

Edge Fluctuations of Eigenvalues of Wigner Matrices

Hanna Döring and Peter Eichelsbacher

Abstract. We establish a moderate deviation principle (MDP) for the number of eigenvalues of a Wigner matrix in an interval close to the edge of the spectrum. Moreover we prove a MDP for the i th largest eigenvalue close to the edge. The proof relies on fine asymptotics of the variance of the eigenvalue counting function of GUE matrices due to Gustavsson. The extension to large families of Wigner matrices is based on the Tao and Vu Four Moment Theorem. Possible extensions to other random matrix ensembles are commented.

Mathematics Subject Classification (2010). Primary 60B20; Secondary 60F10, 15A18.

Keywords. Large deviations, moderate deviations, Wigner random matrices, Gaussian ensembles, Four Moment Theorem.

1. Introduction

Recently, in [5] and [4] the Central Limit Theorem (CLT) for the eigenvalue counting function of *Wigner matrices*, that is the number of eigenvalues falling in an interval, was established. This *universality result* relies on fine asymptotics of the variance of the eigenvalue counting function, on the Fourth Moment Theorem due to Tao and Vu as well as on recent localization results due to Erdős, Yau and Yin. There are many random matrix ensembles of interest, but to focus our discussion and to clear the exposition we shall restrict ourselves to the most famous model class of ensembles, the Wigner Hermitian matrix ensembles. For an integer $n \geq 1$ consider an $n \times n$ Wigner Hermitian matrix $M_n = (Z_{ij})_{1 \leq i, j \leq n}$: Consider a family of jointly independent complex-valued random variables $(Z_{ij})_{1 \leq i, j \leq n}$ with $Z_{ji} = \bar{Z}_{ij}$, in particular the Z_{ii} are real valued. For $1 \leq i < j \leq n$ require that the random variables have mean zero and variance one and the $Z_{ij} \equiv Z$ are identically

distributed, and for $1 \leq i = j \leq n$ require that $Z_{ii} \equiv Z'$ are also identically distributed with mean zero and variance one. The distributions of Z and Z' are called atom distributions. An important example of a Wigner Hermitian matrix M_n is the case where the entries are Gaussian, that is Z_{ij} is distributed according to a complex standard Gaussian $N(0, 1)_{\mathbb{C}}$ for $i \neq j$ and Z_{ii} is distributed according to a real standard Gaussian $N(0, 1)_{\mathbb{R}}$, giving rise to the so-called Gaussian Unitary Ensembles (GUE). GUE matrices will be denoted by M'_n . In this case, the joint law of the eigenvalues is known, allowing a good description of their limiting behavior both in the global and local regimes (see [1]). In the Gaussian case, the distribution of the matrix is invariant by the action of the group $SU(n)$. The eigenvalues of the matrix M_n are independent of the eigenvectors which are Haar distributed. If $(Z_{i,j})_{1 \leq i < j}$ are real valued the *symmetric Wigner matrix* is defined analogously and the case of Gaussian variables with $\mathbb{E}Z_{ii}^2 = 2$ is of particular importance, since their law is invariant under the action of the orthogonal group $SO(n)$, known as Gaussian Orthogonal Ensembles (GOE).

The matrix $W_n := \frac{1}{\sqrt{n}}M_n$ is called the coarse-scale normalized Wigner Hermitian matrix, and $A_n := \sqrt{n}M_n$ is called the fine-scale normalized Wigner Hermitian matrix. For any $n \times n$ Hermitian matrix A we denote by $\lambda_1(A), \dots, \lambda_n(A)$ the real eigenvalues of A . We introduce the *eigenvalue counting function*

$$N_I(A) := |\{1 \leq i \leq n : \lambda_i(A) \in I\}|$$

for any interval $I \subset \mathbb{R}$. We will consider $N_I(W_n)$ as well as $N_I(A_n)$. Remark that $N_I(W_n) = N_{nI}(A_n)$. The global *Wigner theorem* states that the empirical measure $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$ on the eigenvalues λ_i of the coarse-scale normalized Wigner Hermitian matrix W_n converges weakly almost surely as $n \rightarrow \infty$ to the semicircle law

$$d\varrho_{sc}(x) = \frac{1}{2\pi} \sqrt{4 - x^2} 1_{[-2,2]}(x) dx,$$

(see [1, Theorem 2.1.21, Theorem 2.2.1]). Consequently, for any interval $I \subset \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} N_I(W_n) = \varrho_{sc}(I) := \int_I \varrho_{sc}(y) dy$$

almost surely. At the fluctuation level, it is well known that for the GUE, $W'_n := \frac{1}{\sqrt{n}}M'_n$ satisfies a CLT (see [17]): Let I_n be an interval in \mathbb{R} . If $\mathbb{V}(N_{I_n}(W'_n)) \rightarrow \infty$ as $n \rightarrow \infty$, then

$$\frac{N_{I_n}(W'_n) - \mathbb{E}[N_{I_n}(W'_n)]}{\sqrt{\mathbb{V}(N_{I_n}(W'_n))}} \rightarrow N(0, 1)_{\mathbb{R}}$$

as $n \rightarrow \infty$ in distribution. In [14] the asymptotic behavior of the expectation and the variance of the counting function $N_{I_n}(W'_n)$ for intervals $I_n = [y(n), \infty)$ with $y(n) = G^{-1}(k/n)$ (where $k = k(n)$ is such that $k/n \rightarrow a \in (0, 1)$ – *strictly in the bulk* –, and G denotes the distribution function of the semicircle law) was

established:

$$\mathbb{E}[N_{I_n}(W'_n)] = n - k(n) + O\left(\frac{\log n}{n}\right) \text{ and } \mathbb{V}(N_{I_n}(W'_n)) = \left(\frac{1}{2\pi^2} + o(1)\right) \log n. \tag{1.1}$$

The proof applied strong asymptotics for orthogonal polynomials with respect to exponential weights, see [6]. These conclusions were extended to non-Gaussian Wigner Hermitian matrices in [5].

