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On the Rate of Convergence to the Semi-circular Law

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Abstract. Let $\mathbf{X} = (X_{jk})_{j,k=1}^{n}$ denote a Hermitian random matrix with entries X_{jk}, which are independent for $1 \leq j \leq k \leq n$. We consider the rate of convergence of the empirical spectral distribution function of the matrix $\mathbf{X}$ to the semi-circular law assuming that $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$ and that the distributions of the matrix elements X_{jk} have a uniform sub exponential decay in the sense that there exists a constant $\varkappa > 0$ such that for any $1 \leq j \leq k \leq n$ and any $t \geq 1$ we have

$$\Pr\{|X_{jk}| > t\} \leq \varkappa^{-1} \exp\{-t^{\varkappa}\}.$$

By means of a short recursion argument it is shown that the Kolmogorov distance between the empirical spectral distribution of the Wigner matrix $\mathbf{W} = \frac{1}{\sqrt{n}}\mathbf{X}$ and the semicircular law is of order $O(n^{-1}\log^b n)$ with some positive constant $b > 0$.

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1. Introduction

Consider a family $\mathbf{X} = \{X_{jk}\}$, $1 \leq j \leq k \leq n$, of independent real random variables defined on some probability space $(\Omega, \mathfrak{M}, \Pr)$, for any $n \geq 1$. Assume that $X_{jk} = X_{kj}$, for $1 \leq k < j \leq n$, and introduce the symmetric matrices

$$\mathbf{W} = \frac{1}{\sqrt{n}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}.$$

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The matrix $\mathbf{W}$ has a random spectrum $\{\lambda_1, \dots, \lambda_n\}$ and an associated spectral distribution function $\mathcal{F}_n(x) = \frac{1}{n}$ card$\{j \le n : \lambda_j \le x\}$, $\quad x \in \mathbb{R}$. Averaging over the random values $X_{ij}(\omega)$, define the expected (non-random) empirical distribution functions $F_n(x) = \mathbf{E}\,\mathcal{F}_n(x)$. Let $G(x)$ denote the semi-circular distribution function with density $g(x) = G'(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{I}_{[-2,2]}(x)$, where $\mathbb{I}_{[a,b]}(x)$ denotes an indicator-function of interval $[a,b]$. We shall study the rate of convergence of $\mathcal{F}_n(x)$ to the semi-circular law under the condition

$$\Pr\{|X_{jk}| > t\} \le \varkappa^{-1} \exp\{-t^{\varkappa}\}, \tag{1.1}$$

for some $\varkappa > 0$ and for any $t \ge 1$. The rate of convergence to the semi-circular law has been studied by several authors. We proved in [7] that the Kolmogorov distance between $\mathcal{F}_n(x)$ and the distribution function $G(x)$, $\Delta_n^* := \sup_x |\mathcal{F}_n(x) - G(x)|$ is of order $O_P(n^{-\frac{1}{2}})$ (i.e., $n^{\frac{1}{2}}\Delta_n^*$ is bounded in probability). Bai [1] and Girko [4] showed that $\Delta_n := \sup_x |F_n(x) - G(x)| = O(n^{-\frac{1}{2}})$. Bobkov, Götze and Tikhomirov [3] proved that Δ_n and $\mathbf{E}\Delta_n^*$ have order $O(n^{-\frac{2}{3}})$ assuming a Poincaré inequality for the distribution of the matrix elements. For the Gaussian Unitary Ensemble respectively for the Gaussian Orthogonal Ensemble, see [6] respectively [12], it has been shown that $\Delta_n = O(n^{-1})$. Denote by $\gamma_{n1} \le \dots \le \gamma_{nn}$, the quantiles of G, i.e., $G(\gamma_{nj}) = \frac{j}{n}$. We introduce the notation $\mathrm{llog}_n := \log\log n$. Erdös, Yau and Yin [10] showed, for matrices with elements X_{jk} which have a uniformly sub exponential decay, i.e., condition (1.1) holds, the following result

$$\Pr\Big\{\exists\, j : |\lambda_j - \gamma_j| \ge (\log n)^{C\,\mathrm{llog}_n}\Big[\min\{(j, N-j+1)\Big]^{-\frac{1}{3}} n^{-\frac{2}{3}}\Big\} \\ \le C\exp\{-(\log n)^{c\,\mathrm{llog}_n}\},$$

for n large enough. It is straightforward to check that this bound implies that

$$\Pr\Big\{\sup_x |\mathcal{F}_n(x) - G(x)| \le Cn^{-1}(\log n)^{C\,\mathrm{llog}_n}\Big\} \ge 1 - C\exp\{-(\log n)^{c\,\mathrm{llog}_n}\}. \tag{1.2}$$

From the last inequality it is follows that $\mathbf{E}\Delta_n^* \le Cn^{-1}(\log n)^{C\,\mathrm{llog}_n}$. In this paper we derive some improvement of the result (1.2) (reducing the power of logarithm) using arguments similar to those used in [10] and provide a self-contained proof based on recursion methods developed in the papers of Götze and Tikhomirov [7], [5] and [13]. It follows from the results of Gustavsson [8] that the best possible bound in the Gaussian case for the rate of convergence in probability is $O(n^{-1}\sqrt{\log n})$. For any positive constants $\alpha > 0$ and $\varkappa > 0$, define the quantities

$$l_{n,\alpha} := \log n\,(\log\log n)^{\alpha} \quad \text{and} \quad \beta_n := (l_{n,\alpha})^{\frac{1}{\varkappa}+\frac{1}{2}}. \tag{1.3}$$

The main result of this paper is the following

Theorem 1.1. *Let* $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$. *Assume that there exists a constant* $\varkappa > 0$ *such that inequality* (1.1) *holds, for any* $1 \le j \le k \le n$ *and any* $t \ge 1$. *Then, for any positive* $\alpha > 0$ *there exist positive constants* C *and* c *depending on* $\varkappa$ *and* α

only such that

$$\Pr\left\{\sup_x |\mathcal{F}_n(x) - G(x)| > n^{-1}\beta_n^4 \ln n\right\} \le C\exp\{-cl_{n,\alpha}\}.$$

We apply the result of Theorem 1.1 to study the eigenvectors of the matrix $\mathbf{W}$. Let $\mathbf{u}_j = (u_{j1}, \ldots, u_{jn})^T$ be eigenvectors of the matrix $\mathbf{W}$ corresponding to the eigenvalues λ_j, $j = 1, \ldots, n$. We prove the following result.

Theorem 1.2. *Under the conditions of Theorem* 1.1*, for any positive* $\alpha > 0$*, there exist positive constants* C *and* c*, depending on* $\varkappa$ *and* α *only such that*

$$\Pr\Big\{\max_{1\le j,k\le n} |u_{jk}|^2 > \frac{\beta_n^2}{n}\Big\} \le C\exp\{-cl_{n,\alpha}\}, \tag{1.4}$$

and

$$\Pr\Big\{\max_{1\le k\le n}\left|\sum_{\nu=1}^k |u_{j\nu}|^2 - \frac{k}{n}\right| > \frac{\beta_n^2}{\sqrt{n}}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{1.5}$$

2. Bounds for the Kolmogorov distance between distribution functions via Stieltjes transforms

To bound the error Δ_n^* we shall use an approach developed in previous work of the authors, see [7].

We modify the bound of the Kolmogorov distance between an arbitrary distribution function and the semi-circular distribution function via their Stieltjes transforms obtained in [7] Lemma 2.1. For $x \in [-2, 2]$ define $\gamma(x) := 2 - |x|$. Given $\frac{1}{2} > \varepsilon > 0$ introduce the interval $\mathbb{J}_\varepsilon := \{x \in [-2, 2] : \gamma(x) \ge \varepsilon\}$ and $\mathbb{J}'_\varepsilon := \mathbb{J}_{\varepsilon/2}$. For a distribution function F denote by $S_F(z)$ its Stieltjes transform,

$$S_F(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dF(x).$$

Proposition 2.1. *Let* $v > 0$ *and* $a > 0$ *and* $\frac{1}{2} > \varepsilon > 0$ *be positive numbers such that*

$$\frac{1}{\pi}\int_{|u|\le a} \frac{1}{u^2 + 1} du = \frac{3}{4} =: \beta, \tag{2.1}$$

and

$$2va \le \varepsilon^{\frac{3}{2}}. \tag{2.2}$$

If G *denotes the distribution function of the standard semi-circular law, and* F *is any distribution function, there exist some absolute constants* C_1 *and* C_2 *such that*

$$\begin{aligned}\Delta(F, G) &:= \sup_x |F(x) - G(x)| \\ &\le 2\sup_{x\in\mathbb{J}'_\varepsilon}\left|\operatorname{Im}\int_{-\infty}^x (S_F(u + i\frac{v}{\sqrt{\gamma}}) - S_G(u + i\frac{v}{\sqrt{\gamma}}))du\right| + C_1 v + C_2\varepsilon^{\frac{3}{2}}.\end{aligned}$$

Remark 2.2. For any $x \in \mathcal{J}_\varepsilon$ we have $\gamma = \gamma(x) \ge \varepsilon$ and according to condition (2.2), $\frac{av}{\sqrt{\gamma}} \le \frac{\varepsilon}{2}$.

