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The Role of S.G. Samko in the Establishing and Development of the Theory of Fractional Differential Equations and Related Integral Operators

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To Stefan Samko on the occasion of his 70th birthday

Abstract. The aim of this work is to describe main aspects of the modern theory of fractional differential equations, to present elements of classification of fractional differential equations, to formulate basic components of investigations related to fractional differential equations, to pose some open problems in the study of fractional differential equations.
A survey of results by S.G. Samko on different problems of modern mathematical analysis is given. Main results of S.G. Samko having an essential influence on the establishing and development of the theory of fractional differential equations are singled out.

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1. Main aspects of the modern theory of fractional differential equations

1.1. Elements of the classification

In this section we present a brief introduction to the theory of fractional differential equations. Since it is not the main goal of the article, we restrict our attention only to the elements of classification of those types of problems which belong to the discussion in the framework of this theory. For wider expositions, which include many historical facts and extended bibliography, we refer to the main book by

S.G. Samko (published jointly with his co-authors A.A. Kilbas and O.I. Marichev) [43], to recent monographs [6], [23], and to survey paper [24].

It should be mentioned that in earlier time (see, e.g., [43]) mainly fractional differential equations with Riemann-Liouville fractional derivative (see definition (R-L) below) were investigated. Whereas later, motivated by needs of applications, the use of the Caputo fractional derivative (see definition (*Caputo*) below) became popular. Equations with Caputo derivatives for the first time in the book form were presented in [44]. Both types of derivatives were discussed in [23], see also [6].

Ordinary fractional differential equations

This collection includes all equations of the form

$$F(x, y(x), \mathcal{D}_{a_1}^{\alpha_1}\omega_1(x)y(x), \mathcal{D}_{a_2}^{\alpha_2}\omega_2(x)y(x), \ldots, \mathcal{D}_{a_n}^{\alpha_n}\omega_n(x)y(x)) = g(x), \tag{1.1}$$

where $\mathcal{D}_{a_j}^{\alpha_j}$ is a right/left-sided derivative of one of the known types (see Sec. 2 below and/or [43]), $\alpha_1, \ldots, \alpha_n$ are arbitrary positive real numbers, and $\omega_1, \ldots, \omega_n$ are certain weight functions. Such equations appeared due to direct generalizations of different applied models (description of a number of these applications can be found in the Proceedings of IFAC Workshops on Fractional Differentiation and its Applications (2004, Bordeaux, France; 2006, Porto, Portugal; 2008, Ankara, Turkey; 2010, Badajoz, Spain; 2012, Nanjing, China)). The main achievements of the theory of ordinary fractional differential equations are connected with basic results of the fractional calculus (see, e.g., [37], [43]).

We have to point out the following kinds of equations, which are particular cases of equation (1.1):

- linear homogeneous fractional ordinary differential equations,
- linear inhomogeneous fractional ordinary differential equations,
- linear homogeneous fractional ordinary differential equations with variable coefficients,
- linear inhomogeneous fractional ordinary differential equations with variable coefficients,
- nonlinear fractional ordinary differential equations,
- sequential differential equations of fractional order,
- ordinary fractional differential equations on the whole real line,
- multidimensional ordinary fractional differential equations.

The multidimensional theory is less developed because of several reasons. First of all, the corresponding elements of the theory of multidimensional fractional integro-differentiation are not completely investigated. Second, researchers are still in the process of discovering of the physical, mechanical, chemical, biological phenomenon, which can be adequately described by certain multidimensional models involving fractional derivatives. Therefore, one can single out only few types of multidimensional ordinary fractional differential equations developed up to the same level as the corresponding one-dimensional equations. For discussion of these problems we refer to the book [25] and references therein.

Fractional partial differential equations

Several models involving fractional partial differential equations are known. Anyway, a complete classification of this class of equations does not exist. The most cited source on this subject is the book [23]. We can also mention the recent books [33], [69] and extended lists of references presented in [23] and in [33]. Here we present several examples of known fractional partial differential equations. It should be mentioned that fractional derivatives with respect to the time variable are taken in these equations either in the form of Riemann-Liouville (see definition (*R-L*) below in Sec. 2) or in the form of Caputo (see definition (*Caputo*) below in Sec. 2). As for fractional derivatives with respect to spatial variables, they are taken in some appropriate sense including Riesz-Feller type fractional derivatives (inverses to fractional potential type operators) (see Sec. 2 below and/or [23], [43]).