In this article we focus on the behaviour of the eigenvalue counting function $N_{I_n}(W_n)$ evaluated *at the edge* of the spectrum. Fine asymptotics of expectation and variance as well as the CLT at the edge of the spectrum of GUE matrices are known, see [14]. Let $I_n = [y_n, \infty)$ where $y_n \rightarrow 2^-$ for $n \rightarrow \infty$ in a certain speed. We prove a global moderate deviation principle (MDP) for $Z_n := \frac{N_{I_n}(W'_n) - \frac{2}{3\pi}n(2-y_n)^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2} \log(n(2-y_n)^{3/2})}}$, that means: For any sequence $(a_n)_n$ with $1 \ll a_n \ll \sqrt{\log n}$, we have

$$P\left(\frac{N_{I_n}(W'_n) - \frac{2}{3\pi}n(2-y_n)^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2} \log(n(2-y_n)^{3/2})}} \sim x\right) \approx e^{-a_n^2 I(x)},$$

see Theorem 2.1 for a precise statement. Using the equivalence

$$N_{[y,\infty)}(W_n) \leq n - i \text{ if and only if } \lambda_i(W_n) \leq y$$

in Theorem 3.1 we prove a local MDP for an eigenvalue near the edge of the spectrum of W'_n . Applying the Four Moment Theorem due to Tao and Vu, the local MDP can be generalized to a large class of Hermitian Wigner matrices. Finally this yields the main theorem of the present article: a universal global MDP of the eigenvalue counting function $N_{[y,\infty)}(W_n)$ for y near the edge of the spectrum. We state it in Theorem 5.1. The last section indicates how to achieve the previous moderate deviation results for symmetric Wigner matrices.

2. Global moderate deviations at the edge of the spectrum

Certain deviations results and concentration properties for Wigner matrices were considered. Our aim is to establish moderate deviation principles. Recall that a sequence of laws $(P_n)_{n \geq 0}$ on a Polish space Σ satisfies a large deviation principle (LDP) with good rate function $I : \Sigma \rightarrow \mathbb{R}_+$ and speed s_n going to infinity with n if and only if the level sets $\{x : I(x) \leq M\}$, $0 \leq M < \infty$, of I are compact and for all closed sets F

$$\limsup_{n \rightarrow \infty} s_n^{-1} \log P_n(F) \leq - \inf_{x \in F} I(x)$$

whereas for all open sets O

$$\liminf_{n \rightarrow \infty} s_n^{-1} \log P_n(O) \geq - \inf_{x \in O} I(x).$$

We say that a sequence of random variables satisfies the LDP when the sequence of measures induced by these variables satisfies the LDP. Formally a moderate

deviation principle is nothing else but the LDP. However, we speak about a moderate deviation principle (MDP) for a sequence of random variables, whenever the scaling of the corresponding random variables is between that of an ordinary Law of Large Numbers (LLN) and that of a CLT.

Large deviation results for the empirical measures of Wigner matrices are still only known for the Gaussian ensembles since their proof is based on the explicit joint law of the eigenvalues, see [2] and [1]. A moderate deviation principle for the empirical measure of the GUE or GOE is also known, see [7]. This moderate deviations result does not have yet a fully universal version for Wigner matrices. It has been generalised to Gaussian divisible matrices with a deterministic self-adjoint matrix added with converging empirical measure [7] and to Bernoulli matrices [9]. Recently we proved in [11] a MDP for the number of eigenvalues *in the bulk* of the spectrum of a GUE matrix. If M'_n is a GUE matrix and $W'_n := \frac{1}{\sqrt{n}}M'_n$ and I_n be an interval in \mathbb{R} . If $\mathbb{V}(N_{I_n}(W'_n)) \rightarrow \infty$ for $n \rightarrow \infty$, then, for any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll \sqrt{\mathbb{V}(N_{I_n}(W'_n))}$, the sequence $(Z_n)_n$ with

$$Z_n = \frac{N_{I_n}(W'_n) - \mathbb{E}[N_{I_n}(W'_n)]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W'_n))}}$$

satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$. Moreover let $I = [y, \infty)$ with $y \in (-2, 2)$ strictly in the bulk, then the sequence $(\hat{Z}_n)_n$ with $\hat{Z}_n = \frac{N_I(W'_n) - n \varrho_{sc}(I)}{a_n \sqrt{\frac{1}{2\pi^2} \log n}}$ satisfies the MDP with the same speed, the same rate function,

and in the regime $1 \ll a_n \ll \sqrt{\log n}$ (called the MDP with numerics; see Theorem 1.1 in [11]). It follows applying (1.1). In [11], these conclusions were extended to non-Gaussian Wigner Hermitian matrices. In [10] we proved a further universal MDP for the logarithm of the determinants of Wigner matrices.

The first observation in this paper is, that the MDP for $(Z_n)_n$ and $(\hat{Z}_n)_n$, respectively, is not restricted to the bulk of the spectrum. To state the result, let $\delta > 0$ and assume that $y_n \in [-2 + \delta, 2)$ and $n(2 - y_n)^{3/2} \rightarrow \infty$ when $n \rightarrow \infty$. Then with [14, Lemma 2.3] the variance of the number of eigenvalues of W'_n in $I_n := [y_n, \infty)$ satisfies

$$\mathbb{V}(N_{I_n}(W'_n)) = \frac{1}{2\pi^2} \log(n(2 - y_n)^{3/2}) (1 + \eta(n)), \tag{2.1}$$

where $\eta(n) \rightarrow 0$ as $n \rightarrow \infty$. Moreover the expected number of eigenvalues of W'_n in I_n , when $y_n \rightarrow 2^-$, is given by [14, Lemma 2.2]:

$$\mathbb{E}(N_{I_n}(W'_n)) = \frac{2}{3\pi} n(2 - y_n)^{3/2} + O(1). \tag{2.2}$$

Hence applying Theorem 1.1 in [11] we immediately obtain:

Theorem 2.1. *Let M'_n be a GUE matrix and $W'_n = \frac{1}{\sqrt{n}}M'_n$. Let $I_n = [y_n, \infty)$ where $y_n \rightarrow 2^-$ for $n \rightarrow \infty$. Assume that $y_n \in [-2 + \delta, 2)$ and $n(2 - y_n)^{3/2} \rightarrow \infty$ when $n \rightarrow \infty$. Then, for any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll$*

$\sqrt{\mathbb{V}(N_{I_n}(W'_n))}$, the sequence $\frac{N_{I_n}(W'_n) - \mathbb{E}[N_{I_n}(W'_n)]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W'_n))}}$ satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$. Moreover the sequence

$$Z_n := \frac{N_{I_n}(W'_n) - \frac{2}{3\pi}n(2 - y_n)^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2} \log(n(2 - y_n)^{3/2})}}$$

satisfies the MDP with the same speed, the same rate function, and in the regime $1 \ll a_n \ll \sqrt{\log(n(2 - y_n)^{3/2})}$ (called the MDP with numerics).

For symmetry reasons an analogous result could be formulated for the counting function $N_{I_n}(W'_n)$ near the left edge of the spectrum.