Proof. The proof of Proposition 2.1 is a straightforward adaptation of the proof of Lemma 2.1 from [7]. We include it here for the sake of completeness. First we note that

$$\sup_x |F(x)-G(x)| = \sup_{x\in[-2,2]} |F(x)-G(x)| \tag{2.3}$$
$$= \max\Big\{ \sup_{x\in\mathcal{J}_\varepsilon} |F(x)-G(x)|, \sup_{x\in[-2,-2+\varepsilon]} |F(x)-G(x)|, \sup_{x\in[2-\varepsilon,2]} |F(x)-G(x)|\Big\}.$$

Furthermore, for $x\in[-2,-2+\varepsilon]$ we have

$$\begin{aligned} -G(-2+\varepsilon) \le F(x)-G(x) &\le F(-2+\varepsilon)-G(-2+\varepsilon)+G(-2+\varepsilon) \\ &\le \sup_{x\in\mathcal{J}_\varepsilon} |F(x)-G(x)| + G(-2+\varepsilon). \end{aligned} \tag{2.4}$$

This inequality yields

$$\sup_{x\in[-2,-2+\varepsilon]} |F(x)-G(x)| \le \sup_{x\in\mathcal{J}_\varepsilon} |F(x)-G(x)| + G(-2+\varepsilon). \tag{2.5}$$

Similarly we get

$$\sup_{x\in[2-\varepsilon,2]} |F(x)-G(x)| \le \sup_{x\in\mathcal{J}_\varepsilon} |F(x)-G(x)| + 1 - G(2-\varepsilon). \tag{2.6}$$

Note that $G(-2+\varepsilon) = 1-G(2-\varepsilon)$ and $G(-2+\varepsilon) \le C\varepsilon^{\frac{3}{2}}$ with some absolute constant $C>0$. Combining all these relations we get

$$\sup_x |F(x)-G(x)| \le \Delta_\varepsilon(F,G) + C\varepsilon^{\frac{3}{2}}, \tag{2.7}$$

where $\Delta_\varepsilon(F,G) = \sup_{x\in\mathbb{J}_\varepsilon} |F(x)-G(x)|$. We denote $v' = \frac{v}{\sqrt{\gamma}}$. For any $x\in\mathbb{J}'_\varepsilon$

$$\begin{aligned} &\Big|\frac{1}{\pi}\mathrm{Im}\Big(\int_{-\infty}^x (S_F(u+iv')-S_G(u+iv'))du\Big)\Big| \\ &\quad\ge \frac{1}{\pi}\mathrm{Im}\Big(\int_{-\infty}^x (S_F(u+iv')-S_G(u+iv'))du\Big) \\ &\quad= \frac{1}{\pi}\Big[\int_{-\infty}^x\int_{-\infty}^\infty \frac{v'd(F(y)-G(y))}{(y-u)^2+v'^2}\Big]du \\ &\quad= \frac{1}{\pi}\int_{-\infty}^x\Big[\int_{-\infty}^\infty \frac{2v'(y-u)(F(y)-G(y))dy}{((y-u)^2+v'^2)^2}\Big] \\ &\quad= \frac{1}{\pi}\int_{-\infty}^\infty (F(y)-G(y))\Big[\int_{-\infty}^x \frac{2v'(y-u)}{((y-u)^2+v'^2)^2}du\Big]dy \\ &\quad= \frac{1}{\pi}\int_{-\infty}^\infty \frac{F(x-v'y)-G(x-v'y)}{y^2+1}dy, \quad \text{by change of variables.} \end{aligned} \tag{2.8}$$

Furthermore, using (2.1) and the definition of $\Delta(F,G)$ we note that

$$\frac{1}{\pi}\int_{|y|>a} \frac{|F(x-v'y)-G(x-v'y)|}{y^2+1}dy \le (1-\beta)\Delta(F,G). \tag{2.9}$$

Since F is non-decreasing, we have

$$\frac{1}{\pi}\int_{|y|\le a}\frac{F(x-v'y)-G(x-v'y)}{y^2+1}dy \ge \frac{1}{\pi}\int_{|y|\le a}\frac{F(x-v'a)-G(x-v'y)}{y^2+1}dy$$
$$\ge (F(x-v'a)-G(x-v'a))\beta - \frac{1}{\pi}\int_{|y|\le a}|G(x-v'y)-G(x-v'a)|dy. \quad (2.10)$$

These inequalities together imply (using a change of variables in the last step)

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}&\frac{F(x-v'y)-G(x-v'y)}{y^2+1}dy\\ &\ge \beta(F(x-v'a)-G(x-v'a))\\ &\quad -\frac{1}{\pi}\int_{|y|\le a}|G(x-v'y)-G(x-v'a)|dy-(1-\beta)\Delta(F,G)\\ &\ge \beta(F(x-v'a)-G(x-v'a))\\ &\quad -\frac{1}{v'\pi}\int_{|y|\le v'a}|G(x-y)-G(x-v'a))|dy-(1-\beta)\Delta(F,G). \quad (2.11)\end{aligned}$$

Note that according to Remark 2.2, $x\pm v'a\in\mathbb{J}'_\varepsilon$ for any $x\in\mathcal{J}_\varepsilon$. Assume first that $x_n\in\mathbb{J}_\varepsilon$ is a sequence such that $F(x_n)-G(x_n)\to\Delta_\varepsilon(F,G)$. Then $x'_n:=x_n+v'a\in\mathbb{J}'_\varepsilon$. Using (2.8) and (2.11), we get

$$\begin{aligned}\sup_{x\in\mathbb{J}'_\varepsilon}&\left|\operatorname{Im}\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\right|\\ &\ge \operatorname{Im}\int_{-\infty}^{x'_n}(S_F(u+iv')-S_G(u+iv'))du\\ &\ge \beta(F(x'_n-v'a)-G(x'_n-v'a))\\ &\quad -\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\sqrt{\gamma}\int_{|y|\le 2v'a}|G(x+y)-G(x)|dy-(1-\beta)\Delta(F,G)\\ &= \beta(F(x_n)-G(x_n))\\ &\quad -\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\sqrt{\gamma}\int_{|y|<2v'a}|G(x+y)-G(x)|dy-(1-\beta)\Delta(F,G). \quad (2.12)\end{aligned}$$

Assume for definiteness that $y>0$. Recall that $\varepsilon\le 2\gamma$, for any $x\in\mathcal{J}'_\varepsilon$. By Remark 2.2 with $\varepsilon/2$ instead ε, we have $0<y\le 2v'a\le\sqrt{2}\varepsilon$, for any $x\in\mathcal{J}'_\varepsilon$. For the semi-circular law we obtain,

$$\begin{aligned}|G(x+y)-G(x)| &\le y\sup_{u\in[x,x+y]}G'(u)\le yC\sqrt{\gamma+y}\\ &\le Cy\sqrt{\gamma+2v'a}\le Cy\sqrt{\gamma+\varepsilon}\le Cy\sqrt{\gamma}. \quad (2.13)\end{aligned}$$

This yields after integrating in y

$$\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\sqrt{\gamma}\int_{0\le y\le 2v'a}|G(x+y)-G(x)|dy\le\frac{C}{v}\sup_{x\in\mathbb{J}'_\varepsilon}\gamma {v'}^2\le Cv. \quad (2.14)$$

Similarly we get that

$$\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\sqrt{\gamma}\int_{0\ge y\ge -2v'a}|G(x+y)-G(x)|dy \le \frac{C}{v}\sup_{x\in\mathbb{J}'_\varepsilon}\gamma {v'}^2 \le Cv. \tag{2.15}$$

By inequality (2.7)

$$\Delta_\varepsilon(F,G) \ge \Delta(F,G) - C\varepsilon^{\frac{3}{2}}. \tag{2.16}$$

The inequalities (2.12), (2.16) and (2.14), (2.15) together yield as n tends to infinity

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\mathrm{Im}\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\right|$$
$$\ge (2\beta-1)\Delta(F,G) - Cv - C\varepsilon^{\frac{3}{2}}, \tag{2.17}$$

for some constant $C>0$. Similar arguments may be used to prove this inequality in case there is a sequence $x_n\in\mathbb{J}_\varepsilon$ such $F(x_n)-G(x_n)\to -\Delta_\varepsilon(F,G)$. In view of (2.17) and $2\beta-1=1/2$ this completes the proof. □

Lemma 2.1. *Under the conditions of Proposition* 2.1, *for any* $V>v$ *and* $0<v\le\frac{\varepsilon^{3/2}}{2a}$ *and* $v'=v/\sqrt{\gamma}, \gamma=2-|x|$, $x\in\mathbb{J}'_\varepsilon$ *as above, the following inequality holds*

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{-\infty}^{x}(\mathrm{Im}\,(S_F(u+iv')-S_G(u+iv'))du\right|$$
$$\le \int_{-\infty}^{\infty}|S_F(u+iV)-S_G(u+iV)|du + \sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{v'}^{V}(S_F(x+iu)-S_G(x+iu))\,du\right|.$$

Proof. Let $x\in\mathbb{J}'_\varepsilon$ be fixed. Let $\gamma=\gamma(x)$. Put $z=u+iv'$. Since $v'=\frac{v}{\sqrt{\gamma}}\le\frac{\varepsilon}{2a}$, see (2.2), we may assume without loss of generality that $v'\le 4$ for $x\in\mathbb{J}'_\varepsilon$. Since the functions of $S_F(z)$ and $S_G(z)$ are analytic in the upper half-plane, it is enough to use Cauchy's theorem. We can write for $x\in\mathcal{J}'_\varepsilon$

$$\int_{-\infty}^{x}\mathrm{Im}\,(S_F(z)-S_G(z))du = \mathrm{Im}\left\{\lim_{L\to\infty}\int_{-L}^{x}(S_F(u+iv')-S_G(u+iv'))du\right\}.$$

By Cauchy's integral formula, we have

$$\int_{-L}^{x}(S_F(z)-S_G(z))du = \int_{-L}^{x}(S_F(u+iV)-S_G(u+iV))du$$
$$+\int_{v'}^{V}(S_F(-L+iu)-S_G(-L+iu))du$$
$$-\int_{v'}^{V}(S_F(x+iu)-S_G(x+iu))du.$$

Denote by $\xi($ resp. $\eta)$ a random variable with distribution function $F(x)$ (resp. $G(x)$). Then we have

$$|S_F(-L+iu)| = \left|\mathbf{E}\frac{1}{\xi+L-iu}\right| \le {v'}^{-1}\,\Pr\{|\xi|>L/2\}+\frac{2}{L},$$

for any $0 < v' \le u \le V$. Similarly,

$$|S_G(-L+iu)| \le {v'}^{-1}\Pr\{|\eta| > L/2\} + \frac{2}{L}.$$

These inequalities imply that

$$\left|\int_{v'}^{V} (S_F(-L+iu) - S_G(-L+iu))du\right| \to 0 \quad \text{as} \quad L \to \infty,$$

which completes the proof. □

Combining the results of Proposition 2.1 and Lemma 2.1, we get

Corollary 2.2. *Under the conditions of Proposition* 2.1 *the following inequality holds*

$$\Delta(F,G) \le 2\int_{-\infty}^{\infty} |S_F(u+iV) - S_G(u+iV)|du + C_1 v + C_2 \varepsilon^{\frac{3}{2}}$$
$$+ 2\sup_{x\in\mathbb{J}'_\varepsilon}\int_{v'}^{V} |S_F(x+iu) - S_G(x+iu)|du,$$

where $v' = \frac{v}{\sqrt{\gamma}}$ *with* $\gamma = 2-|x|$ *and* $C_1, C_2 > 0$ *denote absolute constants.*

We shall apply the last inequality. We denote the Stieltjes transform of $\mathcal{F}_n(x)$ by $m_n(z)$ and the Stieltjes transform of the semi-circular law by $s(z)$. Let $\mathbf{R} = \mathbf{R}(z)$ be the resolvent matrix of $\mathbf{W}$ given by $\mathbf{R} = (\mathbf{W} - z\mathbf{I}_n)^{-1}$, for all $z = u+iv$ with $v \ne 0$. Here and in what follows $\mathbf{I}_n$ denotes the identity matrix of dimension n. Sometimes we shall omit the sub index in the notation of an identity matrix. It is well known that the Stieltjes transform of the semi-circular distribution satisfies the equation

$$s^2(z) + zs(z) + 1 = 0 \tag{2.18}$$

(see, for example, equality (4.20) in [7]). Furthermore, the Stieltjes transform of an empirical spectral distribution function $\mathcal{F}_n(x)$, say $m_n(z)$, is given by

$$m_n(z) = \frac{1}{n}\sum_{j=1}^{n} R_{jj} = \frac{1}{2n}\operatorname{Tr}\mathbf{R}.$$

