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66

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High Dimensional Probability VI

The Banff Volume

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Contents

Preface

The *High Dimensional Probability* assemblage of probabilists grew out of a group of mathematicians, who had a common interest in doing probability on Banach spaces. There were nine *International Conferences in Probability in Banach Spaces* beginning with Oberwolfach in 1975. An earlier conference on Gaussian processes with many of the same participants as the 1975 meeting was held in Strasbourg, France in 1973. The last Banach space meeting took place in Sandjberg, Denmark in 1993. It was decided in 1994, that in order to reflect the widening interests of the members of the group, to change the name of this conference series to the *International Conference on High Dimensional Probability.*

The present volume is an outgrowth of the *Sixth High Dimensional Probability Conference* (HDP VI) held at the *Banff International Research Station* (BIRS), Banff, Canada, October 9–14, 2011. The scope and the quality of the contributed papers show very well that high dimensional probability (HDP) remains a vibrant and expanding area of mathematical research. Four of the participants of the first Probability on Banach Spaces meeting-Jørgen Hoffmann-Jørgensen, Jim Kuelbs, Mike Marcus and Jan Rosiński-have contributed papers to this volume.

HDP deals with a set of ideas and techniques whose origin can largely be traced back to the theory of Gaussian processes and, in particular, the study of their paths properties. The original impetus was to characterize boundedness or continuity via geometric structures associated with random variables in high dimensional or infinite dimensional spaces. More precisely, these are geometric characteristics of the parameter space, equipped with the metric induced by the covariance structure of the process, described via metric entropy, majorizing measures and generic chaining.

This original set of ideas and techniques turned out to be particularly fruitful in extending the classical limit theorems in probability, such as laws of large numbers, laws of iterated logarithm and central limit theorems, to the context of Banach spaces and in the study of empirical processes.

Similar developments took place in other mathematical subfields such as convex geometry, asymptotic geometric analysis, additive combinatorics and random matrices, to name but a few topics. Moreover, the methods of HDP, and especially its offshoot, the concentration of measure phenomenon, were found to have a number of important applications in these areas as well as in Statistics and Computer

Science. This breadth is very well illustrated by the contributions in the present volume.

Most of the papers in this volume were presented at HDP VI. The participants of this conference are grateful for the support of the BIRS and the editors thank Springer Verlag for agreeing to publish the resulting HDP VI volume.

The papers in this volume aptly display the methods and breadth of HDP. They use a variety of techniques in their analysis that should be of interest to advanced students and researchers. We have organized the papers into five general areas: Inequalities and Convexity, Limit Theorems, Stochastic Processes, Random Matrices and High Dimensional Statistics. To give an idea of their scope, we shall now briefly describe them by subject area.

Inequalities and Convexity:

- Bracketing entropy of high dimensional distributions, by Fuchang Gao
- Slepian's inequality, modularity and integral orderings, by Jørgen Hoffmann-Jørgensen
- A more general maximal Bernstein-type inequality, by Péter Kevei and David M. Mason
- Maximal inequalities for centered norms of sums of independent random vectors, by Rafał Latała
- A probabilistic inequality related to negative definite functions, by Mikhail Lifshits, René L. Schilling and Ilya Tyurin
- Optimal re-centering bound, with applications to Rosenthal-type concentration of measure inequalities, by Iosif Pinelis
- Strong log-concavity is preserved by convolution, by Jon A. Wellner
- On some Gaussian concentration inequality for non-Lipschitz functions, by Paweł Wolff

Gao considers the family of all distribution functions on $[0,1]^d$ and obtains bounds on the bracketing entropies of these classes for L_p metrics with $p \geq 1$. These bounds have important implications for rates of convergence of nonparametric estimators in a number of statistical problems.

Jørgen Hoffmann-Jørgensen proves a unified version of Slepian's inequality under minimal regularity conditions and points to the subtleties of the assumptions of this inequality. This basically covers all forms of Slepian's inequality known in the literature. Then the author explores the connection of Slepian's inequality to integral orderings in general, and to the supermodular ordering in particular, and also corrects some results in the theory of integral orderings.

Kevei and Mason show that under very weak assumptions a general version of Bernstein's exponential inequality for sums of random variables, which are not necessarily independent, extends to a maximal version.

Latała proves a Lévy-Ottaviani type inequality for sums of independent random variables with arbitrary centering. The vector case is explored and a modified inequality proved in the Hilbert space framework.

Lifshits, Schilling and Tyurin prove an inequality comparing the expectations of a negative definite function applied to either the difference or the sum of two iid random vectors. A particular case which involves lower powers of the Euclidean norm is linked to bifractional Brownian motion.

Pinelis considers optimal bounds between expectations of a nonnegative function of centered and non-centered random variables. Applications to Rosenthal-type concentration of measure inequalities are given.

Although it is well known that log-concavity of distributions is preserved under convolution (addition of independent random variables), preservation of the stronger notion of ultra log-concavity under convolution in the setting of integer-valued random variables was first proved by Liggett (1997). Wellner's paper shows that a recent proof of Liggett's result by Johnson (2007) carries over to a proof of the preservation of strong log-concavity under convolution for real-valued random variables.

Wolff's paper establishes a concentration inequality for functions of a pair of Gaussian random vectors. The bounded Lipschitz assumption present in the classical Gaussian concentration inequality is now replaced by the boundedness of second-order derivatives.

Limit Theorems:

- Rates of convergence in the strong invariance principle for non adapted sequences: application to ergodic automorphisms of the torus, by Jérôme Dedecker, Florence Merlevède and Françoise Pène
- On the rate of convergence to the semi-circular law, by Friedrich Götze and Alexandre Tikhomirov
- Empirical quantile CLTs for time dependent data, by James Kuelbs and Joel Zinn
- Asymptotic properties for linear processes of functionals of reversible or normal Markov chains, by Magda Peligrad

Dedecker, Merlevède and Pène establish strong invariance principles for non-adapted sequences and apply them to iterates of ergodic automorphisms of the d-dimensional torus. Their main theorems are proved using an approximating martingale introduced by Gordin (1969).

Applying a bound for the Kolmogorov distance between distribution functions via Stieltjes transforms, Götze and Tikhomirov derive under side conditions a rate of convergence in the semi-circular law.

Peligrad proves central limit theorems for linear processes of functionals of reversible or normal Markov chains. The proofs are based on a result of Peligrad and Utev (2006) concerning the asymptotic behavior of a class of linear processes and spectral calculus.

Kuelbs and Zinn develop central limit theorems for quantile processes defined in terms of empirical processes of time dependent data. The key to their proofs is an important extension of a method of Vervaat (1972).

Stochastic Processes:

- First exit of Brownian motion from a one-sided moving boundary, by Frank Aurzada and Tanja Kramm
- On Lévy's equivalence theorem in the Skorohod space, by Andreas Basse-O'Connor and Jan Rosiński
- Continuity conditions for a class of second-order permanental chaoses, by Michael B. Marcus and Jay Rosen

Aurzada and Kramm give a short proof of the celebrated Uchiyama's result on the first exit time of Brownian motion from a moving boundary for the case of decreasing boundary. As a consequence, a relatively simple proof of Uchiyama's result for monotone boundaries can now be obtained.

Basse-O'Connor and Rosiński present a new and simple proof of Lévy's Equivalence Theorem in the space of càdlàg functions equipped with the Skorohod topology. In the proof, the authors use their recent result on the uniform convergence of jump processes, which removes major difficulties of working with the Skorohod topology in this context.

Marcus and Rosen establish a sufficient condition for the continuity of permanental fields. Permanental fields are defined by a real-valued kernel and a positive parameter; when the kernel is symmetric and the parameter equals 1/2, a permanental field becomes the second-order Gaussian chaos.

Random Matrices and Applications:

- On the operator norm of random rectangular Toeplitz matrices, by Radosław Adamczak
- Edge fluctuations of eigenvalues of Wigner matrices, by Hanna Döring and Peter Eichelsbacher
- On the limiting shape of Young diagrams associated with inhomogeneous random words, by Christian Houdré and Hua Xu

Adamczak's paper presents sharp estimates on the operator norm of rectangular random Toeplitz matrices. The entries of the matrix are generated by centered and independent random variables with moments of order strictly higher than two.

Döring and Eichelsbacher obtain a moderate deviation principle for the eigenvalue counting function of a Wigner matrix in a interval close to the edge of the spectrum. Possible extensions to other random matrix ensembles are briefly discussed.

Houdré and Xu obtain the limiting shape of the random Young diagrams associated with an inhomogeneous random word as a multidimensional Brownian functional. This functional has the same law as the spectrum of a Gaussian random matrix.

High Dimensional Statistics:

- Low rank estimation of similarities on graphs, by Vladimir Koltchinskii and Pedro Rangel
- Sparse principal component analysis with missing observations, by Karim Lounici
- High dimensional CLT and its applications, by Dragan Radulovic

Koltchinskii and Rangel study an estimation problem concerning similarities defined on graphs. They study a class of modified least squares estimators with complexity penalization based on both the nuclear norm and Sobolev type norms of symmetric kernels on the graph, and provide upper bounds on L_2-errors of such estimators.

Lounici studies sparse principal component analysis in a high dimensional setting. This paper focuses on estimation of the first principal component in settings involving data that is only partially observed, and provides both information theoretic lower bounds an analysis of an estimation procedure based on a Bayes Information Criterion (BIC) which achieves the lower bounds up to logarithmic factors.

Radulovic investigates bootstrap methods for finite classes of functions $\mathcal{F}_n$ with finite cardinality k_n increasing with the sample size n. He shows that such classes have statistical applicability in testing situations even though they are not Donsker classes of functions.

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Dedication

This volume is dedicated to the memory of our friend and colleague Wenbo Li, who recently passed away in the prime of his life. He was a very talented, energetic and vital force in our field. In addition to being an extremely dynamic and welcome mover and shaker in probability theory activities worldwide, he was imbedded in the heart and soul of the *High Dimensional Probability* meetings. He had actively participated in all of them, and he had served on the organizing committee of the HDP IV meeting. He had also been a member of the HDP VII meeting committee. His enthusiastic and warm presence among us will be sorely missed.

Part I

Inequalities and Convexity

Progress in Probability, Vol. 66, 3–17

Bracketing Entropy of High Dimensional Distributions

Fuchang Gao

Abstract. Let $\mathcal{F}_d$ be the class of probability distribution functions on $[0,1]^d$, $d \geq 2$. The following estimate for the bracketing entropy of $\mathcal{F}_d$ in the L^p norm, $1 \leq p < \infty$, is obtained:

$$\log N_{[\,]}(\varepsilon, \mathcal{F}_d, \|\cdot\|_p) = O(\varepsilon^{-1}|\log \varepsilon|^{2(d-1)}).$$

Based on this estimate, a general relation between bracketing entropy in the L^p norm and metric entropy in the L^1 norm for multivariate smooth functions is established.

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Keywords. Bracketing entropy, metric entropy, high dimensional distribution.

1. Introduction

Given a probability measure μ on $[0,1]^d$, $d \geq 2$, we denote by F_μ the distribution function of μ, that is

$$F_\mu(x_1, x_2, \ldots, x_d) = \mu([0, x_1] \times [0, x_2] \times \cdots \times [0, x_d]), 0 \leq x_1, x_2, \ldots, x_d \leq 1.$$

Clearly, F_μ belongs to $L^p([0,1]^d) \equiv L^p([0,1]^d, \mathcal{B}_d, \lambda)$ for all $1 \leq p \leq \infty$, where λ is Lebesgue measure on the Borel sigma-field $\mathcal{B}_d$ of $[0,1]^d$. We denote

$$\mathcal{F}_d := \{F_\mu : \mu([0,1]^d) = 1\}.$$

The complexity of the function class $\mathcal{F}_d$ is well known. For example, it is the main object of an active research area in number theory, see, e.g., [1] and the references therein.

In this paper, we are interested the bracketing entropy of $\mathcal{F}_d$ in the L^p norm, $1 \leq p < \infty$. Recall that for a function class $\mathcal{F}$ in a space equipped with a metric ρ, the ε-bracketing entropy of $\mathcal{F}$ is defined as the quantity $\log N_{[\,]}(\varepsilon, \mathcal{F}, \rho)$, where

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$N_{[]}(\varepsilon, \mathcal{F}, \rho)$ is the minimum number of ε-brackets need to cover $\mathcal{F}$, that is,

$$N_{[]}(\varepsilon, \mathcal{F}, \rho) := \min\left\{n : \exists f_1, f_1^*, \dots, f_n, f_n^* \text{ s.t. } \rho(f_k^*, f_k) \le \varepsilon, \mathcal{F} \subset \bigcup_{k=1}^n [f_k, f_k^*]\right\},$$

where the bracket $[f_k, f_k^*]$ is defined by

$$[f_k, f_k^*] = \{g \in \mathcal{F}_d : f_k \le g \le f_k^*\}.$$

The class $\mathcal{F}_d$ is a natural and important object, and the question of studying its bracketing entropy is motivated by its applications in statistics and probability, where the bracketing entropy controls the rate of convergence of uniform limit theorem, and is of central importance in design and analysis of statistical estimators. In fact, the question of determining the bracketing entropy was motivated by its application to rates of convergence of maximum likelihood estimators for high dimensional interval censoring models [9].

Bracketing entropy is closely related to the metric entropy $\log N(\varepsilon, \mathcal{F}, \rho)$, where $N(\varepsilon, \mathcal{F}, \rho)$ is the minimal number of balls of radius ε in ρ-distance needed to cover the function class $\mathcal{F}$. It is clear that

$$N(\varepsilon, \mathcal{F}, \rho) \le N_{[]}(\varepsilon, \mathcal{F}, \rho).$$

However, a reverse inequality is not true in general. Of course, when ρ is dominated by the $\|\cdot\|_\infty$ norm, the following relation is trivial:

$$N_{[]}(\varepsilon, \mathcal{F}, \rho) \le N_{[]}(\varepsilon, \mathcal{F}, \|\cdot\|_\infty) \le N(\varepsilon/2, \mathcal{F}, \|\cdot\|_\infty). \tag{1.1}$$

In fact, (1.1) is the primary tool to study bracketing entropy. For example, by using this relation, Nickl et al. [12] derived bracketing entropy estimates for Besov- and Sobolev-type spaces; and earlier, van der Vaart [13] obtained results of this type for multivariate functions that have bounded derivatives up to order $|\alpha| \le k$. However, if the functions are not smooth enough, then (1.1) is no longer useful. For example, for the class of block monotone functions [7]. In such cases, to estimate bracketing entropy often requires constructive arguments, and thus is usually more difficult to study than metric entropy. As an application of this work, we will develop a general method to estimate the upper bound of bracketing entropy of some multivariate function classes in the L^p norm by the metric entropy of their derivatives in the L^1 norm.

The problem of determining the metric entropy of d-dimensional distributions is a longstanding open problem, even for the simplest case where $p = 2$. It is observed in [3] that when $p = 2$, the problem is equivalent to the rate of the small ball probability of d-dimensional Brownian sheets in the supremum norm. By using an earlier result of Dunker et al. [4] on the small ball probability of d-dimensional Brownian sheets, Blei et al. [3] proved that

$$\log N(\varepsilon, \mathcal{F}_d, \|\cdot\|_2) = O(\varepsilon^{-1}|\log \varepsilon|^{d-\frac{1}{2}}). \tag{1.2}$$

When $d = 2$, the estimate above is sharp. However, for $d > 2$, the best known lower bound is $C\varepsilon^{-1}|\log \varepsilon|^{d-1+\delta}$ for some $\delta > 0$, which can be derived from the work of Bilyk and et al. [2]. We remark that this problem is also equivalent to the longstanding open problem on the discrepancy of distribution in number theory, see, e.g., [1].

When $p \neq 2$, the difficulty of the problem further increases. This is because many familiar tools in Hilbert space are then no longer available. For example, in [3] the upper bound estimate of $\log N(\varepsilon, \mathcal{F}_d, \|\cdot\|_2)$ was obtained using Gaussian processes and the Kuelbs-Li connection [11] between metric entropy and small ball probability. When $p \neq 2$, there is no similar connection between $\log N(\varepsilon, \mathcal{F}_d, \|\cdot\|_p)$ and small ball probability of random processes that has been discovered.

In studying bracketing entropy, even fewer tools are available. For example, the powerful tool of metric entropy duality and standard methods from Fourier analysis are no longer applicable for bracketing entropy. Thus, it is not a surprise that not much is known about the bracketing entropy $\log N_{[\,]}(\varepsilon, \mathcal{F}_d, \|\cdot\|_p)$.

Of course, when $d = 1$, the problem is much simpler. In that case, the class of probability distribution functions is the same as the class of non-decreasing right continuous functions with range $[0, 1]$, and the bracketing entropy estimate is known, [10]. Specifically, [14, p. 159] contains a proof of the following upper bound estimate

$$\log N(\varepsilon, \mathcal{F}_1, \|\cdot\|_p) \sim \log N_{[\,]}(\varepsilon, \mathcal{F}_1, \|\cdot\|_p) \sim \varepsilon^{-1},$$

where $A \sim B$ means that there exist constants positive C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$. A simple proof was given in [3]. In fact, for all m-monotone functions, $m \geq 1$, and complete monotone functions on $[0, 1]$, the metric entropy and bracketing entropy estimates are known; cf. [5, 6, 8].

It is true that a probability distribution function on $[0, 1]^d$ is block non-decreasing, that is, non-decreasing in each variable. However, the class $\mathcal{M}_d$ of d-dimensional block non-decreasing continuous functions on $[0, 1]^d$ is much larger than $\mathcal{F}_d$. In fact, it is proved in [7] that for $p \neq \frac{d}{d-1}$,

$$\log N_{[\,]}(\varepsilon, \mathcal{M}_d, \|\cdot\|_p) \sim \varepsilon^{-\alpha},$$

where $\alpha = \max\{d, (d-1)p\}$; however, for $\mathcal{F}_d$, we are looking for an estimate of the order

$$\log_{[\,]} N(\varepsilon, \mathcal{F}_d, \|\cdot\|_p) \sim \varepsilon^{-1} |\log \varepsilon|^{\beta}$$

for some β.

In this paper, we will prove

Theorem 1.1. *Let $\mathcal{F}_d$ be the set of probability distributions on $[0, 1]^d$. For $1 \leq p < \infty$ and $d \geq 2$,*

$$\log N_{[\,]}(\varepsilon, \mathcal{F}_d, \|\cdot\|_p) = O(\varepsilon^{-1} |\log \varepsilon|^{2(d-1)}).$$

In comparison with the metric entropy estimate (1.2) for $\mathcal{F}_d$ in the case $p = 2$, we see there is some discrepancy on the exponent of the logarithmic term. While it is possible that our result is not optimal for $d > 2$, we believe that in the case $d = 2$, our result is sharp. In other words, we believe the discrepancy is intrinsic.

Note that many examples (such as Besov spaces) witness that when the function class is smoother than the distribution functions, there is no discrepancy between bracketing entropy and metric entropy. Thus, it becomes an interesting question to determine if there is indeed any discrepancy for the class of distribution functions.

As an application, we show that Theorem 1.1 can be used to obtain bracketing entropy estimate for classes of smooth functions using the metric entropy of their derivatives in the L^1 norm. More precisely, we show that

Theorem 1.2. *Let $\mathcal{F}$ be a class of smooth functions f on $[0,1]^d$ bounded by 1, $d>1$. Let $\mathcal{H}=\{D^{\alpha_1,\alpha_2,\dots,\alpha_d}f : 0\le \alpha_1,\alpha_2,\dots,\alpha_d\le 1, f\in\mathcal{F}\}$. Suppose*

$$\log N(\varepsilon,\mathcal{H},\|\cdot\|_1)\le \phi(\varepsilon),$$

where ϕ is a decreasing function on $(0,1)$, then for any $0<\varepsilon<1$,

$$\log N_{[\,]}(\varepsilon,\mathcal{F},\|\cdot\|_p)\le K(d,p)\cdot r(\varepsilon)[\log r(\varepsilon)]^{2(d-1)},$$

where $K(d,p)$ is a positive constant depending only on d and p, and $r(\varepsilon)$ is the solution to the equation

$$\phi(\varepsilon r)=r[\log r]^{2(d-1)}.$$

Remark 1.3. Theorem 1.2 concerns the case $d>1$. When $d=1$, it was proved in Gao and Wellner [8] that the estimate is

$$\log N_{[\,]}(\varepsilon,\mathcal{F},\|\cdot\|_p)\le K(d,p)\cdot r(\varepsilon),$$

where $r(\varepsilon)$ is the solution to the equation $\phi(\varepsilon r)=r$. While we do not further study the sharpness of these estimates in this paper, we remark that at least for all k-monotone functions on $[0,1]$, $k=1,2,\dots$, such estimates are sharp; see [8].

Results of this nature are very useful, because metric entropy in the L^1 norm is much easier to estimate than bracketing entropy in the L^p norm. We use the following example to demonstrate the usefulness of Theorem 1.2.

Example. Let $\mathcal{D}_d$ be the class of probability distribution functions with bounded block monotone density, that is, all the functions in $\mathcal{M}_d := \{D^{1,1,\dots,1}g : g\in\mathcal{D}_d\}$ are integrable and bounded monotone in each variable. Then according to Gao and Wellner [7], $\log N(\varepsilon,\mathcal{M}_d,\|\cdot\|_1)=O(\varepsilon^{-d})$. Thus, by Theorem 1.2, we immediately obtain

$$\log N_{[\,]}(\varepsilon,\mathcal{D}_d,\|\cdot\|_p)=O\left(\varepsilon^{-\frac{d}{d+1}}|\log\varepsilon|^{\frac{2d(d-1)}{d+1}}\right),$$

for all $d\ge 2$ and $1\le p<\infty$. One may compare this estimate with that of the class $\mathcal{C}_1^{\alpha}$ with $\alpha=(1,1,\dots,1)$ – that is, a class of multivariate functions with bounded derivative $D^{1,1,\dots,1}g$. For the latter, van der Vaart [13] proved that the bracketing entropy is of the rate ε^{-1}.

While we do not pursue statistics applications of our results in this paper, we would like to point out that our estimate for the bracketing entropy can be used to derive convergence rates of density estimators in statistics, such as MLE estimators, cf. [7, 8, 14].

2. Bracketing entropy estimate

To simplify the presentation, we first make some observations. Note that, if μ is a measure on $[0,1]^d$ such that $\theta := \mu([0,1]^d) < 1$, then we can define a probability measure

$$\nu = \mu + (1-\theta)\delta_{(1,1,\ldots,1)},$$

so that $F_\mu = F_\nu$ on $[0,1]^d$ except at $(1,1,\ldots,1)$. Thus, $\mathcal{F}_d$ and the function class $\mathcal{F}'_d := \{F_\nu : \nu([0,1]^d) \le 1\}$ have the same bracketing number in the L^p norm for all $1 \le p < \infty$, that is,

$$N_{[\,]}(\varepsilon, \mathcal{F}_d, \|\cdot\|_p) = N_{[\,]}(\varepsilon, \mathcal{F}'_d, \|\cdot\|_p). \tag{2.1}$$

Next, we observe that if we define

$$\mathcal{F}''_d := \{F_\mu : \mu([0,1]^d) \le 1, \mu \ll \lambda\},$$

where λ is the Lebesgue measure on $[0,1]^d$, then $\mathcal{F}''_d$ and $\mathcal{F}_d$ have the same rate of bracketing entropy. More precisely, we have

Lemma 2.1. *For all* $0 < \varepsilon < 1$,

$$\log N_{[\,]}(\varepsilon, \mathcal{F}''_d, \|\cdot\|_p) \le \log N_{[\,]}(\varepsilon, \mathcal{F}_d, \|\cdot\|_p) \le 2\log N_{[\,]}(\varepsilon/3, \mathcal{F}''_d, \|\cdot\|_p).$$

Remark 2.2. The idea of the proof of Lemma 2.1 is simple: by redistributing a point mass to a nearby area in the upper right side of the point, we can approximate the distribution function of a singular measure from below by a distribution function that has a density. Approximation from above is done similarly. The detailed verification is however somewhat tedious. Readers may choose to skip the proof of Lemma 2.1. Though Lemma 2.1 can help readers to better understand the definition of the cutting point in the proof of Theorem 1.1, the latter will be presented without relying on Lemma 2.1.

Proof. In view of (2.1), we can replace the $N_{[\,]}(\varepsilon, \mathcal{F}_d, \|\cdot\|_p)$ in the lemma by $N_{[\,]}(\varepsilon, \mathcal{F}'_d, \|\cdot\|_p)$. Then, the first inequality becomes trivial because $\mathcal{F}''_d \subset \mathcal{F}'_d$.

To prove the second inequality, we first show that for any $F_\mu \in \mathcal{F}_d$, and any $\delta > 0$, we can find measures ν_1 and ν_2 on $[-\delta, 1+\delta]^d$ that are absolutely continuous with respect to the Lebesgue measure, such that when restricted on $[0,1]^d$, we have

$$F_{\nu_1} \le F_\mu \le F_{\nu_2}, \tag{2.2}$$

$$\|F_{\nu_2} - F_{\nu_1}\|_{L^p([0,1]^d)} \le 2(2d\delta)^{1/p}. \tag{2.3}$$

To see this, we write $\mu = \nu + \sum_i c_i \delta_{x_i}$ where $\nu \ll \lambda$, $c_i > 0$ and $x_i = (x_i(1), x_i(2), \ldots, x_i(d)) \in [0,1]^d$. Because $\sum c_i \le \mu([0,1]^d) \le 1$, there exists N such that $\sum_{i>N} c_i < (2d\delta)^{1/p}$. For each $1 \le i \le N$, let u_i be the uniform measure on $x_i + [0,\delta]^d$ with total mass c_i, and v_i be the uniform measure on $x_i + [-\delta, 0]^d$ with total mass c_i. Further, we denote by u_0 the uniform measure on $[1, 1+\delta]^d$

with total mass $\sum_{i>N} c_i$, and v_0 the uniform measure on $[-\delta, 0]^d$ with total mass $\sum_{i>N} c_i$. Now, we define

$$\nu_1 = \nu + \sum_{i=1}^{N} u_i + u_0, \quad \nu_2 = \nu + \sum_{i=1}^{N} v_i + v_0.$$

It is then clear that ν_1 and ν_2 are absolutely continuous with respect to the Lebesgue measure on $[-\delta, 1+\delta]^d$. Furthermore, when restricted on $[0,1]^d$, we have $F_{\nu_1} \leq F_\mu \leq F_{\nu_2}$.

Now, we estimate $\|F_{\nu_2} - F_{\nu_1}\|_{L^p([0,1]^d)}$. For notational simplicity, we denote $\|\cdot\|_{L^p([0,1]^d)}$ by $\|\cdot\|_p$ for short, even if the function involved maybe defined in a larger domain. Note that for each $1 \leq i \leq N$, and for all $(t_1, t_2, \ldots, t_d) \in [0,1]^d$,

$$0 \leq F_{v_i}(t_1, t_2, \ldots, t_d) - F_{u_i}(t_1, t_2, \ldots, t_d) \leq c_i.$$

Furthermore, if $t_j \leq x_i(j) - \delta$ for all $1 \leq j \leq d$, then

$$F_{u_i}(t_1, t_2, \ldots, t_d) = F_{v_i}(t_1, t_2, \ldots, t_d) = 0;$$

if $t_j \geq x_i(j) + \delta$ for all $1 \leq j \leq d$, then

$$F_{u_i}(t_1, t_2, \ldots, t_d) = F_{v_i}(t_1, t_2, \ldots, t_d) = c_i.$$

Hence, for all $1 \leq i \leq N$

$$\|F_{v_i} - F_{u_i}\|_p^p \leq \sum_{j=1}^{d} \int_0^1 \cdots \int_0^1 \int_{x_i(j)-\delta}^{x_i(j)+\delta} \int_0^1 \cdots \int_0^1 c_i^p = 2d\delta c_i^p.$$

Together with the relation

$$0 \leq F_{v_0} - F_{u_0} \leq \sum_{i>N} c_i \leq (2d\delta)^{1/p}$$

we obtain

$$\begin{aligned}
\|F_{\nu_2} - F_{\nu_1}\|_p &\leq \sum_{i=1}^{N} \|F_{v_i} - F_{u_i}\|_p + \|F_{v_0} - F_{u_0}\|_p \\
&\leq \sum_{i=1}^{N} c_i (2d\delta)^{1/p} + (2d\delta)^{1/p} \\
&\leq 2(2d\delta)^{1/p}.
\end{aligned}$$

This proves the statements (2.2) and (2.3).

Note that because ν_1 and ν_2 are not supported on $[0,1]^d$, F_{ν_1} and F_{ν_2} do not belong to $\mathcal{F}_d''$. However, if we define functions G_1 and G_2 on $[0,1]^d$ by

$$G_1(t_1, t_2, \ldots, t_d) = F_{\nu_1}((1+2\delta)t_1 - \delta, (1+2\delta)t_2 - \delta, \ldots, (1+2\delta)t_d - \delta),$$

$$G_2(t_1, t_2, \ldots, t_d) = F_{\nu_2}((1+2\delta)t_1 - \delta, (1+2\delta)t_2 - \delta, \ldots, (1+2\delta)t_d - \delta),$$

for $(t_1, t_2, \ldots, t_d) \in [0,1]^d$, then $G_1, G_2 \in \mathcal{F}_d''$.

Suppose $\mathcal{F}_d''$ can be covered by $\frac{\varepsilon}{3}$-brackets $[g_1, g_1^*], [g_2, g_2^*], \ldots, [g_N, g_N^*]$, then there exists i and j such that $G_1 \in [g_i, g_i^*]$ and $G_2 \in [g_j, g_j^*]$. For $1 \le k \le N$, define f_k and f_k^* on $[-\delta, 1+\delta]^d$ by

$$f_k((1+2\delta)t_1 - \delta, (1+2\delta)t_2 - \delta, \ldots, (1+2\delta)t_d - \delta) = g_k(t_1, t_2, \ldots, t_d),$$
$$f_k^*((1+2\delta)t_1 - \delta, (1+2\delta)t_2 - \delta, \ldots, (1+2\delta)t_d - \delta) = g_k^*(t_1, t_2, \ldots, t_d),$$

for $(t_1, t_2, \ldots, t_d) \in [0,1]^d$. Then we have $F_{\nu_1} \in [f_i, f_i^*]$ and $F_{\nu_2} \in [f_j, f_j^*]$. Thus, restricted on $[0,1]^d$, we have

$$f_i \le F_{\nu_1} \le F_\mu \le F_{\nu_2} \le f_j^*.$$

That is, $F_\mu \in [f_i, f_j^*]$. Note that

$$\begin{aligned}\|f_j^* - f_i\|_p &\le \|f_j^* - F_{\nu_2}\|_p + \|F_{\nu_2} - F_{\nu_1}\|_p + \|F_{\nu_1} - f_i\|_p \\ &\le \|f_j^* - f_j\|_p + \|F_{\nu_2} - F_{\nu_1}\|_p + \|f_i^* - f_i\|_p \\ &\le (1+2\delta)^{d/p}\|g_j^* - g_j\|_p + \|F_{\nu_2} - F_{\nu_1}\|_p + (1+2\delta)^{d/p}\|g_i^* - g_i\|_p \\ &\le (1+2\delta)^{d/p}\varepsilon/3 + 2(2d\delta)^{1/p} + (1+2\delta)^{d/p}\varepsilon/3,\end{aligned}$$

where in the third inequality comes from changing variables, and the last inequality follows from (2.3) and the fact that $\|g_k^* - g_k\|_p \le \frac{\varepsilon}{3}$ for $k = i, j$. In particular, if we choose δ small enough, we have $\|f_j^* - f_i\|_p < \varepsilon$. That is, $[f_i, f_j^*]$ is an ε-bracket that covers F_μ. Because $F_\mu \in \mathcal{F}_d$ is arbitrary, we conclude that $\mathcal{F}_d$ can be covered by the following set of ε-brackets:

$$\{[f_k, f_l^*] : \|f_l^* - f_k\|_p \le \varepsilon, 1 \le k, l \le N\}$$

restricted on $[0,1]^d$. Therefore,

$$N_{[\,]}(\varepsilon, \mathcal{F}_d', \|\cdot\|_p) \le N^2 = [N_{[\,]}(\varepsilon/3, \mathcal{F}_d'', \|\cdot\|_p)]^2.$$

This implies the second inequality in the lemma. □

Now, we return to the proof of Theorem 1.1. Note that if for $I = (a,b] \times [0,1]^{d-1} \subset [0,1]^d$ or $I = [a,b) \times [0,1]^{d-1} \subset [0,1]^d$ we denote

$$\mathcal{F}_{t,I} = \{F_\nu 1_I : \nu(I) \le t\},$$

then by changing variable in the integration and using (2.1), we have

$$N_{[\,]}(t(b-a)^{1/p}\varepsilon, \mathcal{F}_{t,I}, \|\cdot\|_p) = N_{[\,]}(\varepsilon, \mathcal{F}_d, \|\cdot\|_p), \tag{2.4}$$

and the same equality holds for metric entropy. Thus, for convenience, we will call $\mathcal{F}_{t,I}$ a compression of $\mathcal{F}_d$ onto I, and call the quantity $t(b-a)^{1/p}$ its compression factor.

This scaling property of compressions is key to our approach in this paper. To explain how we will use this property. Let us consider a probability distribution F on the unit square $[0,1]^2$. If we cut the square into two rectangles $[0, \frac{1}{2}] \times [0,1]$ and $(\frac{1}{2}, 1] \times [0,1]$, then the restriction of F on the left rectangle belongs to a compression of $\mathcal{F}_2$ onto the left rectangle, while the restriction of F on the right rectangle is a function that belongs to a compression of $\mathcal{F}_2$ onto the right rectangle, plus a lower-dimensional distribution function (which is described fully in the sentence following (2.10)). The scaling property of compressions then enables us to do iteration, and relate the bracketing entropy of $\mathcal{F}_2$ to that of a class of lower-dimensional distributions ($\mathcal{F}_1$). In what follows, we will carefully develop this idea in $[0,1]^d$.

First, let us see how we should cut the cube $[0,1]^d$. There are a few ways we can cut it. For example, we cut it into two identical rectangular boxes. However, for the convenience of later iteration, the best way is to cut $[0,1]^d$ into $[0,c_\mu]\times[0,1]^{d-1}$ and $(c_\mu,1]\times[0,1]^{d-1}$, where the cutting point c_μ is defined by

$$c_\mu = \inf\{x\in[0,1] : \mu([0,x]\times[0,1]^{d-1}) \ge h(x)\},$$

where

$$h(x) = \frac{(1-x)^{1/p}}{x^{1/p}+(1-x)^{1/p}}.$$

(If we use Lemma 2.1, we only need to consider the case when $\mu([0,x])$ is a continuous function, in which case, c_μ is the unique solution to the equation $\mu([0,x]\times[0,1]^{d-1}) = h(x)$.) With the cutting point c_μ defined this way, we have

$$\mu([0,c_\mu)\times[0,1]^{d-1})c_\mu^{1/p} \le h(c_\mu)c_\mu^{1/p},$$
$$[1-\mu([0,c_\mu]\times[0,1]^{d-1})](1-c_\mu)^{1/p} \le h(c_\mu)c_\mu^{1/p}.$$

(Note that in the first inequality, the interval is $[0,c_\mu)$, not $[0,c_\mu]$. Of course, this makes no difference when $\mu\ll\lambda$.)

It is straightforward to check that $x^{\frac{1}{p}}h(x)\le 2^{-\frac{1}{p}-1}$. Indeed, because

$$x^{\frac{1}{p}}h(x) = \frac{x^{\frac{1}{p}}(1-x)^{\frac{1}{p}}}{x^{\frac{1}{p}}+(1-x)^{\frac{1}{p}}} = \frac{1}{x^{-\frac{1}{p}}+(1-x)^{-\frac{1}{p}}}.$$

The inequality follows immediately from the convexity of the function $x^{-\frac{1}{p}}$. Thus,

$$\mu([0,c_\mu)\times[0,1]^{d-1})c_\mu^{1/p} \le 2^{-\frac{1}{p}-1}, \tag{2.5}$$

$$[1-\mu([0,c_\mu]\times[0,1]^{d-1})](1-c_\mu)^{1/p} \le 2^{-\frac{1}{p}-1}. \tag{2.6}$$

The location of c_μ depends on the specific distribution function F_μ, and could be anywhere in $[0,1]$. In order to obtain information about the location, we need to divide $\mathcal{F}_d$ into distinct groups according to the location of c_μ. Let n be a large positive integer to be determined later. For each $F_\mu\in\mathcal{F}_d$, if $c_\mu<1$, then there exists a unique k, $1\le k\le n$, such that $c_\mu\in[\frac{k-1}{n},\frac{k}{n})$. Thus, if we define

$$\mathcal{G}_k = \left\{F_\mu\in\mathcal{F}_d : \frac{k-1}{n}\le c_\mu<\frac{k}{n}\right\},\ \ k=1,2,\dots,n-1,$$

and

$$\mathcal{G}_n = \left\{F_\mu\in\mathcal{F}_d : \frac{n-1}{n}\le c_\mu\le 1\right\},$$

then, $\mathcal{G}_k$, $k=1,2,\dots,n$ form a partition of $\mathcal{F}_d$.

For each $1 \le k \le n$, we denote

$$I_k = \left[0, \frac{k-1}{n}\right) \times [0,1]^{d-1},$$
$$J_k = \left[\frac{k-1}{n}, \frac{k}{n}\right] \times [0,1]^{d-1},$$
$$K_k = \left(\frac{k}{n}, 1\right] \times [0,1]^{d-1}.$$

(I_1 and K_n are empty sets.) When n is large, J_k is a thin slab. It separates the two rectangular boxes I_k and K_k.

For each $F_\mu \in \mathcal{G}_k$, if we define μ_1, μ_2, μ_3 as the restriction of μ on I_k, J_k and K_k respectively, then we can write

$$F_\mu = F_{\mu_1} 1_{I_k} + F_{\mu_1+\mu_2} 1_{J_k} + F_{\mu_3} 1_{K_k} + F_{\mu_1+\mu_2} 1_{K_k}.$$

We denote

$$\mathcal{I}_k = \{F_{\mu_1} 1_{I_k} : F_\mu \in \mathcal{G}_k\}, \quad \mathcal{J}_k = \{F_{\mu_1+\mu_2} 1_{J_k} : F_\mu \in \mathcal{G}_k\},$$
$$\mathcal{K}_k = \{F_{\mu_3} 1_{K_k} : F_\mu \in \mathcal{G}_k\}, \quad \mathcal{M}_k = \{F_{\mu_1+\mu_2} 1_{K_k} : F_\mu \in \mathcal{G}_k\},$$

where $\mathcal{I}_1$ and $\mathcal{K}_n$ are defined by $\{0\}$. If we denote $\mathcal{S}_k = \mathcal{I}_k + \mathcal{J}_k + \mathcal{K}_k + \mathcal{M}_k$, then $\mathcal{G}_k \subset \mathcal{S}_k$, and consequently, $\mathcal{F}_d \subset \cup_{k=1}^n \mathcal{S}_k$. Hence,

$$N_{[\,]}(\varepsilon, \mathcal{F}_d, \|\cdot\|_p) \le \sum\nolimits_{k=1}^n N_{[\,]}(\varepsilon, \mathcal{S}_k, \|\cdot\|_p). \tag{2.7}$$

Now, we take a closer look at $\mathcal{I}_k, \mathcal{J}_k, \mathcal{K}_k$ and $\mathcal{M}_k$.

Firstly, for $k = 2, 3, \ldots, n$, by the definitions of I_k and (2.5), we see that for every $F_\mu \in \mathcal{G}_k$,

$$\mu_1(I_k) \le \mu_1([0, c_\mu) \times [0,1]^{d-1}) \le 2^{-\frac{1}{p}-1}[(k-1)/n]^{-1/p}.$$

Thus, $\mathcal{I}_k$ is a subset of $\mathcal{F}_{t,I_k}$ with $t = 2^{-\frac{1}{p}-1}[(k-1)/n]^{-1/p}$. That is, $\mathcal{I}_k$ is contained in the compression of $\mathcal{F}_d$ onto I_k with the compression factor $2^{-\frac{1}{p}-1}$. Hence, by (2.4), for any $0 < \eta < 1$,

$$N_{[\,]}(2^{-\frac{1}{p}-1}\eta, \mathcal{I}_k, \|\cdot\|_p) \le N_{[\,]}(\eta, \mathcal{F}_d, \|\cdot\|_p). \tag{2.8}$$

The inequality trivially holds for the case $k = 1$.

Similarly, for $k = 1, 2, \ldots, n-1$, and every $F_\mu \in \mathcal{G}_k$, by using definitions of K_k and (2.6), we have

$$\mu_3(K_k) \le [1 - \mu([0, c_\mu] \times [0,1]^{d-1})] \le 2^{-\frac{1}{p}-1}[1 - k/n]^{-1/p}$$

Thus, $\mathcal{K}_k$ is a subset of $\mathcal{F}_{s,K_k}$ with $s = 2^{-\frac{1}{p}-1}[1-k/n]^{-1/p}$. That is, $\mathcal{K}_k$ is contained in the compression of $\mathcal{F}_d$ onto K_k with the compression factor $2^{-\frac{1}{p}-1}$. Hence, by (2.4), for any $0 < \eta < 1$,

$$N_{[\,]}(2^{-\frac{1}{p}-1}\eta, \mathcal{K}_k, \|\cdot\|_p) \le N_{[\,]}(\eta, \mathcal{F}_d, \|\cdot\|_p). \tag{2.9}$$

The inequality trivially holds for $k = n$.

Secondly, because the functions in $\mathcal{J}_k$ are non-negative, bounded by 1, and supported on J_k, the class $\mathcal{J}_k$ can be covered by a single bracket $[0, 1_{J_k}]$. Since J_k is a thin slab of thickness $1/n$, we have $\|1_{J_k} - 0\|_p = n^{-1/p}$. Thus, $[0, 1_{J_k}]$ is an $n^{-1/p}$-bracket, and consequently we have

$$N_{[]}(n^{-\frac{1}{p}}, \mathcal{J}_k, \|\cdot\|_p) = 1. \tag{2.10}$$

Finally, $\mathcal{M}_k$ can be viewed as a class of lower-dimensional distributions. Indeed, if we define ν on $[0,1]^{d-1}$ such that for any Borel set in $[0,1]^{d-1}$,

$$\nu(A) = (\mu_1 + \mu_2)([0, k/n] \times A),$$

then for any $(x_1, x_2, \ldots, x_d) \in K_k$,

$$F_{\mu_1+\mu_2}(x_1, x_2, \ldots, x_d) = F_\nu(x_2, x_3, \ldots, x_d).$$

Thus, by using (2.1), we have for any $\delta, \eta > 0$,

$$\begin{aligned} N_{[]}(2^{-\frac{1}{p}-1}\delta\eta, \mathcal{M}_k, \|\cdot\|_p) &\le N_{[]}(2^{-\frac{1}{p}-1}\delta\eta, \mathcal{F}'_{d-1}, \|\cdot\|_p) \\ &= N_{[]}(2^{-\frac{1}{p}-1}\delta\eta, \mathcal{F}_{d-1}, \|\cdot\|_p). \end{aligned} \tag{2.11}$$

Now we use (2.8), (2.9), (2.10) and (2.11) to estimate the number of brackets needed to cover $\mathcal{S}_k$. For convenience, we denote

$$\begin{aligned} N_I &= N_{[]}(2^{-\frac{1}{p}-1}\eta, \mathcal{I}_k, \|\cdot\|_p), \\ N_K &= N_{[]}(2^{-\frac{1}{p}-1}\eta, \mathcal{K}_k, \|\cdot\|_p), \\ N_M &= N_{[]}(2^{-\frac{1}{p}-1}\delta\eta, \mathcal{M}_k, \|\cdot\|_p). \end{aligned}$$

Suppose $\{[f_i, f_i^*]\}_{i=1}^{N_I}$ and $\{[g_j, g_j^*]\}_{j=1}^{N_K}$ are $2^{-\frac{1}{p}-1}\eta$-brackets that cover $\mathcal{I}_k$, and $\mathcal{K}_k$, respectively. Let $[h, h^*] = [0, 1_{J_k}]$ be the $n^{-\frac{1}{p}}$-bracket that cover $\mathcal{J}_k$, and $\{[m_i, m_i^*]\}_{i=1}^{N_M}$ be $2^{-\frac{1}{p}-1}\delta\eta$-brackets that cover $\mathcal{M}_k$. Define

$$\begin{aligned} \mathcal{B}_k = \{[f_i + g_j + h + m_l, f_i^* + g_j^* + h^* + m_l^*] : \\ 1 \le i \le N_I, 1 \le j \le N_J, 1 \le l \le N_M\}. \end{aligned}$$

Then, $\mathcal{B}_k$ cover $\mathcal{S}_k$. Indeed, for any $F_\mu \in \mathcal{S}_k$, we can write

$$F_\mu = F_I + F_J + F_K + F_M, \; F_I \in \mathcal{I}_k, F_J \in \mathcal{J}_k, F_K \in \mathcal{K}_k, F_M \in \mathcal{M}_k.$$

By the assumptions given above, we have some i, j and l such that

$$F_I \in [f_i, f_i^*], \; F_J \in [h, h^*], \; F_K = [g_j, g_j^*], \; F_M \in [m_l, m_l^*].$$

Hence

$$F_\mu \in [f_i + g_j + h + m_l, f_i^* + g_j^* + h^* + m_l^*].$$

To estimate the width of each bracket in $\mathcal{B}_k$, we note that $\mathcal{I}_k$, $\mathcal{J}_k$ and $\mathcal{K}_k + \mathcal{M}_k$ have disjoint supports, so we can assume that the brackets cover them also have disjoint supports. Thus,

$$\begin{aligned}\|(f_i^*+g_j^*+h^*+m_l^*)-(f_i+g_j+h+m_l)\|_p &= \left(\|f_i^*-f_i\|_p^p+\|h^*-h\|_p^p+\|(g_j^*-g_j)+(m_l^*-m_l)\|_p^p\right)^{1/p}\\ &\le \left(\frac{\eta^p}{2^{p+1}}+\frac{1}{n}+\frac{\eta^p}{2^{p+1}}(1+\delta)^p\right)^{1/p}\\ &\le \frac{1+\delta}{2}\eta,\end{aligned}$$

provided that we choose δ so that $n^{-1}\le 2^{-p-1}(\eta\delta)^p$. Thus, $\mathcal{B}_k$ are $\frac{1+\delta}{2}\eta$-brackets that cover $\mathcal{S}_k$.

Since $\mathcal{B}_k$ contains $N_I N_K N_M$ brackets, by using (2.8), (2.9) and (2.11) we have

$$\begin{aligned}N_{[\,]}(\frac{1+\delta}{2}\eta,\mathcal{S}_k,\|\cdot\|_p) &\le N_I N_K N_M\\ &\le \left[N_{[\,]}(\eta,\mathcal{F}_d,\|\cdot\|_p)\right]^2 N_{[\,]}(2^{-\frac{1}{p}-1}\delta\eta,\mathcal{F}_{d-1},\|\cdot\|_p).\end{aligned} \tag{2.12}$$

Now, we choose $\delta=\frac{1}{m}$ and $\eta=\eta_m:=m2^{-m+1}$, where m is any positive integer satisfying the inequality $n^{-1}\le 2^{-p-1}(\eta\delta)^p=2^{-p-1}(2^{-m+1})^p$, that is

$$1\le m\le r:=\left\lfloor\frac{\log_2 n-1}{p}\right\rfloor. \tag{2.13}$$

Then,

$$N_{[\,]}(\eta_{m+1},\mathcal{S}_k,\|\cdot\|_p)\le\left[N_{[\,]}(\eta_m,\mathcal{F}_d,\|\cdot\|_p)\right]^2 N_{[\,]}(2^{-\frac{1}{p}-m},\mathcal{F}_{d-1},\|\cdot\|_p).$$

Therefore, for $1\le m\le r$, by using (2.7) we obtain the following iteration relation:

$$N_{[\,]}(\eta_{m+1},\mathcal{F}_d,\|\cdot\|_p)\le n\left[N_{[\,]}(\eta_m,\mathcal{F}_d,\|\cdot\|_p)\right]^2 N_{[\,]}(2^{-\frac{1}{p}-m},\mathcal{F}_{d-1},\|\cdot\|_p).$$

To find the bracketing entropy of $\mathcal{F}_d$, we take logarithm to this iteration relation, then multiply it by 2^{-m}, and denote

$$f(m)=2^{-m}\log N_{[\,]}(\eta_{m+1},\mathcal{F}_d,\|\cdot\|_p),$$

we obtain

$$f(m)\le f(m-1)+2^{-m}\log n+2^{-m}\log N_{[\,]}(2^{-\frac{1}{p}-m},\mathcal{F}_{d-1},\|\cdot\|_p),$$

which is valid for all $1\le m\le r$. Telescoping, and using the fact that $f(0)=0$, we obtain

$$f(r)\le\log n+\sum_{m=1}^{r}2^{-m}\log N_{[\,]}(2^{-\frac{1}{p}-m},\mathcal{F}_{d-1},\|\cdot\|_p).$$

Multiplying by 2^r, using the definition of $f(r)$ and recalling $\eta_r=r2^{-r+1}$, we obtain

$$\begin{aligned}&\log N_{[\,]}((r+1)2^{-r},\mathcal{F}_d,\|\cdot\|_p)\\ &\qquad\le 2^r\log n+2^r\sum_{m=1}^{r}2^{-m}\log N_{[\,]}(2^{-\frac{1}{p}-m},\mathcal{F}_{d-1},\|\cdot\|_p).\end{aligned} \tag{2.14}$$

By the definition of r in(2.13), we have $\log n \le \log_2 n \le pr+p+1$. Hence,

$$\begin{aligned}&\log N_{[\,]}((r+1)2^{-r}, \mathcal{F}_d, \|\cdot\|_p)\\ &\qquad \le (pr+p+1)2^r + 2^r \sum_{m=1}^{r} 2^{-m} \log N_{[\,]}(2^{-\frac{1}{p}-m}, \mathcal{F}_{d-1}, \|\cdot\|_p). \qquad (2.15)\end{aligned}$$

As mentioned in the introduction, it is known that there exists a constant C_1 depending only on p such that for all $0<t<1$,

$$\log N_{[\,]}(t, \mathcal{F}_1, \|\cdot\|_p) \le C_1 t^{-1}$$

Applying this to (2.15) for $d=2$, we obtain,

$$\log N_{[\,]}((r+1)2^{-r}, \mathcal{F}_2, \|\cdot\|_p) \le (pr+p+1)2^r + Cr2^r,$$

where C is a constant depending only on p. For any $0<\varepsilon<1$, we choose r to be the smallest integer such that $(r+1)2^{-r} \le \varepsilon$. Then, we have

$$\log N_{[\,]}(\varepsilon, \mathcal{F}_2, \|\cdot\|_p) \le C_2 \varepsilon^{-1} |\log \varepsilon|^2. \qquad (2.16)$$

Note that by using (2.15) and induction, we immediately obtain

$$\log N_{[\,]}(\varepsilon, \mathcal{F}_d, \|\cdot\|_p) \le C_d \varepsilon^{-1} |\log \varepsilon|^{2(d-1)},$$

where C_d is a constant depending only on p and d.

3. Bounding bracketing entropy using metric entropy

Estimating bracketing entropy is typically difficult. Unlike metric entropy for which Fourier analytic methods are standard tools, for bracketing entropy, besides the connection with the $\|\cdot\|_\infty$ norm that has been discussed in the introduction, few tools are available. Based on our results in the previous section, here we develop a useful general method to estimate bracketing entropy for smooth functions. We say that a class $\mathcal{F}$ of multivariate functions on $[0,1]^d$ is smooth if for each $f \in \mathcal{F}$ and for all $0 \le \alpha_1, \alpha_2, \ldots, \alpha_d \le 1$, the derivative

$$D^{\alpha_1,\alpha_2,\ldots,\alpha_d} f := \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_d} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}$$

exists everywhere in $[0,1]^d$, and is integrable. Clearly, many functions fall into this category. Theorem 1.2 stated in the introduction says that in order to estimate the bracketing entropy of $\mathcal{F}$ in the L^p norm, $1 \le p < \infty$, one only needs to study the metric entropy of its derivative $\mathcal{H} := \{D^{\alpha_1,\alpha_2,\ldots,\alpha_d} f : 0 \le \alpha_1, \alpha_2, \ldots, \alpha_d \le 1, f \in \mathcal{F}\}$ in the L^1 norm, which is much easier.

Now, we turn to the proof of Theorem 1.2.

Proof of Theorem 1.2: We first estimate the bracketing entropy of the class $\mathcal{G} \subset \mathcal{F}$ consisting of all the functions in $\mathcal{F}$ that can be expressed as

$$g(x_1, x_2, \ldots, x_d) = \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_d} D^{1,1,\ldots,1} g(t_1, t_2, \ldots, t_d) dt_d \cdots dt_2 dt_1. \qquad (3.1)$$

By the assumption on the metric entropy of $\mathcal{H}$, we can find a set of functions h_i, $1 \le i \le m = N(\varepsilon, \mathcal{H}, \|\cdot\|_1) \le e^{\phi(\varepsilon)}$ that forms an ε-net of $\mathcal{H}$ in the $\|\cdot\|_1$ norm. For any $g \in \mathcal{G}$, $D^{1,1,\dots,1}g \in \mathcal{H}$, so there exists h_i such that $\|D^{1,1,\dots,1}g - h_i\|_1 < \varepsilon$. Of course, such h_i may not be unique. We will always choose the one with the smallest index i. Thus, $i = i(g)$ is a mapping from $\mathcal{G}$ to $\{1, 2, \dots, m\}$. Denote $T^+g = \max\{D^{1,1,\dots,1}g - h_i, 0\}$ and $T^-g = \max\{h_i - D^{1,1,\dots,1}g, 0\}$. Then, T^+g and T^-g are non-negative and have L^1 norms bounded by ε, and $T^+g - T^-g = D^{1,1,\dots,1}g - h_i$. Now, for each $1 \le k \le m$, we define

$$\mathcal{G}_k^+ = \left\{ \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_d} T^+g(t_1, t_2, \dots, t_d) dt_d \cdots dt_2 dt_1 : i(g) = k, g \in \mathcal{G} \right\},$$

$$\mathcal{G}_k^- = \left\{ \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_d} T^-g(t_1, t_2, \dots, t_d) dt_d \cdots dt_2 dt_1 : i(g) = k, g \in \mathcal{G} \right\}.$$

Then

$$\mathcal{G} \subset \bigcup_{k=1}^{m} \left(\mathcal{G}_k^+ - \mathcal{G}_k^- + \mathcal{I}h_k \right),$$

where

$$\mathcal{I}h_i(x_1, x_2, \dots, x_d) = \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_d} h_i(t_1, t_2, \dots, t_d) dt_d \cdots dt_2 dt_1.$$

This implies that for any $\eta < \varepsilon$,

$$N_{[\,]}(\eta, \mathcal{G}, \|\cdot\|_p) \le \sum_{k=1}^{m} N_{[\,]}(\eta/2, \mathcal{G}_k^+, \|\cdot\|_p) N_{[\,]}(\eta/2, \mathcal{G}_k^-, \|\cdot\|_p).$$

A crucial observation is that for each $1 \le k \le m$, $\mathcal{G}_k^+$ and $\mathcal{G}_k^-$ are subsets of $\{F_\mu : \mu([0,1]^d) \le \varepsilon\} = \varepsilon \mathcal{F}_d'$. Thus, by using (2.1), we have for all $1 \le k \le m$

$$N_{[\,]}(\eta, \mathcal{G}_k^+, \|\cdot\|_p) \le N_{[\,]}(\eta, \varepsilon\mathcal{F}_d', \|\cdot\|_p) = N_{[\,]}(\frac{\varepsilon}{\eta}, \mathcal{F}_d, \|\cdot\|_p),$$

$$N_{[\,]}(\eta, \mathcal{G}_k^-, \|\cdot\|_p) \le N_{[\,]}(\eta, \varepsilon\mathcal{F}_d', \|\cdot\|_p) = N_{[\,]}(\frac{\varepsilon}{\eta}, \mathcal{F}_d, \|\cdot\|_p).$$

Hence

$$N_{[\,]}(\eta, \mathcal{G}, \|\cdot\|_p) \le m \left[N_{[\,]}(\frac{\eta}{\varepsilon}, \mathcal{F}_d, \|\cdot\|_p) \right]^2.$$

Applying Theorem 1.1, we then obtain

$$\log N_{[\,]}(\eta, \mathcal{G}, \|\cdot\|_p) \le \phi(\varepsilon) + C(p,d) \frac{\varepsilon}{\eta} \left[\log \frac{\varepsilon}{\eta} \right]^{2(d-1)}, \tag{3.2}$$

where $K(d,p)$ is a positive constant depending only on d and p. In particular, if we choose η and ε so that

$$\phi(\varepsilon) = \frac{\varepsilon}{\eta} \left[\log \frac{\varepsilon}{\eta} \right]^{2(d-1)},$$

Then, we obtain

$$\log N_{[\,]}(\eta, \mathcal{G}, \|\cdot\|_p) \leq K(d,p) \cdot r(\eta)[\log r(\eta)]^{2(d-1)}, \tag{3.3}$$

where $K(d,p)$ is a positive constant depending only on d and p, and $r(\eta)$ is the solution to the equation

$$\phi(r\eta) = r[\log r]^{2(d-1)}.$$

This gives the estimate for the bracketing entropy of the subclass $\mathcal{G}$. Note that the functions in $\mathcal{G}$ has d-variables. For clarity, we denote $\mathcal{G}$ by $\mathcal{G}_d$.

To estimate the bracketing entropy of $\mathcal{F}$, we notice that for any $f \in \mathcal{F}$, we can write

$$f = g_d + c_{d-1}g_{d-1} + c_{d-2}g_{d-2} + \cdots + c_1 g_1 + c_0,$$

where $g_i \in \mathcal{G}_i$, $1 \leq i \leq d$, and the coefficients $c_0, c_1, \ldots, c_{d-1}$ are bounded. Since for all $1 \leq i \leq d-1$, the bracketing entropy of $\mathcal{G}_i$ has a lower order than $r(\eta)[\log r(\eta)]^{2(d-1)}$, a standard argument shows that $\mathcal{F}$ and $\mathcal{G}_d$ have the same rate of bracketing entropy. This finishes the proof of Theorem 1.2. $\square$

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Slepian's Inequality, Modularity and Integral Orderings

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Abstract. Slepian's inequality comes in many variants under different sets of regularity conditions. Unfortunately, some of these variants are wrong and other variants are imposing to strong regularity conditions. The first part of this paper contains a unified version of Slepian's inequality under minimal regularity conditions, covering all the variants I know about. It is well known that Slepian's inequality is closely connected to integral orderings in general and the supermodular ordering in particular. In the last part of the paper I explore this connection and corrects some results in the theory of integral orderings.

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1. Introduction

Throughout this paper, we let $(\Omega, \mathcal{F}, P)$ denote a fixed probability space. If $k \geq 1$ is an integer, we set $[k] := \{1, \ldots, k\}$. If $X = (X_1, \ldots, X_k)$ is a random vector such that $X_1, \ldots, X_k \in L^2(P)$, we let $\bar{X}_i := X_i - EX_i$ denote *the centered random variables* for $i \in [k]$ and we let $\Sigma^X = \{\sigma_{ij}^X\}$ and $\Pi^X = \{\pi_{ij}^X\}$ denote the *covariance matrix* and *intrinsic metric* of X; that is:

$$\sigma_{ij}^X := E(\bar{X}_i \bar{X}_j) \text{ and } \pi_{ij}^X := E(\bar{X}_i - \bar{X}_j)^2 \quad \forall\, i,j \in [k].$$

Note that $\pi_{ij}^X = \sigma_{ii}^X + \sigma_{jj}^X - 2\,\sigma_{ij}^X$ for all $i,j \in [k]$ and that $d(i,j) := \sqrt{\pi_{ij}^X}$ is a Hilbertian pseudo-metric on $[k]$.

It is well known that Slepian's inequality is an important tool in the theory of Gaussian processes. Let $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_k)$ be k-dimensional Gaussian vectors with zero means. Slepian's inequality comes in many variants; see [2, 6, 7, 9, 10, 12, 16, 18], but in essence it states that $Ef(Y) \leq Ef(X)$ for all

$f : \mathbf{R}^k \to \mathbf{R}$ satisfying

$$(\sigma_{ij}^X - \sigma_{ij}^Y)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \quad \text{or} \quad (\pi_{ij}^Y - \pi_{ij}^X)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \quad \forall i,j \tag{1.1}$$

plus some regularity conditions. Condition (1.1) indicates that f should be sufficiently smooth (at least twice differentiable), but Slepian's inequality is often used for indicator functions which are not even continuous. In most of the literature the indicator case and the smooth case are treated separately. The most general form of Slepian's inequality is found in [7] and [9] where (1.1) is interpreted in the sense of Schwartz distributions. However, Theorem 3.11 on p. 74 in [9] is false as it stands:

Example A: Let $k \geq 2$ be an integer, let $U, U_1, \ldots, U_k$ be independent $N(0,1)$-distributed random variables and set $X = (U, \ldots, U)$ and $Y = (U_1, \ldots, U_k)$. Then X and Y are Gaussian random vectors such that $\sigma_{ii}^Y = \sigma_{ii}^X$ and $\sigma_{ij}^Y = 0 < 1 = \sigma_{ij}^X$ for $1 \leq i \neq j \leq 1$. Let $D := \{x \in \mathbf{R}^k \mid x_1 = \cdots = x_k\}$ denote the diagonal in $\mathbf{R}^k$ and set $f = -1_D$. Since $f = 0$ Lebesgue a.e. we have $Ef(Y) = 0$ and $\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$ in distribution sense for all $1 \leq i,j \leq k$ and since $P(X \in D) = 1$, we have $Ef(X) = -1$ showing that Theorem 3.11 on p. 74 in [9] fails in this case. Many other counterexamples can be constructed in a similar manner.

This observation calls for a closer glance at the validity of Slepian's inequality and Section 2 of this paper will be devoted to establish a unified form of Slepian's inequality under minimal regularity conditions on f.

Slepian's inequality is intimately connected with integral orderings in general and the supermodular ordering in particular. If S and T are sets, we let 2^S denote the set of all subsets of S, we let T^S denote the set of all functions from S into T, and we let $B(S)$ denote the set of all bounded, real-valued functions on S. Recall that $(T, \leq)$ is a *proset* if T is a non-empty set and $\leq$ is a relation on T such that $\leq$ is *reflexive* ($t \leq t \; \forall t \in T$) and *transitive* ($t \leq u \;,\; u \leq v \;\Rightarrow\; t \leq v$).

Let $(S, \mathcal{A})$ be a measurable space; that is, S is a non-empty set and $\mathcal{A}$ is a σ-algebra on S. Then we let $M(S, \mathcal{A})$ denote the set of all $\mathcal{A}$-measurable functions from S into $\mathbf{R}$ and we let $\Pr(S, \mathcal{A})$ denote the set of all probability measures on $(S, \mathcal{A})$. Let $\Phi \subseteq M(S, \mathcal{A})$ be a given set of functions. Then it is customary to define *the* Φ*-integral ordering* on $\Pr(S, \mathcal{A})$, denoted $\leq_\Phi$, as follows: $\mu \leq_\Phi \nu$ if and only if $\int_S \phi \, d\mu \leq \int_S \phi \, d\nu$ for all $\phi \in \Phi \cap L^1(\mu) \cap L^1(\nu)$. Then $\leq_\Phi$ is a relation on $\Pr(S, \mathcal{B})$ which is reflexive but not transitive and exhibits strange properties:

Example B: Let $S = \mathbf{R}$ and let $\mathcal{B}$ denote the Borel σ-algebra on $\mathbf{R}$. Let Φ denote the set of all increasing, convex functions $\phi : \mathbf{R} \to \mathbf{R}$. Let μ be a Borel probability measure such that $\int_{\mathbf{R}} x^+ \, \mu(dx) = \infty$. Then $\Phi \cap L^1(\mu)$ is the set of all constant functions and so we have $\mu \leq_\Phi \nu$ and $\nu \leq_\Phi \mu$ for all $\nu \in \Pr(\mathbf{R}, \mathcal{B})$. In particular, we see that $\leq_\Phi$ is not transitive and that the integral ordering $\leq_\Phi$ is *not* a preordering.

To avoid such peculiarities, I shall introduce a slight modification of the Φ-integral ordering. If $\Phi \subseteq \mathbf{R}^S$, we define *the* Φ*-integral ordering* on $\Pr(S, \mathcal{A})$, denoted $\preceq_\Phi$, as follows $\mu \preceq_\Phi \nu$ if and only if $\int^* \phi \, d\mu \leq \int^* \phi \, d\nu$ for all $\phi \in \Phi$, where $\int^* f\,d\mu$ denotes the upper μ-integral of f. Then $(\mathrm{Pr}(S,\mathcal{A}),\preceq_\Phi)$ is a proset. If $(\Omega,\mathcal{F},P)$ is a probability space and $X,Y,Z:(\Omega,\mathcal{F})\to(S,\mathcal{A})$ are measurable functions, we let $P_Z(A)=P(Z\in A)$ for $A\in\mathcal{A}$ denote *the distribution* of Z and we write $X\preceq_\Phi Y$ if $P_X\preceq_\Phi P_Y$. Note that $\mu\preceq_\Phi\nu\ \Rightarrow\mu\leq_\Phi\nu$ and that the converse implication holds if $\Phi\subseteq L^1(\mu)\cap L^1(\nu)$. In Section 3 we shall take a closer look at integral orderings,

The classical *stochastic ordering* on $\mathbf{R}$, usually denoted $\preceq_{\mathrm{st}}$, is the integral ordering induced by the indicator functions $\{1_{[a,\infty)}\mid a\in\mathbf{R}\}$; that is $\mu\preceq_{\mathrm{st}}\nu$ if and only if $\mu([a,\infty))\leq\nu([a,\infty))$ for all $a\in\mathbf{R}$. More generally, let $(S,\leq)$ be a proset. Then we let $\mathrm{In}(S,\leq)$ denote the set of all increasing functions from S into $\mathbf{R}$ and we say that $A\subseteq S$ is an *upper interval* if $1_A\in\mathrm{In}(S,\leq)$. We define *the stochastic ordering on* S, denoted $\preceq_{\mathrm{st}}$, to be the integral ordering induced by indicators of upper intervals; that is, $\mu\preceq_{\mathrm{st}}\nu$ if and only if $\mu^*(A)\leq\nu^*(A)$ for every upper interval $A\subseteq S$. If $u\in S$, we define the upper and lower intervals $[u,*]:=\{s\in S\mid s\geq u\}$ and $[*,u]:=\{s\in S\mid s\leq u\}$ and we define *the orthant ordering*, denoted $\preceq_{\mathrm{or}}$, to be the integral ordering induced by $\{1_{[u,*]}\mid u\in S\}$; that is, $\mu\preceq_{\mathrm{or}}\nu$ if and only if $\mu^*([u,*])\leq\nu^*([u,*])$ for all $u\in S$.

Let $k\geq 1$ be an integer. Then we let $\leq$ denote *the product ordering* on $\mathbf{R}^k$; that is, $(x_1,\dots,x_k)\leq(y_1,\dots,y_k)$ if and only if $x_i\leq y_i$ for all $i=1,\dots,k$. If $x=(x_1,\dots,x_k)$ and $y=(y_1,\dots,y_k)$ are vectors, we define the lattice infimum and supremum as usual $x\wedge y:=(\min(x_1,y_1),\dots,\min(x_k,y_k))$ and $x\vee y:=(\max(x_1,y_1),\dots,\max(x_k,y_k))$, and we define $[x,y]=\{z\in\mathbf{R}^k\mid x\leq z\leq y\}$. We let $\mathcal{B}^k$ denote *the Borel σ-algebra* on $\mathbf{R}^k$. Let $f:\mathbf{R}^k\to\mathbf{R}$ be a given function. Then we say that f is *increasing (decreasing)* if f is increasing (decreasing) with respect to the product ordering $\leq$. We say that f is *supermodular* if $f(x)+f(y)\leq f(x\vee y)+f(x\wedge y)$ for all $x,y\in\mathbf{R}^k$, we say that f is *submodular* if $(-f)$ is supermodular, and we say that f is *modular* if f is supermodular and submodular. We define the following function spaces

$$\mathrm{sm}(\mathbf{R}^k)=\{f\in M(\mathbf{R}^k,\mathcal{B}^k)\mid f\text{ is supermodular }\}$$
$$\mathrm{m}(\mathbf{R}^k)=\{f\in M(\mathbf{R}^k,\mathcal{B}^k)\mid f\text{ is modular }\}\ ,\quad \mathrm{bm}(\mathbf{R}^k)=B(\mathbf{R}^k)\cap\mathrm{m}(\mathbf{R}^k)$$
$$\mathrm{bsm}(\mathbf{R}^k)=B(\mathbf{R}^k)\cap\mathrm{sm}(\mathbf{R}^k)\ ,\quad \mathrm{ism}(\mathbf{R}^k)=\mathrm{In}(\mathbf{R}^k,\leq)\cap\mathrm{sm}(\mathbf{R}^k)$$

and we let $\preceq_{\mathrm{sm}}$, $\preceq_{\mathrm{bm}}$, $\preceq_{\mathrm{m}}$, $\preceq_{\mathrm{bsm}}$ and $\preceq_{\mathrm{ism}}$ denote the integral orderings induced by $\mathrm{sm}(\mathbf{R}^k)$, $\mathrm{bm}(\mathbf{R}^k)$, $\mathrm{m}(\mathbf{R}^k)$, $\mathrm{bsm}(\mathbf{R}^k)$ and $\mathrm{ism}(\mathbf{R}^k)$, respectively. If $k=1$, then every function is supermodular and every increasing function is Borel measurable. Hence, in all dimensions there exists non-measurable supermodular functions and if $k\geq 2$, there exists non-measurable increasing functions. However, in Prop.4.3 below we shall see that an increasing supermodular function is Borel measurable.

Let μ be a Borel probability measure on $\mathbf{R}^k$ and let $F_1,\dots,F_k:\mathbf{R}\to[0,1]$ denote the one-dimensional marginal distribution functions of μ. Then

$$\overline{F}(x_1,\dots,x_k):=\min(F_1(x_1),\dots F_k(x_k))$$

is a k-dimensional distribution function, and if $\lambda_{\overline{F}}$ is the associated Lebesgue-Stieltjes measure, then $\lambda_{\overline{F}}$ is a Borel probability measure on $\mathbf{R}^k$ with the same one-dimensional marginals as μ. By a theorem of A.H. Tchen (see [19]), we have $\int_{\mathbf{R}^k} f\,d\mu \le \int_{\mathbf{R}^k} f\,d\lambda_{\overline{F}}$ for every supermodular function which is continuous, and satisfies a certain (uniform) integrability condition. In Theorem 4.7, we shall see that $\mu \preceq_{\text{bsm}} \lambda_{\overline{F}}$. In the modern literature it is frequently claimed that $\mu \le_{\text{sm}} \lambda_{\overline{F}}$ and that $\le_{\text{sm}}$ coincide with $\le_{\text{bsm}}$; see for instance [10]. The following example shows that both claims fail when $k \ge 3$.

Example C: (see [17]). Let U be a strictly positive random variable with a one-sided Cauchy distribution; that is, with distribution function F given by:

$$F(x) = \tfrac{2}{\pi}\arctan(x) \text{ if } x > 0 \quad \text{and} \quad F(x) = 0 \text{ if } x \le 0$$

Since U is strictly positive, we may define $V := \frac{1}{U}$ and $W := \frac{1}{2}|U - V|$. A straightforward computation shows that U, V and W all have distribution function F and so we have $F_U(x) = F_V(x) = F_W(x) = F(x)$ and $F_{(U,U,U)}(x,y,z) = \min(F(x),F(y),F(z))$ for all $x,y,z \in \mathbf{R}$. By [19] and Theorem 4.7 below, we have that $(U,V,W) \preceq_{\text{bsm}} (U,U,U)$ and $(U,V,W) \preceq_{\text{ism}} (U,U,U)$. Set $f(x,y,z) = x + y - 2z$. Then f is continuous, linear and modular and we have

$$f(U,U,U) = 0\ , \qquad f(U,V,W) = 2U1_{\{U<1\}} + \tfrac{2}{U}1_{\{U\ge 1\}}$$
$$0 < f(U,V,W) \le 2\ , \quad Ef(U,U,U) = 0 < Ef(U,V,W) = \tfrac{2\log 2}{\pi}.$$

Hence, we see that $(U,V,W) \not\le_{\text{sm}} (U,U,U)$ and $(U,V,W) \not\le_{\text{m}} (U,U,U)$ which shows the integrability condition in Theorem 5 of [19] *cannot* be removed and that $\mu \preceq_{\text{bsm}} \nu$ does *not* imply $\mu \le_{\text{m}} \nu$.

Let $X = (X_1,\dots,X_k)$ and $Y = (Y_1,\dots,Y_k)$ be k-dimensional Gaussian vectors with zero means and covariances $\{\sigma_{ij}^X\}$ and $\{\sigma_{ij}^Y\}$ such that $\sigma_{ii}^Y = \sigma_{ii}^X$ for all $1 \le i \le k$ and $\sigma_{ij}^Y \le \sigma_{ij}^X$ for all $1 \le i \ne j \le k$. Let f be a supermodular, locally Lebesgue integrable function. Then we have $\frac{\partial^2 f}{\partial x_i \partial x_j} \ge 0$ in distribution sense for all $1 \le i \ne j \le k$. So it is tempting to infer that Slepian's inequality implies $Y \preceq_{\text{sm}} X$. However Slepian's inequality only shows that $Ef(Y) \le Ef(Y)$ if f satisfies some additional regularity conditions. It can be shown that $Y \preceq_{\text{bsm}} X$; see Theorem 2.8 and Theorem 4.7, but Example C shows that $Y \preceq_{\text{bsm}} X$ does *not* imply $Y \preceq_{\text{sm}} X$ in general, and I don't know if we really have $Y \preceq_{\text{sm}} X$ if X and Y are Gaussian vectors satisfying the above hypotheses. However, Theorem 2.8 and Theorem 4.8 shows that $Ef(Y) \le Ef(X)$ for a large classs of unbounded supermodular functions. Section 4 is devoted the study of the modular orderings introduced above.

2. Slepian's inequality

In this section I shall prove a general version of Slepian's inequality where the partial derivatives are understood in the sense of Schwartz distributions. The idea is to approximate the function $f : \mathbf{R}^k \to \mathbf{R}$ with infinitely often differentiable functions $f_1, f_2, \ldots$ satisfying Slepian's inequality. The approximating sequence will taken as the convolution integrals $f_n(x) = \int_K f(x + \frac{y}{n})g(y)\,dy$ where $K \subseteq \mathbf{R}^k$ is a compact starshaped set and g is a nonnegative infinitely often differentiable function satisfying $\{g \neq 0\} \subseteq K$ and $\int_K g(y)\,dy = 1$. Below we shall see that if f is locally Lebesgue integrable, then f_n is an infinitely often differentiable function inheriting many properties of f and that $f_n(x) \to f(x)$ for all x in a large subset of $\mathbf{R}^k$. However, this requires some preparatory definitions and lemmas.

Let S be a set and let $\kappa : S \to [0, \infty]$ be a given function. If $f \in \mathbf{R}^S$, we let $||f||_\kappa := \inf\{c \in \mathbf{R}_+ \mid |f(s)| \leq c\,\kappa(s)\ \forall\, s \in S\}$ denote *the weighted sup-norm* of $f \in \mathbf{R}^S$ with the usual convention $\inf \emptyset := \infty$. If $\Phi \subseteq \mathbf{R}^S$ is a set of functions, we let $\Phi_+ := \Phi \cap \mathbf{R}^S_+$ denote the set of all nonnegative functions in Φ. If S and T are topological spaces and $\phi : S \to T$ is a given function, we let $C(\phi)$ denote *the continuity set* of ϕ; that is, the set of all $s \in S$ such that ϕ is continuous at s.

Let $k \geq 1$ be an integer and set $[k] := \{1, \ldots, k\}$. We let $e_1, \ldots, e_k$ denote the standard unit vectors in $\mathbf{R}^k$. If $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$ and $y = (y_1, \ldots, y_k) \in \mathbf{R}^k$, we let $\langle x, y\rangle = \sum_{i=1}^k x_i y_i$ denote *the inner product* and we let $||x|| = \langle x, x\rangle^{1/2}$ denote *the Euclidian norm.* We let λ_k denote the k-dimensional *Lebesgue measure* on $\mathbf{R}^k$. We say that $f : \mathbf{R}^k \to \mathbf{R}$ is *locally bounded* if f is bounded on every compact subset of $\mathbf{R}^k$, we say that f is *locally λ_k-integrable* if $1_C f \in L^1(\lambda_k)$ for every compact set $C \subseteq \mathbf{R}^k$, and we let $L^1_{\text{loc}}(\lambda_k)$ denote the set of all locally λ_k-integrable functions.

Let $f : \mathbf{R}^k \to \mathbf{R}$ be a given function. If $i \in [k]$ and $t \in \mathbf{R}$, we let $\Delta_i^t f(x) := f(x + te_i) - f(x)$ for $x \in \mathbf{R}^k$ denote the usual *difference operator.* If $\theta \in \mathbf{R}^k$, we say that f is *θ-differentiable at x* if $t \curvearrowright f(x + t\theta)$ is differentiable at 0 and if so we let $\frac{\partial f}{\partial \theta}(x) := \lim_{t\to 0} t^{-1}\,(f(x + t\theta) - f(x))$ denote *the directional θ-derivative* of f at x. In particular, we let $\frac{\partial f}{\partial x_i}(x) := \lim_{t\to 0} t^{-1}\,\Delta_i^t f(x)$ denote the partial derivative whenever it exists. We say that f is *partially differentiable at* x if the partial derivatives $\frac{\partial f}{\partial x_i}(x)$ exists for all $i \in [k]$ and if so we let $\nabla f(x) := (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x))$ denote the *gradient of f*. We say that f is *θ-differentiable* if f is θ-differentiable at all $x \in \mathbf{R}^k$ and we say that f is *continuously θ-differentiable* if f is θ-differentiable and $x \curvearrowright \frac{\partial f}{\partial \theta}(x)$ is continuous on $\mathbf{R}^k$. We say that f is *partially differentiable* if f is partially differentiable at all $x \in \mathbf{R}^k$. Recall that f is *Fréchet differentiable at x* if the directional derivative $\frac{\partial f}{\partial \theta}(x)$ exists for all $\theta \in \mathbf{R}^k$ and $\frac{\partial f}{\partial \theta}(x) = \langle\theta, \nabla f(x)\rangle$ for all $\theta \in \mathbf{R}^k$. Recall that f is *differentiable at x with differential $D \in \mathbf{R}^k$* if $\lim_{\theta\to 0} \|\theta\|^{-1}|f(x + \theta) - f(x) - \langle D, \theta\rangle| = 0$.

If $i_1, \ldots, i_p \in [k]$, we let $\frac{\partial^p f}{\partial x_{i_1} \cdots \partial x_{i_p}}(x)$ denote the pth order partial derivative whenever it exists. We let $C^\infty(\mathbf{R}^k)$ denote the set of all infinitely often differentiable functions $f : \mathbf{R}^k \to \mathbf{R}$ and if $\kappa : \mathbf{R}^k \to [0, \infty]$ is a nonnegative function we let $C^\infty_\kappa(\mathbf{R}^k)$ denote the set of all $f \in C^\infty(\mathbf{R}^k)$ satisfying

$$\|f\|_\kappa < \infty \quad \text{and} \quad \left\|\frac{\partial^p f}{\partial x_{i_1} \cdots \partial x_{i_p}}\right\|_\kappa < \infty \ \ \forall p \geq 1\ \forall i_1, \ldots, i_p \in [k].$$

In particular, we let $C_b^\infty(\mathbf{R}^k)$ denote the set of all bounded, infinitely often differentiable functions with bounded derivatives of all orders. We let $C_{\circ\circ}^\infty(\mathbf{R}^k)$ denote the set of all $f \in C^\infty(\mathbf{R}^k)$ with compact support and we let ϖ denote the usual inductive limit topology on $C_{\circ\circ}^\infty(\mathbf{R}^k)$; see [13]. We let $\mathcal{D}(\mathbf{R}^k)$ denote the set all *Schwartz distributions*; that is, the set of all ϖ-continuous linear functionals $\zeta : C_{\circ\circ}^\infty(\mathbf{R}^k) \to \mathbf{R}$. If $\zeta \in \mathcal{D}(\mathbf{R}^k)$, we write $\zeta \geq 0$ if and only if $\zeta(\phi) \geq 0$ for all $\phi \in C_{\circ\circ}^\infty(\mathbf{R}^k)_+$. If $f \in L^1_{\text{loc}}(\lambda_k)$, then $f(\phi) := \int_{\mathbf{R}^k} f(x)\,\phi(x)\,dx$ for $\phi \in C_{\circ\circ}^\infty(\mathbf{R}^k)$ defines a Schwartz distribution corresponding to f and if $i_1, \ldots, i_p \in [k]$, then

$$\partial_{i_1,\ldots,i_p} f(\phi) := (-1)^p \int_{\mathbf{R}^k} f(x)\,\frac{\partial^p \phi}{\partial x_{i_1} \cdots \partial x_{i_p}}(x)\,dx \text{ for } \phi \in C_{\circ\circ}^\infty(\mathbf{R}^k)$$

defines a Schwartz distributions, which corresponds to the "the partial derivative" $\frac{\partial^p f}{\partial x_{i_1} \cdots \partial x_{i_p}}$.

Recall that $K \subseteq \mathbf{R}^k$ is *starshaped* if $0 \in K$ and $\alpha x \in K$ for all $x \in K$ and all $0 \leq \alpha \leq 1$. Let $K \subseteq \mathbf{R}^k$ be a bounded, starshaped Borel set and let $x \in \mathbf{R}^k$ be a given vector. Then we say that f is *continuous at x along K* if

$$\lim_{n\to\infty} \Big\{ \sup_{y \in K} |f(x + \tfrac{y}{n}) - f(x)| \Big\} = 0 \tag{2.1}$$

and we let $C^K(f)$ denote the set of all $x \in \mathbf{R}^k$ satisfying (2.1). If 0 belongs to the interior of K, then continuity along K coincides with ordinary continuity. We say that f is *right continuous at x* if f is continuous at x along the unit cube $[0,1]^k$, say that f is *left continuous at x* if f is continuous at x along the negative unit cube $[-1,0]^k$.

Let $K \subseteq \mathbf{R}^k$ be a bounded, starshaped Borel set. Then we say that f is *approximately continuous at x along K* if f is locally λ_k-integrable and

$$\lim_{n\to\infty} \int_K |f(x + \tfrac{y}{n}) - f(x)|\,dy = 0. \tag{2.2}$$

We let $C^K_{\text{ap}}(f)$ denote the set of all $x \in \mathbf{R}^k$ satisfying (2.2). Let $f \in L^1_{\text{loc}}(\lambda_k)$ be a Borel function. By the Fubini-Tonelli theorem, we see that $C^K_{\text{ap}}(f)$ is a Borel set containing $C^K(f)$ and by Theorem III.12.8 p. 217 in [1] we have $\lambda_k(\mathbf{R}^k \setminus C^K_{\text{ap}}(f)) = 0$.

If $f, g : \mathbf{R}^k \to \mathbf{R}$ are λ_k-measurable and $\int_{\mathbf{R}^k} |f(x-y)\,g(y)|\,dy < \infty$ for all $x \in \mathbf{R}^k$, we say that *the convolution* exists and we let $(f \star g)(x) := \int_{\mathbf{R}^k} f(x-y)\,g(y)\,dy$ denote *the convolution* of f and g.