- Gerasimov's equation:

$$\rho \frac{\partial^2 u}{\partial t^2} = k \left(D^{\alpha}_{-,t} \left[\frac{\partial^2 u}{\partial x^2} \right] \right) (x,t).$$

- "Hyperbolic" fractional differential equation:

$$\left(D^{\alpha}_{0+,x} D^{\beta}_{0+,y} u \right) (x,y) = f \left[x, y, u, D^{\alpha}_{0+,x} u, D^{\alpha-1}_{0+,x} D^{\beta}_{0+,y} u, D^{\beta}_{0+,y} u, D^{\alpha-1}_{0+,x} u \right].$$

- Differential equation of semi-integer order:

$$\sum_{k=0}^{n} a_k \left(D^{k/2}_{0+,x} D^{(n-k)/2}_{0+,y} u \right) (x,y) = f(x,y).$$

- Fractional diffusion-wave equation:

$$\left(D^{\alpha}_{0+,t} u \right) (x,t) = \left(D^{\beta}_{x} u \right) (x,t) \quad (1 \le \alpha \le 2,\ 0 \le \beta \le 2).$$

- Multidimensional diffusion-wave equation:

$$\left(D^{\alpha}_{0+,t} u \right) (\mathbf{x},t) = \lambda^2 \left(\Delta_{\mathbf{x}} u \right) (\mathbf{x},t),\ \mathbf{x} \in \mathbb{R}^n \quad (0 < \alpha < 2).$$

1.2. Methods of investigation

In this subsection we mention problems which are usually considered for fractional differential equations. We also single out the types of solutions as well as indicate methods for the solution.

Treating problems:

- Cauchy problem: values of standard derivatives at some points are given.
- Cauchy-type problem: values of fractional derivatives and/or integrals at some points are given.
- Dirichlet-type problem (for partial fractional differential equations): values of standard and/or fractional derivatives at end-points of an interval are given.
- Initial-boundary value problem with local and/or non-local initial conditions (i.e., Cauchy-type initial conditions).

Types of solutions:

- Solutions in classical functional spaces such as Schauder-type spaces or Lebesgue-type spaces, or corresponding weighted spaces.
- Generalised or weak solutions (i.e., distributions on certain spaces of test functions).

Methods of solution:

- Compositional methods based on certain known formulas for different kind of fractional integrals and derivatives.
- Methods of integral transforms, which allow to reduce fractional differential equations to functional equations.
- Integral equations method. Since, by definition, the fractional derivatives are compositions of differential operators and certain integral operators, then it is possible to find (for special types of equations) integral equations equivalent to considered fractional differential equations.
- Numerical methods. Numerical methods for fractional differential equations differ essentially as from the numerical methods for classical differential equations as from those performed for integral equations. Anyway one can find in the literature several numerical algorithms which are especially worked up for fractional equations (see, *e.g.*, [44]).

2. Basic components of investigations related to fractional differential equations

In this section we briefly outline certain investigations which form a base for further development of the theory of fractional differential equations.

2.1. Development of fractional calculus

Starting point for any investigation concerning fractional differential equations is to establish the corresponding results for fractional derivatives and integrals. Below we single out some collections of the properties which are usually in discussion.

- Formal properties (including semi-group properties), composition formulas for different types of fractional derivatives, calculation of fractional derivatives of elementary and special functions. Below one can find a list of some definitions of fractional derivatives. Surely this list is not complete and contains only most frequently cited notations.
  - Riemann-Liouville derivatives

$$\left(D_{a+}^{\alpha}u\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int\limits_{a}^{x}\frac{u(t)dt}{(x-t)^{\alpha-n+1}} \quad (n = [\operatorname{Re}\alpha]+1, \quad \alpha > 0); \tag{R-L}$$

– Caputo (or Gerasimov-Caputo) derivatives

$$\left({}^{C}D^{\alpha}_{a+}u\right)(x) = \left(D^{\alpha}_{a+}\left[u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k\right]\right)(x); \qquad \text{(Caputo)}$$

– Erdelyi-Kober derivatives

$$\left(D^{\alpha}_{a+;\sigma,\eta}u\right)(x) = x^{-\sigma\eta}\left(\frac{1}{\sigma x^{\sigma-1}}\frac{d}{dx}\right)^n \frac{\sigma x^{\sigma(n-\alpha)}}{\Gamma(\alpha)}\int_a^x \frac{t^{\sigma(\eta+\alpha+1)-1}u(t)dt}{(x^\sigma - t^\sigma)^{n-\alpha}}; \qquad \text{(E-K)}$$