3. Local moderate deviations at the edge of the spectrum

Under certain conditions on i it was proved in [14] that the i th eigenvalue λ_i of the GUE W'_n satisfies a CLT. Consider $t(x) \in [-2, 2]$ defined for $x \in [0, 1]$ by

$$x = \int_{-2}^{t(x)} \varrho_{sc}(t) dt = \frac{1}{2\pi} \int_{-2}^{t(x)} \sqrt{4 - x^2} dx.$$

Then for $i = i(n)$ such that $i/n \rightarrow a \in (0, 2)$ as $n \rightarrow \infty$ (i.e., λ_i is eigenvalue in the bulk), $\lambda_i(W'_n)$ satisfies a CLT:

$$X_n := \sqrt{\frac{4 - t(i/n)^2}{2}} \frac{\lambda_i(W'_n) - t(i/n)}{\frac{\sqrt{\log n}}{n}} \rightarrow N(0, 1) \tag{3.1}$$

for $n \rightarrow \infty$. Remark that $t(i/n)$ is sometimes called the *classical or expected location* of the i th eigenvalue. The standard deviation is $\frac{\sqrt{\log n}}{\pi\sqrt{2}} \frac{1}{n\varrho_{sc}(t(i/n))}$. Note that from the semicircular law, the factor $\frac{1}{n\varrho_{sc}(t(i/n))}$ is the mean eigenvalue spacing. Informally, (3.1) asserts in the GUE case, that each eigenvalue $\lambda_i(W'_n)$ typically deviates by $O(\sqrt{\log n}/(n\varrho(t(i/n))))$ around its classical location. This result can be compared with the so-called *eigenvalue rigidity property* $\lambda_i(W'_n) = t(i/n) + O(n^{-1+\varepsilon})$ established in [12], which has a slightly worse bound on the deviation but which holds with overwhelming probability and for general Wigner ensembles. See also discussions in [20, Section 3]. We proved in [11, Theorem 4.1] a MDP for $(\frac{1}{a_n}X_n)_n$ with X_n in (3.1), for any $1 \ll a_n \ll \sqrt{\log n}$, with speed a_n^2 and rate $x^2/2$. Moreover in [11, Theorem 4.2], these conclusions were extended to non-Gaussian Wigner Hermitian matrices. The proofs are achieved by the tight relation between eigenvalues and the counting function expressed by the elementary equivalence, for $I(y) = [y, \infty)$, $y \in \mathbb{R}$,

$$N_{I(y)}(W_n) \leq n - i \text{ if and only if } \lambda_i(W_n) \leq y. \tag{3.2}$$

This relation is true for any eigenvalue $\lambda_i(W_n)$, independent of sitting being in the bulk of the spectrum or very close to the right edge of the spectrum. Hence the next goal is to transport the MDP for the counting function of eigenvalues close

to the (right) edge, Theorem 2.1, to a MDP for any singular eigenvalue close to the right edge of the spectrum. Consider $i = i(n)$ with $i \rightarrow \infty$ but $i/n \rightarrow 0$ as $n \rightarrow \infty$ and define $\lambda_{n-i}(W'_n)$ as eigenvalue number $n - i$ in the GUE. An example is $i(n) = n - \log n$. In [14, Theorem 1.2] a CLT was proven, which is

$$Z_{n,i} := \frac{\lambda_{n-i}(W'_n) - \left(2 - \left(\frac{3\pi}{2} \frac{i}{n}\right)^{2/3}\right)}{\text{const} \left(\frac{\log i}{i^{2/3} n^{4/3}}\right)^{1/2}} \rightarrow N(0, 1)_{\mathbb{R}} \tag{3.3}$$

in distribution with $\text{const} = ((3\pi)^{2/3} 2^{1/3})^{-1/2}$. Remark that the formulation in [14, Theorem 1.2] is different, since first of all the GUE in [14] was defined such that the limiting semicircular law has support $[-1, 1]$ and, second the CLT in [14] is formulated for $\lambda_{n-i}(M'_n)$ instead of $\lambda_{n-i}(W'_n)$. The choice of the asymptotic expectation and variance in (3.3) can be explained as follows. Let $g(y_n)$ be the expected number of eigenvalues in $I_n = [y_n, \infty)$. Then with (3.2)

$$\begin{aligned} P(\lambda_{n-i}(W'_n) \leq y_n) &= P(N_{I_n}(W'_n) \leq i) \\ &= P\left(\frac{N_{I_n}(W'_n) - g(y_n)}{\sqrt{\mathbb{V}(N_{I_n}(W'_n))}} \leq \frac{i - g(y_n)}{\sqrt{\mathbb{V}(N_{I_n}(W'_n))}}\right). \end{aligned}$$

Trying to apply the CLT for N_{I_n} is choosing y_n such that $\frac{i - g(y_n)}{\sqrt{\mathbb{V}(N_{I_n}(W'_n))}} \rightarrow x$ for $n \rightarrow \infty$, because this would imply $P(\lambda_{n-i}(W'_n) \leq y_n) \rightarrow \int_{-\infty}^x \varphi_{0,1}(t) dt$, where $\varphi_{0,1}(\cdot)$ denotes the density of the standard normal distribution. The candidate for y_n can be found as in the proof of [14, Theorem 1.2], with $g(y_n) = \frac{2}{3\pi} n(2 - y_n)^{3/2} + O(1)$ and $h(y_n) = \mathbb{V}(N_{I_n}(W'_n))^{1/2} = \frac{1}{\sqrt{2\pi}} \log^{1/2}(n(2 - y_n)^{3/2}) + o(\log^{1/2}(n(2 - y_n)^{3/2}))$. Applying the same heuristic as on page 157 in [14], we obtain

$$y_n \approx 2 - \left(\frac{3\pi}{2} \frac{i}{n}\right)^{2/3} + x \left(((3\pi)^{2/3} 2^{1/3})^{-1/2} \frac{\log i}{i^{2/3} n^{4/3}} \right)^{1/2}.$$