Lemma 2.1. *Let $f : \mathbf{R}^k \to \mathbf{R}$ be a locally λ_k-integrable function and let $g : \mathbf{R}^k \to \mathbf{R}$ be a bounded Lebesgue measurable function with compact support. Then the convolution $h(x) := (f \star g)(x)$ exists and is continuous on $\mathbf{R}^k$ and if $\theta \in \mathbf{R}^k$ is a given vector, we have*

(1) *If f is θ-differentiable and $\frac{\partial f}{\partial \theta} \in L^1_{\text{loc}}(\lambda_k)$, then h is continuously θ-differentiable and we have $\frac{\partial h}{\partial \theta}(x) = (\frac{\partial f}{\partial \theta} \star g)(x) \quad \forall x \in \mathbf{R}^k$.*

(2) *If g is θ-differentiable and $\frac{\partial g}{\partial\theta}$ is bounded, then h is continuously θ-differentiable and we have $\frac{\partial h}{\partial\theta}(x) = (f \star \frac{\partial g}{\partial\theta})(x) \ \ \forall x \in \mathbf{R}^k$.*

(3) *If f and g are θ-differentiable, $\frac{\partial f}{\partial\theta} \in L^1_{\text{loc}}(\lambda_k)$ and $\frac{\partial g}{\partial\theta}$ is bounded, then we have*

$$\int_{\mathbf{R}^k} \tfrac{\partial f}{\partial\theta}(x) \cdot g(y)\, dy = -\int_{\mathbf{R}^k} f(y) \cdot \tfrac{\partial g}{\partial\theta} g(y)\, dy.$$

Proof. Set $B_r := \{x \in \mathbf{R}^k \mid ||x|| \le r\}$ for $r \ge 0$. Since g is bounded with compact support, there exist $a, \rho > 0$ such that $|g(x)| \le a$ for all $x \in \mathbf{R}^k$ and $g(x) = 0$ for all $x \notin B_\rho$. Since f is locally λ_k-integrable and $|f(x-y)\, g(y)| \le a\, |f(x-y)|\, 1_{B_\rho}(y)$, we see that the convolution $h(x) = (f \star g)(x)$ exists for all $x \in \mathbf{R}^k$. Let $r > 0$ and $x \in B_r$ be given. Then $f_r := f\, 1_{B_{r+\rho}} \in L^1(\lambda_k)$ and we have $f_r(x-y)\, g(y) = f(x-y)\, g(y)$ for all $y \in \mathbf{R}^k$. Hence, we have $(f_r \star g)(x) = h(x)$ for all $x \in B_r$ and by Theorem 1.1.6 p. 4 in [14], we have that $f_r \star g$ is continuous on $\mathbf{R}^k$. Hence, we see that h is continuous on $\mathbf{R}^k$.

Suppose that f is θ-differentiable and that $\frac{\partial f}{\partial\theta} \in L^1_{\text{loc}}(\lambda_k)$. Let $x \in \mathbf{R}^k$ be given. By the argument above we have that the convolutions $\frac{\partial f}{\partial\theta} \star g$ and $|\frac{\partial f}{\partial\theta}| \star 1_{B_r}$ exist and are continuous on $\mathbf{R}^k$ for all $r \ge 0$. Let $x \in \mathbf{R}^k$ be given. By the Fubini-Tonelli theorem and locally boundedness of $|\frac{\partial f}{\partial\theta}| \star 1_{B_r}$, there exists a λ_k-null set N_x such that $s \curvearrowright \frac{\partial f}{\partial\theta}(x - y + s\theta)$ is locally λ_1-integrable on $\mathbf{R}$ for all $y \notin N_x$ and we have

$$\begin{aligned}\int_0^t (\tfrac{\partial f}{\partial\theta} \star g)(x + s\theta)\, ds &= \int_0^t ds \int_{\mathbf{R}^k} \tfrac{\partial f}{\partial\theta}(x - y + s\theta) g(y)\, dy \\ &= \int_{\mathbf{R}^k} dy \int_0^t \tfrac{\partial f}{\partial\theta}(x - y + s\theta)\, g(y)\, ds.\end{aligned}$$

Let $y \in \mathbf{R}^k \backslash N_x$ be given and set $F_{x,y}(s) := f(x-y+s\theta)$. Then $F_{x,y}$ is differentiable with derivative $F'_{x,y}(s) = \frac{\partial f}{\partial\theta}(x - y + s\theta)$ and $F'_{x,y}$ is locally λ_1-integrable. By a classical theorem of Denjoy and Banach (see Thm. IX.4.5 p. 271 and Thm. IX.7.4 p. 284 in [15]), we see that $F_{x,y}$ is absolutely continuous with Lebesgue derivative $F'_{x,y}$. In particular, we have $F_{x,y}(t) - F_{x,y}(0) = \int_0^t F'_{x,y}(s)\, ds$ and since $\int_{\mathbf{R}^k} F_{x,y}(s)\, g(y)\, dy = h(x + s\theta)$, we have

$$\int_0^t (\tfrac{\partial f}{\partial\theta} \star g)(x + s\theta)\, ds = \int_{\mathbf{R}^k} (F_{x,y}(t) - F_{x,y}(0))\, g(y)\, dy = h(x + t\theta) - h(x).$$

Since $\frac{\partial f}{\partial\theta} \star g$ is continuous, we see that h is continuously θ-differentiable with $\frac{\partial h}{\partial\theta}(x) = (\frac{\partial f}{\partial\theta} \star g)(x)$ for all $x \in \mathbf{R}^k$. Thus, (1) is proved and (2) follows in the same manner. Applying (1) and (2) on $f(y)$ and $g(-y)$ with $x = 0$, we obtain (3). □

Lemma 2.2. *Let $f : \mathbf{R}^k \to \mathbf{R}$ be a locally λ_k-integrable function and let $K \subseteq \mathbf{R}^k$ be starshaped, bounded Borel set. Let $g \in C_+^\infty(\mathbf{R}^k)$ be given such that $\{g \ne 0\} \subseteq K$ and $\int_K g(y)\, dy = 1$. Set $g_n(x) := n^k\, g(-nx)$ and $f_n(x) := (f \star g_n)(x)$ for $n \ge 1$*

and $x \in \mathbf{R}^k$; see Lemma 2.1. Let $p \geq 1$ and $i_1, \ldots, i_p \in [k]$ be given integers and let us define

$$\kappa(x) = \sup_{y \in K} |f(x+y)| \quad \forall x \in \mathbf{R}\ , \ c_{i_1 \ldots i_p} = \sup_{y \in K} |\tfrac{\partial^p g}{\partial x_{i_1} \cdots \partial x_{i_p}}(y)|.$$

Then $0 \leq c_{1_1, \ldots, i_p} < \infty$ and we have

(1) $f_n \in C^\infty(\mathbf{R}^k)$ *and* $f_n(x) = \int_{\mathbf{R}^k} f(x + \frac{y}{n}) g(y) dy \quad \forall x \in \mathbf{R}^k\ \forall n \geq 1.$

(2) $\frac{\partial^p f_n}{\partial x_{i_1} \cdots \partial x_{i_p}}(x) = (-n)^p \int_{\mathbf{R}^k} f(x + \frac{y}{n}) \frac{\partial^p g}{\partial x_{i_1} \cdots \partial x_{i_p}}(y)\, dy \quad \forall x \in \mathbf{R}^k\ \forall n \geq 1.$

(3) $|f_n(x)| \leq \kappa(x)$ *and* $|\frac{\partial^p f_n}{\partial x_{i_1} \cdots \partial x_{i_p}}(x)| \leq c_{i_1 \ldots i_p} n^p \kappa(x) \quad \forall x \in \mathbf{R}^k\ \forall n \geq 1.$

(4) $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in C^K_{\mathrm{ap}}(f).$

(5) $\lim_{n\to\infty} \left\{ \sup_{y \in C} \int_C |f(x + \frac{y}{n}) - f(x)| dx \right\} = 0$ *for all compact sets $C \subseteq \mathbf{R}^k$.*

(6) *If f is bounded with compact support, we have*

$$\lim_{n\to\infty} \int_{\mathbf{R}^k} |(f_n(x) - f(x))\psi(x)| dx = 0 \quad \forall \psi \in L^1_{\mathrm{loc}}(\lambda_k)$$

.

Proof. (1)–(2): Note that g, g_n, $G = \frac{\partial^p g}{\partial x_{i_1} \cdots \partial x_{i_p}}$ and $G_n = \frac{\partial^p g_n}{\partial x_{i_1} \cdots \partial x_{i_p}}$ are infinitely often differentiable with compact supports and we have

$$G_n(x) = (-1)^p\, n^{k+p}\, G(-nx).$$

So by Lemma 2.1 we see that $f_n \in C^\infty(\mathbf{R}^k)$ and that $\frac{\partial^p f_n}{\partial x_{i_1} \cdots \partial x_{i_p}}(x) = f \star G_n$. Hence, we see that (1)–(2) follows from the substitution $z = -\frac{y}{n}$.

(3): Let $n \geq 1$ be given. Since K is starshaped, we have $|f(x + \frac{y}{n})| \leq \kappa(x)$ for all $(x,y) \in \mathbf{R}^k \times K$ and since $g \geq 0$ and $\int_K g(y)\, dy = 1$, we see that (3) follows from (1)–(2).

(4): By (1), we have $|f_n(x) - f(x)| \leq a \int_K |f(x + \frac{y}{n}) - f(x)|\, dy$ where $a := \sup_{y \in K} g(y)$. Since $a < \infty$, we see that (4) holds.

(5): Let $r, \varepsilon > 0$ be given and set $B_r := \{x \in \mathbf{R}^k \mid ||x|| < r\}$. Since $f \in L^1_{\mathrm{loc}}(\lambda_k)$, we have $f_r := 1_{B_{1+r}} f \in L^1(\lambda_k)$. By Theorem 1.1.5 in [14], there exists $0 < \delta < 1$ such that $\int_{\mathbf{R}^k} |f_r(x+u) - f_r(x)|\, dx < \varepsilon$ for all $||u|| \leq \delta$. Let $x, y \in B_r$ and $n \geq \frac{r}{\delta}$ be given. Since $||\frac{y}{n}|| \leq \delta < 1$, we have $x \in B_{r+1}$ and $x + \frac{y}{n} \in B_{r+1}$ and so we have $f(x) = f_r(x)$ and $f(x + \frac{y}{n}) = f_r(x + \frac{y}{n})$. Hence, we have

$$\int_{B_r} |f(x + \tfrac{y}{n}) - f(x)| dx \leq \int_{\mathbf{R}^k} |f_r(x + \tfrac{y}{n}) - f_r(x)| dx \leq \varepsilon$$

for all $n \geq \frac{r}{\delta}$ and all $y \in B_r$. Since $r > 0$ is arbitrary, we see that (5) holds.

(6): Suppose that f is bounded with compact support. Then

$$b := \sup_{x \in \mathbf{R}^k} |f(x)| < \infty$$

and there exists $r > 0$ such that $\{f \neq 0\} \cup \{g \neq 0\} \subseteq B_r$. By (1), we see that $\{f_n \neq 0\} \subseteq B_{2r}$ and that $|f_n(x)| \leq b\, 1_{B_{2r}}(x)$. By (4), we see that $f_n(x) \to f(x)$ λ_k-a.e. and so we see that (6) follows from Lebesgue's convergence theorem. □

Lemma 2.3. *Let $f : \mathbf{R}^k \to \mathbf{R}$ be a locally λ_k-integrable function and let $K \subseteq \mathbf{R}^k$ be a starshaped, bounded Borel set. Let $g \in C_+^\infty(\mathbf{R}^k)$ be given such that $\{g \neq 0\} \subseteq K$ and $\int_K g(y)\, dy = 1$ and set $f_n(x) = \int_K f(x + \frac{y}{n})\, g(y)\, dy$ for $n \geq 1$ and $x \in \mathbf{R}^k$; see Lemma 2.2*

Let $A = \{a_{ij}\}$ be a $(k \times k)$-matrix and let $\mathcal{H}^A$ denote the set of all twice partially differentiable functions $h : \mathbf{R}^k \to [0, \infty)$ with compact support such that h, $\frac{\partial h}{\partial x_i}$ and $\frac{\partial^2 h}{\partial x_i \partial x_j}$ are locally λ_k-integrable for all $i, j \in [k]$ and $\sum_{i=1}^k \sum_{j=1}^k a_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j}$ is bounded. Let $\epsilon_1, \delta_1, \epsilon_2, \delta_2, \ldots > 0$ be positive numbers such that $\epsilon_n \to 0$ and $\delta_n \to 0$. Then the following four statements are equivalent:

(1) $\sum_{i=1}^k \sum_{j=1}^k a_{ij} \partial_{ij} f \geq 0.$

(2) $\sum_{i=1}^k \sum_{j=1}^k a_{ij} \int_{\mathbf{R}^k} f(x) \frac{\partial^2 h}{\partial x_i \partial x_j}(x) dx \geq 0 \quad \forall h \in \mathcal{H}^A.$

(3) $\sum_{i=1}^k \sum_{j=1}^k a_{ij} \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x) \geq 0 \quad \forall x \in \mathbf{R}^k \ \forall n \geq 1.$

(4) $\sum_{i=1}^k \sum_{j=1}^k a_{ij} \Delta_i^{\epsilon_n} \Delta_j^{\delta_n} f(x) \geq 0 \quad \lambda_k\text{-a.e.} \quad \forall n \geq 1.$

In particular, we have

(5) *If f is convex and A is nonnegative definite, then $\sum_{i=1}^k \sum_{j=1}^k a_{ij} \partial_{ij} f \geq 0$*

and if f is twice partially differentiable and f, $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are locally λ_k-integrable for all $i, j \in [k]$, then we have

(6) $\sum_{i=1}^k \sum_{j=1}^k a_{ij} \partial_{ij} f \geq 0 \ \Leftrightarrow \ \sum_{i=1}^k \sum_{j=1}^k a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \quad \lambda_k\text{-a.e.}$

Proof. Set $\zeta = \sum_{i=1}^k \sum_{j=1}^k a_{ij}\, \partial_{ij} f$, $F_n(x) = \sum_{i=1}^k \sum_{j=1}^k a_{ij} \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x)$ and $g_n(x) = n^k g(-nx)$ for all $x \in \mathbf{R}^k$ and all $n \geq 1$.

(1) ⇒ (2): Suppose that $\zeta \geq 0$ and let $h \in \mathcal{H}^A$ be given. Set $H(x) = \sum_{i=1}^k \sum_{j=1}^k a_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j}(x)$. By Lemma 2.2 and local λ_k-integrability of h and H, we have that the convolutions $h_n := h \star g_n$ and $H_n := H \star g_n$ exist and satisfies (1)–(6) in Lemma 2.2. Since h is nonnegative with compact support, we have $h_n, H_n \in$

$C_{\circ\circ}^\infty(\mathbf{R}^k)$ and $h_n \geq 0$ and by Lemma 2.1, we have $H_n = \sum_{i,j} a_{ij} \frac{\partial^2 h_n}{\partial x_i \partial x_j}$. Since $\zeta \geq 0$, we have $0 \leq \zeta(h_n) = \int_{\mathbf{R}^k} f H_n \, d\lambda_k$ for all $n \geq 1$. Recall that H is bounded with compact support and that f is locally λ_k-integrable. So by Lemma 2.2.(6) applied to the pair $(f, \psi) := (H, f)$, we have $\int_{\mathbf{R}^k} f H \, d\lambda_k = \lim_{n\to\infty} \zeta(h_n) \geq 0$.

(2) ⇒ (3): Suppose that (2) holds and let $x \in \mathbf{R}^k$ and $n \geq 1$ be given. Set $g_{nx}(y) = g_n(x-y)$ for $y \in \mathbf{R}^k$. By Lemma 2.1, we have

$$\frac{\partial^2 f_n}{\partial x_i \partial x_j}(x) = \int_{\mathbf{R}^k} f(y) \frac{\partial^2 g_n}{\partial x_i \partial x_j}(x-y)dy = \int_{\mathbf{R}^k} f(y) \frac{\partial^2 g_{nx}}{\partial x_i \partial x_j}(y)dy$$

and since $g_{nx} \in \mathcal{H}^A$, we see that (2) implies (3).

(3) ⇒ (4): Suppose that (3) holds and let $n \geq 1$ and $u, v > 0$ be given. Since $f_n \in C^\infty(\mathbf{R}^k)$, we have

$$\Delta_i^u \Delta_j^v f_n(x) = \int_0^u ds \int_0^v \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x + se_i + te_j) \, dt \quad \forall x \in \mathbf{R}^k.$$

So by (3) we have $\sum_{i,j} a_{ij} \, \Delta_i^u \Delta_j^v f_n(x) \geq 0$ for all $x \in \mathbf{R}^k$ and by Lemma 2.2.(4) we have that $f_n \to f$ λ_k-a.e. Hence, we see that (3) implies (4).

(4) ⇒ (1): Suppose that (4) holds. As above, we see that (4) implies $F_n(x) \geq 0$ for all $x \in \mathbf{R}^k$. Let $h \in C_{\circ\circ}^\infty(\mathbf{R}^k)_+$ be given and set $H = \sum_{i,j} a_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j}$. Then we have $\zeta(h) = \int_{\mathbf{R}^k} f H \, d\lambda_k$ and observe that H is bounded with compact support. So by Lemma 2.1.(3) and Lemma 2.2.(6) we have

$$\zeta(h) = \lim_{n\to\infty} \int_{\mathbf{R}^k} f_n(x) H(x) \, dx = \lim_{n\to\infty} \int_{\mathbf{R}^k} F_n(x) h(x) \, dx \geq 0.$$

Hence we see that (4) implies (1).

(5): Suppose that f is convex and A is nonnegative definite. Let $n \geq 1$ be a given integer. By nonnegativity of g, we see that f_n is convex and infinitely often differentiable. Let $x \in \mathbf{R}^k$ and $n \geq 1$ be given and set $b_{ij} = \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x)$ for $i, j \in [k]$. Then $B = (b_{ij})$ is the Hessian of f_n and since f_n is convex, we have that B is a nonnegative definite $(k \times k)$-matrix. By Schur's product theorem (see Thm. 7.5.3 p. 458 in [5]) we have that the Hadamard product $(c_{ij}) = (a_{ij} \, b_{ij})$ is nonnegative definite. In particular, we have

$$\sum_{i=1}^k \sum_{j=1}^k a_{ij} \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x) = \sum_{i=1}^k \sum_{j=1}^k c_{ij} \geq 0.$$

Hence, we see that (5) follows from the equivalence of (3) and (1).

(6): Suppose that f is twice partially differentiable such that $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are locally λ_k-integrable for all $i, j \in [k]$. Then $F := \sum_{i,j} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}$ belongs to $L^1_{\text{loc}}(\lambda_k)$ and by Lemma 2.1 we have

$$\zeta(\phi) = \sum_{i=1}^k \sum_{j=1}^k a_{ij} \int_{\mathbf{R}^k} f(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) dx = \int_{\mathbf{R}^k} F(x) \phi(x) \, dx \; \forall \phi \in C_{\circ\circ}^\infty(\mathbf{R}^k).$$

Since $F \geq 0$ λ_k-a.e. if and only if $\int_{\mathbf{R}^k} F\phi \, d\lambda_k \geq 0$ for all $\phi \in C_{oo}^{\infty}(\mathbf{R}^k)_+$, we see that (6) holds. □

Lemma 2.4. *Let $f : \mathbf{R}^k \to \mathbf{R}$ be a locally λ_k-integrable Borel function and let $\{a_{ij}\}_{1\leq i,j\leq k}$ be a $(k \times k)$-matrix. Let $\theta = (\theta_1, \ldots, \theta_k)$, $b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_k)$ be a given vectors. If there exist functions $h : \mathbf{R}^k \to \mathbf{R}$ and $\psi : \mathbf{R} \to \mathbf{R}$ satisfying $f(x + t\theta) = h(x) + \psi(t)$ for all $x \in \mathbf{R}^k$ and all $t \in \mathbf{R}$, then we have*

(1) $f(x + t\theta) = f(x) + \gamma t \quad \forall x \in \mathbf{R}^k \ \forall t \in \mathbf{R} \quad$ *where* $\gamma := f(\theta) - f(0)$.

(2) $\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}\partial_{ij} f = \sum_{i=1}^{k}\sum_{j=1}^{k}(a_{ij} + b_i\theta_j + c_j\theta_i)\partial_{ij} f$.

Proof. (1): Set $\psi_0(t) = \psi(t) - \psi(0)$ for all $t \in \mathbf{R}$. Since $f(x) = h(x) + \psi(0)$, we have $f(x + t\theta) = f(x) + \psi_0(t)$ and so we have

$$\psi_0(s + t) = f(s\theta + t\theta) - f(0) = f(s\theta) + \psi_0(t) - f(0) = \psi_0(s) + \psi_0(t)$$

for all $s, t \in \mathbf{R}$. Since f is Borel measurable and $\psi_0(t) = f(t\theta) - f(0)$, we see that ψ_0 is a Borel function satisfying $\psi_0(0) = 0$ and $\psi_0(s + t) = \psi_0(s) + \psi_0(t)$ for all $s, t \in \mathbf{R}$. So by [11] we have $\psi_0(t) = \gamma t$ for all $t \in \mathbf{R}$ where $\gamma = \psi_0(1) = f(\theta) - f(0)$. Since $f(x + t\theta) = f(x) + \psi_0(t)$, we see that (1) holds.

(2): Let $g \in C_{oo}^{\infty}(\mathbf{R}^k)_+$ be a nonnegative function with $\int_{\mathbf{R}^k} g(y)\,dy = 1$ and set $f_n(x) = \int_{\mathbf{R}^k} f(x + \frac{y}{n})\, g(y)\, dy$ for all $x \in \mathbf{R}^k$ and all $n \in \mathbf{N}$ (see Lem. 2.2). Then $f_n \in C^{\infty}(\mathbf{R}^k)$ and by (1), we have $f_n(x + t\theta) = f_n(x) + \gamma t$. In particular, we see that $\gamma = \frac{\partial f_n}{\partial\theta}(x) = \sum_{i=1}^{k} \theta_i \frac{\partial f_n}{\partial x_i}(x)$ for all $x \in \mathbf{R}^k$. Hence, we have

$$\sum_{j=1}^{k}\sum_{i=1}^{k} \theta_i c_j \frac{\partial^2 f_n}{\partial x_j \partial x_i}(x) = \sum_{j=1}^{k} c_j \frac{\partial}{\partial x_j}\left\{\sum_{i=1}^{k} \theta_i \frac{\partial f_n}{\partial x_i}(x)\right\} = 0 \quad \forall x \in \mathbf{R}^k.$$

In the same manner we see that $\sum_{i=1}^{k}\sum_{j=1}^{k} b_i\theta_j \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x) = 0$ for all $x \in \mathbf{R}^k$ and so we see that (2) follows from Proposition 2.3. □

Lemma 2.5. *Let $\phi, \psi : \mathbf{R} \to \mathbf{R}$ be absolutely continuous functions with Lebesgue derivatives $\dot\phi(t)$ and $\dot\psi(t)$. Set $\psi_*(t) := \int_t^{\infty} |\dot\psi(s)|\, ds$ for $t \geq 0$ and $\psi_*(t) := \int_{-\infty}^{t} |\dot\psi(s)|\, ds$ for $t < 0$. If (ϕ, ψ) satisfies the following condition*

(1) $\dot\psi \in L^1(\lambda_1)$, $\dot\phi \cdot \psi_* \in L^1(\lambda_1)$ *and* $\lim_{x\to\infty} \psi(x) = 0 = \lim_{x\to-\infty} \psi(x)$

then $\dot\phi \cdot \psi$ and $\phi \cdot \dot\psi$ are λ-integrable and we have

(2) $\int_{-\infty}^{\infty} \dot\phi(s)\psi(s)\, ds = -\int_{-\infty}^{\infty} \phi(s)\dot\psi(s)\, ds.$

Proof. Since $\dot\psi$ is λ_1-integrable and $\lim_{x\to\pm\infty} \psi(x) = 0$, we have

$$\psi(t) = \int_{-\infty}^{t} \dot\psi(s)\, ds = -\int_t^{\infty} \dot\psi(s)\, ds$$

for all $t \in \mathbf{R}$. In particular, we see that $|\psi(t)| \le \psi_*(t)$ for all $t \in \mathbf{R}$ and so by (1) we have $\dot\phi(t)\,\psi(t) \in L^1(\lambda)$. By the Fubini-Tonelli theorem, we have

$$\int_0^\infty |\phi(t)-\phi(0)|\cdot|\dot\psi(t)|dt \le \int_0^\infty dt\int_0^t |\dot\phi(s)\dot\psi(t)|\,ds$$
$$= \int_0^\infty ds\int_s^\infty |\dot\phi(s)\dot\psi(t)|dt = \int_0^\infty |\dot\phi(s)|\psi_*(s)\,ds < \infty$$

and in the same manner, we see that $\int_{-\infty}^0 |\phi(t)-\phi(0)|\cdot|\dot\psi(t)|\,dt < \infty$. Since $\dot\psi \in L^1(\lambda)$, we see that $\phi(t)\,\dot\psi(t) \in L^1(\lambda)$. So by the Fubini-Tonelli theorem we have

$$\int_0^\infty \dot\phi(t)\psi(t)\,dt = -\int_0^\infty dt\int_t^\infty \dot\phi(t)\dot\psi(s)\,ds = -\int_0^\infty ds\int_0^s \dot\phi(t)\dot\psi(s)\,dt$$
$$= \int_0^\infty (\phi(0)-\phi(s))\dot\psi(s)\,ds = -\phi(0)\psi(0) - \int_0^\infty \phi(s)\dot\psi(s)\,ds.$$

In the same manner, we see that $\int_{-\infty}^0 \dot\phi(t)\,\psi(t)\,dt = \phi(0)\psi(0) - \int_{-\infty}^0 \phi(t)\,\dot\psi(t)\,dt$. Adding the two equalities we obtain (2). □

Lemma 2.6. *Let $(U_0, U_1, \ldots, U_k)$ be a $(k+1)$-dimensional Gaussian random vector with mean zero and set $U := (U_1, \ldots, U_k)$ and $\theta := (\theta_1, \ldots, \theta_k)$ where $\theta_i := E(U_0\,U_i)$ for $i = 1, \ldots, k$. Let $h : \mathbf{R}^k \to \mathbf{R}$ be a θ-differentiable Borel function satisfying $E|U_0 h(U)| < \infty$ and $E|\frac{\partial h}{\partial\theta}(U)| < \infty$. Then we have*

(1) $E\{U_0 h(U)\} = E\{\frac{\partial h}{\partial\theta}(U)\}$.

Proof. Set $\sigma^2 = EU_0^2$. If $\sigma^2 = 0$, then we have $U_0 = 0$ a.s. and so we have $\theta = 0$ and $\frac{\partial h}{\partial\theta}(x) = 0$ for all $x \in \mathbf{R}^k$. Hence, we see that (1) holds if $\sigma^2 = 0$. So suppose that $\sigma^2 > 0$ and set $\phi_z(t) := h(z + t\theta)$ for all $t \in \mathbf{R}$ and all $z \in \mathbf{R}^k$. Then ϕ_z is differentiable on $\mathbf{R}$ with derivative $\phi_z'(t) = \frac{\partial h}{\partial\theta}(z + t\theta)$. Let us define

$$B = \{z \in \mathbf{R}^k \mid \int_{\mathbf{R}} |\tfrac{\partial h}{\partial\theta}(z+t\theta)|e^{-(\sigma t)^2/2}dt < \infty\}$$

and set $V_0 := \sigma^{-2}\,U_0$ and $V := (V_1, \ldots, V_k)$ where $V_i := U_i - \theta_i\,V_0$ for $i = 1, \ldots, k$. Then we have $V_0 \sim N(0, \sigma^{-2})$ and $E(V_0 V_i) = 0$ for all $1 \le i \le k$. Since $(V_0, V_1, \ldots, V_k)$ is Gaussian with mean zero, we see that V_0 and V are independent and since $U = V + V_0\theta$ and $\frac{\partial h}{\partial\theta}(U) \in L^1(P)$, we see that $P(V \in B) = 1$.

Let $z \in B$ be given and set $\psi(t) := -e^{-(\sigma t)^2/2}$ for all $t \in \mathbf{R}$. Then we have $\psi'(t) = \sigma^2 t\, e^{-(\sigma t)^2/2}$ and $\psi_*(t) = e^{-(\sigma t)^2/2}$ where $\psi_*(t)$ is defined as in Lemma 2.4. Since $z \in B$, we see that ϕ_z' is locally λ-integrable on $\mathbf{R}$. So by Theorem IX.4.5 p. 271 and Theorem IX.7.4 p. 284 in [15] we have that ϕ_z is absolutely continuous on $\mathbf{R}$ with Lebesgue derivative ϕ_z' and that (ϕ_z, ψ) satisfies condition

(1) in Lemma 2.5. Hence, we have that $\phi_z' \, \psi$ and $\phi_z \, \psi'$ are λ_1-integrable and

$$\int_{\mathbf{R}} h(z+t\theta)\sigma^2 t e^{-(\sigma t)^2/2} dt = \int_{\mathbf{R}} \phi_z(t)\psi'(t)\,dt = -\int_{\mathbf{R}} \phi_z'(t)\psi(t)\,dt$$
$$= \int_{\mathbf{R}} \tfrac{\partial h}{\partial \theta}(z+t\theta)e^{-(\sigma t)^2/2} dt.$$

Recall that V_0 and V are independent with $U = V + V_0\theta$ and $V_0 \sim N(0,\sigma^{-2})$. Since $U_0 h(U) \in L^1(P)$ and $\frac{\partial h}{\partial \theta}(U) \in L^1(P)$, we have $P(V \in B) = P_V(B) = 1$ and

$$\begin{aligned} E(U_0 h(U)) &= \sigma^2 E\{V_0 h(V+V_0\theta)\} \\ &= \tfrac{\sigma}{\sqrt{2\pi}} \int_B P_V\,(dz) \int_{\mathbf{R}} h(z+t\theta)\sigma^2 t e^{-(\sigma t)^2/2} dt \\ &= \tfrac{\sigma}{\sqrt{2\pi}} \int_B P_V(dz) \int_{\mathbf{R}} \tfrac{\partial h}{\partial \theta}(z+t\theta) e^{-(\sigma t)^2/2}\, dt \\ &= E(\tfrac{\partial h}{\partial \theta}(V+V_0\theta)) = E(\tfrac{\partial h}{\partial \theta}(U)) \end{aligned}$$

which proves the lemma. □

Lemma 2.7. *Let $Z = (Z_1, \ldots, Z_n)$ be an n-dimensional Gaussian random vector with mean zero and a non-zero covariance matrix $\Sigma^Z = \{\sigma_{ij}^Z\}$. Let λ denote the largest eigenvalue of Σ^Z and let ν denote the multiplicity of the eigenvalue λ. Let r denote the rank of Σ^Z and let $\phi : [0,\infty) \to [0,\infty)$ be an essentially increasing Borel function (see the remark below). Then $\lambda > 0$ and $1 \le \nu \le r \le n$ and we have*

(1) $\displaystyle\int_0^\infty t^{r-1}\phi(t)e^{-\frac{1}{2\lambda}t^2}\, dt < \infty \;\Rightarrow\; E\phi(\|Z\|) < \infty.$

(2) $\displaystyle E\phi(\|Z\|) < \infty \;\Rightarrow\; \int_0^\infty (1+t)^{\nu-1}\phi(t)e^{-\frac{1}{2\lambda}t^2}\, dt < \infty.$

(3) $E\phi(\|Z\|) < \infty \;\Rightarrow\; \exists c > 0$ *so that* $\phi(t) \le c(1+t)^{2-\nu} \exp\big(\frac{1}{2\lambda}t^2\big)\ \forall t \ge 0.$

Remark. We say that $\phi : [0,\infty) \to [0,\infty)$ is *essentially increasing* if there exists $C \ge 0$ such that $\phi(s) \le C\,(1+\phi(t))$ for all $0 \le s \le t$.

Proof. Since $\Sigma^Z \ne 0$, we have $\lambda > 0$ and $1 \le \nu \le r \le n$. Let $v_1, \ldots, v_n \in \mathbf{R}^n$ be an orthonormal basis of eigenvectors of Σ^Z ordered such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ where λ_i is the eigenvalue associated to v_i. Then we have $\lambda = \lambda_i \ge \lambda_r > 0$ for $1 \le i \le \nu$ and $\lambda_i = 0$ for $r < i \le n$. Set $U_i := \lambda_i^{-1/2}\langle Z, v_i\rangle$ for $i = 1, \ldots, r$. Then $U_1, \ldots, U_r$ are independent $N(0,1)$-distributed random variables such that $Z = \sum_{i=1}^r \lambda_i^{1/2} U_i\, v_i$ a.s. and $||Z||^2 = \sum_{i=1}^r \lambda_i U_i^2$ a.s. Let $1 \le d \le r$ be a given integer and set $U^d = (U_1, \ldots, U_d)$.

Recall that $2\pi^{d/2}\,\Gamma(\frac{d}{2})^{-1}$ is the $(d-1)$-dimensional volume of the $(d-1)$-dimensional unit sphere $\{u \in \mathbf{R}^d \mid ||u|| = 1\}$; see (8.43.9) p. 60 in vol. 2 of [3]. So by (8.24.1) p. 27 in vol. 2 of [3] with $T(x) = ||x||$, we have

$$E\phi(\sqrt{\lambda}\|U^d\|) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \phi(\sqrt{\lambda}\|x\|) e^{-\frac{1}{2}\|x\|^2} dx$$
$$= \frac{2}{2^{d/2}\Gamma(\frac{d}{2})} \int_0^\infty t^{d-1}\phi(t\sqrt{\lambda}) e^{-\frac{1}{2}t^2}\, dt = \frac{2}{(2\lambda)^{d/2}\Gamma(\frac{d}{2})} \int_0^\infty s^{d-1}\phi(s) e^{-\frac{1}{2\lambda}s^2}\, ds.$$

Since $||Z||^2 = \sum_{i=1}^r \lambda_i U_i^2$, we have $\lambda\,||U^\nu||^2 \le ||Z||^2 \le \lambda\,||U^r||^2$ and since ϕ is essentially increasing, there exists a constant $C > 0$ such that $0 \le \phi(s) \le C\,(1 + \phi(t))$ for all $0 \le s \le t$. In particular, we have

$$\phi(\sqrt{\lambda}\|U^\nu\|) \le C(1 + \phi(\|Z\|))\ ,\ \phi(\|Z\|) \le C(1 + \phi(\sqrt{\lambda}\|U^r\|))$$

and since $\phi(t) \le C(1 + \phi(1))$ for $0 \le t \le 1$ and $(1+t)^{\nu-1} \le 2^{\nu-1}\, t^{\nu-1}$ for $t \ge 1$, we see that (1) and (2) follow from the equality above. Since $\phi(s) \le C\,(1 + \phi(t))$ for all $t \ge s$, we have

$$\phi(s)(1+s)^{\nu-1} \int_s^\infty e^{-\frac{1}{2\lambda}t^2}\, dt \le C \int_0^\infty (1+t)^{\nu-1}(1+\phi(t)) e^{-\frac{1}{2\lambda}t^2} dt$$

and so we see that (3) follows from (2) and Exercise 2.51 p. 148 in vol. 1 of [3]. □

Theorem 2.8 (Slepian's inequality). *Let $k \ge 1$ be an integer and let $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_k)$ be Gaussian random vectors with zero means and covariance matrices $\Sigma^X = \{\sigma_{ij}^X\}$ and $\Sigma^Y = \{\sigma_{ij}^Y\}$. Let $K \subseteq \mathbf{R}^k$ be a bounded, starshaped Borel set with non-empty interior. Let $\phi : [0,\infty) \to [0,\infty)$ be an essentially increasing Borel function and let $f : \mathbf{R}^k \to \mathbf{R}$ be a locally λ_k-integrable Borel function satisfying*

(1) $P(X \in C_{\text{ap}}^K(f)) = 1 = P(Y \in C_{\text{ap}}^K(f))$.
(2) $E\phi(\|X\|) + E\phi(\|Y\|) < \infty$ *and* $\sup_{y\in K} |f(x+\delta y)| \le c\,\phi(\|x\|)\ \forall x \in \mathbf{R}^k$

for some positive numbers $c, \delta > 0$. Then $E|f(X)| < \infty$ and $E|f(Y)| < \infty$ and we have

(3) $\sum_{i=1}^k \sum_{j=1}^k (\sigma_{ij}^X - \sigma_{ij}^Y)\, \partial_{ij} f \ge 0 \ \Rightarrow\ Ef(Y) \le Ef(X)$.

Set $e = (1,1,\ldots,1)$ and suppose that there exist functions $h : \mathbf{R}^k \to \mathbf{R}$ and $\psi : \mathbf{R} \to \mathbf{R}$ such that $f(x+te) = h(x) + \psi(t)$ for all $(t,x) \in \mathbf{R} \times \mathbf{R}^k$. Set $\pi_{ij}^X = E(X_i - X_j)^2$ and $\pi_{ij}^Y = E(Y_i - Y_j)^2$ for all $i, j \in [k]$. Then we have

(4) $\sum_{i=1}^k \sum_{j=1}^k (\pi_{ij}^Y - \pi_{ij}^X)\partial_{ij} f = 2 \sum_{i=1}^k \sum_{j=1}^k (\sigma_{ij}^X - \sigma_{ij}^Y)\partial_{ij} f$.

(5) $\sum_{i=1}^k \sum_{j=1}^k (\pi_{ij}^Y - \pi_{ij}^X)\partial_{ij} f \ge 0 \ \Rightarrow\ Ef(Y) \le Ef(X)$.

Proof. Since K is starshaped, we have $0 \in K$. Hence, by (2), we have $|f(x)| \le c\,\phi(||x||)$ and $E|f(X)| < \infty$ and $E|f(Y)| < \infty$.

So suppose that $\sum_{i=1}^k \sum_{j=1}^k \theta_{ij}\, \partial_{ij} f \ge 0$, where $\theta_{ij} := \sigma_{ij}^X - \sigma_{ij}^Y$ and let me show that $Ef(Y) \le Ef(X)$. Without loss of generality we may assume that

X and Y are independent. If $X = Y = 0$ a.s. then (3) holds trivially. So let us assume that $P(X \neq 0) + P(Y \neq 0) > 0$ and set $\tilde{\phi}(t) := \sup_{s\in[0,t]} \phi(s)$. Then $\tilde{\phi}$ is increasing with $\phi \leq \tilde{\phi}$ and since ϕ is essentially increasing, there exists $C > 0$ such that $\tilde{\phi}(t) \leq C\,(1 + \phi(t))$ for all $t \geq 0$. Let $c, \delta > 0$ be chosen according to (2). Replacing (ϕ, K) by $(c\,\tilde{\phi}, \delta K)$ we may, without loss of generality, assume that ϕ is increasing and that $|f(x + y)| \leq \phi(||x||)$ for all $x \in \mathbf{R}^k$ and all $y \in K$. Let λ_X and λ_Y denote the largest eigenvalues of Σ_X and Σ_Y, respectively, and set $\lambda = \max(\lambda_X, \lambda_Y)$ and $\kappa(x) = \phi(||x||)$ for $x \in \mathbf{R}^k$. Then $\lambda > 0$ and by (2) and Lemma 2.7 there exists $b > 0$ such that $\phi(t) \leq b(1 + t)\,\exp(\frac{1}{2\lambda}\,t^2)$ for all $t \geq 0$.

Let $0 < r < 1$ and $n \in \mathbf{N}$ be fixed for a while. Since K has non-empty interior, there exists $g \in C^\infty_{\circ\circ}(\mathbf{R}^k)$ such that $g \geq 0$, $\{g \neq 0\} \subseteq K$ and $\int_K g(y)\,dy = 1$. Set $f_n(x) = \int_{\mathbf{R}^k} f(x + \frac{y}{n})\,g(y)\,dy$ for $x \in \mathbf{R}^k$ (see Lemma 2.2). By Lemma 2.2 we have that $f_n \in C^\infty_\kappa(\mathbf{R}^k)$ and that (f_n) satisfies (1)–(6) in Lemma 2.2. Since $0 < r < 1$ and $\phi(rt) \leq b(1 + rt)\,\exp(\frac{r^2}{2\lambda}t^2)$ we have

$$\text{(i)}\quad A := E\left\{\left(1 + \sqrt{\|X\|^2 + \|Y\|^2}\right)\phi\left(r\,\sqrt{\|X\|^2 + \|Y\|^2}\right)\right\} < \infty.$$

Let $x, y \in \mathbf{R}^k$ be given and set $U_{x,y}(t) = r(t^{1/2}\,x + (1 - t)^{1/2}\,y)$ and $V_{x,y}(t) = f_n(U_{x,y}(t))$ for $t \in [0, 1]$. Then $U_{x,y}$ and $V_{x,y}$ are continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$ with derivatives

$$U'_{x,y}(t) = \tfrac{r}{2}(t^{-1/2}x - (1 - t)^{-1/2}y) = \tfrac{r}{2\sqrt{t(1-t)}}((1 - t)^{1/2}x - t^{1/2}y)$$
$$V'_{x,y}(t) = \langle U'_{x,y}(t), \nabla f_n(U_{x,y}(t))\rangle$$

for all $0 < t < 1$. By the Cauchy-Schwartz inequality, we have

$$\|U_{x,y}(t)\| \leq rt^{1/2}\|x\| + r(1 - t)^{1/2}\|y\| \leq r\sqrt{\|x\|^2 + \|y\|^2} \quad \forall 0 \leq t \leq 1$$
$$\|U'_{x,y}(t)\| \leq \tfrac{r}{2\sqrt{t(1-t)}}\sqrt{\|x\|^2 + \|y\|^2} \quad \forall 0 < t < 1.$$

In particular, we see that $V_{x,y}$ is absolutely continuous on $[0, 1]$ and so we have

$$\text{(ii)}\quad f_n(rx) - f_n(ry) = V_{x,y}(1) - V_{x,y}(0) = \int_0^1 V'_{x,y}(t)\,dt.$$

Let $0 < t < 1$ be given and let us define $U(t) = (U_1(t), \ldots, U_k(t)) := U_{X,Y}(t)$ and $V'(t) := V'_{X,Y}(t)$. Since $V'(t) = \langle U'(t), \nabla f_n(U(t))\rangle$, we have

$$\|U(t)\| \leq r\sqrt{\|X\|^2 + \|Y\|^2}\ ,\ |V'(t)| \leq \|U'(t)\| \cdot \|\nabla f_n(U(t))\|$$
$$\|U'(t)\| \leq \tfrac{r}{2\sqrt{t(1-t)}}\sqrt{\|X\|^2 + \|Y\|^2}.$$

Set $\vartheta_j = \frac{r^2}{2}\,(\theta_{1j}, \ldots, \theta_{kj})$ and $h_j(x) = \frac{\partial f_n}{\partial x_j}(x)$ for $x \in \mathbf{R}^k$ and $1 \leq j \leq k$. Since $f_n \in C^\infty_\kappa(\mathbf{R}^k)$, we have $h_j \in C^\infty_\kappa(\mathbf{R}^k)$ and since $|h_j(x)| \leq ||\nabla f_n(x)||$ and $\kappa(x) =$

$\phi(\|x\|)$, we have

$$|U_j'(t)h_j(U(t))| \le \|U'(t)\| \cdot \|\nabla f_n(U(t))\| \le \|\nabla f_n\|_\kappa \cdot \|U'(t)\| \cdot \phi(\|U(t)\|)$$
$$|\tfrac{\partial h_j}{\partial \vartheta_j}(U(t))| \le \|\vartheta_j\| \cdot \|\nabla h_j(U(t))\| \le \|\vartheta_j\| \cdot \|\nabla h_j\|_\kappa \cdot \phi(\|U(t)\|).$$

Hence, by (i) we see that $U_j'(t)\, h_j(U(t))$, $\frac{\partial h_j}{\partial \vartheta_j}(U(t))$ and $V'(t)$ are P-integrable and

$$\int_0^1 E|V'(t)|\, dt \le A\|\nabla f_n\|_\kappa \int_0^1 \tfrac{1}{\sqrt{t(1-t)}} dt < \infty.$$

So by (ii) and the Fubini-Tonelli theorem, we have

(iii) $Ef_n(rX) - Ef_n(rY) = \displaystyle\int_0^1 EV'(t)\, dt.$

Since X and Y are independent Gaussian random vector with zero means, we see that $(U_j'(t), U_1(t), \ldots, U_k(t))$ is a $(k+1)$-dimensional Gaussian random vector with zero mean and $E(U_j'(t)\, U_i(t)) = \frac{r^2}{2}\, \theta_{ij}$. So by Lemma 2.6 we have

$$E\big\{U_j'(t)h_j(U(t))\big\} = E\Big\{\tfrac{\partial h_j}{\partial \vartheta_j}(U(t))\Big\} = \tfrac{r^2}{2} E\Big\{\sum_{i=1}^k \theta_{ij}\, \tfrac{\partial^2 f_n}{\partial x_i \partial x_j}(U(t))\Big\}$$

and by Proposition 2.3, we have $\sum_{i=1}^k \sum_{j=1}^k \theta_{ij}\, \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x) \ge 0$ for all $x \in \mathbf{R}^k$. Since

$$V'(t) = \langle U'(t), \nabla f_n(U(t)\rangle = \sum_{j=1}^k U_j'(t) h_j(U(t))$$

we see that $EV'(t) \ge 0$ for all $0 < t < 1$ and so by (iii) we have $Ef_n(rY) \le Ef_n(rX)$ for all $n \in \mathbf{N}$ and all $0 < r < 1$. Since ϕ is increasing and $\sup_{y \in K} |f(x+y)| \le \phi(||x||)$, we have $|f_n(rx)| \le \phi(||x||)$ for all $0 < r \le 1$, all $x \in \mathbf{R}^k$ and all $n \in \mathbf{N}$; see Lemma 2.2. So by (2), continuity of f_n and Lebesgue's convergence theorem, we have

$$Ef_n(Y) = \lim_{r \uparrow 1} Ef_n(rY) \le \lim_{r \uparrow 1} Ef_n(rX) = Ef_n(X)$$

for all $n \ge 1$. By (1) and Lemma 2.2, we have $f_n(x) \to f(x)$ $(P_X + P_Y)$-a.s. and recall that $|f_n(x)| \le \phi(\|x|)$. So by (2) and Lebesgue's convergence theorem, we have

$$Ef(Y) = \lim_{n \to \infty} Ef_n(Y) \le \lim_{n \to \infty} Ef_n(X) = Ef(X)$$

which completes the proof of (3).

Suppose that there exist functions $h : \mathbf{R}^k \to \mathbf{R}$ and $\psi : \mathbf{R} \to \mathbf{R}$ such that $f(x + te) = h(x) + \psi(t)$ for all $(t, x) \in \mathbf{R} \times \mathbf{R}^k$. Note that $\pi_{ii}^X = \pi_{jj}^Y = 0$ and

$$\pi_{ij}^Y - \pi_{ij}^X = 2(\sigma_{ij}^X - \sigma_{ij}^Y) + (\sigma_{ii}^Y - \sigma_{ii}^X) + (\sigma_{jj}^Y - \sigma_{jj}^X)$$

Hence, we see (4)–(5) follow from (3) and Lemma 2.4. □

Remark 2.9. (a): Condition (1) is a weak smoothness restriction on f. Note that condition (1) holds if f is right or left continuous and since $\mathbf{R}^k \setminus C^K_{\text{ap}}(f)$ is a λ_k-null set, we see that (1) holds if Σ^X and Σ^Y are non-singular. However, Example A in introduction shows that some smoothness condition on f is needed.

(b): Condition (2) is a growth condition on f. Let $a, \epsilon > 0$ be positive numbers and let $\psi : [0, \infty) \to [0, \infty)$ be an essentially increasing function satisfying $E\psi(\epsilon + ||X||) + E\psi(\epsilon + ||Y||) < \infty$ and $|f(x)| \leq a\,\psi(||x||)$ for all $x \in \mathbf{R}^k$. Since K is bounded and ψ is essentially increasing, it follows easily that f satisfies condition (2) with $\phi(t) := \psi(\epsilon + t)$.

(c): Let λ_X denote the largest eigenvalues of Σ^X, let ν_X denote the multiplicity of λ_X and let r_X denote the rank of the covariance matrix Σ^X. Let $p > 0$ be a given number and set $\phi(t) = (1+t)^{-p}\, e^{\frac{1}{2\lambda}\, t^2}$ for all $t \geq 0$. By Lemma 2.7, we have $p > r_X \;\Rightarrow E\phi(||X||) < \infty \;\Rightarrow\; p > \nu_X$. Since, the Slepian inequality implicitly requires finiteness of $E|f(X)|$ and $E|f(Y)|$, we see that the growth condition (2) is close to be optimal.

(d): Let $\varphi : [0, \infty) \to \mathbf{R}$ be an increasing convex function and set $Q(x) := \max_{1 \leq i,j \leq k} |x_i - x_j|$ and $f(x) = \varphi(Q(x))$ for all $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$. Fernique (see Theorem 2.1.2 p. 18 in [2]) has shown that $\pi^Y_{ij} \leq \pi^X_{ij}\ \forall\, i, j \in [k]$ implies $Ef(Y) \leq Ef(X)$. Note that $\pi^X_{ii} = \pi^Y_{ii} = 0$ and $f(x + te) = f(x)$ for all $(t, x) \in \mathbf{R} \times \mathbf{R}^k$ and in Corollary 4.5 below, we shall see that $\partial_{ij} f \leq 0$ for all $i \neq j$. Hence, we see that (5) is an extension of Fernique's version of Slepian's inequality.

3. Integral orderings

In this section we shall study an extension of the integral ordering to the set of finitely additive contents. Let $(S, \mathcal{A}, \mu)$ be a *content space*; i.e., $\mathcal{A} \subseteq 2^S$ is an algebra on the set S and $\mu : \mathcal{A} \to [0, \infty]$ is a finitely additive set function satisfying $\mu(\emptyset) = 0$. If $D \subseteq S$, we let $\mu^*(D) = \inf_{A \in \mathcal{A}, A \supseteq D} \mu(A)$ denote *the outer content* of D. We let $TM(\mu)$ denote the set of all *totally μ-measurable* real-valued functions (see Def. III.2.10 p. 106 in [1]), and we let $L^1(\mu)$ denote the set of all μ-integrable functions (see Def. III.2.17 p. 112 in [1]). If $f, f_1, f_2, \ldots \in \mathbf{R}^S$, we write $f_n \to^\mu f$ if $\mu^*(|f - f_n| > \varepsilon) \to 0$ for all $\varepsilon > 0$ (see Lem. III.2.7 p. 104 in [1]). If $f, g : S \to \bar{\mathbf{R}}$, we write $f \leq g$ μ-a.e. if $\mu^*(f > \epsilon + g) = 0$ for all $\epsilon > 0$. Note that $\mu^*(f > g) = 0$ implies $f \leq g$ μ-a.e. and the converse implication holds if μ is a measure. If $f : S \to \bar{\mathbf{R}}$, we let

$$\int^* f d\mu := \inf\{\textstyle\int_S \phi\, d\mu \mid \phi \in L^1(\mu), f \leq \phi\ \mu\text{ -a.e. }\} \quad (\inf\emptyset := \infty)$$
$$\int_* f\, d\mu := \sup\{\textstyle\int_S \phi d\mu \mid \phi \in L^1(\mu), \phi \leq f\ \mu\text{ -a.e. }\} \quad (\sup\emptyset := -\infty)$$

denote *the upper and lower μ-integral* of f; see [4]. We say that $\Phi \subseteq \mathbf{R}^S$ is *uniformly μ-integrable* if for every $\varepsilon > 0$ there exists $h \in L^1(\mu)$ such that $\int^* (|\phi| - h)^+ \, d\mu < \varepsilon$ for all $\phi \in \Phi$. If μ is a measure and $\Phi \subseteq L^1(\mu)$, then the reader easily verifies that uniform integrability as defined here coincides with the usual definition of uniform integrability; see for instance, (3.22.34) p. 187 in vol. 1 of [3]. Let $f, f_1, f_2, \ldots \in \bar{\mathbf{R}}^S$ be given functions. Then we write $f \leq_\mu \liminf f_n$ if and only if

$$\mu^*(A \cap \{f > t\}) \leq \liminf_{n\to\infty} \mu^*(A \cap \{f_n > s\}) \ \ \forall s < t \ \forall A \in \mathcal{A}.$$

We define $\mathcal{A}^\circ := \{A \in \mathcal{A} \mid \mu(A) < \infty\}$ and we write $\limsup f_n \leq_\mu f$ if and only if

$$\limsup_{n\to\infty} \mu^*(A \cap \{f_n > t\}) \leq \mu^*(A \cap \{f > s\}) \ \ \forall s < t \ \forall A \in \mathcal{A}^\circ.$$

Recall that $L^1(\mu) \subseteq TM(\mu)$ and if $\phi \in TM(\mu)$, then we have

(3.1) $(f - f_n)^+ \to^\mu 0 \Rightarrow f \leq_\mu \liminf f_n \Rightarrow (f + \phi) \leq_\mu \liminf(f_n + \phi)$.

(3.2) $(f_n - f)^+ \to^\mu 0 \Rightarrow \limsup f_n \leq_\mu f \Rightarrow \limsup(f_n + \phi) \leq_\mu (f + \phi)$.

Set $f_*(s) = \liminf_{n\to\infty} f_n(s)$ and $f^*(s) = \limsup_{n\to\infty} f_n(s)$ for all $s \in S$. If $(S, \mathcal{A}, \mu)$ is a measure space, then we have

(3.3) $f_* \leq_\mu \liminf f_n$.

(3.4) If $f_1, f_2, \ldots$ are μ-measurable, then $\limsup f_n \leq_\mu f^*$.

If $\mathcal{L} \subseteq 2^S$, we let $W(S, \mathcal{L})$ denote the set of all functions $f : S \to \mathbf{R}$ such that for all $y > x$ there exists a set $L \in \mathcal{L} \cup \{\emptyset, S\}$ satisfying $\{f > y\} \subseteq L \subseteq \{f > x\}$, and we let $W_\circ^+(S, \mathcal{L})$ denote the set of all functions $f : S \to [0, \infty)$ such that for all $y > x > 0$ there exists a set $L \in \mathcal{L} \cup \{\emptyset\}$ satisfying $\{f > y\} \subseteq L \subseteq \{f > x\}$. We say that $\Phi \subseteq \mathbf{R}^S$ is $(\uparrow)$-*stable* if $\sup_{n\geq 1} \phi_n \in \Phi$ for every increasing sequence $(\phi_n) \subseteq \Phi$ satisfying $\sup_{n\geq 1} \phi_n(s) < \infty$ for all $s \in S$, and we say that Φ is *sequentially closed* if for every pointwise convergent sequence $\phi_1, \phi_2, \ldots \in \Phi$ we have $\phi \in \Phi$ where $\phi(s) = \lim_{n\to\infty} \phi_n(s)$ for all $s \in S$. We let $\mathcal{L}_\uparrow$ denote the set of all sets of the form $\cup_{n=1}^\infty L_n$ for some increasing sequence $(L_n) \subseteq \mathcal{L} \cup \{\emptyset\}$ and we let $\mathcal{L}_\downarrow$ denote the set of all sets of the form $\cap_{n=1}^\infty L_n$ for some decreasing sequence $(L_n) \subseteq \mathcal{L} \cup \{\emptyset\}$. If $f \in W(S, \mathcal{L})$, then we have

(3.5) $\{f > t\} \in \mathcal{L}_\uparrow \ \ \forall t \geq \inf_{s\in S} f(s)$, $\{f \geq t\} \in \mathcal{L}_\downarrow \ \forall t > \inf_{s\in S} f(s)$.

If $\mathcal{L}$ is a σ-algebra on S, then $W(S, \mathcal{L}) = M(S, \mathcal{L})$. If S is a topological space and $\mathcal{L}$ is the set of all open (closed) subsets of S, then $W(S, \mathcal{L})$ is the set of all lower (upper) semicontinuous functions. If $(S, \leq)$ is a proset and $\mathcal{L}$ is the set of all upper intervals, then $W(S, \mathcal{L}) = \mathrm{In}(S, \leq)$.

Lemma 3.1. *Let S be a non-empty set and let $\Phi \subseteq \mathbf{R}_+^S$ be a $(\uparrow)$-stable, convex cone. Let $J \subseteq \mathbf{R}$ be an interval with interior J° and let $h : J \to \mathbf{R}$ be a continuous, increasing, convex function such that $\inf_{x\in J} h(x) = 0$. Let $f : S \to J$ be a given function satisfying $(f - c1_S)^+ \in \Phi$ for all $c \in J^\circ$. Then we have $h \circ f \in \Phi$.*

Proof. Let $a = \inf J$ and $b = \sup J$ denote the endpoints of J. If $a = b$, then the lemma holds trivially. So suppose that $a < b$ and set $\theta_c(t) = (t - c)^+$ for all $t \in J$ and all $c \in \mathbf{R}$. Let Γ denote the convex cone generated by $\{\theta_c \mid c \geq a\}$ and let us define

$$h_0(t) = \sup\{\gamma(t) \mid \gamma \in \Gamma, \gamma(s) \leq h(s) \ \ \forall s \in J\} \ \ \forall t \in J.$$

Then h_0 is an increasing, convex function satisfying $0 \le h_0(t) \le h(t)$ for all $t \in J$ and since h_0 is lower semicontinuous on J, we have that h_0 is continuous on J. Let $a < c < b$ be given. Since h is convex, we have that the right-hand derivative $r := \lim_{x \downarrow c} \frac{h(x)-h(c)}{x-c}$ exists and is finite and satisfies $h(c) + r\,(x-c) \le h(x)$ for all $x \in J$. Set $\gamma(x) := (h(c) + r\,(x-c))^+$ for all $x \in J$. If $r = 0$, we have $h(x) \ge h(c)$ for all $x \in J$ and since $\inf_{x \in J} h(x) = 0$, we have $h(c) = 0 = \gamma(x)$ for all $x \in J$. If $r > 0$, we have $\gamma = r\,\theta_u$ where $u = c - \frac{h(c)}{r}$ and since $\inf_{x \in J} h(x) = 0$ and $c + \frac{h(x)-h(c)}{r} \ge x \ge a$ for all $x \in J$, we have $u \ge a$. Hence, in either case, we have $\gamma \in \Gamma$ and $\gamma(x) \le h(x)$ for all $x \in J$. Since $\gamma(c) = h(c) \ge h_0(c)$, we have $h_0(c) = h(c)$ for all $c \in J^\circ$ and so by continuity of h and h_0, we have $h = h_0$.

By Lindelöf's theorem there exist $\gamma_1, \gamma_2, \ldots \in \Gamma$ such that $h(x) = \sup_{n \ge 1} \gamma_n(x)$ for all $x \in J$. Set $h_n(x) = \max(\gamma_1(x), \ldots, \gamma_n(x))$ for $x \in J$ and $n \ge 1$. Note that Γ is the set of all increasing, continuous, convex, piecewise linear functions $\gamma : J \to \mathbf{R}$ satisfying $\inf_{x \in J} \gamma(x) = 0$. In particular, we see that $h_n \in \Gamma$ and that $h_n(x) \uparrow h(x)$ for all $x \in J$. By assumption, we have $\theta_c \circ f \in \Phi$ for all $c \in J^\circ$ and since $\theta_c = 0$ for $c \ge b$ and Φ is an $(\uparrow)$-stable convex cone, we have $\gamma \circ f \in \Phi$ for all $\gamma \in \Gamma$. In particular, we have $h_n \circ f \in \Phi$ for all $n \ge 1$ and since $h_n(f(s)) \uparrow h(f(s))$ for all $s \in S$, we have $h \circ f \in \Phi$ □

Lemma 3.2. *Let $(S, \mathcal{A}, \mu)$ be a content space and let $f, f_1, f_2, \ldots : S \to \bar{\mathbf{R}}$ be given functions. If $\{f_n^+) \mid n \ge 1\}$ is uniformly μ-integrable, then we have*

(1) $\limsup f_n \le_\mu f \Rightarrow \limsup_{n\to\infty} \int^* f_n d\mu \le \int^* f d\mu.$

If $\{f_n^-) \mid n \ge 1\}$ is uniformly μ-integrable, then we have

(2) $f \le_\mu \liminf f_n \Rightarrow \int^* f d\mu \le \liminf_{n\to\infty} \int^* f_n d\mu.$

Proof. Suppose that $\limsup f_n \le_\mu f$ and that $\{f_n^+ \mid n \ge 1\}$ is uniformly μ-integrable. If $\int^* f\,d\mu = \infty$, then (1) holds trivially. So suppose that $\int^* f\,d\mu < \infty$ and let $\phi \in L^1(\mu)$ be given function satisfying $f \le \phi$ μ-a.e. Set $g = f - \phi$ and $g_n = f_n - \phi$. Let $\varepsilon > 0$ be given. Since $g_n^+ \le f_n^+ + \phi^-$, we see that (g_n^+) is uniformly μ-integrable. Hence, there exists $\psi \in L^1(\mu)$ such that $\psi \ge 0$ and $\int^* (g_n^+ - \psi)^+\,d\mu < \frac{\varepsilon}{2}$ for all $n \ge 1$. Since $\psi \in L^1_+(\mu)$, there exist positive numbers $\delta, c > 0$ such that $\int_S (\psi \wedge \delta)\,d\mu < \frac{\varepsilon}{2}$ and $\int_S (\psi - c)^+\,d\mu < \frac{\varepsilon}{2}$. Since $\mu^*(\psi > \delta) < \infty$, there exists $F \in \mathcal{A}$ such that $\{\psi > \delta\} \subseteq F$ and $\mu(F) < \infty$. Let $n \ge 1$ be given. Since $S \setminus F \subseteq \{\psi \le \delta\}$ and $g_n^+ \le \psi + (g_n^+ - \psi)^+$, we have

$$\int^* 1_{S\setminus F}\, g_n^+ d\mu \le \int^* (\psi \wedge \delta) d\mu + \int^* (g_n^+ - \psi) d\mu \le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Set $h_n = 1_F\, g_n^+$ and $Q_n(t) = \mu^*(h_n > t)$ for $t \in \mathbf{R}_+$ and $n \ge 1$. Let $t > 0$ be given. Since $f \le \phi$ μ-a.e., we have $\mu^*(g > t) = 0$ and by (3.2) we have $\limsup g_n \le_\mu g$. Since $\{h_n > t\} = F \cap \{g_n > t\}$ and $\mu(F) < \infty$, we have $Q_n(t) \to 0$ for all $t > 0$.

Let $n \ge 1$ be given. Then we have $0 \le h_n \le g_n^+ \le (g_n^+ - \psi)^+ + \psi$ and so by Theorem 2.1.(7) in [4] we have

$$\int_0^\infty Q_n(t)\,dt = \int^* h_n d\mu \le \int^* (g_n^+ - \psi)^+\,d\mu + \int_S \psi\,d\mu \le \tfrac{\varepsilon}{2} + \int_S \psi\,d\mu.$$

Since $0 \le h_n \le g_n^+$, we have $(h_n - c)^+ \le (g_n^+ - \psi)^+ + (\psi - c)^+$ and so by Theorem 2.1.(7) in [4] we have

$$\int_c^\infty Q_n(t)dt = \int_0^\infty Q_n(t+c)dt = \int^* (h_n - c)^+ \, d\mu$$
$$\le \int^* (g_n^+ - \psi)^+ \, d\mu + \int_S (\psi - c)^+ d\mu \le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Since $0 \le h_n \le g_n^+ \le (g_n^+ - \psi)^+ + \psi$, we have $h_n \wedge \delta \le (g_n^+ - \psi)^+ + (\psi \wedge \delta)$ and so by Theorem 2.1.(7) in [4] we have

$$\int_0^\delta Q_n(t)\,dt = \int^* (h_n \wedge \delta)\,d\mu \le \int^* (g_n^+ - \psi)^+ \, d\mu + \int^* (\psi^+ \wedge \delta)\,d\mu \le \varepsilon.$$

Hence, for every $\varepsilon > 0$ there exist positive numbers $c, \delta > 0$ satisfying

$$\sup_{n\ge 1} \int_0^\infty Q_n(t)dt < \infty \ , \ \sup_{n\ge 1} \int_c^\infty Q_n(t)dt \le \varepsilon \ , \ \sup_{n\ge 1} \int_0^\delta Q_n(t)\,dt \le \varepsilon.$$

Since Q_n is decreasing, it follows easily that $\{Q_n \mid n \ge 1\}$ is uniformly λ_1-integrable and recall that $Q_n(t) \to 0$ for all $t > 0$. Hence, by the Dunford-Pettis theorem (see (3.23) p. 189 in [3]) we have

$$0 = \lim_{n\to\infty} \int_0^\infty Q_n(t)\,dt = \lim_{n\to\infty} \int^* h_n \, d\mu$$

and since $g_n^+ = h_n + 1_{S\setminus F}\, g_n^+$, we have

$$\textstyle\int^* f_n \, d\mu \le \int_S \phi \, d\mu + \int^* g_n^+ \, d\mu \le \int_S \phi \, d\mu + \int^* h_n \, d\mu + \int^* 1_{S\setminus F} g_n^+ \, d\mu.$$

Since $\int^* 1_{S\setminus F}\, g_n^+ \, d\mu \le \varepsilon$ for all $n \ge 1$, we have

$$\limsup_{n\to\infty} \textstyle\int^* f_n \, d\mu \le \int_S \phi \, d\mu + \varepsilon \quad \forall \varepsilon > 0.$$

Letting $\varepsilon \downarrow 0$, we see that $\limsup \int^* f_n \, d\mu \le \int_S \phi \, d\mu$ for all $\phi \in L^1(\mu)$ with $f \le \phi$ μ-a.e. Taking infimum over ϕ, we obtain (1).

Suppose that $f \le_\mu \liminf f_n$ and that $\{f_n^- \mid n \ge 1\}$ is uniformly integrable. Let $\varepsilon > 0$ be given. Then there exists $\phi \in L^1(\mu)$ such that $\phi \ge 0$ and $\int^* (f_n^- - \phi)^+ \, d\mu \le \varepsilon$ for all $n \ge 1$. Set $g = f + \phi$ and $g_n = f_n + \phi$ and let us define $Q(t) = \mu^*(g > t)$ and $Q_n(t) = \mu^*(g_n > t)$ for $t \in \mathbf{R}$ and $n \ge 1$. By (3.1), we have $Q(t) \le \liminf Q_n(s)$ for all $0 < s < t$ and so we have $Q(t) \le \liminf Q_n(t)$ for all $t \in C(Q) \cap (0, \infty)$. Since Q is decreasing, we have $\mathbf{R} \setminus C(Q)$ is at most countable. Hence, by Theorem 2.1.(7) in [4] and Fatou's lemma we have

$$\int^* g^+ \, d\mu = \int_0^\infty Q(t)\,dt \le \liminf_{n\to\infty} \int_0^\infty Q_n(t)\,dt = \liminf_{n\to\infty} \int^* g_n^+ \, d\mu.$$

Since $g_n^- = (-f_n - \phi)^+ \le (f_n^- - \phi)^+$, we have (see Thm. 2.1.(5) in [4])

$$\textstyle\int^* g_n^+ \, d\mu = \int^* g_n \, d\mu + \int_* g_n^- \, d\mu \le \varepsilon + \int^* g_n \, d\mu$$

and since $\phi \in L^1(\mu)$ and $f = g + \phi$, we have

$$\int^* f\, d\mu \leq \int^* g^+ \, d\mu + \int_S \phi \, d\mu \leq \liminf_{n\to\infty} (\int^* g_n^+ \, d\mu + \int_S \phi \, d\mu)$$
$$\leq \varepsilon + \liminf_{n\to\infty} (\int^* g_n \, d\mu + \int_S \phi \, d\mu) = \varepsilon + \liminf_{n\to\infty} \int^* f_n \, d\mu.$$

Letting $\varepsilon \downarrow 0$, we obtain (2). □

Theorem 3.3. *Let $(S, \mathcal{A}_1, \mu)$ and $(S, \mathcal{A}_2, \nu)$ be content spaces and let us define*

$$\Lambda = \{f \in \mathbf{R}^S \mid \int^* f\, d\mu \leq \int^* f\, d\nu\} \ , \ \mathcal{L} = \{L \subseteq S \mid \mu^*(L) \leq \nu^*(L)\}.$$

Let $f, g \in \mathbf{R}^S$ be given functions. Then we have

(1) $f \in \Lambda$ *and* $g \in \Lambda \cap L^1(\nu) \ \Rightarrow \ af + g \in \Lambda \ \ \forall a \in \mathbf{R}_+$.

(2) $f \vee (-n1_S) \in \Lambda \ \ \forall n \in \mathbf{N} \ \Rightarrow \ f \in \Lambda$.

(3) $\nu(S) < \infty$, $\nu(S) \leq \mu(S)$ *and* $(f + n1_S)^+ \in \Lambda \ \ \forall n \in \mathbf{N} \ \Rightarrow \ f \in \Lambda$.

(4) $W_\circ^+(S, \mathcal{L}) \subseteq \Lambda$ *and if* $\mu(S) = \nu(S) < \infty$, *then* $W(S, \mathcal{L}) \subseteq \Lambda$.

(5) *If $f_1, f_2, \ldots \in \Lambda$ are given functions such that $(f_n^- \mid n \geq 1)$ is uniformly μ-integrable, $(f_n^+ \mid n \geq 1)$ is uniformly ν-integrable, $f \leq_\mu \liminf f_n$ and $\limsup f_n \leq_\nu f$, then we have $f \in \Lambda$.*

(6) *If μ is a measure and $\int_* f\, d\mu > -\infty$, then $\{h \in \Lambda \mid h \geq f\}$ is $(\uparrow)$-stable.*

Let $J \subseteq \mathbf{R}$ be an interval with interior J° such that $f(S) \subseteq J$ and let $G : J \to \mathbf{R}$ be an increasing, continuous, convex function such that $\inf_{x\in J} G(x) = 0$. If $(S, \mathcal{A}_1, \mu)$ and $(S, \mathcal{A}_2, \nu)$ and $(T, \mathcal{B}, \eta)$ are measure spaces and $\mathcal{A} := \mathcal{A}_1 \cap \mathcal{A}_2$, then we have

(7) *If μ, ν and η are σ-finite and $h \in L^1(\mu \otimes \eta) \cap L^1(\nu \otimes \eta)$ is a given function such that $h(\cdot, t) \in \Lambda \ \forall t \in T$ and $h(s, \cdot) \in L^1(\eta) \ \forall s \in S$, then we have $h^\eta \in \Lambda \cap L^1(\mu) \cap L^1(\nu)$ where $h^\eta(s) := \int_T h(s,t)\, \eta(dt)$.*

(8) *If $f \in M(S, \mathcal{A}_2)$ and $(f - c1_S)^+ \in \Lambda \ \ \forall c \in J^\circ$, then $G \circ f \in \Lambda$.*

(9) *If $\Phi \subseteq M(S, \mathcal{A})$ is sequentially closed, $\{\phi^- \mid \phi \in \Phi\}$ is uniformly μ-integrable and $\{\phi^+ \mid \phi \in \Phi\}$ is uniformly ν-integrable, then $\Lambda \cap \Phi$ is sequentially closed.*

Proof. (1): Let $f \in \Lambda$ and $g \in \Lambda \cap L^1(\nu)$ be given functions and let $a \geq 0$ be a nonnegative number. Then we have (see [4]):

$$\int^* (af + g) d\mu \leq a \int^* f\, d\mu \dot{+} \int^* g\, d\mu \leq a \int^* f\, d\nu + \int_S g\, d\nu \leq \int^* (af + g)\, d\nu$$

which proves (1).

(2) is an immediate consequence of Theorem 2.1.(6) in [4] and since

$$f \vee (-n1_S) = (f + n1_S)^+ - n1_S,$$

we see that (3) follows from (1) and (2).

(4): Let $f \in W_\circ^+(S, \mathcal{L})$ be given. Let $y > x > 0$ be given. Since $\emptyset \in \mathcal{L}$, there exists $L \in \mathcal{L}$ such that $\{f > y\} \subseteq L \subseteq \{f > x\}$. Hence, we have $\mu^*(f > y) \leq \mu^*(L) \leq \nu^*(L) \leq \nu^*(f > x)$ for all $0 < x < y$ and so we have $\mu^*(f > x) \leq \nu^*(f > x)$ for all $x \in \mathbf{R}_+ \setminus D$ where D is the set of all discontinuity points of $y \curvearrowright \mu^*(f > y)$. Since $y \curvearrowright \mu^*(f > y)$ is decreasing, we have that D is at most countable and so by Theorem 2.1.(7) in [4] we have

$$\int^* f\,d\mu = \int_0^\infty \mu^*(f > t)\,dt \le \int_0^\infty \nu^*(f > t)\,dt = \int^* f\,d\nu$$

which proves the first inclusion in (4). So suppose that $\mu(S) = \nu(S) < \infty$ and let $f \in W(S, \mathcal{L})$ be given. Since $\{\emptyset, S\} \in \mathcal{L}$, we have $(f + n1_S)^+ \in W_\circ^+(S, \mathcal{L}) \subseteq \Lambda$ for all $n \ge 1$ and so by (3) we have $f \in \Lambda$ which completes the proof of (4).

(5): Suppose that the hypotheses of (5) hold. Then we have $\int^* f_n\,d\mu \le \int^* f_n\,d\nu$ and so by Lemma 3.2 we have

$$\int^* f\,d\mu \le \liminf_{n\to\infty} \int^* f_n d\mu \le \liminf_{n\to\infty} \int^* f_n d\nu \le \limsup_{n\to\infty} \int^* f_n d\nu \le \int^* f\,d\nu$$

which proves (5).

(6): Suppose that μ is a measure and $\int_* f\,d\mu > -\infty$. Let $(h_n) \subseteq \Lambda$ be an increasing sequence such that $h_1 \ge f$ and $h_n \uparrow h \in \mathbf{R}^S$. Since $\int_* f\,d\mu > -\infty$, we have $\int^* f^-\,d\mu < \infty$ and since $0 \le h_n^- \le h_1^- \le f^-$, we see that $\{h_n^- \mid n \ge 1\}$ is uniformly μ-integrable. Since $h_n \le h$ and $h_n \in \Lambda$, we have $\int^* h_n\,d\mu \le \int^* h_n\,d\nu \le \int^* h\,d\nu$ for all $n \ge 1$ and by (3.3), we have $h \le_\mu \liminf h_n$. So by Lemma 3.2, we see that $\int^* h\,d\mu \le \int^* h\,d\nu$ which proves (6).

(7): Suppose that the hypotheses of (7) hold. Since $h \in L^1(\mu\otimes\eta) \cap L^1(\nu\otimes\eta)$, there exists a η-null set $N \subseteq T$ such that $h(\cdot\,, t) \in L^1(\mu) \cap L^1(\nu)$ for all $t \in T \setminus N$. Since $h(\cdot, t) \in \Lambda$, we have $\int_S h(s,t)\,\mu(ds) \le \int_S h(s,t)\,\nu(ds)$ for all $t \in T \setminus N$ and by the Fubini-Tonelli theorem we have $h^\eta \in L^1(\mu) \cap L^1(\nu)$ and

$$\int_S h^\eta\,d\mu = \int_{T\setminus N} \eta(dt) \int_S h(s,t)\,\mu(ds) \le \int_{T\setminus N} \eta(dt) \int_S h(s,t)\,\nu(ds) = \int_S h^\eta d\nu.$$

Hence, we see that $h^\eta \in \Lambda \cap L^1(\mu) \cap L^1(\nu)$.

(8): Suppose that $f \in M(S, \mathcal{A}_2)$ and $(f - c1_S)^+ \in \Lambda$ for all $c \in J^\circ$. Set $\Phi = \Lambda \cap M_+(S, \mathcal{A}_2)$. By (1) and (6), we have that Φ is an $(\uparrow)$-stable cone. Let $\phi, \psi \in \Phi$ be given and let me show that $\phi + \psi \in \Lambda$. If $\int^*(\phi+\psi)\,d\nu = \infty$, this is evident. So suppose that $\int^*(\phi+\psi)\,d\nu < \infty$. Since $\psi \ge 0$, we have $\int^* \phi\,d\nu < \infty$ and since ϕ is nonnegative and ν-measurable, we have $\phi \in L^1(\nu)$, So by (1) we see that $\phi + \psi \in \Phi$. Hence, we see that Φ is an $(\uparrow)$-stable convex cone containing $(f - c1_S)^+ \in \Lambda$ for all $c \in J^\circ$ and so by Lemma 3.1 we have $G \circ f \in \Phi \subseteq \Lambda$.

(9): Suppose that the hypotheses of (9) hold and let $h \in \mathbf{R}^S$ and $(h_n) \subseteq \Lambda \cap \Phi \cap M(S, \mathcal{A})$ be given functions satisfying $h_n(s) \to h(s)$ for all $s \in S$. Since Φ is sequentially closed, we have $h \in \Phi \cap M(S, \mathcal{A})$. By (3.3)–(3.4), we have $h \le_\mu \liminf h_n$ and $h \le_\nu \limsup h_n$ and since $\{h_n^- \mid n \ge 1\}$ is uniformly μ-integrable and $\{h_n^+ \mid n \ge 1\}$ is uniformly ν-integrable, we have $h \in \Lambda$ by (5). □

Theorem 3.4. *Let $\Lambda \subseteq M(\mathbf{R}^k, \mathcal{B}^k)$ be a non-empty set, let $\kappa : \mathbf{R}^k \to [0,\infty]$ be a Borel function and let μ and ν be Borel measures on $\mathbf{R}^k$ satisfying*

(1) $\phi \star g \in \Lambda \quad \forall \phi \in \Lambda \cap L^1_{\mathrm{loc}}(\lambda_k)\ \forall g \in C^\infty_{\circ\circ}(\mathbf{R}^k)_+$ *with* $\int_{\mathbf{R}^k} g(y)dy = 1.$

(2) $\kappa \in L^1(\mu) \cap L^1(\nu)$ *and* $\int_{\mathbf{R}^k} f d\mu \le \int_{\mathbf{R}^k} f\, d\nu \quad \forall f \in \Lambda \cap C_\kappa^\infty(\mathbf{R}^k)$.

Let $K \subseteq \mathbf{R}^k$ *be a bounded, starshaped, Borel set with non-empty interior and let* $f \in \Lambda$ *be a locally* λ_k*-integrable Borel function satisfying*

(3) $\exists c, \delta > 0$ *so that* $\sup_{y \in K} |f(x + \delta y)| \le c\, \kappa(x) \quad \forall x \in \mathbf{R}^k$.

(4) $\mu(\mathbf{R}^k \setminus C_{\mathrm{ap}}^K(f)) = 0 = \nu(\mathbf{R}^k \setminus C_{\mathrm{ap}}^K(f))$.

Then $f \in L^1(\mu) \cap L^1(\nu)$ *and we have* $\int_{\mathbf{R}^k} f\, d\mu \le \int_{\mathbf{R}^k} f\, d\nu$.

Proof. Let $f \in \Lambda \cap L_{\mathrm{loc}}^1(\lambda_k)$ be a given function satisfying (3)–(4) and let $c, \delta > 0$ be chosen according to (3). Since K has non-empty interior, there exists a function $g \in C_{\circ\circ}^\infty(\mathbf{R}^k)_+$ such that $\{g \neq 0\} \subseteq \delta K$ and $\int_K g\, d\lambda_k = 1$. Let $n \ge 1$ be given and set $f_n(x) = \int_{\mathbf{R}^k} f(x + \frac{y}{n})\, g(y)\, dy$ for $x \in \mathbf{R}^k$; see Lemma 2.2. By (1) and Lemma 2.2, we have $f_n \in \Lambda \cap C_\kappa^\infty(\mathbf{R}^k)$. So by (2), we have $\int f_n\, d\mu \le \int f_n\, d\nu$ for all $n \ge 1$. By (3) and Lemma 2.2.(3), we have $|f_n(x)| \le c\, \kappa(x)$ for all $x \in \mathbf{R}^k$ and all $n \ge 1$ and by (4) and Lemma 2.2.(4), we have $f_n \to f$ μ-a.e. and ν-a.e. By (2) we have $\kappa \in L^1(\mu) \cap L^1(\nu)$ and so by Lebesgue's convergence theorem we have $f \in L^1(\mu) \cap L^1(\nu)$ and

$$\int_{\mathbf{R}^k} f d\mu = \lim_{n\to\infty} \int_{\mathbf{R}^k} f_n d\mu \le \lim_{n\to\infty} \int_{\mathbf{R}^k} f_n\, d\nu = \int_{\mathbf{R}^k} f\, d\nu$$

which proves the theorem. □

4. Modular orderings

Let μ and ν be Borel probability measures on $\mathbf{R}^k$ such that $\mu \le_{\mathrm{bsm}} \nu$. In the modern literature it is frequently claimed that this implies $\mu \le_{\mathrm{sm}} \nu$; see for instance [10]. Theorem 4.8 below shows that we do have $\int f\, d\mu \le \int f\, d\nu$ for a large class of unbounded, supermodular Borel functions, and that we do have $\mu \le_{\mathrm{sm}} \nu$ if μ and ν are discrete measures with finitely many mass points. However, Example C of the introduction shows that this inequality may fail for some continuous, linear, modular function f satisfying $0 \le f \le 2$ μ-a.s. and ν-a.s. This shows that a closer glance at the supermodular ordering is needed. This section will be devoted to the study of supermodular functions and the modular orderings introduced in the introduction. Recall that $f : \mathbf{R}^k \to \mathbf{R}$ is supermodular if and only if $f(x \vee y) + f(x \wedge y) \le f(x) + f(y)$ for all $x, y \in \mathbf{R}^k$. Here we shall use an equivalent definition: f is supermodular if and only if $\Delta_i^s \Delta_j^t f(x) \ge 0$ for all $1 \le i \neq j \le k$, all $x \in \mathbf{R}^k$ and all $s, t \in \mathbf{R}_+$; see [8], where Δ_i^s for $i \in [k]$ is the difference operator $\Delta_i^s f(x) = f(x + se_i) - f(x)$ and $\Delta_i^s \Delta_j^t$ is the second-order difference operator:

$$\Delta_i^s \Delta_j^t f(x) = f(x + se_i + te_j) - f(x + se_i) - f(x + te_j) + f(x).$$

Let $k \ge 1$ be a given integer and let $f : \mathbf{R}^k \to \mathbf{R}$ be a given function. If $i \in [k]$, we write $\Delta_i f \ge 0$ if and only if $\Delta_i^s f(x) \ge 0$ for all $s \in \mathbf{R}_+$ and all $x \in \mathbf{R}^k$.

If $i, j \in [k]$, we write $\Delta_{ij} f \geq 0$ if and only if $\Delta_i^s \Delta_j^t f(x) \geq 0$ for all $s, t \in \mathbf{R}_+$ and all $x \in \mathbf{R}^k$, and we write $\Delta_{ij} f \leq 0$ if and only if $\Delta_{ij}(-f) \geq 0$. If $x \in \mathbf{R}^k$ and $i \in [k]$, we let

$$f_i^x(t) = f(x + (t - x_i)e_i) = f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_k) \quad \forall t \in \mathbf{R}$$

denote the partial function. Let "xxx" be a given property of a function of one variable (such as "increasing" or "continuous" or "differentiable"). If $i \in [k]$ and $f : \mathbf{R}^k \to \mathbf{R}$ is a function of k variables, we say that f has "xxx" *in the ith coordinate* if the partial functions f_i^x has "xxx" for all $x \in \mathbf{R}^k$. Note that f is increasing if and only if f is increasing in each coordinate and that we have

(4.1) $\Delta_j f \geq 0 \Leftrightarrow f$ is increasing in the jth coordinate.