– Hadamard derivatives

$$\left(\mathcal{D}^{\alpha}_{a+}u\right)(x) = \left(x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha+1}\frac{u(t)dt}{t}; \qquad \text{(Hadamard)}$$

– Grünwald-Letnikov derivatives

$$u^{\alpha)}_{+}(x) = \lim_{h\to+0}\frac{\left(\Delta^{\alpha}_h u\right)(x)}{h^\alpha} = \lim_{h\to+0}\frac{\sum\limits_{k=0}^{\infty}(-1)^k\left(\begin{array}{c}\alpha\\ k\end{array}\right)u(x-kh)}{h^\alpha}; \qquad \text{(G-L)}$$

– Riesz derivatives

$$\mathbf{D}^{\alpha}u(\mathbf{x}) = \mathcal{F}^{-1}|\boldsymbol{\omega}|^{\alpha}\mathcal{F}u(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^n; \qquad \text{(Riesz)}$$

– other definitions of fractional derivatives via pseudo-differential operators method (see, e.g., [20]).

- Acting properties of operators of fractional integration (differentiation) in different functional spaces.
- Representation of functions by fractional integrals of functions which belong to different functional spaces (in particular, characterization of spaces $I^{\alpha}(L_p)$).
- Operator properties (relations to singular integral operators, inversion formulas, relations to operators of integral transforms).

2.2. Development of the theory of first-order integral equations

Here we mention several kinds of integral equations which are closely related to the problems of fractional differential equations. The study of their solvability as well as their solution in closed form are important components of the theory of fractional differential equations.

– Abel integral equations and their applications.
– Integral equations with weak singularities.
– Volterra integral equations.
– Convolution type integral equations.

One of the most import aims here is to prove an equivalence of differential equations and integral equations.

2.3. Development of methods of integral transforms

As it was already mentioned, the method of integral transforms is one of the basic methods at the investigation of fractional differential equations. The most used integral transforms in this branch of mathematics are the following transforms:

- Fourier integral transforms.
- Laplace integral transforms.
- Mellin integral transforms (see, *e.g.*, [34]).
- Integral transforms with special functions in the kernel.
- Integral **G**-transforms

$$(\mathbf{G}f)(x) = \int_0^\infty G_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix} (a_i)_{1,p} \\ (b_j)_{1,q} \end{matrix}\right] f(t)dt,$$

where

$$G_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i)_{1,p} \\ (b_j)_{1,q} \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+s)\prod\limits_{i=1}^{n}\Gamma(1-a_i-s)}{\prod\limits_{i=n+1}^{p}\Gamma(a_i+s)\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-s)} z^{-s}ds.$$

- Integral **H**-transforms (see, *e.g.*, [21])

$$(\mathbf{H}f)(x) = \int_0^\infty H_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] f(t)dt,$$

where

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod\limits_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod\limits_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)} z^{-s}ds.$$

2.4. Development of the theory of special functions

Methods and results of special function theory are very important for the problems which we are discussing. First of all this is due to the fact that a number of fractional differential equations admits closed form solutions. These solutions are represented usually via special functions of specific types. Special functions mostly applied in the theory of fractional differential equations are the following (see, *e.g.*, [35]):

- generalizations of Mittag-Leffler special functions

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)};$$

– generalizations of Wright special functions

$$ {}_p\Psi_q \left[\begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod\limits_{i=1}^{p} \Gamma(a_i+\alpha_i k)}{\prod\limits_{j=1}^{q} \Gamma(b_j+\beta_j k)} \frac{z^k}{k!}; $$

– general types of special functions:

– G-function: $G^{m,n}_{p,q}\left[z \middle| \begin{array}{c} (a_i)_{1,p} \\ (b_j)_{1,q} \end{array} \right]$,

– H-function (see, *e.g.*, [36]): $H^{m,n}_{p,q}\left[z \middle| \begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \right]$.