(see, for instance, equality (4.3) in [7]). Introduce the matrices $\mathbf{W}^{(j)}$, which are obtained from $\mathbf{W}$ by deleting the jth row and the jth column, and the corresponding resolvent matrix $\mathbf{R}^{(j)}$ defined by $\mathbf{R}^{(j)} := (\mathbf{W}^{(j)} - z\mathbf{I}_{n-1})^{-1}$ and let $m_n^{(j)}(z) := \frac{1}{n-1}\operatorname{Tr}\mathbf{R}^{(j)}$. Consider the index sets $\mathbb{T}_j := \{1,\ldots,n\}\setminus\{j\}$. We shall use the representation

$$R_{jj} = \frac{1}{-z + \frac{1}{\sqrt{n}}X_{jj} - \frac{1}{n}\sum_{k,l\in\mathbb{T}_j} X_{jk}X_{jl}R_{kl}^{(j)}},$$

(see, for example, equality (4.6) in [7]). We may rewrite it as follows

$$R_{jj} = -\frac{1}{z+m_n(z)} + \frac{1}{z+m_n(z)}\varepsilon_j R_{jj}, \tag{2.19}$$

where $\varepsilon_j := \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3} + \varepsilon_{j4}$ with

$$\varepsilon_{j1} := \frac{1}{\sqrt{n}} X_{jj}, \quad \varepsilon_{j2} := \frac{1}{n} \sum_{k \in \mathbb{T}_j} (X_{jk}^2 - 1) R_{kk}^{(j)},$$

$$\varepsilon_{j3} := \frac{1}{n} \sum_{k \neq l \in \mathbb{T}_j} X_{jk} X_{jl} R_{kl}^{(j)}, \quad \varepsilon_{j4} := \frac{1}{n} (\operatorname{Tr} \mathbf{R}^{(j)} - \operatorname{Tr} \mathbf{R}). \tag{2.20}$$

This relation immediately implies the following two equations

$$R_{jj} = -\frac{1}{z + m_n(z)} - \sum_{\nu=1}^{3} \frac{\varepsilon_{j\nu}}{(z + m_n(z))^2} + \sum_{\nu=1}^{3} \frac{1}{(z + m_n(z))^2} \varepsilon_{j\nu} \varepsilon_j R_{jj} + \frac{1}{z + m_n(z)} \varepsilon_{j4} R_{jj},$$

and

$$m_n(z) = -\frac{1}{z + m_n(z)} - \frac{1}{(z + m_n(z))} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j R_{jj} \tag{2.21}$$

$$= -\frac{1}{z + m_n(z)} - \frac{1}{(z + m_n(z))^2} \frac{1}{n} \sum_{\nu=1}^{3} \sum_{j=1}^{n} \varepsilon_{j\nu} + \frac{1}{(z + m_n(z))^2} \frac{1}{n} \sum_{\nu=1}^{3} \sum_{j=1}^{n} \varepsilon_{j\nu} \varepsilon_j R_{jj} + \frac{1}{z + m_n(z)} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j4} R_{jj}. \tag{2.22}$$

3. Large deviations I

In the following lemmas we shall bound $\varepsilon_{j\nu}$, for $\nu = 1, \ldots, 4$ and $j = 1, \ldots, n$. Using the exponential tails of the distribution of X_{jk} we shall replace quantities like, e.g., $\mathbf{E}|X_{jk}|^p I(|X_{jk}| > l_{n,\alpha}^{\frac{1}{\varkappa}})$ and others by a uniform error bound $C \exp\{-c l_{n,\alpha}\}$ with constants $C, c > 0$ depending on $\varkappa$ and α varying from one instance to the next.

Lemma 3.1. *Assuming the conditions of Theorem* 1.1 *there exist positive constants* C *and* c*, depending on* $\varkappa$ *and* α *such that*

$$\Pr\{|\varepsilon_{j1}| \geq 2 l_{n,\alpha}^{\frac{1}{\varkappa}} n^{-\frac{1}{2}}\} \leq C \exp\{-c l_{n,\alpha}\},$$

for any $j = 1, \ldots, n$.

Proof. The result follows immediately from the hypothesis (1.1). □

Lemma 3.2. *Assuming the conditions of Theorem* 1.1 *we have, for any* $z = u + iv$ *with* $v > 0$ *and any* $j = 1, \ldots, n$,

$$|\varepsilon_{j4}| \leq \frac{1}{nv}.$$

Proof. The conclusion of Lemma 3.2 follows immediately from the obvious inequality $|\mathrm{Tr}\,\mathbf{R} - \mathrm{Tr}\,\mathbf{R}^{(j)}| \le v^{-1}$ (see Lemma 4.1 in [7]). □

Lemma 3.3. *Assuming the conditions of Theorem* 1.1*, for all* $z = u+iv$ *with* $u \in \mathbb{R}$ *and* $v > 0$*, the following inequality holds*

$$\Pr\Big\{|\varepsilon_{j2}| > 3 l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12} n^{-\frac12} (n^{-1} \sum_{l\in\mathbb{T}_j} |R_{ll}^{(j)}|^2)^{\frac12}\Big\} \le C \exp\{-c l_{n,\alpha}\},$$

for some positive constants $c > 0$ *and* C*, depending on* $\varkappa$ *and* α *only.*

Proof. We use the following well-known inequality for sums of independent random variables. Let $\xi_1, \dots, \xi_n$ be independent random variables such that $\mathbf{E}\xi_j = 0$ and $|\xi_j| \le \sigma_j$. Then, for some numerical constant $c > 0$,

$$\Pr\Big\{\Big|\sum_{j=1}^n \xi_j\Big| > x\Big\} \le c(1 - \Phi(x/\sigma)) \le \frac{c\sigma}{x} \exp\Big\{-\frac{x^2}{2\sigma^2}\Big\}, \tag{3.1}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\{-\frac{y^2}{2}\} dy$ and $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$. The last inequality holds for $x \ge \sigma$. (See, for instance [2], p.1, first inequality.) We put $\eta_l = X_{jl}^2 - 1$, and define,

$$\xi_l = \big(\eta_l \mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\} - \mathbf{E}\eta_l \mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}\big) R_{ll}^{(j)}.$$

Note that $\mathbf{E}\xi_l = 0$ and $|\xi_l| \le 2 l_{n,\alpha}^{\frac{2}{\varkappa}} |R_{ll}^{(j)}|$. Introduce the σ-algebra $\mathfrak{M}^{(j)}$ generated by the random variables X_{kl} with $k, l \in \mathbb{T}_j$. Let $\mathbf{E}_j$ and $\Pr_j$ denote the conditional expectation and the conditional probability with respect to $\mathfrak{M}^{(j)}$. Note that the random variables X_{jl} and the σ-algebra $\mathfrak{M}^{(j)}$ are independent. Applying inequality (3.1) with $x := l_{n,\alpha}^{\frac12} \sigma$ and with

$$\sigma^2 = 4n l_{n,\alpha}^{\frac{4}{\varkappa}} \Big(\frac{1}{n} \sum_{l\in\mathbb{T}_j} |R_{ll}^{(j)}|^2\Big),$$

we get

$$\begin{aligned}\Pr\Big\{\Big|\sum_{l\in\mathbb{T}_j} \xi_j\Big| > x\Big\} &= \mathbf{E}\Pr_j\Big\{\Big|\sum_{l\in\mathbb{T}_j} \xi_j\Big| \ge x\Big\} \\ &\le \mathbf{E}\exp\Big\{-\frac{x^2}{\sigma^2}\Big\} \le C\exp\{-c l_{n,\alpha}\}.\end{aligned} \tag{3.2}$$

Furthermore, note that

$$\mathbf{E}_j \eta_l \mathbb{I}\Big\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\Big\} = -\mathbf{E}_j \eta_l \mathbb{I}\Big\{|X_{jl}| \ge l_{n,\alpha}^{\frac{1}{\varkappa}}\Big\}.$$

This implies

$$\begin{aligned}|\mathbf{E}_j \eta_l \mathbb{I}\Big\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\Big\}| &\le \mathbf{E}_j^{\frac12} |\eta_l|^2 \Pr_j^{\frac12}\Big\{|X_{jl}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\Big\} \\ &\le \mathbf{E}^{\frac12} |\eta_l|^2 \exp\Big\{-\frac12 l_{n,\alpha}\Big\} \le C\exp\Big\{-\frac12 l_{n,\alpha}\Big\}.\end{aligned}$$

The last inequality implies that

$$\Big|\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathbf{E}_j\eta_l\mathbb{I}\left\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}})\right\}R_{ll}^{(j)}\Big|$$

$$\le\left(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|\mathbf{E}_j\eta_l\mathbb{I}\left\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}})\right\}|^2\right)^{\frac{1}{2}}\left(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\right)^{\frac{1}{2}}$$

$$\le C\exp\{-cl_{n,\alpha}\}\left(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\right)^{\frac{1}{2}}. \tag{3.3}$$

Furthermore, we note that if $|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}}$ for all $l\in\mathbb{T}_j$, (which holds with probability at least $1-\varkappa^{-1}\exp\{-cl_{n,\alpha}\}$)

$$|\varepsilon_{j2}|\le\Big|\frac{1}{n}\sum_{l\in\mathbb{T}_j}\xi_l\Big|+\Big|\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathbf{E}_j\eta_l\mathbb{I}\left\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}}\right\}R_{ll}^{(j)}\Big|. \tag{3.4}$$

The inequalities (3.2), (3.3) and (3.4) together conclude the proof of Lemma 3.3. Thus Lemma 3.3 is proved. □

Corollary 3.4. *Assuming the conditions of Theorem* 1.1 *for any* $\alpha>0$ *there exist positive constants* c *and* C, *depending on* $\varkappa$ *and* α *such that for any* $z=u+iv$ *with* $u\in\mathbb{R}$ *and* $v>0$

$$\Pr\left\{|\varepsilon_{j2}|>3l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}(nv)^{-\frac{1}{2}}(\operatorname{Im}m_n^{(j)}(z))^{\frac{1}{2}}\right\}\le C\exp\{-cl_{n,\alpha}\}.$$

Proof. Note that

$$n^{-1}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\le n^{-1}\operatorname{Tr}|\mathbf{R}^{(j)}|^2=\frac{1}{v}\operatorname{Im}m_n^{(j)}(z),$$

where $|\mathbf{R}^{(j)}|^2=\mathbf{R}^{(j)}\mathbf{R}^{(j)^*}$. The result follows now from Lemma 3.3. □

Lemma 3.5. *Assuming the conditions of Theorem* 1.1, *for any* $j=1,\dots,n$ *and for any* $z=u+iv$ *with* $u\in\mathbb{R}$ *and* $v>0$, *the following inequality holds,*

$$\Pr\left\{|\varepsilon_{j3}|>\beta_n^2n^{-\frac{1}{2}}\left(\frac{1}{n}\sum_{k\ne l\in\mathbb{T}_j}|R_{kl}^{(j)}|^2\right)^{\frac{1}{2}}\right\}\le C\exp\{-cl_{n,\alpha}\}.$$

Proof. We shall use a large deviation bound for quadratic forms which follows from results by Ledoux (see [11]).

