^{1/2}},$ (4.4)

of the differential equation and a scalar multiple of the

eyl solution for H_{∞} , viz. $u_2(r, z) + m_{\infty}(z) u_1(r, z)$, as $z \downarrow \lambda$.
 $\rightarrow \infty$ as $z \downarrow \lambda \in \mathbb{$ pointwise limit of the Weyl solution for H_{∞} , viz. $u_2(r, z) + m_{\infty}(z) u_1(r, z)$, as $z \downarrow \lambda$.
Note also that if $m_{\infty}(z) \to \infty$ as $z \downarrow \lambda \in \mathbf{R}$, it follows from (4.3) that $\sigma_{11} = 1$ and $\sigma_{12} = 0$, so that $v(r, \lambda$ Note also that if $m_{\infty}(z) \to \infty$ as $z \downarrow \lambda \in \mathbf{R}$, it follows from (4.3) that $\sigma_{11} = 1$ and $\sigma_{12} = 0$, so that $v(r, \lambda) = u_1(r, \lambda)$, which is an eigenfunction of H_{∞} ; this is also directly evident from (4.4) abo σ_{12} = 0, so that $v(r, \lambda) = u_1(r, \lambda)$, which is an eigenfunction of H_{∞} ; this is also
ctly evident from (4.4) above. Thus we see that for all λ in S_1 , $v(r, \lambda)$ is a real
tion of $Lu = \lambda u$ which is subordinate at ∞ . directly evident from (4.4) above. Thus we see that for all λ in S_1 , $v(r, \lambda)$ is a real
solution of $Lu = \lambda u$ which is subordinate at ∞ . Also, since $m_{-\infty}(\lambda)$ is strictly
complex for $\lambda \in S_1$, we infer from the solution of $Lu = \lambda u$ which is subordinate at ∞. Also, since $m_{-\infty}(\lambda)$ is strictly
complex for $\lambda \in S_1$, we infer from the theory of subordinacy that all solutions are
of comparable asymptotic size as $r \to -\infty$ [11]. Sub complex for $\lambda \in S_1$, we infer from the theory of subordinacy that all solutions are
of comparable asymptotic size as $r \to -\infty$ [11]. Substituting for $v(r, \lambda)$ from (4.4)
into (4.1) now yields for $f(r) \in L_2(\mathbf{R})$,
 $f(r) =$ into (4.1) now yields for $f(r) \in L_2(\mathbf{R})$,

of comparable asymptotic size as
$$
r \to -\infty
$$
 [11]. Substituting for $v(r, \lambda)$ from (4.4)
\ninto (4.1) now yields for $f(r) \in L_2(\mathbf{R})$,
\n
$$
f(r) = \frac{\lim_{\omega \to \infty} \int_{-\omega}^{\omega} \frac{u_2(r, \lambda) + m_\infty(\lambda) u_1(r, \lambda)}{(1 + (m_\infty(\lambda))^2)^{1/2}} G(\lambda) d\rho_r(\lambda),
$$
\n(4.5)
\nwhere
\n
$$
G(\lambda) = \frac{\lim_{\eta \to \infty} \int_{-\eta}^{\eta} \frac{u_2(r, \lambda) + m_\infty(\lambda) u_1(r, \lambda)}{(1 + (m_\infty(\lambda))^2)^{1/2}} f(r) dr,
$$
\nwith convergence as before.
\nWe illustrate the process outlined above by considering two operators, both
\nof which are in the limit exist, see at the end have much absolutely continuous

$$
G(\lambda) = \lim_{\eta \to \infty} \int_{-\eta}^{\eta} \frac{u_2(r, \lambda) + m_{\infty}(\lambda) u_1(r, \lambda)}{(1 + (m_{\infty}(\lambda))^{2})^{1/2}} f(r) dr,
$$

with c

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of whi

spectr $(\lambda) = \lim_{\eta \to \infty} \int_{-\eta}^{\eta}$

; as before.

se the process of the process of the process of the process of the dimit point of the set $\frac{(r, \lambda) + m_{\infty}(\lambda) u_1(r, \lambda)}{(1 + (m_{\infty}(\lambda))^{2})^{1/2}} f(r) dr,$
ined above by considering two at $\pm \infty$ and have purely absolu We illustrate the procent of which are in the limit pospectrum.
Spectrum.
Example 1. Let ich are in the limit point case at $\pm \infty$ and have purely absolutely continuous
rum.
 ple 1. Let
 $q(r) = \begin{cases} 0 & -\infty < r < 0, \\ 1 & 0 \le r < \infty, \end{cases}$ (4.6) of which are in the limit point case at $\pm \infty$ and have purely absolutely continuous
spectrum.
Example 1. Let
 $q(r) = \begin{cases} 0 & -\infty < r < 0, \\ 1 & 0 \le r < \infty, \end{cases}$ (4.6)
and define a fundamental set of solutions $\{u_1(r, z), u_2(r, z$

Example 1.

$$
q(r) = \begin{cases} 0 & -\infty < r < 0, \\ 1 & 0 \le r < \infty, \end{cases} \tag{4.6}
$$

Example :
and define
satisfy the fun
ndit
we l $(r) = \begin{cases} 0 & -\infty < r < 0, \\ 1 & 0 \le r < \infty, \end{cases}$ (4.6)

et of solutions $\{u_1(r, z), u_2(r, z)\}$ of $Lu = zu$ on **R** to

c). By choosing the \sqrt{z} -plane to have positive imaginary
 $z > 0,$
 $L_2(\mathbf{R}^-), \qquad \exp(i\sqrt{z-1}r) \in L_2(\mathbf{R}^+),$
 and define a fundamental set of solutions $\{u_1(r, z), u_2(r, z)\}$ of $Lu = zu$ on **R** to
satisfy the conditions in (2.1). By choosing the \sqrt{z} -plane to have positive imaginary
part in \mathbb{C}^+ , we have for $\text{Im} z > 0$,
 $\exp(-i$ satisfy the conditions in (2.1). By choosing the \sqrt{z} -plane to have positive imaginary part in \mathbb{C}^+ , we have for $\text{Im} z > 0$,

$$
\exp(-i\sqrt{z}r) \in L_2(\mathbf{R}^-), \qquad \exp(i\sqrt{z-1}r) \in L_2(\mathbf{R}^+),
$$

is straightforward to show that for $z \in \mathbf{C}^+$,

$$
m_{-\infty}(z) = -i\sqrt{z}, \qquad m_{\infty}(z) = i\sqrt{z-1},
$$

from which it is straightforward to show that for $z \in \mathbb{C}^+$,
 $m_{-\infty}(z) = -i\sqrt{z}$, $m_{\infty}(z) = i\sqrt{z}$.

$$
m_{-\infty}(z) = -\mathrm{i}\sqrt{z}, \qquad m_{\infty}(z) = \mathrm{i}\sqrt{z-1},
$$

$$
m_{-\infty}(\lambda) = -i\sqrt{\lambda}, \qquad m_{\infty}(\lambda) = i\sqrt{\lambda - 1}, \tag{4.7}
$$

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so that the boundary values for $z = \lambda \in \mathbf{R}$ are given by
 $m_{-\infty}(\lambda) = -i\sqrt{\lambda}, \qquad m_{\infty}(\lambda) = i\sqrt{\lambda - 1},$ (4.7)
respectively. We remark that $m_{-\infty}(\lambda)$ is real for $\lambda \le$ so that the boundary values for $z = \lambda \in \mathbf{R}$ are given by
 $m_{-\infty}(\lambda) = -i\sqrt{\lambda}, \qquad m_{\infty}(\lambda) = i\sqrt{\lambda}$

respectively. We remark that $m_{-\infty}(\lambda)$ is real for $\lambda \leq 1$

while $m_{\infty}(\lambda)$ is real for $\lambda \leq 1$ and complex for (λ) = $-i\sqrt{\lambda}$, $m_{\infty}(\lambda) = i\sqrt{\lambda - 1}$, (4.7)

that $m_{-\infty}(\lambda)$ is real for $\lambda \le 0$ and complex for $\lambda > 0$,
 ≤ 1 and complex for $\lambda > 1$. Thus we have $S_1 = (0, 1]$,
 $S_d = (1, \infty)$, where S_d is as in Corollary 1. S respectively. We remark that $m_{-\infty}(\lambda)$ is real for $\lambda \le 0$ and complex for $\lambda > 0$,
while $m_{\infty}(\lambda)$ is real for $\lambda \le 1$ and complex for $\lambda > 1$. Thus we have $S_1 = (0, 1]$,
 $S_2 = S_3 = \emptyset$ and $S_4 = S_d = (1, \infty)$, where while $m_{\infty}(\lambda)$ is real for $\lambda \le 1$ and complex for $\lambda > 1$. Thus we have $S_1 = (0, 1]$,
 $S_2 = S_3 = \emptyset$ and $S_4 = S_d = (1, \infty)$, where S_d is as in Corollary 1. Since both
 $m_{-\infty}(\lambda)$ and $m_{\infty}(\lambda)$ are real for $\lambda \le 0$ S_2 = $S_3 = \emptyset$ and $S_4 = S_d = (1, \infty)$, where S_d is as in Corollary 1. Since both $\infty(\lambda)$ and $m_\infty(\lambda)$ are real for $\lambda \leq 0$, we may take $(\sigma_{ij}(\lambda))$ to be the zero matrix n (3.5) for $-\infty < \lambda \leq 0$, and note that the resolv $m_{-\infty}(\lambda)$ and $m_{\infty}(\lambda)$ are real for $\lambda \leq 0$, we may take $(\sigma_{ij}(\lambda))$ to be the zero matrix as in (3.5) for $-\infty < \lambda < 0$, and note that the resolvent set is $(-\infty, 0)$.

$$
m_{-\infty}(\lambda)
$$
 and $m_{\infty}(\lambda)$ are real for $\lambda \leq 0$, we may take $(\sigma_{ij}(\lambda))$ to be the zero matrix as in (3.5) for $-\infty < \lambda \leq 0$, and note that the resolvent set is $(-\infty, 0)$. It is straightforward to determine the spectral density matrix explicitly for $\lambda > 0$, using (3.4)–(3.8). For $0 < \lambda \leq 1$, we note that $m_{\infty}(\lambda) = -\sqrt{1-\lambda}$ to obtain $\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = \frac{1}{2-\lambda}\left(\begin{array}{cc}1-\lambda & -\sqrt{1-\lambda}\\-\sqrt{1-\lambda} & 1\end{array}\right),\right.$ (4.8) which has rank 1, from which it follows by Theorem 1 that the spectrum of H is the $0 < \lambda < 1$.

 $\lambda > 0$, using (3.4)–(3.8). For $0 < \lambda \le 1$, we note that $m_{\infty}(\lambda) = -\sqrt{1-\lambda}$ to obtain
 $\left(\frac{d\rho_{ij}}{d\rho_{\tau}}(\lambda)\right) = \frac{1}{2-\lambda}\left(\begin{array}{cc}1-\lambda & -\sqrt{1-\lambda}\\-\sqrt{1-\lambda}&1\end{array}\right)$, (4.8)

which has rank 1, from which it follows by Theorem 1 simple on $0 < \lambda \leq 1$. The spectral density matrix for $\lambda > 1$ is obtained in a similar way and has full rank (see [10], Example 5.10), so that $S_2 = (1, \infty) = \mathcal{M}_2$. which has rank 1, from which it follows by Theorem 1 that the spectrum of H is (λ)
where
see
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ity $=\frac{1}{2-\lambda}$
h it follo
cetral de:
0], Exar
atrix fo $1 - \lambda$ - $\sqrt{1 - \lambda}$ 1

y Theorem 1 that

matrix for $\lambda > 1$ is

5.10), so that $S_2 =$
 S_1 in (4.8) may Theorem 1 that the spectrum of *H* is
atrix for $\lambda > 1$ is obtained in a similar
(10), so that $S_2 = (1, \infty) = \mathcal{M}_2$.
 S_1 in (4.8) may be decomposed as in which has rank 1, from which it follows by Theorem 1 that the spectrum of *H* is
simple on $0 < \lambda \le 1$. The spectral density matrix for $\lambda > 1$ is obtained in a similar
way and has full rank (see [10], Example 5.10), so th

simple on
$$
0 < \lambda \le 1
$$
. The spectral density matrix for $\lambda > 1$ is obtained in a similar
way and has full rank (see [10], Example 5.10), so that $S_2 = (1, \infty) = M_2$.
The spectral density matrix for $\lambda \in S_1$ in (4.8) may be decomposed as in
(3.10), to give

$$
D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P = \frac{1}{\sqrt{2-\lambda}} \begin{pmatrix} -\sqrt{1-\lambda} & 1 \\ 0 & 0 \end{pmatrix}, \qquad (4.9)
$$

and hence from (4.7) and (4.9),

$$
PU = \begin{pmatrix} -\sqrt{1-\lambda} & 1 \\ -\sqrt{2-\lambda} & u_1(r,\lambda) + \frac{1}{\sqrt{2-\lambda}}u_2(r,\lambda) \end{pmatrix}
$$

$$
= \frac{u_2(r,\lambda) + m_{\infty}(\lambda)u_1(r,\lambda)}{(4.10)}
$$

 $\frac{1}{2}$ and hence from

$$
D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P = \frac{1}{\sqrt{2 - \lambda}} \begin{pmatrix} -\sqrt{1 - \lambda} & 1 \\ 0 & 0 \end{pmatrix}, \qquad (4.9)
$$

and hence from (4.7) and (4.9),

$$
PU = \left(-\frac{\sqrt{1 - \lambda}}{\sqrt{2 - \lambda}} u_1(r, \lambda) + \frac{1}{\sqrt{2 - \lambda}} u_2(r, \lambda) \right)
$$

$$
= \frac{u_2(r, \lambda) + m_{\infty}(\lambda) u_1(r, \lambda)}{(1 + (m_{\infty}(\lambda))^2)^{\frac{1}{2}}}
$$

$$
= v(r, \lambda),
$$

for $\lambda \in (0, 1]$, where $u_1(r, \lambda)$, $u_2(r, \lambda)$ satisfy

$$
u_1(r, \lambda) = \begin{cases} \frac{1}{2\sqrt{1 - \lambda}} \left(e^{\sqrt{1 - \lambda}r} - e^{-\sqrt{1 - \lambda}r} \right), & r \ge 0 \\ \frac{\sin(\sqrt{\lambda}r)}{r} & r < 0 \end{cases}
$$
(4.11)

$$
= v(r,\lambda),
$$

for $\lambda \in (0,1]$, where $u_1(r,\lambda)$, $u_2(r,\lambda)$ satisfy

$$
u_1(r,\lambda) = \begin{cases} \frac{1}{2\sqrt{1-\lambda}} \left(e^{\sqrt{1-\lambda}r} - e^{-\sqrt{1-\lambda}r} \right), & r \ge 0 \\ \frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}}, & r < 0 \end{cases}
$$
(4.11)

$$
u_2(r,\lambda) = \begin{cases} \frac{1}{2} \left(e^{\sqrt{1-\lambda}r} + e^{-\sqrt{1-\lambda}r} \right), & r \ge 0 \\ \cos(\sqrt{\lambda}r) & r < 0 \end{cases}
$$
(4.12)

$$
u_1(r,\lambda) = \begin{cases} 2\sqrt{1-\lambda} & \text{if } r < 0 \\ \frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}}, & r < 0 \end{cases} \tag{4.11}
$$
\n
$$
u_2(r,\lambda) = \begin{cases} \frac{1}{2} \left(e^{\sqrt{1-\lambda}r} + e^{-\sqrt{1-\lambda}r} \right), & r \ge 0 \\ \cos\left(\sqrt{\lambda}r\right), & r < 0 \end{cases} \tag{4.12}
$$
\n
$$
u_1(4.11) \text{ and } (4.12) \text{ that for } \lambda \in (0,1], \text{ no solution of } Lu = \lambda u \text{ at } -\infty, \text{ and that the real-valued eigenfunction } v(r,\lambda) \text{ in } (4.10)
$$

 $(r, \lambda) =$
4.11) a:
 ∞ , is subo
nd (3.2) + $e^{-\sqrt{1-\lambda}r}$
 $(\sqrt{\lambda}r)$,

for $\lambda \in (0$

ceal-valued ϵ

Moreover, si 0
 $\frac{1}{10}$ cos $(\sqrt{\lambda}r)$, $r < 0$

that for $\lambda \in (0,1]$, no solution of $Lu = \lambda u$

the real-valued eigenfunction $v(r, \lambda)$ in (4.10)
 ∞ . Moreover, since $m_{\infty}(\lambda)$ is real on (0, 1], we

(λ) $(1 + (m_{\infty}(\lambda))^2)$ It follows from (4.11) and (4.12) that for $\lambda \in (0, 1]$, no solution of $Lu = \lambda u$
is subordinate at $-\infty$, and that the real-valued eigenfunction $v(r, \lambda)$ in (4.10)
satisfies (4.5) and is subordinate at ∞ . Moreover, sin

is subordinate at
$$
-\infty
$$
, and that the real-valued eigenfunction $v(r, \lambda)$ in (4.10)
satisfies (4.5) and is subordinate at ∞ . Moreover, since $m_{\infty}(\lambda)$ is real on (0, 1], we
have from (3.1) and (3.2),

$$
\rho'_{\tau}(\lambda) = -\frac{1}{\pi} \frac{\text{Im } m_{-\infty}(\lambda) (1 + (m_{\infty}(\lambda))^2)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|^2}, \quad 0 < \lambda \le 1,
$$
 (4.13)

$$
= (2 - \lambda) \rho'_{-\infty}(\lambda),
$$

noting that on this
$$
\lambda
$$
-interval,
\n
$$
\rho'_{-\infty}(\lambda) = -\frac{1}{\pi} \operatorname{Im} m_{-\infty}(\lambda), \qquad 1 + (m_{\infty}(\lambda))^2 = 2 - \lambda,
$$
\nand\n
$$
|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|^2 = 1.
$$
\nSince $m_{\infty}(\lambda)$ is finite for all values of λ , we may multiply (4.10) by (1 and divide (4.13) by (1 + (m_0(\lambda))^2) to obtain a simpler form of

$$
|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|^2 = 1.
$$

 $(\lambda) = -\frac{1}{\pi} \operatorname{Im} m_{-\infty}(\lambda), \quad 1 + (m_{\infty}(\lambda))^2 = 2 - \lambda,$
 $| m_{-\infty}(\lambda) - m_{\infty}(\lambda) |^2 = 1.$

te for all values of λ , we may multiply (4.10) by (1

by $(1 + (m_{\infty}(\lambda))^2)$ to obtain a simpler form of

on $S_1 = (0, 1]$. The part of $|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|^2 = 1.$
Since $m_{\infty}(\lambda)$ is finite for all values of λ , we may multipl
and divide (4.13) by $(1 + (m_{\infty}(\lambda))^2)$ to obtain a sin
(4.5) which holds on $S_1 = (0, 1]$. The part of the expa-
simple spectr Since $m_{\infty}(\lambda)$ is finite for all values of λ , we may multiply (4.10) by $(1 + (m_{\infty}(\lambda))^2)$
and divide (4.13) by $(1 + (m_{\infty}(\lambda))^2)$ to obtain a simpler form of the expansic
(4.5) which holds on $S_1 = (0, 1]$. The part o $\frac{1}{2}$ and divide (4.13) by $(1 + (m_{\infty}(\lambda))^2)$ to obtain a simpler form of the expansion
(4.5) which holds on $S_1 = (0,1]$. The part of the expansion corresponding to the
simple spectrum of H on $S_1(\lambda)$ then reduces to the follow (4.5) which holds on $S_1 = (0, 1]$. The part of the expansion corresponding to the simple spectrum of H on $S_1(\lambda)$ then reduces to the following spectral projection:
 $\mathcal{P}_{(0,1]}f(r) = \int_0^1 e^{-\sqrt{1-\lambda}r} G(\lambda) \sqrt{\lambda} d\lambda$,

whe

simple spectrum of *H* on
$$
S_1(\lambda)
$$
 then reduces to the following spectral projection:
\n
$$
\mathcal{P}_{(0,1]}f(r) = \int_0^1 e^{-\sqrt{1-\lambda}r} G(\lambda) \sqrt{\lambda} d\lambda,
$$
\nwhere
\n
$$
G(\lambda) = \int_{-\infty}^{\infty} e^{-\sqrt{1-\lambda}r} f(r) dr,
$$
\nand the integrals converge absolutely. Note that the full expansion for *H* would also include the set $S_4 = (1, \infty)$, on which the spectrum is purely absolutely continuous

$$
e^{-\sqrt{1-\lambda}r} G(\lambda) \sqrt{\lambda}
$$

$$
G(\lambda) = \int_{-\infty}^{\infty} e^{-\sqrt{1-\lambda}r} f(r) dr,
$$
absolutely. Note that the full e:
0, on which the spectrum is pu

 $G(\lambda) = \int_{-\infty}^{\infty} e^{-\sqrt{1-\lambda}r} f(r) dr$,
and the integrals converge absolutely. Note that the full e
include the set $S_4 = (1, \infty)$, on which the spectrum is pu
with multiplicity 2.
Example 2. Consider the Airy operator assoc and the integrals converge absolutely. Note that the full expansion for *H* would also
include the set $S_4 = (1, \infty)$, on which the spectrum is purely absolutely continuous
with multiplicity 2.
Example 2. Consider the Ai

include the set $S_4 = (1, \infty)$, on which the spectrum is purely absolutely continuous
with multiplicity 2.
Example 2. Consider the Airy operator associated with the singular Sturm-Liou-
ville equation,
 $-u''(r, \lambda) + ru(r, \lambda) = \$ **Example 2.** Consider the Airy operator associated with the singular Sturm-Liou-

$$
-u''(r,\lambda) + ru(r,\lambda) = \lambda u(r,\lambda), \quad -\infty < r < \infty. \tag{4.14}
$$

Example 2. Considering $-u$
It is well known that λ , $Bi(r - \lambda)$, and ville equation,
 $-u''(r,\lambda) + r u(r,\lambda) = \lambda u(r,\lambda), \quad -\infty < r < \infty.$ (4.14)

It is well known that a fundamental set of solutions for (4.14) is given by $\{Ai(r - \lambda), Bi(r - \lambda)\}$, and that the differential equation is in the limit point case a $(r, \lambda) + r u(r, \lambda) = \lambda u(r, \lambda), \quad -\infty < r < \infty.$ (4.14)

i a fundamental set of solutions for (4.14) is given by $\{Ai(r - \text{hat the differential equation is in the limit point case at both})\}$

hat the differential equation is in the limit point case at both

wer, Ai and Bi can be expr It is well known that a fundamental set of solutions for (4.14) is given by $\{Ai(r - \lambda), Bi(r - \lambda)\}\$, and that the differential equation is in the limit point case at both endpoints [5]. Moreover, Ai and Bi can be expressed in t λ endpoints [5]. Moreover, Ai and Bi can be expressed in terms of Bessel functions [1], [15], and the solution $Ai(r - \lambda)$, $\lambda \in \mathbf{R}$, is real valued and square integrable at infinity with respect to r for all $\lambda \in \mathbf{R}$. [1], [15], and the solution $Ai(r - \lambda)$, $\lambda \in \mathbf{R}$, is real valued and square integrable at [1], [15], and the solution $Ai(r - \lambda)$, $\lambda \in \mathbf{R}$, is real valued and square integrable at infinity with respect to r for all $\lambda \in \mathbf{R}$. It follows that $Ai(r - \lambda)$ is subordinate at infinity [11], from which it may be i infinity with respect to *r* for all $\lambda \in \mathbf{R}$. It follows that $Ai(r - \lambda)$ is subordinate at infinity [11], from which it may be inferred that
 $m_{\infty}(\lambda) = \lim_{z \downarrow \lambda} \frac{Ai'(r - z)}{Ai(r - z)} \Big|_{r=0}$, (4.15)

for all $\lambda \in \mathbf{R}$,

$$
m_{\infty}(\lambda) = \lim_{z \downarrow \lambda} \frac{Ai'(r-z)}{Ai(r-z)}\bigg|_{r=0},\tag{4.15}
$$

for all $\lambda \in \mathbf{R}$, where ' denotes differentiation with respect to r.
To investigate the asymptotic behaviour of solutions of (4.14) at $-\infty$, we first note that the conditions for validity of the Liouville-Green appro $(\lambda) = \lim_{z \downarrow \lambda} \frac{Ai'(r-z)}{Ai(r-z)}$
differentiation with
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, Chapter 6). Hence
 (r, λ) , of solutions |
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| 31 i($r - z$) |_{r=0},

on with respect to *r*.

viour of solutions of (4.14) at $-\infty$, we

f the Liouville-Green approximation are

. Hence the asymptotic behaviour of a

olutions of (4.14) as $r \to -\infty$ is given for all $\lambda \in \mathbf{R}$, where ' denotes differentiation with respect to r.
To investigate the asymptotic behaviour of solutions of
first note that the conditions for validity of the Liouville-Green
satisfied in this case (s To investigate the asymptotic behaviour of solutions of (4.14) at $-\infty$, we
note that the conditions for validity of the Liouville-Green approximation are
ied in this case (see [15], Chapter 6). Hence the asymptotic behav satisfied in this case (see [15], Chapter 6). Hence the asymptotic behaviour of a
fundamental set, $\{u_+(r,\lambda), u_-(r,\lambda)\}$, of solutions of (4.14) as $r \to -\infty$ is given
by:
 $u_{\pm}(r,\lambda) = \frac{1}{(r-\lambda)^{\frac{1}{4}}} \exp\left(\int^r \pm (r-\lambda)^{\frac{1}{2}} dr\$

fundamental set,
$$
\{u_{+}(r,\lambda), u_{-}(r,\lambda)\}\
$$
, of solutions of (4.14) as $r \to -\infty$ is given
by:

$$
u_{\pm}(r,\lambda) = \frac{1}{(r-\lambda)^{\frac{1}{4}}}\exp\left(\int^r \pm (r-\lambda)^{\frac{1}{2}} dr\right) (1+o(1)). \qquad (4.16)
$$

Since for each fixed $\lambda \in \mathbf{R}$, $(r-\lambda)$ is eventually negative as $r \to -\infty$, we may write

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wri

$$
u_{\pm}(r,\lambda) = \frac{1}{(r-\lambda)^{\frac{1}{4}}}\exp\left(\int^{r} \pm (r-\lambda)^{\frac{1}{2}} dr\right) (1+o(1)).\tag{4.16}
$$

Since for each fixed $\lambda \in \mathbf{R}$, $(r-\lambda)$ is eventually negative as $r \to -\infty$, we may write

$$
u_{\pm}(r,\lambda) = K \frac{1}{(\lambda-r)^{\frac{1}{4}}}\exp\left(\int^{r} \pm i(\lambda-r)^{\frac{1}{2}} dr\right) (1+o(1)),\tag{4.17}
$$

or each $\lambda \in \mathbf{R}$, where K is a constant which is independent of r
be deduced from (4.17) that for every real value of λ , no solution
bordinate at $-\infty$, so that $m_{-\infty}(\lambda) \in \mathbf{C}^-$ for all $\lambda \in \mathbf{R}$ [11]. This, as $r \to -\infty$ for each $\lambda \in \mathbf{R}$, where K is a constant which is independent of r and λ . It may be deduced from (4.17) that for every real value of λ , no solution of (4.14) is subordinate at $-\infty$, so that $m_{-\infty}$ and λ . It may be deduced from (4.17) that for every real value of λ , no solution
of (4.14) is subordinate at $-\infty$, so that $m_{-\infty}(\lambda) \in \mathbb{C}^-$ for all $\lambda \in \mathbb{R}$ [11]. This,
together with (4.15), which implies t

together with (4.15), which implies that $m_{\infty}(\lambda) \in \mathbf{R} \cup \{\infty\}$ for all $\lambda \in \mathbf{R}$, shows that $S_1 = \mathbf{R}$.
Thus we have shown that in the case of the Airy operator, the spectrum of *H* is purely absolutely conti together with (4.15), which implies that $m_{\infty}(\lambda) \in \mathbf{R} \cup \{\infty\}$ for all $\lambda \in \mathbf{R}$, shows
that $S_1 = \mathbf{R}$.
Thus we have shown that in the case of the Airy operator, the spectrum of
H is purely absolutely contin that $S_1 = \mathbf{R}$.
Thus we
H is purely a
that the abso
are given by *H* is purely absolutely continuous with multiplicity one on the whole real line, and
that the absolutely continuous eigenfunctions which feature in the expansion (4.1)
are given by
 $v(r, \lambda) = \frac{u_2(r, \lambda) + m_{\infty}(\lambda)u_1(r, \lambda)}{(1$ is purely absolutely continuous eigenfunctions which feature in the expansion (4.1)

given by
 $v(r,\lambda) = \frac{u_2(r,\lambda) + m_\infty(\lambda)u_1(r,\lambda)}{(1 + (m_\infty(\lambda))^2)^{1/2}},$ (4.18)

ere $m_\infty(\lambda)$ is as in (4.15). Note that $v(r,\lambda)$ is a scalar multi

$$
v(r,\lambda) = \frac{u_2(r,\lambda) + m_\infty(\lambda)u_1(r,\lambda)}{(1 + (m_\infty(\lambda))^2)^{1/2}},
$$
\n(4.18)

are given by
 $v(r,\lambda) = \frac{u_2(r,\lambda) + m_{\infty}(\lambda)u_1(r,\lambda)}{(1 + (m_{\infty}(\lambda))^2)^{1/2}},$ (4.18)

where $m_{\infty}(\lambda)$ is as in (4.15). Note that $v(r,\lambda)$ is a scalar multiple of $Ai(r - \lambda)$,

and that when λ is an eigenvalue of H_{∞} , so tha $v(r, \lambda) = \frac{u_2(r, \lambda) + m_{\infty}(\lambda)u_1(r, \lambda)}{(1 + (m_{\infty}(\lambda))^2)^{1/2}},$ (4.18)
where $m_{\infty}(\lambda)$ is as in (4.15). Note that $v(r, \lambda)$ is a scalar multiple of $Ai(r - \lambda)$,
and that when λ is an eigenvalue of H_{∞} , so that $m_{\infty}(z) \$

where $m_{\infty}(\lambda)$ is as in (4.15). Note that $v(r, \lambda)$ is a scalar multiple of $Ai(r - \lambda)$,
and that when λ is an eigenvalue of H_{∞} , so that $m_{\infty}(z) \to \infty$ as $z \downarrow \lambda$, it follows
from (4.18) that $v(r, \lambda) = u_1(r, \lambda)$.
 and that when λ is an eigenvalue of H_{∞} , so that $m_{\infty}(z) \to \infty$ as $z \downarrow \lambda$, it follows
from (4.18) that $v(r, \lambda) = u_1(r, \lambda)$.
To determine the scalar Herglotz function, $M(z)$, explicitly it is first necessary
to id from (4.18) that $v(r, \lambda) = u_1(r, \lambda)$.
To determine the scalar Hergl
to identify (up to a scalar multip
 $L_2(\mathbf{R}^-)$ for $z \in \mathbf{C}^+$. Then $m_{-\infty}$
derivative of the Weyl solution at r
These examples confirm that where
ti To determine the scalar Herglotz function, $M(z)$, explicitly it is first necessary
entify (up to a scalar multiple) the Weyl solution of $Lu = zu$, which is in
 \Box for $z \in \mathbb{C}^+$. Then $m_{-\infty}$ can be obtained by evaluati $L_2(\mathbf{R}^-)$ for $z \in \mathbf{C}^+$. Then $m_{-\infty}$ can be obtained by evaluating the logarithmic

to identify (up to a scalar multiple) the Weyl solution of $Lu = zu$, which is in $L_2(\mathbf{R}^-)$ for $z \in \mathbf{C}^+$. Then $m_{-\infty}$ can be obtained by evaluating the logarithmic derivative of the Weyl solution at $r = 0$, as in ((**R**[−]) for $z \in \mathbb{C}^+$. Then $m_{-\infty}$ can be obtained by evaluating the logarithmic rivative of the Weyl solution at $r = 0$, as in (4.15). We omit the technical details.
nese examples confirm that when part or all of derivative of the Weyl solution at $r = 0$, as in (4.15). We omit the technical details.
These examples confirm that when part or all of the spectrum is absolutely con-
tinuous with multiplicity one, the corresponding eige tinuous with multiplicity one, the corresponding eigenfunctions in the expansion (4.1) are solutions of $Lu = \lambda u$ which are subordinate at one of the limit point endpoints, and are of comparable asymptotic size to all linea (4.1) are solutions of $Lu = \lambda u$ which are subordinate at one of the limit point endpoints, and are of comparable asymptotic size to all linearly independent solutions of the same equation at the other endpoint. In cases wh lutions of the same equation at the other endpoint. In cases where both S_1 and S_2 have positive μ_{τ} -measure, but S_3 and S_4 are empty, the simplified expansions will still be valid on the relevant λ -set S_2 have positive μ_{τ} -measure, but S_3 and S_4 are empty, the simplified expansions have positive μ_{τ} -measure, but S_3 and S_4 are empty, the simplified expansions
l still be valid on the relevant λ -sets, but non-overlapping spectral projections
needed to separate the corresponding parts of t will still be valid on the relevant λ -sets, but non-overlapping spectral projections
are needed to separate the corresponding parts of the expansion. Thus we expect
the absolutely continuous eigenfunctions associated w the absolutely continuous eigenfunctions associated with S_1 and S_2 respectively
to be real scalar multiples of $u_2(r, \lambda) + m_{\infty} u_1(r, \lambda)$ and $u_2(r, \lambda) + m_{-\infty} u_1(r, \lambda)$,
and the corresponding spectral densities to be to be real scalar multiples of $u_2(r,\lambda) + m_\infty u_1(r,\lambda)$ and $u_2(r,\lambda) + m_\infty u_1(r,\lambda)$, and the corresponding spectral densities to be positive scalar multiples of $\rho_{-\infty}(\lambda)$ We have for $f(r) \in L_2(-\infty, \infty)$,

and the corresponding spectral densities to be positive scalar multiples of
$$
\rho_{-\infty}(\lambda)
$$

\nand $\rho_{\infty}(\lambda)$, noting that there may be some λ dependence in the scalar multiples.
\nWe have for $f(r) \in L_2(-\infty, \infty)$,
\n
$$
f(r) = \frac{\ln \text{Im}}{\omega \to \infty} \left\{ \int_{(-\omega,\omega) \cap S_1} \frac{u_2(r,\lambda) + m_{\infty}(\lambda)u_1(r,\lambda)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|} G_+(\lambda) d\rho_{-\infty}(\lambda) \right\}
$$
\n
$$
+ \int_{(-\omega,\omega) \cap S_2} \frac{u_2(r,\lambda) + m_{-\infty}(\lambda)u_1(r,\lambda)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|} G_-(\lambda) d\rho_{\infty}(\lambda) \right\} \qquad (4.19)
$$
\nwhere
\n
$$
G_+(r,\lambda) = \lim_{\eta \to \infty} \int_{-\eta}^{\eta} \frac{u_2(r,\lambda) + m_{\infty}(\lambda)u_1(r,\lambda)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|} f(r) dr,
$$
\n
$$
G_-(r,\lambda) = \lim_{\eta \to \infty} \int_{-\eta}^{\eta} \frac{u_2(r,\lambda) + m_{-\infty}(\lambda)u_1(r,\lambda)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|} f(r) dr.
$$

where

+
$$
\int_{(-\omega,\omega)\cap S_2} \frac{u_2(r,\lambda) + m_{-\infty}(\lambda) u_1(r,\lambda)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|} G_{-}(\lambda) d\rho_{\infty}(\lambda)
$$

$$
G_{+}(r,\lambda) = \frac{\lim_{\eta \to \infty}}{\eta} \int_{-\eta}^{\eta} \frac{u_2(r,\lambda) + m_{\infty}(\lambda) u_1(r,\lambda)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|} f(r) dr,
$$

$$
G_{-}(r,\lambda) = \frac{\lim_{\eta \to \infty}}{\eta} \int_{-\eta}^{\eta} \frac{u_2(r,\lambda) + m_{-\infty}(\lambda) u_1(r,\lambda)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|} f(r) dr.
$$

with convergence in $L_2(\mathbf{R})$, $L_2(\mathbf{R}; d\rho_{-\infty})$ and
that for $\lambda \in S_1(\lambda)$,
 $\rho'_\tau(\lambda) = -\frac{1}{\pi} \frac{\text{Im } m_{-\infty}(\lambda) (1 + (m_{\infty} \lambda) - m_{\infty}(\lambda))}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|}$

with convergence in
$$
L_2(\mathbf{R})
$$
, $L_2(\mathbf{R}; d\rho_{-\infty})$ and $L_2(\mathbf{R}; d\rho_{\infty})$ respectively. Noting
that for $\lambda \in S_1(\lambda)$,

$$
\rho'_{\tau}(\lambda) = -\frac{1}{\pi} \frac{\text{Im } m_{-\infty}(\lambda) (1 + (m_{\infty}(\lambda))^2)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|^2}, \quad 0 < \lambda \le 1,
$$
(4.20)
we may write

$$
d\rho_{\tau}(\lambda) = \frac{(1 + (m_{\infty}(\lambda))^2)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|^2} d\rho_{-\infty}(\lambda), \quad 0 < \lambda \le 1.
$$

We see that the form of the first integrand in (4.19) is derived from (4.5), by
cancellation of the term $(1 + (m_{-\lambda}(\lambda))^2)$ in the numerator of the expression for

$$
\pi \quad | m_{-\infty}(\lambda) - m_{\infty}(\lambda) |^2
$$
\ne
\ne
\n
$$
d\rho_{\tau}(\lambda) = \frac{\left(1 + (m_{\infty}(\lambda))^2\right)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|^2} d\rho_{-\infty}(\lambda), \quad 0 < \lambda \le 1
$$
\nthe form of the first integrand in (4.19) is derived for
\nof the term $(1 + (m_{\infty}(\lambda))^2)$ in the numerator of the ϵ

We see that
cancellation $\rho'_{\tau}(\lambda)$ in (4.20 $(\lambda) = \frac{(1 + (m_{\infty}(\lambda))^2)}{|m_{-\infty}(\lambda) - m_{\infty}(\lambda)}$

Exercise form of the first integries

the term $(1 + (m_{\infty}(\lambda))^2)$

with the denominator of the $\infty(\lambda) - m_{\infty}(\lambda)$ |², which is
 $\omega'_\tau(\lambda)$ in (4.20) to the denominator $(\lambda) - m_{\infty}(\lambda)$ |² $d\rho_{-\infty}(\lambda)$, $0 < \lambda \leq 1$.
 \sum_{i} first integrand in (4.19) is derived from $(m_{\infty}(\lambda))^2$ in the numerator of the exponent of the eigenfunctions in (4.5), and (4.5) and (4.5) is strictly positive fo cancellation of the term $(1 + (m_{\infty}(\lambda))^2)$ in the numerator of the expression for $\rho'_{\tau}(\lambda)$ in (4.20) with the denominator of the eigenfunctions in (4.5), and by removal of the term $|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|^2$, which is st ρ'_{τ} (λ) in (4.20) with the denominator of the eigenfunctions in (4.5), and by removal
the term $|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|^2$, which is strictly positive for $\lambda \in S_1 \cup S_2$, from the
nominator of $\rho'_{\tau}(\lambda)$ in (4.20) to the den of the term $|m_{-\infty}(\lambda) - m_{\infty}(\lambda)|$
denominator of $\rho'_{\tau}(\lambda)$ in (4.20)
The argument is entirely analog
Note that the expansion in (4.19)
 λ in S_1 or S_2 respectively, in w
The form of the expansion
tween propertie 2, which is strictly positive for $\lambda \in S_1 \cup S_2$, from the other denominators of the eigenfunction in (4.19).

bus for $\lambda \in S_2(\lambda)$, with ∞ and $-\infty$ interchanged.

is still valid if $m_{\infty}(z)$ or $m_{-\infty}(z) \downarrow \infty$ as denominator of ρ' .
The argument is a
Note that the exp
 λ in S_1 or S_2 resp
The form of
tween properties a
asymptotic behav τ (λ) in (4.20) to the denominators of the eigenfunction in (4.19).
ntirely analogous for $\lambda \in S_2(\lambda)$, with ∞ and $-\infty$ interchanged.
nnsion in (4.19) is still valid if $m_{\infty}(z)$ or $m_{-\infty}(z) \downarrow \infty$ as $z \downarrow \lambda$ for λ

The argument is entirely analogous for $\lambda \in S_2(\lambda)$, with ∞ and $-\infty$ interchanged.
Note that the expansion in (4.19) is still valid if $m_{\infty}(z)$ or $m_{-\infty}(z) \downarrow \infty$ as $z \downarrow \lambda$ for λ in S_1 or S_2 respectivel Note that the expansion in (4.19) is still valid if $m_{\infty}(z)$ or $m_{-\infty}(z) \downarrow \infty$ as $z \downarrow \lambda$ for λ in S_1 or S_2 respectively, in which case the eigenfunctions are given by $u_1(r, \lambda)$.
The form of the expansion i The form of the expansion in (4.19) clearly demonstrates the relationships be-
tween properties of the simple part of the absolutely continuous spectrum and the
asymptotic behaviour of the corresponding absolutely continu tween properties of the simple part of the absolutely continuous spectrum and the asymptotic behaviour of the corresponding absolutely continuous eigenfunctions; it also exposes the contribution of the Titchmarsh-Weyl m-f also exposes the contribution of the Titchmarsh-Weyl *m*-functions and associated
spectral densities of the half-line operators, $H_{-\infty}$ and H_{∞} , to the structure of the
expansion in this case. We expect to extend t spectral densities of the half-line operators, $H_{-\infty}$ and H_{∞} , to the structure of the expansion in this case. We expect to extend this work to cases where the spectrum includes non-trivial singular and/or degenera spectral densities of the half-line operators, $H_{-\infty}$ and H_{∞} , to the structure of the expansion in this case. We expect to extend this work to cases where the spectrum includes non-trivial singular and/or degenera

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includes non-trivial singular and/or degenerate parts in a separate publication.
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Spectral Minimal Partitions for a Thin Strip on a Cylinder or a Thin Annulus like Domain with Neumann Condition

Bernard Helffer and Thomas Hoffmann-Ostenhof

Abstract. We analyze "Neumann" spectral minimal partitions for a thin strip on a cylinder or for the thin annulus.

Mathematics Subject Classification (2010). Primary 35B05.

Keywords. Spectral theory, Courant's nodal theorem, spectral minimal partitions.

1. Introduction

In previous papers, sometimes in collaborations with other colleagues, we have analyzed spectral minimal partitions for some specific open subsets of \mathbb{R}^2 and for the sphere \mathbb{S}^2 . See [4] for some of the basic for the sphere \mathbb{S}^2 . See [4] for some of the basic results and for more detailed definitions. In contrast with two-dimensional eigenvalue problems for which a few examples exist where the eigenvalues and the eigenfu – rectangles, the disk, sectors, the equilateral triangle, \mathbb{S}^2 and the torus – explicit examples exist where the eigenvalues and the eigenfunctions are explicitly known – rectangles, the disk, sectors, the equilateral triangle, \mathbb{S}^2 and the torus – explicit non-nodal examples for minimal partitions are

– rectangles, the disk, sectors, the equilateral triangle, \mathbb{S}^2 and the torus – explicit non-nodal examples for minimal partitions are lacking. Up to now we only have been able to work out explicitly \mathfrak{L}_3 for been able to work out explicitly \mathfrak{L}_3 for the 2-sphere, [5].

Here we find other examples of non-nodal minimal partitions for problems for

which the circle \mathbb{S}^1 is a deformation retract. Note that the Laplaci which the circle \mathbb{S}^1 is a deformation retract. Note that the Laplacian on the circle \mathbb{S}^1_* (with perimeter 1) can be interpreted as the Laplacian on an interval $(0, 1)$ with periodic boundary conditions. For \mathbb{S}^1 (with perimeter 1) can be interpreted as the Laplacian on an interval $(0,1)$ with periodic boundary conditions. For this one-dimensional problem we can work out (with perimeter 1) can be interpreted as the Laplacian on an interval $(0, 1)$ with riodic boundary conditions. For this one-dimensional problem we can work out partition eigenvalues (see below for a definition) $\mathcal{L}_k(\$ the partition eigenvalues (see below for a definition) $\mathfrak{L}_k(\mathbb{S}^1_*)$ explicitly. We have $\mathfrak{L}_k = \pi^2 k^2$. Observe that for odd $k \geq 3$ the \mathfrak{L}_k are not eigenvalues, whereas for k even they are. The corres the partition eigenvalues (see below for a definition) $\mathfrak{L}_k(\mathbb{S}_*^1)$ explicitly. We have $\mathfrak{L}_k = \pi^2 k^2$. Observe that for odd $k \geq 3$ the \mathfrak{L}_k are not eigenvalues, whereas for k even they are. The corres $\mathfrak{L}_k = \pi^2 k^2$. Observe that for odd $k \geq 3$ the \mathfrak{L}_k are not eigen into k equal parts, hence $D_1 = (0, 1/k), D_2 = (1/k, 2/k), \dots, D_k = ((k-1)/k, 1)$
identifying 0 with 1.
We will consider a strip on a cylinder or the annulus with suitable boundary

into *k* equal parts, hence $D_1 = (0, 1/k), D_2 = (1/k, 2/k),..., D_k = ((k - 1)/k, 1)$
identifying 0 with 1.
We will consider a strip on a cylinder or the annulus with suitable boundary
conditions. All these domains are homotopic to $\mathbb{S$ We will conside
conditions. All these
to investigate the co conditions. All these domains are homotopic to \mathbb{S}^1_* . For those domains we are going to investigate the corresponding minimal 3-partitions. to investigate the corresponding minimal 3-partitions. . For the domains we are going
ions. We recall some notation and definitions. Consider a l

..., D_k), i.e., k disjoint open subsets D_i of some Ω . Here

domain in \mathbb{R}^2 or in a 2-dimensional C^{∞} Riemannian manife

Consider first $-\Delta$ on Ω ...

We recall some notation and definitions. Consider a k-partition $\mathcal{D}_k = (D_1, D_k)$, i.e., k disjoint open subsets D_i of some Ω . Here Ω can be a bounded in in \mathbb{R}^2 or in a 2-dimensional C^{∞} Riemannian ma , D_k), i.e., *k* disjoint open subsets D_i of some Ω. Here Ω can be a bounded main in \mathbb{R}^2 or in a 2-dimensional C^{∞} Riemannian manifold.
Consider first $-\Delta$ on Ω where Δ can be the usual Laplacian or domain in ℝ² or in a 2-dimensional C^{∞} Riemannian manifold.
Consider first $-\Delta$ on Ω where Δ can be the usual Lapla
of a manifold (with boundary or without boundary) the corre
Beltrami operator. For the case Consider first $-\Delta$ on Ω where Δ can be the usual Laplacian of in the case
manifold (with boundary or without boundary) the corresponding Laplace-
ami operator. For the case with boundary we can impose Dirichlet or Beltrami operator. For the case with boundary we can impose Dirichlet or Neu-
mann but we could also have mixed boundary conditions.
We associate with \mathcal{D}_k
 $\Lambda(\mathcal{D}_k) = \sup_{1 \leq i \leq k} \lambda_1(D_i)$
where $\lambda_1(D_i)$ denotes mann but we could also have mixed boundary conditions.

We associate with D_k
 $\Lambda(\mathcal{D}_k) = \sup_{1 \leq i \leq k} \lambda_1(D_i)$

where $\lambda_1(D_i)$ denotes

$$
\Lambda(\mathcal{D}_k) = \sup_{1 \le i \le k} \lambda_1(D_i)
$$

- We associate with \mathcal{D}_k
 $\Lambda(\mathcal{D}_k) = \sup_{1 \le i \le k} \lambda_1(D_i)$

where $\lambda_1(D_i)$ denotes

 either the lowest eigenvalue of the Dirichlet Laplacia ∙
- we associate with D_k
e $\lambda_1(D_i)$ denotes
either the lowest eigenvalues $\Lambda(\mathcal{D}_k) = \sup_{1 \leq i \leq k} \lambda_1(D_i)$

lue of the Dirichlet La

of the Laplacian in *L*
 $\partial D_i \subset \Omega$ and the Neu where $\lambda_1(D_i)$ denotes

• either the lowest eig

• or the lowest eig

boundary condit
 $\partial D_i \cap \partial \Omega$.

It is probably we

the case of measurabl either the lowest eigenvalue of the Dirichlet Laplacian in D_i
or the lowest eigenvalue of the Laplacian in D_i where we p
boundary condition on $\partial D_i \subset \Omega$ and the Neumann bound:
 $\partial D_i \cap \partial \Omega$.
It is probably worth to ex • or the lowest eigenvalue of the Laplacian in D_i where we put the Dirichlet or the lowest eigenvalue of the Laplacian in D_i where we put the Dirichlet
boundary condition on $\partial D_i \subset \Omega$ and the Neumann boundary condition on
 $\partial D_i \cap \partial \Omega$.
It is probably worth to explain rigorously what we mean abo $\partial D_i \cap \partial \Omega$.
It is probably worth to explain rigorously what we mean above by $\lambda_1(D_i)$ in

boundary condition on $\partial D_i \subset \Omega$ and the Neumann boundary condition on $\partial D_i \cap \partial \Omega$.
It is probably worth to explain rigorously what we mean above by $\lambda_1(D_i)$ in
ase of measurable D_i 's.
ition 1.1. For any measurable ba
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ge It is probably worth to explain rigorously what we mean above by $\lambda_1(D_i)$ in
ase of measurable D_i 's.
ition 1.1. For any measurable $\omega \subset \Omega$, let $\lambda_1^D(\omega)$ (resp. $\lambda_1^N(\omega)$) denotes the
eigenvalue of the Dirichlet the case of measurable D_i 's.
 Definition 1.1. For any mea

first eigenvalue of the Dirich

the following generalized ser

if $\{u \in W^1(\Omega) \mid u \equiv 0 \text{ a.e. } \text{on} \}$ **Definition 1.1.** For any measurable $\omega \subset \Omega$, let $\lambda_1^D(\omega)$ (resp. λ_1^D For any measurable $\omega \subset \Omega$, let $\lambda_1^D(\omega)$ (resp. $\lambda_1^N(\omega)$) denotes the
of the Dirichlet realization (resp. $\partial\Omega$ -Neumann) of the operator in
neralized sense. We define
 $\lambda_1^{D \text{ or } N}(\omega) = +\infty$,
 $u \equiv 0$ a.e. on $\Omega \set$ first eigenvalue of the Dirichlet realization (resp. ∂Ω-Neumann) of the operator in

the following generalized sense. We define
 $\lambda_1^{D \text{ or } N}(\omega) = +\infty$,

if $\{u \in W^1(\Omega), u \equiv 0$ a.e. on $\Omega \setminus \omega\} = \{0\}$,
 $\lambda_1^D(\omega) = \inf \left\$

$$
\lambda_1^{D\ or\ N}(\omega)=+\infty\ ,
$$

if {

$$
\lambda_1^{D \text{ or } N}(\omega) = +\infty ,
$$

if $\{u \in W^1(\Omega), u \equiv 0 \text{ a.e. on } \Omega \setminus \omega\} = \{0\},$

$$
\lambda_1^D(\omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u(x)|^2 dx}{\int_{\Omega} |u(x)|^2 dx} : u \in W_0^1(\Omega) \setminus \{0\}, u \equiv 0 \text{ a.e. on } \Omega \setminus \omega \right\} ,
$$

$$
\lambda_1^N(\omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u(x)|^2 dx}{\int_{\Omega} |u(x)|^2 dx} : u \in W^1(\Omega) \setminus \{0\}, u \equiv 0 \text{ a.e. on } \Omega \setminus \omega \right\} ,
$$

otherwise.
We call grounds
that any function ϕ achieving the above infimum.
Of course, if $\omega \subset\subset \Omega$, we have $\lambda_1^D(\omega) = \lambda_1^N(\omega)$.
The *k*th partition-eigenvalue $\mathfrak{L}_k(\Omega)$ is then defined by

$$
\mathfrak{L}_k(\Omega) = \inf \Lambda(\mathcal{D}),
$$

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Of co
The Of course, if $ω ⊂ ⊂ Ω$, we have $λ_1^D(ω) = λ_1^N$
The *k*th partition-eigenvalue $\mathfrak{L}_k(Ω)$ is then
 $\mathfrak{L}_k(Ω) = \inf_{\mathcal{D}} Λ(Ω)$
e the infimum is considered¹ over the *k*-par
Any *k*-partition D for which The kth partition-eigenvalue $\mathfrak{L}_k(\Omega)$ is then defined by
 $\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}} \Lambda(\mathcal{D})$,

e the infimum is considered¹ over the k-partitions.

Any k-partition \mathcal{D} for which
 $\mathfrak{L}_k(\Omega) = \Lambda(\mathcal{D})$

We call groundstate any function
$$
\phi
$$
 achieving the above infimum.
\nOf course, if $\omega \subset\subset \Omega$, we have $\lambda_1^D(\omega) = \lambda_1^N(\omega)$.
\nThe *k*th partition-eigenvalue $\mathfrak{L}_k(\Omega)$ is then defined by
\n
$$
\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}} \Lambda(\mathcal{D}), \qquad (1.1)
$$
\ne the infimum is considered¹ over the *k*-partitions.
\nAny *k*-partition \mathcal{D} for which
\n
$$
\mathfrak{L}_k(\Omega) = \Lambda(\mathcal{D}) \qquad (1.2)
$$
\nled spectral minimal *k*-partition, for short minimal *k*-partition.

$$
\mathfrak{L}_k(\Omega) = \Lambda(\mathcal{D})\tag{1.2}
$$

where the infimum is considered¹ over the *k*-partitions.

Any *k*-partition D for which
 $\mathcal{L}_k(\Omega) = \Lambda(D)$

is called spectral minimal *k*-partition, for short minima

¹We refer to [4] for a more precise definition Any k -partition D for which
led spectral minimal k -partitiefer to [4] for a more precise definitially
ular representatives. (1.2)

n, for short minimal k-partition.

1 of the considered class of k-partitions and the notion is called spectral minimal k -partition, for short minimal k -partition.

¹We refer to [4] for a more precise definition of the considered class of k -partitions of regular representatives. ¹We refer to [4] for a more precise definition of the considered class of k -partitions and the notion of regular representatives.

Spectral Minimal Particular Minimal Particular the $\partial \Omega$ -Neumann condition in the above definitions.

Spectral Minimal Particular Minimal Particular and the case of Neumann. In particular, minimal partitions are non-non-If needed we will write $\mathcal{L}_k^D(\Omega)$ or \mathcal{L}_k^N
tion or the ∂Ω-Neumann condition
Although not explicitly written in [
hlet are also true in the case of Neu
and have regular representatives.
One of the main results condition or the $\partial\Omega$ -Neumann condition in the above definitions.
Although not explicitly written in [4], all the results obtained in the case of
Dirichlet are also true in the case of Neumann. In particular, minimal pa condition or the ∂ Ω -Neumann condition in the above definitions.
Although not explicitly written in [4], all the results obtain
Dirichlet are also true in the case of Neumann. In particular, mi
exist and have regular re

hlet are also true in the case of Neumann. In particular, minimal partitions
and have regular representatives.
One of the main results in [4] concerns the characterization of the case of
ity in Courant's nodal theorem. Co exist and have regular representatives.

One of the main results in [4] concerns the characterization of the case of

equality in Courant's nodal theorem. Consider an eigenvalue problem $-\Delta u_k = \lambda_k$

with suitable homogeneo One of the main results in [4] co
equality in Courant's nodal theorem. C
with suitable homogeneous boundary c
the eigenvalues in increasing order λ_1
that u_k is real, then Courant's nodal
domains $\mu(u_k)$ satisfies $\$ ity in Courant's nodal theorem. Consider an eigenvalue problem $-\Delta u_k = \lambda_k$
suitable homogeneous boundary conditions (as previously defined) and order
igenvalues in increasing order $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \ldots$. If we equality in Courant's nodal theorem. Consider an eigenvalue problem $-\Delta u_k = \lambda_k$
with suitable homogeneous boundary conditions (as previously defined) and order
the eigenvalues in increasing order $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq$ the eigenvalues in increasing order $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \dots$. If we assume
that u_k is real, then Courant's nodal theorem says that the number of its nodal
domains $\mu(u_k)$ satisfies $\mu(u_k) \leq k$. Note that Courant the eigenvalues in increasing order $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k$ If we assume
that u_k is real, then Courant's nodal theorem says that the number of its nodal
domains $\mu(u_k)$ satisfies $\mu(u_k) \leq k$. Note that Couran that u_k is real, then Courant's nodal theorem says that the number of its nodal
domains $\mu(u_k)$ satisfies $\mu(u_k) \leq k$. Note that Courant's nodal theorem holds in
greater generality, in higher dimensions and with a poten domains $\mu(u_k)$ satisfies $\mu(u_k) \leq k$. Note that Courant's nodal theorem holds in greater generality, in higher dimensions and with a potential. Here a nodal domain is a component of $\Omega \setminus N(u_k)$ where Ω is the domain i main is a component of $\Omega \setminus N(u_k)$ where Ω is the domain in Ω or the manifold and $N(u_k) = \{x \in \Omega \mid u_k(x) = 0\}$. We call u_k and λ_k **Courant sharp** if $\mu(u_k) = k$. In [4] we have also described some properties of mini $N(u_k) = \overline{\{x \in \Omega \mid u_k(x) = 0\}}$. We call u_k and λ_k **Courant sharp** if $\mu(u_k) = k$. In $(u_k) = {x \in \Omega | u_k(x) = 0}$. We call u_k and λ_k **Courant sharp** if $\mu(u_k) = k$. In we have also described some properties of minimal partitions. In many respects ey are related to nodal domains. Nodal domains have many intere they are related to nodal domains. Nodal domains have many interesting proper-
ties. In particular in neighboring nodal domains the corresponding eigenfunction
has different signs. Thereby two nodal domains D_i, D_j are sa ties. In particular in neighboring nodal domains the corresponding eigenfunction
has different signs. Thereby two nodal domains D_i , D_j are said to be neighbors if
 $\overline{\text{Int } D_i \cup D_j}$ is connected. We can associate with Int $D_i \cup D_j$ is connected. We can associate with any (not necessarily nodal) parti-
tion, say $\mathcal{D}_k = (D_1, \ldots, D_k)$, a simple graph in the following way: we associate to
each D_i a vertex and draw an edge between two ve tion, say $\mathcal{D}_k = (D_1, \ldots, D_k)$, a simple graph in the following way: we associate to each D_i a vertex and draw an edge between two vertices i, j if the corresponding D_i, D_j are neighbors. This amounts to say that no each D_i a vertex and draw an edge between two vertices i, j if the corresponding D_i, D_j are neighbors. This amounts to say that nodal graphs $\mathcal{G}(\mathcal{D}_k)$ are **bipartite** graphs.
The relation with Courant's nodal t D_i, D_j

are neighbors. This amounts to say that nodal graphs $\mathcal{G}(\mathcal{D}_k)$ are **bipartite**
the relation with Courant's nodal theorem is now the following, which is
the Dirichlet or Neumann case:
m 1.2 (Dirichlet). If for a b T

valid in
 Theore
 $\text{imal } k - \mathcal{G}(\mathcal{D}), t$
 function in the Dirichlet or Neumann case:
 rem 1.2 (Dirichlet). If for a bounded domain Ω with smooth boundary a min-
 k -partition D with associated partition eigenvalue \mathfrak{L}_k^D has a bipartite graph

, then this mi **Theorem 1.2 (Dirichlet).** If for a bound
imal k-partition D with associated part
 $\mathcal{G}(D)$, then this minimal k-partition is p
function u which is **Courant sharp** so t.
Theorem 1.3 ($\partial \Omega$ **-Neumann).** If for a **Theorem 1.2 (Dirichlet).** If for a bounded domain Ω with smooth boundary a min- Ω with smooth boundary a min-
value \mathfrak{L}_k^D has a bipartite graph
the nodal domains of an eigen-
 $= \lambda_k^D u$ in Ω and $\lambda_k^D = \mathfrak{L}_k^D$.
omain Ω with smooth boundary
neigenvalue \mathfrak{L}_k^N has a bipartite
u imal k-partition \mathcal{D} with associated partition eigenvalue \mathcal{L}_k^D has a bipartite graph $\mathcal{G}(\mathcal{D})$, then this minimal k-partition is produced by the nodal domains of an eigenfunction *u* which is **Courant sharp** so that $-\Delta^D u = \lambda_k^D u$ in Ω and $\lambda_k^D = \mathfrak{L}_k^D$.

(D), then this minimal k-partition is produced by the nodal domains of an eigen-
unction u which is **Courant sharp** so that $-\Delta^D u = \lambda_k^D u$ in Ω and $\lambda_k^D = \mathfrak{L}_k^D$.
heorem 1.3 ($\partial\Omega$ **-Neumann).** If for a bounded do $\Delta^D u = \lambda_k^D u$ in Ω and $\lambda_k^D = \mathfrak{L}_k^D$
ded domain Ω with smooth bour
tition eigenvalue \mathfrak{L}_k^N has a bip
produced by the nodal domains
 $at - \Delta^N u = \lambda_k^N u$ in Ω and $\lambda_k^N =$
 $\lambda_k^N \leq \mathfrak{L}_k^D$, resp. $\lambda_k^$ **Theorem 1.3 (** $\partial\Omega$ **-Neumann).** If for a bounded domain Ω with smooth boundary Ω with smooth boundary
value \mathfrak{L}_k^N has a bipartite
the nodal domains of an
 $\lambda_k^N u$ in Ω and $\lambda_k^N = \mathfrak{L}_k^N$.
 $\lambda_k^N \leq \mathfrak{L}_k^N$ and that, by
at for each $k > k(\Omega)$ any a minimal k-partition \mathcal{D} with associated partition eigenvalue \mathcal{L}_k^N has a bipartite graph $\mathcal{G}(\mathcal{D})$, then this minimal k-partition is produced by the nodal domains of an eigenfunction u which is **Courant sharp** so that $-\Delta^N u = \lambda_k^N u$ in Ω and $\lambda_k^N = \mathfrak{L}_k^N$.

(D), then this minimal k-partition is produced by the nodal domains of an
nction u which is **Courant sharp** so that $-\Delta^N u = \lambda_k^N u$ in Ω and $\lambda_k^N = \mathfrak{L}_k^N$.
the that by the minimax principle $\lambda_k^D \leq \mathfrak{L}_k^D$, re $\Delta^N u = \lambda_k^N u$ in Ω and $\lambda_k^N = \mathfrak{L}_k^N$
 \mathfrak{L}_k^D , resp. $\lambda_k^N \leq \mathfrak{L}_k^N$ and that, b
such that for each $k > k(\Omega)$ an
k nodal domains. That implie
k-partitions are non-nodal.
ss, when either Dirichlet or Neu Note that by the minimax principle $\lambda_k^D \leq \mathfrak{L}_k^D$, resp. $\lambda_k^N \leq \mathfrak{L}_k^N$
el's Theorem [10], for each Ω there is a $k(\Omega)$ such that for each
iated eigenfunction u has strictly less than k nodal domains.
for s Pleijel's Theorem [10], for each Ω there is a $k(\Omega)$ such that for each $k > k(\Omega)$ any associated eigenfunction *u* has strictly less than *k* nodal domains. That implies that, for sufficiently high *k*, the spectral mi that, for sufficiently high k , the spectral minimal k -partitions are non-nodal.
Note also that we will also meet mixed cases, when either Dirichlet or Neu-

associated eigenfunction *u* has strictly less than *k* nodal domains. That implies that, for sufficiently high *k*, the spectral minimal *k*-partitions are non-nodal.
Note also that we will also meet mixed cases, when ei Note also that we will also that we will also meet that will also meet that when either $\partial \Omega$. mann boundary conditions are assumed on different components of $\partial\Omega$.

2. Neumann problem for a strip on the cylinder

$$
C(b) = \mathbb{S}^1_* \times (0, b). \tag{2.1}
$$

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2. **Neumann problem for a strip on the cylinder**

We start with the a strip $C(1, b) = C(b)$ on a cylinder where
 $C(b) = \mathbb{S}^1_* \times (0, b)$. (2.1)

If needed, we can represent the strip b identification of $x = 0$ and $x = 1$. But the open sets of the partition are always considered as open sets on the strip.
We consider Neumann boundary conditions at $y = 0$ and $y = b$. The spectrum (b) = $\mathbb{S}_{*}^{1} \times (0, b)$. (2.1)

strip by a rectangle $R(1, b) = (0, 1) \times (0, b)$ with

1. But the open sets of the partition are always

rip.

dary conditions at $y = 0$ and $y = b$. The spectrum

boundary conditions is discre If needed, we can represent the strip by a rectangle $R(1, b) = (0, 1) \times (0, b)$ with
identification of $x = 0$ and $x = 1$. But the open sets of the partition are always
considered as open sets on the strip.
We consider Neumann

identification of $x = 0$ and $x = 1$. But the open sets of the partition are always
considered as open sets on the strip.
We consider Neumann boundary conditions at $y = 0$ and $y = b$. The spectrum
for the Laplacian Δ^N w We consider Neumann boundar
for the Laplacian Δ^N with these bou $\sigma(-\Delta^N) = \begin{cases} \pi^2 \end{cases}$.
Note that eigenvalues for $m \geq$

$$
\sigma(-\Delta^N) = \left\{ \pi^2 \left(4m^2 + \frac{n^2}{b^2} \right)_{(m,n) \in \mathbb{N}^2} \right\}.
$$
 (2.2)

We consider Neumann boundary conditions at $y = 0$ and $y = b$. The spectrum

i.e Laplacian Δ^N with these boundary conditions is discrete and is given by
 $\sigma(-\Delta^N) = \left\{ \pi^2 \left(4m^2 + \frac{n^2}{b^2} \right)_{(m,n) \in \mathbb{N}^2} \right\}$. (2 for the Laplacian Δ^N with these boundary conditions is discrete and is given by
 $\sigma(-\Delta^N) = \left\{ \pi^2 \left(4m^2 + \frac{n^2}{b^2} \right)_{(m,n) \in \mathbb{N}^2} \right\}$. (2.2

Note that eigenvalues for $m \ge 1$ have at least multiplicity two. I $(-\Delta^N) = \left\{ \pi^2 \left(4m^2 + \frac{n^2}{b^2} \right)_{(m,n) \in \mathbb{N}^2} \right\}.$ (2.2)
alues for $m \ge 1$ have at least multiplicity two. Identifying
1, b)), a corresponding orthonormal basis of eigenfunctions
ons on $R(1, b)$ $(x, y) \mapsto \cos(2\pi mx) \$ Note that eigenvalues for $m \ge 1$ have at least multiplicity two. Identifying $(1, b)$ and $L^2(R(1, b))$, a corresponding orthonormal basis of eigenfunctions
ren by the functions on $R(1, b)$ $(x, y) \mapsto \cos(2\pi mx) \cos(\pi n \frac{y}{b})$ $((m$ L^2 $(C(1, b))$ and $L^2(R(1, b))$, a corresponding orthonormal basis of eigenfunctions
given by the functions on $R(1, b)$ $(x, y) \mapsto \cos(2\pi mx) \cos(\pi n \frac{y}{b})$ $((m, n) \in \mathbb{N}^2)$
d $(x, y) \mapsto \sin(2\pi mx) \cos(\pi n \frac{y}{b})$ $((m, n) \in \mathbb{N}^* \times \mathbb{N})$, is given by the functions on $R(1, b)$ $(x, y) \mapsto \cos(2\pi mx) \cos(\pi n \frac{y}{b})$ $((m, n) \in \mathbb{N}^2)$
and $(x, y) \mapsto \sin(2\pi mx) \cos(\pi n \frac{y}{b})$ $((m, n) \in \mathbb{N}^* \times \mathbb{N})$, where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We can
now distinguish the following cas and $(x, y) \mapsto \sin(2\pi mx) \cos(\pi n \frac{y}{b})$ $((m, n) \in \mathbb{N}^* \times \mathbb{N})$, where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We can
now distinguish the following cases:
(i) If $b < \frac{1}{2}$,
 $\lambda_1^N = 0$, $\lambda_2^N = \lambda_3^N = 4\pi^2 < \lambda_4^N$.
(ii) If $\frac{1}{2} < b <$

(i) If $b < \frac{1}{2}$,

$$
\lambda_1^N = 0
$$
, $\lambda_2^N = \lambda_3^N = 4\pi^2 < \lambda_4^N$.