(4.2) $\Delta_{ij} f \geq 0 \Leftrightarrow x \curvearrowright \Delta_i^s f(x)$ is increasing in the jth coordinate for all $s > 0$.

(4.3) f is supermodular if and only if $\Delta_{ij} f \geq 0$ for all $1 \leq i \neq j \leq k$; see [8].

(4.4) f is convex in the ith coordinate if and only if $\Delta_{ii} f \geq 0$ and f has the Baire property in the ith coordinate; see [20].

If $i \in [k]$ and $x \in \mathbf{R}^k$, we let

$$\overline{D}_i f(x) := \limsup_{u \to 0} \frac{f(x+ue_i)-f(x)}{u}, \quad \underline{D}_i f(x) := \liminf_{u \to 0} \frac{f(x+ue_i)-f(x)}{u}$$
$$\overline{D}_i^r f(x) := \limsup_{u \downarrow 0} \frac{f(x+ue_i)-f(x)}{u}, \quad \overline{D}_i^\ell f(x) := \limsup_{u \uparrow 0} \frac{f(x+ue_i)-f(x)}{u}$$
$$\underline{D}_i^r f(x) := \liminf_{u \downarrow 0} \frac{f(x+ue_i)-f(x)}{u}, \quad \underline{D}_i^\ell f(x) := \liminf_{u \uparrow 0} \frac{f(x+ue_i)-f(x)}{u}$$

denote *the right / left / upper / lower partial Dini derivatives* of f at x; see [15].

Proposition 4.1. *Let $f : \mathbf{R}^k \to \mathbf{R}$ and $\phi_1, \ldots, \phi_k : \mathbf{R} \to \mathbf{R}$ be given functions and set $\phi(x) := (\phi_1(x_1), \ldots, \phi_k(x_k))$ and $\zeta(x) := \prod_{i=1}^k \phi_i(x_i)$ for all $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$. Let $J \subseteq \mathbf{R}$ be an interval and let $\xi : J \to \mathbf{R}$ be an increasing and convex function. Let $i, j \in [k]$ be given integers. Then we have*

(1) *If $\phi_1, \ldots, \phi_k$ are nonnegative, $i \neq j$ and ϕ_i and ϕ_j are both increasing (decreasing) on $\mathbf{R}$, then we have $\Delta_{ij} \zeta \geq 0$.*

(2) *If $\Delta_{ij} f \geq 0$, $i \neq j$ and ϕ_i and ϕ_j are both increasing (decreasing) on $\mathbf{R}$, then we have $\Delta_{ij}(f \circ \phi) \geq 0$.*

(3) *Let $h_1, \ldots, h_n : \mathbf{R}^k \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$ be increasing functions and set $h(x) := (h_1(x), \ldots, h_n(x))$ for all $x \in \mathbf{R}^k$. Then we have*

$$\Delta_{ij} h_\ell \geq 0 \text{ and } \Delta_{\ell m} g \geq 0 \quad \forall 1 \leq \ell, m \leq n \Rightarrow \Delta_{ij}(g \circ h) \geq 0.$$

(4) *If $f(\mathbf{R}^k) \subseteq J$ and $\Delta_{ij}(f \vee a) \geq 0 \; \forall a \in J$, then $\Delta_{ij}(\xi \circ f) \geq 0$.*

(5) *If $f(\mathbf{R}^k) \subseteq J$, f is increasing and $\Delta_{ij} f \geq 0$, then $\Delta_{ij}(\xi \circ f) \geq 0$.*

Proof. (1) and (2) are easy and well known. Let g and $h_1,\dots,h_n$ be increasing functions such that $\Delta_{ij}h_\ell \geq 0$ and $\Delta_{\ell m}g \geq 0$ for all $1 \leq \ell, m \leq n$. Set $\psi(x) := g(h(x))$ for $x \in \mathbf{R}^k$. Let $u, v > 0$ and $x \in \mathbf{R}^k$ be given and let us define $y = x + ve_j$ and

$$\begin{aligned} &x^0 = h(x)\ ,\ x^n = h(x+ue_i)\ ,\ y^0 = h(y)\ ,\ y^n = h(y+ue_i) \\ &x^\ell = (h_1(x+ue_i),\dots,h_\ell(x+ue_i),h_{\ell+1}(x),\dots,h_n(x))\ \ \forall 1 \leq \ell < n \\ &y^\ell = (h_1(y+ue_i),\dots,h_\ell(y+ue_i),h_{\ell+1}(y),\dots,h_n(y))\ \ \forall 1 \leq \ell < n. \end{aligned}$$

Let $1 \leq \ell \leq n$ be given and set $u_\ell := \Delta_i^u h_\ell(x)$ and $v_\ell := \Delta_i^u h_\ell(y)$. Then we have $g(x^\ell) - g(x^{\ell-1}) = \Delta_\ell^{u_\ell} g(x^{\ell-1})$ and $g(y^\ell) - g(y^{\ell-1}) = \Delta_\ell^{v_\ell} g(y^{\ell-1})$. Since h_ℓ is increasing, we have $u^\ell \geq 0$ and recall that $\Delta_{ij}h_\ell \geq 0$. Hence, by (4.2) we have that $z \curvearrowright \Delta_i^u h_\ell(z)$ is increasing in the jth coordinate and so we have $0 \leq u_\ell \leq v_\ell$. Recall that $\Delta_{\ell m}g \geq 0$ for all $m = 1,\dots,n$. Hence, by (4.2) we have that $z \curvearrowright \Delta_\ell^{u_\ell} g(z)$ is increasing on $\mathbf{R}^n$ and since $x \leq y$ and $h_1,\dots,h_n$ are increasing, we have $x^{\ell-1} \leq y^{\ell-1}$. Thus, we have $\Delta_\ell^{u_\ell} g(x^{\ell-1}) \leq \Delta_\ell^{u_\ell} g(y^{\ell-1})$ and since $0 \leq u_\ell \leq v_\ell$ and g is increasing, we have $\Delta_\ell^{u_\ell} g(y^{\ell-1}) \leq \Delta_\ell^{v_\ell} g(y^{\ell-1})$. Hence, we have

$$\begin{aligned} \Delta_i^u \psi(x) &= g(x^n) - g(x^0) = \textstyle\sum_{\ell=1}^n (g(x^\ell) - g(x^{\ell-1})) = \sum_{\ell=1}^n \Delta_\ell^{u_\ell} g(x^{\ell-1}) \\ &\leq \textstyle\sum_{\ell=1}^n \Delta_\ell^{v_\ell} g(y^{\ell-1}) = \sum_{\ell=1}^n (g(y^\ell) - g(y^{\ell-1})) = g(y^n) - g(y^0) = \Delta_i^u \psi(x+ve_j) \end{aligned}$$

for all $x \in \mathbf{R}^k$ and all $u, v > 0$. In particular, we see that $x \curvearrowright \Delta_i^u \psi(x)$ is increasing in the jth coordinate for all $u > 0$ and so by (4.2) we conclude that $\Delta_{ij}\psi \geq 0$. Thus, (3) is proved.

(4): Suppose that $\Delta_{ij}(f \vee a) \geq 0$ for all $a \in J$. Let Φ_{ij} denote the set of all functions $F : \mathbf{R}^k \to \mathbf{R}$ such that $\Delta_{ij}F \geq 0$. Then Φ_{ij} is a pointwise closed, convex cone containing all constant functions and since $(f(x) - a)^+ = (f \vee a)(x) - a$, we see that $(f(\,\cdot\,) - a)^+ \in \Phi_{ij}$ for all $a \in J$. Since ξ is increasing and convex on J, there exist increasing, continuous, convex functions $\xi_1, \xi_2, \ldots : \mathbf{R} \to \mathbf{R}$ such that $\xi_m(t) \to \xi(t)$ for all $t \in J$ and $c_m := \inf_{t\in J} \xi_m(t) > -\infty$ for all $m \geq 1$. Then $\eta_m(t) = \xi_m(t) - c_m$ is an increasing, continuous, convex function on J with $\inf_{t\in J} \eta_m(t) = 0$. So by Lemma 3.1, we see that $\eta_m \circ f \in \Phi_{ij}$ for all $m \geq 1$ and since $\eta_m(t) + c_m = \xi_m(t) \to \xi(t)$, we have $\Delta_{ij}(\xi \circ f) \geq 0$.

(5): Suppose that f is increasing with $\Delta_{ij}f \geq 0$ and let $\xi_1, \xi_2, \ldots : \mathbf{R} \to \mathbf{R}$ be chosen as above. By (4.4), we have $\Delta_{11}\xi_m \geq 0$ and so by (3) applied with $n := 1$ and $(g, h_1) := (\xi_m, f)$, we see that $\Delta_{ij}(\xi_m \circ f) \geq 0$ for all $m \geq 1$ and since $\xi_m(f(x)) \to \xi(f(x))$, we have $\Delta_{ij}(\xi \circ f) \geq 0$. □

Proposition 4.2. *Let $f : \mathbf{R}^k \to \mathbf{R}$ be a supermodular function. If $i \in [k]$ and $s, t \in \mathbf{R}$, then we have*

(1) *$x \curvearrowright f_i^x(t) - f_i^x(s)$ is increasing on $\mathbf{R}^k$ if $s \leq t$ and decreasing on $\mathbf{R}^k$ if $t \leq s$.*

Let us define $\sigma(x,y)=\{i\in[k]\mid x_i<y_i\}$ *for* $x=(x_1,\dots,x_k)\in\mathbf{R}^k$ *and* $y=(y_1,\dots,y_k)\in\mathbf{R}^k$*. Let us define*

$$F_a(x)=f(x\vee a)-\sum_{i=1}^n f_i^a(x_i\vee a_i)\ ,\ F^a(x)=f(x\wedge a)-\sum_{i=1}^n f_i^a(x_i\wedge a_i)$$

for all $a=(a_1,\dots,a_k)\in\mathbf{R}^k$ *and all* $x=(x_1,\dots,x_k)\in\mathbf{R}^k$*. If* $a\le b$ *and* $x,y\in[a,b]$ *are given vectors, we have*

(2) $f(x)-f(y)\le\sum_{i\in\sigma(x,y)}(f_i^a(x_i)-f_i^a(y_i))+\sum_{i\in\sigma(y,x)}(f_i^b(x_i)-f_i^b(y_i))$.

(3) $|f(x)-f(y)|\le\sum_{i=1}^k|f_i^a(x_i)-f_i^a(y_i)|+\sum_{i=1}^k|f_i^b(x_i)-f_i^b(y_i)|$.

(4) f *is modular if and only if there exist functions* $f_1,\dots,f_k:\mathbf{R}\to\mathbf{R}$ *such that* $f(u)=f_1(u_1)+\cdots+f_k(u_k)$ *for all* $u=(u_1,\dots,u_k)\in\mathbf{R}^k$*.*

(5) F_a *is increasing and supermodular and* F^a *is decreasing and supermodular.*

Proof. (1): Let $1\le i\ne j\le k$ and $s,t\in\mathbf{R}$ be given such that $s\le t$. Then $u:=t-s\ge 0$ and by (4.2)–(4.3) and supermodularity of f, we have that $x\curvearrowright\Delta_i^u f(x)$ is increasing in the jth coordinate. Since $f_i^x(t)-f_i^x(s)=\Delta_i^u f(x+(s-x_i)e_i)$, we see that $x\curvearrowright f_i^x(t)-f_i^x(s)$ is increasing in the jth coordinate for $j\ne i$ and since $x\curvearrowright f_i^x(t)-f_i^x(s)$ is constant in the ith coordinate, we see that $x\curvearrowright f_i^x(t)-f_i^x(s)$ is increasing if $s\le t$. Interchanging s and t, we see that $x\curvearrowright f_i^x(t)-f_i^x(s)$ is decreasing if $t\le s$.

(2)–(3): Let $x,y\in\mathbf{R}^k$ be given vectors and set $z_0=x$, $z_n=y$ and $z_i=(y_1,\dots,y_i,x_{i+1},\dots,x_k)$ for $1\le i<n$. Then we have

(i) $f(x)-f(y)=\sum_{i=1}^n(f(z_{i-1})-f(z_i))=\sum_{i=1}^n(f_i^{z_i}(x_i)-f_i^{z_i}(y_i))$.

Let $a,b\in\mathbf{R}^k$ be given vectors such that $x,y\in[a,b]$. Since $a\le z_i\le b$, we have $f_i^{z_i}(x_i)-f_i^{z_i}(y_i)\le f_i^a(x_i)-f_i^a(y_i)$ for all $i\in\sigma(x,y)$ and $f_i^{z_i}(x_i)-f_i^{z_i}(y_i)\le f_i^b(x_i)-f_i^b(y_i)$ for all $i\in\sigma(y,x)$. Since $f_i^{z_i}(x_i)-f_i^{z_i}(y_i)=0$ for $i\notin\sigma(x,y)\cup\sigma(y,x)$, we see that (2) holds. (3) is an immediate consequence of (2).

(4): So suppose that f is modular. By (1), there exist functions $g_1,\dots,g_k:\mathbf{R}\to\mathbf{R}$ such that $f_i^x(t)-f_i^x(0)=g_i(t)$ for all $(t,x)\in\mathbf{R}\times\mathbf{R}^k$ and all $i\in[k]$. So by (i) with $y=(0,\dots,0)$, we see that $f(x)=f(0,\dots,0)+\sum_{i=1}^k g_i(x_i)$ which proves the "only if" in (4). The "if" part is evident.

(5): Let $x\le y$ be given. Then we have $a\le x\vee a\le y\vee a$ and so by (2) with $b:=y\vee a$, we have

$$f(x\vee a)-f(y\vee a)\le\textstyle\sum_{i=1}^k(f_i^a(x_i\vee a_i)-f_i^a(y_i\vee a_i)).$$

Hence, we see that F_a is increasing. In the same manner we see that F^a is decreasing. By Proposition 4.1.(2) we see that $f(x\vee a)$ and $f(x\wedge a)$ are supermodular. So by (4) we see that F_a and F^a are supermodular. □

Proposition 4.3. *Let* $f:\mathbf{R}^k\to\mathbf{R}$ *be a supermodular function. Let* $D\subseteq\mathbf{R}^k$ *be a given set satisfying* $\cup_{u\in D}[u,*]=\mathbf{R}^k=\cup_{u\in D}[*,u]$*. Let* $\mathcal{A}_1,\dots,\mathcal{A}_k$ *be* σ*-algebras* *on* $\mathbf{R}$*, let* μ_i *be a finite measure on* $(\mathbf{R}, \mathcal{A}_i)$ *for* $i \in [k]$ *and let* $\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k$ *denote the product* σ*-algebra on* $\mathbf{R}^k$*. Let* $\tau_1, \ldots, \tau_k$ *be topologies on* $\mathbf{R}$ *and let* $\tau = \tau_1 \times \cdots \times \tau_k$ *denote the product topology on* $\mathbf{R}^k$*. If* $c = (c_i, \ldots, c_k)$ *is a given vector such that* c_i *admits a bounded* τ_i*-neighborhood for all* $i \in [k]$*, then we have*

(1) *If* f_i^u *is* $\mathcal{A}_i$*-measurable* $\forall\, u \in D\ \forall\, i \in [k]$*, then* f *is* $\mathcal{A}$*-measurable.*

(2) *If* f_i^u *is* τ_i*-continuous at* c_i $\ \forall\, u \in D\ \forall\, i \in [k]$*, then* f *is* τ*-continuous at* $c = (c_1, \ldots, c_k)$*.*

(3) *If* $f_i^u \in L^1(\mu_i)\ \forall\, u \in D\ \forall\, i \in [k]$*, then* $f_i^x \in L^1(\mu_i)\ \forall\, x \in \mathbf{R}^k\ \forall\, i \in [k]$*.*

Proof. Since $\mathbf{R}^k = \cup_{a \in D}\, [a, *]$ there exists $a^n = (a_1^n, \ldots, a_k^n) \in D$ for $n \geq 1$ such that $a_i^{n+1} < a_i^n \leq -n$ for all $n \in \mathbf{N}$ and all $i \in [k]$ and since $\mathbf{R}^k = \cup_{a \in D}\, [*, a]$ there exists $b^n = (b_1^n, \ldots, b_k^n) \in D$ for $n \geq 1$ such that $n \leq b_i^n < b_i^{n+1}$ for all $n \in \mathbf{N}$ and all $i \in [k]$. Set $C_n = [a^n, b^n]$ for all $n \geq 1$. Then we have $C_n \uparrow \mathbf{R}^k$ and by Proposition 4.2.(3) we have

(i) $|f(x) - f(y)| \leq \sum_{i=1}^k |f_i^{a^n}(x_i) - f_i^{a^n}(y_i)| + \sum_{i=1}^k |f_i^{b^n}(x_i) - f_i^{b^n}(y_i)|$

for all $x, y \in C_n$. Suppose that f_i^u is $\mathcal{A}_i$-measurable for all $u \in D$ and all $i \in [k]$. Since $f_i^{a^n}$ and $f_i^{b^n}$ are $\mathcal{A}_i$-measurable for all $n \in \mathbf{N}$ and all $i \in [k]$, it follows easily that f is $(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k)$-measurable.

(2): Suppose that f_i^u is continuous at c_i for all $u \in D$ and all $i \in [k]$. By assumption, we have that c_i admits a bounded τ_i-neighborhood G_i for $i = 1, \ldots, k$. Since $a_i^n \leq -n < 0 < n \leq b_i^n$, there exists $q \in \mathbf{N}$ such that $G_i \subseteq [a_i^q, b_i^q]$ for all $i \in [k]$. Then $G := G_1 \times \cdots \times G_k$ is a τ-neighborhood of c such that $G \subseteq [a^q, b^q]$ and $f_i^{a^q}$ and $f_i^{b^q}$ are τ_i-continuous at c_i for all $i \in [k]$. So by (i) we see that f is τ-continuous at c.

(3): Suppose that $f_i^u \in L^1(\mu_i)$ for all $u \in D$ and all $i \in [k]$, Let $x \in \mathbf{R}^k$ and $i \in [k]$ be given. Then there exists $n \geq 1$ such that $x \in [a^n, b^n]$. So by Proposition 4.2 (1) we have

$$|f_i^x(t) - f_i^x(0)| \leq |f_i^{a^n}(t) - f_i^{a^n}(0)| + |f_i^{b^n}(t) - f_i^{b^n}(0)| \quad \forall t \in \mathbf{R}$$

and by (1), we see that f_i^x is μ_i-measurable. Since $(\mathbf{R}^k, \mathcal{A}_i, \mu_i)$ is a finite measure space and $f_i^{a^n}$ and f^{b^n} belong to $L^1(\mu_i)$, we see that $f_i^x \in L^1(\mu_i)$. □

Theorem 4.4. *Let* $f : \mathbf{R}^k \to \mathbf{R}$ *be a given function and let* $i, j \in [k]$ *be given integers such that* f *is continuous in the* ith *coordinate. Then we have*

(1) *If* $\lambda_1(t \in \mathbf{R} \mid \overline{D}_i f(x + te_i) < 0) = 0$ *and* $\{t \in \mathbf{R} \mid \overline{D}_i f(x + te_i) = -\infty\}$ *is at most countable for all* $x \in \mathbf{R}^k$*, then* f *is increasing in the* ith *coordinate.*

Let $D_i^\diamond$ *denote one of the six Dini operators* $\overline{D}_i^r$, $\underline{D}_i^r$, $\overline{D}_i^\ell$, $\underline{D}_i^\ell$, $\overline{D}_i$ *or* $\underline{D}_i$*. and let us define*

$$I_{i,j}^x := \{t \in \mathbf{R} \mid D_i^\diamond f(x + te_j) > 0\}\ ,\ \ J_{i,j}^x := \{t \in \mathbf{R} \mid D_i^\diamond f(x + te_j) < 0\}$$

for all $x \in \mathbf{R}^k$. Suppose that $D_i^\diamond f(x)$ is finite for all $x \in \mathbf{R}^k$. If $J \subseteq \mathbf{R}$ is an interval containing $f(\mathbf{R}^k)$ and $\xi : J \to \mathbf{R}$ is an increasing convex function, then we have

(2) $\Delta_{ij} f \geq 0 \Leftrightarrow D_i^\diamond f(x)$ *is increasing in the jth coordinate.*

(3) *If $\Delta_{ij} f \geq 0$ and $t \curvearrowright f(x+te_j)$ is increasing on $I_{i,j}^x$ and decreasing on $J_{i,j}^x$ for all $x \in \mathbf{R}^k$, then $\Delta_{ij}(\xi \circ f) \geq 0$.*

(4) *If $\Delta_{ij} f \leq 0$ and $t \curvearrowright f(x+te_j)$ is decreasing on $I_{i,j}^x$ and increasing on $J_{i,j}^x$ for all $x \in \mathbf{R}^k$, then $\Delta_{ij}(\xi \circ f) \leq 0$.*

Proof. (1) follows from Theorem VI.7.3 p. 204 in [15]. So let $D_i^\diamond$ be one of the six Dini derivatives and suppose that $D_i^\diamond f(x)$ is finite for all $x \in \mathbf{R}^k$.

(2): Suppose that $\Delta_{ij} f \geq 0$ and let $x \in \mathbf{R}^k$ and $s > 0$ be given. By (4.2) we have that $t \curvearrowright \Delta_i^s f(x+te_j)$ is increasing on $\mathbf{R}$ and so we see that $D_i^\diamond f(x)$ is increasing in the jth coordinate. Conversely, suppose that $D_i^\diamond f(x)$ is increasing in the jth coordinate. Let $u > 0$ and $x \in \mathbf{R}^k$ be given and set $g(t) := \Delta_j^u f(x+te_i) = f(x+ue_j+te_i) - f(x+te_i)$ for all $t \in \mathbf{R}$. Since f is continuous in the ith coordinate, we have that g is continuous on $\mathbf{R}$ and since $D_i^\diamond f(x)$ is finite and increasing in the jth coordinate, it follows easily that we have

$$\overline{D} g(t) \geq D_i^\diamond f(x+ue_j+te_i) - D_i^\diamond f(x+te_i) \geq 0 \quad \forall t \in \mathbf{R}.$$

Hence, by Theorem VI.7.3 p. 204 in [15] we see that g is increasing; that is, $\Delta_i^u f$ is increasing in the jth coordinate and so by (4.2) we have $\Delta_{ij} f \geq 0$.

Let $\phi : \mathbf{R} \to \mathbf{R}$ be a continuously differentiable, increasing, convex function and set $h = \phi \circ f$. Let $x \in \mathbf{R}^k$ be given. Then we have

$$\Delta_i^s h(x) = \Delta_i^s f(x) \cdot \int_0^1 \phi'(f(x) + t \cdot \Delta_i^s f(x))\, dt \quad \forall s \in \mathbf{R}$$

and since f is continuous in the ith coordinate and ϕ' is increasing, nonnegative and continuous we have

(i) $D_i^\diamond h(x+te_j) = D_i^\diamond f(x+te_j)\phi'(f(x+te_j)) \quad \forall t \in \mathbf{R}$.

Suppose that $\Delta_{ij} f \geq 0$ and that $t \curvearrowright f(x+te_j)$ is increasing on $I_{i,j}^x$ and decreasing on $J_{i,j}^x$ for all $x \in \mathbf{R}^k$. By (2), we have that $t \curvearrowright D_i^\diamond f(x+te_j)$ is increasing on $\mathbf{R}$. Let $x \in \mathbf{R}^k$ be a given vector, let $s < t$ be given numbers and let me show that $D_i^\diamond h(x+se_j) \leq D_i^\diamond h(x+te_j)$.

Suppose that $s \in I_{i,j}^x$. Then we have $0 < D_i^\diamond f(x+se_j) \leq D_i^\diamond f(x+te_j)$ and so we have $s, t \in I_{i,j}^x$ and $f(x+se_j) \leq f(x+te_j)$. Since ϕ' is increasing and nonnegative, we have $0 \leq \phi'(f(x+se_j)) \leq \phi'(f(x+te_j))$. So by (i) we have $D_i^\diamond h(x+se_j) \leq D_i^\diamond h(x+te_j)$.

Suppose that $t \in J_{i,j}^x$. Then we have $D_i^\diamond f(x+se_j) \leq D_i^\diamond f(x+te_j) < 0$ and so we have $s, t \in J_{i,j}^x$ and $f(x+te_j) \leq f(x+se_j)$. Since ϕ' is increasing and nonnegative, we have $0 \leq \phi'(f(x+te_j)) \leq \phi'(f(x+se_j))$. So by (i) we have $D_i^\diamond h(x+se_j) \leq D_i^\diamond h(x+te_j)$.

Suppose that $s \notin I^x_{i,j}$ and $t \notin J^x_{i,j}$. Then we have $D^\diamond_i f(x+se_j) \le 0 \le D^\diamond_i f(x+te_j)$ and so by (i) and nonnegativity of ϕ', we have $D^\diamond_i h(x+se_j) \le 0 \le D^\diamond_i h(x+te_j)$.

Hence, in all cases we have $D^\diamond_i h(x+se_j) \le D^\diamond_i h(x+te_j)$ and so we see that $D^\diamond_i h(x)$ is increasing in the jth coordinate. So by (2), we have $\Delta_{ij}(\phi \circ f) \ge 0$. Since $\xi : J \to \mathbf{R}$ is increasing and convex, there exist continuously differentiable, increasing convex functions $\xi_1, \xi_2, \ldots : \mathbf{R} \to \mathbf{R}$ such that $\xi_m(t) \to \xi(t)$ for all $t \in J$. By the argument above, we have $\Delta_{ij}(\xi_m \circ f) \ge 0$ for all $m \ge 1$ and since $\xi_m(f(x)) \to \xi(f(x))$, we see that $\Delta_{ij}(\xi \circ f) \ge 0$. Thus (3) is proved and (4) follows in the same manner. □

Corollary 4.5. *Let $\phi_1, \ldots, \phi_k : \mathbf{R} \to \mathbf{R}$ be given functions which are either all increasing or all decreasing and let us define $\phi(x) = (\phi_1(x_1), \ldots, \phi_k(x_k))$ and*

$$M_k(x) = \max_{i \in [k]} x_i \ , \ m_k(x) = \min_{i \in [k]} x_i \ , \ Q_k(x) = \max_{i,j \in [k]} |x_i - x_j|$$

for all $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$. Let $\psi : \mathbf{R} \to \mathbf{R}$ be an increasing function and let $\xi : [0,\infty) \to \mathbf{R}$ be an increasing convex function. Then we have

(1) *$M_k(x)$ and $G(x) := \psi(M_k(\phi(x)))$ are submodular.*

(2) *$m_k(x)$ and $F(x) := \psi(m_k(\phi(x)))$ are supermodular.*

(3) *$Q_k(x)$ and $H(x) := \xi(Q_k(\phi(x)))$ are submodular.*

Proof. Let $x, y \in \mathbf{R}^k$ be given. Then we have $M_k(x \vee y) = M_k(x) \vee M_k(y)$ and since M_k is increasing, we have $M_k(x \wedge y) \le M_k(x) \wedge M_k(y)$.

Hence we have

$$M_k(x \vee y) + M_k(x \wedge y) \le M_k(x) \vee M_k(y) + M_k(x) \wedge M_k(y) = M_k(x) + M_k(y).$$

Hence, we see that (1) follows from Proposition 4.1 and since $m_k(x) = -M_k(-x)$, we see that (2) follows from (1) and Proposition 4.1. Since $Q_k(x) = M_k(x) - m_k(x)$, we see that Q_k is submodular. Let $1 \le i \ne j \le k$ be given and set $\pi_{ij} = [k] \setminus \{i,j\}$. Let $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$ be given and set $M_{ij} = \max_{\nu \in \pi_{ij}} x_\nu$ and $m_{ij} = \min_{\nu \in \pi_{ij}} x_\nu$ with the usual conventions $\sup \emptyset := -\infty$ and $\inf \emptyset := +\infty$. Then we have $\overline{D}^r_i Q_k(x) = 1$ if $x_i \ge x_j \vee M_{ij}$, $\overline{D}^r_i Q_k(x) = 0$ if $x_j \wedge m_{ij} \le x_i < x_j \vee M_{ij}$ and $\overline{D}^r_i Q_k(x) = -1$ if $x_i < x_j \wedge m_{ij}$. So by Theorem 4.3.(4), we have $\Delta_{ij}(\xi \circ Q_k) \le 0$. Hence, we see that (3) follows from (4.3) and Proposition 4.1. □

Lemma 4.6. *Let $D \subseteq \mathbf{R}$ be a countable set and let ℓ_D denote the topology on $\mathbf{R}$ generated by $\{(a,b] \mid a, b \in \mathbf{R}\} \cup \{G \mid G \subseteq D\}$. Let $\Theta(\mathbf{R}^k)$ denote the set of all $\theta : \mathbf{R}^k \to \mathbf{R}^k$ of the form $\theta(x_1, \ldots, x_k) = (\theta_1(x_1), \ldots, \theta_k(x_k))$ for some increasing, right continuous step functions $\theta_1, \ldots, \theta_k : \mathbf{R} \to \mathbf{R}$. Let $\Phi \subseteq \mathbf{R}^{\mathbf{R}^k}$ be a non-empty set and let $f : \mathbf{R}^k \to \mathbf{R}$ be a given function satisfying*

(1) *Φ is sequentially closed and $f \circ \theta \in \Phi \quad \forall \theta \in \Theta(\mathbf{R}^k)$.*

(2) *f is ℓ_D-continuous in the ith coordinate for all $i = 1, \ldots, k$.*

Then we have $f \in \Phi \cap M(\mathbf{R}^k, \mathcal{B}^k)$. More precisely, $f \in \Phi$ and f is of Baire class $k+1$. Let $F, H \in M(\mathbf{R}^k, \mathcal{B}^k)$ be given functions satisfying

(3) $\exists \delta > 0$ *so that* $F(x) \le f(x-y) \le H(x) \quad \forall x \in \mathbf{R}^k \ \forall y \in [0,\delta]^k$

and let $\mathfrak{F}$ *denote the set of all locally bounded, right continuous functions* $h : \mathbf{R}^k \to \mathbf{R}$ *such that* $F(x) \le h(x) \le H(x)$ *for all* $x \in \mathbf{R}^k$*. Let* μ *and* ν *be Borel measures on* $\mathbf{R}^k$ *satisfying*

(4) $F \in L^1(\mu)$, $H \in L^1(\nu)$ *and* $\int^* \phi \, d\mu \le \int^* \phi \, d\nu \quad \forall \phi \in \Phi \cap \mathfrak{F}$.

Then we have $f \in L^1(\mu)$ *and* $\int_{\mathbf{R}^k} f \, d\mu \le \int^* f \, d\nu < \infty$.

Remark. (a): Note that $g : \mathbf{R} \to \mathbf{R}$ is ℓ_D-continuous if and only if g is left continuous at x for all $x \in \mathbf{R} \setminus D$. Recall that $g : \mathbf{R} \to \mathbf{R}$ is a *right continuous step function* if and only if there exist numbers $(c_i \mid i \in \mathbf{Z})$ such that $c_i < c_{i+1}$ for all $i \in \mathbf{Z}$, $\sup_{i \in \mathbf{Z}} c_i = \infty$, $\inf_{i \in \mathbf{Z}} c_i = -\infty$ and $g(t) = g(c_i)$ for all $t \in [c_i, c_{i+1})$ and all $i \in \mathbf{Z}$.

(b): Let T be a topological space and let $h : T \to \mathbf{R}$ be a function. Recall that h is of *Baire class* 0 if h is continuous and that h is of *Baire class* α for some ordinal $\alpha > 0$ if and only if h is a pointwise limit of a sequence of functions of Baire class $< \alpha$.

Proof. Let $E_1 \subseteq E_2 \subseteq \cdots \subseteq \mathbf{R}$ be an increasing sequence of finite sets such that $E_n \uparrow D$ and set $D_n = E_n \cup \{i\, 2^{-n} \mid i \in \mathbf{Z}\}$ and $\theta_n(t) = \sup(D_n \cap (-\infty, t])$ for all $n \ge 1$ and all $t \in \mathbf{R}$. Then θ_n is an increasing, right continuous, step function and we have $t - 2^{-n} \le \theta_n(t) \le \theta_{n+1}(t) \le t$ for all $n \ge 1$ and all $t \in \mathbf{R}$. Let $\sigma_1, \ldots, \sigma_k \in \mathbf{N}$ and $x = (x_1, \ldots, x_h) \in \mathbf{R}^k$ be given and set $f^k_{\sigma_1,\ldots,\sigma_k}(x) = f(\theta_{\sigma_1}(x_1), \ldots, \theta_{\sigma_k}(x_k))$ and

$$f^i_{\sigma_1,\ldots,\sigma_i}(x) = f(\theta_{\sigma_1}(x_1), \ldots, \theta_{\sigma_i}(x_i), x_{i+1}, \ldots, x_k) \quad \text{for } 1 \le i < k.$$

Note that $\theta_n(t) = t$ for all $t \in E_m$ and all $n \ge m$. Since $E_n \uparrow D$, we see that $\theta_n(t) \to t$ in ℓ_D and so by (2) we have

$$f^{i-1}_{\sigma_1,\ldots,\sigma_{i-1}}(x) = \lim_{\sigma_i \to \infty} f^i_{\sigma_1,\ldots,\sigma_i}(x) \ \ \forall 1 < i \le k \ , \ f(x) = \lim_{\sigma_1 \to \infty} f^1_{\sigma_1}(x).$$

By (1), we have $f^k_{\sigma_1,\ldots,\sigma_k} \in \Phi$ and since Φ is sequentially closed we see that $f \in \Phi$. Since θ_n is a right continuous step function, we see that $f^k_{\sigma_1,\ldots,\sigma_k}$ is a right continuous, locally bounded Borel function on $\mathbf{R}^k$. So by Lemma 2.2 we see that $f^k_{\sigma_1,\ldots,\sigma_k}$ is of Baire class 1. Hence, we see that f is of Baire class $k+1$.

Let μ and ν be Borel measures on $\mathbf{R}^k$ and let $F \in L^1(\mu)$ and $H \in L^1(\nu)$ be given functions satisfying (3)–(4). Let $\delta > 0$ be chosen according to (3) and let $q \in \mathbf{N}$ be chosen such that $2^{-q} < \delta$. Let Λ denote the set of all functions $h : \mathbf{R}^k \to \mathbf{R}$ satisfying $\int^* h \, d\mu \le \int^* h \, d\nu$ and set $\Psi = \{\psi \in \Phi \mid F \le \phi \le H\}$. By (1), we see that Ψ is sequentially closed and since $F \in L^1(\mu)$ and $H \in L^1(\nu)$, we see that $\{\psi^- \mid \psi \in \Psi\}$ is uniformly μ-integrable and that $\{\psi^+ \mid \psi \in \Psi\}$ is uniformly ν-integrable. By Theorem 3.3.(9), we have that $\Lambda \cap \Psi \cap M(\mathbf{R}^k, \mathcal{B}^k)$ is sequentially closed. Let $\sigma_1, \ldots, \sigma_k \ge q$ be given integers. Since θ_n is a right continuous, step function satisfying $t - \delta \le \theta_n(t) \le t$ for all $n \ge q$ and all $t \in \mathbf{R}$, we have $f^k_{\sigma_1,\ldots,\sigma_k} \in \Phi \cap \mathfrak{F} \cap M(\mathbf{R}^k, \mathcal{B}^k)$. So by (3)–(4), we have $f^k_{\sigma_1,\ldots,\sigma_k} \in \Lambda \cap \Psi \cap M(\mathbf{R}^k, \mathcal{B}^k)$

for all $\sigma_1, \dots, \sigma_k \geq q$ and since $\Lambda \cap \Psi \cap M(\mathbf{R}^k, \mathcal{B}^k)$ is sequentially closed, we see that $f \in \Lambda \cap \Psi \cap M(\mathbf{R}^k, \mathcal{B}^k)$. Hence, we have

$$-\infty < \int_{\mathbf{R}^k} F \, d\mu \leq \int_* f \, d\mu \leq \int^* f \, d\mu \leq \int^* f \, d\nu \leq \int_{\mathbf{R}^k} H \, d\nu < \infty.$$

So by Theorem 2.1.(8) in [4] we have $f \in L^1(\mu)$ and $\int_{\mathbf{R}^k} f \, d\mu \leq \int^* f \, d\nu < \infty$ □

Theorem 4.7. *Let μ and ν be Borel probability measures on $\mathbf{R}^k$ with one-dimensional marginals $\mu_1, \dots, \mu_k$ and $\nu_1, \dots, \nu_k$, respectively. Then we have*

(1) $\mu \preceq_{\text{bm}} \nu \Leftrightarrow \mu_i = \nu_i$ *for* $i = 1, \dots, k$.

(2) $\int_{\mathbf{R}^k} f \, d\mu \leq \int_{\mathbf{R}^k} f \, d\nu \; \forall f \in C_b^\infty(\mathbf{R}^k) \cap \text{ism}(\mathbf{R}^k) \;\; \Leftrightarrow \mu \preceq_{\text{ism}} \nu$.

(3) $\int_{\mathbf{R}^k} f \, d\mu \leq \int_{\mathbf{R}^k} f \, d\nu \; \forall f \in C_b^\infty(\mathbf{R}^k) \cap \text{m}(\mathbf{R}^k) \;\; \Leftrightarrow \mu \preceq_{\text{bm}} \nu$.

(4) $\mu \preceq_{\text{bsm}} \nu \Leftrightarrow \mu \preceq_{\text{ism}} \nu$ *and* $\mu \preceq_{\text{bm}} \nu$.

(5) $\int_{\mathbf{R}^k} f \, d\mu \leq \int_{\mathbf{R}^k} f \, d\nu \; \forall f \in C_b^\infty(\mathbf{R}^k) \cap \text{sm}(\mathbf{R}^k) \;\; \Leftrightarrow \mu \preceq_{\text{bsm}} \nu$.

Proof. Throughout the proof we let Λ denote the set of all functions $f : \mathbf{R}^k \to \mathbf{R}$ satisfying $\int^* f \, d\mu \leq \int^* f \, d\nu$. (1) follows easily from Proposition 4.2.(4).

(2): Suppose that $C_b^\infty(\mathbf{R}^k) \cap \text{ism}(\mathbf{R}^k) \subseteq \Lambda$. Let C_b^r denote the set of all bounded, right continuous functions on $\mathbf{R}^k$ and set $\kappa(x) \equiv 1$. Then $(\text{ism}(\mathbf{R}^k), \kappa)$ satisfies conditions (1)–(2) in Theorem 3.4 and so we conclude that $C_b^r \cap \text{ism}(\mathbf{R}^k) \subseteq \Lambda$. Let $f \in B(\mathbf{R}^k) \cap \text{ism}(\mathbf{R}^k)$ be given. Then there exists $c > 0$ such that $|f(x)| \leq c$ for all $x \in \mathbf{R}^k$. Let $A \subseteq \mathbf{R}^k$ be a countable dense subset of $\mathbf{R}^k$. Since f_i^u is increasing, there exists a countable set $D \subseteq \mathbf{R}$ such that $\mathbf{R} \setminus D \subseteq C(f_i^u)$ for all $i \in [k]$ and all $u \in A$. Hence, by Proposition 4.2.(3), we have $\mathbf{R} \setminus D \subseteq C(f_i^u)$ for all $i \in [k]$ and all $u \in \mathbf{R}^k$ and so we see the (f, D) satisfies condition (2) in Lemma 4.6. By Proposition 4.1.(2) we see that $(\text{ism}(\mathbf{R}^k, f)$ satisfies condition (1) in Lemma 4.6. Since $C_b^r \cap \text{ism}(\mathbf{R}^k) \subseteq \Lambda$ we see that f satisfies condition (3)–(4) in Lemma 4.6 with $F(x) \equiv -c$ and $H(x) \equiv c$. Hence, we have $f \in \Lambda$ for all $f \in B(\mathbf{R}^k) \cap \text{ism}(\mathbf{R}^k)$. Let $f \in \text{ism}(\mathbf{R}^k)$ be given. Set $b^m = (m, \dots, m)$ and $f_{m,n}(x) = f(x \wedge b^m) \vee (-n)$ for all $m, n \in \mathbf{N}$ and all $x \in \mathbf{R}^k$. Then $f_{m,n}$ is increasing and bounded and by Proposition 4.1, we have that $f_{m,n}$ is supermodular. Hence, we have $f_{m,n} \in \Lambda$. Since f is increasing, we see that $-n \leq f_{1,n} \leq f_{2.n} \leq \cdots$ and $\sup_{m \geq 1} f_{m,n}(x) = f(x) \vee (-n)$ for all $x \in \mathbf{R}^k$. So by Theorem 3.3.(8) we see that $f(x) \vee (-n) \in \Lambda$ for all $n \in \mathbf{N}$ and so by Theorem 3.3.(2) we have $f \in \Lambda$; that is, $\mu \preceq_{\text{ism}} \nu$ which completes the proof of (2).

(3): Suppose that $C_b^\infty(\mathbf{R}^k) \cap \text{m}(\mathbf{R}^k) \subseteq \Lambda$ and let $i \in [k]$ be given. By Proposition 4.2.(4), we have $\int_{\mathbf{R}} \phi \, \mu_i = \int_{\mathbf{R}} \phi \, \nu_i$ for all $\phi \in C_b^\infty(\mathbf{R})$ and so we have $\mu_i = \nu_i$. Hence, we see that (3) follows from (1).

(4): The implication "⇒" in (4) follows directly from (2). Suppose that $\mu \preceq_{\text{ism}} \nu$ and $\mu \preceq_{\text{bm}} \nu$ and let $f \in \text{bsm}(\mathbf{R}^k)$ be given. Set $a^n = (-n, \dots, -n)$ for $n \geq 1$.

By Proposition 4.2.(1) we have that the limits

$$\alpha_i(t) := \lim_{n\to\infty} (f_i^{a^n}(t) - f_i^{a^n}(0))$$

exist and are finite for all $t \in \mathbf{R}$ and all $i \in [k]$. Since f is a bounded Borel function, we see that α_i is bounded Borel function. So by Proposition 4.2.(4), we have that $G(x) := \sum_{i=1}^k \alpha_i(x_i)$ is a bounded, modular, Borel function. By Proposition 4.2.(5), we see that

$$x \curvearrowright f(x \vee a) - \sum_{i=1}^k (f_i^a(x_i \vee a_i) - f_i^a(0))$$

is increasing and supermodular for all $a \in \mathbf{R}^k$. Hence, we see that $F(x) := f(x) - G(x)$ is a bounded, increasing, supermodular, Borel function. Since $\mu \preceq_{\text{ism}} \nu$, we have $F \in \Lambda \cap L^1(\mu) \cap L^1(\nu)$ and since $\mu \preceq_{\text{bm}} \nu$, we have $G \in \Lambda \cap L^1(\mu) \cap L^1(\nu)$. So by Theorem 3.3.(1) we have $f = F + G \in \Lambda$ for all $f \in \text{bsm}(\mathbf{R}^k)$ which completes the proof of (4).

(5): Suppose that $f \in C_b^\infty(\mathbf{R}^k) \cap \text{sm}(\mathbf{R}^k) \subseteq \Lambda$. By (2) and (3), we have $\mu \preceq_{\text{bm}} \nu$ and $\mu \preceq_{\text{ism}} \nu$. So by (4) we have $\mu \preceq_{\text{bsm}} \nu$. □

Theorem 4.8. *Let μ and ν be Borel probability measures on $\mathbf{R}^k$ such that $\mu \preceq_{\text{bsm}} \nu$ and let $\mu_1, \ldots, \mu_k$ denote the one-dimensional marginals of μ. Let $f : \mathbf{R}^k \to \mathbf{R}$ be a supermodular Borel function and let us define $f_{\vee c}(x) = f(x \vee c)$ and $f_{\wedge c}(x) = f(x \wedge c)$ for all $c, x \in \mathbf{R}^k$. If $c \in \mathbf{R}^k$ is a given vector, then we have*

(1) *If f is either increasing or decreasing, then $\int^* f\,d\mu \le \int^* f\,d\nu$.*

(2) *If $f_i^c \in L^1(\mu_i)$ for all $i \in [k]$, then we have $\int^* f_{\vee c} d\mu \le \int^* f_{\vee c}\,d\nu$ and $\int^* f_{\wedge c} d\mu \le \int^* f_{\wedge c}\,d\nu$.*

Let $A, B \subseteq \mathbf{R}^k$ be given sets satisfying $\cup_{a\in A}[a, *] = \mathbf{R}^k = \cup_{b\in B}[*, b]$. Then we have $\int^* f\,d\mu \le \int^* f\,d\nu$ if just one of the following three conditions holds:

(A) *$f_i^a \in L^1(\mu_i)\ \forall i \in [k]\ \forall a \in A$, $\{f_{\vee a}^- \mid a \in A\}$ is uniformly μ-integrable and $\{f_{\vee a}^+ \mid a \in A\}$ is uniformly ν-integrable.*

(B) *$f_i^b \in L^1(\mu_i)\ \forall i \in [k]\ \forall b \in B$, $\{f_{\wedge b}^- \mid b \in B\}$ is uniformly μ-integrable and $\{f_{\wedge b}^+ \mid b \in B\}$ is uniformly ν-integrable.*

(C) *There exist functions $h_1 \in L_+^1(\mu_1), \ldots, h_k \in L_+^1(\mu_k)$ such that $|f(x)| \le \sum_{i=1}^k h_i(x_i)$ for all $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$.*

Proof. Throughout the proof, we let Λ denote the set of all functions $h : \mathbf{R}^k \to \mathbf{R}$ satisfying $\int^* h\,d\mu \le \int^* h\,d\nu$. We set $\tilde{f}(x) = f(-x)$ for all $x \in \mathbf{R}^k$ and we set $\tilde{\mu}(B) = \mu(-B)$ and $\tilde{\nu}(B) - \nu(-B)$ for all $B \in \mathcal{B}^k$. Then $\tilde{\mu}$ and $\tilde{\nu}$ are Borel probability measures on $\mathbf{R}^k$ such that $\int \tilde{f}\,d\tilde{\mu} = \int f\,d\mu$ and $\int \tilde{f}\,d\tilde{\nu} = \int f\,d\nu$. By Proposition 4.1, we see that $\tilde{f}$ is supermodular and that $\tilde{\mu} \preceq_{\text{bsm}} \tilde{\nu}$.

(1): Since $\mu \preceq_{\text{bsm}} \nu$, we have $\text{bsm}(\mathbf{R}^k) \subseteq \Lambda$ and by Theorem 4.7, we have $\text{ism}(\mathbf{R}^k) \subseteq \Lambda$ and $\mu_i = \nu_i$ for $i = 1, \ldots, k$ where $\nu_1, \ldots, \nu_k$ are the one-dimensional marginals of ν. Hence, we have $f \in \Lambda$ if f is increasing. Applying this to the triple $(\tilde{f}, \tilde{\mu}, \tilde{\nu})$, we see that $f \in \Lambda$ if f is decreasing.

(2): Suppose that $f_i^c \in L^1(\mu_i)$ for all $i = 1, \ldots, k$ and set $G(x) = \sum_{i=1}^k f_i^c(x_i \vee c_i)$ for all $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$. By Proposition 4.2.(4), we have that G is a modular, Borel function and since μ_i is finite, we have that $f_i^c(t \vee c_i)$ belongs to $L^1(\mu_i)$. Since $\nu_i = \mu_i$, we have $G \in \Lambda \cap L^1(\mu) \cap L^1(\nu)$ and by Proposition 4.2.(5), we have $f_{\vee c} - G \in \text{ism}(\mathbf{R}^k) \subseteq \Lambda$. So by Theorem 3.3.(1) we have $f_{\vee c} \in \Lambda$. Applying this to triple $(\tilde{f}, \tilde{\mu}, \tilde{\nu})$ with $c := -c$, we see that $f_{\wedge c} \in \Lambda$.

Suppose that condition (A) holds. By (2), we see that $f_{\vee a} \in \Lambda$ for all $a \in A$ and since $\mathbf{R}^k = \cup_{a \in A} [a, *]$, there exists $a^n = (a_1^n, \ldots, a_k^n) \in A$ such that $a_i^{n+1} \leq a_i^n \leq -n$ for all $n \in \mathbf{N}$ and all $i \in [k]$. Then we have $f_{\vee a^n}(x) \to f(x)$ and by (2) and (A), we have $f_{\vee a^n} \in \Lambda$, $\{f_{\vee a^n}^- \mid n \geq 1\}$ is uniformly μ-integrable and $\{f_{\vee a^n}^+ \mid n \geq 1\}$ is uniformly ν-integrable. So by (3.3)–(3.4) and Theorem 3.3.(5) we have $f \in \Lambda$.

Suppose that condition (B) holds. Applying case (A) on the triple $(\tilde{f}, \tilde{\mu}, \tilde{\nu})$ with $A := -B$, we see that $f \in \Lambda$. Suppose that condition (C) holds. Since $h_i \in L_+^1(\mu_i)$, we see that $f_i^x \in L^1(\mu_i)$ for all $x \in \mathbf{R}^k$ and all $i \in [k]$. Let ξ be a finite Borel measure on $\mathbf{R}$. Then I claim that we have

(i) $\displaystyle\liminf_{u \to -\infty} \int_{\mathbf{R}} |h(t) - h(t \vee u)| \xi(dt) = 0 \quad \forall h \in L^1(\xi).$

Proof of (i). Suppose that (i) fails. Then there exist $h \in L^1(\xi)$ and a positive number $\delta > 0$ such that $\liminf_{u \downarrow -\infty} \int_{\mathbf{R}} |h(t) - h(t \vee u)| \, \xi(dt) > 2\delta$. Since $h \in L^1(\xi)$, there exists $q \in \mathbf{R}$ such that

$$\int_{(-\infty, q]} |h| \, d\xi < \delta \text{ and } \int_{\mathbf{R}} |h(t) - h(t \vee u)| \, \xi(dt) > 2 \, \delta \quad \forall u \leq q.$$

Let $u \leq q$ be given and set $F_\xi(u) := \xi((-\infty, u])$. Then we have

$$\begin{aligned} 2\delta &< \int_{\mathbf{R}} |h(t) - h(t \vee u)| \, \xi(dt) \leq \int_{(-\infty, q]} |h(t)| \xi(dt) + |h(u)| F_\xi(u) \\ &\leq \delta + |h(u)| F_\xi(u) \end{aligned}$$

and so we see that $|h(u)| \, F_\xi(u) > \delta$ for all $u \leq q$. Set $m = \inf_{s \leq q} |h(s)|$ and let $s \leq q$ be given. Since F_ξ is increasing. we have $\delta \leq |h(s)| \, F_\xi(s) \leq |h(s)| \, F_\xi(q)$ and so we have $\delta \leq m \, F_\xi(q)$ and

$$\delta \leq m F_\xi(q) = \textstyle\int_{(-\infty, q]} m \, \xi(ds) \leq \displaystyle\int_{(-\infty, q]} |h(s)| \, \xi(ds) < \delta$$

which is impossible. Thus, we see that (i) holds.

Let $i \in [k]$ be given. By (i) there exist numbers $a_i^1 > a_i^2 > \cdots$ such that $a_i^n < -n$ for all $n \geq 1$ and

$$\lim_{n \to \infty} \int_{\mathbf{R}} |h_i(t) - h_i(t \vee a_i^n)| \, \mu_i(dt) = 0 \quad \forall i = 1, \ldots, k.$$

Set $a^n = (a_1^n, \ldots, a_k^n)$ and $H(x) = \sum_{i=1}^k h_i(x_i)$ for $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$. Since $\mu_i = \nu_i$ and

$$|H(x) - H(x \vee a^n)| \le \sum_{i=1}^n |h_i(x_i) - h(x_i \vee a_i^n)|$$

we have

$$\int_{\mathbf{R}^k} |H(x) - H(x \vee a^n)| \, \mu(dx) \to 0 \ , \ \int_{\mathbf{R}^k} |H(x) - H(x \vee a^n)| \, \nu(dx) \to 0.$$

In particular, we see that $\{H_{\vee a^n} \mid n \ge 1\}$ is uniformly μ-integrable and uniformly ν-integrable. Since $|f(x \vee a^n)| \le H(x \vee a^n)$, we see that $\{f_{\vee a^n} \mid n \ge 1\}$ is uniformly μ-integrable and uniformly ν-integrable. So by case (A) we have $f \in \Lambda$ $\square$

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A More General Maximal Bernstein-type Inequality

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Abstract. We extend a general Bernstein-type maximal inequality of Kevei and Mason (2011) for sums of random variables.

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1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of random variables, and for any choice of $1 \leq k \leq l < \infty$ we denote the partial sum $S(k,l) = \sum_{i=k}^{l} X_i$, and define $M(k,l) = \max\{|S(k,k)|, \ldots, |S(k,l)|\}$. It turns out that under a variety of assumptions the partial sums $S(k,l)$ will satisfy a generalized Bernstein-type inequality of the following form: for suitable constants $A > 0$, $a > 0$, $b \geq 0$ and $0 < \gamma < 2$ for all $m \geq 0$, $n \geq 1$ and $t \geq 0$,

$$P\{|S(m+1, m+n)| > t\} \leq A \exp\left\{-\frac{at^2}{n + bt^\gamma}\right\}. \tag{1.1}$$

Kevei and Mason [2] provide numerous examples of sequences of random variables $X_1, X_2, \ldots,$ that satisfy a Bernstein-type inequality of the form (1.1). They show, somewhat unexpectedly, without any additional assumptions, a modified version of it also holds for $M(1+m, n+m)$ for all $m \geq 0$ and $n \geq 1$. Here is their main result.

Theorem 1.1. *Assume that for constants $A > 0$, $a > 0$, $b \geq 0$ and $\gamma \in (0,2)$, inequality* (1.1) *holds for all $m \geq 0, n \geq 1$ and $t \geq 0$. Then for every $0 < c < a$*

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there exists a $C > 0$ *depending only on* A, a, b, c *and* γ *such that for all* $n \geq 1$, $m \geq 0$ *and* $t \geq 0$,

$$P\{M(m+1, m+n) > t\} \leq C \exp\left\{-\frac{ct^2}{n + bt^\gamma}\right\}. \tag{1.2}$$

There exists an interesting class of Bernstein-type inequalities that are not of the form (1.1). Here are two motivating examples.

Example 1. Assume that $X_1, X_2, \ldots$ is a stationary Markov chain satisfying the conditions of Theorem 6 of Adamczak [1] and let f be any bounded measurable function such that $Ef(X_1) = 0$. His theorem implies that for some constants $D > 0$, $d_1 > 0$ and $d_2 > 0$ for all $t \geq 0$ and $n \geq 1$,

$$P\{|S_n(f)| \geq t\} \leq D^{-1} \exp\left(-\frac{Dt^2}{nd_1 + td_2 \log n}\right), \tag{1.3}$$

where $S_n(f) = \sum_{i=1}^n f(X_i)$, and D/d_1 is related to the limiting variance in the central limit theorem.

Example 2. Assume that $X_1, X_2, \ldots$ is a strong mixing sequence with mixing coefficients $\alpha(n)$, $n \geq 1$, satisfying for some $d > 0$, $\alpha(n) \leq \exp(-2dn)$. Also assume that $EX_i = 0$ and for some $M > 0$, $|X_i| \leq M$, for all $i \geq 1$. Theorem 2 of Merlevède, Peligrad and Rio [4] implies that for some constant $D > 0$ for all $t \geq 0$ and $n \geq 1$,

$$P\{|S_n| \geq t\} \leq D \exp\left(-\frac{Dt^2}{nv^2 + M^2 + tM(\log n)^2}\right), \tag{1.4}$$

where $S_n = \sum_{i=1}^n X_i$ and $v^2 = \sup_{i>0}\left(Var(X_i) + 2\sum_{j>i}|cov(X_i, X_j)|\right)$.

The purpose of this note to establish the following extended version of Theorem 1.1 that will show that a maximal version of inequalities (1.3) and (1.4) also holds.

Theorem 1.2. *Assume that there exist constants* $A > 0$ *and* $a > 0$ *and a sequence of non-decreasing non-negative functions* $\{g_n\}_{n\geq 1}$ *on* $(0, \infty)$, *such that for all* $t > 0$ *and* $n \geq 1$, $g_n(t) \leq g_{n+1}(t)$ *and for all* $0 < \rho < 1$

$$\lim_{n\to\infty} \inf\left\{\frac{t^2}{g_n(t)\log t} : g_n(t) > \rho n\right\} = \infty, \tag{1.5}$$

where the infimum of the empty set is defined to be infinity, such that for all $m \geq 0$, $n \geq 1$ *and* $t \geq 0$,

$$P\{|S(m+1, m+n)| > t\} \leq A \exp\left\{-\frac{at^2}{n + g_n(t)}\right\}. \tag{1.6}$$

Then for every $0 < c < a$ *there exists a* $C > 0$ *depending only on* A, a, c *and* $\{g_n\}_{n\geq 1}$ *such that for all* $n \geq 1$, $m \geq 0$ *and* $t \geq 0$,

$$P\{M(m+1, m+n) > t\} \leq C \exp\left\{-\frac{ct^2}{n + g_n(t)}\right\}. \tag{1.7}$$

Note that condition (1.5) trivially holds when the functions g_n are bounded, since the corresponding sets are empty sets. However, in the interesting cases g_n's are not bounded, and in this case the condition basically says that $g_n(t)$ increases slower than t^2.

Essentially the same proof shows that the statement of Theorem 1.2 remains true if in the numerator of (1.6) and (1.7) the function t^2 is replaced by a regularly varying function at infinity $f(t)$ with a positive index. In this case the t^2 in condition (1.5) must be replaced by $f(t)$. Since we do not know any application of a result of this type, we only mention this generalization.

Proof. Choose any $0 < c < a$. We prove our theorem by induction on n. Notice that by the assumption, for any integer $n_0 \geq 1$ we may choose $C > An_0$ to make the statement true for all $1 \leq n \leq n_0$. This remark will be important, because at some steps of the proof we assume that n is large enough. Also since the constants A and a in (1.6) are independent of m, we can without loss of generality assume $m = 0$.

Assume the statement holds up to some $n \geq 2$. (The constant C will be determined in the course of the proof.)

Case 1. Fix a $t > 0$ and assume that

$$g_{n+1}(t) \leq \alpha\, n, \tag{1.8}$$

for some $0 < \alpha < 1$ be specified later. (In any case, we assume that $\alpha n \geq 1$.) Using an idea of [5], we may write for arbitrary $1 \leq k < n$, $0 < q < 1$ and $p + q = 1$ the inequality

$$\begin{aligned} P\{M(1, n+1) > t\} \leq P\{M(1, k) > t\} &+ P\{|S(1, k+1)| > pt\} \\ &+ P\{M(k+2, n+1) > qt\}. \end{aligned}$$

Let

$$u = \frac{n + g_{n+1}(qt) - q^2 g_{n+1}(t)}{1 + q^2}.$$

Note that $u \leq n - 1$ if $0 < \alpha < 1$ is chosen small enough depending on q, for n large enough. Notice that

$$\frac{t^2}{u + g_{n+1}(t)} = \frac{q^2 t^2}{n - u + g_{n+1}(qt)}. \tag{1.9}$$

Set

$$k = \lceil u \rceil\,. \tag{1.10}$$

Using the induction hypothesis and (1.6), keeping in mind that $1 \leq k \leq n - 1$, we obtain

$$\begin{aligned} P\{M(1, n+1) > t\} \leq C \exp\left\{ -\frac{ct^2}{k + g_k(t)} \right\} &+ A \exp\left\{ -\frac{ap^2 t^2}{k + 1 + g_{k+1}(pt)} \right\} \\ &+ C \exp\left\{ -\frac{cq^2 t^2}{n - k + g_{n-k}(qt)} \right\} \end{aligned}$$

$$\leq C \exp\left\{-\frac{ct^2}{k+g_{n+1}(t)}\right\} + A \exp\left\{-\frac{ap^2t^2}{k+1+g_{n+1}(pt)}\right\}$$
$$+ C \exp\left\{-\frac{cq^2t^2}{n-k+g_{n+1}(qt)}\right\}. \tag{1.11}$$

Notice that we chose k to make the first and third terms in (1.11) almost equal, and since by (1.10)

$$\frac{t^2}{k+g_{n+1}(t)} \leq \frac{q^2t^2}{n-k+g_{n+1}(qt)}$$

the first term is greater than or equal to the third.

First we handle the second term in formula (1.11), showing that whenever $g_{n+1}(t) \leq \alpha n$,

$$\exp\left\{-\frac{ap^2t^2}{k+1+g_{n+1}(pt)}\right\} \leq \exp\left\{-\frac{ct^2}{n+1+g_{n+1}(t)}\right\}.$$

For this we need to verify that for $g_{n+1}(t) \leq \alpha n$,

$$\frac{ap^2}{k+1+g_{n+1}(pt)} > \frac{c}{n+1+g_{n+1}(t)}, \tag{1.12}$$

which is equivalent to

$$ap^2(n+1+g_{n+1}(t)) > c(k+1+g_{n+1}(pt)).$$

Using that

$$k = \lceil u \rceil \leq u+1 = 1 + \frac{1}{1+q^2}\left[n + g_{n+1}(qt) - q^2 g_{n+1}(t)\right],$$

it is enough to show

$$n\left(ap^2 - \frac{c}{1+q^2}\right) + ap^2 - 2c$$
$$+ \left[g_{n+1}(t)ap^2 - g_{n+1}(pt)c - \frac{c}{1+q^2}\left(g_{n+1}(qt) - q^2 g_{n+1}(t)\right)\right] > 0.$$

Note that if the coefficient of n is positive, then we can choose α in (1.8) small enough to make the above inequality hold. So in order to guarantee (1.12) (at least for large n) we only have to choose the parameter p so that $ap^2 - c > 0$, which implies that

$$ap^2 - \frac{c}{1+q^2} > 0 \tag{1.13}$$

holds, and then select α small enough, keeping mind that we assume $\alpha n \geq 1$ and $k \leq n-1$.

Next we treat the first and third terms in (1.11). Because of the remark above, it is enough to handle the first term. Let us examine the ratio of

$$C \exp\left\{\frac{-ct^2}{k+g_{n+1}(t)}\right\} \quad \text{and} \quad C \exp\left\{\frac{-ct^2}{n+1+g_{n+1}(t)}\right\}.$$

Notice again that since $u+1 \geq k$, the monotonicity of $g_{n+1}(t)$ and $g_{n+1}(t) \leq \alpha n$ implies

$$\begin{aligned} n+1-k \geq n-u &= n - \frac{n+g_{n+1}(qt)-q^2 g_{n+1}(t)}{1+q^2} \\ &\geq \frac{q^2 n - (1-q^2) g_{n+1}(t)}{1+q^2} \\ &\geq n \frac{q^2 - \alpha(1-q^2)}{1+q^2} \\ &=: c_1 n. \end{aligned}$$

At this point we need that $0 < c_1 < 1$. Thus we choose α small enough so that

$$q^2 - \alpha(1-q^2) > 0. \tag{1.14}$$

Also we get using $g_{n+1}(t) \leq \alpha n$ the bound

$$(n+1+g_{n+1}(t))(k+g_{n+1}(t)) \leq 2n^2(1+\alpha)^2 =: c_2 n^2,$$

which holds if n large enough. Therefore, we obtain for the ratio

$$\exp\left\{-ct^2\left(\frac{1}{k+g_{n+1}(t)} - \frac{1}{n+1+g_{n+1}(t)}\right)\right\} \leq \exp\left\{-\frac{cc_1 t^2}{c_2 n}\right\} \leq \mathrm{e}^{-1},$$

whenever $cc_1 t^2/(c_2 n) \geq 1$, that is $t \geq \sqrt{c_2 n/(cc_1)}$. Substituting back into (1.11), for $t \geq \sqrt{c_2 n/(cc_1)}$ and $g_{n+1}(t) \leq \alpha n$ we obtain

$$P\{M(1,n+1) > t\}$$

$$\leq \left(\frac{2}{\mathrm{e}}C + A\right) \exp\{-ct^2/(n+1+g_{n+1}(t))\} \leq C \exp\{-ct^2/(n+1+g_{n+1}(t))\},$$

where the last inequality holds for $C > A\mathrm{e}/(\mathrm{e}-2)$.

Next assume that $t < \sqrt{c_2 n/(cc_1)}$. In this case choosing C large enough we can make the bound > 1, namely

$$C \exp\left\{-\frac{ct^2}{n+1+g_{n+1}(t)}\right\} \geq C \exp\left\{-\frac{cc_2 n}{cc_1 n}\right\} = C\mathrm{e}^{-c_2/c_1} \geq 1,$$

if $C > \mathrm{e}^{c_2/c_1}$.

Case 2. Now we must handle the case $g_{n+1}(t) > \alpha n$. Here we apply the inequality

$$P\{M(1,n+1) > t\} \leq P\{M(1,n) > t\} + P\{|S(1,n+1)| > t\}.$$

Using assumption (1.6) and the induction hypothesis, we have

$$\begin{aligned} P\{M(1,n+1) > t\} &\leq C \exp\left\{-\frac{ct^2}{n+g_n(t)}\right\} + A \exp\left\{-\frac{at^2}{n+1+g_{n+1}(t)}\right\} \\ &\leq C \exp\left\{-\frac{ct^2}{n+g_{n+1}(t)}\right\} + A \exp\left\{-\frac{at^2}{n+1+g_{n+1}(t)}\right\}. \end{aligned}$$

We will show that the right side $\leq C \exp\{-ct^2/(n+1+g_{n+1}(t))\}$. For this it is enough to prove

$$\exp\left\{-ct^2\left(\frac{1}{n+g_{n+1}(t)}-\frac{1}{n+1+g_{n+1}(t)}\right)\right\} + \frac{A}{C}\exp\left\{-\frac{t^2(a-c)}{n+1+g_{n+1}(t)}\right\} \leq 1. \tag{1.15}$$

Using the bound following from $g_{n+1}(t) > \alpha n$ and recalling that $\alpha n \geq 1$ and $0 < \alpha < 1$, we get

$$\frac{t^2}{(n+g_{n+1}(t))(n+1+g_{n+1}(t))} \geq \frac{\alpha^2 t^2}{(1+\alpha)(1+2\alpha)g_{n+1}(t)^2} =: c_3 \frac{t^2}{g_{n+1}(t)^2},$$

and

$$\frac{t^2(a-c)}{n+1+g_{n+1}(t)} \geq \frac{t^2}{g_{n+1}(t)}\frac{\alpha(a-c)}{1+2\alpha} =: \frac{t^2}{g_{n+1}(t)}c_4.$$

Choose $\delta > 0$ so small such that $0 < x \leq \delta$ implies $\mathrm{e}^{-cc_3x^2} \leq 1 - \frac{cc_3}{2}x^2$. For $t/g_{n+1}(t) \geq \delta$ the left-hand side of (1.15) is less then

$$\mathrm{e}^{-cc_3\delta^2} + \frac{A}{C},$$

which is less than 1, for C large enough.

For $t/g_{n+1}(t) \leq \delta$ by the choice of δ the left-hand side of (1.15) is less then

$$1 - \frac{cc_3}{2}\frac{t^2}{g_{n+1}(t)^2} + \frac{A}{C}\exp\left\{-\frac{t^2}{g_{n+1}(t)}c_4\right\},$$

which is less than 1 if

$$\frac{cc_3}{2}\frac{t^2}{g_{n+1}(t)^2} > \frac{A}{C}\exp\left\{-\frac{t^2}{g_{n+1}(t)}c_4\right\}.$$

By (1.5), for any $0 < \eta < 1$ and all large enough n, $g_{n+1}(t)1\{g_{n+1}(t) > \alpha n\} \leq \eta t^2$, so that for all large n, whenever $g_{n+1}(t) > \alpha n$, we have

$$\frac{t^2}{g_{n+1}(t)^2} \geq t^{-2},$$

and again by (1.5) for all large n, whenever

$$g_{n+1}(t) > \alpha n, \quad t^2/g_{n+1}(t) \geq (3/c_4)\log t.$$

Therefore for all large n, whenever $g_{n+1}(t) > \alpha n$,

$$\exp\left\{-\frac{t^2}{g_{n+1}(t)}c_4\right\} \leq t^{-3},$$

which is smaller than $t^{-2}\frac{Ccc_3}{2A}$, for t large enough, i.e., for n large enough. The proof is complete. □

By choosing $g_n(t) = bt^\gamma$ for all $n \geq 1$ we see that Theorem 1.2 gives Theorem 1.1 as a special case. Also note that Theorem 1.2 remains valid for sums of Banach space-valued random variables with absolute value $|\cdot|$ replaced by norm $||\cdot||$. Theorem 1.2 permits us to derive the following maximal versions of inequalities (1.3) and (1.4).

Application 1. In Example 1 one readily checks that the assumptions of Theorem 1.2 are satisfied with $A = D^{-1}$ and $a = D/d_1$

$$g_n(t) = \frac{td_2}{d_1}\log n.$$

We get the maximal version of inequality (1.3) holding for any $0 < c < 1$ and all $n \geq 1$ and $t > 0$

$$P\left\{\left|\max_{1\leq m\leq n} S_m(f)\right| \geq t\right\} \leq C\exp\left(-\frac{cDt^2}{nd_1 + td_2\log n}\right), \tag{1.16}$$

for some constant $C \geq D^{-1}$ depending on c, D^{-1}, D/d_1 and $\{g_n\}_{n\geq 1}$.

Application 2. In Example 2 one can verify that the assumptions of the Theorem 1.2 hold with $A = D$ and $a = D/v^2$ and

$$g_n(t) = \frac{M^2}{v^2} + \frac{tM}{v^2}(\log n)^2,$$

which leads to the maximal version of inequality (1.4) valid for any $0 < c < 1$ and all $n \geq 1$ and $t > 0$

$$P\left\{\max_{1\leq m\leq n}|S_m| \geq t\right\} \leq C\exp\left(-\frac{cDt^2}{nv^2 + M^2 + tM(\log n)^2}\right) \tag{1.17}$$

for some constant $C \geq D$ depending on c, D/v^2 and $\{g_n\}_{n\geq 1}$. See Corollary 24 of Merlevède and Peligrad [3] for a closely related inequality that holds for all $n \geq 2$ and $t > K\log n$ for some $K > 0$.

Remark. There is a small oversight in the published version of the Kevei and Mason paper. Here are the corrections that fix it.

1. Page 1057, line −9: Replace "$1 \leq k \leq n$" by "$1 \leq k < n$".
2. Page 1057, line −7: Replace this line with
$$\leq \mathbf{P}\{M(1,k) > t\} + \mathbf{P}\{S(1,k+1) > pt\} + \mathbf{P}\{M(k+2,n+1) > qt\}.$$
3. Page 1058: Replace "$k + bp^\gamma t^\gamma$" by "$k + 1 + bp^\gamma t^\gamma$" in equations (2.4) and (2.5), as well as in line −13.
4. Page 1058: Replace "$ap^2 - c$" by "$ap^2 - 2c$" in line −9.

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Maximal Inequalities for Centered Norms of Sums of Independent Random Vectors

Rafał Latała

Abstract. Let $X_1, X_2, \dots, X_n$ be independent random variables and $S_k = \sum_{i=1}^{k} X_i$. We show that for any constants a_k,

$$\mathbb{P}(\max_{1\le k\le n} ||S_k| - a_k| > 11t) \le 30 \max_{1\le k\le n} \mathbb{P}(||S_k| - a_k| > t).$$

We also discuss similar inequalities for sums of Hilbert and Banach space-valued random vectors.

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Keywords. Sums of independent random variables, random vectors, maximal inequality.

1. Introduction and main results

Let $X_1, X_2, \dots$ be independent random vectors in a separable Banach space F. The Lévy-Ottaviani maximal inequality (see, e.g., Proposition 1.1.1 in [2]) states that for any $t > 0$,

$$\mathbb{P}\Big(\max_{1\le k\le n} \|S_k\| > 3t\Big) \le 3 \max_{1\le k\le n} \mathbb{P}(\|S_k\| > t), \tag{1.1}$$

where here and in the rest of this note,

$$S_k = \sum_{i=1}^{k} X_i \quad \text{for } k = 1, 2, \dots.$$

If, additionally, variables X_i are symmetric then the classical Lévy inequality gives the sharper bound

$$\mathbb{P}\Big(\max_{1\le k\le n} \|S_k\| > t\Big) \le 2\mathbb{P}(\|S_n\| > t).$$

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Montgomery-Smith [4] showed that if we replace symmetry assumptions by the identical distribution then

$$\mathbb{P}\Big(\max_{1\le k\le n}\|S_k\| > C_1 t\Big) \le C_2\mathbb{P}(\|S_n\| > t), \tag{1.2}$$

where one may take $C_1 = 30$ and $C_2 = 9$.

Maximal inequalities are fundamental tools in the study of convergence of random series and limit theorems for sums of independent random vectors (see, e.g., [2] and [3]).

In some applications one needs to investigate asymptotic behaviour of centered norms of sums, i.e., random variables of the form $(\|S_n\| - a_n)/b_n$ (cf. [1]). For such purpose it is natural to ask whether in (1.1) or (1.2) one may replace variables $\|S_k\|$ by $|\|S_k\| - a_k|$. The answer turns out to be positive in the real case.

Theorem 1.1. *Let $X_1, X_2, \dots, X_n$ be independent real r.v.'s. Then for any numbers $a_1, a_2, \dots, a_n$ and $t > 0$,*

$$\mathbb{P}\Big(\max_{1\le k\le n}||S_k| - a_k| > 11t\Big) \le 30\max_{1\le k\le n}\mathbb{P}(||S_k| - a_k| > t). \tag{1.3}$$

Example. Let $Y_1, Y_2, \dots$ be i.i.d. r.v.'s such that $\mathbb{E}Y_i^2 = 1$ and $\mathrm{Var}(Y_i^2) < \infty$. Let $S_k = \sum_{i=1}^k X_i$, where $X_i = e_i Y_i$ and (e_i) is an orthonormal system in a Hilbert space $\mathcal{H}$; also let $|x|$ denote the norm of a vector $x \in \mathcal{H}$. Then for $t > 0$,

$$\mathbb{P}(||S_k| - \sqrt{k}| \ge t) \le \mathbb{P}(||S_k|^2 - k| \ge t\sqrt{k}) \le \frac{\mathrm{Var}(|S_k|^2)}{t^2 k} = \frac{\mathrm{Var}(Y_1^2)}{t^2}.$$

On the other hand if we choose j_0 such that $2^{j_0/2} \ge t$, then for $n \ge 2^{j_0}$,

$$\begin{aligned} p_n := \mathbb{P}\Big(\max_{1\le k\le n}||S_k| - \sqrt{k}| \ge t\Big) &\ge \mathbb{P}\Big(\max_{2^{j_0}\le k\le n}(|S_k|^2 - k) \ge 3t\sqrt{k}\Big) \\ &\ge \mathbb{P}\Big(\bigcup_{j_0\le j\le \log_2 n}\Big\{|S_{2^j}|^2 - 2^j \ge 3\cdot 2^{j/2}t\Big\}\Big) \\ &\ge \mathbb{P}\Big(\bigcup_{j_0+1\le j\le \log_2 n}\Big\{2^{-j/2}\sum_{i=2^{j-1}+1}^{2^j}(Y_i^2 - 1) \ge 6t\Big\}\Big) \end{aligned}$$

and $\lim_{n\to\infty} p_n = 1$ for any $t > 0$ by the CLT. It is not hard to modify this example in such a way that X_i be an i.i.d. sequence.

Hence Theorem 1.1 does not hold in infinite dimensional Hilbert spaces even if we assume that X_i are symmetric and identically distributed. However a modification of (1.3) is satisfied in Hilbert spaces.

Proposition 1.2. *Let $X_1, \dots, X_n$ be independent symmetric r.v.'s with values in a separable Hilbert space $(\mathcal{H}, |\ |)$. Then for any sequence of real numbers $a_1, \dots, a_n$ and $t \ge 0$,*

$$\mathbb{P}\Big(\max_{1\le k\le n}\big||S_k|^2 - a_k\big| \ge 3t\Big) \le 6\max_{1\le k\le n}\mathbb{P}\big(\big||S_k|^2 - a_k\big| \ge t\big).$$

A first consequence of Proposition 1.2 is the following Hilbert-space version of (1.3) under a regularity assumption on coefficients (a_k).

Corollary 1.3. *Let $X_1, \ldots, X_n$ be as in Proposition 1.2, $1 \le i \le n$ and nonnegative real numbers $a_i, \ldots, a_n$, α, β and t satisfy the condition*

$$a_k \le \alpha a_l + \beta t \quad \text{for all } i \le k, l \le n. \tag{1.4}$$

Then

$$\mathbb{P}\Big(\max_{i \le k \le n} ||S_k| - a_k| \ge (6\alpha + 2\beta + 1)t \Big) \le 6 \max_{i \le k \le n} \mathbb{P}\big(||S_k| - a_k| \ge t\big).$$

In proofs of limit theorems one typically applies maximal inequalities to uniformly estimate $\|S_k\|$ for $cn \le k \le n$, where c is some constant. Next two corollaries show that if we restrict k to such a group of indices then, under i.i.d. and symmetry assumptions, (1.3) holds in Hilbert spaces.

Corollary 1.4. *Let $X_1, X_2, \ldots, X_n$ be symmetric i.i.d. r.v.'s with values in a separable Hilbert space $(\mathcal{H}, |\ |)$. Then for any integer i such that $\frac{n}{2} \le i \le n$ and any sequence of positive numbers $a_i, \ldots, a_n$ and $t \ge 0$ we have*

$$\mathbb{P}\Big(\max_{i \le k \le n} ||S_k| - a_k| \ge 19t \Big) \le 6 \max_{i \le k \le n} \mathbb{P}(||S_k| - a_k| \ge t).$$

Proof. We may obviously assume that

$$\max_{i \le k \le n} \mathbb{P}(||S_k| - a_k| \ge t) \le \frac{1}{6}.$$

Observe that for any $k < l$, the random variable $S_{k,l} := \sum_{i=k}^{l} X_i$ has the same distribution as S_{l-k+1}.

Take $k, l \in \{i, \ldots, n\}$, then

$$\begin{aligned} \mathbb{P}(|S_{2k}| \ge 2a_k + 2t) &\le \mathbb{P}(|S_k| \ge a_k + t) + \mathbb{P}(|S_{k+1,2k}| \ge a_k + t) \\ &= 2\mathbb{P}(|S_k| \ge a_k + t) \le \frac{1}{3}. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbb{P}(a_l - t \le |S_l| \le 2a_k + 2t) \\ &\quad \ge \mathbb{P}(a_l - t \le |S_l|,\ |S_l + S_{l+1,2k}| \le 2a_k + 2t,\ |S_l - S_{l+1,2k}| \le 2a_k + 2t) \\ &\quad \ge 1 - \mathbb{P}(|S_l| < a_l - t) - 2\mathbb{P}(|S_{2k}| > 2a_k + 2t) \ge 1 - \frac{1}{6} - \frac{2}{3} > 0, \end{aligned}$$

where in the second inequality we used the symmetry of X_i. Hence we get $a_l \le 2a_k + 3t$ and we may apply Corollary 1.3 with $\alpha = 2$ and $\beta = 3$. □

Corollary 1.5. *Let $X_1, X_2, \ldots, X_n$ be as before. Then for any $\frac{n}{2^j} \le i \le n$ and any sequence of positive numbers $a_i, \ldots, a_n$ and $t \ge 0$ we have*

$$\mathbb{P}\Big(\max_{i \le k \le n} ||S_k| - a_k| \ge 19t \Big) \le 6j \max_{i \le k \le n} \mathbb{P}(||S_k| - a_k| \ge t).$$

Corollary 1.4 naturally leads to the formulation of the following open question.

Question. Characterize all separable Banach spaces $(E, \| \ \|)$ with the following property. There exist constants $C_1, C_2 < \infty$ such that for any symmetric i.i.d. r.v.'s $X_1, X_2, \ldots, X_n$ with values in E, any $\frac{n}{2} \leq i \leq n$, any positive constants $a_i, \ldots, a_n$ and $t > 0$,

$$\mathbb{P}\Big(\max_{i \leq k \leq n} |\|S_k\| - a_k| \geq C_1 t\Big) \leq C_2 \max_{i \leq k \leq n} \mathbb{P}(|\|S_k\| - a_k| \geq t). \tag{1.5}$$

In particular does the above inequality hold in L^p with $1 < p < \infty$?

In the last section of the paper we present an example showing that in a general separable Banach space estimate (1.5) does not hold.

2. Proofs

Below we will use the following notation. By $\tilde{X}_1, \tilde{X}_2, \ldots$ we will denote the independent copy of the random sequence $X_1, X_2, \ldots$. We put

$$\tilde{S}_k := \sum_{i=1}^{k} \tilde{X}_i, \quad S_{k,n} := S_n - S_{k-1} = \sum_{i=k}^{n} X_i.$$

We start with the following simple lemma.