2.5. Development of multidimensional fractional calculus

At last we have to mention several multidimensional operators whose properties are now forming the base for developing a theory of multidimensional fractional differential equations:

– polypotential type operator

$$ (\mathcal{K}^\alpha\varphi)(\mathbf{x}) = \frac{1}{2^n \prod\limits_{k=1}^{n} \Gamma(\alpha_k)\cos \alpha_k\pi/2} \int\limits_{\mathbb{R}^n} \frac{\varphi(\mathbf{t})d\mathbf{t}}{\prod\limits_{k=1}^{n} |x_k-t_k|^{\alpha_k}}; $$

– Riesz potential

$$ (\mathbf{I}^\alpha\varphi)(\mathbf{x}) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{\varphi(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^\alpha}; $$

– hypersingular operator

$$ (\mathbf{D}^\alpha\varphi)(\mathbf{x}) = \frac{1}{d_{n,l}(\alpha)} \int\limits_{\mathbb{R}^n} \frac{\left(\Delta^l_{\mathbf{y}}\varphi\right)(\mathbf{x})}{|\mathbf{y}|^{n+\alpha}} d\mathbf{y}, \ \left(\Delta^l_{\mathbf{y}}\varphi\right)(\mathbf{x}) = \sum_{k=0}^{l}(-1)^k \left(\begin{array}{c} l \\ k \end{array} \right) \varphi(\mathbf{x}-k\mathbf{y}). $$

3. The role of Professor S.G. Samko in the creation and development of the theory of fractional differential equations

First of all we have to point out that it is problematic to mention all results by S.G. Samko having influence on the development of the discussed theory. Therefore, we restrict ourselves only on the main results which we will briefly outline. In particular, the list of S.G. Samko's books and articles, cited here, is surely not complete.

Main directions in which S.G. Samko obtained results having an essential influence on the establishing and development of the theory of fractional differential equations are the following:

– singular integral equations and boundary value problems;
– Abel integral equations and their applications;
– integral equations with weak singularities;

– integral equations of convolution type;
– fractional calculus;
– fractional powers of operators;
– one- and multi-dimensional theory of potential type operators;
– hypersingular operators;
– functional spaces with variable exponents.

One can compare these results with the above-described aims of the theory and applications of fractional differential equations. It makes the reader an impression on the great importance of the results by S.G. Samko.

The rest of this section is organized as follows. We mention certain directions as titles of subsections, specify some results in these directions and present the corresponding references.

3.1. Singular integral equations and boundary value problems

– Abstract theory (Noether theory) of complete (or general) singular integral equations and singular integral equations with a shift [15], [16], [17] (including equations with integrals along an open arc Γ [7], [14])

$$a(t)\varphi(t) + b(t)\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t}d\tau + \int\limits_{\Gamma} k(t,\tau)\varphi(\tau)d\tau = f(t).$$

– Classes of complete singular integral equations solvable in closed form [53], [57].
– Generalized argument principle [8], [9].
– Acting properties of singular integral operators (SIO) (including SIO on Carleson curves) in spaces with variable exponents [27], [29], [30], [31].
– Applications of the theory of singular integral equations to the investigation of boundary value problems in classes of analytic functions represented by the Cauchy type integrals [26], [28].

3.2. Abel integral equations and their generalizations

– Solution in closed form of generalized Abel type integral equations [49], [50], [55].
– Proof of the relation formulas of fractional integration operators with singular integral operators [51], in particular,

$$I_{-}^{\alpha}\varphi = \cos\,\alpha\pi I_{+}^{\alpha}\varphi + \sin\,\alpha\pi S I_{+}^{\alpha}\varphi.$$

– Noether theory of generalized Abel type integral equations [52], [54]:

$$u(x)\int\limits_{a}^{x} \frac{\varphi(t)dt}{(x-t)^{\mu}} + v(x)\int\limits_{x}^{b} \frac{\varphi(t)dt}{(t-x)^{\mu}} = f(x).$$

3.3. Integral equations with weak singularities

- Solution in closed form of integral equations with logarithmic kernel [60].
- Normal solvability, asymptotic method of solution to integral equations with logarithmic and homogeneous kernel [22].
- Asymptotic behaviour of singular values of integral operators with weak singularities [10].
- Inversion of integral equation

$$\int_{-\infty}^{x} k(x-t)\varphi(t)dt = f(x)$$

  with Sonine's kernel $k \in L_1^{loc}(\mathbb{R}^1_+)$ (i.e., such that there exists $l \in L_1^{loc}(\mathbb{R}^1_+)$ such that $\int_0^x k(x-t)l(t)dt \equiv 1$) [3], [4].
- Classes of correctness in the weighted Hölder spaces for Sonine's integral equation [5].

3.4. Convolution type integral equations

- Solvability of convolution type integral equations [12], [13]

$$a_0(t)\varphi(t) + \sum_{j=1}^{n} a_j(t) \int_{-\infty}^{\infty} b_j(\tau)h_j(t-\tau)\varphi(\tau)d\tau = f(t).$$

- Solution in a closed form of special kinds of convolution type integral equations [18].
- Noether theory of convolution type integral equations [19].
- Convolution type operators in spaces with variable exponent [64], [65].
- Nonlinear convolution type operators and equations [11].