(i) If $b < \frac{1}{2}$
(ii) If $\frac{1}{2} < b$
(ii) If $b = 1$, (ii) If $\frac{1}{2} < b <$
(iii) If $b = 1$,
(iv) If $b > 1$

(i) If
$$
b < \frac{1}{2}
$$
,
\n(i) If $\frac{1}{2} < b < 1$,
\n $\lambda_1^N = 0$, $\lambda_2^N = \lambda_3^N = 4\pi^2 < \lambda_4^N$.
\n(ii) If $\frac{1}{2} < b < 1$,
\n $\lambda_1^N = 0$, $\lambda_2^N = \frac{\pi^2}{b^2}$, $\lambda_3^N = \lambda_4^N = 4\pi^2 < \lambda_5^N$.
\n(iii) If $b = 1$,
\n $\lambda_1^N = 0$, $\lambda_2^N = \pi^2$, $\lambda_3^N = \lambda_4^N = \lambda_5^N = 4\pi^2 < \lambda_4^N$.
\n(iv) If $b > 1$,
\n $\lambda_1^N = 0$, $\lambda_2^N = \pi^2$, $\lambda_3^N = 4\pi^2 < \lambda_4^N$.

$$
\lambda_1^N = 0, \lambda_2^N = \frac{\pi^2}{b^2}, \lambda_3^N = \lambda_4^N = 4\pi^2 < \lambda_5^N.
$$
\n
$$
\lambda_1^N = 0, \lambda_2^N = \pi^2, \lambda_3^N = \lambda_4^N = \lambda_5^N = 4\pi^2 < \lambda_6^N.
$$
\n
$$
\lambda_1^N = 0, \lambda_2^N = \pi^2, \lambda_3^N = 4\pi^2 < \lambda_4^N.
$$

$$
\lambda_1^N = 0 \, , \, \lambda_2^N = \pi^2 \, , \, \lambda_3^N = 4 \pi^2 < \lambda_4^N \, .
$$

(iii) If $b = 1$,
(iv) If $b > 1$,
(in particular,
also that, for = 0, $\lambda_2^N = \pi^2$, $\lambda_3^N = \lambda_4^N = \lambda_5^N = 4\pi^2 < \lambda_6^N$
 $\lambda_1^N = 0$, $\lambda_2^N = \pi^2$, $\lambda_3^N = 4\pi^2 < \lambda_4^N$.
 λ_1^N that $\lambda_3^N(C(1, b))$ is Courant sharp if and c

1], $\lambda_4^N(C(1, b))$ cannot be Courant sharp, and

sharp (iv) If $b > 1$,
In particular,
also that, for α
cannot be Co
Before v
its double cov $N_1 = 0$, $\lambda_2^N = \pi^2$, $\lambda_3^N = 4\pi^2 < \lambda_4^N$.

at $\lambda_3^N(C(1, b))$ is Courant sharp if a
 $\lambda_4^N(C(1, b))$ cannot be Courant sharp

p.

e main result for the strip on the cy

b), whose associated rectangle is g. In particular, we see that $\lambda_3^N(C(1, b))$ is Courant sharp if and only if $b \ge 1$. Note
also that, for $b \in (\frac{1}{2}, 1], \lambda_4^N(C(1, b))$ cannot be Courant sharp, and that $\lambda_5^N(C(1, 1))$
cannot be Courant sharp.
Before we state 3 also that, for $b \in (\frac{1}{2}, 1], \lambda_4^N(C(1, b))$ cannot be Courant sharp, and that $\lambda_5^N(C(1, 1))$
cannot be Courant sharp.
Before we state the main result for the strip on the cylinder, we look also at
its double covering $C(2$

Before we state the intervals double covering $C(2, b)$
Lemma 2.1. The Neuman
bu

$$
b \le 1/3,\tag{2.3}
$$

by

its double covering
$$
C(2, b)
$$
, whose associated rectangle is given by $(0, 2) \times (0, b)$.
\n**Lemma 2.1.** The Neumann eigenvalues for $C(2, b)$ are given, assuming that
\n $b \le 1/3$, (2.3)
\nby
\n $\lambda_1^N = 0, \lambda_2^N = \lambda_3^N = \pi^2, \lambda_4^N = \lambda_5^N = 4\pi^2, \lambda_6^N = \lambda_7^N = 9\pi^2$. (2.4)
\nNote that $\lambda_6^N(C(2, b))$ is **Country** sharp if $b \le 1/3$.

1/3, (2.3)
 $= \lambda_5^N = 4\pi^2, \ \lambda_6^N = \lambda_7^N = 9\pi^2.$ (2.4)
 if $b \le 1/3.$

that $\mathfrak{L}_3^N > \lambda_3^N$ and that by Theorem 1.3 = 0, $\lambda_2^N = \lambda_3^N = \pi^2$, $\lambda_4^N = \lambda_5^N = 4\pi^2$, $\lambda_6^N = \lambda_7^N = 9\pi^2$. (2.4)
 $\mathcal{L}(2, b)$ is **Courant sharp** if $b \le 1/3$.

ote that $\lambda_2^N = \lambda_3^N$ implies that $\mathfrak{L}_3^N > \lambda_3^N$ and that by Theorem 1.3
 \mathcal{D}_3 $(C(2, b))$ is **Courant sharp** if $b \le 1/3$.
Note that $\lambda_2^N = \lambda_3^N$ implies that \mathfrak{L}_3^N is **100-nodal**.
at for $b \ge 1$, we get by the same theor **Remark 2.2.** Note that $\lambda_2^N = \lambda_3^N$ implies that $\mathcal{L}_3^N > \lambda_3^N$ **Remark 2.2.** Note that $\lambda_2^N = \lambda_3^N$ implies that $\mathfrak{L}_3^N > \lambda_3^N$ and that by Theorem 1.3 the associated \mathcal{D}_3 is **non-nodal**.
Note that for $b \ge 1$, we get by the same theorem that $\lambda_3^N(b) = \mathfrak{L}_3^N(b) = \frac{4\$

Note that for $b \ge 1$, we get by the same theorem that $\lambda_3^N(b) = \mathfrak{L}_3^N(b) = \frac{4\pi^2}{b^2}$ b^2

Spectral Minimal Partitions for a Thin Strip on a Cylinder 111
 Remark 2.3. The Neumann boundary conditions imply that the zero's hit the

boundary not as in the Dirichlet case. More precisely consider the cylinder $C(1$ **Remark 2.3.** The Neumann boundary conditions imply that the zero's hit the boundary not as in the Dirichlet case. More precisely consider the cylinder $C(1, b)$ with associated rectangle $R(1, b)$ and assume that the zero hits at $(x_0, 0)$. In polar coordinates (r, ω) centered at this point the z with associated rectangle $R(1, b)$ and assume that the zero hits at $(x_0, 0)$. In polar coordinates (r, ω) centered at this point the zeroset looks locally like the zeroset of $r^m \cos m\omega$, $m = 1, 2, \ldots$. This is in contra coordinates (r, ω) centered at this point the zeroset looks locally like the zeroset of $r^m \cos m\omega$, $m = 1, 2, \ldots$. This is in contrast with the Dirichlet case where we would have $r^m \sin m\omega$. The r^m factor is just incl r^m cos $m\omega$, $m = 1, 2, \ldots$ This is in contrast with the Dirichlet case where we would cos $m\omega$, $m = 1, 2, \dots$. This is in contrast with the Dirichlet case where we would
ve $r^m \sin m\omega$. The r^m factor is just included to point out that eigenfunctions
is zero's behave to leading order as harmonic homogeneo have $r^m \sin m\omega$. The r^m factor is just included to point out that eigenfunctions
near zero's behave to leading order as harmonic homogeneous polynomials.
Here comes the main result for the minimal 3-partition for the s

near zero's behave to leading order as harmonic homogeneous polynomials.
Here comes the main result for the minimal 3-partition for the strip on the cylinder.

Theorem
we have
 $\sum_{i=1}^{n}$ **Theorem 2.4.** For

der.

\n
$$
\mathbf{rem\ 2.4.} \text{ For}
$$
\n
$$
b \le b_0 = \frac{1}{2\sqrt{5}},
$$
\n
$$
\mathfrak{L}_2^N(C(b)) = 9\pi^2.
$$
\n(2.6)

we have

$$
\mathfrak{L}_3^N(C(b)) = 9\pi^2. \tag{2.6}
$$

= $\frac{1}{2\sqrt{5}}$, (2.5)

(2.6)

(2.6)

(b) = (D_1, D_2, D_3) is up to rotation repre-

(3, $\ell/3$) × (0, b), (2.7) $\frac{1}{2}$)7
]
3) $(C(b)) = 9\pi^2$. (2.6)
 $\mathcal{D}_3(b) = (D_1, D_2, D_3)$ is up to rotation repre-
 $-1)/3$, $\ell/3$ × (0, b), (2.7)

at be appropriate to consider the case $b_0 < b \leq$ sented by

$$
D_{\ell} = ((\ell - 1)/3, \ell/3) \times (0, b), \tag{2.7}
$$

in $R(1,b)$.

The associated minimal 3-partition $\mathcal{D}_3(b) = (D_1, D_2, D_3)$ is up to rotation repre-
sented by
 $D_{\ell} = ((\ell - 1)/3, \ell/3) \times (0, b),$ (2.7)
in $R(1, b)$.
Before giving the proof it might be appropriate to consider the case $b_0 < b \$ = ((ℓ − 1)/3, ℓ/3) × (0, b), (2.7)

t might be appropriate to consider the case $b_0 < b \le$,

(1) the spectral minimal 3-partition $\mathcal{D}_3(b)$ is not the

(b)) < $9\pi^2$ for 2/3 < b < 1.

 $(1, b).$
Befor
this
positio
given
f It is **Proposition 2.5.** For $b \in$ one given by (2.7) and \mathfrak{L}_3^p

Before giving the proof it might be appropriate to consider the case $b_0 < b \le$
this direction, we have:
osition 2.5. For $b \in [2/3, 1)$ the spectral minimal 3-partition $\mathcal{D}_3(b)$ is not the
iven by (2.7) and $\mathfrak{L}_$ **Proposition 2.5.** For $b \in [2/\frac{1}{2})$
one given by (2.7) and \mathfrak{L}_3^N (
Proof. It is immediate that
 $(x, y) \mapsto \cos(2\pi \frac{y}{b})$ has three
 $9\pi^2$. [2/3, 1) the spectral minimal 3-partition $\mathcal{D}_3(b)$ is not the $\frac{N}{3}(C(b)) < 9\pi^2$ for $2/3 < b < 1$.

hat the eigenfunction associated with $m = 0$ and $n = 2$

are nodal domains with energy $\frac{4\pi^2}{b^2}$ which is less th (2.7) and $\mathfrak{L}_3^N(C(b)) < 9\pi^2$ for $2/3 < b < 1$.
mmediate that the eigenfunction associated
 $2\pi \frac{y}{b}$ has three nodal domains with energ
heorem. Note that by definition of \mathfrak{L}_3^N , we *Proof.* It is immediate that the eigenfunction associated with $m = 0$ and $n = 2$ It is immediate that the eigenfunction associated with $m = 0$ and $n = 2$
 $\rightarrow \cos(2\pi \frac{y}{b})$ has three nodal domains with energy $\frac{4\pi^2}{b^2}$ which is less than

If the theorem. Note that by definition of \mathfrak{L}_3^N , $(x, y) \mapsto \cos(2\pi \frac{y}{b})$
 $9\pi^2$.
 Proof of the theor

We first sket

will be crucial for) has three nodal domains with energy $\frac{4\pi^2}{b^2}$
 em. Note that by definition of \mathfrak{L}_3^N , we have
 $\mathfrak{L}_3^N(C(b)) \leq 9\pi^2$.

ch the main ideas for the proof. There are the

the proof: b^2 f of the theorem. Note that by definition of \mathfrak{L}_3^N , we have in any case
 $\mathfrak{L}_3^N(C(b)) \leq 9\pi^2$. (2.8)
We first sketch the main ideas for the proof. There are two arguments which

Proof of the theorem. Note that by definition of \mathfrak{L}_3^P

$$
\mathfrak{L}_3^N(C(b)) \le 9\pi^2. \tag{2.8}
$$

9 π^2 .
 Proof of the theorem. Note that by definition of \mathfrak{L}_3^N , we have in any case
 $\mathfrak{L}_3^N(C(b)) \leq 9\pi^2$. (2.8)

We first sketch the main ideas for the proof. There are two arguments which

will be crucial

- Note that by definition of \mathfrak{L}_3^N , we have in any case
 $\mathfrak{L}_3^N(C(b)) \leq 9\pi^2$. (2.8)

he main ideas for the proof. There are two arguments which

proof:

ate for a minimal 3-partition, $\mathcal{D}_3 = (D_1, D_2, D_3)$. If De crucial for the proof:

Take any candidate for a minimal 3-partition, $\mathcal{D}_3 = (D_1, D_2, D_3)$. If we can

show that $\Lambda(\mathcal{D}_3) > 9\pi^2$ then this \mathcal{D}_3 cannot be a minimal partition due to

the definition of $\mathfrak{$ (1) Take any candidate for
show that $\Lambda(\mathcal{D}_3) > 9\pi$
the definition of $\mathfrak{L}_3^N(C)$
(2) Assume $b \leq 1/3$. A 3-1
on the double covering
Assume that there is **n** Take any candidate for a minimal 3-partition, $\mathcal{D}_3 = (D_1, D_2, D_3)$. If we can
show that $\Lambda(\mathcal{D}_3) > 9\pi^2$ then this \mathcal{D}_3 cannot be a minimal partition due to
the definition of $\mathfrak{L}_3^N(C(b))$.
Assume $b \leq 1/3$. the definition of $\mathfrak{L}_3^N(C(b))$.
- (2) Assume $b \leq 1/3$. A 3-partition \mathcal{D}_3 is said to have **property B** if it becomes

show that $\Lambda(\mathcal{D}_3) > 9\pi^2$ then this \mathcal{D}_3 cannot be a minimal partition due to
the definition of $\mathfrak{L}_3^N(C(b))$.
Assume $b \le 1/3$. A 3-partition \mathcal{D}_3 is said to have **property B** if it becomes
on the double the definition of $\mathfrak{L}_3^N(C(b))$.
Assume $b \leq 1/3$. A 3-partion the double covering $C(2)$
Assume that there is **minitially** nergy of this partition is lander that by Lemma 2.1, λ and 6-partition $\mathcal{D}_6 = (D_1, \ldots, D_$ Assume $b \le 1/3$. A 3-partition D_3 is said to have **property B** if it becomes
on the double covering $C(2, b)$ a 6-partition.
Assume that there is **minimal** 3-partition D_3 with **property B**. Then $\Lambda(\mathcal{D}_3)$,
nergy on the double covering $C(2, b)$ a 6-partition.
Assume that there is **minimal** 3-partition D_3
nergy of this partition is larger than or eq
note that by Lemma 2.1, $\lambda_6^N(C(2, b))$ is Co
mal 6-partition $D_6 = (D_1, ..., D_6)$ is Assume that there is **minimal** 3-partition \mathcal{D}_3 with **property B**. Then $\Lambda(\mathcal{D}_3)$, nergy of this partition is larger than or equal to $\lambda_6^N(C(2, b))$. To see this note that by Lemma 2.1, $\lambda_6^N(C(2, b))$ is Courant s the energy of this partition is larger than or equal to $\lambda_6^N(C(2, b))$. To see this just note that by Lemma 2.1, $\lambda_6^N(C(2, b))$ is Courant sharp. The corresponding minimal 6-partition $\mathcal{D}_6 = (D_1, \ldots, D_6)$ is given by the energy of this partition is larger than or equal to $\lambda_6^N(C(2, b))$. To see this just note that by Lemma 2.1, $\lambda_6^N(C(2, b))$ is Courant sharp. The corresponding
minimal 6-partition $\mathcal{D}_6 = (D_1, \dots, D_6)$ is given by
 $D_\ell = ((\ell - 1)/3, \ell/3) \times (0, b), \ell = 1, 2, \dots, 6.$ (2.9) minimal 6-partition $\mathcal{D}_6 = (D_1, \ldots, D_6)$ is given by

$$
D_{\ell} = ((\ell - 1)/3, \ell/3) \times (0, b), \quad \ell = 1, 2, ..., 6.
$$
 (2.9)

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Furthermore \mathcal{D}_6 for $C(2, b)$ is just the lifted 3-partition of $C(1, b)$ given in Theorem 2.4.
 Hence it suffices to show that the candidates for minimal partitions
 have t

Hence it suffices to show that the candidates for minimal partitions have the property B

Furthermore \mathcal{D}_6 for $C(2, b)$ is just the lifted 3-partition of $C(1, b)$ given in Theo-
rem 2.4.
Hence it suffices to show that the candidates for minimal partitions
have the property **B**.
We also observe that the $\begin{aligned} \mathbf{He} \\ \mathbf{hav} \\ \mathbf{We} \\ b < 1, \, \text{ar} \\ \text{for a mi} \end{aligned}$.
.
.
. 1, are non-nodal and further that, if we have a candidate for $\mathcal{D}_3 = (D_1, D_2, D_3)$

1, a minimal partition, each D_i is nice, that means
 $\text{Int}(\overline{D}_i) = D_i.$ (2.10)

2.10)

2.10)

2.10 of we could lower the energy by $b<$

$$
Int(\overline{D}_i) = D_i.
$$
\n(2.10)

for a minimal partition, each D_i is nice, that means
 $\text{Int}(\overline{D}_i) = D_i.$

If not we could lower the energy by removing an an

the number of the D_i . Here we neglect sets of capac

The assumption on D_i implies by mon Int $(D_i) = D_i$. (2.10)
by removing an arc inside \overline{D}_i without reducing
glect sets of capacity 0.
es by monotonicity that
 $> \lambda_1^D(C(1, b)) = \pi^2/b^2$.
d partition must already lead to a $\Lambda_3(\mathcal{D}_3) > 9\pi^2$,

$$
\lambda_1^D(D_i) > \lambda_1^D(C(1, b)) = \pi^2/b^2.
$$

If not we could lower the energy by removing an arc inside D_i without reducing
the number of the D_i . Here we neglect sets of capacity 0.
The assumption on D_i implies by monotonicity that
 $\lambda_1^D(D_i) > \lambda_1^D(C(1, b)) = \pi^2$ the number of the D_i . Here we neglect sets of capacity 0.

The assumption on D_i implies by monotonicity that
 $\lambda_1^D(D_i) > \lambda_1^D(C(1, b)) = \pi^2/b^2$.

Hence if $\pi^2/b^2 > 9\pi^2$ the associated partition must already

so (1) The assumption on D_i implies by monotonicity that
 $\lambda_1^D(D_i) > \lambda_1^D(C(1, b)) = \pi^2/b^2$.

e if $\pi^2/b^2 > 9\pi^2$ the associated partition must already

) applies.

We proceed by showing that any minimal 3-partition

t suffic

 $(D_i) > \lambda_1^D(C(1, b)) = \pi^2/b^2.$
ociated partition must alread
that any minimal 3-partition
inimal 3-partition $\mathcal{D}_3 = (L$
hat lifting this partition to t Hence if $\pi^2/b^2 > 9\pi^2$ the associated partition must already lead to a $\Lambda_3(\mathcal{D}_3) > 9\pi^2$,
so (1) applies.
We proceed by showing that any minimal 3-partition has property **B**. To show
this it suffices that in any mi so (1) applies. \square
We proceed by showing that any minimal 3-partition has property **B**. To show
this it suffices that in any minimal 3-partition $D_3 = (D_1, D_2, D_3)$ all the D_i are
0-homotopic. This implies that liftin We proceed by showing that any minimal 3-partition has property **B**. To show
this it suffices that in any minimal 3-partition $\mathcal{D}_3 = (D_1, D_2, D_3)$ all the D_i are
0-homotopic. This implies that lifting this partition this it suffices that in any minimal 3-partition $\mathcal{D}_3 = (D_1, D_2, D_3)$ all the D_i are 0-homotopic. This implies that lifting this partition to the double covering yields a 6-partition and the argument (2) applies. We a 6-partition and the argument **(2)** applies. We assume for contradiction that D_3 is not contractible, hence contains a path of index 1. We first observe that D_1 and D_2 must be neighbours (if not the partition wo is not contractible, hence contains a path of index 1. We first observe that D_1 and D_2 must be neighbours (if not the partition would be nodal). Then let us introduce $D_{12} = \text{Int} (\bar{D}_1 \cup \bar{D}_2)$. Because D_3 cont D_2 must be neighbours (if not the partition would be nodal). Then let us introduce $m_l = \text{Int}(\bar{D}_1 \cup \bar{D}_2)$. Because D_3 contains a path of index $1, \overline{D_{12}}$ cannot touch
component of the boundary of the cylinder and we have $\lambda_2^N(D_{12}) = \mathfrak{L}_3^N$. By
nain monotonicity (this is not the Dirichlet one component of the boundary of the cylinder and we have $\lambda_2^N(D_{12}) = \mathfrak{L}_3^N$. By
domain monotonicity (this is not the Dirichlet monotonicity result but the proof
can be done either by reflection or by a density argu $2(\nu_{12}) - 23$ can be done either by reflection or by a density argument), the second eigenvalue $\lambda_2^N(D_{12}) = \lambda_1^N(D_1)$ must be be larger than the second eigenvalue of the Dirichlet-Neumann problem of the cylinder. But we have, with $\lambda_2^N(D_{12}) = \lambda_1^N(D_1)$ must be be larger than the second eigenvalue of the Dirichlet-Neumann problem of the cylinder. But we have, with λ_i^{ND} denoting the eigenvalues with Neumann and Dirichlet boundary conditions on $\lambda_2^N(D_{12})=\lambda_1^N$ Neumann problem of the cylinder. But we have, with λ_i^{ND}
with Neumann and Dirichlet boundary conditions on th
boundary of the strip,
 $\lambda_1^{ND} = \frac{\pi^2}{4b^2}, \ \lambda_2^{ND} = \pi^2 \min\left(\frac{1}{b^2}, \ \frac{1}{4b^2}\right)$
Hence with Neumann and Dirichlet boundary conditions on the two components of the

boundary of the strip,
\n
$$
\lambda_1^{ND} = \frac{\pi^2}{4b^2}, \ \lambda_2^{ND} = \pi^2 \min\left(\frac{1}{b^2}, \ \frac{1}{4b^2} + 4\right).
$$
\n(2.11)
\nHence
\n
$$
\lambda_2^N(D_{1,2}) > \lambda_2^{ND},
$$
\n(2.12)

$$
\lambda_2^N(D_{1,2}) > \lambda_2^{ND},\tag{2.12}
$$

between $\lambda_1^{ND} = \frac{\pi^2}{4b^2}, \lambda_2^{ND} = \pi^2 \min\left(\frac{1}{b^2}, \frac{1}{4b^2} + 4\right).$ (2.11)

Hence $\lambda_2^N(D_{1,2}) > \lambda_2^{ND},$ (2.12)

and we get a contradiction if $\lambda_2^{ND} \ge 9\pi^2$. We just have to work out the condition

on b such that $\frac{1}{6}b$ $\lambda_2^N(D_{1,2}) > \lambda_2^{ND}$, (2.12)
and we get a contradiction if $\lambda_2^{ND} \ge 9\pi^2$. We just have to work out the condition
on b such that
 $\min\left(\frac{1}{b^2}, \frac{1}{4b^2} + 4\right) \ge 9.$ (2.13)
This is achieved for $b \le (2\sqrt{5})^{-1}$ as cla and we get a contradiction if λ_2^{ND}
on b such that
min
This is achieved for $b \leq (2\sqrt{5})^{-1}$ a
Remark 2.6. We recall that, altho
on the double covering), the minim ${}_{2}^{ND} \geq 9\pi^2$. We just have to work out the condition
 $\sin\left(\frac{1}{b^2}, \frac{1}{4b^2} + 4\right) \geq 9.$ (2.13)

⁻¹ as claimed in (2.5) in Theorem 2.4.

though there is a natural candidate (which is nodal

nimal 3-partition pr

$$
\min\left(\frac{1}{b^2}, \frac{1}{4b^2} + 4\right) \ge 9. \tag{2.13}
$$

on b such that
This is achieve
Remark 2.6. V
on the double of
for the annulu min $\left(\frac{1}{b^2}, \frac{1}{4b^2} + 4\right)$
 $\left(\frac{1}{b^2}, \frac{1}{4b^2} + 4\right)$
and in although there is a ninimal 3-partition
ase of a thin annu $\frac{1}{2}b$ m
at a 9. (2.13)
5) in Theorem 2.4.
tural candidate (which is nodal
oblem with Dirichlet conditions
or the disk is still open. This is achieved for $b \leq (2\sqrt{5})^{-1}$ as claimed in (2.5) in Theorem 2.4.
Remark 2.6. We recall that, although there is a natural candidate (won the double covering), the minimal 3-partition problem with Dirichl for the **Remark 2.6.** on the double covering), the minimal 3-partition problem with Dirichlet conditions for the annulus (also in the case of a thin annulus) or the disk is still open.

Spectral Minimal Partitions for a Thin Strip on a Cylinder 113
 Remark 2.7. In view of the considerations above and of Proposition 2.5 the ques-

tion arises whether for some $b < 1$ the corresponding minimal partition h **Remark 2.7.** In view of the considerations above and of Proposition 2.5 the question arises whether for some $b < 1$ the corresponding minimal partition has the property that it has one or two points in its zero set where 3 arcs meet, hence having locally a **Y**-structure as discussed for instance in [having locally a **Y**-structure as discussed for instance in [5]. As in the case of the rectangle considered in [1], we can observe that in the case $b = 1$, the eigenfunction $\cos 2\pi x - \cos 2\pi y$ has a nodal set described in rectangle considered in [1], we can observe that in the case $b = 1$, the eigenfunc-
tion $\cos 2\pi x - \cos 2\pi y$ has a nodal set described in $R(1,1)$ by the two diagonals
of the square and that it determines indeed a nodal 3-p of the square and that it determines indeed a nodal 3-partition. The guess is
then that when $1 - \epsilon < b < 1$ (with $\epsilon > 0$ small enough) this nodal 3-partition
will be deformed into a non nodal minimal 3-partition keeping the then that when $1-\epsilon < b < 1$ (with $\epsilon > 0$ small enough) this nodal 3-partition. will be deformed into a non nodal minimal 3-partition keeping the symmetry

3. Extension to minimal *k*-partitions of $C(b)$

(*x*, *y*) → (*x*, 1 − *y*). We expect two critical points from which three arcs start.

3. Extension to minimal *k*-partitions of $C(b)$

One can also consider for $\Omega = C(b)$ minimal *k*-partitions with *k* odd (*k* ≥ 3) a

$$
b < \frac{1}{k} \,. \tag{3.1}
$$

Neumann condition and assume
 $b < \frac{1}{k}$. (3.1)

The theorem of the previous section can be extended to the case $k > 3$.

First one observes that, if the closure of one open set of the minimal k-partition

contains a lin The theorem of the previous sec
First one observes that, if the c
contains a line joining the two co
double covering and obtain a (2) k or s ic s (1)
ition
(6) is
ition The theorem of the previous section can be extended to the case $k > 3$.
First one observes that, if the closure of one open set of the minimal k contains a line joining the two components of the boundary, then one car
dou First one observes that, if the closure of one open set of the minimal *k*-partition contains a line joining the two components of the boundary, then one can go to the double covering and obtain a $(2k)$ -partition. If $(3$ double covering and obtain a $(2k)$ -partition. If (3.1) is satisfied, $\lambda_k^N(C(0, 2b))$ is
Courant sharp, and get as in the previous section that the energy of this partition
is necessarily higher than $k^2 \pi^2$.
So there \cdot

double covering and obtain a $(2k)$ -partition. If (3.1) is satisfied, $\lambda_k^N(C(0, 2b))$ is Courant sharp, and get as in the previous section that the energy of this partition is necessarily higher than $k^2 \pi^2$.
So there is necessarily higher than $k^2 \pi^2$.

So there is no D_i whose boundary has nonempty intersection with both parts

of the boundary of the strip. Hence there exists one component of $\partial \Omega$ and at least
 $\frac{k+1}{2} D_i$ of t is necessarily higher than $k^2 \pi^2$.
So there is no D_i whose bo
of the boundary of the strip. He
 $\frac{k+1}{2}$ D_i of the k-partition such
this component. We immediate
 $\min(\mathfrak{L})$ So there is no D_i whose boundary has nonempty intersection with both parts

a boundary of the strip. Hence there exists one component of $\partial\Omega$ and at least
 D_i of the k-partition such that their boundaries ∂D_i 's d of the boundary of the strip. Hence there exists one component of $\partial\Omega$ and at least $\frac{k+1}{2}$ D_i of the k-partition such that their boundaries ∂D_i 's do not intersect with this component. We immediately deduce that $\frac{k+1}{2}$ D_i of the *k*-partition such that their boundaries ∂D_i 's do not intersect with
ponent. We immediately deduce that if $b \leq \frac{1}{k}$:
 $\min \left(\mathfrak{L}_{\frac{k+1}{2}}^{DN}(\Omega), \mathfrak{L}_{\frac{k+1}{2}}^{ND}(\Omega) \right) \geq k^2 \pi^2$, (3.2)
 $\mathfrak{L}_k^N(\Omega) = k^2 \$ \boldsymbol{k}

$$
\min\left(\mathfrak{L}_{\frac{k+1}{2}}^{DN}(\Omega),\mathfrak{L}_{\frac{k+1}{2}}^{ND}(\Omega)\right) \geq k^2 \pi^2\,,\tag{3.2}
$$

 \boldsymbol{k}

this component. We immediately deduce that if $b \leq \frac{1}{k}$
 $\min \left(\mathfrak{L}_{\frac{k+1}{2}}^{DN}(\Omega), \mathfrak{L}_{\frac{k+1}{2}}^{ND}(\Omega) \right) \geq k^2$

we have $\mathfrak{L}_k^N(\Omega) = k^2 \pi^2$.

Here \mathfrak{L}_ℓ^{DN} corresponds to the *l*th spectral particuith ่
:
:
: min $\left(\mathfrak{L}_{\frac{k+1}{2}}^{DN}(\Omega), \mathfrak{L}_{\frac{k+1}{2}}^{ND}(\Omega)\right) \geq k^2 \pi^2$, (3.2)

onds to the *l*th spectral partition eigenvalue for the strip

y condition on $y = 0$ and Neumann boundary condition

efined by exchange of the bound we have $\mathcal{L}_k^N(\Omega) = k^2 \pi^2$.

Here \mathcal{L}_ℓ^{DN} corresp

with Dirichlet boundar

for $y = b$ and \mathcal{L}_ℓ^{ND} is de

boundaries. In our spec

have actually $\mathcal{L}_\ell^{DN}(C(b))$

Having in mind th Here \mathfrak{L}_{ℓ}^{DN} corresponds to the ℓ th spectral partition eigenvalue for the strip
Dirichlet boundary condition on $y = 0$ and Neumann boundary condition
= b and \mathfrak{L}_{ℓ}^{ND} is defined by exchange of the boun ℓ with Dirichlet boundary condition on $y = 0$ and Neumann boundary condition
for $y = b$ and \mathfrak{L}_{ℓ}^{ND} is defined by exchange of the boundary conditions on the two
boundaries. In our special case, due to the symmetry wi for $y = b$ and \mathfrak{L}_{ℓ}^{ND} is defined by exchange of the boundary conditions on the two
boundaries. In our special case, due to the symmetry with respect to $y = \frac{b}{2}$, we
have actually $\mathfrak{L}_{\ell}^{DN}(C(b)) = \mathfrak{L}_{\ell}^{ND}($ for $y = b$ and \mathfrak{L}_{ℓ}^{ND} is defined by exchange of the boundary conditions on the two boundaries. In our special case, due to the symmetry with respect to $y = \frac{b}{2}$, we , we
|
|

 $\ell^{DN} \leq \mathfrak{L}^{DN}_\ell$

$$
\lambda_{\frac{k+1}{2}}^{DN} \ge k^2 \pi^2.
$$

have actually $\mathfrak{L}_{\ell}^{DN}(C(b)) = \mathfrak{L}_{\ell}^{ND}(C(b)).$

Having in mind that² $\lambda_{\ell}^{DN} \leq \mathfrak{L}_{\ell}^{DN}$
 $\lambda_{\frac{k+1}{2}}^{DN}$

If $b < \frac{1}{k}$, we get in the case when $k = 4p$
 $\frac{1}{4k^2} + 4(p+1)$ Having in mind that² $\lambda_{\ell}^{DN} \leq \mathfrak{L}_{\ell}^{DN}$, (3.2) is a consequence of
 $\lambda_{\frac{k+1}{2}}^{DN} \geq k^2 \pi^2$.
 $\lambda_{\frac{k+1}{2}}^{1} \geq k^2 \pi^2$.
 $\frac{1}{k}$, we get in the case when $k = 4p + 3$ $(p \in \mathbb{N})$ the addition.
 $\frac{1}{4b^2}$ If $b < \frac{1}{k}$
²Note th $\frac{1}{k}$, we get in the case when $k = 4p + 3$ $(p \in \mathbb{N})$ the additional condition
 $\frac{1}{4b^2} + 4(p+1)^2 \ge (4p+3)^2$.
hat we have equality for ℓ even and $b < \frac{1}{\ell}$.

$$
\frac{1}{4b^2} + 4(p+1)^2 \ge (4p+3)^2.
$$

or ℓ even and $b < \frac{1}{\ell}$.

²Note that we have equality for ℓ even and $b < \frac{1}{\ell}$.

Similarly, we get in the case when
$$
k = 4p + 1
$$
 $(p \in \mathbb{N}^*)$

$$
\frac{1}{4b^2} + 4p^2 \ge (4p + 1)^2.
$$

We have consequently proven:

Theorem 3.1. If

Similarly, we get in the case when $k = 4p + 1$ ($p \in \mathbb{N}^*$)
 $\frac{1}{4b^2} + 4p^2 \ge (4p + 1)^2$.

We have consequently proven:
 Theorem 3.1. If
 $\bullet k = 4p + 3$ ($p \in \mathbb{N}$) and $b \le 1/\sqrt{(3k+1)(k-1)}$, $\frac{1}{1}$ $\frac{1}{4b^2} + 4p^2 \ge (4p + 1)^2.$
:
 $b \le 1/\sqrt{(3k+1)(k-1)}$
 $b \le 1/\sqrt{(3k-1)(k+1)}$ Theorem 3.1. If
 $\bullet k = 4p + 3 \ (p \in \mathbb{N}) \ and \ b$

or
 $\bullet k = 4p + 1 \ (p \in \mathbb{N}^*) \ and$

then • $k = 4p + 3 \ (p \in \mathbb{N}) \ and \ b \leq 1/\sqrt{2}$ = $4p + 3$ ($p \in \mathbb{N}$) and $b \leq 1/\sqrt{(3k+1)(k-1)}$,

= $4p + 1$ ($p \in \mathbb{N}^*$) and $b \leq 1/\sqrt{(3k-1)(k+1)}$
 $\mathfrak{L}_k(C(b)) = k^2 \pi^2$,

minimal k-partition is given by $D_\ell = ((\ell - 1)^k)$. or • $k = 4p + 1$ $(p \in \mathbb{N}^*)$ and $b \leq 1/\sqrt{ }$

then

$$
\mathfrak{L}_k(C(b)) = k^2 \pi^2 ,
$$

= $4p + 1$ ($p \in \mathbb{N}^*$) and $b \le 1/\sqrt{(3k-1)(k+1)}$,
 $\mathfrak{L}_k(C(b)) = k^2 \pi^2$,

minimal k-partition is given by $D_\ell = ((\ell - 1))$. $(C(b)) = k^2 \pi^2$,

n by $D_{\ell} = ((\ell$

in domains

al than it seem

4. Generalization to other thin domains

and a minimal k-partition is given by $D_{\ell} = ((\ell - 1)/k, \ell/k) \times (0, b)$, for $\ell = 1, ..., k$.
4. **Generalization to other thin domains**
The previous proof is more general than it seems at the first look. At the price
of less explic $1, \ldots, k.$
4. Gen
The pre
of less exemention of less explicit results we have a similar result for an annulus like domain Ω . We
mention first the case $k = 3$ where the conditions read

• The eigenfunction associated with $\lambda_0^N(\Omega^R)$ is Courant sharp and antisym

-
- λ_6^N

mention first the case $k = 3$ where the conditions read

• The eigenfunction associated with $\lambda_6^N(\Omega^R)$ is Courant sharp and antisymmetric with respect to the deck transformation from Ω^R onto Ω ,

• $\lambda_6^N(\Omega^R$ • The eigenfunction associated with $\lambda_6^N(\Omega^R)$ is Courant sharp and antisymmetric with respect to the deck transformation from Ω^R onto Ω ,

• $\lambda_6^N(\Omega^R) \leq \inf(\lambda_2^{DN}(\Omega), \lambda_2^{ND}(\Omega))$. (4.1)

Here Ω^R is the dou metric with respect to the deck transformation from Ω^R onto Ω ,
 $\lambda_6^N(\Omega^R) \leq \inf(\lambda_2^{DN}(\Omega), \lambda_2^{ND}(\Omega))$.

Here Ω^R is the double covering of Ω and (DN) (respectively (

ds to the Dirichlet-Neumann problem (Dir ${}_{6}^{N}(\Omega^{R}) \leq \inf(\lambda_{2}^{DN}(\Omega), \lambda_{2}^{ND})$
ble covering of Ω and (DN
Veumann problem (Dirichle
tside, Neumann inside). In
nore explicit.
sult which can be expected
on the circle such that h_1 (Ω)). (4.1)

(i) (respectively (ND)) correttinside, Neumann outside),

the case of the annulus, these

(i). For $b > 0$ and two regular
 $\langle h_2, w \rangle$ consider an annulus Here Ω^R is the double covering of Ω and (DN) (respectively (ND)) corre-
ds to the Dirichlet-Neumann problem (Dirichlet inside, Neumann outside),
ctively (Dirichlet outside, Neumann inside). In the case of the annul sponds to the Dirichlet-Neumann problem (Dirichlet inside, Neumann outside), respectively (Dirichlet outside, Neumann inside). In the case of the annulus, these conditions can be made more explicit.
Here is a typical resu

conditions can be made more explicit.

Here is a typical result which can be expected. For $b > 0$ and two regular

functions $h_1(\theta)$ and $h_2(\theta)$ on the circle such that $h_1 < h_2$, we consider an annulus

like domain aro Here is a typical result which can be expected. For $b > 0$ and two regular
functions $h_1(\theta)$ and $h_2(\theta)$ on the circle such that $h_1 < h_2$, we consider an annulus
like domain around the unit circle defined in polar coor

$$
A(b) = \{(x, y) : 1 + bh_1(\theta) < r < 1 + bh_2(\theta)\}.
$$

functions $h_1(\theta)$ and $h_2(\theta)$ on the circle such that $h_1 < h_2$, we consider an annulus
like domain around the unit circle defined in polar coordinates by
 $A(b) = \{(x, y) : 1 + bh_1(\theta) < r < 1 + bh_2(\theta)\}\.$
It is clear from [8] togeth

 $A(b) = \{(x, y) : 1 + bh_1(\theta) < r < 1 + bh_2(\theta)\}$.
It is clear from [8] together with Poincaré's inequality that there ex
that, if $0 < b \le b_0$, condition (4.1) is satisfied.
One must verify the condition for Courant sharpness, which
six $(b) = \{(x, y) : 1 + bh_1(\theta) < r < 1 + bh_2(\theta)\}\.$

ogether with Poincaré's inequality that there condition (4.1) is satisfied.

ify the condition for Courant sharpness, which the lifted Laplacian on the double coverir

true for our mor It is clear from [8] together with Poincaré's inequality that there exists $b_0 > 0$ such that, if $0 < b \le b_0$, condition (4.1) is satisfied.
One must verify the condition for Courant sharpness, which is true for the sixth that, if $0 < b \le b_0$, condition (4.1) is satisfied.
One must verify the condition for Coura
sixth eigenvalue of the lifted Laplacian on t
and should be also true for our more general
references, (see however [2] for thin c eigenvalue of the lifted Laplacian on the double covering of the annulus
should be also true for our more general situation but for which we have no
ences, (see however [2] for thin curved tubes and [9]).
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and should be also true for our more general situation but for which we have no
references, (see however [2] for thin curved tubes and [9]).
Remark 4.1. Although not explicit, condition (4.1) can be analyzed by pertur references, (see however [2] for thin curved tubes and [9]).
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tive method. This is indeed a purely spectral question. There is a huge lite references, (see however [2] for thin curved tubes and [9]).
 Remark 4.1. Although not explicit, condition (4.1) can be analyzed by perturba-

tive method. This is indeed a purely spectral question. There is a huge lite **Remark 4.1.** Although not explicit, condition (4.1) can be analyzed by perturba-

tive method. This is indeed a purely spectral question. There is a huge literature concerning thin domains, see for example [3, 8] (and references therein).
Remark 4.2. Similar considerations lead also to extensions to **Remark 4.2.** Similar considerations lead also to extensions to higher k or thin annulus with Neumann boundary conditions. **Remark 4.2.** Similar considerations lead also to extensions to higher *k* odd for the thin annulus with Neumann boundary conditions.

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Acknowledgement
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Jacobi and CMV Matrices with Coefficients of Generalized Bounded Variation

Milivoje Lukic

Abstract. We consider Jacobi and CMV matrices with coefficients satisfying an ℓ^p condition and a generalized bounded variation condition. This includes discrete Schrödinger operators on a half-line or line with finite linear combinations of Wigner–von Neumann type potentials $\cos(n\phi + \alpha)/n^{\gamma}$ with $\gamma > 0$.

Our results show preservation of the absolutely continuous spectrum, absence of singular continuous spectrum, and that embedded pure points in the continuous spectrum can only occur in an explicit finite set.

Mathematics Subject Classification (2010). Primary 42C05,47B36.

Keywords. Jacobi matrix, CMV matrix, bounded variation, Wigner–von Neumann potential, almost periodic.

Jacobi matrix is the tridiagonal matrix acting on $\ell^2(\mathbb{N}),$

For a bounded sequence
$$
\{a_n, b_n\}_{n=1}^{\infty}
$$
 with $a_n > 0$, $b_n \in \mathbb{R}$, the corresponding
Jacobi matrix is the tridiagonal matrix acting on $\ell^2(\mathbb{N})$,
\n
$$
J(a, b) = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 & a_2 \\ & & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}
$$
\nFor a sequence $\{\alpha_n\}_{n=0}^{\infty}$ with $|\alpha_n| < 1$, the corresponding CMV matrix [1] is a five-diagonal unitary matrix which acts on $\ell^2(\mathbb{N})$, defined by

For a sequence $\{\alpha_n\}_{n=\infty}^{\infty}$
five-diagonal unitary m
 $\mathcal{C}(\alpha) =$

For a sequence
$$
\{\alpha_n\}_{n=0}^{\infty}
$$
 with $|\alpha_n| < 1$, the corresponding CMV matrix [1] is a five-diagonal unitary matrix which acts on $\ell^2(\mathbb{N})$, defined by
\n
$$
\mathcal{C}(\alpha) = \begin{pmatrix} \Theta_0 & & \\ & \Theta_2 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & \\ & \Theta_1 & \\ & & \ddots \end{pmatrix}
$$
\nwhere 1 stands for a single entry of 1 and Θ_j are unitary 2×2 blocks
\n
$$
\Theta_j = \begin{pmatrix} \overline{\alpha}_j & \sqrt{1-|\alpha_j|^2} \\ & & -\alpha_j \end{pmatrix}
$$
\n(3)

where 1 stands for a single entry of 1 and Θ_i are unitary 2×2 blocks

where 1 stands for a single entry of 1 and
$$
\Theta_j
$$
 are unitary 2×2 blocks
\n
$$
\Theta_j = \begin{pmatrix} \frac{\bar{\alpha}_j}{\sqrt{1-|\alpha_j|^2}} & \sqrt{1-|\alpha_j|^2} \\ 0 & -\alpha_j \end{pmatrix}
$$
\n(3)

118 $\,$ M. Lukic $\,$ CMV matrices arise naturally in the study of orthogonal polynomials on the unit circle (see [1, 11]). It is well known that bounded variation combined with decay to the free case implies preservatio circle (see [1, 11]).
It is well known that bounded variation combined with decay to the free case
implies preservation of a.c. spectrum. These results are often cited as Weidmann's
theorem, who proved the result for Schr It is well know
implies preservation
theorem, who produced theorem, who produced
Jacobi and CMV r
Interest in op
work of Wigner and es preservation of a.c. spectrum. These results are often cited as Weidmann's em, who proved the result for Schrödinger operators [13]. The analogous bi and CMV results are due to Máté–Nevai [7] and Peherstorfer–Steinbaue

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Jacobi and CMV results are due to Máté-Nevai [7] and Peherstorfer-Steinbauer [9].
Interest in operators with decaying harmonic oscillations date Jacobi and CMV results are due to Máté–Nevai [7] and Peherstorfer–Steinbauer [9].

Interest in operators with decaying harmonic oscillations dates at least to the

work of Wigner and von Neumann [8] (see also [10, XIII.13 Interest in operators with decaying harmonic oscillations dates at least to the work of Wigner and von Neumann [8] (see also [10, XIII.13]), who constructed on of Wigner and von Neumann [8] (see also [10, XIII.13]), who constructed on
radial potential $V(r)$ with the asymptotic behavior
 $V(r) = -8 \frac{\sin(2r)}{r} + O(r^{-2}), \quad r \to \infty$ (4)
the peculiar property that the Schrödinger operator $-\Delta +$ \mathbb{R}^3 a radial potential $V(r)$ with the asymptotic behavior

$$
V(r) = -8\frac{\sin(2r)}{r} + O(r^{-2}), \quad r \to \infty
$$
 (4)

 \mathbb{R}^3 a radial potential $V(r)$ with the asymptotic behavior
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at +1 embedded in the a.c. spectr $(r) = -8 \frac{\sin(2r)}{r} + O(r^{-2}), \quad r \to \infty$ (4)
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a.c. spectrum [0, + ∞). We are interested in a class of Jacobi
h similar behavior. This motivates our use of the notion of $\frac{\sin(2r)}{r}$
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at +1 embedded in the a.c. spectrum [0, + ∞). We are interested in a class of Jacobi
and CMV matrices with similar behavior. This motiva at +1 embedded in the a.c. spectrum [0, +∞). We are interested in a class of Jacobi
and CMV matrices with similar behavior. This motivates our use of the notion of
generalized bounded variation:
Definition 1. A sequen generalized bounded variation

generalized bounded variation:
 Definition 1. A sequence $\beta = {\beta_n}_{n=N}^{\infty}$ (*N* can be finite or $-\infty$) has *rotated* bounded variation with phase ϕ if
 $\sum_{n=1}^{\infty} |e^{i\phi} \beta_{n+1} - \beta_n| < \infty.$ (5) **Definition 1.** A sequence $\beta = {\beta_n}_{n=N}^{\infty}$ bounded variation with phase ϕ if

A sequence
$$
\beta = {\beta_n}_{n=N}^{\infty}
$$
 (*N* can be finite or $-\infty$) has *rotated*
tion with phase ϕ if

$$
\sum_{n=N}^{\infty} |e^{i\phi}\beta_{n+1} - \beta_n| < \infty.
$$
 (5)
 $= {\alpha_n}_{n=N}^{\infty}$ has *generalized bounded variation* with the set of phases
 β_L if it can be expressed as a finite sum

with phase ϕ if
 $\sum_{n=N}^{\infty}$ |
 $\sum_{n=N}^{\infty}$ has *gen*
f it can be expi A sequence $\alpha = {\alpha_n}_{n=0}^{\infty}$
 $A = {\phi_1, \ldots, \phi_L}$ if it divides $\beta^{(1)}, \ldots$ $n = N$ \boldsymbol{A}

has generalized bounded variation with the set of phases
be expressed as a finite sum

$$
\alpha_n = \sum_{l=1}^{L} \beta_n^{(l)}
$$
(6)

$$
\alpha_n^{(L)}
$$
, such that the *l*th sequence $\beta^{(l)}$ has rotated bounded

= { ϕ_1, \ldots, ϕ_L } if it can be expressed as a finite sum
 $\alpha_n = \sum_{l=1}^L \beta_n^{(l)}$

L sequences $\beta^{(1)}, \ldots, \beta^{(L)}$, such that the *l*th sequence riation with phase ϕ_l . We will denote by $GBV(A)$ $=\sum_{l=1}$
at the note b
et of p
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of *L* sequences $\beta^{(1)}, \ldots, \beta^{(L)}$, such that the *l*th sequence $\beta^{(l)}$ has rotated bounded
variation with phase ϕ_l . We will denote by $GBV(A)$ the set of sequences with
generalized bounded variation with set of phase variation with phase ϕ_l . We will denote by $GBV(A)$ the set of sequences with
generalized bounded variation with set of phases A.
For an example of rotated bounded variation with phase ϕ , take $\beta_n = e^{-i(n\phi + \alpha)}\gamma_n$,
wit generalized bounded variation with set of phases A.
For an example of rotated bounded variation with ph
with $\{\gamma_n\}_{n=N}^{\infty}$ any sequence of bounded variation. C
may seem like an unnatural condition for real-valu-
ing ro For an example of rotated bounded variation with phase ϕ , take $\beta_n = e^{-i(n\phi + \alpha)}\gamma_n$,
with $\{\gamma_n\}_{n=N}^{\infty}$ any sequence of bounded variation. Generalized bounded variation
may seem like an unnatural condition for real-v with $\{\gamma_n\}_{n=N}^{\infty}$ any sequence of bounded variation. Generalized bounded variation
may seem like an unnatural condition for real-valued sequences, but by combin-
ing rotated bounded variation with phases ϕ and $-\phi$ $n = N$ ing rotated bounded variation with phases ϕ and $-\phi$, one gets $e^{-i(n\phi+\alpha)}\gamma_n + e^{+i(n\phi+\alpha)}\gamma_n = \cos(n\phi+\alpha)\gamma_n$. It is then clear that a linear combination of Wigner-
von Neumann type potentials plus an ℓ^1 part,
 $V_n = \sum_{k=1}$

$$
e^{+i(n\phi+\alpha)}\gamma_n = \cos(n\phi+\alpha)\gamma_n.
$$
 It is then clear that a linear combination of Wigner–
von Neumann type potentials plus an ℓ^1 part,

$$
V_n = \sum_{k=1}^K \lambda_k \cos(n\phi_k + \delta_k)/n^{\gamma_k} + q_n
$$
(7)
with $\gamma_k > 0$ and $\{q_n\} \in \ell^1$, has generalized bounded variation.
Wong [14] has the first result for CMV matrices with generalized bounded

= $\sum_{k=1} \lambda_k \cos(n\phi_k + \delta_k)/n^{\gamma_k} + q_n$ (7)

, has generalized bounded variation.

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m 2 in the case $\{\alpha_n\}_{n=0}^{\infty} \in \ell^2$. For discrete Schrödinger

[3] analyzed the poten with $\gamma_k > 0$ and $\{q_n\} \in \ell^1$, has generalized bounded variation.
Wong [14] has the first result for CMV matrices with govariation, proving Theorem 2 in the case $\{\alpha_n\}_{n=0}^{\infty} \in \ell^2$. For doperators, Janas-Simonov [tion, proving Theorem 2 in the case $\{\alpha_n\}_{n=0}^{\infty} \in \ell^2$. For discrete Schrödinger
ttors, Janas–Simonov [3] analyzed the potential $V_n = \cos(\phi n + \delta)/n^{\gamma} + q_n$, variation, proving Theorem 2 in the case $\{\alpha_n\}_{n=0}^{\infty} \in \ell^2$. For discrete Schrödinger operators, Janas–Simonov [3] analyzed the potential $V_n = \cos(\phi n + \delta)/n^{\gamma} + q_n$,

 $\mathcal{E}_{t-1} \in \ell^1$, and obtained for this potential the same spectral
3.
he main results:
 $\det J$ be a Jacobi matrix with coefficients $\{a_n, b_n\}_{n=1}^{\infty}$. Let $\sum_{n=1}^{\infty} \in \ell^1$

with $\gamma > 1/3$ and $\{q_n\}_{n=1}^{\infty} \in \ell^1$, and obtained for this potential the same spectral
results as our Corollary 3.
We can now state the main results:
Theorem 1 (Lukic [6]). Let J be a Jacobi matrix with coefficie We can now state th
 Theorem 1 (Lukic [6]). Le

p be a positive integer, A α

assumptions:
 $1^{\circ} \{a_n^2 - 1\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ **rem 1 (Lukic [6]).** Let *J* be a *Jacoba*
a positive integer, $A \subset \mathbb{R}$ a finite set
nptions:
 $\{a_n^2 - 1\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in \ell^p \cap GBV$
 $\{a_n - 1\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in \ell^p \cap GBV$ **Theorem 1 (Lukic [6]).** Let J be a Jacobi matrix with coefficients $\{a_n, b_n\}_{n=1}^{\infty}$. Let p be a positive integer, $A \subset \mathbb{R}$ a finite set of phases, and make one of these sets of assumptions:

 $\sum_{n=1}^{\infty}$, ${b_n}_{n=1}^{\infty} \in \ell^p \cap GBV$

Then

1° { $a_n^2 - 1$ }_{n[∞]-1}, { b_n }_{n[∞]-1} ∈ $\ell^p \cap GBV$

2° { $a_n - 1$ }_{n−1}, { b_n }_{n−1} ∈ $\ell^p \cap GBV$

hen

(i) $\sigma_{ac}(J) = [-2, 2]$

ii) $\sigma_{sc}(J) = \emptyset$

ii) $\sigma_{pp}(J) \cap (-2, 2)$ is the subset of a (A)
 (A)
 $n e$ 2° { $a_n - 1$ }_{n=}

hen

(i) $\sigma_{ac}(J) = [\cdot]$

(i) $\sigma_{sc}(J) = \emptyset$

(ii) $\sigma_{pp}(J) \cap (\cdot)$ (A)
n es (i) $\sigma_{ac}(J) = [-2, 2]$

(ii) $\sigma_{sc}(J) = \emptyset$

(iii) $\sigma_{pp}(J) \cap (-2, 2)$
 $\sigma_{pp}(J)$ (ii) $\sigma_{\rm sc}(J) = \emptyset$
(iii) $\sigma_{\rm pp}(J) \cap ($
where $\tilde{A} = A \cup$

(iii)
$$
\sigma_{\text{pp}}(J) \cap (-2, 2)
$$
 is the subset of an explicit finite set,
\n
$$
\sigma_{\text{pp}}(J) \cap (-2, 2) \subset \{ \pm 2 \cos(\eta/2) | \eta \in \underbrace{\tilde{A} + \cdots + \tilde{A}}_{p-1 \text{ times}} \}
$$
\n(8)
\nwhere $\tilde{A} = A \cup \{0\}$ in case 1° and $\tilde{A} = (A + A) \cup A \cup \{0\}$ in case 2° .
\n**Theorem 2 (Lukic [6]).** Let $C = C(\alpha)$ be a CMV matrix with coefficients $\{\alpha_n\}_{n=0}^{\infty}$
\nsuch that

where A

 ∞
 $n=0$ where $A = A \cup \{0\}$ in case 1° and $A = (A + A) \cup A \cup \{0\}$ in case 2° .
 Theorem 2 (Lukic [6]). Let $C = C(\alpha)$ be a CMV matrix with coefficie

such that
 $\{\alpha_n\}_{n=0}^{\infty} \in \ell^p \cap GBV(A)$

for a positive odd integer $p = 2q + 1$ **Theorem 2 (Lukic [6]).** Let $C = C(\alpha)$ be a CMV matrix with coefficients $\{\alpha_n\}_{n=0}^{\infty}$
such that
 $\{\alpha_n\}_{n=0}^{\infty} \in \ell^p \cap GBV(A)$
for a positive odd integer $p = 2q + 1$ and a finite set $A \subset \mathbb{R}$. Then
(i) $\sigma_{\text{ac}}(C) = \$ such that

$$
\{\alpha_n\}_{n=0}^{\infty} \in \ell^p \cap GBV(A)
$$

 (A)
set .

- (i) $\sigma_{\text{ac}}(\mathcal{C}) = \partial \mathbb{D}$, where $\partial \mathbb{D}$ is the unit circle
(ii) $\sigma_{\text{sc}}(\mathcal{C}) = \emptyset$
(iii) $\sigma_{\text{DD}}(\mathcal{C})$ is the subset of an explicit finite set,
-
-

$$
\sigma_{\rm pp}(\mathcal{C}) \subset \left\{ \exp(i\eta) \middle| \eta \in \underbrace{(A + \dots + A)}_{q \text{ times}} - \underbrace{(A + \dots + A)}_{q-1 \text{ times}} \right\} \tag{9}
$$

(ii) $\sigma_{\rm sc}(\mathcal{C}) = \emptyset$

(iii) $\sigma_{\rm pp}(\mathcal{C})$ is a
 $\sigma_{\rm F}$

Remark 1. If a (iii) $\sigma_{\text{pp}}(\mathcal{C})$ is the subset of an explicit finite set,
 $\sigma_{\text{pp}}(\mathcal{C}) \subset \{\exp(i\eta) | \eta \in (\underbrace{A + \cdots + A}_{q \text{ times}}\})$

Remark 1. If a sequence $\{\beta_n\}$ has rotated q-bourged
 $\beta_n | \langle \infty \rangle$, then it also has generalized boun $(C) \subset \{ \exp(i\eta) | \eta \in (\underbrace{A + \cdots + A}_{q \text{ times}}) - (\underbrace{A + \cdots + A}_{q-1 \text{ times}})$

equence $\{\beta_n\}$ has rotated *q*-bounded variation, i.

t also has generalized bounded variation so our

equences (with the appropriate adjustment of the $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\frac{q}{q}$ – ally Remark 1. If a sequence $\{\beta_n\}$ has rotated q-bounded variation, i.e., $\sum |e^{i\phi}\beta_{n+q} - \beta_n| < \infty$, then it also has generalized bounded variation so our results trivially extend to such sequences (with the appropriate ad β_n < ∞ , then it also has generalized bounded variation so our results trivially extend to such sequences (with the appropriate adjustment of the set A).

In the special case $a_n = 1$, Theorem 1 becomes a result on discrete Schrödinger operators on a half-line. By a standard pasting argument, this also implies a result for discrete Schrödinger operators on a line. extend to such sequences (with the appropriate adjustment of the set A).
In the special case $a_n = 1$, Theorem 1 becomes a result on discrete S
ger operators on a half-line. By a standard pasting argument, this also is
r In the special case $a_n = 1$, Theorem 1 becomes a result on discrete Schrödin-
perators on a half-line. By a standard pasting argument, this also implies a
t for discrete Schrödinger operators on a line.

 $\lim_{n \to \infty} 3. Let$

Corollary 3. Let

$$
(Hx)_n = x_{n+1} + x_{n-1} + V_n x_n \tag{10}
$$

result for discrete Schrödinger operators on a line.
 Corollary 3. Let
 $(Hx)_n = x_{n+1} + x_{n-1} + V_n x_n$ (10)

be a discrete Schrödinger operator on a half-line or line, with $\{V_n\}$ in ℓ^p with **Corollary 3.** Let
 $(Hx)_n = x_{n+1} + x_{n-1} +$

be a discrete Schrödinger operator on a half-line

generalized bounded variation with set of phases A $(Hx)_n = x_{n+1} + x_{n-1} + V_n x_n$ (10)

converted to the phase of phases A.

(10)

the phases A. be a discrete Schrödinger operator on a half-line or line, with ${V_n}$ in ℓ^p with generalized bounded variation with set of phases A.

Then

Then

(i) $\sigma_{ac}(H) = [-2, 2]$

(ii) $\sigma_{sc}(H) = \emptyset$

(iii) $\sigma_{pp}(H) \cap (-2, 2)$ is a finite set, (i) $\sigma_{ac}(H) = [-2, 2]$

(ii) $\sigma_{sc}(H) = \emptyset$

(iii) $\sigma_{pp}(H) \cap (-2, 2)$
 $\sigma_{pp}(H) \cap$ (ii) $\sigma_{\rm sc}(H) = \emptyset$

(iii) $\sigma_{\rm pp}(H) \cap (\sigma_{\rm pp}(H))$

(iii)
$$
\sigma_{\text{pp}}(H) \cap (-2,2)
$$
 is a finite set,
\n
$$
\sigma_{\text{pp}}(H) \cap (-2,2) \subset \left\{ \pm 2 \cos(\eta/2) \middle| \eta \in \bigcup_{k=1}^{p-1} (\underbrace{A + \dots + A}_{k \text{ times}}) \right\}
$$
\nThis corollary applies in particular to linear combinations of
\nNeumann potentials (7)

(*H*) ∩ (-2, 2) ⊂ { $\pm 2 \cos(\eta/2) | \eta \in \bigcup_{k=1}^{\infty} \underbrace{(A + \cdots + A)}_{k \text{ times}}$
ary applies in particular to linear combinations
tials (7).
ults discussed so far concern perturbation of the f
nded variation. For perturbations of o $\frac{1}{\pi}$ nann potentials (7).
All the results discussed so far concern perturbation of the free operator by
alized bounded variation. For perturbations of other operators the situation
re complicated. For instance, in contrast to All the results disc
generalized bounded var
is more complicated. For
shown that for some clas
operator $-\Delta + V_0$ by a
spectrum. alized bounded variation. For perturbations of other operators the situation
re complicated. For instance, in contrast to Weidmann's theorem, Last [5] has
n that for some classes of potentials V_0 , perturbing the discre is more complicated. For instance, in contrast to Weidmann's theorem, Last [5] has
shown that for some classes of potentials V_0 , perturbing the discrete Schrödinger
operator $-\Delta + V_0$ by a perturbation V of bounded vari

shown that for some classes of potentials V_0 , perturbing the discrete Schrödinger
operator $-\Delta + V_0$ by a perturbation V of bounded variation can destroy a.c.
spectrum.
In another direction, one can relax the bounded va operator $-\Delta + V_0$ by a perturbation *V* of bounded variation can destroy a.c.
spectrum.
In another direction, one can relax the bounded variation condition to an
 ℓ^2 condition on *q*-variation, namely $\sum_n |x_{n+q} - x_n|^2 <$ In an ℓ^2 conditions in probability of the spectrum.

In that this late of the spectrum. ℓ^2 condition on q-variation, namely $\sum_n |x_{n+q} - x_n|^2 < \infty$. Kaluzhny-Shamis [4],
using in part ideas from Denisov [2] who studied the case $a_n \equiv 1$, have shown
that this kind of perturbation with $x_n \to 0$ preserves the

 ℓ^2 condition on q-variation, namely $\sum_n |x_{n+q} - x_n|^2 < \infty$. Kaluzhny–Shamis [4],
using in part ideas from Denisov [2] who studied the case $a_n \equiv 1$, have shown
that this kind of perturbation with $x_n \to 0$ preserves the that this kind of perturbation with $x_n \to 0$ preserves the a.c. spectrum of periodic
Jacobi operators.
In yet another direction, Stolz [12] takes Δ to be the forward difference
operator $(\Delta x)_n = x_{n+1} - x_n$ and analyzes d In yet anot
operator $(\Delta x)_n =$
 $\Delta^j V \in \ell^{k/j}$ for 1
interval $[-2 + \text{lim}$
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in a different way

operator $(\Delta x)_n = x_{n+1} - x_n$ and analyzes discrete Schrödinger potentials with $\Delta^j V \in \ell^{k/j}$ for $1 \leq j \leq k$, showing that a.c. spectrum persists precisely on the interval $[-2 + \limsup_{n \to \infty} V_n, 2 + \liminf_{n \to \infty} V_n]$.
As communic $\Delta^j V \in \ell^{k/j}$ for $1 \le j \le k$, showing that a.c. spectrum persists precisely on the
interval $[-2 + \limsup_{n \to \infty} V_n, 2 + \liminf_{n \to \infty} V_n]$.
As communicated to us by Yoram Last, this problem can also be motivated
in a different way interval $[-2 + \limsup_{n \to \infty} V_n, 2 + \liminf_{n \to \infty} V_n]$.
As communicated to us by Yoram Last, this
in a different way: let $V_n = \lambda_n W_n$, with $\lambda_n > 0$ mo
be given by (10). For different classes of potential
need to ensure preservatio lifferent way: let $V_n = \lambda_n W_n$, with $\lambda_n > 0$ monotone decaying to 0, and let *H* ven by (10). For different classes of potentials *W*, what kind of decay do we to ensure preservation of a.c. spectrum? For *W* from a large in a different way: let $V_n = \lambda_n W_n$, with $\lambda_n > 0$ monotone decaying to 0, and let H
be given by (10). For different classes of potentials W , what kind of decay do we
need to ensure preservation of a.c. spectrum? For

be given by (10). For different classes of potentials W, what kind of decay do we
need to ensure preservation of a.c. spectrum? For W from a large class of almost
periodic potentials, our results state that any $\{\lambda_n\} \in \$ need to ensure preservation of a.c. spectrum? For *W* from a large class of almost
periodic potentials, our results state that any $\{\lambda_n\} \in \ell^p$, $p < \infty$, suffices:
Corollary 4. Let (10) be a discrete Schrödinger opera periodic potentials, our results state that any $\{\lambda_n\} \in \ell^p$, $p < \infty$, suffices:
 Corollary 4. Let (10) be a discrete Schrödinger operator on a half-line or $V_n = \lambda_n W_n$, $\{\lambda_n\} \in \ell^p$ of bounded variation (with $p < \infty$ **Corollary 4.** Let (10) be a discrete Schrödinger operator on a half-line or line with (10) be a discrete Schrödinger operator on a half-line or line with
 $\} \in \ell^p$ of bounded variation (with $p < \infty$) and W a trigonometric
 $W_n = \sum_{l=1}^L a_l \cos(2\pi\alpha_l n + \phi_l)$
 $+2\pi\alpha_l + 2\pi\alpha_l$ all conclusions of Gamllony 2 h $V_n = \lambda_n W_n$, $\{\lambda_n\} \in \ell^p$ of bounded variation (with $p < \infty$) and W a trigonometric
polynomial,
 $W_n = \sum_{l=1}^L a_l \cos(2\pi \alpha_l n + \phi_l)$
Then with $A = \{\pm 2\pi \alpha_1, \ldots, \pm 2\pi \alpha_l\}$, all conclusions of Corollary 3 hold. polynomial,

$$
W_n = \sum_{l=1}^{L} a_l \cos(2\pi \alpha_l n + \phi_l)
$$

., $\pm 2\pi \alpha_l$, all conclusions of

$$
\sum_{l=1}^{L} a_l \cos(2\pi \alpha_l n + \phi_l)
$$

where a_l is the same as a_l

Then with $A = \{\pm 2\pi\alpha_1, \ldots, \pm 2\pi\alpha_l\}$, all conclusions of Corollary 3 hold.