Lemma 2.1. *Suppose that real numbers x, y, a, b and u satisfy the conditions $||x| - a| \leq u$, $||y| - a| \leq u$, $||x + s| - b| \leq u$, $||y + s| - b| \leq u$ and $|x - y| > 2u$. Then $|a - b| \leq 2u$ and $|s| \leq 4u$.*

Proof. If $a < 0$ then $|x|, |y| < u$ and $|x - y| < 2u$. So $a \geq 0$ and in the same way we show that $b \geq 0$. Without loss of generality we may assume $x < y$, hence $x \in (-a-u, -a+u)$, $y \in (a-u, a+u)$, $x+s \in (-b-u, -b+u)$, $y+s \in (b-u, b+u)$. Thus $2a - 2u \leq y - x \leq 2a + 2u$ and $2b - 2u \leq (y + s) - (x + s) \leq 2b + 2u$ and therefore $|a - b| \leq 2u$. Moreover, $-b + a - 2u \leq s \leq -b + a + 2u$ and we get $|s| \leq |a - b| + 2u \leq 4u$. □

Proof of Theorem 1.1. We may and will assume that

$$p := \max_{1 \leq k \leq n} \mathbb{P}(||S_k| - a_k| > t) \in (0, 1/30).$$

Let

$$I_1 := \{k\colon\ a_k \leq 2t\}, \quad I_2 := \{k\colon\ \mathbb{P}(|S_k - \tilde{S}_k| > 2t) > 5p\}$$

and

$$I_3 := \{1, \ldots, n\} \setminus (I_1 \cup I_2).$$

First we show that

$$\mathbb{P}\Big(\max_{k \in I_1} ||S_k| - a_k| > 11t\Big) \leq 3p. \tag{2.1}$$

Indeed, notice that for all k, $a_k > -t$ (otherwise $p = 1$). Therefore by the Lévy-Ottaviani inequality (1.1),

$$\mathbb{P}\Big(\max_{k\in I_1} ||S_k| - a_k| > 11t\Big) \leq \mathbb{P}(\max_{k\in I_1} |S_k| > 9t) \leq 3\max_{k\in I_1}\mathbb{P}(|S_k| > 3t)$$
$$\leq 3\max_{k\in I_1}\mathbb{P}(||S_k| - a_k| > t) \leq 3p.$$

Next we prove that

$$\mathbb{P}\Big(\max_{k\in I_2} ||S_k| - a_k| > 11t\Big) \leq 5p. \tag{2.2}$$

Let us take $k \in I_2$ and define the following events

$$A_1 := \{|S_k - \tilde{S}_k| > 2t\}, \quad A_2 := A_1 \cap \{|S_{k+1,n}| > 4t\}$$

and

$$B := \{||S_k| - a_k| \leq t, ||\tilde{S}_k| - a_k| \leq t, ||S_n| - a_n| \leq t, ||\tilde{S}_k + S_{k+1,n}| - a_n| \leq t\}.$$

We have $\mathbb{P}(A_1) + \mathbb{P}(B) > 5p + 1 - 4p > 1$, hence $A_1 \cap B \neq \emptyset$ and by Lemma 2.1, $|a_k - a_n| \leq 2t$. Also by Lemma 2.1, $A_2 \cap B = \emptyset$, hence $\mathbb{P}(A_2) + \mathbb{P}(B) \leq 1$. Therefore $5p\mathbb{P}(|S_{k+1,n}| > 4t) \leq \mathbb{P}(A_2) \leq 4p$. Thus for all $k \in I_2$, $|a_k - a_n| \leq 2t$ and $\mathbb{P}(|S_{k+1,n}| \leq 4t) \geq 1/5$. Let

$$\tau := \inf\{k \in I_2 : ||S_k| - a_k| > 11t\}.$$

Then

$$\begin{aligned}\frac{1}{5}\mathbb{P}(\tau = k) &\leq \mathbb{P}(\tau = k, |S_{k+1,n}| \leq 4t)\\ &\leq \mathbb{P}(\tau = k, ||S_n| - a_n| > 11t - 4t - |a_k - a_n|)\\ &\leq \mathbb{P}(\tau = k, ||S_n| - a_n| > t)\end{aligned}$$

and

$$\mathbb{P}\Big(\max_{k\in I_2} ||S_k| - a_k| > 11t\Big) = \sum_{k\in I_2}\mathbb{P}(\tau = k) \leq 5\sum_{k\in I_2}\mathbb{P}(\tau = k, ||S_n| - a_n| > t)$$
$$\leq 5\mathbb{P}(||S_n| - a_n| > t) \leq 5p.$$

Finally we show

$$\mathbb{P}\Big(\max_{k\in I_3} ||S_k| - a_k| > 11t\Big) \leq 21p. \tag{2.3}$$

To this end take any $k \in I_3$ and notice that

$$\begin{aligned}2\max\{\mathbb{P}(|S_k - a_k| \leq t),&\mathbb{P}(|S_k + a_k| \leq t)\}\\ &\geq \mathbb{P}(|S_k - a_k| \leq t) + \mathbb{P}(|S_k + a_k| \leq t)\\ &\geq \mathbb{P}(||S_k| - a_k| \leq t) \geq 1 - p \geq \frac{29}{30}.\end{aligned}$$

If $|x-a_k|\le t$ and $|y+a_k|\le t$ then $|x-y|\ge 2a_k-2t>2t$. Therefore

$$\begin{aligned}5p &\ge \mathbb{P}(|S_k-\tilde{S}_k|>2t)\\ &\ge \mathbb{P}(|S_k-a_k|\le t, |\tilde{S}_k+a_k|\le t)+\mathbb{P}(|S_k+a_k|\le t, |\tilde{S}_k-a_k|\le t)\\ &= 2\mathbb{P}(|S_k-a_k|\le t)\mathbb{P}(|S_k+a_k|\le t).\end{aligned}$$

So for any $k\in I_3$ we may choose $b_k=\pm a_k$ such that

$$\mathbb{P}(|S_k-b_k|\le t)\le \frac{30}{29}5p\le 6p.$$

Therefore

$$\mathbb{P}(|S_k+b_k|>t)\le \mathbb{P}(||S_k|-a_k|>t)+\mathbb{P}(|S_k-b_k|\le t)\le 7p$$

and by the Lévy-Ottaviani inequality (1.1),

$$\begin{aligned}\mathbb{P}(\max_{k\in I_3}||S_k|-a_k|>11t) &\le \mathbb{P}(\max_{k\in I_3}|S_k+b_k|>11t)\\ &\le 3\max_{k\in I_3}\mathbb{P}(|S_k+b_k|>\frac{11}{3}t)\le 21p.\end{aligned}$$

This shows (2.3).

Inequalities (2.1), (2.2) and (2.3) imply (1.3). □

Proof of Proposition 1.2. It is enough to consider the case when

$$p:=\max_{1\le k\le n}\mathbb{P}\big(\big||S_k|^2-a_k\big|\ge t\big)<\frac{1}{6}.$$

Notice that

$$\mathbb{P}\big(\big||S_n|^2-|S_k|^2-(a_n-a_k)\big|\ge 2t\big)\le \mathbb{P}\big(\big||S_n|^2-a_n\big|\ge t\big)+\mathbb{P}\big(\big||S_k|^2-a_k\big|\ge t\big)$$

Therefore

$$\mathbb{P}\big(\big||S_{k+1,n}|^2+2\langle S_k,S_{k+1,n}\rangle-(a_n-a_k)\big|\ge 2t)\le 2p$$

and by the symmetry

$$\mathbb{P}\big(\big||S_{k+1,n}|^2-2\langle S_k,S_{k+1,n}\rangle-(a_n-a_k)\big|\ge 2t\big)\le 2p.$$

Thus by the triangle inequality

$$\mathbb{P}\big(\big||S_{k+1,n}|^2-(a_n-a_k)\big|\ge 2t\big)\le 4p.$$

Now let $x\in\mathcal{H}$ be such that $||x|^2-a_k|\ge 3t$ then by the triangle inequality and symmetry

$$\begin{aligned}1-4p &\le \mathbb{P}\big(\big||x|^2+|S_{k+1,n}|^2-a_n\big|\ge t\big)\\ &\le \mathbb{P}\big(\big||x|^2+|S_{k+1,n}|^2+2\langle x,S_{k+1,n}\rangle-a_n\big|\ge t\big)\\ &\quad+\mathbb{P}\big(\big||x|^2+|S_{k+1,n}|^2-2\langle x,S_{k+1,n}\rangle-a_n\big|\ge t\big)\\ &= 2\mathbb{P}\big(\big||x|^2+|S_{k+1,n}|^2+2\langle x,S_{k+1,n}\rangle-a_n\big|\ge t\big)\\ &= 2\mathbb{P}\big(\big||x+S_{k+1,n}|^2-a_n\big|\ge t\big).\end{aligned}$$

So for any $x \in \mathcal{H}$ and $k = 1, 2, \ldots, n$,

$$\big||x|^2 - a_k\big| \geq 3t \ \Rightarrow\ \mathbb{P}\big(\big||x + S_{k+1,n}|^2 - a_n\big| \geq t\big) \geq \frac{1}{2}(1 - 4p) \geq \frac{1}{6}. \tag{2.4}$$

Now let

$$\tau := \inf\big\{k \leq n\colon\ \big||S_k|^2 - a_k\big| \geq 3t\big\},$$

then since $\{\tau = k\} \in \sigma(X_1, \ldots, X_k)$ we get by (2.4),

$$\mathbb{P}\big(\tau = k, \big||S_n|^2 - a_n\big| \geq t\big) \geq \frac{1}{6}\mathbb{P}(\tau = k).$$

Hence

$$\mathbb{P}\big(\big||S_n|^2 - a_n\big| \geq t\big) \geq \frac{1}{6}\sum_{k=1}^{n}\mathbb{P}(\tau = k) = \frac{1}{6}\mathbb{P}\Big(\max_{1\leq k\leq n}\big||S_k|^2 - a_k\big| \geq 3t\Big)$$

and the proposition follows. □

Proof of Corollary 1.3. We may consider variables $S_i, X_{i+1}, \ldots, X_n$ instead of $X_1, \ldots, X_n$ and assume that $i = 1$. Let $a := \min_{1\leq k\leq n} a_k$. We will analyze two cases.

Case 1. $a \leq 3t$. Then by (1.4) we get $a_k \leq (3\alpha + \beta)t$ for all k. Thus by the Lévy inequality,

$$\begin{aligned}\mathbb{P}\Big(\max_k ||S_k| - a_k| \geq (6\alpha + 2\beta + 1)t\Big) &\leq \mathbb{P}\Big(\max_k |S_k| \geq (3\alpha + \beta + 1)t\Big)\\ &\leq 2\mathbb{P}(|S_n| \geq (3\alpha + \beta + 1)t)\\ &\leq 2\mathbb{P}(||S_n| - a_n| \geq t)\\ &\leq 2\max_k \mathbb{P}(||S_k| - a_k| \geq t).\end{aligned}$$

Case 2. $a \geq 3t$. Notice first that for any $s > 0$ we have

$$\{||S_k|^2 - a_k^2| \geq s(2a_k + s)\} \subset \{||S_k| - a_k| \geq s\} \subset \{||S_k|^2 - a_k^2| \geq sa_k\}. \tag{2.5}$$

Indeed, the last inclusion follows since $||S_k|^2 - a_k^2| = (|S_k| + a_k)||S_k| - a_k| \geq a_k||S_k| - a_k|$. To see the first inclusion in (2.5) observe that

$$\begin{aligned}\{||S_k|^2 - a_k^2| \geq s(2a_k + s)\} &\subset \{||S_k| - a_k| \geq s\} \cup \{|S_k| + a_k \geq 2a_k + s\}\\ &\subset \{||S_k| - a_k| \geq s\}.\end{aligned}$$

Now by (2.5) we get

$$\begin{aligned}&\mathbb{P}\Big(\max_k ||S_k| - a_k| \geq (6\alpha + 2\beta + 1)t\Big)\\ &\qquad\leq \mathbb{P}\Big(\max_k ||S_k|^2 - a_k^2| \geq (6\alpha + 2\beta + 1)at\Big).\end{aligned}$$

Hence by Proposition 1.2,

$$\mathbb{P}\Big(\max_k ||S_k| - a_k| \geq (6\alpha + 2\beta + 1)t\Big) \leq 6\max_k \mathbb{P}\big(||S_k|^2 - a_k^2| \geq \frac{1}{3}(6\alpha + 2\beta + 1)at\big).$$

But $\frac{1}{3}(6\alpha+2\beta+1)a \geq 2(\alpha a+\beta t)+t \geq 2a_k+t$ for all k by (1.4). Therefore by (2.5),

$$\mathbb{P}\Big(\max_k ||S_k|-a_k| \geq (6\alpha+2\beta+1)t\Big) \leq 6\max_k \mathbb{P}(||S_k|^2-a_k^2| \geq t(2a_k+t)) \\ \leq 6\max_k \mathbb{P}(||S_k|-a_k| \geq t). \qquad \square$$

3. Example

Let us fix a positive integer n and put

$$I_n = \Big\{j \in \mathbb{Z}\colon\ \frac{n}{2} \leq j \leq n\Big\}.$$

Let $t_j = \frac{n^2+j}{j}$ for $j=1,2,\dots,n$, then

$$jt_j = n^2+j \quad \text{and} \quad (j-1)t_j \leq n^2 \quad \text{for } j \in I_n. \tag{3.1}$$

Let N be a large integer (to be fixed later) and let F be the space of all double-indexed sequences $a=(a_{i,j})_{0\leq i\leq N, j\in I_n}$ with the norm

$$\Big\|(a_{i,j})_{0\leq i\leq N, j\in I_n}\Big\| = \max_{j\in I_n}\Big(|a_{0,j}| + t_j \sum_{1\leq i_1<i_2<\dots<i_j\leq N} \sum_{s=1}^{j} |a_{i_s,j}|\Big).$$

Let $(e_{i,j})$ be a standard basis of F, so that $(a_{i,j}) = \sum_{i,j} a_{i,j}e_{i,j}$.

Define random vectors $X_1, X_2, \dots, X_n$ by the formula

$$X_l = \sum_{j\in I_n} (Y_{l,j}e_{0,j} + R_{l,j}e_{N_l,j}),$$

where $(Y_{l,j}, R_{l,j})_{l\leq n, j\in I_n}$ and $(N_l)_{l\leq n}$ are independent r.v's, $\mathbb{P}(R_{l,j}=\pm 1)=1/2$, $Y_{l,j}$ are symmetric $\mathbb{P}(|Y_{l,j}| = \frac{1}{2n}) = 1-\mathbb{P}(Y_{k,j}=0) = p_n$ (with p_n a small positive number to be specified later) and N_l are uniformly sampled from the set $\{1,\dots,N\}$.

Obviously $X_1, X_2, \dots, X_n$ are i.i.d. and symmetric. As usual we set $S_k = X_1+X_2+\dots+X_k$. Let

$$A = \{N_1, N_2, \dots, N_n \text{ are pairwise distinct}\}.$$

Notice that $\mathbb{P}(A^c) \to 0$ when $N\to\infty$. On the set A we have for $k \leq n$,

$$\|S_k\| = \max_{j\in I_n}\Big(\Big|\sum_{l=1}^{k} Y_{l,j}\Big| + t_j \min\{k,j\}\Big).$$

For $j > k$ we have by (3.1),

$$\Big|\sum_{l=1}^{k} Y_{l,j}\Big| + t_j\min\{k,j\} < 1 + t_j(j-1) \leq n^2+1,$$

hence on the set A, for $k \in I_n$ we get

$$\|S_k\| = \max_{j \in I_n, j \le k} \Big(\Big| \sum_{l=1}^{k} Y_{l,j} \Big| + n^2 + j \Big) = \Big| \sum_{l=1}^{k} Y_{l,k} \Big| + n^2 + k.$$

Take $0 < t < \frac{1}{2nC_1}$ then for $k \in I_n$,

$$\mathbb{P}(|\|S_k\| - (n^2 + k)| \ge t) \le \mathbb{P}(A) + \mathbb{P}\Big(\sum_{l=1}^{k} Y_{l,k} \ne 0 \Big) \le \mathbb{P}(A^c) + kp_n$$

and

$$\mathbb{P}\Big(\max_{k \in I_n} |\|S_k\| - (n^2 + k)| \ge tC_1 \Big) \ge \mathbb{P}\Big(\max_{k \in I_n} \Big| \sum_{l=1}^{k} Y_{l,k} \Big| \ne 0 \Big) - \mathbb{P}(A^c).$$

The last number is of order $n^2 p_n$ if N is large and p_n is small. This shows that if (1.5) holds for $i = \lceil n/2 \rceil$ in F then C_2 must be of order n. So (1.5) cannot hold with absolute constants C_1 and C_2 in (infinite dimensional) separable Banach spaces.

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A Probabilistic Inequality Related to Negative Definite Functions

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Abstract. We prove that for any pair of i.i.d. random vectors X, Y in $\mathbb{R}^n$ and any real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ the inequality

$$\mathbb{E}\,\psi(X - Y) \leqslant \mathbb{E}\,\psi(X + Y).$$

holds. In particular, for $\alpha \in (0, 2]$ and the Euclidean norm $\|\cdot\|_2$ one has

$$\mathbb{E}\|X - Y\|_2^\alpha \leqslant \mathbb{E}\|X + Y\|_2^\alpha.$$

The latter inequality is due to A. Buja et al. [4] where it is used for some applications in multivariate statistics. We show a surprising connection with bifractional Brownian motion and provide some related counter-examples.

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1. Introduction

Let X, Y be i.i.d. random variables with finite expectations. Then one has

$$\mathbb{E}|X - Y| \leqslant \mathbb{E}|X + Y|. \tag{1.1}$$

The inequality (1.1) appeared recently in an analytic context (properties of integrable functions) [8]. Since (1.1) is a nice fact in itself and since it seems not to be well known in the probabilistic community, it is desirable to search for adequate proofs and to explore possible extensions of it. For instance, for which values of α do we have

$$\mathbb{E}|X - Y|^\alpha \leqslant \mathbb{E}|X + Y|^\alpha\,? \tag{1.2}$$

As before, we assume that X and Y are i.i.d. and $\mathbb{E}|X|^\alpha < \infty$.

Proving (1.1) is a non-trivial exercise for a probability course. If X, Y are real valued, one way to see this inequality is to use the identity

$$\mathbb{E}|X+Y| - \mathbb{E}|X-Y| = 2\int_0^\infty \left[\mathbb{P}(X>r) - \mathbb{P}(X<-r)\right]^2 dr.$$

For (1.2) we are, however, not aware of a similar elementary approach. On the other hand, A. Buja et al. prove in [4] even a multivariate version of (1.2): for any pair of i.i.d. random vectors X, Y in $\mathbb{R}^n$, any $\alpha \in (0,2]$ and for a class of norms $\|\cdot\|$ on $\mathbb{R}^n$ including the Euclidean norm $\|\cdot\|_2$ the estimate

$$\mathbb{E}\|X-Y\|^\alpha \leqslant \mathbb{E}\|X+Y\|^\alpha \tag{1.3}$$

holds true. The elegance of this inequality is obvious; at the same time we stress that it arises from statistical applications. In any case it merits to be better known in the probabilistic community!

In Section 2 we give an extension of (1.3) by replacing the norm with an arbitrary negative definite function. Moreover, we show how this fact extends to an arbitrary number of i.i.d. random vectors. In Sections 3 and 4 we establish a surprising connection to some recent advances in the theory of random processes related to bifractional Brownian motion. A counterexample to (1.2) with $\alpha \in (2,\infty)$ is given in Section 5.

2. Main result

Consider the class of *continuous real-valued negative definite functions*, i.e., characteristic exponents of symmetric Lévy processes. The notion of negative definite function goes back to Schoenberg; good sources are the books [3] and [11]. Recall that a continuous real-valued negative definite function is uniquely given by its Lévy-Khintchine representation

$$\psi(\xi) = a + \frac{1}{2}\langle Q\xi, \xi\rangle + \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos\langle \xi, u\rangle)\,\nu(du), \qquad \xi \in \mathbb{R}^n, \tag{2.1}$$

where $a \geqslant 0$ is a constant, $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix and ν is the Lévy measure, i.e., a measure on $\mathbb{R}^n \setminus \{0\}$ satisfying the integrability condition

$$\int_{\mathbb{R}^n\setminus\{0\}} \min\{\|u\|_2^2, 1\}\nu(du) < \infty. \tag{2.2}$$

Without loss of generality, we will always assume that $a = 0$, i.e., $\psi(0) = 0$. For our discussion it is worth noticing that $(\xi, \eta) \mapsto \sqrt{\psi(\xi-\eta)}$ is always a metric. A deep theorem of Schoenberg states that a metric space $(\mathbb{R}^n, d)$ can be isometrically embedded into an (in general infinite dimensional) Hilbert space $\mathcal{H}$ if, and only if, $d(\xi,\eta)$ is of the form $d_\psi(\xi,\eta) = \sqrt{\psi(\xi-\eta)}$, cf. [12], [2, p. 187] as well as [7] for a discussion of metric measure spaces related to the metric d_ψ.

An important subclass of continuous negative definite functions are the spherically symmetric negative definite functions. These are of the form

$$\xi \mapsto f(\|\xi\|_2^2) \qquad \text{where } f \text{ is a Bernstein function.} \tag{2.3}$$

Recall that a *Bernstein function* is a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which admits the following Lévy-Khintchine representation

$$f(\lambda) = a + b\lambda + \int_0^\infty \left(1 - e^{-t\lambda}\right) \mu(dt);$$

here $a, b \geqslant 0$ are constants and μ is a measure on $(0, \infty)$ satisfying the integrability condition $\int_0^\infty \min\{t, 1\}\, \mu(dt) < \infty$. In probability theory Bernstein functions arise as the characteristic exponents of the Laplace transform of subordinators, i.e., increasing one-dimensional Lévy processes. Bernstein functions, many examples and their connections to various fields of mathematics are discussed in the monograph [11]. It is easy to see that Bernstein functions are infinitely many times differentiable, increasing, concave; moreover, they grow at most linearly. Typical examples are $\lambda \mapsto \log(1 + \lambda)$ and $\lambda \mapsto f_\beta(\lambda) := \lambda^\beta$ for $0 < \beta \leqslant 1$. Note that the composition $f \circ \psi$ of a Bernstein function f with a continuous real-valued negative definite function ψ is again a continuous real-valued negative definite function. At the level of stochastic processes this corresponds to *Bochner's subordination* of the Lévy process with characteristic exponent ψ by the subordinator with the Laplace exponent f.

Using the Bernstein functions f_β with $\beta = \alpha/2$ and $0 < \alpha \leqslant 2$ we obtain

$$\xi \mapsto \|\xi\|_2^\alpha = f_{\alpha/2}(\|\xi\|^2), \qquad 0 < \alpha \leqslant 2,$$
$$\xi \mapsto d_\psi(\xi, 0)^\alpha = \sqrt{\psi(\xi)}^{\,\alpha} = f_{\alpha/2}(\psi(\xi)), \qquad 0 < \alpha \leqslant 2,$$

as examples for real-valued continuous negative definite functions. Note that the functions defined by (2.3) are characteristic exponents of subordinate Brownian motions.

We prove the following result extending (1.3).

Theorem 2.1. *Let ψ be a real-valued continuous negative definite function on $\mathbb{R}^n$. For any pair of i.i.d. random vectors X, Y in $\mathbb{R}^n$ it is true that*

$$\mathbb{E}\,\psi(X - Y) \leqslant \mathbb{E}\,\psi(X + Y). \tag{2.4}$$

Proof. Without loss of generality we may assume that $a = 0$ and $Q = 0$ – in both cases the inequality (2.4) is elementary.

Using the Lévy-Khintchine representation of ψ we get

$$\begin{aligned}\mathbb{E}\,\psi(X + Y) &= \mathbb{E} \int_{\mathbb{R}^n \setminus \{0\}} \big(1 - \cos\langle X + Y, u\rangle\big)\, \nu(du) \\ &= \mathbb{E} \int_{\mathbb{R}^n \setminus \{0\}} \big(1 - \operatorname{Re} \exp(i\langle X + Y, u\rangle)\big)\, \nu(du) \\ &= \int_{\mathbb{R}^n \setminus \{0\}} \big(1 - \operatorname{Re}\, \mathbb{E} \exp(i\langle X + Y, u\rangle)\big)\, \nu(du)\end{aligned}$$

$$= \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - \operatorname{Re}\left[\mathbb{E}\exp(i\langle X, u\rangle)\right]^2\right) \nu(du).$$

A similar calculation yields

$$\begin{aligned}
\mathbb{E}\,\psi(X-Y) &= \mathbb{E}\int_{\mathbb{R}^n\setminus\{0\}} \big(1 - \cos\langle X-Y, u\rangle\big)\,\nu(du)\\
&= \mathbb{E}\int_{\mathbb{R}^n\setminus\{0\}} \big(1 - \operatorname{Re}\exp(i\langle X-Y, u\rangle)\big)\,\nu(du)\\
&= \int_{\mathbb{R}^n\setminus\{0\}} \big(1 - \operatorname{Re}\,\mathbb{E}\exp(i\langle X-Y, u\rangle)\big)\,\nu(du)\\
&= \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - |\mathbb{E}\exp(i\langle X, u\rangle)|^2\right)\nu(du).
\end{aligned}$$

Using the elementary estimate $\operatorname{Re}(z^2) \leqslant |z^2| = |z|^2$ we obtain (2.4). □

Remark 2.2. Let $X_1, \dots, X_{2m}$ be i.i.d. random variables in $\mathbb{R}^n$ and $\varepsilon_j = \pm 1$ (non-random, or even random but independent of the $X_1, \dots, X_{2m}$) constants satisfying $\sum_{j=1}^{2m} \varepsilon_j = 0$. Then

$$\mathbb{E}\,\psi\left(\sum_{j=1}^{2m} \varepsilon_j X_j\right) \leqslant \mathbb{E}\,\psi\left(\sum_{j=1}^{2m} X_j\right). \tag{2.5}$$

This follows if we use Theorem 2.1 for $X = \sum_{j=1}^{2m} \varepsilon_j^+ X_j$ and $Y = \sum_{j=1}^{2m} \varepsilon_j^- X_j$.

Remark 2.3. Essentially the same calculations as in the proof of Theorem 2.1 show that we also have

$$\mathbb{E}\psi(X) \leqslant \mathbb{E}\psi(X+Y). \tag{2.6}$$

This follows from the elementary inequality $\operatorname{Re}(z^2) \leqslant \operatorname{Re} z$ for $|z| \leqslant 1$ and the fact that

$$\mathbb{E}\psi(X) = \int_{\mathbb{R}^n\setminus\{0\}} \Big(1 - \operatorname{Re}\mathbb{E}\exp(i\langle X, u\rangle)\Big)\,\nu(du).$$

A special case of the inequality (2.6) with $\psi(\xi) = |\xi|$ and $\nu(du) = \frac{1}{\pi}\,u^{-2}\,du$ appeared in the 2003 Putnam competition, cf. [10, Problem B6, p. 783 and p. 790] where the task was to show that

$$\int_0^1\int_0^1 |f(x) + f(y)|\,dx\,dy \geqslant \int_0^1 |f(x)|\,dx$$

for a continuous real-valued function f defined on the interval $[0,1]$.

Using the distance function $d_\psi(\xi, \eta) := \sqrt{\psi(\xi-\eta)}$ related to a real-valued continuous negative definite function ψ we get the following counterpart of (1.3).

Corollary 2.4. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous negative definite function, $d_\psi(\xi, \eta) = \sqrt{\psi(\xi - \eta)}$ the associated metric and $0 < \alpha \leqslant 2$. For any pair of i.i.d. random vectors X, Y in $\mathbb{R}^n$ it is true that*

$$\mathbb{E}\, d_\psi^\alpha(X - Y) \leqslant \mathbb{E}\, d_\psi^\alpha(X + Y). \tag{2.7}$$

Remark 2.5. Assume that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that $\psi(0) = 0$ and $\psi(\xi) = \psi(-\xi)$. If (2.4) holds for this ψ and *any* random variable X (and an independent copy Y of X), then one can show that the kernel $K_\psi(\xi, \eta) := \psi(\xi + \eta) - \psi(\xi - \eta)$ is positive definite. We wonder whether this already entails that ψ is a continuous negative definite function.

3. A relation to random processes

We will show now that the inequality (2.4) has an interesting relation to Gaussian processes. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous negative definite function defined on $\mathbb{R}^n$.

Lemma 3.1. *The kernel $K^\psi(\xi, \eta) = \psi(\xi + \eta) - \psi(\xi - \eta)$ is positive definite.*

Proof. By the Lévy-Khintchine formula (2.1) we get

$$K^\psi(\xi, \eta) = 2\langle Q\xi, \eta\rangle + \int_{\mathbb{R}^n \setminus \{0\}} (\cos(\langle \xi - \eta, u\rangle) - \cos(\langle \xi + \eta, u\rangle))\; \nu(du).$$

Using the elementary trigonometric identity

$$\cos\langle \xi - \eta, u\rangle - \cos\langle \xi + \eta, u\rangle = 2 \sin\langle \xi, u\rangle \sin\langle \eta, u\rangle,$$

we see that

$$K^\psi(\xi, \eta) = 2\langle Q\xi, \eta\rangle + 2\int_{\mathbb{R}^n \setminus \{0\}} \sin\langle \xi, u\rangle \sin\langle \eta, u\rangle\, \nu(du).$$

Now let S be a finite set and $(\lambda_\xi,\ \xi \in S)$ be complex numbers. Then

$$\begin{aligned}
&\sum_{\xi,\eta \in S} K^\psi(\xi, \eta)\lambda_\xi \overline{\lambda_\eta} \\
&\quad = 2 \sum_{\xi,\eta \in S} \lambda_\xi \overline{\lambda_\eta} \langle Q\xi, \eta\rangle + 2\int_{\mathbb{R}^n \setminus \{0\}} \left(\sum_{\xi,\eta \in S} \lambda_\xi \sin\langle \xi, u\rangle\, \overline{\lambda_\eta \sin\langle \eta, u\rangle} \right) \nu(du) \\
&\quad = 2 \left\langle Q \sum_{\xi \in S} \lambda_\xi\, \xi, \sum_{\xi \in S} \lambda_\xi\, \xi \right\rangle + 2\int_{\mathbb{R}^n \setminus \{0\}} \left| \sum_{\xi \in S} \lambda_\xi \sin\langle \xi, u\rangle \right|^2 \nu(du) \geqslant 0,
\end{aligned}$$

which means that $K^\psi(\cdot, \cdot)$ is positive definite. □

Remark 3.2. A special case of Lemma 3.1 for powers of ℓ_p-norms is proved in [4].

Probabilistic proof of Theorem 2.1. Since $K^\psi(\xi, \eta)$ is positive definite, there is a centered Gaussian process $\left(G_\xi^\psi,\ \xi \in \mathbb{R}^n\right)$ whose covariance function is $K^\psi(\xi, \eta)$.

For given i.i.d. random vectors $X, Y \in \mathbb{R}^n$ set

$$Z^\psi := \int_{\mathbb{R}^n} G_\xi^\psi \, P(d\xi),$$

where P stands for the common distribution of X and Y. Then

$$\begin{aligned} 0 \leqslant \operatorname{Var}(Z^\psi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K^\psi(\xi, \eta) \, P(d\xi) \, P(d\eta) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\psi(\xi + \eta) - \psi(\xi - \eta)\right) \, P(d\xi) \, P(d\eta) \\ &= \mathbb{E}\,\psi(X + Y) - \mathbb{E}\,\psi(X - Y), \end{aligned}$$

and we obtain again $\mathbb{E}\,\psi(X - Y) \leqslant \mathbb{E}\,\psi(X + Y)$. □

4. Relation to bifractional Brownian motion

In some most important cases it is possible to identify the Gaussian process $\left(G_\xi^\psi, \ \xi \in \mathbb{R}^n\right)$ of Section 3 with *bifractional Brownian motion* (bBm). The latter process was introduced by Houdré and Villa in [6] as a centered Gaussian process $B^{H,K} = \left(B_t^{H,K}, \ t \in \mathbb{R}^n\right)$ with covariance function

$$R^{H,K}(t, s) := \mathbb{E}\left(B_t^{H,K} B_s^{H,K}\right) = 2^{-K}\left((||t||_2^{2H} + ||s||_2^{2H})^K - ||t - s||_2^{2HK}\right),$$

where $s, t \in \mathbb{R}^n$. For $n = 1$, $K = 1$ we get the usual fractional Brownian motion B^H with Hurst index H. Originally, the process was defined for the parameters $H \in (0, 1]$ and $K \in (0, 1]$. Bardina and Es-Sebaiy [1] recently proved that $B^{H,K}$ exists for all $(H, K) \in \mathcal{D}$, where

$$\mathcal{D} := \{H, K : 0 < H \leqslant 1, 0 < K \leqslant 2, H \cdot K \leqslant 1\}.$$

(The possibility of such an extension was already indicated in the earlier work by Lei and Nualart [9] who established an integral representation relating $B^{H,K}$ with fractional Brownian motion B^{HK}.)

For $\psi(\xi) := |\xi|^\alpha$, $0 < \alpha \leqslant 2$, and

$$G_\xi^\psi := 2^{\alpha/2} \operatorname{sgn}(\xi) \, B_{|\xi|}^{\frac{1}{2}, \alpha}, \qquad \xi \in \mathbb{R},$$

it is trivial to see that

$$\mathbb{E}\left(G_\xi^\psi G_\eta^\psi\right) = \operatorname{sgn}(\xi\eta) \, 2^\alpha \, \mathbb{E}\left(B_{|\xi|}^{\frac{1}{2}, \alpha}, B_{|\xi|}^{\frac{1}{2}, \alpha}\right) = |\xi + \eta|^\alpha - |\xi - \eta|^\alpha = K^\psi(\xi, \eta).$$

Therefore, we are led to a probabilistic interpretation of the inequality (1.2) through $B^{\frac{1}{2}, \alpha}$.

Remark 4.1. In higher dimensions bi-fractional Brownian motion does not show up in the context of our inequalities (nor do we rely on bBm with $H \neq \frac{1}{2}$); therefore it becomes natural to search for the extensions of bBm based upon general negative definite functions. This will be done elsewhere.

5. A counterexample

The inequality (1.2) trivially extends to the case $\alpha = \infty$ in the following sense. Let

$$M = \sup\{r : \mathbb{P}(X < r) < 1\} = \operatorname{ess\,sup} X;$$
$$m = \sup\{r : \mathbb{P}(X < r) = 0\} = \operatorname{ess\,inf} X.$$

Then

$$\|X - Y\|_\infty = M - m \leqslant 2 \max\{|M|, |m|\} = \|X + Y\|_\infty.$$

Without further assumptions the inequality (1.2) will, in general, not hold, for $2 < \alpha < \infty$. To see this, fix $\alpha \in (2, \infty)$ and $c > 0$. For any $M \geqslant c$ set $q := c/M$ and $p := 1 - q$. Let X_M, Y_M be i.i.d. random variables such that

$$\mathbb{P}(X_M = 1) = \mathbb{P}(Y_M = 1) = p;$$
$$\mathbb{P}(X_M = -M) = \mathbb{P}(Y_M = -M) = q.$$

If $M \geqslant 1$, then

$$\begin{aligned}
\mathbb{E}|X_M - Y_M|^\alpha - \mathbb{E}|X_M + Y_M|^\alpha
&= 2pq\,[(M+1)^\alpha - (M-1)^\alpha] - 2^\alpha M^\alpha q^2 - 2^\alpha p^2 \\
&\geqslant 4pq\alpha M^{\alpha-1} - 2^\alpha M^\alpha q^2 - 2^\alpha p^2 \\
&= M^{\alpha-2}(4p\alpha c - 2^\alpha c^2) - 2^\alpha p^2.
\end{aligned}$$

Hence, whenever $c < 2^{2-\alpha}\alpha$ and M is large enough,

$$\mathbb{E}|X_M - Y_M|^\alpha - \mathbb{E}|X_M + Y_M|^\alpha > 0,$$

and (1.2) fails.

Remark 5.1. Further counterexamples are presented in [4].

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Optimal Re-centering Bounds, with Applications to Rosenthal-type Concentration of Measure Inequalities

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Abstract. For any nonnegative Borel-measurable function f such that $f(x) = 0$ if and only if $x = 0$, the best constant c_f in the inequality $\mathsf{E}\, f(X - \mathsf{E}\, X) \leqslant c_f\, \mathsf{E}\, f(X)$ for all random variables X with a finite mean is obtained. Properties of the constant c_f in the case when $f = |\cdot|^p$ for $p > 0$ are studied. Applications to concentration of measure in the form of Rosenthal-type bounds on the moments of separately Lipschitz functions on product spaces are given.

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1. Introduction

In many situations (as, e.g., in [18]), one starts with zero-mean random variables (r.v.'s), which need to be truncated in some manner, and then the means no longer have to be zero. So, to utilize such tools as the Rosenthal inequality for sums of independent zero-mean r.v.'s, one has to re-center the truncated r.v.'s. Then one will usually need to bound moments of the re-centered truncated r.v.'s in terms of the corresponding moments of the original r.v.'s. To be more specific, let Z be a given r.v., possibly (but not necessarily) of zero mean. Next, let $\tilde{Z}$ be a truncated version of Z such that $|\tilde{Z}| \leqslant |Z|$; possibilities here include letting $\tilde{Z}$ equal $Z\,\mathbf{I}\{Z \leqslant z\}$ or $Z\,\mathbf{I}\{|Z| \leqslant z\}$ or $Z \wedge z$, for some $z > 0$; cf. [16, 26]. Assume that $\mathsf{E}\,|\tilde{Z}| < \infty$. Then for any $p \geqslant 1$ one can use the inequalities $|x - y|^p \leqslant 2^{p-1}(|x|^p + |y|^p)$ and $(\mathsf{E}\,|\tilde{Z}|)^p \leqslant \mathsf{E}\,|\tilde{Z}|^p$, to write

$$\mathsf{E}\,|\tilde{Z} - \mathsf{E}\,\tilde{Z}|^p \leqslant 2^p\,\mathsf{E}\,|\tilde{Z}|^p \leqslant 2^p\,\mathsf{E}\,|Z|^p, \tag{1.1}$$

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as it is often done. However, the factor 2^p in (1.1) can be significantly improved, especially for $p \geqslant 2$. For instance, it is clear that for $p = 2$ this factor can be reduced from $2^2 = 4$ to 1. More generally, for every real $p > 1$ we shall provide the best constant factor C_p in the inequality

$$\mathsf{E}\,|X - \mathsf{E}\,X|^p \leqslant C_p\,\mathsf{E}\,|X|^p \tag{1.2}$$

for all r.v.'s X with a finite mean $\mathsf{E}\,X$. In particular, C_p improves the factor 2^p more than 6 times for $p = 3$, and for large p this improvement is asymptotically $\sqrt{8ep}$ times; see parts (vi) and (iv) of Theorem 2.3 and the left panel in Figure 2 in this paper. In fact, in Theorem 2.1 below we shall present an extended version of the exact inequality (1.2), for a quite general class of moment functions f in place of the power functions $|\cdot|^p$.

Another natural application of these results is to concentration of measure for separately Lipschitz functions on product spaces. In Section 3 of this paper, we shall give Rosenthal-type bounds on the moments of such functions. Similar extensions of the von Bahr–Esseen inequality were given in [17].

2. Summary and discussion

Let $f\colon \mathbb{R} \to \mathbb{R}$ be any nonnegative Borel-measurable function such that $f(x) = 0$ if and only if $x = 0$. Let X stand for any random variable (r.v.) with a finite mean $\mathsf{E}\,X$.

Theorem 2.1. *One has*

$$\mathsf{E}\,f(X - \mathsf{E}\,X) \leqslant c_f\,\mathsf{E}\,f(X), \tag{2.1}$$

where

$$c_f := \sup\Big\{\frac{af(b) + bf(-a)}{af(b-t) + bf(-a-t)} : a \in (0,\infty), b \in (0,\infty), t \in \mathbb{R}\Big\} \tag{2.2}$$

is the best possible constant factor in (2.1) (*over all r.v.'s X with a finite mean*).

All necessary proofs will be given in Section 4.

Note that for all $a \in (0,\infty)$, $b \in (0,\infty)$, and $t \in \mathbb{R}$ both the numerator and the denominator of the ratio in (2.2) are strictly positive (since f is nonnegative and vanishes only at 0). So, c_f is correctly defined, with possible values in $(0,\infty]$.

It is possible to say much more about the optimal constant factor c_f in the important case when f is the power function $|\cdot|^p$. To state the corresponding result, let us introduce more notation.

Take any $a \in (0,\infty)$ and $b \in (0,\infty)$, and let $X_{a,b}$ be any zero-mean r.v. with values $-a$ and b, so that

$$\mathsf{P}(X_{a,b} = b) = \frac{a}{a+b} = 1 - \mathsf{P}(X_{a,b} = -a).$$

Note that

$$X_{b,a} \overset{\mathrm{D}}{=} -X_{a,b},$$

where $\overset{\mathrm{D}}{=}$ denotes the equality in distribution.

Take any

$$p \in (1, \infty) \tag{2.3}$$

and introduce

$$R(p,b) := (b^{p-1} + (1-b)^{p-1})\big(b^{\frac{1}{p-1}} + (1-b)^{\frac{1}{p-1}}\big)^{p-1} \quad \text{for any } b \in [0,1]. \tag{2.4}$$

Proposition 2.2. *If $p \neq 2$ then there exists $b_p \in (0, \frac{1}{2})$ such that*

(i) $\partial_b R(p,b) > 0$ *for* $b \in (0, b_p)$ *and hence* $R(p,b)$ *is (strictly) increasing in* $b \in [0, b_p]$;

(ii) $\partial_b R(p,b) < 0$ *for* $b \in (b_p, \frac{1}{2})$ *and hence* $R(p,b)$ *is decreasing in* $b \in [b_p, \frac{1}{2}]$.

So, b_p is the unique maximizer of $R(p,b)$ over all $b \in [0, \frac{1}{2}]$.

In Proposition 2.2 and in the sequel, $\partial_\cdot$ denotes the partial differentiation with respect to the argument in the subscript.

Theorem 2.3.

(i) *Inequality* (1.2) *holds with the constant factor*

$$C_p := c_{|\cdot|^p} = \sup_{b \in [0,1]} R(p,b) = \max_{b \in (0,1/2)} R(p,b) = R(p, b_p), \tag{2.5}$$

where $R(p,b)$ is as in (2.4) *and b_p is as in Proposition* 2.2*. In particular, $C_2 = R(2,b) = 1$ for all $b \in [0,1]$.*

(ii) *C_p is the best possible constant factor in* (1.2)*. More specifically, the equality in* (1.2) *obtains if and only if one of the following three conditions holds:*

  (a) $\mathsf{E}\,|X|^p = \infty$;

  (b) $p = 2$, $\mathsf{E}\,X^2 < \infty$, *and* $\mathsf{E}\,X = 0$;

  (c) $p \neq 2$ *and* $X \overset{\mathrm{D}}{=} \lambda(X_{1-b_p, b_p} - t_{b_p})$ *for some* $\lambda \in \mathbb{R}$, *where*

$$t_b := b - \frac{b^{1/(p-1)}}{b^{1/(p-1)} + (1-b)^{1/(p-1)}} \tag{2.6}$$

  for all $b \in (0,1)$, and b_p is as in Proposition 2.2.

(iii) *One has the symmetries*

$$C_p^{1/\sqrt{p-1}} = C_q^{1/\sqrt{q-1}} \quad \text{and} \quad b_p = b_q, \tag{2.7}$$

where q is dual to p in the sense of L^p-spaces:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(iv) *For $p \to \infty$,*

$$C_p \sim \frac{2^p}{\sqrt{8ep}}; \tag{2.8}$$

as usual, $A \sim B$ means that $A/B \to 1$.

(v) *C_p is strictly log-convex and hence continuous in $p \in (1,\infty)$; moreover, C_p decreases in $p \in (1,2]$ from 2 to 1 and increases in $p \in [2,\infty)$ from 1 to ∞.*

(vi) *The values of C_p, b_p, and t_{b_p} are algebraic whenever p is rational; in particular,* $C_3 = \frac{1}{27}(17+7\sqrt{7}) = 1.315\dots$, $b_3 = \frac{1}{2} - \frac{1}{6}\sqrt{1+2\sqrt{7}} = 0.0819\dots$, *and* $t_{b_3} = -\frac{1}{3}\sqrt{\frac{1}{2}\left(13\sqrt{7}-34\right)} = -0.148\dots$.

By parts (vi) and (v) of Theorem 2.3, C_p can in principle be however closely bracketed for any real $p \in (1,\infty)$. However, such a calculation may in many cases be inefficient. On the other hand, Proposition 2.2 allows one to bracket the maximizer b_p of $R(b,p)$ however closely and thus, perhaps more efficiently, compute C_p with any degree of accuracy.

(A part of) the graph of C_p is shown in Figure 1, and those of $2^p/C_p$ and b_p are shown in Figure 2.

Remark 2.4. What if, instead of the condition (2.3), one has $p \in (0,1]$? It is easy to see that the inequality (1.2) holds for $p=1$ with $C_1 = 2$ (cf. (1.1)), which is then the best possible factor, as seen by letting

$$X = X_{1-b,b} - b \text{ with } b \downarrow 0. \tag{2.9}$$

However, the equality $\mathsf{E}\,|X - \mathsf{E}\,X| = 2\,\mathsf{E}\,|X|$ obtains only if $X \overset{\mathrm{D}}{=} 0$; one may also note here that, by part (v) of Theorem 2.3, $C_{1+} = 2 = C_1$. As to $p \in (0,1)$, for each such value of p the best possible factor C_p in (1.2) is ∞; indeed, consider X as in (2.9).

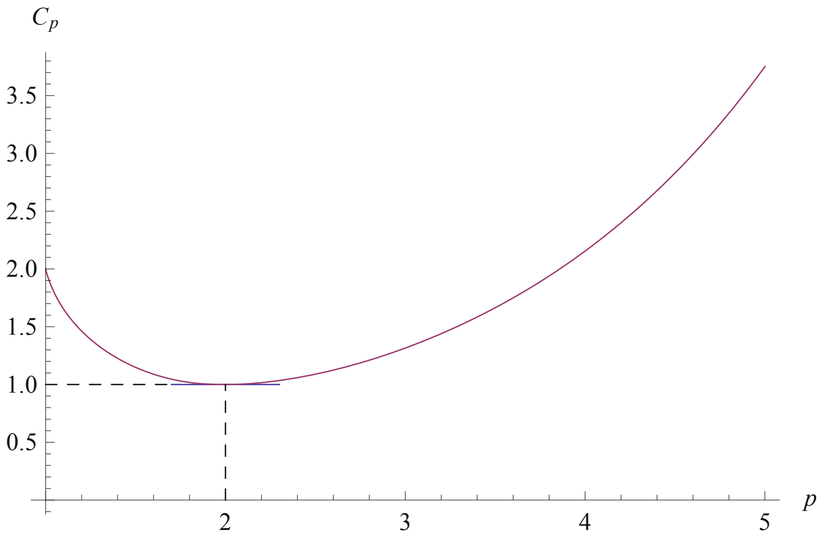

Figure 1. C_p decreases in $p \in (1,2]$ from 2 to 1 and increases in $p \in [2,\infty)$ from 1 to ∞.

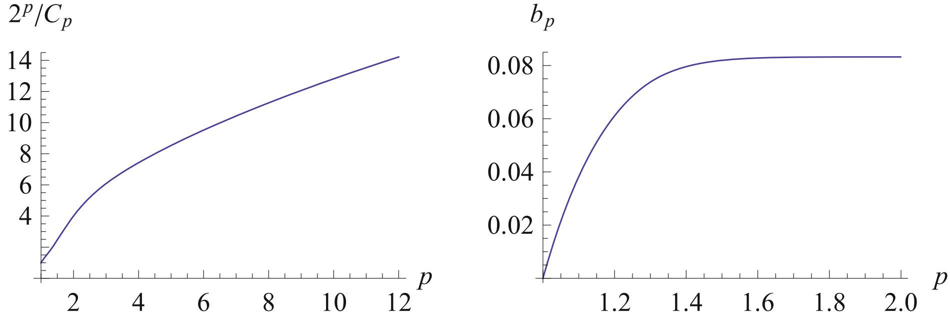

FIGURE 2. By (2.8), $2^p/C_p \sim \sqrt{8ep}$ as $p \to \infty$. By (2.7), $b_p = b_q$; note here also that $p \in (1,2] \iff q \in [2,\infty)$; by (4.16), $b_p \sim (p-1)/2$ as $p \downarrow 1$.

3. Application: Rosenthal-type concentration inequalities for separately Lipschitz functions on product spaces

It is well known that for every $p \in [2,\infty)$ there exist finite positive constants $c_1(p)$ and $c_2(p)$, depending only on p, such that for any independent real-valued zero-mean r.v.'s $X_1,\dots,X_n$

$$\mathsf{E}\,|Y|^p \leqslant c_1(p)A_p + c_2(p)B^p,$$

where $Y := X_1 + \cdots + X_n$, $A_p := \mathsf{E}\,|X_1|^p + \cdots + \mathsf{E}\,|X_n|^p$, and $B := (\mathsf{E}\,X_1^2 + \cdots + \mathsf{E}\,X_n^2)^{1/2}$. An inequality of this form was first proved by Rosenthal [27], and has since been very useful in many applications. It was generalized to martingales [4, (21.5)], including martingales in Hilbert spaces [19] and, further, in 2-smooth Banach spaces [23]. The constant factors $c_1(p)$ and $c_2(p)$ were actually allowed in [19] and [23] to depend on certain freely chosen parameters, which provided for optimal in a certain sense sizes of $c_1(p)$ and $c_2(p)$, for any given positive value of the Lyapunov ratio A_p/B^p. Best possible Rosenthal-type bounds for sums of independent real-valued zero-mean r.v.'s were given, under different conditions, by Utev [28] and Ibragimov and Sharakhmetov [6, 7]. Also for sums of independent real-valued zero-mean r.v.'s $X_1,\dots,X_n$, Latała [10] obtained an expression $\mathcal{E}$ in terms of p and the individual distributions of the X_i's such that $a_1\mathcal{E} \leqslant \|Y\|_p \leqslant a_2\mathcal{E}$ for some positive absolute constants a_1 and a_2.

Given a Rosenthal-type upper bound for real-valued martingales, one can use the Yurinskiĭ martingale decomposition [8] and (say) Theorem 2.3 to obtain a corresponding upper bound on the pth absolute *central* moment of the norm of the sum of independent random vectors in an arbitrary separable Banach space; even more generally, one can obtain such a measure-concentration inequality for separately Lipschitz functions on product spaces.

To state such a result, let $X_1,\dots,X_n$ be independent r.v.'s with values in measurable spaces $\mathfrak{X}_1,\dots,\mathfrak{X}_n$, respectively. Let $g\colon \mathfrak{P}\to\mathbb{R}$ be a measurable function on the product space $\mathfrak{P}:=\mathfrak{X}_1\times\dots\times\mathfrak{X}_n$. Let us say (cf. [1, 24]) that g is *separately Lipschitz* if it satisfies a Lipschitz-type condition in each of its arguments:

$$|g(x_1,\dots,x_{i-1},\tilde{x}_i,x_{i+1},\dots,x_n)-g(x_1,\dots,x_n)|\leqslant \rho_i(\tilde{x}_i,x_i) \tag{3.1}$$

for some measurable functions $\rho_i\colon \mathfrak{X}_i\times\mathfrak{X}_i\to\mathbb{R}$ and all $i\in\overline{1,n}$, $(x_1,\dots,x_n)\in\mathfrak{P}$, and $\tilde{x}_i\in\mathfrak{X}_i$. Take now any separately Lipschitz function g and let

$$Y:=g(X_1,\dots,X_n).$$

Suppose that the r.v. Y has a finite mean.

On the other hand, take any $p\in[2,\infty)$ and suppose that positive constants $c_1(p)$ and $c_2(p)$ are such that for all real-valued martingales $(\zeta_j)_{j=0}^n$ with $\zeta_0=0$ and differences $\xi_i:=\zeta_i-\zeta_{i-1}$

$$\mathsf{E}\,|\zeta_n|^p\leqslant c_1(p)\sum_1^n\mathsf{E}\,|\xi_i|^p+c_2(p)\Big(\sum_1^n\|\,\mathsf{E}_{i-1}\,\xi_i^2\|_\infty\Big)^{p/2}, \tag{3.2}$$

where E_j denotes the expectation given $\zeta_0,\dots,\zeta_j$.

Then one has

Corollary 3.1. *For each $i\in\overline{1,n}$, take any x_i and y_i in $\mathfrak{X}_i$. Then*

$$\mathsf{E}\,|Y-\mathsf{E}\,Y|^p\leqslant C_pc_1(p)\sum_1^n\mathsf{E}\,\rho_i(X_i,x_i)^p+c_2(p)\Big(\sum_1^n\mathsf{E}\,\rho_i(X_i,y_i)^2\Big)^{p/2}, \tag{3.3}$$

where C_p is as in (2.5).

An example of separately Lipschitz functions $g:\mathfrak{X}^n\to\mathbb{R}$ is given by the formula

$$g(x_1,\dots,x_n)=\|x_1+\dots+x_n\| \tag{3.4}$$

for all $x_1,\dots,x_n$ in a separable Banach space $(\mathfrak{X},\|\cdot\|)$. In this case, one may take $\rho_i(\tilde{x}_i,x_i)\equiv\|\tilde{x}_i-x_i\|$. Thus, one immediately obtains

Corollary 3.2. *Let $X_1,\dots,X_n$ be independent random vectors in a separable Banach space $(\mathfrak{X},\|\cdot\|)$. Let here $Y:=\|X_1+\dots+X_n\|$. For each $i\in\overline{1,n}$, take any x_i and y_i in $\mathfrak{X}_i$. Then*

$$\mathsf{E}\,|Y-\mathsf{E}\,Y|^p\leqslant C_pc_1(p)\sum_1^n\mathsf{E}\,\|X_i-x_i\|^p+c_2(p)\Big(\sum_1^n\mathsf{E}\,\|X_i-y_i\|^2\Big)^{p/2}. \tag{3.5}$$

Particular cases of separately Lipschitz functions more general than the norm of the sum as in (3.4) were discussed earlier in [21] and [20, pages 20–23].

For $p=2$, it is obvious that the inequality (3.2) holds with $c_1(2)=1$ and $c_2(2)=0$, and then the inequalities (3.3) and (3.5) do so. Thus, for $p=2$ (3.5) becomes

$$\mathsf{Var}\,Y\leqslant\sum_1^n\mathsf{E}\,\|X_i-x_i\|^2, \tag{3.6}$$

since $C_2 = 1$. The inequality (3.6) was presented in [20, page 29] and [22, Theorem 4], based on an improvement of the method of Yurinskiĭ [8]; cf. [1, 14, 15], [23, Proposition 2.5], and [24, Section 4]. The proof of Corollary 3.1 is based in part on the same kind of improvement.

The case $p = 3$ is also of particular importance in applications, especially to Berry–Esseen-type bounds; cf., e.g., [2, Lemma A1], [5, Lemma 6.3], and [18]. It follows from the main result of [19] that (3.2) holds for $p = 3$ with $c_1(3) = 1$ and $c_2(3) = 3$, whereas, by part (vi) of Theorem 2.3, $C_3 < 1.316$. Thus, one has an instance of (3.5) with rather small constant factors:

$$\mathsf{E}\,|Y - \mathsf{E}\,Y|^3 \leqslant 1.316 \sum_1^n \mathsf{E}\,\|X_i - x_i\|^3 + 3\Big(\sum_1^n \mathsf{E}\,\|X_i - y_i\|^2\Big)^{3/2}.$$

Similarly, the more general inequality (3.3) holds for $p = 3$ with 1.316 and 3 in place of $C_p c_1(p)$ and $c_2(p)$.

As can be seen from the proof given in Section 4, both Corollaries 3.1 and 3.2 will hold even if the separately-Lipschitz condition (3.1) is relaxed to

$$|\,\mathsf{E}\,g(x_1, \dots, x_{i-1}, \tilde{x}_i, X_{i+1}, \dots, X_n) - \mathsf{E}\,g(x_1, \dots, x_i, X_{i+1}, \dots, X_n)| \leqslant \rho_i(\tilde{x}_i, x_i). \tag{3.7}$$

Note also that in Corollaries 3.1 and 3.2 the r.v.'s X_i do not have to be zero-mean, or even to have any definable mean; at that, the arbitrarily chosen x_i's and y_i's may act as the centers, in some sense, of the distributions of the corresponding X_i's.

Other inequalities for the distributions of separately Lipschitz functions on product spaces were given in [1, 17, 24].

Clearly, the separate-Lipschitz (sep-Lip) condition (3.1) is easier to check than a joint-Lipschitz one. Also, sep-Lip (especially in the relaxed form (3.7)) is more generally applicable. On the other hand, when a joint-Lipschitz condition is satisfied, one can generally obtain better bounds. Literature on the concentration of measure phenomenon, almost all of it for joint-Lipschitz settings, is vast; let us mention here only [3, 9, 11–13].

4. Proofs

Proof of Theorem 2.1. It is well known that any zero-mean probability distribution on $\mathbb{R}$ is a mixture of zero-mean distributions on sets of at most two elements; see, e.g., [25, Proposition 3.18]. So, there exists a Borel probability measure μ on the set

$$S := \mathbb{R} \times (0, 1/2]$$

such that

$$\mathsf{E}\,g(X - \mathsf{E}\,X) = \int_S \mathsf{E}\,g(\lambda X_{1-b,b})\,\mu(\mathrm{d}\lambda \times \mathrm{d}b) \tag{4.1}$$

for all nonnegative Borel functions g; the measure μ depends on the distribution of the r.v. $X - \mathsf{E}\,X$. Letting now

$$S_0 := (\mathbb{R} \setminus \{0\}) \times (0, 1/2] \tag{4.2}$$

and using the condition $f(0) = 0$, one has

$$\begin{aligned}
\mathsf{E}\,f(X - \mathsf{E}\,X) &= \int_S \mathsf{E}\,f(\lambda X_{1-b,b})\,\mu(\mathrm{d}\lambda \times \mathrm{d}b) \\
&= \int_{S_0} \mathsf{E}\,f(\lambda X_{1-b,b})\,\mu(\mathrm{d}\lambda \times \mathrm{d}b) \\
&\leqslant \tilde{c}_f \int_{S_0} \mathsf{E}\,f(\lambda X_{1-b,b} + \mathsf{E}\,X)\,\mu(\mathrm{d}\lambda \times \mathrm{d}b) \qquad (4.3) \\
&\leqslant \tilde{c}_f \int_{S} \mathsf{E}\,f(\lambda X_{1-b,b} + \mathsf{E}\,X)\,\mu(\mathrm{d}\lambda \times \mathrm{d}b) \qquad (4.4) \\
&= \tilde{c}_f\,\mathsf{E}\,f\big((X - \mathsf{E}\,X) + \mathsf{E}\,X\big) = \tilde{c}_f\,\mathsf{E}\,f(X),
\end{aligned}$$

where

$$\tilde{c}_f := \sup\{\tilde{\rho}_f(\lambda, b, t)\colon (\lambda, b) \in S_0, t \in \mathbb{R}\} \quad \text{and} \tag{4.5}$$

$$\tilde{\rho}_f(\lambda, b, t) := \frac{\mathsf{E}\,f(\lambda X_{1-b,b})}{\mathsf{E}\,f\big(\lambda(X_{1-b,b} - t)\big)}, \tag{4.6}$$

so that

$$\tilde{c}_f = c_f. \tag{4.7}$$

Now the inequality in (2.1) follows from the above multi-line display and (4.7), and (4.7) (together with (4.5) and (4.6)) also shows that c_f is the best possible constant factor in (2.1). □

Proof of Proposition 2.2. It is straightforward to check the symmetry

$$R(p, b)^{1/\sqrt{p-1}} = R(q, b)^{1/\sqrt{q-1}} \tag{4.8}$$

for all $b \in [0, 1]$, where q is dual to p.

So, it remains to consider $p \in (1, 2)$. Also assume that $b \in (0, 1/2)$ and introduce

$$r := p - 1, \quad x := \frac{b}{1-b}, \quad \text{and} \quad z := -\frac{\ln x}{r}, \tag{4.9}$$

so that

$$r \in (0, 1), \quad x \in (0, 1), \quad \text{and} \quad z \in (0, \infty).$$

Now introduce

$$D_1(x) := D_1(r, x) := (1-b)\frac{x^r + 1}{x^{r-1} - 1}\,\partial_b \ln R(p, b) = r - \frac{(x - x^{1/r})(1 + x^r)}{(x^r - x)(1 + x^{1/r})} \tag{4.10}$$

and

$$D_2(x) := D_2(r, x) := r x^3 (1 + x^{1/r})^2 (x^{r-1} - 1)^2\,D_1'(x), \tag{4.11}$$

so that $D_1(x)$ and $D_2(x)$ are equal in sign to $\partial_b \ln R(p,b)$ and $D_1'(x)$, respectively. One can verify the identity

$$D_2(x)e^{(1+r+r^2)z}/2 = D_{21}(z) + (1-r)D_{22}(z), \tag{4.12}$$

where

$$D_{21}(z) := r^2 \operatorname{sh}((1-r)z) + \operatorname{sh}(r(1-r)z) - r\operatorname{sh}((1-r^2)z),$$
$$D_{22}(z) := h(z) - h(rz), \quad h(u) := \operatorname{sh} ru - r\operatorname{sh} u;$$

we use sh and ch for sinh and cosh. Note that $h'(u) = r(\operatorname{ch} ru - \operatorname{ch} u) < 0$ for $u > 0$ and hence

$$D_{22}(z) < 0.$$

Next,

$$\frac{D_{21}'(z)}{(1-r)r} = \big(\operatorname{ch}[(1-r)rz] - \operatorname{ch}[(1-r^2)z]\big) + r\big(\operatorname{ch}[(1-r)z] - \operatorname{ch}[(1-r^2)z]\big) < 0,$$

since $(1-r)r < 1-r < 1-r^2$. So, $D_{21}(z)$ is decreasing (in $z > 0$) and, obviously, $D_{21}(0+) = 0$. Hence, $D_{21}(z) < 0$ as well. Thus, by (4.12), $D_2(x) < 0$, which shows that $D_1'(x) < 0$ and $D_1(x)$ is decreasing – in $x \in (0,1)$. Moreover, $D_1(0+) = r > 0 > r - 1/r = D_1(1-)$. It follows, in view of (4.11), that $D_1(x)$ changes in sign exactly once, from $+$ to $-$, as x increases from 0 to 1. Equivalently, by (4.10), $\partial_b \ln R(p,b)$ changes in sign exactly once, from $+$ to $-$, as b increases from 0 to 1/2. This completes the proof of Proposition 2.2. □

Proof of Theorem 2.3. (i) To begin the proof of part (i) of Theorem 2.3, note that the last two inequalities in (2.5) follow by the obvious symmetry

$$R(p,b) = R(p,1-b) \quad \text{for all } b \in [0,1] \tag{4.13}$$

and Proposition 2.2.

Next, in view of the definition of C_p in (2.5), inequality (1.2) is a special case of (2.1). Moreover, by the definition of $\tilde\rho$ in (4.6) and the homogeneity of the power function $|\cdot|^p$,

$$\tilde\rho_{|\cdot|^p}(\lambda,b,t) = \rho_p(b,t) := \tilde\rho_{|\cdot|^p}(1,b,t) = \frac{\mathsf{E}\,|X_{1-b,b}|^p}{\mathsf{E}\,|X_{1-b,b}-t|^p} \tag{4.14}$$

for all $(\lambda,b) \in S_0$ and $t \in \mathbb{R}$, where S_0 is as in (4.2). Next, the denominator $\mathsf{E}\,|X_{1-b,b}-t|^p$ decreases in $t \in (-\infty, b-1]$, increases in $t \in [b,\infty)$, and attains its minimum over all $t \in [b-1,b]$ $\big($and thus over all $t \in \mathbb{R}\big)$ only at $t = t_b$, where t_b is as in (2.6). So,

$$\max_{\lambda\in\mathbb{R}\setminus\{0\},\, t\in\mathbb{R}} \tilde\rho_{|\cdot|^p}(\lambda,b,t) = \max_{t\in\mathbb{R}} \rho_p(b,t) = \rho_p(b,t_b) = R(p,b) \tag{4.15}$$

for all $b \in (0,1/2]$, in view of (2.4). Now (4.7), (4.5), and (4.13) yield

$$c_{|\cdot|^p} = \sup_{b\in(0,1/2]} R(p,b) = \sup_{b\in[0,1]} R(p,b).$$

Thus, the proof of (2.5) and all of part (i) of Theorem 2.3 is complete.

(ii) That the equality in (1.2) obtains under either of the conditions (a) or (b) in part (ii) of Theorem 2.3 is trivial. If the condition (c) of part (ii) holds with $\lambda = 0$, then $X \overset{\mathrm{D}}{=} 0$, and again the equality in (1.2) is trivial. If now (c) holds with some $\lambda \in \mathbb{R} \setminus \{0\}$ – so that $X \overset{\mathrm{D}}{=} \lambda(X_{1-b_p,b_p} - t_{b_p})$, then (2.5), (4.15), and (4.14) imply

$$C_p = R(p, b_p) = \rho_p(b_p, t_{b_p}) = \frac{\mathsf{E}\,|X_{1-b_p,b_p}|^p}{\mathsf{E}\,|X_{1-b_p,b_p} - t_{b_p}|^p} = \frac{\mathsf{E}\,|X - \mathsf{E}\,X|^p}{\mathsf{E}\,|X|^p},$$

whence the equality in (1.2) follows. Thus, for the equality in (1.2) to hold it is sufficient that one of the conditions (a), (b), or (c) be satisfied.

Let us now verify the necessity of one of these three conditions. W.l.o.g. condition (a) fails to hold, so that $\mathsf{E}\,|X|^p < \infty$. If now $p = 2$ then $C_p = C_2 = 1$, and the necessity of the condition $\mathsf{E}\,X = 0$ for the equality in (1.2) is obvious. It remains to consider the case when $p \neq 2$ and $\mathsf{E}\,|X|^p < \infty$. Suppose that one has the equality in (1.2) and let $f = |\cdot|^p$. Then, by the definition of C_p in (2.5) and the equality (4.7), equalities take place in (4.3) and (4.4). In view of the condition $\mathsf{E}\,|X|^p < \infty$, the integrals in (4.3) and (4.4) are both finite and equal to each other. So, the equality in (4.4) means that $|\,\mathsf{E}\,X|^p\,\mu\big(\{0\} \times (0, 1/2]\big) = 0$. If now $\mu\big(\{0\} \times (0, 1/2]\big) \neq 0$ then $\mathsf{E}\,X = 0$, and the equality in (1.2) takes the form $\mathsf{E}\,|X|^p = C_p\,\mathsf{E}\,|X|^p$; but, by part (v) of Theorem 2.3 (to be proved a bit later), the condition $p \neq 2$ implies $C_p > 1$, which yields $\mathsf{E}\,|X|^p = 0$, and so, $X \overset{\mathrm{D}}{=} \lambda(X_{1-b_p,b_p} - t_{b_p})$ for $\lambda = 0$. It remains to consider the case when $p \neq 2$, $\mathsf{E}\,|X|^p < \infty$, and $\mu\big(\{0\} \times (0, 1/2]\big) = 0$. Then $\mu(S_0) = \mu(S) = 1$, and the equality in (4.3) (again with $f = |\cdot|^p$), together with (2.5) and (4.7), will imply that $\mathsf{E}\,|\lambda X_{1-b,b}|^p = C_p\,\mathsf{E}\,|\lambda X_{1-b,b} + \mathsf{E}\,X|^p$ for μ-almost all $(\lambda, b) \in S_0$. In view of (4.14), (2.5), Proposition 2.2, and (4.15), this in turn yields

$$\rho_p(b, -\mathsf{E}\,X/\lambda) = R(p, b_p) \geqslant R(p, b) = \rho_p(b, t_b)$$

for μ-almost all $(\lambda, b) \in S_0$. Now recall that for each $b \in (0, 1/2]$ the maximum of $\rho_p(b, t)$ in $t \in \mathbb{R}$ is attained only at $t = t_b$. It follows that for μ-almost all $(\lambda, b) \in S_0$ one has

(i) $R(p, b_p) = R(p, b)$ and hence, by Proposition 2.2, $b = b_p$ and
(ii) $-\mathsf{E}\,X/\lambda = t_b = t_{b_p}$ or, equivalently, $\lambda = -\mathsf{E}\,X/t_b = -\mathsf{E}\,X/t_{b_p} =: \lambda_p$.

Therefore, $(\lambda, b) = (\lambda_p, b_p)$ for μ-almost all $(\lambda, b) \in S_0$ and thus for μ-almost all $(\lambda, b) \in S$. Now (4.1) shows that $X + \lambda_p t_{b_p} = X - \mathsf{E}\,X \overset{\mathrm{D}}{=} \lambda_p X_{1-b_p,b_p}$ or, equivalently, $X \overset{\mathrm{D}}{=} \lambda_p(X_{1-b_p,b_p} - t_{b_p})$, which completes the proof of part (ii) of Theorem 2.3.

(iii) Part (iii) of Theorem 2.3 follows immediately by the symmetry (4.8) of $R(p, b)$ in p and the definitions of C_p and b_p in (2.5) and Proposition 2.2, respectively.

(iv) As in (4.9), let $r := p - 1$, so that $r \to \infty$. For a moment, take any $k \in (0, \infty)$ and choose $b = \frac{k}{r}$. Then, by (4.9), $x \sim b = \frac{k}{r}$, and now (4.10) yields $D_1(r, x) \sim (1 - \frac{1}{2k})r$, whence $D_1(r, x)$ is eventually (i.e., for all large enough r) positive or negative according as k is greater or less than $\frac{1}{2}$. So, again by (4.9), for any real $\check{k}$ and $\hat{k}$ such that $0 < \check{k} < \frac{1}{2} < \hat{k}$, eventually $\partial_b R(p,b)\big|_{b=\check{k}/r} > 0 > \partial_b R(p,b)\big|_{b=\hat{k}/r}$. It follows by Proposition 2.2 that

$$b_p \sim \frac{1}{2r}, \tag{4.16}$$

that is, $b_p = \kappa/r$ for some κ varying with r so that $\kappa \to 1/2$. Hence,

$$(1-b_p)^r + b_p^r = (1-\kappa/r)^r + (\kappa/r)^r \to e^{-1/2}. \tag{4.17}$$

Next, $b_p^{1/r} = (\kappa/r)^{1/r} = \exp\left(\frac{1}{r}\ln\frac{\kappa}{r}\right) = 1 + \frac{1}{r}\ln\frac{\kappa}{r} + O\left(\left(\frac{1}{r}\ln\frac{\kappa}{r}\right)^2\right)$ and $(1-b_p)^{1/r} = 1 + O(1/r^2)$, whence

$$\begin{aligned}\left((1-b_p)^{1/r} + b_p^{1/r}\right)^r &= \left[2\left(1 + \frac{1}{2r}\ln\frac{\kappa}{r} + O\left(\frac{\ln^2 r}{r^2}\right)\right)\right]^r \\ &= \left[2\exp\left\{\frac{1}{2r}\ln\frac{\kappa}{r} + o\left(\frac{1}{r}\right)\right\}\right]^r \sim 2^r\sqrt{\frac{\kappa}{r}} \sim \frac{2^p}{\sqrt{8p}}.\end{aligned}$$

Recalling now (2.5), (2.4), and (4.17), one obtains (2.8).
(v) Take any $b \in (0,1/2)$. Then

$$d_{2,1}(r) := \partial_r\,\partial_r\,\ln\left(b^r + (1-b)^r\right) = \frac{(1-b)^r b^r}{\left(b^r + (1-b)^r\right)^2}\ln^2\frac{1-b}{b} > 0$$

for all $r > 0$. Moreover,

$$d_{2,2}(r) := \partial_r\,\partial_r\,\ln\left[\left(b^{1/r} + (1-b)^{1/r}\right)^r\right] = d_{2,1}(1/r)/r^3 > 0$$

for all $r > 0$. So, $\partial_p\,\partial_p \ln R(p,b) = d_{2,1}(p-1) + d_{2,2}(p-1) > 0$, which shows that $R(p,b)$ is strictly log-convex in $p \in (1,\infty)$. Also, $\partial_p \ln R(p,b)\big|_{p=2} = 0$, so that $R(p,b)$ decreases in $p \in (1,2]$ and increases in $p \in [2,\infty)$, with $R(2,b) = 1$. Therefore and in view of (2.5) – note in particular the attainment of the supremum there, C_p is strictly log-convex and hence continuous in $p \in (1,\infty)$, and it also follows that C_p decreases in $p \in (1,2]$ and increases in $p \in [2,\infty)$, with $C_p = 1$. Next, (2.8) shows that $C_p \to \infty$ as $p \to \infty$. Letting now $p \downarrow 1$ and using (2.7), one has $q \to \infty$ and hence $C_p = C_q^{1/(q-1)} = \left(2^q/\sqrt{(8+o(1))eq}\right)^{1/(q-1)} \to 2$. This completes the proof of part (v) of Theorem 2.3.
(vi) The proof of part (vi) of Theorem 2.3 is straightforward, in view of (2.5), Proposition 2.2, (2.4), and (2.6). □

Proof of Corollary 3.1. The proof is based on ideas presented in [20, 22] concerning the use of the mentioned Yurinskiĭ martingale decomposition; similar ideas were also used, e.g., in [1, 17, 24]. Consider the martingale defined by the formula $\zeta_j := \mathsf{E}_j(Y - \mathsf{E}\,Y)$ for $j \in \overline{0,n}$, where E_j stands for the conditional expectation given the σ-algebra generated by $(X_1,\dots,X_j)$, with $\mathsf{E}_0 := \mathsf{E}$, and then consider the differences $\xi_i := \zeta_i - \zeta_{i-1}$. Next, for each $i \in \overline{1,n}$ introduce the r.v.

$$\eta_i := \mathsf{E}_i(Y - \tilde{Y}_i),$$

where $\tilde{Y}_i := g(X_1, \dots, X_{i-1}, x_i, X_{i+1}, \dots, X_n)$, so that $\xi_i = \eta_i - \mathsf{E}_{i-1}\,\eta_i$, since the r.v.'s $X_1, \dots, X_n$ are independent. Also, in view of (3.1) or (3.7), for all $i \in \overline{1,n}$ and $z_i \in \mathfrak{X}_i$ one has $|\eta_i| \leqslant \rho_i(X_i, z_i)$, whence, by (1.2),

$$\begin{aligned}\mathsf{E}_{i-1}\,|\xi_i|^r = \mathsf{E}_{i-1}\,|\eta_i - \mathsf{E}_{i-1}\,\eta_i|^r \leqslant C_r\,\mathsf{E}_{i-1}\,|\eta_i|^r \leqslant C_r\,\mathsf{E}_{i-1}\,\rho_i(X_i, z_i)^r \\ = C_r\,\mathsf{E}\,\rho_i(X_i, z_i)^r\end{aligned}$$

for all $r \in (1, \infty)$. Now (3.3) follows from (3.2), since $\zeta_n = Y - \mathsf{E}\,Y$ and $C_2 = 1$. □

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Strong Log-concavity is Preserved by Convolution

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Abstract. We review and formulate results concerning strong-log-concavity in both discrete and continuous settings. Although four different proofs of preservation of strong log-concavity are known in the discrete setting (where strong log-concavity is known as "ultra-log-concavity"), preservation of strong log-concavity under convolution has apparently not been investigated previously in the continuous case.

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1. Log-concavity and ultra-log-concavity for discrete distributions

We begin with a discussion of log-concavity and ultra-log-concavity in the setting of discrete random variables. This material is from [13] and [11].

A sequence $\{a_i : i \in \mathbb{Z}_+\}$ of non-negative real numbers is *log-concave* if

$$a_i^2 \geq a_{i-1}a_{i+1} \quad \text{for} \quad i \geq 1$$

and the set $\{i \geq 0 : a_i > 0\}$ is an interval of integers. A non-negative integer-valued random variable X with probability mass function $\{p_x : x \in \mathbb{Z}_+\}$ is *log-concave* if $\{p_x\}$ is a log-concave sequence with $\sum_{x=0}^{\infty} p_x = 1$. A stronger notion, analogous to strong log-concavity in the case of continuous random variables, is that of *ultra-log-concavity*: for any $\lambda > 0$ define $\mathbf{ULC}(\lambda)$ to be the class of non-negative integer-valued random variables X with mean $EX = \lambda$ such that the probability mass function p_x satisfies

$$xp_x^2 \geq (x+1)p_{x+1}p_{x-1} \quad \text{for all} \quad x \geq 1. \tag{1.1}$$

Then the class of ultra log-concave random variables is $\mathbf{ULC} = \cup_{\lambda>0}\mathbf{ULC}(\lambda)$. Note that (1.1) is equivalent to log-concavity of $x \mapsto p_x/\pi_{\lambda,x}$ where $\pi_{\lambda,x} =$

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$e^{-\lambda}\lambda^x/x!$ is the Poisson distribution on $\mathbb{N}$, and hence ultra-log-concavity corresponds to p being *log-concave relative to* π_λ (or $p \le_{\rm lc} \pi_\lambda$) in the sense defined by [23] pages 625–626: $p \le_{\rm lc} q$ if p/q is log-concave. Thus $X \in \mathbf{ULC}(\lambda)$ if and only if $EX = \lambda$ and $p_x = h_x \pi_{\lambda,x}$ where h is log-concave. When we want to emphasize that the mass function $\{p_x\}$ corresponds to X, we also write $p_X(x)$ instead of p_x.

Our main interest here is the preservation of ultra-log-concavity under convolution.

Theorem 1.1 ([17]). *The class of ultra-log-concave distributions on $\mathbb{Z}$ is closed under convolution. More precisely, these classes are closed under convolution in the following sense: if $U \in \mathbf{ULC}(\lambda)$ and $V \in \mathbf{ULC}(\mu)$ are independent, then $U + V \in \mathbf{ULC}(\lambda + \mu)$.*

Liggett's proof proceeds by direct calculation. For recent alternative proofs of this property of ultra-log-concave distributions, see [8], [14], and [18]. A relatively simple proof is given by [11] using results from [16] and [6], and that is the proof we will summarize here.

Proposition 1.2. $X \in \mathbf{ULC}(\lambda)$ *if and only if the relative score function*

$$\rho_X(i) \equiv \frac{(i+1)p_X(i+1)}{\lambda p_X(i)} - 1 = \frac{(i+1)p_X(i+1)}{\lambda p_X(i)} - \frac{(i+1)\pi_{\lambda,i+1}}{\lambda \pi_{\lambda,i}}$$

is a decreasing function of i.

Proof. This follows immediately from the definitions and elementary rearrangement of terms. □

Proposition 1.3.

(i) *If X and Y are independent discrete random variables with means $\mu = E(X)$ and $\nu = E(Y)$ respectively, then*

$$\rho_{X+Y}(z) = E\left\{ \frac{\mu}{\mu+\nu}\rho_X(X) + \frac{\nu}{\mu+\nu}\rho_Y(Y) \middle| X + Y = z \right\}.$$

(ii) *If $X \in \mathbf{ULC}(\mu)$, $Y \in \mathbf{ULC}(\nu)$ are independent, then $X + Y \in \mathbf{ULC}(\mu + \nu)$.*

Proof. (i) This projection formula is proved in the Lemma on page 471 of [16] by direct calculation.

(ii) This claim follows from (i) and Theorem 1 of [6], upon noting Efron's remark 1, page 278, concerning the discrete case of his theorem: for independent log-concave random variables X and Y and a measurable function Φ monotone (decreasing here) in each argument, $E\{\Phi(X,Y)|X+Y=z\}$ is a monotone decreasing function of z: note that ultra-log-concavity of X and Y implies that

$$\Phi(x,y) = \frac{\mu}{\mu+\nu}\rho_X(x) + \frac{\nu}{\mu+\nu}\rho_Y(y)$$

is a monotone decreasing function of x and y (separately) by Proposition 1.2. Thus ρ_{X+Y} is a decreasing function of z by Proposition 1.3 and Efron's theorem, and hence $X + Y \in \mathbf{ULC}(\mu + \nu)$ by Proposition 1.2 again. This completes the proof of (ii) of Proposition 1.3 and hence Theorem 1.1. □

Here are some facts concerning the entropy of discrete random variables, Bernoulli sums, and ultra-log-concavity.

For any probability distribution $\{p_x : x \in \mathbb{Z}\}$, the entropy $H(p)$ is given by

$$H(p) \equiv -\sum_x p_x \log p_x.$$

If $X_1, \ldots, X_n$ are independent Bernoulli$(p_1), \ldots,$ Bernoulli(p_n) random variables, then $S_n = \sum_{i=1}^n X_i$ is called a *Bernoulli sum*, and we write $b_{\mathbf{p}}(x) \equiv P_{\mathbf{p}}(S_n = x)$ for $x \in \mathbb{N}$ for its probability mass function where $\mathbf{p} = (p_1, \ldots, p_n) \in [0,1]^n$. Furthermore, for each $\lambda > 0$ set

$$\mathcal{P}_n(\lambda) = \{\mathbf{p} \in [0,1]^n : p_1 + \cdots + p_n = \lambda\},$$
$$\mathcal{P}_\infty(\lambda) = \cup_{n=1}^\infty \mathcal{P}_n(\lambda).$$

Fact 1.4 ([20]). *For each fixed $n \geq 1$, the Bernoulli sum b_p which has maximal entropy among all Bernoulli sums with mean λ is Binomial $(n, \lambda/n)$, the Binomial with parameters n and λ/n. In other words,*

$$H(\mathit{Binomial}(n, \lambda/n)) = \max\{H(b_{\mathbf{p}}) : \mathbf{p} \in \mathcal{P}_n(\lambda)\}.$$

This was extended to the Poisson distribution by Harremoës: If Po(λ) denotes the Poisson distribution with mean λ on $\mathbb{N}$ then we have:

Fact 1.5 ([10]).

$$H(Po(\lambda)) = \sup\{H(b_{\mathbf{p}}) : \mathbf{p} \in \mathcal{P}_\infty(\lambda)\}.$$

Fact 1.6. *The Poisson distribution and all Bernoulli sums are ultra-log-concave. (This is trivial for Poisson, easily verified for any Bernoulli variable, and hence true for Bernoulli sums by Proposition* 2*. Also see* [7] *for a direct proof that Bernoulli sums are ultra-log-concave in the terminology of* [17]*.)*

Fact 1.7 ([11]).

$$H(Po(\lambda)) = \max\{H(p) : p \in \mathbf{ULC} \ \ \textit{with mean } \lambda\}.$$

Theorem 1.8. *For any $\lambda \geq 0$, if $X \in \mathbf{ULC}(\lambda)$ then the entropy $H(X)$ of X satisfies $H(X) \leq H(Z_\lambda)$ with equality if and only if $X \stackrel{d}{=} Z_\lambda \sim Poisson(\lambda)$.*

Proposition 1.9. *For any $\lambda \geq 0$ and $\mu \geq 0$:*

(i) *If $V \in \mathbf{ULC}(\lambda)$ then it is log-concave.*
(ii) *The Poisson random variable $Z_\lambda \in \mathbf{ULC}(\lambda)$.*
(iii) *The classes are closed under convolution in the following sense: if $U \in \mathbf{ULC}(\lambda)$ and $V \in \mathbf{ULC}(\mu)$ are independent, then $U + V \in \mathbf{ULC}(\lambda + \mu)$.*
(iv) *$\mathcal{P}_\infty(\lambda) \subset \mathbf{ULC}(\lambda)$.*

Fact 1.10. *It follows from* [7] *that the hypergeometric distribution* (*sampling without replacement count of "successes"*) *is equal in distribution to a Bernoulli sum; hence the hypergeometric distribution is ultra-log-concave.*

Question 1.11. Is there an analogue of Chernoff's density in the discrete case which is ultra-log-concave? (See [1] for Chernoff's density in the continuous case. Possible connections to Polya frequency sequences as treated in [15], Chapter 8?)

2. Log-concavity and strong-log-concavity for continuous distributions on $\mathbb{R}$

Despite considerable interest in strong log-concavity as a hypothesis for correlation inequalities, log-Sobolev inequalities, and various results in transportation theory (see, e.g., [9], [22], and [4]), I am unaware of any previous proof that strong log-concavity is preserved by convolution. Here we give a proof of this preservation property along the lines of the proof by [11] in the discrete case discussed in Section 1.

A non-negative function g on $\mathbb{R}$ (or $\mathbb{R}^d$) is *log-concave* if for all $x, y \in \mathbb{R}$ (respectively $\mathbb{R}^d$) and $\theta \in (0,1)$

$$g(\theta x + (1-\theta)y) \ge g(x)^\theta g(y)^{1-\theta}.$$

A density function g on $\mathbb{R}$ (respectively $\mathbb{R}^d$) is *log-concave* if it is a non-negative log-concave function with $\int_{\mathbb{R}} g(x)dx = 1$ (respectively $\int_{\mathbb{R}^d} g(x)dx = 1$).

Definition 2.1. For any $\sigma^2 > 0$ define the class *strongly-log-concave* with variance parameter σ^2, denoted $\mathbf{SLC}(\boldsymbol{\sigma^2})$, to be the collection of random variables X (or their corresponding density functions $f = f_X$) with $EX = 0$, $\mathrm{Var}(X) = \sigma^2$, $P(X \in dx) = f(x)dx$, such that

$$f(x) = g(x)\frac{1}{\sigma}\phi(x/\sigma) \quad \text{with} \quad g \quad \text{log-concave} \tag{2.1}$$

where $\phi(z) = (2\pi)^{-1/2}\exp(-z^2/2)$ is the standard Gaussian density.

Thus strong log-concavity of f is equivalent to f being *log-concave relative to* $\phi(\cdot/\sigma)/\sigma$ (or $f \le_{\mathrm{lc}} \sigma^{-1}\phi(\cdot/\sigma)$) in the terminology of [23]: $f \le_{\mathrm{lc}} g$ if and only if f/g is log-concave. When $(-\log f)$ is twice differentiable, a useful sufficient condition is

$$(-\log f)''(x) \ge \frac{1}{\sigma^2} \quad \text{for all} \quad x \in \mathbb{R}.$$

For X with $\mathrm{Var}(X) = \sigma^2$, define the *relative score*

$$\rho_X(x) \equiv \rho_f(x) \equiv -\frac{f'}{f}(x) - \frac{x}{\sigma^2} = -\frac{f'}{f}(x) - \left(-\log\{\sigma^{-1}\phi(x/\sigma)\}\right)'$$

where $\phi(z) = (2\pi)^{-1/2}\exp(-z^2/2)$ is the standard Gaussian density.