With respect to the statement in Theorem 2.1 one might expect a MDP for $(\frac{1}{a_n} Z_{n,i})_n$ for certain growing sequences $(a_n)_n$. We have

$$\begin{aligned} P(Z_{n,i}/a_n \leq x) &= P(\lambda_{n-i}(W'_n) \leq y_n(a_n)) = P(N_{I_n}(W'_n) \leq i) \\ &= P\left(\frac{N_{I_n}(W'_n) - \mathbb{E}(N_{I_n}(W'_n))}{a_n \sqrt{\mathbb{V}(N_{I_n}(W'_n))}} \leq \frac{i - \mathbb{E}(N_{I_n}(W'_n))}{a_n \sqrt{\mathbb{V}(N_{I_n}(W'_n))}}\right) \end{aligned}$$

with

$$y_n(a_n) := 2 - \left(\frac{3\pi}{2} \frac{i}{n}\right)^{2/3} + a_n x \left(((3\pi)^{2/3} 2^{1/3})^{-1/2} \left(\frac{\log i}{i^{2/3} n^{4/3}}\right) \right)^{1/2} \tag{3.4}$$

and $I_n = [y_n(a_n), \infty)$. Since $i \rightarrow \infty$ and $i/n \rightarrow 0$ for $n \rightarrow \infty$, we have that $y_n(a_n) \rightarrow 2^-$ for every a_n such that $a_n \ll (\log i)^{1/2}$. Hence we can apply (2.2), that is $\mathbb{E}(N_{I_n}(W'_n)) = \frac{2}{3\pi} n(2 - y_n(a_n))^{3/2} + O(1)$. With

$$2 - y_n(a_n) = \left(\frac{3\pi}{2} \frac{i}{n}\right)^{2/3} \left(1 - \frac{a_n x \log^{1/2} i}{(3\pi/\sqrt{2})i}\right)$$

by Taylor’s expansion we obtain

$$\frac{2}{3\pi}n(2 - y_n(a_n))^{3/2} = i - \frac{1}{\sqrt{2\pi}}a_n x \log^{1/2} i + o(a_n x \log^{1/2} i), \tag{3.5}$$

and therefore $i - \mathbb{E}(N_{I_n}(W'_n)) = \frac{1}{\sqrt{2\pi}}a_n x \log^{1/2} i + o(a_n x \log^{1/2} i) + O(1)$. From (3.5) we obtain that $n(2 - y_n(a_n))^{3/2} \rightarrow \infty$ for $n \rightarrow \infty$ for every $a_n \ll (\log i)^{1/2}$. Hence we can apply (2.1), that is $\mathbb{V}(N_{I_n}(W'_n)) = \frac{1}{2\pi^2} \log(n(2 - y_n(a_n))^{3/2}) (1 + o(1))$. With (3.5) the variance $\mathbb{V}(N_{I_n}(W'_n))$ equals

$$\left(\frac{1}{2\pi^2} \log\left(\frac{3\pi}{2}i\right) + \frac{1}{2\pi^2} \log\left(1 - \frac{a_n x (\log i)^{1/2}}{\sqrt{2\pi}i} + o\left(\frac{a_n x (\log i)^{1/2}}{i}\right)\right) \right) (1 + o(1)).$$

Summarizing we have proven that for any growing sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll (\log i)^{1/2}$

$$\frac{i - \mathbb{E}(N_{I_n}(W'_n))}{a_n \mathbb{V}(N_{I_n}(W'_n))^{1/2}} = x + o(1).$$

By Theorem 2.1 we obtain for every $x < 0$ that $\lim_{n \rightarrow \infty} \frac{1}{a_n^2} \log P(Z_{n,i}/a_n \leq x) = -\frac{x^2}{2}$. With $P(Z_{n,i}/a_n \geq x) = P(N_{I_n}(W'_n) \geq i - 1)$ the same calculations lead, for every $x > 0$, to

$$\lim_{n \rightarrow \infty} \frac{1}{a_n^2} \log P(Z_{n,i}/a_n \geq x) = -\frac{x^2}{2}. \tag{3.6}$$

Next we choose all open intervals (a, b) , where at least one of the endpoints is finite and where none of the endpoints is zero. Denote the family of such intervals by \mathcal{U} . Now it follows for each $U = (a, b) \in \mathcal{U}$,

$$\mathcal{L}_U := \lim_{n \rightarrow \infty} \frac{1}{a_n^2} \log P(Z_{n,i}/a_n \in U) = \begin{cases} b^2/2 & : a < b < 0 \\ 0 & : a < 0 < b \\ a^2/2 & : 0 < a < b \end{cases}$$

By [8, Theorem 4.1.11], $(Z_{n,i}/a_n)_n$ satisfies a weak MDP (see definition in [8, Section 1.2]) with speed a_n^2 and rate function $t \mapsto \sup_{U \in \mathcal{U}: t \in U} \mathcal{L}_U = \frac{t^2}{2}$. With (3.6), it follows that $(Z_{n,i}/a_n)_n$ is exponentially tight (see definition in [8, Section 1.2]), hence by Lemma 1.2.18 in [8], $(Z_{n,i}/a_n)_n$ satisfies the MDP with the same speed and the same good rate function. Hence we have proven:

Theorem 3.1. *Let M'_n be a GUE matrix and $W'_n = \frac{1}{\sqrt{n}}M'_n$. Consider $i = i(n)$ such that $i \rightarrow \infty$ but $i/n \rightarrow 0$ as $n \rightarrow \infty$. If λ_{n-i} denotes the eigenvalue number $n - i$ in the GUE matrix W'_n it holds that for any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll (\log i)^{1/2}$, the sequence $(\frac{1}{a_n}Z_{n,i})_n$ with $Z_{n,i}$ given by (3.3) satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$.*

4. Universal local moderate deviations near the edge

Our next goal is to check whether the precise distribution of the atom variables Z_{ij} of a Hermitian random matrix M_n are relevant for the conclusion of the MDP stated in Theorems 2.1 and 3.1, so long as they are normalized to have mean zero and variance one, and are jointly independent on the upper-triangular portion of M_n . It is a remarkable feature of our MDP results that they are *universal*, hence the distribution of the atom variables are irrelevant in some sense. The arguments used above relied heavily on the special structure of the GUE ensemble, in particular on the determinantal structure of the joint probability distribution (see [11, Theorem 1.1 and 1.3]) and on the fine asymptotics of the expectation and the variance of the eigenvalue counting function of GUE presented in [14]. We apply the swapping method due to Tao and Vu, in which one replaces the entries of one Wigner Hermitian matrix M_n with another matrix M'_n which are close in the sense of matching moments. This goes back to Lindeberg’s exchange strategy for proving the classical CLT, [15], and applied to Wigner matrices, e.g., in [3]. The precise statement of the so-called Four Moment Theorem needs some preparation. We will use the notation as in [20].

We say that two complex random variables η_1 and η_2 *match to order k* with $k \in \mathbb{N}$ if

$$\mathbb{E}[\operatorname{Re}(\eta_1)^m \operatorname{Im}(\eta_1)^l] = \mathbb{E}[\operatorname{Re}(\eta_2)^m \operatorname{Im}(\eta_2)^l]$$

for all $m, l \geq 0$ such that $m + l \leq k$. We will consider the case when the real and the imaginary parts of η_1 or of η_2 are independent, then the matching moment condition simplifies to the assertion that $E[\operatorname{Re}(\eta_1)^m] = E[\operatorname{Re}(\eta_2)^m]$ and $E[\operatorname{Im}(\eta_1)^l] = E[\operatorname{Im}(\eta_2)^l]$ for all $m, l \geq 0$ such that $m + l \leq k$.