Acknowledgement

= { $\pm 2\pi\alpha_1,\ldots,\pm 2\pi\alpha_l$ }, all conclusions of Corollary 3 hold.
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helpful discussions. It is my pleasure to thank Barry Simon for suggesting the problem and for his
guidance and helpful discussions.

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Order Convergence Ergodic Theorems in Rearrangement Invariant Spaces

Mustafa Muratov, Julia Pashkova and Ben-Zion Rubshtein

Abstract. We find necessary and sufficient conditions for order convergence of Ces´aro averages of positive absolute contractions in rearrangement invariant spaces. We study the case, when the considered measure is infinite. The investigation of order convergence includes both Dominated and Individual Ergodic Theorems.

Mathematics Subject Classification (2010). Primary 37A30; Secondary 46B42.

Keywords. Ergodic theorems, rearrangement invariant spaces, positive absolute contraction, order convergence.

1. Introduction

 $p \leq$

Let (Ω, μ) be an infinite σ -finite non-atomic measure space, $\mathbf{L}_p = \mathbf{L}_p(\Omega, \mu)$, $1 \leq p \leq +\infty$ and $\mathbf{L}_0 = \mathbf{L}_0(\Omega, \mu)$ be the set of all μ -measurable functions $f : \Omega \to \mathbf{R}$.
We write: $\mathbf{L}_0 = \mathbf{L}_0(\$ +∞ and $\mathbf{L}_0 = \mathbf{L}_0(\mathbf{\Omega}, \mu)$ be the set of all μ -measurable functions $f : \mathbf{\Omega} \to \mathbf{R}$.
We write: $\mathbf{L}_0 = \mathbf{L}_0(\mathbf{R}_+, \mathbf{m})$ in the particular case, when $\mathbf{\Omega} = \mathbf{R}_+ = [0, \infty$
 $\mu = \mathbf{m}$ is the usual Leb We write: $\mathbf{L}_0 = \mathbf{L}_0(\mathbf{R}_+,\mathbf{m})$ in the particular case, when $\mathbf{\Omega} = \mathbf{R}_+ = [0,\infty)$
 $\iota = \mathbf{m}$ is the usual Lebesgue measure on $[0, +\infty)$.

A linear operator $T : \mathbf{L}_1 + \mathbf{L}_{\infty} \to \mathbf{L}_1 + \mathbf{L}_{\infty}$ is sai and $\mu = \mathbf{m}$ is the usual Lebesgue measure on $[0, +\infty)$.

A linear operator $T : \mathbf{L}_1 + \mathbf{L}_{\infty} \to \mathbf{L}_1 + \mathbf{L}_{\infty}$ is said to $or (\mathbf{L}_1, \mathbf{L}_{\infty})$ -contraction if T is a contraction in \mathbf{L}_1 and i
 T is sa A linear operator $T : \mathbf{L}_1 + \mathbf{L}_{\infty} \to \mathbf{L}_1 + \mathbf{L}_{\infty}$ is said to be an absolute contraction $_1, \mathbf{L}_{\infty}$)-contraction if T is a contraction in \mathbf{L}_1 and in \mathbf{L}_{∞} as well. The operator said to be *po* or (**L**₁, **L**_∞)-contraction if *T* is a contraction in **L**₁ and in **L**_∞ as well. The operator is said to be *positive* if $Tf \ge 0$ for all $f \ge 0$. Let us denote by \mathcal{PAC} the set of positive absolute contraction *T* is said to be *positive* if $Tf \ge 0$ for all $f \ge 0$. Let us denote by \mathcal{PAC} the set of all positive absolute contractions.
For any $T \in \mathcal{PAC}$ and $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ we consider the Cesáro averages $A_{n,T}f =$ all positive absolute contractions.
For any $T \in \mathcal{PAC}$ and $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ we consider the Cesáro averages

$$
A_{n,T}f = \frac{1}{n} \sum_{k=1}^{n} T^{k-1} f
$$

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For any $T \in \mathcal{PAC}$ and $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ we consider the Cesáro averages
 $A_{n,T} f = \frac{1}{n} \sum_{k=1}^{n} T^{k-1} f$

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and the corresponding dominant function
\n
$$
B_T f = \sup_{n \ge 1} A_{n,T} |f| = \sup_{n \ge 1} \frac{1}{n} \sum_{k=1}^n T^{k-1} f.
$$
\nNotice that B for all $f \in \mathbf{I}$, for all $f \in \mathbf{I}$, $|A| = \int_{a}^{b} f(x) \, dx$, so that B

 $B_T f = \sup_{n \ge 1} A_{n,T} |f|$
Notice that $B_T f \in \mathbf{L}_0$ for all $f \in \mathbf{L}_1 + \mathbf{L}_\infty$
A Banach space **E** of measurable fun
invariant (r.i.) if $n \ge 1$
all $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ (see
measurable function
E, $f^* \le g^* \implies f$ $\frac{n}{\epsilon}$ *invariant* $(r.i.)$ if

$$
f \in \mathbf{L}_0
$$
, $g \in \mathbf{E}$, $f^* \leq g^* \implies f \in \mathbf{E}$, $||f||_{\mathbf{E}} \leq ||g||_{\mathbf{E}}$.

Notice that $B_T f \in \mathbf{L}_0$ for all $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ (see [9], Ch. 8, §4 and [22], Ch. 1, §6).

A Banach space **E** of measurable functions on $(\mathbf{\Omega}, \mu)$ is called *rearrangement*
 invariant (r.i.) if
 $f \in \mathbf{L}_0$ A Banach space **E** of measurable functions on (Ω, μ) is called *rearrangement*
 iant (r.i.) if
 $f \in \mathbf{L}_0$, $g \in \mathbf{E}$, $f^* \leq g^* \implies f \in \mathbf{E}$, $||f||_{\mathbf{E}} \leq ||g||_{\mathbf{E}}$.
 f^* denotes the decreasing right-contin $\begin{aligned} f \in \text{denotes} \\ \text{the ring} \end{aligned}$ \implies $f \in \mathbf{E}$, $||f||_{\mathbf{E}} \le ||g||_{\mathbf{E}}$.

pontinuous rearrangement of

zed inverse
 $:\mathbf{n}_f(y) \le x\}, \quad x \in [0, \infty)$

hich is Here f^* denotes the decreasing right-continuous rearrangement of |f|. It can be
defined as the right-continuous generalized inverse
 $f^*(x) := \inf\{y \in [0, +\infty) : \mathbf{n}_f(y) \le x\}, \quad x \in [0, \infty)$
of the distribution function \mathbf{n}_f

$$
f^*(x) := \inf\{y \in [0, +\infty) \colon \mathbf{n}_f(y) \le x\}, \qquad x \in [0, \infty)
$$

$$
\mathbf{n}_f(x) = \mu \left\{ u \in \mathbf{\Omega} : |f(u)| > x \right\},\,
$$

 $f^*(x) := \inf\{y \in [0, +\infty) \colon \mathbf{n}_f(y) \leq x\}$
of the distribution function \mathbf{n}_f of $|f|$, which is
 $\mathbf{n}_f(x) = \mu\{u \in \mathbf{\Omega} \colon |f(u)|\}$
The function f^* is well defined if $\mathbf{n}_f(x) < +\infty$ for

of the distribution function \mathbf{n}_f of $|f|$, which is
 $\mathbf{n}_f(x) = \mu \{u \in \Omega : |f(u)| > x\}$,

The function f^* is well defined if $\mathbf{n}_f(x) < +\infty$ for some $x \ge 0$.

In the case $(\Omega, \mu) = (\mathbf{R}_+, \mathbf{m})$ r.i. spaces $\mathbf{E} = \mathbf$ of the distribution function **n**_f of |f|, which is
 $\mathbf{n}_f(x) = \mu \{u \in \mathbf{\Omega} : |f$

The function f^* is well defined if $\mathbf{n}_f(x) < +\infty$

In the case $(\mathbf{\Omega}, \mu) = (\mathbf{R}_+, \mathbf{m})$ r.i. spaces
 dard. For any r.i. space $\mathbf{$ $(x) = \mu \{ u \in \mathbf{\Omega} : |f(u)| > x \},$

ned if $\mathbf{n}_f(x) < +\infty$ for some
 \mathbf{R}_+ , **m**) r.i. spaces $\mathbf{E} = \mathbf{E}(\mathbf{R}_-, \mathbf{\Omega}, \mu)$ on an arbitrary measur
 $\mathbf{E}(\mathbf{R}_+,\mathbf{m})$ on $(\mathbf{R}_+,\mathbf{m})$ (calle The function f^* is well defined if $\mathbf{n}_f(x) < +\infty$ for some $x \ge 0$.

In the case $(\mathbf{\Omega}, \mu) = (\mathbf{R}_+,\mathbf{m})$ r.i. spaces $\mathbf{E} = \mathbf{E}(\mathbf{R}_+,\mathbf{m})$ *dard*. For any r.i. space $\mathbf{E}(\mathbf{\Omega}, \mu)$ on an arbitrary measure In the case $(\mathbf{\Omega}, \mu) = (\mathbf{R}_+,\mathbf{m})$ r.i. spaces $\mathbf{E} = \mathbf{E}(\mathbf{R}_+,\mathbf{m})$ will be called *stan*-
For any r.i. space $\mathbf{E}(\mathbf{\Omega}, \mu)$ on an arbitrary measure space $(\mathbf{\Omega}, \mu)$ there is a
te standard r.i. space $\mathbf{E$ dardunique standard r.i. space $\mathbf{E}(\mathbf{R}_+,\mathbf{m})$ on $(\mathbf{R}_+,\mathbf{m})$ (called standard realization of \mathbf{E}) such that
 $f \in \mathbf{E}(\mathbf{\Omega}, \mu) \Longleftrightarrow f^* \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})$
(see [21], Ch. II, §8). In general we do not assume t E) such that

$$
f \in \mathbf{E}(\mathbf{\Omega}, \mu) \Longleftrightarrow f^* \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})
$$

unique standard r.i. space $\mathbf{E}(\mathbf{R}_+,\mathbf{m})$ on $(\mathbf{R}_+,\mathbf{m})$ (called standard realization of $\mathbf{E})$ such that
 $f \in \mathbf{E}(\Omega, \mu) \iff f^* \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})$

(see [21], Ch. II, §8). In general we do not assume that $f \in \mathbf{E}(\mathbf{\Omega}, \mu) \Longleftrightarrow f^* \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})$

ee [21], Ch. II, §8). In general we do not assume that t

separable and isomorphic to the standard measure spa

It is known (see [21], Ch. II, §4.1 or [24], Ch. 2.a), t

er

(see [21], Ch. II, §8). In general we do not assume that the measure space (Ω, μ)
is separable and isomorphic to the standard measure space $(\mathbf{R}_{+}, \mathbf{m})$.
It is known (see [21], Ch. II, §4.1 or [24], Ch. 2.a), that f is separable and isomorphic to the standard measure space $(\mathbf{R}_+,\mathbf{m})$.
It is known (see [21], Ch. II, §4.1 or [24], Ch. 2.a), that for every
there exist continuous inclusions
 $\mathbf{L}_1 \cap \mathbf{L}_{\infty} \subseteq \mathbf{E} \subseteq \mathbf{L}_1 + \mathbf{$

$$
\mathbf{L}_1\cap\mathbf{L}_{\infty}\subseteq\mathbf{E}\subseteq\mathbf{L}_1+\mathbf{L}_{\infty}\subseteq\mathbf{L}_0,
$$

It is known (see [21], Ch. II, §4.1 or [24], Ch. 2.a), that for every r.i. space **E** exist continuous inclusions
 $\mathbf{L}_1 \cap \mathbf{L}_{\infty} \subseteq \mathbf{E} \subseteq \mathbf{L}_1 + \mathbf{L}_{\infty} \subseteq \mathbf{L}_0$,
 \mathbf{L}_0 is considered as a complete topologi

L₁ \cap **L**₂ where **L**₀ is considered as a com stochastic convergence, i.e., the \circ On the other hand, every sublattice of the lattice **L**₀, equ $+ \mathbf{L}_{\infty} \subseteq \mathbf{L}_0$,
gical linear s
ivergence on
i space **E** is
the usual pa
ttice \mathbf{L}_0 is on where \mathbf{L}_0 is considered as a complete topological linear space with respect to the stochastic convergence, i.e., the measure convergence on all finite measure sets.
On the other hand, every r.i. Banach space **E** is On the other hand, every r.i. Banach space **E** is a Banach lattice and a sublattice of the lattice \mathbf{L}_0 , equipped with the usual partial order on functions (see [17], Ch. 10 and [24], Ch. 1.c.). The lattice \mathbf{L}_0

sublattice of the lattice **L**₀, equipped with the usual partial order on functions (see [17], Ch. 10 and [24], Ch. 1.c.). The lattice **L**₀ is order σ -complete and also order complete, since the measure μ is σ (see [17], Ch. 10 and [24], Ch. 1.c.). The lattice \mathbf{L}_0 is order σ -complete and also order complete, since the measure μ is σ -finite.

Remind that a subset F_0 of a partially ordered set F is said to be order complete, since the measure μ is σ -finite.
Remind that a subset F_0 of a partially
bounded in F if $f \leq g$ for all $f \in F_0$ and s
order complete if every order bounded subset .
sup $F_0 \in F$ and the greates Remind that a subset F_0 of a partially ordered set F is said to be *order*
ded in F if $f \leq g$ for all $f \in F_0$ and some $g \in F$. The set F is called
complete if every order bounded subset $F_0 \subseteq F$ has the lea bounded

in *F* if $f \leq g$ for all $f \in F_0$ and some $g \in F$. The set *F* is called
nplete if every order bounded subset $F_0 \subseteq F$ has the least upper bound
F and the greatest lower bound inf $F_0 \in F$ in *F*.
ther, a sequence $\{f$ order complete if every order bounded subset $F_0 \subseteq F$ has the least upper bound
sup $F_0 \in F$ and the greatest lower bound inf $F_0 \in F$ in F .
Further, a sequence $\{f_n\}_{n=1}^{\infty}$ of elements of a partially ordered set sup $F_0 \in F$ and the greatest lower bound inf $F_0 \in F$ in F .
Further, a sequence $\{f_n\}_{n=1}^{\infty}$ of elements of a partial
to be *order convergent* to $f \in F$ in $F (f_n \xrightarrow{(o)} f)$, if there e
such that
 $g_n \uparrow f$, $h_n \downarrow f$, Further, a sequence ${f_n}_{n=1}^{\infty}$ of elements of a partially ordered set F is said order convergent to $f \in F$ in F $(f_n \xrightarrow{(o)} f)$, if there exist $g_n \in F$ and $h_n \in F$ that
 $g_n \uparrow f$, $h_n \downarrow f$, $f = \sup_{n \geq 1} g_n = \inf_{n \geq$ to be *order convergent* to $f \in F$ in $F(f_n \xrightarrow{(o)} f)$, if there exist $g_n \in F$ and $h_n \in F$
such that
 $g_n \uparrow f$, $h_n \downarrow f$, $f = \sup_{n \ge 1} g_n = \inf_{n \ge 1} h_n \in F$.

$$
g_n \uparrow f , h_n \downarrow f , f = \sup_{n \ge 1} g_n = \inf_{n \ge 1} h_n \in F.
$$

order σ -complete lattice then $f_n \xrightarrow{(o)} f \in F$ iff $\{f_n, n \ge 1\}$
 $f = \sup_{n \ge 1} \inf_{m \ge n} f_m = \inf_{n \ge 1} \sup_{m \ge n} f_m \in F$. If, in addition, F is an order σ -complete lattice then $f_n \xrightarrow{\sim} f \in F$ iff $\{f_n, n \ge 1\}$

is order bounded in F and
 $f = \sup_{n \ge 1} \inf_{m \ge n} f_m = \inf_{n \ge 1} \sup_{m \ge n} f_m \in F$.

Let $\mathbf{E} \subseteq \mathbf{L}_0(\Omega, \mu)$ be an r.i. Banach sp $\xrightarrow{(o)} f \in F$

$$
f = \sup_{n \ge 1} \inf_{m \ge n} f_m = \inf_{n \ge 1} \sup_{m \ge n} f_m \in F.
$$

- **E** is a order complete sublattice of the order complete lattice **L**₀.
- is order bounded in F and
 $f =$

Let $\mathbf{E} \subseteq \mathbf{L}_0(\Omega, \mu)$ be

 E is a order complet

 A sequence $\{f_n\}_{n=1}^{\infty}$ $n \geq 1$ $m \geq n$ $n \geq 1$ $m \geq r$
be an r.i. Banach space or
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der bounded in **E** and t $n \geq$
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lat
den
und Let $\mathbf{E} \subseteq \mathbf{L}_0(\Omega, \mu)$ be an r.i. Banach space on (Ω, μ) . Then
 E is a order complete sublattice of the order complete lat

A sequence $\{f_n\}_{n=1}^{\infty}$ is order convergent in **E** ($f_n \xrightarrow{(o)}$;
 $\{f_n, n \ge 1\}$ is is a order complete sublattice of the order complete lattice \mathbf{L}_0 .
sequence $\{f_n\}_{n=1}^{\infty}$ is order convergent in \mathbf{E} ($f_n \xrightarrow{(o)} f \in \mathbf{E}$), $n \geq 1\}$ is order bounded in \mathbf{E} and the sequence $\{f_n\}_{n=1$ • A sequence $\{f_n\}_{n=1}^{\infty}$ is order convergent in **E** ($f_n \xrightarrow{(o)} f \in \mathbf{E}$) iff the set $\{f_n\}_{n=1}^{\infty}$ is order convergent in **L**₀.

• $f_n \xrightarrow{(o)} f \in \mathbf{L}_0$) iff $f_n \to f$ almost everywhere on $(\mathbf{\Omega}, \mu)$).

Keeping 1} is order bounded in **E** and the sequence $\{f_n\}_{n=1}^{\infty}$
in **L**₀.
E **L**₀) iff $f_n \to f$ almost everywhere on (Ω, μ) .
mind these facts we can formulate the following two
er our consideration:
E be an r.i. Banach
	- $f_n \xrightarrow{(o)} f \in \mathbf{L}_0$

Keeping in mind these facts we can formulate the following two problems convergent in **L**₀.
 $f_n \xrightarrow{(o)} f \in \mathbf{L}_0$ if

Keeping in mind

will be under our
 lem 1. Let **E** be) iff $f_n \to f$ almost everywhere on (Ω, μ) .

Ind these facts we can formulate the follow

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be an r.i. Banach space and $T \in \mathcal{PAC}$. Wh
 $\{f \in \mathbf{E} : \{A_{n,T}f\}_{n=1}^{\infty} \}$ is order convergent in

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Problem 1. Let **E** be an r.i. Banach space and $T \in \mathcal{PAC}$. What is the subset

 $\mathbf{E}^T := \{ f \in \mathbf{E} \colon \{ A_{n,T} f \}_{n=1}^{\infty}$

will be under our consideration:
 lem 1. Let **E** be an r.i. Banach space and $T \in \mathcal{PAC}$. What is the subset
 $\mathbf{E}^T := \{f \in \mathbf{E} : \{A_{n,T}f\}_{n=1}^{\infty} \text{ is order convergent in } \mathbf{E} \}$?
 lem 2. What is the subclass of all r.i. Bana **Problem 1.** Let **E** be an r.i. Banach
 $\mathbf{E}^T := \{f \in \mathbf{E} : \{A_{n,T}f\}\}$
 Problem 2. What is the subclass of all $T \in \mathcal{PAC}$, i.e., the sequence of Ces

in **E** for all $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$? Let **E** be an r.i. Banach space and $T \in \mathcal{PAC}$. What is the subset
 $\mathbf{E}^T := \{ f \in \mathbf{E} : \{ A_{n,T} f \}_{n=1}^{\infty} \}$ is order convergent in **E** $\}$?

What is the subclass of all r.i. Banach spaces such that $\mathbf{E}^T = \mathbf{$:= { $f \in \mathbf{E}$: { $A_{n,T} f$ }[∞]₂ is order convergent in **E** } ?
at is the subclass of all r.i. Banach spaces such that
., the sequence of Cesáro averages { $A_{n,T} f$ }[∞]₂ is orde
E and $T \in \mathcal{PAC}$?
n that the sequen **Problem 2.** What is the subclass of all r.i. Banach spaces such that $\mathbf{E}^T = \mathbf{E}$ for
i.e., the sequence of Cesáro averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent
 $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$?
again that the sequence $\{A_{n,T$ $n=1$

all $T \in \mathcal{PAC}$, i.e., the sequence of Cesáro averages $\{A_{n,T}f\}_{n=1}^{\infty}$
in **E** for all $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$?
Notice again that the sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ of Cesáro a
gent in **E** iff the corresponding rages is (*o*)-conver-
gs to **E**, while the
2, μ).
lic Theorem ($\mathcal{D} \mathcal{E} \mathcal{T}$)
sical case of spaces in **E** for all *f* ∈ **E** and *T* ∈ *PAC* ?

Notice again that the sequence

gent in **E** iff the corresponding do

sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ itself conver

This means that the setup inc

and Individual (Pointwise) Er $n=1$ $n=1$

Notice again that the sequence $\{A_{n,T}f\}_{n=1}^{\infty}$
in **E** iff the corresponding dominant func
nce $\{A_{n,T}f\}_{n=1}^{\infty}$ itself converges almost e
This means that the setup includes both D
ndividual (Pointwise) Ergodic T gent in **E** iff the corresponding dominant function $B_T f$ belongs to **E**, while the
sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ itself converges almost everywhere on (Ω, μ) .
This means that the setup includes both Dominated Ergodic gent in **E** iff the corresponding dominant function $B_T f$ belongs to **E**, while the
sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ itself converges almost everywhere on (Ω, μ) .
This means that the setup includes both Dominated Ergodic sequence $\{A_{n,T}f\}_{n=1}^{\infty}$
This means that
and Individual (Point)
 L_p and Zygmund's
§1.6).
First individua
and [18] for measure This means that the setup includes both Dominated Ergodic Theorem ($\mathcal{D}\mathcal{E}\mathcal{T}$)
and Individual (Pointwise) Ergodic Theorem ($\mathcal{I}\mathcal{E}\mathcal{T}$) in the classical case of spaces
 \mathbf{L}_p and Zygmund's classes \mathcal{Z}_r This means that the setup includes both Dominated Ergodic Theorem ($D\mathcal{E}$) in dividual (Pointwise) Ergodic Theorem ($\mathcal{I}\mathcal{E}\mathcal{T}$) in the classical case of spaces of Zygmund's classes $\mathcal{Z}_r = \mathbf{L} \log^r \mathbf{L}$ (se $$1.6$).

L_p and Zygmund's classes $Z_r = L \log^r L$ (see, e.g., [9], Ch. VIII, §6 or [22], Ch. I, §1.6).

First individual and dominated ergodic theorems where proved in [12], [40],[2]

and [18] for measure preserving transformation $\text{and }} [1]$
ariou
ase considers 18] for measure preserving transformations. One can find detailed explanation,
us generalizations and relevant references in [22], and also in [1], where the
of infinite measure is treated in more details. Dunford and Sch various generalizations and relevant references in [22], and also in [1], where the case of infinite measure is treated in more details. Dunford and Schwartz ([8], [9] considered ergodic theorems for positive absolute con case of infinite measure is treated in more details. Dunford and Schwartz ([8], [9] considered ergodic theorems for positive absolute contractions in spaces \mathbf{L}_p , $1 \leq p < \infty$. The Converse Dominated Ergodic Theorem i considered ergodic theorems for positive absolute contractions in spaces \mathbf{L}_p , $1 \leq p < \infty$. The Converse Dominated Ergodic Theorem in \mathbf{L}_1 was proved by Ornstein [34] for finite measure preserving transformation $p < \infty$

. The Converse Dominated Ergodic Theorem in \mathbf{L}_1 was proved by Ornstein
r finite measure preserving transformations (see [22], pp. 54–56, where the
e measure case is also described).
.. Veksler and A. Fedorov began t infinite measure case is also described).
A. Veksler and A. Fedorov began to study Ergodic Theorems for general
rearrangement invariant spaces. They investigated in [38] and [39] conditions of
strong operator convergence A. Veksler and A. Fedorov began
rearrangement invariant spaces. They
strong operator convergence of the Cesa
spaces **E**. In particular they described t
tistical (Mean) Ergodic Theorem holds
 $\{A_{n,T}f\}_{n=1}^{\infty}$ is conver angement invariant spaces. They investigated in [38] and [39] conditions of g operator convergence of the Cesáro averages $A_{n,T}$, $T \in \mathcal{PAC}$ in r.i. Banach s **E**. In particular they described the class of r.i. spaces strong operator convergence of the Cesáro averages $A_{n,T}$, $T \in \mathcal{PAC}$ in r.i. Banach spaces **E**. In particular they described the class of r.i. spaces **E**, for which the Statistical (Mean) Ergodic Theorem holds on **E**, strong operator convergence of the Cesáro averages $A_{n,T}$, $T \in \mathcal{PAC}$ in r.i. Banach
spaces **E**. In particular they described the class of r.i. spaces **E**, for which the Sta-
tistical (Mean) Ergodic Theorem holds on **E** spaces **E**. In particular they described the class of r.i. spaces **E**, for which the Sta-
tistical (Mean) Ergodic Theorem holds on **E**, i.e., the sequence of Cesáro averages
 $\{A_{n,T}f\}_{n=1}^{\infty}$ is convergent in norm $\|\$ ${A_{n,T}}f}_{n=1}^{\infty}$

tistical (Mean) Ergodic Theorem holds on **E**, i.e., the sequence of Cesáro averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is convergent in norm $\|\cdot\|_{\mathbf{E}}$ for all $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$.
Dominated Ergodic Theorems in r.i. space Dominated Ergodic Theorems in r.i. spaces on finite measure spaces were studied in [4]. Some recent results on Ergodic Theorems in Orlicz and Lorentz spaces one can find in [30] [31], [32] and [33]. ed in [4]. Some recent results on Ergodic Theorems in Orlicz and Lorentz
s one can find in [30] [31], [32] and [33]. spaces one can find in [30] [31], [32] and [33].

In this paper we solve Problems 1 and 2. Our main results
and some their consequences are formulated in Section 2.
These theorems follow in turn immediately from correspo
Ergodic Theorem, Converse Dominated Ergodic Theorem In their consequences are formulated in Section 2.
These theorems follow in turn immediately from corresponding Dominated
dic Theorem, Converse Dominated Ergodic Theorem and Individual Ergodic
rem, which are proved in Sec These theorems follow in turn immediately from co
Ergodic Theorem, Converse Dominated Ergodic Theorem
Theorem, which are proved in Section 3, 4, and 5, respec
In the important particular cases, when the r.i. spa
 $\mathbf{E} = \$

dic Theorem, Converse Dominated Ergodic Theorem and Individual Ergodic
rem, which are proved in Section 3, 4, and 5, respectively.
In the important particular cases, when the r.i. space **E** is an Orlicz space
 L_{Φ} or a Theorem, which are proved in Section 3, 4, and 5, respectively.

In the important particular cases, when the r.i. space **E** is an Orlicz space
 E = L_{Φ} or a Lorentz space Λ_W , Problems 1 and 2 are solved in Sectio In the important particular cases, when the r.i. space **E** is an Orlicz space $\mathbf{E} = \mathbf{L}_{\Phi}$ or a Lorentz space $\mathbf{\Lambda}_W$, Problems 1 and 2 are solved in Section 6 in terms of corresponding Orlicz functions Φ and w **E** of corresponding Orlicz functions Φ and weight functions W .
The corresponding classical results for space \mathbf{L}_p and for Zygmund classes

of corresponding Orlicz functions Φ and weight functions W .

The corresponding classical results for space \mathbf{L}_p and $\mathbf{L} \log^r \mathbf{L}$ also follow.
 2. Main results

To estimate the dominant functions $B_T f \in \math$ $\mathbf{L} \log^r \mathbf{L}$ also follow.

2. Main results

To estimate the dominant functions $B_T f$, $f \in \mathbf{E}$, we use the *maximal Hardy*log^r L also follow.
 Main results

b estimate the doutline *dittlewood function* To estimate the dominant functions $B_T f$, $f \in \mathbf{E}$, we use the *maximal Hardy-*
Littlewood function f^{**} , which is defined for any function $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ by
 $f^{**}(x) := \frac{1}{x} \int_{0}^{x} f^{*}(u) du$, $x \in (0, \infty)$.
Le $Littlewood\ function\ f^{**}$

$$
f^{**}
$$
, which is defined for any function $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ by
\n
$$
f^{**}(x) := \frac{1}{x} \int_{0}^{x} f^{*}(u) du, \qquad x \in (0, \infty).
$$

\n u) be an r.i. space on a measure space (Ω, μ) , and $\mathbf{E}(I)$
\ng standard r.i. space. The Hardy core of **E** is
\n $\Omega, \mu) := \{ f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\Omega, \mu) : f^{**} \in \mathbf{E}(\mathbf{R}_{+}, \mathbf{m}) \}$.
\nfor every $f \in \mathbf{E}$, whence $\mathbf{E}_{\mathbf{H}} \subset \mathbf{E}$. Moreover, it can be

$$
\mathbf{E}_{\mathbf{H}} = \mathbf{E}_{\mathbf{H}}(\mathbf{\Omega}, \mu) := \{ f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\mathbf{\Omega}, \mu) \colon f^{**} \in \mathbf{E}(\mathbf{R}_+, \mathbf{m}) \} .
$$

Let $\mathbf{E} = \mathbf{E}(\mathbf{\Omega}, \mu)$ be an r.i. space on a measure space $(\mathbf{\Omega}, \mu)$, and $\mathbf{E}(\mathbf{R}_+,\mathbf{m})$
e corresponding standard r.i. space. The Hardy core of \mathbf{E} is
 $\mathbf{E}_{\mathbf{H}} = \mathbf{E}_{\mathbf{H}}(\mathbf{\Omega}, \mu) := \{f \in (\mathbf{L}_1 + \$ be the corresponding standard r.i. space. The Hardy core of **E** is
 $\mathbf{E}_{\mathbf{H}} = \mathbf{E}_{\mathbf{H}}(\mathbf{\Omega}, \mu) := \{f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\mathbf{\Omega}, \mu) : f^{**} \in \mathbf{E}(\mathbf{R}_+),\}$

Evidently, $f^* \leq f^{**}$ for every $f \in \mathbf{E}$, whence $\mathbf{$ $\mathbf{E} = \mathbf{E}_{\mathbf{H}}(\mathbf{\Omega}, \mu) := \{ f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\mathbf{\Omega}, \mu) : f^{**} \in \mathbf{E}(\mathbf{R}_+, \mathbf{m}) \}$.
 $\leq f^{**}$ for every $f \in \mathbf{E}$, whence $\mathbf{E}_{\mathbf{H}} \subseteq \mathbf{E}$. Moreover, it can bundle in the norm
 $||f||_{\mathbf{E}_{\mathbf{H}}} := ||f^{**}||_{\$

$$
||f||_{\mathbf{E}_{\mathbf{H}}} := ||f^{**}||_{\mathbf{E}} , f \in \mathbf{E}_{\mathbf{H}},
$$

Evidently, $f^* \leq f^{**}$ for every $f \in \mathbf{E}$, whence $\mathbf{E}_{\mathbf{H}} \subseteq \mathbf{E}$. Moreover, it can be verified
that $\mathbf{E}_{\mathbf{H}}$ is an r.i. space with the norm
 $||f||_{\mathbf{E}_{\mathbf{H}}} := ||f^{**}||_{\mathbf{E}}$, $f \in \mathbf{E}_{\mathbf{H}}$,
provided t that **E_H** is an r.i. space with the norm
 $||f||_{\mathbf{E}_{\mathbf{H}}} := ||f$

provided that $\mathbf{E}_{\mathbf{H}} \neq \{0\}$.

Notice that in the standard case

largest r.i. space, for which the Hardy provided that $\mathbf{E}_{\mathbf{H}} \neq \{0\}$.

Notice that in the

largest r.i. space, for whi
 (Hf) Notice that in the standard case $(\Omega, \mu) = (\mathbf{R}_+,\mathbf{m})$, the space $\mathbf{E}_{\mathbf{H}}$ is the
st r.i. space, for which the Hardy operator
 $(Hf)(x) := \frac{1}{x} \int_{0}^{x} f(u) du$, $x \in (0, \infty)$
ositive contraction from $\mathbf{E}_{\mathbf{H}}$ to \math

$$
||f||_{\mathbf{E}_{\mathbf{H}}} := ||f^{**}||_{\mathbf{E}} , f \in \mathbf{E}_{\mathbf{H}},
$$
\nprovided that $\mathbf{E}_{\mathbf{H}} \neq \{0\}.$
\nNotice that in the standard case $(\mathbf{\Omega}, \mu) = (\mathbf{R}_{+}, \mathbf{m}),$
\nlargest r.i. space, for which the Hardy operator
\n
$$
(Hf)(x) := \frac{1}{x} \int_{0}^{x} f(u) du , x \in (0, \infty)
$$
\nis a positive contraction from $\mathbf{E}_{\mathbf{H}}$ to $\mathbf{E}.$
\nIt should be mentioned that for any r.i. space \mathbf{E} and T
\n
$$
T(\mathbf{E}_{\mathbf{H}}) \subseteq \mathbf{E}_{\mathbf{H}}
$$
\nand the restriction $T|_{\mathbf{E}_{\mathbf{H}}} : \mathbf{E}_{\mathbf{H}} \to \mathbf{E}_{\mathbf{H}}$ is a contraction.

It should be mentioned that for any r.i. space **E** and $T \in \mathcal{PAC}$
 $T(\mathbf{E}_{\mathbf{H}}) \subseteq \mathbf{E}_{\mathbf{H}}$

he restriction $T|_{\mathbf{E}_{\mathbf{H}}} : \mathbf{E}_{\mathbf{H}} \to \mathbf{E}_{\mathbf{H}}$ is a contraction.

Indeed, by Calderon-Mityagin theorem
 $(Tf$

$$
T(\mathbf{E}_{\mathbf{H}}) \subseteq \mathbf{E}_{\mathbf{H}}
$$

is a positive contraction from $\mathbf{E}_{\mathbf{H}}$ to \mathbf{E} .
It should be mentioned that for are $T(\mathbf{E}_{\mathbf{H}})$
and the restriction $T|_{\mathbf{E}_{\mathbf{H}}}$: $\mathbf{E}_{\mathbf{H}} \to \mathbf{E}_{\mathbf{H}}$ is Indeed, by Calderon-Mityagin thee

$$
T(\mathbf{E}_{\mathbf{H}}) \subseteq \mathbf{E}_{\mathbf{H}}
$$

: $\mathbf{E}_{\mathbf{H}} \to \mathbf{E}_{\mathbf{H}}$ is a contraction.
n-Mityagin theorem

$$
(Tf)^{**}(x) \le f^{**}(x), 0 < x < \infty
$$

$$
T \in \mathcal{PAC} \text{ (see [5], [26] or [21], C)}
$$

and the restriction $T|_{\mathbf{E}_{\mathbf{H}}}$: $\mathbf{E}_{\mathbf{H}} \to \mathbf{E}_{\mathbf{H}}$ is a contraction.

Indeed, by Calderon-Mityagin theorem
 $(Tf)^{**}(x) \le f^{**}(x)$, $0 < x <$

for all $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ and $T \in \mathcal{PAC}$ (see [5], [26] or [$(Tf)^{*_s}(x) \leq f^{*_s}(x)$

I $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ and $T \in \mathcal{PAC}$ (see [5], $||Tf||_{\mathbf{E_H}} := ||(Tf)^{**}||_{\mathbf{E}} \leq$ for all $f \in L_1 + L_\infty$ and $T \in \mathcal{PAC}$ (see [5], [26] or [21], Ch.II, §3.4). Whence
 $||Tf||_{\mathbf{E_H}} := ||(Tf)^{**}||_{\mathbf{E}} \le ||f^{**}||_{\mathbf{E}} := ||f||_{\mathbf{E_H}}$.

$$
(Tf)^{**}(x) \le f^{**}(x), 0 < x < \infty
$$

\n
$$
\in \mathcal{PAC} \text{ (see [5], [26] or [21], Ch.II, {\mathcal{E}})}
$$

\n
$$
||Tf||_{\mathbf{E_H}} := ||(Tf)^{**}||_{\mathbf{E}} \le ||f^{**}||_{\mathbf{E}} := ||f||_{\mathbf{E_H}}.
$$

Order Convergence Ergodic Theorems 127
Thus every r.i. space of the form $\mathbf{E}_{\mathbf{H}}$, is an interpolation space with respect to the
pair $(\mathbf{L}_1 \cap \mathbf{L}_{\infty}, \mathbf{L}_1 + \mathbf{L}_{\infty})$.
On the other hand, r.i. spaces need not

Thus every r.i. space of the form $\mathbf{E}_{\mathbf{H}}$, is an interpolation space with respect to the
pair $(\mathbf{L}_1 \cap \mathbf{L}_{\infty}, \mathbf{L}_1 + \mathbf{L}_{\infty})$.
On the other hand, r.i. spaces need not to be interpolational. There exist r.i. pair $(\mathbf{L}_1 \cap \mathbf{L}_{\infty}, \mathbf{L}_1 + \mathbf{L}_{\infty})$.

On the other hand, r

spaces **E** such that $T\mathbf{E} \nsubseteq A_{n,T} f$ do not *a priori* bel

The r.i. space $\mathcal{R}_0 = \mathcal{R}_0 = \{f \in \mathcal{R}_0\}$ is **E** such that $T\mathbf{E} \not\subseteq \mathbf{E}$ for some $T \in \mathcal{PAC}$ [21], Ch. II §5.7). Thus Tf and
f do not a priori belong to **E** for $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$.
The r.i. space $\mathcal{R}_0 = \mathcal{R}_0(\Omega, \mu)$ can be defined by
 $\mathcal{$ spaces **E** such that $T\mathbf{E} \not\subseteq \mathbf{E}$ for some $T \in \mathcal{PAC}$ ([21], Ch. II §5.7). Thus Tf and $A_{n,T}f$ do not a priori belong to **E** for $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$.
The r.i. space $\mathcal{R}_0 = \mathcal{R}_0(\Omega, \mu)$ can be defi

$$
\mathcal{R}_0 = \{ f \in \mathbf{L}_1 + \mathbf{L}_{\infty} \colon f^*(+\infty) := \lim_{x \to +\infty} f(x) = 0 \} .
$$

 $A_{n,T}f$ do not a priori belong to **E** for $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$.

The r.i. space $\mathcal{R}_0 = \mathcal{R}_0(\mathbf{\Omega}, \mu)$ can be defined by
 $\mathcal{R}_0 = \{f \in \mathbf{L}_1 + \mathbf{L}_{\infty} : f^*(+\infty) := \lim_{x \to +\infty} f(x)$

It plays an important role i The r.i. space $\mathcal{R}_0 = \mathcal{R}_0(\Omega, \mu)$ can be defined by
 $\mathcal{R}_0 = \{ f \in \mathbf{L}_1 + \mathbf{L}_{\infty} : f^*(+\infty) := \lim_{x \to +}$

ys an important role in our exposition. This spa

ent ways:
 $\{ f \in \mathbf{L}_1 + \mathbf{L}_{\infty} : \mathbf{n}_f(x) < +\infty, x > 0 \} = cl_{\math$ = { $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$: $f^*(+\infty) := \lim_{x \to +\infty} f(x) = 0$ }.

ant role in our exposition. This space may be des
 ∞ : $\mathbf{n}_f(x) < +\infty$, $x > 0$ } = $cl_{\mathbf{L}_1 + \mathbf{L}_{\infty}}(\mathbf{L}_1 \cap \mathbf{L}_{\infty})$ =

t of $\mathbf{L}_1 + \mathbf{L}_{\infty}$ consid

different ways:
\n
$$
\mathcal{R}_0 = \{f \in \mathbf{L}_1 + \mathbf{L}_{\infty} : \mathbf{n}_f(x) < +\infty, x > 0\} = cl_{\mathbf{L}_1 + \mathbf{L}_{\infty}}(\mathbf{L}_1 \cap \mathbf{L}_{\infty}) = cl_{\mathbf{L}_1 + \mathbf{L}_{\infty}}(\mathbf{L}_1),
$$
\nand \mathcal{R}_0 is the heart of $\mathbf{L}_1 + \mathbf{L}_{\infty}$ considered as an Orlicz space.
\nNow we can formulate our main result (solving Problem 1):
\n**Theorem 2.1.** Let **E** be an r.i. space. Then for all $f \in \mathbf{E}_{\mathbf{H}} \cap \mathcal{R}_0$ and all $T \in \mathcal{PAC}$
\nthe sequence of averages $\{A_{\infty}, f\}^{\infty}$ is order convergent in **E**

 $= { f ∈ L_1 + L_∞ : n_f(x) < +∞, x > 0 } = cl_{L_1 + L_∞}(L_1 ∩ L_∞) = cl_{L_1 + L_∞}(L_1),$
 R_0 is the heart of $L_1 + L_∞$ considered as an Orlicz space.

Now we can formulate our main result (solving Problem 1):
 oorem 2.1. Let **E** be an r.i. and \mathcal{R}_0 is the heart of $\mathbf{L}_1 + \mathbf{L}_{\infty}$ considered as an Orlicz space.

Now we can formulate our main result (solving Problem in **Theorem 2.1.** Let **E** be an r.i. space. Then for all $f \in \mathbf{E}_{\mathbf{H}} \cap \mathcal{R}$
 Theorem 2.1. Let **E** be an r.i. space. Then for all $f \in \mathbf{E}_{\mathbf{H}} \cap \mathcal{R}_0$ and all $T \in \mathcal{PAC}$ the sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in **E**.

rem 2.1. Let **E** be an r.i. space. Then for all $f \in \mathbf{E}_{\mathbf{H}} \cap \mathcal{R}_0$ dequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in **E**.
Condition $f \in \mathbf{E}_{\mathbf{H}}$ implies that the sequence $\{A_{n,T}f\}_{n=1}^{\infty$ Condition $f \in \mathbf{E_H}$ implies that the sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ is order bounded in

, i.e., $B_T f \in \mathbf{E}$. Thus Dominated Ergodic Theorem $(\mathcal{D}\mathcal{E}\mathcal{T})$ in **E** holds on Hardy

re $\mathbf{E_H}$ of **E**.

Condition f Condition $f \in \mathbf{E_H}$ implies that the sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ is order bounded in **E**, i.e., $B_T f \in \mathbf{E}$. Thus Dominated Ergodic Theorem ($\mathcal{D} \mathcal{E} \mathcal{T}$) in **E** holds on Hardy

 $n=1$

core E_H of E .
Conditio
in L_0 , i.e., Ind
The conv
Theorem 2.2.
 $f \in E$ and T Condition $f \in \mathcal{R}_0$ implies that the sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent
in **L**₀, i.e., Individual Ergodic Theorem (*IET*) holds on \mathcal{R}_0 .
The converse is also true. Namely,
Theorem 2.2. Let **E** The convertible in the convertible convertible convertible and $T \in \mathcal{PAC}$ such that the sequence of Both \mathcal{DET} and \mathcal{IET} parts in the contract and \mathcal{IET} parts in the contract and \mathcal{IET} parts in the contr **Theorem 2.2.** Let **E** be an r.i. space such that $\mathbf{E} \neq \mathbf{E}_{\mathbf{H}} \cap \mathcal{R}_{0}$. Then there exist $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$ such that the sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ is not order convergent in **E**.

 \neq **EH** ∩ R₀. Then there exist
 f _{${}_{n=1}^{\infty}$} is not order convergent

orem can be made more precise

aation on (**Ω**, μ) and $T = T_{\theta} \in$
 ${}^{2}AC$ and T_{θ} **E** = **E** for each r.i. Both *DC f* and *LC f* parts in the converse theorem can be made more precise
separately. Let *θ* be a measure preserving transformation on $(Ω, μ)$ and $T = T_θ ∈ ΡAC$ is of the form $T_θ f = f ∘ θ$. Evidently, $T_θ ∈ ΡAC$ PAC is of the form $T_{\theta}f = f \circ \theta$. Evidently, $T_{\theta} \in \mathcal{PAC}$ and $T_{\theta}E = E$ for each r.i.
 E. Let \mathcal{PAC}_0 consists of all operators T_{θ} , where θ is an *conservative ergodic*
 ire preserving transformation on space **E**. Let \mathcal{PAC}_0 consists of all operators T_{θ} , where θ is an conservative ergodic
measure preserving transformation on (Ω, μ) .
Theorem 2.3. Let **E** be an r.i. space and $T = T_{\theta} \in \mathcal{PAC}_0$. Then
1) If measure preserving transformation on (Ω, μ) .

Theorem 2.3. Let **E** be an r.i. space and $T = T_{\theta} \in \mathcal{PAC}_0$. Then

-
- on (Ω, μ) .
 and $T =$
 $\{A_{n,T}f\}_{n=0}^{\infty}$

area the fo = $T_{\theta} \in \mathcal{PAC}_0$. Then
 $\sum_{n=1}^{\infty}$ is not order controllary

following corollary 1) If $B_T f \in \mathbf{E}$ then $f \in \mathbf{E_H}$.

2) If $\mathbf{E} \nsubseteq \mathcal{R}_0$ then the seque
 $f \in \mathbf{E}$.

Turning to Problem 2, v

d 2.2.
 heorem 2.4. Let \mathbf{E} be an r.i. 2) If $\mathbf{E} \nsubseteq \mathcal{R}_0$ then the sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ is not order convergent for some $f \in \mathbf{E}$.

Turning to Problem 2, we have the following corollary of Theorems 2.1

d 2.2.
 heorem 2.4. Let \mathbf{E} be $f \in \mathbf{E}$.

EXECUTE:

Turning to Problem 2.4. Let **E** be an r.i. space. The following conditions are equivalent:

The sequence $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent for all $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$.
 E = **E_H** $\cap \mathcal{R}_0$

Theorem 2.4. Let \mathbf{E} be an r.i. space. The following conditions are equivalent:

- **Theorer**
(i) Th
(ii) \mathbf{E} (i) The sequence ${A_{n,T}}f_{n=1}^{\infty}$ is order convergent for all $f \in \mathbf{E}$ and $T \in \mathcal{PAC}$.

(ii) $\mathbf{E} = \mathbf{E}_{\mathbf{H}} \cap \mathcal{R}_0$.
- (ii) $\mathbf{E} = \mathbf{E_H} \cap \mathcal{R}_0$.

Thus

• An r.i. space **E** satisfies Order Ergodic Theorem (**E** $\in \mathcal{O}$.

and **E** $\subseteq \mathcal{R}_0$.

Condition **E_H** = **E** means that the Hardy operator *H* a ● An r.i. space **E** satisfies Order Ergodic Theorem (**E** ∈ OET), iff **EH** and $\mathbf{E} \subseteq \mathcal{R}_0$.
Condition $\mathbf{E}_{\mathbf{H}} = \mathbf{E}$ means that the Hardy operator H acts as a bounded

An r.i. space **E** satisfies Order Ergodic Theorem (**E** \in OET), iff **E**_H = **E** and **E** \subseteq R₀.
Condition **E_H** = **E** means that the Hardy operator *H* acts as a bounded tor on **E**. To clarify the condition one can Condition $\mathbf{E}_{\mathbf{H}} = \mathbf{E}$ means that the Hardy operator *H* acts as a bounded
ttor on **E**. To clarify the condition one can use the dilation group $\{D_t, 0 < \infty\}$ and the lower and upper indexes $1 \leq p_{\mathbf{E}} \leq q_{\math$ operator on **E**. To clarify the condition one can use the dilation group $\{D_t, 0 < t < +\infty\}$ and the lower and upper indexes $1 \leq p_{\mathbf{E}} \leq q_{\mathbf{E}} \leq +\infty$ of an r.i. space **E**,
which are defined as follows.
Let for any

$$
D_t f(x) := f(x/t) , 0 < x, t < \infty.
$$

 $t < +\infty$ } and the lower and upper indexes $1 \le p_{\mathbf{E}} \le q_{\mathbf{E}} \le +\infty$ of an r.i. space **E**,
which are defined as follows.
Let for any $f \in \mathbf{L}_0 = \mathbf{L}_0(\mathbf{R}_+,\mathbf{m})$:
 $D_t f(x) := f(x/t)$, $0 < x, t < \infty$.
Then $\{D_t, 0 < t < \infty\}$ which are defined as follows.

Let for any $f \in \mathbf{L}_0 = \mathbf{L}_0(\mathbf{R}_+,\mathbf{m})$:
 $D_t f(x) := f(x/t)$, $0 < x, t < \infty$.

Then $\{D_t, 0 < t < \infty\}$ is a group of bounded linear operators $D_t : \mathbf{E} \to \mathbf{E}$ on the standard r.i. space $\mathbf{$ Let for any $f \in \mathbf{L}_0 = \mathbf{L}_0(\mathbf{R}_+,\mathbf{m})$:
 $D_t f(x) := f(x$

{ $D_t, 0 < t < \infty$ } is a group of botard r.i. space $\mathbf{E} = \mathbf{E}(\mathbf{R}_+,\mathbf{m})$, cor

The function $d_{\mathbf{E}}(t) := ||D_t||_{\mathbf{E} \to \mathbf{E}}$ if
 $d_{\mathbf{E}}(s) d_{\mathbf{E}}(t)$ fo $(x) := f(x/t)$, $0 < x, t < \infty$.
group of bounded linear oper
 $(+, \mathbf{m})$, corresponding to $\mathbf{E}(\mathbf{\Omega})$
 $|D_t||_{\mathbf{E} \to \mathbf{E}}$ is semi-multiplicatif
Hence, there exist the limits
 $\frac{\log(t)}{\log(t)}, \quad q_{\mathbf{E}} := \lim_{t \to 0} \frac{\log t}{\log t}$

 t $\leq d_{\mathbf{E}}(s) d_{\mathbf{E}}(t)$ for all s, t. Hence, there exist the limits

Then
$$
\{D_t, 0 < t < \infty\}
$$
 is a group of bounded linear operators $D_t : \mathbf{E} \to \mathbf{E}$ on the
standard r.i. space $\mathbf{E} = \mathbf{E}(\mathbf{R}_+, \mathbf{m})$, corresponding to $\mathbf{E}(\mathbf{\Omega}, \mu)$.
The function $d_{\mathbf{E}}(t) := ||D_t||_{\mathbf{E} \to \mathbf{E}}$ is semi-multiplicative on $(0, \infty)$, i.e., $d_{\mathbf{E}}(s +$
 $t) \leq d_{\mathbf{E}}(s) d_{\mathbf{E}}(t)$ for all s, t . Hence, there exist the limits
 $p_{\mathbf{E}} := \lim_{t \to \infty} \frac{\log t}{\log d_{\mathbf{E}}(t)} = \inf_{1 < t} \frac{\log(t)}{\log d_{\mathbf{E}}(t)}, \quad q_{\mathbf{E}} := \lim_{t \to 0} \frac{\log t}{\log d_{\mathbf{E}}(t)} = \inf_{0 < t < 1} \frac{\log t}{\log d_{\mathbf{E}}(t)},$
and they are called *lower and upper Boyd indices* of the r.i. space \mathbf{E} (see [3], [21],
Ch. II, §4.3, [24], Ch. 2.b).
Proposition 2.5 ([21], Chap. II, §6.1). Let \mathbf{E} be an r.i. space and $d_{\mathbf{E}}(t) = ||D_t||_{\mathbf{E} \to \mathbf{E}}$
and $p_{\mathbf{E}}$ be the lower Boyd index. Then the following conditions are equivalent:
 \bullet $\mathbf{E}_{\mathbf{H}} = \mathbf{E}$.

log $d_{\mathbf{E}}(t)$ is its called *lower an*
called *lower an*
24], Ch. 2.b).
5 ([21], Chap. lower Boyd in E. $\log d_{\mathbf{E}}(t)$ = $0 < t <$
of the r.i. space
i. space and $d_{\mathbf{E}}$
g conditions are and they are called *lower and upper Boyd indices* of the r.i. space **E** (see [3], [21],
Ch. II, §4.3, [24], Ch. 2.b).
Proposition 2.5 ([21], Chap. II, §6.1). *Let* **E** *be an r.i. space and* $d_{\mathbf{E}}(t) = ||D_t||_{\mathbf{E} \to \$

Ch. 11, §4.3, [24], Ch. 2.b).
 Proposition 2.5 ([21], Chap

and $p_{\mathbf{E}}$ be the lower Boyd

• E_H = E.

• $p_{\mathbf{E}} > 1$.

• $\int_0^1 d_{\mathbf{E}}(1/t) dt < \infty$. **Proposition 2.5 ([21], Chap. II,** §**6.1).** Let **E** be an r.i. space and $d_{\mathbf{E}}(t) = ||D_t||_{\mathbf{E} \to \mathbf{E}}$

and $p_{\mathbf{E}}$ be the lower Boyd index. Then the following conditions are equivalent:

• **E**_H = **E**.

• \int_0^1 and $p_{\mathbf{r}}$ be the lower Boyd index. Then the following conditions are equivalent:

- \bullet **E**_H = **E**.
- \bullet $p_{\mathbf{E}} >$
-
- $d_{\mathbf{E}}(t) = o(t)$ as $t \to +\infty$.

 $=$ **E**.
 > 1 .
 $\binom{1}{E}$
 $\binom{1}{E}$
 ≥ 0
 ≥ 0
 ≤ 0
 ≤ 0
 ≤ 0 1.
 $(1 = \text{cc} \text{)}$
 $\text{ct} \text{}$
 $\text{ct} \text{}$ • $\int_0^1 d\mathbf{E}(1/t) dt < \infty$.

• $d\mathbf{E}(t) = o(t)$ as $t \rightarrow$

The second conditio
 $\mathbf{E} \supseteq \mathbf{L}_{\infty}$

For instance, $p_{\mathbf{L}_p} =$

• 1, while $(\mathbf{L}_1)_{\mathbf{H}} = \mathcal{Z}_1$ The second condition $\mathbf{E} \subseteq \mathcal{R}_0$ is well verifiable since $\mathbf{E} \nsubseteq \mathcal{R}_0$ implies $1 \in \mathbf{E}$
and $\mathbf{E} \supseteq \mathbf{L}_{\infty}$ The second condition $\mathbf{E} \subseteq \mathcal{R}_0$ is well verifiable since $\mathbf{E} \nsubseteq \mathcal{R}_0$ implies $1 \in \mathbf{E}$
 $\mathbf{E} \supseteq \mathbf{L}_{\infty}$

For instance, $p_{\mathbf{L}_p} = q_{\mathbf{L}_p} = p$ for each $1 \leq p \leq +\infty$, whence $(\mathbf{L}_p)_{\mathbf{H}} = \mathbf{$

and **E** \supseteq **L**_∞
For inst
 $p > 1$, while
• **L**_p \in (
while
• **L**_p \in 1 For instance, $p_{\mathbf{L}_p} = q_{\mathbf{L}_p} = p$ for each $1 \leq p \leq +\infty$, whence $(\mathbf{L}_p)_{\mathbf{H}} = \mathbf{L}_p$ for
, while $(\mathbf{L}_1)_{\mathbf{H}} = \mathcal{Z}_1 \nsubseteq \mathbf{L}_1$ (see below). Thus
 $\mathbf{L}_p \in \mathcal{O}\mathcal{E}\mathcal{T}$ iff $1 < p < +\infty$.
 $\mathbf{L}_p \in \mathcal{$ *p* > 1, while (L_1) **H** = $Z_1 \nsubseteq L_1$ (see below). Thus

● $L_p \in \mathcal{OET}$ iff $1 < p < +\infty$.

while

● $L_p \in \mathcal{DET}$ iff $1 < p \leq +\infty$.

● $L_p \in \mathcal{IET}$ iff $1 \leq p < +\infty$.

• $\mathbf{L}_p \in \mathcal{O} \mathcal{E} \mathcal{T}$ iff $1 < p < +\infty$.
while iff $1 < p < +\infty$.

iff $1 < p \leq +\infty$.

iff $1 \leq p < +\infty$.
 \therefore
 \therefore \therefore

- L_p ∈ DE T
- $\mathbf{L}_p \in \mathcal{I} \mathcal{E} \mathcal{T}$ iff $1 \leq p < +\infty$.

3. Dominated Ergodic Theorem

• $\mathbf{L}_p \in \mathcal{DET}$ iff $1 < p \leq +\infty$.

• $\mathbf{L}_p \in \mathcal{IET}$ iff $1 \leq p < +\infty$.
 3. Dominated Ergodic Theore:

Let $T \in \mathcal{PAC}$ and let $A_{n,T} = \frac{1}{n} \sum_{k=1}^n$

Consider the *dominant function* iff $1 \leq p < +\infty$.
 Ergodic Theore

let $A_{n,T} = \frac{1}{n} \sum_{k=1}^{n}$

mant function
 $B_{\mathcal{F}} f(\omega) = \sup_{k=1}^{\infty}$ Let $T \in \mathcal{PAC}$ and let $A_{n,T} = \frac{1}{n} \sum_{k=1}^{n} T^{k-1}$
Consider the *dominant function*
 $B_T f(\omega) = \sup_{n \geq 1} (A_{n,T}$
Notice that $B_T f(\omega) < \infty$ for all $f \in \mathbb{R}$
also follows from Lemma 3.3 (below).

$$
B_Tf(\omega)=\sup_{n\geq 1}(A_{n,T}|f|)(\omega), \ \ \omega\in\Omega.
$$

 $f|(\omega)$, $\omega \in \Omega$.
 $\mathbf{L}_1 + \mathbf{L}_{\infty}$ ([9], Ch. VIII, §4). This fact Consider the *dominant function*
 $B_T f(\omega) = \sup_{n \ge 1}$
Notice that $B_T f(\omega) < \infty$ for all
also follows from Lemma 3.3 (below). $(\omega) = \sup_{n \ge 1} (A_{n,T}|f|)(\omega), \quad \omega \in \Omega.$
 $\langle \infty \text{ for all } f \in \mathbf{L}_1 + \mathbf{L}_{\infty} \quad [9], \text{ C.}$

3 (below). Notice that $B_T f(\omega) < \infty$ for all $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ ([9], Ch. VIII, §4). This fact collows from Lemma 3.3 (below).

• Let E be an r.i. Banach space and $T \in \mathcal{PAC}$. What is the subset

$$
\mathbf{E}_{\mathcal{D}\mathcal{E}\mathcal{T}}^T := \{ f \in \mathbf{E} : B_T f \in \mathbf{E} \} ?
$$

For consider two following problems

Banach space and $T \in \mathcal{PAC}$. What is the subset
 $\mathbf{E}_{\mathcal{DET}}^T := \{ f \in \mathbf{E} : B_T f \in \mathbf{E} \}$?

Class of all r.i. Banach spaces E such that $\mathbf{E}_{\mathcal{DET}}^T = \mathbf{E}$ for all Let **E** be an r.i. Banach space and $T \in \mathcal{PAC}$. What is the subset
 $\mathbf{E}_{\mathcal{DET}}^T := \{f \in \mathbf{E} : B_T f \in \mathbf{E}\}\n$?

What is the subclass of all r.i. Banach spaces E such that $\mathbf{E}_{\mathcal{DET}}^T$
 $T \in \mathcal{PAC}$, i.e., sequenc $:= { f ∈ \mathbf{E} : B_T f ∈ \mathbf{E} }$?

1 r.i. Banach spaces *E* sue

of Cesáro averages {*A_{n,T}*,

odic **Theorem**). Let **E**(Ω

s $B_T f ∈ \mathbf{E}$ and ∙ What is the subclass of all r.i. Banach spaces E such that \mathbf{E}_T^T
 $T \in \mathcal{PAC}$, i.e., sequences of Cesáro averages $\{A_{n,T}f\}_{n=1}^{\infty}$ are

in **E** for every $T \in \mathcal{PAC}$?
 rem 3.1 (Dominated Ergodic Theorem). $\frac{T}{D} \varepsilon \tau = \mathbf{E}$ for all
 $\frac{T}{D} \varepsilon$ order bounded
 $\frac{T}{D} \tau$. space and

 $T \in \mathcal{PAC}$, i.e., sequences of Cesáro averages $\{A_{n,T}f\}_{n=1}^{\infty}$
in **E** for every $T \in \mathcal{PAC}$?
rem 3.1 (Dominated Ergodic Theorem). Let $\mathbf{E}(\Omega, \mu)$ b
 \mathcal{PAC} . Then $f \in \mathbf{E_H}$ implies $B_T f \in \mathbf{E}$ and
 $||B_T$ an r.i. space and in **E** for every $T \in \mathcal{PAC}$?
 rem 3.1 (Dominated Erg
 \mathcal{PAC} . Then $f \in \mathbf{E}_{\mathbf{H}}$ implies

Theorem 3.1 follows from **Theorem 3.1 (Dominated Ergodic Theorem).** Let **E** (Ω, μ) be an r.i. space and
mas.
 $f \geq 0, g \geq 0,$ satisfy the $T \in \mathcal{PAC}$. Then $f \in \mathbf{E_H}$ implies $B_T f \in \mathbf{E}$ and

$$
||B_Tf||_{\mathbf{E}} \leqslant ||f||_{\mathbf{E}_{\mathbf{H}}}.
$$

na 3.2. Let functions $f, g \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\Omega, \mu), f$

ving condition:
 $g^*(\infty) = 0;$

For every $t > 0$
 $\mu\{g > t\} \leq \frac{1}{t} \int f d\mu.$ **Lemma 3.2.** Let functions $f, g \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\Omega, \mu), f \ge 0, g \ge 0$, satisfy the following condition:

1) $g^*(\infty) = 0$;

2) For every $t > 0$
 $\mu\{g > t\} \le \frac{1}{t} \int_{\{g > t\}} f d\mu.$ following condition:

-
-

$$
\mu\{g > t\} \le \frac{1}{t} \int\limits_{\{g > t\}} f d\mu.
$$

s > 0.
wing "maximal" property

1) $g^*(\infty) = 0;$
2) For every t
hen $g^*(s) \leq f^*$
roof. We shall Then g^*

2) For every $t > 0$
hen $g^*(s) \leqslant f^{**}(s)$
roof. We shall use t $(s) \leq f^{**}(s)$ for all $s > 0$.
Ve shall use the following '
 $\int_{0}^{s} f^{*}(x) dx$ *Proof.* We shall use the following "maximal" property of the function f^* ([21], Ch.

II, §2),
 $\int_{0}^{s} f^*(x) dx = \sup_{G:\mu G = s} \int_{G} f d\mu.$

Let $t > 0$ and $s = \mu\{g > t\}$. Then

$$
\int_{0}^{s} f^*(x) dx = \sup_{G: \mu G = s} \int_{G} f d\mu.
$$

II, 3^2),
Let $t >$

$$
(x) dx = \sup_{G:\mu G=s} \int_G
$$

then

$$
s \leq \frac{1}{t} \int_{\{g>t\}} f d\mu,
$$

Let
$$
t > 0
$$
 and $s = \mu\{g > t\}$. Then
\n
$$
s \leq \frac{1}{t} \int_{\{g > t\}} f d\mu,
$$
\nand hence
\n
$$
t \leq \frac{1}{s} \int_{\{g > t\}} f d\mu \leq \frac{1}{s} \sup_{G : \mu G = s} \int_{G} f d\mu = \frac{1}{s} \int_{0}^{s} f^*(x) dx = f^{**}(s).
$$
\nThus
\n
$$
t \leq f^{**}(\mu\{g > t\}) = f^{**}(s).
$$
\nSince $s = \mu\{g > t\} = \mathbf{m}\{g^* > t\}$, then $g^*(s) \leq t$ and, in addition, g^* the case when g^* is continuous at the point s .

Thus $\frac{a}{a}$

$$
t \leqslant f^{**}(\mu\{g > t\}) = f^{**}(s).
$$

 $:\mu \dot{G}$ = $**(\mu \rightarrow)$
 $:\}$, t $t \leqslant f^{**}(\mu\{g > t\}) = f^{**}(s).$

Since $s = \mu\{g > t\} = \mathbf{m}\{g^* > t\}$, then $g^*(s) \leqslant t$ and

the case when g^* is continuous at the point s .