Strong log-concavity of f implies that $(-\log f)'$ exists at all but countably many points; see, e.g., [21], Theorem 1.26, page 19. By using the left derivative of

f we can define ρ_X for every strongly log-concave X with density f. Note that the "standardized Fisher Information" $J(X)$ of [2] is given by

$$J(X) = \sigma^2 E_f \rho_f^2(X).$$

Furthermore

$$\begin{aligned} D(X) \equiv K(f, \phi) &\equiv \int f(x) \log \frac{f(x)}{\phi(x)} dx \\ &= \int_0^1 J(\sqrt{t} X + \sqrt{1-t} Z) \frac{1}{2t} dt \\ &= \int_0^\infty J(e^{-v} X + \sqrt{1 - e^{-2v}} Z) dv \end{aligned}$$

where $Z \sim N(0, \sigma^2)$. Compare with (1.14) of [5].

To prove Theorem 2.3 below, we first prove an analogue of Proposition 1.2 in the preceding section:

Proposition 2.2. *$X \in \mathbf{SLC}(\sigma^2)$ if and only if $\rho_f(x)$ is nondecreasing in x.*

Proof. Suppose that ρ_f is nondecreasing where we take f' to be the left derivative of f. Then for $x > x_0$,

$$\begin{aligned} \int_{x_0}^x \rho_f(y) dy &= \int_{x_0}^x \frac{-f'(y)}{f(y)} dy - \int_{x_0}^x \frac{y}{\sigma^2} dy \\ &= -\log f(x) - (-\log f(x_0)) - \frac{1}{2\sigma^2}(x^2 - x_0^2) \end{aligned}$$

is a convex function of x. Thus

$$-\log f(x) - \frac{1}{2\sigma^2} x^2,$$

is convex, and it follows that $f(x) = g(x)\sigma^{-1}\phi(x/\sigma)$ where g is log-concave.

On the other hand suppose that (2.1) holds. Then

$$-\log f(x) = -\log g(x) - \log(\phi(x/\sigma)/\sigma)$$

is convex and its derivative (which exists at all but countably many x's; see, e.g., [21], page 19) is

$$-\frac{f'}{f}(x) = -\frac{g'}{g}(x) + \frac{x}{\sigma^2}$$

where

$$-\frac{g'}{g}(x) = -\frac{f'}{f}(x) - \frac{x}{\sigma^2} = \rho_f(x)$$

is non-decreasing since $-\log g$ is convex; see [21], Theorem 1.26, page 19, or [19], Exercise 12.59, page 565. $\square$

From [3] (see (2.5) on page 142) or [12], Lemma 3.1 we know that for independent random variables X and Y with absolutely continuous densities f and g respectively we have

$$E\left\{-\frac{f'_Y}{f_Y}(Y)|X+Y=z\right\} = -\frac{f'_{X+Y}}{f_{X+Y}}(z). \tag{2.2}$$

This yields the following theorem:

Theorem 2.3.

(i) *If X and Y are independent random variables with variances $\sigma^2 = \mathrm{Var}(X)$ and $\tau^2 = \mathrm{Var}(Y)$, and absolutely continuous densities f and g respectively, then*

$$\rho_{X+Y}(z) = E\left\{\frac{\sigma^2}{\sigma^2+\tau^2}\rho_X(X) + \frac{\tau^2}{\sigma^2+\tau^2}\rho_Y(Y)\middle| X+Y=z\right\}.$$

(ii) *If $X \in \mathbf{SLC}(\sigma^2)$ and $Y \in \mathbf{SLC}(\tau^2)$ are independent, then $X+Y \in \mathbf{SLC}(\sigma^2+\tau^2)$.*

Proof. (i) follows immediately from the projection formula (2.2) and linearity of conditional expectation: here is a detailed calculation.

$$\begin{aligned}
E&\left\{\frac{\sigma^2}{\sigma^2+\tau^2}\left\{-\frac{f'_X}{f_X}(X) - \frac{X}{\sigma^2}\right\} + \frac{\tau^2}{\sigma^2+\tau^2}\left\{-\frac{f'_Y}{f_Y}(Y) - \frac{Y}{\tau^2}\right\}\middle| X+Y=z\right\} \\
&= \frac{\sigma^2}{\sigma^2+\tau^2}\left\{-\frac{f'_{X+Y}}{f_{X+Y}}(z)\right\} + \frac{\tau^2}{\sigma^2+\tau^2}\left\{-\frac{f'_{X+Y}}{f_{X+Y}}(z)\right\} \\
&\quad - E\left\{\frac{X+Y}{\sigma^2+\tau^2}\middle| X+Y=z\right\} \\
&= -\frac{f'_{X+Y}}{f_{X+Y}}(z) - \frac{z}{\sigma^2+\tau^2} \\
&= \rho_{X+Y}(z).
\end{aligned}$$

(ii) follows from (i) and Efron's ([6]) observation that for independent log-concave random variables X and Y and a measurable function Φ increasing in each argument, $E\{\Phi(X,Y)|X+Y=z\}$ is an increasing function of z: take

$$\Phi(x,y) = \frac{\sigma^2}{\sigma^2+\tau^2}\rho_X(x) + \frac{\tau^2}{\sigma^2+\tau^2}\rho_Y(y).$$

Thus ρ_{X+Y} is an increasing function of z and hence $X+Y \in \mathbf{SLC}(\sigma^2+\tau^2)$ by Proposition 2.2. □

Question 2.4. Are there alternative proofs of (ii) of Theorem 2.3 paralleling the alternative proofs by [8] and [14] that ultra-log-concavity is preserved by convolution?

Question 2.5. Is multivariate strong log-concavity preserved under convolution?

Question 2.6. Can the result of Theorem 2.3 be used to prove the strong log-concavity of Chernoff's density conjectured in [1]?

3. Appendix: strong convexity and strong log-concavity

Following [19], page 565, we say that a proper convex function $h : \mathbb{R}^d \to \overline{\mathbb{R}}$ is *strongly convex* if there exists a positive number c such that

$$h(\theta x + (1-\theta)y) \le \theta h(x) + (1-\theta)h(y) - \frac{1}{2}c\theta(1-\theta)\|x-y\|^2$$

for all $x, y \in \mathbb{R}^d$and $\theta \in (0,1)$. It is easily seen that this is equivalent to convexity of $h(x) - (1/2)c\|x\|^2$ (see [19], Exercise12.59, page 565).

Now f is strongly log-concave if and only

$$f(x) \quad = g(x)\sigma^{-d}\prod_{j=1}^{d}\phi(x_j/\sigma)$$

for some $\sigma > 0$ where g is log-concave. But this agrees with the definition of strong convexity given above since,

$$h(x) \equiv -\log f(x) = -\log g(x) + d\log(\sigma\sqrt{2\pi}) + \frac{\|x\|^2}{2\sigma^2},$$

so that

$$-\log f(x) - \frac{\|x\|^2}{2\sigma^2} = -\log g(x) + d\log(\sigma\sqrt{2\pi})$$

is convex; i.e., $-\log f(x)$ is strongly convex with $c = 1/(2\sigma^2)$.

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On Some Gaussian Concentration Inequality for Non-Lipschitz Functions

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Abstract. A concentration inequality for functions of a pair of Gaussian random vectors is established. Instead of the usual Lipschitz condition some boundedness of second-order derivatives is assumed. This result can be viewed as an extension of a well-known tail estimate for Gaussian random bi-linear forms to the non-linear case.

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1. Introduction

The Gaussian concentration inequality in one of its forms states that for a standard Gaussian random vector X in $\mathbb{R}^n$ and a function $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfying the Lipschitz condition, with a constant $L > 0$,

$$\mathbb{P}(|f(X) - \mathbb{E}f(X)| \geq t) \leq 2\exp\left(-\frac{t^2}{2L^2}\right) \tag{1.1}$$

for all $t > 0$ (see, e.g., [5, Ch. 2.3]).

In order to motivate further considerations, let us make the following trivial observation. Let $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function satisfying

$$\left|\frac{\partial^2 f}{\partial x \partial y}(x,y)\right| \leq L$$

for all $(x,y) \in \mathbb{R}\times\mathbb{R}$ and put $g(x,y) = f(x,0) + f(0,y) - f(0,0)$. Then for any $t > 0$,

$$(\gamma_1 \otimes \gamma_1)\big(\{(x,y) \in \mathbb{R}\times\mathbb{R}\colon |f(x,y) - g(x,y)| \geq t\}\big) \leq Ce^{-ct/L}, \tag{1.2}$$

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where γ_1 is the standard Gaussian distribution on $\mathbb{R}$. This fact is as trivial as the Gaussian concentration on the real line. Just write

$$f(x,y) - g(x,y) = f(x,y) - f(x,0) - f(0,y) + f(0,0) = \int_0^x \int_0^y \frac{\partial^2 f}{\partial x \partial y}(u,v)\, du dv$$

to see that

$$|f(x,y) - g(x,y)| \leq L|xy|.$$

Now the desired inequality follows from the fact that the random variable $|g_1 g_2|$, where g_1, g_2 are independent $\mathcal{N}(0,1)$, roughly behaves like an exponential random variable, i.e., $\mathbb{P}(|g_1 g_2| \geq t) \leq Ce^{-ct}$. The example $f(x,y) = f_1(x) + f_2(y)$ shows that under the assumption of boundedness of the second-order mixed derivative, one cannot control the deviation of f from anything "simpler" than a linear combination of functions depending on a single variable only.

This observation leads to natural questions: Does (1.2) have a multidimensional counterpart in the spirit of the usual Gaussian concentration inequality? What should one assume on the second-order derivatives of $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and what is a natural choice for the function g?

2. The result

Before we formulate the main result, we introduce some notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X, Y be independent random vectors defined on that space. For any integrable random variable V we define

$$\Pi_1 V = \mathbb{E}[V|X] + \mathbb{E}[V|Y] - \mathbb{E}V.$$

Note that the operator Π_1 restricted to $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is an orthogonal projection onto the subspace spanned by random variables which are either $\sigma(X)$- or $\sigma(Y)$-measurable, i.e., $L^2(\Omega, \sigma(X), \mathbb{P}) + L^2(\Omega, \sigma(Y), \mathbb{P})$. In the case $V = f(X,Y)$, $\Pi_1 V = \mathbb{E}_Y f(X,Y) + \mathbb{E}_X f(X,Y) - \mathbb{E} f(X,Y)$, where, e.g., by $\mathbb{E}_X f(X,Y)$ we mean the integration w.r.t. to X only. The operator Π_1 is related to so-called Hoeffding projection from the theory of U-statistics.

For a $\mathcal{C}^2$ function $f\colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and a point $(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we shall consider a matrix of second-order mixed derivatives of f at (x,y):

$$\partial^2_{xy} f(x,y) = \left(\frac{\partial^2 f}{\partial x_i \partial y_j}(x,y) \right)_{\substack{i \leq n_1 \\ j \leq n_2.}}$$

Note that apart from special situations, $\partial^2_{xy} f(x,y)$ is usually not symmetric.

For an $n_1 \times n_2$ matrix $A = (a_{ij})$ we shall consider the operator norm and the Hilbert-Schmidt norm of A:

$$\|A\|_{\mathrm{op}} = \sup \left\{ \sum_{i \le n_1, j \le n_2} a_{ij} x_i y_j \colon \sum_{i \le n_1} x_i^2 \le 1, \sum_{j \le n_2} y_j^2 \le 1 \right\},$$

$$\|A\|_{\mathrm{HS}} = \left(\sum_{i \le n_1, j \le n_2} a_{ij}^2 \right)^{1/2}.$$

For two $n_1 \times n_2$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we shall write $\langle A, B \rangle = \sum_{i,j} a_{ij} b_{ij}$. For $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$, $x \otimes y$ is the $n_1 \times n_2$ matrix with the entries $x_i y_j$, for $i \le n_1$ and $j \le n_2$. With the above notation $\langle A, x \otimes y \rangle = \sum_{i,j} a_{ij} x_i y_j$.

By C, c, etc. we denote positive numerical constants which do not depend on any parameters involved. At each occurrence a value of such constant may be different. Finally, for a random variable Z, $\|Z\|_p$ denotes its L^p-norm, i.e., $(\mathbb{E}|Z|^p)^{1/p}$.

The main result of this note is the following

Theorem 2.1. *Let X and Y be independent standard Gaussian random vectors in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively, $f \colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be a $\mathcal{C}^2$ function and $\mathbb{E}|f(X,Y)| < \infty$. If for any $(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,*

$$\|\partial^2_{xy} f(x,y)\|_{\mathrm{op}} \le a \quad \text{and} \quad \|\partial^2_{xy} f(x,y)\|_{\mathrm{HS}} \le b,$$

then for all $t > 0$,

$$\mathbb{P}\big(|f(X,Y) - \Pi_1 f(X,Y)| \ge t\big) \le C \exp\big(-c \min\big(t/a, t^2/b^2\big)\big).$$

Let us first discuss the optimality of the above estimate. If f is bi-linear, i.e., $f(x,y) = \sum_{i \le n_1, j \le n_2} a_{ij} x_i y_j$, then $\Pi_1 f(X,Y) = 0$ a.s. and in this case the inequality from Theorem 2.1 matches (up to numerical constants) the well-known upper bound for the tail of a (decoupled) Gaussian chaos of order 2, i.e., the random variable $S = \sum_{i \le n_1, j \le n_2} a_{ij} g_i g'_j$, where g_i, g'_j are i.i.d. $\mathcal{N}(0,1)$ random variables [2]:

$$\mathbb{P}(|S| \ge t) \le C \exp\left(-c \min\left(t/\|(a_{ij})\|_{\mathrm{op}}, t^2/\|(a_{ij})\|^2_{\mathrm{HS}}\right)\right). \tag{2.1}$$

The estimate (2.1) is optimal in a sense that with different numerical constants it is also a lower bound for the tail of $|S|$ (cf. [3]). Therefore, Theorem 2.1 can be considered as an extension of (2.1) to the non-linear case. In fact, the bi-linear case will be one of the ingredients of the proof of Theorem 2.1. More precisely, we shall use the following estimate for moments of $S = \sum_{i \le n_1, j \le n_2} a_{ij} g_i g'_j$: for any $p \ge 2$,

$$\|S\|_p \le C \max\left(p\|(a_{ij})\|_{\mathrm{op}}, \sqrt{p}\|(a_{ij})\|_{\mathrm{HS}}\right). \tag{2.2}$$

In the form stated above, (2.2) is an easy consequence of the usual Gaussian concentration, e.g., in the form of (1.1).

The other ingredient in the proof of Theorem 2.1 is the following Sobolev-type inequality due to Maurey and Pisier [7]:

Theorem 2.2 (Maurey-Pisier). *Let $X, \bar{X}$ be independent standard Gaussian vectors in $\mathbb{R}^n$, $f\colon \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function satisfying $\mathbb{E}|f(X)| < \infty$, and $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Then*

$$\mathbb{E}\Phi(f(X) - \mathbb{E}f(X)) \leq \mathbb{E}\Phi\left(\frac{\pi}{2}\langle \nabla f(X), \bar{X}\rangle\right). \tag{2.3}$$

Actually, we shall use the following "tensorized" version of the above inequality:

Proposition 2.3. *Let $X, \bar{X}$ be independent standard Gaussian vectors in $\mathbb{R}^{n_1}$ and $Y, \bar{Y}$ be independent standard Gaussian vectors in $\mathbb{R}^{n_2}$, independent of $(X, \bar{X})$. Further, let $f\colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be $\mathcal{C}^2$ and $\mathbb{E}|f(X,Y)| < \infty$, and $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Then*

$$\mathbb{E}\,\Phi\big(f(X,Y) - \Pi_1 f(X,Y)\big) \leq \mathbb{E}\,\Phi\left((\frac{\pi}{2})^2 \langle \partial^2_{xy} f(X,Y), \bar{X} \otimes \bar{Y}\rangle\right). \tag{2.4}$$

Proof. We just follow the proof of Theorem 2.2 as presented in [7]. For $(\theta_1, \theta_2) \in [0, \pi/2]^2$, we define

$$X(\theta_1) = X \sin\theta_1 + \bar{X}\cos\theta_1,$$
$$Y(\theta_2) = Y \sin\theta_2 + \bar{Y}\cos\theta_2.$$

The crucial property of $X(\theta_1)$ and $Y(\theta_2)$ is

$$\begin{aligned} (X(\theta_1), X'(\theta_1)) &\stackrel{d}{=} (X, \bar{X}), \\ (Y(\theta_2), Y'(\theta_2)) &\stackrel{d}{=} (Y, \bar{Y}) \end{aligned} \tag{2.5}$$

for any $\theta_1, \theta_2 \in [0, \pi/2]$ (by, e.g., $X'(\theta_1)$ we mean $\frac{d}{d\theta_1} X(\theta_1) = X\cos\theta_1 - \bar{X}\sin\theta_1$). By the smoothness assumption and the chain rule,

$$\begin{aligned} &f(X,Y) - f(\bar{X},Y) - f(X,\bar{Y}) + f(\bar{X},\bar{Y}) \\ &\quad= \int_0^{\pi/2}\int_0^{\pi/2} \frac{\partial^2_{xy} f(X(\theta_1), Y(\theta_2))}{\partial\theta_1\partial\theta_2}\, d\theta_1 d\theta_2 \\ &\quad= \int_0^{\pi/2}\int_0^{\pi/2} \langle \partial^2_{xy} f(X(\theta_1), Y(\theta_2)), X'(\theta_1) \otimes Y'(\theta_2)\rangle\, d\theta_1 d\theta_2 \quad \text{a.s.} \end{aligned}$$

Applying Φ to the both sides and using the Jensen inequality yield

$$\begin{aligned} &\Phi\left(f(X,Y) - f(\bar{X},Y) - f(X,\bar{Y}) + f(\bar{X},\bar{Y})\right) \\ &\quad\leq \int_0^{\pi/2}\int_0^{\pi/2} \Phi\left((\pi/2)^2 \langle \partial^2_{xy} f(X(\theta_1), Y(\theta_2)), X'(\theta_1) \otimes Y'(\theta_2)\rangle\right) \\ &\quad\quad \times (2/\pi)^2\, d\theta_1 d\theta_2 \quad \text{a.s.} \end{aligned}$$

Integrating the both sides, applying the Fubini theorem and using (2.5), we arrive with

$$\begin{aligned}&\mathbb{E}\,\Phi\left(f(X,Y)-f(\bar X,Y)-f(X,\bar Y)+f(\bar X,\bar Y)\right)\\&\quad\le\int_0^{\pi/2}\int_0^{\pi/2}\mathbb{E}\,\Phi\left((\pi/2)^2\langle\partial^2_{xy}f(X,Y),\bar X\otimes\bar Y\rangle\right)(2/\pi)^2\,d\theta_1 d\theta_2,\end{aligned}$$

hence the two outer integrals on the right-hand side can be omitted. We finish with the Jensen inequality applied conditionally to the left-hand side:

$$\begin{aligned}&\mathbb{E}\Phi\big(f(X,Y)-\Pi_1 f(X,Y)\big)\\&\quad\le\mathbb{E}\,\mathbb{E}\Big[\Phi\big(f(X,Y)-f(\bar X,Y)-f(X,\bar Y)+f(\bar X,\bar Y)\big)\Big|X,Y\Big].\end{aligned}\qquad\square$$

Proof of Theorem 2.1*.* Taking $\Phi(u)=|u|^p$ for $p\ge 2$ in Proposition 2.3 we obtain

$$\|f(X,Y)-\Pi_1 f(X,Y)\|_p\le\left(\mathbb{E}_{X,Y}\mathbb{E}_{\bar X,\bar Y}\left((\frac{\pi}{2})^2\langle\partial^2_{xy}f(X,Y),\bar X\otimes\bar Y\rangle\right)^p\right)^{1/p}.$$

Since for a given (X,Y), $\langle\partial^2_{xy}f(X,Y),\bar X\otimes\bar Y\rangle$ is a decoupled Gaussian chaos of order 2, we can use (2.2):

$$\begin{aligned}&\mathbb{E}_{\bar X,\bar Y}\left((\frac{\pi}{2})^2\langle\partial^2_{xy}f(X,Y),\bar X\otimes\bar Y\rangle\right)^p\\&\quad\le C^p\max\left(p^p\|\partial^2_{xy}f(X,Y)\|^p_{\mathrm{op}},p^{p/2}\|\partial^2_{xy}f(X,Y)\|^p_{\mathrm{HS}}\right)\quad\text{a.s.},\end{aligned}$$

so plugging it into the right-hand side of the previous inequality gives

$$\|f(X,Y)-\Pi_1 f(X,Y)\|_p\le C\max\left(pa,\sqrt{p}b\right).$$

By the Chebyshev inequality, for any $p\ge 2$,

$$\mathbb{P}\big(|f(X,Y)-\Pi_1 f(X,Y)|\ge e\|f(X,Y)-\Pi_1 f(X,Y)\|_p\big)\le e^{-p},$$

hence

$$\mathbb{P}\big(|f(X,Y)-\Pi_1 f(X,Y)|\ge eC\max\left(pa,\sqrt{p}b\right)\big)\le e^{-p}.$$

The observation that $t=eC\max\left(pa,\sqrt{p}b\right)$ iff $p=\min\left(t/(eCa),t^2/(eCb)^2\right)$ concludes the proof. $\square$

Remark 2.4. Another way to obtain Proposition 2.3 is to iterate the inequality (2.3) twice. First use it conditionally on Y for the function $g_Y(x)=f(x,Y)-\mathbb{E}f(x,Y)$ and obtain

$$\begin{aligned}&\mathbb{E}\Phi\big(f(X,Y)-\mathbb{E}_Y f(X,Y)-\mathbb{E}_X(f(X,Y)-\mathbb{E}_Y f(X,Y))\big)\le\mathbb{E}\Phi\left(\frac{\pi}{2}\langle\nabla g_Y(X),\bar X\rangle\right)\\&\quad=\mathbb{E}\,\mathbb{E}\Big[\Phi\left(\frac{\pi}{2}\left(\langle\nabla_x f(X,Y),\bar X\rangle-\mathbb{E}_Y\langle\nabla_x f(X,Y),\bar X\rangle\right)\right)\Big|X,\bar X\Big].\end{aligned}$$

Now use (2.3) conditionally on $\sigma(X,\bar X)$ for $h_{X,\bar X}(y)=\frac{\pi}{2}\langle\nabla_x f(X,y),\bar X\rangle$ and note that

$$\langle\nabla_y h_{X,\bar X}(y),\bar y\rangle=\frac{\pi}{2}\langle\partial^2_{xy}f(X,y),\bar X\otimes\bar y\rangle.$$

A different iteration scheme of the same Maurey-Pisier inequality (2.3) is proposed in [1].

Proposition 2.3 has a straightforward generalization to functions of more than two Gaussian vectors. Namely, given d independent random vectors $X_1, \ldots, X_d$, one defines

$$\Pi_{d-1} V = \sum_{\emptyset \neq K \subseteq \{1,\ldots,d\}} (-1)^{\# K - 1} \mathbb{E}\left[V | \sigma(X_i \colon i \notin K)\right].$$

On the L^2 space, Π_{d-1} is an orthogonal projection onto the subspace spanned by functions each of which depends on at most $d-1$ vectors among $X_1, \ldots, X_d$. Next, let X_i be a standard Gaussian vector in $\mathbb{R}^{n_i}$ and $(\bar{X}_1, \ldots, \bar{X}_d)$ be an independent copy of the sequence $(X_1, \ldots, X_d)$. Then, for any convex $\Phi \colon \mathbb{R} \to \mathbb{R}$ and $\mathcal{C}^d$ function $f \colon \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d} \to \mathbb{R}$,

$$\begin{aligned} &\mathbb{E}\Phi\big(f(X_1, \ldots, X_d) - \Pi_{d-1} f(X_1, \ldots, X_d)\big) \\ &\quad \leq \mathbb{E}\Phi\left((\frac{\pi}{2})^d \langle \partial^d_{x_1 \ldots x_d} f(X_1, \ldots, X_d), \bar{X}_1 \otimes \cdots \otimes \bar{X}_d \rangle \right), \end{aligned}$$

where

$$\begin{aligned} &\langle \partial^d_{x_1 \ldots x_d} f(x^{(1)}, \ldots, x^{(d)}), y^{(1)} \otimes \cdots \otimes y^{(d)} \rangle \\ &\quad = \sum_{i_1 \leq n_1, \ldots, i_d \leq n_d} \frac{\partial^d f}{\partial x^{(1)}_{i_1} \cdots \partial x^{(d)}_{i_d}} (x^{(1)}, \ldots, x^{(d)}) y^{(1)}_{i_1} \cdot \cdots \cdot y^{(d)}_{i_d}. \end{aligned}$$

Clearly, the higher-order analog of Theorem 2.1 requires moments estimates for (decoupled) Gaussian chaoses of order d,

$$S = \sum_{i_1 \leq n_1, \ldots, i_d \leq n_d} a_{i_1 \cdots i_d} g^{(1)}_{i_1} \cdot \cdots \cdot g^{(d)}_{i_d},$$

where $G^{(j)} = (g^{(j)}_1, \ldots, g^{(j)}_{n_j})$, $j = 1, \ldots, d$ are independent standard Gaussian vectors. Optimal estimates were found by Latała [4] and they are expressed explicitly in terms of some norms of the multi-indexed matrix $A = (a_{i_1 \cdots i_d})$. For example, in the case $d = 3$, for any $p \geq 2$,

$$c m_p(A) \leq \|S\|_p \leq C m_p(A),$$

where

$$\begin{aligned} m_p(A) = {} & \|A\|_{\{1\}\{2\}\{3\}} p^{3/2} + (\|A\|_{\{12\}\{3\}} + \|A\|_{\{13\}\{2\}} + \|A\|_{\{23\}\{1\}}) p \\ & + \|A\|_{\{123\}} p^{1/2} \end{aligned}$$

and

$$\|A\|_{\{1\}\{2\}\{3\}} = \sup \left\{ \sum_{i_1, i_2, i_3} a_{i_1 i_2 i_3} x^{(1)}_{i_1} x^{(2)}_{i_2} x^{(3)}_{i_3} \colon \sum_{i_j} \left(x^{(j)}_{i_j} \right)^2 \leq 1 \text{ for } j = 1, 2, 3 \right\}$$

is the operator norm of A considered as a 3-linear functional on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$, $\|A\|_{\{123\}}$ is the Hilbert-Schmidt norm of A (the sum of squares of all the entries of A), and, e.g.,

$$\|A\|_{\{12\}\{3\}} = \sup\left\{ \sum_{i_1,i_2,i_3} a_{i_1 i_2 i_3} x^{(12)}_{i_1 i_2} x^{(3)}_{i_3} \colon \sum_{i_1,i_2} \left(x^{(12)}_{i_1 i_2}\right)^2 \le 1 \text{ and } \sum_{i_3} \left(x^{(3)}_{i_3}\right)^2 \le 1 \right\}.$$

Let us also mention that since the Maurey-Pisier inequality behaves well under Lipschitz maps (see [7, p. 181]), one can have a version of Proposition 2.3 for random vectors $X = T_1(G_1)$ and $Y = T_2(G_2)$, where $T_i \colon \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is Lipschitz with a constant L_i and G_1, G_2 are independent standard Gaussian vectors:

$$\mathbb{E}\Phi\big(f(X,Y) - \Pi_1 f(X,Y)\big) \le \mathbb{E}\Phi\left(L_1 L_2 (\frac{\pi}{2})^2 \langle \partial^2_{xy} f(X,Y), \bar{G}_1 \otimes \bar{G}_2 \rangle\right),$$

where $(\bar{G}_1, \bar{G}_2)$ is an independent copy of (G_1, G_2),

3. Application to U-statistics

Consider a U-statistic over an i.i.d. sample of $\mathcal{N}(0,1)$ random variables

$$Z = \sum_{i,j \le n,\ i \ne j} h_{i,j}(g_i, g_j).$$

Assume that all kernels $h_{i,j}$ are symmetric and completely degenerate, i.e., $h(x,y) = h(y,x)$ and $\mathbb{E}_{g_i} h_{i,j}(g_i, g_j) = 0$ a.s. and $\mathbb{E}_{g_j} h_{i,j}(g_i, g_j) = 0$ a.s. Further, assume for all $x, y \in \mathbb{R}$,

$$\left| \frac{\partial^2 h}{\partial x \partial y}(x,y) \right| \le a_{ij}.$$

By the decoupling inequalities for U-statistics [6], an estimate for tails of Z follows from a corresponding estimate for a decoupled version of Z:

$$\tilde{Z} = \sum_{i,j \le n,\ i \ne j} h_{i,j}(g_i, g'_j),$$

where $Y = (g'_1, \ldots, g'_n)$ is an independent copy of $X = (g_1, \ldots, g_n)$. Note that the complete degeneracy of the kernels $h_{i,j}$ implies $\Pi_1 \tilde{Z} = 0$ a.s. Also note that $\tilde{Z} = f(X,Y)$ with f satisfying

$$\|\partial^2_{xy} f\|_{\mathrm{op}} \le \|(a_{ij})\|_{\mathrm{op}} \quad \text{and} \quad \|\partial^2_{xy} f\|_{\mathrm{HS}} \le \|(a_{ij})\|_{\mathrm{HS}}.$$

Therefore we obtain

Corollary 3.1. *In the setting described above, for all $t > 0$,*

$$\mathbb{P}(|Z| \ge t) \le C \exp\left(-c \min\left(t/\|(a_{ij})\|_{\mathrm{op}}, t^2/\|(a_{ij})\|^2_{\mathrm{HS}}\right)\right).$$

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Part II

Limit Theorems

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Rates of Convergence in the Strong Invariance Principle for Non-adapted Sequences. Application to Ergodic Automorphisms of the Torus

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Abstract. In this paper, we give rates of convergence in the strong invariance principle for non-adapted sequences satisfying projective criteria. The results apply to the iterates of ergodic automorphisms T of the d-dimensional torus $\mathbb{T}^d$, even in the non hyperbolic case. In this context, we give a large class of unbounded function f from $\mathbb{T}^d$ to $\mathbb{R}$, for which the partial sum $f \circ T + f \circ T^2 + \cdots + f \circ T^n$ satisfies a strong invariance principle with an explicit rate of convergence.

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1. Introduction and notations

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. For a σ-algebra $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, we define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. The $\mathbb{L}^p$ norm of a random variable X is denoted by $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p}$.

Let X_0 be a real-valued and square integrable random variable such that $\mathbb{E}(X_0) = 0$, and define the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$. Define then the partial sum by $S_n = X_1 + X_2 + \cdots + X_n$. According to the Birkhoff-Khinchine theorem, S_n satisfies a strong law of large numbers. One can go further in the study of the statistical properties of S_n. We study here the rate of convergence in the almost sure invariance principle (ASIP). More precisely, we give conditions

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under which there exists a sequence of independent identically distributed (iid) Gaussian random variables $(Z_i)_{i\geq 1}$ such that

$$\sup_{1\leq k\leq n}\Big|\sum_{i=1}^{k}(X_i - Z_i)\Big| = o(n^{1/p}L(n)) \quad \text{almost surely,} \tag{1.1}$$

for $p \in]2,4]$ and L an explicit slowly varying function. Let us recall that, in the iid case, Komlós, Major and Tusnády [13] and Major [18] obtained an ASIP with the optimal rate $o(n^{1/p})$ in (1.1) as soon as the random variables admit a moment of order p, for $p > 2$.

Since the seminal paper by Philipp and Stout [25], many authors have considered this problem in a dependent context, but most of the papers deal with the adapted case, when X_0 is $\mathcal{F}_0$ measurable (for instance, $\mathcal{F}_0$ is the past σ-algebra $\sigma(X_i, i \leq 0)$). Unfortunately, it is quite common to encounter dynamical systems for which the natural filtration does not permit the control of any quantity involving terms of the type $\|\mathbb{E}(X_n|\mathcal{F}_0)\|_p$.

In this paper, we shall not assume that X_0 is $\mathcal{F}_0$-measurable, and we shall give conditions on the quantities $\|\mathbb{E}(X_n|\mathcal{F}_0)\|_p$, $\|X_{-n}-\mathbb{E}(X_{-n}|\mathcal{F}_0)\|_p$ and $\|\mathbb{E}(S_n^2|\mathcal{F}_{-n})-\mathbb{E}(S_n^2)\|_{p/2}$ for (1.1) to hold (see Theorems 3.1 and 3.2 of Section 3). These conditions are in the same spirit as those given by Gordin [7] for $p = 2$ to get the usual central limit theorem. Our proof is based on the approximation

$$\sum_{i=1}^{n} X_i = M_n + R_n$$

by the martingale $M_n = d_1 + d_2 + \cdots + d_n$, where d_i is the martingale difference

$$d_i = \sum_{k\in\mathbb{Z}}(\mathbb{E}(X_k|\mathcal{F}_i) - \mathbb{E}(X_k|\mathcal{F}_{i-1}))$$

introduced by Gordin [7] and Heyde [10]. In the adapted case, similar conditions are given in the recent paper [2], together with a long list of applications.

In the non-adapted case, it is easy to see that our results apply to a large class of two-sided functions of iid sequences, or two-sided functions of absolutely regular sequences. But they also apply to very complicated dynamical systems, for which such a representation by functions of absolutely regular sequences is not available. In the next section, we consider the case where T is an ergodic automorphism of the d-dimensional torus $\mathbb{T}^d$, and $\mathbb{P}$ is the Lebesgue measure on $\mathbb{T}^d$. In this context, we use the σ-algebras $\mathcal{F}_i$ considered by Le Borgne [14]. As a consequence of Theorem 2.1, we obtain that (1.1) holds for $p = 4$ and $X_i = f\circ T^i$, where $f : \mathbb{T}^d \to \mathbb{R}$, as soon as the Fourier coefficients $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ of f are such that

$$|c_{\mathbf{k}}| \leq A\prod_{i=1}^{d}\frac{1}{(1+|k_i|)^{3/4}\log^{\alpha}(2+|k_i|)} \quad \text{for some } \alpha > 13/8.$$

We also get that there exists a positive ε such that

$$\sup_{1\le k\le n}\Big|\sum_{i=1}^{k}(X_i - Z_i)\Big| = o(n^{1/2-\varepsilon}) \quad \text{almost surely,}$$

as soon as

$$|c_{\mathbf{k}}| \le A\prod_{i=1}^{d}\frac{1}{(1+|k_i|)^{\delta}} \quad \text{for some } \delta > 1/2.$$

These rates of convergence in the almost sure invariance principle complement the results by Leonov [16] and Le Borgne [14] for the central limit theorem and the almost sure invariance principle respectively. Let us mention that Dolgopyat [5] established an ASIP with the rate $o(n^{1/2-\varepsilon})$ (for some $\varepsilon > 0$) valid for ergodic automorphisms of the torus and f a Hölder continuous function. Thanks to the decorrelation estimates obtained in [15], the rate for Hölder observables can be improved by applying the general result of Gouëzel in [8] to get the rate $o(n^{1/4+\varepsilon})$ for every $\varepsilon > 0$, and by applying the results of the present paper to get the rate $o(n^{1/4}L(n))$. To our knowledge, the present work gives the first strong approximation results for such partially hyperbolic transformations T for unbounded (and then non continuous) functions f.

To conclude, let us mention some previous works in the context of dynamical systems: several results have been established with the rate $o(n^{1/2-\varepsilon})$ for some $\varepsilon > 0$ (see [4, 5, 11, 19, 24]). Results giving a rate in $o(n^{1/4+\varepsilon})$ for every $\varepsilon > 0$ can be found in [6, 8, 20, 21]. Most of these results hold for bounded functions f.

Let us reiterate that we can reach the rate $o(n^{1/4}L(n))$ instead of $o(n^{1/4+\varepsilon})$ for every $\varepsilon > 0$. Moreover, our conditions giving the rate $o(n^{1/p}L(n))$ are related to moments of order p of f. Such results are not very common in the context of dynamical systems (let us mention [8] in the particular case of Gibbs-Markov maps, and [3, 23] for generalized Pommeau-Manneville maps).

2. ASIP with rates for ergodic automorphisms of the torus

A probability dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is given by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable $\mathbb{P}$-preserving transformation $T : \Omega \to \Omega$. Such a dynamical system is said to be ergodic if the only $A \in \mathcal{F}$ such that $T^{-1}A = A$ a.s. are the sets of probability 0 or 1.

If $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is ergodic, the study of the stochastic properties of the stationary sequence $(f \circ T^k)_{k\ge 1}$ starts with the Birkhoff-Khintchine theorem [1, 12]. This theorem ensures that, for every integrable function $f : \Omega \to \mathbb{R}$, the sequence $n^{-1}\sum_{k=1}^{n} f \circ T^k$ converges almost surely to $\mathbb{E}(f)$. This means that $(f \circ T^k)_{k\ge 1}$ satisfies a strong law of large number. A natural question is then to investigate further stochastic properties of $(f \circ T^k)_{k\ge 1}$.

We illustrate our general Theorems 3.1 and 3.2 by a concrete example of invertible non hyperbolic dynamical system, which is actually partially hyperbolic.

We shall prove a strong invariance principle for a large class of unbounded functions f, with a rate depending on the rate of convergence to zero of the Fourier coefficients of f. In this context, we use the σ-algebras $\mathcal{F}_i$ considered by Le Borgne [14]. The stationary sequence $(T^i)_{i\in\mathbb{Z}}$ is non-adapted to this filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$, in the sense that T^i is not $\mathcal{F}_i$ measurable. This is an important difference with the classical probabilistic situation, where the study of stationary sequences can often be done with the help of a natural "past" filtration (think of stationary Markov chains, or of causal linear processes).

Let $d \geq 2$. We consider a group automorphism T of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For every $x \in \mathbb{R}^d$, we write $\bar{x}$ its class in $\mathbb{T}^d$. We recall that T is the quotient map of a linear map $\tilde{T} : \mathbb{R}^d \to \mathbb{R}^d$ given by $\tilde{T}(x) = S \cdot x$, where S is a $d \times d$-matrix with integer entries and with determinant 1 or -1. The map $x \mapsto S \cdot x$ preserves the infinite Lebesgue measure λ on $\mathbb{R}^d$ and T preserves the probability Lebesgue measure $\bar{\lambda}$. We suppose that T is ergodic, which is equivalent to the fact that no eigenvalue of S is a root of the unity. In this case, it is known that the spectral radius of S is larger than one (and so S admits at least an eigenvalue of modulus larger than one and at least an eigenvalue of modulus smaller than one). This hypothesis holds true in the case of hyperbolic automorphisms of the torus (i.e., in the case when no eigenvalue of S has modulus one) but is much weaker. Indeed, as mentioned in [14], the following matrix gives an example of an ergodic non hyperbolic automorphism of $\mathbb{T}^4$:

$$S := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

When T is ergodic and non hyperbolic, the dynamical system $(\mathbb{T}^d, T, \bar{\lambda})$ has no Markov partition. However, it is possible to construct some measurable partition [17], to prove a central limit theorem [16]. Moreover, in [14], Le Borgne proved the functional central limit theorem and the Strassen strong invariance principle for $(X_k = f \circ T^k)_k$ under weak hypotheses on f, thanks to Gordin's method and to the partitions studied by Lind in [17].

We give here rates of convergence in the strong invariance principle for $(X_k = f \circ T^k)_k$ under conditions on the Fourier coefficients of $f : \mathbb{T}^d \to \mathbb{R}$. In what follows, for $\mathbf{k} \in \mathbb{Z}^d$, we denote by $|\mathbf{k}| = \max_{i\in\{1,\dots,d\}} |k_i|$.

Theorem 2.1. *Let T be an ergodic automorphism of $\mathbb{T}^d$ with the notations as above. Let $p \in]2,4]$ and q be its conjugate exponent. Let $f : \mathbb{T}^d \to \mathbb{R}$ be a centered function with Fourier coefficients $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ satisfying, for any integer $b \geq 2$,*

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^q \leq R \log^{-\theta}(b) \quad \textit{for some } \theta > \frac{p^2-2}{p(p-1)}, \tag{2.2}$$

and

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^2 \leq R\log^{-\beta}(b) \quad \textit{for some } \beta > \frac{3p-4}{p}\,. \tag{2.3}$$

Then the series

$$\sigma^2 = \bar\lambda((f-\bar\lambda(f))^2) + 2\sum_{k>0} \bar\lambda((f-\bar\lambda(f))f\circ T^k)$$

converges absolutely and, enlarging $\mathbb{T}^d$ *if necessary, there exists a sequence* $(Z_i)_{i\geq 1}$ *of iid Gaussian random variables with zero mean and variance* σ^2 *such that, for any* $t > 2/p$,

$$\sup_{1\leq k\leq n}\Big|\sum_{i=1}^k f\circ T^i - \sum_{i=1}^k Z_i\Big| = o\big(n^{1/p}(\log n)^{(t+1)/2}\big) \ \textit{almost surely, as } n\to\infty. \tag{2.4}$$

Observe that (2.3) follows from (2.2) provided that $\theta > (3p-4)/(2p-2)$. Hence, (2.2) and (2.3) are both satisfied as soon as

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^q \leq R\log^{-\theta}(b) \quad \text{for some } \theta > \frac{3p-4}{2(p-1)}\,.$$

Example. Let $p\in]2,4]$. If we assume that the Fourier coefficients of f are such that

$$|c_{\mathbf{k}}| \leq A\prod_{i=1}^d \frac{1}{(1+|k_i|)^{1/q}\log^{\alpha}(2+|k_i|)}\,, \tag{2.5}$$

for some positive constant A, then the conditions (2.2) and (2.3) are both satisfied provided that $\alpha > (2p^2-p-2)/p^2$.

Let us now compare our hypotheses on Fourier coefficients with those appearing in other works. In [16], Leonov proved a central limit theorem (possibly degenerated) when

$$|c_{\mathbf{k}}| \leq A\prod_{i=1}^d \frac{1}{(1+|k_i|)^{1/2}\log^{\alpha}(2+|k_i|)} \quad \text{for some } \alpha > 3/2. \tag{2.6}$$

In [14], Le Borgne proved the functional central limit theorem and the Strassen strong invariance principle when (2.3) holds true with $\beta > 2$ (and when f is not a coboundary), which is a weaker condition than (2.6). Observe that, as p converges to 2, $(p^2-2)/(p(p-1))$ and $(3p-4)/p$ both converge to 1.

3. Probabilistic results

In the rest of the paper, we shall use the following notations: $\mathbb{E}_k(X) = \mathbb{E}(X|\mathcal{F}_k)$, and $a_n \ll b_n$ means that there exists a numerical constant C not depending on n such that $a_n \leq Cb_n$, for all positive integers n.

In this section, we give rates of convergence in the strong invariance principle under projective criteria for stationary sequences that are non necessarily adapted to $\mathcal{F}_i$.

Theorem 3.1. *Let $2 < p < 4$ and $t > 2/p$. Assume that X_0 belongs to $\mathbb{L}^p$, that*

$$\sum_{n\geq 2} \frac{n^{p-1}}{n^{2/p}(\log n)^{(t-1)p/2}} \big(\|\mathbb{E}_0(X_n)\|_p^p + \|X_{-n} - \mathbb{E}_0(X_{-n})\|_p^p\big) < \infty\,, \tag{3.1}$$

and that

$$\sum_{n\geq 2} \frac{n^{3p/4}}{n^{2}(\log n)^{(t-1)p/2}} \big(\|\mathbb{E}_0(X_n)\|_2^{p/2} + \|X_{-n} - \mathbb{E}_0(X_{-n})\|_2^{p/2}\big) < \infty\,. \tag{3.2}$$

Assume in addition that there exists a positive integer m such that

$$\sum_{n\geq 2} \frac{1}{n^{2}(\log n)^{(t-1)p/2}} \big\|\mathbb{E}_{-nm}(S_n^2) - \mathbb{E}(S_n^2)\big\|_{p/2}^{p/2} < \infty\,. \tag{3.3}$$

Then $n^{-1}\mathbb{E}(S_n^2)$ converges to $\sigma^2 = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k)$ and, enlarging Ω if necessary, there exists a sequence $(Z_i)_{i\geq 1}$ of iid Gaussian random variables with zero mean and variance σ^2 such that

$$\sup_{1\leq k\leq n} \Big|S_k - \sum_{i=1}^{k} Z_i\Big| = o\big(n^{1/p}(\log n)^{(t+1)/2}\big) \quad \text{almost surely, as } n\to\infty. \tag{3.4}$$

Theorem 3.2. *Let $t > 1/2$. Assume that X_0 belongs to $\mathbb{L}^4$ and that the conditions* (3.1) *and* (3.3) *hold with $p = 4$. Assume in addition that*

$$\sum_{n\geq 2} n(\log n)^{4-2t}\big(\|\mathbb{E}_0(X_n)\|_2^2 + \|X_{-n} - \mathbb{E}_0(X_{-n})\|_2^2\big) < \infty\,. \tag{3.5}$$

Then the conclusion of Theorem 3.1 *holds with $p = 4$.*

Of course, if $(X_i)_{i\in\mathbb{Z}}$ is a sequence of iid random variables in $\mathbb{L}^p$, then, taking $\mathcal{F}_i = \sigma(X_k, k \leq i)$, all the conditions (3.1), (3.2), (3.3) and (3.5) are satisfied. In that particular case, we obtain an extra power of $\log(n)$ compared to the optimal rate $n^{1/p}$.

The conditions (3.1), (3.2) and (3.5) are similar to the conditions given by Gordin [7] when $p = 2$ for the central limit theorem, and hence Theorem 3.1 and 3.2 have the same range of applicability as Gordin's result.

For the proof of Theorems 3.1 and 3.2, we shall use the approximating martingale M_n introduced by Gordin [7], and we shall give an appropriate upper bound on $R_n = S_n - M_n$. The next step is to get a strong approximation result for the martingale M_n. This will be done by applying Proposition 5.1 in [2], which itself is based on the Skorohod embedding for martingales, as in [26].

3.1. Proofs of Theorems 3.1 and 3.2

Proof. We first notice that since $p > 2$, (3.1) implies that

$$\sum_{n>0} n^{-1/p}\|\mathbb{E}_0(X_n)\|_p < \infty \quad \text{and} \quad \sum_{n>0} n^{-1/p}\|X_{-n} - \mathbb{E}_0(X_{-n})\|_p < \infty$$

(apply Hölder's inequality to see this). Let $P_k(X) = \mathbb{E}_k(X) - \mathbb{E}_{k-1}(X)$. Using Lemma 5.1 of the appendix with $q = 1$, we infer that

$$\sum_{k\in\mathbb{Z}} \|P_0(X_k)\|_p < \infty \,. \tag{3.6}$$

In addition the condition (3.6) implies that $n^{-1}\mathbb{E}(S_n^2)$ converges to the quantity $\sigma^2 = \sum_{k\in\mathbf{Z}} \operatorname{Cov}(X_0, X_k)$.

Let now $d_0 := \sum_{j\in\mathbb{Z}} P_0(X_j)$. Then d_0 belongs to $\mathbb{L}^p$ and $\mathbb{E}(d_0|\mathcal{F}_{-1}) = 0$. Let $d_i := d_0 \circ T^i$ for all $i \in \mathbb{Z}$. Then $(d_i)_{i\in\mathbb{Z}}$ is a stationary sequence of martingale differences in $\mathbb{L}^p$. Let

$$M_n := \sum_{i=1}^{n} d_i \quad \text{and} \quad R_n := S_n - M_n \,.$$

The theorems will be proven if we can show that

$$R_n = o\big(n^{1/p}(\log n)^{(t+1)/2}\big) \quad \text{almost surely as } n \to \infty, \tag{3.7}$$

and that (3.4) holds true with M_k replacing S_k. Since $\mathbb{E}(d_0^2) = \sigma^2$ and $t > p/2$, according to Proposition 5.1 in [2] (applied with $\psi(n) := n^{2/p}(\log n)^t$), to prove that (3.4) holds true with M_k replacing S_k, it suffices to prove that

$$\sum_{n\geq 2} \frac{1}{n^2(\log n)^{(t-1)p/2}} \big\|\mathbb{E}_0(M_n^2) - \mathbb{E}(M_n^2)\big\|_{p/2}^{p/2} < \infty \,. \tag{3.8}$$

By standard arguments, (3.7) will be satisfied if we can show that

$$\sum_{r>0} \frac{\|\max_{1\leq \ell\leq 2^r} |R_\ell|\|_p^p}{2^r\, r^{(t+1)p/2}} < \infty \,. \tag{3.9}$$

Now, by stationarity, $\|\max_{1\leq\ell\leq 2^r}|R_\ell|\|_p \ll 2^{r/p}\sum_{k=0}^{r} 2^{-k/p}\|R_{2^k}\|_p$ (see for instance inequality (6) in [27]) and for all $i, j \geq 0$, $\|R_{i+j}\|_q \leq \|R_i\|_q + \|R_j\|_q$. Applying then Item 1 of Lemma 37 in [22], we derive that for any integer n in $[2^r, 2^{r+1}[$,

$$\Big\|\max_{1\leq\ell\leq 2^r} |R_\ell|\Big\|_p \ll n^{1/p}\sum_{k=1}^{n} k^{-(1+1/p)}\|R_k\|_p \,. \tag{3.10}$$

Therefore using (3.10) followed by an application of Hölder's inequality, we get that for any $\alpha < 1$,

$$\sum_{r>0} \frac{\| \max_{1\le \ell \le 2^r} |R_\ell| \|_p^p}{2^r\, r^{(t+1)p/2}} \ll \sum_{n\ge 2} \frac{1}{n\,(\log n)^{(t+1)p/2}} \Big(\sum_{k=1}^n k^{-(1+1/p)} \|R_k\|_p \Big)^p$$
$$\ll \sum_{n\ge 2} \frac{(\log n)^{(p-1)(1-\alpha)}}{n\,(\log n)^{(t+1)p/2}} \sum_{k=1}^n k^{-2} (\log k)^{\alpha(p-1)} \|R_k\|_p^p .$$

Hence taking $\alpha \in]1 - p/(2(p-1)), 1[$ and changing the order of summation, we infer that (3.9) and then (3.7) hold provided that

$$\sum_{n\ge 2} \frac{\|R_n\|_p^p}{n^2 (\log n)^{(t-1)p/2}} < \infty . \tag{3.11}$$

On an other hand, we shall prove that condition (3.8) is implied by: there exists a positive finite integer m such that

$$\sum_{n\ge 2} \frac{1}{n^2 (\log n)^{(t-1)p/2}} \big\| \mathbb{E}_{-nm}(M_n^2) - \mathbb{E}(M_n^2) \big\|_{p/2}^{p/2} < \infty . \tag{3.12}$$

For any nonnegative integer i, we set $V_i := \|\mathbb{E}_0(M_i^2) - \mathbb{E}(M_i^2)\|_{p/2}$. Using that M_n is a martingale, we infer that, for any nonnegative integers i and j,

$$V_{i+j} \le V_i + V_j . \tag{3.13}$$

Let now $n \in [2^k, 2^{k+1} - 1] \cap \mathbb{N}$, and write its binary expansion:

$$n = \sum_{\ell=0}^k 2^\ell b_\ell \quad \text{where } b_k = 1 \text{ and } b_j \in \{0,1\} \text{ for } j = 0, \dots, k-1 .$$

Inequality (3.13) combined with Hölder's inequality implies that, for any $\eta > 0$,

$$V_n^{p/2} \le \Big(\sum_{\ell=0}^k V_{2^\ell} \Big)^{p/2} \ll 2^{\eta p(k+1)/2} \sum_{\ell=0}^k \Big(\frac{V_{2^\ell}}{2^{\eta \ell}} \Big)^{p/2} . \tag{3.14}$$

Therefore

$$\sum_{n\ge 2} \frac{1}{n^2 (\log n)^{(t-1)p/2}} V_n^{p/2} \ll \sum_{k>0} \frac{2^{\eta p(k+1)/2}}{2^k k^{(t-1)p/2}} \sum_{\ell=0}^k \Big(\frac{V_{2^\ell}}{2^{\eta \ell}} \Big)^{p/2} .$$

Changing the order of summation and taking $\eta \in]0, 2/p[$, it follows that (3.8) is implied by

$$\sum_{k\ge 1} \frac{1}{2^k k^{(t-1)p/2}} \big\| \mathbb{E}_0(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2) \big\|_{p/2}^{p/2} < \infty \tag{3.15}$$

(actually due to the subadditivity of the sequence (V_i) both conditions are equivalent, see the proof of item 1 of Lemma 37 in [22] to prove that (3.8) entails (3.15)).

Now, since (M_n) is a martingale,

$$\mathbb{E}_0(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2) = \sum_{j=1}^{k} \big(\mathbb{E}_0((M_{2^j} - M_{2^{j-1}})^2) - \mathbb{E}((M_{2^j} - M_{2^{j-1}})^2)\big) + \mathbb{E}_0(d_1^2) - \mathbb{E}(d_1^2)\,,$$

which implies by stationarity that

$$\big\|\mathbb{E}_0(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2)\big\|_{p/2} \le \sum_{j=0}^{k-1} \big\|\mathbb{E}_{-2^j}(M_{2^j}^2) - \mathbb{E}(M_{2^j}^2)\big\|_{p/2} + \big\|\mathbb{E}_0(d_1^2) - \mathbb{E}(d_1^2)\big\|_{p/2}\,.$$

Therefore by using Hölder's inequality as done in (3.14) with $\eta \in]0, 2/p[$, we infer that (3.15) is implied by

$$\sum_{k\ge1} \frac{1}{2^k k^{(t-1)p/2}} \big\|\mathbb{E}_{-2^k}(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2)\big\|_{p/2}^{p/2} < \infty\,. \tag{3.16}$$

Notice now that the sequence $(W_n)_{n>0}$ defined by

$$W_n := \big\|\mathbb{E}_{-n}(M_n^2) - \mathbb{E}(M_n^2)\big\|_{p/2}$$

is subadditive. Indeed, for any non negative integers i and j, using that M_n is a martingale together with the stationarity, we derive that

$$\begin{aligned} W_{i+j} &= \big\|\mathbb{E}_{-(i+j)}(M_i^2) - \mathbb{E}(M_i^2) + \mathbb{E}_{-(i+j)}((M_{i+j} - M_i)^2) - \mathbb{E}((M_{i+j} - M_i)^2)\big\|_{p/2} \\ &\le \big\|\mathbb{E}_{-i}(M_i^2) - \mathbb{E}(M_i^2)\big\|_{p/2} + \big\|\mathbb{E}_{-j}(M_j)^2) - \mathbb{E}(M_j)^2)\big\|_{p/2} \\ &\le W_i + W_j\,. \end{aligned}$$

Therefore $W_{i+j}^{p/2} \le 2^{p/2} W_i^{p/2} + 2^{p/2} W_j^{p/2}$. This implies that, for any integer ℓ and any integer $0 \le j \le \ell$, $W_\ell^{p/2} \le 2^{p/2}(W_j^{p/2} + W_{\ell-j}^{p/2})$ in such a way that

$$(\ell+1) W_\ell^{p/2} \le 2^{1+p/2} \sum_{j=1}^{\ell} W_j^{p/2}\,. \tag{3.17}$$

Therefore using (3.17) with $\ell = 2^k$, we infer that condition (3.16) is implied by

$$\sum_{n\ge2} \frac{1}{n^2 (\log n)^{(t-1)p/2}} \big\|\mathbb{E}_{-n}(M_n^2) - \mathbb{E}(M_n^2)\big\|_{p/2}^{p/2} < \infty\,. \tag{3.18}$$

It remains to prove that (3.12) implies (3.18). With this aim, we have, for any positive integer m,

$$M_n = \sum_{k=1}^{m} \big(M_{k[nm^{-1}]} - M_{(k-1)[nm^{-1}]}\big) + M_n - M_{m[nm^{-1}]}\,.$$

Using that M_n is a martingale together with the stationarity, we then infer that

$$\big\|\mathbb{E}_{-n}(M_n^2)-\mathbb{E}(M_n^2)\big\|_{p/2}^{p/2}\le 2^{p/2}m^{p/2}\big\|\mathbb{E}_{-n}(M_{[nm^{-1}]}^2)-\mathbb{E}(M_{[nm^{-1}]}^2)\big\|_{p/2}^{p/2} + 2^{p/2}\big\|\mathbb{E}_{-n}(M_{n-m[nm^{-1}]}^2)-\mathbb{E}(M_{n-m[nm^{-1}]}^2)\big\|_{p/2}^{p/2},$$

which, together with the fact that $n-m[nm^{-1}]<m$, implies that

$$\begin{aligned}\big\|\mathbb{E}_{-n}(M_n^2)-\mathbb{E}(M_n^2)\big\|_{p/2}^{p/2}\\ &\le 2^{p/2}m^{p/2}\Big(2^{p/2}\|d_0\|_p^p+\big\|\mathbb{E}_{-n}(M_{[nm^{-1}]}^2)-\mathbb{E}(M_{[nm^{-1}]}^2)\big\|_{p/2}^{p/2}\Big) \qquad (3.19)\\ &\le 2^{p/2}m^{p/2}\Big(2^{p/2}\|d_0\|_p^p+\big\|\mathbb{E}_{-m[nm^{-1}]}(M_{[nm^{-1}]}^2)-\mathbb{E}(M_{[nm^{-1}]}^2)\big\|_{p/2}^{p/2}\Big),\end{aligned}$$

where for the last line we have used the fact that $n\ge m[nm^{-1}]$. We notice now that due to the martingale property of (M_n) and to stationarity, the sequence $(U_i)_{i\ge0}$ defined for any non negative integer i by

$$U_i:=\big\|\mathbb{E}_{-mi}(M_i^2)-\mathbb{E}(M_i^2)\big\|_{p/2}^{p/2}$$

satisfies, for any positive integers i and j,

$$\begin{aligned}U_{i+j}&\le\Big(\big\|\mathbb{E}_{-m(i+j)}(M_i^2)-\mathbb{E}(M_i^2)\big\|_{p/2}\\ &\qquad+\big\|\mathbb{E}_{-m(i+j)}((M_{i+j}-M_i)^2)-\mathbb{E}((M_{i+j}-M_i)^2)\big\|_{p/2}\Big)^{p/2}\\ &\le 2^{p/2}U_i+2^{p/2}U_j\,.\end{aligned}$$

Hence by (3.17) applied with $W_i^{p/2}=U_i$,

$$U_{[nm^{-1}]}\le 2^{1+p/2}([nm^{-1}]+1)^{-1}\sum_{k=1}^{[nm^{-1}]}U_k\le 2^{1+p/2}\sum_{k=1}^{[nm^{-1}]}\frac{U_k}{k}\,. \qquad (3.20)$$

Therefore starting from (3.19), considering (3.20) and changing the order of summation, we infer that (3.18) (and so (3.8)) holds provided that (3.12) does. To end the proof, it remains to show that under the conditions of Theorems 3.1 and 3.2, the conditions (3.11) and (3.12) are satisfied. This is achieved by using the two following lemmas. □

Lemma 3.1. *Let* $p\in[2,4]$. *Assume that* (3.1) *holds. Then*

$$\sum_{n\ge2}\frac{\max_{1\le\ell\le n}\|R_\ell\|_p^p}{n^2(\log n)^{(t-1)p/2}}<\infty\,,$$

and (3.11) *holds.*

Proof. Since (3.1) implies (3.6), Item 2 of Proposition 5.1 given in the appendix implies that, for any positive integers ℓ and N,

$$\|R_\ell\|_p\ll\max_{k=\ell,N}\|\mathbb{E}_0(S_k)\|_p+\max_{k=\ell,N}\|S_k-\mathbb{E}_k(S_k)\|_p+\ell^{1/2}\sum_{|j|\ge N}\|P_0(X_j)\|_p\,.$$

Next, applying Lemma 5.1 given in the appendix with $q = 1$, and using the fact that by stationarity, for any positive integer k,

$$\|\mathbb{E}_0(S_k)\|_p \leq \sum_{\ell=1}^{k} \|\mathbb{E}_0(X_\ell)\|_p \quad \text{and} \quad \|S_k - \mathbb{E}_k(S_k)\|_p \leq \sum_{\ell=0}^{k-1} \|X_{-\ell} - \mathbb{E}_0(X_{-\ell})\|_p\,, \tag{3.21}$$

we derive that for, any positive integers $N \geq n \geq 2$,

$$\max_{1\leq \ell \leq n} \|R_\ell\|_p \ll \sum_{k=1}^{N} \|\mathbb{E}_0(X_k)\|_p + \sum_{k=0}^{N-1} \|X_{-k} - \mathbb{E}_0(X_{-k})\|_p \tag{3.22}$$
$$+ n^{1/2} \sum_{k \geq [N/2]} \frac{\|\mathbb{E}_0(X_k)\|_p}{k^{1/p}} + n^{1/2} \sum_{k\geq [N/2]} \frac{\|X_{-k} - \mathbb{E}_0(X_{-k})\|_p}{k^{1/p}}\,.$$

The lemma follows from (3.22) with $N = [n^{p/2}]$ by using Hölder's inequality (see the computations in the proof of Proposition 2.2 in [2]). □

Lemma 3.2. *Let $p \in [2, 4]$ and assume that* (3.1) *and* (3.3) *are satisfied. Assume in addition that* (3.2) *holds when $2 < p < 4$ and* (3.5) *does when $p = 4$. Then* (3.12) *is satisfied.*

Proof. Let m be a positive integer such that (3.3) is satisfied. We first write that

$$\|\mathbb{E}_{-nm}(M_n^2) - \mathbb{E}(M_n^2)\|_{p/2} \leq \|\mathbb{E}_{-nm}(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2}$$
$$+ 2\|\mathbb{E}_{-nm}(S_nR_n) - \mathbb{E}(S_nR_n)\|_{p/2} + 2\|R_n\|_p^2\,.$$

By using Lemma 3.1, and since (3.3) holds, Lemma 3.2 will follow if we can prove that

$$\sum_{n\geq 1} \frac{1}{n^2(\log n)^{(t-1)p/2}} \|\mathbb{E}_{-nm}(S_nR_n)\|_{p/2}^{p/2} < \infty\,. \tag{3.23}$$

With this aim we shall prove the following inequality. For any non negative integer r and any positive integer u_n such that $u_n \leq n$, we have that

$$\|\mathbb{E}_{-r}(S_nR_n)\|_{p/2} \tag{3.24}$$
$$\ll \sqrt{u_n}\big(\|\mathbb{E}_0(S_n)\|_2 + \|S_n - \mathbb{E}_n(S_n)\|_2\big)$$
$$+ \max_{k=\{n,n-u_n\}} \|R_k\|_p^2 + \sqrt{n}\big(\|\mathbb{E}_{-u_n}(S_n)\|_2 + \|S_n - \mathbb{E}_{n+u_n}(S_n)\|_2\big)$$
$$+ \max_{k=\{n,u_n\}} \|\mathbb{E}_{-r}(S_k^2) - \mathbb{E}(S_k^2)\|_{p/2} + \sqrt{n}\Big(\sum_{k=1}^{n} \Big\| \sum_{|j|\geq k+n} P_0(X_j)\Big\|_2^2\Big)^{1/2}\,.$$

Let us show how, thanks to (3.24), the convergence (3.23) can be proven. Let us first consider the case where $2 < p < 4$. Notice that the following elementary claim is valid:

Claim 3.1. *If $\mathcal{F}$ and $\mathcal{G}$ are two σ-algebras such that $\mathcal{G} \subset \mathcal{F}$, then for any random variable X in $\mathbb{L}^q$ for $q \geq 1$, $\|X - \mathbb{E}(X|\mathcal{F})\|_q \leq 2\|X - \mathbb{E}(X|\mathcal{G})\|_q$.*

Starting from (3.24) with $r = nm$ and $u_n = n$, and using Claim 3.1, we derive that

$$\begin{aligned}\|\mathbb{E}_{-nm}(S_nR_n)\|_{p/2} \ll \|\mathbb{E}_{-nm}(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} \\ + \sqrt{n}\big(\|\mathbb{E}_0(S_n)\|_2 + \|S_n - \mathbb{E}_n(S_n)\|_2\big) \\ + \|R_n\|_p^2 + n\sum_{|j|\geq n}\|P_0(X_j)\|_2\,.\end{aligned}$$

This last inequality combined with condition (3.3) and Lemma 3.1 shows that (3.23) will be satisfied if we can prove that

$$\sum_{n\geq 2}\frac{n^{p/4}}{n^2(\log n)^{(t-1)p/2}}\big(\|\mathbb{E}_0(S_n)\|_2 + \|S_n - \mathbb{E}_n(S_n)\|_2\big)^{p/2} < \infty\,, \tag{3.25}$$

and

$$\sum_{n\geq 2}\frac{n^{p/2}}{n^2(\log n)^{(t-1)p/2}}\bigg(\sum_{|j|\geq n}\|P_0(X_j)\|_2\bigg)^{p/2} < \infty\,. \tag{3.26}$$

To prove (3.25), we use the inequalities (3.21) with $p = 2$. Hence setting

$$a_\ell = \|\mathbb{E}_0(X_\ell)\|_2 + \|X_{-\ell+1} - \mathbb{E}_0(X_{-\ell+1})\|_2\,, \tag{3.27}$$

and using Hölder's inequality, we derive that for any $\alpha < 1$,

$$\begin{aligned}&\sum_{n\geq 2}\frac{n^{p/4}}{n^2(\log n)^{(t-1)p/2}}\big(\|\mathbb{E}_0(S_n)\|_2 + \|S_n - \mathbb{E}_n(S_n)\|_2\big)^{p/2} \\ &\ll \sum_{n\geq 2}\frac{n^{p/4}}{n^2(\log n)^{(t-1)p/2}}\bigg(\sum_{\ell=1}^n a_\ell\bigg)^{p/2} \ll \sum_{n\geq 2}\frac{n^{p/4}n^{(1-\alpha)(p/2-1)}}{n^2(\log n)^{(t-1)p/2}}\sum_{\ell=1}^n \ell^{\alpha(p/2-1)}a_\ell^{p/2}\,.\end{aligned}$$

Taking $\alpha \in](3p-8)/(2p-4), 1[$ (this is possible since $p < 4$) and changing the order of summation, we infer that (3.25) holds provided that (3.2) does. It remains to show that (3.26) is satisfied. Using Lemma 5.1 and the notation (3.27), we first observe that

$$\begin{aligned}&\sum_{n\geq 2}\frac{n^{p/2}}{n^2(\log n)^{(t-1)p/2}}\bigg(\sum_{|j|\geq n}\|P_0(X_j)\|_2\bigg)^{p/2} \\ &\qquad \ll \sum_{n\geq 2}\frac{n^{p/2}}{n^2(\log n)^{(t-1)p/2}}\bigg(\sum_{\ell\geq [n/2]}\ell^{-1/2}a_\ell\bigg)^{p/2}.\end{aligned}$$

Therefore by Hölder's inequality, it follows that for any $\alpha < 1$,

$$\begin{aligned}&\sum_{n\geq 2}\frac{n^{p/2}}{n^2(\log n)^{(t-1)p/2}}\bigg(\sum_{|j|\geq n}\|P_0(X_j)\|_2\bigg)^{p/2} \\ &\qquad \ll \sum_{n\geq 2}\frac{n^{p/2}n^{(1-\alpha)(p/2-1)}}{n^2(\log n)^{(t-1)p/2}}\sum_{\ell\geq [n/2]}\ell^{\alpha(p/2-1)}\ell^{-p/4}a_\ell^{p/2}\,.\end{aligned}$$

Therefore taking $\alpha \in]1,2[$ and changing the order of summation, we infer that (3.25) holds provided that (3.2) does. This ends the proof of (3.23) when $p \in]2,4[$.

Now, we prove (3.23) when $p=4$. With this aim we start from (3.24) with $r = nm$ and $u_n = [\sqrt{n}]$. This inequality combined with condition (3.3), Lemma 3.1 and the arguments developed to prove (3.25) and (3.26) shows that (3.23) will be satisfied for $p=4$ if we can prove that

$$\sum_{n\geq 2} \frac{1}{n(\log n)^{2(t-1)}} \big(\|\mathbb{E}_{-[\sqrt{n}]}(S_n)\|_2 + \|S_n - \mathbb{E}_{n+[\sqrt{n}]}(S_n)\|_2\big)^2 < \infty \,, \tag{3.28}$$

and

$$\sum_{n\geq 2} \frac{1}{n^2(\log n)^{2(t-1)}} \big\|\mathbb{E}_{-nm}(S^2_{[\sqrt{n}]}) - \mathbb{E}(S^2_{[\sqrt{n}]})\big\|_2^2 < \infty \,. \tag{3.29}$$

We start by proving (3.28). With this aim, using the notation (3.27), we first write that

$$\|\mathbb{E}_{-[\sqrt{n}]}(S_n)\|_2 + \|S_n - \mathbb{E}_{n+[\sqrt{n}]}(S_n)\|_2 \leq \sum_{k=[\sqrt{n}]+1}^{n+[\sqrt{n}]} a_k \,.$$

Therefore by Cauchy-Schwarz's inequality

$$\sum_{n\geq 2} \frac{1}{n(\log n)^{2(t-1)}} \big(\|\mathbb{E}_{-[\sqrt{n}]}(S_n)\|_2 + \|S_n - \mathbb{E}_{n+[\sqrt{n}]}(S_n)\|_2\big)^2$$
$$\ll \sum_{n\geq 2} \frac{\log n}{n(\log n)^{2(t-1)}} \sum_{k=[\sqrt{n}]+1}^{n+[\sqrt{n}]} k a_k^2 \ll \sum_{n\geq 1} \frac{1}{n} \sum_{k=[\sqrt{n}]+1}^{n+[\sqrt{n}]} \frac{k \log k}{(\log k)^{2(t-1)}} a_k^2 \,.$$

Changing the order of summation, this proves that (3.28) holds provided that (3.5) does. It remains to prove (3.29). With this aim, we set for any positive real x,

$$h([x]) = \big\|\mathbb{E}_{-m[x]}(S^2_{[x]}) - \mathbb{E}(S^2_{[x]})\big\|_2^2 \,,$$

and we notice that, for any integer $n \geq 0$,

$$\big\|\mathbb{E}_{-nm}(S^2_{[\sqrt{n}]}) - \mathbb{E}(S^2_{[\sqrt{n}]})\big\|_2^2 \leq h([\sqrt{n}]) \,.$$

In addition, if $x \in [n, n+1[$, then $[\sqrt{n}] = [\sqrt{x}]$ or $[\sqrt{n}] = [\sqrt{x}] - 1$. Therefore

$$\sum_{n\geq 3} \frac{1}{n^2(\log n)^{(t-1)p/2}} h([\sqrt{n}]) \ll \sum_{n\geq 3} h([\sqrt{n}]) \int_{[n,n+1[} \frac{1}{x^2(\log x)^{(t-1)p/2}} dx$$
$$\ll \int_3^\infty \frac{1}{x^2(\log x)^{(t-1)p/2}} h([\sqrt{x}]) dx + \int_3^\infty \frac{1}{x^2(\log x)^{(t-1)p/2}} h([\sqrt{x}]-1) dx$$
$$\ll \int_2^\infty \frac{1}{y^3(\log y)^{(t-1)p/2}} h([y]) dy \ll \sum_{n\geq 2} \frac{1}{n^3(\log n)^{(t-1)p/2}} h(n) dy \,.$$

For the last inequality, we have used that if $y \in [n, n+1[$, then $[y] = n$. Therefore condition (3.3) implies (3.29). This ends the proof of (3.23) when $p=4$.

It remains to prove (3.24). With this aim, we start with the decomposition of R_n given in Proposition 5.1 of the appendix with $N = n$. Therefore setting

$$A_n := \sum_{k=1}^{n} \sum_{j \geq 2n+1} P_k(X_j) + \sum_{k=1}^{n} \sum_{j \geq n} P_k(X_{-j}) ,$$

we write that

$$\begin{aligned} R_n = \mathbb{E}_0(S_n) - \mathbb{E}_0(S_n) \circ T^n + \mathbb{E}_{-n}(S_n) \circ T^n + S_n - \mathbb{E}_n(S_n) \\ - (\mathbb{E}_{2n}(S_n - \mathbb{E}_n(S_n)) \circ T^{-n} - A_n . \end{aligned} \tag{3.30}$$

Starting from (3.30) and noticing that

$$\begin{aligned} \|\mathbb{E}_{-r}(S_n(\mathbb{E}_{-n}(S_n) \circ T^n)\|_{p/2} &\leq \|\mathbb{E}_0(S_n(\mathbb{E}_{-n}(S_n) \circ T^n)\|_{p/2} \\ &\leq \|\mathbb{E}_0(S_n)\|_p \|\mathbb{E}_0(S_{2n} - S_n)\|_p , \end{aligned}$$

and that $\mathbb{E}_{-r}(S_n(S_n - \mathbb{E}_n(S_n)) = \mathbb{E}_{-r}((S_n - \mathbb{E}_n(S_n))^2)$, we first get

$$\begin{aligned} \|\mathbb{E}_{-r}(S_n R_n)\|_{p/2} \leq 2\|\mathbb{E}_0(S_n)\|_p^2 + \|S_n - \mathbb{E}_n(S_n)\|_p^2 \\ + \|\mathbb{E}_{-r}(S_n \mathbb{E}_n(S_{2n} - S_n))\|_{p/2} \\ + \|\mathbb{E}_{-r}(S_n \mathbb{E}_n(S_n \circ T^{-n} - \mathbb{E}_0(S_n \circ T^{-n})))\|_{p/2} \\ + \big\|\mathbb{E}_{-r}(S_n A_n)\big\|_{p/2} . \end{aligned} \tag{3.31}$$

Next, we use the following fact: if X and Y are two variables in $\mathbb{L}^p$ with $p \in [2, 4]$, then for any integer u,

$$\|\mathbb{E}_u(XY)\|_{p/2} \leq \|\mathbb{E}_u(X^2) - \mathbb{E}(X^2)\|_{p/2} + \|Y\|_p^2 + \sqrt{\mathbb{E}(X^2)}\|Y\|_2 . \tag{3.32}$$

Indeed, it suffices to write that

$$\begin{aligned} \|\mathbb{E}_u(XY)\|_{p/2} &\leq \|\mathbb{E}_u^{1/2}(X^2)\mathbb{E}_u^{1/2}(Y^2)\|_{p/2} \\ &\leq \||\mathbb{E}_u(X^2) - \mathbb{E}(X^2)|^{1/2}\mathbb{E}_u^{1/2}(Y^2)\|_{p/2} + (\mathbb{E}(X^2))^{1/2}\|\mathbb{E}_u^{1/2}(Y^2)\|_{p/2} \\ &\leq \|\mathbb{E}_u(X^2) - \mathbb{E}(X^2)\|_{p/2} + \|Y\|_p^2 + (\mathbb{E}(X^2))^{1/2}\|\mathbb{E}_u^{1/2}(Y^2)\|_{p/2} , \end{aligned}$$

and to notice that, since $p \in [2, 4]$, $\|\mathbb{E}_u^{1/2}(Y^2)\|_{p/2} \leq \|\mathbb{E}_u^{1/2}(Y^2)\|_2 = \|Y\|_2$. Therefore, starting from (3.31) and using (3.32) together with $\mathbb{E}(S_n^2) \ll n$, we infer that

$$\begin{aligned} \|\mathbb{E}_{-r}(S_n R_n)\|_{p/2} \ll \|\mathbb{E}_0(S_n)\|_p^2 + \|S_n - \mathbb{E}_n(S_n)\|_p^2 \\ + \|\mathbb{E}_{-r}(S_n \mathbb{E}_n(S_{2n} - S_n))\|_{p/2} \\ + \|\mathbb{E}_{-r}(S_n \mathbb{E}_n(S_n \circ T^{-n} - \mathbb{E}_0(S_n \circ T^{-n})))\|_{p/2} \\ + \big\|\mathbb{E}_{-r}(S_n^2) - \mathbb{E}(S_n^2)\big\|_{p/2} + \|A_n\|_p^2 + n^{1/2}\|A_n\|_2 , \end{aligned}$$

and since $\|\mathbb{E}_0(S_n)\|_p \le \|R_n\|_p$, $\|S_n - \mathbb{E}_n(S_n)\|_p \le 2\|R_n\|_p$ and $\|A_n\|_p \le 8\|R_n\|_p$, we have overall that

$$\begin{aligned}\|\mathbb{E}_{-r}(S_nR_n)\|_{p/2} \ll \|R_n\|_p^2 &+ \|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\ &+ \|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n \circ T^{-n} - \mathbb{E}_0(S_n \circ T^{-n})))\|_{p/2} \\ &+ \big\|\mathbb{E}_{-r}(S_n^2) - \mathbb{E}(S_n^2)\big\|_{p/2} + n^{1/2}\|A_n\|_2 \,. \end{aligned} \tag{3.33}$$

By orthogonality and by stationarity,

$$\begin{aligned}\|A_n\|_2 &\le \Big(\sum_{k=1}^n \Big\|\sum_{j\ge 2n+1} P_k(X_j)\Big\|_2^2\Big)^{1/2} + \Big(\sum_{k=1}^n \Big\|\sum_{j\ge n} P_k(X_{-j})\Big\|_2^2\Big)^{1/2} \\ &\le \Big(\sum_{k=1}^n \Big\|\sum_{\ell\ge k+n} P_0(X_\ell)\Big\|_2^2\Big)^{1/2} + \Big(\sum_{k=1}^n \Big\|\sum_{\ell\ge k+n} P_0(X_{-\ell})\Big\|_2^2\Big)^{1/2} . \end{aligned} \tag{3.34}$$

Now for any integer u_n such that $u_n \le n$,

$$\begin{aligned}\|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} &\le \|\mathbb{E}_{-r}((S_n - S_{n-u_n})\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\ &\quad + \|\mathbb{E}_{-r}(S_{n-u_n}\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\ &\ll \big\|\mathbb{E}_{-r}(S_{u_n}^2) - \mathbb{E}(S_{u_n}^2)\big\|_{p/2} \\ &\quad + \|\mathbb{E}_0(S_n)\|_p^2 + \sqrt{u_n}\|\mathbb{E}_0(S_n)\|_2 \\ &\quad + \|\mathbb{E}_{-r}(S_{n-u_n}\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \,, \end{aligned} \tag{3.35}$$

where for the last inequality we have used (3.32) together with $\mathbb{E}(S_{u_n}^2) \ll u_n$. Next, we write that

$$\begin{aligned}&\|\mathbb{E}_{-r}(S_{n-u_n}\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\ &\quad \le \|\mathbb{E}_{-r}((S_{n-u_n} - \mathbb{E}_{n-u_n}(S_{n-u_n}))\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\ &\qquad + \|\mathbb{E}_{-r}(\mathbb{E}_{n-u_n}(S_{n-u_n})\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\ &\quad \le \|S_{n-u_n} - \mathbb{E}_{n-u_n}(S_{n-u_n})\|_p\|\mathbb{E}_0(S_n)\|_p \\ &\qquad + \|\mathbb{E}_{-r}(\mathbb{E}_{n-u_n}(S_{n-u_n})\mathbb{E}_{n-u_n}(S_{2n}-S_n))\|_{p/2} \\ &\quad \le \|S_{n-u_n} - \mathbb{E}_{n-u_n}(S_{n-u_n})\|_p^2 + \|\mathbb{E}_0(S_n)\|_p^2 \\ &\qquad + \|\mathbb{E}_{-r}(S_n\mathbb{E}_{n-u_n}(S_{2n}-S_n))\|_{p/2} \\ &\qquad + \|\mathbb{E}_{-r}((S_n - S_{n-u_n})\mathbb{E}_{n-u_n}(S_{2n}-S_n))\|_{p/2} \,. \end{aligned}$$

Therefore using (3.32), we infer that

$$\begin{aligned}&\|\mathbb{E}_{-r}(S_{n-u_n}\mathbb{E}_n(S_{2n}-S_n))\|_{p/2} \\ &\quad \ll \max_{k=\{n,n-u_n\}} \|R_k\|_p^2 + \sqrt{n}\|\mathbb{E}_{-u_n}(S_n)\|_2 + \max_{k=\{n,u_n\}} \|\mathbb{E}_{-r}(S_k^2) - \mathbb{E}(S_k^2)\|_{p/2} \,. \end{aligned} \tag{3.36}$$

We deal now with the third term in the right-hand side of (3.33). With this aim, we first write that

$$\begin{aligned}&\|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2}\\&\quad\le\|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n}-\mathbb{E}_{u_n}(S_n\circ T^{-n})))\|_{p/2}\\&\qquad+\|\mathbb{E}_{-r}(S_n\mathbb{E}_{u_n}(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2}.\end{aligned}\tag{3.37}$$

By using (3.32) together with $\mathbb{E}(S_{u_n}^2)\ll n$, stationarity and the fact that $\|S_n-\mathbb{E}_{n+u_n}(S_n)\|_2\le 2\|R_n\|_p$, we infer that

$$\begin{aligned}&\|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n}-\mathbb{E}_{u_n}(S_n\circ T^{-n})))\|_{p/2}\\&\quad\ll\|\mathbb{E}_{-r}(S_n^2)-\mathbb{E}(S_n^2)\|_{p/2}+\|R_n\|_p^2+\sqrt{n}\|S_n-\mathbb{E}_{n+u_n}(S_n)\|_2\,.\end{aligned}\tag{3.38}$$

On the other hand,

$$\begin{aligned}&\|\mathbb{E}_{-r}(S_n\mathbb{E}_{u_n}(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2}\\&\quad\le\|\mathbb{E}_{-r}(S_{u_n}\mathbb{E}_{u_n}(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2}\\&\qquad+\|\mathbb{E}_{-r}(\mathbb{E}_{u_n}(S_n-S_{u_n})\mathbb{E}_{u_n}(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n}))\|_{p/2}\,.\end{aligned}$$

We apply (3.32) to the first term of the right-hand side together with the fact that $\mathbb{E}(S_{u_n}^2)\ll n$. Hence by stationarity and since $\|S_n-\mathbb{E}_n(S_n)\|_p\le 2\|R_n\|_p$, we derive that

$$\begin{aligned}&\|\mathbb{E}_{-r}(S_{u_n}\mathbb{E}_{u_n}(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2}\\&\quad\ll\|\mathbb{E}_{-r}(S_{u_n}^2)-\mathbb{E}(S_{u_n}^2)\|_{p/2}+\|R_n\|_p^2+\sqrt{u_n}\|S_n-\mathbb{E}_n(S_n)\|_2\,.\end{aligned}$$

On the other hand, by stationarity,

$$\begin{aligned}&\|\mathbb{E}_{-r}(\mathbb{E}_{u_n}(S_n-S_{u_n})\mathbb{E}_{u_n}(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2}\\&\qquad\le\|\mathbb{E}_{u_n}(S_n-S_{u_n})\|_p\|\mathbb{E}_{u_n}(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n}))\|_p\\&\qquad\le\|\mathbb{E}_0(S_{n-u_n})\|_p\|S_n-\mathbb{E}_n(S_n))\|_p\,.\\&\qquad\le\|\mathbb{E}_0(S_{n-u_n})\|_p^2+\|R_n\|_p^2\,.\end{aligned}$$

Therefore we get overall that

$$\begin{aligned}&\|\mathbb{E}_{-r}(S_n\mathbb{E}_{u_n}(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2}\\&\quad\ll\|R_n\|_p^2+\|\mathbb{E}_0(S_{n-u_n})\|_p^2+\|\mathbb{E}_{-r}(S_{u_n}^2)-\mathbb{E}(S_{u_n}^2)\|_{p/2}+\sqrt{u_n}\|S_n-\mathbb{E}_n(S_n)\|_2\,.\end{aligned}\tag{3.39}$$

Starting from (3.37) and taking into account (3.38) and (3.39), we get that

$$\begin{aligned}&\|\mathbb{E}_{-r}(S_n\mathbb{E}_n(S_n\circ T^{-n}-\mathbb{E}_0(S_n\circ T^{-n})))\|_{p/2}\\&\quad\ll\sqrt{u_n}\|S_n-\mathbb{E}_n(S_n)\|_2+\sqrt{n}\|S_n-\mathbb{E}_{n+u_n}(S_n)\|_2\\&\qquad+\max_{k=\{n,u_n\}}\|\mathbb{E}_{-r}(S_k^2)-\mathbb{E}(S_k^2)\|_{p/2}+\max_{k=\{n,n-u_n\}}\|R_k\|_p^2\,.\end{aligned}\tag{3.40}$$

Finally, starting from (3.33) and considering (3.34), (3.35), (3.36) and (3.40), we conclude that (3.24) holds. □

4. Proof of Theorem 2.1

4.1. Preparatory material

Let us denote by E_u, E_e and E_s the S-stable vector spaces associated to the eigenvalues of S of modulus respectively larger than one, equal to one and smaller than one. Let d_u, d_e and d_s be their respective dimensions. Let $v_1, \dots, v_d$ be a basis of $\mathbb{R}^d$ in which S is represented by a real Jordan matrix. Suppose that $v_1, \dots, v_{d_u}$ are in E_u, $v_{d_u+1}, \dots, v_{d_u+d_e}$ are in E_e and $v_{d_u+d_e+1}, \dots, v_d$ are in E_s. We suppose moreover that $\det(v_1|v_2|\cdots|v_d) = 1$. Let us write $||\cdot||$ the norm on $\mathbb{R}^d$ given by

$$\left\|\sum_{i=1}^d x_i v_i\right\| = \max_{i=1,\dots,d} |x_i|$$

and $d_0(\cdot,\cdot)$ the metric induced by $||\cdot||$ on $\mathbb{R}^d$. Let also d_1 be the metric induced by d_0 on $\mathbb{T}^d$. We define now $B_u(\delta) := \{y \in E_u \ : \ ||y|| \le \delta\}$, $B_e(\delta) := \{y \in E_e \ : \ ||y|| \le \delta\}$ and $B_s(\delta) = \{y \in E_s \ : \ ||y|| \le \delta\}$. Let $|\cdot|$ be the usual euclidean norm on $\mathbb{R}^d$.