Proposition 3.1. *Let* $\xi_1,\dots,\xi_n$ *be independent random variables such that* $|\xi_j|\le 1$. *Let* a_{ij} *denote real numbers such that* $a_{ij}=a_{ji}$ *and* $a_{jj}=0$. *Let* $Z=\sum_{l,k=1}^n\xi_l\xi_ka_{lk}$. *Let* $\sigma^2=\sum_{l,k=1}^n|a_{lk}|^2$. *Then for every* $t>0$ *there exists some positive constant* $c>0$ *such that the following inequality holds*

$$\Pr\left\{|Z|\ge\frac{3}{2}\mathbf{E}^{\frac{1}{2}}|Z|^2+t\right\}\le\exp\left\{-\frac{ct}{\sigma}\right\}.$$

Proof. Proposition 3.1 follows from Theorem 3.1 in [11]. □

Remark 3.2. Proposition 3.1 holds for complex a_{ij} as well. Here we should consider two quadratic forms with coefficients $\mathrm{Re}\, a_{jk}$ and $\mathrm{Im}\, a_{jk}$.

In order to bound ε_{j3} we use Proposition 3.1 with

$$\xi_l = \Big(X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\} - \mathbf{E}X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}\Big)/(2\, l_{n,\alpha}^{\frac{1}{\varkappa}}).$$

Note that the random variables X_{jl}, $l \in \mathbb{T}_j$ and the matrix $\mathbf{R}^{(j)}$ are mutually independent for any fixed $j = 1, \ldots, n$. Moreover, we have $|\xi_l| \le 1$. Put $Z := \sum_{k \ne l \in \mathbb{T}_j} \xi_l \xi_k R_{kl}^{(j)}$. Note that $\mathbf{R}^{(j)} = {\mathbf{R}^{(j)}}^T$. We have $\mathbf{E}_j|Z|^2 = 2\sum_{k,l \in \mathbb{T}_j} |R_{kl}^{(j)}|^2$. Applying Proposition 3.1 with $t = l_{n,\alpha}(\sum_{l \ne k \in \mathbb{T}_j} |R_{lk}^{(j)}|^2)^{\frac{1}{2}}$, we get

$$\mathbf{E}\mathrm{Pr}_j\Big\{|Z| \ge l_{n,\alpha}(\sum_{l \ne k \in \mathbb{T}_j} |R_{lk}^{(j)}|^2)^{\frac{1}{2}}\Big\} \le C\exp\{-c l_{n,\alpha}\}. \tag{3.5}$$

Furthermore, for some appropriate $c > 0$ and for $n \ge 2$

$$\mathrm{Pr}\{\exists j, l \in [1, \ldots, n] : \, |X_{jl}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\} \le \varkappa^{-1} n^2 \exp\{-l_{n,\alpha}\} \le C\,\exp\{-c l_{n,\alpha}\}$$

and similarly since $\mathbf{E}X_{jl} = 0$,

$$|\mathbf{E}X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}| \le \mathrm{Pr}^{\frac{1}{2}}\{\exists j, l \in [1, \ldots, n] : \, |X_{jl}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\} \le C\exp\{-c l_{n,\alpha}\}. \tag{3.6}$$

Introduce the random variables

$$\widehat{\xi}_l = X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}/(2 l_{n,\alpha}^{\frac{1}{\varkappa}}) \quad \text{and} \quad \widehat{Z} = \sum_{l,k \in \mathbb{T}_j} \widehat{\xi}_l \widehat{\xi}_k R_{lk}^{(j)}.$$

Note that

$$\mathrm{Pr}\Big\{\sum_{l,k \in \mathbb{T}_j} X_{jk}X_{jl}R_{kl}^{(j)} \ne 4 l_{n,\alpha}^{\frac{2}{\varkappa}}\widehat{Z}\Big\} \le C\exp\{-c l_{n,\alpha}\}. \tag{3.7}$$

Furthermore, by (3.6) we have

$$\left|\frac{1}{n}\sum_{l,k \in \mathbb{T}_j} R_{kl}^{(j)}\mathbf{E}\widehat{\xi}_l\mathbf{E}\widehat{\xi}_k\right| \le C\exp\{-c l_{n,\alpha}\}\left(\frac{1}{n}\sum_{k \ne l \in \mathbb{T}_j} |R_{kl}^{(j)}|^2\right)^{\frac{1}{2}}. \tag{3.8}$$

Finally, inequalities (3.5)–(3.8) together imply

$$\mathrm{Pr}\Big\{|\varepsilon_{j3}| > 4\beta_n^2 n^{-\frac{1}{2}} bigg(\frac{1}{n}\sum_{k \ne l \in \mathbb{T}_j} |R_{kl}^{(j)}|^2\Big)^{\frac{1}{2}}\Big\} \le C\exp\{-c l_{n,\alpha}\}.$$

Thus Lemma 3.5 is proved. □

Corollary 3.6. *Under the conditions of Theorem* 1.1 *there exist positive constants* c *and* C *depending on* $\varkappa$ *and* α *such that for any* $z = u + iv$ *with* $u \in \mathbb{R}$ *and with* $v > 0$

$$\Pr\{|\varepsilon_{j3}| > 4\beta_n^2 (nv)^{-\frac{1}{2}} (\operatorname{Im} m_n^{(j)}(z))^{\frac{1}{2}}\} \le C \exp\{-c l_{n,\alpha}\}.$$

Proof. Note that as above

$$n^{-1} \sum_{k \ne l \in \mathbb{T}_j} |R_{kl}^{(j)}|^2 \le n^{-1} \operatorname{Tr} |\mathbf{R}^{(j)}|^2 = \frac{1}{v} \operatorname{Im} m_n^{(j)}(z). \tag{3.9}$$

The result now follows from Lemma 3.5. □

To summarize these results we recall $\beta_n = (l_{n,\alpha})^{\frac{1}{\varkappa}+\frac{1}{2}}$, defined previously in (1.3). Without loss of generality we may assume that $\beta_n \ge 1$ and $l_{n,\alpha} \ge 1$. Then Lemmas 3.1, 3.2, Lemma 3.3 (with $l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}$ replaced by β_n^2), and Lemma 3.5 together imply

$$\Pr\Big\{|\varepsilon_j| > \frac{\beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n^{(j)}(z)}{\sqrt{v}} + \frac{1}{\sqrt{v}\sqrt{nv}}\Big)\Big\} \le C \exp\{-c l_{n,\alpha}\}.$$

Using that

$$0 < \operatorname{Im} m_n^{(j)}(z) \le \operatorname{Im} m_n(z) + \frac{1}{nv}, \tag{3.10}$$

we may rewrite the last inequality

$$\Pr\Big\{|\varepsilon_j| > \frac{\beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{v}\sqrt{nv}}\Big)\Big\} \le C \exp\{-c l_{n,\alpha}\}. \tag{3.11}$$

Denote by

$$\Omega_n(z,\theta) = \Big\{\omega \in \Omega : \, |\varepsilon_j| \le \frac{\theta\beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{nv}}\Big)\Big\}, \tag{3.12}$$

for any $\theta \ge 1$. Let

$$v_0 := \frac{d\beta_n^4}{n} \tag{3.13}$$

with a sufficiently large positive constant $d > 0$. We introduce the region $\mathcal{D} = \{z = u + iv \in \mathbb{C} : |u| \le 2, \, v_0 < v \le 2\}$. Furthermore, we introduce the sequence $z_l = u_l + v_l$ in $\mathcal{D}$, recursively defined via $u_{l+1} - u_l = \frac{4}{n^8}$ and $v_{l+1} - v_l = \frac{2}{n^8}$. Using a union bound, we have

$$\Pr\{\cap_{z_l \in \mathcal{D}} \Omega_n(z_l, \theta)\} \ge 1 - C(\theta) \exp\{-c(\theta)\, l_{n,\alpha}\} \tag{3.14}$$

with some constant $C(\theta)$ and $c(\theta)$ depending on $\alpha, \varkappa$ and θ. Using the resolvent equality $\mathbf{R}(z) - \mathbf{R}(z') = -(z - z')\mathbf{R}(z)\mathbf{R}'(z)$, we get

$$|R_{k+n,l+n}^{(j)}(z) - R_{k+n,l+n}^{(j)}(z')| \le \frac{|z - z'|}{vv'}.$$

This inequality and the definition of ε_j together imply

$$\Pr\left\{|\varepsilon_j(z)-\varepsilon_j(z')|\le\frac{nl_{n,\alpha}^{\frac{2}{\varkappa}}|z-z'|}{v_0^2}\quad\text{for all } z,z'\in\mathcal{D}\right\}\ge 1-C\exp\{-cl_{n,\alpha}\}. \tag{3.15}$$

For any $z\in\mathcal{D}$ there exists a point z_l such that $|z-z_l|\le Cn^{-8}$. This together with inequalities (3.14) and (3.15) immediately implies that

$$\begin{aligned}\Pr\{\cap_{z\in\mathcal{D}}\Omega_n(z,2)\}&\ge\Pr\{\cap_{z_l\in\mathcal{D}}\Omega_n(z_l,1)\}-C\exp\{-cl_{n,\alpha}\}\\&\ge 1-C\exp\{-cl_{n,\alpha}\},\end{aligned} \tag{3.16}$$

with some constants C and c depending on α and $\varkappa$ only. Let

$$\Omega_n:=\cap_{z\in\mathcal{D}}\Omega_n(z,2). \tag{3.17}$$

Put now

$$v_0':=v_0'(z)=\frac{\sqrt{2}v_0}{\sqrt{\gamma}}, \tag{3.18}$$

where $\gamma:=2-|u|$, $z=u+iv$ and v_0 is given by (3.13). Note that $0\le\gamma\le2$, for $u\in[-2,2]$ and $v_0'\ge v_0$. Denote $\mathcal{D}':=\{z\in\mathcal{D}:\ v\ge v_0'\}$.