Consider the partition $(0, +\infty) = A \cup B$, where
 $\mathcal{A} = \{s \in (0, +\infty) : s = \mu\{g > t$ Since $s = \mu\{g > t\} = \mathbf{m}\{g^* > t\}$, then $g^*(s) \leq t$ and, in addition, $g^*(s) = t$ in
the case when g^* is continuous at the point s .
Consider the partition $(0, +\infty) = A \cup B$, where
 $\mathcal{A} = \{s \in (0, +\infty) : s = \mu\{g > t\}$, for the case when g^* is continuous at the point s.
Consider the partition $(0, +\infty) = A \cup B$
 $A = \{s \in (0, +\infty) : s = \mu\{g > \text{and} \}$
 $B = (0, +\infty)$

Consider the partition
$$
(0, +\infty) = A \cup B
$$
, where
\n
$$
A = \{s \in (0, +\infty) : s = \mu\{g > t\}, \text{ for some } t > 0\}
$$
\n
$$
\mathcal{B} = (0, +\infty) \setminus \mathcal{A}.
$$

$$
\mathcal{B} = (0, +\infty) \backslash \mathcal{A}.
$$

Consider two cases.
\n1) Let
$$
s_0 \in A
$$
. Then
\n
$$
\{t > 0 : \mu\{g > t\} = s_0\} \neq \emptyset.
$$
\nDenote

$$
t_0 = \sup\{t > 0 : \mu\{g > t\} = s_0\} = \sup\{t > 0 : \mathbf{m}\{g^* > t\} = s_0\}.
$$

\n*s*) is a non-increasing right continuous function, we have
\n
$$
t_0 = g^*(s_0 - 0) = \lim_{s \to s_0 - 0} g^*(s).
$$

\nthe inequality $t \le f^{**}(s_0)$ holds for all $t > 0$ such that
\n
$$
\mu\{g > t\} = s_0.
$$

1) Let $s_0 \in \mathcal{A}$. Then

Denote
 $t_0 = \sup\{t > 0 :$

Since $g^*(s)$ is a non-incre Since $g^*(s)$ is a non-increasing right continuous function, we have
 $t_0 = g^*(s_0 - 0) = \lim_{s \to s_0 - 0} g^*(s)$.

Further, the inequality $t \leqslant f^{**}(s_0)$ holds for all $t > 0$ such that
 $\mu\{g > t\} = s_0$.

Hence $t_0 \leqslant f^{**}(s_0$

$$
\{t > 0 : \mu\{g > t\} = s_0\} \neq \emptyset.
$$

\n
$$
\mu\{g > t\} = s_0\} = \sup\{t > 0 : \mathbf{n}\}
$$

\n
$$
t_0 = g^*(s_0 - 0) = \lim_{s \to s_0 - 0} g^*(s).
$$

\n
$$
\leq f^{**}(s_0) \text{ holds for all } t > 0 \text{ su}
$$

\n
$$
\mu\{g > t\} = s_0.
$$

\n
$$
g^*(s_0 - 0) \leq f^{**}(s_0),
$$

Since g
Further Further, the inequality $t \leq f^{**}(s_0)$ holds for all $t > 0$ such that
 $\mu\{g > t\} = s_0$.

Hence $t_0 \leq f^{**}(s_0)$ and
 $g^*(s_0 - 0) \leq f^{**}(s_0)$,

whence
 $g^*(s_0) \leq g^*(s_0 - 0) \leq f^{**}(s_0)$.

$$
\mu\{g > t\} = s_0
$$

$$
g^*(s_0 - 0) \leqslant f^{**}(s_0),
$$

$$
\mu\{g > t\} = s_0.
$$
\n
$$
g^*(s_0 - 0) \leqslant f^{**}(s_0),
$$
\n
$$
g^*(s_0) \leqslant g^*(s_0 - 0) \leqslant f^{**}(s_0).
$$
\n
$$
g^*(s_0) = \sup\{u > s : g^*(u) = g^*(s)\}
$$
\nsubcases

Hence $t_0 \leq f^{**}(s_0)$ and
whence
2) Let $s \in \mathcal{B}$. Conside: 2) Let
There a
2.

$$
g^*(s_0) \leqslant g^*(s_0 - 0) \leqslant f^{**}(s_0).
$$
\n
$$
s_0 = \sup\{u > s : g^*(u) = g^*(s)\}.
$$
\nsubcases\n
$$
s_0
$$
\n
$$
s_0
$$
\n
$$
s_0
$$
\n
$$
s_0
$$
\n
$$
s_0
$$
\nThen\n
$$
m\{g^* > t\} = s_0
$$
\n
$$
s_0 - 0)
$$
.\nHence\n
$$
s_0 \in \mathcal{A} \text{ and}
$$

There are three possible subcases
2.1) g^* is not continuous at s_0 . Then

$$
\mathbf{m}{g^*}>t} = s_0
$$

$$
g^*(s_0) = g^*(s_0 - 0) \leqslant f^{**}(s_0) \leqslant f^{**}(s).
$$

2.1) g^* is not continuous at s_0 . Then
 $\mathbf{m}{g^* > t}$

for every $t \in [g^*(s_0), g^*(s_0 - 0)]$. Hence s_0
 $g^*(s_0) = g^*(s_0 - 0) \le$

2.2) g^* is continuous at s_0 . Let s_1

decreasing on $[s_0, s_1]$. Then for every $= s_0$
 $\in \mathcal{A}$
 $f^{**}(s)$
 $> s_0$
 $\leq t$ for every $t \in [g^*(s_0), g^*(s_0 - 0)]$. Hence $s_0 \in \mathcal{A}$ and
 $g^*(s_0) = g^*(s_0 - 0) \leqslant f^{**}(s_0)$:

2.2) g^* is continuous at s_0 . Let $s_1 > s_0$ and

decreasing on $[s_0, s_1]$. Then for every $s' \in (s_0, s_1)$ t
 $\mathbf{m}{g^* >$ $(s_0) = g^*(s_0 - 0) \leqslant f^{**}(s_0) \leqslant f^{**}(s).$
 nuous at s_0 . Let $s_1 > s_0$ and the fi

Then for every $s' \in (s_0, s_1)$ there exis
 $\mathbf{m}{g^* > t} = \mu{g > t} = s'.$
 $g^*(s') \leqslant f^{**}(s').$ 2.2) g^* is continuous at s_0 . Let $s_1 > s_0$ and the function g^* is strictly
asing on $[s_0, s_1]$. Then for every $s' \in (s_0, s_1)$ there exists $t > 0$ such that
 $\mathbf{m}{g^* > t} = \mu\{g > t\} = s'$.
e $s' \in \mathcal{A}$ and
 $g^*(s') \$ decreasing on $[s_0, s_1]$. Then for every $s' \in (s_0, s_1)$ there exists $t > 0$ such that
 $\mathbf{m}\lbrace g^* > t \rbrace = \mu\lbrace g > t \rbrace = s'$.

Hence $s' \in \mathcal{A}$ and
 $g^*(s') \leq f^{**}(s')$.

By passing to the limit with $s' \to s_0 - 0$ we have
 $g^*(s) =$

$$
\mathbf{m}{g^*} > t
$$
 = $\mu{g > t}$ = s'.

$$
g^*(s') \leqslant f^{**}(s').
$$

$$
\mathbf{m}\{g^* > t\} = \mu\{g > t\} = s'.
$$

\n
$$
\in \mathcal{A} \text{ and}
$$

\n
$$
g^*(s') \leq f^{**}(s').
$$

\nng to the limit with $s' \to s_0 - 0$ we have
\n
$$
g^*(s) = g^*(s_0) = g^*(s_0 + 0) \leq f^{**}(s_0 + 0) \leq f^{**}(s_0) \leq f^{**}(s).
$$

\n
$$
s_0 = +\infty.
$$

\nen for each $x \in [s, +\infty)$ we have
\n
$$
g^*(x) = g^*(s) = g^*(\infty) = 0,
$$

\n
$$
g^*(s) = 0 \leq f^{**}(s).
$$

Hence $s' \in \mathcal{A}$ and
By passing to the
 $g^*(s) =$
2.3) $s_0 = +$
Then for ead $(s'$
 (0)
 (0)
 (e)
 (h) By passing to the limit with $s' \to s_0 - 0$ we have
 $g^*(s) = g^*(s_0) = g^*(s_0 + 0) \leq f^{**}(s_0 +$

2.3) $s_0 = +\infty$.

Then for each $x \in [s, +\infty)$ we have
 $g^*(x) = g^*(s) = g^*(\infty)$ 2.3) $s_0 = +\infty$.
Then for each $x \in [s, +\infty)$ we have
 $g^*(x) = g^*(s)$
and hence

$$
g^*(x) = g^*(s) = g^*(\infty) = 0,
$$

$$
g^*(s) = 0 \leqslant f^{**}(s).
$$

Then for each $x \in [s, +\infty)$ we have
 $g^*(x) = g^*(s)$

nence
 $g^*(s) = 0$

na 3.3. Let $T \in \mathcal{PAC}$ and $f \in (\mathbf{L}_1 (x) = g^*(s) = g^*(\infty) = 0,$
 $g^*(s) = 0 \le f^{**}(s).$
 $id f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\mathbf{\Omega}, \mu).$
 $(B_T f)^*(s) \le f^{**}(s).$ Lemma 3. $(s) = 0 \leq f^{**}(s).$
 $\in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\Omega, \mu)$. Then
 $\tau f)^*(s) \leq f^{**}(s).$ **Lemma 3.3.** Let $T \in \mathcal{PAC}$ and $f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\Omega, \mu)$. Then
 $(B_T f)^*(s) \leqslant f^{**}(s)$. $(B_T f)$ $(B_T f)^*(s) \leq f^{**}(s).$ $(s) \leq f^{**}(s).$

Proof. We may assume without loss of generality that $f = |f| \geq 0$.

are without loss of generality that $f = |f| \ge 0$.

case $f^*(\infty) = 0$. Then $f^{**}(\infty) = 0$. It is not hard to show,
 $(B_T f)^*(\infty) = 0$. We may assume without loss of generality that $f = |f| \ge 0$.

st consider the case $f^*(\infty) = 0$. Then $f^{**}(\infty) = 0$. It is not
 $(B_T f)^*(\infty) = 0$.

Maximal Ergodic Inequality for $T \in \mathcal{PAC}$ ([22], Ch. 1, §1.6 1) First consider the case $f^*(\infty) = 0$. Then $f^{**}(\infty) = 0$. It is not hard to show,

that
 $(B_T f)^*(\infty) = 0$.

We use Maximal Ergodic Inequality for $T \in \mathcal{PAC}$ ([22], Ch. 1, §1.6):
 $\mu\{B_T f > t\} \leq \frac{1}{t} \int f d\mu, \quad t > 0$

$$
(B_Tf)^*(\infty) = 0.
$$

$$
(B_T f)^*(\infty) = 0.
$$
\nWe use Maximal Ergodic Inequality for $T \in \mathcal{PAC}$ ([22], Ch

\n
$$
\mu\{B_T f > t\} \leq \frac{1}{t} \int f d\mu, \quad t > 0
$$
\nand put $g = B_T f$.

\nThe functions f and g satisfy the conditions of Lemma

\n
$$
g^*(s) = (B_T f)^* \leq f^{**}(s), \quad s > 0.
$$

and put $g = B_T f$.
The functions f and g satisfy the conditions of Lemma 3.2 and hence

$$
g^*(s) = (B_T f)^* \leqslant f^{**}(s), \quad s > 0.
$$

and put $g = Brf$.
The function
2) Let now $f^*(\infty)$ The functions f and g satisfy the conditions of Lemma 3.2 and hence
 $g^*(s) = (B_T f)^* \leq f^{**}(s), \quad s > 0.$

et now $f^*(\infty) = \lambda > 0$. Then $f = f_{\lambda} + \lambda$, where $f_{\lambda}^*(\infty) = 0$. Since
 $(f_{\lambda} + \lambda)^*(t) = f_{\lambda}^*(t) + \lambda$ $(s) = (B_T f)^* \leq f^{**}(s), \quad s > 0.$

0. Then $f = f_{\lambda} + \lambda$, where f_{λ}^*
 $(f_{\lambda} + \lambda)^*(t) = f_{\lambda}^*(t) + \lambda$

$$
(f_{\lambda} + \lambda)^{*}(t) = f_{\lambda}^{*}(t) + \lambda
$$

2) Let now
$$
f^*(\infty) = \lambda > 0
$$
. Then $f = f_{\lambda} + \lambda$, where $f_{\lambda}^*(\infty) = 0$. Since
\n
$$
(f_{\lambda} + \lambda)^*(t) = f_{\lambda}^*(t) + \lambda
$$
\nand
\n
$$
(f_{\lambda} + \lambda)^{**}(t) = \frac{1}{t} \int_0^t (f_{\lambda} + \lambda)^*(s) ds = \frac{1}{t} \int_0^t (f_{\lambda}^* + \lambda)(s) ds
$$
\n
$$
= \frac{1}{t} \int_0^t f_{\lambda}^*(s) ds + \lambda = f_{\lambda}^{**}(t) + \lambda,
$$
\nwe have

$$
= \frac{1}{t} \int_{0}^{t} f_{\lambda}^{*}(s) ds + \lambda = f_{\lambda}^{**}(t) + \lambda,
$$

\n
$$
(B_T f)^{*}(s) = [B_T(f_{\lambda} + \lambda)]^{*}(s) \leq [B_T(f_{\lambda})]^{*}(s) + \lambda
$$

\n
$$
\leq f_{\lambda}^{**}(s) + \lambda = (f_{\lambda} + \lambda)^{**}(s) = f^{**}(s),
$$

\n
$$
(B_T f)^{*}(s) \leq f^{**}(s).
$$

\n*m* 3.1. Let $f \in \mathbf{E_H}$, then $f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\Omega, \mu)$ and

i.e.,

$$
(B_T f)^*(s) \leqslant f^{**}(s).
$$

 $(s) + \lambda = (f_{\lambda} + \lambda)^{**}(s) = f^{**}(s),$
 $B_T f)^*(s) \leq f^{**}(s).$
 H, then $f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\Omega, \mu)$ and $(B_T f)^*(s) \leqslant f^{**}(s).$

Froof of Theorem 3.1. Let $f \in \mathbf{E}_{\mathbf{H}}$, then $f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\Omega, \mu)$ and $f^{**} \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})$.

By Lemma 3.3,
 $(B_T f)^*(s) \leqslant f^{**}(s), \ s > 0.$

Therefore $(B_T f)^* \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})$ and Proof of Theorem 3.1. Let $f \in \mathbf{E}_{\mathbf{H}}$, then $f \in (\mathbf{L}_1 + \mathbf{L}_{\infty})(\Omega, \mu)$ and $f^{**} \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})$.
By Lemma 3.3,
 $(B_T f)^*(s) \leqslant f^{**}(s), s > 0$.
Therefore $(B_T f)^* \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})$ and
 $||(B_T f)^*||_{\mathbf{E}(\mathbf{R}_+,\mathbf$

$$
(B_T f)^*(s) \leqslant f^{**}(s), \ s > 0.
$$

$$
|| (B_T f)^* ||_{\mathbf{E}(\mathbf{R}_+,\mathbf{m})} \leq || f^{**} ||_{\mathbf{E}(\mathbf{R}_+,\mathbf{m})}.
$$

 $(B_T f)^*(s) \leqslant f^{**}(s), \ s > 0.$

efore $(B_T f)^* \in \mathbf{E}(\mathbf{R}_+, \mathbf{m})$ and
 $||(B_T f)^*||_{\mathbf{E}(\mathbf{R}_+, \mathbf{m})} \leqslant ||f^{**}||_{\mathbf{E}(\mathbf{R}_+)}$
 $[(B_T f)^*]^* = (B_T f)^*$, we have also $B_T f \in \mathbf{E}$ and
 $||B_T f||_{\mathbf{F}} = ||(B_T f)^*||_{\mathbf{F}(\mathbf{R}_+, \mathbf{m})} \$ Therefore $(B_T f)^* \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})$ and
 $||(B_T f)^*||_{\mathbf{E}(\cdot)}$

Since $[(B_T f)^*]^* = (B_T f)^*$, we hav
 $||B_T f||_{\mathbf{E}} = ||(B_T f)^*||_{\mathbf{E}}$ $(B_T f)$
f)*, we
 $\|(B_T f$ Since $[(B_T f)^*]^* = (B_T f)^*$, we have also $B_T f \in \mathbf{E}$ and
 $||B_T f||_{\mathbf{E}} = ||(B_T f)^*||_{\mathbf{E}(\mathbf{R}_+,\mathbf{m})} \le ||f^{**}||_{\mathbf{E}(\mathbf{R}_+)}$

$$
||B_Tf||_{\mathbf{E}} = ||(B_Tf)^*||_{\mathbf{E}(\mathbf{R}_+,\mathbf{m})} \le ||f^{**}||_{\mathbf{E}(\mathbf{R}_+,\mathbf{m})} = ||f||_{\mathbf{E}_{\mathbf{H}}}.
$$

4. Converse Dominated Ergodic Theorem

4. Converse Dominated Ergodic Theorem
Theorem 4.1 (Converse Dominated Ergodic Theorem). Let E
space, θ be an ergodic conservative measure preserving transfor
and $T = T_{\theta} \in \mathcal{PAC}_0$ is of the form $T_{\theta}f = f \circ \theta$. The **Theorem 4.1 (Converse Dominated Ergodic Theorem).** Let $\mathbf{E}(\Omega,\mu)$ be an r.i. $(\mathbf{\Omega}, \mu)$ be an r.i.
mation on $(\mathbf{\Omega}, \mu)$
implies $f \in \mathbf{E}_{\mathbf{H}}$.
alities.
> 1. Then space, θ be an ergodic conservative measure preserving transformation on (Ω, μ) (Ω, μ)
 \in **E**_H.

Proposition 4.2. Let $f \in L_1$

and
$$
T = T_{\theta} \in \mathcal{PAC}_0
$$
 is of the form $T_{\theta}f = f \circ \theta$. Then $B_T f \in \mathbf{E}$ implies $f \in \mathbf{E_H}$.
\nIn order to prove the theorem we need the following inequalities.
\n**Proposition 4.2.** Let $f \in \mathbf{L}_1(\Omega, \mu) + \mathbf{L}_{\infty}(\Omega, \mu)$, $f \ge 0$ and $C > 1$. Then
\n $(C - 1)\mathbf{m}\lbrace f^{**} \ge Ct \rbrace \le \frac{1}{t} \int f d\mu \le \mathbf{m}\lbrace f^{**} > t \rbrace$
\nfor all $t > f^*(\infty)$.
\n*Proof.* 1) The first inequality. Since $\mathbf{n}_f = \mathbf{n}_{f^*}$ we may suppose without loss of generality that $f = f^*$.
\nLet $s = \mathbf{m}\lbrace f^{**} \ge Ct \rbrace$. Since $t > f^*(\infty)$ and $C > 1$, we have $Ct > f^*(\infty)$.

for all $t > f^*(\infty)$.

 $(\infty).$
first
t $f = \mathbf{m}\{f^*$
 $f^*(c)$ 1) The first inequality. Since $\mathbf{n}_f = \mathbf{n}_{f^*}$ we may suppose without loss of
ity that $f = f^*$.
et $s = \mathbf{m}\{f^{**} \geq Ct\}$. Since $t > f^*(\infty)$ and $C > 1$, we have $Ct > f^*(\infty)$.
ry $u > f^*(\infty)$
 $f^{**}(\mathbf{m}\{f^{**} > u\}) = u$,
re generality that $f = f^*$.

Let $s = m\{f^{**}\}\$

For every $u > f^*(\infty)$

therefore $f^{**}(s) = Ct$.
 $\frac{1}{s} \int_0^s$ Let $s = \mathbf{m} \{f^{**} \geq Ct\}$. Since $t > f^*(\infty)$ and $C > 1$, we have $Ct > f^*(\infty)$.

very $u > f^*(\infty)$
 $f^{**}(\mathbf{m} \{f^{**} > u\}) = u$,

fore $f^{**}(s) = Ct$. Thus
 $\frac{1}{s} \int_s^s f^*(\tau) d\tau = Ct$, and $s = \frac{1}{Ct} \int_s^s f^* d\mathbf{m}$.

$$
f^{**}(\mathbf{m}{f^{**}}>u\})=u,
$$

For every $u > f^*(\infty)$
therefore $f^{**}(s) = C$;
 $\frac{1}{s}$

therefore
$$
f^{**}(\mathbf{m}\lbrace f^{**} > u \rbrace) = u
$$
,
\ntherefore $f^{**}(s) = Ct$. Thus
\n
$$
\frac{1}{s} \int_{0}^{s} f^{*}(\tau) d\tau = Ct, \text{ and } s = \frac{1}{Ct} \int_{0}^{s} f^{*} d\mathbf{m}.
$$
\nIf $s \le \mathbf{m}\lbrace f^{*} > t \rbrace$, Then
\n
$$
s \le \frac{1}{Ct} \int_{0}^{\mathbf{m}\lbrace f^{*} > t \rbrace} f^{*} d\mathbf{m} = \frac{1}{Ct} \int_{0}^{s} f^{*} d\mathbf{m}.
$$

If $s \leqslant m\{f^* > t\}$, Then
 $s \leqslant$
Whence
 $(C - \frac{1}{\sqrt{C}})$

$$
\int_{0}^{t^{*}} f^{*}(\tau) d\tau = Ct, \text{ and } s = \frac{1}{Ct} \int_{0}^{t^{*}} f^{*} d\mathbf{r}
$$
\nThen

\n
$$
s \leq \frac{1}{Ct} \int_{0}^{\mathbf{m}\{f^{*} > t\}} f^{*} d\mathbf{m} = \frac{1}{Ct} \int_{\{f^{*} > t\}} f^{*} d\mathbf{m}.
$$

$$
s \leq \frac{Ct}{Ct} \int_{0}^{t^{*}} f^{*} d\mathbf{m} = \frac{Ct}{Ct} \int_{\{f^{*} > t\}} f^{*} d\mathbf{m}.
$$

$$
(C-1)s < Cs \leq \frac{1}{t} \int_{\{f^{*} > t\}} f^{*} d\mathbf{m} = \int_{\{f > t\}} f d\mu.
$$

is non-increasing, hence $s > \mathbf{m}\{f^{*} > t\}$ in
 $\frac{1}{\sqrt{t}} \int_{0}^{s} f^{*} d\mathbf{m} \leq \frac{1}{Ct} \int_{0}^{s} t d\mathbf{m} \leq \frac{1}{Ct} st =$

(C − 1)s < Cs ≤

The function f^* is non-increasin

Then
 $\frac{1}{Ct} \int_{m\{f^*\geq t\}}^{s} f^* d\mathbf{m}$ ≤

The function
$$
f^*
$$
 is non-increasing, hence $s > \mathbf{m} \{f^* > t\}$ implies $f^*(s) \leq t$.
\nThen
\n
$$
\frac{1}{Ct} \int_{\mathbf{m} \{f^* > t\}}^{s} f^* d\mathbf{m} \leq \frac{1}{Ct} \int_{\mathbf{m} \{f^* > t\}}^{s} t d\mathbf{m} \leq \frac{1}{Ct} st = \frac{s}{C}
$$
\nand hence
\n
$$
s - \frac{s}{C} \leq \frac{1}{Ct} \int_{0}^{\mathbf{m} \{f^* > t\}} f^* d\mathbf{m}.
$$

 $\frac{1}{\pi}$ and $\frac{1}{\pi}$

$$
f^* d\mathbf{m} \leq \frac{Ct}{Ct} \int_{\mathbf{m}\{f^*>t\}} t \, d\mathbf{m} \leq
$$

$$
s - \frac{s}{C} \leq \frac{1}{Ct} \int_{0}^{\mathbf{m}\{f^*>t\}} f^* \, d\mathbf{m}.
$$

Thus \Box

$$
s - \overline{C} \leq \overline{Ct} \int_{0}^{t^{*}} \int_{\{f^{*} > t\}} f^{*} d\mathbf{m}.
$$

(*C* – 1) $s \leq \frac{1}{t} \int_{\{f^{*} > t\}} f^{*} d\mathbf{m}.$

ality. Let $u = \mathbf{m} \{f^{**} > t\}$. Since $t > f^{**}(\infty)$, we have
 $t = \frac{1}{u} \int_{0}^{u} f^{*}(s) ds$. 2) The second inequality. Let $u = \mathbf{m} \{f^{**} > t\}$. Since $t > f^{**}(\infty)$, we have
 $f^{**}(u) = t$ and
 $t = \frac{1}{u} \int_0^u f^*(s) ds$.

Whence
 $1 \int_0^u f^*(s) ds = 1 \int_0^{t} f^*(s) ds$. $f^{**}(u) = t$ and $(u) = t$ and
ence

$$
t = \frac{1}{u} \int\limits_0^u f^*(s) \, ds.
$$

$$
t = \frac{1}{u} \int_{0}^{u} f^{*}(s) ds.
$$

$$
u = \frac{1}{t} \int_{0}^{u} f^{*}(s) ds \ge \frac{1}{t} \int_{\{f^{*} > f^{*}(u)\}} f^{*} d\mathbf{m}.
$$

$$
f^{*}(u)
$$
, we have also

Since t :

$$
u = \frac{1}{t} \int_{0}^{t} f^{*}(s) ds \geq \frac{1}{t} \int_{\{f^{*} > f^{*}(u)\}} f^{*} d\mathbf{m}
$$

Since $t = f^{**}(u) \geq f^{*}(u)$, we have also

$$
\frac{1}{t} \int_{\{f^{*} > f^{*}(u)\}} f^{*} d\mathbf{m} \geq \frac{1}{t} \int_{\{f^{*} > t\}} f^{*} d\mathbf{m},
$$

i.e.,
$$
\mathbf{m}\{f^{**} > t\} = u \geq \frac{1}{t} \int_{\{f^{*} > t\}} f^{*} d\mathbf{m}.
$$

$$
\int_{\{f^*>f^*(u)\}} f^* d\mathbf{m} \ge \frac{1}{t} \int_{\{f^*>t\}} f^* d\mathbf{m},
$$

\n
$$
\mathbf{m}\{f^{**}>t\} = u \ge \frac{1}{t} \int_{\{f^*>t\}} f^* d\mathbf{m}.
$$

\nIn ergodic conservative measure preserving transformation
\n $\in \mathcal{PAC}$ is of the form $T_{\theta}f = f \circ \theta$. Then
\n
$$
f^{**}(s) \le 2(B_T f)^* \left(\frac{s}{2}\right)
$$

i.
Len
on **Lemma 4.3.** Let θ be an ergodic conservative measure preserving transformation on

$$
f^{**}(s) \leq 2(B_T f)^*\left(\frac{s}{2}\right)
$$

for all $s > 0$ and for every $f \in \mathcal{R}_0$.

 (Ω, μ) and $T = T_{\theta} \in \mathcal{PAC}$ is of the form $T_{\theta}f = f \circ \theta$. Then
 $f^{**}(s) \leq 2(B_T f)^* \left(\frac{s}{2}\right)$

all $s > 0$ and for every $f \in \mathcal{R}_0$.

oof. We may suppose without loss of generality that $f = |f| \geq$

By Proposition 4.2 Proof.

$$
f^{**}(s) \leq 2(B_T f)^* \left(\frac{1}{2}\right)
$$

0 and for every $f \in \mathcal{R}_0$.
may suppose without loss of generality that
oposition 4.2 with $C = 2$ we have

$$
\mathbf{m}\lbrace f^{**} \geq 2t \rbrace \leq \frac{1}{t} \int_{\{f > t\}} f d\mu
$$

$$
f^*(\infty). \text{ Here } f^*(\infty) = 0 \text{ since } f \in \mathcal{R}_0. \text{ It is}
$$

$$
\frac{1}{2t} \int_{\mathcal{I}} f d\mu \leq \frac{1}{2t} \int_{\mathcal{I}} f d\mu \leq \mu\lbrace L \rbrace
$$

We may suppose without loss of generality that $f = |f| \ge 0$.

y Proposition 4.2 with $C = 2$ we have
 $\mathbf{m}\lbrace f^{**} \ge 2t \rbrace \le \frac{1}{t} \int f d\mu$
 $\lbrace f > t \rbrace$
 $\lbrace f > t \rbrace$
 $\lbrace f \rbrace \ge f^*(\infty)$. Here $f^*(\infty) = 0$ since $f \in \mathcal{R}_0$. It is By Proposition 4.2 with $C = 2$ we have
 $\mathbf{m}\lbrace f^{**} \geq 2t \rbrace \leq \frac{1}{t}$
 $1 \ t > f^*(\infty)$. Here $f^*(\infty) = 0$ since $f \in$
 $\frac{1}{\int f \ d\mu} \leq \frac{1}{\int f}$ for all $t > f^*(\infty)$. Here $f^*(\infty) = 0$ since $f \in \mathcal{R}_0$. It is proved in [22], Lemma 6.7,
that
 $\frac{1}{2t} \int_{\{f > t\}} f d\mu \leq \frac{1}{2t} \int_{\{B_T f > t\}} f d\mu \leq \mu \{B_T f > t\}$.
Thus
 $\mathbf{m} \{f^{**} \geq 2t\} \leq 2\mu \{B_T f > t\}$ \overline{c} $\overline{1}$ \overline{c}

$$
\mathbf{m}\lbrace f^{**} \geq 2t \rbrace \leq \frac{1}{t} \int f d\mu
$$

\n
$$
\lbrace f > t \rbrace
$$

\nHere $f^*(\infty) = 0$ since $f \in \mathcal{R}_0$. It is proved in
\n
$$
\frac{1}{2t} \int f d\mu \leq \frac{1}{2t} \int f d\mu \leq \mu \lbrace B_T f > t \rbrace.
$$

\n
$$
\mathbf{m}\lbrace f^{**} \geq 2t \rbrace \leq 2\mu \lbrace B_T f > t \rbrace
$$

\n
$$
\mathbf{m}\lbrace f^{**} \geq 2t \rbrace \leq 2\mu \lbrace B_T f > t \rbrace
$$

\nis inequality holds for every $T \in \mathcal{PAC}$ of the for
\ninsertvature measure preserving transformation

$$
\mathbf{m}{f^{**}} \geq 2t \} \leq 2\mu\{B_T f > t\}
$$

Thu for ϵ 2t
 $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$
 $\frac{1}{2}$
> 2t
 \geqslant \geqslant s

urv

$$
\mathbf{m}\{f^{**} \ge 2t\} \le 2\mu\{B_T f > t\}
$$
\nfor all $t > 0$. This inequality holds for every $T \in \mathcal{PAC}$ of the form $T = T_{\theta}$, where θ is an ergodic conservative measure preserving transformation of (Ω, μ) .

\nFor every $s > 0$ we have now

\n
$$
(B_T f)^* \left(\frac{s}{2}\right) = \inf \{t > 0 : \mu\{B_T f > t\} \le \frac{s}{2}\}
$$
\n
$$
\ge \inf \{t > 0 : \mathbf{m}\{f^{**} \ge 2t\} \le s\} \ge \inf \{t > 0 : \mathbf{m}\{f^{**} > 2t\} \le s\}
$$
\n
$$
= \frac{1}{2} \inf \{2t > 0 : \mathbf{m}\{f^{**} > 2t\} \le s\} = \frac{1}{2}(f^{**})^*(s) = \frac{1}{2}f^{**}(s).
$$

$$
(D_2g)(t) = g\left(\frac{t}{2}\right), \quad t > 0,
$$

Proof of Theorem 4.1. Let $B_T f \in \mathbf{E}$. By using the dilation opera
 $(D_2 g)(t) = g\left(\frac{t}{2}\right)$, $t > 0$,

and putting $g = B_T f$, we can write the inequality in Lemm Proof of Theorem 4.1. Let $B_T f \in \mathbf{E}$. By using the dilation operator D_t with $t = 2$,
 $(D_2 g)(t) = g\left(\frac{t}{2}\right)$, $t > 0$,

and putting $g = B_T f$, we can write the inequality in Lemma 4.3 as $f^{**} \le 2D_2(B_T f)^*$. It is known and putting $g = B_T f$, we can write the inequality in Lemma 4.3 as $f^{**} \leq$ e ['
}
} 0,
ity
2.1
D2 $2D_2(B_Tf)^*$. It is known ([21], Ch. II, §4.3, [24], Ch. 2.b) that the dilation operator D_t act as a bounded liner operator in each of standard rearrangement invariant space $\mathbf{E}(\mathbf{R}_+,\mathbf{m})$. Hence,
 $B_Tf \in \mathbf{E}(\mathbf{\$ $2D_2(B_Tf)^*$. It is known ([21], Ch. II, §4.3, [24], Ch. 2.b) that the dilation operator
 D_t act as a bounded liner operator in each of standard rearrangement invariant

space $\mathbf{E}(\mathbf{R}_+,\mathbf{m})$. Hence,
 $B_Tf \in \mathbf{E}$ D_t ce $\mathbf{E}(\mathbf{R}_{+}, \mathbf{m})$. Hence,
 $B_T f \in \mathbf{E}(\mathbf{\Omega}, \mu) \Leftrightarrow (B_T f)^* \in \mathbf{E}(\mathbf{R}_{+}, \mathbf{m}) \Leftrightarrow D_2(B_T f)^* \in \mathbf{E}(\mathbf{R}_{+}, \mathbf{m})$.

expectively integrables with the standard rearrangement in the standard real integration of

space
$$
\mathbf{E}(\mathbf{R}_+,\mathbf{m})
$$
. Hence,
\n $B_T f \in \mathbf{E}(\mathbf{\Omega},\mu) \Leftrightarrow (B_T f)^* \in \mathbf{E}(\mathbf{R}_+,\mathbf{m}) \Leftrightarrow D_2(B_T f)^* \in \mathbf{E}(\mathbf{R}_+,\mathbf{m})$.
\nTherefore
\n $B_T f \in \mathbf{E}(\mathbf{\Omega},\mu) \Rightarrow f^{**} \in \mathbf{E}(\mathbf{R}_+,\mathbf{m}) \Leftrightarrow f \in \mathbf{E}_{\mathbf{H}}(\mathbf{\Omega},\mu)$. \square
\nWe shall say that an r.i. space **E** has Hardy-Littlewood property (**E** \in $H\mathcal{LP}$)

$$
B_T f \in \mathbf{E}(\Omega, \mu) \Rightarrow f^{**} \in \mathbf{E}(\mathbf{R}_+,\mathbf{m}) \Leftrightarrow f \in \mathbf{E}_{\mathbf{H}}(\Omega, \mu) . \Box
$$

 $(P, \mu) \Leftrightarrow (B_T f)^* \in \mathbf{E}(\mathbf{R}_+, \mathbf{m}) \Leftrightarrow D_2(B_T f)^* \in \mathbf{E}(\mathbf{R}_+, \mathbf{m}).$
 $\forall f \in \mathbf{E}(\mathbf{\Omega}, \mu) \Rightarrow f^{**} \in \mathbf{E}(\mathbf{R}_+, \mathbf{m}) \Leftrightarrow f \in \mathbf{E}_{\mathbf{H}}(\mathbf{\Omega}, \mu).$

say that an r.i. space **E** has Hardy-Littlewood property (**E** \in We set $\text{if } \mathbf{E}_{\mathbf{H}} = \mathbf{I}$
all $f \in \mathbf{E}$ $(\Omega, \mu) \Rightarrow f^{**} \in \mathbf{E}(\mathbf{R}_+,\mathbf{m}) \Leftrightarrow f \in \mathbf{E}_{\mathbf{H}}(\Omega, \mu)$. □
t an r.i. space **E** has Hardy-Littlewood property (**E** $\in \mathcal{HLP}$)
 $\in \mathbf{E} \Leftrightarrow f^{**} \in \mathbf{E}$. We shall write $\mathbf{E} \in \mathcal{DET}$ if $B_T f \in \mathbf{E}$ for
 $\mathcal{H$ We shall say that an r.i. space **E** has Hardy-Littlewood property ($\mathbf{E} \in \mathcal{HLP}$)
 $\mathbf{E} \in \mathbf{E}$, that is $f \in \mathbf{E} \Leftrightarrow f^{**} \in \mathbf{E}$. We shall write $\mathbf{E} \in \mathcal{DET}$ if $B_T f \in \mathbf{E}$ for
 $\in \mathbf{E}$ and $T \in \mathcal{$ if **E_H** = **E**, that is $f \in \mathbf{E} \Leftrightarrow f^{**} \in \mathbf{E}$. We shall write **E** ∈ $\mathcal{D}\mathcal{E}\mathcal{T}$ if $B_T f \in \mathbf{E}$ for all $f \in \mathbf{E}$ and $T \in \mathcal{P}\mathcal{A}\mathcal{C}$.
 Corollary 4.4. E ∈ $\mathcal{D}\mathcal{E}\mathcal{T} \Leftrightarrow \mathbf{E} \in \mathcal{H}\mathcal{$

 $\mathbf{E} \in \mathcal{DET} \Leftrightarrow \mathbf{E} \in \mathcal{HLP}.$

5. Pointwise Ergodic Theorem

all *f* ∈ **E** and *T* ∈ *PAC*.
 Corollary 4.4. E ∈ *DET* \Leftrightarrow **E** ∈ *HLP*.
 5. Pointwise Ergodic Theorem

Recall that $\mathcal{R}_0 = cl_{\mathbf{L}_1 + \mathbf{L}_{\infty}}(\mathbf{L}_1 \cap \mathbf{L}_{\infty})$ is the minimal part of the space $\mathbf{L}_1 + \math$ $\mathbf{n}_f(x) < \infty$ for all $x > 0$.

Recall that $\mathcal{R}_0 = cl_{\mathbf{L}_1 + \mathbf{L}_\infty}(\mathbf{L}_1 \cap \mathbf{L}_\infty)$ is the minimal part of the space $\mathbf{L}_1 + \mathbf{L}_\infty$.

It consists of all $f \in \mathbf{L}_1 + \mathbf{L}_\infty$ such that $f^*(\infty) = 0$, or (an equivalent condition)
 $\mathbf{n}_f(x$ It consists of all $f \in \mathbf{L}_1 + \mathbf{L}_{\infty}$ such that $f^*(\infty) = 0$, or (an equivalent condition)
 $\mathbf{n}_f(x) < \infty$ for all $x > 0$.
 Theorem 5.1. Let $T \in \mathcal{PAC}$ and $f \in \mathcal{R}_0$, then $\{A_{n,T}f\}_{n=1}^{\infty}$ converges almos $(x) < \infty$ for all $x > 0$.
 eorem 5.1. Let $T \in \mathcal{P}$.
 ere on (Ω, μ) . Conver
 msformation on (Ω, μ)
 $\in \mathbf{L}_{\infty}$ such that $\{A_{n,T}$.
 eof. The first part of **Theorem 5.1.** Let $T \in \mathcal{PAC}$ and $f \in \mathcal{R}_0$, then $\{A_{n,T}f\}_{n=1}^{\infty}$ converges almost everywhere on (Ω, μ) . Conversely, let θ be an ergodic conservative measure preserving (Ω, μ) . Conversely, let θ be an ergodic conservative measure preserving
ation on (Ω, μ) and $T = T_{\theta}$ is of the form $T_{\theta}f = f \circ \theta$. Then there exists
uch that $\{A_{n,T}f\}_{n=1}^{\infty}$ is not a.e. convergent on $(\Omega, \mu$ transformation on (Ω, μ) and $T = T_{\theta}$ is of the form $T_{\theta}f = f \circ \theta$. Then there exists $\{A_{n,T}f\}_{n=1}^{\infty}$ is not a.e. convergent on (Ω, μ) .

art of the theorem is an improved version of Dunford-Schwartz

Theorem (see, for example, $f \in \mathbf{L}_{\infty}$ such that $\{A_{n,T}f\}_{n=1}^{\infty}$ is not a.e. convergent on

 (Ω, μ) .
sion of
3, §8.2.
hat θ b
satisfies
for eve: *Proof.* The first part of the theorem is an improved version of Dunford-Schwartz

ise Ergodic Theorem (see, for example, [10], Ch. 8, §8.2.6 and §8.6.11).

order to prove the "converse" part we suppose that θ be an ergodic con-
 μ -preserving transformation on Ω and $T = T_{\theta}$ satisfies the cond I only see Ergodic Theorem (see, for example, [10], Ch. 8, §8.2.6 and §8.0.11).
In order to prove the "converse" part we suppose that θ be an ergodic convertive μ -preserving transformation on Ω and $T = T_{\theta}$ sati In order to prove the "converse" part we suppose that θ be an ergodic con-
tive μ -preserving transformation on Ω and $T = T_{\theta}$ satisfies the condition:
 $A_{n,T} f\}_{n=1}^{\infty}$ converges almost everywhere on (Ω, μ) servative μ -preserving transformation on **Ω** and $T = T_{\theta}$ satisfies the condition:
 $\{A_{n,T}f\}_{n=1}^{\infty}$ converges almost everywhere on $(\mathbf{\Omega}, \mu)$ for every $f \in \mathbf{L}_{\infty}$.

Then for any probability measure $\nu \sim \mu$ ${A_{n,T}}f_{n=1}^{\infty}$

$$
\{A_{n,T}f\}_{n=1}^{\infty}
$$
 converges almost everywhere on (Ω, μ) for every $f \in \mathbf{L}_{\infty}$. (*)
Then for any probability measure $\nu \sim \mu$ and all measurable sets A there exist
limits

$$
\widetilde{\nu}(A) = \lim_{n \to \infty} \int_{\Omega} A_{n,T} \chi_A d\nu = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \nu(\theta^{-k} A).
$$
(**)
It can be shown by means Theorems 4.3.1–4.3.3 from [22], that the weak convergence

$$
\frac{1}{n} \sum_{k=0}^{n-1} \nu \circ \theta^{-k} \to \widetilde{\nu}
$$
(***)

It can
gence gence $\frac{1}{n}\sum_{k=0}^{n-1}\nu\circ\theta$ in (**) yields in fact convergence in norm.

$$
\widetilde{\nu}(A) = \lim_{n \to \infty} \int_{\Omega} A_{n,T} \chi_A \, d\nu = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0} \nu(\theta^{-k} A). \tag{**}
$$
\nIt can be shown by means Theorems 4.3.1–4.3.3 from [22], that the weak convergence

\n
$$
\frac{1}{n} \sum_{k=0}^{n-1} \nu \circ \theta^{-k} \to \widetilde{\nu} \tag{***}
$$
\nin (**) yields in fact convergence in norm.

in (∗∗) yields in fact convergence in norm.

$$
h = \frac{d\nu}{d\mu} \in \mathbf{L}_1(\nu)
$$
 be Radon-Nikodym derivative and

$$
T_\theta^o: \mathbf{L}_1(\nu) \ni g \to T_\theta^o g = g \circ \theta^{-1} \frac{h \circ \theta^{-1}}{h} \in \mathbf{L}_1(\nu).
$$

Indeed, let
$$
h = \frac{d\nu}{d\mu} \in \mathbf{L_1}(\nu)
$$
 be Radon-Nikodym derivative and
\n
$$
T_\theta^o: \quad \mathbf{L_1}(\nu) \ni g \to T_\theta^o g = g \circ \theta^{-1} \frac{h \circ \theta^{-1}}{h} \in \mathbf{L_1}(\nu).
$$
\nThen
\n
$$
\int_{\Omega} (T_\theta^o g)(\omega) d\nu(\omega) = \int_{\Omega} g(\theta^{-1} w) \frac{h(\theta^{-1} w)}{h(\omega)} d\nu(\omega) = \int_{\Omega} g(\theta^{-1} w) h(\theta^{-1} w) d\mu(\omega)
$$
\n
$$
= \int_{\Omega} g(\omega) h(\omega) d\mu(\omega) = \int_{\Omega} g(\omega) d\nu(\omega),
$$
\ni.e., the operator T_θ^o is a positive isometry in $\mathbf{L_1}(\nu)$ as well as its dual T_θ is a positive
\nisometry in $\mathbf{L_\infty}(\nu) = \mathbf{L_\infty}(\widetilde{\nu})$. Setting $g = 1 \in \mathbf{L_1}(\nu)$ and using Mean Erg

= $(\omega)h(\omega) d\mu(\omega) = \int_{\Omega} g(\omega) d\nu(\omega),$
itive isometry in $\mathbf{L}_1(\nu)$ as well as
 $\tilde{\nu}$). Setting $g = \mathbf{1} \in \mathbf{L}_1(\nu)$ and (1.1) , we get that the weak convents as functionally i.e., the operator T^{σ}_{θ}
isometry in $\mathbf{L}_{\infty}(\nu)$
Theorem ([22], The
(provided by (*)) in
is nothing more than
Thus $\widetilde{\nu} = \frac{d\widetilde{\nu}}{d\nu} \nu$
Since θ is ergodic a is a positive isometry in $\mathbf{L}_1(\nu)$ as well as its dual T_{θ} is a positive
 $= \mathbf{L}_{\infty}(\tilde{\nu})$. Setting $g = 1 \in \mathbf{L}_1(\nu)$ and using Mean Ergodic

orem 2.1.1), we get that the weak convergence of $\frac{1}{n} \sum_{k=0}^{n$ isometry in $\mathbf{L}_{\infty}(\nu) = \mathbf{L}_{\infty}(\tilde{\nu})$. Setting $g = \mathbf{1} \in \mathbf{L}_1(\nu)$ and using Mean Ergodic
Theorem ([22], Theorem 2.1.1), we get that the weak convergence of $\frac{1}{n} \sum_{k=0}^{n-1} T_{\theta}^o \mathbf{1}$
(provided by (*)) i $\frac{1}{n}\sum_{k=0}^{n-1}T_{\theta}^{o}\mathbf{1}$ $h \in \mathbf{L}_1$ $d\nu$

Theorem ([22], Theorem 2.1.1), we get that the weak convergence of $\frac{1}{n}$
(provided by (*)) implies the strong convergence to a function $\tilde{h} \in \mathbf{L}$
is nothing more than $\frac{d\tilde{\nu}}{d\nu}$.
Thus $\tilde{\nu} = \frac{d\tilde{\nu}}{$ (provided by (*)) implies the strong convergence to a function $h \in \mathbf{L}_1(\nu)$, which
is nothing more than $\frac{d\tilde{\nu}}{d\nu}$.
Thus $\tilde{\nu} = \frac{d\tilde{\nu}}{d\nu} \nu$ is a θ -invariant measure on Ω such that $\tilde{\nu} \sim \mu$ and (provided by (*)) implies

is nothing more than $\frac{d\tilde{\nu}}{d\nu}$

Thus $\tilde{\nu} = \frac{d\tilde{\nu}}{d\nu} \nu$ is a

Since θ is ergodic and constant $c >$

not hold. ϵ
 $\frac{1}{2}$ Thus $\widetilde{\nu} = \frac{d\widetilde{\nu}}{d\nu}$
Thus $\widetilde{\nu} = \frac{d\widetilde{\nu}}{d\nu}$
 θ is ergodic
th a constant old. $\frac{d\nu}{d\nu}\nu$ Since θ is ergodic and conservative, every such θ -invariant measure is of the form $c\mu$ with a constant $c > 0$, i.e., $\tilde{\nu}(\Omega) = \infty$. The contradiction shows that $(*)$ does not hold. Since θ is ergodic and conservative, every such θ -invariant measure is of the form $c\mu$ with a constant $c > 0$, i.e., $\tilde{\nu}(\Omega) = \infty$. The contradiction shows that (*) does not hold.
 6. Consequences for Orlicz an $c\mu$ with a constant $c > 0$, i.e., $\tilde{\nu}(\Omega) = \infty$. The contradiction shows that (*) does not hold. □
6. Consequences for Orlicz and Lorentz spaces
Orlicz spaces not hold.
 6. Consequences for Orlicz and Lorentz spaces
 0rlicz spaces
 0rlicz spaces
 10. $+\infty$ \rightarrow $[0, +\infty]$ be an Orlicz function, i.e., $\Phi(0) = 0$, Φ is increasing

6. Consequences for Orlicz and Lorentz spaces

Orlicz spaces

Left-continuous and convex. Assume also that Φ is nontrivial, i.e., $\Phi(x) > 0$ and $\Phi(y) < \infty$ for some $x, y > 0$. The derivative Φ' exists a.e., and it is assumed to be left-continuous with $\Phi'(x) = +\infty$ iff $\Phi(x) = +\infty$ left-continuous and convex. Assume also that Φ is nontrivial, i.e., $\Phi(x) > 0$ and $\Phi(y) < \infty$ for some $x, y > 0$. The derivative Φ' exists a.e., and it is assumed to be left-continuous with $\Phi'(x) = +\infty$ iff $\Phi(x) = +\infty$ $\Phi(y) < \infty$ for some $x, y > 0$. The derivative Φ' exists a.e., and it is assumed to be left-continuous with $\Phi'(x) = +\infty$ iff $\Phi(x) = +\infty$.
The corresponding conjugate Orlicz function Ψ is defined by its derivative Ψ'

 Ψ' , which is the left-continuous inverse of Φ' .

it-continuous with
$$
\Phi'(x) = +\infty
$$
 iff $\Phi(x) = +\infty$. The corresponding conjugate Orlicz function Ψ is defined by its derivative, which is the left-continuous inverse of Φ' . The *Orlicz space* $\mathbf{L}_{\Phi} = \mathbf{L}_{\Phi}(\Omega, \mu)$ is the set defined as follows\n
$$
\mathbf{L}_{\Phi} := \left\{ f \in \mathbf{L}_0 : \int_{\Omega} \Phi(f/a) \, d\mu < \infty \text{ for some } a > 0 \right\},
$$
\n\nsupped with the norm\n
$$
\|f\|_{\mathbf{L}_{\Phi}} := \inf \left\{ a > 0 : \int_{\Omega} \Phi(|f|/a) \, d\mu \leq 1 \right\}, \ f \in \mathbf{L}_0,
$$

Ψ′

$$
\mathbf{L}_{\Phi} := \left\{ f \in \mathbf{L}_0 \colon \int_{\Omega} \Phi(f/a) \, d\mu < \infty \text{ for some } a > 0 \right\}
$$
\nthe norm

\n
$$
\|f\|_{\mathbf{L}_{\Phi}} := \inf \left\{ a > 0 \colon \int_{\Omega} \Phi(|f|/a) \, d\mu \le 1 \right\}, \ f \in \mathbf{L}_0 \ ,
$$
\n
$$
\infty.
$$

 $||f||_{\mathbf{L}_{\Phi}} := i$
where $\inf \emptyset := \infty$.
Notice that this "sland also $\mathbf{L}_1 \cap \mathbf{L}_{\infty}$, $\mathbf{L}_1 + i$:= inf { $a > 0$: $\int_{\Omega} \Phi(|f|/a) d\mu \le 1$

s "slightly generalized" definition in

1 + **L**_∞ as the smallest and largest

nd also [35], [36]). where $\lim_{\nu \to \infty} \infty$.

Notice that

and also $L_1 \cap L_{\infty}$

2, [10], Ch. 2, §2. Notice that this "slightly generalized" definition includes the spaces \mathbf{L}_1 , \mathbf{L}_{∞}
lso $\mathbf{L}_1 \cap \mathbf{L}_{\infty}$, $\mathbf{L}_1 + \mathbf{L}_{\infty}$ as the smallest and largest Orlicz spaces (see [19], Ch.
9, Ch. 2, §2.1 and al and also $\mathbf{L}_1 \cap \mathbf{L}_{\infty}$, $\mathbf{L}_1 + \mathbf{L}_{\infty}$ as the smallest and largest Orlicz spaces (see [19], Ch. 2, [10], Ch. 2, [2.1] and also [35], [36]). 2, [10], Ch. 2, 32.1 and also [35], [36]).

We use also the heart
$$
\mathbf{H}_{\Phi} = \mathbf{H}_{\Phi}(\Omega, \mu)
$$
 of the Orlicz space \mathbf{L}_{Φ} , $\mathbf{H}_{\Phi} := \left\{ f \in \mathbf{L}_0 \colon \int_{\Omega} \Phi(|f|/a) \, d\mu < \infty \text{ for all } a > 0 \right\}$, which is a closed subspace of \mathbf{L}_{Φ} . If $\Phi(x) < +\infty$ for all $x > 0$, the

We use also the heart $\mathbf{H}_{\Phi} = \mathbf{H}_{\Phi}(\Omega, \mu)$ of the Orlicz space \mathbf{L}_{Φ} ,
 $\mathbf{H}_{\Phi} := \left\{ f \in \mathbf{L}_0 : \int_{\Omega} \Phi(|f|/a) d\mu < \infty \text{ for all } a > 0 \right\}$,

1 is a closed subspace of \mathbf{L}_{Φ} . If $\Phi(x) < +\infty$ for all $x > 0$,

i := { $f \in \mathbf{L}_0$: $\int_{\mathbf{\Omega}} \Phi(|f|/a) d\mu < \infty$ for all $a > 0$
subspace of \mathbf{L}_{Φ} . If $\Phi(x) < +\infty$ for all $x >$
ex closure $cl_{\mathbf{L}_{\Phi}}(\mathbf{L}_1 \cap \mathbf{L}_\infty)$ of \mathbf{L}_{Φ} . If $\mathbf{L}_{\Phi}(x) = +\infty$
cz function Φ we use f
أو:
[4 coincides with the closure $cl_{\mathbf{L}_{\Phi}}(\mathbf{L}_1 \cap \mathbf{L}_{\infty})$ of \mathbf{L}_{Φ} . If $\mathbf{L}_{\Phi}(x) = +\infty$ for some $x > 0$
then $\mathbf{H}_{\Phi} = \{0\}$.
For any Orlicz function Φ we use the function ξ_{Φ} defined by

$$
\xi_{\Phi}(x) = \Psi(\Phi'(x)) .
$$

coincides with the closure $cl_{\mathbf{L}_{\Phi}}(\mathbf{L}_1 \cap \mathbf{L}_{\infty})$ of \mathbf{L}_{Φ} . If $\mathbf{L}_{\Phi}(x) = +\infty$ for some $x > 0$
then $\mathbf{H}_{\Phi} = \{0\}$.
For any Orlicz function Φ we use the function ξ_{Φ} defined by
 $\xi_{\Phi}(x) = \Psi(\Phi$ then $\mathbf{H}_{\Phi} = \{0\}$.

For any O₁

If $y = \Phi'(x) < -xy \leq \Phi(x) + \Psi(x)$ For any Orlicz function Φ we use the function ξ_{Φ} defined by
 $\xi_{\Phi}(x) = \Psi(\Phi'(x))$.

= $\Phi'(x) < +\infty$, there is equality $xy = \Phi(x) + \Psi(y)$ in the Yo
 $\Phi(x) + \Psi(y)$, i.e.,
 $\xi_{\Phi}(x) = x\Phi'(x) - \Phi(x)$.

any important cases (but n $(x) = \Psi(\Phi'(x))$.

lity $xy = \Phi(x) +$
 $= x\Phi'(x) - \Phi(x)$
 \therefore always) ξ_{Φ} is anses: Does there e $xy \leq \Phi(x) + \Psi(y)$, i.e.,

$$
\xi_{\Phi}(x) = x\Phi'(x) - \Phi(x) .
$$

If $y = \Phi'$
 $xy \leq \Phi(x)$

In many

A c

that ξ_{Φ_H}
 $\xi_{\Phi_H} = \Phi$ (x) < +∞, there is equality $xy = \Phi(x) + \Psi(y)$ in the Young's inequality
 $\Phi(x) + \Psi(y)$, i.e.,
 $\xi_{\Phi}(x) = x\Phi'(x) - \Phi(x)$.

important cases (but not always) ξ_{Φ} is an Orlicz function.

onverse question thus arises: Does ther $\Phi(x) + \Psi(y)$, i.e.,
my important cas
A converse question
 $\tilde{\Phi}_H = \Phi$, for a given $\Phi = \Phi$ for $(x) = x\Phi'(x) - \Phi(x)$.
not always) ξ_{Φ} is an C
arises: Does there exis
z function Φ ? It is eas
 $= x \int_0^x \frac{\Phi'(u)}{u} du - \Phi$ In many important cases (but not always) ξ_{Φ} is an Orlicz function.

A converse question thus arises: Does there exist an Orlicz function

that $\xi_{\Phi_H} = \Phi$, for a given Orlicz function Φ ? It is easy to show, for
 A converse question thus arises: Does there exist an Orlicz function Ψ_H such
 $\xi_{\Phi_H} = \Phi$, for a given Orlicz function Φ ? It is easy to show, for instance, that
 $= \Phi$ for
 $\Phi_H(x) = x \int_0^x \frac{\Phi'(u)}{u} du - \Phi(x)$,

ded tha that $\xi_{\Phi_H} = \Phi$, for a given Orlicz function Φ ? It is easy to show, for instance, that
 $\xi_{\Phi_H} = \Phi$ for
 $\Phi_H(x) = x \int_0^x \frac{\Phi'(u)}{u} du - \Phi(x)$,

provided that the integral exists for some $x > 0$ small enough.
 Propositio $\xi_{\Phi_H} = \Phi$ for

$$
\Phi_H(x) = x \int_0^x \frac{\Phi'(u)}{u} du - \Phi(x) ,
$$

ral exists for some $x > 0$ small en
and Φ_H be two Orlicz functions s

$$
L_{\Phi})_H = L_{\Phi_H}
$$
 and $(H_{\Phi})_H = H_{\Phi_H}$
 $p_{L_{\phi}}$ and $q_{L_{\phi}}$ of Orlicz spaces c

ided tha
 position Proposition 6.1. Let

provided that the integral exists for some $x > 0$ small enough.
 Proposition 6.1. Let Φ and Φ_H be two Orlicz functions such t
 $(\mathbf{L}_{\Phi})_{\mathbf{H}} = \mathbf{L}_{\Phi_H}$ and $(\mathbf{H}_{\Phi})_{\mathbf{H}} = \mathbf{H}_{\Phi_H}$.

The Boyd indexes $p_{$ Φ and Φ_H be two Orlicz functions such that $\xi_{\Phi_H} = \Phi$. Then
 $(\mathbf{L}_{\Phi})_{\mathbf{H}} = \mathbf{L}_{\Phi_H}$ and $(\mathbf{H}_{\Phi})_{\mathbf{H}} = \mathbf{H}_{\Phi_H}$.

xes $p_{\mathbf{L}_{\phi}}$ and $q_{\mathbf{L}_{\phi}}$ of Orlicz spaces can be computed by the

ncide wit The Boyd indexes $p_{\mathbf{L}_{\phi}}$ and $q_{\mathbf{L}_{\phi}}$ of Orlicz spaces can be computed by the
ion Φ . They coincide with the dilation indexes of Φ ,
 $p_{\mathbf{L}_{\Phi}} = \lim_{x \to \infty} \frac{\log M_{\Phi}(x)}{\log x}$, where $M_{\Phi}(x) = \sup_{0 \le y \le \infty} \frac{\Phi$

$$
(\mathbf{L}_{\Phi})_{\mathbf{H}} = \mathbf{L}_{\Phi_{H}} \quad \text{and} \quad (\mathbf{H}_{\Phi})_{\mathbf{H}} = \mathbf{H}_{\Phi_{H}}.
$$

\nSoyd indexes $p_{\mathbf{L}_{\phi}}$ and $q_{\mathbf{L}_{\phi}}$ of Orlicz spaces can be comp
\nThey coincide with the dilation indexes of Φ ,
\n $p_{\mathbf{L}_{\Phi}} = \lim_{x \to \infty} \frac{\log M_{\Phi}(x)}{\log x}$, where $M_{\Phi}(x) = \sup_{0 < y < \infty} \frac{\Phi(xy)}{\Phi(y)}.$
\ncondition $(\mathbf{L}_{\Phi})_{\mathbf{H}} = \mathbf{L}_{\Phi}$, (which is equivalent to $p_{\mathbf{L}_{\Phi}} > 1$),
\n $p(\Phi) := \sup \left\{ p > 0 : \inf_{x > 0, y > 1} \frac{\Phi(xy)}{x^{p} \Phi(x)} > 0 \right\}.$
\nthe index $p(\Phi)$ was studied in [4] in the case $\mu(\Omega) < +\infty$,
\ntional Δ_2 -conditions.

 $p_{\mathbf{L}_{\Phi}} = \lim_{x \to \infty} \frac{\log M_{\Phi}(x)}{\log x}$, where $M_{\Phi}(x) = \lim_{0 < x \to 0}$
To check condition $(\mathbf{L}_{\Phi})_{\mathbf{H}} = \mathbf{L}_{\Phi}$, (which is equivalent to use $p(\Phi) := \sup \left\{ p > 0 : \inf_{x > 0} \frac{\Phi(xy)}{x^p \Phi(x)} \right\}$

$$
p(\Phi) := \sup \left\{ p > 0 \colon \inf_{x > 0, y > 1} \frac{\Phi(xy)}{x^p \Phi(x)} > 0 \right\}
$$
\n
$$
\text{Ex } p(\Phi) \text{ was studied in [4] in the case } \mu(\Omega)
$$
\n
$$
\text{A}_2\text{-conditions.}
$$
\n
$$
\text{Let } \Phi \text{ be an Orlicz function. Then}
$$
\n
$$
\iff p(\Phi) > 1 \iff (\mathbf{H}_\Phi)_\mathbf{H} = \mathbf{H}_\Phi.
$$
\n
$$
\text{A}_0 \iff \Phi(x) > 0 \text{ for all } x > 0.
$$

.

To check condition $(\mathbf{L}_{\Phi})_{\mathbf{H}} = \mathbf{L}_{\Phi}$, (which is equivalent to $p_{\mathbf{L}_{\Phi}} > 1$), one can also
use
 $p(\Phi) := \sup \left\{ p > 0 : \inf_{x > 0, y > 1} \frac{\Phi(xy)}{x^p \Phi(x)} > 0 \right\}.$
Note that the index $p(\Phi)$ was studied in [4] in the case Not
und
Pro $\frac{(\frac{\mathbf{a}}{\Phi(x)})}{\Phi(x)} > 0$
e case $\mu(\mathbf{a})$
 n
 \mathbf{H}_{Φ} . Note that the index $p(\Phi)$ was studied in [4] in the case $\mu(\Omega) < +\infty$, and in [11]
under additional Δ_2 -conditions.
Proposition 6.2. Let Φ be an Orlicz function. Then
1) $(\mathbf{L}_{\Phi})_{\mathbf{H}} = \mathbf{L}_{\Phi} \iff p(\Phi) > 1 \iff (\mathbf{$

Proposition 6.2. Let Φ be an Orlicz function. Then

-
- **Proposition 6.2.** Let Φ be an O:

1) $(\mathbf{L}_{\Phi})_{\mathbf{H}} = \mathbf{L}_{\Phi} \iff p(\Phi) >$

2) a) $\mathbf{L}_{\Phi} \subseteq \mathcal{R}_0 \iff \Phi(x)$

b) $\mathbf{H}_{\Phi} \subseteq \mathcal{R}_0$

These results together with

 Φ be an Orlicz function. Then
 $\Rightarrow p(\Phi) > 1 \iff (\mathbf{H}_{\Phi})_{\mathbf{H}} = 1$
 $\iff \Phi(x) > 0$ for all $x > 0$.

gether with Theorems 2.1 and

sir hearts. 1) $(\mathbf{L}_{\Phi})_{\mathbf{H}} = \mathbf{L}_{\Phi} \iff p(\Phi) > 1 \iff (\mathbf{H}_{\Phi})_{\mathbf{H}} = \mathbf{H}_{\Phi}$.

2) a) $\mathbf{L}_{\Phi} \subseteq \mathcal{R}_0 \iff \Phi(x) > 0$ for all $x > 0$.

b) $\mathbf{H}_{\Phi} \subseteq \mathcal{R}_0$

These results together with Theorems 2.1 and 2.4 s

rlicz spaces and the

- 2) a) $\mathbf{L}_{\Phi} \subseteq \mathcal{R}_0 \iff \Phi(x) > 0$ for all $x > 0$.

b) $\mathbf{H}_{\Phi} \subseteq \mathcal{R}_0$

These results together with Theorems 2.1 and

rlicz spaces and their hearts.

 The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is ord

and $f^*($ b) $\mathbf{H}_{\Phi} \subseteq \mathcal{R}_0$
hese results to
spaces and th
he sequence o
he sequence o The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in \mathbf{L}_{Φ} if $f \in \mathbf{L}_{\Phi_H}$
and $f^*(+\infty) = 0$.
The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in \mathbf{H}_{Φ} if $f \in \mathbf{H}_{\Phi_H}$. • The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in \mathbf{L}_{Φ} if $f \in \mathbf{L}_{\Phi_H}$
and $f^*(+\infty) = 0$.
• The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in \mathbf{H}_{Φ} if $f \in \mathbf{H}_{\Phi_H$ and $f^*(+\infty) = 0$. $(+\infty)=0.$ quence of a
	- The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in \mathbf{H}_{Φ} if $f \in \mathbf{H}_{\Phi_H}$.
- \mathbf{L}_{Φ} satisfies Order Ergodic Theorem ($\mathbf{L}_{\Phi} \in \mathcal{O}\mathcal{E}\mathcal{T}$) iff $p(\Phi) >$
for all $x > 0$.
 \mathbf{H}_{Φ} satisfies Order Ergodic Theorem ($\mathbf{L}_{\Phi} \in \mathcal{O}\mathcal{E}\mathcal{T}$) iff $p(\Phi) > 1$. • An Orlicz space \mathbf{L}_{Φ} satisfies Order Ergodic Theorem ($\mathbf{L}_{\Phi} \in \mathcal{OET}$) iff $p(\Phi) > 1$ and $\Phi(x) > 0$ for all $x > 0$. $(\mathbf{L}_{\Phi} \in \mathcal{O}\mathcal{E}\mathcal{T}) \text{ iff } p(\Phi) > 1.$
 $\mathbf{L}_{\Phi} \in \mathcal{O}\mathcal{E}\mathcal{T}) \text{ iff } p(\Phi) > 1.$

is defined by the follow-
- An Orlicz heart **H**_Φ satisfies Order Ergodic Theorem

Zygmund classes

 $\mathcal{Z}_r = \mathbf{L} \log^r \mathbf{L}$, $0 \leq r < +\infty$.

1 and $\Phi(x) > 0$ for all $x > 0$.
An Orlicz heart \mathbf{H}_{Φ} satisfies (
 uund classes
 $\mathbf{L} \log^{r} \mathbf{L}$, $0 \leq r < +\infty$.

This important class of Orlicz

pricz functions: $(\mathbf{L}_{\Phi} \in \mathcal{O} \mathcal{E} \mathcal{T})$ *iff* $p(\Phi) > 1$.
is defined by the follow-
 $\langle r \rangle$ + ∞ = **L** log^r **L**, 0 ≤ *r* < +∞.
This important class of
Orlicz functions:
 $\Phi_r(x) := \begin{cases} 0 \\ x \text{ lo} \end{cases}$
set This important class of Orlicz spaces $\mathcal{Z}_r = \mathcal{Z}_r(\Omega, \mu)$ is defined by the follow-
 $\Phi_r(x) := \begin{cases} 0 & , 0 \leq x \leq 1 \\ x \log^r x & , 1 < x < \infty \end{cases}$, $0 < r < +\infty$.

et
 $\mathcal{Z}_r := \mathbf{L}_{\Phi_r}$, $\mathcal{R}_r := \mathbf{H}_{\Phi_r}$, $0 < r < \infty$,

$$
\Phi_r(x) := \begin{cases}\n0 & , & 0 \le x \le 1 \\
x \log^r x & , & 1 < x < \infty\n\end{cases}, \quad 0 < r < +\infty.
$$
\nWe set

\n
$$
\mathcal{Z}_r := \mathbf{L}_{\Phi_r} \quad , \quad \mathcal{R}_r := \mathbf{H}_{\Phi_r} \quad , \quad 0 < r < \infty,
$$
\nand also

\n
$$
\mathcal{Z}_0 = \mathbf{L}_1 + \mathbf{L}_{\infty}, \text{ having the heart } \mathcal{R}_0.
$$
\nProposition 6.3. For all

\n
$$
0 \le r < +\infty:
$$
\n
$$
1) \quad (\mathcal{Z}_r)_{\mathbf{H}} = \mathcal{Z}_{r+1} \quad and \quad (\mathcal{R}_r)_{\mathbf{H}} = \mathcal{R}_{r+1}.
$$

$$
\mathcal{Z}_r := \mathbf{L}_{\Phi_r} , \quad \mathcal{R}_r := \mathbf{H}_{\Phi_r} , \quad 0 < r < \infty ,
$$
\n
$$
\mathbf{L}_{\infty}, \text{ having the heart } \mathcal{R}_0.
$$
\n
$$
r \text{ all } 0 \le r < +\infty:
$$
\n
$$
\text{and } (\mathcal{R}_r)_{\mathbf{H}} = \mathcal{R}_{r+1}.
$$
\n
$$
= cl_{\mathcal{Z}_r}(\mathbf{L}_1 \cap \mathbf{L}_{\infty}).
$$
\n
$$
\le r < +\infty:
$$

and als
 Propos
 $\begin{bmatrix} 1 \end{bmatrix}$ (
 $\begin{bmatrix} 2 \end{bmatrix}$ 2

Proposition 6.3. For all $0 \le r \le +\infty$.