We say that the Wigner Hermitian matrix M_n obeys Condition **(C0)** if we have the exponential decay condition

$$P(|Z_{ij}| \geq t^C) \leq e^{-t}$$

for all $1 \leq i, j \leq n$ and $t \geq C'$, and some constants C, C' independent of i, j, n . We say that the Wigner Hermitian matrix M_n obeys Condition **(C1)** with constant C_0 if one has

$$\mathbb{E}|Z_{ij}|^{C_0} \leq C$$

for some constant C independent of n . Of course, Condition **(C0)** implies Condition **(C1)** for any C_0 , but not conversely. The statement of the Four Moment Theorem for eigenvalues is:

Theorem 4.1 (Four Moment Theorem due to Tao and Vu). *Let $c_0 > 0$ be a sufficiently small constant. Let $M_n = (Z_{ij})$ and $M'_n = (Z'_{ij})$ be two $n \times n$ Wigner Hermitian matrices satisfying Condition **(C1)** for some sufficiently large constant C_0 . Assume furthermore that for any $1 \leq i < j \leq n$, Z_{ij} and Z'_{ij} match to order 4 and for any $1 \leq i \leq n$, and Z_{ii} and Z'_{ii} match to order 2. Set $A_n := \sqrt{n}M_n$ and $A'_n := \sqrt{n}M'_n$, let $1 \leq k \leq n^{c_0}$ be an integer, and let $G : \mathbb{R}^k \rightarrow \mathbb{R}$ be a smooth function obeying the derivative bounds $|\nabla^j G(x)| \leq n^{c_0}$ for all $0 \leq j \leq 5$ and $x \in \mathbb{R}^k$.*

Then for any $1 \leq i_1 < i_2 < \dots < i_k \leq n$, and for n sufficiently large we have

$$|\mathbb{E}(G(\lambda_{i_1}(A_n), \dots, \lambda_{i_k}(A_n))) - \mathbb{E}(G(\lambda_{i_1}(A'_n), \dots, \lambda_{i_k}(A'_n)))| \leq n^{-c_0}. \tag{4.1}$$

The preliminary version of this Theorem was first established in the case of bulk eigenvalues and assuming Condition **(C0)**, [19]. Later the restriction to the bulk was removed and the Condition **(C0)** was relaxed to Condition **(C1)** for a sufficiently large value of C_0 , [18]. Moreover, a natural question is whether the requirement of four matching moments is necessary. As far as the distribution of individual eigenvalues $\lambda_i(A_n)$ are concerned, the answer is essentially yes. For this see the discussions in [20].

Applying this Theorem for the special case when M'_n is GUE, we obtain the following MDP:

Theorem 4.2. *The MDP for $(\frac{1}{a_n}Z_{n,i})_n$, Theorem 3.1, hold for Wigner Hermitian matrices obeying Condition **(C1)** for a sufficiently large C_0 , and whose atom distributions match that of GUE to second order on the diagonal and fourth order off the diagonal. Given $i = i(n)$ such that $i \rightarrow \infty$ and $i/n \rightarrow 0$ as $n \rightarrow \infty$ we have: The sequence $(\frac{1}{a_n}Z_{n,i})_n$ with*

$$Z_{n,i} := \frac{\lambda_{n-i}(W_n) - (2 - (\frac{3\pi}{2} \frac{i}{n})^{2/3})}{\text{const}(\frac{\log i}{i^{2/3}n^{4/3}})^{1/2}} \tag{4.2}$$

satisfies the MDP for any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll (\log i)^{1/2}$ with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$.

Proof. Let M_n be a Wigner Hermitian matrix whose entries satisfy Condition **(C1)** and match the corresponding entries of GUE up to order 4. Let i be as in the statement of the Theorem, and let c_0 be as in Theorem 4.1. Then [19, (18)] says that

$$P(\lambda_i(A'_n) \in I_-) - n^{-c_0} \leq P(\lambda_i(A_n) \in I) \leq P(\lambda_i(A'_n) \in I_+) + n^{-c_0} \tag{4.3}$$

for all intervals $I = [b, c]$, and n sufficiently large, where $I_+ := [b - n^{-c_0/10}, c + n^{-c_0/10}]$ and $I_- := [b + n^{-c_0/10}, c - n^{-c_0/10}]$. We present the argument of proof of (4.3) just to make the presentation more self-contained. One can find a smooth bump function $G : \mathbb{R} \rightarrow \mathbb{R}_+$ which is equal to one on the smaller interval I and vanishes outside the larger interval I_+ . It follows that $P(\lambda_i(A_n) \in I) \leq \mathbb{E}G(\lambda_i(A_n))$ and $\mathbb{E}G(\lambda_i(A'_n)) \leq P(\lambda_i(A'_n) \in I_+)$. One can choose G to obey the condition $|\nabla^j G(x)| \leq n^{c_0}$ for $j = 0, \dots, 5$ and hence by Theorem 4.1 one gets

$$|\mathbb{E}G(\lambda_i(A_n)) - \mathbb{E}G(\lambda_i(A'_n))| \leq n^{-c_0}.$$

Therefore the second inequality in (4.3) follows from the triangle inequality. The first inequality is proven similarly.

Now for n sufficiently large we consider the interval $I_n := [b_n, c_n]$ with

$$b_n := b a_n n \text{const}\left(\frac{\log i}{i^{2/3}n^{4/3}}\right)^{1/2} + n\left(2 - \left(\frac{3\pi}{2} \frac{i}{n}\right)^{2/3}\right),$$

$$c_n := c a_n n \operatorname{const} \left(\frac{\log i}{i^{2/3} n^{4/3}} \right)^{1/2} + n \left(2 - \left(\frac{3\pi i}{2 n} \right)^{2/3} \right)$$

with $b, c \in \mathbb{R}$, $b \leq c$ and $\operatorname{const} = ((3\pi)^{2/3} 2^{1/3})^{-1/2}$. Then for $\frac{1}{a_n} Z_{n,i}$ defined as in the statement of the Theorem we have $P(Z_{n,i}/a_n \in [b, c]) = P(\lambda_{n-i}(A_n) \in (I_n))$. With (4.3) and [8, Lemma 1.2.15] we obtain

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{a_n^2} \log P(Z_{n,i}/a_n \in [b, c]) \\ & \leq \max \left(\limsup_{n \rightarrow \infty} \frac{1}{a_n^2} \log P(\lambda_{n-i}(A'_n) \in (I_n)_+); \limsup_{n \rightarrow \infty} \frac{1}{a_n^2} \log n^{-c_0} \right). \end{aligned}$$