4. Bounds for $|m_n(z)|$

In this section we bound the probability that $\operatorname{Im} m_n(z)\le C$ for some numerical constant C and for any $z\in\mathcal{D}$. We shall derive auxiliary bounds for the difference between the Stieltjes transforms $m_n(z)$ of the empirical spectral measure of the matrix $\mathbf{X}$ and the Stieltjes transform $s(z)$ of the semi-circular law. Introduce the additional notations

$$\delta_n:=\frac{1}{n}\sum_{j=1}^n\varepsilon_jR_{jj}.$$

Recall that $s(z)$ satisfies the equation

$$s(z)=-\frac{1}{z+s(z)}. \tag{4.1}$$

For the semi-circular law the following inequalities hold

$$|s(z)|\le1\text{ and }|z+s(z)|\ge1. \tag{4.2}$$

Introduce $g_n(z):=m_n(z)-s(z)$. Equality (4.1) implies that

$$1-\frac{1}{(z+s(z))(z+m_n(z))}=\frac{z+m_n(z)+s(z)}{z+m_n(z)}. \tag{4.3}$$

The representation (2.21) implies

$$g_n(z)=\frac{g_n(z)}{(z+s(z))(z+m_n(z))}+\frac{\delta_n}{z+m_n(z)}. \tag{4.4}$$

From here it follows by solving for $g_n(z)$ that

$$g_n(z) = \frac{\delta_n(z)}{z + m_n(z) + s(z)}. \tag{4.5}$$

Lemma 4.1. *Let*

$$|g_n(z)| \le \frac{1}{2}. \tag{4.6}$$

Then $|z + m_n(z)| \ge \frac{1}{2}$ *and* $\operatorname{Im} m_n(z) \le |m_n(z)| \le \frac{3}{2}$.

Proof. This is an immediate consequence of inequalities (4.2) and of

$$|z+m_n(z)| \ge |z+s(z)| - |g_n(z)| \ge \frac{1}{2}, \quad \text{and} \quad |m_n(z)| \le |s(z)| + |g_n(z)| \le \frac{3}{2}. \ \Box$$

Lemma 4.2. *Assume condition* (4.6) *for* $z = u + iv$ *with* $v \ge v_0$. *Then for any* $\omega \in \Omega_n$, *defined in* (3.17), *we obtain* $|R_{jj}| \le 4$.

Proof. By definition of Ω_n in (3.17), we have

$$|\varepsilon_j| \le \frac{\beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{v}\sqrt{nv}}\Big). \tag{4.7}$$

Applying Lemmas 4.1 and (3.13), we get $|\varepsilon_j| \le \frac{A\beta_n^2}{\sqrt{nv}}$ with some $A > 0$ depending on the parameter $d \ge 1$ in (3.13) which we may choose such that

$$|\varepsilon_j| \le \frac{1}{200}, \tag{4.8}$$

for any $\omega \in \Omega_n$, $n \ge 2$, and $v \ge v_0$. Using representation (2.19) and applying Lemma 4.1, we get $|R_{jj}| \le 4$. $\Box$

Lemma 4.3. *Assume condition* (4.6). *Then, for any* $\omega \in \Omega_n$ *and* $v \ge v_0$,

$$|g_n(z)| \le \frac{1}{100}. \tag{4.9}$$

Proof. Lemma 4.2, inequality (4.8), and representation (4.5) together imply

$$|\delta_n| \le \frac{4}{n}\sum_{j=1}^{n}|\varepsilon_j| \le \frac{4\beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{v}\sqrt{nv}}\Big) \tag{4.10}$$

Note that

$$|z + m_n(z) + s(z)| \ge \operatorname{Im} z + \operatorname{Im} m_n(z) + \operatorname{Im} s(z) \ge \operatorname{Im}(z + s(z)) \ge \frac{1}{2}\operatorname{Im}\{\sqrt{z^2 - 4}\}. \tag{4.11}$$

For $z \in \mathcal{D}$ we get $\operatorname{Re}(z^2 - 4) \le 0$ and $\frac{\pi}{2} \le \arg(z^2 - 4) \le \frac{3\pi}{2}$. Therefore,

$$\operatorname{Im}\{\sqrt{z^2 - 4}\} \ge \frac{1}{\sqrt{2}}|z^2 - 4|^{\frac{1}{2}} \ge \frac{1}{4}\sqrt{\gamma + v}, \tag{4.12}$$

where $\gamma = 2 - |u|$. These relations imply that

$$\frac{|\delta_n|}{|z + m_n(z) + s(z)|} \le \frac{\beta_n^2}{\sqrt{nv}} + \frac{\beta_n^2}{\sqrt{n}\sqrt{v\sqrt{\gamma}}} + \frac{\beta_n^2}{(nv)^{\frac{3}{2}}\sqrt{\gamma}}. \tag{4.13}$$

For $v\sqrt{\gamma} \geq v_0$, we get

$$|g_n(z)| \leq \frac{8\beta_n^2}{\sqrt{nv_0}} \leq \frac{1}{100} \tag{4.14}$$

by choosing the constant $d \geq 1$ in v_0 appropriately large. Thus the lemma is proved. □

Lemma 4.4. *Assume that condition* (4.6) *holds, for some* $z = u + iv \in \mathcal{D}'$ *and for any* $\omega \in \Omega_n$, *(see* (3.17) *and the subsequent notions). Then* (4.6) *holds as well for* $z' = u + i\widehat{v} \in \mathcal{D}'$ *with* $v \geq \widehat{v} \geq v - n^{-8}$, *for any* $\omega \in \Omega_n$.

Proof. First of all note that

$$|m_n(z) - m_n(z')| = \frac{1}{n}(v - \widehat{v})|\mathrm{Tr}\,\mathbf{R}(z)\mathbf{R}(z')| \leq \frac{v - \widehat{v}}{v\widehat{v}} \leq \frac{C}{n^4} \leq \frac{1}{100}$$

and $|s(z) - s(z')| \leq \frac{|z-z'|}{v\widehat{v}} \leq \frac{1}{100}$. By Lemma 4.3, we have $|g_n(z)| \leq \frac{1}{100}$. All these inequalities together imply $|g_n(z')| \leq \frac{3}{100} < \frac{1}{2}$. Thus, Lemma 4.4 is proved. □

Proposition 4.1. *Assuming the conditions of Theorem* 1.1 *there exist constants* $C > 0$ *and* $c > 0$ *depending on* $\varkappa$ *and* α *only such that*

$$\Pr\left\{|m_n(z)| \leq \frac{3}{2} \textit{ for any } z \in \mathcal{D}'\right\} \leq C\exp\{-cl_{n,\alpha}\}. \tag{4.15}$$

Proof. First we note that $|g_n(z)| \leq \frac{1}{2}$ a.s., for $z = u+4i$. By Lemma 4.4, $|g_n(z')| \leq \frac{1}{2}$ for any $\omega \in \Omega_n$. Applying Lemma 4.1 and a union bound, we get

$$\Pr\left\{|m_n(z)| \leq \frac{3}{2} \text{ for any } z \in \mathcal{D}'\right\} \leq C\exp\{-cl_{n,\alpha}\}. \tag{4.16}$$

Thus the proposition is proved. □

5. Large deviations II

In this section we improve the bounds for δ_n. We shall use bounds for large deviation probabilities of the sum of ε_j. We start with

$$\delta_{n1} = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_{j1}. \tag{5.1}$$

Lemma 5.1. *There exist constants* c *and* C *depending on* $\varkappa$ *and* α *and such that*

$$\Pr\left\{|\delta_{n1}| > n^{-1}\beta_n\right\} \leq C\exp\{-cl_{n,\alpha}\}.$$

Proof. We repeat the proof of Lemma 3.1. Consider the truncated random variables $\widehat{X}_{jj} = X_{jj}\mathbb{I}\{|X_{jj}| \leq l_{n,\alpha}^{\frac{1}{\varkappa}}\}$. By assumption (1.1),

$$\Pr\left\{|X_{jj}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\right\} \leq \varkappa^{-1}\exp\{-l_{n,\alpha}\}.$$

Moreover,

$$|\mathbf{E}\widehat{X}_{jj}| \leq C\exp\{-cl_{n,\alpha}\}.$$

We define $\widetilde{X}_{jj} = \widehat{X}_{jj} - \mathbf{E}\widehat{X}_{jj}$ and consider the sum

$$\widetilde{\delta}_{n1} := \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \widetilde{X}_{jj}.$$

Since $|\widetilde{X}_{jj}| \le 2l_{n,\alpha}^{\frac{1}{\varkappa}}$, by inequality (3.1), we have

$$\Pr\left\{|\widetilde{\delta}_{n1}| > n^{-1} l_{n,\alpha}^{\frac{1}{\varkappa}+\frac{1}{2}}\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.2}$$

Note that

$$|\widetilde{\delta}_{n1} - \delta_{n1}| \le \frac{1}{n}\sum_{j=1}^{n} |\mathbf{E}\widehat{X}_{jj}| \le C\exp\{-cl_{n,\alpha}\}.$$

This inequality and inequality (5.2) together imply

$$\Pr\left\{|\delta_{n1}| > n^{-1} l_{n,\alpha}^{\frac{1}{\varkappa}+\frac{1}{2}}\right\} \le C\exp\{-cl_{n,\alpha}\}.$$

Thus, Lemma 5.1 is proved. □

Consider now the quantity

$$\delta_{n2} := \frac{1}{n^2}\sum_{j=1}^{n}\sum_{l\in\mathbb{T}_j} (X_{jl}^2 - 1)R_{ll}^{(j)}. \tag{5.3}$$

We prove the following lemma

Lemma 5.2. *Let* $v_0 = \frac{d\beta_n^4}{n}$ *with some numerical constant* $d \ge 1$. *Under the conditions of Theorem* 1.1 *there exist constants* c *and* C, *depending on* $\varkappa$ *and* α *only, such that*

$$\Pr\left\{|\delta_{n2}| > 2n^{-1}\beta_n^2 \frac{1}{\sqrt{v}}\left(\frac{3}{2} + \frac{1}{nv}\right)^{\frac{1}{2}}\right\} \le C\exp\{-cl_{n,\alpha}\},$$

for any $z \in \mathcal{D}'$.

Proof. Introduce the truncated random variables $\xi_{jl} = \widehat{X}_{jl}^2 - \mathbf{E}\widehat{X}_{jl}^2$, where $\widehat{X}_{jl} = X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}$. It is straightforward to check that

$$0 \le 1 - E\widehat{X}_{jl}^2 \le C\exp\{-cl_{n,\alpha}\}. \tag{5.4}$$

We shall need the following quantities as well

$$\widehat{\delta}_{n2} = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{l\in\mathbb{T}_j}(\widehat{X}_{jl}^2 - 1)R_{ll}^{(j)} \quad \text{and} \quad \widetilde{\delta}_{n2} = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{l\in\mathbb{T}_j}\xi_{jl}R_{ll}^{(j)}.$$

By assumption (1.1),

$$\Pr\{\delta_{n2} \ne \widehat{\delta}_{n2}\} \le \sum_{j=1}^{n}\sum_{l\in\mathbb{T}_j}\Pr\left\{|X_{jl}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\right\} \le C\exp\{-cl_{n,\alpha}\}.$$

By inequality (5.4),

$$|\widehat{\delta}_{n2}-\widetilde{\delta}_{n2}|\le\frac{1}{n^2}\sum_{j=1}^{n}\sum_{l\in\mathbb{T}_j}|\mathbf{E}\widehat{X}_{jl}^2-1||R_{ll}^{(j)}|\le Cv_0^{-1}\exp\{-cl_{n,\alpha}\}$$
$$\le C\exp\{-cl_{n,\alpha}\},$$

for $v\ge v_0$ and $C,c>0$ which are independent of $d\ge 1$.