- 1) $(\mathcal{Z}_r)_{\mathbf{H}} = \mathcal{Z}_{r+1}$ and $(\mathcal{R}_r)_{\mathbf{H}} = \mathcal{R}_{r+1}$.
2) $\mathcal{Z}_r \cap \mathcal{R}_0 = \mathcal{R}_r = cl_{\mathcal{Z}_r}(\mathbf{L}_1 \cap \mathbf{L}_{\infty})$.
Thus for all $0 \le r \le +\infty$: $0 \le r < +\infty$:
 l $(\mathcal{R}_r)_{\mathbf{H}} = \mathcal{R}$
 $z_r(\mathbf{L}_1 \cap \mathbf{L}_{\infty})$
 $< +\infty$:

erages $\{A_{n,T}\}$

rages $\{A_{n,T}\}$ 1) $(\mathcal{Z}_r)_{\mathbf{H}} = \mathcal{Z}_{r+1}$ and $(\mathcal{R}_r)_{\mathbf{H}} = \mathcal{R}_{r+1}$.

2) $\mathcal{Z}_r \cap \mathcal{R}_0 = \mathcal{R}_r = cl_{\mathcal{Z}_r}(\mathbf{L}_1 \cap \mathbf{L}_{\infty})$.

Thus for all $0 \le r < +\infty$:

• The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$

• The sequence
- 2) $\mathcal{Z}_r \cap \mathcal{R}_0 = \mathcal{R}_r = cl_{\mathcal{Z}_r}(\mathbf{L}_1 \cap \mathbf{L}_{\infty}).$

Thus for all $0 \le r < +\infty$:

 The sequence of averages $\{A_{n,T}f\}$

 The sequence of averages $\{A_{n,T}f\}$

prentz spaces

tt W be an increasing function on [(

- The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order bounded in \mathcal{Z}_r if $f \in \mathcal{Z}_{r+1}$.
- The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in \mathcal{Z}_r if $f \in \mathcal{R}_{r+1}$.

Lorentz spaces

Thus for all $0 \le r < +\infty$:
The sequence of averages
The sequence of averages
ntz spaces
W be an increasing function, $+\infty$), and $W(x) > 0$ for Let *W* be an increasing function on $[0, +\infty)$ such that: $W(0) = 0$, *W* is concave
on $(0, +\infty)$, and $W(x) > 0$ for some $x > 0$. Then *W* is absolutely continuous on
the open interval $(0, \infty)$ with the decreasing densit the open interval $(0, \infty)$ with the decreasing density function $W'(x)$, $x > 0$, while the open interval $(0, \infty)$ with the decreasing density function W'
 $W(0+)$ may be positive.

The Lorentz space $\mathbf{\Lambda}_W = \mathbf{\Lambda}_W(\mathbf{\Omega}, \mu)$ is defined as
 $\mathbf{\Lambda}_W := \{f \in \mathbf{L}_0 : ||f||_{\mathbf{\Lambda}_W} < +\infty\}$

with the norm
 $||f||_{\$ (x) , $x>0,$ while
 $(x)\,dx<\infty\;.$ $W(0+)$ may be positive.

The Lorentz space $\Lambda_W = \Lambda_W(\Omega, \mu)$ is defined as

$$
\mathbf{\Lambda}_W := \{ f \in \mathbf{L}_0 \colon \|f\|_{\mathbf{\Lambda}_W} < +\infty \}
$$

The Lorentz space
$$
\Lambda_W = \Lambda_W(\Omega, \mu)
$$
 is defined as
\n
$$
\Lambda_W := \{ f \in \mathbf{L}_0 : ||f||_{\mathbf{\Lambda}_W} < +\infty \}
$$
\nthe norm
\n
$$
||f||_{\mathbf{\Lambda}_W} := \int_0^\infty f^*(x) dW(x) = f^*(0)W(0+) + \int_0^\infty f^*(x) W'(x) dx < \infty,
$$
\nre $+\infty \cdot 0 = 0$ (see [21], Ch. II, §5.1, and also [24], Ch. 2, [25] and refer
\nein).

 $||f||_{\mathbf{\Lambda}_W} := \int_0^\infty f^*(x) dW(x) = f^*(0)W(0+) + \int_0^\infty f^*(x) W'(x) dx < \infty$,

where $+\infty \cdot 0 = 0$ (see [21], Ch. II, §5.1, and also [24], Ch. 2, [25] and refere

therein).

The Stieltjes integral $\int_0^\infty f^*(x) dW(x)$ has an atomic part where $+\infty \cdot 0 = 0$ (see [21], Ch. 11, §5.1, and also [24], Ch. 2, [25] and references
therein).
The Stieltjes integral $\int_{0}^{\infty} f^*(x) dW(x)$ has an atomic part $f^*(0) W(0+)$ in the
case $W(0+) > 0$. The Lorentz spaces are max The
case $W($
norm $\parallel \cdot$ By
 $\lim_{M \to \infty} W($ The Stieltjes integral \int_{0}^{∞}
 $W(0+) > 0$. The Loren
 $\|\cdot\|_{\mathbf{\Lambda}_W}$.

By this definition $\mathbf{\Lambda}_W$
 $W(x) < +\infty$. Whence
 ∞) < $+\infty$ hold. 0 $f^*(x) dW(x)$ has an atomic part $f^*(0) W(0+)$ in the
tz spaces are maximal r.i. spaces with respect to the
 $\subseteq \mathbf{L}_{\infty}$ if $W(0+) > 0$, and $\mathbf{\Lambda}_W \supseteq \mathbf{L}_{\infty}$ if $W(+\infty) :=$
 $\mathbf{\Lambda}_W = \mathbf{L}_{\infty}$ if both the conditions $W($

case $W(0+) > 0$. The Lorentz spaces are maximal r.i. spaces with respect to the
norm $\|\cdot\|_{\mathbf{\Lambda}_W}$.
By this definition $\mathbf{\Lambda}_W \subseteq \mathbf{L}_{\infty}$ if $W(0+) > 0$, and $\mathbf{\Lambda}_W \supseteq \mathbf{L}_{\infty}$ if $W(+\infty) :=$
 $\lim_{x \to \infty} W(x) < +\infty$. W norm $|| \cdot || \mathbf{A}_W \cdot$
By this or
 $\lim_{x \to \infty} W(x) <$
 $W(+\infty) < +\infty$
The Har
are easily com By this definition $\Lambda_W \subseteq \mathbf{L}_{\infty}$ if $W(0+) > 0$, and $\Lambda_W \supseteq \mathbf{L}_{\infty}$ if $W(+\infty) := W(x) < +\infty$. Whence $\Lambda_W = \mathbf{L}_{\infty}$ if both the conditions $W(0+) > 0$ and ∞) < $+\infty$ hold.
The Hardy core $(\Lambda_W)_H$ of the Lorentz spac lim_{x→∞} $W(x) < +\infty$. Whence $\Lambda_W = \mathbf{L}_{\infty}$ if both the conditions $W(0+) > 0$ and $W(+\infty) < +\infty$ hold.
The Hardy core $(\Lambda_W)_H$ of the Lorentz space Λ_W and its lower index p_{Λ_W} are easily computed by the weight functi $W(+\infty)<+\infty$ hold.

 $(+\infty) < +\infty$ hold.
The Hardy core
easily computed l The Hardy core (Λ_W) **H** of the Lorentz space Λ_W and its lower index p_{Λ_W} assily computed by the weight function W. are easily computed by the weight function W .
Proposition 6.4.

138 M. Muratov, J. Pashkova and B.-Z. Rubshtein
 Proposition 6.4.

1) Let $\Lambda_{W_H} = \Lambda_{W_H}(\Omega, \mu)$ be the Lorentz space, where its weight function W_H is 1) Let $\Lambda_{W_H} = \Lambda_{W_H}(\Omega, \mu)$ be the Lorentz space, where its weight function W_H is

uniquely defined by the conditions: $W_H(0) = W(0) = 0$, $W_H(0+) = W(0+)$

and
 $W'_H(x) = \int_x^{+\infty} \frac{W'(u)}{u} du < +\infty$, $x \in (0, +\infty)$.

Then $(\Lambda_W)_H = \$ uniquely defined by the conditions: $W_H(0) = W(0) = 0$, $W_H(0+) = W(0+)$ (0) = $W(0) = 0$, $W_H(0+) = W(0+)$
+ ∞ , $x \in (0, +\infty)$.
= $+\infty$ then $(\mathbf{\Lambda}_W)_\mathbf{H} = \{0\}$.)
here the dilation index $\beta_{\mathbf{W}}$ of W is and

′ () = ∫ +∞ +[∞] , [∈] (0, +∞) . ′ () <

Then $(\mathbf{\Lambda}_W)_{\mathbf{H}} = \mathbf{\Lambda}_{W_H}$. (If $\int_{1}^{+\infty} \frac{W'(x)}{x} dx = +\infty$ then $(\mathbf{\Lambda}_W)_{\mathbf{H}} = \{0\}$.)

The index $p_{\mathbf{\Lambda}_w}$ is equal to $(\beta_W)^{-1}$, where the dilation index β_W

defined by
 $\beta_W = \lim_{x \to \infty} \frac{\log M_W(x)}{\log x}$, wh

2) The index p_{Λ_w} is equal to $(\beta_W)^{-1}$, where the dilation index β_W of W is defined by
 $\beta_W = \lim_{x \to \infty} \frac{\log M_W(x)}{\log x}$, where $M_W(x) = \sup_{0 \le y \le \infty} \frac{W(xy)}{W(y)}$.

3) $\Lambda_W \subseteq \mathcal{R}_0$ iff $W(+\infty) := \lim_{x \to +\infty} W(x) = +\infty$. defined by

$$
\beta_W = \lim_{x \to \infty} \frac{\log M_W(x)}{\log x}, \text{ where } M_W(x) = \sup_{0 < y < \infty} \frac{W(xy)}{W(y)}.
$$
\n
$$
\exists R_0 \text{ iff } W(+\infty) := \lim_{x \to +\infty} W(x) = +\infty.
$$
\nfor any Lorentz space Λ_W ,

\nsequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in Λ_W ;

\n $\Gamma^*(+\infty) = 0.$

\nrentz space Λ_W satisfies the Order-Ferodic Theorem (Λ_W)

- 3) $\Lambda_W \subseteq \mathcal{R}_0$ *iff* $W(+\infty) := \lim_{x \to +\infty} W(x) = +\infty$.

Thus for any Lorentz space Λ_W ,

 The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order

and $f^*(+\infty) = 0$.

 A Lorentz space Λ_W satisfies the Order Ergc
 β Thus for any Lorentz space Λ_W ,
The sequence of averages $\{A_{n,T}f$
and $f^*(+\infty) = 0$.
A Lorentz space Λ_W satisfies th
 $\beta_W < 1$ and $W(+\infty) = +\infty$.
More general Lorentz spaces Λ_W • The sequence of averages $\{A_{n,T}f\}_{n=1}^{T}$
- The sequence of averages $\{A_{n,T}f\}_{n=1}^{\infty}$ is order convergent in Λ_W if $f \in \Lambda_{W_H}$
and $f^*(+\infty) = 0$.
A Lorentz space Λ_W satisfies the Order Ergodic Theorem $(\Lambda_W \in \mathcal{OET})$ if
 $\beta_W < 1$ and $W(+\infty) = +\infty$.
More gen and $f^*(+\infty) = 0$.

A Lorentz space .
 $\beta_W < 1$ and $W(+$

More general Lorentz . $\beta_W < 1$ and $W(+\infty) = +\infty$.

\n- A Lorentz space
$$
\Lambda_W
$$
 satisfies the Order Ergodic Theorem $(\Lambda_W \in \mathcal{O}\mathcal{ET})$ iff $\beta_W < 1$ and $W(+\infty) = +\infty$. More general Lorentz spaces $\Lambda_{W,q} = \Lambda_{W,q}(\Omega, \mu)$ are defined by
\n- \n
$$
\Lambda_{W,q} := \left\{ f \in \mathcal{S}_0 \colon \|f\|_{\mathbf{L}_{W,q}} := \left(\int_0^\infty (f^*(x))^q \, dW(x) \right)^{1/q} < \infty \right\}
$$
\n for $1 \leq q < \infty$, where $\Lambda_{W,1} = \Lambda_W$. The classical Lorentz spaces are defined as $\mathbf{L}_{p,q} := \Lambda_{W,q}$ with $W(x) = x^{q/p}$, and for $1 \leq q \leq p \leq \infty$,\n
$$
\mathbf{L}_{p,\infty} := \left\{ f \in \mathbf{L}_0 \colon \|f\|_{\mathbf{L}_{p,\infty}} := \sup \left(x^{1/p} f^*(x) \right) < \infty \right\}.
$$
\n
\n

for $1 \le q < \infty$, where $\Lambda_{W,1} = \Lambda_W$.
The classical Lorentz spaces a
and for $1 \le q \le p \le \infty$,
 $\mathbf{L}_{p,\infty} := \left\{ f \in \mathbf{L}_0 : ||f||_{\mathbf{L}_1} \right\}$
The cases when $q > p$ (i.e., W is no

The classical Lorentz spaces are defined as
$$
\mathbf{L}_{p,q} := \Lambda_{W,q}
$$
 with $W(x) = x^{q/p}$, for $1 \leq q \leq p \leq \infty$, $\mathbf{L}_{p,\infty} := \left\{ f \in \mathbf{L}_0 \colon \|f\|_{\mathbf{L}_{p,\infty}} := \sup_{0 < x < \infty} \left(x^{1/p} f^*(x) \right) < \infty \right\}$.

and for $1 \le q \le p \le \infty$,
 $\mathbf{L}_{p,\infty} := \left\{ f$

The cases when $q > p$ (

is not a norm there (see

Order Ergodic Th

 $:= \left\{ f \in \mathbf{L}_0 : ||f||_{\mathbf{L}_{p,\infty}} := \sup_{0 < x < p}$
 q > p (i.e., *W* is not concave) are (see [37], Ch. V, §3).

dic Theorems in the spaces *l* see of **Λ***w*. Roughly speaking, the vergence in **Λ***w*,*q*. (x)
nt a
 1
inc The cases when $q > p$ (i.e., W is not concave) are relevant as well, while $|| \cdot ||_{\mathbf{L}_{W,q}}$
is not a norm there (see [37], Ch. V, §3).
Order Ergodic Theorems in the spaces $\Lambda_{W,q}$ with $1 < q < +\infty$ are quite
similar to th is not a norm there (see [37], Ch. V, g3).

Order Ergodic Theorems in the sp

similar to the case of Λ_W . Roughly speal

on the order convergence in $\Lambda_{W,q}$.

7. Additional comments and rem Order Ergodic Theorems in the spaces $\Lambda_{W,q}$ with $1 < q < +\infty$ are quite similar to the case of Λ_W . Roughly speaking, the second index q has no influence on the order convergence in $\Lambda_{W,q}$. similar to the case of Λ_W . Roughly speaking, the second index q has no influence
on the order convergence in $\Lambda_{W,q}$.
7. Additional comments and remarks
Orlicz-Lorentz spaces

7. Additional comments and remarks

Orlicz-Lorentz spaces

7. Additional comments and remarks
Orlicz-Lorentz spaces
The situation becomes more intricate if we turn to general Orlicz-Lorentz spaces.

These r.i. spaces $\mathbf{\Lambda}_{W,\Phi} = \mathbf{\Lambda}_{W,\Phi}(\mathbf{\Omega},\mu)$, can be defined by

These r.i. spaces
$$
\Lambda_{W,\Phi} = \Lambda_{W,\Phi}(\Omega,\mu)
$$
, can be defined by
\n
$$
\Lambda_{W,\Phi} := \{ f \in \mathcal{S}_0(\Omega,\mu) : \mathcal{I}_{W,\Phi}(f/a) < \infty \text{ for some } a > 0 \},
$$

with the norm

$$
||f||_{\mathbf{\Lambda}_{W,\Phi}} := \inf \left\{ a > 0 \colon \mathcal{I}_{W,\Phi}(f/a) \le 1 \right\} ,
$$

where $\frac{1}{\sqrt{2}}$

$$
||f||_{\mathbf{\Lambda}_{W,\Phi}} := \inf \{ a > 0 : \mathcal{I}_{W,\Phi}(f/a) \le 1 \},
$$

$$
\mathcal{I}_{W,\Phi}(f) := \int_{0}^{\infty} \Phi(f^*(x)) dW(x), \ f \in \mathbf{L}_0 = \mathbf{L}_0(\mathbf{\Omega}, \mu).
$$

$$
\Phi \text{ and } W \text{ in this definition are not usually assumed to be convex}
$$

y wide class of r.i. spaces (including original Orlicz and Lorentz
een intensively studied for the last two décades. We can refer to [27],
[15] [16] [20] and to the references cited therein as well

The f

 $\Phi(f^*(x)) dW(x)$, $f \in \mathbf{L}_0 = \mathbf{L}_0(\Omega, \mu)$.
his definition are not usually assumed
f r.i. spaces (including original Orlic
studied for the last two décades. We can to the references cited therein as The functions Φ and W in this definition are not usually assumed to be convex
and concave.
This very wide class of r.i. spaces (including original Orlicz and Lorentz
spaces) has been intensively studied for the last This ve
spaces) has b
[28], [13], [14]
The concave E_H for the s
[14], [15], [23]

s) has been intensively studied for the last two décades. We can refer to [27], [13], [14], [15], [16], [20] and to the references cited therein as well.
The computation of Boyd indexes $p_{\mathbf{E}}$, $q_{\mathbf{E}}$ and the desc [28], [13], [14], [15], [16], [20] and to the references cited therein as well.
The computation of Boyd indexes $p_{\mathbf{E}}$, $q_{\mathbf{E}}$ and the description of Hardy core
 $\mathbf{E}_{\mathbf{H}}$ for the spaces $\mathbf{E} = \mathbf{L}_{\Phi,W}$ is The computation of Boyd indexes $p_{\mathbf{E}}$, $q_{\mathbf{E}}$ and the description of Hardy core $\mathbf{E}_{\mathbf{H}}$ for the spaces $\mathbf{E} = \mathbf{L}_{\Phi,W}$ is a hard partly open problem. (See, e.g., [6], [7], [14], [15], [23], [29].)
Any **E_H** for the spaces **E** = **L**_{Φ,*W*} is a hard partly open problem. (See, e.g., [6], [7], [14], [15], [23], [29].)

Any progress in this direction will imply new results on Problems 1 and 2.
 Order convergence of condi

Order convergence of conditional expectations

[14], [15], [23], [29].)
Any progress in this direction will imply new results on Problems 1 and 2.
Order convergence of conditional expectations
Order convergence results (similar to Theorems 2.1, 2.2, 2.4) can be prov

Let F denote the σ -algebra of all μ -measurable subsets of Ω and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq$
 $\mathcal{F}_3 \subseteq \cdots$ be an increasing sequence of σ -subalgebras \mathcal{F}_n of F. Since $\mu(\Omega) = +\infty$, sequences of conditional expectations $T_n = E^{\mathcal{F}_n}$.

Let *F* denote the *σ*-algebra of all *μ*-measurable subsets of **Ω** and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq$
 $\mathcal{F}_3 \subseteq \cdots$ be an increasing sequence of *σ*-subalgebras \mathcal{F}_n Let *F* denote the σ -algebra of all μ -measurable subsets of Ω and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be an increasing sequence of σ -subalgebras \mathcal{F}_n of *F*. Since $\mu(\Omega) = +\infty$, onditional expectation operators $T_n = E$ $\mathcal{F}_3 \subseteq \cdots$ be an increasing sequence of σ -subalgebras \mathcal{F}_n of \mathcal{F} . Since $\mu(\Omega) = +\infty$,
the conditional expectation operators $T_n = E^{\mathcal{F}_n}$ (with respect to \mathcal{F}_n) need not to
be well defined. Howeve \mathcal{F}_n is σ -finite.

the conditional expectation operators $T_n = E^{\mathcal{F}_n}$ (with respect to \mathcal{F}_n) need not to
be well defined. However, $T_n \in \mathcal{PAC}$ if the restriction $\mu|_{\mathcal{F}_n}$ of the measure μ on
 \mathcal{F}_n is σ -finite.
The be well defined. However, $T_n \in \mathcal{PAC}$ if the restriction $\mu|_{\mathcal{F}_n}$ of the measure μ on \mathcal{F}_n is σ -finite.
 Theorem 7.1. Let $T_n = E^{\mathcal{F}_n}$, $n \ge 1$, be the conditional expectations with respect to \mathcal is σ -finite.
 orem 7.1.
 \mathcal{F}_n and as
 uence of th
 $f \in \mathbf{E}_{\mathbf{H}} \cap \mathcal{F}$
 Convers **Theorem 7.1.** Let T_n = $E^{\mathcal{F}_n}$, $n \ge 1$, be the conditional expectations with respect
at $T_n \in \mathcal{PAC}$ for all n. Let **E** be an r.i. space. Then the
ional expectations $T_n = E^{\mathcal{F}_n} f$ is order convergent in **E** for
every $f \notin \mathbf{E}_{\mathbf{H$ to \mathcal{F}_n and assume that $T_n \in \mathcal{PAC}$ for all n. Let **E** be an r.i. space. Then the sequence of the conditional expectations $T_n = E^{\mathcal{F}_n} f$ is order convergent in **E** for $all f \in \mathbf{E_H} \cap \mathcal{R}_0$.

 E^{f_n} is order convergent in **E** for
there exists a sequence of conditional
ch that the sequence $\{T_n f\}_{n=1}^{\infty}$ is not
al 4.3 with the dominant function Conversely, for every $f \notin \mathbf{E}_{\mathbf{H}} \cap \mathcal{R}_{0}$ there exists a sequence of conditional expectations $T_n = E^{\mathcal{F}_n} \in \mathcal{PAC}$, $n \ge 1$, such that the sequence $\{T_n f\}_{n=1}^{\infty}$ is not
order convergent in **E**.
The proof is based on Lemmas 3.3 and 4.3 with the dominant function
 $g = Bf := \sup_{n \ge 1} E^{\mathcal{F}_n} f$.
Su order convergent in **E**.

$$
g = Bf := \sup_{n \ge 1} E^{\mathcal{F}_n} f.
$$

 $g = Bf := \sup_{n\geq 1} E^{\mathcal{F}_n} f$.
ble versions of Doob maximal inequality (adapted to the case of infinite
are used to this end. It should be mentioned that analogous estimate
 $\sup T_n f$ were obtained in [10], Ch. 3, in the case = $Bf := \sup_{n \ge 1}$
nal inequality
hould be me
n [10], Ch. 3,
 $\rightarrow \xi_{\Phi}$ was in sure) are used to this end. It should be mentioned that analogous estimates for $Bf = \sup_{n\geq 1} T_n f$ were obtained in [10], Ch. 3, in the case of Orlicz spaces $\mathbf{E} = \mathbf{L}_{\Phi}$.
The mentioned above map $\Phi \to \xi_{\Phi}$ was in $n \geq 1$
ment The mentioned above map $\Phi \to \xi_{\Phi}$ was introduced and treated therein to this end.

퓞퓔퓣 **on finite measure spaces**

140 M. Muratov, J. Pashkova and B.-Z. Rubshtein
 \mathcal{OET} on finite measure spaces

Main Order convergence results (Theorems 2.1–2.4) hold if $m(\Omega) < +\infty$. The

restriction that $f \in \mathcal{R}_0$ is evidently unnecessary in th

Main Order convergence results (Theorems 2.1–2.4) hold if $\mathbf{m}(\mathbf{\Omega}) < +\infty$. The restriction that $f \in \mathcal{R}_0$ is evidently unnecessary in this case.
In other words, order boundedness in **E** of the Cesáro sums $\{A_{n,T}f$ restriction that $f \in \mathcal{R}_0$ is evidently unnecessary in this case.
In other words, order boundedness in **E** of the Cesáro sur \mathcal{PAC} implies their order convergence in the r.i. space **E**. In p
implies **E** \in *OET*. In other words, order boundedness in **E** of the Cesáro sums { $A_{n,T} f$ }[∞]_{$n=1$}, $T \in$ implies their order convergence in the r.i. space **E**. In particular, **E** \in *DET* es **E** \in *OET*.
Dominated Ergodic Theorems i $n=1$ \mathcal{F} AC

implies their order convergence in the r.i. space **E**. In particular, $\mathbf{E} \in \mathcal{DET}$
s $\mathbf{E} \in \mathcal{OET}$.
Dominated Ergodic Theorems in r.i. spaces on finite measure spaces are stud
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tical (Mean) E implies $\mathbf{E} \in \mathcal{O} \mathcal{E} \mathcal{T}$.

Dominated E

ied in [4].
 Convergence in no

Statistical (Mean)

Cesáro sums { A.

Convergence in norm

Convergence in norm

Statistical (Mean) Ergodic Theorems $(\mathcal{S}\mathcal{E}\mathcal{T})$ deal with the norm convergence of

Cesáro sums $\{A_{n,T}f\}_{n=1}^{\infty}$, where T is a bounded linear operator on a Banach space.

The operator is ass $n=1$

Cesaro sums $\{A_{n,T}f\}_{n=1}^{\infty}$, where T is a bounded linear operator on a Banach space.
The operator is assumed to be Cesaro bounded (see, e.g., [22], Ch. 2 §2.1).
For rearrangement invariant Banach spaces a natural re The operator is assumed to be Cesáro bounded (see, e.g., [22], Ch. 2 §2.1).
For rearrangement invariant Banach spaces a natural related problem is:
When does the sequence $\{A_{n,T}\}_{n=1}^{\infty}$ converge in the norm $\|\cdot\|_{\math$, where *T* is a bounded linear operator on a Banach space.
to be Cesáro bounded (see, e.g., [22], Ch. 2 §2.1).
invariant Banach spaces a natural related problem is:
 $\{A_{n,T}\}_{n=1}^{\infty}$ converge in the norm $\|\cdot\|_{\mathbf{E}}$ The operator is assumed to be Cesaro bounded (see, e.g., [22], Ch. 2 §2.1).
For rearrangement invariant Banach spaces a natural related problem
When does the sequence $\{A_{n,T}\}_{n=1}^{\infty}$ converge in the norm $\|\cdot\|_{\mathbf{E}}$ 1 does the sequence $\{A_{n,T}\}_{n=1}^{\infty}$ converge in the norm $\|\cdot\|_{\mathbf{E}}$ of the r.i. space
r all $f \in \mathbf{E}$ and all $T \in \mathcal{PAC}$ ($\mathbf{E} \in \mathcal{SET}$)?
The following result can be proved in the case of finite measure ([**E**

The following result can be proved in the case of finite measure (138) and

When does the sequence $\{A_{n,T}\}\$ **E**, for all $f \in \mathbf{E}$ and all $T \in \mathcal{PA}$.
The following result can b [39]):
Theorem 7.2. Let $\mathbf{E} = \mathbf{E}(\mathbf{\Omega}, \mu)$ iff **E** has order continuous norm for all *f* ∈ **E** and all *T* ∈ *PAC* (**E** ∈ *SET*)?

The following result can be proved in the

9]):
 heorem 7.2. Let **E** = **E**(Ω, μ) be an r.i. space
 f **E** has order continuous norm, i.e.,
 $0 \le f$, ∈ **E The following result can be provided in the following result can be proved in the case of finite measure (f)** and $\mu(\Omega) < \infty$. Then $\mathbf{E} \in \mathcal{S}\mathcal{E}\mathcal{T}$
 $0 \le f_n \in \mathbf{E}$, $f_n \downarrow 0$, $||f_n||_{\mathbf{E}} \to 0$. \sum_{i}
 \sum_{i} **Theorem 7.2.** Let **E** iff **E** has order continuous norm, i.e.,

$$
0 \le f_n \in \mathbf{E} , f_n \downarrow 0 , \|f_n\|_{\mathbf{E}} \to 0 .
$$

 $\mathbf{E} = \mathbf{E}(\mathbf{\Omega}, \mu)$ be an r.i. space and $\mu(\mathbf{\Omega}) < \infty$. Then $\mathbf{E} \in \mathcal{S}\mathcal{E}\mathcal{T}$
innous norm, i.e.,
 $0 \le f_n \in \mathbf{E}$, $f_n \downarrow 0$, $||f_n||_{\mathbf{E}} \to 0$.
nnous norm property can be described in some different way
[24 $0 \le f_n \in \mathbf{E}$, $f_n \downarrow 0$, $||f_n||_{\mathbf{E}} \to 0$.
 uous norm property can be described

24], Ch. I).
 $\mathbf{E} = \mathbf{E}(\mathbf{\Omega}, \mu)$ be an r.i. space. Then the The *order continuous norm* property can be described in some different ways

21], Ch. II, §4, [24], Ch. I).
 osition 7.3. Let $\mathbf{E} = \mathbf{E}(\mathbf{\Omega}, \mu)$ be an r.i. space. Then the following conditions

quivalent:
 E ha

(see [21], Ch. 11, §4, [24], Ch. 1).
 Proposition 7.3. Let $\mathbf{E} = \mathbf{E}(\Omega)$, are equivalent:

1) **E** has an order continuous

2) **E** is minimal (i.e., $\mathbf{E} = c$ fundamental function of **E**. **Proposition 7.3.** Let $\mathbf{E} = \mathbf{E}(\Omega, \mu)$ be an r.i. space. Then the following conditions $= \mathbf{E}(\mathbf{\Omega}, \mu)$ be an r.i. space. Then the following conditions
ntinuous norm.
 $\mathbf{E} = cl_{\mathbf{E}}(\mathbf{L}_1 \cap \mathbf{L}_{\infty})$ and $\varphi_{\mathbf{E}}(0+) = 0$, where $\varphi_{\mathbf{E}}$ is the
ion of **E**.
associated (Köthe dual) space of **E** coi are equivalent:

-
- fundamental function of **E**.
- 1) **E** has an order continuous norm.

2) **E** is minimal (i.e., $\mathbf{E} = cl_{\mathbf{E}}(\mathbf{L}_1 \cap \text{fundamental function of } \mathbf{E})$.

3) $\mathbf{E}' = \mathbf{E}^*$, i.e., the associated (Köispace.

4) The standard r.i. space $\mathbf{E}(\mathbf{R}_+,\mathbf{m})$. 2) **E** is minimal (i.e., $\mathbf{E} = c l_{\mathbf{E}}(\mathbf{L}_1 \cap \mathbf{L}_{\infty})$) and $\varphi_{\mathbf{E}}(0+) = 0$, where $\varphi_{\mathbf{E}}$ is the fundamental function of **E**.
3) **E**' = **E**^{*}, i.e., the associated (Köthe dual) space of **E** coincides wit 3) $\mathbf{E}' = \mathbf{E}^*$, *i.e.*, the associated (Köthe dual) space of \mathbf{E} coincides with its dual space.
4) The standard r.i. space $\mathbf{E}(\mathbf{R}_+,\mathbf{m})$ corresponding to \mathbf{E} is separable.
It should be noted that t space.
-

4) The standard r.i. space $\mathbf{E}(\mathbf{R}_+,\mathbf{m})$ corresponding to \mathbf{E} is separable.

It should be noted that there is no connection between the condi
 $\mathcal{E}\mathcal{T}$ and $\mathbf{E} \in \mathcal{S}\mathcal{E}\mathcal{T}$. For instance, (provid It should be noted that there is no connection between the conditions $\mathbf{E} \in$
and $\mathbf{E} \in \mathcal{SET}$. For instance, (provided that $\mu(\Omega) < \infty$), one has:
 $\mathbf{L}_p \in \mathcal{OET}$ and $\mathbf{L}_p \in \mathcal{SET}$ for $1 < p < +\infty$.
 $\mathbf{L}_1 \$ σ

- $\mathbf{L}_n \in \mathcal{O} \mathcal{E} \mathcal{T}$ and $\mathbf{L}_n \in \mathcal{S} \mathcal{E} \mathcal{T}$ for $1 < p < +\infty$.
- $\mathbf{L}_1 \notin \mathcal{O} \mathcal{E} \mathcal{T}$ and $\mathbf{L}_1 \in \mathcal{S} \mathcal{E} \mathcal{T}$.
- $\mathbf{L}_{p,\infty} \in \mathcal{OET}$

and $\mathbf{E} \in \mathcal{S}\mathcal{E}\mathcal{T}$. For instance, (provided that $\mu(\mathbf{\Omega}) < \infty$), one has:
 $L_p \in \mathcal{O}\mathcal{E}\mathcal{T}$ and $\mathbf{L}_p \in \mathcal{S}\mathcal{E}\mathcal{T}$ for $1 < p < +\infty$.
 $L_1 \notin \mathcal{O}\mathcal{E}\mathcal{T}$ and $\mathbf{L}_1 \in \mathcal{S}\mathcal{E}\mathcal{T}$.
 $L_{p,\infty}$ and $\mathbf{L}_p \in \mathcal{S}\mathcal{E}\mathcal{T}$ for $1 < p < +\infty$.

and $\mathbf{L}_1 \in \mathcal{S}\mathcal{E}\mathcal{T}$.
 ∇ and $\mathbf{L}_{p,\infty} \notin \mathcal{S}\mathcal{E}\mathcal{T}$ for $1 < p < \mu(\Omega) = \infty$ the space \mathbf{L}_1 has an or
 \vdots $\mathbf{L}_p \in \mathcal{S}\mathcal{E}\mathcal{T}$ for $1 < p < \infty$, and $\mathbf{L}_1 \in \mathcal{S}\mathcal{E}\mathcal{T}$.
 \mathcal{T} and $\mathbf{L}_{p,\infty} \notin \mathcal{S}$
 $\mu(\Omega) = \infty$ the sp
 \vdots $\mathbf{L}_p \in \mathcal{S}\mathcal{E}\mathcal{T}$ for 1 and $\mathbf{L}_{p,\infty} \notin \mathcal{SET}$ for $1 < p < +\infty$.
 Ω) = ∞ the space \mathbf{L}_1 has an order c
 $\Omega_p \in \mathcal{SET}$ for $1 < p < \infty$, since the sp In the case $\mu(\Omega) = \infty$ the space \mathbf{L}_1 has an order continuous norm, however \mathcal{SET} . While $\mathbf{L}_p \in \mathcal{SET}$ for $1 < p < \infty$, since the spaces are reflexive. $\mathbf{L}_1 \notin \mathcal{SET}$. While $\mathbf{L}_p \in \mathcal{SET}$ for $1 < p < \infty$, since the spaces are reflexive.

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Local Approximation of Observables and Commutator Bounds

Bruno Nachtergaele, Volkher B. Scholz and Reinhard F. Werner

Abstract. We discuss conditional expectations that can be used as generalizations of the partial trace for quantum systems with an infinite-dimensional Hilbert space of states.

Mathematics Subject Classification (2010). Primary 46L10; Secondary 46L53. **Keywords.** Property P, quantum conditional expectation, support of observables, small commutator.

1. Introduction

We denote by $\mathcal{B}(\mathcal{H})$ the bounded linear operators on a complex Hilbert space \mathcal{H} ,
equipped with the operator norm, and for $A, B \in \mathcal{B}(\mathcal{H})$, $[A, B] = AB - BA$ is the
commutator of A and B . Let \mathcal{H}_1 and $\$ equipped with the operator norm, and for $A, B \in \mathcal{B}(\mathcal{H})$, $[A, B] = AB - BA$ is the
commutator of A and B. Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces and $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$
their tensor product. In this note we commutator of *A* and *B*. Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces and $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$
their tensor product. In this note we consider the following situation. Suppose
 $A \in \mathcal{B}(\mathcal{H})$ and $\epsilon \ge 0$ ar $A \in \mathcal{B}(\mathcal{H})$ and $\epsilon \geq 0$ are such that

$$
\left\| [A, \mathbb{1} \otimes B] \right\| \le \epsilon \|A\| \|B\| \quad \text{for all } B \in \mathcal{B}(\mathcal{H}_2). \tag{1.1}
$$

 $A \in \mathcal{B}(\mathcal{H})$ and $\epsilon \ge 0$ are such that
 $||[A, \mathbb{1} \otimes B]|| \le \epsilon ||A|| ||B||$ for all $B \in \mathcal{B}(\mathcal{H}_2)$. (1.1)

We will prove that there exists $A' \in \mathcal{B}(\mathcal{H}_1)$ such that $||A - A' \otimes \mathbb{1}|| \le \epsilon ||A||$. The

case $\epsilon = 0$ is [$A, \mathbb{I} \otimes B$] $|| \leq \epsilon ||A|| ||B||$ for all $B \in \mathcal{B}(\mathcal{H}_2)$. (1.1)
there exists $A' \in \mathcal{B}(\mathcal{H}_1)$ such that $||A - A' \otimes \mathbb{I}|| \leq \epsilon ||A||$. The
, since in that case we have $A \in (\mathbb{I} \otimes \mathcal{B}(\mathcal{H}_2))' = \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{$ We will prove that there exists $A' \in \mathcal{B}(\mathcal{H}_1)$ such that $||A - A' \otimes 1|| \le \epsilon ||A||$. The case $\epsilon = 0$ is trivial, since in that case we have $A \in (1 \otimes \mathcal{B}(\mathcal{H}_2))' = \mathcal{B}(\mathcal{H}_1) \otimes 1$, and therefore there exists $A' \in \$ therefore there exists $A' \in \mathcal{B}(\mathcal{H}_1)$ such that $A = A' \otimes \mathbb{1}$. If \mathcal{H}_2 is finite dimensional, the result is also well known. In that case one can take for A' the normalized partial trace of A : the result is also well known. In that case one can take for A' the normalized partial the result is also well known. In that case one can take for A' the normalized partial
trace of A :
 $A' = \frac{1}{\dim \mathcal{H}_2} \text{Tr}_{\mathcal{H}_2} A$.
To see that this choice for A' does the job, it suffices to note that
 $A' \otimes 1 = \$

$$
A' = \frac{1}{\dim \mathcal{H}_2} \text{Tr}_{\mathcal{H}_2} A.
$$

trace of A :
To see that
where dU

$$
A' = \frac{\dim \mathcal{H}_2}{\dim \mathcal{H}_2} \text{Tr}_{\mathcal{H}_2} A.
$$

we for A' does the job, it suffices to not

$$
A' \otimes \mathbb{1} = \int_{\mathcal{U}(\mathcal{H}_2)} dU \left(\mathbb{1} \otimes U^* \right) A (\mathbb{1} \otimes U),
$$
alized Haar measure on the unitary gr
in part by the National Science Foundation.

To see that this choice for A' does the job, it suffices to note that
 $A' \otimes \mathbb{1} = \int_{\mathcal{U}(\mathcal{H}_2)} dU (\mathbb{1} \otimes U^*) A (\mathbb{1} \otimes U),$

where dU is the normalized Haar measure on the unitary group, if

Based on work support H
by
IT $(1 \otimes U^*)A(1 \otimes U),$
e on the unitary group
l Science Foundation under where dU is the normalized Haar measure on the unitary group, $\mathcal{U}(\mathcal{H}_2)$, of \mathcal{H}_2 .
Based on work supported in part by the National Science Foundation under grant DMS-10095 and the European project COQUIT. Based on work supported in part by the National Science Foundation under grant DMS-1009502 and the European project COQUIT.

Then, by the assumption (1.1) one has
\n
$$
||A' \otimes 1 - A|| \le \int_{\mathcal{U}(\mathcal{H}_2)} dU || (\mathbb{1} \otimes U^*)[A, (\mathbb{1} \otimes U)] || \le \epsilon ||A||.
$$
\n(1.2)
\nOur direct motivation for extending this result to the general case, in which \mathcal{H}_2

 $||A' \otimes 1 - A|| \leq \int_{\mathcal{U}(\mathcal{H}_2)} dU || (1 \otimes$
direct motivation for extending this resoved to be infinite dimensional, stems
nson bounds [8] to obtaining local approximation in the model [1, 2, 0, $(1 \otimes U^*)[A, (1 \otimes U)] \mid \leq \epsilon ||A||.$ (1.2)

is result to the general case, in which \mathcal{H}_2

tems from the recent applications of Lieb-

approximations of time-evolved observables

2, 9, 10]. is allowed to be infinite dimensional, stems from the recent applications of Lieb-Robinson bounds $[8]$ to obtaining local approximations of time-evolved observables in quantum mechanics in the works $[1, 2, 9, 10]$.

2. The main lemma

The existence of an approximation $A' \in \mathcal{B}(\mathcal{H}_1)$ satisfying the same error bound in quantum mechanics in the works $[1, 2, 9, 10]$.
 2. The main lemma

The existence of an approximation $A' \in \mathcal{B}(\mathcal{H}_1)$ satisfying the same error bound $\epsilon ||A||$ as in (1.2) is shown in the following lemma. The lemm **2. The main lemma**
The existence of an approximation $A' \in \mathcal{B}(\mathcal{H}_1)$
 $\epsilon ||A||$ as in (1.2) is shown in the following lemma
finite-dimensional case, one can take A' to give The existence of an approximation $A' \in \mathcal{B}(\mathcal{H}_1)$ satisfying the same error bound $\epsilon \|A\|$ as in (1.2) is shown in the following lemma. The lemma shows that, as in the finite-dimensional case, one can take A' to gi $\frac{\epsilon \|A\|}{c}$ finite-dimensional case, one can take A' to given by a completely positive linear
map $\mathbb{E}: \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ which has the defining properties of a conditional
expectation.
Lemma 2.1. Let \math map $\mathbb{E}: \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ which has the defining properties of a conditional

map $\mathbb{E}: \mathcal{D}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{D}(\mathcal{H}_1)$ which has the defining properties of a conditional
expectation.
 Lemma 2.1. Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces. Then there is a completely positive

linear **Lemma 2.1.**

linear map I

1. For all

2. Whene **Lemma 2.1.** Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces. Then there is a completely positive linear map $\mathbb{E} : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$

-
- : $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1)$ with the following properties:
 $A \in \mathcal{B}(\mathcal{H}_1), \mathbb{E}(A \otimes 1) = A;$
 $\text{er } A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfies the commutator bound
 $|| [A, 1 \otimes B] || \leq \epsilon ||A|| ||B||$ for all $B \in \math$ 1. For all $A \in \mathcal{B}(\mathcal{H}_1)$, $\mathbb{E}(A \otimes \mathbb{1})$

2. Whenever $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$
 $|| [A, \mathbb{1} \otimes B] || \le$
 $\mathbb{E}(A) \in \mathcal{B}(\mathcal{H}_1)$ satisfies the ϵ
 $|| \mathbb{E}(A \otimes \mathcal{H}_1) ||$

3. For all $C, D \in \mathcal{B}(\mathcal{H}_1)$ $) = A;$
satisfie
 $\leq \epsilon \|A\|$
stimat
 $) \otimes 1 - \epsilon$ 2. Whenever $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfies the commutator bound
 $|| [A, \mathbb{1} \otimes B] || \leq \epsilon ||A|| ||B||$ for all $B \in \mathcal{B}(\mathcal{H}_2)$,
 $\mathbb{E}(A) \in \mathcal{B}(\mathcal{H}_1)$ satisfies the estimate
 $||\mathbb{E}(A) \otimes \mathbb{1} - A|| \leq \epsilon ||A||$;

3.

 $\left\| [A, \mathbb{1} \otimes B] \right\| \leq \epsilon \|A\| \|B\|$ for all $B \in \mathcal{B}$

 $\mathbb{E}(A) \in \mathcal{B}(\mathcal{H}_1)$ satisfies the estimate $\frac{1}{n}$

$$
\|\mathbb{E}(A)\otimes 1 - A\| \leq \epsilon \|A\|;
$$

 $||[A, 1 \otimes B]|| \le \epsilon ||A|| ||B||$ for all $B \in \mathcal{B}(\mathcal{H}_2)$,
 $\mathbb{E}(A) \in \mathcal{B}(\mathcal{H}_1)$ satisfies the estimate
 $||\mathbb{E}(A) \otimes 1 - A|| \le \epsilon ||A||;$

For all $C, D \in \mathcal{B}(\mathcal{H}_1)$ and $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we have
 $\mathbb{E}((C \otimes$

$$
\mathbb{E}((C \otimes \mathbb{1})A(D \otimes \mathbb{1})) = C\mathbb{E}(A)D
$$

 $(A) \in \mathcal{B}(\mathcal{H}_1)$ satisfies the estimate
 $\Vert \mathbb{E}(A) \otimes \mathbb{1} -$

or all $C, D \in \mathcal{B}(\mathcal{H}_1)$ and $A \in \mathcal{B}(\mathcal{H}_1)$
 $\mathbb{E}((C \otimes \mathbb{1})A(D \otimes$

For any finite-dimensional project

coup of unitary operators of the fo $(A) \otimes 1$
 $A \in \mathcal{B}$
 $\in \mathcal{B}$
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 $\in \mathcal{B}$;
;
r 3. For all $C, D \in \mathcal{B}(\mathcal{H}_1)$ and $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we have
 $\mathbb{E}((C \otimes \mathbb{1})A(D \otimes \mathbb{1})) = C\mathbb{E}(A)D$.

coof. For any finite-dimensional projection $P \in \mathcal{B}(\mathcal{H}_2)$ d

ct group of unitary operators of $((C \otimes \mathbb{1})A(D \otimes \mathbb{1})) = C \mathbb{E}(A)D.$

ensional projection $P \in \mathcal{B}(\mathcal{H}_2)$ d

rators of the form $U = (\mathbb{1} - P)$

he normalized Haar measure dU
 $(A) = \int_{\mathcal{U}(P)} dU \; (\mathbb{1} \otimes U^*)A(\mathbb{1} \otimes U)$ Proof. For any finite-dimensional projection $P \in \mathcal{B}(\mathcal{H}_2)$ denote by $\mathcal{U}(P)$ the com-
oup of unitary operators of the form $U = (1 - P) + PUP$, and by \mathbb{E}_P the
ng operator with the normalized Haar measure dU on $\mathcal{U}(P$ pact group of unitary operators of the form $U = (1 - P) + PUP$, and by \mathbb{E}_P the averaging operator with the normalized Haar measure dU on $\mathcal{U}(P)$:

$$
\mathbb{E}_P(A) = \int_{\mathcal{U}(P)} dU \ (1 \otimes U^*) A(1 \otimes U). \tag{2.1}
$$

pact group of unitary operators of the form $U = (\mathbb{1} - P) + PUP$, and by \mathbb{E}_P the
averaging operator with the normalized Haar measure dU on $\mathcal{U}(P)$:
 $\mathbb{E}_P(A) = \int_{\mathcal{U}(P)} dU (\mathbb{1} \otimes U^*) A(\mathbb{1} \otimes U)$. (2.1)
By the argu averaging operator with the normalized Haar measure dU on $\mathcal{U}(P)$:
 $\mathbb{E}_P(A) = \int_{\mathcal{U}(P)} dU (1 \otimes U^*) A (1 \otimes U)$.

By the argument given in the introduction we have $||A - \mathbb{E}_P(A)|| \le$
 $C, D \in \mathcal{B}(\mathcal{H}_1)$ and $A \in \mathcal{B}$ $(A) = \int_{\mathcal{U}(P)} dU (1 \otimes U^*) A(1 \otimes U).$ (2.1)

the introduction we have $||A - \mathbb{E}_P(A)|| \le \epsilon ||A||$ and for
 $(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we have $\mathbb{E}_P((C \otimes 1)A(D \otimes 1)) = C \mathbb{E}_P(A)D.$

ave $\mathcal{U}(P) \supset \mathcal{U}(Q)$, and hence
 $\mathbb{E}_P(A)] = 0$ ∞
 ∞
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 ∞
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 ∞ By the argument given in the introduction we have $||A - \mathbb{E}_P(A)|| \le \epsilon ||A||$ and for $C, D \in \mathcal{B}(\mathcal{H}_1)$ and $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we have $\mathbb{E}_P((C \otimes \mathbb{1})A(D \otimes \mathbb{1})) = C \mathbb{E}_P(A)D$.
Moreover, if $P \ge Q$, we have $\$ $C, D \in \mathcal{B}$

$$
[\mathbb{1} \otimes U, \mathbb{E}_P(A)] = 0 \quad \text{for } P \ge Q \text{ and } U \in \mathcal{U}(Q). \tag{2.2}
$$

(*H*₁) and $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we have $\mathbb{E}_P((C \otimes \mathbb{1})A(D \otimes \mathbb{1})) = C \mathbb{E}_P(A)D$.
 \therefore if $P \geq Q$, we have $\mathcal{U}(P) \supset \mathcal{U}(Q)$, and hence
 $[\mathbb{1} \otimes U, \mathbb{E}_P(A)] = 0$ for $P \geq Q$ and $U \in \mathcal{U}(Q)$. (2.2)
 Moreover, if $P \ge Q$, we have $\mathcal{U}(P) \supset \mathcal{U}(Q)$, and hence
 $[\mathbb{1} \otimes U, \mathbb{E}_P(A)] = 0$ for $P \ge Q$ and \mathcal{U}

Now let $(P(\alpha))_{\alpha \in I}$ be a universal subnet of the net of fin

over some directed index set I . Then since $I \neq I$
edd
thals (A)] = 0 for $P \ge Q$ and $U \in U(Q)$. (2.2)
ersal subnet of the net of finite-dimensional projections
et I. Then since $\|\mathbb{E}_{P(\alpha)}(A)\| \le \|A\|$, the universal
refore must be weak-*-convergent. We call the limit
is linear, com Now let $(P(\alpha))_{\alpha \in I}$ be a universal subnet of the net of finite-dimensional projections
over some directed index set *I*. Then since $\|\mathbb{E}_{P(\alpha)}(A)\| \le \|A\|$, the universal
subnet is bounded and therefore must be weak-*-c over some directed index set *I*. Then since $||E_{P(\alpha)}(A)|| \le ||A||$, the universal subnet is bounded and therefore must be weak-*-convergent. We call the limit $\mathbb{E}_{\infty}(A)$. Clearly then, \mathbb{E}_{∞} is linear, completely p $\mathbb{E}_{\infty}(A)$. Clearly then, \mathbb{E}_{∞} is linear, completely positive, leaves every operator $A \otimes \mathbb{1}$ fixed, and also satisfies the property 3 of the statement of the lemma, since it is defined as a weak limit of $\mathbb{E}_{\infty}(A)$. Clearly then, \mathbb{E}_{∞} is linear, completely positive, leaves every operator $A \otimes \mathbb{1}$ fixed, and it is defined as a weak limit of a map with these properties. Moreover, if A satisfies it is defined as a weak limit of a map with these properties. Moreover, if A satisfies

d each $\mathbb{E}_P(A)$ lies in the compact $(\epsilon ||A||)$ -ball around A and
ove that we can write $\mathbb{E}_{\infty}(A) = \mathbb{E}(A) \otimes \mathbb{1}$, i.e., that $[\mathbb{1} \otimes B \in \mathcal{B}(\mathcal{H}_2)$. By taking the limit of Eq. (2.2) over P along the
at this the commutator bound each $\mathbb{E}_P(A)$ lies in the compact $(\epsilon||A||)$ -ball around A and
so does the limit.
It remains to prove that we can write $\mathbb{E}_{\infty}(A) = \mathbb{E}(A) \otimes \mathbb{1}$, i.e., that $[\mathbb{1} \otimes B, \mathbb{E}_{\infty}(A)] = 0$ for It remains to prove that we can write $\mathbb{E}_{\infty}(A) = \mathbb{E}(A) \otimes \mathbb{1}$, i.e., that [I
 $B, \mathbb{E}_{\infty}(A) = 0$ for all $B \in \mathcal{B}(\mathcal{H}_2)$. By taking the limit of Eq. (2.2) over P along t

chosen net, we find that this is tru ⊗ chosen net, we find that this is true for any $B \in \mathcal{U}(Q)$ for any finite-dimensional Q . But these sets generate a weakly dense subalgebra of $\mathcal{B}(\mathcal{H}_2)$, which concludes the proof.

Note that the map E of Lemma Q. But these sets generate a weakly dense subalgebra of $\mathcal{B}(\mathcal{H}_2)$, which concludes

chosen net, we find that this is true for any $B \in \mathcal{U}(Q)$ for any finite-dimensional Q . But these sets generate a weakly dense subalgebra of $\mathcal{B}(\mathcal{H}_2)$, which concludes the proof.

Note that the map E of Lemma . But these sets generate a weakly dense subalgebra of $\mathcal{B}(\mathcal{H}_2)$, which concludes
in eproof.
Note that the map \mathbb{E} of Lemma 2.1 is completely positive and unit preserv-
g and therefore bounded (with $\|\mathbb{E}\| =$ the proof.

Note that the map E of Lemma 2.1 is completely positive and unit preserv-

ing and therefore bounded (with $||E|| = 1$) and hence norm-continuous. Norm-

continuity is however not always sufficient in applicati nd therefore bounded (with $||\mathbb{E}|| = 1$) and hence norm-continuous. Norm-
nuity is however not always sufficient in applications. It is sometimes impor-
that the map $A \mapsto A'$ is continuous with respect to a different, mo ing and therefore bounded (with $||\mathbb{D}|| = 1$) and hence norm-continuous. Norm-continuity is however not always sufficient in applications. It is sometimes important that the map $A \mapsto A'$ is continuous with respect to a d tant that the map $A \mapsto A'$ is continuous with respect to a different, more suitable
topology. In [1], e.g., the local approximations appear in an integral and continu-
ity is relied on to insure the integrability of the i Ity is relied on to insure the integrability of the integrand. Since in Lemma 2.1
 A' is obtained as a weak cluster point, its continuity properties are not obvious.

Therefore, we consider other maps with the propertie Therefore, we consider other maps with the properties of a conditional expectation, namely 1 and 3 of Lemma 2.1, but with a slightly worse approximation property (to be precise, with the ϵ in property 2 of Lemma 2.1 re namely 1 and 3 of Lemma 2.1, but with a slightly worse approximation property
(to be precise, with the ϵ in property 2 of Lemma 2.1 replaced by 2 ϵ) and which
is continuous with respect to the weak (and σ -weak) op

(to be precise, with the ϵ in property 2 of Lemma 2.1 replaced by 2 ϵ) and which
is continuous with respect to the weak (and σ -weak) operator topology.
Proposition 2.2. Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spa is continuous with respect to the weak (and σ -weak) operator topology.
 Proposition 2.2. Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces and let ρ be a norma
 $\mathcal{B}(\mathcal{H}_2)$. Define the map $\mathbb{E}_{\rho} = id \otimes \rho$ by **Proposition 2.2.** Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces and let ρ be a normal state on $\mathcal{B}(\mathcal{H}_2)$. Define the map $\mathbb{E}_{\rho} = id \otimes \rho$ by $\mathbb{E}_{\rho}(A \otimes B) = \rho(B)A$ for all $A \in \mathcal{B}(\mathcal{H}_1)$
and $B \in \mathcal{B}(\mathcal{H}_2)$. Then, \mathbb{E}_{ρ} has the properties 1 and 3 of Lemma 2.1 and, whenever
 $A \in \mathcal{B}(\mathcal{H}_1$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Then, \mathbb{E}_{ρ} has the properties 1 and 3 of Lemma 2.1 and, whenever
 $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfies the commutator bound
 $||(A, \mathbb{1} \otimes B)|| \le \epsilon ||A|| ||B||$ for all $B \in \mathcal{B}(\mathcal{H}_2)$.

we h $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfies the commutator bound
 $||[A, \mathbb{1} \otimes B]|| \le \epsilon ||A|| ||B||$ for we have
 $||\mathbb{E}_{\rho}(A) - A|| \le 2\epsilon|$
 Proof. By Lemma 2.1 we have

 $\|[A, \mathbb{1} \otimes B] \| \leq \epsilon$

we have

$$
\|\mathbb{E}_{\rho}(A) - A\| \le 2\epsilon \|A\|.\tag{2.3}
$$

Proof.

$$
||[A, \mathbb{1} \otimes B]|| \le \epsilon ||A|| ||B|| \text{ for all } B \in \mathcal{B}(\mathcal{H}_2).
$$

we

$$
||\mathbb{E}_{\rho}(A) - A|| \le 2\epsilon ||A||. \tag{2.3}
$$

$$
\therefore \text{ By Lemma 2.1 we have}
$$

$$
||\mathbb{E}(A) - \mathbb{E}_{\rho}(A)|| = ||\mathbb{E}_{\rho} (E(A) \otimes \mathbb{1} - A) || \le ||\mathbb{E}(A) \otimes \mathbb{1} - A|| \le \epsilon ||A||.
$$

efore, it follows that

$$
|\mathbb{E}_{\rho}(A) \otimes \mathbb{1} - A|| \le ||(\mathbb{E}(A) - \mathbb{E}_{\rho}(A)) \otimes \mathbb{1}|| + ||\mathbb{E}(A) \otimes \mathbb{1} - A|| \le 2\epsilon ||A||. \square
$$

It is unclear whether the factor 2 in equation (2.3) is really needed. Numerical

$$
\|\mathbb{E}_{\rho}(A)\otimes 1 - A\| \le \|(\mathbb{E}(A) - \mathbb{E}_{\rho}(A))\otimes 1\| + \|\mathbb{E}(A)\otimes 1 - A\| \le 2\epsilon \|A\|.
$$

 $|(A) - \mathbb{E}_{\rho}(A)|| = ||\mathbb{E}_{\rho}(\mathbb{E}(A) \otimes \mathbb{1} - A)|| \le ||\mathbb{E}(A) \otimes \mathbb{1}||$

e, it follows that
 $(A) \otimes \mathbb{1} - A|| \le ||(\mathbb{E}(A) - \mathbb{E}_{\rho}(A)) \otimes \mathbb{1}|| + ||\mathbb{E}(A) \otimes \mathbb{1}||$

sunclear whether the factor 2 in equation (2.3) is reall

s $\|\mathbb{E}_{\rho}(A) \otimes 1 - A\| \leq$
It is unclear whether
evidence suggests that ma
By an approximation arg
We tried low-dimensiona $(A) \otimes 1 - A \| \leq \| (\mathbb{E}(A) - \mathbb{E}_{\rho}(A)) \otimes 1 \| + \| \mathbb{E}(A) \otimes 1 - A \| \leq 2\epsilon \|A\|.$

sunclear whether the factor 2 in equation (2.3) is really needed. Numerical

suggests that maybe it is even true with the same bound as in the Lemma nce suggests that maybe it is even true with the same bound as in the Lemma.

In approximation argument it would suffice to show this in finite dimension.

Fied low-dimensional (21 × 21) random matrices A, choosing for \r By an approximation argument it would suffice to show this in finite dimension.
We tried low-dimensional (21×21) random matrices A, choosing for ρ the state
farthest removed from the tracial state, namely a pure on We tried low-dimensional (21×21) random matrices A, choosing for ρ the state farthest removed from the tracial state, namely a pure one. The random matrices were drawn from the unitarily invariant ensemble. Then $\$ We tried low-dimensional (21 × 21) random matrices A , choosing for ρ the state farthest removed from the tracial state, namely a pure one. The random matrices were drawn from the unitarily invariant ensemble. Then $\$ were drawn from the unitarily invariant ensemble. Then $\delta = ||\mathbb{E}_{\rho}(A) - A||$ is
readily computed, and in all cases we found unitary operators $U \in \mathcal{B}(\mathcal{H}_2)$ such
that $||A - (1 \otimes U^*)A(1 \otimes U)|| \ge \delta$. We are, of course aware readily computed, and in all cases we found unitary operators $U \in \mathcal{B}(\mathcal{H}_2)$ such that $||A - (\mathbb{1} \otimes U^*)A(\mathbb{1} \otimes U)|| \ge \delta$. We are, of course aware, that this is far from conclusive, since by measure concentration rand that $||A - (1 \otimes U^*)A(1 \otimes U)|| \ge \delta$. We are, of course aware, that this is far from conclusive, since by measure concentration random matrices in high dimension might easily avoid the regions of counterexample with high probab $\otimes U^*$) $A(1 \otimes U)$ || \geq might easily avoid the regions of counterexample with high probability. That is, 146 B. Nachtergaele, V.B. Scholz and R.F. Werner
for most of the cases with respect to the unitarily invariant measure the factor 2
is not needed, but counterexamples nevertheless might exist.

3. Application to infinite systems

for most of the cases with respect to the cases with respect to the form
 3. Application to infinite systems

So far, we have discussed two-component systems with a Hilbert space of the form
 $\mathcal{H}_1 \otimes \mathcal{H}_2$. In app **3. Application to infinite systems**
So far, we have discussed two-component systems with a Hilb
 $\mathcal{H}_1 \otimes \mathcal{H}_2$. In applications the decomposition into two compone
to selecting a finite subsystem of an infinite syste $\mathcal{H}_1 \otimes \mathcal{H}_2$. In applications the decomposition into two components often corresponds to selecting a finite subsystem of an infinite system [1].

 $\mathcal{H}_1 \otimes \mathcal{H}_2$. In applications the decomposition into two components often corresponds
to selecting a finite subsystem of an infinite system [1].
Consider a collection of systems labeled by a countable set Γ (e.g ing a finite subsystem of an infinite system [1].

nsider a collection of systems labeled by a countable set Γ (e.g., Γ is often

b be the d-dimensional hypercubic lattice \mathbb{Z}^d .) Associated with each site

her Consider a collection of systems labeled by a countaken to be the *d*-dimensional hypercubic lattice \mathbb{Z}^d .)
 $x \in \Gamma$, there is a quantum system with a Hilbert space

define
 $\mathcal{H}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ and $\mathcal{A$ Consider a consider a column appearance and \mathbb{Z}^d .) Associated with each site

Consider a countable system with a Hilbert space \mathcal{H}_x . For finite $\Lambda \subset \Gamma$, we
 $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ and $\mathcal{A}_\Lambda = \bigotimes_{x \in$ taken to be the *d*-dimensional hypercubic lattice \mathbb{Z}^d .) Associated with each site $x \in \Gamma$, there is a quantum system with a Hilbert space \mathcal{H}_x . For finite $\Lambda \subset \Gamma$, we define
 $\mathcal{H}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ a $x \in$

$$
\mathcal{H}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{H}_x \quad \text{and} \quad \mathcal{A}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x) \tag{3.1}
$$

F, there is a quantum system with a Hilbert space $π_x$. For limite Λ ⊂ Γ, we

let $\mathcal{H}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ and $\mathcal{A}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x)$ (3.1)
 Λ are $\mathcal{B}(\mathcal{H}_x)$ denotes the bounded linear oper $\mathcal{H}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ and $\mathcal{A}_{\Lambda} = \bigotimes_{x \in \Lambda}$
where $\mathcal{B}(\mathcal{H}_x)$ denotes the bounded linear operators combe identified in the natural way with $\mathcal{A}_{\Lambda_0} \otimes \mathbb{1}_{\Lambda \setminus \Lambda}$
 $\mathcal{A}_{\text{loc}} = \bigcup_{\Lambda \subset \Gamma} \mathcal{A}_{$ where $\mathcal{B}(\mathcal{H}_x)$ denotes the bounded linear operators on \mathcal{H}_x . For $\Lambda_0 \subset \Lambda \subset \Gamma$, \mathcal{A}_{Λ_0}
can be identified in the natural way with $\mathcal{A}_{\Lambda_0} \otimes \mathbb{1}_{\Lambda \setminus \Lambda_0} \subset \mathcal{A}_{\Lambda}$. One then defines
 \mathcal{A}_{\text

$$
\mathcal{A}_{\rm loc} = \bigcup_{\Lambda \subset \Gamma} \mathcal{A}_{\Lambda} \tag{3.2}
$$

where $\mathcal{B}(\mathcal{H}_x)$ denotes the bounded linear operators on \mathcal{H}_x . For $\Lambda_0 \subset \Lambda \subset \Gamma$, \mathcal{A}_{Λ_0}
can be identified in the natural way with $\mathcal{A}_{\Lambda_0} \otimes \mathbb{I}_{\Lambda \setminus \Lambda_0} \subset \mathcal{A}_{\Lambda}$. One then defines
 \mathcal{A}_{\mathrm can be identified in the natural way with $A_{\Lambda_0} \otimes \mathbb{I}_{\Lambda \setminus \Lambda_0} \subset A_{\Lambda}$. One then defines
 $A_{\text{loc}} = \bigcup_{\Lambda \subset \Gamma} A_{\Lambda}$ (3

as an inductive limit taken over the net of all finite subsets of Γ . The completion
 A_{\text \mathcal{A}_{loc}

= $\bigcup_{\Lambda \subset \Gamma} A_{\Lambda}$ (3.2)
t of all finite subsets of Γ . The completion of
is a C^* -algebra, which we will denote by A_{Γ} .
now allows us to define a family of maps
 $A_{\Gamma} \to A_{\Lambda}$, in a way compatible with the
e., A_{loc} with respect to the operator norm is a *C*^{*}-algebra, which we will denote by *A*_Γ.