For the first object we have

$$\begin{aligned} & P(\lambda_{n-i}(A'_n) \in (I_n)_+) \\ & = P \left(\frac{\lambda_{n-i}(A'_n) - n \left(2 - \left(\frac{3\pi i}{2 n} \right)^{2/3} \right)}{a_n n \operatorname{const} \left(\frac{\log i}{i^{2/3} n^{4/3}} \right)^{1/2}} \in [b - \eta(n), c + \eta(n)] \right) \end{aligned}$$

with $\eta(n) = n^{-c_0/10} (a_n n \operatorname{const} \left(\frac{\log i}{i^{2/3} n^{4/3}} \right)^{1/2})^{-1} \rightarrow 0$ as $n \rightarrow \infty$. Since $c_0 > 0$ and $\log n/a_n^2 \rightarrow \infty$ for $n \rightarrow \infty$ by assumption, applying Theorem 3.1 we have

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n^2} \log P(Z_{n,i}/a_n \in [b, c]) \leq - \inf_{x \in [b,c]} \frac{x^2}{2}.$$

Applying the first inequality in (4.3) in the same manner we also obtain the upper bound

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n^2} \log P(Z_{n,i}/a_n \in [b, c]) \geq - \inf_{x \in [b,c]} \frac{x^2}{2}.$$

Finally the argument in the last part of the proof of Theorem 3.1 can be repeated to obtain the MDP for $(Z_{n,i}/a_n)_n$. □

5. Universal global moderate deviations near the edge

Next we show the MDP for the eigenvalue counting function near the edge of the spectrum for Wigner Hermitian matrices matching moments with GUE up to order 4:

Theorem 5.1. *The MDP for $(Z_n)_n$, Theorem 2.1, hold for Wigner Hermitian matrices M_n obeying Condition (C1) for a sufficiently large C_0 , and whose atom distributions match that of GUE to second order on the diagonal and fourth order off the diagonal. Let $W_n = \frac{1}{\sqrt{n}} M_n$, let $I_n = [y_n, \infty)$ where $y_n \rightarrow 2^-$ for $n \rightarrow \infty$. Assume that $y_n \in [-2 + \delta, 2)$ and $n(2 - y_n)^{3/2} \rightarrow \infty$ when $n \rightarrow \infty$. Then the sequence*

$$Z_n = \frac{N_{I_n}(W_n) - \frac{2}{3\pi} n(2 - y_n)^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2} \log(n(2 - y_n)^{3/2})}}$$

satisfies the MDP with speed a_n^2 , rate function $x^2/2$ and in the regime $1 \ll a_n \ll \sqrt{\log(n(2 - y_n)^{3/2})}$.

Proof. For every $\xi \in \mathbb{R}$ and k_n defined by

$$k_n := \xi a_n \sqrt{\frac{1}{2\pi^2} \log(n(2 - y_n)^{3/2})} + \frac{2}{3\pi} n(2 - y_n)^{3/2}$$

we obtain that $P(Z_n \leq \xi) = P(N_{I_n}(W_n) \leq k_n)$. Hence using (3.2) it follows

$$P(Z_n \leq \xi) = P(\lambda_{n-k_n}(W_n) \leq y_n) = P(\lambda_{n-k_n}(A_n) \leq n y_n).$$

With (4.3) we have

$$P(\lambda_{n-k_n}(A_n) \leq n y_n) \leq P(\lambda_{n-k_n}(A'_n) \leq n y_n + n^{-c_0/10}) + n^{-c_0}$$

and

$$\begin{aligned} P(\lambda_{n-k_n}(A'_n) \leq n y_n + n^{-c_0/10}) &= P(\lambda_{n-k_n}(W'_n) \leq y_n + n^{-1-c_0/10}) \\ &= P(N_{J_n}(W'_n) \leq k_n), \end{aligned}$$

where $J_n = [y_n + n^{-1-c_0/10}, \infty)$. With $y'_n := y_n + n^{-1-c_0/10}$ we consider

$$Z'_n = \frac{N_{J_n}(W'_n) - \frac{2}{3\pi} n(2 - y'_n)^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2} \log(n(2 - y'_n)^{3/2})}}.$$

In order to apply Theorem 2.1 for $(Z'_n)_n$, we have to check if $y'_n \rightarrow 2^-$ and $n(2 - y'_n)^{3/2} \rightarrow \infty$ when $n \rightarrow \infty$. For a proof see [4, Section 2]. We present the arguments just to make the presentation more self-contained. By assumption we take $y_n \in [-2 + \delta, 2)$ with $y_n \rightarrow 2^-$. Suppose that $y'_n > 2$ for some n , then $y_n - 2 + n^{-1-c_0/10} > 0$, hence $2 - y_n < n^{-1-c_0/10}$, which implies $n(2 - y_n)^{3/2} < n n^{-3/2-3c_0/20}$, but the left-hand side is growing by assumption, a contradiction. We have proven $y'_n \rightarrow 2^-$. Moreover we have

$$\begin{aligned} (2 - y'_n)^{3/2} &= (2 - y_n)^{3/2} \left(1 - \frac{n^{-1-c_0/10}}{2 - y_n} \right)^{3/2} \\ &= (2 - y_n)^{3/2} \left(1 - \frac{3}{2} \frac{n^{-1-c_0/10}}{2 - y_n} + o\left(\frac{n^{-1-c_0/10}}{2 - y_n} \right) \right). \end{aligned}$$

Notice that $\frac{n^{-1-c_0/10}}{2 - y_n} = \frac{n^{-1/3-c_0/10}}{(n(2 - y_n)^{3/2})^{2/3}} \rightarrow 0$ and $n(2 - y_n)^{3/2} \rightarrow \infty$ when $n \rightarrow \infty$ by assumption. Hence we can apply Theorem 2.1, which is the MDP for $(Z'_n)_n$. Summarizing we have

$$P(Z_n \leq \xi) \leq P(Z'_n \leq \xi_n) + n^{-c_0}$$

with

$$\begin{aligned} \xi_n &= \frac{k_n - \frac{2}{3\pi}n(2 - y'_n)^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2} \log(n(2 - y'_n)^{3/2})}} \\ &= \frac{\frac{2}{3\pi}n((2 - y_n)^{3/2} - (2 - y'_n)^{3/2})}{a_n \sqrt{\frac{1}{2\pi^2} \log(n(2 - y'_n)^{3/2})}} + \xi \left(\frac{\log(n(2 - y_n)^{3/2})}{\log(n(2 - y'_n)^{3/2})} \right)^{1/2}. \end{aligned}$$