Let $\zeta_j:=\frac{1}{\sqrt{n}}\sum_{l\in\mathbb{T}_j}\xi_{jl}R_{ll}^{(j)}$. Then $\widetilde{\delta}_{n2}=\frac{1}{n^{\frac{3}{2}}}\sum_{j=1}^{n}\zeta_j$. Let $\mathfrak{R}_j$, for $j=1,\dots n$, denote the σ-algebras generated by the random variables X_{lk} with $1\le l\le j$ and $1\le k\le j$. Let $\mathfrak{R}_0$ denote the trivial σ-algebra. Note that the sequence $\widetilde{\delta}_{n2}$ is a martingale with respect to the σ-algebras $\mathfrak{R}_j$. In fact,

$$\mathbf{E}\{\zeta_j|\mathfrak{R}_{j-1}\}=\mathbf{E}\{\mathbf{E}\{\zeta_j|\mathfrak{M}^{(j)}\}|\mathfrak{R}_{j-1}\}=0.$$

In order to use large deviation bounds for $\widetilde{\delta}_{n2}$ we replace the differences ζ_j by truncated random variables. We put $\widehat{\zeta}_j=\zeta_j\mathbb{I}\{|\zeta_j|\le l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}(\frac{3}{2}+\frac{1}{nv})^{\frac{1}{2}}\}$. Denote by $t_{nv}^2=\frac{3}{2}+\frac{1}{nv}$. Since ζ_j is a sum of independent bounded random variables with mean zero (conditioned on $\mathfrak{M}^{(j)}$), similar as in Lemma (3.3) we get

$$\Pr{}_j\bigg\{|\zeta_j|>l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}\bigg(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\bigg)^{\frac{1}{2}}\bigg\}\le C\exp\{-cl_{n,\alpha}\}.$$

Using (3.9) and (3.10), we have

$$\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\le\frac{1}{v}t_{nv}^2. \tag{5.5}$$

By Proposition 4.1, we have

$$\Pr{}_j\Big\{|\zeta_j|>l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}v^{-\frac{1}{2}}t_{nv}\Big\}\le C\exp\{-cl_{n,\alpha}\}. \tag{5.6}$$

This implies that

$$\Pr\bigg\{\sum_{j=1}^{n}\zeta_j\ne\sum_{j=1}^{n}\widehat{\zeta}_j\bigg\}\le C\exp\{-cl_{n,\alpha}\}. \tag{5.7}$$

Furthermore, introduce the random variables $\widetilde{\zeta}_j=\widehat{\zeta}_j-\mathbf{E}\{\widehat{\zeta}_j|\mathfrak{R}_{j-1}\}$. First we note that

$$\mathbf{E}\{\widehat{\zeta}_j|\mathfrak{R}_{j-1}\}=-\mathbf{E}\Big\{\zeta_j\mathbb{I}\{|\zeta_j|>l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}v^{-\frac{1}{2}}t_{nv}\}\Big|\mathfrak{R}_{j-1}\Big\}.$$

Applying Cauchy-Schwartz, $E_j\xi_{jl}\xi_{jl'}R^{(j)}_{ll}R^{(j)}_{l'l'}=0$ for $l\neq l'$, $l,l'\in\mathbb{T}_j$ and $|R^{(j)}_{ll}|\leq v^{-1}$ as well as $\mathbf{E}\{\mathbf{E}_j\{|\zeta_j|^2\}|\mathfrak{R}_{j-1}\}\leq\frac{1}{nv}\sum_{l\in\mathbb{T}_j}\mathbf{E}|\xi_{jl}|^2$ we get

$$\begin{aligned}|\mathbf{E}\{\widehat{\zeta_j}|\mathfrak{R}_{j-1}\}|&\leq C\mathbf{E}^{\frac12}\{|\zeta_j|^2|\mathfrak{R}_{j-1}\}\mathrm{Pr}^{\frac12}\Big\{|\zeta_j|>l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12}v^{-\frac12}t_{nv}\}\Big|\mathfrak{R}_{j-1}\Big\}\\&=C\mathbf{E}^{\frac12}\{\mathbf{E}_j\{|\zeta_j|^2\}|\mathfrak{R}_{j-1}\}\mathbf{E}^{\frac12}\Big\{\mathrm{Pr}_j\Big\{|\zeta_j|>l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12}v^{-\frac12}t_{nv}\}\Big\}\Big|\mathfrak{R}_{j-1}\Big\}\\&\leq Cv^{-1}\Big(\frac1n\sum_{l\in\mathbb{T}_j}\mathbf{E}|\xi_{jl}|^2\Big)^{\frac12}\exp\{-cl_{n,\alpha}\}\leq C\exp\{-cl_{n,\alpha}\},\end{aligned}\tag{5.8}$$

for $v\sqrt{\gamma}\geq v_0$ with constants C and c depending on α and $\varkappa$.

Furthermore, we may use a martingale bound due to Bentkus, [2], Theorem 1.1. It provides the following result. Let $\mathfrak{R}_0=\{\emptyset,\Omega\}\subset\mathfrak{R}_1\subset\cdots\subset\mathfrak{R}_n\subset\mathfrak{M}$ be a family of σ-algebras of a measurable space $\{\Omega,\mathfrak{M}\}$. Let $M_n=\xi_1+\cdots+\xi_n$ be a martingale with bounded differences $\xi_j=M_j-M_{j-1}$ such that $\Pr\{|\xi_j|\leq b_j\}=1,\quad$ for $\quad j=1,\dots,n$. Then, for $x>\sqrt{8}$

$$\Pr\{|M_n|\geq x\}\leq c(1-\Phi(\frac{x}{\sigma}))=\int_{\frac{x}{\sigma}}^{\infty}\varphi(t)dt,\quad\varphi(t)=\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{t^2}{2}\Big\},$$

with some numerical constant $c>0$ and $\sigma^2=b_1^2+\cdots+b_n^2$. Note that for $t>C$

$$1-\Phi(t)\leq\frac1C\varphi(t).$$

Thus, this leads to the inequality

$$\Pr\{|M_n|\geq x\}\leq\exp\Big\{-\frac{x^2}{2\sigma^2}\Big\},\tag{5.9}$$

which we shall use to bound $\widetilde{\delta}_{n2}$. Take $M_n=\sum_{j=1}^n\widetilde{\delta}_j$ with $|\widetilde{\delta}_j|$ bounded by $b_j=2l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12}v^{-\frac12}t_{nv}$. By Proposition 4.1 obtain

$$\sigma^2=4nv^{-1}l_{n,\alpha}^{\frac{4}{\varkappa}+1}t_{nv}^2.\tag{5.10}$$

Inequalities (5.9) with $x=l_{n,\alpha}^{\frac12}\sigma$ and (5.10) together imply

$$\Pr\Big\{|\widetilde{\delta}_{n2}|>2n^{-1}\beta_n^2\frac{1}{\sqrt{v}}t_{nv}\Big\}\leq C\exp\{-cl_{n,\alpha}\}.\tag{5.11}$$

Inequalities (5.7)–(5.11) together conclude the proof of Lemma 5.2. □

Let

$$\delta_{n3}:=\frac{1}{n^2}\sum_{j=1}^n\sum_{l\neq k\in\mathbb{T}_j}X_{jl}X_{jk}R^{(j)}_{lk}.\tag{5.12}$$

Lemma 5.3. *Let* $v_0 = \frac{d\beta_n^4}{n}$ *with some numerical constant* $d > 1$. *Under condition of Theorem* 1.1 *there exist constants* c *and* C, *depending on* $\varkappa$, α *only such that*

$$\Pr\left\{|\delta_{n3}| > \frac{4\beta_n^2 l_{n,\alpha}^{\frac{1}{2}}}{n\sqrt{v}}\left(\frac{3}{2} + \frac{1}{nv}\right)^{\frac{1}{2}}\right\} \le C\exp\{-cl_{n,\alpha}\},$$

for any $z \in \mathcal{D}'$.

Proof. The proof of this lemma is similar to the proof of Lemma 5.2. We introduce the random variables $\eta_j = \frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j} X_{jk}X_{jl}R_{lk}^{(j)}$ and note that the sequence $M_j = \frac{1}{n}\sum_{m=1}^{j}\eta_m$ is martingale with respect to the σ-algebras $\mathfrak{R}_j$, for $j = 1,\dots,n$. By Proposition 4.1, using inequality (5.5), we get

$$\Pr\left\{\frac{1}{n}\sum_{l,k\in\mathbb{T}_j}|R_{lk}^{(j)}|^2 \le \frac{1}{v}t_{nv}^2 \text{ for any } z\in\mathcal{D}'\right\} \ge 1 - C\exp\{-cl_{n,\alpha}\}. \tag{5.13}$$

At first we apply Proposition 3.1 replacing η_j by truncated random variables and then apply the martingale bound of Bentkus (5.9). Introduce the random variables $\widehat{X}_{jk} = X_{jk}\mathbb{I}\{|X_{jk}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}$ and $\widetilde{X}_{jk} = \widehat{X}_{jk} - \mathbf{E}\widehat{X}_{jk}$. By condition (1.1), we have

$$\Pr\{X_{jk} \ne \widehat{X}_{jk}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.14}$$

The same condition yields

$$|\mathbf{E}\widehat{X}_{jk}| = |\mathbf{E}X_{jk}\mathbb{I}\{|X_{jk}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\}| \le C\exp\{-cl_{n,\alpha}\} \tag{5.15}$$

Let

$$\widehat{\eta}_j = \frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}\widehat{X}_{jk}\widehat{X}_{jl}R_{lk}^{(j)}, \quad \text{and} \quad \widetilde{\eta}_j = \frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}\widetilde{X}_{jk}\widetilde{X}_{jl}R_{lk}^{(j)}. \tag{5.16}$$

Inequality (5.14) implies that

$$\Pr\{\eta_j \ne \widehat{\eta}_j\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.17}$$