The strategy of Proposition 2.2 now allows us to define a family of maps
 E_A , for finite $Λ ⊂ Γ$, such that $E_A : A_\Gamma \to A_\$ $^{\perp\!\!\!\perp\!\!\!\perp}$

$$
\mathbb{E}_{\Lambda_0} = \mathbb{E}_{\Lambda_0} \circ \mathbb{E}_{\Lambda}, \text{ if } \Lambda_0 \subset \Lambda. \tag{3.3}
$$

for finite $\Lambda \subset \Gamma$, such that $\mathbb{E}_{\Lambda} : \mathcal{A}_{\Gamma} \to \mathcal{A}_{\Lambda}$, in a way compatible with the
ddings $\mathcal{A}_{\Lambda_0} \subset \mathcal{A}_{\Lambda}$, for $\Lambda_0 \subset \Lambda$, i.e., such that
 $\mathbb{E}_{\Lambda_0} = \mathbb{E}_{\Lambda_0} \circ \mathbb{E}_{\Lambda}$, if $\Lambda_0 \subset \Lambda$. (3.3)
vil beddings $A_{\Lambda_0} \subset A_{\Lambda}$, for $\Lambda_0 \subset \Lambda$, i.e., such that
 $\mathbb{E}_{\Lambda_0} = \mathbb{E}_{\Lambda_0} \circ \mathbb{E}_{\Lambda}$, if $\Lambda_0 \subset \Lambda$. (3.3)

will therefore choose a family of normal states on $\mathcal{B}(\mathcal{H}_x)$, or equivalently, a

nily of dens E_{Λ₀} ⊂ *A*_Λ, for *Λ*₀ ⊂ *Λ*, i.e., such that
 $\mathbb{E}_{\Lambda_0} = \mathbb{E}_{\Lambda_0} \circ \mathbb{E}_{\Lambda}$, if Λ_0

We will therefore choose a family of normal state

family of density matrices, $(\rho_x)_{x \in \Gamma}$ and let ρ_{Γ} be th
 = $\mathbb{E}_{\Lambda_0} \circ \mathbb{E}_{\Lambda_1}$ if $\Lambda_0 \subset \Lambda$. (3.3)

ily of normal states on $\mathcal{B}(\mathcal{H}_x)$, or equivalently, a
 $\mathcal{E}_{\epsilon \in \Gamma}$ and let ρ_{Γ} be the corresponding a product state

denote the restriction of ρ_{Γ} to We will therefore choose a family of normal states on $\mathcal{B}(\mathcal{H}_x)$, or equivalently, a
family of density matrices, $(\rho_x)_{x \in \Gamma}$ and let ρ_{Γ} be the corresponding a product state
on \mathcal{A}_{Γ} . For each $\Lambda \subset \Gamma$,

$$
\mathbb{E}_{\Lambda} = \mathrm{id}_{\mathcal{A}_{\Lambda}} \otimes \rho_{\Lambda^c} \,. \tag{3.4}
$$

family of density matrices, $(\rho_x)_{x \in \Gamma}$ and let ρ_{Γ} be the corresponding a product state
on \mathcal{A}_{Γ} . For each $\Lambda \subset \Gamma$, let ρ_{Λ^c} denote the restriction of ρ_{Γ} to $\mathcal{A}_{\Gamma \setminus \Lambda}$. On \mathcal{A}_{loc} , on \mathcal{A}_{Γ} . For each $\Lambda \subset \Gamma$, let ρ_{Λ^c} denote the restriction of ρ_{Γ} to $\mathcal{A}_{\Gamma \backslash \Lambda}$. On \mathcal{A}_{loc} , \mathbb{E}_{Λ}
is then defined by setting
 $\mathbb{E}_{\Lambda} = id_{\mathcal{A}_{\Lambda}} \otimes \rho_{\Lambda^c}$. (3.4)
and it is straightf $\mathbb{E}_{\Lambda} = id_{\mathcal{A}_{\Lambda}} \otimes \rho_{\Lambda^c}$. (3.4)
and it is straightforward to see that the \mathbb{E}_{Λ} defined in this way satisfy the compat-
ibility property (3.3). All these maps are contractions and extend uniquely to $\mathcal{A$ and it is straightforward to see that the E_A defined in this way satisfy the compatibility property (3.3). All these maps are contractions and extend uniquely to A_{Γ} by continuous extension, with preservation of the ibility property (3.3). All these maps are contractions and extend uniquely to A_{Γ}
by continuous extension, with preservation of the compatibility property. Clearly,
 \mathbb{E}_{Λ} can be considered as a map $A_{\Gamma} \to A_{\Gamma$ \mathbb{E}_{Λ} can be considered as a map $\mathcal{A}_{\Gamma} \to \mathcal{A}_{\Gamma}$ with $\text{ran} \mathbb{E}_{\Lambda} = \mathcal{A}_{\Lambda} \subset \mathcal{A}_{\Gamma}$. Note that the maps \mathbb{E}_{Λ} depend on the choice of normal states ρ_x . Since the properties we are interested \mathbb{E}_{Λ} can be considered as a map $\mathcal{A}_{\Gamma} \to \mathcal{A}_{\Gamma}$ with ran $\mathbb{E}_{\Lambda} = \mathcal{A}_{\Lambda} \subset \mathcal{A}_{\Gamma}$. Note that the can be considered as a map $A_{\Gamma} \to A_{\Gamma}$ with rank $A = A_{\Lambda} \subset A_{\Gamma}$. Note that the
is \mathbb{E}_{Λ} depend on the choice of normal states ρ_x . Since the properties we are
rested in here do not explicitly depend on this c maps \mathbb{E}_{Λ} depend on the choice of normal states ρ_x . Since the properties we are
interested in here do not explicitly depend on this choice, we suppress it in the
notation. The following property is a direct cons interested in the following property is a direct consequence of the construction of the \mathbb{E}_{Λ} and Proposition 2.2.
 Corollary 3.1. Let $\Lambda \subset \Gamma$ be finite. Suppose $\epsilon \ge 0$ and $A \in \mathcal{A}_{\Gamma}$ are such that
 $\| [A, \$ \mathbb{E}_{Λ} and Proposition 2.2.
 Corollary 3.1. Let $\Lambda \subset \Gamma$ be finite. Suppose $\epsilon \geq 0$ and $A \in \mathcal{A}_{\Gamma}$ are such that
 $\left|\left|[A, \mathbb{1} \otimes B]\right|\right| \leq \epsilon \|A\| \|B\|$ for all $B \in \mathcal{A}_{\Gamma \setminus \Lambda}$.

Then, with \mathbb{E}_{Λ} the \mathbb{E}_{Λ} and Proposition 2.2.

bollary 3.1. Let $\Lambda \subset \Pi$
 $|| [A, \Pi]$
n, with \mathbb{E}_{Λ} the map **Corollary 3.1.** Let $\Lambda \subset \Gamma$ be finite. Suppose $\epsilon \ge 0$ and $A \in \mathcal{A}_{\Gamma}$ are such that
 $||[A, \mathbb{1} \otimes B]|| \le \epsilon ||A|| ||B||$ for all $B \in \mathcal{A}_{\Gamma \setminus \Lambda}$.

Then, with \mathbb{E}_{Λ} the map defined in (3.4), we have $\mathbb{E}_{\Lambda}(A) \$

$$
\left| \left[A, \mathbb{1} \otimes B \right] \right| \le \epsilon \|A\| \|B\| \text{ for all } B \in \mathcal{A}_{\Gamma \setminus \Lambda}.
$$

$$
\left\| \mathbb{E}_{\Lambda}(A) - A \right\| \le 2\epsilon \|A\|.
$$

$$
\left\| \mathbb{E}_{\Lambda}(A) - A \right\| \le 2\epsilon \|A\|.
$$

Then, with \mathbb{E}_{Λ} the map defined in (3.4), we have $\mathbb{E}_{\Lambda}(A) \in \mathcal{A}_{\Lambda}$ and
 $\|\mathbb{E}_{\Lambda}(A) - A\| \leq 2\epsilon \|A\|$.

$$
\text{med in (3.4), we have } \mathbb{E}_{\Lambda}(A) \in \mathcal{A}_{\Lambda} \text{ and}
$$
\n
$$
\|\mathbb{E}_{\Lambda}(A) - A\| \le 2\epsilon \|A\|.
$$
\n
$$
(3.5)
$$

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We remark that if dim $\mathcal{H}_x < \infty$, for all $x \in \Gamma$, i.e., when \mathcal{A}_{Γ} is a UHF
algebra, we can take the normalized partial trace (maximally mixed state) for We remark that if dim $\mathcal{H}_x < \infty$, for all $x \in \Gamma$, i.e., when \mathcal{A}_{Γ} is a UHF ra, we can take the normalized partial trace (maximally mixed state) for of the ρ_x and replace 2ϵ by ϵ by the argument given i each of the ρ_x and replace 2ϵ by ϵ by the argument given in the introduction.
In any case, it is easy to construct representations of \mathcal{A}_{Γ} in which the maps \mathbb{E}_{Λ} are represented by weakly continuous In any case, it is easy to construct representations of A_{Γ} in which the maps \mathbb{E}_{Λ} are represented by weakly continuous maps. Again, it is an interesting question whether the replacement of the 'error' ϵ by are represented by weakly continuous maps. Again, it is an interesting question whether the replacement of the 'error' ϵ by 2ϵ is really necessary in order to be able to treat the situation with infinite-dimensional

whether the replacement of the 'error' ϵ by 2ϵ is really necessary in order to be able to treat the situation with infinite-dimensional component systems.
In the next section we discuss the relation of our construct In the next section we discuss the relation of our construction of
tional expectations E , with the property P introduced by Schwartz alr
years ago [12].
4. Extension to general von Neumann algebras tional expectations \mathbb{E} , with the property P introduced by Schwartz almost fifty
years ago [12].
4. Extension to general von Neumann algebras
The ideas in Lemma 2.1 can be extended to the wider setting of von Neu

4. Extension to general von Neumann algebras

The ideas in Lemma 2.1 can be extended to the wider setting of von Neumann 4. Extension
The ideas in 1
algebras, when
 M on the Hil algebras, when we replace the algebra $\mathcal{B}(\mathcal{H}_2)$ by a general von Neumann algebra \mathcal{M} on the Hilbert space \mathcal{H} , which replaces $\mathcal{H}_1 \otimes \mathcal{H}_2$. As usual, \mathcal{M}' denotes the commutant of \mathcal{M} , i.e algebras, when we replace the algebra $B(\mathcal{H}_2)$ by a general von Neumann algebra M on the Hilbert space \mathcal{H} , which replaces $\mathcal{H}_1 \otimes \mathcal{H}_2$. As usual, M' denotes the commutant of M , i.e., the von Neumann ℳ

commutant of M , i.e., the von Neumann algebra of bounded operators commuting
with M .
Some of the following equivalences are known deep results. Our addition is the
last item. Let us mention that while some implication with *M*.
Some of the following equivalences are known deep results. Our addition is the
last item. Let us mention that while some implications in the following proposition
are only valid in the case of *H* being separabl with \mathcal{M} .
Son
last item
are only
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Proposit tem. Let us mention that while some implications in the following proposition
nly valid in the case of $\mathcal H$ being separable, the others do not depend on this
nption. This will be made clear in the proof.
osition 4.1. are only valid in the case of $\mathcal H$ being separable, the others do not depend on this
assumption. This will be made clear in the proof.
Proposition 4.1. Let $M \subset \mathcal B(\mathcal H)$ be a von Neumann algebra with trivial center.

Proposition 4.1. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neum
Then the following properties are equivalent:
1. M is hyperfinite, i.e., the weak closure of
algebras all sharing the same identity.
2. M has property P [11], i.e., f **Proposition 4.1.** Let $M \subset B$ Then the following properties are equivalent:

- are only valid in the case of *H* being separable, the others do not depend on this
assumption. This will be made clear in the proof.
Proposition 4.1. Let $M \subset B(H)$ be a von Neumann algebra with trivial center.
Then the algebras all sharing the same identity.
	- 1. M is hyperfinite, i.e., the weak closure of an increasing family of matrix
algebras all sharing the same identity.
2. M has property P [11], i.e., for every $X \in \mathcal{B}(\mathcal{H})$ the weak*-closed convex hull
of $\{U^*XU \$ of $\{U^*XU \mid U \in \mathcal{M} \text{ unitary}\}\)$ contains an element of \mathcal{M}' .
	- 1. M is hyperfinite, i.e., the weak closure of an increasing family of matrix
algebras all sharing the same identity.
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of $\{U^*XU \$ 2. M has property P [11], i.e., for every $X \in \mathcal{B}(\mathcal{H})$ the weak*-closed convex hull
of $\{U^*XU \mid U \in \mathcal{M}$ unitary} contains an element of \mathcal{M}' .
3. M' is injective, i.e., there is a linear map $\mathbb{E} : \mathcal{B}(\math$ 3. M' is injective, i.e., there is a linear map $\mathbb{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$ such that
 $\|\mathbb{E}(X)\| \le \|X\|$, and $\mathbb{E}(A) = A$ for $A \in \mathcal{M}'$.

	4. There is a linear map $\mathbb{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$ such that for all $X \in$ $\Vert E(X) \Vert \leq \Vert X \Vert$, and $E(A) = A$ for $A \in \mathcal{M}'$.
	-

$$
\|\mathbb{E}(X)\| \le \|X\|, \text{ and } \mathbb{E}(A) = A \text{ for } A \in \mathcal{M}'.
$$

4. There is a linear map $\mathbb{E}: \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$ such that for all $X \in \mathcal{B}(\mathcal{H})$

$$
\|\mathbb{E}(X) - X\| \le \sup \{ \| [X, U] \| \mid U \in \mathcal{M} \text{ unitary} \}.
$$

Furthermore we have that \mathbb{E} is completely positive with norm 1 an

$$
\mathbb{E}(AXB) = A\mathbb{E}(X)B \text{ for } A, B \in \mathbb{E}(\mathcal{M}).
$$

coof. The next three implications also apply to the case where \mathcal{H} is not

Furthermore we have that E is completely positive with norm 1 and fulfills

$$
\mathbb{E}(AXB) = A\mathbb{E}(X)B \text{ for } A, B \in \mathbb{E}(\mathcal{M}).
$$

Proof.

 $|(X) - X|| \le \sup \{ ||[X, U]||$
 $we \text{ have that } \mathbb{E} \text{ is complete}$
 $\mathbb{E}(AXB) = A\mathbb{E}(X)B \text{ for}$
 $\text{reemplications also apply}$
 $\text{follows by an easy applica}$
 $\text{or} \text{otherwise } M$ and each of the 1 and fulfills
not separabl
M is the weak
y has proper
section of tl $(AXB) = A \mathbb{E}(X)B$ for $A, B \in \mathbb{E}(\mathcal{M})$.
implications also apply to the case when
llows by an easy application of the factoras \mathcal{M}_{α} , and each of these algebras ob
property P since its commutant is the also [11], C The next three impircations also apply to the case where π is not separable.

) implies (2) follows by an easy application of the fact that M is the weak

of matrix algebras M_{α} , and each of these algebras obviou (1) implies (2) follows by an easy application of the fact that *M* is the weak
re of matrix algebras M_{α} , and each of these algebras obviously has property
at then *M* has property P since its commutant is the inters closure of matrix algebras M_{α} , and each of these algebras obviously has property
P. But then M has property P since its commutant is the intersection of the
commutants M'_{α} , see also [11], Corollary 4.4.17.
That commutants \mathcal{M}'_{α} , see also [11], Corollary 4.4.17.

F. But then \mathcal{M} has property P since its commutant is the intersection of the commutants \mathcal{M}'_{α} , see also [11], Corollary 4.4.17.
That (1) implies the first equation in (4) is proven along the lines of the proo commutants \mathcal{M}'_c
That (1) is
proof of Lemma plies the first equation in (4)
2.1, using again the fact that That (1) implies the first equation in (1) is proven along the lines of the lines of the veak closure of matrix of Lemma 2.1, using again the fact that M is the weak closure of matrix proof of Lemma 2.1, using again the fact that *M* is the weak closure of matrix 148 B. Nachtergaele, V.B. Scholz and R.F. Werner
algebras M_{α} , for which the bound is immediate. But if we choose $X \in M'$ we
find that $||E(X) - X|| = 0$. The second identity then follows from the fact that
if $E : \mathcal{N} \to \math$ algebras M_{α} , for which the bound is immediate. But if we choose $X \in \mathcal{M}'$ we find that $\|\mathbb{E}(X) - X\| = 0$. The second identity then follows from the fact that if $\mathbb{E}: \mathcal{N} \to \mathcal{N}$ is a projection on a von Neum find that $||\mathbb{E}(X) - X|| = 0$. The second identity then follows from the fact that if $\mathbb{E}: \mathcal{N} \to \mathcal{N}$ is a projection on a von Neumann algebra such that the range $\mathbb{E}(\mathcal{N})$ is a von Neumann subalgebra containing $E(N)$ is a von Neumann subalgebra containing the identity, then E has to be completely positive with norm 1, and satisfies the identity $E(AXB) = A E(N)B$ for $A, B \in E(N)$ [5, 14]. The implication (4) to (3) follows immediately f $\mathbb{E}(\mathcal{N})$ is a von Neumann subalgebra containing the identity, then \mathbb{E} has to be $E(V)$ is a von Neumann subalgebra containing the identity, then E has to be completely positive with norm 1, and satisfies the identity $E(AXB) = AE(X)B$ for $A, B \in E(\mathcal{N})$ [5, 14]. The implication (4) to (3) follows immediate

for $A, B \in \mathbb{E}(\mathcal{N})$ [5, 14]. The implication (4) to (3) follows immediately from the last argument.
The last two implications do require a separable Hilbert space \mathcal{H} .
The equivalence of the notions of hyperfinit The last
The equi
result by Connective if and
injective if and
also implies (1)
The missi The last two implications do require a separable Hilbert space Λ .
The equivalence of the notions of hyperfiniteness and injectivity
t by Connes [4] (see [7] for a simpler proof). It is easily seen
ive if and only if $\$ t by Connes [4] (see [7] for a simpler proof). It is easily seen that M is ive if and only if M' is injective, see [13], Proposition XV.3.2. Hence, (3) mplies (1).
The missing implication, *i.e.*, (2) implies (3), was injective if and only if \mathcal{M}' is injective, see [13], Proposition XV.3.2. Hence, (3) also implies (1).
The missing implication, *i.e.*, (2) implies (3), was proven by Schwartz, in the same paper where he also defined

injective if and only if M' is injective, see [13], Proposition XV.3.2. Hence, (3) also implies (1).

The missing implication, *i.e.*, (2) implies (3), was proven by Schwartz, in the same paper where he also defined prope

The missing implication, *i.e.*, (2) implies (3), was proven by Schwartz, in the
same paper where he also defined property P [12].
In the above situation we have that, for $A \in \mathcal{M}$ and $B \in \mathcal{M}'$, we get $B\mathbb{E}(A) = \math$ same paper where he also defined property P [12].

In the above situation we have that, for $A \in \mathcal{M}$ and $B \in \mathcal{M}'$, we get $B\mathbb{E}(A) = \mathbb{E}(BA) = \mathbb{E}(AB) = \mathbb{E}(A)B$, i.e., $\mathbb{E}(A) \in \mathcal{M}' \cap \mathcal{M}'' = \mathbb{C}1$. Hence the In the above situation we have that, for $A \in \mathcal{M}$ and $B \in \mathcal{M}'$, we get $B\mathbb{E}(A) =$
 $A) = \mathbb{E}(AB) = \mathbb{E}(A)B$, i.e., $\mathbb{E}(A) \in \mathcal{M}' \cap \mathcal{M}'' = \mathbb{C}1$. Hence there is a state
 h that $\mathbb{E}(A) = \rho(A)$, and thus
 $\$ ρ such that $\mathbb{E}(A) = \rho(A)$, and thus

$$
\mathbb{E}(AB) = \rho(A)B \quad \text{for } A \in \mathcal{M}, \ B \in \mathcal{M}'.
$$
 (4.1)

 $(BA) = \mathbb{E}(AB) = \mathbb{E}(A)B$, i.e., $\mathbb{E}(A) \in \mathcal{M}' \cap \mathcal{M}'' = \mathbb{C}\mathbb{1}$. Hence there is a state
such that $\mathbb{E}(A) = \rho(A)$, and thus
 $\mathbb{E}(AB) = \rho(A)B$ for $A \in \mathcal{M}$, $B \in \mathcal{M}'$. (4.1)
nce the linear hull of the set of el such that $\mathbb{E}(A) = \rho(A)$, and thus
 $\mathbb{E}(AB) = \rho(A)$
nce the linear hull of the set of ϵ
em that via this formula the stat
ccause \mathbb{E} need not be normal (i.e $(AB) = \rho(A)B$ for $A \in \mathcal{M}$, $B \in \mathcal{M}'$. (4.1)

f the set of elements AB is weak^{*}-dense in $\mathcal{B}(\mathcal{H})$ it would

mula the state ρ determines \mathbb{E} . However, that is deceptive,

e normal (i.e., weak^{*}-continu Since the linear hull of the set of elements AB is weak^{*}-dense in $\mathcal{B}(\mathcal{H})$ it would
seem that via this formula the state ρ determines \mathbb{E} . However, that is deceptive,
because \mathbb{E} need not be normal (seem that via this formula the state ρ determines $\mathbb E$. However, that is deceptive,
because $\mathbb E$ need not be normal (i.e., weak*-continuous). Indeed, the *only* case in
which $\mathbb E$ is normal, is the case described because E need not be normal (i.e., weak*-continuous). Indeed, the *only* case in
which E is normal, is the case described in Proposition 2.2. The state ρ is then
obviously also normal. That $\mathcal{M}' = E(\mathcal{B}(\mathcal{H}))$ because E need not be normal (i.e., weak*-continuous). Indeed, the *only* case in which E is normal, is the case described in Proposition 2.2. The state ρ is then
obviously also normal. That $\mathcal{M}' = \mathbb{E}(\mathcal{B}(\mathcal{H})))$ must be type one follows from a
general result of Tomiyama that the von Neuman obviously also normal. That $\mathcal{M}' = \mathbb{E}(\mathcal{B}(\mathcal{H}))$ must be type one follows from a general result of Tomiyama that the von Neumann type (I, II, or III) cannot increase under normal conditional expectations (see also increase under normal conditional expectations (see also [6, Example 1.1] and [15, Theorem IV.2.2]). Note also that by evaluating with a normal state σ we can obtain product states $AB \mapsto \rho(A)\sigma(\mathbb{E}(B))$ between $\mathcal M$ an

Theorem IV.2.2]). Note also that by evaluating with a normal state σ we can obtain product states $AB \mapsto \rho(A)\sigma(E(B))$ between M and M' when E is normal, such product states could also be made normal, which also entail product states $AB \mapsto \rho(A)\sigma(\mathbb{E}(B))$ between M and M' when E is normal, such
product states could also be made normal, which also entails that M is type I [3].
It follows from this discussion that one can, in general, not u product states could also be made normal, which also entails that *M* is type I [3].
It follows from this discussion that one can, in general, not use (4.1) to define
E with a normal state ρ taking the place of the par It follows from the place of the partial trace: the map E densely ed by (4.1) cannot have a continuous extension to $\mathcal{B}(\mathcal{H})$, except in the type e. with a normal state ρ taking the place of the partial trace: the map E densely
fined by (4.1) cannot have a continuous extension to $\mathcal{B}(\mathcal{H})$, except in the type
ase.
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ne authors gratefully acknowled

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defined by (4.1) cannot have a continuous extension to $\mathcal{D}(\mathcal{H})$, except in the type
I case.
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The an
sphere
obtain sphere of Institut Mittag-Leffler, where the results reported in this paper were
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The Spectra of Selfadjoint Extensions of Entire Operators with Deficiency Indices (1,1)

Luis O. Silva and Julio H. Toloza

Abstract. We give necessary and sufficient conditions for real sequences to be the spectra of selfadjoint extensions of an entire operator whose domain may be non-dense. For this spectral characterization we use de Branges space techniques and a generalization of Krein's functional model for simple, regular, closed, symmetric operators with deficiency indices (1,1). This is an extension of our previous work in which similar results were obtained for densely defined operators.

Mathematics Subject Classification (2010). 46E22, 47A25, 47B25, 47N99. **Keywords.** Symmetric operators, entire operators, de Branges spaces, spectral analysis.

1. Introduction

of entire operators given in [18]. This generalization is realized by extending the notion of entire operators to a subclass of symmetric operators with deficiency indices $(1,1)$ that may have non-dense domain. The spect notion of entire operators to a subclass of symmetric operators with deficiency
indices $(1,1)$ that may have non-dense domain. The spectral characterization of
a given operator in the class is based on the distribution o indices $(1, 1)$ that may have non-dense domain. The spectral characterization of
a given operator in the class is based on the distribution of the spectra of its
selfadjoint extensions within the Hilbert space. More conc indices $(1, 1)$ that may have non-dense domain. The spectral characterization of a given operator in the class is based on the distribution of the spectra of its selfadjoint extensions within the Hilbert space. More conc selfadjoint extensions within the Hilbert space. More concretely, for a given simple, regular, closed symmetric (possibly not densely defined) operator with deficiency indices (1, 1) to be entire it is necessary and suffic regular, closed symmetric (possibly not densely defined) operator with deficiency
indices $(1, 1)$ to be entire it is necessary and sufficient that the spectra of two of its
selfadjoint extensions satisfy conditions which indices $(1, 1)$ to be entire it is necessary and sufficient that the spectra of two of its selfadjoint extensions satisfy conditions which reduce to the convergence of certain series (the precise statement is Proposition selfadjoint extensions satisfy conditions which reduce to the convergence of certain
series (the precise statement is Proposition 5.2).
The class of entire operators was concocted by M.G. Krein as a tool for

series (the precise statement is Proposition 5.2).
The class of entire operators was concocted by M.G. Krein as a tool for
treating in a unified way several classical problems in analysis [10, 11, 12, 14]. The
L.O. Silva w The class of entire operators was concoct
treating in a unified way several classical problem
L.O. Silva was partially supported by CONACYT (Méxi
J.H. Toloza was partially supported by CONICET (Arg
01741. ing in a unified way several classical problems in analysis $[10, 11, 12, 14]$. The
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Noloza was partially supported by CONICET (Argentina) t L.O. Silva was partially supported by CONACYT (México) through grant CB-2008-01-99100.
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L.O. Silva and J.H. Toloza
entire operators form a subclass of the closed, densely defined, symmetric, regular
operators with equal deficiency indices. They have many remarkable properties as
is accounted for in the review operators with equal deficiency indices. They have many remarkable properties as
is accounted for in the review book [7]. Krein's definition of entire operators hinges
on his functional model for symmetric operators and it is accounted for in the review book [7]. Krein's definition of entire operators hinges
on his functional model for symmetric operators and it requires the existence of
an element of the Hilbert space with very peculiar pro is accounting to the Hilbert space with very peculiar properties. As first discussed
in [18] it is possible to determine whether an operator is entire by conditions that
rely exclusively on the distribution of the spectra an element of the Hilbert space with very peculiar properties. As first discussed
in [18] it is possible to determine whether an operator is entire by conditions that
rely exclusively on the distribution of the spectra of in [18] it is possible to determine whether an operator is entire by conditions that rely exclusively on the distribution of the spectra of selfadjoint extensions of the operator.
Although Krein's original work considers

rely exclusively on the distribution of the spectra of selfadjoint extensions of the operator.
Although Krein's original work considers only densely defined symmetric operators, it is clear that the definition of entire op operator. Although Krein's original work considers only densely defined symmetric operators, it is clear that the definition of entire operators can be extended to the case of not necessarily dense domain with no formal ch Alth
erators, it
of not nee
non-dens
machiner
One erators, it is clear that the definition of entire operators can be extended to the case
of not necessarily dense domain with no formal changes (see Definition 2.5). Since
non-densely and densely defined symmetric operator

non-densely and densely defined symmetric operators share certain properties, the machinery developed in $[18]$ carries over with some mild modifications.
One ingredient of our discussion is an extension of the functional machinery developed in [18] carries over with some mild modifications.
One ingredient of our discussion is an extension of the functional model devel-
oped in [18]. This functional model associates a de Branges space to e One ingredient of our discussion is an extension of the functional m
oped in [18]. This functional model associates a de Branges space to eve
regular, closed symmetric operator with deficiency indices (1,1). It is
marking oped in [18]. This functional model associates a de Branges space to every simple, regular, closed symmetric operator with deficiency indices (1,1). It is worth remarking that functional models for this and for related cla marking that functional models for this and for related classes of operators have
been implemented before; see for instance [5, 20]. However, the functional model
proposed in [18] has shown to be particularly suitable for

proposed in [18] has shown to be particularly suitable for us. Here we deem appropriate to mention [16] for a related kind of results.
This paper is organized as follows. In Section 2 we recall some of the properties held propriate to mention [16] for a related kind of results.
This paper is organized as follows. In Section 2 we recall some of the prop-
erties held by operators that are closed, simple, symmetric with deficiency indices
(1,1 erties held by operators that are closed, simple, symmetric with denciency indices (1, 1); the notion of entire operator is also introduced here. Section 3 provides a short review on the theory of de Branges Hilbert spaces relevant to this work, in particular, a slightly modified version of a theorem due to Woracek (Proposition 3.1). In Section 4 we introduce a functional model for any operator of the class under consideration so that the mo Woracek (Proposition 3.1). In Section 4 we introduce a functional model for any operator of the class under consideration so that the model space is always a de Branges space. Finally, in Section 5 we single out the class operator of the class under consideration so that the model space is always a de
Branges space. Finally, in Section 5 we single out the class of de Branges spaces
corresponding to entire operators and provide necessary and Branges space. Finally, in Section 5 we single out the class of de Branges spaces corresponding to entire operators and provide necessary and sufficient conditions on the spectra of two selfadjoint extensions of an entire corresponding to entire operators and provide necessary and sufficient conditions
on the spectra of two selfadjoint extensions of an entire operator.
2. On symmetric operators with not necessarily dense domain

2. On symmetric operators with not necessarily dense domain

Let H be a separable Hilbert space whose inner product $\langle \cdot, \cdot \rangle$ is assumed antilinear **2. On symmetric operators with not necessarily dense**
Let $\mathcal H$ be a separable Hilbert space whose inner product $\langle \cdot, \cdot \rangle$ is ass
in its first argument. In this space we consider a closed, symmetric in its first argument. In this space we consider a closed, symmetric operator A with deficiency indices (1, 1). It is not assumed that its domain is dense in H , therefore one should deal with the case when the adjoint one should deal with the case when the adjoint of A is a linear relation. That is,
in general,
 $A^* := \{ \{\eta, \omega\} \in \mathcal{H} \oplus \mathcal{H} : \langle \eta, A\varphi \rangle = \langle \omega, \varphi \rangle \text{ for all } \varphi \in \text{dom}(A) \}.$ (1)
Whenever the orthogonal complement of dom(A deficiency indices (1, 1). It is not assumed that its domain is dense in \mathcal{H} , therefore
one should deal with the case when the adjoint of A is a linear relation. That is,
in general,
 $A^* := \{ \{\eta, \omega\} \in \mathcal{H} \oplus \mathcal{H} :$

$$
A^* := \{ \{\eta, \omega\} \in \mathcal{H} \oplus \mathcal{H} : \langle \eta, A\varphi \rangle = \langle \omega, \varphi \rangle \text{ for all } \varphi \in \text{dom}(A) \}.
$$
 (1)

one should deal with the case when the adjoint of A is a linear relation. That is,
in general,
 $A^* := \{ \{ \eta, \omega \} \in \mathcal{H} \oplus \mathcal{H} : \langle \eta, A\varphi \rangle = \langle \omega, \varphi \rangle \text{ for all } \varphi \in \text{dom}(A) \}.$ (1)
Whenever the orthogonal complement of dom($\begin{aligned} A \\ \text{Whenever } \mathcal{H}: \{0, \omega\} \\ A^* \text{ is a pre} \end{aligned}$:= { $\{\eta, \omega\} \in \mathcal{H} \oplus \mathcal{H} : \langle \eta, A\varphi \rangle = \langle \omega, \varphi \rangle$ for all $\varphi \in \text{dom}(A)$ }. (1)
e orthogonal complement of dom(A) is trivial, the set $A^*(0) := \{\omega \in A^*\}$ is also trivial, i.e. $A^*(0) = \{0\}$, so A^* is an operator; oth Whenever the orthogonal complement of dom(A) is trivial, the set $A^*(0) := \{\omega \in \mathcal{H} : \{0, \omega\} \in A^*\}$ is also trivial, i.e. $A^*(0) = \{0\}$, so A^* is an operator; otherwise A^* is a proper closed linear relation. $\mathcal{H}: \{0, \omega\} \in A^* \}$ is also trivial, i.e. $A^*(0) = \{0\}$, so A^* is an operator; otherwise A^* is a proper closed linear relation. A^* is a proper closed linear relation.

$$
\in \mathbb{C} \text{ one has}
$$
\n
$$
A^* - zI := \{ \{\eta, \omega - z\eta\} \in \mathcal{H} \oplus \mathcal{H} : \{\eta, \omega\} \in A^* \},
$$
\n
$$
\text{gly}
$$
\n
$$
\ker(A^* - zI) := \{ n \in \mathcal{H} : \{ n, 0 \} \in A^* - zI \}
$$
\n
$$
(3)
$$

$$
\ker(A^* - zI) := \{ \eta \in \mathcal{H} : \{ \eta, 0 \} \in A^* - zI \}.
$$
 (3)

For $z \in \mathbb{C}$ one has
 $A^* - z$

cordingly
 $\ker(A^* - zI) = \mathcal{F}$

es dim kor $(A^* - zI)$:= { $\{\eta, \omega - z\eta\} \in \mathcal{H} \oplus \mathcal{H} : \{\eta, \omega\} \in A^*\},$ (2)
* - zI) := { $\eta \in \mathcal{H} : \{\eta, 0\} \in A^* - zI\}.$ (3)
 $\ominus \operatorname{ran}(A - \overline{z}I)$, our assumption on the deficiency indices
= 1 for all $z \in \mathbb{C} \setminus \mathbb{R}$. Also, since
 \mathcal{H} Since $\ker(A^* - \text{implies dim } \ker(A^*)$ ker($A^* - zI$) := { $\eta \in \mathcal{H} : {\eta, 0} \in A^* - zI$ }. (3)

= $\mathcal{H} \ominus \text{ran}(A - \overline{z}I)$, our assumption on the deficiency indices
 $-zI$) = 1 for all $z \in \mathbb{C} \setminus \mathbb{R}$. Also, since

(0) = { $\omega \in \mathcal{H} : \langle \omega, \psi \rangle = 0$ for all

$$
A^*(0) = \{ \omega \in \mathcal{H} : \langle \omega, \psi \rangle = 0 \text{ for all } \psi \in \text{dom}(A) \},
$$

Since ker($A^* - zI$) = $\mathcal{H} \ominus \text{ran}(A - \overline{z}I)$, our assumption on the deficiency indices
implies dim ker($A^* - zI$) = 1 for all $z \in \mathbb{C} \setminus \mathbb{R}$. Also, since
 $A^*(0) = \{ \omega \in \mathcal{H} : \langle \omega, \psi \rangle = 0 \text{ for all } \psi \in \text{dom}(A) \}$,
it i implies dim ker($A^* - zI$) = 1 for all $z \in \mathbb{C} \setminus \mathbb{R}$. Also, since
 $A^*(0) = \{ \omega \in \mathcal{H} : \langle \omega, \psi \rangle = 0 \text{ for all } \psi \in \text{dc}$

it is obvious that $A^*(0) = \text{dom}(A)^{\perp}$.

The selfadjoint extensions within \mathcal{H} of a closed, n $(0) = {\omega \in \mathcal{H} : \langle \omega, \psi \rangle = 0$ for all $\psi \in \text{dom}(A)}$,
 $A^*(0) = \text{dom}(A)^{\perp}$.

the extensions within \mathcal{H} of a closed, non-densely of

the selfadjoint linear relations that extend the

relation B is selfadjoint if $B = B$ it is obvious that $A^*(0) = \text{dom}(A)^{\perp}$.
The selfadjoint extensions withincic operator A are the selfadjoint line recall that a linear relation B is selfation.
The following assertion follows Theorem 2.4.]. The senation extensions within *H* of a closed, non-densely defined symmet-
berator *A* are the selfadjoint linear relations that extend the graph of *A*. We
that a linear relation *B* is selfadjoint if $B = B^*$ (as subset

recall that a linear relation *B* is selfadjoint if $B = B^*$ (as subsets of $\mathcal{H} \oplus \mathcal{H}$).
The following assertion follows easily from [8, Section 1, Lemma 2.2 and
Theorem 2.4].
Proposition 2.1. Let *A* be a closed, recall that a linear relation *B* is selfadjoint if $B = B^*$ (as subsets of $\mathcal{H} \oplus \mathcal{H}$).
The following assertion follows easily from [8, Section 1, Lemma 2.2
Theorem 2.4].
Proposition 2.1. Let *A* be a closed, non

The following asset of the action of domestic and the following defined, symmetric operator in $\mathcal H$
deficiency indices (1, 1). Then:
The codimension of dom (A) equals one.
All except one of the selfadjoint extensions of **Proposition 2.**
with deficienc
(i) The codi
(ii) All excep
(iii) Let A_{γ} b **Proposition 2.1.** Let A be a closed, non-densely defined, symmetric operator in H with deficiency indices $(1,1)$. Then:

-
- (i) The codimension of dom(A) equals one.

(ii) All except one of the selfadjoint extension

(iii) Let A_{γ} be one of the selfadjoint extension
 $I + (z w)(A_{\gamma} zI)^{-1}, \quad z \in$

maps ker($A^* wI$) injectively onto ker(

In

\n
$$
cy\ indices\ (1,1)
$$
. Then:
\n $imension\ of\ dom(A)\ equals\ one.$
\n $pt\ one\ of\ the\ selfadjoint\ extensions\ of\ A\ within\ H.\ The\ be\ one\ of\ the\ selfadjoint\ extensions\ of\ A\ within\ H.\ The\ $I + (z - w)(A_{\gamma} - zI)^{-1}, \quad z \in \mathbb{C} \setminus \mathrm{spec}(A_{\gamma}), \quad w \in \mathbb{C}$
\n $rr(A^* - wI)\ injectively\ onto\ ker(A^* - zI).$ \n$

maps ker $(A^* - wI)$ injectively onto ker $(A^* - zI)$.

(ii) All except one of the selfadjoint extensions of A within *H* are operators.

(iii) Let A_{γ} be one of the selfadjoint extensions of A within *H*. Then the oper
 $I + (z - w)(A_{\gamma} - zI)^{-1}, \quad z \in \mathbb{C} \setminus \text{spec}(A_{\gamma}), \quad w \in \mathbb$ (iii) Let A_{γ} be one of the selfadjoint extensions of A within \mathcal{H} . Then the operator
 $I + (z - w)(A_{\gamma} - zI)^{-1}, \quad z \in \mathbb{C} \setminus \text{spec}(A_{\gamma}), \quad w \in \mathbb{C}$

maps ker $(A^* - wI)$ injectively onto ker $(A^* - zI)$.

In connection w + $(z - w)(A_{\gamma} - zI)^{-1}$, $z \in \mathbb{C} \setminus \text{spec}(A_{\gamma})$, $w \in \mathbb{C}$
 $A^* - wI$ injectively onto ker($A^* - zI$).

ion with this proposition we remind the reader that

ar relation *B* is the complement of the set of all *z*

bounded In connection with this proposition we remind the reader that the spectrum
of a closed linear relation B is the complement of the set of all $z \in \mathbb{C}$ such that
 $(B - zI)^{-1}$ is a bounded operator defined on all \mathcal{H} . closed linear relation B is the complement of the set of all $z \in \mathbb{C}$ such that $zI)^{-1}$ is a bounded operator defined on all H. Moreover, spec $(B) \subset \mathbb{R}$ when a selfadjoint linear relation [6].
Given $\psi_{w_0} \in \ker(A^*$ of a closed linear relation *B* is the complement of the set of all $z \in \mathbb{C}$ such that $(B - zI)^{-1}$ is a bounded operator defined on all *H*. Moreover, spec $(B) \subset \mathbb{R}$ when *B* is a selfadjoint linear relation [6].
Give B is a selfadjoint linear relation [6].

$$
\psi(z) := \left[I + (z - w_0)(A_\gamma - zI)^{-1}\right]\psi_{w_0},\tag{4}
$$

 $(B - zI)^{-1}$ is a bounded operator defined on all *H*. Moreover, spec(*B*) ⊂ ℝ when
 B is a selfadjoint linear relation [6].

Given $\psi_{w_0} \in \ker(A^* - w_0I)$, with $w_0 \in \mathbb{C} \setminus \mathbb{R}$, let us define
 $\psi(z) := [I + (z - w_0)(A_{\gamma}$ Given $\psi_{w_0} \in \ker(A^* - w_0 I)$, with $w_0 \in \mathbb{C} \setminus \mathbb{R}$, let us define
 $\psi(z) := [I + (z - w_0)(A_\gamma - zI)^{-1}] \psi_{w_0}$,

the that $I + (z - w_0)(A_\gamma - zI)^{-1}$ is the generalized Cayley tran
 $w_0) = \psi_{w_0}$. Moreover, a computation involvi $(z) := [I + (z - w_0)(A_{\gamma} - zI)]$
 $(A_{\gamma} - zI)^{-1}$ is the generalized
 $w(t) = [I + (z - v)(A_{\gamma} - zI)]^{-1}$
 R. This identity will be used if some concepts that will be $\frac{1}{\text{sky}},$
(5) Note that $I + (z - w_0)(A_\gamma - zI)^{-1}$ is the generalized Cayley transform. Obviously,
 $\psi(w_0) = \psi_{w_0}$. Moreover, a computation involving the resolvent identity yields
 $\psi(z) = \left[I + (z - v)(A_\gamma - zI)^{-1}\right]\psi(v)$, (5)

for any pair $z, v \in$

$$
\psi(z) = [I + (z - v)(A_{\gamma} - zI)^{-1}] \psi(v), \tag{5}
$$

 $\psi(w_0) = \psi_{w_0}$. Moreover, a computation involving the resolvent identity yields
 $\psi(z) = [I + (z - v)(A_{\gamma} - zI)^{-1}] \psi(v)$,

for any pair $z, v \in \mathbb{C} \setminus \mathbb{R}$. This identity will be used later on.

Let us now recall some concep (z) = [$I + (z - v)(A_{\gamma} - zI)$

R. This identity will be used

some concepts that will be

tors with deficiency indices

ric operator A is called simp
 \bigcap ran($A - zI$) = {(Let us now recall some concepts that will be used to single out a class of closed symmetric operators with deficiency indices $(1, 1)$.
A closed, symmetric operator A is called *simple* if

$$
\bigcap_{z \in \mathbb{C} \backslash \mathbb{R}} \operatorname{ran}(A - zI) = \{0\}
$$

closed symmetric operators with deficiency indices (1, 1).

A closed, symmetric operator A is called *simple* if
 $\bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \text{ran}(A - zI) = \{0\}.$

Equivalently, A is simple if there exists no non-trivial subspace A closed, symmetric operator A is called *simple* if
 $\bigcap_{z \in \mathbb{C} \backslash \mathbb{R}} \text{ran}(A - zI) = \{0\}.$

walently, A is simple if there exists no non-trivial su

d whose restriction to $\mathcal L$ yields a selfadjoint operation ran($A - zI$) = {0}.

xists no non-trivial s

ls a selfadjoint oper Equivalently, *A* is simple if there exists no non-trivial subspace $\mathcal{L} \subset \mathcal{H}$ that reduces *A* and whose restriction to \mathcal{L} yields a selfadjoint operator [15, Proposition 1.1]. A and whose restriction to $\mathcal L$ yields a selfadjoint operator [15, Proposition 1.1]. and whose restriction to $\mathcal L$ yields a senadjoint operator [15, Proposition 1.1].

154 L.O. Silva and J.H. Toloza

There is one property specific to simple, closed symmetric operators with

deficiency indices $(1,1)$, that is of interest to us. It concerns their commutativity

with involutions. We say t deficiency indices (1, 1), that is of interest to us. It concerns their commutativity
with involutions. We say that an involution J commutes with a selfadjoint relation
 B if
 $J(B - zI)^{-1}\varphi = (B - \overline{z}I)^{-1}J\varphi$,
for every B if

$$
J(B - zI)^{-1}\varphi = (B - \overline{z}I)^{-1}J\varphi,
$$

with involutions. We say that an involution J commutes with a selfadjoint relation B if
 $J(B - zI)^{-1}\varphi = (B - \overline{z}I)^{-1}J\varphi$,

for every $\varphi \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. If B is moreover an operator this is equi if
 $\frac{1}{2}$ $(B - zI)^{-1} \varphi = (B - \overline{z}I)$
 $\sum \setminus \mathbb{R}$. If *B* is moreover at

utativity, that is,
 $m(B) \subset dom(B)$, *JB* for every $\varphi \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. If *B* is moreover an operator this is equivalent to
the usual notion of commutativity, that is,
 $J \text{ dom}(B) \subset \text{dom}(B)$, $JB\varphi = BJ\varphi$
for every $\varphi \in \text{dom}(B)$.
Proposition 2.

$$
J \operatorname{dom}(B) \subset \operatorname{dom}(B), \qquad JB\varphi = BJ\varphi
$$

 $J \text{ dom}(B) \subset \text{dom}(B),$
for every $\varphi \in \text{dom}(B)$.
Proposition 2.2. Let A be a simple, closed
dices (1, 1). Then there exists an involution
extensions within H dom(B) ⊂ dom(B), $JB\varphi = BJ\varphi$
be a simple, closed symmetric operat
exists an involution J that commutes
light extension A, and consider ψ for every $\varphi \in \text{dom}(B)$.
Proposition 2.2. Let A
dices (1, 1). Then there
extensions within H.
Proof. Choose a selfa
Recalling (5) along wit **Proposition 2.2.** Let A be a simple, closed symmetric operator with deficiency indices extensions within H.

(1, 1). Then there exists an involution *J* that commutes with all its selfadjoint
sions within \mathcal{H} .
Choose a selfadjoint extension A_{γ} and consider $\psi(z)$ as defined by (4).
ling (5) along with the unitary chara Proof. Recalling (5) along with the unitary character of the generalized Cayley transform,
and applying the resolvent identity, one can verify that
 $\langle \psi(\overline{z}), \psi(\overline{v}) \rangle = \langle \psi(v), \psi(z) \rangle$ (6)
for every pair $z, v \in \mathbb{C} \setminus \mathbb{R}$. and applying the resolvent identity, one can verify that
 $\langle \psi(\overline{z}), \psi(\overline{v}) \rangle = \langle \psi(v), \psi(z) \rangle$ (6)

for every pair $z, v \in \mathbb{C} \setminus \mathbb{R}$.

Now define the action of J on the set $\{\psi(z) : z \in \mathbb{C} \setminus \mathbb{R}\}$ by the rule
 $J\$

$$
\langle \psi(\overline{z}), \psi(\overline{v}) \rangle = \langle \psi(v), \psi(z) \rangle
$$
\n(6)

\nof J on the set $\{\psi(z) : z \in \mathbb{C} \setminus \mathbb{R}\}$ by the rule

\n
$$
J\psi(z) = \psi(\overline{z}),
$$
\nlinear combinations of such elements as

 $\langle \psi(\overline{z}), \psi(\overline{v}) \rangle = \langle \psi(v), \psi(z) \rangle$
for every pair $z, v \in \mathbb{C} \setminus \mathbb{R}$.
Now define the action of J on the set $\{\psi(z) : z \in \mathbb{R}\}$
 $J\psi(z) = \psi(\overline{z}),$

$$
J\psi(z)=\psi(\overline{z})
$$

for every pair $z, v \in \mathbb{C} \setminus \mathbb{R}$.
Now define the action
and on the set \mathcal{D} of finite J

Now define the action of
$$
J
$$
 on the set $\{\psi(z) : z \in \mathbb{C} \setminus \mathbb{R}\}$ by the rule
\n
$$
J\psi(z) = \psi(\overline{z}),
$$
\nonthe set \mathcal{D} of finite linear combinations of such elements as\n
$$
J\left(\sum_n c_n \psi(z_n)\right) := \sum_n \overline{c_n} \psi(\overline{z_n}).
$$
\n, on one hand, (6) implies that J is an involution on \mathcal{D} which can be a \mathcal{U} because of the simplicity of A . On the other hand, since by the i

and on the set *D* of finite linear combinations of such elements as
 $J\left(\sum_n c_n \psi(z_n)\right) := \sum_n \overline{c_n} \psi(\overline{z_n}).$

Then, on one hand, (6) implies that *J* is an involution on *D* which to all *H* because of the simplicity of *A* (z_n) = $\sum_{n} \overline{c_n} \psi(\overline{z_n}).$
 J is an involution on 2

if *A*. On the other han
 $\psi(z) = \frac{\psi(z) - \psi(w)}{z - w}$ to all H because of the simplicity of A . On the other hand, since by the resolvent identity to all *H* because of the simplicity of *A*. On the other hand, since by the resolvent
identity
 $(A_{\gamma} - wI)^{-1}\psi(z) = \frac{\psi(z) - \psi(w)}{z - w}$,
one obtains the identity
 $J(A_{\gamma} - wI)^{-1}\psi(z) = (A_{\gamma} - \overline{w}I)^{-1}J\psi(z)$

$$
(A_{\gamma} - wI)^{-1}\psi(z) = \frac{\psi(z) - \psi(w)}{z - w},
$$

$$
A_{\gamma} - wI)^{-1}\psi(z) = (A_{\gamma} - \overline{w}I)^{-1}J\psi(z)
$$

so \mathcal{D} and in turn it extends to al
at J commutes with A_{γ} . By reso
 K rein (see [8: Theorem 3.2] for a g

one obtains the identity
\n
$$
J(A_{\gamma}-wI)^{-1}\psi(z)=(A_{\gamma}-\overline{w}I)^{-1}J\psi(z)
$$
\nwhich by linearity holds on \mathcal{D} and in turn it extends to all \mathcal{H} .

 $J(A)$
which by linearity holds
So far we know the
resolvent formula due to
one immediately obtains $(A_{\gamma} - wI)^{-1} \psi(z) = (A_{\gamma} - \overline{w}I)$
ds on D and in turn it extends
that J commutes with A_{γ} . By
to Krein (see [8, Theorem 3.2] for
ins the commutativity of J with (z)
 $\frac{1}{t}$ which by linearity holds on ν and in turn it extends to an n .

So far we know that J commutes with A_{γ} . By resorting

resolvent formula due to Krein (see [8, Theorem 3.2] for a gener

one immediately obtains th So far we know that J commutes with A_{γ} . By resorting to the well-known
vent formula due to Krein (see [8, Theorem 3.2] for a generalized formulation),
mmediately obtains the commutativity of J with all the selfad one immediately obtains the commutativity of *J* with all the selfadjoint extensions

of *A* within \mathcal{H} .
 \Box

of *A* within *H*. \Box

A closed, symmetric operator is called *regular* if for every $z \in \mathbb{C}$ there exists $d_z > 0$ such that $\|(A - zI)\psi\| \ge d_z \|\psi\|$, (7) A closed, symmetric operator is called *regular* if for every $z \in \mathbb{C}$ there exists 0 such that
 $||(A - zI)\psi|| \ge d_z ||\psi||$, (7) $d_{z} > 0$ such that 0 such that

$$
\|(A - zI)\psi\| \ge d_z \|\psi\|,\tag{7}
$$

The Spectra of Selfadjoint Extensions of Entire Operators 155
for all $\psi \in \text{dom}(A)$. In other words, A is regular if every point of the complex
plane is a point of regular type.
Definition 2.3. Let $\mathcal{S}(\mathcal{H})$ be the

Definition 2.3.

for all $\psi \in \text{dom}(A)$. In other words, A is regular if every point of the complex
plane is a point of regular type.
Definition 2.3. Let $S(\mathcal{H})$ be the class of simple, regular, closed symmetric operator
in \mathcal{H} , **Definition 2.3.** Let $S(H)$ be the in H , whose deficiency indices a
In [17, 18] we deal with the defined. In the present work we above. At this point it is conve Let $\mathcal{O}(k)$ be the class of simple, regular, closed symmetric operator

in \mathcal{H} , whose deficiency indices are $(1, 1)$.

In [17, 18] we deal with the subclass of operators in $\mathcal{S}(\mathcal{H})$ that are densely

define defined. In the present work we extend the results of [18] to the larger class defined
above. At this point it is convenient to touch upon some well-known properties
shared by the operators in $S(\mathcal{H})$ that are densely d above. At this point it is convenient to touch upon some well-known properties
shared by the operators in $S(\mathcal{H})$ that are densely defined, and whose generaliza-
tions to the whole class is rather straightforward. The f shared by the operators in $S(\mathcal{H})$ that are densely defined, and whose generaliza-
tions to the whole class is rather straightforward. The following statement is one
of such generalizations which we believe may have bee tions to the whole class is rather straightforward. The following statement is one

Proposition 2.4. For $A \in \mathcal{S}$

- (i) The spectrum of every selfadjoint extension of A within H consists solely of **Proposition 2.4.** For $A \in \mathcal{S}(\mathcal{H})$ the following assertions hold true:

(i) The spectrum of every selfadjoint extension of A within H con

isolated eigenvalues of multiplicity one.

(ii) Every real number is part of (*H*) the following assertions hold true:
 μ selfadjoint extension of A within H c

i multiplicity one.

part of the spectrum of one, and only
 μ .

djoint extensions of A within H are pair

were similar to the are u isolated eigenvalues of multiplicity one.
- (ii) Every real number is part of the spectrum of one, and only one, selfadjoint (i) The spectrum of every selfadjoint extension of A within *H* consists solely of
isolated eigenvalues of multiplicity one.
(ii) Every real number is part of the spectrum of one, and only one, selfadjoint
extension of A extension of A within H .
-

(ii) Every real number is part of the spectrum of one, and only one, selfadjoint extension of A within H .

(iii) The spectra of the selfadjoint extensions of A within H are pairwise interlaced.

Proof. Let us prove (i *Proof.* Let us prove (i) in a way similar to the one used to prove [7, Propositions

3.1 and 3.2], but taking into account that the operator is not necessarily densely defined.
For $A \in \mathcal{S}(\mathcal{H})$ and any $r \in \mathbb{R}$ consider the constant d_r of (7). Thus, the symmetric operator $(A - rI)^{-1}$, defined on 1 3.2], but taking into account that the operator is not necessarily densely

or $A \in \mathcal{S}(\mathcal{H})$ and any $r \in \mathbb{R}$ consider the constant d_r of (7). Thus, the

tric operator $(A - rI)^{-1}$, defined on the subspace $\text{ran}($ defined.

For $A \in \mathcal{S}(\mathcal{H})$ and any $r \in \mathbb{R}$ consider the constant d_r of (7). Thus, the

symmetric operator $(A - rI)^{-1}$, defined on the subspace $\text{ran}(A - rI)$, is such that
 $||(A - rI)^{-1}|| \leq d_r^{-1}$. By [13, Theorem 2] For $A \in \mathcal{S}(\mathcal{H})$ and any $r \in \mathbb{R}$ consider the constant d_r of (7). Thus, the
symmetric operator $(A - rI)^{-1}$, defined on the subspace $\text{ran}(A - rI)$, is such that
 $||(A - rI)^{-1}|| \leq d_r^{-1}$. By [13, Theorem 2] there is a symmetric operator $(A - rI)^{-1}$, defined on the subspace ran $(A - rI)$, is such that $||(A - rI)^{-1}|| \leq d_r^{-1}$. By [13, Theorem 2] there is a selfadjoint extension *B* of $(A - rI)^{-1}$ defined on the whole space and such that $||B|| \leq d$ $||(A - rI)^{-1}|| \leq d_r^{-1}$ $(A - rI)^{-1} \leq d_r^{-1}$. By [13, Theorem 2] there is a selfadjoint extension *B* of $A - rI)^{-1}$ defined on the whole space and such that $||B|| \leq d_r^{-1}$. Now, B^{-1} is selfadjoint extension of $A - rI$ and $||B^{-1}f|| \geq d_r ||f||$ for an $(A - rI)^{-1}$ defined on the whole space and such that $||B|| \leq d_r^{-1}$. Now, B^{-1} is
a selfadjoint extension of $A - rI$ and $||B^{-1}f|| \geq d_r ||f||$ for any $f \in \text{dom}(B^{-1})$,
which implies that the interval $(-d_r, d_r) \cap \text{spec}(B^{-1}) = \emptyset$. B a selfadjoint extension of $A - rI$ and $||B^{-1}f|| \ge d_r ||f||$ for any $f \in \text{dom}(B^{-1})$,
which implies that the interval $(-d_r, d_r) \cap \text{spec}(B^{-1}) = \emptyset$. By shifting B^{-1} one
obtains a selfadjoint extension of A with no spectrum in the $||B^{-1}f|| \geq d_r ||f||$ which implies that the interval $(-d_r, d_r) \cap \text{spec}(B^{-1}) = \emptyset$. By shifting B^{-1} one
obtains a selfadjoint extension of A with no spectrum in the spectral lacuna $(r - d_r, r + d_r)$. By perturbation theory any selfadjoint extension obtains a selfadjoint extension of *A* with no spectrum in the spectral lacuna $(r - d_r, r + d_r)$. By perturbation theory any selfadjoint extension of *A* which is an operator has no points of the spectrum in this spectral lacu d_r,r operator has no points of the spectrum in this spectral lacuna other than one eigenvalue of multiplicity one. When dom(A) \neq H , the same is also true for the spectrum of the selfadjoint extension which is not an operator. This follows from a generalization of the Aronzajn-Krein formula (see spectrum of the selfadjoint extension which is not an operator. This follows from a generalization of the Aronzajn-Krein formula (see [8, Equation 3.17]) after noting that the Weyl function is Herglotz and meromorphic for generalization of the Aronzajn-Krein formula (see [8, Equation 3.17]) after noting
that the Weyl function is Herglotz and meromorphic for any selfadjoint extension
being an operator. Now, for proving (i) consider any clos that the Weyl function is Herglotz and meromorphic for any selfadjoint extension

being an operator. Now, for proving (i) consider any closed interval of \mathbb{R} , cover it with spectral lacunae and take a finite subcover.
Once (i) has been proven, the assertions (ii) and (iii) follow from [8, Equation being an operator. Now, for proving (i) consider any closed interval of ℝ, cover it
with spectral lacunae and take a finite subcover.
Once (i) has been proven, the assertions (ii) and (iii) follow from [8, Equa-
tion 3.17 tion 3.17 and the properties of Herglotz meromorphic functions.

Definition 2.5. An operator $A \in \mathcal{S}(\mathcal{H})$ is called *entire* if there exists $\mu \in \mathcal{H}$ such that $\mathcal{H} = \text{ran}(A - zI) \dot{+} \text{span}\{\mu\}$ for all $z \in \mathbb{C}$. Such μ is called an *entire gauge*. **Definition 2.5.** An operator $A \in \mathcal{S}(\mathcal{H})$ is called *entire* if there exists $\mu \in \mathcal{H}$ such that $\mathcal{H} = \text{ran}(A - zI) \dot{+} \text{span}\{\mu\}$ for all $z \in \mathbb{C}$. Such μ is called an *entire gauge*.

$$
\mathcal{H} = \text{ran}(A - zI) + \text{span}\{\mu\}
$$

led an *entire gauge.*

for all $z \in \mathbb{C}$. Such μ is called an *entire qauge*. for all $z \in \mathbb{C}$. Such μ is called an *entire gauge*.

If $A \in \mathcal{S}(\mathcal{H})$ turns out to be densely defined, the Krein's [12, Section 1]. There are various densely defientire [7, Chapter 3], [12, Section 4]. On the other hand in the subsequent sections, there are also entire If $A \in \mathcal{S}(\mathcal{H})$ turns out to be densely defined, then Definition 2.5 reduces to

i's [12, Section 1]. There are various densely defined operators known to be
 $[\mathbf{7}, \mathbf{Chapter 3}]$, [12, Section 4]. On the other hand, for entire [7, Chapter 3], [12, Section 4]. On the other hand, for what will be explained
in the subsequent sections, there are also entire operators with non-dense domain.
Let us outline how one may construct an entire opera in the subsequent sections, there are also entire operators with non-dense domain.
Let us outline how one may construct an entire operator which is not densely
defined. The details of this construction will be expounded i Let us outline how one may construct an entire operator which is not densely
defined. The details of this construction will be expounded in a further paper.
Consider the semi-infinite Jacobi matrix
 $\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cd$

defined. The details of this construction will be expounded in a further paper.
\nConsider the semi-infinite Jacobi matrix
\n
$$
\begin{pmatrix}\nq_1 & b_1 & 0 & 0 & \cdots \\
b_1 & q_2 & b_2 & 0 & \cdots \\
0 & b_2 & q_3 & b_3 & \cdots \\
0 & 0 & b_3 & q_4 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots\n\end{pmatrix},
$$
\nwhere $b_k > 0$ and $q_k \in \mathbb{R}$ for $k \in \mathbb{N}$. Fix an orthonormal basis $\{\delta_k\}_{k \in \mathbb{N}}$ in \mathcal{H} . Let B be the operator in \mathcal{H} whose matrix representation with respect to $\{\delta_k\}_{k \in \mathbb{N}}$ is (8) (cf. [2, Section 47]). We assume that $B \neq B^*$, equivalently, that B has deficiency

be the operator in $\mathcal H$ whose matrix representation with respect to $\{\delta_k\}_{k \in \mathbb N}$ is (8) (cf. [2, Section 47]). We assume that $B \neq B^*$, equivalently, that B has deficiency indices (1, 1) [1, Chapter 4, Section 1.2] ... :
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|} . : $\in \mathbb{N}$. Fix an orthonomological extension
tatrix representation
that $B \neq B^*$, equ
ction 1.2]. Let B_0 b
follows from (1), (2)
quation .
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ה where $b_k > 0$ and $q_k \in \mathbb{R}$ for $k \in \mathbb{N}$. Fix an orthonormal basis $\{\delta_k\}_{k \in \mathbb{N}}$ in *H*. Let *B* be the operator in *H* whose matrix representation with respect to $\{\delta_k\}_{k \in \mathbb{N}}$ is (8) (cf. [2, Section 47]). be the operator in *H* whose matrix representation with respect to $\{\delta_k\}_{k \in \mathbb{N}}$ is (8) (cf. [2, Section 47]). We assume that $B \neq B^*$, equivalently, that *B* has deficiency indices (1, 1) [1, Chapter 4, Section 1.2] (cf. [2, Section 47]). We assume that $B \neq B^*$, equivalently, that B has deficiency indices (1, 1) [1, Chapter 4, Section 1.2]. Let B_0 be the restriction of B to the set $\{\phi \in \text{dom}(B) : \langle \phi, \delta_1 \rangle = 0\}$. It follows indices (1, 1) [1, Chapter 4, Section 1.2]. Let B_0 be the restriction of B to the set $\{\phi \in \text{dom}(B) : \langle \phi, \delta_1 \rangle = 0\}$. It follows from (1), (2) and (3) that $\eta \in \text{ker}(B_0^* - zI)$ if and only if it satisfies the equati { $\phi \in \text{dom}(B) : \langle \phi, \delta_1 \rangle = 0$ }. It follows from (1), (2) and (3) that $\eta \in \text{ker}(B_0^* - zI)$
if and only if it satisfies the equation
 $\langle B\phi, \eta \rangle = \langle \phi, z\eta \rangle \qquad \forall \phi \in \text{dom}(B_0)$.
Thus ker($B_0^* - zI$) is the set of η 's in)
)

$$
\langle B\phi, \eta \rangle = \langle \phi, z\eta \rangle \qquad \forall \phi \in \text{dom}(B_0) \, .
$$

$$
b_{k-1} \langle \delta_{k-1}, \eta \rangle + q_k \langle \delta_k, \eta \rangle + b_k \langle \delta_{k+1}, \eta \rangle = z \langle \delta_k, \eta \rangle \quad \forall k > 1 \tag{9}
$$

 $\langle B\phi, \eta \rangle = \langle \phi, z \rangle$
Thus ker $(B_0^* - zI)$ is the set of η 's in
 $b_{k-1} \langle \delta_{k-1}, \eta \rangle + q_k \langle \delta_k, \eta \rangle$
Hence dim ker $(B_0^* - zI) \leq 2$. Now, le Thus ker $(B_0^*$
b
Hence dim k $\int_0^* - zI$

$$
\langle B\phi, \eta \rangle = \langle \phi, z\eta \rangle \qquad \forall \phi \in \text{dom}(B_0).
$$

\n
$$
_{0}^{*} - zI \text{ is the set of } \eta \text{'s in } \mathcal{H} \text{ that satisfy}
$$

\n
$$
b_{k-1} \langle \delta_{k-1}, \eta \rangle + q_k \langle \delta_k, \eta \rangle + b_k \langle \delta_{k+1}, \eta \rangle = z \langle \delta_k, \eta \rangle \quad \forall k > 1
$$

\n
$$
\text{ker}(B_0^* - zI) \le 2. \text{ Now, let}
$$

\n
$$
\pi(z) := \sum_{k=1}^{\infty} P_{k-1}(z)\delta_k \qquad \theta(z) := \sum_{k=1}^{\infty} Q_{k-1}(z)\delta_k,
$$

\n
$$
\text{superscript{3}}, \text{respectively } Q_k(z), \text{ is the } k \text{th polynomial of first, respectively second,}
$$

Hence dim ker $(B_0^* - zI) \leq 2$. Now, let
 $\pi(z) := \sum_{k=1}^{\infty} P_{k-1}(z) \delta_k$

where $P_k(z)$, respectively $Q_k(z)$, is the

kind associated to (8). By the definit

Chapter 1, Section 2.1], $\pi(z)$ and $\theta(z)$ $(z) := \sum_{k=1}^{\infty} P_{k-1}(z)\delta_k$ $\theta(z) := \sum_{k=1}^{\infty} Q_{k-1}(z)\delta_k$,
ectively $Q_k(z)$, is the *k*th polynomial of first, resp
o (8). By the definition of the polynomials $P_k(z)$
n 2.1], $\pi(z)$ and $\theta(z)$ are linearly independent so
. kind associated to (8). By the definition of the polynomials $P_k(z)$ and $Q_k(z)$ [1, Chapter 1, Section 2.1], $\pi(z)$ and $\theta(z)$ are linearly independent solutions of (9) for every fixed $z \in \mathbb{C}$. Moreover, since $B \neq B^*$ kind associated to (8). By the definition of the polynomials $P_k(z)$ and $Q_k(z)$ [1,
Chapter 1, Section 2.1], $\pi(z)$ and $\theta(z)$ are linearly independent solutions of (9) for
every fixed $z \in \mathbb{C}$. Moreover, since $B \neq B^*$ Chapter 1, Section 2.1], $\pi(z)$ and $\theta(z)$ are linearly independent solutions of (9) for
every fixed $z \in \mathbb{C}$. Moreover, since $B \neq B^*$, $\pi(z)$ and $\theta(z)$ are in \mathcal{H} for all $z \in \mathbb{C}$
[1, Theorems 1.3.1, 1.3.2] [1, Theorems 1.3.1, 1.3.2], [19, Theorem 3]. So one arrives at the conclusion that,
for every fixed $z \in \mathbb{C}$,
 $\ker(B_0^* - zI) = \text{span}\{\pi(z), \theta(z)\}\.$
Any symmetric non-selfadjoint extension of B_0 has deficiency indices (1,1).

$$
\ker(B_0^*-zI)=\mathrm{span}\{\pi(z),\theta(z)\}.
$$

for every fixed $z \in \mathbb{C}$,
 $\ker(B_0^* - zI) = \text{span}\{\pi(z), \theta(z)\}\.$

Any symmetric non-selfadjoint extension of B_0 has deficiency indices (1,1). Fur-

thermore, if $\kappa(z)$ is a (z-dependent) linear combination of $\pi(z)$ and $\$ ker $(B_0^* - zI) = \text{span}\{\pi(z), \theta(z)\}\,$.

adjoint extension of B_0 has defic

dependent) linear combination of
 $\in \mathbb{C} \setminus \mathbb{R}$, then (by a parametrized

so an appropriately chosen isomet

joint symmetric extension \widetilde{B} Any symmetric non-selfadjoint extension of B_0 has deficiency indices (1,1). Fur-
thermore, if $\kappa(z)$ is a (z-dependent) linear combination of $\pi(z)$ and $\theta(z)$ such that
 $\langle \kappa(z), \theta(z) \rangle = 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ thermore, if $\kappa(z)$ is a (z-dependent) linear combination of $\pi(z)$ and $\theta(z)$ such that $\langle \kappa(z), \theta(z) \rangle = 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, then (by a parametrized version of [19, Theorem 2.4]) there corresponds to an appr $\frac{\langle \kappa \rangle}{\gamma}$ $(z), \theta(z)$ = 0 for all $z \in \mathbb{C} \setminus \mathbb{R}$, then (by a parametrized version of [19, Theorem 4]) there corresponds to an appropriately chosen isometry from span $\{\kappa(z)\}$ onto an $\{\kappa(\overline{z})\}$ a non-selfadjoint symmetric ex 2.4]) there corresponds to an appropriately chosen isometry from span $\{\kappa(z)\}$ onto span $\{\kappa(\overline{z})\}$ a non-selfadjoint symmetric extension \widetilde{B} of B_0 such that dom (\widetilde{B}) is not dense and ker $(\widetilde{B}^* - zI) = \text{$ dense and ker $(\widetilde{B}^* - zI) = \text{span}\{\theta(z)\}\.$ We claim that \widetilde{B} is a non-densely defined dense and ker($B^* - zI$) = span $\{\theta(z)\}$. We claim that B is a non-densely defined
entire operator. Indeed, $\widetilde{B} \in \mathcal{S}(\mathcal{H})$ (the simplicity follows from the properties of entire operator. Indeed, $B \in \mathcal{S}(\mathcal{H})$ (the simplicity follows from the properties of

The Spectra of Selfadjoint Extensions of Entire Operators 157
the associated polynomials [1, Chapter 1, Addenda and Problems 7]). Moreover, since $\langle \theta(z), \delta_2 \rangle = b_1^{-1}, \quad \forall z \in \mathbb{C},$

3. A review on de Branges spaces with zero-free functions

$$
\langle \theta(z), \delta_2 \rangle = b_1^{-1} \,, \qquad \forall z \in \mathbb{C} \,,
$$

 δ_2 is an entire gauge.