3.5. Fractional integro-differentiation

One of the most essential achievements in this area were gathered in the books written by S.G. Samko jointly with his friends and colleagues A.A. Kilbas and O.I. Marichev [42], [43]. These books are now standard reference works on fractional calculus. They constitute the very precise and careful analysis of all essential questions of this subject. It is not a surprise that the English version [43] is called by many scientists as "THE BIBLE OF FRACTIONAL CALCULUS". Inspite of the great importance of [43] and its encyclopedic nature, investigations of fractional differential equations, conducted after this book and inspired by it, still deserve a separate survey article.

Below we briefly describe a number of precise results obtained by S.G. Samko in this area.

- Description of the image $\mathbf{I}^\alpha(L_p)$ of the Riesz potential [56], [58].
- Isomorphism of weighted Hölder spaces under the acting of the operator of fractional derivative [39].

– Marchaud-type formula for the operator of fractional derivative in domains $\Omega \subset \mathbb{R}^n$ [45]

$$D_{\Omega}^{\alpha}\varphi(\mathbf{x}) = c(\alpha)\left[\varphi(\mathbf{x})\int\limits_{\mathbb{R}^n\setminus\Omega}\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}} + \int\limits_{\Omega}\frac{\varphi(\mathbf{x})-\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}}d\mathbf{y}\right].$$

– Description of functions that have no first-order derivative, but have fractional derivatives of all orders less than one [32].
– Approximative definition of a fractional differentiation [48].
– Characterization of the range of one-dimensional fractional integration in the space with variable exponent [47].
– Description of the acting properties of generalized Riemann-Liouville operator of fractional integration of variable order

$$I_{a+}^{\alpha(x)}\varphi(x) = \frac{1}{\Gamma(\alpha(x))}\int\limits_{a}^{x}\frac{\varphi(t)dt}{(x-t)^{1-\alpha(x)}}$$

in generalized Hölder spaces [38], [39].
– Formula for a left inverse to Liouville operator of fractional integration (i.e., with $a = -\infty$) of variable order [63].
– Fractional differentiation and integration of variable order in spaces with variable exponent [41], [62].

3.6. Fractional powers of operators

– Application of the hypersingular integrals' method to construction of effective formulas for fractional powers of the classical operators of mathematical physics

$$I - \Delta, \qquad -\Delta_x + \frac{\partial}{\partial t}, \qquad I - \Delta_x + \frac{\partial}{\partial t},$$

where I is an identity operator, and Δ is the Laplace operator [1], [66], [67].

3.7. The theory of (one- and multidimensional) potential type operators

– Inversion of the Riesz type fractional potential [40], [59]

$$\left(\mathcal{K}^{\alpha}\varphi\right)(x) = \frac{1}{\gamma_{n-1}(\alpha)}\int\limits_{|\mathbf{x}|=1}\frac{\varphi(\sigma)d\sigma}{|\mathbf{x}-\sigma|^{n-1-\alpha}}.$$

– Investigation of properties of operators generalizing fractional integration operators

$$\left(I_m^{\alpha(\cdot)}\varphi\right)(x) = \int\limits_{\Omega}\frac{\varphi(\mathbf{y})d\mu(\mathbf{y})}{d(\mathbf{x},\mathbf{y})^{m-\alpha(x)}}$$

in the Lebesgue spaces with variable exponents [2].
– Application to multidimensional integral equations of the first kind [68], [46].
– Relation to Grünwald-Letnikov's approach to fractional calculus [61].

4. Conclusion

The theory of fractional differential equations is a highly developing branch of mathematics. These equations are arising either from theoretical considerations and generalizations or from many applications for which the fractional machinery is a natural tool for the description of the corresponding phenomenon.

In Section 2 we tried to describe briefly the main aims and ideas which are used at the formulation and investigation of fractional differential equations. Surely such a description is far from completeness since the subject is too wide.

In order to establish and develop the theory of fractional differential equations one needs to create background, to develop instruments for investigation and to introduce new innovative ideas. In all these directions Professor S.G. Samko obtained many important results. In Section 3 we have seen only few of these results, which lay in the core of the considered theory. A lot of fresh and innovative ideas can be found in the cited papers and books by S.G. Samko.

We hope that our article helps the reader to find new ideas and to show the ways for further development of a very interesting branch of science – the theory and applications of fractional differential equations.

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