Inequality (5.15) and $|\widetilde{X}_{jk}| \le 2l_{n,\alpha}$ together imply

$$\Pr\{|\widehat{\eta}_j - \widetilde{\eta}_j| \le Cl_{n,\alpha}^{\frac{1}{\varkappa}}\exp\{-cl_{n,\alpha}\}v^{-\frac{1}{2}}t_{nv}\} = 1. \tag{5.18}$$

Applying now Propositions 4.1 and 3.1, and inequality (5.5), similar to Lemma 3.5 we get, introducing $r_{v,n} := v^{-\frac{1}{2}}\beta_n^2 t_{nv}$,

$$\Pr\{|\widetilde{\eta}_j| > n^{-\frac{1}{2}}r_{v,n}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.19}$$

Now we introduce

$$\theta_j = \eta_j\mathbb{I}\{|\eta_j| \le n^{-\frac{1}{2}}r_{v,n}\} - \mathbf{E}\eta_j\mathbb{I}\{|\eta_j| \le n^{-\frac{1}{2}}r_{v,n}\}. \tag{5.20}$$

Furthermore, we consider the random variables $\widetilde{\theta}_j = \theta_j - \mathbf{E}\{\theta_j|\mathfrak{R}_{j-1}\}$. The sequence $\widehat{M}_s$, defined by $\widehat{M}_s = \sum_{m=1}^{s}\widetilde{\theta}_m$, is a martingale with respect to the σ-algebras $\mathfrak{R}_s$, for $s = 1,\dots,n$. Similar to the proof of Lemma 5.1 we get

$$\Pr\{|\widehat{M}_n - M_n| > 4l_{n,\alpha}^{\frac{1}{2}}r_{v,n}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.21}$$

Applying inequality (5.9) to $\widehat{M}_n$ with $\sigma^2 = 16r_{v,n}^2$ and $x = l_{n,\alpha}^{\frac{1}{2}}\sigma$, we get

$$\Pr\left\{|\widehat{M}_n| > 4l_{n,\alpha}^{\frac{1}{2}}r_{v,n}\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.22}$$

Thus the lemma is proved. □

Finally, we shall bound

$$\delta_{n4} := \frac{1}{n^2}\sum_{j=1}^{n}(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)})R_{jj}. \tag{5.23}$$

Lemma 5.4. *For any $z = u + iv$ with $v > 0$ the following inequality*

$$|\delta_{n4}| \le \frac{1}{nv}\operatorname{Im} m_n(z) \ \textit{a. s.} \tag{5.24}$$

holds.

Proof. By formula (5.4) in [7], we have

$$(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)})R_{jj} = \left(1 + \frac{1}{n}\sum_{l,k\in T_j} X_{jl}X_{jk}(R^{(j)})^2_{lk}\right)R_{jj}^2 = \frac{d}{dz}R_{jj}. \tag{5.25}$$

From here it follows that

$$\frac{1}{n^2}\sum_{j=1}^{n}(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)})R_{jj} = \frac{1}{n^2}\frac{d}{dz}\operatorname{Tr}\mathbf{R} = \frac{1}{n^2}\operatorname{Tr}\mathbf{R}^2. \tag{5.26}$$

Finally, we note that

$$|\frac{1}{n^2}\operatorname{Tr}\mathbf{R}^2| \le \frac{1}{nv}\operatorname{Im} m_n(z).$$

The last inequality concludes the proof. Thus, Lemma 5.4 is proved. □

6. Stieltjes transforms

We shall derive auxiliary bounds for the difference between the Stieltjes transforms $m_n(z)$ of the empirical spectral measure of the matrix $\mathbf{X}$ and the Stieltjes transform $s(z)$ of the semi-circular law. Recalling the definitions of $\varepsilon_j, \varepsilon_{j\mu}$ in (2.20) and of $\delta_{n\nu}$ in (5.1), (5.3), (5.12) as well as (5.23), we introduce the additional notations

$$\delta_n' := \delta_{n1} + \delta_{n2} + \delta_{n3}, \quad \widehat{\delta}_n := \delta_{n4}, \quad \overline{\delta}_n := \frac{1}{n}\sum_{\nu=1}^{3}\sum_{j=1}^{n}\varepsilon_{j\nu}\varepsilon_j R_{jj}. \tag{6.1}$$

Recall that $g_n(z) := m_n(z) - s(z)$. The representation (2.22) implies

$$g_n(z) = \frac{g_n(z)}{(z + s(z))(z + m_n(z))} - \frac{\delta_n'}{(z + m_n(z))^2} + \frac{\widehat{\delta}_n}{z + m_n(z)} + \frac{\overline{\delta}_n}{(z + m_n(z))^2}. \tag{6.2}$$

The equalities (6.2) and (4.3) together yield

$$|g_n(z)| \le \frac{|\delta_n'| + |\overline{\delta}_n|}{|z + m_n(z)||z + s(z) + m_n(z)|} + \frac{|\widehat{\delta}_n|}{|z + s(z) + m_n(z)|}. \tag{6.3}$$

For any $z \in \mathcal{D}'$ introduce the events

$$\widehat{\Omega}_n(z) := \Big\{\omega \in \Omega : |\delta'_n| \le \Big(\frac{\beta_n}{n} + \frac{\beta_n^2 l_{n,\alpha}^{\frac12}\sqrt{\frac32}}{n\sqrt{v}} + \frac{\beta_n^2 l_{n,\alpha}^{\frac12}}{n^{\frac32} v}\Big)\Big\}, \tag{6.4}$$

$$\widetilde{\Omega}_n(z) := \Big\{\omega \in \Omega : |\widehat{\delta}_n| \le \frac{C\,\mathrm{Im}\, m_n(z)}{nv}\Big\}, \tag{6.5}$$

$$\overline{\Omega}_n(z) := \Big\{\omega \in \Omega : |\overline{\delta}_n| \le \Big(\frac{\beta_n^2}{n} + \frac{\beta_n^4(\mathrm{Im}\, m_n(z) + \frac{1}{nv})}{nv} + \frac{1}{n^2v^2}\Big)\frac{1}{n}\sum_{j=1}^n |R_{jj}|\Big\}.$$

Put $\Omega_n^*(z) := \widehat{\Omega}_n(z) \cap \widetilde{\Omega}_n(z) \cap \overline{\Omega}_n(z)$. By Lemmas 5.1–5.3, we have

$$\Pr\{\widehat{\Omega}_n(z)\} \ge 1 - C\exp\{-cl_{n,\alpha}\}.$$

The proof of the last relation is similar to the proof of inequality (3.16). By Lemma 5.4,

$$\Pr\{\widetilde{\Omega}_n(z)\} = 1.$$

Note that

$$|\varepsilon_{j\nu}\varepsilon_{j4}| \le \frac12(|\varepsilon_{j\nu}|^2 + |\varepsilon_{j4}|^2).$$

By Lemmas 3.3 and 3.5, we have, for $\nu = 2, 3$,

$$\Pr\Big\{|\varepsilon_{j\nu}|^2 > \frac{\beta_n^4}{nv}\Big(\mathrm{Im}\, m_n(z) + \frac{1}{nv}\Big)\Big\} \le C\exp\{-cl_{n,\alpha}\}.$$

According to Lemma 3.1,

$$\Pr\Big\{|\varepsilon_{j1}|^2 > \frac{\beta_n^2}{n}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{6.6}$$

and, by Lemma 3.2

$$\Pr\Big\{|\varepsilon_{j4}|^2 \le \frac{1}{n^2v^2}\Big\} = 1.$$

Similarly as in (3.16) we may show that

$$\Pr\{\cap_{z\in\mathcal{D}}\Omega_n^*(z) \cap \Omega_n\} \ge 1 - C\exp\{-cl_{n,\alpha}\}.$$

Let

$$\Omega_n^* := \cap_{z\in\mathcal{D}}\Omega_n^*(z) \cap \Omega_n,$$

where Ω_n was defined in (3.17). We prove now some auxiliary lemmas.

Lemma 6.1. *Let $z = u + iv \in \mathcal{D}$ and $\omega \in \Omega_n^*$. Assume that*

$$|g_n(z)| \le \frac12. \tag{6.7}$$

Then the following bound holds

$$|g_n(z)| \le \frac{C\beta_n^4}{nv} + \frac{C\beta_n^4}{n^2v^2\sqrt{\gamma+v}}.$$

Proof. Inequality (6.3) implies that for $\omega \in \Omega_n^*$

$$|g_n(z)| \le \frac{\beta_n^2 l_{n,\alpha}^{\frac{1}{2}}\left(1+\sqrt{\frac{3}{2}}\right)}{n\sqrt{v}|z+m_n(z)||z+s(z)+m_n(z)|} + \frac{C\,\mathrm{Im}\,m_n(z)}{nv|z+s(z)+m_n(z)|}$$
$$+ \frac{\beta_n^2 l_{n,\alpha}^{\frac{1}{2}}}{n^{\frac{3}{2}}v|z+m_n(z)||z+s(z)+m_n(z)|}$$
$$+ \frac{\beta_n^4}{nv|z+m_n(z)||z+s(z)+m_n(z)|}\Big(\mathrm{Im}\,m_n(z)+\frac{1}{nv}\Big)\frac{1}{n}\sum_{j=1}^n |R_{jj}|. \tag{6.8}$$

Inequality (6.8) and Lemmas 4.1, inequalities (4.11), (4.12) together imply

$$|g_n(z)| \le \frac{C\beta_n^4}{nv}\Big(1+\frac{1}{nv\sqrt{\gamma+v}}\Big). \tag{6.9}$$

This inequality completes the proof of lemma. □

Put now

$$v_0' := v_0'(z) = \frac{\sqrt{2}v_0}{\sqrt{\gamma}}, \tag{6.10}$$

where $\gamma := 2-|u|$, $z = u+iv$ and v_0 given by (3.13). Note that $0 \le \gamma \le 2$, for $u \in [-2,2]$ and $v_0' \ge v_0$. Denote $\widehat{\mathcal{D}} := \{z \in \mathcal{D} : \, v \ge v_0'\}$.

Corollary 6.2. *Assume that* $|g_n(z)| \le \frac{1}{2}$, *for* $\omega \in \Omega_n^*$ *and* $z = u+iv \in \widehat{\mathcal{D}}$. *Then* $|g_n(z)| \le \frac{1}{100}$, *for sufficiently large* d *in the definition of* v_0.

Proof. Note that for $v \ge v_0'$

$$\frac{C\beta_n^4}{nv} + \frac{C\beta_n^4}{n^2v^2\sqrt{\gamma+v}} \le \frac{C\sqrt{\gamma}}{d} + \frac{C\sqrt{\gamma}}{d^2\beta_n^4} \le \frac{1}{100}, \tag{6.11}$$

for an appropriately large constant d in the definition of v_0. Thus, the corollary is proved. □

Remark 6.1. In what follows we shall assume that $d \ge 1$ is chosen and fixed such that inequality (6.11) holds.