3. A review on de Branges spaces with zero-free functions

 $\langle \theta(z), \delta_2 \rangle = b_1^{-1}$
 3. A review on de Branges spaces

Let *B* denote a nontrivial Hilbert space
 $\langle \cdot, \cdot \rangle$ *B* is a de Branges space when for **A review on det**
 \mathcal{B} denote a nont
 $\left\langle \right\rangle_{\mathcal{B}}$. \mathcal{B} is a de Branditions holds: Let B denote a nontrivial Hilbert space of entire functions with limer product $\langle \cdot, \cdot \rangle_B$. $\mathcal B$ is a de Branges space when, for every function $f(z)$ in $\mathcal B$, the following conditions holds:

(A1) For every $w \in \mathbb$ $\langle \cdot, \cdot \rangle_B$. B is a de Branges space when, for every function $f(z)$ in B, the following conditions holds:

(A1) For every $w \in \mathbb{C} \setminus \mathbb{R}$, the linear functional $f(\cdot) \mapsto f(w)$ is continuous;

(A2) for every non-real zero w of $f(z)$, the function $f(z)(z - \overline{w})(z - w)^{-1}$

to $\mathcal B$ and has the same norm as $f(z)$;

(A3) the fun (A2) for every non-real zero w of $f(z)$, the function $f(z)(z - \overline{w})(z - w)^{-1}$ belongs
to $\mathcal B$ and has the same norm as $f(z)$;
(A3) the function $f^*(z) := \overline{f(\overline{z})}$ also belongs to $\mathcal B$ and has the same norm as $f(z)$.
 to *B* and has the same norm as $f(z)$;
the function $f^{\#}(z) := \overline{f(\overline{z})}$ also belon
It follows from (A1) that for every
such that $\langle k(\cdot, w), f(\cdot) \rangle_B = f(w)$ for
 w , $k(\cdot, w) \rangle_B \ge 0$ where, as a consec
very non-real *w* unles (A3) the function $f^{\#}(z) := f(\overline{z})$ also belongs to \mathcal{B} and has the same norm as $f(z)$.
It follows from (A1) that for every non-real w there is a function $k(z, w)$
in \mathcal{B} such that $\langle k(\cdot, w), f(\cdot) \rangle_{\mathcal{B}} = f(w)$ f It follows from (A1) that for every non-real *w* there is a function $k(z, w)$
such that $\langle k(\cdot, w), f(\cdot) \rangle_B = f(w)$ for every $f(z) \in \mathcal{B}$. Moreover, $k(w, w) = w$, $k(\cdot, w) \rangle_B \ge 0$ where, as a consequence of (A2), the positivity is in B such that $\langle k(\cdot, w), f(\cdot) \rangle_B = f(w)$ for every $f(z) \in B$. Moreover, $k(w, w) =$
 $\langle k(\cdot, w), k(\cdot, w) \rangle_B \ge 0$ where, as a consequence of (A2), the positivity is strict

for every non-real w unless B is C; see the proof of Theorem $\binom{k}{c}$ (·, *w*), $k(\cdot, w)$ _{*B*} ≥ 0 where, as a consequence of (A2), the positivity is strict
r every non-real *w* unless *B* is *C*; see the proof of Theorem 23 in [4]. Note
at $k(z, w) = \langle k(\cdot, z), k(\cdot, w) \rangle_B$ whenever *z* and *w* are for every non-real *w* unless \mathcal{B} is ℂ; see the proof of Theorem 23 in [4]. Note
that $k(z, w) = \langle k(\cdot, z), k(\cdot, w) \rangle_B$ whenever *z* and *w* are both non-real, therefore
 $k(w, z) = \overline{k(z, w)}$. Furthermore, due to (A3) it can be s that $k(z, w) = \langle k(\cdot, z), k(\cdot, w) \rangle_B$ whenever z and w are both non-real, therefore $k(w, z) = \overline{k(z, w)}$. Furthermore, due to (A3) it can be shown that $\overline{k(z, w)} = k(z, \overline{w})$ for every non-real w ; we refer again to the proof of \boldsymbol{k} for every non-real w; we refer again to the proof of Theorem 23 in [4]. Also note
that $k(z, w)$ is entire with respect to its first argument and, by (A3), it is anti-
entire with respect to the second one (once $k(z, w)$, as

There is another way of defining a de Branges space. One starts by considering that $k(z, w)$ is entire with respect to its first argument and, by (A3), it is anti-
entire with respect to the second one (once $k(z, w)$, as a function of its second
argument, has been extended to the whole complex plane [entire with respect to the second one (once $k(z, w)$, as a function of its second
argument, has been extended to the whole complex plane [4, Problem 52]).
There is another way of defining a de Branges space. One starts by There is another way of defining a de Branges space. One starts by consi
an entire function $e(z)$ of the Hermite-Biehler class, that is, an entire fu
without zeros in the upper half-plane \mathbb{C}^+ that satisfies the ine an entire function $e(z)$ of the Hermite-Biehler class, that is, an entire function
without zeros in the upper half-plane \mathbb{C}^+ that satisfies the inequality $|e(z)| >$
 $|e^{\#}(z)|$ for $z \in \mathbb{C}^+$. Then, the de Branges without zeros in the upper half-plane \mathbb{C}^+
 $|e^{\#}(z)|$ for $z \in \mathbb{C}^+$. Then, the de Brange
linear manifold of all entire functions $f(z)$ st
belong to the Hardy space $H^2(\mathbb{C}^+)$, and ec
 $\langle f(\cdot), g(\cdot) \rangle_{\mathcal{B}(e)} := \int$ $\overline{}$ that satisfies the inequality $|e(z)| >$
space $\mathcal{B}(e)$ associated to $e(z)$ is the
ch that both $f(z)/e(z)$ and $f^{\#}(z)/e(z)$
nipped with the inner product
 $\frac{\overline{f(x)}g(x)}{|e(x)|^2}dx$. $|e^{\#}(z)|$ for $z \in \mathbb{C}^+$. Then, the de Branges space $\mathcal{B}(e)$ associated to $e(z)$ is the linear manifold of all entire functions $f(z)$ such that both $f(z)/e(z)$ and $f^{\#}(z)/e(z)$ belong to the Hardy space $H^2(\mathbb{C}^+)$ $\frac{\text{(}z\text{)}}{\text{and}}$ linear manifold of all entire functions $f(z)$ such that both $f(z)/e(z)$ and $f^*(z)/e(z)$
belong to the Hardy space $H^2(\mathbb{C}^+)$, and equipped with the inner product
 $\langle f(\cdot), g(\cdot) \rangle_{\mathcal{B}(e)} := \int_{-\infty}^{\infty} \frac{\overline{f(x)}g(x)}{|e(x)|^2} dx$.
I

$$
\langle f(\cdot), g(\cdot) \rangle_{\mathcal{B}(e)} := \int_{-\infty}^{\infty} \frac{\overline{f(x)}g(x)}{|e(x)|^2} dx.
$$

belong to the Hardy space $H^2(\mathbb{C}^+)$, and equipped with the inner product
 $\langle f(\cdot), g(\cdot) \rangle_{\mathcal{B}(e)} := \int_{-\infty}^{\infty} \frac{\overline{f(x)}g(x)}{|e(x)|^2} dx$.

It turns out that $\mathcal{B}(e)$ is complete.

Both definitions of de Branges spaces are $(\cdot), g(\cdot)\rangle_{\mathcal{B}(e)} := \int_{-\infty}^{\infty}$
complete.
de Branges spaces ε
y, given a space \mathcal{B} t
cides with $\mathcal{B}(e)$ as
 $(\cdot)\|_{\mathcal{B}(e)}$ [4, Chapter Both definitions of de Branges spaces are equivalent, viz., every space $\mathcal{B}(e)$
obeys (A1–A3); conversely, given a space \mathcal{B} there exists an Hermite-Biehler func-
tion $e(z)$ such that \mathcal{B} coincides with $\mathcal{$ $\begin{align} |(x)| \leq \text{equi} \leq \text{ex} \ \text{and} \ \text{The} \end{align}$ It turns out that $\mathcal{B}(e)$ is complete.
Both definitions of de Brang
obeys (A1–A3); conversely, given a
tion $e(z)$ such that $\mathcal B$ coincides wit
the equality $||f(\cdot)||_{\mathcal B} = ||f(\cdot)||_{\mathcal B(e)}$ [
a choice for it is Both definitions of de Branges spaces are equivalent, viz., every space $\mathcal{B}(e)$

s (A1-A3); conversely, given a space \mathcal{B} there exists an Hermite-Biehler func-
 $e(z)$ such that \mathcal{B} coincides with $\mathcal{B}(e)$ a obeys (A1–A3); conversely, given a space B there exists an Hermite-Biehler func-
tion $e(z)$ such that B coincides with $B(e)$ as sets and the respective norms satisfy
the equality $||f(\cdot)||_B = ||f(\cdot)||_{B(e)}$ [4, Chapter 2]. Th the equality $||f(\cdot)||_{\mathcal{B}} = ||f(\cdot)||_{\mathcal{B}(e)}$ [4, Chapter 2]. The function $e(z)$ is not unique;
a choice for it is
 $e(z) = -i\sqrt{\frac{\pi}{k(w_0, w_0)\operatorname{im}(w_0)}} (z - \overline{w_0}) k(z, w_0),$
where w_0 is some fixed complex number in \mathbb{C}^+ .

$$
e(z) = -i\sqrt{\frac{\pi}{k(w_0, w_0)\operatorname{im}(w_0)}}(z - \overline{w_0}) k(z, w_0),
$$

ifixed complex number in \mathbb{C}^+ .

where w_0 is som where w_0 is some fixed complex number in \mathbb{C}^+ . An entire function $g(z)$ is said to be associated
for every $f(z) \in \mathcal{B}$ and $w \in \mathbb{C}$,
 $\frac{g(z)f(w) - g(w)f(z)}{z - w} \in \mathcal{B}$. An entire function $g(z)$ is said to be associated to a de Branges space \mathcal{B} if
 $\text{very } f(z) \in \mathcal{B} \text{ and } w \in \mathbb{C},$
 $\frac{g(z)f(w) - g(w)f(z)}{z - w} \in \mathcal{B}.$

Set of associated functions is denoted assoc \mathcal{B} . It is well known

$$
\frac{g(z)f(w)-g(w)f(z)}{z-w}\in\mathcal{B}.
$$

for every $f(z) \in \mathcal{B}$ and $w \in \mathbb{C}$,

The set of associated functions

see [4, Theorem 25] and [9, Le The set of associated functions is denoted assoc \mathcal{B} . It is well known that

$$
assoc\mathcal{B} = \mathcal{B} + z\mathcal{B};
$$

 $\frac{(z)f(w) - g(w)f(z)}{z - w}$

ns is denoted assoc,

assoc $B = B + zE$

emma 4.5] for alter

assoc $B(e) \setminus B(e)$; set of associated functions is denoted assoc \mathcal{B} . It is well known that
assoc $\mathcal{B} = \mathcal{B} + z\mathcal{B}$;
see [4, Theorem 25] and [9, Lemma 4.5] for alternative characterizations
ing, let us note that $e(z) \in \operatorname{assoc} \mathcal{B}($ assoc $\mathcal{B} = \mathcal{B} + z\mathcal{B}$;

ama 4.5] for altern

ssoc $\mathcal{B}(e) \setminus \mathcal{B}(e)$; t

ains a distinctive : ing, let us note that $e(z) \in \operatorname{assoc} \mathcal{B}(e) \setminus \mathcal{B}(e)$; this fact follows easily from [4,
Theorem 25].
The space assoc $\mathcal{B}(e)$ contains a distinctive family of entire functions. They
are given by
 $s_{\beta}(z) := \frac{i}{2} [e^{i\beta}$

ing, let us note that $e(z) \in \operatorname{assoc} \mathcal{B}(e) \setminus \mathcal{B}(e)$; this fact follows easily from [4,
Theorem 25].
The space assoc $\mathcal{B}(e)$ contains a distinctive family of entire functions. They
are given by
 $s_{\beta}(z) := \frac{i}{2} [e^{i\beta}$ are given by

$$
s_{\beta}(z) := \frac{i}{2} \left[e^{i\beta} e(z) - e^{-i\beta} e^{\#}(z) \right], \quad \beta \in [0, \pi).
$$

The space assoc $\mathcal{B}(e)$ contains a distinctive family of entire functions. They
iven by
 $s_{\beta}(z) := \frac{i}{2} \left[e^{i\beta} e(z) - e^{-i\beta} e^{\#}(z) \right], \quad \beta \in [0, \pi).$
e real entire functions are related to the selfadjoint extensions of the plication operator S defined by

$$
dom(S) := \{ f(z) \in \mathcal{B} : zf(z) \in \mathcal{B} \}, \quad (Sf)(z) = zf(z). \tag{10}
$$

 $(z) := \frac{i}{2} \left[e^{i\beta} e(z) - e^{-i\beta} e^{\#}(z) \right], \quad \beta \in [0, \pi).$
notions are related to the selfadjoint extens
i defined by
 $S) := \{ f(z) \in \mathcal{B} : zf(z) \in \mathcal{B} \}, \quad (Sf)(z) = zf$
gular, closed symmetric operator with deficially densely defined $\frac{2}{3}$
 $\frac{1}{3}$ plication operator *S* defined by
 $dom(S) := \{f(z) \in \mathcal{B} : zf(z) \in \mathcal{B}\}, (Sf)(z) = zf(z).$ (10)

This is a simple, regular, closed symmetric operator with deficiency indices (1, 1)

which is not necessarily densely defined [9, Proposi dom(S) := { $f(z) \in \mathcal{B} : z f(z) \in \mathcal{B}$ }, (Sf)(z) = $zf(z)$. (10)

ble, regular, closed symmetric operator with deficiency indices (1, 1)

ecessarily densely defined [9, Proposition 4.2, Corollary 4.3, Corollary

out that This is a simple, regular, closed symmetric operator with deficiency indices (1, 1)
which is not necessarily densely defined [9, Proposition 4.2, Corollary 4.3, Corollary
4.7]. It turns out that $\overline{\text{dom}(S)} \neq \mathcal{B}$ if a 4.7]. It turns out that $\overline{\text{dom}(S)} \neq \mathcal{B}$ if and only if there exists $\gamma \in [0, \pi)$ such that $s_{\gamma}(z) \in \mathcal{B}$. Furthermore, $\text{dom}(S)^{\perp} = \text{span}\{s_{\gamma}(z)\}$ [4, Theorem 29] and [9, Corollary 6.3]; compare with (i) of Pro that $s_{\gamma}(z) \in \mathcal{B}$. Furthermore, $\text{dom}(S)^{\perp} = \text{span}\{s_{\gamma}(z)\}$ [4, Theorem 29] and [9, Corollary 6.3]; compare with (i) of Proposition 2.1.
For any selfadjoint extension S_{\sharp} of S there exists a unique β in [0,

For any selfadjoint extension S_{H} of S there exists a unique β in $[0, \pi)$ such

For any selfadjoint extension
$$
S_{\sharp}
$$
 of *S* there exists a unique β in $[0, \pi)$ such
that

$$
(S_{\sharp} - wI)^{-1}f(z) = \frac{f(z) - \frac{s_{\beta}(z)}{s_{\beta}(w)}f(w)}{z - w}, \quad w \in \mathbb{C} \setminus \text{spec}(S_{\sharp}), \quad f(z) \in \mathcal{B}. \tag{11}
$$
Moreover, $\text{spec}(S_{\sharp}) = \{x \in \mathbb{R} : s_{\beta}(x) = 0\}$. [9, Propositions 4.6 and 6.1]. If S_{\sharp} is a
selfadjoint operator extension of *S*, then (11) is equivalent to

$$
\text{dom}(S_{\sharp}) = \left\{g(z) = \frac{f(z) - \frac{s_{\beta}(z)}{s_{\beta}(z_0)}f(z_0)}{z - z_0}, \quad f(z) \in \mathcal{B}, \quad z_0 : s_{\beta}(z_0) \neq 0\right\},
$$

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$$
(S_{\sharp} - wI)^{-1}f(z) = \frac{f(z) - \frac{s_{\beta}(z)}{s_{\beta}(w)}f(w)}{z - w}, \quad w \in \mathbb{C} \setminus \text{spec}(S_{\sharp}), \quad f(z) \in \mathcal{B}. \tag{11}
$$

Moreover, $\text{spec}(S_{\sharp}) = \{x \in \mathbb{R} : s_{\beta}(x) = 0\}$. [9, Propositions 4.6 and 6.1]. If S_{\sharp} is a
selfadjoint operator extension of *S*, then (11) is equivalent to

$$
\text{dom}(S_{\sharp}) = \left\{g(z) = \frac{f(z) - \frac{s_{\beta}(z)}{s_{\beta}(z_0)}f(z_0)}{z - z_0}, \quad f(z) \in \mathcal{B}, \quad z_0 : s_{\beta}(z_0) \neq 0\right\},
$$

$$
(S_{\sharp}g)(z) = zg(z) + \frac{s_{\beta}(z)}{s_{\beta}(z_0)}f(z_0).
$$
The eigenfunction g_x corresponding to $x \in \text{spec}(S_{\sharp})$ is given (up to normalization) by

$$
g_x(z) = \frac{s_{\beta}(z)}{z - x}.
$$
Thus, since *S* is regular and simple, every $s_{\beta}(z)$ has only real zeros of multiplicity
one and the (sets of) zeros of any pair $s_{\beta}(z)$ and $s_{\beta'}(z)$ are always interlaced.

$$
g_x(z) = \frac{s_\beta(z)}{z - x}.
$$

The eigenfunction g_x corresponding to $x \in \text{spec}(S_\sharp)$ is given (up to normaliza-
tion) by
 $g_x(z) = \frac{s_\beta(z)}{z - x}$.
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one and the (sets of

 $g_x(z) = \frac{s_\beta(z)}{z - x}$.
Thus, since *S* is regular and simple, every $s_\beta(z)$ one and the (sets of) zeros of any pair $s_\beta(z)$ and The proof of the following result can be for selfadjoint extensions of *S*. Another proof, when Thus, since *S* is regular and simple, every $s_{\beta}(z)$ has only real zeros of multiplicity
one and the (sets of) zeros of any pair $s_{\beta}(z)$ and $s_{\beta'}(z)$ are always interlaced.
The proof of the following result can be f one and the (sets of) zeros of any pair $s_{\beta}(z)$ and $s_{\beta'}(z)$ are always interlaced.
The proof of the following result can be found in [21] for a particular pair selfadjoint extensions of S. Another proof, when the ope selfadjoint extensions of *S*. Another proof, when the operator *S* is densely defined, is given in [18, Proposition 3.9].

3.1. Suppose $e(x) \neq 0$ for $x \in \mathbb{R}$ and $e(0) = (\sin \gamma)^{-1}$ for some fixed
Let $\{x_n\}_{n \in \mathbb{N}}$ be the sequence of zeros of the function $s_{\gamma}(z)$. Also, let
nd $\{x_n^-\}_{n \in \mathbb{N}}$ be the sequences of positive, respecti **Proposition 3.1.** Suppose $(x) \neq 0$ for $x \in \mathbb{R}$ and $e(0) = (\sin \gamma)^{-1}$ for some fixed
 e the sequence of zeros of the function $s_{\gamma}(z)$. Also, let

the sequences of positive, respectively negative, zeros

and only if the following conditions h $\begin{array}{c} \gamma \in \\ 0 \end{array}$ (0, π). Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence of zeros of the function $s_\gamma(z)$. Also, let $\{x_n\}_{n\in\mathbb{N}}$ be the sequences of positive, respectively negative, zeros (z) , arranged according to increasing modulus. The ${x_n^+}_{n \in \mathbb{N}}$ and ${x_n^-}_{n \in \mathbb{N}}$ be the sequences of positive, respectively negative, zeros of $s_{\gamma}(z)$, arranged according to increasing modulus. Then a zero-free, real entire
function belongs to $\mathcal{B}(e)$ if and only if the following conditions hold true:
(C1) The limit $\lim_{r\to\infty} \sum_{0<|x_n|\leq r} \frac{1}{x_n}$ exis

function belongs to
$$
\mathcal{B}(e)
$$
 if and only if the following conditions hold true:\n(C1) The limit $\lim_{r \to \infty} \sum_{0 < |x_n| \le r} \frac{1}{x_n}$ exists:\n(C2) $\lim_{n \to \infty} \frac{n}{x_n^+} = -\lim_{n \to \infty} \frac{n}{x_n^-} < \infty;$ \n(C3) Assuming that $\{b_n\}_{n \in \mathbb{N}}$ are the zeros of $s_\beta(z)$, define\n
$$
\left(\lim_{r \to \infty} \prod_{v \in \mathbb{N}} \left(1 - \frac{z}{b_n}\right) \right) \quad \text{if } 0 \text{ is not a root of } s_\beta(z)
$$

(C2)
$$
\lim_{n \to \infty} \frac{1}{x_n^+} = -\lim_{n \to \infty} \frac{1}{x_n^-} < \infty;
$$

\n- \n (C2)
$$
\lim_{n \to \infty} \frac{1}{x_n^+} = -\lim_{n \to \infty} \frac{1}{x_n^-} < \infty
$$
,\n
\n- \n (C3) *Assuming that* $\{b_n\}_{n \in \mathbb{N}}$ are the zeros of $s_{\beta}(z)$, define\n
$$
h_{\beta}(z) :=\n \begin{cases}\n \lim_{r \to \infty} \prod_{|b_n| \leq r} \left(1 - \frac{z}{b_n}\right) & \text{if } 0 \text{ is not a root of } s_{\beta}(z), \\
 z \lim_{r \to \infty} \prod_{0 < |b_n| \leq r} \left(1 - \frac{z}{b_n}\right) & \text{otherwise.} \\
 \end{cases}
$$
\n*The series* $\sum_{n \in \mathbb{N}} \left| \frac{1}{h_0(x_n) h'_\gamma(x_n)} \right|$ is convergent.
\n
\nProof. Combine Theorem 3.2 of [21] with Lemmas 3.3 and 3.4 of [18].

\n\n- \n**4. A functional model for operators in** \n $\mathcal{S}(\mathcal{H})$ \n
\n- \n The functional model given in this section follows the construction, develop the condition of the image.\n
\n

Proof.

4. A functional model for operators in $\mathcal{S}(\mathcal{H})$

Proof. Combine Theorem 3.2 of [21] with Lemmas 3.3 and 3.4 of [18]. □
 4. A functional model for operators in $S(\mathcal{H})$

The functional model given in this section follows the construction developed in [18], now adapt [18], now adapted to include all the operators in the class $S(\mathcal{H})$. This functional
model is based on (the properties of) the operator mentioned in (iii) of Proposi-
tion 2.1 with the following addition.
Proposition 4

[16], now adapted to include all the operators in the class $O(\mathcal{H})$. This functional
model is based on (the properties of) the operator mentioned in (iii) of Proposi-
tion 2.1 with the following addition.
Proposition 4 Proposition 4.1. Given $A \in S(\mathcal{H})$, let J be an involution that commutes with one
of its selfadjoint extensions within \mathcal{H} (hence with all of them), say, A_{γ} . Choose
 $v \in \text{spec}(A_{\gamma})$. Then, there exists $\psi_v \$ **Proposition 4.1.** Given $A \in S(\mathcal{H})$, l
of its selfadjoint extensions within $v \in \text{spec}(A_{\gamma})$. Then, there exists ψ_v
Proof. Let ϕ_v be an element of kerement is the immediately obtains that $J\phi_v \in \mathcal{H}$ **Proposition 4.1.** Given $A \in \mathcal{S}$ of its selfadjoint extensions within H (hence with all of them), say, A_{γ} . Choose $v \in \text{spec}(A_{\gamma})$. Then, there exists $\psi_v \in \text{ker}(A^* - vI)$ such that $J\psi_v = \psi_v$.

(*H*), let *J* be an involution that commutes with one

sithin *H* (hence with all of them), say, A_{γ} . Choose

sts $\psi_v \in \ker(A^* - vI)$ such that $J\psi_v = \psi_v$.

of $\ker(A_{\gamma} - vI)$. Since *J* commutes with A_{γ} , one
 $v \in \ker$ (hence with all of them), say, A_{γ} . Choose
ker($A^* - vI$) such that $J\psi_v = \psi_v$.
 $L_{\gamma} - vI$). Since J commutes with A_{γ} , one
 $A_{\gamma} - vI$). But, by our assumption on the
ty, ker($A^* - vI$) is a one-dimensional space
 r spec(A_{γ}). Then, there exists $\psi_v \in \ker(A^* - vI)$ such that $J\psi_v = \psi_v$.

of. Let ϕ_v be an element of $\ker(A_{\gamma} - vI)$. Since J commutes with

nediately obtains that $J\phi_v \in \ker(A_{\gamma} - vI)$. But, by our assumptic

ciency in *Proof.* Let ϕ_v be an element of ker $(A_\gamma - vI)$. Since J commutes with A_γ , one Let ϕ_v be an element of ker($A_{\gamma} - vI$). Since J commutes with A_{γ} , one
iately obtains that $J\phi_v \in \text{ker}(A_{\gamma} - vI)$. But, by our assumption on the
icy indices of A and its regularity, ker($A^* - vI$) is a one-dime immediately obtains that $J\phi_v \in \text{ker}(A_{\gamma} - vI)$. But, by our assumption on the
deficiency indices of A and its regularity, $\text{ker}(A^* - vI)$ is a one-dimensional space
and it contains $\text{ker}(A_{\gamma} - vI)$. So, in $\text{ker}(A_{\gamma} - vI)$ deficiency indices of *A* and its regularity, ker($A^* - vI$) is a one-dimensional space
and it contains ker($A_{\gamma} - vI$). So, in ker($A_{\gamma} - vI$), *J* reduces to multiplication by
a scalar α and the properties of the invo and it contains ker($A_{\gamma} - vI$). So, in ker($A_{\gamma} - vI$), J reduces to multiplication by
a scalar α and the properties of the involution imply that $|\alpha| = 1$. Now, $\psi_v := (1 + \alpha)\phi_v$ has the required properties.
Given $A \in S(\$

a scalar α and the properties of the involution imply that $|\alpha| = 1$. Now, $\psi_v := (1 + \alpha)\phi_v$ has the required properties.

Given $A \in S(\mathcal{H})$ and an involution J that commutes with its selfadjoint extensions within \math (1+ α) ϕ_v has the required properties. \Box

Given $A \in \mathcal{S}(\mathcal{H})$ and an involution J that commutes with its selfadjoint extensions within \mathcal{H} , define $\xi_{\gamma,v}(z) := h_{\gamma}(z) [I + (z - v)(A_{\gamma} - zI)^{-1}] \psi_v$, (12)

where

$$
\xi_{\gamma,v}(z) := h_{\gamma}(z) \left[I + (z - v)(A_{\gamma} - zI)^{-1} \right] \psi_v, \tag{12}
$$

Given $A \in \mathcal{S}(\mathcal{H})$ and an involution J that commutes with its selfadjoint
sions within \mathcal{H} , define
 $\xi_{\gamma,v}(z) := h_{\gamma}(z) [I + (z - v)(A_{\gamma} - zI)^{-1}] \psi_v$, (12)
e v and ψ_v are chosen as in the previous proposition, an extensions within π , define
 $\xi_{\gamma,v}(z) :=$

where v and ψ_v are chosen

entire function whose zero s $(z) := h_{\gamma}(z) [I + (z - v)(A_{\gamma} - zI)^{-1}] \psi_{v}$, (12)
chosen as in the previous proposition, and $h_{\gamma}(z)$ is a real
zero set is spec(A_{γ}) (see Proposition 2.4 (i)). Clearly, up to where v and ψ_v are chosen as in the previous proposition, and $h_{\gamma}(z)$ is a real
entire function whose zero set is spec(A_{γ}) (see Proposition 2.4 (i)). Clearly, up to
entire function whose zero set is spec(A_{γ} entire function whose zero set is spec() (see Proposition 2.4 (i)). Clearly, up to

160 L.O. Silva and J.H. Toloza
a zero-free real entire function, $\xi_{\gamma,\nu}(z)$ is completely determined by the choice of
the selfadjoint extension A_{γ} and ν . Actually, as it is stated more precisely below, a zero-free real entire function, $\xi_{\gamma,v}(z)$ is completely determined by the choice of
the selfadjoint extension A_{γ} and v. Actually, as it is stated more precisely below,
 $\xi_{\gamma,v}(z)$ does not depend on A_{γ} nor o $\xi_{\gamma,v}(z)$ does not depend on A_{γ} nor on v.

Proposition 4.2.

- the selfadjoint extension A_{γ} and v. Actually, as it is stated more precisely below,
 $\xi_{\gamma,v}(z)$ does not depend on A_{γ} nor on v.
 Proposition 4.2.

(i) The vector-valued function $\xi_{\gamma,v}(z)$ is zero-free and e (z) does not depend on A_{γ} nor on v .
 position 4.2.

The vector-valued function $\xi_{\gamma,v}(z)$ a

zI) for every $z \in \mathbb{C}$.
 $J\xi_{\gamma,v}(z) = \xi_{\gamma,v}(\overline{z})$ for all $z \in \mathbb{C}$.

Given $\xi_{\gamma_1,v_1}(z)$ and $\xi_{\gamma_2,v_2}(z)$ zI) for every $z \in \mathbb{C}$.
-
- (i) The vector-valued function $\xi_{\gamma,\upsilon}(z)$ is zero-free and entire. It lies in ker($A^* zI$) for every $z \in \mathbb{C}$.

(ii) $J\xi_{\gamma,\upsilon}(z) = \xi_{\gamma,\upsilon}(\overline{z})$ for all $z \in \mathbb{C}$.

(iii) Given $\xi_{\gamma_1,\upsilon_1}(z)$ and ξ_{γ_2,\ups such that $\xi_{\gamma_2,v_2}(z) = g(z)\xi_{\gamma_1,v_1}(z)$.

) for every $z \in \mathbb{C}$.
 $\zeta_{\gamma,\nu}(z) = \xi_{\gamma,\nu}(\overline{z})$ for $\zeta_{\gamma_1,\nu_1}(z)$ and

ch that $\xi_{\gamma_2,\nu_2}(z)$ =

Due to (iii) of Pr

one should only f (ii) $J\xi_{\gamma,\nu}(z) = \xi_{\gamma,\nu}(\overline{z})$ for all $z \in \mathbb{C}$.

(iii) Given $\xi_{\gamma_1,\nu_1}(z)$ and $\xi_{\gamma_2,\nu_2}(z)$, th

such that $\xi_{\gamma_2,\nu_2}(z) = g(z)\xi_{\gamma_1,\nu_1}(z)$

Proof. Due to (iii) of Proposition 2.

in fact, one should only f (iii) Given $\xi_{\gamma_1,v_1}(z)$ and $\xi_{\gamma_2,v_2}(z)$, there exists a zero-free real entire function $g(z)$
such that $\xi_{\gamma_2,v_2}(z) = g(z)\xi_{\gamma_1,v_1}(z)$.
Proof. Due to (iii) of Proposition 2.1, the proof of (i) is rather straightfo $(z) = g(z)\xi_{\gamma_1,v_1}(z)$.
of Proposition 2.1,
only follow the first
ows easily from ou
(iii), one first uses
1 to obtain that ξ
ion Then the reali *Proof.* Due to (iii) of Proposition 2.1, the proof of (i) is rather straightforward. one should only follow the first part of the proof of [18, Lemma 4.1]. The f (ii) also follows easily from our choice of ψ_w and $h_{\gamma}(z)$ in the definition (z) . To prove (iii), one first uses (iii) of Proposition 2.1 proof of (ii) also follows easily from our choice of ψ_w and $h_{\gamma}(z)$ in the definition
of $\xi_{\gamma,w}(z)$. To prove (iii), one first uses (iii) of Proposition 2.1 and the fact that
dim ker($A^* - wI$) = 1 to obtain that ξ of $\xi_{\gamma,w}(z)$. To prove (iii), one first uses (iii) of Proposition 2.1 and the fact that
dim ker($A^* - wI$) = 1 to obtain that $\xi_{\gamma_2,w_2}(z)$ and $\xi_{\gamma_1,w_1}(z)$ differ by a nonzero
scalar complex function. Then the real

scalar complex function. Then the reality of this function follows from (ii). \Box
For the reason already explained, from now on the function $\xi_{\gamma,v}(z)$ will be denoted by $\xi(z)$. Now define scalar complex function. Then the reality of this function follows from (ii). □

For the reason already explained, from now on the function $\xi_{\gamma,\nu}(z)$ will be

denoted by $\xi(z)$. Now define
 $(\Phi\varphi)(z) := \langle \xi(\overline{z}), \varphi \rangle, \q$

$$
(\Phi \varphi)(z) := \langle \xi(\overline{z}), \varphi \rangle, \qquad \varphi \in \mathcal{H}.
$$

For the reason already explained, from now on the function $\xi_{\gamma,v}(z)$ will be
ted by $\xi(z)$. Now define
 $(\Phi\varphi)(z) := \langle \xi(\overline{z}), \varphi \rangle$, $\varphi \in \mathcal{H}$.
pps $\mathcal H$ onto a certain linear manifold $\hat{\mathcal H}$ of entire functions. denoted by $\xi(z)$. Now define
 $(\Phi \varphi)$
 Φ maps $\mathcal H$ onto a certain lin

it follows that Φ is injective

a reminder of the fact that i

The linear space $\hat{\mathcal H}$ is t $(\Phi \varphi) (z) := \langle \xi(\overline{z}), \varphi \rangle, \qquad \varphi \in \mathcal{H}.$

a linear manifold $\widehat{\mathcal{H}}$ of entire functive. A generic element of $\widehat{\mathcal{H}}$ will

nat it is the image under Φ of a u

is turned into a Hilbert space by
 $\langle \widehat{\eta}(\cdot), \wide$ it follows that Φ is injective. A generic element of $\hat{\mathcal{H}}$ will be denoted by $\hat{\varphi}(z)$, as
a reminder of the fact that it is the image under Φ of a unique element $\varphi \in \mathcal{H}$.
The linear space $\hat{\mathcal{H}}$ is a reminder of the fact that it is the image under Φ of a unique element $\varphi \in \mathcal{H}$.

a reminder of the fact that it is the image under Φ of a unique element $\varphi \in \mathcal{H}$.
The linear space $\hat{\mathcal{H}}$ is turned into a Hilbert space by defining
 $\langle \hat{\eta}(\cdot), \hat{\varphi}(\cdot) \rangle := \langle \eta, \varphi \rangle$.
Clearly, Φ is an isom

$$
\langle \widehat{\eta}(\cdot), \widehat{\varphi}(\cdot) \rangle := \langle \eta, \varphi \rangle \; .
$$

Proposition 4.3. H is a de Branges space.

The linear space \mathcal{H} is turned into a Hilbert space by defining
 $\langle \hat{\eta}(\cdot), \hat{\varphi}(\cdot) \rangle := \langle \eta, \varphi \rangle$.

ly, Φ is an isometry from \mathcal{H} onto $\hat{\mathcal{H}}$.
 osition 4.3. $\hat{\mathcal{H}}$ is a de Branges space.

f. It su $\langle \cdot \rangle, \widehat{\varphi}(\cdot) \rangle := \langle \eta, \varphi \rangle.$
 ℓ onto $\widehat{\mathcal{H}}$.
 $ges\ space.$

e axioms given at the state of the st **Proposition 4.3.** $\hat{\mathcal{H}}$ is a de Branges space
Proof. It suffices to show that the axioms
for $\hat{\mathcal{H}}$.
It is straightforward to verify that
kernel for $\hat{\mathcal{H}}$. This accounts for (A1). Proof.

for *H*.
It is straightforward to verify that $k(z, w) := \langle \xi(\overline{z}), \xi(\overline{w}) \rangle$ is a reproducing
kernel for $\hat{\mathcal{H}}$. This accounts for (A1).
Suppose $\hat{\varphi}(z) \in \hat{\mathcal{H}}$ has a zero at $z = w$. Then its preimage $\varphi \in \mathcal{H}$

is straightforward to verify that $k(z, w) := \langle \xi(\overline{z}), \xi(\overline{w}) \rangle$ is a reproducing
for $\widehat{\mathcal{H}}$. This accounts for (A1).
uppose $\widehat{\varphi}(z) \in \widehat{\mathcal{H}}$ has a zero at $z = w$. Then its preimage $\varphi \in \mathcal{H}$ lies in
 $-wI$). Th It is straightforward to verify that $k(z, w) := \langle \xi(\overline{z}), \xi(\overline{w}) \rangle$ is a reproducing

If for $\widehat{\mathcal{H}}$. This accounts for (A1).

Suppose $\widehat{\varphi}(z) \in \widehat{\mathcal{H}}$ has a zero at $z = w$. Then its preimage $\varphi \in \mathcal{H}$ lies in
 kernel for \mathcal{H} . This accounts for $(A1)$.

Suppose $\hat{\varphi}(z) \in \hat{\mathcal{H}}$ has a zero $\operatorname{ran}(A - wI)$. This allows one to set $\eta = (A - \overline{w}I)(A - wI)^{-1}$

Now, recalling (12) and applying the Suppose $\widehat{\varphi}(z) \in \mathcal{H}$ has a zero at $z = w$. Then its preimage $\varphi \in \mathcal{H}$ lies in $A - wI$). This allows one to set $\eta \in \mathcal{H}$ by
 $\eta = (A - \overline{w}I)(A - wI)^{-1}\varphi = \varphi + (w - \overline{w})(A_{\gamma} - wI)^{-1}\varphi$.

recalling (12) and applyin

$$
\eta = (A - \overline{w}I)(A - wI)^{-1}\varphi = \varphi + (w - \overline{w})(A_{\gamma} - wI)^{-1}\varphi.
$$

Now, recalling (12) and applying the resolvent identity one obtains

$$
\langle \xi(\overline{z}), \eta \rangle = \frac{z - w}{z - w} \langle \xi(\overline{z}), \varphi \rangle.
$$

= $(A - \overline{w}I)(A - wI)^{-1}\varphi = \varphi + (w - \overline{w})(A_{\gamma} - wI)$

; (12) and applying the resolvent identity one obta
 $\langle \xi(\overline{z}), \eta \rangle = \frac{z - \overline{w}}{z - w} \langle \xi(\overline{z}), \varphi \rangle$.
 w are related by a Cayley transform, the equality of the sequality $\langle \xi(\overline{z}), \eta \rangle = \frac{z - \overline{w}}{z - w} \langle \xi(\overline{z}), \varphi \rangle$.
Since η and φ are related by a Cayley transform, the equality of This proves (A2). Since η and φ are related by a Cayley transform, the equality of norms follows.
This proves (A2). The Spectra of Selfadjoint Extensions of Entire Operators 161

As for (A3), consider any $\hat{\varphi}(z) = \langle \xi(\overline{z}), \varphi \rangle$. Then, as a consequence of (ii) of

Proposition 4.2, one has $\hat{\varphi}^{\#}(z) = \langle \xi(\overline{z}), J\varphi \rangle$. \square

It is

As for (A3), consider any $\hat{\varphi}(z) = \langle \xi(\overline{z}), \varphi \rangle$. Then, as a consequence of (ii) of osition 4.2, one has $\hat{\varphi}^{\#}(z) = \langle \xi(\overline{z}), J\varphi \rangle$.
It is worth remarking that the last part of the proof given above shows that $\$ Proposition 4.2, one has $\hat{\varphi}^{\#}(z) = \langle \xi(\overline{z}), J\varphi \rangle$.

It is worth remarking that the last part of the proof given above shows that
 $^{\#} = \Phi J \Phi^{-1}$.

The following obvious assertion is the key of (every) functional m $^# = \Phi J \Phi^{-1}.$

= ΦJΦ⁻¹.

The following obvious assertion is the key of (every) functional model; we
 ate it for the sake of completeness.
 coposition 4.4. Let S be the multiplication operator on $\hat{\mathcal{H}}$ given by (10).

(i) S

Proposition 4.4. Let S be the multiplication operator on H given by

-
- it for the sake of completeness.
 osition 4.4. Let S be the multiplication operator on $\hat{\mathcal{H}}$ given by (10).
 $S = \Phi A \Phi^{-1}$ and dom(S) = Φ dom(A).

The selfadjoint extensions of S within $\hat{\mathcal{H}}$ are in one-one **Proposition 4.4.** Let S be the multip
(i) $S = \Phi A \Phi^{-1}$ and dom $(S) = \Phi$ d
(ii) The selfadjoint extensions of S
the selfadjoint extensions of A $(10).$ (i) $S = \Phi A \Phi^{-1}$ and dom(S) = Φ dom(A).

(ii) The selfadjoint extensions of S within

the selfadjoint extensions of A within

Then (ii) above can be stated more suc
 $\Phi(A_{\beta} - zI)^{-1}\Phi^{-1} = (S_{\beta} - zI)$ the selfadjoint extensions of A within H .

$$
\Phi(A_{\beta}-zI)^{-1}\Phi^{-1}=(S_{\beta}-zI)^{-1}, \quad z\in\mathbb{C}\setminus\operatorname{spec}(A_{\gamma}),
$$

(ii) above can be stated more succinctly by saying that
 $\Phi(A_{\beta} - zI)^{-1}\Phi^{-1} = (S_{\beta} - zI)^{-1}, \quad z \in \mathbb{C} \setminus \text{spec}(A_{\gamma}),$

for all β of a certain (common) parametrization of the selfadjoint extensions of

both A and S. This $\Phi(A_{\beta} - zI)^{-1}\Phi^{-1} = (S_{\beta} - zI)^{-1}, \quad z \in \mathbb{C} \setminus \text{spec}(A_{\gamma})$
Il β of a certain (common) parametrization of the selfadjoin A and S. This expression is of course valid even for the experator) selfadjoint extension of A. In $\Phi(A_{\beta} - zI)^{-1}\Phi^{-1} = (S_{\beta} - zI)^{-1}, \quad z \in \mathbb{C} \setminus \text{spec}(A_{\gamma}),$
certain (common) parametrization of the selfadjoint
S. This expression is of course valid even for the exc
selfadjoint extension of A. In passing we note that tl
 for all β of a certain (common) parametrization of the selfadjoint extensions of both A and S . This expression is of course valid even for the exceptional (i.e., non-operator) selfadjoint extension of A . In passi both *A* and *S*. This expression is of course valid even for the exceptional (i.e., non-operator) selfadjoint extension of *A*. In passing we note that the exceptional selfadjoint extension of a non-densely defined opera non-operator) selfadjoint extension of A. In passing we note that the exceptional
selfadjoint extension of a non-densely defined operator in $S(\mathcal{H})$ corresponds to the
selfadjoint extension of the operator S whose assoc selfadjoint extension of the operator S whose associated function lies in $\hat{\mathcal{H}}$.

5. Spectral characterization

In the previous section we constructed a functional model that associates a de Branges space to every operator A in $\mathcal{S}(\mathcal{H})$ in such a way that the operator of multiplication in the de Branges space is unitarily equ Branges space to every operator A in $S(\mathcal{H})$ in such a way that the operator of multiplication in the de Branges space is unitarily equivalent to A . The first task in this section is to single out the class of de Br multiplication in the de Branges space is unitarily equivalent to A. The first task
in this section is to single out the class of de Branges spaces corresponding to entire
operators in our functional model. Having found th operators in our functional model. Having found this class, we use the theory of de Branges spaces to give a spectral characterization of the multiplication operator for the class we found. This is how we give necessary an Branges spaces to give a spectral characterization of the multiplication operator for the class we found. This is how we give necessary and sufficient conditions on the spectra of two selfadjoint extensions of an entire o for the class we found. This is how we give necessary and sufficient conditions on
the spectra of two selfadjoint extensions of an entire operator.
The following proposition gives a characterization of the class of de Bra

the spectra of two selfadjoint extensions of an entire operator.
The following proposition gives a characterization of the class of de Branges
spaces corresponding to entire operators in our functional model.
Proposition The following proposition gives a characterization of the
spaces corresponding to entire operators in our functional model
Proposition 5.1. $A \in \mathcal{S}(\mathcal{H})$ is entire if and only if $\hat{\mathcal{H}}$ contain
function. s corresponding to entire operators in our functional model.
 osition 5.1. $A \in \mathcal{S}(\mathcal{H})$ is entire if and only if $\hat{\mathcal{H}}$ contains a zero-free entire

ion.
 $f: \text{Let } g(z) \in \hat{\mathcal{H}}$ be the function whose existence **Proposition 5.1.** $A \in \mathcal{S}$ function.

Proposition 5.1. $A \in \mathcal{S}(\mathcal{H})$ is entire if and only if $\hat{\mathcal{H}}$ contains a function.
Proof. Let $g(z) \in \hat{\mathcal{H}}$ be the function whose existence is assume exists (a unique) $\mu \in \mathcal{H}$ such that $g(z) \equiv \langle \xi(\overline{z$ (*H*) is entire if and only if *H* contains a zero-free entire

e the function whose existence is assumed. Clearly there

such that $g(z) \equiv \langle \xi(\overline{z}), \mu \rangle$. Therefore, μ is never orthogonal
 $\kappa \in \mathbb{C}$. That is, μ Proof. exists (a unique) $\mu \in \mathcal{H}$ such that $g(z) \equiv \langle \xi(\overline{z}), \mu \rangle$. Therefore, μ is never orthogonal
to ker($A^* - zI$) for all $z \in \mathbb{C}$. That is, μ is an entire gauge for the operator A.
The necessity is established b to ker $(A^* - zI)$ for all $z \in \mathbb{C}$. That is, μ is an entire gauge for the operator A.

The necessity is established by noting that the image of the entire gauge The necessity is established by noting that the image of the entire gauge
of Φ is a zero-free function. under Φ is a zero-free function. \Box

Proposition 5.2. For $A \in S(H)$, consider the selfadjoin
and A_{γ} , with $0 < \gamma < \pi$. Then A is entire with rea
if and only if $\text{spec}(A_0)$ and $\text{spec}(A_{\gamma})$ obey condition
Proposition 3.1. **Proposition 5.2.** For $A \in \mathcal{S}(\mathcal{H})$, consider the selfadjoint extensions (within H) A_0 (*H*), consider the selfadjoint extensions (within *H*) A_0
Then *A* is entire with real entire gauge μ ($J\mu = \mu$)
ad spec(A_{γ}) obey conditions (C1), (C2) and (C3) o_j
3.1 along with Proposition 5.1. \Box
A is and A_{γ} , with $0 \leq \gamma \leq \pi$. Then A is entire with real entire gauge μ ($J\mu = \mu$) $0 < \gamma < \pi$. Then A is entire with real entire gauge μ ($J\mu = \mu$)
spec(A_0) and spec(A_{γ}) obey conditions (C1), (C2) and (C3) of
l.
Proposition 3.1 along with Proposition 5.1. \Box
ck that when A is an entire opera if and only if Proposition 3.1.

Proof. Apply Proposition 3.1 along with Proposition 5.1.

spec(A_0) and spec(A_{γ}) obey conditions (C1), (C2) and (C3) of

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Troposition 3.1 along with Proposition 5.1. \square

k that when A is an entire operator with non-dense domain, it may
 A_0 or A_{γ} is not an oper 3.1.
y P
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case
llow Apply Proposition 3.1 along with Proposition 5.1. □
 \Box
 \Box remark that when A is an entire operator with non-dense domain, it may either A_0 or A_γ is not an operator (see Proposition 2.1 (ii)). Nevertheless, thi be that either A_0 or A_γ is not an operator (see Proposition 2.1 (ii)). Nevertheless,
even in this case, $spec(A_0)$ and $spec(A_\gamma)$ satisfy (C1), (C2) and (C3).
The following proposition shows, among other things, that the be that either A_0 or A_γ is not an operator (see Proposition 2.1 (ii)). Nevertheless,
even in this case, $spec(A_0)$ and $spec(A_\gamma)$ satisfy (C1), (C2) and (C3).
The following proposition shows, among other things, that the

even in this case, spec(A_0) and spec(A_{γ}) satisfy (C1), (C2) and (C3).

The following proposition shows, among other things, that the or-

tional model by Krein is a particular case of our functional model.
 Propo

Proposition 5.3. Assume

$$
h_{\gamma}(z) = \left\langle \psi_v + (z - v)(A_{\gamma} - zI)^{-1} \psi_v, \mu \right\rangle^{-1}
$$

The following proposition 5.3. Assume $1 \in \hat{\mathcal{H}}$. Then there exists $\mu \in \mathcal{H}$ such that
 $h_{\gamma}(z) = \langle \psi_v + (z - v)(A_{\gamma} - zI)^{-1} \psi_v, \mu \rangle^{-1}$
 $J\mu = \mu$. Moreover, μ is the unique entire gauge of A modulo a real scalar
 Proposition 5.3. Assume $1 \in \hat{\mathcal{H}}$. Then there exists $\mu \in \mathcal{H}$ such that
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and $J\mu = \mu$. Moreover, μ is the unique entire gauge of A modu-

factor. 1 ∈ *H*. Then there exists $\mu \in \mathcal{H}$ such that
= $\langle \psi_v + (z - v)(A_\gamma - zI)^{-1} \psi_v, \mu \rangle^{-1}$
u is the unique entire gauge of *A* modul
(\overline{z}), μ) for some $\mu \in \mathcal{H}$. By (12), and takin
obtains the stated expression and $J\mu$ factor.

 $(z) = \langle \psi_v + (z - v)(A_{\gamma} - zI) \rangle$

er, μ is the unique entire ga
 $\equiv \langle \xi(\overline{z}), \mu \rangle$ for some $\mu \in \mathcal{H}$. By

one obtains the stated exprecis shown. = μ . Moreover, μ is the unique entire gauge of A modulo a real scalar
lecessarily, $1 \equiv \langle \xi(\overline{z}), \mu \rangle$ for some $\mu \in \mathcal{H}$. By (12), and taking into account
urrence of J, one obtains the stated expression for $h_{\$ *Proof.* Necessarily, $1 \equiv \langle \xi(\overline{z}), \mu \rangle$ for some $\mu \in \mathcal{H}$. By (12), and taking into account

Necessarily, $1 \equiv \langle \xi(\overline{z}), \mu \rangle$ for some $\mu \in \mathcal{H}$. By (12), and taking into account
currence of *J*, one obtains the stated expression for $h_{\gamma}(z)$. By the same
the reality of μ is shown.
uppose that there are tw the occurrence of J , one obtains the stated expression for $h_{\gamma}(z)$. By the same
token, the reality of μ is shown.
Suppose that there are two real entire gauges μ and μ' . The discussion in
Paragraph 5.2 of [7] token, the reality of μ is shown.
Suppose that there are two
Paragraph 5.2 of [7] shows that
the assumed reality, one conclude
6. Concluding remarks Suppose that there are two real entire gauges μ and μ' . The discussion in
graph 5.2 of [7] shows that $(\Phi_{\mu}\mu')(z) = ae^{ibz}$ with $a \in \mathbb{C}$ and $b \in \mathbb{R}$. Due to
ssumed reality, one concludes that $b = 0$ and $a \in \mathbb$

6. Concluding remarks

the assumed reality, one concludes that $b = 0$ and $a \in \mathbb{R}$. \Box
 6. Concluding remarks

We would like to add some few comments concerning further extensions of the present work. the assumed reality, one concludes that $b = 0$ and $a \in \mathbb{R}$. \Box
 Concluding remarks

would like to add some few comments concerning further extensions of the

mt work.

First, since there are de Branges spaces that

First, since there are de Branges spaces that contain the constant functions
but whose multiplication operator is not densely defined, it follows that, apart
from the example given in Section 2, there should be other opera First, sin
but whose m
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entire operate
of our results whose multiplication operator is not densely defined, it follows that, apart
the example given in Section 2, there should be other operators in the class
duced in this work that are not comprised in the original Krein's no from the example given in Section 2, there should be other operators in the class introduced in this work that are not comprised in the original Krein's notion of entire operators. The details of our example as well as oth introduced in this work that are not comprised in the original Krein's notion of entire operators. The details of our example as well as other ones and applications of our results will be studied elsewhere. Second, it is

entire operators. The details of our example as well as other ones and applications
of our results will be studied elsewhere.
Second, it is possible to define a notion of a (possibly non-densely defined)
operator that is e end our results will be studied elsewhere.
Second, it is possible to define a notion of a (possibly non-densely defined)
operator that is entire in a generalized sense, much in the same vein as the original
definition by K Second, it is possible to define a 1
operator that is entire in a generalized s
definition by Krein for densely defined
Following [18, Section 5], operators ent
characterized by the spectra of their sel
Finally, it is know to that is entire in a generalized sense, much in the same vein as the original
ition by Krein for densely defined operators (see [7, Chapter 2, Section 9]).
wing [18, Section 5], operators entire in this generalized sense definition by Krein for densely defined operators (see [7, Chapter 2, Section 9]).
Following [18, Section 5], operators entire in this generalized sense could also be characterized by the spectra of their selfadjoint exte

Following [18, Section 5], operators entire in this generalized sense could also be characterized by the spectra of their selfadjoint extensions.
Finally, it is known that the set of selfadjoint operator extensions within characterized by the spectra of their selfadjoint extensions.
Finally, it is known that the set of selfadjoint operator extensions within $\mathcal H$ of a non-densely defined operator are in one-one correspondence with a set o Finally, it is known that the set of selfadjoint opera $\mathcal H$ of a non-densely defined operator are in one-one corresponding the spectra of the selfadjoint operator 2. This set of rank-one perturbations is generated by el a non-densely defined operator are in one-one correspondence with a set of one perturbations of one of these selfadjoint operator extensions [8, Section his set of rank-one perturbations is generated by elements in H so $\overline{}$ is a non-dependent operator extensions [8, Section This set of rank-one perturbations is generated by elements in $\mathcal H$ so it seems eresting to study the relation (if any) between these elements and the gauges of set of 2]. This set of rank-one perturbations is generated by elements in $\mathcal H$ so it seems interesting to study the relation (if any) between these elements and the gauges of interesting to study the relation (if any) between these elements and the gauges of

 $S(\mathcal{H})$. Ultimately, we believe that a suitable characterization of the erturbations could provide another necessary and sufficient condition ensely defined operator in $S(\mathcal{H})$ to be entire. This problem, as well as s operators in $\mathcal{O}(\mathcal{H})$. Ultimately, we believe that a suitable characterization of the
rank-one perturbations could provide another necessary and sufficient condition
for a non-densely defined operator in $\mathcal{S}(\mathcal{$ for a non-densely defined operator in $S(\mathcal{H})$ to be entire. This problem, as well as
the previous one, will be discussed in a subsequent work.
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for a non-densely defined operator in $O(\kappa)$ to be entire. This problem, as well as
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Part of this work was done while the second author (JUNAM in January 2011. He sincerely thanks them for the
Performance UNAM in January 2011. He sincerely thanks them for their kind hospitality.

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Asymptotics of Eigenvalues of an Energy Operator in a Problem of Quantum Physics

Eduard A. Yanovich

Abstract. In this paper we consider eigenvalues asymptotics of the energy operator in one of the most interesting models of quantum physics, describing an interaction between two-level system and harmonic oscillator. The energy operator in this model can be reduced to a class of infinite Jacobi matrices. Discrete spectrum of this class of operators represents the perturbed spectrum of harmonic oscillator. The perturbation is an unbounded operator compact with respect to unperturbed one. We use slightly modified Janas-Naboko successive diagonalization approach and some new compactness criteria for infinite matrices. First two terms of eigenvalues asymptotics and the estimation of remainder are found.

Mathematics Subject Classification (2010). Primary 47A75; Secondary 47B36. **Keywords.** Spectral theory, asymptotics of eigenvalues.

1. Introduction and main results

$$
\hat{\mathbf{H}} = \frac{\hbar\omega_0}{2}\,\hat{\sigma}_z + \hbar\omega\,\hat{\mathbf{a}}^+\,\hat{\mathbf{a}} + \hbar\lambda\,(\hat{\sigma}_+ + \hat{\sigma}_-)(\,\hat{\mathbf{a}} + \hat{\mathbf{a}}^+) \,,
$$

$$
\hat{\mathbf{H}} = \frac{\hbar\omega_0}{2}\,\hat{\sigma}_z + \hbar\omega\,\hat{\mathbf{a}}^+ \,\hat{\mathbf{a}} + \hbar\lambda\,(\hat{\sigma}_+ + \hat{\sigma}_-) (\hat{\mathbf{a}} + \hat{\mathbf{a}}^+) \,,
$$
\nwhere $\hat{\sigma}_z, \hat{\sigma}_+, \hat{\sigma}_-$ are the 2 × 2 matrices of form\n
$$
\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,, \quad \hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \,, \quad \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \,,
$$
\n
$$
\hat{\mathbf{a}} \text{ and } \hat{\mathbf{a}}^+ \text{ are the creation and annihilation operators of the harmonic } \lambda \text{ is the interaction constant, } \omega \text{ is the oscillator frequency, } \omega_0 \text{ is the transformation.}
$$

where $\hat{\sigma}_z$, $\hat{\sigma}_+$, $\hat{\sigma}_-$ are the 2 × 2 matrices of form
 $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
 \hat{a} and \hat{a}^+ are the creation and annihilation ope
 λ is the interaction c $\hat{\sigma}_z = \begin{pmatrix} 0 & -1 \end{pmatrix}$

are the creation

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the two-level synchrons 0 −1 \vec{a} , $\hat{\sigma}_+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
reation and annihilation of constant, ω is the oscillate
evel system. These matrice
ons
 \vec{b}_z , $[\hat{\sigma}_z, \hat{\sigma}_+] = 2 \hat{\sigma}_+$, $[\hat{\sigma}_z]$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\hat{\sigma}_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

on operators of the harm

illator frequency, ω_0 is th

trices and operators satis
 $[\hat{\sigma}_z, \hat{\sigma}_+] = -2 \hat{\sigma}_-, \quad [\hat{\mathbf{a}}, \hat{\sigma}_+]$ $\begin{pmatrix} 0 & 0 \ 1 & 0 \end{pmatrix}$
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satisfy
 $[\hat{\mathbf{a}}, \hat{\mathbf{a}}^+]$ $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^+$ are the creation and annihilation operators of the harmonic oscillator, **â** and $\hat{\mathbf{a}}^+$ are the creation and annihilation operators of the harmonic oscillator,
 λ is the interaction constant, ω is the oscillator frequency, ω_0 is the transition frequency in the two-level system. λ quency in the two-level system. These matrices and operators satisfy the following commutative relations commutative relations
 $[\hat{\sigma}_+,\hat{\sigma}_-]=\hat{\sigma}_z$, $[\hat{\sigma}_z,\hat{\sigma}_+] = 2\hat{\sigma}_+$, $[\hat{\sigma}_z,\hat{\sigma}_-] = -2\hat{\sigma}_-$, $[\hat{\mathbf{a}},\hat{\mathbf{a}}^+] = 1$

$$
[\hat{\sigma}_{+}, \hat{\sigma}_{-}] = \hat{\sigma}_{z}, \quad [\hat{\sigma}_{z}, \hat{\sigma}_{+}] = 2 \hat{\sigma}_{+}, \quad [\hat{\sigma}_{z}, \hat{\sigma}_{-}] = -2 \hat{\sigma}_{-}, \quad [\hat{\mathbf{a}}, \hat{\mathbf{a}}^{+}] = 1
$$

E.A. Yanovich
It was shown [13] that the hamiltonian of this model is represented by two Jacobi
matrices. These matrices have the following general form

matrices. These matrices have the following general form
\n
$$
A = \begin{pmatrix}\nc_1 & g\sqrt{1} & 0 & 0 & 0 & \dots \\
g\sqrt{1} & 1+c_2 & g\sqrt{2} & 0 & 0 & \dots \\
0 & g\sqrt{2} & 2+c_1 & g\sqrt{3} & 0 & \dots \\
0 & 0 & g\sqrt{3} & 3+c_2 & g\sqrt{4} & \dots \\
0 & 0 & 0 & g\sqrt{4} & 4+c_1 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots\n\end{pmatrix}
$$
\nwhere g , c_1 , c_2 are real parameters. It is well known [1, 2] that the matrix A defines a selfadjoint operator with simple spectrum and the domain $D(A)$ is dense in the space $l_a(\mathbb{N})$. Since the operator A can be considered as relatively compact in the space $l_b(\mathbb{N})$.

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bir
\{|} 0 0 $g\sqrt{3}$ $3+c_2$ $g\sqrt{4}$...

0 0 0 $g\sqrt{4}$ $4+c_1$...

...

l parameters. It is well known [1, 2] therator with simple spectrum and the dom-

e the operator A can be considered as represent and i 0 0 0 $g\sqrt{4}$ $4+c_1$...

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ul parameters. It is well known [1, 2] therator with simple spectrum and the dom-

e the operator A can be considered as represent in diagonal, its spectrum is disc $\begin{array}{c} \text{fix} \ A \ \text{lense} \ \text{update} \end{array}$ where g , c_1 , c_2 are real parameters. It is well known [1, 2] that the matrix A defines a selfadjoint operator with simple spectrum and the domain $D(A)$ is dense in the space $l_2(\mathbb{N})$. Since the operator A c

defines a selfadjoint operator with simple spectrum and the domain $D(A)$ is dense
in the space $l_2(\mathbb{N})$. Since the operator A can be considered as relatively compact
perturbation of the main diagonal, its spectrum is d in the space $l_2(\mathbb{N})$. Since the operator A can be considered as relatively compact
perturbation of the main diagonal, its spectrum is discrete.
The main goal of this paper is the investigation of the eigenvalues $\lambda_n(A$ The main goal of this paper is the investigation of behavior for large values of *n* with other parameters fixed. T
concerning similar problems [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].
The result of this paper is given by the f The main goal of this paper is the investigation of the eigenvalues $\lambda_n(A)$
vior for large values of *n* with other parameters fixed. There are many articles
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The

behavior for large values of *n* with other parameters fixed. There are many articles
concerning similar problems [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].
The result of this paper is given by the following asymptotic formula (T The result of this paper is given by the following asymone (3.3)
 $\lambda_n(A) = n - g^2 + \frac{c_1 + c_2}{2} + O\left(\frac{1}{n^{1/16}}\right)$, $n \to \infty$

From (3.3)

\n
$$
\lambda_n(A) = n - g^2 + \frac{c_1 + c_2}{2} + O\left(\frac{1}{n^{1/16}}\right), \quad n \to \infty \quad (g \neq 0).
$$
\n**2. Selection of the main component in the asymptotics**

\nLet us present the operator A in (1.1) in the form

\n
$$
A = A_0 + \frac{c_1 + c_2}{2} I + \frac{c_1 - c_2}{2} R,\tag{2.1}
$$

2. Selection of the main component in the asymptotics

$$
A = A_0 + \frac{c_1 + c_2}{2} I + \frac{c_1 - c_2}{2} R,
$$
\n(2.1)

Let us present the operator A in (1.1) in the form
\n
$$
A = A_0 + \frac{c_1 + c_2}{2} I + \frac{c_1 - c_2}{2} R,
$$
\nwhere I is the identical matrix, A_0 and R are defined in the following way
\n
$$
A_0 = \begin{pmatrix} 0 & g\sqrt{1} & 0 & 0 & \dots \\ g\sqrt{1} & 1 & g\sqrt{2} & 0 & \dots \\ 0 & g\sqrt{2} & 2 & g\sqrt{3} & \dots \\ 0 & 0 & g\sqrt{3} & 3 & \dots \end{pmatrix}, R = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}
$$
\nThe matrix A_0 represents so-called "shifted oscillator" operator: $a^+ a + g(a + a^+)$, where a^+ and a are the creation and annihilation operators. If we use the matrix representation of a^+ and a, we obtain exactly the matrix A_0 .
\nEigenvalues problem for the operator A, has an exact solution. This solution

 $\begin{array}{c}\n\mathbf{T} \\
\mathbf{W} \\
\mathbf{X} \\
\mathbf{X}\n\end{array}$ 0 0 $g\sqrt{3}$ 3 ... $\left[\begin{array}{cccc} 0 & 0 & 0 & -1 & \dots \\ \dots & \dots & \dots & \dots \\ a^+ & a^+ & \text{and } a \text{ are the creation and annihilation operators. If we use
\nessentiality of a^+ and a , we obtain exactly the matrix A_0 .
\nvalues problem for the operator A_0 has an exact solution. This solu-
\nsinged in different ways. For example, with the help of Bogolu-
\n*h*$ os
hi
ac a^+), where a^+ and a are the creation and annihilation operators. If we use the

The matrix A_0 represents so-called "shifted oscillator" operator: a^+a+g ($a+$ where a^+ and a are the creation and annihilation operators. If we use the ix representation of a^+ and a , we obtain exactly the), where a^+ and a are the creation and annihilation operators. If we use the trix representation of a^+ and a , we obtain exactly the matrix A_0 .
Eigenvalues problem for the operator A_0 has an exact solution. matrix representation of a^+ and a , we obtain exactly the matrix A_0 .