Let r_u be the spectral radius of $S^{-1}_{|E_u}$. For every $\rho_u \in (r_u, 1)$, there exists $K > 0$ such that, for every integer $n \ge 0$, we have

$$\forall h_u \in E_u, \quad ||S^n h_u|| \ge K \rho_u^{-n} ||h_u|| \tag{4.41}$$

and

$$\forall (h_e, h_s) \in E_e \times E_s, \quad ||S^n(h_e + h_s)|| \le K(1+n)^{d_e} ||h_e + h_s||. \tag{4.42}$$

Let $\rho_u \in (r_u, 1)$ and K satisfying (4.41) and (4.42). Let us denote by m_u, m_e, m_s the Lebesgue measure on E_u (in the basis $v_1, \dots, v_{d_u}$), E_e (in the basis $v_{d_u+1}, \dots, v_{d_u+d_e}$) and E_s (in the basis $v_{d_u+d_e+1}, \dots, v_d$) respectively. Observe that $d\lambda(h_u + h_e + h_s) = dm_u(h_u) dm_e(h_e) dm_s(h_s)$.

The properties satisfied by the filtration considered in [14, 17] and enabling the use of a martingale approximation method à la Gordin will be crucial here. Given a finite partition $\mathcal{P}$ of $\mathbb{T}^d$, we define the measurable partition $\mathcal{P}_0^\infty$ by :

$$\forall \bar{x} \in \mathbb{T}^d, \quad \mathcal{P}_0^\infty(\bar{x}) := \bigcap_{k \ge 0} T^k \mathcal{P}(T^{-k}(\bar{x}))$$

and, for every integer n, the σ-algebra $\mathcal{F}_n$ generated by

$$\forall \bar{x} \in \mathbb{T}^d, \quad \mathcal{P}_{-n}^\infty(\bar{x}) := \bigcap_{k \ge -n} T^k \mathcal{P}(T^{-k}(\bar{x})) = T^{-n}(\mathcal{P}_0^\infty(T^n(\bar{x})).$$

These definitions coincide with the ones of [14] applied to the ergodic torus automorphism T^{-1}. We obviously have $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} = T^{-1}\mathcal{F}_n$. Let $r_0 > 0$ be such that $(h_u, h_e, h_s) \mapsto \overline{h_u + h_e + h_s}$ defines a diffeomorphism from $B_u(r_0) \times B_e(r_0) \times B_s(r_0)$ on its image in $\mathbb{T}^d$. Observe that, for every $\bar{x} \in \mathbb{T}^d$, on the set $\bar{x} + B_u(r_0) + B_e(r_0) + B_s(r_0)$, we have $d\bar{\lambda}(\bar{x} + \overline{h_u} + \overline{h_e} + \overline{h_s}) = dm_u(h_u) dm_e(h_e) dm_s(h_s)$.

Proposition 4.1 ([14, 17] applied to T^{-1}). *There exist some $Q > 0$, $K_0 > 0$, $\alpha \in (0,1)$ and some finite partition $\mathcal{P}$ of $\mathbb{T}^d$ whose elements are of the form*

$\sum_{i=1}^{d} I_i\overline{v_i}$ *where the* I_i *are intervals with diameter smaller than* $\min(r_0, K)$ *such that, for almost every* $\bar{x} \in \mathbb{T}^d$*,*

1. *the local leaf* $\mathcal{P}_0^\infty(\bar{x})$ *of* $\mathcal{P}_0^\infty$ *containing* $\bar{x}$ *is a bounded convex set* $\bar{x} + \overline{F(\bar{x})}$*, with* $0 \in F(\bar{x}) \subseteq E_u$*,* $F(\bar{x})$ *having non-empty interior in* E_u*,*
2. *we have*

$$\mathbb{E}_n(f)(\bar{x}) = \frac{1}{m_u(S^{-n}F(T^n\bar{x}))} \int_{S^{-n}F(T^n\bar{x})} f(\bar{x} + \overline{h_u})\, dm_u(h_u), \tag{4.43}$$

3. *for every* $\gamma > 0$*, we have*

$$m_u(\partial(F(\bar{x}))(\gamma)) \le Q\gamma, \tag{4.44}$$

where

$$\partial F(\beta) := \{y \in F \ : \ d(y, \partial F) \le \beta\},$$

4. *for every* $\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}$*, for every integer* $n \ge 0$*,*

$$\left|\mathbb{E}_{-n}(e^{2i\pi\langle \mathbf{k},\cdot\rangle})(\bar{x})\right| \le \frac{K_0}{m_u(F(T^{-n}(x)))} |\mathbf{k}|^{d_e+d_s}\alpha^n, \tag{4.45}$$

5. *for every* $\beta \in (0,1)$*,*

$$\exists L > 0,\ \forall n \ge 0,\ \bar{\lambda}(m_u(F(\cdot)) < \beta^n) \le L\beta^{n/d_u}. \tag{4.46}$$

Proof. The first item comes from Proposition II.1 of [14]. Item 2 comes from the formula given after Lemma II.2 of [14]. Item 3 follows from Lemma III.1 of [14] and from the fact that the numbers $a(\mathcal{P}_0^\infty(\cdot))$ considered in [14] are uniformly bounded. Item 4 comes from Proposition III.3 of [14] and from the uniform boundedness of $a(\mathcal{P}_0^\infty(\cdot))$. Item 5 comes from the proof of Proposition II.1 of [14]. □

According to the first item of Proposition 4.1 and to (4.41), there exists $c_u > 0$ such that, for almost every $\bar{x} \in \mathbb{T}^d$ and every $n \ge 1$, we have

$$\sup_{h_u \in S^{-n}F(T^n(\bar{x}))} |h_u| \le c_u \rho_u^n. \tag{4.47}$$

Proposition 4.2. *Let* $p \ge 2$ *and* q *be its conjugate exponent. Let* $\theta > 0$ *and* $f : \mathbb{T}^d \to \mathbb{R}$ *be a centered function with Fourier coefficients* $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ *satisfying*

$$\sum_{|\mathbf{k}|\ge b} |c_{\mathbf{k}}|^q \le R\log^{-\theta}(b)\,. \tag{4.48}$$

Then

$$\|\mathbb{E}_0(f \circ T^n)\|_p = \|\mathbb{E}_{-n}(f)\|_p = O(n^{-\theta(p-1)/p})\,.$$

Proof. Recall first that $\mathbb{E}_0(f \circ T^n) = \mathbb{E}_{-n}(f) \circ T^n$. Let us consider α satisfying (4.45). Let $\beta := \alpha^{1/2}$, $\gamma := \max(\alpha^{p/2}, \beta^{1/d_u})$ and

$$\mathcal{V}_n := \left\{\bar{x} \in \mathbb{T}^d \ : \ m_u(F(T^{-n}(\bar{x})) \ge \beta^n\right\}\,.$$

Let $b(n) := \left[\gamma^{-n/(2p(d+d_e+d_s))}\right]$. Let us write

$$f = f_{1,n} + f_{2,n} \quad \text{where} \quad f_{1,n} := \sum_{|\mathbf{k}|<b(n)} c_{\mathbf{k}} e^{2i\pi\langle \mathbf{k},\cdot\rangle} \quad \text{and} \quad f_{2,n} := \sum_{|\mathbf{k}|\geq b(n)} c_{\mathbf{k}} e^{2i\pi\langle \mathbf{k},\cdot\rangle}. \tag{4.49}$$

We have

$$\begin{aligned}\int_{\mathcal{V}_n} |\mathbb{E}_{-n}(f_{1,n})|^p \, d\bar{\lambda} &\leq \operatorname*{\mathbf{esssup}}_{\bar{x}\in\mathcal{V}_n} \Big(\sum_{|\mathbf{k}|\leq b(n)} |c_{\mathbf{k}}| \big|\mathbb{E}_{-n}(e^{2i\pi\langle \mathbf{k},\cdot\rangle})(\bar{x})\big| \Big)^p \\ &\leq \Big(\sum_{|\mathbf{k}|\leq b(n)} |c_{\mathbf{k}}| K_0 \beta^{-n} |\mathbf{k}|^{d_e+d_s} \alpha^n \Big)^p,\end{aligned}$$

according to (4.45) and thanks to the definition of $\mathcal{V}_n$. Now, since $\beta = \alpha^{1/2}$, we get

$$\int_{\mathcal{V}_n} |\mathbb{E}_{-n}(f_{1,n})|^p \, d\bar{\lambda} \leq 3^{dp} ||f||_1^p K_0^p \alpha^{\frac{np}{2}} (b(n))^{p(d+d_e+d_s)}.$$

Hence

$$\int_{\mathcal{V}_n} |\mathbb{E}_{-n}(f_{1,n})|^p \, d\bar{\lambda} = O(\gamma^n (b(n))^{p(d+d_e+d_s)}) = O(\gamma^{n/2}). \tag{4.50}$$

Moreover, thanks to (4.46), we have

$$\begin{aligned}\int_{\mathcal{V}_n^c} |\mathbb{E}_{-n}(f_{1,n})|^p \, d\bar{\lambda} &\leq \bar{\lambda}(\mathcal{V}_n^c) \Big(\sum_{|\mathbf{k}|\leq b(n)} |c_{\mathbf{k}}| \Big)^p \\ &= O((b(n))^{dp} \beta^{n/d_u}) = O((b(n))^{dp} \gamma^n) = O(\gamma^{n/2}).\end{aligned} \tag{4.51}$$

Since $p \geq 2$ and since $p/q = p-1$, thanks to (4.48), we have

$$\begin{aligned}\|\mathbb{E}_{-n}(f_{2,n})\|_p^p \leq \|f_{2,n}\|_p^p &\leq \Big(\sum_{|\mathbf{k}|\geq b(n)} |c_{\mathbf{k}}|^q \Big)^{p/q} \\ &\leq R^{p-1} (\log(b(n)))^{-\theta(p-1)} \ll n^{-\theta(p-1)}.\end{aligned} \tag{4.52}$$

Combining (4.50), (4.51) and (4.52), the proposition follows. □

Proposition 4.3. *Under the assumptions of Proposition* 4.2*,*

$$\big\|\mathbb{E}_0(f \circ T^{-n}) - f \circ T^{-n}\big\|_p = \|\mathbb{E}_n(f) - f\|_p = O(n^{-\theta(p-1)/p}).$$

Proof. We consider the decomposition (4.49) with $b(n)$ defined by

$$b(n) = \left[\rho_u^{-n/(2(d+1))}\right].$$

We have

$$\begin{aligned}\|\mathbb{E}_n(f_{1,n}) - f_{1,n}\|_p &\le \|\mathbb{E}_n(f_{1,n}) - f_{1,n}\|_\infty \\ &\le \sum_{|\mathbf{k}|\le b(n)} |c_{\mathbf{k}}| \|\mathbb{E}_n(e^{2i\pi\langle \mathbf{k},\cdot\rangle}) - e^{2i\pi\langle \mathbf{k},\cdot\rangle}\|_\infty \\ &\le \sum_{|\mathbf{k}|\le b(n)} |c_{\mathbf{k}}| 2\pi |\mathbf{k}| c_u \rho_u^n ,\end{aligned}$$

according to (4.43) and to (4.47). Therefore

$$\left\|\mathbb{E}_n(f_{1,n}) - f_{1,n}\right\|_p \ll (b(n))^{d+1}\rho_u^n \ll \rho_u^{n/2}. \tag{4.53}$$

Moreover, thanks to (4.48), we have

$$\begin{aligned}\|\mathbb{E}_n(f_{2,n}) - f_{2,n}\|_p^p \le 2^p\|f_{2,n}\|_p^p &\le 2^p\Bigg(\sum_{|\mathbf{k}|\ge b(n)} |c_{\mathbf{k}}|^q\Bigg)^{p/q} \\ &\le 2^p R^{p-1}(\log(b(n)))^{-\theta(p-1)} \ll n^{-\theta(p-1)} .\end{aligned} \tag{4.54}$$

Considering (4.53) and (4.54), the proposition follows. □

Proposition 4.4. *Let $p \in [2,4]$ and set $S_n(f) := \sum_{k=1}^n f\circ T^k$ with $f : \mathbb{T}^d \to \mathbb{R}$ be a centered function with Fourier coefficients satisfying* (4.48) *with $\theta > 0$ and*

$$\sum_{|\mathbf{k}|\ge b} |c_{\mathbf{k}}|^2 \le R\log^{-\eta}(b) \quad \textit{for some } \eta > 1 . \tag{4.55}$$

Set

$$m := \Big[-\frac{4(d_e+d_s)\log(r)}{\log(\alpha)}\Big] + 1 . \tag{4.56}$$

where r is the spectral radius of S. Then

$$\|\mathbb{E}_{-nm}(S_n^2(f)) - \mathbb{E}(S_n^2(f))\|_{p/2} \ll n^{2-2\theta(p-1)/p} + n^{(3-\eta)/2} .$$

Proof. Let

$$\beta := \alpha^{1/2},\ \mathcal{V}_{nm} := \left\{\bar{x}\in\mathbb{T}^d : m_u(F(T^{-nm}(\bar{x})) \ge \beta^{nm}\right\},\ \gamma := \max(\alpha^{p/8},\beta^{1/d_u})$$

and

$$b(n) := \left[\gamma^{n\,m/(p(2d+d_e+d_s))}\right]. \tag{4.57}$$

We consider the decomposition (4.49) with $b(n)$ defined by (4.57) and we set

$$S_{1,n}(f) := \sum_{k=1}^n f_{1,n}\circ T^k \quad \text{and} \quad S_{2,n}(f) := \sum_{k=1}^n f_{2,n}\circ T^k .$$

First, we note that

$$\begin{aligned}
&\|\mathbb{E}_{-nm}(S_n^2(f)) - \mathbb{E}(S_n^2(f))\|_{p/2} \\
&\qquad \le \|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} \\
&\qquad\quad + \|\mathbb{E}_{-nm}(S_{2,n}^2(f)) - \mathbb{E}(S_{2,n}^2(f))\|_{p/2} \\
&\qquad\quad + 2\|\mathbb{E}_{-nm}(S_{1,n}(f)S_{2,n}(f)) - \mathbb{E}(S_{1,n}(f)S_{2,n}(f))\|_{p/2} \\
&\qquad \le \|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} \\
&\qquad\quad + 2\|S_{2,n}(f)\|_p^2 + 4\|\mathbb{E}_{-nm}(S_{1,n}(f)S_{2,n}(f))\|_{p/2}\,.
\end{aligned}$$

Next using (3.32), we get that

$$\begin{aligned}
\|\mathbb{E}_{-nm}(S_{1,n}(f)S_{2,n}(f))\|_{p/2} \le &\|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} \\
&+ \|S_{2,n}(f)\|_p^2 + \|S_{1,n}(f)\|_2\|S_{2,n}(f)\|_2 \\
\le &\|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} \\
&+ 2\|S_{2,n}(f)\|_p^2 + \|S_n(f)\|_2\|S_{2,n}(f)\|_2\,.
\end{aligned}$$

By Propositions 4.2 and 4.3, (4.55) implies that

$$\sum_{n>0} \frac{\|\mathbb{E}_{-n}(f)\|_2}{n^{1/2}} < \infty \quad \text{and} \quad \sum_{n>0} \frac{\|f - \mathbb{E}_n(f)\|_2}{n^{1/2}} < \infty\,,$$

which yields (3.6) with $p = 2$, and then $\|S_n(f)\|_2 \ll \sqrt{n}$. Therefore, we get overall that

$$\begin{aligned}
&\|\mathbb{E}_{-nm}(S_n^2(f)) - \mathbb{E}(S_n^2(f))\|_{p/2} \\
&\ll \|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} + \|S_{2,n}(f)\|_p^2 + \sqrt{n}\|S_{2,n}(f)\|_2\,. \qquad (4.58)
\end{aligned}$$

Since $p \ge 2$ and $p/q = p - 1$, (4.48) implies that

$$\begin{aligned}
\|S_{2,n}(f)\|_p \le n\|f_{2,n}\|_p \le n\Big(\sum_{|\mathbf{k}|\ge b(n)} |c_{\mathbf{k}}|^q\Big)^{1/q} \\
\le nR^{(p-1)/p}(\log(b(n)))^{-\theta(p-1)/p} \ll n^{1-\theta(p-1)/p}\,. \qquad (4.59)
\end{aligned}$$

Similarly using (4.55), we get that

$$\|S_{2,n}(f)\|_2 \le n\|f_{2,n}\|_2 \ll n^{1-\eta/2}\,. \qquad (4.60)$$

We deal now with the first term in the right-hand side of (4.58). With this aim, we first observe that, for any non negative integer ℓ, $e^{2i\pi\langle \mathbf{k}, T^\ell(\cdot)\rangle} = e^{2i\pi\langle {}^tS^\ell\mathbf{k},\cdot\rangle}$, where

${}^tS^\ell$ is the transposed matrix of S^ℓ. Therefore,

$$\int_{\mathcal{V}_{nm}} \left|\mathbb{E}_{-nm}(f_{1,n}.f_{1,n}\circ T^\ell) - \mathbb{E}(f_{1,n}.f_{1,n}\circ T^\ell)\right|^{p/2} d\bar\lambda$$
$$\le \operatorname*{\mathbf{esssup}}_{\bar x\in\mathcal{V}_{nm}}\Big(\sum_{|\mathbf{k}|,|\mathbf{m}|\le b(n):\mathbf{k}+{}^tS^\ell\mathbf{m}\neq 0} |c_\mathbf{k}||c_\mathbf{m}|\left|\mathbb{E}_{-nm}(e^{2i\pi\langle \mathbf{k}+{}^tS^\ell\mathbf{m},\cdot\rangle})(\bar x)\right|\Big)^{p/2}$$
$$\le \Big(\sum_{|\mathbf{k}|,|\mathbf{m}|\le b(n)} |c_\mathbf{k}||c_\mathbf{m}|K_0\beta^{-nm}|\mathbf{k}+{}^tS^\ell\mathbf{m}|^{d_e+d_s}\alpha^{nm}\Big)^{p/2},$$

according to (4.45) and to the definition of $\mathcal{V}_{nm}$. It follows that

$$\int_{\mathcal{V}_{nm}} \left|\mathbb{E}_{-nm}(f_{1,n}.f_{1,n}\circ T^\ell) - \mathbb{E}(f_{1,n}.f_{1,n}\circ T^\ell)\right|^{p/2} d\bar\lambda$$
$$\le \Big(\sum_{|\mathbf{k}|,|\mathbf{m}|\le b(n)} \|f\|_1^2 K_0\beta^{-nm}(|\mathbf{k}|+r^\ell|\mathbf{m}|)^{d_e+d_s}\alpha^{nm}\Big)^{p/2}$$
$$\ll \alpha^{\frac{nmp}{4}} r^{p\ell(d_e+d_s)/2}(b(n))^{p(2d+d_e+d_s)/2}.$$

Hence, since $\gamma \ge \alpha^{p/8}$, $m \ge 4(d_e+d_s)\log(r)/\log(1/\alpha)$, and according to the definition of $b(n)$, we have

$$\sup_{\ell\in\{0,\dots,n\}}\int_{\mathcal{V}_{nm}} \left|\mathbb{E}_{-nm}(f_{1,n}.f_{1,n}\circ T^\ell) - \mathbb{E}(f_{1,n}.f_{1,n}\circ T^\ell)\right|^{p/2} d\bar\lambda$$
$$\ll \alpha^{3nmp/16} r^{pn(d_e+d_s)/2} \ll \gamma^{nm/2}. \tag{4.61}$$

Moreover, for any non negative integer ℓ,

$$\int_{\mathcal{V}_{nm}^c} \left|\mathbb{E}_{-nm}(f_{1,n}.f_{1,n}\circ T^\ell)\right|^{p/2} d\bar\lambda \le \bar\lambda(\mathcal{V}_{nm}^c)\Big(\sum_{|\mathbf{k}|,|\mathbf{m}|\le b(n)}|c_\mathbf{k}||c_\mathbf{m}|\Big)^{p/2} \tag{4.62}$$
$$\ll (b(n))^{dp}\beta^{nm/d_u} \ll (b(n))^{dp}\gamma^{nm} \ll \gamma^{nm/2},$$

according to (4.46) and to the definition of $b(n)$ and of γ. Combining (4.61) and (4.62), we then derive that

$$\|\mathbb{E}_{-nm}(S_{1,n}^2(f)) - \mathbb{E}(S_{1,n}^2(f))\|_{p/2} \tag{4.63}$$
$$\le 2\sum_{i=1}^{n}\sum_{j=0}^{n-i}\|\mathbb{E}_{-nm}(f_{1,n}\circ T^i f_{1,n}\circ T^{i+j}) - \mathbb{E}(f_{1,n}\circ T^i f_{1,n}\circ T^{i+j})\|_{p/2}$$
$$\le n^2 \sup_{\ell\in\{0,\dots,n\}}\|\mathbb{E}_{-nm}(f_{1,n}f_{1,n}\circ T^\ell) - \mathbb{E}(f_{1,n}f_{1,n}\circ T^\ell)\|_{p/2} \ll n^2\gamma^{nm/p}.$$

Considering (4.59), (4.60) and (4.63) in (4.58), the proposition follows. □

4.2. End of the proof of Theorem 2.1

Proof. Propositions 4.2 and 4.3 give (3.1) provided (2.2) is satisfied. Propositions 4.2 and 4.3 give (3.2) (when $p \in]2,4[$) and (3.5) (when $p = 4$), provided (2.3) is satisfied. Finally, Proposition 4.4 gives (3.3) provided (2.2) and (2.3) are satisfied. The proof follows now from Theorem 3.1 when $p \in]2,4[$ and from Theorem 3.2 when $p = 4$. □

5. Appendix

As in Section 3, let $P_k(X) = \mathbb{E}_k(X) - \mathbb{E}_{k-1}(X)$.

Lemma 5.1. *Let $p \in [2,\infty[$. Then, for any real $1 \le q \le p$ and any positive integer n,*

$$\sum_{k \ge 2n} \|P_0(X_k)\|_p^q \ll \sum_{k \ge n} \frac{\|\mathbb{E}_0(X_k)\|_p^q}{k^{q/p}}$$

and

$$\sum_{k \ge 2n} \|P_0(X_{-k})\|_p^q \ll \sum_{k \ge n} \frac{\|X_{-k} - \mathbb{E}_0(X_{-k})\|_p^q}{k^{q/p}} .$$

Proof. The first inequality is Lemma 5.1 in [2]. To prove the second one, we first consider the case $p > q$ and we follow the lines of the proof Lemma 5.1 in [2] with $P_k(X_0)$ replacing $P_{-k}(X_0)$. We get that

$$\sum_{k \ge 2n} \|P_0(X_{-k})\|_p^q \ll \sum_{k \ge n+1} k^{-\frac{q}{p}} \Big(\sum_{\ell \ge k} \|P_0(X_{-\ell})\|_p^p \Big)^{q/p} .$$

Now, we notice that, by the Rosenthal's inequality given in Theorem 2.12 of [9], there exists a constant c_p depending only on p such that

$$\sum_{\ell \ge k} \|P_0(X_{-\ell})\|_p^p = \sum_{\ell \ge k} \|P_\ell(X_0)\|_p^p \le c_p \Big\| \sum_{\ell \ge k} P_\ell(X_0) \Big\|_p^p$$
$$= c_p \|X_0 - \mathbb{E}_k(X_0)\|_p^p = c_p \|X_{-k} - \mathbb{E}_0(X_{-k})\|_p^p . \qquad (5.1)$$

Now when $p = q$, inequality (5.1) together with the fact that by Claim 3.1, for any integer k in $[n+1, 2n]$, $\|X_0 - \mathbb{E}_{2n}(X_0)\|_p^p \le 2^p \|X_0 - \mathbb{E}_k(X_0)\|_p^p$ imply the result. Indeed we have

$$\sum_{k \ge 2n} \|P_0(X_{-\ell})\|_p^p \le c_p \|X_0 - \mathbb{E}_{2n}(X_0)\|_p^p \ll \sum_{k=n+1}^{2n} k^{-1} \|X_0 - \mathbb{E}_k(X_0)\|_p^p .$$

The proof is complete. □

Proposition 5.1. *Let $p \in [1,\infty[$ and assume that*

$$\text{the series} \quad d_0 = \sum_{i \in \mathbb{Z}} P_0(X_i) \quad \text{converges in } \mathbb{L}^p. \qquad (5.2)$$

Let $M_n := \sum_{i=1}^n d_0 \circ T^i$ *and* $R_n := S_n - M_n$*. Then, for any positive integers* n *and* N*,*

$$R_n = \mathbb{E}_0(S_n) - \mathbb{E}_0(S_N)\circ T^n + \mathbb{E}_{-n}(S_N)\circ T^n - \sum_{k=1}^n \sum_{j\ge n+N+1} P_k(X_j)$$
$$+ S_n - \mathbb{E}_n(S_n) - (\mathbb{E}_{n+N}(S_N - \mathbb{E}_N(S_N))\circ T^{-N} - \sum_{k=1}^n \sum_{j\ge N} P_k(X_{-j}),$$

and

$$\|R_n\|_p^{p'} \ll \|\mathbb{E}_0(S_n)\|_p^{p'} + \|\mathbb{E}_0(S_N)\|_p^{p'} + \|S_n - \mathbb{E}_n(S_n)\|_p^{p'} + \|S_N - \mathbb{E}_N(S_N)\|_p^{p'}$$
$$+ \sum_{k=1}^n \Big\| \sum_{j\ge k+N} P_0(X_j)\Big\|_p^{p'} + \sum_{k=1}^n \Big\| \sum_{j\ge k+N} P_0(X_{-j})\Big\|_p^{p'},$$

where $p' = \min(2,p)$.

Proof. Notice first that the following decomposition is valid: for any positive integer n,

$$R_n = \sum_{k=1}^n\Big(X_k - \sum_{j=1}^n P_j(X_k)\Big) - \sum_{k=1}^n \sum_{j\ge n+1} P_k(X_j) - \sum_{k=1}^n \sum_{j=0}^\infty P_k(X_{-j})$$
$$= R_{n,1} + R_{n,2}\,, \tag{5.3}$$

where

$$R_{n,1} := \mathbb{E}_0(S_n) - \sum_{k=1}^n \sum_{j\ge n+1} P_k(X_j)\,, \quad R_{n,2} := S_n - \mathbb{E}_n(S_n) - \sum_{k=1}^n \sum_{j=0}^\infty P_k(X_{-j})\,. \tag{5.4}$$

Let N be a positive integer. According to item 1 of Proposition 2.1 in [2],

$$R_{n,1} = \mathbb{E}_0(S_n) - \mathbb{E}_n(S_{n+N} - S_n) + \mathbb{E}_0(S_{n+N} - S_n) - \sum_{k=1}^n \sum_{j\ge n+N+1} P_k(X_j)\,. \tag{5.5}$$

On an other hand, we write that

$$\sum_{j=0}^\infty P_k(X_{-j}) = \sum_{j=0}^{N-1} P_k(X_{-j}) + \sum_{j\ge N} P_k(X_{-j})\,.$$

Therefore

$$R_{n,2} = S_n - \mathbb{E}_n(S_n) - (\mathbb{E}_{n+N}(S_N - \mathbb{E}_N(S_N))\circ T^{-N} - \sum_{k=1}^n \sum_{j\ge N} P_k(X_{-j})\,. \tag{5.6}$$

Starting from (5.3) and considering (5.5) and (5.6), the first part follows. We turn now to the second part of the proposition. Applying Burkholder's inequality and using stationarity, we obtain that there exists a positive constant c_p such that, for any positive integer n,

$$\Big\|\sum_{k=1}^{n}\sum_{j\geq n+N+1}P_k(X_j)\Big\|_p^{p'}\leq c_p\sum_{k=1}^{n}\Big\|\sum_{j\geq n+N+1}P_k(X_j)\Big\|_p^{p'}=c_p\sum_{k=1}^{n}\Big\|\sum_{j\geq N+k}P_0(X_j)\Big\|_p^{p'}, \tag{5.7}$$

and

$$\Big\|\sum_{k=1}^{n}\sum_{j\geq N}P_k(X_{-j})\Big\|_p^{p'}\leq c_p\sum_{k=1}^{n}\Big\|\sum_{j\geq N}P_k(X_{-j})\Big\|_p^{p'}=c_p\sum_{k=1}^{n}\Big\|\sum_{j\geq N+k}P_0(X_{-j})\Big\|_p^{p'}. \tag{5.8}$$

The second part of the proposition follows from item 1 by taking into account the stationarity and by considering the bounds (5.7) and (5.8). □

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On the Rate of Convergence to the Semi-circular Law

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Abstract. Let $\mathbf{X} = (X_{jk})_{j,k=1}^n$ denote a Hermitian random matrix with entries X_{jk}, which are independent for $1 \le j \le k \le n$. We consider the rate of convergence of the empirical spectral distribution function of the matrix $\mathbf{X}$ to the semi-circular law assuming that $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$ and that the distributions of the matrix elements X_{jk} have a uniform sub exponential decay in the sense that there exists a constant $\varkappa > 0$ such that for any $1 \le j \le k \le n$ and any $t \ge 1$ we have

$$\Pr\{|X_{jk}| > t\} \le \varkappa^{-1} \exp\{-t^{\varkappa}\}.$$

By means of a short recursion argument it is shown that the Kolmogorov distance between the empirical spectral distribution of the Wigner matrix $\mathbf{W} = \frac{1}{\sqrt{n}}\mathbf{X}$ and the semicircular law is of order $O(n^{-1}\log^b n)$ with some positive constant $b > 0$.

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1. Introduction

Consider a family $\mathbf{X} = \{X_{jk}\}$, $1 \le j \le k \le n$, of independent real random variables defined on some probability space $(\Omega, \mathfrak{M}, \Pr)$, for any $n \ge 1$. Assume that $X_{jk} = X_{kj}$, for $1 \le k < j \le n$, and introduce the symmetric matrices

$$\mathbf{W} = \frac{1}{\sqrt{n}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}.$$

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The matrix $\mathbf{W}$ has a random spectrum $\{\lambda_1, \dots, \lambda_n\}$ and an associated spectral distribution function $\mathcal{F}_n(x) = \frac{1}{n}\,\mathrm{card}\{j \le n : \lambda_j \le x\}, \quad x \in \mathbb{R}$. Averaging over the random values $X_{ij}(\omega)$, define the expected (non-random) empirical distribution functions $F_n(x) = \mathbf{E}\,\mathcal{F}_n(x)$. Let $G(x)$ denote the semi-circular distribution function with density $g(x) = G'(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{I}_{[-2,2]}(x)$, where $\mathbb{I}_{[a,b]}(x)$ denotes an indicator-function of interval $[a,b]$. We shall study the rate of convergence of $\mathcal{F}_n(x)$ to the semi-circular law under the condition

$$\Pr\{|X_{jk}| > t\} \le \varkappa^{-1}\exp\{-t^{\varkappa}\}, \tag{1.1}$$

for some $\varkappa > 0$ and for any $t \ge 1$. The rate of convergence to the semi-circular law has been studied by several authors. We proved in [7] that the Kolmogorov distance between $\mathcal{F}_n(x)$ and the distribution function $G(x)$, $\Delta_n^* := \sup_x |\mathcal{F}_n(x) - G(x)|$ is of order $O_P(n^{-\frac12})$ (i.e., $n^{\frac12}\Delta_n^*$ is bounded in probability). Bai [1] and Girko [4] showed that $\Delta_n := \sup_x |F_n(x) - G(x)| = O(n^{-\frac12})$. Bobkov, Götze and Tikhomirov [3] proved that Δ_n and $\mathbf{E}\Delta_n^*$ have order $O(n^{-\frac23})$ assuming a Poincaré inequality for the distribution of the matrix elements. For the Gaussian Unitary Ensemble respectively for the Gaussian Orthogonal Ensemble, see [6] respectively [12], it has been shown that $\Delta_n = O(n^{-1})$. Denote by $\gamma_{n1} \le \dots \le \gamma_{nn}$, the quantiles of G, i.e., $G(\gamma_{nj}) = \frac{j}{n}$. We introduce the notation $\mathrm{llog}_n := \log\log n$. Erdös, Yau and Yin [10] showed, for matrices with elements X_{jk} which have a uniformly sub exponential decay, i.e., condition (1.1) holds, the following result

$$\Pr\Big\{\exists\, j : |\lambda_j - \gamma_j| \ge (\log n)^{C\,\mathrm{llog}_n}\Big[\min\{(j, N-j+1)\Big]^{-\frac13} n^{-\frac23}\Big\}$$
$$\le C\exp\{-(\log n)^{c\,\mathrm{llog}_n}\},$$

for n large enough. It is straightforward to check that this bound implies that

$$\Pr\Big\{\sup_x |\mathcal{F}_n(x) - G(x)| \le Cn^{-1}(\log n)^{C\,\mathrm{llog}_n}\Big\} \ge 1 - C\exp\{-(\log n)^{c\,\mathrm{llog}_n}\}. \tag{1.2}$$

From the last inequality it is follows that $\mathbf{E}\Delta_n^* \le Cn^{-1}(\log n)^{C\,\mathrm{llog}_n}$. In this paper we derive some improvement of the result (1.2) (reducing the power of logarithm) using arguments similar to those used in [10] and provide a self-contained proof based on recursion methods developed in the papers of Götze and Tikhomirov [7], [5] and [13]. It follows from the results of Gustavsson [8] that the best possible bound in the Gaussian case for the rate of convergence in probability is $O(n^{-1}\sqrt{\log n})$. For any positive constants $\alpha > 0$ and $\varkappa > 0$, define the quantities

$$l_{n,\alpha} := \log n\,(\log\log n)^{\alpha} \quad \text{and} \quad \beta_n := (l_{n,\alpha})^{\frac{1}{\varkappa}+\frac12}. \tag{1.3}$$

The main result of this paper is the following

Theorem 1.1. *Let* $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$. *Assume that there exists a constant* $\varkappa > 0$ *such that inequality* (1.1) *holds, for any* $1 \le j \le k \le n$ *and any* $t \ge 1$. *Then, for any positive* $\alpha > 0$ *there exist positive constants* C *and* c *depending on* $\varkappa$ *and* α

only such that

$$\Pr\left\{\sup_x |\mathcal{F}_n(x) - G(x)| > n^{-1}\beta_n^4 \ln n\right\} \le C\exp\{-cl_{n,\alpha}\}.$$

We apply the result of Theorem 1.1 to study the eigenvectors of the matrix $\mathbf{W}$. Let $\mathbf{u}_j = (u_{j1}, \dots, u_{jn})^T$ be eigenvectors of the matrix $\mathbf{W}$ corresponding to the eigenvalues λ_j, $j = 1, \dots, n$. We prove the following result.

Theorem 1.2. *Under the conditions of Theorem* 1.1*, for any positive* $\alpha > 0$*, there exist positive constants* C *and* c*, depending on* $\varkappa$ *and* α *only such that*

$$\Pr\Big\{\max_{1\le j,k\le n} |u_{jk}|^2 > \frac{\beta_n^2}{n}\Big\} \le C\exp\{-cl_{n,\alpha}\}, \tag{1.4}$$

and

$$\Pr\Big\{\max_{1\le k\le n}\left|\sum_{\nu=1}^k |u_{j\nu}|^2 - \frac{k}{n}\right| > \frac{\beta_n^2}{\sqrt{n}}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{1.5}$$

2. Bounds for the Kolmogorov distance between distribution functions via Stieltjes transforms

To bound the error Δ_n^* we shall use an approach developed in previous work of the authors, see [7].

We modify the bound of the Kolmogorov distance between an arbitrary distribution function and the semi-circular distribution function via their Stieltjes transforms obtained in [7] Lemma 2.1. For $x \in [-2,2]$ define $\gamma(x) := 2 - |x|$. Given $\frac{1}{2} > \varepsilon > 0$ introduce the interval $\mathbb{J}_\varepsilon := \{x \in [-2,2] : \gamma(x) \ge \varepsilon\}$ and $\mathbb{J}'_\varepsilon := \mathbb{J}_{\varepsilon/2}$. For a distribution function F denote by $S_F(z)$ its Stieltjes transform,

$$S_F(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dF(x).$$

Proposition 2.1. *Let* $v > 0$ *and* $a > 0$ *and* $\frac{1}{2} > \varepsilon > 0$ *be positive numbers such that*

$$\frac{1}{\pi}\int_{|u|\le a} \frac{1}{u^2+1} du = \frac{3}{4} =: \beta, \tag{2.1}$$

and

$$2va \le \varepsilon^{\frac{3}{2}}. \tag{2.2}$$

If G *denotes the distribution function of the standard semi-circular law, and* F *is any distribution function, there exist some absolute constants* C_1 *and* C_2 *such that*

$$\Delta(F,G) := \sup_x |F(x) - G(x)|$$
$$\le 2\sup_{x\in\mathbb{J}'_\varepsilon}\left|\operatorname{Im}\int_{-\infty}^{x} (S_F(u + i\frac{v}{\sqrt{\gamma}}) - S_G(u + i\frac{v}{\sqrt{\gamma}}))du\right| + C_1 v + C_2\varepsilon^{\frac{3}{2}}.$$

Remark 2.2. For any $x \in \mathcal{J}_\varepsilon$ we have $\gamma = \gamma(x) \ge \varepsilon$ and according to condition (2.2), $\frac{av}{\sqrt{\gamma}} \le \frac{\varepsilon}{2}$.

Proof. The proof of Proposition 2.1 is a straightforward adaptation of the proof of Lemma 2.1 from [7]. We include it here for the sake of completeness. First we note that

$$\sup_x |F(x)-G(x)| = \sup_{x\in[-2,2]} |F(x)-G(x)| \tag{2.3}$$
$$= \max\Big\{ \sup_{x\in\mathcal{J}_\varepsilon} |F(x)-G(x)|, \sup_{x\in[-2,-2+\varepsilon]} |F(x)-G(x)|, \sup_{x\in[2-\varepsilon,2]} |F(x)-G(x)|\Big\}.$$

Furthermore, for $x\in[-2,-2+\varepsilon]$ we have

$$\begin{aligned} -G(-2+\varepsilon) \le F(x)-G(x) &\le F(-2+\varepsilon)-G(-2+\varepsilon)+G(-2+\varepsilon)\\ &\le \sup_{x\in\mathcal{J}_\varepsilon} |F(x)-G(x)| + G(-2+\varepsilon). \end{aligned} \tag{2.4}$$

This inequality yields

$$\sup_{x\in[-2,-2+\varepsilon]} |F(x)-G(x)| \le \sup_{x\in\mathcal{J}_\varepsilon} |F(x)-G(x)| + G(-2+\varepsilon). \tag{2.5}$$

Similarly we get

$$\sup_{x\in[2-\varepsilon,2]} |F(x)-G(x)| \le \sup_{x\in\mathcal{J}_\varepsilon} |F(x)-G(x)| + 1 - G(2-\varepsilon). \tag{2.6}$$

Note that $G(-2+\varepsilon)=1-G(2-\varepsilon)$ and $G(-2+\varepsilon)\le C\varepsilon^{\frac{3}{2}}$ with some absolute constant $C>0$. Combining all these relations we get

$$\sup_x |F(x)-G(x)| \le \Delta_\varepsilon(F,G) + C\varepsilon^{\frac{3}{2}}, \tag{2.7}$$

where $\Delta_\varepsilon(F,G)=\sup_{x\in\mathbb{J}_\varepsilon}|F(x)-G(x)|$. We denote $v'=\frac{v}{\sqrt{\gamma}}$. For any $x\in\mathbb{J}'_\varepsilon$

$$\begin{aligned} &\Big|\frac{1}{\pi}\mathrm{Im}\Big(\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\Big)\Big| \\ &\quad\ge \frac{1}{\pi}\mathrm{Im}\Big(\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\Big) \\ &\quad= \frac{1}{\pi}\Big[\int_{-\infty}^{x}\int_{-\infty}^{\infty}\frac{v'd(F(y)-G(y))}{(y-u)^2+v'^2}\Big]du \\ &\quad= \frac{1}{\pi}\int_{-\infty}^{x}\Big[\int_{-\infty}^{\infty}\frac{2v'(y-u)(F(y)-G(y))dy}{((y-u)^2+v'^2)^2}\Big] \\ &\quad= \frac{1}{\pi}\int_{-\infty}^{\infty}(F(y)-G(y))\Big[\int_{-\infty}^{x}\frac{2v'(y-u)}{((y-u)^2+v'^2)^2}du\Big]dy \\ &\quad= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F(x-v'y)-G(x-v'y)}{y^2+1}dy, \quad \text{by change of variables.} \end{aligned} \tag{2.8}$$

Furthermore, using (2.1) and the definition of $\Delta(F,G)$ we note that

$$\frac{1}{\pi}\int_{|y|>a}\frac{|F(x-v'y)-G(x-v'y)|}{y^2+1}dy \le (1-\beta)\Delta(F,G). \tag{2.9}$$

Since F is non-decreasing, we have

$$\frac{1}{\pi}\int_{|y|\le a}\frac{F(x-v'y)-G(x-v'y)}{y^2+1}dy \ge \frac{1}{\pi}\int_{|y|\le a}\frac{F(x-v'a)-G(x-v'y)}{y^2+1}dy$$
$$\ge (F(x-v'a)-G(x-v'a))\beta - \frac{1}{\pi}\int_{|y|\le a}|G(x-v'y)-G(x-v'a)|dy. \quad (2.10)$$

These inequalities together imply (using a change of variables in the last step)

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}&\frac{F(x-v'y)-G(x-v'y)}{y^2+1}dy\\ &\ge \beta(F(x-v'a)-G(x-v'a))\\ &\quad -\frac{1}{\pi}\int_{|y|\le a}|G(x-v'y)-G(x-v'a)|dy-(1-\beta)\Delta(F,G)\\ &\ge \beta(F(x-v'a)-G(x-v'a))\\ &\quad -\frac{1}{v'\pi}\int_{|y|\le v'a}|G(x-y)-G(x-v'a))|dy-(1-\beta)\Delta(F,G). \end{aligned} \quad (2.11)$$

Note that according to Remark 2.2, $x \pm v'a \in \mathbb{J}'_\varepsilon$ for any $x \in \mathcal{J}_\varepsilon$. Assume first that $x_n \in \mathbb{J}_\varepsilon$ is a sequence such that $F(x_n)-G(x_n) \to \Delta_\varepsilon(F,G)$. Then $x'_n := x_n + v'a \in \mathbb{J}'_\varepsilon$. Using (2.8) and (2.11), we get

$$\begin{aligned}\sup_{x\in\mathbb{J}'_\varepsilon}&\left|\operatorname{Im}\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\right|\\ &\ge \operatorname{Im}\int_{-\infty}^{x'_n}(S_F(u+iv')-S_G(u+iv'))du\\ &\ge \beta(F(x'_n-v'a)-G(x'_n-v'a))\\ &\quad -\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\sqrt{\gamma}\int_{|y|\le 2v'a}|G(x+y)-G(x)|dy-(1-\beta)\Delta(F,G)\\ &= \beta(F(x_n)-G(x_n))\\ &\quad -\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\sqrt{\gamma}\int_{|y|<2v'a}|G(x+y)-G(x)|dy-(1-\beta)\Delta(F,G). \end{aligned} \quad (2.12)$$

Assume for definiteness that $y > 0$. Recall that $\varepsilon \le 2\gamma$, for any $x \in \mathcal{J}'_\varepsilon$. By Remark 2.2 with $\varepsilon/2$ instead ε, we have $0 < y \le 2v'a \le \sqrt{2}\varepsilon$, for any $x \in \mathcal{J}'_\varepsilon$. For the semi-circular law we obtain,

$$\begin{aligned}|G(x+y)-G(x)| &\le y\sup_{u\in[x,x+y]}G'(u) \le yC\sqrt{\gamma+y}\\ &\le Cy\sqrt{\gamma+2v'a} \le Cy\sqrt{\gamma+\varepsilon} \le Cy\sqrt{\gamma}. \end{aligned} \quad (2.13)$$

This yields after integrating in y

$$\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\sqrt{\gamma}\int_{0\le y\le 2v'a}|G(x+y)-G(x)|dy \le \frac{C}{v}\sup_{x\in\mathbb{J}'_\varepsilon}\gamma v'^2 \le Cv. \quad (2.14)$$

Similarly we get that

$$\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\sqrt{\gamma}\int_{0\ge y\ge -2v'a}|G(x+y)-G(x)|dy \le \frac{C}{v}\sup_{x\in\mathbb{J}'_\varepsilon}\gamma {v'}^2 \le Cv. \tag{2.15}$$

By inequality (2.7)

$$\Delta_\varepsilon(F,G)\ge \Delta(F,G)-C\varepsilon^{\frac{3}{2}}. \tag{2.16}$$

The inequalities (2.12), (2.16) and (2.14), (2.15) together yield as n tends to infinity

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\operatorname{Im}\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\right|$$
$$\ge (2\beta-1)\Delta(F,G)-Cv-C\varepsilon^{\frac{3}{2}}, \tag{2.17}$$

for some constant $C>0$. Similar arguments may be used to prove this inequality in case there is a sequence $x_n\in\mathbb{J}_\varepsilon$ such $F(x_n)-G(x_n)\to-\Delta_\varepsilon(F,G)$. In view of (2.17) and $2\beta-1=1/2$ this completes the proof. □

Lemma 2.1. *Under the conditions of Proposition* 2.1*, for any* $V>v$ *and* $0<v\le\frac{\varepsilon^{3/2}}{2a}$ *and* $v'=v/\sqrt{\gamma},\gamma=2-|x|$, $x\in\mathbb{J}'_\varepsilon$ *as above, the following inequality holds*

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{-\infty}^{x}(\operatorname{Im}(S_F(u+iv')-S_G(u+iv'))du\right|$$
$$\le\int_{-\infty}^{\infty}|S_F(u+iV)-S_G(u+iV)|du+\sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{v'}^{V}(S_F(x+iu)-S_G(x+iu))\,du\right|.$$

Proof. Let $x\in\mathbb{J}'_\varepsilon$ be fixed. Let $\gamma=\gamma(x)$. Put $z=u+iv'$. Since $v'=\frac{v}{\sqrt{\gamma}}\le\frac{\varepsilon}{2a}$, see (2.2), we may assume without loss of generality that $v'\le 4$ for $x\in\mathbb{J}'_\varepsilon$. Since the functions of $S_F(z)$ and $S_G(z)$ are analytic in the upper half-plane, it is enough to use Cauchy's theorem. We can write for $x\in\mathcal{J}'_\varepsilon$

$$\int_{-\infty}^{x}\operatorname{Im}(S_F(z)-S_G(z))du=\operatorname{Im}\left\{\lim_{L\to\infty}\int_{-L}^{x}(S_F(u+iv')-S_G(u+iv'))du\right\}.$$

By Cauchy's integral formula, we have

$$\int_{-L}^{x}(S_F(z)-S_G(z))du=\int_{-L}^{x}(S_F(u+iV)-S_G(u+iV))du$$
$$+\int_{v'}^{V}(S_F(-L+iu)-S_G(-L+iu))du$$
$$-\int_{v'}^{V}(S_F(x+iu)-S_G(x+iu))du.$$

Denote by ξ(resp. η) a random variable with distribution function $F(x)$ (resp. $G(x)$). Then we have

$$|S_F(-L+iu)|=\left|\mathbf{E}\frac{1}{\xi+L-iu}\right|\le {v'}^{-1}\Pr\{|\xi|>L/2\}+\frac{2}{L},$$

for any $0 < v' \le u \le V$. Similarly,

$$|S_G(-L+iu)| \le {v'}^{-1} \Pr\{|\eta| > L/2\} + \frac{2}{L}.$$

These inequalities imply that

$$\left| \int_{v'}^{V} (S_F(-L+iu) - S_G(-L+iu))du \right| \to 0 \quad \text{as} \quad L \to \infty,$$

which completes the proof. □

Combining the results of Proposition 2.1 and Lemma 2.1, we get

Corollary 2.2. *Under the conditions of Proposition* 2.1 *the following inequality holds*

$$\Delta(F,G) \le 2\int_{-\infty}^{\infty} |S_F(u+iV) - S_G(u+iV)|du + C_1 v + C_2 \varepsilon^{\frac{3}{2}}$$
$$+ 2 \sup_{x\in\mathbb{J}'_\varepsilon} \int_{v'}^{V} |S_F(x+iu) - S_G(x+iu)|du,$$

where $v' = \frac{v}{\sqrt{\gamma}}$ *with* $\gamma = 2 - |x|$ *and* $C_1, C_2 > 0$ *denote absolute constants.*

We shall apply the last inequality. We denote the Stieltjes transform of $\mathcal{F}_n(x)$ by $m_n(z)$ and the Stieltjes transform of the semi-circular law by $s(z)$. Let $\mathbf{R} = \mathbf{R}(z)$ be the resolvent matrix of $\mathbf{W}$ given by $\mathbf{R} = (\mathbf{W} - z\mathbf{I}_n)^{-1}$, for all $z = u + iv$ with $v \neq 0$. Here and in what follows $\mathbf{I}_n$ denotes the identity matrix of dimension n. Sometimes we shall omit the sub index in the notation of an identity matrix. It is well known that the Stieltjes transform of the semi-circular distribution satisfies the equation

$$s^2(z) + zs(z) + 1 = 0 \tag{2.18}$$

(see, for example, equality (4.20) in [7]). Furthermore, the Stieltjes transform of an empirical spectral distribution function $\mathcal{F}_n(x)$, say $m_n(z)$, is given by

$$m_n(z) = \frac{1}{n}\sum_{j=1}^{n} R_{jj} = \frac{1}{2n}\operatorname{Tr}\mathbf{R}.$$

(see, for instance, equality (4.3) in [7]). Introduce the matrices $\mathbf{W}^{(j)}$, which are obtained from $\mathbf{W}$ by deleting the jth row and the jth column, and the corresponding resolvent matrix $\mathbf{R}^{(j)}$ defined by $\mathbf{R}^{(j)} := (\mathbf{W}^{(j)} - z\mathbf{I}_{n-1})^{-1}$ and let $m_n^{(j)}(z) := \frac{1}{n-1}\operatorname{Tr}\mathbf{R}^{(j)}$. Consider the index sets $\mathbb{T}_j := \{1,\dots,n\} \setminus \{j\}$. We shall use the representation

$$R_{jj} = \frac{1}{-z + \frac{1}{\sqrt{n}}X_{jj} - \frac{1}{n}\sum_{k,l\in\mathbb{T}_j} X_{jk}X_{jl}R_{kl}^{(j)}},$$

(see, for example, equality (4.6) in [7]). We may rewrite it as follows

$$R_{jj} = -\frac{1}{z + m_n(z)} + \frac{1}{z+m_n(z)}\varepsilon_j R_{jj}, \tag{2.19}$$

where $\varepsilon_j := \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3} + \varepsilon_{j4}$ with

$$\varepsilon_{j1} := \frac{1}{\sqrt{n}} X_{jj}, \quad \varepsilon_{j2} := \frac{1}{n} \sum_{k\in\mathbb{T}_j} (X_{jk}^2 - 1) R_{kk}^{(j)},$$
$$\varepsilon_{j3} := \frac{1}{n} \sum_{k\neq l\in\mathbb{T}_j} X_{jk} X_{jl} R_{kl}^{(j)}, \quad \varepsilon_{j4} := \frac{1}{n}(\operatorname{Tr}\mathbf{R}^{(j)} - \operatorname{Tr}\mathbf{R}). \tag{2.20}$$

This relation immediately implies the following two equations

$$R_{jj} = -\frac{1}{z+m_n(z)} - \sum_{\nu=1}^{3} \frac{\varepsilon_{j\nu}}{(z+m_n(z))^2} + \sum_{\nu=1}^{3} \frac{1}{(z+m_n(z))^2} \varepsilon_{j\nu}\varepsilon_j R_{jj} + \frac{1}{z+m_n(z)} \varepsilon_{j4} R_{jj},$$

and

$$m_n(z) = -\frac{1}{z+m_n(z)} - \frac{1}{(z+m_n(z))} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j R_{jj} \tag{2.21}$$
$$= -\frac{1}{z+m_n(z)} - \frac{1}{(z+m_n(z))^2} \frac{1}{n} \sum_{\nu=1}^{3} \sum_{j=1}^{n} \varepsilon_{j\nu} + $$
$$+ \frac{1}{(z+m_n(z))^2} \frac{1}{n} \sum_{\nu=1}^{3} \sum_{j=1}^{n} \varepsilon_{j\nu}\varepsilon_j R_{jj} + \frac{1}{z+m_n(z)} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j4} R_{jj}. \tag{2.22}$$

3. Large deviations I

In the following lemmas we shall bound $\varepsilon_{j\nu}$, for $\nu = 1, \dots, 4$ and $j = 1, \dots, n$. Using the exponential tails of the distribution of X_{jk} we shall replace quantities like, e.g., $\mathbf{E}|X_{jk}|^p I(|X_{jk}| > l_{n,\alpha}^{\frac{1}{\varkappa}})$ and others by a uniform error bound $C\exp\{-c l_{n,\alpha}\}$ with constants $C, c > 0$ depending on $\varkappa$ and α varying from one instance to the next.

Lemma 3.1. *Assuming the conditions of Theorem* 1.1 *there exist positive constants C and c, depending on $\varkappa$ and α such that*

$$\Pr\{|\varepsilon_{j1}| \geq 2 l_{n,\alpha}^{\frac{1}{\varkappa}} n^{-\frac{1}{2}}\} \leq C\exp\{-c l_{n,\alpha}\},$$

for any $j = 1, \dots, n$.

Proof. The result follows immediately from the hypothesis (1.1). □

Lemma 3.2. *Assuming the conditions of Theorem* 1.1 *we have, for any $z = u + iv$ with $v > 0$ and any $j = 1, \dots, n$,*

$$|\varepsilon_{j4}| \leq \frac{1}{nv}.$$

Proof. The conclusion of Lemma 3.2 follows immediately from the obvious inequality $|\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)}| \le v^{-1}$ (see Lemma 4.1 in [7]). □

Lemma 3.3. *Assuming the conditions of Theorem* 1.1*, for all $z = u+iv$ with $u \in \mathbb{R}$ and $v > 0$, the following inequality holds*

$$\Pr\Big\{|\varepsilon_{j2}| > 3l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}n^{-\frac{1}{2}}(n^{-1}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2)^{\frac{1}{2}}\Big\} \le C\exp\{-cl_{n,\alpha}\},$$

for some positive constants $c > 0$ and C, depending on $\varkappa$ and α only.

Proof. We use the following well-known inequality for sums of independent random variables. Let $\xi_1,\ldots,\xi_n$ be independent random variables such that $\mathbf{E}\xi_j = 0$ and $|\xi_j| \le \sigma_j$. Then, for some numerical constant $c > 0$,

$$\Pr\Big\{\Big|\sum_{j=1}^n \xi_j\Big| > x\Big\} \le c(1-\Phi(x/\sigma)) \le \frac{c\sigma}{x}\exp\Big\{-\frac{x^2}{2\sigma^2}\Big\}, \tag{3.1}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \exp\{-\frac{y^2}{2}\}dy$ and $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$. The last inequality holds for $x \ge \sigma$. (See, for instance [2], p.1, first inequality.) We put $\eta_l = X_{jl}^2 - 1$, and define,

$$\xi_l = \big(\eta_l\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\} - \mathbf{E}\eta_l\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}\big)R_{ll}^{(j)}.$$

Note that $\mathbf{E}\xi_l = 0$ and $|\xi_l| \le 2l_{n,\alpha}^{\frac{2}{\varkappa}}|R_{ll}^{(j)}|$. Introduce the σ-algebra $\mathfrak{M}^{(j)}$ generated by the random variables X_{kl} with $k,l \in \mathbb{T}_j$. Let $\mathbf{E}_j$ and $\Pr_j$ denote the conditional expectation and the conditional probability with respect to $\mathfrak{M}^{(j)}$. Note that the random variables X_{jl} and the σ-algebra $\mathfrak{M}^{(j)}$ are independent. Applying inequality (3.1) with $x := l_{n,\alpha}^{\frac{1}{2}}\sigma$ and with

$$\sigma^2 = 4nl_{n,\alpha}^{\frac{4}{\varkappa}}\bigg(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\bigg),$$

we get

$$\begin{aligned}\Pr\Big\{\Big|\sum_{l\in\mathbb{T}_j}\xi_j\Big| > x\Big\} &= \mathbf{E}\Pr_j\Big\{\Big|\sum_{l\in\mathbb{T}_j}\xi_j\Big| \ge x\Big\} \\ &\le \mathbf{E}\exp\Big\{-\frac{x^2}{\sigma^2}\Big\} \le C\exp\{-cl_{n,\alpha}\}.\end{aligned} \tag{3.2}$$

Furthermore, note that

$$\mathbf{E}_j\eta_l\mathbb{I}\Big\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\Big\} = -\mathbf{E}_j\eta_l\mathbb{I}\Big\{|X_{jl}| \ge l_{n,\alpha}^{\frac{1}{\varkappa}}\Big\}.$$

This implies

$$\begin{aligned}|\mathbf{E}_j\eta_l\mathbb{I}\Big\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\Big\}| &\le \mathbf{E}_j^{\frac{1}{2}}|\eta_l|^2\Pr_j^{\frac{1}{2}}\Big\{|X_{jl}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\Big\} \\ &\le \mathbf{E}^{\frac{1}{2}}|\eta_l|^2\exp\Big\{-\frac{1}{2}l_{n,\alpha}\Big\} \le C\exp\Big\{-\frac{1}{2}l_{n,\alpha}\Big\}.\end{aligned}$$

The last inequality implies that

$$\Big|\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathbf{E}_j\eta_l\mathbb{I}\left\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}})\right\}R_{ll}^{(j)}\Big|$$
$$\le\left(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|\mathbf{E}_j\eta_l\mathbb{I}\left\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}})\right\}|^2\right)^{\frac12}\left(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\right)^{\frac12}$$
$$\le C\exp\{-cl_{n,\alpha}\}\left(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\right)^{\frac12}. \tag{3.3}$$

Furthermore, we note that if $|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}}$ for all $l\in\mathbb{T}_j$, (which holds with probability at least $1-\varkappa^{-1}\exp\{-cl_{n,\alpha}\}$)

$$|\varepsilon_{j2}|\le\Big|\frac{1}{n}\sum_{l\in\mathbb{T}_j}\xi_l\Big|+\Big|\frac{1}{n}\sum_{l\in\mathbb{T}_j}\mathbf{E}_j\eta_l\mathbb{I}\left\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}}\right\}R_{ll}^{(j)}\Big|. \tag{3.4}$$

The inequalities (3.2), (3.3) and (3.4) together conclude the proof of Lemma 3.3. Thus Lemma 3.3 is proved. □

Corollary 3.4. *Assuming the conditions of Theorem* 1.1 *for any* $\alpha>0$ *there exist positive constants* c *and* C*, depending on* $\varkappa$ *and* α *such that for any* $z=u+iv$ *with* $u\in\mathbb{R}$ *and* $v>0$

$$\Pr\left\{|\varepsilon_{j2}|>3l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12}(nv)^{-\frac12}(\operatorname{Im}m_n^{(j)}(z))^{\frac12}\right\}\le C\exp\{-cl_{n,\alpha}\}.$$

Proof. Note that

$$n^{-1}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2\le n^{-1}\operatorname{Tr}|\mathbf{R}^{(j)}|^2=\frac{1}{v}\operatorname{Im}m_n^{(j)}(z),$$

where $|\mathbf{R}^{(j)}|^2=\mathbf{R}^{(j)}\mathbf{R}^{(j)^*}$. The result follows now from Lemma 3.3. □

Lemma 3.5. *Assuming the conditions of Theorem* 1.1*, for any* $j=1,\dots,n$ *and for any* $z=u+iv$ *with* $u\in\mathbb{R}$ *and* $v>0$*, the following inequality holds,*

$$\Pr\left\{|\varepsilon_{j3}|>\beta_n^2n^{-\frac12}\left(\frac{1}{n}\sum_{k\ne l\in\mathbb{T}_j}|R_{kl}^{(j)}|^2\right)^{\frac12}\right\}\le C\exp\{-cl_{n,\alpha}\}.$$

Proof. We shall use a large deviation bound for quadratic forms which follows from results by Ledoux (see [11]).

Proposition 3.1. *Let* $\xi_1,\dots,\xi_n$ *be independent random variables such that* $|\xi_j|\le1$*. Let* a_{ij} *denote real numbers such that* $a_{ij}=a_{ji}$ *and* $a_{jj}=0$*. Let* $Z=\sum_{l,k=1}^n\xi_l\xi_ka_{lk}$*. Let* $\sigma^2=\sum_{l,k=1}^n|a_{lk}|^2$*. Then for every* $t>0$ *there exists some positive constant* $c>0$ *such that the following inequality holds*

$$\Pr\left\{|Z|\ge\frac32\mathbf{E}^{\frac12}|Z|^2+t\right\}\le\exp\left\{-\frac{ct}{\sigma}\right\}.$$

Proof. Proposition 3.1 follows from Theorem 3.1 in [11]. □

Remark 3.2. Proposition 3.1 holds for complex a_{ij} as well. Here we should consider two quadratic forms with coefficients $\mathrm{Re}\, a_{jk}$ and $\mathrm{Im}\, a_{jk}$.

In order to bound ε_{j3} we use Proposition 3.1 with

$$\xi_l = \Big(X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\} - \mathbf{E}X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}\Big)/(2\,l_{n,\alpha}^{\frac{1}{\varkappa}}).$$

Note that the random variables X_{jl}, $l \in \mathbb{T}_j$ and the matrix $\mathbf{R}^{(j)}$ are mutually independent for any fixed $j = 1, \dots, n$. Moreover, we have $|\xi_l| \le 1$. Put $Z := \sum_{k \ne l \in \mathbb{T}_j} \xi_l \xi_k R_{kl}^{(j)}$. Note that $\mathbf{R}^{(j)} = {\mathbf{R}^{(j)}}^T$. We have $\mathbf{E}_j|Z|^2 = 2\sum_{k,l \in \mathbb{T}_j} |R_{kl}^{(j)}|^2$. Applying Proposition 3.1 with $t = l_{n,\alpha}(\sum_{l \ne k \in \mathbb{T}_j} |R_{lk}^{(j)}|^2)^{\frac{1}{2}}$, we get

$$\mathbf{E}\mathrm{Pr}_j\Big\{|Z| \ge l_{n,\alpha}(\sum_{l \ne k \in \mathbb{T}_j} |R_{lk}^{(j)}|^2)^{\frac{1}{2}}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.5}$$

Furthermore, for some appropriate $c > 0$ and for $n \ge 2$

$$\mathrm{Pr}\{\exists j, l \in [1, \dots, n] : |X_{jl}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\} \le \varkappa^{-1} n^2 \exp\{-l_{n,\alpha}\} \le C\exp\{-cl_{n,\alpha}\}$$

and similarly since $\mathbf{E}X_{jl} = 0$,

$$|\mathbf{E}X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}| \le \mathrm{Pr}^{\frac{1}{2}}\{\exists j, l \in [1, \dots, n] : |X_{jl}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.6}$$

Introduce the random variables

$$\widehat{\xi}_l = X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}/(2l_{n,\alpha}^{\frac{1}{\varkappa}}) \quad \text{and} \quad \widehat{Z} = \sum_{l,k \in \mathbb{T}_j} \widehat{\xi}_l \widehat{\xi}_k R_{lk}^{(j)}.$$

Note that

$$\mathrm{Pr}\Big\{\sum_{l,k \in \mathbb{T}_j} X_{jk}X_{jl}R_{kl}^{(j)} \ne 4l_{n,\alpha}^{\frac{2}{\varkappa}}\widehat{Z}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.7}$$

Furthermore, by (3.6) we have

$$\Big|\frac{1}{n}\sum_{l,k \in \mathbb{T}_j} R_{kl}^{(j)}\mathbf{E}\widehat{\xi}_l\mathbf{E}\widehat{\xi}_k\Big| \le C\exp\{-cl_{n,\alpha}\}\Big(\frac{1}{n}\sum_{k \ne l \in \mathbb{T}_j} |R_{kl}^{(j)}|^2\Big)^{\frac{1}{2}}. \tag{3.8}$$

Finally, inequalities (3.5)–(3.8) together imply

$$\mathrm{Pr}\Big\{|\varepsilon_{j3}| > 4\beta_n^2 n^{-\frac{1}{2}} bigg(\frac{1}{n}\sum_{k \ne l \in \mathbb{T}_j} |R_{kl}^{(j)}|^2\Big)^{\frac{1}{2}}\Big\} \le C\exp\{-cl_{n,\alpha}\}.$$

Thus Lemma 3.5 is proved. □

Corollary 3.6. *Under the conditions of Theorem* 1.1 *there exist positive constants* c *and* C *depending on* $\varkappa$ *and* α *such that for any* $z = u + iv$ *with* $u \in \mathbb{R}$ *and with* $v > 0$

$$\Pr\{|\varepsilon_{j3}| > 4\beta_n^2 (nv)^{-\frac{1}{2}} (\operatorname{Im} m_n^{(j)}(z))^{\frac{1}{2}}\} \le C \exp\{-c l_{n,\alpha}\}.$$

Proof. Note that as above

$$n^{-1} \sum_{k \ne l \in \mathbb{T}_j} |R_{kl}^{(j)}|^2 \le n^{-1} \operatorname{Tr} |\mathbf{R}^{(j)}|^2 = \frac{1}{v} \operatorname{Im} m_n^{(j)}(z). \tag{3.9}$$

The result now follows from Lemma 3.5. □

To summarize these results we recall $\beta_n = (l_{n,\alpha})^{\frac{1}{\varkappa}+\frac{1}{2}}$, defined previously in (1.3). Without loss of generality we may assume that $\beta_n \ge 1$ and $l_{n,\alpha} \ge 1$. Then Lemmas 3.1, 3.2, Lemma 3.3 (with $l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}$ replaced by β_n^2), and Lemma 3.5 together imply

$$\Pr\Big\{|\varepsilon_j| > \frac{\beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n^{(j)}(z)}{\sqrt{v}} + \frac{1}{\sqrt{v}\sqrt{nv}}\Big)\Big\} \le C \exp\{-c l_{n,\alpha}\}.$$

Using that

$$0 < \operatorname{Im} m_n^{(j)}(z) \le \operatorname{Im} m_n(z) + \frac{1}{nv}, \tag{3.10}$$

we may rewrite the last inequality

$$\Pr\Big\{|\varepsilon_j| > \frac{\beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{v}\sqrt{nv}}\Big)\Big\} \le C \exp\{-c l_{n,\alpha}\}. \tag{3.11}$$

Denote by

$$\Omega_n(z,\theta) = \Big\{\omega \in \Omega : \, |\varepsilon_j| \le \frac{\theta \beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{nv}}\Big)\Big\}, \tag{3.12}$$

for any $\theta \ge 1$. Let

$$v_0 := \frac{d\beta_n^4}{n} \tag{3.13}$$

with a sufficiently large positive constant $d > 0$. We introduce the region $\mathcal{D} = \{z = u + iv \in \mathbb{C} : |u| \le 2, \, v_0 < v \le 2\}$. Furthermore, we introduce the sequence $z_l = u_l + v_l$ in $\mathcal{D}$, recursively defined via $u_{l+1} - u_l = \frac{4}{n^8}$ and $v_{l+1} - v_l = \frac{2}{n^8}$. Using a union bound, we have

$$\Pr\{\cap_{z_l \in \mathcal{D}} \Omega_n(z_l, \theta)\} \ge 1 - C(\theta) \exp\{-c(\theta) \, l_{n,\alpha}\} \tag{3.14}$$

with some constant $C(\theta)$ and $c(\theta)$ depending on $\alpha, \varkappa$ and θ. Using the resolvent equality $\mathbf{R}(z) - \mathbf{R}(z') = -(z - z')\mathbf{R}(z)\mathbf{R}'(z)$, we get

$$|R_{k+n,l+n}^{(j)}(z) - R_{k+n,l+n}^{(j)}(z')| \le \frac{|z - z'|}{vv'}.$$

This inequality and the definition of ε_j together imply

$$\Pr\left\{|\varepsilon_j(z)-\varepsilon_j(z')| \le \frac{n l_{n,\alpha}^{\frac{2}{\varkappa}}|z-z'|}{v_0^2} \quad \text{for all } z,z'\in\mathcal{D}\right\} \ge 1 - C\exp\{-c l_{n,\alpha}\}. \tag{3.15}$$

For any $z\in\mathcal{D}$ there exists a point z_l such that $|z-z_l|\le Cn^{-8}$. This together with inequalities (3.14) and (3.15) immediately implies that

$$\begin{aligned}\Pr\{\cap_{z\in\mathcal{D}}\Omega_n(z,2)\} &\ge \Pr\{\cap_{z_l\in\mathcal{D}}\Omega_n(z_l,1)\} - C\exp\{-c l_{n,\alpha}\}\\ &\ge 1 - C\exp\{-c l_{n,\alpha}\},\end{aligned} \tag{3.16}$$

with some constants C and c depending on α and $\varkappa$ only. Let

$$\Omega_n := \cap_{z\in\mathcal{D}}\Omega_n(z,2). \tag{3.17}$$

Put now

$$v_0' := v_0'(z) = \frac{\sqrt{2}v_0}{\sqrt{\gamma}}, \tag{3.18}$$

where $\gamma := 2-|u|$, $z=u+iv$ and v_0 is given by (3.13). Note that $0\le\gamma\le 2$, for $u\in[-2,2]$ and $v_0'\ge v_0$. Denote $\mathcal{D}' := \{z\in\mathcal{D} :\ v\ge v_0'\}$.

4. Bounds for $|m_n(z)|$

In this section we bound the probability that $\operatorname{Im} m_n(z)\le C$ for some numerical constant C and for any $z\in\mathcal{D}$. We shall derive auxiliary bounds for the difference between the Stieltjes transforms $m_n(z)$ of the empirical spectral measure of the matrix $\mathbf{X}$ and the Stieltjes transform $s(z)$ of the semi-circular law. Introduce the additional notations

$$\delta_n := \frac{1}{n}\sum_{j=1}^n \varepsilon_j R_{jj}.$$

Recall that $s(z)$ satisfies the equation

$$s(z) = -\frac{1}{z+s(z)}. \tag{4.1}$$

For the semi-circular law the following inequalities hold

$$|s(z)|\le 1 \text{ and } |z+s(z)|\ge 1. \tag{4.2}$$

Introduce $g_n(z) := m_n(z)-s(z)$. Equality (4.1) implies that

$$1-\frac{1}{(z+s(z))(z+m_n(z))} = \frac{z+m_n(z)+s(z)}{z+m_n(z)}. \tag{4.3}$$

The representation (2.21) implies

$$g_n(z) = \frac{g_n(z)}{(z+s(z))(z+m_n(z))} + \frac{\delta_n}{z+m_n(z)}. \tag{4.4}$$

From here it follows by solving for $g_n(z)$ that

$$g_n(z) = \frac{\delta_n(z)}{z + m_n(z) + s(z)}. \tag{4.5}$$

Lemma 4.1. *Let*

$$|g_n(z)| \le \frac{1}{2}. \tag{4.6}$$

Then $|z + m_n(z)| \ge \frac{1}{2}$ *and* $\operatorname{Im} m_n(z) \le |m_n(z)| \le \frac{3}{2}$.

Proof. This is an immediate consequence of inequalities (4.2) and of

$$|z+m_n(z)| \ge |z+s(z)| - |g_n(z)| \ge \frac{1}{2}, \quad \text{and} \quad |m_n(z)| \le |s(z)| + |g_n(z)| \le \frac{3}{2}. \ \square$$

Lemma 4.2. *Assume condition* (4.6) *for* $z = u + iv$ *with* $v \ge v_0$. *Then for any* $\omega \in \Omega_n$, *defined in* (3.17), *we obtain* $|R_{jj}| \le 4$.

Proof. By definition of Ω_n in (3.17), we have

$$|\varepsilon_j| \le \frac{\beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{v}\sqrt{nv}}\Big). \tag{4.7}$$

Applying Lemmas 4.1 and (3.13), we get $|\varepsilon_j| \le \frac{A\beta_n^2}{\sqrt{nv}}$ with some $A > 0$ depending on the parameter $d \ge 1$ in (3.13) which we may choose such that

$$|\varepsilon_j| \le \frac{1}{200}, \tag{4.8}$$

for any $\omega \in \Omega_n$, $n \ge 2$, and $v \ge v_0$. Using representation (2.19) and applying Lemma 4.1, we get $|R_{jj}| \le 4$. $\square$

Lemma 4.3. *Assume condition* (4.6). *Then, for any* $\omega \in \Omega_n$ *and* $v \ge v_0$,

$$|g_n(z)| \le \frac{1}{100}. \tag{4.9}$$

Proof. Lemma 4.2, inequality (4.8), and representation (4.5) together imply

$$|\delta_n| \le \frac{4}{n}\sum_{j=1}^{n} |\varepsilon_j| \le \frac{4\beta_n^2}{\sqrt{n}}\Big(1 + \frac{\operatorname{Im}^{\frac{1}{2}} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{v}\sqrt{nv}}\Big) \tag{4.10}$$

Note that

$$|z + m_n(z) + s(z)| \ge \operatorname{Im} z + \operatorname{Im} m_n(z) + \operatorname{Im} s(z) \ge \operatorname{Im}(z + s(z)) \ge \frac{1}{2}\operatorname{Im}\{\sqrt{z^2-4}\}. \tag{4.11}$$

For $z \in \mathcal{D}$ we get $\operatorname{Re}(z^2-4) \le 0$ and $\frac{\pi}{2} \le \arg(z^2-4) \le \frac{3\pi}{2}$. Therefore,

$$\operatorname{Im}\{\sqrt{z^2-4}\} \ge \frac{1}{\sqrt{2}}|z^2-4|^{\frac{1}{2}} \ge \frac{1}{4}\sqrt{\gamma + v}, \tag{4.12}$$

where $\gamma = 2 - |u|$. These relations imply that

$$\frac{|\delta_n|}{|z + m_n(z) + s(z)|} \le \frac{\beta_n^2}{\sqrt{nv}} + \frac{\beta_n^2}{\sqrt{n}\sqrt{v\sqrt{\gamma}}} + \frac{\beta_n^2}{(nv)^{\frac{3}{2}}\sqrt{\gamma}}. \tag{4.13}$$

For $v\sqrt{\gamma} \ge v_0$, we get

$$|g_n(z)| \le \frac{8\beta_n^2}{\sqrt{nv_0}} \le \frac{1}{100} \tag{4.14}$$

by choosing the constant $d \ge 1$ in v_0 appropriately large. Thus the lemma is proved. □

Lemma 4.4. *Assume that condition* (4.6) *holds, for some* $z = u + iv \in \mathcal{D}'$ *and for any* $\omega \in \Omega_n$, *(see* (3.17) *and the subsequent notions). Then* (4.6) *holds as well for* $z' = u + i\widehat{v} \in \mathcal{D}'$ *with* $v \ge \widehat{v} \ge v - n^{-8}$, *for any* $\omega \in \Omega_n$.

Proof. First of all note that

$$|m_n(z) - m_n(z')| = \frac{1}{n}(v - \widehat{v})|\mathrm{Tr}\,\mathbf{R}(z)\mathbf{R}(z')| \le \frac{v-\widehat{v}}{v\widehat{v}} \le \frac{C}{n^4} \le \frac{1}{100}$$

and $|s(z) - s(z')| \le \frac{|z-z'|}{v\widehat{v}} \le \frac{1}{100}$. By Lemma 4.3, we have $|g_n(z)| \le \frac{1}{100}$. All these inequalities together imply $|g_n(z')| \le \frac{3}{100} < \frac{1}{2}$. Thus, Lemma 4.4 is proved. □

Proposition 4.1. *Assuming the conditions of Theorem* 1.1 *there exist constants* $C > 0$ *and* $c > 0$ *depending on* $\varkappa$ *and* α *only such that*

$$\Pr\left\{|m_n(z)| \le \frac{3}{2} \textit{ for any } z \in \mathcal{D}'\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{4.15}$$

Proof. First we note that $|g_n(z)| \le \frac{1}{2}$ a.s., for $z = u+4i$. By Lemma 4.4, $|g_n(z')| \le \frac{1}{2}$ for any $\omega \in \Omega_n$. Applying Lemma 4.1 and a union bound, we get

$$\Pr\left\{|m_n(z)| \le \frac{3}{2} \text{ for any } z \in \mathcal{D}'\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{4.16}$$

Thus the proposition is proved. □

5. Large deviations II

In this section we improve the bounds for δ_n. We shall use bounds for large deviation probabilities of the sum of ε_j. We start with

$$\delta_{n1} = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_{j1}. \tag{5.1}$$

Lemma 5.1. *There exist constants* c *and* C *depending on* $\varkappa$ *and* α *and such that*

$$\Pr\left\{|\delta_{n1}| > n^{-1}\beta_n\right\} \le C\exp\{-cl_{n,\alpha}\}.$$

Proof. We repeat the proof of Lemma 3.1. Consider the truncated random variables $\widehat{X}_{jj} = X_{jj}\mathbb{I}\{|X_{jj}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}$. By assumption (1.1),

$$\Pr\left\{|X_{jj}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\right\} \le \varkappa^{-1}\exp\{-l_{n,\alpha}\}.$$

Moreover,

$$|\mathbf{E}\widehat{X}_{jj}| \le C\exp\{-cl_{n,\alpha}\}.$$

We define $\widetilde{X}_{jj} = \widehat{X}_{jj} - \mathbf{E}\widehat{X}_{jj}$ and consider the sum

$$\widetilde{\delta}_{n1} := \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \widetilde{X}_{jj}.$$

Since $|\widetilde{X}_{jj}| \le 2l_{n,\alpha}^{\frac{1}{\varkappa}}$, by inequality (3.1), we have

$$\Pr\left\{|\widetilde{\delta}_{n1}| > n^{-1} l_{n,\alpha}^{\frac{1}{\varkappa}+\frac{1}{2}}\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.2}$$

Note that

$$|\widetilde{\delta}_{n1} - \delta_{n1}| \le \frac{1}{n} \sum_{j=1}^{n} |\mathbf{E}\widehat{X}_{jj}| \le C\exp\{-cl_{n,\alpha}\}.$$

This inequality and inequality (5.2) together imply

$$\Pr\left\{|\delta_{n1}| > n^{-1} l_{n,\alpha}^{\frac{1}{\varkappa}+\frac{1}{2}}\right\} \le C\exp\{-cl_{n,\alpha}\}.$$

Thus, Lemma 5.1 is proved. □

Consider now the quantity

$$\delta_{n2} := \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l \in \mathbb{T}_j} (X_{jl}^2 - 1) R_{ll}^{(j)}. \tag{5.3}$$

We prove the following lemma

Lemma 5.2. *Let* $v_0 = \frac{d\beta_n^4}{n}$ *with some numerical constant* $d \ge 1$. *Under the conditions of Theorem* 1.1 *there exist constants* c *and* C, *depending on* $\varkappa$ *and* α *only, such that*

$$\Pr\left\{|\delta_{n2}| > 2n^{-1}\beta_n^2 \frac{1}{\sqrt{v}}\left(\frac{3}{2} + \frac{1}{nv}\right)^{\frac{1}{2}}\right\} \le C\exp\{-cl_{n,\alpha}\},$$

for any $z \in \mathcal{D}'$.

Proof. Introduce the truncated random variables $\xi_{jl} = \widehat{X}_{jl}^2 - \mathbf{E}\widehat{X}_{jl}^2$, where $\widehat{X}_{jl} = X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}$. It is straightforward to check that

$$0 \le 1 - E\widehat{X}_{jl}^2 \le C\exp\{-cl_{n,\alpha}\}. \tag{5.4}$$

We shall need the following quantities as well

$$\widehat{\delta}_{n2} = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l \in \mathbb{T}_j} (\widehat{X}_{jl}^2 - 1) R_{ll}^{(j)} \quad \text{and} \quad \widetilde{\delta}_{n2} = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l \in \mathbb{T}_j} \xi_{jl} R_{ll}^{(j)}.$$

By assumption (1.1),

$$\Pr\{\delta_{n2} \ne \widehat{\delta}_{n2}\} \le \sum_{j=1}^{n} \sum_{l \in \mathbb{T}_j} \Pr\left\{|X_{jl}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\right\} \le C\exp\{-cl_{n,\alpha}\}.$$

By inequality (5.4),

$$|\widehat{\delta}_{n2} - \widetilde{\delta}_{n2}| \le \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l \in \mathbb{T}_j} |\mathbf{E}\widehat{X}_{jl}^2 - 1||R_{ll}^{(j)}| \le C v_0^{-1} \exp\{-c l_{n,\alpha}\}$$
$$\le C \exp\{-c l_{n,\alpha}\},$$

for $v \ge v_0$ and $C, c > 0$ which are independent of $d \ge 1$.