We will prove that $\xi_n = \xi + o(1)$. Using the preceding representation we have

$$n((2 - y_n)^{3/2} - (2 - y'_n)^{3/2}) = \frac{3}{2}n^{-c_0/10}(2 - y_n)^{1/2} + o(n^{-c_0/10}) \rightarrow 0$$

and $a_n \sqrt{\frac{1}{2\pi^2} \log(n(2 - y'_n)^{3/2})} \rightarrow \infty$ when $n \rightarrow \infty$. Moreover

$$\frac{\log(n(2 - y_n)^{3/2})}{\log(n(2 - y'_n)^{3/2})} = \frac{\log(n(2 - y_n)^{3/2})}{\log(n(2 - y_n)^{3/2}) + \frac{3}{2} \log(1 - \frac{n^{-1-c_0/10}}{2-y_n})} \rightarrow 1.$$

Applying Theorem 2.1, it follows that $\lim_{n \rightarrow \infty} \frac{1}{a_n^2} \log P(Z_n \leq \xi) = -\frac{\xi^2}{2}$ for all $\xi < 0$. Similarly we obtain for any $\xi > 0$ that $\lim_{n \rightarrow \infty} \frac{1}{a_n^2} \log P(Z_n \geq \xi) = -\frac{\xi^2}{2}$ and the MDP for $(Z_n)_n$ follows along the lines of the proof of Theorem 2.1. \square

Remark 5.2. In a next step one could ask whether the statement of Theorem 5.1 is true also for the sequence

$$\frac{N_{I_n}(W_n) - \mathbb{E}[N_{I_n}(W_n)]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n))}}.$$

Hence the question is whether the asymptotic behavior of the expectation and the variance of $N_{I_n}(W_n)$ is identical to the one for GUE matrices, given in (2.1) and (2.2). The answer is yes, but only for Wigner matrices obeying Condition **(C0)**. The reason for this is that the Four Moment Theorem 4.1 deals with a finite number of eigenvalues, whereas $N_{I_n}(W_n)$ involves all the eigenvalues of the Wigner matrix M_n . Theorem 4.1 does not give the asymptotics (2.1) and (2.2) for Wigner matrices. A recent result of Erdős, Yau and Yin [12] describe strong localization of the eigenvalues of Wigner matrices and this result provides the additional step necessary to obtain (2.1) and (2.2) for Wigner matrices M_n obeying Condition **(C0)**. The result in [12] is that for M_n being a Wigner Hermitian matrix obeying Condition **(C0)**, there is a constant $C > 0$ such that for any $i \in \{1, \dots, n\}$

$$P(|\lambda_i(W_n) - t(i/n)| \geq (\log n)^{C \log \log n} \min(i, n - i + 1)^{-1/3} n^{-2/3}) \leq n^{-3}.$$

Along the lines of the proof of [5, Lemma 5] one obtains (2.1) and (2.2). We will not present the details.

6. Further random matrix ensembles

In this section, we indicate how the preceding results for Wigner Hermitian matrices can be stated and proved for *real Wigner symmetric* matrices. Real Wigner matrices are random symmetric matrices M_n of size n such that, for $i < j$, $(M_n)_{ij}$ are i.i.d. with mean zero and variance one, $(M_n)_{ii}$ are i.i.d. with mean zero and variance 2. The case where the entries are Gaussian is the GOE mentioned in the introduction. As in the Hermitian case, the main issue is to establish our conclusions for the GOE. On the level of CLT, this was developed in [16] by means of the famous *interlacing formulas* due to Forrester and Rains, [13], that relates the eigenvalues of different matrix ensembles.

Theorem 6.1 (Forrester, Rains, 2001). *The following relation holds between GUE and GOE matrix ensembles:*

$$\text{GUE}_n = \text{even}(\text{GOE}_n \cup \text{GOE}_{n+1}). \quad (6.1)$$

The statement is: Take two independent (!) matrices from the GOE: one of size $n \times n$ and one of size $(n+1) \times (n+1)$. Superimpose the $2n+1$ eigenvalues on the real line and then take the n even ones. They have the same distribution as the eigenvalues of a $n \times n$ matrix from the GUE. Let $M_n^{\mathbb{R}}$ denote a GOE matrix and let $W_n^{\mathbb{R}} := \frac{1}{\sqrt{n}} M_n^{\mathbb{R}}$. In [11, Theorem 4.2] we have proved a MDP for

$$Z_n^{\mathbb{R}} := \frac{N_{I_n}(W_n^{\mathbb{R}}) - \mathbb{E}[N_{I_n}(W_n^{\mathbb{R}})]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))}} \quad (6.2)$$

for any $1 \ll a_n \ll \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))}$, I_n an interval in \mathbb{R} , with speed a_n^2 and rate $x^2/2$. If $M_n^{\mathbb{C}}$ denotes a GUE matrix and $W_n^{\mathbb{C}}$ the corresponding normalized matrix, the nice consequences of (6.1) were already suitably developed in [16]: applying Cauchy's interlacing theorem one can write

$$N_{I_n}(W_n^{\mathbb{C}}) = \frac{1}{2} \left[N_{I_n}(W_n^{\mathbb{R}}) + N_{I_n}(\widehat{W}_n^{\mathbb{R}}) + \eta'_n(I_n) \right], \quad (6.3)$$

where one obtains GOE'_n in $N_{I_n}(\widehat{W}_n^{\mathbb{R}})$ from GOE_{n+1} by considering the principle sub-matrix of GOE_{n+1} and $\eta'_n(I_n)$ takes values in $\{-2, -1, 0, 1, 2\}$. Note that $N_{I_n}(W_n^{\mathbb{R}})$ and $N_{I_n}(\widehat{W}_n^{\mathbb{R}})$ are independent because GOE_{n+1} and GOE_n denote independent matrices from the GOE. Now the same arguments as in [11, Section 4] and Theorem 2.1 lead to the MDP for $(Z_n^{\mathbb{R}})_n$, if we consider intervals $I_n = [y_n, \infty)$ where $y_n \rightarrow 2^-$ for $n \rightarrow \infty$. Remark that the interlacing formula (6.3) leads to $2\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) + O(1) = \mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))$ if $\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) \rightarrow \infty$. Next the proof of Theorem 3.1 can be adapted to obtain an MDP for $\lambda_{n-i}(W_n^{\mathbb{R}})$: Consider

$$Z_{n,i}^{\mathbb{R}} := \frac{\lambda_{n-i}(W_n^{\mathbb{R}}) - \left(2 - \left(\frac{3\pi}{2} \frac{i}{n}\right)^{2/3}\right)}{\text{const}\left(\frac{2 \log i}{i^{2/3} n^{4/3}}\right)^{1/2}}.$$

With

$$\mathbb{E}[N_{I_n}(W_n^{\mathbb{R}})] = \mathbb{E}[N_{I_n}(W_n^{\mathbb{C}})] + O(1) \quad \text{and} \quad 2\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) + O(1) = \mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))$$

if $\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) \rightarrow \infty$ we get a MDP along the lines of the proof of Theorem 3.1. We omit the details. The Four Moment Theorem also applies for real symmetric matrices. The proof of the next Theorem is nearly identical to the proofs of Theorem 4.2 and Theorem 5.1.