Eigenvalues problem for the operator A_0 has an exact solution.

can be obtained in different ways. For example, with the help of

transformat Eigenvalues problem for the operator A_0 has an exact solution. This solution
be obtained in different ways. For example, with the help of Bogolubov's
formation [14] or by using continued fractions [15, 16]. In the work transformation [14] or by using continued fractions [15, 16]. In the work [7] the operator A_0 was considered using the Bargmann space.
The solution of the eigenvalues problem for the operator A_0 has the form $A_0 a_n =$ operator A_0 was considered using the Bargmann space.
The solution of the eigenvalues problem for the operator A_0 has the form
 $A_0 a_n = \mu_n a_n$, $\mu_n = n - g^2$, $n = 0, 1, 2, ...$ (2.2)

The solution of the eigenvalues problem for the operator
$$
A_0
$$
 has the form
\n
$$
A_0 a_n = \mu_n a_n, \quad \mu_n = n - g^2, \quad n = 0, 1, 2, ...
$$
\n(2.2)

Asymptotics of Eigenvalues of an Energy Operator 167
Here a_n are normalized eigenvectors of the operator A_0 . Its expansion through the
basis vectors e_n of the matrix representation (1.1) has the form

Here
$$
a_n
$$
 are normalized eigenvectors of the operator A_0 . Its expansion through the
basis vectors e_n of the matrix representation (1.1) has the form
\n
$$
a_m = \sum_{n=0}^{\infty} U_{n,m} e_n,
$$
\n
$$
U_{n,m} \equiv \begin{bmatrix}\n\exp\{-g^2/2\} \sqrt{n!m!} \, g^{m-n} \\
\times \sum_{i=0}^n (-1)^i \frac{(g^2)^i}{i! (n-i)! (i+m-n)!}, \ m \ge n\n\end{bmatrix}
$$
\n
$$
U_{n,m} \equiv \begin{cases}\n\exp\{-g^2/2\} \sqrt{n!m!} \, g^{n-m} \\
\exp\{-g^2/2\} \sqrt{n!m!} \, g^{n-m} (-1)^{n-m} \\
\times \sum_{i=0}^m (-1)^i \frac{(g^2)^i}{i! (m-i)! (i+n-m)!}, \ n \ge m.\n\end{cases}
$$
\nUsing the definition of generalized Chebyshev-Laguerre polynomials $L_n^{(s)}(x)$ [19]

Using the definition of generalized Chebyshev-Laguerre polynomials
$$
L_n^{(s)}(x)
$$
 [19]
\n
$$
L_n^{(s)}(x) = \frac{(n+s)!}{n!} \sum_{i=0}^n C_n^i (-1)^i \frac{x^i}{(i+s)!}, \quad C_n^i = \frac{n!}{i!(n-i)!}, \quad (s \ge 0)
$$
\nand its property\n
$$
L_n^{(-s)}(x) = (-x)^s \frac{(n-s)!}{n!} L_{n-s}^{(s)}(x), \quad (s \ge 0), \quad (2.4)
$$
\nwe obtain

$$
\frac{C_{n-1}}{n!} \sum_{i=0}^{n} C_{n}^{i} (-1)^{i} \frac{1}{(i+s)!}, \quad C_{n}^{i} = \frac{1}{i! (n-i)!}, \quad (s \ge 0)
$$

$$
L_{n}^{(-s)}(x) = (-x)^{s} \frac{(n-s)!}{n!} L_{n-s}^{(s)}(x), \quad (s \ge 0), \quad (2.4)
$$

$$
L_n^{(-s)}(x) = (-x)^s \frac{(n-s)!}{n!} L_{n-s}^{(s)}(x), \quad (s \ge 0),
$$
\n(2.4)

\nwe obtain

\n
$$
U_{n,m} \equiv U_{n,m}(g) = \exp\{-g^2/2\} \sqrt{\frac{n!}{m!}} g^{m-n} L_n^{(m-n)}(g^2).
$$
\n(2.5)

\nThe simplest way to obtain (2.2) is to use Bogolubov's transformation. Its idea is following. It is well known that the spectrum of harmonic oscillator can

idea is following. It is well known that the spectrum of harmonic oscillator can The idea is following the obtained $\sum_{n=1}^{\infty}$ $(g) = \exp\{-g^2/2\} \sqrt{\frac{n!}{m!}} g^{m-n} L_n^{(m-n)}(g^2).$ (2.5)

obtain (2.2) is to use Bogolubov's transformation. Its

ll known that the spectrum of harmonic oscillator can

commutative property $[a, a^+] = 1$ of operators a and a^+
 ! is following. It is well known that the spectrum of harmonic oscillator can
tained through the commutative property $[a, a^+] = 1$ of operators a and a^+
Assume that $a = b + C$ $(a^+ = b^+ + C)$, where C is some real constant. It
d be obtained through the commutative property $[a, a^+] = 1$ of operators a and a^+
only. Assume that $a = b + C$ $(a^+ = b^+ + C)$, where C is some real constant. It
is evident that $[b, b^+] = 1$, and the spectrum of b^+b is the spec be obtained through the commutative property $[a, a^+] = 1$ of operators a and a^+
only. Assume that $a = b + C$ $(a^+ = b^+ + C)$, where C is some real constant. It
is evident that $[b, b^+] = 1$, and the spectrum of b^+b is the only. Assume that $a = b + C$ $(a^+ = b^+ + C)$, where *C* is some real constant. It
is evident that $[b, b^+] = 1$, and the spectrum of b^+b is the spectrum of harmonic
oscillator. On the other hand
 $A_0 = a^+a + g(a + a^+) = b^+b + (C + g)(b + b^+)$

$$
A_0 = a^+a + g(a + a^+) = b^+b + (C + g)(b + b^+) + C^2 + 2gC.
$$

is evident that $[b, b^+] = 1$, and the spectrum of b^+b is the spectrum of harmonic
oscillator. On the other hand
 $A_0 = a^+a + g(a + a^+) = b^+b + (C + g)(b + b^+) + C^2 + 2gC$.
If $C = -g$, $A_0 = b^+b - g^2$, i.e., A_0 is shifted oscillator. Fr $A_0 = a^+a + g(a+a^+) = b^+b + (C+g)(b+b^+) + C^2 + 2gC.$

If $C = -g$, $A_0 = b^+b - g^2$, i.e., A_0 is shifted oscillator. From that (2.2) once.

To receive (2.3) let us notice that the transition $a \to (a - g)$ can b by orthogonal transformat If $C = -g$, $A_0 = b^+b - g^2$, i.e., A_0 is shifted oscillator. From that (2.2) follows at
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by orthogonal transformation U
 $a - g = U^+aU$, U

once.

To receive (2.3) let us notice that the transition $a \to (a - g)$ can be obtained

by orthogonal transformation U
 $a - g = U^+ aU$, $U = e^{g(a - a^+)} = e^{-g^2/2} e^{-ga^+} e^{ga}$.

This transformation is well known in the theory of cohe

$$
a - g = U^+ a U
$$
, $U = e^{g(a - a^+)} = e^{-g^2/2} e^{-g a^+} e^{g a}$.

To receive (2.3) let us notice that the transition $a \to (a - g)$ can be obtained
thogonal transformation U
 $a - g = U^+ a U$, $U = e^{g(a-a^+)} = e^{-g^2/2} e^{-ga^+} e^{ga}$.
transformation is well known in the theory of coherent states (see fo by orthogonal transformation U
 $a - g = U^+ a U$,

This transformation is well know

ple [17]). Thus we have
 $U^+ A_0 U = a^+ a - g$ $= U^+ a U,$ $U = e^{g(a-a^+)} = e^{-g^2/2} e^{-ga^+}$
is well known in the theory of coherent s
ve
 $= a^+ a - g^2,$ $a_m = U e_m = e^{-g^2/2} e^{-ga^+}$

ple [I7]). Thus we have
\n
$$
U^{+}A_{0}U = a^{+}a - g^{2}, \quad a_{m} = Ue_{m} = e^{-g^{2}/2}e^{-ga^{4}}e^{ga}e_{m}.
$$

168 E.A. Yanovich
Expanding exponents to the series and using relations

$$
ae_m = \sqrt{m} e_{m-1}, \quad a^+ e_m = \sqrt{m+1} e_{m+1}, \quad m = 0, 1, 2, \dots,
$$
after simple algebraic transformations, we obtain (2.3) and hence (2.5).

tions (in coordinate representation) for complex values of the parameter g is considered in [18]. = $\sqrt{m} e_{m-1}$, $a^+ e_m = \sqrt{m+1} e_{m+1}$, $m = 0, 1, 2, \ldots$,
gebraic transformations, we obtain (2.3) and hence (2.5).
te here that the completeness of the "shifted oscillator"
inate representation) for complex values of the Let us note here that the completeness of the "shifted oscillator"
tions (in coordinate representation) for complex values of the paramete
sidered in [18].
Let us find the matrix of the operator R in the basis of the op
e

Let us find the matrix of the operator R in the basis of the operator A_0 tions (in coordinate representation) for complex values of the parameter g is con-
sidered in [18].
Let us find the matrix of the operator R in the basis of the operator A_0
eigenvectors. Denoting the elements of the tr Let us fin
eigenvectors. Γ
into account the
 \tilde{R} Let us find the matrix of the operator R in the basis of the operator A_0
vectors. Denoting the elements of the transformed matrix as $\tilde{R}_{k,m}$ and taking
account that $R_{n,m} = (-1)^n \delta_{n,m}$, we obtain
 $\tilde{R}_{k,m} = (Re_m, e_k$

eigenvectors. Denoting the elements of the transformed matrix as
$$
R_{k,m}
$$
 and taking
into account that $R_{n,m} = (-1)^n \delta_{n,m}$, we obtain

$$
\tilde{R}_{k,m} = (Re_m, e_k) = (U^T R U)_{k,m} = \sum_{n=0}^{\infty} (-1)^n U_{n,k} U_{n,m}.
$$
(2.6)
Let us represent matrix elements $U_{n,m} = U_{n,m}(g)$ as contour integral

$$
\tilde{R}_{k,m} = (Re_m, e_k) = (U^T RU)_{k,m} = \sum_{n=0}^{\infty} (-1)^n U_{n,k} U_{n,m}.
$$
\nLet us represent matrix elements $U_{n,m} = U_{n,m}(g)$ as contour integral

\n
$$
U_{n,m}(g) = \exp\{-g^2/2\} \sqrt{\frac{m!}{n!}} g^{n-m} \frac{1}{2\pi i} \oint_C x^{m-1} \left(\frac{1}{x} - 1\right)^n \exp\left\{\frac{g^2}{x}\right\} dx, \quad (2.7)
$$
\nwhere C is a unit circle centered in the origin of the complex plane x (C is positively oriented). This expression can be easily checked by calculation of integral with the

(g) = exp{ $-g^2/2$ } $\sqrt{\frac{m!}{n!}} g^{n-m} \frac{1}{2\pi i} \oint_C x^{m-1} \left(\frac{1}{x} - 1\right)^n \exp\left\{\frac{g^2}{x}\right\} dx$, (2.7)
 C is a unit circle centered in the origin of the complex plane *x* (*C* is positively

d). This expression can be easily !
}
3 $\frac{x}{\text{a}}$ er
1
1 where *C* is a unit circle centered in the origin of the complex plane *x* (*C* is positively oriented). This expression can be easily checked by calculation of integral with the help of residues.
Substituting (2.7) in (2

Substituting (2.7) in (2.6) and summing up over *n*, we find

help of residues.
\nSubstituting (2.7) in (2.6) and summing up over *n*, we find
\n
$$
\tilde{R}_{k,m} = \exp\{-2g^2\} \sqrt{k! m!} g^{-m-k} \frac{1}{(2\pi i)^2}
$$
\n
$$
\times \oint_C f(x)^{m-1} (x')^{k-1} \exp\left\{g^2 \left(\frac{2}{x} + \frac{2}{x'} - \frac{1}{xx'}\right)\right\} dx dx'.
$$
\nContour integrals in this expression can be calculated consistently with the help of residues as before. As a result, using (2.5), we obtain
\n
$$
\tilde{R}_{k,m} = (-1)^k U_{k,m}(2g).
$$
\n(2.8)

)
)]
;
; .
|
|).
!

$$
\tilde{R}_{k,m} = (-1)^k U_{k,m}(2g). \tag{2.8}
$$

of residues as before. As a result, using (2.5), we obtain
 $\tilde{R}_{k,m} = (-1)^k U_{k,m}(2g)$. (2.8)

In spite of seeming asymmetry, the matrix $\tilde{R}_{k,m}$ is symmetric $(\tilde{R}_{k,m} = \tilde{R}_{m,k})$.

the easily verified by means of the p $\tilde{R}_{k,m} = (-1)^k U_{k,m}(2g).$
In spite of seeming asymmetry, the matrix $\tilde{R}_{k,m}$ is symmetry.
It can be easily verified by means of the property (2.4).
Using the asymtotics of the generalized Chebyshev-Lagu $\tilde{R}_{k,m} = (-1)^k U_{k,m}(2g).$ (2.8)

metry, the matrix $\tilde{R}_{k,m}$ is symmetric $(\tilde{R}_{k,m} = \tilde{R}_{m,k}).$ eans of the property (2.4).

the generalized Chebyshev-Laguerre polynomials [19]
 $\frac{1}{x}x^{-s/2-1/4}e^{x/2}$ (2.9)

In spite of seemingly verified by means of the property
$$
(2.4)
$$
. It can be easily verified by means of the property (2.4) . Using the asymptotics of the generalized Chebyshev-Laguerre polynomials [19] $L_n^s(x) = \pi^{-1/2} n^{s/2-1/4} x^{-s/2-1/4} e^{x/2}$ \n $\times \left\{ \cos \left(2\sqrt{n}x - s\pi/2 - \pi/4 \right) + O(n^{-1/2}) \right\}, \quad n \to \infty,$ \n\nwe find\n
$$
\lim_{n \to \infty} \tilde{R}_{n,n+p} = 0, \quad \forall p \in Z.
$$
\n(2.10)

$$
\overline{nx} - s\pi/2 - \pi/4 + O(n^{-1/2}) \}, \quad n \to \infty,
$$

$$
\lim_{n \to \infty} \tilde{R}_{n,n+p} = 0, \quad \forall p \in Z.
$$
 (2.10)

bllows we will need the following result [5]
 Janas-S. Naboko). Let D be a selfadjoint operator in a Hilbert space

discrete spectrum $(De_n = \mu_n e_n)$, where $\{e_n\}$ is an orthonormal

ectors in H and μ_n are simple eige **na 2.1 (J. Janas-S. Naboko).** Let D be a selfadjoint of
th simple discrete spectrum $(De_n = \mu_n e_n)$, where
of eigenvectors in H and μ_n are simple eigenvalue
 $\subseteq |\mu_{i+1}|$. Assume that $|\mu_i - \mu_k| \ge \epsilon_0 > 0$, $\forall i \ne k$. If
then **Lemma 2.1 (J. Janas-S. Naboko).** Let D be a selfadjoint operator in a Hilbert space H with simple discrete spectrum $(De_n = \mu_n e_n)$, where $\{e_n\}$ is an orthonormal $(De_n = \mu_n e_n)$, where $\{e_n\}$ is an orthonormal μ_n are simple eigenvalues $(\mu_n \to \infty)$, ordered by $\kappa | \geq \epsilon_0 > 0$, $\forall i \neq k$. If R is a compact operator f the operator $T = D + R$ (with discrete spectrum s of n and satis basis of eigenvectors in H and μ_n are simple eigenvalues $(\mu_n \to \infty)$, ordered by
 Ω is a compact operator

(with discrete spectrum

asymptotic formula

, (2.11) $|\mu_i| \leq |\mu_{i+1}|$. Assume that $|\mu_i - \mu_k| \geq \epsilon_0 > 0$, $\forall i \neq k$. If R is a compact operator
in H then the eigenvalues $\lambda_n(T)$ of the operator $T = D + R$ (with discrete spectrum
too) become simple for large values of n and sati in H then the eigenvalues $\lambda_n(T)$ of the operator $T = D + R$ (with discrete spectrum (*T*) of the operator $T = D + R$ (with discrete spectrum
values of *n* and satisfy to the asymptotic formula
 $= \mu_n + O(||R^*e_n||)$, $n \to \infty$, (2.11)
ator with respect to *R*.
 $T^T R U$ represent bounded noncompact operator (pro-
 T too) become simple for large values of n and satisfy to the asymptotic formula

$$
\lambda_n(T) = \mu_n + O(\|R^*e_n\|), \quad n \to \infty,
$$
\n(2.11)

where R^* is the adjoint operator with respect to R.

) become simple for large values of *n* and satisfy to the asymptotic formula
 $\lambda_n(T) = \mu_n + O(||R^*e_n||)$, $n \to \infty$, (2)

ere R^* is the adjoint operator with respect to R .

Matrices R and $\tilde{R} = U^T R U$ represent bounde $(T) = \mu_n + O(||R^*e_n||), \quad n \to \infty,$ (2.11)

operator with respect to R.
 $= U^T RU$ represent bounded noncompact operator (pro-

erefore we can't apply here at once Lemma 2.1.

llowing theorem:

Matrices *R* and $R = U^T R U$ represent bounded noncompact operator (pro-
r) since $R^2 = I$. Therefore we can't apply here at once Lemma 2.1.
Let us prove the following theorem:
rem 2.2. Let *D* be a selfadjoint operator in jector) since $R^2 = I$. Therefore we can't apply here at once Lemma 2.1.
Let us prove the following theorem:
Theorem 2.2. Let D be a selfadjoint operator in a Hilbert space H with e:
 $\mu_n = n$, $(n = 0, 1, 2, ...)$ and complete **The Figure 12.2.** Let D be a selfadjoint oper
 $n, (n = 0, 1, 2, ...)$ and complete sy
 Ω be a bounded, selfadjoint, noncom

of operator D eigenvectors satisfy t
 $\lim_{n, n+p}$ = **Theorem 2.2.** Let D be a selfadjoint operator in a Hilbert space H with eigenvalues μ_n = n, (n = 0, 1, 2,...) and complete system of corresponding eigenvectors in H.

R be a bounded, selfadjoint, noncompact operator and its matrix $R_{n,k}$ in the

is of operator D eigenvectors satisfy to the condition
 \lim_{n Let R be a bounded, selfadjoint, noncompact operator and its matrix $R_{n,k}$ in the basis of operator D eigenvectors satisfy to the condition

$$
\lim_{n \to \infty} R_{n, n+p} = 0, \quad \forall p \in Z. \tag{2.12}
$$

lim_{n→∞} $R_{n,n+p} = 0$, $\forall p \in Z$. (2.12)

(2.12)

(a) of the operator $T = D + R$ (having a discrete spectrum

(2.13)
 $\Rightarrow p = n + R_{n,n} + O(s_n)$, $n \to \infty$, (2.13) Then the eigenvalues $\lambda_n(T)$ of the operator $T = D + R$ (having a discrete spectrum
too) become simple for large values of n and satisfy to the following asymptotic
estimation
 $\lambda_n(T) = n + R_{n,n} + O(s_n)$, $n \to \infty$, (2.13)
where
 s_n too) become simple for large values of n and satisfy to the following asymptotic) become simple for large values of *n* and satisfy to the following asymptotic
imation
 $\lambda_n(T) = n + R_{n,n} + O(s_n) , \quad n \to \infty ,$ (2.13)
ere
 $s_n = \sqrt{\sum_{k \neq n} \frac{|R_{k,n}|^2}{(n-k)^2}} ,$ estimation

$$
\lambda_n(T) = n + R_{n,n} + O(s_n) , \quad n \to \infty , \tag{2.13}
$$

where

$$
(T) = n + R_{n,n} + O(s_n), \quad n \to \infty,
$$
\n
$$
(2.13)
$$
\n
$$
s_n = \sqrt{\sum_{k \neq n} \frac{|R_{k,n}|^2}{(n-k)^2}},
$$
\nhis theorem we need the following compactness criteria for

\n*bounded, noncompact operator in a Hilbert space H. Let*

and $s_n \to 0$ at $n \to \infty$.

 $\frac{1}{2}$ 0 at $n \to \infty$.

he proof of titrices.

3. Let V be
 $V_{i,j}$ $(i, j = 0)$

Example 12.3. Let V be a bounded, noncompact operator in a Hilbert space H. Let a
trix $V_{i,j}$ $(i, j = 0, 1, ...)$ in some orthonormal basis satisfy to the condition
 $\lim_{n \to \infty} V_{n, n+p} = 0$, $\forall p \in Z.$ (2.14) **Lemma 2.3.** Let V be a bounded, noncomparties matrix $V_{i,j}$ $(i, j = 0, 1, ...)$ in some ortho
 $\lim_{n \to \infty} V_{n, n+p} = 0$,
Let $b = \{b_i\}_{i=-\infty}^{\infty}$ is an arbitrary l_2 -sequence **Lemma 2.3.** Let V be a bounded, noncompact operator in a Hilbert space H . Let

$$
\lim_{n \to \infty} V_{n, n+p} = 0, \quad \forall p \in Z. \tag{2.14}
$$

its matrix
$$
V_{i,j}
$$
 $(i, j = 0, 1, ...)$ in some orthonormal basis satisfy to the condition
\n
$$
\lim_{n \to \infty} V_{n, n+p} = 0, \quad \forall p \in Z.
$$
\n(2.14)
\nLet $b = \{b_i\}_{i=-\infty}^{\infty}$ is an arbitrary l_2 -sequence
\n
$$
||b||^2 = \sum_{i=-\infty}^{\infty} |b_i|^2 < \infty.
$$
\n(2.15)
\nThen the operator K with matrix $K_{i,j} = b_{i-j}V_{i,j}$ $(i, j = 0, 1, ...)$ is compact in H.

 $=\{b_i\}_{i=1}^{\infty}$
the oper Then the operator K with matrix $K_{i,j} = b_{i-j} V_{i,j}$ $(i, j = 0, 1, \ldots)$ is compact in H .
 \vdots *Proof.* Let us show at first that the operator K
prove the estimates [20]
 $\sum_{i=0}^{\infty} |K_{i,j}| < A, \ \forall i; \qquad \sum_{i=0}^{\infty} |K_{i,j}|$ Proof.

Let us show at first that the operator
$$
K
$$
 is bounded. For that we need to
\nthe estimates [20]
\n
$$
\sum_{j=0}^{\infty} |K_{i,j}| < A, \ \forall i; \quad \sum_{i=0}^{\infty} |K_{i,j}| < A, \ \forall j, \qquad (2.16)
$$
\n4 is a constant independent of i and $j (||K|| \leq A)$. Using Cauchy's inequal-
\nhave
\n
$$
\sum_{j=0}^{\infty} |K_{i,j}| = \sum_{j=0}^{\infty} |b_{i-j}V_{i,j}| \leq \left(\sum_{j=0}^{\infty} |b_{i-j}|^2\right)^{1/2} \left(\sum_{j=0}^{\infty} |V_{i,j}|^2\right)^{1/2}
$$

prove the extending $\sum_{j=0}^{\infty}$ | i where A is a constant in ity, we have

where *A* is a constant independent of *i* and *j* (
$$
||K|| \le A
$$
). Using Cauchy's inequality, we have\n
$$
\sum_{j=0}^{\infty} |K_{i,j}| = \sum_{j=0}^{\infty} |b_{i-j}V_{i,j}| \le \left(\sum_{j=0}^{\infty} |b_{i-j}|^2\right)^{1/2} \left(\sum_{j=0}^{\infty} |V_{i,j}|^2\right)^{1/2}
$$
\n
$$
\le ||b||\sqrt{(VV^*)_{i,i}} \le ||b|| \cdot ||V||
$$
\nDue to (2.15) the first estimate in (2.16) is fulfilled. By the same way the validity of the second estimate in (2.16) is established. Thus the operator *K* is bounded. Let us prove now its compactness.

 $(VV^*)_{i,i} \le ||b|| \cdot ||V||$
fulfilled. By the sam
ned. Thus the operat
= ${b_i^{(n)}}_{i=-\infty}^{\infty}$ of the
 $|i| > n$ of the second estimate in (2.16) is established. Thus the operator K is bounded.

Let us prove now its compactness.

Let us define the cut-off function $b^{(n)} = \{b_i^{(n)}\}_{i=-\infty}^{\infty}$ of the sequence $\{b_i\}$
 $b_i^{(n)} = \begin{$

 $\{n \atop i\}^{\infty}_{i=-\infty}$

$$
b_i^{(n)} = \begin{cases} 0, & |i| > n \\ b_i, & |i| \le n. \end{cases}
$$

Let us define the cut-off function $b^{(n)} = \{b_i^{(n)}\}_{i=-\infty}^{\infty}$ of the sequence $\{b_i\}$
 $b_i^{(n)} = \begin{cases} 0, & |i| > n \\ b_i, & |i| \le n. \end{cases}$

Let us define the sequence of operators $K^{(n)}$ by the formula $K_{i,j}^{(n)} = b_{i-j}^{(n)}$

It f d
fi
us
(n $\begin{align} 0 \,, & & |i| > n \ b_i \,, & & |i| \leq n \ \end{align}$
 $\begin{align} \text{rators } K^{(n)} \,. \end{align}$
 $\begin{align} \text{r} & & \geq 1 \,, \end{align}$ Let us define the sequence of operators $K^{(n)}$ by the formula $K_{i,j}^{(n)}$
lows from this definition and from (2.14) that $K^{(n)}$ is a compact of
rary n. We proceed further as in the proof of (2.16)
 $||K - K^{(n)}|| \le ||b - b^{(n)}|| \$ i,j $= b$
pers $y_{i-j}^{(n)} V_{i,j}$ **...** It follows from this definition and from (2.14) that $K^{(n)}$ is a compact operator for arbitrary *n*. We proceed further as in the proof of (2.16)
 $||K - K^{(n)}|| \le ||b - b^{(n)}|| \cdot ||V||$.
Therefore

$$
||K - K^{(n)}|| \le ||b - b^{(n)}|| \cdot ||V||.
$$

$$
||K - K^{(n)}|| \to 0, \quad n \to \infty
$$

arbitrary *n*. We proceed further as in the proof of (2.16)
 $||K - K^{(n)}|| \le ||b - b^{(n)}|| \cdot ||V||.$

Therefore
 $||K - K^{(n)}|| \to 0, \quad n \to \infty,$

and K is compact as a limit by norm of compact operat

 $||K - K^{(n)}|| \to 0, \quad n \to \infty$,
and K is compact as a limit by norm of compact opera
Proof of Theorem 2.2. Keeping the same notations, let
with a matrix in the basis of the operator D eigenvecta
Following the main ideas of th *Proof of Theorem 2.2.* Keeping the same notations, let us associate each operator

with a matrix in the basis of the operator D eigenvectors.
Following the main ideas of the work [5] let us show that there exist such anti-hermitian operator $K (K^* = -K)$ that

$$
(I + K)T - D_1(I + K) = B, \t(2.17)
$$

where B is compact operator and $D_1 = D + \text{diag}\{R_{n,n}\}\$. (So D_1 is the diagonal with a matrix in the basis of the operator D eigenvectors.

Following the main ideas of the work [5] let us show

anti-hermitian operator $K (K^* = -K)$ that
 $(I + K)T - D_1(I + K) = B$,

where B is compact operator and $D_1 = D + \text$ anti-hermitian operator $K (K^* = -K)$ that
 $(I + K)T - D_1(I + K) = B$, (2.17)

where B is compact operator and $D_1 = D + \text{diag}\{R_{n,n}\}$. (So D_1 is the diagonal

matrix with elements $(D_1)_{n,n} = n + R_{n,n}$.) Suppose that such operator K $(I + K)T - D_1(I + K) = B$, (2.17)

tor and $D_1 = D + \text{diag}\{R_{n,n}\}$. (So D_1 is the diagonal
 $n_n = n + R_{n,n}$.) Suppose that such operator K have
 (T) means that
 $+ K)^{-1}(D_1 + B(I + K)^{-1})(I + K)$. where *B* is compact operator and $D_1 = D + \text{diag}\{R_{n,n}\}$. (So D_1 is the diagonal
matrix with elements $(D_1)_{n,n} = n + R_{n,n}$.) Suppose that such operator *K* have
found. The condition (2.17) means that
 $T = (I + K)^{-1}(D_1 + B(I + K)^{-1$

$$
T = (I + K)^{-1} (D_1 + B(I + K)^{-1}) (I + K)
$$

matrix with elements $(D_1)_{n,n} = n + R_{n,n}$.) Suppose that such operator *K* have
found. The condition (2.17) means that
 $T = (I + K)^{-1}(D_1 + B(I + K)^{-1})(I + K)$.
(The existence of the inverse operator $(I + K)^{-1}$ follows from the anti-her $T = (I + K)^{-1}(D_1 +$
(The existence of the inverse operator (*1*) of *K*.) Thus the operators *T* and $D_1 + I$ = $(I + K)^{-1}(D_1 + B(I + K)^{-1})(I + K)$.
inverse operator $(I + K)^{-1}$ follows from th
ators T and $D_1 + B(I + K)^{-1}$ are similar (The existence of the inverse operator $(I+K)^{-1}$ follows from the anti-hermitianess
of K.) Thus the operators T and $D_1 + B(I+K)^{-1}$ are similar and have the same of K.) Thus the operators T and $D_1 + B(I + K)^{-1}$ are similar and have the same Asymptotics of Eigenvalues of an Energy Operator 171
spectrum. But the operator *B* is compact and the eigenvalues of D_1 due to (2.12)
satisfy the requirements of Lemma 2.1. Applying Lemma 2.1 we obtain
 $\lambda_n(T) = n + R_{n,n} +$

$$
\lambda_n(T) = n + R_{n,n} + O\left(\|B^*e_n\|\right). \tag{2.18}
$$

spectrum. But the operator *B* is compact and the eigenvalues of D_1 due to (2.12)
satisfy the requirements of Lemma 2.1. Applying Lemma 2.1 we obtain
 $\lambda_n(T) = n + R_{n,n} + O(||B^*e_n||)$. (2.18)
Therefore for the proof of the the satisfy $\lambda_n(T) = n + R_{n,n} + O(||B^* e_n||)$.
Therefore for the proof of the theorem we should establish the existen
the operator K and find the matrix of the compact operator B. Substitutes
expressions for the matrices T and D_1 (*T*) = $n + R_{n,n} + O(||B^*e_n||)$. (2.18)
the theorem we should establish the existence of such
ne matrix of the compact operator *B*. Substituting the
ss *T* and *D*₁ in (2.17) we obtain
 K) = $R_1 - [D, K] + KR - \text{diag}\{R_{n,n}\}K$, (the operator *K* and find the matrix of the compact operator *B*. Substituting the expressions for the matrices *T* and D_1 in (2.17) we obtain
 $(I + K)T - D_1(I + K) = R_1 - [D, K] + KR - \text{diag}\{R_{n,n}\}K$, (2.19)

where $[\cdot, \cdot]$ is the c

$$
(I + K)T - D_1(I + K) = R_1 - [D, K] + KR - diag\{R_{n,n}\}K, \qquad (2.19)
$$

the operator *K* and find the matrix of the compact operator *B*. Substituting the
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where $[\cdot, \cdot]$ is the com expressions for the matrices T and D_1 in (2.17) we obtain
 $(I + K)T - D_1(I + K) = R_1 - [D, K] + KR - \text{diag}$

where [.,.] is the commutator and R_1 is the matrix of th

main diagonal $(R_1 = R - \text{diag}\{R_{n,n}\})$.

This expression will be

 $(I + K)T - D_1(I + K) = R_1 - [D, K] + KR - \text{diag}\{R_{n,n}\}K,$ (2.19)
[.,.] is the commutator and R_1 is the matrix of the operator R with zero
liagonal $(R_1 = R - \text{diag}\{R_{n,n}\})$.
This expression will be the matrix of compact operator if we c where [⋅, ⋅] is the commutator and R_1 is the matrix of the operator R with zero
main diagonal $(R_1 = R - \text{diag}\{R_{n,n}\})$.
This expression will be the matrix of compact operator if we can find such a
compact operator K th main diagonal $(R_1 = R - \text{diag}\{R_{n,n}\})$.
This expression will be the matrix
compact operator K that the conditio
 $K_{i,j}$ $(i - j) = (R_1)_{i,j}$. It follows from th
 $K_{i,j} = \frac{R_{i,j}}{i - j}, \quad i \neq j$; act operator K that the condition $[D, K] = R_1$ is valid, or in matrix form:
 $(i - j) = (R_1)_{i,j}$. It follows from that
 $K_{i,j} = \frac{R_{i,j}}{i - j}, \quad i \neq j; \qquad K_{i,i} = 0, \quad i = 0, 1, \dots$ (2.20)

As the operator R is selfadjoint the corresp $K_{i,j}$

$$
K_{i,j} = \frac{R_{i,j}}{i-j}, \quad i \neq j; \qquad K_{i,i} = 0, \ i = 0, 1, \tag{2.20}
$$

compact operator *K* that the condition $[D, K] = R_1$ is valid, or in matrix form:
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 $K_{i,j} = \frac{R_{i,j}}{i - j}, \quad i \neq j;$ $K_{i,i} = 0, i = 0, 1, \dots$ (2.20)

As the operator *R* is selfadjoint $(i - j) = (R_1)_{i,j}$. It follows from that
 $K_{i,j} = \frac{R_{i,j}}{i - j}, \quad i \neq j$;

As the operator R is selfadjoint the

or K is anti-hermitian. Its compactne

hoose the sequence $\{b_i\}$ as $\{1/i\}$ ($i = j$ erator K is anti-hermitian. Its compactness follows from Lemma 2.3. Actually, if
we choose the sequence ${b_i}$ as ${1/i}$ $(i \neq 0)$ then from (2.12) it follows that all
conditions of Lemma 2.3 are fulfilled.
Now from (2.19 $\neq j$; $K_{i,i} = 0, i = 0, 1, \ldots$ (2.20)
oint the corresponding to the matrix (2.20) op-
mpactness follows from Lemma 2.3. Actually, if
 $1/i$ $(i \neq 0)$ then from (2.12) it follows that all
led.
form of the compact operator *B* As the operator *R* is selfadjoint the corresponding to the matrix (2.20) op-
r *K* is anti-hermitian. Its compactness follows from Lemma 2.3. Actually, if
noose the sequence $\{b_i\}$ as $\{1/i\}$ $(i \neq 0)$ then from (2.12 erator *K* is anti-hermitian. Its compactness follows from Lemma 2.3. Actually, if
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Now from we choose the sequence ${b_i}$ as ${1/i}$ $(i \neq 0)$ then from (2.12) it follows that all
conditions of Lemma 2.3 are fulfilled.
Now from (2.19) we find the form of the compact operator *B*:
 $B = KR - \text{diag}\{R_{n,n}\} K$.
Since $||B^*e$

$$
B = KR - \text{diag}\{R_{n,n}\}\,K.
$$

Now from (2.19) we find the for
 $B = KR$

Since $||B^*e_n|| \leq C||K^*e_n||$, wh
 $O(||B^*e_n||)$ on $O(||K^*e_n||)$. Taking is Now from (2.19) we find the form of the compact operator B :
 $B = KR - \text{diag}\{R_{n,n}\} K$.

Since $||B^*e_n|| \leq C||K^*e_n||$, where C is constant, we can rep
 $B^*e_n||)$ on $O(||K^*e_n||)$. Taking into account (2.20) we obtain Since $||B^*e_n|| \le C||K^*e_n||$, where C is constant, we can replace in (2.18)
 $B^*e_n||)$ on $O(||K^*e_n||)$. Taking into account (2.20) we obtain
 $O(||K^*e_n||) = O\left(\sqrt{\sum_{k \ne n} \frac{|R_{k,n}|^2}{(n-k)^2}}\right)$.

Substituting this estimate to (2.18

$$
B = KR - \text{diag}\{R_{n,n}\}K.
$$

Since $||B^*e_n|| \leq C||K^*e_n||$, where C is constant, we can r
 $O(||B^*e_n||)$ on $O(||K^*e_n||)$. Taking into account (2.20) we obtain

$$
O(||K^*e_n||) = O\left(\sqrt{\sum_{k \neq n} \frac{|R_{k,n}|^2}{(n-k)^2}}\right).
$$

Substituting this estimate to (2.18), we obtain the formula (

Substituting this estimate to (2.18), we obtain the formula (2.13). The theo-
is proved. \square
Due to (2.10), the condition (2.12) of Theorem 2.2 is fulfilled. Hence, applying $(n - k)^2$
tain the
em 2.2 is
em 2.2 is

From is proved.

Substitution (2.12) of Theorem 2.2 is fulfilled. Hence, applying

Theorem 2.2 and taking into account (2.2), (2.1) and (2.8) we have the following

result
 $\lambda_n(A) = n - g^2 + \frac{c_1 + c_2}{2} + O(s_n) , \quad n \to \infty ,$ rem 2.2 and taking into account (2.2), (2.1) and (2.8) we have the following
 $\lambda_n(A) = n - g^2 + \frac{c_1 + c_2}{2} + O(s_n)$, $n \to \infty$,

$$
\lambda_n(A) = n - g^2 + \frac{c_1 + c_2}{2} + O(s_n) , \quad n \to \infty ,
$$

where

result
\n
$$
\lambda_n(A) = n - g^2 + \frac{c_1 + c_2}{2} + O(s_n), \quad n \to \infty,
$$
\nwhere
\n
$$
s_n = \sqrt{\sum_{k \neq n} \frac{|\tilde{R}_{k,n}|^2}{(n-k)^2}} = \sqrt{\sum_{k \neq n} \frac{|\omega_k^{(n-k)}(4g^2)|^2}{(n-k)^2}}.
$$
\n(2.21)

We use here the normalized Laguerre functions
$$
\omega_n^{(s)}(x)
$$

\n
$$
\omega_n^{(s)}(x) = \sqrt{\frac{n!}{(n+s)!}} e^{-x/2} x^{s/2} L_n^{(s)}(x), \quad \int_0^{+\infty} \omega_m^{(s)}(x) \omega_n^{(s)}(x) dx = \delta_{n,m}.
$$
\nFrom (2.5) and (2.8) it follows that\n
$$
|\tilde{R}_{k,n}| = |\omega_k^{(n-k)}(4g^2)|.
$$
\n\n**Estimation of the remainder**

$$
|\tilde{R}_{k,n}| = |\omega_k^{(n-k)}(4g^2)|
$$

3. Estimation of the remainder

From (2.5) and (2.8) it follows that
 $|\tilde{R}_{k,n}| = |\omega_k^{(n-k)}(4g^2)|$.
 3. Estimation of the remainder

To estimate the decreasing rate of the sequence s_n we should have another estima- $\tilde{R}_{k,n}| = |\omega_k^{(n-k)}(4g^2)|.$

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and the sequence s_n w

and Massuld not find To estimate the decreasing rate of the sequence s_n we should have another estima-
tion for Laguerre's functions $\omega_n^{(s)}(x)$ rather than the estimation following from (2.9)
(in (2.9) the parameter s is fixed). We could (in (2.9) the parameter s is fixed). We could not find this result among known one and therefore we provide here not only the formulation but also the proof of it.

(in (2.9) the parameter *s* is fixed). We could not find this result among known one
and therefore we provide here not only the formulation but also the proof of it.
Lemma 3.1. Suppose that $x > 0$, $s \in Z_+$. Then the fo **Lemma 3.1.** Suppose that $x > 0$, $s \in Z_+$. Then the following estimate for the *Bessel functions* $J_s(x)$ is valid $|J_s(x)| \leq 2\sqrt{\frac{2}{\pi x}} \left(1 + \frac{s}{x}\right)^s$. **Lemma 3.1.** Suppose that $x > 0$, $s \in Z_+$. Then the following estimate for the
Bessel functions $J_s(x)$ is valid $|J_s(x)| \leq 2\sqrt{\frac{2}{\pi x}} \left(1 + \frac{s}{x}\right)^s$.
Proof. Let us use known representation [22] Bessel functions $J_s(x)$ is valid

$$
|J_s(x)| \le 2\sqrt{\frac{2}{\pi x}} \left(1 + \frac{s}{x}\right)^s.
$$

Proof.

\n
$$
|J_s(x)| \leq 2 \sqrt{\frac{2}{\pi x}} \left(1 + \frac{s}{x}\right)^s.
$$
\n

\n\n $u(f, x) = \sqrt{\frac{2}{\pi x}} \left(1 + \frac{s}{x}\right)^s.$ \n

\n\n $u(f, x) = \sqrt{\frac{2}{\pi x}} \left(P(x, s) \cos(x - s\pi/2 - \pi/4) - Q(x, s) \sin(x - s\pi/2 - \pi/4) \right),$ \n

\n\n Here\n

\n\n $\frac{1}{2}(x, s) = \frac{1}{2 \pi \sqrt{2 \left(1 + \frac{1}{2}\right)^2}} \int_0^\infty e^{-u} u^{s - 1/2} \left\{ \left(1 + \frac{iu}{2\pi}\right)^{s - 1/2} + \left(1 - \frac{iu}{2\pi}\right)^{s - 1/2} \right\} du$ \n

$$
J_s(x) = \sqrt{\frac{2}{\pi x}} \left(P(x, s) \cos(x - s\pi/2 - \pi/4) - Q(x, s) \sin(x - s\pi/2 - \pi/4) \right),
$$

where

$$
P(x, s) = \frac{1}{2\Gamma(s + 1/2)} \int_0^\infty e^{-u} u^{s - 1/2} \left\{ \left(1 + \frac{iu}{2x} \right)^{s - 1/2} + \left(1 - \frac{iu}{2x} \right)^{s - 1/2} \right\} du
$$

$$
Q(x, s) = \frac{1}{2i\Gamma(s + 1/2)} \int_0^\infty e^{-u} u^{s - 1/2} \left\{ \left(1 + \frac{iu}{2x} \right)^{s - 1/2} - \left(1 - \frac{iu}{2x} \right)^{s - 1/2} \right\} du.
$$
It is evident that

$$
|J_s(x)| \le \sqrt{\frac{2}{\pi x}} \left(|P(x, s)| + |Q(x, s)| \right),
$$
(3.1)
and everything reduces to the estimation of the integrals $P(x, s)$ and $Q(x, s)$. Le

It is evident that

$$
|J_{s}(x)| \leq \sqrt{\frac{2}{\pi x}} \left(|P(x, s)| + |Q(x, s)| \right),
$$
\n(3.1)\n\nas to the estimation of the integrals $P(x, s)$ and $Q(x, s)$. Let

and everything responsider the is $s = 0$ we have $(x)| \leq \sqrt{\frac{2}{\pi s}}$

(o the estimator $P(x, s)$)
 $| \leq 1$, $|Q(s)|$

(o that s $(|P(x, s)| + |Q(x, s)|)$, (3.1)
tion of the integrals $P(x, s)$ and $Q(x, s)$. Let
The estimation for $Q(x, s)$ is the same. At
 $|s| \leq 1$ and the estimation (3.1) gives the
N. In this case we have and everything reduces to the estimation of the integrals $P(x, s)$ and $Q(x, s)$. Let
us consider the integral for $P(x, s)$. The estimation for $Q(x, s)$ is the same. At
 $s = 0$ we have $|P(x, s)| \le 1$, $|Q(x, s)| \le 1$ and the estimat us consider the integral for $P(x, s)$. The estimation for $Q(x, s)$ is the same. At $s = 0$ we have $|P(x, s)| \leq 1$, $|Q(x, s)| \leq 1$ and the estimation (3.1) gives the

$$
s = 0
$$
 we have $|P(x, s)| \leq 1$, $|Q(x, s)| \leq 1$ and the estimation (3.1) gives the required inequality. Suppose that $s \in N$. In this case we have $|P(x, s)| \leq \frac{1}{\Gamma(s + 1/2)} \int_{0}^{\infty} e^{-u} u^{s-1/2} \left(1 + \frac{u}{2x}\right)^{s-1/2} du.$

Expanding the binomial in this integral in the series on
$$
u/2x
$$

\n
$$
\left(1+\frac{u}{2x}\right)^{s-1/2} = 1+\sum_{k=1}^{p-1} \frac{(s-1/2)\cdot \cdot \cdot \cdot (s-1/2-(k-1))}{k!} \left(\frac{u}{2x}\right)^k + \frac{(s-1/2)\cdot \cdot \cdot \cdot (s-1/2-(p-1))}{k!} (1+\theta)^{s-p-1/2} \left(\frac{u}{2x}\right)^p,
$$
\n
$$
\theta \in (0, u/2x)
$$
\nand putting $p = s$ we have $(1+\theta)^{s-p-1/2} < 1$ and therefore\n
$$
\left(1+\frac{u}{2x}\right)^{s-1/2} < 1+\sum_{k=1}^{s} \frac{(s-1/2)\cdot \cdot \cdot \cdot (s-1/2-(k-1))}{k!} \left(\frac{u}{2x}\right)^k.
$$

$$
\theta \in (0, u/2x)
$$

and putting $p = s$ we have $(1 + \theta)^{s-p-1/2} < 1$ and therefore

$$
\left(1 + \frac{u}{2x}\right)^{s-1/2} < 1 + \sum_{k=1}^{s} \frac{(s-1/2) \cdot \cdots \cdot (s-1/2-(k-1))}{k!} \left(\frac{u}{2x}\right)^k.
$$

Integrating by terms we obtain

$$
|P(x,s)| \le 1 + \sum_{k=1}^{s} \frac{\Gamma(s+k+1/2)}{\Gamma(s+1/2)} \frac{(s-1/2) \cdot \cdots \cdot (s-1/2-(k-1))}{k!} \frac{1}{k!}
$$

Integrating by terms we obtain

$$
\left(1 + \frac{u}{2x}\right)^{s-1/2} < 1 + \sum_{k=1}^{s} \frac{(s-1/2) \cdot \dots \cdot (s-1/2 - (k-1))}{k!} \left(\frac{u}{2x}\right)^k.
$$
\nIntegrating by terms we obtain\n
$$
|P(x,s)| \leq 1 + \sum_{k=1}^{s} \frac{\Gamma(s+k+1/2)}{\Gamma(s+1/2)} \cdot \frac{(s-1/2) \cdot \dots \cdot (s-1/2 - (k-1))}{k!} \cdot \frac{1}{(2x)^k}
$$
\n
$$
< 1 + \sum_{k=1}^{s} (2s)^k \frac{s \cdot \dots \cdot (s-(k-1))}{k!} \cdot \frac{1}{(2x)^k} = \left(1 + \frac{s}{x}\right)^s.
$$
\nFor $Q(x, s)$ the same estimate is valid and the formula (3.1) leads again to the required inequality. The lemma is proved.\n\n**Lemma 3.2.** If $x > 0$; $n, s \in Z_+$ and $s^{16} \leq n$ then\n
$$
\left|\omega_n^{(s)}(x)\right| \leq \frac{C(x)}{(n+1)^{1/4}},\tag{3.
$$

 $\frac{1}{3}$ For $Q(x, s)$ the same estimate is valid and the formula (3.1) leads again to the
required inequality. The lemma is proved.
Lemma 3.2. If $x > 0$; $n, s \in Z_+$ and $s^{16} \le n$ then
 $\left|\omega_n^{(s)}(x)\right| \le \frac{C(x)}{(n+1)^{1/4}}$, (3.2)
wh

required inequality. The lemma is proved.
\n**Lemma 3.2.** If
$$
x > 0
$$
; $n, s \in Z_+$ and $s^{16} \le n$ then
\n
$$
\left|\omega_n^{(s)}(x)\right| \le \frac{C(x)}{(n+1)^{1/4}},
$$
\nwhere the constant $C(x)$ depends on x only.
\nProof. Let us use Laguerre's functions integral representation through the Bessel functions [19]

where the constant $C(x)$ depends on x only.

Lemma 3.2. If $x > 0$; $n, s \in Z_+$ and $s^{16} \le n$ then
 $\left|\omega_n^{(s)}(x)\right| \le \frac{C(x)}{(n+1)^1}$,

where the constant $C(x)$ depends on x only.

Proof. Let us use Laguerre's functions integral refunctions [19] (x)
on
tion
 ∞ $\frac{(3.2)}{(n+1)^{1/4}}$, (3.2)
 ldy.

ttegral representation through the Bessel
 $\frac{n+\frac{s}{2}}{J_s(2\sqrt{tx}) dt}$, $n, s \in Z_+$. Proof.

where the constant
$$
C(x)
$$
 depends on x only.
\nProof. Let us use Laguerre's functions integral representation through the Bessel functions [19]
\n
$$
\omega_n^{(s)}(x) = \frac{e^{x/2}}{\sqrt{n! (n+s)!}} \int_0^\infty e^{-t} t^{n+\frac{s}{2}} J_s(2\sqrt{tx}) dt, \quad n, s \in Z_+.
$$
\nLet us split this integral into two one
\n
$$
\omega_n^{(s)}(x) = \frac{e^{x/2}}{\sqrt{n! (n+s)!}} \left(\int_0^{t_0} + \int_{t_0}^\infty \right),
$$

$$
\frac{e^{x/2}}{\sqrt{n! (n+s)!}} \int_{0}^{\infty} e^{-t} t^{n+\frac{s}{2}} J_{s}(2\sqrt{tx}) dt,
$$

ral into two one

$$
\omega_{n}^{(s)}(x) = \frac{e^{x/2}}{\sqrt{n! (n+s)!}} \left(\int_{0}^{t_{0}} + \int_{t_{0}}^{\infty} \right),
$$

itrary now.
f the first integral let us use the know

$$
|J_{s}(x)| \le 1, \quad s \in Z_{+}, \ x \in R.
$$

second integral we use more precise est

where $t_0 \geq 0$ is an arbitrary now.

$$
|J_s(x)| \le 1, \quad s \in Z_+, \ x \in R
$$

where
$$
t_0 \ge 0
$$
 is an arbitrary now.
\nFor estimation of the first integral let us use the known inequality [22]
\n $|J_s(x)| \le 1$, $s \in Z_+$, $x \in R$.
\nFor estimation of the second integral we use more precise estimate from Lemma 3.1
\n $|J_s(x)| \le 2\sqrt{\frac{2}{\pi x}} \left(1 + \frac{s}{x}\right)^s < 2e\sqrt{\frac{2}{\pi x}}, \quad x \ge s^2$.

Putting
$$
t_0 = s^4/4x
$$
 (so that at $t \ge t_0$ one can use the last estimate) we have
\n
$$
|\omega_n^{(s)}(x)| \le \frac{e^{x/2}}{\sqrt{n! (n+s)!}} \left[\int_0^{t_0} e^{-t} t^{n+\frac{s}{2}} dt + \frac{2e}{\sqrt{\pi \sqrt{x}}} \int_0^{\infty} e^{-t} t^{n+\frac{s}{2}-1/4} dt \right]
$$
\n
$$
\le \frac{e^{x/2}}{\sqrt{n! (n+s)!}} \left[t_0 \max_{t \ge 0} \left\{ e^{-t} t^{n+\frac{s}{2}} \right\} + \frac{2e}{\sqrt{\pi \sqrt{x}}} \Gamma(n+s/2+3/4) \right]
$$
\n
$$
= e^{x/2} \left[\frac{s^4}{4x} \frac{e^{-(n+\frac{s}{2})} (n+s/2)^{n+\frac{s}{2}}}{\sqrt{n! (n+s)!}} + \frac{2e}{\sqrt{\pi \sqrt{x}}} \frac{\Gamma(n+s/2+3/4)}{\sqrt{n! (n+s)!}} \right].
$$
\nUsing known inequalities for Gamma-function following from Stirling's
\nmula
\n $C_1 z^{z-1/2} e^{-z} \le \Gamma(z) \le C_2 z^{z-1/2} e^{-z}, \quad z \ge \delta > 0,$
\nwhere C_1, C_2 are some constants independent of z. Let us estimate each term
\nsquare brackets. We have

$$
C_1 z^{z-1/2} e^{-z} \le \Gamma(z) \le C_2 z^{z-1/2} e^{-z}, \quad z \ge \delta > 0,
$$

 $4x$
equ $\frac{1}{2}$
le e
nav ! $(n + s)!$
r Gamma-functi
 $(z) \leq C_2 z^{z-1/2} e$
independent of

mula
\n
$$
C_1 z^{z-1/2} e^{-z} \le \Gamma(z) \le C_2 z^{z-1/2} e^{-z}, \quad z \ge \delta > 0,
$$
\nwhere C_1, C_2 are some constants independent of z. Let us estimate each term in
\nsquare brackets. We have
\n
$$
\frac{2e}{\sqrt{\pi \sqrt{x}}} \frac{\Gamma(n+s/2+3/4)}{\sqrt{n!(n+s)!}} \le C(x) \frac{(n+s/2+3/4)^{n+s/2+1/4}}{(n+1)^{n/2+1/4} (n+s+1)^{n/2+s/2+1/4}}
$$
\n
$$
\le \frac{C(x)}{(n+1)^{1/4}} \frac{(1+\frac{s}{2(n+1)})^n}{(1+\frac{s}{n+1})^{n/2}} \le \frac{C(x)}{(n+1)^{1/4}} \left(1+\frac{s^2}{4(n+1)^2}\right)^{n/2}
$$
\n
$$
\le \frac{C(x)}{(n+1)^{1/4}} \exp\left\{\frac{s^2}{8(n+1)}\right\}.
$$
\nAt last, if we put $n \ge s^2$ then
\n
$$
\frac{2e}{\sqrt{\pi \sqrt{x}}} \frac{\Gamma(n+s/2+3/4)}{\sqrt{n!(n+s)!}} \le \frac{C(x)}{(n+1)^{1/4}}, \quad (n \ge s^2).
$$
\n(3.3)

$$
\frac{C(x)}{(n+1)^{1/4}} \exp\left\{\frac{s^2}{8(n+1)}\right\}.
$$

if we put $n \ge s^2$ then

$$
\frac{2e}{\sqrt{\pi\sqrt{x}}} \frac{\Gamma(n+s/2+3/4)}{\sqrt{n!(n+s)!}} \le \frac{C(x)}{(n+1)^{1/4}}, \quad (n \ge s^2).
$$
(3.3)
can estimate the second term
 $+\frac{s}{2}(n+s/2)^{n+\frac{s}{2}}$ $s^4(n+s/2)^{n+s/2}$

At last, if we put
$$
n \ge s^2
$$
 then
\n
$$
\frac{2e}{\sqrt{\pi \sqrt{x}}} \frac{\Gamma(n+s/2+3/4)}{\sqrt{n! (n+s)!}} \le \frac{C(x)}{(n+1)^{1/4}}, \quad (n \ge s^2).
$$
\n(Similarly, we can estimate the second term
\n
$$
\frac{s^4}{4x} \frac{e^{-(n+\frac{s}{2})} (n+s/2)^{n+\frac{s}{2}}}{\sqrt{n! (n+s)!}} \le C(x) \frac{s^4 (n+s/2)^{n+s/2}}{(n+1)^{n/2+1/4} (n+s+1)^{n/2+s/2+1/4}}
$$
\n
$$
\le \frac{C(x)}{(n+1)^{1/4}} \frac{s^4}{(n+s+1)^{1/4}} \frac{(1+\frac{s}{2(n+1)})^n}{(1+\frac{s}{n+1})^{n/2}}
$$
\n
$$
\le \frac{C(x)}{(n+1)^{1/4}} \frac{s^4}{(n+s+1)^{1/4}} \exp\left\{\frac{s^2}{8(n+1)}\right\}.
$$
\nIf $n \ge s^{16}$ then
\n
$$
\frac{s^4}{4x} \frac{e^{-(n+\frac{s}{2})} (n+s/2)^{n+\frac{s}{2}}}{\sqrt{n! (n+s)!}} \le \frac{C(x)}{(n+1)^{1/4}}, \quad (n \ge s^{16}).
$$
\n(3.4)

$$
\leq \frac{C(x)}{(n+1)^{1/4}} \frac{s^4}{(n+s+1)^{1/4}} \exp\left\{\frac{s^2}{8(n+1)}\right\}.
$$

If $n \geq s^{16}$ then

$$
\frac{s^4}{4x} \frac{e^{-(n+\frac{s}{2})} (n+s/2)^{n+\frac{s}{2}}}{\sqrt{n!(n+s)!}} \leq \frac{C(x)}{(n+1)^{1/4}}, \quad (n \geq s^{16}).
$$
(3.4)
From (3.3), (3.4) it follows that

$$
\left|\omega_n^{(s)}(x)\right| \leq \frac{C(x)}{(n+1)^{1/4}}, \quad (n \geq s^{16}),
$$

From (3.3), (3.4) it follows that\n
$$
\left| \frac{\omega_n^{(s)}(n+s)}{\omega_n^{(s)}(x)} \right| \leq \frac{C(x)}{(n+1)^{1/4}}, \quad (n \geq s^{16}).
$$
\n
$$
(3.4)
$$
\n
$$
\left| \frac{\omega_n^{(s)}(x)}{(n+1)^{1/4}}, \quad (n \geq s^{16}), \quad \Box
$$
Asymptotics of Eigenvalues of an Energy Operator 175
Having the estimate (3.2) and the orthogonality condition of the transforma-

$$
\sum_{k=0}^{\infty} |U_{k,n}|^2 = \sum_{k=0}^{\infty} \left| \omega_k^{(n-k)} \right|^2 = \sum_{k=0}^n \left| \omega_{n-k}^{(k)} \right|^2 + \sum_{k=1}^{\infty} \left| \omega_n^{(k)} \right|^2 = 1, \quad \forall n \in \mathbb{Z}_+ \quad (3.5)
$$

$$
(|\omega_k^{(n-k)}| = |\omega_n^{(k-n)}|) \text{ one can give the estimate of the remainder which is defined by the sum}
$$

$$
\sum_{k=0}^{\infty} \frac{|\omega_k^{(n-k)}|^2}{(n-k)^2} = \sum_{k=0}^n \frac{|\omega_{n-k}^{(k)}|^2}{k^2} + \sum_{k=0}^{\infty} \frac{|\omega_n^{(k)}|^2}{k^2}.
$$
 (3.6)

$$
(|\omega_k^{(n-k)}| = |\omega_n^{(k-n)}|) \text{ one can give the estimate of the remainder which is defined by the sum}
$$
\n
$$
\sum_{k \neq n} \frac{|\omega_k^{(n-k)}|^2}{(n-k)^2} = \sum_{k=1}^n \frac{|\omega_{n-k}^{(k)}|^2}{k^2} + \sum_{k=1}^\infty \frac{|\omega_n^{(k)}|^2}{k^2}.
$$
\n(3.6)

\nFrom Lemma 3.2 it follows that

\n
$$
|\omega_{n-k}^{(k)}|^2 \leq \frac{C}{(n-k+1)^{1/2}}, \quad n-k \geq k^{16} \quad (n \geq k^{16} + k)
$$
\n
$$
|\omega_k^{(k)}|^2 < \frac{C}{(n-k+1)^{1/2}}, \quad n \geq k^{16} \quad (k < n^{1/16}).
$$

$$
\sum_{k \neq n} \frac{|\omega_k^{(n-k)}|^2}{(n-k)^2} = \sum_{k=1}^n \frac{|\omega_{n-k}^{(k)}|^2}{k^2} + \sum_{k=1}^\infty \frac{|\omega_n^{(k)}|^2}{k^2}.
$$

From Lemma 3.2 it follows that

$$
|\omega_{n-k}^{(k)}|^2 \leq \frac{C}{(n-k+1)^{1/2}}, \quad n-k \geq k^{16} \quad (n \geq k^{16} + k)
$$

$$
|\omega_n^{(k)}|^2 \leq \frac{C}{(n+1)^{1/2}}, \quad n \geq k^{16} \quad (k \leq n^{1/16}).
$$

Let $k_n \geq 0$ be a maximal nonnegative integer of k, satisfying to
 $n \geq k^{16} + k$. It is evident that $k_n \leq n^{1/16}$. Hence

$$
|\omega_n^{(k)}|^2 \le \frac{C}{(n+1)^{1/2}}, \qquad n \ge k^{16} \quad (k \le n^{1/16}).
$$

Let $k_n \ge 0$ be a maximal nonnegative integer of k , satisfying to the equation

$$
n \ge k^{16} + k.
$$
 It is evident that $k_n \le n^{1/16}$. Hence

$$
|\omega_{n-k}^{(k)}|^2 \le \frac{C}{(n-n^{1/8}+1)^{1/2}}, \qquad k \le k_n
$$

$$
|\omega_n^{(k)}|^2 \le \frac{C}{(n+1)^{1/2}}, \qquad k \le k_n.
$$

Let us present the sum (3.6) in the form

$$
\sum \frac{|\omega_n^{(n-k)}|^2}{(n+1)^{1/2}} = \left[\sum_{n=1}^{k_n} \frac{|\omega_{n-k}^{(k)}|^2}{(n+1)^{1/2}} + \sum_{n=1}^{k_n} \frac{|\omega_n^{(k)}|^2}{(n+1)^{1/2}}\right]
$$

$$
|\omega_n^{(k)}|^2 \le \frac{1}{(n+1)^{1/2}}, \qquad k \le k_n.
$$

Let us present the sum (3.6) in the form

$$
\sum_{k \ne n} \frac{|\omega_k^{(n-k)}|^2}{(n-k)^2} = \left[\sum_{k=1}^{k_n} \frac{|\omega_{n-k}^{(k)}|^2}{k^2} + \sum_{k=1}^{k_n} \frac{|\omega_n^{(k)}|^2}{k^2} \right] + \left[\sum_{k=k_n+1}^n \frac{|\omega_{n-k}^{(k)}|^2}{k^2} + \sum_{k=k_n+1}^\infty \frac{|\omega_n^{(k)}|^2}{k^2} \right].
$$

Due to last inequalities we have

Due to last inequalities we have

$$
+ \left[\sum_{k=k_n+1}^{n} \frac{|\omega_{n-k}^{(k)}|^2}{k^2} + \sum_{k=k_n+1}^{\infty} \frac{|\omega_{n}^{(k)}|^2}{k^2} \right].
$$

Due to last inequalities we have

$$
\left[\sum_{k=1}^{k_n} \frac{|\omega_{n-k}^{(k)}|^2}{k^2} + \sum_{k=1}^{k_n} \frac{|\omega_{n}^{(k)}|^2}{k^2} \right] \le \frac{2C}{(n - n^{1/8} + 1)^{1/2}} \sum_{k=1}^{k_n} \frac{1}{k^2} = O\left(\frac{1}{n^{1/2}}\right).
$$

Since $k_n \sim n^{1/16}$, we have using (3.5)

$$
\left[\sum_{k=k+1}^{n} \frac{|\omega_{n-k}^{(k)}|^2}{k^2} + \sum_{k=k+1}^{\infty} \frac{|\omega_{n}^{(k)}|^2}{k^2} \right] \le \frac{1}{(k_n + 1)^2} \left[\sum_{k=k+1}^{n} |\omega_{n-k}^{(k)}|^2 + \sum_{k=k+1}^{\infty} |\omega_{n-k}^{(k)}|^2 \right].
$$

$$
\left[\sum_{k=1}^{\lfloor \frac{\omega_{n-k}}{k^2} \rfloor} + \sum_{k=1}^{\lfloor \frac{\omega_{n}}{k^2} \rfloor} \frac{\lfloor \frac{\omega_{n}}{k^2} \rfloor}{k^2} \right] \leq \frac{2C}{(n - n^{1/8} + 1)^{1/2}} \sum_{k=1}^{\lfloor \frac{\omega}{k^2} \rfloor} \frac{1}{k^2} = O\left(\frac{1}{n^{1/2}}\right).
$$

Since $k_n \sim n^{1/16}$, we have using (3.5)

$$
\left[\sum_{k=k_n+1}^{n} \frac{|\omega_{n-k}^{(k)}|^2}{k^2} + \sum_{k=k_n+1}^{\infty} \frac{|\omega_n^{(k)}|^2}{k^2}\right] \leq \frac{1}{(k_n+1)^2} \left[\sum_{k=k_n+1}^{n} |\omega_{n-k}^{(k)}|^2 + \sum_{k=k_n+1}^{\infty} |\omega_n^{(k)}|^2\right]
$$

$$
\leq \frac{1}{(k_n+1)^2} = O\left(\frac{1}{n^{1/8}}\right).
$$

Combining both estimates, we obtain
\n
$$
\sum_{k \neq n} \frac{|\omega_k^{(n-k)}|^2}{(n-k)^2} = O\left(\frac{1}{n^{1/2}}\right) + O\left(\frac{1}{n^{1/8}}\right) = O\left(\frac{1}{n^{1/8}}\right).
$$
\nTaking into account formula (2.21), we arrive at the following main
\n**Theorem 3.3.** The eigenvalues $\lambda_n(A)$ of the operator A (1.1) at g
\nfollowing asymptotics
\n
$$
\lambda_n(A) = n - a^2 + \frac{c_1 + c_2}{n} + O\left(\frac{1}{n^{1/8}}\right), \quad n \to \infty.
$$

 $\frac{1}{4}$ $\frac{1}{r}$ $\frac{1}{1}$ **Theorem 3.3.** The eigenvalues $\lambda_n(A)$ of the operator A (1.1) at $g \neq 0$ h
following asymptotics
 $\lambda_n(A) = n - g^2 + \frac{c_1 + c_2}{2} + O\left(\frac{1}{n^{1/16}}\right)$, $n \to \infty$. **Theorem 3.3.** The eigenvalues λ_n (A) of the operator A (1.1) at $g \neq 0$ have the
 $\frac{n_1 + c_2}{2} + O\left(\frac{1}{n^{1/16}}\right)$, $n \to \infty$.

b for the discussion of the results. Special gratifollowing asymptotics

$$
\lambda_n(A) = n - g^2 + \frac{c_1 + c_2}{2} + O\left(\frac{1}{n^{1/16}}\right), \quad n \to \infty.
$$

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o Prof. S.N. Naboko for the discussion of the results
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References

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