Let $\zeta_j := \frac{1}{\sqrt{n}} \sum_{l \in \mathbb{T}_j} \xi_{jl} R_{ll}^{(j)}$. Then $\widetilde{\delta}_{n2} = \frac{1}{n^{\frac{3}{2}}} \sum_{j=1}^{n} \zeta_j$. Let $\mathfrak{R}_j$, for $j = 1, \dots n$, denote the σ-algebras generated by the random variables X_{lk} with $1 \le l \le j$ and $1 \le k \le j$. Let $\mathfrak{R}_0$ denote the trivial σ-algebra. Note that the sequence $\widetilde{\delta}_{n2}$ is a martingale with respect to the σ-algebras $\mathfrak{R}_j$. In fact,

$$\mathbf{E}\{\zeta_j | \mathfrak{R}_{j-1}\} = \mathbf{E}\{\mathbf{E}\{\zeta_j | \mathfrak{M}^{(j)}\} | \mathfrak{R}_{j-1}\} = 0.$$

In order to use large deviation bounds for $\widetilde{\delta}_{n2}$ we replace the differences ζ_j by truncated random variables. We put $\widehat{\zeta}_j = \zeta_j \mathbb{I}\{|\zeta_j| \le l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}} (\frac{3}{2} + \frac{1}{nv})^{\frac{1}{2}}\}$. Denote by $t_{nv}^2 = \frac{3}{2} + \frac{1}{nv}$. Since ζ_j is a sum of independent bounded random variables with mean zero (conditioned on $\mathfrak{M}^{(j)}$), similar as in Lemma (3.3) we get

$$\Pr_j \left\{ |\zeta_j| > l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}} \left(\frac{1}{n} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^2 \right)^{\frac{1}{2}} \right\} \le C \exp\{-c l_{n,\alpha}\}.$$

Using (3.9) and (3.10), we have

$$\frac{1}{n} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^2 \le \frac{1}{v} t_{nv}^2. \tag{5.5}$$

By Proposition 4.1, we have

$$\Pr_j \left\{ |\zeta_j| > l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}} v^{-\frac{1}{2}} t_{nv} \right\} \le C \exp\{-c l_{n,\alpha}\}. \tag{5.6}$$

This implies that

$$\Pr \left\{ \sum_{j=1}^{n} \zeta_j \ne \sum_{j=1}^{n} \widehat{\zeta}_j \right\} \le C \exp\{-c l_{n,\alpha}\}. \tag{5.7}$$

Furthermore, introduce the random variables $\widetilde{\zeta}_j = \widehat{\zeta}_j - \mathbf{E}\{\widehat{\zeta}_j | \mathfrak{R}_{j-1}\}$. First we note that

$$\mathbf{E}\{\widehat{\zeta}_j | \mathfrak{R}_{j-1}\} = -\mathbf{E}\left\{ \zeta_j \mathbb{I}\{|\zeta_j| > l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}} v^{-\frac{1}{2}} t_{nv}\} \Big| \mathfrak{R}_{j-1} \right\}.$$

Applying Cauchy-Schwartz, $E_j\xi_{jl}\xi_{jl'}R^{(j)}_{ll}R^{(j)}_{l'l'}=0$ for $l\neq l'$, $l,l'\in\mathbb{T}_j$ and $|R^{(j)}_{ll}|\leq v^{-1}$ as well as $\mathbf{E}\{\mathbf{E}_j\{|\zeta_j|^2\}|\mathfrak{R}_{j-1}\}\leq\frac{1}{nv}\sum_{l\in\mathbb{T}_j}\mathbf{E}|\xi_{jl}|^2$ we get

$$\begin{aligned}|\mathbf{E}\{\widehat{\zeta_j}|\mathfrak{R}_{j-1}\}|&\leq C\mathbf{E}^{\frac12}\{|\zeta_j|^2|\mathfrak{R}_{j-1}\}\mathrm{Pr}^{\frac12}\Big\{|\zeta_j|>l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12}v^{-\frac12}t_{nv}\}\Big|\mathfrak{R}_{j-1}\Big\}\\&=C\mathbf{E}^{\frac12}\{\mathbf{E}_j\{|\zeta_j|^2\}|\mathfrak{R}_{j-1}\}\mathbf{E}^{\frac12}\Big\{\mathrm{Pr}_j\Big\{|\zeta_j|>l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12}v^{-\frac12}t_{nv}\}\Big\}\Big|\mathfrak{R}_{j-1}\Big\}\\&\leq Cv^{-1}\Big(\frac1n\sum_{l\in\mathbb{T}_j}\mathbf{E}|\xi_{jl}|^2\Big)^{\frac12}\exp\{-cl_{n,\alpha}\}\leq C\exp\{-cl_{n,\alpha}\},\end{aligned}\tag{5.8}$$

for $v\sqrt{\gamma}\geq v_0$ with constants C and c depending on α and $\varkappa$.

Furthermore, we may use a martingale bound due to Bentkus, [2], Theorem 1.1. It provides the following result. Let $\mathfrak{R}_0=\{\emptyset,\Omega\}\subset\mathfrak{R}_1\subset\cdots\subset\mathfrak{R}_n\subset\mathfrak{M}$ be a family of σ-algebras of a measurable space $\{\Omega,\mathfrak{M}\}$. Let $M_n=\xi_1+\cdots+\xi_n$ be a martingale with bounded differences $\xi_j=M_j-M_{j-1}$ such that $\Pr\{|\xi_j|\leq b_j\}=1$, for $j=1,\ldots,n$. Then, for $x>\sqrt{8}$

$$\Pr\{|M_n|\geq x\}\leq c(1-\Phi(\frac{x}{\sigma}))=\int_{\frac{x}{\sigma}}^{\infty}\varphi(t)dt,\quad\varphi(t)=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{t^2}{2}\right\},$$

with some numerical constant $c>0$ and $\sigma^2=b_1^2+\cdots+b_n^2$. Note that for $t>C$

$$1-\Phi(t)\leq\frac{1}{C}\varphi(t).$$

Thus, this leads to the inequality

$$\Pr\{|M_n|\geq x\}\leq\exp\left\{-\frac{x^2}{2\sigma^2}\right\},\tag{5.9}$$

which we shall use to bound $\widetilde{\delta}_{n2}$. Take $M_n=\sum_{j=1}^n\widetilde{\delta}_j$ with $|\widetilde{\delta}_j|$ bounded by $b_j=2l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12}v^{-\frac12}t_{nv}$. By Proposition 4.1 obtain

$$\sigma^2=4nv^{-1}l_{n,\alpha}^{\frac{4}{\varkappa}+1}t_{nv}^2.\tag{5.10}$$

Inequalities (5.9) with $x=l_{n,\alpha}^{\frac12}\sigma$ and (5.10) together imply

$$\Pr\left\{|\widetilde{\delta}_{n2}|>2n^{-1}\beta_n^2\frac{1}{\sqrt{v}}t_{nv}\right\}\leq C\exp\{-cl_{n,\alpha}\}.\tag{5.11}$$

Inequalities (5.7)–(5.11) together conclude the proof of Lemma 5.2. □

Let

$$\delta_{n3}:=\frac{1}{n^2}\sum_{j=1}^n\sum_{l\neq k\in\mathbb{T}_j}X_{jl}X_{jk}R^{(j)}_{lk}.\tag{5.12}$$

Lemma 5.3. *Let* $v_0 = \frac{d\beta_n^4}{n}$ *with some numerical constant* $d > 1$. *Under condition of Theorem* 1.1 *there exist constants* c *and* C, *depending on* $\varkappa$, α *only such that*

$$\Pr\left\{|\delta_{n3}| > \frac{4\beta_n^2 l_{n,\alpha}^{\frac{1}{2}}}{n\sqrt{v}}\left(\frac{3}{2}+\frac{1}{nv}\right)^{\frac{1}{2}}\right\} \le C\exp\{-cl_{n,\alpha}\},$$

for any $z \in \mathcal{D}'$.

Proof. The proof of this lemma is similar to the proof of Lemma 5.2. We introduce the random variables $\eta_j = \frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j} X_{jk}X_{jl}R_{lk}^{(j)}$ and note that the sequence $M_j = \frac{1}{n}\sum_{m=1}^{j}\eta_m$ is martingale with respect to the σ-algebras $\mathfrak{R}_j$, for $j = 1,\dots,n$. By Proposition 4.1, using inequality (5.5), we get

$$\Pr\left\{\frac{1}{n}\sum_{l,k\in\mathbb{T}_j}|R_{lk}^{(j)}|^2 \le \frac{1}{v}t_{nv}^2 \text{ for any } z\in\mathcal{D}'\right\} \ge 1 - C\exp\{-cl_{n,\alpha}\}. \tag{5.13}$$

At first we apply Proposition 3.1 replacing η_j by truncated random variables and then apply the martingale bound of Bentkus (5.9). Introduce the random variables $\widehat{X}_{jk} = X_{jk}\mathbb{I}\{|X_{jk}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}$ and $\widetilde{X}_{jk} = \widehat{X}_{jk} - \mathbf{E}\widehat{X}_{jk}$. By condition (1.1), we have

$$\Pr\{X_{jk} \ne \widehat{X}_{jk}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.14}$$

The same condition yields

$$|\mathbf{E}\widehat{X}_{jk}| = |\mathbf{E}X_{jk}\mathbb{I}\{|X_{jk}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\}| \le C\exp\{-cl_{n,\alpha}\} \tag{5.15}$$

Let

$$\widehat{\eta}_j = \frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}\widehat{X}_{jk}\widehat{X}_{jl}R_{lk}^{(j)}, \quad \text{and} \quad \widetilde{\eta}_j = \frac{1}{n}\sum_{l\ne k\in\mathbb{T}_j}\widetilde{X}_{jk}\widetilde{X}_{jl}R_{lk}^{(j)}. \tag{5.16}$$

Inequality (5.14) implies that

$$\Pr\{\eta_j \ne \widehat{\eta}_j\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.17}$$

Inequality (5.15) and $|\widetilde{X}_{jk}| \le 2l_{n,\alpha}$ together imply

$$\Pr\{|\widehat{\eta}_j - \widetilde{\eta}_j| \le Cl_{n,\alpha}^{\frac{1}{\varkappa}}\exp\{-cl_{n,\alpha}\}v^{-\frac{1}{2}}t_{nv}\} = 1. \tag{5.18}$$

Applying now Propositions 4.1 and 3.1, and inequality (5.5), similar to Lemma 3.5 we get, introducing $r_{v,n} := v^{-\frac{1}{2}}\beta_n^2 t_{nv}$,

$$\Pr\{|\widetilde{\eta}_j| > n^{-\frac{1}{2}}r_{v,n}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.19}$$

Now we introduce

$$\theta_j = \eta_j\mathbb{I}\{|\eta_j| \le n^{-\frac{1}{2}}r_{v,n}\} - \mathbf{E}\eta_j\mathbb{I}\{|\eta_j| \le n^{-\frac{1}{2}}r_{v,n}\}. \tag{5.20}$$

Furthermore, we consider the random variables $\widetilde{\theta}_j = \theta_j - \mathbf{E}\{\theta_j|\mathfrak{R}_{j-1}\}$. The sequence $\widehat{M}_s$, defined by $\widehat{M}_s = \sum_{m=1}^{s}\widetilde{\theta}_m$, is a martingale with respect to the σ-algebras $\mathfrak{R}_s$, for $s = 1,\dots,n$. Similar to the proof of Lemma 5.1 we get

$$\Pr\{|\widehat{M}_n - M_n| > 4l_{n,\alpha}^{\frac{1}{2}}r_{v,n}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.21}$$

Applying inequality (5.9) to $\widehat{M}_n$ with $\sigma^2 = 16r_{v,n}^2$ and $x = l_{n,\alpha}^{\frac{1}{2}}\sigma$, we get

$$\Pr\left\{|\widehat{M}_n| > 4l_{n,\alpha}^{\frac{1}{2}}r_{v,n}\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.22}$$

Thus the lemma is proved. □

Finally, we shall bound

$$\delta_{n4} := \frac{1}{n^2}\sum_{j=1}^{n}(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)})R_{jj}. \tag{5.23}$$

Lemma 5.4. *For any $z = u + iv$ with $v > 0$ the following inequality*

$$|\delta_{n4}| \le \frac{1}{nv}\operatorname{Im} m_n(z) \ a.\ s. \tag{5.24}$$

holds.

Proof. By formula (5.4) in [7], we have

$$(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)})R_{jj} = \left(1 + \frac{1}{n}\sum_{l,k\in T_j} X_{jl}X_{jk}(R^{(j)})_{lk}^2\right)R_{jj}^2 = \frac{d}{dz}R_{jj}. \tag{5.25}$$

From here it follows that

$$\frac{1}{n^2}\sum_{j=1}^{n}(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)})R_{jj} = \frac{1}{n^2}\frac{d}{dz}\operatorname{Tr}\mathbf{R} = \frac{1}{n^2}\operatorname{Tr}\mathbf{R}^2. \tag{5.26}$$

Finally, we note that

$$|\frac{1}{n^2}\operatorname{Tr}\mathbf{R}^2| \le \frac{1}{nv}\operatorname{Im} m_n(z).$$

The last inequality concludes the proof. Thus, Lemma 5.4 is proved. □

6. Stieltjes transforms

We shall derive auxiliary bounds for the difference between the Stieltjes transforms $m_n(z)$ of the empirical spectral measure of the matrix $\mathbf{X}$ and the Stieltjes transform $s(z)$ of the semi-circular law. Recalling the definitions of $\varepsilon_j, \varepsilon_{j\mu}$ in (2.20) and of $\delta_{n\nu}$ in (5.1), (5.3), (5.12) as well as (5.23), we introduce the additional notations

$$\delta_n' := \delta_{n1} + \delta_{n2} + \delta_{n3}, \quad \widehat{\delta}_n := \delta_{n4}, \quad \overline{\delta}_n := \frac{1}{n}\sum_{\nu=1}^{3}\sum_{j=1}^{n}\varepsilon_{j\nu}\varepsilon_j R_{jj}. \tag{6.1}$$

Recall that $g_n(z) := m_n(z) - s(z)$. The representation (2.22) implies

$$g_n(z) = \frac{g_n(z)}{(z+s(z))(z+m_n(z))} - \frac{\delta_n'}{(z+m_n(z))^2} + \frac{\widehat{\delta}_n}{z+m_n(z)} + \frac{\overline{\delta}_n}{(z+m_n(z))^2}. \tag{6.2}$$

The equalities (6.2) and (4.3) together yield

$$|g_n(z)| \le \frac{|\delta_n'| + |\overline{\delta}_n|}{|z+m_n(z)||z+s(z)+m_n(z)|} + \frac{|\widehat{\delta}_n|}{|z+s(z)+m_n(z)|}. \tag{6.3}$$

For any $z \in \mathcal{D}'$ introduce the events

$$\widehat{\Omega}_n(z) := \Big\{\omega \in \Omega : |\delta_n'| \le \Big(\frac{\beta_n}{n} + \frac{\beta_n^2 l_{n,\alpha}^{\frac{1}{2}}\sqrt{\frac{3}{2}}}{n\sqrt{v}} + \frac{\beta_n^2 l_{n,\alpha}^{\frac{1}{2}}}{n^{\frac{3}{2}}v}\Big)\Big\}, \tag{6.4}$$

$$\widetilde{\Omega}_n(z) := \Big\{\omega \in \Omega : |\widehat{\delta}_n| \le \frac{C\,\mathrm{Im}\, m_n(z)}{nv}\Big\}, \tag{6.5}$$

$$\overline{\Omega}_n(z) := \Big\{\omega \in \Omega : |\overline{\delta}_n| \le \Big(\frac{\beta_n^2}{n} + \frac{\beta_n^4(\mathrm{Im}\, m_n(z) + \frac{1}{nv})}{nv} + \frac{1}{n^2v^2}\Big)\frac{1}{n}\sum_{j=1}^n |R_{jj}|\Big\}.$$

Put $\Omega_n^*(z) := \widehat{\Omega}_n(z) \cap \widetilde{\Omega}_n(z) \cap \overline{\Omega}_n(z)$. By Lemmas 5.1–5.3, we have

$$\Pr\{\widehat{\Omega}_n(z)\} \ge 1 - C\exp\{-cl_{n,\alpha}\}.$$

The proof of the last relation is similar to the proof of inequality (3.16). By Lemma 5.4,

$$\Pr\{\widetilde{\Omega}_n(z)\} = 1.$$

Note that

$$|\varepsilon_{j\nu}\varepsilon_{j4}| \le \frac{1}{2}(|\varepsilon_{j\nu}|^2 + |\varepsilon_{j4}|^2).$$

By Lemmas 3.3 and 3.5, we have, for $\nu = 2, 3$,

$$\Pr\Big\{|\varepsilon_{j\nu}|^2 > \frac{\beta_n^4}{nv}\Big(\mathrm{Im}\, m_n(z) + \frac{1}{nv}\Big)\Big\} \le C\exp\{-cl_{n,\alpha}\}.$$

According to Lemma 3.1,

$$\Pr\Big\{|\varepsilon_{j1}|^2 > \frac{\beta_n^2}{n}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{6.6}$$

and, by Lemma 3.2

$$\Pr\Big\{|\varepsilon_{j4}|^2 \le \frac{1}{n^2v^2}\Big\} = 1.$$

Similarly as in (3.16) we may show that

$$\Pr\{\cap_{z\in\mathcal{D}}\Omega_n^*(z) \cap \Omega_n\} \ge 1 - C\exp\{-cl_{n,\alpha}\}.$$

Let

$$\Omega_n^* := \cap_{z\in\mathcal{D}}\Omega_n^*(z) \cap \Omega_n,$$

where Ω_n was defined in (3.17). We prove now some auxiliary lemmas.

Lemma 6.1. *Let $z = u + iv \in \mathcal{D}$ and $\omega \in \Omega_n^*$. Assume that*

$$|g_n(z)| \le \frac{1}{2}. \tag{6.7}$$

Then the following bound holds

$$|g_n(z)| \le \frac{C\beta_n^4}{nv} + \frac{C\beta_n^4}{n^2v^2\sqrt{\gamma + v}}.$$

Proof. Inequality (6.3) implies that for $\omega \in \Omega_n^*$

$$|g_n(z)| \le \frac{\beta_n^2 l_{n,\alpha}^{\frac{1}{2}}\Big(1+\sqrt{\frac{3}{2}}\Big)}{n\sqrt{v}|z+m_n(z)||z+s(z)+m_n(z)|} + \frac{C\,\mathrm{Im}\, m_n(z)}{nv|z+s(z)+m_n(z)|}$$
$$+ \frac{\beta_n^2 l_{n,\alpha}^{\frac{1}{2}}}{n^{\frac{3}{2}}v|z+m_n(z)||z+s(z)+m_n(z)|}$$
$$+ \frac{\beta_n^4}{nv|z+m_n(z)||z+s(z)+m_n(z)|}\Big(\mathrm{Im}\, m_n(z)+\frac{1}{nv}\Big)\frac{1}{n}\sum_{j=1}^n |R_{jj}|. \tag{6.8}$$

Inequality (6.8) and Lemmas 4.1, inequalities (4.11), (4.12) together imply

$$|g_n(z)| \le \frac{C\beta_n^4}{nv}\Big(1+\frac{1}{nv\sqrt{\gamma+v}}\Big). \tag{6.9}$$

This inequality completes the proof of lemma. □

Put now

$$v_0' := v_0'(z) = \frac{\sqrt{2}v_0}{\sqrt{\gamma}}, \tag{6.10}$$

where $\gamma := 2-|u|$, $z = u+iv$ and v_0 given by (3.13). Note that $0 \le \gamma \le 2$, for $u \in [-2,2]$ and $v_0' \ge v_0$. Denote $\widehat{\mathcal{D}} := \{z \in \mathcal{D} :\ v \ge v_0'\}$.

Corollary 6.2. *Assume that $|g_n(z)| \le \frac{1}{2}$, for $\omega \in \Omega_n^*$ and $z = u+iv \in \widehat{\mathcal{D}}$. Then $|g_n(z)| \le \frac{1}{100}$, for sufficiently large d in the definition of v_0.*

Proof. Note that for $v \ge v_0'$

$$\frac{C\beta_n^4}{nv} + \frac{C\beta_n^4}{n^2v^2\sqrt{\gamma+v}} \le \frac{C\sqrt{\gamma}}{d} + \frac{C\sqrt{\gamma}}{d^2\beta_n^4} \le \frac{1}{100}, \tag{6.11}$$

for an appropriately large constant d in the definition of v_0. Thus, the corollary is proved. □

Remark 6.1. In what follows we shall assume that $d \ge 1$ is chosen and fixed such that inequality (6.11) holds.

Assume that N_0 is sufficiently large number such that for any $n \ge N_0$ and for any $v \in \mathcal{D}$ the right-hand side of inequality (6.9) is smaller then $\frac{1}{100}$. In the what follows we shall assume that $n \ge N_0$ is fixed. We repeat here Lemma 4.4. It is similar to Lemma 3.4 in [9].

Lemma 6.3. *Assume that condition* (6.7) *holds, for some $z = u+iv \in \mathcal{D}'$ and for any $\omega \in \Omega_n^*$. Then* (6.7) *holds for $z' = u+i\widehat{v} \in \mathcal{D}$ as well with $v \ge \widehat{v} \ge v - n^{-8}$, for any $\omega \in \Omega_n^*$.*

Proof. To prove this lemma is enough to repeat the proof of Lemma 4.4. □

Proposition 6.2. *There exist positive constants C, c, depending on α and $\varkappa$ only such that*

$$\Pr\left\{|g_n(z)| > \frac{C\beta_n^4}{nv} + \frac{C\beta_n^4}{n^2v^2\sqrt{\gamma+v}}\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{6.12}$$

for all $z \in \mathcal{D}'$

Proof. Note that for $v = 4$ we have, for any $\omega \in \Omega_n^*$, $|g_n(z)| \ge \frac{1}{2}$. By Lemma 6.1, we obtain inequality (6.12) for $v = 4$. By Lemma 6.3, this inequality holds for any v with $v_0 \le v \le 4$ as well. Thus Proposition 6.2 is proved. □

7. Proof of Theorem 1.1

To conclude the proof of Theorem 1.1 we shall now apply the result of Corollary 2.2 with $v = \sqrt{2}v_0$ and $V = 4$ to the empirical spectral distribution function $\mathcal{F}_n(x)$ of the random matrix $\mathbf{X}$. At first we bound the integral over the line $V = 4$. Note that in this case we have $|z + m_n(z)| \ge 1$ and $|g_n(z)| \le \frac{1}{2}$ a.s. Moreover, $\operatorname{Im} m_n^{(j)}(z) \le \frac{1}{V} \le \frac{1}{2}$. In this case the results of Lemmas 5.1–5.3 hold for any $z = u + 4i$ with $u \in \mathbb{R}$. We apply inequality (6.8):

$$\begin{aligned}|g_n(z)| \le{}& \frac{\beta_n^2(1+\operatorname{Im}^{\frac{1}{2}} m_n(z))}{n\sqrt{v}|z+m_n(z)||z+s(z)+m_n(z)|} + \frac{C\operatorname{Im} m_n(z)}{nv|z+s(z)+m_n(z)|}\\ &+ \frac{\beta_n^2}{n^{\frac{3}{2}}v|z+m_n(z)||z+s(z)+m_n(z)|}\\ &+ \frac{\beta_n^4}{nv|z+m_n(z)||z+s(z)+m_n(z)|}\Big(\operatorname{Im} m_n(z) + \frac{1}{nv}\Big)\frac{1}{n}\sum_{j=1}^n |R_{jj}|,\end{aligned} \tag{7.1}$$

which holds for any $z = u+4i$, $u \in \mathbb{R}$, with probability at least $1 - C\exp\{-cl_{n,\alpha}\}$. Note that for $V = 4$

$$|z+m_n(z)||z+m_n(z)+s(z)| \ge \begin{cases} 4 & \text{for} \quad |u| \le 2, \\ \frac{1}{4}|z|^2 & \text{for} |u| > 2 \end{cases} \qquad \text{a.s.}$$

We may rewrite the bound (7.1) as follows

$$|g_n(z)| \le \frac{C\beta_n^4}{n(|z|^2+1)} + \frac{C\operatorname{Im} m_n(z)}{nV}.$$

Note that for any distribution function $F(x)$ we have

$$\int_{-\infty}^{\infty} \operatorname{Im} s_F(u+iv)du \le \pi$$

Moreover, for any random variable ξ with distribution function $F(x)$ and $\mathbf{E}\xi = 0$, $E\xi^2 = h^2$ we have

$$\operatorname{Im} s_F(u+iV) \le \frac{C(1+h^2)}{u^2}$$

with some numerical constant C. From here it follows that, for $V = 4$,

$$\int_{|u|\geq n^2} |m_n(z) - s(z)|du \leq \frac{C(1+h_n^2)}{n^2} \text{ a.s.} \tag{7.2}$$

with $h_n^2 = \int_{-\infty}^{\infty} x^2 d\mathcal{F}_n(x)$. Furthermore, note that

$$h_n^2 = \frac{1}{n^2}\sum_{j,k=1}^{n} X_{jk}^2 \leq \frac{2}{n^2}\sum_{1\leq j\leq k\leq n} X_{jk}^2.$$

Using inequality (3.1), we get

$$\Pr\{h_n^2 > Cn\} \leq C\exp\{-l_{n,\alpha}\}.$$

The last inequality and inequality (7.2) together imply that

$$\int_{|u|>n^2} |m_n(u+iV) - s(u+iV)|du \leq \frac{C}{n}$$

with probability at least $1 - C\exp\{-cl_{n,\alpha}\}$. Denote $\overline{\mathcal{D}}_n := \{z = u + 2i : |u| \leq n^2\}$ and

$$\overline{\Omega}_n := \left\{\cap_{z\in\overline{\mathcal{D}}_n}\left\{\omega\in\Omega:\ |g_n(z)| \leq \frac{C\beta_n^2}{n(|z|^2+1)}\right\}\right\}\cap\Omega_n^*.$$

Using a union bound, similar to (3.16) we may show that

$$\Pr\{\overline{\Omega}_n\} \geq 1 - C\exp\{-cl_{n,\alpha}\}.$$

It is straightforward to check that for $\omega\in\overline{\Omega}_n$

$$\int_{-\infty}^{\infty} |m_n(z) - s(z)|du \leq \frac{C\beta_n^4}{n}. \tag{7.3}$$

Furthermore, we put $\varepsilon = (2av_0)^{\frac{2}{3}}$ and $v_0 = \frac{d\beta_n^4}{n}$ with the constant d as introduced in (6.11). To conclude the proof we need to consider the "vertical" integrals, for $z = x + iv'$ with $x \in \mathbb{J}'_\varepsilon$, $v' = \frac{v_0}{\sqrt{\gamma}}$ and $\gamma = 2 - |x|$. Note that

$$\int_{v'}^{2} \frac{\beta_n^4}{nv}dv \leq \frac{C\beta_n^4\ln n}{n}.$$

Furthermore,

$$\int_{v'}^{2} \frac{1}{n^2v^2\sqrt{\gamma+v}}dv \leq \frac{1}{n^2v'\sqrt{\gamma}} \leq \frac{1}{n^2v_0} \leq \frac{\beta_n^4\ln n}{n}.$$

Finally, we get, for any $\omega\in\overline{\Omega}_n$,

$$\Delta(\mathcal{F}_n, G) = \sup_x |\mathcal{F}_n(x) - G(x)| \leq \frac{\beta_n^4\ln n}{n}.$$

Thus Theorem 1.1 is proved. □

8. Proof of Theorem 1.2

We may express the diagonal entries of the resolvent matrix **R** as follows

$$R_{jj} = \sum_{k=1}^{n} \frac{1}{\lambda_k - z} |u_{jk}|^2. \tag{8.1}$$

Consider the distribution function, say $F_{nj}(x)$, of the probability distribution of the eigenvalues λ_k

$$F_{nj}(x) = \sum_{k=1}^{n} |u_{jk}|^2 \mathbb{I}\{\lambda_k \le x\}.$$

Then we have

$$R_{jj} = R_{jj}(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dF_{nj}(x),$$

which means that R_{jj} is the Stieltjes transform of the distribution $F_{nj}(x)$. Note that, for any $\lambda > 0$,

$$\max_{1\le k\le n} |u_{jk}|^2 \le \sup_x (F_{nj}(x+\lambda) - F_{nj}(x)) =: Q_{nj}(\lambda).$$

On the other hand, it is easy to check that

$$Q_{nj}(\lambda) \le 2 \sup_u \lambda \operatorname{Im} R_{jj}(u + i\lambda). \tag{8.2}$$

By relations (3.12) and (3.16), we obtain, for any $v \ge v_0$ with $v_0 = \frac{d\beta_n^4}{n}$ with a sufficiently large constant d,

$$\Pr\Big\{\frac{|\varepsilon_j|}{|z + m_n(z)|} \le \frac{1}{2}\Big\} \le C \exp\{-c l_{n,\alpha}\} \tag{8.3}$$

with constants C and c depending on $\varkappa$, α and d. Furthermore, the representation (2.19) and inequality (8.3) together imply, for $v \ge v_0$, $\operatorname{Im} R_{jj} \le |R_{jj}| \le C_1$ with some positive constant $C_1 > 0$ depending on $\varkappa$ and α. This implies that

$$\Pr\Big\{\max_{1\le k\le n} |u_{jk}|^2 \le \frac{\beta_n^4}{n}\Big\} \le C \exp\{-c l_{n,\alpha}\}.$$

By a union bound we arrive at the inequality (1.4). To prove inequality (1.5), we consider the quantity $r_j := R_{jj} - s(z)$. Using equalities (2.19) and (4.1), we get

$$r_j = -\frac{s(z) g_n(z)}{z + m_n(z)} + \frac{\varepsilon_j}{z + m_n(z)} R_{jj}.$$

By inequalities (6.12), (3.11) and (3.16), we have

$$\Pr\{|r_j| \le \frac{c\beta_n^2}{\sqrt{nv}}\} \ge 1 - C \exp\{-c l_{n,\alpha}\}.$$

From here it follows that

$$\sup_{x \in \mathbb{J}_\varepsilon} \int_{v'}^{V} |r_j(x + iv)| dv \le \frac{C}{\sqrt{n}}.$$

Similar to (7.3) we get

$$\int_{-\infty}^{\infty} |r_j(x+iV)|dx \le \frac{C\beta_n^2}{\sqrt{n}}.$$

Applying Corollary 2.2, we get

$$\Pr\{\sup_x |F_{nj}(x) - G(x)| \le \frac{\beta_n^2}{\sqrt{n}}\} \ge 1 - C\exp\{-cl_{n,\alpha}\}.$$

Using now that

$$\Pr\Big\{\sup_x |\mathcal{F}_n(x) - G(x)| \le \frac{\beta_n^4 \ln n}{n}\Big\} \ge 1 - C\exp\{-cl_{n,\alpha}\},$$

we get

$$\Pr\Big\{\sup_x |F_{nj}(x) - \mathcal{F}_n(x)| \le \frac{\beta_n^2}{\sqrt{n}}\Big\} \ge 1 - C\exp\{-cl_{n,\alpha}\}.$$

Thus, Theorem 1.2 is proved. □

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Empirical Quantile CLTs for Time-dependent Data

James Kuelbs and Joel Zinn

Abstract. We establish empirical quantile process CLTs based on n independent copies of a stochastic process $\{X_t : t \in E\}$ that are uniform in $t \in E$ and quantile levels $\alpha \in I$, where I is a closed sub-interval of $(0, 1)$. The process $\{X_t : t \in E\}$ may be chosen from a broad collection of Gaussian processes, compound Poisson processes, stationary independent increment stable processes, and martingales.

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1. Introduction

Let $X = \{X(t)\colon\ t \in E\}$ be a stochastic process with $P(X(\cdot) \in D(E)) = 1$, where E is a set and $D(E)$ is a collection of real-valued functions on E. Also, let $\mathcal{C} = \{C_{s,x}\colon\ s \in E, x \in \mathbb{R}\}$, where $C_{s,x} = \{z \in D(E)\colon\ z(s) \le x\}, s \in E, x \in \mathbb{R}$. If $\{X_j\}_{j=1}^{\infty}$ are i.i.d. copies of the stochastic process X and $F_t(x) := F(t,x) := P(X(t) \le x) = P(X(\cdot) \in C_{t,x})$, then the empirical distributions built on $\mathcal{C}$ (or built on the process X) are defined by

$$F_n(t,x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x]}(X_i(t)) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \in C_{t,x}\}}, C_{t,x} \in \mathcal{C},$$

and we say X is the input process.

The empirical processes indexed by $\mathcal{C}$ (or just $E \times \mathbb{R}$) and built from the process, X, are given by

$$\nu_n(t,x) := \sqrt{n}\big(F_n(t,x) - F(t,x)\big).$$

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In [4] we studied the central limit theorem in this setting, that is, we found sufficient conditions for a pair $(\mathcal{C}, P)$, where P is the law of X on $D(E)$, ensuring that the sequence of empirical processes $\{\nu_n(t,x) : (t,x) \in E \times \mathbb{R}\}, n \geq 1$, converge to a centered Gaussian process, $G = \{G_{t,x} \colon\ (t,x) \in E \times \mathbb{R}\}$ with covariance

$$\mathbb{E}(G(s,x)G(t,y)) = \mathbb{E}([I(X_s \leq x) - P(X_s \leq x)][I(X_t \leq y) - P(X_t \leq y)]).$$

This requires that the law of G on $\ell_\infty(E \times \mathbb{R})$ (with the usual sup-norm) be Radon, or equivalently, (see Example 1.5.10 in [10]), that G has sample paths which are bounded and uniformly continuous on $E \times \mathbb{R}$ with respect to the psuedo-metric

$$d((s,x),(t,y)) = (\mathbb{E}[G(s,x) - G(t,y)]^2)^{1/2}. \tag{1}$$

It also requires that for every bounded, continuous $F\colon\ \ell_\infty(E \times \mathbb{R}) \longrightarrow \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}^* F(\nu_n) = \mathbb{E}F(G),$$

where $\mathbb{E}^*$ denotes the upper expectation (see, e.g., p. 94 in [2]).

The quantiles and empirical quantiles are defined as the left-continuous inverses of $F(t,x)$ and $F_n(t,x)$ in the variable x, respectively:

$$\tau_\alpha(t) = F^{-1}(t,\alpha) = \inf\{x \colon F(t,x) \geq \alpha\} \tag{2}$$

and

$$\tau_\alpha^n(t) = F_n^{-1}(t,\alpha) = \inf\{x \colon F_n(t,x) \geq \alpha\}. \tag{3}$$

The empirical quantile processes are defined as

$$\sqrt{n}\big(F_n^{-1}(t,\alpha) - F^{-1}(t,\alpha)\big),$$

and we also use the more compact notation

$$\sqrt{n}\big(\tau_\alpha^n(t) - \tau_\alpha(t)\big),$$

for these processes. Since we are seeking limit theorems with non-degenerate Gaussian limits, it is appropriate to mention that for $\alpha \in (0,1)$ and t fixed, that is, for a one-dimensional situation, a necessary condition for the weak convergence of

$$\sqrt{n}\big(\tau_\alpha^n(t) - \tau_\alpha(t)\big) \Longrightarrow \xi, \tag{4}$$

where ξ has a strictly increasing, continuous distribution, is that the distribution function $F(t,\cdot)$ be differentiable at $\tau_\alpha(t)$ and $F'(t,\tau_\alpha(t)) > 0$. Hence $F(t,\cdot)$ is strictly increasing near $\tau_\alpha(t)$ as a function of x, and if we keep t fixed, but ask that (4) holds for all $\alpha \in (0,1)$, then $F(t,x)$ will be differentiable, with strictly positive derivative $F'(t,x)$ on the set $J_t = \{x : 0 < F(t,x) < 1\}$. Moreover, by Theorem 8.21, p. 168, of [6], if $F'(t,x)$ is locally in L_1 with respect to Lebesgue measure on J_t, then $F'(t,x)$ is the density of $F(t,\cdot)$ and it is strictly positive on J_t. For many of the base processes we study here, $J_t = \mathbb{R}$ for all $t \in E$, but should that not be the case, it can always be arranged by adding an independent random variable Z with strictly positive density to our base process in order to have a suitable input process. In particular, the reader should consider a base process as one which, after possibly some modification, will be a suitable input process. At first glance perhaps this may seem like a convenient shortcut, but we know from

[4] that when $E = [0, T]$ and the base process is a fractional Brownian motion starting at zero when $t = 0$, then the empirical CLT over $\mathcal{C}$ fails, but by adding Z as indicated above it will hold. Of course, in these cases adding Z is just starting the process with the distribution of Z, and hence a typical assumption throughout paper is that the distributions $F(t, \cdot)$ are continuous and strictly increasing on $\mathbb{R}$.

Section 2 provides statements of our main theoretical results providing CLTs for empirical quantile processes. In particular, Theorem 2 provides applications for a broad range of input processes that includes Gaussian processes, compound Poisson processes, symmetric stationary increment stable processes, and certain martingales. In Section 3 we provide a proof of Theorem 1, using an idea of Vervaat from [11] on the relation between empirical and quantile processes. This relationship is based on almost sure convergence results for weak convergence of empirical measures, and since we are seeking results that are uniform in quantile levels and the parameter set E that indexes our input process, its proof involves a number of ideas from empirical process theory as presented in [2] and [10]. Section 4 then turns to the proof of Theorem 2.

The papers of Swanson [8] and [9] were the first to motivate our interest in this set of problems, but the techniques we use are quite different and apply to a much broader set of input processes. In the first of these papers Swanson obtained a central limit theorem for the median process, when in our terminology the input process $\{X_t : t \geq 0\}$ is a sample continuous Brownian motion tied down to have value 0 at time 0. In the second he establishes a CLT for the empirical quantile process for each fixed $\alpha \in (0, 1)$, but now $\{X_t : t \geq 0\}$ is assumed to be a sample continuous Brownian motion whose distribution at time zero is assumed to have a density with a unique α quantile. In particular, his results are uniform in $t \in [0, T]$ for $T \in (0, \infty)$, but only for fixed α, and only for empirical quantile processes of Brownian motion. Hence our results are more general in most ways, but because the results for empirical quantile CLTs we present here depend on the empirical CLT over $\mathcal{C}$, and this fails when X is tied down Brownian motion on $[0, T]$, they apply to Brownian motion on $[0, T]$ only when we start with a nice density at time 0. On the other hand, as can be seen in Theorem 2, our results apply to general classes of processes, including symmetric stable processes and fractional Brownian motions, as long as they have a nice density at time 0, and our quantile CLTs are uniform in $t \in [0, T]$ and also in $\alpha \in I$, where I is a closed subinterval of $(0, 1)$. Furthermore, as can be seen from [4] and Corollary 3 below, these processes fail the empirical CLT over $\mathcal{C}$ on $[0, T]$ when they start at 0 when $t = 0$, so to apply our quantile CLTs to such processes we must find a way to circumvent the assumption of the empirical CLT holding over $\mathcal{C}$ on $[0, T]$. This has been done for a broad class of processes, and is not at all immediate or trivial, and will appear in a future paper. Finally, it is perhaps worth mentioning that as we learned more about the related statistical literature, the idea of studying these problem for a wide range of input processes and seeking results uniform in the quantile levels as well as in the parameter $t \in E$ became an interesting and natural goal.

2. Statement of results

Throughout we assume the notation of Section 1. In particular, we are assuming that for all $t \in E$, $F(t,x)$ is strictly increasing and continuous in $x \in \mathbb{R}$. Our goal in this section is to present two results. The first is an empirical quantile CLT which is uniform in both the parameters t and α, and the second indicates specific applications of the general result to a broad class of processes.

The proof of our first result uses a method of Vervaat. In particular, we prove an analogue of Vervaat's Lemma 1 in [11], depending on an almost sure version of the empirical CLT in the setting of [4]. This approach has been used by others, see [3] and in [7] for two examples, but since we require uniformity in the parameters t and α, the application in this setting is more general and requires a bit of care to handle the necessary measurability issues required to obtain the results. Its proof is in Section 3.

Up to this point we have only assumed that the distribution functions $\{F(t,\cdot)\colon\ t \in E\}$ are continuous and strictly increasing on $\mathbb{R}$. As explained in Section 1, since we are seeking non-degenerate Gaussian limits for all $\alpha \in (0,1), t \in E$ in our quantile CLT's, the distribution functions $F(t,x)$ must be differentiable in x with strictly positive and finite derivative on $J_t = \{x : 0 < F(t,x) < 1\}$. By smoothing if necessary, $J_t = \mathbb{R}$ for all $t \in E$, and hence we typically assume that these distribution functions have densities $\{f(t,\cdot)\colon\ t \in E\}$ such that

$$\lim_{\delta \to 0} \sup_{t \in E} \sup_{|u-v| \le \delta} |f(t,u) - f(t,v)| = 0, \tag{5}$$

for every closed interval I in $(0,1)$ there is an $\theta(I) > 0$ such that

$$\inf_{t \in E, \alpha \in I, |x - \tau_\alpha(t)| \le \theta(I)} f(t,x) \equiv c_{I,\theta(I)} > 0, \tag{6}$$

and

$$\sup_{t \in E, x \in \mathbb{R}} f(t,x) < \infty. \tag{7}$$

The assumptions (5) and (6) suffice for our quantile CLTs when the empirical CLT over $\mathcal{C}$ holds, but otherwise (7), perhaps after smoothing, is a typical additional assumption. In particular, Corollaries 2 and 3 show that the empirical CLT over $\mathcal{C}$ may fail for important classes of input processes when (7) fails.

Our first quantile CLT requires the empirical CLT holds over the sets $\mathcal{C}$, with Gaussian limit $\{G(t,x) : (t,x) \in E \times \mathbb{R}\}$ as given in Section 1, and is given in the following result.

Theorem 1. *Assume for all $t \in E$ that the distribution functions $F(t,x)$ are strictly increasing, their densities $f(t,\cdot)$ satisfy* (5) *and* (6)*, and the CLT holds on $\mathcal{C}$ with centered Gaussian limit $\{G(t,x)\colon\ (t,x) \in E \times \mathbb{R}\}$. Then, for I any closed subinterval of* $(0,1)$*, the quantile processes $\{\sqrt{n}(\tau_\alpha^n(t) - \tau_\alpha(t))f(t,\tau_\alpha(t))\colon\ n \ge 1\}$ satisfy the CLT in $\ell_\infty(E \times I)$ with Gaussian limit process $\{G(t,\tau_\alpha(t))\colon\ (t,\alpha) \in E \times I\}$. Moreover, the quantile processes $\{\sqrt{n}(\tau_\alpha^n(t) - \tau_\alpha(t))\colon\ n \ge 1\}$ also satisfy the CLT in $\ell_\infty(E \times I)$ with Gaussian limit process $\{\frac{G(t,\tau_\alpha(t))}{f(t,\tau_\alpha(t))}\colon\ (t,\alpha) \in E \times I\}$.*

Our next result yields a large number of specific input processes $\{X(t) : t \in E\}$ where the empirical quantile CLT holds by applying Theorem 1. In particular, since an important assumption in Theorem 1 is that an empirical CLT over $\mathcal{C}$ holds for $\{X(t) : t \in E\}$, an important part of the proof will be to verify this for the processes considered by applying results in [4].

The typical empirical quantile CLT we establish next starts with a base process $\{Y_t : t \in E\}$, and we define $X_t = Y_t + Z, t \in E$, where Z is independent of $\{Y_t : t \in E\}$ and Z has density $g(\cdot)$ on $\mathbb{R}$. For the empirical process CLT's over $\mathcal{C}$ we need only assume $g(\cdot)$ is uniformly bounded on $\mathbb{R}$, or in $L_a(\mathbb{R})$ for some $a > 1$. In order to prove our empirical quantile results, we assume a bit more about $g(\cdot)$, but these assumptions are not unusual, even for real-valued quantile CLT's. Moreover, keeping in mind possible application to a diverse collection of base processes, we have chosen to put the assumptions we require on $g(\cdot)$, but the reader should also note that if the distributions of $Y_t, t \in E$, have densities with similar properties, then we could assume less about $g(\cdot)$. This is easily seen from the proofs, and basic facts about convolutions, and are left for the reader to implement should the occasion arise. It should also be emphasized that there are many interesting base processes $\{Y_t : t \in E\}$ where the empirical CLT over $\mathcal{C}$ fails, but it holds for the smoothed process $\{X_t = Y_t + Z : t \in E\}$ as above. Such examples can be found in [4], and include all fractional Brownian motions on $E = [0, T]$ that start at zero at time zero with probability one. Others will appear in the results of Section 4 below.

The assumptions we impose on the input process $\{X_t : t \in E\}$ will be sufficent for the empirical CLT over $\mathcal{C}$ with centered Gaussian limit on $\ell_\infty(E \times \mathbb{R})$ given by $\{G(t, x) : t \in E, x \in \mathbb{R}\}$, where $G(\cdot, \cdot)$ is sample bounded on $E \times \mathbb{R}$, and uniformly continuous with respect to its L_2-distance there. Of course, as in Section 1 a typical point $(t, x) \in E \times \mathbb{R}$ has been identified with $C_{t,x}$. Our empirical quantile CLT's in this setting will then be of two types, and in these results I will always be a closed subinterval of $(0, 1)$. The first is that the quantile processes

$$\{\sqrt{n}(\tau_\alpha^n(t) - \tau_\alpha(t))f(t, \tau_\alpha(t))\colon \ n \geq 1\} \tag{8}$$

satisfy the CLT in $\ell_\infty(E \times I)$ with Gaussian limit process

$$\{G(t, \tau_\alpha(t))\colon \ (t, \alpha) \in E \times I\}, \tag{9}$$

and the second asserts that the quantile processes

$$\{\sqrt{n}(\tau_\alpha^n(t) - \tau_\alpha(t))\colon \ n \geq 1\} \tag{10}$$

satisfy the CLT in $\ell_\infty(E \times I)$ with Gaussian limit process

$$\left\{\frac{G(t, \tau_\alpha(t))}{f(t, \tau_\alpha(t))}\colon \ (t, \alpha) \in E \times I\right\}. \tag{11}$$

The base processes $\{Y_t : t \in E\}$ we consider are of three general types, and there is some overlap between these types. For example, the compound Poisson processes in (iii) below could also be martingales, but it is easy to check that there are examples which fit into one and only one of the classes we study.

(i) $\{Y_t : t \in E\}$ is a centered sample continuous Gaussian process on a compact subset E of $[0,T]^d$ whose L_2-distance d_Y is such that for some $k_1 < \infty$, $s,t \in E$,

$$d_Y(s,t) = [\mathbb{E}((Y_t - Y_s)^2)]^{1/2} \leq k_1 ||t-s||_{\mathbb{R}^d}^{\gamma}, \tag{12}$$

where $||t-s||_{\mathbb{R}^d}$ is the usual L_2-distance on $\mathbb{R}^d$ and $0 < \gamma \leq 1$. Of course, when $X_t = Y_t + Z, t \in E$, the L_2-distance d_X of X also satisfies (12).

(ii) $E = [0,T]$ and $\{Y(t) : t \geq 0\}$ is a stochastic process with cadlag sample paths on $[0,\infty)$ such that $P(Y(0) = 0) = 1$. In addition, $\{Y(t) \colon\ t \in E\}$ is a martingale whose L_1-increments are Lip-β for some $\beta \in (0,1]$, or a stationary independent increments process whose L_p-increments are Lip-β on E for some $p \in (0,1]$, i.e., there is a $\beta \in (0,1]$ and $C < \infty$ such that for all $s,t \in E$

$$\mathbb{E}(|Y(t) - Y(s)|^p) \leq C|t-s|^{\beta}. \tag{13}$$

(iii) $E = [0,T]$ and $\{Y_t : t \in E\}$ is a compound Poisson process built from the i.i.d. random variables $\{Y_k : k \geq 1\}$ having no mass at zero and Poisson process $\{N(t) : t \geq 0\}$ with cadlag paths and parameter $\lambda \in (0,\infty)$ providing the jump times for $\{Y_t : t \in E\}$.

Remark 1. If $\{Y(t) : t \geq 0\}$ is a strictly stable process with stationary independent increments and index $r \in (0,2]$, then for $r \in (1,2]$ we have $\mathbb{E}(|Y(t)|) = t^{1/r}\mathbb{E}(|Y(1)|)$ and hence

$$\mathbb{E}(|Y(t) - Y(s)|) = |t-s|^{1/r}\mathbb{E}(|Y(1)|),$$

which implies it has L_1-increments that are Lip-$\frac{1}{r}$. Of course, it is also a martingale when $r \in (1,2]$. If $0 < r \leq 1$, then for $0 < p < r$ we have

$$\mathbb{E}(|Y(t) - Y(s)|^p) = |t-s|^{p/r}\mathbb{E}(|Y(1)|^p),$$

which implies it has L_p-increments that are Lip-$\frac{p}{r}$. If $\{Y(t) : t \geq 0\}$ is a square integrable martingale with $\lambda(t) = \mathbb{E}(Y^2(t)), t \geq 0$, then for $0 \leq s \leq t$ the orthogonality of the increments of $\{Y(t) : t \geq 0\}$ implies

$$\mathbb{E}((Y(t) - Y(s))^2) = \lambda(t) - \lambda(s). \tag{14}$$

Hence, if $\lambda(\cdot)$ is Lip-γ on E, then (14) implies (13) with $p = 1, \beta = \gamma/2$. In addition, if $\{Y(t) : t \geq 0\}$ also has stationary, independent increments with $P(Y(0) = 0) = 1$ and $\lambda(t) = \mathbb{E}(|Y(t)|) < \infty, t \geq 0$, then for $s,t \in E$ we have

$$\mathbb{E}(|Y(t) - Y(s)|) = \mathbb{E}(|Y(|t-s|)|) = \lambda(|t-s|). \tag{15}$$

Therefore, if $\lambda(t) \leq Ct^{\beta}$ for $t \in [0,\delta]$ and some $\delta > 0, \beta \in (0,1]$, then it is easy to check that (15) implies (13) with $p = 1$ and the given β for all $s,t \in E$, and a possibly larger constant C, depending on δ.

Theorem 2. *Let* $\{Y_t : t \in E\}$ *satisfy* (i), (ii), *or* (iii). *In addition, assume* $X_t = Y_t + Z$, *where* Z *is independent of* $\{Y_t \colon\ t \in E\}$, *and* Z *has a strictly positive,* *uniformly bounded, uniformly continuous density function* g *on* $\mathbb{R}$. *If* I *is any closed subinterval of* $(0,1)$, *and we also assume that*

$$\lim_{b\to\infty}\sup_{t\in E} P(|Y_t|\geq b)=0, \tag{16}$$

then the quantile processes of (8) *and* (10) *built from the input process* $\{X_t : t\in E\}$ *satisfy the empirical quantile CLT with corresponding Gaussian limit as in* (9) *and* (11).

Remark 2. It is easy to see at this point that Theorem 2 implies empirical quantile results of both types for fractional Brownian motions, the Brownian sheet, strictly stable stationary independent increment processes, martingales, and compound Poisson processes. The precise corollaries are easy to formulate, and hence are not included. The details of the proof of Theorem 2 will show that the empirical process CLT's over $\mathcal{C}$ will follow under the slightly weaker assumptions that $g(\cdot)$ be uniformly bounded on $\mathbb{R}$, or in $L_a(\mathbb{R})$ for some $a>1$.

3. The proof of Theorem 1: Vervaat's approach

3.1. Notation and some lemmas

As before, we assume for all $t\in E$ that $F(t,x)$ is strictly increasing and continuous in $x\in\mathbb{R}$. Our first task of significance is to prove an analogue of Vervaat's Lemma 1 in [11]. We follow Vervaat's idea of using an almost sure version of the empirical CLT, but given the generality of our setting the implementation of these ideas in the following three sub-sections is not as immediate as one might like. In particular, to obtain uniformity in the parameters (t,x) requires a general approach to such issues, and one that also can handle the measurability problems that arise. Showing that weak convergence, or convergence in law, can be expressed in terms of almost sure convergence has a long history, and for the task here we use Theorem 3.5.1 in [2]. Its statement below is slightly less general than that in [2].

Notation 1. For a function $f: S\longrightarrow\bar{\mathbb{R}}$ we use the notation f^* to denote a measurable cover function (see Lemma 1.2.1 [10]).

Theorem 3 ([2]). *Let* (D,d_∞) *be a metric space,* $(\Omega,\mathcal{A},Q)$ *be a probability space and* $f_n\colon\ \Omega\to D$ *for each* $n=0,1,\dots$. *Suppose* f_0 *has separable range,* D_0, *and is measurable with respect to the Borel sigma algebra on* D_0. *Then* $\{f_n\colon\ n\geq 1\}$ *converges weakly, or in law, to* f_0 *iff there exists a probability space* $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P})$ *and perfect measurable functions* g_n *from* $(\widehat{\Omega},\widehat{\mathcal{F}})$ *to* $(\Omega,\mathcal{A})$ *for* $n=0,1,\dots$, *such that*

$$\widehat{P}\circ g_n^{-1}=Q \text{ on } \mathcal{A} \tag{17}$$

for each n, *and*

$$d^*_\infty(f_n\circ g_n, f_0\circ g_0)\underset{a.s}{\rightarrow} 0. \tag{18}$$

where $d^*_\infty(f_n\circ g_n,f_0\circ g_0)$ *denotes the measurable cover function for* $d_\infty(f_n\circ g_n,f_0\circ g_0)$ *and the a.s. convergence is with respect to* $\widehat{P}$.

In our setting the metric space D is $\ell_\infty(E\times\mathbb{R})$, with distance d_∞ the usual sup-norm there, and the probability space $(\Omega,\mathcal{A},Q)$ supports the i.i.d. sequence $\{X_j\colon\ j\geq 1\}$ and the Gaussian process G. Then, for $\omega\in\Omega, n\geq 1$, the f_n of Dudley's result is our ν_n,

$$f_n(\omega)=\sqrt{n}(F_n(\cdot,\cdot)(\omega)-F(\cdot,\cdot))\in\ell_\infty(E\times\mathbb{R}), \tag{19}$$

and $\{f_n\colon\ n\geq 1\}$ converges in law to

$$f_0(\omega)=G(\cdot,\cdot)(\omega)\in\ell_\infty(E\times\mathbb{R}). \tag{20}$$

That is, we are assuming the empirical CLT over $\mathcal{C}$, and therefore Theorem 3 implies there is a suitable probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P})$ and a set $\widehat{\Omega}_1\subset\widehat{\Omega}$ with $\widehat{P}(\widehat{\Omega}_1)=1$ such that for all $\hat{\omega}\in\widehat{\Omega}_1$,

$$||(f_n\circ g_n)(\hat{\omega})-(f_0\circ g_0)(\hat{\omega})||^*\equiv\Big(\sup_{t\in E,x\in\mathbb{R}}|(f_n\circ g_n)(\hat{\omega})-(f_0\circ g_0)(\hat{\omega})|\Big)^*\to 0. \tag{21}$$

Hence, if

$$\widehat{F}_n(t,x)(\hat{\omega})=F_n(t,x)(g_n(\hat{\omega}))\quad\text{and}\quad\widehat{G}(t,x)(\hat{\omega})=G(t,x)(g_0(\hat{\omega})),$$

then on $\widehat{\Omega_1}$ we have the empirical distribution functions $\{\widehat{F}_n\colon\ n\geq 1\}$ satisfying

$$\|\sqrt{n}(\widehat{F}_n-F)-\widehat{G}\|^*\equiv\Big(\sup_{t\in E,x\in\mathbb{R}}|\sqrt{n}(\widehat{F}_n(t,x)(\hat{\omega})-F(t,x))-\widehat{G}(t,x)(\hat{\omega})|\Big)^*\to 0. \tag{22}$$

Remark 3. The functions $\widehat{F}_n$ are still distribution functions as functions of x, and on $\widehat{\Omega}$ we have $\sqrt{n}(\widehat{F}_n-F)-\widehat{G}\in\ell_\infty(E\times\mathbb{R})$. In addition, since the functions $\{g_n\colon\ n\geq 0\}$ are perfect and (17) holds, it follows for every bounded, real-valued function h on $\ell_\infty(E\times\mathbb{R})$, and $n\geq 1$, that

$$\mathbb{E}^*_{\widehat{P}}[h(\sqrt{n}(\widehat{F}_n-F))]=\mathbb{E}^*_Q[h(\sqrt{n}(F_n-F))]\quad\text{and}\quad\mathbb{E}_{\widehat{P}}[h(\widehat{G})]=\mathbb{E}_Q[h(G)]. \tag{23}$$

Since we are assuming $\{\sqrt{n}(F_n-F)\colon\ n\geq 1\}$ converges weakly to the Gaussian limit G, and G has separable support in $\ell_\infty(E\times\mathbb{R})$, then (23) immediately implies $\{\sqrt{n}(\widehat{F}_n-F)\colon\ n\geq 1\}$ also converges weakly to G.

The generalized inverse of $\widehat{F}_n(t,\cdot)$ in the second variable is given by

$$\widehat{\tau}^n_\alpha(t)\equiv\widehat{F}_n^{-1}(t,\alpha)=\inf\big\{x\colon\ \widehat{F}_n(t,x)\geq\alpha\big\},\ t\in E,\alpha\in(0,1),n\geq 1, \tag{24}$$

and as before for each $t\in E,\alpha\in(0,1)$, the inverse function

$$\tau_\alpha(t)\equiv F^{-1}(t,\alpha)=\inf\{x:F(t,x)\geq\alpha\}. \tag{25}$$

Of course, since we are assuming $F_t(x):=F(t,x)$ is strictly increasing, this is a classical inverse function, and to emphasize that the inverse is only on the second variable we also will write $\widehat{F}^{-1}_{n,t}$ and F_t^{-1} for these inverses. Then, for each $t\in E$, since $X_1(t),\ldots,X_n(t)$ are real numbers, we have $\widehat{F}^{-1}_{n,t}(\cdot)\colon\ [0,1]\xrightarrow{\text{into}}\mathbb{R}$ and since F_t is assumed continuous and strictly increasing we have $F_t^{-1}(\cdot)\colon\ (0,1)\xrightarrow{\text{onto}}\mathbb{R}$. It is also useful to define $F_t^{-1}(0)=-\infty$, $F_t(-\infty)=\widehat{F}_{n,t}(-\infty)=0$, $F_t^{-1}(1)=\infty$,

$F_t(+\infty) = \widehat{F}_{n,t}(+\infty) = 1$, and $\widehat{G}_t(-\infty) = \widehat{G}_t(+\infty) = 0$. We also set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

To use (22) we will need the function $\widehat{F}_{n,t} \circ F_t^{-1}$ and its inverse, which is determined in the next lemma.

Lemma 1. *For each $t \in E$*

$$(\widehat{F}_{n,t} \circ F_t^{-1})^{-1} = F_t \circ \widehat{F}_{n,t}^{-1} \tag{26}$$

where the inverses are defined as in (24) *and* (25).

Proof. For each $t \in E, \alpha \in [0,1]$, we have, since we are assuming $F_t(\cdot)$ is strictly increasing and continuous, that

$$\begin{aligned}(\widehat{F}_{n,t} \circ F_t^{-1})^{-1}(\alpha) &= \inf\{\beta\colon\ \widehat{F}_{n,t} \circ F_t^{-1}(\beta) \geq \alpha\} \\ &= \inf\{F_t(x)\colon\ \widehat{F}_{n,t}(x) \geq \alpha\} \\ &= F_t(\inf\{x\colon\ \widehat{F}_{n,t}(x) \geq \alpha\}) \\ &= (F_t \circ \widehat{F}_{n,t}^{-1})(\alpha).\end{aligned}$$

□

The next lemma is our modification of Lemma 1 in [11] applicable to the present situation.

Lemma 2. *Let $a_n \to 0$, and that*

$$\left(\sup_{t\in E, x\in\mathbb{R}} \left|\frac{\widehat{F}_{n,t}(x) - F_t(x)}{a_n} - \widehat{G}_t(x)\right|\right)^* \to 0 \tag{27}$$

as n tends to infinity. Then, setting $I_t(\alpha) = \alpha$ for $t \in E, \alpha \in [0,1]$, we have

$$\left(\sup_{t\in E, \alpha\in[0,1]} \left|\frac{(\widehat{F}_{n,t} \circ F_t^{-1})(\alpha) - I_t(\alpha)}{a_n} - (\widehat{G}_t \circ F_t^{-1})(\alpha)\right|\right)^* \to 0. \tag{28}$$

Furthermore, uniformly in $t \in E$, $\widehat{G}_t(F_t^{-1}(\alpha))$ is a uniformly continuous function of $\alpha \in [0,1]$, and

$$\left(\sup_{t\in E, u\in[0,1]} \left|\frac{(F_t \circ \widehat{F}_{n,t}^{-1})(u) - I_t(u)}{a_n} + (\widehat{G}_t \circ F_t^{-1})(u)\right|\right)^* \to 0. \tag{29}$$

Proof. Since we are assuming for each $t \in E$ that $F(t,\cdot)$ is strictly increasing and continuous on $\mathbb{R}$, it follows that $\{F_t^{-1}(\alpha)\colon\ \alpha \in (0,1)\} = \mathbb{R}$. Therefore, if one restricts α in (28) to be in $(0,1)$, then (28) follows immediately from (27). To obtain (28) for $\alpha = 0$ and $\alpha = 1$, then follows from the conventions we made prior to the statement of the lemma involving $\pm\infty$.

To show (28) implies (29) we define for each $t \in E$ the completed graph of $(\widehat{F}_{n,t} \circ F_t^{-1})(\cdot)$ on $[0,1]$ to be given by

$$\Gamma_{n,t} = \{(\alpha, u)\colon\ \alpha \in [0,1], (\widehat{F}_{n,t} \circ F_t^{-1})(\alpha - 0) \leq u \leq (\widehat{F}_{n,t} \circ F_t^{-1})(\alpha + 0)\}.$$

Here $(\widehat{F}_{n,t} \circ F_t^{-1})(\alpha \pm 0)$ denotes the left- and right-hand limits of $(\widehat{F}_{n,t} \circ F_t^{-1})(\cdot)$ when $\alpha \in (0,1)$, and are given through the conventions above when $\alpha = 0$ or 1, i.e., we understand the left- and right-hand limits at zero to both be zero, and the left- and right-hand limits at one to both be one. Now (28) implies that

$$\lim_{n\to\infty} \left(\sup\left\{\left|\tfrac{u-\alpha}{a_n} - (\widehat{G}_t \circ F_t^{-1})(\alpha)\right| : \ (t,\alpha) \in E \times [0,1], (\alpha,u) \in \Gamma_{n,t}\right\}\right)^* = 0,$$

which also implies that

$$\lim_{n\to\infty} \left(\sup\left\{\left|\tfrac{\alpha-u}{a_n} + (\widehat{G}_t \circ F_t^{-1})(\alpha)\right| : \ (t,\alpha) \in E \times [0,1], (\alpha,u) \in \Gamma_{n,t}\right\}\right)^* = 0. \tag{30}$$

For each $t \in E$ we set

$$\Gamma_{n,t}^{-1} = \{(u,\alpha) : \ u \in [0,1], (F_t \circ \widehat{F}_{n,t}^{-1})(u-0) \le \alpha \le (F_t \circ \widehat{F}_{n,t}^{-1})(u+0)\},$$

where one can check that the left-hand limit of $F_t \circ \widehat{F}_{n,t}^{-1}(\cdot)$ at zero is zero and we take the right-hand limit at one to be one.

Then one can check that

$$(\alpha,u) \in \Gamma_{n,t} \text{ if and only if } (u,\alpha) \in \Gamma_{n,t}^{-1}. \tag{31}$$

Moreover, (29) is implied by

$$\lim_{n\to\infty} \left(\sup\left\{\left|\tfrac{\alpha-u}{a_n} + (\widehat{G}_t \circ F_t^{-1})(u)\right| : \ (t,u) \in E \times [0,1], (u,\alpha) \in \Gamma_{n,t}^{-1}\right\}\right)^* = 0, \tag{32}$$

and (30) and (31) implies (32) provided we show

$$\lim_{n\to\infty} (\sup\{|(\widehat{G}_t \circ F_t^{-1})(u) - (\widehat{G}_t \circ F_t^{-1})(\alpha)| : \ t \in E, (u,\alpha) \in \Gamma_{n,t}^{-1}\})^* = 0. \tag{33}$$

Since we are assuming the empirical CLT holds over $\mathcal{C}$ with Gaussian limit process $\{G(t,x) : (t,x) \in E \times \mathbb{R}\}$, it follows that G has a version which is sample uniformly continuous on $E \times \mathbb{R}$ with respect to its L_2-distance, $d(\cdot,\cdot)$, given in (1). This is a consequence of the addendum to Theorem 1.5.7 of [10], p. 37. When referring to G we will mean this version. By the total boundedness of the distance, the associated space of uniformly continuous functions is separable in the uniform topology. This space is a closed subspace of $\ell_\infty(E \times \mathbb{R})$, which then implies that this version of G is measurable with respect to the Borel sets of $\ell_\infty(E \times \mathbb{R})$. Using the definition of $\widehat{G}$ following (21), and (17) with $n = 0$, we have the laws of G and $\widehat{G}$ are equal on $\ell_\infty(E \times \mathbb{R})$. In particular, $\widehat{G}$ is also measurable with respect to the Borel sets of $\ell_\infty(E \times \mathbb{R})$ with separable support there, it has the same covariance as G and its L_2-distance is d, and it is sample continuous on $(E \times \mathbb{R}, d)$ with $\widehat{P}$-probability one.

Now for each $t \in E$ and $\alpha, \beta \in (0,1)$ we have

$$d^2((t,F_t^{-1}(\alpha)),(t,F_t^{-1}(\beta))) = |\alpha-\beta| - |\alpha-\beta|^2 \le |\alpha-\beta|. \tag{34}$$

Thus for each $t \in E$ we have $d((t,F_t^{-1}(\alpha)),(t,F_t^{-1}(\beta))) \to 0$ as $\alpha,\beta \to 0$ or $\alpha,\beta \to 1$.

We also have

$$d^2((t,F_t^{-1}(\alpha)),(t,F_t^{-1}(0))) = \alpha-\alpha^2 \le |\alpha-0|,$$
$$d^2((t,F_t^{-1}(\alpha)),(t,F_t^{-1}(1))) = \alpha-\alpha^2 \le |\alpha-1|,$$

and

$$d((t,F_t^{-1}(0)),(t,F_t^{-1}(1))) = 0,$$

and hence the uniform continuity of $\widehat{G}$ along with $\widehat{G}(t,F_t^{-1}(0))=\widehat{G}(t,F_t^{-1}(1))=0$ implies that uniformly in $t\in E$ we have $\widehat{G}_t(F_t^{-1}(\alpha))$ is uniformly continuous in $\alpha\in[0,1]$ with probability one. Moreover, the process $\{\widehat{G}(t,x)\colon\ (t,x)\in E\times\mathbb{R}\}$ has separable support in $\ell_\infty(E\times\mathbb{R})$, and hence the upper cover used in (33) is unnecessary as the function there is measurable. We also have with $\widehat{P}$-probability one that

$$\sup_{t\in E,\alpha\in[0,1]} |\widehat{G}_t(F_t^{-1}(\alpha))| < \infty, \tag{35}$$

and hence a_n converging to zero, and (30) implies

$$\lim_{n\to\infty} \sup\{|u-\alpha|\colon\ t\in E, (\alpha,u)\in\Gamma_{n,t}\} = 0. \tag{36}$$

Therefore, (33) follows from (31) and that uniformly in $t\in E$ we have $\widehat{G}_t\circ F_t^{-1}(\alpha)$ uniformly continuous in $\alpha\in[0,1]$. Therefore, (33) holds, and this implies (29), so the lemma is proven. □

3.2. Applying Lemma 2 to obtain an empirical quantile CLT

Assuming the empirical CLT holds over $\mathcal{C}$, the conclusions of Lemma 2 hold with $a_n=\frac{1}{\sqrt{n}}$, and we have proved the following lemma.

Lemma 3. *For all $t\in E$ assume the distribution function $F(t,x)$ is strictly increasing and continuous in $x\in\mathbb{R}$, and that the CLT holds on $\mathcal{C}$ with limit*

$$\{G(t,x)\colon\ (t,x)\in E\times\mathbb{R}\}.$$

Then, with $\widehat{P}$-probability one, we have

$$\Big(\sup_{t\in E,\alpha\in[0,1]} |\sqrt{n}[(F_t\circ\widehat{F}_{n,t}^{-1})(\alpha)-I_t(\alpha)]+(\widehat{G}_t\circ F_t^{-1})(\alpha)|\Big)^* \to 0. \tag{37}$$

Up to this point we have only assumed that the distribution functions $\{F_t(\cdot)\colon\ t\in E\}$ are continuous and strictly increasing on $\mathbb{R}$, and that the empirical processes satisfy the CLT over $\mathcal{C}$. Now we add the assumptions that these distribution functions have densities $\{f(t,\cdot)\colon\ t\in E\}$ satisfying (5) and (6).

Lemma 4. *Assume for all $t\in E$ that the distribution functions $F(t,x)$ are strictly increasing and continuous, and that their densities $f(t,\cdot)$ satisfy* (6). *If the CLT holds on $\mathcal{C}$, then for every closed subinterval I of* $(0,1)$

$$\lim_{n\to\infty}[\sup_{t\in E,\alpha\in I}|\widehat{\tau}_\alpha^n(t)-\tau_\alpha(t)|]^* = 0. \tag{38}$$

in $\widehat{P}$ probability.

Proof. Since we are assuming (6), fix I a closed subinterval of $(0,1)$, and take $0<\epsilon<\theta(I)$. Then, since we have the CLT over $\mathcal{C}$ with respect to $\widehat{P}$, Lemma 2.10.14 on page 194 of [10] implies there exists $\delta(\epsilon)\in(0,\epsilon c_{I,\theta((I)}/2)$ such that for $\delta\in(0,\delta(\epsilon))$ there is $n_\delta<\infty$ such that $n\geq n_\delta$ implies

$$\widehat{P}(B_n)<\epsilon, \tag{39}$$

where

$$B_n=\{[\sup_{t\in E,x\in\mathbb{R}}|\widehat{F}_n(t,x)-F(t,x)|]^*>\delta\}. \tag{40}$$

In addition, (6) implies we also have

$$\sup_{\alpha\in I,t\in E}(F(t,\tau_\alpha(t)-\epsilon)-\alpha)<-\delta, \tag{41}$$

and

$$\inf_{\alpha\in I,t\in E}(F(t,\tau_\alpha(t)+\epsilon)-\alpha)>\delta. \tag{42}$$

That is, (41) holds by (6) since

$$\sup_{t\in E,\alpha\in I}(F(t,\tau_\alpha(t)-\epsilon)-\alpha)=-\inf_{t\in E,\alpha\in I}\int_{\tau_\alpha(t)-\epsilon}^{\tau_\alpha(t)}f(t,x)dx,$$

and if $0<\delta<\delta(\epsilon)<\frac{\epsilon c_{I,\theta(I)}}{2}, 0<\epsilon<\theta(I)$, we then have

$$\inf_{t\in E,\alpha\in I}\int_{\tau_\alpha(t)-\epsilon}^{\tau_\alpha(t)}f(t,x)dx\geq\epsilon c_{I,\theta(I)}>\delta.$$

Thus on B_n^c, for all $t\in E,\alpha\in I$,

$$F(t,\widehat{\tau}_\alpha^n(t))\geq\widehat{F}_n(t,\widehat{\tau}_\alpha^n(t))-\delta\geq\alpha-\delta,$$

where the second inequality follows by definition of $\widehat{\tau}_\alpha^n(t)$. Combined with (41), on B_n^c this implies that for all $t\in E$, all $\alpha\in I$

$$\widehat{\tau}_\alpha^n(t)\geq\tau_\alpha(t)-\epsilon. \tag{43}$$

Similarly, if $0<\delta<\delta(\epsilon)<\frac{\epsilon c_{I,\theta(I)}}{2}, 0<\epsilon<\theta(I)$, (42) holds by (6), and so on B_n^c we have

$$\tau_\alpha(t)+\epsilon\geq\widehat{\tau}_\alpha^n(t)$$

for all $t\in E,\alpha\in I$. Combining this with (43), on B_n^c we have for all $t\in E$, all $\alpha\in I$, that

$$\tau_\alpha(t)-\epsilon\leq\widehat{\tau}_\alpha^n(t)\leq\tau_\alpha(t)+\epsilon. \tag{44}$$

Hence on B_n^c

$$[\sup_{t\in E,\alpha\in I}|\widehat{\tau}_\alpha^n(t)-\tau_\alpha(t)|]^*\leq\epsilon.$$

Since B_n is measurable, we thus have for $n\geq n_\delta$ that

$$\widehat{P}([\sup_{t\in E,\alpha\in I}|\widehat{\tau}_\alpha^n(t)-\tau_\alpha(t)|]^*>\epsilon)\leq\widehat{P}(B_n)\leq\epsilon.$$

Since $\epsilon>0$ can be taken arbitrarily small, letting $n\to\infty$ implies (38). Thus the lemma is proven. □

Proposition 1. *Assume for all $t \in E$ that the distribution functions $F(t,x)$ are strictly increasing, their densities $f(t,\cdot)$ satisfy* (5) *and* (6), *and the CLT holds on $\mathcal{C}$ with limit $\{G(t,x)\colon\ (t,x) \in E \times \mathbb{R}\}$. Then, for I any closed subinterval of* $(0,1)$ *we have*

$$\Big(\sup_{t\in E,\alpha\in I} |\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))f(t,\tau_\alpha(t)) + \widehat{G}(t,\tau_\alpha(t))|\Big)^* \to 0 \tag{45}$$

in $\widehat{P}$ probability, and therefore the quantile processes

$$\{\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))\ f(t,\tau_\alpha(t))\colon\ n \geq 1\}$$

satisfy the CLT in $\ell_\infty(E \times I)$ with Gaussian limit process $\{\widehat{G}(t,\tau_\alpha(t))\colon\ (t,\alpha) \in E \times I\}$. Moreover, the quantile processes $\{\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))\colon\ n \geq 1\}$ also satisfy the CLT in $\ell_\infty(E \times I)$ with Gaussian limit process $\{\frac{\widehat{G}(t,\tau_\alpha(t))}{f(t,\tau_\alpha(t))}\colon\ (t,\alpha) \in E \times I\}$.

Proof. Applying Theorem 3.6.1 of [2], the first CLT asserted follows immediately from (45) and symmetry of the process $\{\widehat{G}(t,\tau_\alpha(t)) : (t,\alpha) \in E \times I\}$. Hence we next turn to the proof of (45).

First we observe that under the given assumptions, we have (37) holding. Furthermore, since the densities are assumed continuous,

$$F(t,y) - F(t,x) = f(t,x)(y-x) + R(t,x,y)(y-x), \tag{46}$$

where $R(t,x,y) = f(t,\xi(t)) - f(t,x)$, and $\xi(t)$ between x and y is determined by the mean value theorem applied to $F(t,\cdot)$. Of course, $R(t,\cdot,\cdot)$ depends on $F(t,\cdot)$, but we suppress that, and simply note that since $\xi(t)$ is between x and y, we have $|R(t,x,y)| \leq \sup_{u\in[x,y]\cup[y,x]} |f(t,u) - f(t,x)|$. Therefore, for $M > 0$ we have

$$\widehat{P}([\sup_{t\in E,\alpha\in I} |\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))|\frac{f(t,\tau_\alpha(t))}{2}]^* \geq M) \leq a_n(M) + b_n,$$

where

$$\begin{aligned}
a_n(M) &= \widehat{P}(A_{n,1} \cap A_{n,2}),\\
A_{n,1} &= \left\{\left[\sup_{t\in E,\alpha\in I} |\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))|(f(t,\tau_\alpha(t)) + R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t)))\right]^* \geq M\right\},\\
A_{n,2} &= \left\{\left[\sup_{t\in E,\alpha\in I} |R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t))|\right]^* \leq \frac{c_{I,\theta(I)}}{2}\right\},\\
b_n &= \widehat{P}\left(\left[\sup_{t\in E,\alpha\in I} |R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t))|\right]^* > \frac{c_{I,\theta(I)}}{2}\right),
\end{aligned}$$

and $c_{I,\theta(I)} > 0$ is given as in (6). Thus

$$\widehat{P}\left(\left[\sup_{t\in E,\alpha\in I} |\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))|\frac{f(t,\tau_\alpha(t))}{2}\right]^* \geq M\right) \leq \widehat{P}(A_{n,1}) + b_n,$$

and by (46) we also have

$$A_{n,1} = \left\{ \left[\sup_{t\in E,\alpha\in I} |\sqrt{n}(F_t(\widehat{\tau}^n_\alpha(t)) - F_t(\tau_\alpha(t)))| \right]^* \geq M \right\},$$

which implies

$$\begin{aligned} &\widehat{P}\left(\left[\sup_{t\in E,\alpha\in I} |\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))| \frac{f(t,\tau_\alpha(t))}{2} \right]^* \geq M \right) \\ &\quad \leq \widehat{P}\left(\left[\sup_{t\in E,\alpha\in I} |\sqrt{n}(F_t(\widehat{\tau}^n_\alpha(t)) - F_t(\tau_\alpha(t)))| \right]^* \geq M \right) + b_n. \end{aligned} \tag{47}$$

Since $I_t(\alpha) = F_t(F_t^{-1}(\alpha)), \alpha \in (0,1), t \in E$, and $I \subseteq (0,1)$

$$\begin{aligned} &\left[\sup_{t\in E,\alpha\in I} |\sqrt{n}(F_t(\widehat{\tau}^n_\alpha(t)) - F_t(\tau_\alpha(t)))| \right]^* \\ &\quad \leq \left[\sup_{t\in E,\alpha\in[0,1]} |\sqrt{n}[(F_t \circ \widehat{F}^{-1}_{n,t})(\alpha) - I_t(\alpha)] + (\widehat{G}_t \circ F_t^{-1}(\alpha))| \right]^* \\ &\quad\quad + \left[\sup_{t\in E,\alpha\in I} |\widehat{G}_t \circ F_t^{-1}(\alpha))| \right]^*, \end{aligned} \tag{48}$$

and since the process $\{\widehat{G}(t,x) : t \in E, x \in \mathbb{R}\}$ is sample continuous on $E \times \mathbb{R}$ in the semi-metric d given in (1) with Radon support in $\ell_\infty(E \times \mathbb{R})$ we also have

$$\left[\sup_{t\in E,\alpha\in I} |\widehat{G}_t \circ F_t^{-1}(\alpha)| \right]^* = \sup_{t\in E,\alpha\in I} |\widehat{G}_t \circ F_t^{-1}(\alpha)|. \tag{49}$$

Therefore, for every $\epsilon > 0$ and all $n \geq 1$, by combining (37), (48) and(49) we have an $M = M(\epsilon)$ sufficiently large that

$$\widehat{P}\left(\left[\sup_{t\in E,\alpha\in I} |\sqrt{n}(F_t(\widehat{\tau}^n_\alpha(t)) - F_t(\tau_\alpha(t)))| \right]^* \geq M \right) \leq \epsilon. \tag{50}$$

We now turn to showing that

$$\left[\sup_{t\in E,\alpha\in I} |\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))| \right]^* \tag{51}$$

is bounded in $\widehat{P}$ probability. That is, let

$$\lambda(t,\delta) = \sup_{|u-v|\leq\delta} |f(t,u) - f(t,v)|.$$

Then

$$|R(t,x,y)| \leq \lambda(t,|x-y|),$$

and hence by (5) for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| \leq \delta$ implies

$$\sup_{t\in E} |R(t,x,y)| < \epsilon.$$

Therefore, for every $\epsilon \in (0, \frac{c_{I,\theta(I)}}{2})$

$$b_n \leq \widehat{P}\left(\left[\sup_{t\in E,\alpha\in I}|R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t))|\right]^* \geq \epsilon\right) = \widehat{P}^*\left(\sup_{t\in E,\alpha\in I}|R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t))| \geq \epsilon\right), \tag{52}$$

and since

$$\begin{aligned} &\widehat{P}^*\left(\sup_{t\in E,\alpha\in I}|R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t))| \geq \epsilon\right) \qquad (53)\\ &\quad\leq \widehat{P}^*\left(\sup_{t\in E,\alpha\in I}|\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))| \geq \delta\right) = \widehat{P}\left(\left[\sup_{t\in E,\alpha\in I}|\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))|\right]^* \geq \delta\right), \end{aligned}$$

Lemma 4 implies for every $\epsilon \in (0, \frac{c_{I,\theta(I)}}{2})$ that

$$\lim_{n\to\infty} b_n = 0. \tag{54}$$

Combining (47), (50), and (54), we have (51), i.e., $[\sup_{t\in E,\alpha\in I}|\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))|]^*$ is bounded in $\widehat{P}$ probability. Furthermore, we then also have that

$$\left[\sup_{t\in E,\alpha\in I}|\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))||R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t))|\right]^* \tag{55}$$

converges in $\widehat{P}$ probability to zero.

Now, by (37) and (46), we have with $\widehat{P}$ probability one that

$$\lim_{n\to\infty}\left(\sup_{t\in E,\alpha\in I}|\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))[f(t,\tau_\alpha(t)) + R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t))] + \widehat{G}(t,\tau_\alpha(t)|\right)^* = 0, \tag{56}$$

and since

$$\left[\sup_{t\in E,\alpha\in I}|\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))f(t,\tau_\alpha(t)) + \widehat{G}(t,\tau_\alpha(t)|\right]^* \leq u_n + v_n,$$

where

$$u_n \leq \left[\sup_{t\in E,\alpha\in I}|\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))[f(t,\tau_\alpha(t)) + R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t))\right] + \widehat{G}(t,\tau_\alpha(t)|]^*$$

and

$$v_n \leq \left[\sup_{t\in E,\alpha\in I}|\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))R(t,\tau_\alpha(t),\widehat{\tau}^n_\alpha(t))|\right]^*,$$

we have by combining (55) and (56) that

$$\left[\sup_{t\in E,\alpha\in I}|\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t))f(t,\tau_\alpha(t)) + \widehat{G}(t,\tau_\alpha(t)|\right]^* \to 0$$

in $\widehat{P}$ probability. Hence (45) is proven.

To finish the proof it remains to check that the quantile processes

$$\left\{\sqrt{n}(\widehat{\tau}^n_\alpha(t) - \tau_\alpha(t)) \colon\ n \geq 1\right\}$$

also satisfy the CLT in $\ell_\infty(E \times I)$ with Gaussian limit process

$$\left\{ \frac{\widehat{G}(t,\tau_\alpha(t))}{f(t,\tau_\alpha(t))} : \ (t,\alpha) \in E \times I \right\}.$$

Since (45) holds, and by (6) we have the non-random quantity

$$\sup_{t\in E,\alpha\in I} \frac{1}{f(t,\tau_\alpha(t))} < \infty,$$

we thus have

$$\left(\sup_{t\in E,\alpha\in I} \left| \sqrt{n}(\widehat{\tau}_\alpha^n(t) - \tau_\alpha(t)) + \frac{\widehat{G}(t,\tau_\alpha(t))}{f(t,\tau_\alpha(t))} \right| \right)^* \to 0$$

in $\widehat{P}$ probability. The CLT then follows from Theorem 3.6.1 of [2], and that the Gaussian process $\widehat{G}$ is symmetric. Hence the proposition is proven. □

3.3. Proof of Theorem 1

In Proposition 1 we proved the convergence in outer probability of quantities very closely related to those we wish to study. The best we could hope for concerning our original quantities is convergence in distribution, so there is a little work to do to obtain Theorem 1. One might think at this point that this should follow with a wave of the hand, but for the reader's benefit, as well as our own, and to appreciate the use of perfect maps in Theorem 3 we present a complete argument. This finally establishes Theorem 1.

Proof. Recall the notation established at the start of this section in connection with the statement of Theorem 3, and the perfect mappings $g_n : \widehat{\Omega} \to \Omega$ such that $Q = \widehat{P} \circ g_n^{-1}$. In particular, equations (17) to (25) are relevant.