Theorem 6.2. *Consider a real symmetric Wigner matrix $W_n = \frac{1}{\sqrt{n}}M_n$ whose entries satisfy Condition (C1) and match the corresponding entries of GOE up to order 4. Consider $i = i(n)$ such that $i \rightarrow \infty$ and $i/n \rightarrow 0$ as $n \rightarrow \infty$. Denote the i th eigenvalue of W_n by $\lambda_i(W_n)$. Let $(a_n)_n$ be a sequence of real numbers such that $1 \ll a_n \ll \sqrt{\log i}$. Then the sequence $(Z_{n,i})_n$ with*

$$Z_{n,i} = \frac{\lambda_{n-i}(W_n) - \left(2 - \left(\frac{3\pi}{2} \frac{i}{n}\right)^{2/3}\right)}{\text{const} \left(\frac{2 \log i}{i^{2/3} n^{4/3}}\right)^{1/2}}$$

universally satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$. Moreover the statement of Theorem 5.1 can be adapted and proved analogously.

Remark that one could consider the Gaussian Symplectic Ensemble (GSE). Quaternion self-dual Wigner Hermitian matrices have not been studied. Due to Forrester and Rains, the following relation holds between matrix ensembles: $\text{GSE}_n = \text{even}(\text{GOE}_{2n+1}) \frac{1}{\sqrt{2}}$. The multiplication by $\frac{1}{\sqrt{2}}$ denotes scaling the $(2n+1) \times (2n+1)$ GOE matrix by the factor $\frac{1}{\sqrt{2}}$. Let $x_1 < x_2 < \dots < x_n$ denote the ordered eigenvalues of an $n \times n$ matrix from the GSE and let $y_1 < y_2 < \dots < y_{2n+1}$ denote the ordered eigenvalues of an $(2n+1) \times (2n+1)$ matrix from the GOE. Then it follows that $x_i = y_{2i}/\sqrt{2}$ in distribution. Hence the MDP for the i th eigenvalue of the GSE follows from the MDP in the GOE case. We omit formulating the result.

Acknowledgement

We would like to thank two anonymous referees for helpful comments.

References

- [1] G.W. Anderson, A. Guionnet, and O. Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics, vol. 118, Cambridge University Press, 2010.
- [2] G. Ben Arous and A. Guionnet, *Large deviations for Wigner's law and Voiculescu's non-commutative entropy*, Probab. Theory Related Fields **108** (1997), no. 4, 517–542.
- [3] S. Chatterjee, *A generalization of the Lindeberg principle*, Ann. Probab. **34** (2006), no. 6, 2061–2076.
- [4] S. Dallaporta, *A note on the central limit theorem for the eigenvalue counting function of Wigner matrices and covariance matrices*, preprint, (2011).
- [5] S. Dallaporta and V. Vu, *A note on the central limit theorem for the eigenvalue counting function of Wigner matrices*, Electron. Commun. Probab. **16** (2011), 314–322.

- [6] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, and X. Zhou, *Strong asymptotics of orthogonal polynomials with respect to exponential weights*, Comm. Pure Appl. Math. **52** (1999), no. 12, 1491–1552.
- [7] A. Dembo, A. Guionnet, and O. Zeitouni, *Moderate deviations for the spectral measure of certain random matrices*, Ann. Inst. H. Poincaré Probab. Statist. **39** (2003), no. 6, 1013–1042.
- [8] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Springer, New York, 1998.
- [9] H. Döring and P. Eichelsbacher, *Moderate deviations in a random graph and for the spectrum of Bernoulli random matrices*, Electronic Journal of Probability **14** (2009), 2636–2656.
- [10] ———, *Moderate deviations for the determinant of Wigner matrices*, to appear in Limit Theorems in Probability, Statistics and Number Theory, Springer Proceedings in Mathematics & Statistics (2013), dedicated to Friedrich Götze on the occasion of his sixtieth birthday.
- [11] ———, *Moderate deviations for the eigenvalue counting function of Wigner matrices*, arXiv:1104.0221, to appear in Lat. Am. J. Probab. Math. Stat. (2013).
- [12] L. Erdős, H.-T. Yau, and J. Yin, *Rigidity of eigenvalues of generalized Wigner matrices*, Advances in Mathematics **229** (2012), no. 3, 1435–1515.
- [13] P.J. Forrester and E.M. Rains, *Interrelationships between orthogonal, unitary and symplectic matrix ensembles*, Random matrix models and their applications, Math. Sci. Res. Inst. Publ., vol. 40, Cambridge Univ. Press, Cambridge, 2001, pp. 171–207.
- [14] J. Gustavsson, *Gaussian fluctuations of eigenvalues in the GUE*, Ann. Inst. H. Poincaré Probab. Statist. **41** (2005), no. 2, 151–178.
- [15] J.W. Lindeberg, *Eine neue Herleitung des Exponentialgesetzes in der Wahrscheinlichkeitsrechnung*, Math. Z. **15** (1922), no. 1, 211–225.
- [16] S. O’Rourke, *Gaussian fluctuations of eigenvalues in Wigner random matrices*, J. Stat. Phys. **138** (2010), no. 6, 1045–1066.
- [17] A.B. Soshnikov, *Gaussian fluctuation for the number of particles in Airy, Bessel, sine, and other determinantal random point fields*, J. Statist. Phys. **100** (2000), no. 3–4, 491–522.
- [18] T. Tao and V. Vu, *Random matrices: universality of local eigenvalue statistics up to the edge*, Comm. Math. Phys. **298** (2010), no. 2, 549–572.
- [19] ———, *Random matrices: universality of local eigenvalue statistics*, Acta Math. **206** (2011), 127–204.
- [20] ———, *Random matrices: The universality phenomenon for Wigner ensembles*, preprint, arXiv:1202.0068 (2012).

Hanna Döring and Peter Eichelsbacher
Ruhr-Universität Bochum
Fakultät für Mathematik
D-44780 Bochum, Germany
e-mail: hanna.doering@ruhr-uni-bochum.de
peter.eichelsbacher@ruhr-uni-bochum.de