Assume that N_0 is sufficiently large number such that for any $n \ge N_0$ and for any $v \in \mathcal{D}$ the right-hand side of inequality (6.9) is smaller then $\frac{1}{100}$. In the what follows we shall assume that $n \ge N_0$ is fixed. We repeat here Lemma 4.4. It is similar to Lemma 3.4 in [9].

Lemma 6.3. *Assume that condition* (6.7) *holds, for some* $z = u+iv \in \mathcal{D}'$ *and for any* $\omega \in \Omega_n^*$. *Then* (6.7) *holds for* $z' = u+i\widehat{v} \in \mathcal{D}$ *as well with* $v \ge \widehat{v} \ge v - n^{-8}$, *for any* $\omega \in \Omega_n^*$.

Proof. To prove this lemma is enough to repeat the proof of Lemma 4.4. □

Proposition 6.2. *There exist positive constants C, c, depending on α and $\varkappa$ only such that*

$$\Pr\left\{|g_n(z)| > \frac{C\beta_n^4}{nv} + \frac{C\beta_n^4}{n^2v^2\sqrt{\gamma+v}}\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{6.12}$$

for all $z \in \mathcal{D}'$

Proof. Note that for $v = 4$ we have, for any $\omega \in \Omega_n^*$, $|g_n(z)| \ge \frac{1}{2}$. By Lemma 6.1, we obtain inequality (6.12) for $v = 4$. By Lemma 6.3, this inequality holds for any v with $v_0 \le v \le 4$ as well. Thus Proposition 6.2 is proved. □

7. Proof of Theorem 1.1

To conclude the proof of Theorem 1.1 we shall now apply the result of Corollary 2.2 with $v = \sqrt{2}v_0$ and $V = 4$ to the empirical spectral distribution function $\mathcal{F}_n(x)$ of the random matrix $\mathbf{X}$. At first we bound the integral over the line $V = 4$. Note that in this case we have $|z + m_n(z)| \ge 1$ and $|g_n(z)| \le \frac{1}{2}$ a.s. Moreover, $\operatorname{Im} m_n^{(j)}(z) \le \frac{1}{V} \le \frac{1}{2}$. In this case the results of Lemmas 5.1–5.3 hold for any $z = u + 4i$ with $u \in \mathbb{R}$. We apply inequality (6.8):

$$\begin{aligned}|g_n(z)| \le{}& \frac{\beta_n^2(1+\operatorname{Im}^{\frac{1}{2}} m_n(z))}{n\sqrt{v}|z+m_n(z)||z+s(z)+m_n(z)|} + \frac{C\operatorname{Im} m_n(z)}{nv|z+s(z)+m_n(z)|}\\ &+ \frac{\beta_n^2}{n^{\frac{3}{2}}v|z+m_n(z)||z+s(z)+m_n(z)|}\\ &+ \frac{\beta_n^4}{nv|z+m_n(z)||z+s(z)+m_n(z)|}\Big(\operatorname{Im} m_n(z) + \frac{1}{nv}\Big)\frac{1}{n}\sum_{j=1}^n |R_{jj}|,\end{aligned} \tag{7.1}$$

which holds for any $z = u + 4i$, $u \in \mathbb{R}$, with probability at least $1 - C\exp\{-cl_{n,\alpha}\}$. Note that for $V = 4$

$$|z+m_n(z)||z+m_n(z)+s(z)| \ge \begin{cases} 4 & \text{for} \quad |u| \le 2,\\ \frac{1}{4}|z|^2 & \text{for} |u| > 2\end{cases} \qquad \text{a.s.}$$

We may rewrite the bound (7.1) as follows

$$|g_n(z)| \le \frac{C\beta_n^4}{n(|z|^2+1)} + \frac{C\operatorname{Im} m_n(z)}{nV}.$$

Note that for any distribution function $F(x)$ we have

$$\int_{-\infty}^{\infty} \operatorname{Im} s_F(u+iv)du \le \pi$$

Moreover, for any random variable ξ with distribution function $F(x)$ and $\mathbf{E}\xi = 0$, $E\xi^2 = h^2$ we have

$$\operatorname{Im} s_F(u+iV) \le \frac{C(1+h^2)}{u^2}$$

with some numerical constant C. From here it follows that, for $V = 4$,

$$\int_{|u|\geq n^2} |m_n(z) - s(z)| du \leq \frac{C(1+h_n^2)}{n^2} \text{ a.s.} \tag{7.2}$$

with $h_n^2 = \int_{-\infty}^{\infty} x^2 d\mathcal{F}_n(x)$. Furthermore, note that

$$h_n^2 = \frac{1}{n^2} \sum_{j,k=1}^{n} X_{jk}^2 \leq \frac{2}{n^2} \sum_{1\leq j\leq k\leq n} X_{jk}^2.$$

Using inequality (3.1), we get

$$\Pr\{h_n^2 > Cn\} \leq C \exp\{-l_{n,\alpha}\}.$$

The last inequality and inequality (7.2) together imply that

$$\int_{|u|>n^2} |m_n(u+iV) - s(u+iV)| du \leq \frac{C}{n}$$

with probability at least $1 - C\exp\{-cl_{n,\alpha}\}$. Denote $\overline{\mathcal{D}}_n := \{z = u + 2i : |u| \leq n^2\}$ and

$$\overline{\Omega}_n := \left\{\cap_{z\in\overline{\mathcal{D}}_n} \left\{\omega \in \Omega : \, |g_n(z)| \leq \frac{C\beta_n^2}{n(|z|^2+1)}\right\}\right\} \cap \Omega_n^*.$$

Using a union bound, similar to (3.16) we may show that

$$\Pr\{\overline{\Omega}_n\} \geq 1 - C\exp\{-cl_{n,\alpha}\}.$$

It is straightforward to check that for $\omega \in \overline{\Omega}_n$

$$\int_{-\infty}^{\infty} |m_n(z) - s(z)| du \leq \frac{C\beta_n^4}{n}. \tag{7.3}$$

Furthermore, we put $\varepsilon = (2av_0)^{\frac{2}{3}}$ and $v_0 = \frac{d\beta_n^4}{n}$ with the constant d as introduced in (6.11). To conclude the proof we need to consider the "vertical" integrals, for $z = x + iv'$ with $x \in \mathbb{J}'_\varepsilon$, $v' = \frac{v_0}{\sqrt{\gamma}}$ and $\gamma = 2 - |x|$. Note that

$$\int_{v'}^{2} \frac{\beta_n^4}{nv} dv \leq \frac{C\beta_n^4 \ln n}{n}.$$

Furthermore,

$$\int_{v'}^{2} \frac{1}{n^2 v^2 \sqrt{\gamma+v}} dv \leq \frac{1}{n^2 v' \sqrt{\gamma}} \leq \frac{1}{n^2 v_0} \leq \frac{\beta_n^4 \ln n}{n}.$$

Finally, we get, for any $\omega \in \overline{\Omega}_n$,

$$\Delta(\mathcal{F}_n, G) = \sup_x |\mathcal{F}_n(x) - G(x)| \leq \frac{\beta_n^4 \ln n}{n}.$$

Thus Theorem 1.1 is proved. □

8. Proof of Theorem 1.2

We may express the diagonal entries of the resolvent matrix $\mathbf{R}$ as follows

$$R_{jj} = \sum_{k=1}^{n} \frac{1}{\lambda_k - z} |u_{jk}|^2. \tag{8.1}$$

Consider the distribution function, say $F_{nj}(x)$, of the probability distribution of the eigenvalues λ_k

$$F_{nj}(x) = \sum_{k=1}^{n} |u_{jk}|^2 \mathbb{I}\{\lambda_k \le x\}.$$

Then we have

$$R_{jj} = R_{jj}(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dF_{nj}(x),$$

which means that R_{jj} is the Stieltjes transform of the distribution $F_{nj}(x)$. Note that, for any $\lambda > 0$,

$$\max_{1\le k\le n} |u_{jk}|^2 \le \sup_x (F_{nj}(x+\lambda) - F_{nj}(x)) =: Q_{nj}(\lambda).$$

On the other hand, it is easy to check that

$$Q_{nj}(\lambda) \le 2 \sup_u \lambda \operatorname{Im} R_{jj}(u + i\lambda). \tag{8.2}$$

By relations (3.12) and (3.16), we obtain, for any $v \ge v_0$ with $v_0 = \frac{d\beta_n^4}{n}$ with a sufficiently large constant d,

$$\Pr\Big\{ \frac{|\varepsilon_j|}{|z + m_n(z)|} \le \frac{1}{2} \Big\} \le C \exp\{-c l_{n,\alpha}\} \tag{8.3}$$

with constants C and c depending on $\varkappa$, α and d. Furthermore, the representation (2.19) and inequality (8.3) together imply, for $v \ge v_0$, $\operatorname{Im} R_{jj} \le |R_{jj}| \le C_1$ with some positive constant $C_1 > 0$ depending on $\varkappa$ and α. This implies that

$$\Pr\Big\{ \max_{1\le k\le n} |u_{jk}|^2 \le \frac{\beta_n^4}{n} \Big\} \le C \exp\{-c l_{n,\alpha}\}.$$

By a union bound we arrive at the inequality (1.4). To prove inequality (1.5), we consider the quantity $r_j := R_{jj} - s(z)$. Using equalities (2.19) and (4.1), we get

$$r_j = -\frac{s(z) g_n(z)}{z + m_n(z)} + \frac{\varepsilon_j}{z + m_n(z)} R_{jj}.$$

By inequalities (6.12), (3.11) and (3.16), we have

$$\Pr\{|r_j| \le \frac{c\beta_n^2}{\sqrt{nv}}\} \ge 1 - C \exp\{-c l_{n,\alpha}\}.$$

From here it follows that

$$\sup_{x \in \mathbb{J}_\varepsilon} \int_{v'}^{V} |r_j(x+iv)| dv \le \frac{C}{\sqrt{n}}.$$

Similar to (7.3) we get

$$\int_{-\infty}^{\infty} |r_j(x+iV)|dx \le \frac{C\beta_n^2}{\sqrt{n}}.$$

Applying Corollary 2.2, we get

$$\Pr\{\sup_x |F_{nj}(x) - G(x)| \le \frac{\beta_n^2}{\sqrt{n}}\} \ge 1 - C\exp\{-cl_{n,\alpha}\}.$$

Using now that

$$\Pr\Big\{\sup_x |\mathcal{F}_n(x) - G(x)| \le \frac{\beta_n^4 \ln n}{n}\Big\} \ge 1 - C\exp\{-cl_{n,\alpha}\},$$

we get

$$\Pr\Big\{\sup_x |F_{nj}(x) - \mathcal{F}_n(x)| \le \frac{\beta_n^2}{\sqrt{n}}\Big\} \ge 1 - C\exp\{-cl_{n,\alpha}\}.$$

Thus, Theorem 1.2 is proved. □

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