For $u_1,\ldots,u_n \in D(E)$ and $n \geq 1, t \in E, \alpha \in (0,1)$ define

$$k_n(u_1,\ldots,u_n,t,\alpha) = \sqrt{n}\left[\inf\left\{x : \sum_{j=1}^{n} I(u_j(t) \leq x) \geq n\alpha\right\} - \tau_\alpha(t)\right] f(t,\tau_\alpha(t)), \tag{57}$$

where $\tau_\alpha(t) = F_t^{-1}(\alpha)$. Hence setting

$$r_n(t,\alpha,\omega) \equiv k_n(X_1,\ldots,X_n,t,\alpha)(\omega) \equiv k_n(X_1(\cdot,\omega),\ldots,X_n(\cdot,\omega),t,\alpha),$$

we then have

$$\sqrt{n}[F_{n,t}^{-1}(\alpha)(\omega) - \tau_\alpha(t)] f(t,\tau_\alpha(t)) = r_n(t,\alpha,\omega) \tag{58}$$

and

$$\sqrt{n}[\widehat{F}_{n,t}^{-1}(\alpha)(\hat{\omega}) - \tau_\alpha(t)] f(t,\tau_\alpha(t)) = r_n(t,\alpha,g_n(\hat{\omega})). \tag{59}$$

Therefore, for $\hat{\omega} \in \widehat{\Omega}$

$$\widehat{F}_{n,t}^{-1}(\alpha)(\hat{\omega}) = F_{n,t}^{-1}(\alpha)(g_n(\hat{\omega})),$$

and for h bounded on $\ell_\infty(E \times I)$ we have

$$h\left(\sqrt{n}\left[\widehat{F}_{n,t}^{-1}(\alpha)(\hat{\omega}) - \tau_\alpha(t)\right] f(t,\tau_\alpha(t))\right) = (h \circ r_n(t,\alpha,\cdot) \circ g_n)(\hat{\omega}),$$

and hence the upper integrals

$$\begin{aligned}\int_{\widehat{\Omega}}^{*} h(\sqrt{n}[\widehat{F}_{n,t}^{-1}(\alpha)(\hat{\omega}) - \tau_\alpha(t)]f(t,\tau_\alpha(t)))d\widehat{P}(\hat{\omega}) \\ = \int_{\widehat{\Omega}}^{*} (h \circ r_n(t,\alpha,\cdot) \circ g_n)(\hat{\omega})d\widehat{P}(\hat{\omega}) \\ = \int_{\widehat{\Omega}} [(h \circ r_n(t,\alpha,\cdot) \circ g_n)]^*(\hat{\omega})d\widehat{P}(\hat{\omega}) \\ = \int_{\widehat{\Omega}} ([h \circ r_n(t,\alpha,\cdot)]^* \circ g_n)(\hat{\omega})d\widehat{P}(\hat{\omega}),\end{aligned} \tag{60}$$

where the last equality holds since g_n is perfect. Now

$$\begin{aligned}\int_{\widehat{\Omega}} ([h \circ r_n(t,\alpha,\cdot)]^* \circ g_n)(\hat{\omega})d\widehat{P}(\omega) &= \int_{\Omega} [h \circ r_n(t,\alpha,\omega)]^* dQ(\omega) \\ &= \int_{\Omega}^{*} (h \circ r_n)(t,\alpha,\omega)dQ(\omega),\end{aligned}$$

and therefore by (58) and (60), for all h bounded on $\ell_\infty(E \times I)$,

$$\begin{aligned}\int_{\widehat{\Omega}}^{*} h(\sqrt{n}[\widehat{F}_{n,t}^{-1}(\alpha)(\hat{\omega}) - \tau_\alpha(t)]f(t,\tau_\alpha(t)))d\widehat{P}(\hat{\omega}) \\ = \int_{\Omega}^{*} h(\sqrt{n}[F_{n,t}^{-1}(\alpha)(\omega) - \tau_\alpha(t)]f(t,\tau_\alpha(t)))dQ(\omega).\end{aligned} \tag{61}$$

Now the equality in (61) implies that the quantile processes

$$\left\{\sqrt{n}\left[\widehat{F}_{n,t}^{-1}(\alpha)(\hat{\omega}) - \tau_\alpha(t)\right] f(t,\tau_\alpha(t)) : n \geq 1, t \in E, \alpha \in I\right\}$$

satisfy the CLT in $\ell_\infty(E \times I)$ if and only if

$$\left\{\sqrt{n}\left[F_{n,t}^{-1}(\alpha)(\omega) - \tau_\alpha(t)\right] f(t,\tau_\alpha(t)) : n \geq 1, t \in E, \alpha \in I\right\}$$

satisfy the CLT there, and they have the same Gaussian limit, namely

$$\{G(t,\tau_\alpha(t)) : t \in E, \alpha \in I\}.$$

A similar argument implies the quantile processes

$$\left\{\sqrt{n}\left[\widehat{F}_{n,t}^{-1}(\alpha)(\hat{\omega}) - \tau_\alpha(t)\right] : n \geq 1, t \in E, \alpha \in I\right\}$$

satisfy the CLT in $\ell_\infty(E \times I)$ if and only if

$$\left\{\sqrt{n}\left[F_{n,t}^{-1}(\alpha)(\omega) - \tau_\alpha(t)\right] : n \geq 1, t \in E, \alpha \in I\right\}$$

satisfy the CLT there, and they have the same Gaussian limit. Since Proposition 1 implies the Gaussian limit of

$$\left\{\sqrt{n}\left[\widehat{F}_{n,t}^{-1}(\alpha)(\hat{\omega}) - \tau_\alpha(t)\right] : n \geq 1, t \in E, \alpha \in I\right\}$$

is given by $\left\{\frac{\widehat{G}(t,\tau_\alpha(t))}{f(t,\tau_\alpha(t))} : t \in E, \alpha \in I\right\}$, which has the same Radon law on $\ell_\infty(E \times I)$ as $\left\{\frac{G(t,\tau_\alpha(t))}{f(t,\tau_\alpha(t))} : t \in E, \alpha \in I\right\}$, the theorem is proven. □

4. Proof of Theorem 2

It is easily seen that Theorem 2 follows from Theorem 1 once we show the distribution functions $F(t,\cdot)$ of $X_t = Y_t + Z, t \in E$, have strictly positive densities $f(t,\cdot)$ satisfying (5), (6), and the CLT holds on $\mathcal{C}$ with centered Gaussian limit $\{G(t,x)\colon\ (t,x) \in E \times \mathbb{R}\}$. The proof of the conditions on the densities $f(t,x)$ follows from the conditions imposed on the density g of Z, and is our next lemma. Hence, once the lemma is proven it only remains to show that each of the three classes of processes in the assumptions of the theorem satisfy the empirical CLT over $\mathcal{C}$. This will be done through a series of propositions, but we will also indicate a few corollaries that make explicit the breadth of the processes in the classes described prior to the statement of Theorem 2. In particular, we point out some situations where the empirical CLT fails when the process is tied down to be a constant on some subset of E.

Lemma 5. *Let $X_t = Y_t + Z, t \in E$, where Z is a real-valued random variable independent of $\{Y_t : t \in E\}$ having probability density function $g(\cdot)$ which is strictly positive, uniformly bounded, and uniformly continuous on $\mathbb{R}$, and also assume* (16) *holds. Then, the distribution functions $F(t,\cdot)$ of $X_t, t \in E$ have strictly positive densities $f(t,\cdot)$ satisfying* (5), (6), *and* (7).

Proof. If $H_t(x), t \in E$, is the distribution function of Y_t, then X_t has probability density function

$$f(t,x) = \int_{\mathbb{R}} g(x-v)dH_t(v), t \in E.$$

Hence if g, the density of Z, is strictly positive, uniformly bounded, and uniformly continuous on $\mathbb{R}$, then it is easy to check that each of the densities $f(t,\cdot), t \in E$, have the same properties. In particular, (7) is obvious, and we have

$$\lim_{\delta \to 0} \sup_{t \in E} \sup_{|u-v| \le \delta} |f(t,u) - f(t,v)| = 0,$$

which implies (5). To show (6) it then suffices to verify that the densities $f(t,\cdot)$ of $\{X_t : t \in E\}$ satisfy

$$\inf_{t \in E, \alpha \in I} f(t, \tau_\alpha(t)) = c_I > 0 \tag{62}$$

for every closed interval I in $(0,1)$.

Now (62) holds if we show that for any closed subinterval I of $(0,1)$ and all $a > 0$ that

$$\inf_{t \in E, |x| \le a} f(t,x) = c_a > 0 \quad \text{and} \quad \sup_{t \in E, \alpha \in I} |\tau_\alpha(t)| < \infty. \tag{63}$$

First we show the left expression in (63) holds, so take $a > 0$. Then, for every $b > 0$

$$\inf_{t \in E, |x| \le a} f(t,x) \ge \inf_{t \in E} \int_{\mathbb{R}} \inf_{|x| \le a} g(x-v)dH_t(v) \quad \ge \inf_{|u| \le a+b} g(u) \inf_{t \in E} \int_{-b}^{b} dH_t(v),$$

and, since (16) implies

$$\lim_{b\to\infty} \sup_{t\in E} P(|Y_t| \geq b) \leq \lim_{b\to\infty} P\left(\sup_{t\in E} |Y_t| \geq b\right) = 0, \tag{64}$$

there exists $b_0 > 0$ sufficiently large that $\inf_{t\in E} \int_{-b_0}^{b_0} dH_t(v) \geq \frac{1}{2}$. Therefore, we have

$$\inf_{t\in E, |x|\leq a} f(t,x) \geq \frac{1}{2} \inf_{|u|\leq a+b_0} g(u) \equiv c_a > 0.$$

Now we turn to the second term in (63). Since I is a closed interval of $(0,1)$ there is a $\theta \in (0, \frac{1}{2})$ such that $I \subset (\theta, 1-\theta)$ and

$$\sup_{t\in E} P(|\langle t, X\rangle + Z| \geq a) \leq \sup_{t\in E} P\left(|\langle t, X\rangle| \geq \frac{a}{2}\right) + P\left(|Z| \geq \frac{a}{2}\right) \leq \frac{\theta}{2},$$

where the second inequality follows from (64) by taking $a > 0$ sufficiently large. Hence for each $t \in [0,T], \alpha \in I$ we have $\tau_\alpha(t) \in [-a,a]$ and the right term of (63) holds. Thus (62) holds, and the lemma is proven. □

4.1. Gaussian process empirical CLT's over $\mathcal{C}$

Throughout this subsection we assume E is a compact subset of the d-fold product of $[0,T]$, which we denote by $[0,T]^d$, and that $\{X_t : t \in E\}$ is a centered Gaussian process whose L_2-distance d_X satisfies (12). Then, by applying Theorem 6.11 and Corollary 6.12 on pages 144–45 of [1] we have from (12) that $\{X_t : t \in E\}$ has a sample continuous version $\{\tilde{X}_t : t \in E\}$ such that for $s, t \in E$

$$|\tilde{X}_t - \tilde{X}_s| \leq \Gamma ||t-s||_{\mathbb{R}^d}^r, \tag{65}$$

where $\Gamma < \infty$ with probability one, and $0 < r < \gamma$, $\gamma \leq 1$ by (12). Hence, without loss of generality, we may also assume throughout the sub-section that $\{X_t : t \in E\}$ is sample continuous with (65) holding.

Proposition 2. *Let E be a compact subset of $[0,T]^d$, and assume $\{X_t : t \in E\}$ is a sample continuous centered Gaussian process such that (12) holds, and there exists $k_2 < \infty$ and $\beta \in (0,1]$ such that for all $x, y \in \mathbb{R}$*

$$\sup_{t\in E} |F_t(x) - F_t(y)| \leq k_2 |x-y|^\beta. \tag{66}$$

Then, the empirical CLT built from the process $\{X_t : t \in E\}$ holds over $\mathcal{C}$. Moreover, if $\{Y_t : t \in E\}$ is a sample continuous centered Gaussian process such that (12) holds, and Z is a random variable independent of $\{Y_t : t \in E\}$ whose density is uniformly bounded on $\mathbb{R}$, or in $L_p(\mathbb{R})$ for some $p \in (1,\infty)$, then the empirical CLT based on the process $\{X_t : t \in E\}$ holds over $\mathcal{C}$, where $X_t = Y_t + Z, t \in E$.

Proof. Using Theorem 5 of [4] we have the empirical CLT over $\mathcal{C}$, or equivalently in $\ell_\infty(E \times \mathbb{R})$ when we identify points $(t,y) \in E \times \mathbb{R}$ with the sets $C_{t,y} \in \mathcal{C}$, provided we verify the following three conditions:

(I) For some $\beta \in (0,1]$, some $k < \infty$, and all $x, y \in \mathbb{R}$

$$\sup_{t\in E} |F(t,x) - F(t,y)| \le k|x-y|^{\beta}. \tag{67}$$

(II) For all $s, t \in E$

$$|X_t - X_s| \le \Gamma\phi(s,t) \tag{68}$$

and some $\eta > 0$, and all $u \ge u_0$

$$P(\Gamma \ge u) \le u^{-\eta}.$$

(III) For β as in (I), and η as in (II), there exists $\alpha \in (0, \beta/2)$ such that

$$\eta(\frac{1}{\alpha} - \frac{2}{\beta}) \ge 2, \tag{69}$$

and $(\phi(s,t))^{\alpha} \le \rho(s,t)$, where $\rho(s,t)$ is the L_2 distance of a sample bounded, uniformly continuous, centered Gaussian process on (E,ρ), which for later use we denote by $\{\lambda(t) : t \in E\}$.

First we assume $\{X_t : t \in E\}$ is a sample continuous centered Gaussian process such that (12), (65), and (66) hold. Then, applying the Landau-Shepp-Fernique result as in Lemma 2.2.5 of [2], we have exponential decay of the tail probability of Γ in (65), and hence assumptions (I) and (II) hold with η in (68) allowed to be arbitrarily large and $\phi(s,t) = ||t-s||^r_{\mathbb{R}^d}$. If (12) and (65) hold for $\{Y_t : t \in E\}$ and $X_t = Y_t + Z$, where the density of Z is uniformly bounded, or in L_p as indicated, then standard convolution formulas imply (66) holds for $\{X_t : t \in E\}$. In particular, if the density of Z is assumed to be uniformly bounded, then (66) holds with $\beta = 1$, and if it is in $L_p(\mathbb{R})$, then $\beta = 1 - 1/p$ suffices. Therefore, under either assumption on the density of Z, we have assumptions (I) and (II) holding for $\{X_t : t \in E\}$.

Therefore, the conclusions of the proposition hold in either situation provided we verify condition (III). Since $\eta > 0$ can be taken arbitrarily large in (69) and (65), it suffices to show that there is a centered Gaussian process $\{\lambda(t) : t \in E\}$ with L_2-distance $\rho(s,t)$, which is sample bounded and uniformly continuous on (E,ρ), and for some $\alpha \in (0, \frac{\beta}{2})$ we have

$$(||t-s||_{\mathbb{R}^d})^{r\alpha} \le \rho(s,t), s,t \in E. \tag{70}$$

Therefore, we take $\{\lambda_t : t \in E\}$ to be Lévy's θ-fractional Brownian motion on E with

$$\rho(s,t) = \mathbb{E}((\lambda_t - \lambda_s)^2)^{1/2} = ||t-s||^{\theta}_{\mathbb{R}^d}, s,t \in E, 0 < \theta < 1. \tag{71}$$

Hence with $\theta = r\alpha$, since $r < 1$ and $\alpha < 1/2$, we have $\{\lambda_t : t \in E\}$ a sample bounded, uniformly continuous centered Gaussian process on (E,ρ). In fact, it is well known that this process can be taken to be sample continuous on all of $\mathbb{R}^d$ with respect to the distance $\rho(\cdot,\cdot)$, and this can be checked by showing Dudley's metric entropy condition $\int_{0+} (\log N(C_a, u, \rho))^{1/2} du < \infty$ holds for cubes $C_a = \prod_{j=1}^d [-a, a]$ with $a > 0$ arbitrarily large. Hence the proof is complete. □

An immediate application of Proposition 2 is to fractional Brownian motions, which was also obtained in [4].

Corollary 1. *Let $E = [0,T]$, and assume $\{Y_t\colon\ t \in E\}$ is a centered sample continuous γ-fractional Brownian motion for $0 < \gamma < 1$ such that $Y_0 = 0$ with probability one and $\mathbb{E}(Y_t^2) = t^{2\gamma}$ for $t \in E$. Set $X_t = Y_t + Z$, where Z is independent of $\{Y_t\colon\ t \in E\}$, and assume Z has a density that is uniformly bounded on $\mathbb{R}$ or is in $L_p(\mathbb{R})$ for some $p \in (1,\infty)$. Then, the empirical CLT holds over $\mathcal{C}$.*

Proof. The L_2-distance for $\{X_t\colon\ t \in E\}$ is $d_X(s,t) = |s-t|^\gamma$, and hence (12) holds with $k_1 = 1$. Also, (65) holds with $0 < r < \gamma$, and the assumptions on the density of Z then imply (66). Therefore, Proposition 2 applies to complete the proof. □

Our next application of Proposition 2 is to the d-dimensional Brownian sheet. A result for $d = 2$ appeared in [4], but once we have Proposition 2 in hand, the general case follows easily.

Corollary 2. *Let $E = [0,T]^d$ for $d \geq 2$, and assume $\{Y_t\colon\ t \in E\}$ is a centered sample continuous Brownian sheet with covariance function*

$$\mathbb{E}(Y_sY_t) = \prod_{j=1}^{d}(s_j \wedge t_j),\ \ s = (s_1,\ldots,s_d), t = (t_1,\ldots,t_d) \in E. \tag{72}$$

For $t \in E$, let $X_t = Y_t + Z$, where Z is independent of $\{Y_t\colon\ t \in E\}$, and assume Z has a density that is uniformly bounded on $\mathbb{R}$ or is in $L_p(\mathbb{R})$ for some $p \in (1,\infty)$. Then, the empirical CLT based on the process $\{X_t : t \in E\}$ holds over $\mathcal{C} = \{C_{t,x}\colon\ (t,x) \in E \times \mathbb{R}\}$, where in this setting $C_{t,x} = \{z \in D(E)\colon\ z(t) \leq x\}$, and $D(E)$ denotes the continuous functions on E. Moreover, the empirical CLT over $\mathcal{C}$ fails for the base process $\{Y(t) : t \in E\}$.

Proof. First we observe that if $0 \leq s_j \leq t_j \leq T$ for $j = 1,\ldots,d$, then for $d \geq 1$ we have $\prod_{j=1}^d t_j - \prod_{j=1}^d s_j \leq T^{d-1}\sum_{j=1}^d |t_j - s_j|$. This elementary fact is obvious for $d = 1$ with $T^0 = 1$, and for $d \geq 2$ it follows by an easy induction argument. Moreover, the L_2-distance for $\{X_t : t \in E\}$ satisfies

$$d_X^2(s,t) = \left|\prod_{j=1}^{d} t_j + \prod_{j=1}^{d} s_j - 2\prod_{j=1}^{d}(s_j \wedge t_j)\right|,$$

and hence we easily have

$$d_X^2(s,t) \leq dT^{d-1}\sum_{j=1}^{d}|t_j - s_j|\frac{1}{d} \leq dT^{d-1}\left(\sum_{j=1}^{d}|t_j - s_j|^2\frac{1}{d}\right)^{1/2} = d^{1/2}T^{d-1}||t-s||_{\mathbb{R}^d}.$$

Thus $d_X(s,t) \leq T^{\frac{d-1}{2}}d^{1/4}||t-s||_{\mathbb{R}^d}^{1/2}$, and hence (12) holds. Either assumption for the density of Z implies (66) for a suitable β, and thus Proposition 2 applies to show the CLT over $\mathcal{C}$ holds for $\{X(t) : t \in [0,T]^d\}$ holds.

To see why this CLT fails for the base process $\{Y(t) : t \in [0,T]^d\}$, observe that the process $W(r) = Y(r^{1/d}(1,\ldots,1)), 0 \le r \le T^d$, is a Brownian motion with $P(W(0) = 0) = 1$. Thus by Lemma 7 of [4] we have $\{Y(t) : t \in [0,T]^d\}$ fails the CLT over the class of sets $\mathcal{C}_1 = \{C_{r,x} : 0 \le r \le T^d, x \in \mathbb{R}\}$, where $C_{r,x} = \{z \in D(E) : z(r^{1/d}(1,\ldots,1)) \le x\}$. Since $\mathcal{C}_1 \subseteq \mathcal{C}$, it follows from that the CLT for Y over $\mathcal{C}$ must also fail. □

4.2. Compound Poisson process empirical CLT's over $\mathcal{C}$

Here we examine the empirical CLT over $\mathcal{C}$ when our base process is an arbitrary compound Poisson process. This will be done in the next proposition by applying Theorem 3 of [4]. We will see from its proof that the Gaussian process needed for this application can be taken to be a sample continuous Brownian motion, and the space of functions $D(E)$, when $E = [0,T]$, is the space of cadlag functions on $[0,T]$. These examples are somewhat interesting since the sample paths of the base process $\{Y(t)\colon\ t \in [0,T]\}$ have jumps, while those of significance in [4] and the previous subsection were all sample path continuous.

To define the base process in these examples we let $\{N(t)\colon\ 0 \le t < \infty\}$ be a Poisson process with parameter $\lambda \in (0,\infty)$, and jump times $\tau_1, \tau_2, \ldots$. As usual we assume $P(N(0) = 0) = 1$, and that the sample paths $\{N(t)\colon\ 0 \le t \le \infty\}$ are right continuous and nondecreasing. Also, let $\{Y_k\colon\ k \ge 1\}$ be i.i.d. real-valued random variables, independent of $\{N(t)\colon\ 0 \le t \le \infty\}$, and without mass at zero. Then, $Y(t)$ is defined to be zero on $[0,\tau_1)$, Y_1 on $[\tau_1,\tau_2)$, and $Y_1 + \cdots + Y_k$ on $[\tau_k, \tau_{k+1})$ for $k \ge 1$.

Proposition 3. *The empirical process built from i.i.d. copies of the compound Poisson process $\{Y(t)\colon\ t \in E\}$ with parameter $\lambda \in (0,\infty)$ and $E = [0,T]$ satisfies the empirical CLT over $\mathcal{C}$. In addition, if $X(t) = Y(t) + Z, t \in [0,T]$, then the process $\{X(t) : t \in [0,T]\}$ also satisfies the empirical CLT over $\mathcal{C}$.*

Proof. Using Theorem 3 of [4] it suffices to show $\{Y(t)\colon\ t \in E\}$ satisfies the L condition of [4] when the Gaussian process involved is Brownian motion and the ρ distance is a multiple of standard Euclidean distance on $[0,T]$. Since the distribution function of $Y(t)$ is not necessarily continuous, the L-condition involves distributional transforms of the distribution functions $F(t,x) = P(Y(t) \le x)$ denoted by $\tilde{F}_t(x)$. They are defined for $t \in E, x \in \mathbb{R}$ as

$$\tilde{F}_t(x) = F(t,x^-) + V(F(t,x) - F(t,x^-)),$$

where V is a uniform random variable on $[0,1]$ independent of the process

$$\{Y(t)\colon\ t \in E\}.$$

To verify the L-condition for the Y process, let $\{H(t)\colon\ 0 \le t < \infty\}$ be a sample continuous Brownian motion with $P(H(0) = 0) = 1$ satisfying

$$\rho^2(s,t) = \mathbb{E}((H(s) - H(t))^2) = 4(\lambda \vee 1)|t-s|.$$

Then, for $\epsilon > 0$

$$\begin{aligned}\Lambda &\equiv \sup_{t\in[0,T]} P\Big(\sup_{\{s:\ \rho(s,t)\le\epsilon\}} |\tilde F_t(Y(s)) - \tilde F_t(Y(t))| > \epsilon^2\Big)\\ &\le \sup_{t\in[0,T]} P\Big(\sup_{\{s:\ \rho(s,t)\le\epsilon\}} |Y(s)-Y(t)| > 0\Big)\\ &= \sup_{t\in[0,T]}\Big[1 - P\Big(\sup_{\{s:\ \rho(s,t)\le\epsilon\}} |Y(s)-Y(t)| = 0\Big)\Big].\end{aligned}$$

Since $Y(s) - Y(t) = 0$ whenever $N(s) - N(t) = 0$, and for $t \in [0,T]$ fixed

$$\begin{aligned}&\Big\{\sup_{\{s:\ \rho(s,t)\le\epsilon\}} |N(s)-N(t)| = 0\Big\}\\ &\quad = \Big\{N\Big(\Big(t+\frac{\epsilon^2}{4(\lambda\vee 1)}\Big)\wedge T\Big) - N\Big(\Big(t-\frac{\epsilon^2}{4(\lambda\vee 1)}\Big)\vee 0\Big) = 0\Big\},\end{aligned}$$

it follows that

$$\begin{aligned}&P\Big(N\Big(\Big(t+\frac{\epsilon^2}{4(\lambda\vee 1)}\Big)\wedge T\Big) - N\Big(\Big(t-\frac{\epsilon^2}{4(\lambda\vee 1)}\Big)\vee 0\Big) = 0\Big)\\ &\quad = P\Big(\sup_{\{s:\ \rho(s,t)\le\epsilon\}} |N(s)-N(t)| = 0\Big) \le P\Big(\sup_{\{s:\ \rho(s,t)\le\epsilon\}} |Y(s)-Y(t)| = 0\Big).\end{aligned}$$

Now

$$P\Big(N\Big(\Big(t+\frac{\epsilon^2}{4(\lambda\vee 1)}\Big)\wedge T\Big) - N\Big(\Big(t-\frac{\epsilon^2}{4(\lambda\vee 1)}\Big)\vee 0\Big) = 0\Big) \ge \exp\Big\{-\frac{\lambda\epsilon^2}{2(\lambda\vee 1)}\Big\},$$

and hence for $0 < \epsilon < \epsilon_0$ we have $\Lambda \le 1 - \exp\left\{-\frac{\epsilon^2}{2}\right\} \le \epsilon^2$. Taking L suitably large we have for all $\epsilon > 0$ that $\Lambda \le L\epsilon^2$, and hence the L-condition holds for the compound Poisson process Y.

To complete the proof we let $F_{X,t}(y) = P(X_t \le y)$, and verify the L-condition for $\{X_t : t \in [0,T]\}$. Since the increments of X and Y are identical, by repeating the above, the proof is the same as before. □

Remark 4. It is interesting to note that in the previous proposition we proved the base process $\{Y_t : t \in [0,T]\}$ satisfied the empirical CLT, and using the proof of that fact, we then showed $X_t = Y_t + Z, t \in [0,T]$, also satisfied the empirical CLT. This is the reverse of the argument in Propositions 2 and 4, but since we are interested in empirical quantile CLTs for such data, some smoothing is eventually necessary, i.e., in Theorem 2 our assumptions require that the smoothed process satisfies the empirical CLT over $\mathcal{C}$ and that its densities are sufficiently smooth.

4.3. Empirical process CLT's over $\mathcal{C}$ for other independent increment processes and martingales

The processes we study here are either martingales, or stationary independent increment processes. They form class (ii) that appeared prior to the statement of Theorem 2, and Remark 1 indicates some specific examples and properties of processes in this class. The base processes are $\{Y(t) : t \geq 0\}$, and $X(t) = Y(t) + Z, t \geq 0$, where Z is a real random variable independent of $\{Y(t) : t \geq 0\}$ with density $g(\cdot)$ such that

$$k = \sup_{x \in \mathbb{R}} |g(x)| < \infty \text{ or } g \in L_a(\mathbb{R}) \text{ for some } a > 1. \tag{73}$$

Then, g is uniformly bounded implies

$$\sup_{t \in E} |F_t(x) - F_t(y)| \leq k|x - y|, x, y \in \mathbb{R}, \tag{74}$$

and if $g \in L_a(\mathbb{R})$ we have a $\tilde{k} < \infty$

$$\sup_{t \in E} |F_t(x) - F_t(y)| \leq \tilde{k}|x - y|^{1-\frac{1}{a}}, x, y \in \mathbb{R}. \tag{75}$$

Proposition 4. *Let $E = [0, T]$, and assume $\{Y(t) : t \geq 0\}$ is a stochastic process whose sample paths are right continuous, with left-hand limits on $[0, \infty)$, and satisfying $P(Y(0) = 0) = 1$. Furthermore, assume $\{Y(t) \colon t \in E\}$ is a martingale whose L_1-increments are Lip-β for some $\beta \in (0, 1]$, or a stationary independent increments process satisfying (13) for some $p \in (0, 1)$ and $\beta \in (0, 1]$. Let $X(t) = Y(t) + Z, t \geq 0$, where Z is a random variable independent of $\{Y(t) : t \geq 0\}$ and having density $g(\cdot)$ on $\mathbb{R}$ satisfying (73). Then, the empirical process built from i.i.d. copies of $\{X(t) \colon t \in E\}$ satisfies the CLT over $\mathcal{C}$.*

Proof. Let $\rho(s, t) = |s - t|^\theta, 0 < \theta < 1$. Then, ρ is the L_2-distance of a θ-fractional Brownian motion on E, and the proposition follows from Theorem 3 of [4] provided we verify the L-condition for $\{X(t) \colon t \in E\}$ with respect to ρ and an appropriately chosen θ. That is, since the distribution functions $F_t(\cdot)$ have a density, they are continuous, and hence it suffices to show for an appropriate $\theta > 0$ there is a constant $L < \infty$ such that for every $\epsilon > 0$

$$\sup_{t \in E} P\Big(\sup_{\{s : s \in E, \rho(s,t) \leq \epsilon\}} |F_t(X_s) - F_t(X_t)| > \epsilon^2\Big) \leq L\epsilon^2. \tag{76}$$

We prove the L-condition holds assuming the density g of Z is uniformly bounded, and hence we have (74) holding. The proof when $g \in L_a(\mathbb{R})$ is essentially the same, only the algebra changes, and hence the details are left to the reader.

First we examine the situation when $\{Y(t) : t \geq 0\}$ is a martingale satisfying (13) with $p = 1$ and some $\beta \in (0, 1]$. Applying (74) to (76) we then have

$$\sup_{t \in E} P\Big(\sup_{\{s : s \in E, \rho(s,t) \leq \epsilon\}} |F_t(X_s) - F_t(X_t)| > \epsilon^2\Big) \leq A_\epsilon + B_\epsilon, \tag{77}$$

where

$$A_\epsilon = \sup_{t \in E} P\Big(\sup_{\{s : s \in [t, (t + \epsilon^{1/\theta}) \wedge T]\}} |X_s - X_t| > \frac{\epsilon^2}{2k}\Big),$$

and

$$B_\epsilon = \sup_{t \in E} P\Big(\sup_{\{s : s \in [(t - \epsilon^{1/\theta}) \vee 0, t]\}} |X_s - X_t| > \frac{\epsilon^2}{2k}\Big).$$

Now

$$A_\epsilon = \sup_{t \in E} P\Bigg(\sup_{\{s: s \in [t,(t+\epsilon^{1/\theta}) \wedge T]\}} |Y_s - Y_t| > \frac{\epsilon^2}{2k} \Bigg),$$

and hence Doob's martingale maximal inequality implies

$$A_\epsilon \le \sup_{t \in E} 2k\epsilon^{-2} \mathbb{E}\Big(|Y_{(t+\epsilon^{1/\theta}) \wedge T} - Y_t| \Big) \le 2kC\epsilon^{-2+\frac{\beta}{\theta}}, \tag{78}$$

where the last inequality follows from (13) with $p = 1$. We also have

$$\begin{aligned} B_\epsilon \le & \sup_{t \in E} P\left(|Y_{(t-\epsilon^{1/\theta}) \vee 0} - Y_t| > \frac{\epsilon^2}{4k} \right) \\ & + \sup_{t \in E} P\Bigg(\sup_{\{s: s \in [(t-\epsilon^{1/\theta}) \vee 0, t]\}} |Y_s - Y_{(t-\epsilon^{1/\theta}) \vee 0}| > \frac{\epsilon^2}{4k} \Bigg), \end{aligned}$$

and using Markov's inequality, the martingale maximal inequality, and (66) with $p = 1$ as before, we have

$$B_\epsilon \le 8kC\epsilon^{-2+\frac{\beta}{\theta}}. \tag{79}$$

Combining (77),(78), and (79) we have

$$\sup_{t \in E} P\Bigg(\sup_{\{s: s \in E, \rho(s,t) \le \epsilon\}} |F_t(X_s) - F_t(X_t)| > \epsilon^2 \Bigg) \le 10kC\epsilon^{-2+\frac{\beta}{\theta}}. \tag{80}$$

Given our assumption that (13) holds with $p = 1$ and some $\beta \in (0,1]$, we take $\theta = \frac{\beta}{4}$, and hence (79) implies we have the L-condition in (76) with $L = 10kC < \infty$.

Now we assume $\{Y(t) : t \ge 0\}$ is a process with stationary independent increments satisfying (13) with $p \in (0,1)$ and some $\beta \in (0,1]$. Applying (74) to (76) we again have (77), and as before

$$A_\epsilon = \sup_{t \in E} P\Bigg(\sup_{\{s: s \in [t,(t+\epsilon^{1/\theta}) \wedge T]\}} |Y_s - Y_t| > \frac{\epsilon^2}{2k} \Bigg), \tag{81}$$

Since $\{Y(t) : t \in E\}$ is a process with stationary independent increments and cadlag sample paths, an application of Montgomery-Smith's maximal inequality in [5] implies

$$A_\epsilon \le 3 \sup_{t \in E} P\Bigg(|Y_{(t+\epsilon^{1/\theta}) \wedge T} - Y_t| > \frac{\epsilon^2}{20k} \Bigg).$$

This maximal inequality is stated for sequences of i.i.d. random variables, but since $\{Y(t) : t \in E\}$ is a process with stationary independent increments and cadlag sample paths, for any integer n we can partition any subinterval I of E into 2^n equal subintervals and apply [5] to the partial sums formed from increments over each of these subintervals. One can add auxiliary i.i.d. increments to form a sequence, but that is unnecessary since for given $n \ge 1$ we need only work with the partial sums of the 2^n increments of that partition. We then use [5] for an upper bound, and then pass via an increasing limit to what is needed, i.e., the desired upper bound is fixed, and hence is an upper bound for the limit.

Thus by Markov's inequality, and our assumption of (13), we have

$$A_\epsilon \le 3(20k\epsilon^{-2})^p \sup_{t\in E} \mathbb{E}\left(|Y_{(t+\epsilon^{1/\theta})\wedge T} - Y_t|^p\right) \le 3C(20k\epsilon^{-2})^p \epsilon^{\beta/\theta}. \tag{82}$$

We also have

$$B_\epsilon \le \sup_{t\in E} P\bigg(|Y_{(t-\epsilon^{1/\theta})\vee 0} - Y_t| > \frac{\epsilon^2}{4k}\bigg) + \sup_{t\in E} P\bigg(\sup_{\{s:s\in[(t-\epsilon^{1/\theta})\vee 0,t]\}} |Y_s - Y_{(t-\epsilon^{1/\theta})\vee 0}| > \frac{\epsilon^2}{4k}\bigg),$$

and using Montgomery-Smith's maximal inequality again we have

$$B_\epsilon \le 4\sup_{t\in E} P\left(|Y_{(t-\epsilon^{1/\theta})\vee 0} - Y_t| > \frac{\epsilon^2}{40k}\right).$$

Thus by Markov's inequality and (13)

$$B_\epsilon \le 4(40k\epsilon^{-2})^p \sup_{t\in E} \mathbb{E}\left(|Y_{(t+\epsilon^{1/\theta})\wedge T} - Y_t|^p\right) \le 4C(40k\epsilon^{-2})^p \epsilon^{\beta/\theta}. \tag{83}$$

Combining (77), (82) and (83) we have

$$\sup_{t\in E} P\bigg(\sup_{\{s:s\in E,\rho(s,t)\le\epsilon\}} |F_t(X_s) - F_t(X_t)| > \epsilon^2\bigg) \le 7C(40k)^p \epsilon^{\frac{\beta}{\theta}-2p},$$

and hence the L-condition holds with $L = 7C(40k)^p$ provided $\theta = \frac{\beta}{2+2p}$. □

Corollary 3. *Let $E = [0,T]$, and assume $\{Y(t) : t \ge 0\}$ is a strictly stable process of index $r \in (0,2]$ with stationary independent increments, cadlag sample paths on $[0,\infty)$, and such that $P(Y(0) = 0) = 1$. Let $X(t) = Y(t) + Z, t \ge 0$, where Z is a random variable independent of $\{Y(t) : t \ge 0\}$ and having density $g(\cdot)$ on $\mathbb{R}$ satisfying (73). Then, the empirical process built from i.i.d. copies of $\{X(t) : t \in E\}$ satisfies the CLT over $\mathcal{C}$. Moreover, except for the degenerate cases when $r = 1$ and $\{Y(t) : t \ge 0\}$ is pure drift, or $Y(t)$ is degenerate at zero for all $t \in E$, the empirical CLT over $\mathcal{C}$ fails for these $\{Y(t) : t \in E\}$.*

Proof. The assertions about the CLT holding are immediate consequences of Proposition 4 once we check that $\{Y(t) : t \ge 0\}$ satisfies (13). This follows from Remark 1, and hence this part of the proof is established.

To show that the CLT fails for the strictly stable stationary independent increment processes $\{Y(t) : t \ge 0\}$ specified follows from an application of the Kolmogorov zero-one law, and the fact that $Y(t)$ is non-degenerate and strictly stable implies $Y(t)$ has a probability density for each $t > 0$. The case $r = 2$ was previously established in [4] using a law of the iterated logarithm argument, which also applied to all fractional Brownian motions.

Now fix $n \ge 1$, and let $Y_1, \dots, Y_n$ be independent copies of Y. Let $\mathbb{Q}$ denote the rational numbers and let $\mathcal{C}_{\mathbb{Q}}$ denote the countable subclass of $\mathcal{C}$ given by

$\mathcal{C}_{\mathbb{Q}} = \{C_{t,y} \in \mathcal{C} : t, y \in \mathbb{Q}\}$. Then,

$$P(\text{card}\{Y_1(t), \dots, Y_n(t)\} = n \text{ for all } t \in \mathbb{Q} \cap (0, \infty)) = 1, \tag{84}$$

and as in the proof of Lemma 7 in [4], to show the empirical CLT fails for $\{Y(t) : t \in E\}$ it suffices to show that

$$P(\Delta^{\mathcal{C}_{\mathbb{Q}}}(Y_1, \dots, Y_n) = 2^n) = 1, \tag{85}$$

where $\Delta^{\mathcal{C}_{\mathbb{Q}}}(Y_1, \dots, Y_n) = \text{ card}\{C \cap \{Y_1, \dots, Y_n\} : C \in \mathcal{C}_{\mathbb{Q}}\}$.

Therefore, a first step is to show for every $r, 0 \leq r \leq n$, and $\{j_1, \dots, j_r\} \subseteq \{1, \dots, n\}$ that

$$P(\{Y_{j_1}, \dots, Y_{j_r}\} \in \{C \cap \{Y_1, \dots, Y_n\} : C \in \mathcal{C}_{\mathbb{Q}}\}) = 1. \tag{86}$$

Hence, fix $\{j_1, \dots, j_r\} \subseteq \{1, 2, \dots, n\}$ and define

$$\Omega(j_1, \dots, j_r) = \{\omega : I_m \text{ holds for infinitely many } m\}$$

where I_m is the condition

$$\max_{1 \leq k \leq r} Y_{j_k}\left(\frac{1}{m}, \omega\right) < \min_{k \notin \{j_1, \dots, j_r\}} Y_k\left(\frac{1}{m}, \omega\right), m = 1, 2, \dots. \tag{87}$$

Now let

$$H_k = \left(Y_1\left(\frac{1}{k}\right) - Y_1\left(\frac{1}{k+1}\right), \dots, Y_n\left(\frac{1}{k}\right) - Y_n\left(\frac{1}{k+1}\right)\right), k \geq 1,$$

and set $\mathcal{E} = \cap_{m=1}^{\infty} \sigma(H_m, H_{m+1}, \dots)$. Then, $\mathcal{E}$ is the tail sigma field for the independent random vectors $\{H_k : k \geq 1\}$, and since

$$Y_i\left(\frac{1}{m}\right) = \sum_{k=m}^{\infty}\left(Y_i\left(\frac{1}{k}\right) - Y_i\left(\frac{1}{k+1}\right)\right),$$

for $m \geq 1$ and $i = 1, \dots, n$, we have $\Omega(j_1, \dots, j_r) \in \mathcal{E}$ for all $r, 1 \leq r \leq n$, and $\{j_1, \dots, j_r\} \subseteq \{1, \dots, n\}$.

Therefore, Kolmogorov's zero-one law implies $P(\Omega(j_1, \dots, j_r)) = 0$ or 1. Since the coordinate processes are i.i.d. these sets all have the same probability for each fixed r and subset $\{j_1, \dots, j_r\}$. Moreover, there are finitely many such sets, and if they all have probability zero, then there is a set of ω's of probability one with $\tau_r(\omega) \uparrow \infty$ such that for all $m \geq \tau_r(\omega)$ we have

$$\max_{1 \leq k \leq r} Y_{j_k}\left(\frac{1}{m}, \omega\right) \geq \min_{k \notin \{j_1, \dots, j_r\}} Y_k\left(\frac{1}{m}, \omega\right) \tag{88}$$

for all choices of $\{j_1, \dots, j_r\}$. Call this set Ω_0. Using (84) there is also a universal set Ω_1 of probability one so that for all $t \in (0, T] \cap \mathbb{Q}$ the numbers $\{Y_1(t, \omega), \dots, Y_n(t, \omega)\}$ are all distinct. Hence for each $\omega \in \Omega_0 \cap \Omega_1$ and $m \geq \tau_r(\omega)$ there must be r strictly smallest values among $\{Y_1(t), \dots, Y_n(t)\}$ for each integer $r \in \{1, \dots, n\}$. Thus we arrive at a contradiction since this violates the previous inequality (88) for some one of the subsets $\{j_1, \dots, j_r\}$.

Hence (86) holds for every $r \in \{1, \ldots, n\}$, and all $(2^n - 1)$ non-empty subsets of $\{Y_1, \ldots, Y_n\}$ are in $\{C \cap \{Y_1, \ldots, Y_n\} : C \in \mathcal{C}_{\mathbb{Q}}\}$ with probability one. To get the empty set with probability one is trivial, i.e., the sample functions are cadlag on $[0, T]$, and the choice of x in $C_{t,x} \in \mathcal{C}_{\mathbb{Q}}$ can be taken arbitrarily negative. Hence (85) holds, and the corollary is proven. □

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Asymptotic Properties for Linear Processes of Functionals of Reversible or Normal Markov Chains

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Abstract. In this paper we study the asymptotic behavior of linear processes having as innovations mean zero, square integrable functions of stationary reversible or normal Markov chains. In doing so we shall preserve the generality of coefficients assuming only that they are square summable. In this way we include in our study the long range dependence case. The only assumption imposed on the innovations for reversible Markov chains is the absolute summability of their covariances. Besides the central limit theorem we also study the convergence to fractional Brownian motion. The proofs are based on general results for linear processes with stationary innovations that have interest in themselves.

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1. Introduction

Let $(\xi_i)_{i\in\mathbb{Z}}$ be a stationary sequence of random variables on a probability space $(\Omega, \mathcal{K}, \mathbb{P})$ with finite second moment and zero mean ($\mathbb{E}\xi_0 = 0$). Let $(a_i)_{i\in\mathbb{Z}}$ be a sequence of real numbers such that $\sum_{i\in\mathbb{Z}} a_i^2 < \infty$ and denote by

$$X_k = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j \ , \ S_n(X) = S_n = \sum_{k=1}^{n} X_k, \tag{1.1}$$

$$b_{n,j} = a_{j+1} + \cdots + a_{j+n} \quad \text{and} \quad b_n^2 = \sum_{j=-\infty}^{\infty} b_{n,j}^2.$$

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The linear process $(X_k)_{k\in\mathbb{Z}}$ is widely used in a variety of applied fields. It is properly defined for any square summable sequence $(a_i)_{i\in\mathbb{Z}}$ if and only if the stationary sequence of innovations $(\xi_i)_{i\in\mathbb{Z}}$ has a bounded spectral density. In general, the covariances of $(X_k)_{k\in\mathbb{Z}}$ might not be summable so that the linear process might exhibit long range dependence.

An important theoretical question with numerous practical implications is to prove stability of the central limit theorem under formation of linear sums. By this we understand that if $\sum_{i=1}^{n}\xi_i/\sqrt{n}$ converges in distribution to a normal variable the same holds for $S_n(X)$ properly normalized. This problem was first studied in the literature by Ibragimov (1962) who proved that if $(\xi_i)_{i\in\mathbb{Z}}$ are i.i.d. centered with finite second moments, then $S_n(X)/b_n$ satisfies the central limit theorem (CLT). The extra condition of finite second moment was removed by Peligrad and Sang (2011). The central limit theorem for $S_n(X)/b_n$ for the case when the innovations are square integrable martingale differences was proved in Peligrad and Utev (1997) and (2006-a), where an extension to generalized martingales was also given.

On the other hand, motivated by applications to unit root testing and to isotonic regression, a related question is to study the limiting behavior of $S_{[nt]}/b_n$ (here and throughout the paper $[x]$ denotes the integer part of x). The first results for i.i.d. random innovations go back to Davydov (1970), who established convergence to fractional Brownian motion. Extensions to dependent settings under certain projective criteria can be found for instance in Wu and Min (2005) and Dedecker et al. (2011), among others.

In this paper we shall address both these questions of CLT and convergence to fractional Brownian motion for linear processes with functions of reversible or normal Markov chains innovations.

Kipnis and Varadhan (1986) considered partial sums S_n (where $a_0 = 1$, and 0 elsewhere) of an additive functional zero mean of a stationary reversible Markov chain and showed that the convergence of $var(S_n)/n$ implies convergence of $\{S_{[nt]}/\sqrt{n},\, 0 \le t \le 1\}$ to the Brownian motion. There is a considerable number of papers that further extend and apply this result to infinite particle systems, random walks, processes in random media, Metropolis-Hastings algorithms. Among others, Kipnis and Landim (1999) considered interacting particle systems, Tierney (1994) discussed the applications to Markov Chain Monte Carlo. Liming Wu (1999) studied the law of the iterated logarithm.

Our first result will show that under the only assumption of absolute summability of covariances of innovations, the partial sums of the linear process $S_n(X)/b_n$ satisfies the central limit theorem provided $b_n \to \infty$. If we only assume the convergence of $var(S_n)/n$ we can also treat a related linear process.

Furthermore, we shall also establish convergence to the fractional Brownian motion under a necessary regularity condition imposed to b_n^2. For a Hurst index larger than 1/2 we obtain a full blown invariance principle. This is not possible without imposing additional conditions for a Hurst index smaller than or equal to 1/2. However we can still get the convergence of finite dimensional distributions.

For a Hurst index of 1/2 we shall also consider the short memory case, when the sequence of constants is absolutely summable, and obtain convergence to the Brownian motion.

In this paper, besides a condition on the covariances, no other assumptions such as irreducibility or aperiodicity are imposed to the reversible Markov chain.

The proofs are based on a result of Peligrad and Utev (2006-a) concerning the asymptotic behavior of a class of linear processes and spectral calculus. In addition, in Section 4.1 we develop several asymptotic results for a class of linear processes with stationary innovations, which is not necessarily Markov or reversible. The innovation satisfy a martingale-like condition. These results have interest in themselves and can be applied to treat other classes of linear processes.

Applications are given to a Metropolis Hastings Markov chain, to instantaneous functions of a Gaussian process and to random walks on compact groups.

Our paper is organized as follows: Section 2 contains the definitions, a short background of the problem and the results. Applications are discussed in Section 3. Section 4 is devoted to the proofs. The Appendix contains some technical results.

2. Definitions, background and results

We assume that $(\gamma_n)_{n\in\mathbb{Z}}$ is a stationary Markov chain defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with values in a general state space $(S,\mathcal{A})$. The marginal distribution is denoted by $\pi(A)=\mathbb{P}(\gamma_0\in A)$. Assume that there is a regular conditional distribution for γ_1 given γ_0 denoted by $Q(x,A)=\mathbb{P}(\gamma_1\in A|\,\gamma_0=x)$. Let Q also denotes the Markov operator acting via $(Qg)(x)=\int_S g(s)Q(x,ds)$. Next, let $\mathbb{L}_0^2(\pi)$ be the set of measurable functions on S such that $\int g^2 d\pi<\infty$ and $\int g d\pi=0$. If $g,h\in\mathbb{L}_0^2(\pi)$, the integral $\int_S g(s)h(s)d\pi$ will sometimes be denoted by $<g,h>$.

For some function $g\in\mathbb{L}_0^2(\pi)$, let

$$\xi_i=g(\gamma_i),\ S_n(\xi)=\sum_{i=1}^n \xi_i,\ \sigma_n(g)=(\mathbb{E}S_n^2(\xi))^{1/2}. \tag{2.1}$$

Denote by $\mathcal{F}_k$ the σ-field generated by γ_i with $i\le k$ and by $\mathcal{I}$ the invariant σ-field.

For any integrable random variable X we denote $\mathbb{E}_k X=\mathbb{E}(X|\mathcal{F}_k)$. With this notation, $\mathbb{E}_0\xi_1=Qg(\gamma_0)=\mathbb{E}(\xi_1|\gamma_0)$. We denote by $||X||_p$ the norm in $\mathbb{L}^p(\Omega,\mathcal{F},\mathbb{P})$.

The Markov chain is called reversible if $Q=Q^*$, where Q^* is the adjoint operator of Q. In this setting, the condition of reversibility is equivalent to requiring that (γ_0,γ_1) and (γ_1,γ_0) have the same distribution. Equivalently

$$\int_A Q(\omega,B)\pi(d\omega)=\int_B Q(\omega,A)\pi(d\omega)$$

for all Borel sets $A, B \in \mathcal{A}$. The spectral measure of Q with respect to g is concentrated on $[-1,1]$ and will be denoted by ρ_g. Then

$$\mathbb{E}(Q^m g(\gamma_0) Q^n g(\gamma_0)) = \langle Q^m g, Q^n g\rangle = \int_{-1}^{1} t^{n+m} \rho_g(dt).$$

Kipnis and Varadhan (1986) assumed that

$$\lim_{n\to\infty} \frac{\sigma_n^2(g)}{n} = \sigma_g^2 \tag{2.2}$$

and proved that for any reversible ergodic Markov chain defined by (1.1) this condition implies

$$W_n(t) = \frac{S_{[nt]}(\xi)}{\sqrt{n}} \Rightarrow |\sigma_g| W(t), \tag{2.3}$$

where $W(t)$ is the standard Brownian motion, $\Rightarrow$ denotes weak convergence.

As shown by Kipnis and Varadhan (1986, relation 1.1) condition (2.2) is equivalent to

$$\int_{-1}^{1} \frac{1}{1-t} \rho_g(dt) < \infty, \tag{2.4}$$

and then

$$\sigma_g^2 = \int_{-1}^{1} \frac{1+t}{1-t} \rho_g(dt).$$

We shall establish the following central limit theorem:

Theorem 2.1. *Assume that $(\xi_j)_{j\in\mathbb{Z}}$ is defined by (2.1) and $Q = Q^*$. Define (X_k), S_n and b_n as in (1.1). Assume that $b_n \to \infty$ as $n \to \infty$ and*

$$\sum_{j\geq 0} |\mathrm{cov}(\xi_0, \xi_j)| < \infty. \tag{2.5}$$

Then, there is a nonnegative random variable η measurable with respect to $\mathcal{I}$ such that $n^{-1}\mathbb{E}((\sum_{k=1}^{n} \xi_k)^2 | \mathcal{F}_0) \to \eta$ in $\mathbb{L}_1(\Omega, \mathcal{F}, \mathbb{P})$ as $n \to \infty$ and $\mathbb{E}\eta = \sigma_g^2$. In addition

$$\lim_{n\to\infty} \frac{\mathrm{Var}(S_n(X))}{b_n^2} = \sigma_g^2$$

and

$$\frac{S_n(X)}{b_n} \Rightarrow \sqrt{\eta}\, N \ \text{as } n \to \infty, \tag{2.6}$$

where N is a standard normal variable independent on η. Moreover if the sequence $(\xi_i)_{i\in\mathbb{Z}}$ is ergodic the central limit theorem in (2.6) holds with $\eta = \sigma_g^2$.

It should be noted that under the conditions of this theorem σ_g^2 also has the following interpretation: the stationary sequence $(\xi_i)_{i\in\mathbb{Z}}$ has a continuous spectral density $f(x)$ and $\sigma_g^2 = 2\pi f(0)$.

In order to present the functional form of the CLT we introduce a regularity assumption which is necessary for this type of result. We denote by $D([0,1])$ the space of functions defined on $[0, 1]$ which are right continuous and have left-hand limits at any point.

Definition 2.2. We say that a positive sequence $(b_n^2)_{n\geq 1}$ is regularly varying with exponent $\beta > 0$ if for any $t \in]0, 1]$,

$$\frac{b_{[nt]}^2}{b_n^2} \to t^\beta \text{ as } n \to \infty. \tag{2.7}$$

We shall separate the case $\beta \in]1, 2]$ from the case $\beta \in]0, 1]$.

Theorem 2.3. *Assume that the conditions of Theorem* 2.1 *are satisfied and in addition b_n^2, defined by* (1.1), *is regularly varying with exponent β for a certain $\beta \in]1, 2]$. Then, the process $\{b_n^{-1} S_{[nt]}(X), t \in [0, 1]\}$ converges in $D([0, 1])$ to $\sqrt{\eta} W_H$ where W_H is a standard fractional Brownian motion independent of η with Hurst index $H = \beta/2$.*

The case $\beta \in]0, 1]$ is more delicate. For this case we only give the convergence of the finite dimensional distributions since there are counterexamples showing that the tightness might not hold without additional assumptions. As a matter of fact, for $\beta = 1$, it is known from counterexamples given in Wu and Woodroofe (2004) and also in Merlevède and Peligrad (2006) that the weak invariance principle may not be true for the partial sums of the linear process with i.i.d. square integrable innovations.

Theorem 2.4. *Assume that the conditions of Theorem* 2.1 *are satisfied and in addition b_n^2 is regularly varying with exponent β for a certain $\beta \in]0, 1]$. Then the finite dimensional distributions of $\{b_n^{-1} S_{[nt]}(X), t \in [0, 1]\}$ converges to the corresponding ones of $\sqrt{\eta} W_H$, where W_H is a standard fractional Brownian motion independent of η with Hurst index $H = \beta/2$.*

In the context of Theorems 2.3 and 2.4, condition (2.7) is necessary for the conclusion of this theorem (see Lamperti, 1962). This condition has been also imposed by Davydov (1970) for studying the weak invariance principle of linear processes with i.i.d. innovations.

The following theorem is obtained under condition (2.2).

Theorem 2.5. *Assume that (ξ_j) is defined by* (2.1) *and condition* (2.2) *is satisfied. Define*

$$X_k' = \sum_{j=-\infty}^{\infty} a_{k+j}(\xi_j + \xi_{j+1}) \ , \ S_n(X') = \sum_{k=1}^{n} X_k', . \tag{2.8}$$

Then the conclusions of Theorems 2.1, 2.3 *and* 2.4 *hold for $S_n(X')$ and $S_{[nt]}(X')$. In this case η is identified as the limit $n^{-1}\mathbb{E}(\sum_{k=1}^n (\xi_k + \xi_{k+1})^2 | \mathcal{F}_0) \to \eta$ in $\mathbb{L}_1(\Omega, \mathcal{F}, \mathbb{P})$ as $n \to \infty$. Furthermore, the stationary sequence $(\xi_k + \xi_{k+1})_{k\in\mathbb{Z}}$ has a continuous spectral density $h(x)$ and*

$$\mathbb{E}\eta = 2\pi h(0) = \lim_{n\to\infty} \mathrm{Var} S_n(X')/b_n^2. \tag{2.9}$$

As a corollary we obtain:

Corollary 2.6. *Assume that* (ξ_j) *is defined by* (2.1)*,* $Q = Q^*$*, and condition* (2.2) *is satisfied. Assume* (X_k) *defined by* (1.1) *exists in* $\mathbb{L}_2(\Omega, \mathcal{F}, \mathbb{P})$*. Then the conclusion of Theorems* 2.1*,* 2.3 *and* 2.4 *hold for* $S_n(X)$ *and* $S_{[nt]}(X)$ *with* $\eta/4$ *where* η *is identified as in Theorem* 2.5*.*

We shall present next the short memory case:

Theorem 2.7. *Assume now* $\sum_{i\in\mathbb{Z}} |a_i| < \infty$ *and let* $(X_k)_{k\geq 1}$ *be as in Theorem* 2.1*. Assume that condition* (2.2) *is satisfied. Then the process* $\{S_{[nt]}(X)/\sqrt{n}, t \in [0,1]\}$ *converges in* $D([0,1])$ *to* $\sqrt{\eta}|A|W$ *where* W *is a standard Brownian motion and* $A = \sum_{i\in\mathbb{Z}} a_i$*.*

Remark 2.8. It is easy to see that Theorem 2.7 extends the Kipnis and Varadhan (1987) result to linear processes.

We give a few examples of sequences (a_n) satisfying the conditions of our theorems. In these examples the notation $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

Example 1. For the selection $a_i \sim i^{-\alpha}\ell(i)$ where ℓ is a slowly varying function at infinity and $1/2 < \alpha < 1$ for $i \geq 1$ and $a_i = 0$ elsewhere, then, $b_n^2 \sim \kappa_\alpha n^{3-2\alpha}\ell^2(n)$ (see for instance Relations (12) in Wang *et al.* (2003)), where κ_α is a positive constant depending on α. Clearly, Theorem 2.3 and the corresponding part of Theorem 2.5 apply.

Example 2. Let us consider now the fractionally integrated processes since they play an important role in financial time series modeling and they are widely studied. Such processes are defined for $0 < d < 1/2$ by

$$X_k = (1-B)^{-d}\xi_k = \sum_{i\geq 0} a_i \xi_{k-i} \quad \text{with } a_i = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}, \tag{2.10}$$

where B is the backward shift operator, $B\varepsilon_k = \varepsilon_{k-1}$.

For this example, by the well-known fact that for any real x, $\lim_{n\to\infty} \Gamma(n+x)/n^x\Gamma(n) = 1$, we have $\lim_{n\to\infty} a_n/n^{d-1} = 1/\Gamma(d)$. Theorem 2.3 and the corresponding part of Theorem 2.5 apply with $\beta = 2d+1$, since for $k \geq 1$ we have $a_k \sim \kappa_d k^{d-1}$ for some $\kappa_d > 0$ and $a_k = 0$ elsewhere.

Example 3. Now, if we consider the following selection of $(a_k)_{k\geq 0}$: $a_0 = 1$ and $a_i = (i+1)^{-\alpha} - i^{-\alpha}$ for $i \geq 1$ with $\alpha \in]0, 1/2[$ and $a_i = 0$ elsewhere, then both Theorem 2.4 and the corresponding part of Theorem 2.5 apply. Indeed for this selection, $b_n^2 \sim \kappa_\alpha n^{1-2\alpha}$, where κ_α is a positive constant depending on α.

Example 4. Finally, if $a_i \sim i^{-1/2}(\log i)^{-\alpha}$ for some $\alpha > 1/2$, then

$$b_n^2 \sim n^2(\log n)^{1-2\alpha}/(2\alpha - 1)$$

(see Relations (12) in Wang et al. (2003)). Hence (2.7) is satisfied with $\beta = 2$.

3. Applications

3.1. Application to a Metropolis Hastings Markov chain

In this subsection we analyze a standardized example of a stationary irreducible and aperiodic Metropolis-Hastings algorithm with uniform marginal distribution. This type of Markov chain is interesting since it can easily be transformed into Markov chains with different marginal distributions. Markov chains of this type are often studied in the literature from different points of view. See, for instance, Doukhan et al. (1994) and Longla et al. (2012) among many others.

Let $E = [-1, 1]$ and let υ be a symmetric atomless law on E. The transition probabilities are defined by

$$Q(x, A) = (1 - |x|)\delta_x(A) + |x|\upsilon(A),$$

where δ_x denotes the Dirac measure. Assume that $\theta = \int_E |x|^{-1}\upsilon(dx) < \infty$. Then there is a unique invariant measure

$$\pi(dx) = \theta^{-1}|x|^{-1}\upsilon\,(dx)$$

and the stationary Markov chain (γ_k) generated by $Q(x, A)$ and π is reversible and positively recurrent, therefore ergodic.

Theorem 3.1. *Let $g(-x) = -g(x)$ for any $x \in E$ and assume*

$$\int_0^1 g^2(x)x^{-2}d\upsilon < \infty.$$

Then, the conclusions of all our theorems in Section 2 hold for (X_k) and $S_n(X)$ defined by (1.1) with

$$\eta = \sigma_g^2 = \theta^{-1}\left(-\int_E g^2(x)|x|^{-1}\upsilon(dx)\ + 2\int_E g^2(x)|x|^{-2}\upsilon(dx)\right).$$

Proof. Since g is an odd function we have

$$\mathbb{E}(g(\gamma_k)|\gamma_0) = (1 - |\gamma_0|)^k g(\gamma_0) \text{ a.s.} \tag{3.1}$$

Therefore, for any $j \geq 0$,

$$\mathbb{E}(X_0X_j) = \mathbb{E}(g(\gamma_0)\mathbb{E}(g(\gamma_j)|\gamma_0)) = \theta^{-1}\int_E g^2(x)(1 - |x|)^j|x|^{-1}\upsilon(dx).$$

Then,

$$\sum_{j=1}^{k-1}|\mathbb{E}(X_0X_j)| \leq 2\theta^{-1}\sum_{j=1}^{k-1}\int_0^1 g^2(x)(1 - x)^j x^{-1}\upsilon(dx) \leq 2\theta^{-1}\int_0^1 g^2(x)x^{-2}\upsilon(dx) \tag{3.2}$$

and therefore condition (2.5) is satisfied. □

3.2. Linear process of instantaneous functions of a Gaussian sequence

Theorem 3.2. *Let* $(\xi_k)_{k\in\mathbb{Z}}$ *be instantaneous functions of a stationary Markov Gaussian sequence* (γ_n), $\xi_k = g(\gamma_n)$ *where* g *is a measurable real function such that* $\mathbb{E}g(\gamma_n) = 0$ *and* $\mathbb{E}g^2(\gamma_n) < \infty$. *Define* X_k *and* $S_n(X)$ *by* (1.1). *Then the conclusions of our theorems in Section* 2 *hold.*

Proof. In order to apply our results, because (γ_n) is reversible, we have only to check condition (2.5). Under our conditions g can be expanded in Hermite polynomials $g(x) = \sum_{j\geq 1} c_j H_j(x)$, where $\sum_{j=1} c_j^2 j! < \infty$.

For computing the covariances we shall apply the following well-known formula: if a and b are jointly Gaussian random variables, $\mathbb{E}a = \mathbb{E}b = 0$, $\mathbb{E}a^2 = \mathbb{E}b^2 = 1$, $r = \mathbb{E}ab$, then

$$\mathbb{E}H_k(a)H_l(b) = \delta(k,l)r^k k!,$$

where δ denotes the Kronecker delta. It follows that

$$cov(\xi_0, \xi_k) = \mathbb{E}\sum_{j\geq 1} c_j^2 H_j(\gamma_0)H_j(\gamma_k) = \sum_{j\geq 1} c_j^2 r_k^j j!.$$

Clearly, because under our condition it is known that $r_k = \exp(-\alpha k/2)$ for some $\alpha > 0$, then

$$|cov(\xi_0, \xi_k)| \leq \exp(-\alpha k/2)\sum_{j\geq 1} c_j^2 j!$$

and the result follows. □

For a particular class of weights of the form in Example 3, we mention that Breuer and Major (1983) studied this problem for Gaussian chains without Markov assumption.

3.3. Application to random walks on compact groups

In this section we shall apply our results to random walks on compact groups.

Let $\mathcal{X}$ be a compact Abelian group, $\mathcal{A}$ a sigma algebra of Borel subsets of $\mathcal{X}$ and π the normalized Haar measure on $\mathcal{X}$. The group operation is denoted by $+$. Let ν be a probability measure on $(\mathcal{X}, \mathcal{A})$. The random walk on $\mathcal{X}$ defined by ν is the stationary Markov chain having the transition function

$$(x, A) \to Q(x, A) = \nu(A - x).$$

The corresponding Markov operator denoted by Q is defined by

$$(Qf)(x) = f * \nu(x) = \int_{\mathcal{X}} f(x + y)\nu(dy).$$

The Haar measure is invariant under Q. We shall assume that ν is not supported by a proper closed subgroup of $\mathcal{X}$, a condition that is equivalent to Q being ergodic. In this context

$$(Q^* f)(x) = f * \nu^*(x) = \int_{\mathcal{X}} f(x - y)\nu(dy),$$

where ν^* is the image of measure ν by the map $x \to -x$. Thus Q is symmetric on $\mathbb{L}_2(\pi)$ if and only if ν is symmetric on $\mathcal{X}$, that is $\nu = \nu^*$.

The dual group of $\mathcal{X}$, denoted by $\hat{\mathcal{X}}$, is discrete. Denote by $\hat{\nu}$ the Fourier transform of the measure ν, that is the function

$$g \to \hat{\nu}(g) = \int_{\mathcal{X}} g(x)\nu(dx) \quad \text{with } g \in \hat{\mathcal{X}}.$$

A function $f \in \mathbb{L}^2(\pi)$ has the Fourier expansion

$$f = \sum_{g \in \hat{\mathcal{X}}} \hat{f}(g) g.$$

Ergodicity of Q is equivalent to $\hat{\nu}(g) \neq 1$ for any non-identity $g \in \hat{\mathcal{X}}$. By arguments in Borodin and Ibragimov (1994, Ch. 4, Sect. 9) and also Derriennic and Lin (2001, Section 8) condition (2.4) takes the form

$$\sum_{1 \neq g \in \hat{\mathcal{X}}} \frac{|\hat{f}(g)|^2}{|1 - \hat{\nu}(g)|} < \infty. \tag{3.3}$$

Combining these considerations with the results in Section 2 we obtain the following result:

Theorem 3.3. *Let ν be ergodic and symmetric on $\mathcal{X}$. Let (ξ_i) be the stationary Markov chain with marginal distribution π and transition operator Q. If for g in $\mathbb{L}_0^2(\pi)$ condition* (3.3) *is satisfied then the conclusions of Theorem* 2.5 *and Corollary* 2.6 *in Section* 2 *hold.*

4. Proofs

4.1. Preliminary general results

This section contains some general results for linear processes of stationary sequences which are not necessarily Markov. We start by mentioning the following theorem which is a variant of a result from Peligrad and Utev (2006-a). See also Proposition 5.1 in Dedecker et al. (2011).

Theorem 4.1. *Let $(\xi_k)_{k\in\mathbb{Z}}$ be a strictly stationary sequence of centered square integrable random variables such that*

$$\Gamma_j = \sum_{k=0}^{\infty} |\mathbb{E}(\xi_{j+k}\mathbb{E}_0\xi_j)| < \infty \textit{ and } \frac{1}{p}\sum_{j=1}^{p} \Gamma_j \to 0 \textit{ as } p \to \infty. \tag{4.1}$$

For any positive integer n, let $(d_{n,i})_{i\in\mathbb{Z}}$ be a triangular array of numbers satisfying, for some positive c,

$$\sum_{i\in\mathbb{Z}} d_{n,i}^2 \to c^2 \textit{ and } \sum_{j\in\mathbb{Z}} (d_{n,j} - d_{n,j-1})^2 \to 0 \textit{ as } n \to \infty. \tag{4.2}$$

In addition assume

$$\sup_{j\in\mathbb{Z}} |d_{n,j}| \to 0 \quad \textit{as} \quad n \to \infty. \tag{4.3}$$

Then $\sum_{j\in\mathbb{Z}} d_{n,j}\xi_j$ converges in distribution to $\sqrt{\eta}cN$ where N is a standard Gaussian random variable independent of η. The variable η is measurable with respect to the invariant sigma field $\mathcal{I}$ and $n^{-1}\mathbb{E}((\sum_{k=1}^{n}\xi_k)^2|\mathcal{F}_0) \to \eta$ in $\mathbb{L}_1(\Omega,\mathcal{F},\mathbb{P})$ as $n\to\infty$. Furthermore $(\xi_i)_{i\in\mathbb{Z}}$ has a continuous spectral density $f(x)$ and $\mathbb{E}\eta = 2\pi f(0)$. If the sequence $(\xi_i)_{i\in\mathbb{Z}}$ is ergodic we have $\eta = 2\pi f(0)$.

Proof. The proof follows the lines of Theorem 1 from Peligrad and Utev (2006-a). We just have to repeat the arguments there with $b_{n,i}/b_n$ replaced by $d_{n,i}$ and take into account that the properties (4.2) and (4.3) are precisely all is needed to complete the proof. □

Next we shall establish the convergence of finite dimensional distributions.

Theorem 4.2. *Define (X_k) and S_n by (1.1) and assume condition (4.1) is satisfied. Then S_n/b_n converges in distribution to $\sqrt{\eta}N$ where N and η are as in Theorem 4.1. If we assume in addition that condition (2.7) is satisfied, then the finite dimensional distributions of $\{W_n(t) = b_n^{-1}S_{[nt]}, t\in[0,1]\}$ converge to the corresponding ones of $\sqrt{\eta}W_H$, where W_H is a standard fractional Brownian motion independent of η with Hurst index $H = \beta/2$.*

Proof. The central limit theorem part requires just to verify the conditions of Theorem 4.1 for $d_{n,j} = b_{n,j}/b_n$ and $c = 1$. Condition (4.3) was verified in Peligrad and Utev (1997, pp. 448–449) while condition (4.2) was verified in Lemma A.1. in Peligrad and Utev (2006-a).

We shall prove next the second part of the theorem. Notice that if we impose (2.7), for each t fixed

$$\mathrm{var}(W_n(t)) \to 2\pi f(0)t^\beta \tag{4.4}$$

and $W_n(t) \Rightarrow \eta t^{\beta/2}N$.

Let $0\le t_1\le\cdots\le t_k\le 1$. By Cramèr-Wold device, in order to find the limiting distribution of $(W_n(t_i))_{1\le i\le k}$ we have to study $V_n = \sum_{i=1}^{k} u_i W_n(t_i)$ where u_i is a real vector. Let us compute its limiting variance. To find it, let $0\le s\le t\le 1$. By using the fact that for any two real numbers a and b we have $a(a-b) = (a^2+(a-b)^2-b^2)/2$, we obtain the representation:

$$\begin{aligned}\mathrm{cov}(W_n(t),W_n(s)) &= \mathrm{var}(W_n(s)) + \mathrm{cov}(W_n(s),W_n(t)-W_n(s))\\ &= \mathrm{var}(W_n(s)) + 1/2\left[\mathrm{var}(W_n(t)-W_n(s)) + \mathrm{var}(W_n(t)) - \mathrm{var}(W_n(s))\right].\end{aligned}$$

By stationarity,

$$\mathrm{var}(W_n(t)-W_n(s)) = \mathrm{var}(W_{[nt]-[ns]}),$$

and by (4.4) and the fact that $b_n\to\infty$ we obtain

$$\lim_{n\to\infty}\mathrm{cov}(W_n(t),W_n(s)) = \pi f(0)(s^\beta + t^\beta - |t-s|^\beta). \tag{4.5}$$

So,

$$\lim_{n\to\infty}\frac{1}{2\pi f(0)}\mathrm{var}(V_n) = \sum_{i=1}^{k}u_i^2 t_i^\beta + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}u_iu_j(t_i^\beta + t_j^\beta - (t_j-t_i)^\beta) = B_k. \tag{4.6}$$

Writing now

$$V_n = \sum_{i=1}^{k} u_i W_n(t_i) = \sum_{j\in\mathbb{Z}} d_{n,j}(k)\xi_j,$$

where $d_{n,j}(k) = \sum_{i=1}^{k} u_i b_{[nt_i],j}/b_n$, we shall apply Theorem 4.1. The second part of (4.2) and (4.3) were verified in Peligrad and Utev (1996 and 2006-a). It remains to verify the first part of condition (4.2). By the point (iii) of Lemma 5.1 in the Appendix we obtain

$$\mathrm{var}(V_n)/\sum_{j\in\mathbb{Z}} d_{n,j}^2(k) \to 2\pi f(0),$$

which combined with (4.6) implies that the first part of (4.2) is verified with $c^2 = \lim_{n\to\infty}\sum_{j\in\mathbb{Z}} d_{n,j}^2(k) = B_k$. In other words, the finite dimensional distributions are convergent to those of a fractional Brownian motion with Hurst index $\beta/2$. □

Discussion on tightness. As we mentioned above, for $\beta \le 1$ the conditions of Theorem 4.2 are not sufficient to imply tightness.

However for $\beta > 1$ we can obtain tightness in $D([0,1])$ endowed with Skorohod topology. By the point (i) of Lemma 5.1 in the Appendix we have the inequality

$$\mathbb{E}|S_k|^2 \le \left(\mathbb{E}[\xi_0^2] + 2\sum_{k\in\mathbb{Z}} |\mathbb{E}(\xi_0\xi_k)|\right)\sum_{j\in\mathbb{Z}} b_{k,j}^2.$$

Therefore, by using (4.1) and (2.7), the conditions of Lemma 2.1 p. 290 in Taqqu (1975) are satisfied when $\beta > 1$, and the tightness follows. □

To treat the short memory case we mention the following result in Peligrad and Utev (2006-b).

Theorem 4.3. *Assume that X_k and S_n are defined by* (1.1) *and* $\sum_{i\in Z}|a_i| < \infty$. *Moreover assume that for some $c_n > 0$ the innovations satisfy the invariance principle*

$$c_n^{-1} S_{[nt]}(\xi) \Rightarrow \eta W(t),$$

where η is $\mathcal{I}$-measurable and W is a standard Brownian motion on $[0,1]$ *independent on $\mathcal{I}$. In addition assume that the following condition holds:*

$$\mathbb{E}\max_{1\le j\le n} |S_j(\xi)| \le Cc_n. \tag{4.7}$$

where C is a positive constant. Then, the linear process also satisfies the invariance principle, i.e., $c_n^{-1}S_{[nt]}(X) \Rightarrow \eta|A|W(t)$ as $n\to\infty$ where $A = \sum_{i\in\mathbb{Z}} a_i$.

4.2. Normal and reversible Markov chains

In this subsection we give the proofs of the theorems stated in Section 2. The goal is to verify condition (4.1) that will assure that all the results in Subsection 4.1 are valid.

We start by applying the general results to normal Markov chains.

Theorem 4.4. *Assume that* $(\xi_j)_{j\in\mathbb{Z}}$ *is defined by* (2.1) *and the Markov chain is normal,* $QQ^* = Q^*Q$*. For this case condition* (4.1) *is implied by*

$$\sum_{k\geq 0} ||Q^k g||_2^2 < \infty. \tag{4.8}$$

and as a consequence all the results obtained in Subsection 4.1 *are valid.*

Proof. Indeed, we start by rewriting (4.1) in operator notation:

$$\begin{aligned}|\mathbb{E}[\xi_{j+k}\mathbb{E}(\xi_j|\mathcal{F}_0)]| &= |\mathbb{E}(\mathbb{E}_0\xi_{k+j}\mathbb{E}_0\xi_j)| = \left|\langle Q^{k+j}g, Q^j g\rangle\right| \\ = &\left|\left\langle Q^{[k/2]+j}g, (Q^*)^{k-[k/2]}Q^j g\right\rangle\right| \leq ||Q^{[k/2]+j}g||_2||(Q^*)^{k-[k/2]}Q^j g||_2.\end{aligned}$$

For a normal operator, by using the properties of conditional expectation, we have

$$||(Q^*)^{k-[k/2]}Q^j g||_2 = ||Q^j(Q^*)^{k-[k/2]}g||_2 \leq ||(Q^*)^{k-[k/2]}g||_2.$$

Since for all $\varepsilon > 0$, and any two numbers a and b we have $|ab| \leq a^2/2\varepsilon + \varepsilon b^2/2$, by the above considerations we easily obtain

$$\begin{aligned}\sum_{k\geq 0}|\mathbb{E}[\xi_{j+k}\mathbb{E}(\xi_j|\mathcal{F}_0)]| &\leq \sum_{k\geq 0}||Q^{[k/2]+j}g||_2||Q^{k-[k/2]}g||_2 \\ &\leq \frac{1}{\varepsilon}\sum_{k\geq j}||Q^k g||_2^2 + \varepsilon\sum_{k\geq 0}||Q^k g||_2^2,\end{aligned}$$

condition (4.1) is verified under (4.8), by letting $j \to \infty$ followed by $\varepsilon \to 0$. □

In terms of spectral measure $\rho_g(dz)$, condition (4.8) is implied by

$$\int_D \frac{1}{1-|z|}\rho_g(dz) < \infty,$$

where D is the unit disk. Note that this condition is stronger than the condition needed for the validity of CLT for the partial sums (i.e., the case $a_1 = 1, a_i = 0$ elsewhere), which requires only the condition $\int_D \frac{1}{|1-z|}\rho_g(dz) < \infty$ (see Gordin and Lifshitz (1981), or in Ch. IV in Borodin and Ibragimov (1994)).

Proof of Theorems 2.1, 2.3 and 2.4

For the reversible Markov chains just notice that

$$\mathbb{E}[\xi_{j+k}\mathbb{E}(\xi_j|\mathcal{F}_0)] = \int_{-1}^{1} t^{2j+k}\rho_g(dz) = \operatorname{cov}(\xi_0, \xi_{2j+k})$$

and then, condition (4.1) is verified under (2.5) because

$$\sum_{k\geq 0}|\mathbb{E}[\xi_{j+k}\mathbb{E}(\xi_j|\mathcal{F}_0)]| = \sum_{k\geq 2j}|\operatorname{cov}(\xi_0, \xi_k)| \to 0 \quad \text{as} \quad j \to \infty.$$

Theorems 2.1, 2.3 and 2.4 follow as simple applications of the results in Subsection 4.1.

Proof of Theorem 2.5. In order to prove this theorem, we shall also apply Theorem 4.2 along to the tightness discussion at the end of Subsection 4.1. We denote $\gamma_j = \xi_j + \xi_{j+1}$ and verify condition (4.1) for this sequence of innovations. We have

$$|\mathbb{E}(\gamma_{k+j}\mathbb{E}_0\gamma_j)| = \left|\langle Q^{k+j}g + Q^{k+j+1}g, Q^j g + Q^{j+1}g\rangle\right|$$

and by spectral calculus

$$\sum_{k\geq 0}\left|\langle Q^{k+j}g + Q^{k+j+1}g, Q^j g + Q^{j+1}g\rangle\right| = \sum_{k\geq 0}\left|\int_{-1}^{1} t^{k+2j}(1+t)^2 d\rho_g\right|.$$

We divide the sum in 2 parts, according to k even or odd. When $k=2u$ the sum has positive terms and it can be written as

$$\sum_{u\geq 0}\int_{-1}^{1} t^{2u+2j}(1+t)^2 d\rho_g \leq \int_{-1}^{1}\frac{t^{2j}}{1-t^2}(1+t)^2 d\rho_g = \int_{-1}^{1}\frac{t^{2j}(1+t)}{1-t}d\rho_g.$$

When k is odd

$$\begin{aligned}\sum_{k\geq 1, k \text{ odd}}\left|\int_{-1}^{1} t^{k+2j}(1+t)^2 d\rho_g\right| &\leq \int_{-1}^{1}\sum_{k\geq 1, k \text{ odd}}|t^{k+2j}(1+t)^2| d\rho_g \\ &\leq \int_{-1}^{1}\sum_{k\geq 1, k \text{ odd}}|t^{k-1+2j}(1+t)^2| d\rho_g \leq \sum_{u\geq 0}|t^{2u+2j}(1+t)^2| d\rho_g,\end{aligned}$$

and we continue the computation as for the case k even. It follows that

$$\frac{1}{m}\sum_{j=1}^{m}\sum_{k\geq 0}|\mathbb{E}(\gamma_{k+j}\mathbb{E}_0\gamma_j)| \leq \frac{2}{m}\sum_{j=1}^{m}\int_{-1}^{1}\frac{t^{2j}(1+t)}{1-t}d\rho_g.$$

Note that (2.4) implies that $\rho_g(1)=0$. We also have $m^{-1}\sum_{j=1}^{m} t^{2j}(1+t)$ is convergent to 0 for all $t \in [-1,1)$. Furthermore, $m^{-1}\sum_{j=1}^{m} t^{2j}(1+t)$ is dominated by 2 and in view of (2.4) and Lebesgue dominated convergence theorem we have

$$\lim_{m\to\infty}\int_{-1}^{1}\frac{1}{m}\sum_{j=1}^{m}\frac{t^{2j}(1+t)}{1-t}d\rho_g = 0,$$

and therefore condition (4.1) is satisfied. □

Proof of Corollary 2.6. We start from the representation given by (1.1),

$$\frac{1}{b_n}(X_1+\cdots+X_n) = \frac{1}{b_n}\sum_{j\in\mathbb{Z}} b_{n,j}\xi_j \quad\text{and}\quad \frac{1}{b_n}(X_0+\cdots+X_{n-1}) = \frac{1}{b_n}\sum_{j\in\mathbb{Z}} b_{n,j}\xi_{j+1}.$$

By adding these relations we obtain

$$\frac{X_0+2S_n-X_n}{b_n} = \frac{1}{b_n}\sum_{j\in\mathbb{Z}} b_{n,j}(\xi_j+\xi_{j+1}). \tag{4.9}$$

Because $b_n \to \infty$, by Theorem 3.1 in Billingsley (1999), the limiting behavior of $2S_n/b_n$ is given by the sequence

$$\frac{1}{b_n}\sum_{j=-\infty}^{\infty} b_{n,j}(\xi_j + \xi_{j+1}) = \frac{1}{b_n}\sum_{j=1}^{n} X_j',$$

with X_j' defined by (2.8). The conclusion of Theorem 4.1 follows by the corresponding part of Theorem 2.5.

To derive the conclusions of Theorems 2.3 and 2.4, note that by (4.9) and (2.8) with the notations $W_n(t) = b_n^{-1}S_{[nt]}$ and $W_n^{'}(t) = \sum_{j=1}^{[nt]} X_j^{'}/b_n$, we have

$$2W_n(t) = W_n^{'}(t) - \frac{X_0}{b_n} + \frac{X_{[nt]}}{b_n}. \tag{4.10}$$

It is well known that for a stationary sequence with finite second moments

$$\frac{1}{n}\mathbb{E}\Big(\max_{1\le i\le n} X_i^2\Big) \to 0.$$

So, by the fact that $1 < \beta < 2$ and by (2.7) we also have

$$\frac{1}{b_n^2}\mathbb{E}\Big(\max_{1\le i\le n} X_i^2\Big) \to 0.$$

and then, by Theorem 3.1 in Billingsley (1999), it follows that the asymptotic behavior of $\{2W_n(t), 0 \le t \le 1\}$ is identical to that of $\{W_n^{'}(t),\, 0 \le t \le 1\}$ and we apply the second part of Theorem 2.5. □

Proof of Theorem 2.7. Theorem 2.7 follows by combining Theorem 4.3 with the invariance principle in Kipnis and Varadhan (1997). We have only to verify condition (4.7). It is known that the maximal inequality required by condition (4.7) holds for partial sums of functions of reversible Markov chains. Indeed, we know from Proposition 4 in Longla et al. (2012) that

$$\mathbb{E}\Big(\max_{1\le i\le n} S_i^2\Big) \le 2\mathbb{E}\Big(\max_{1\le i\le n} X_i^2\Big) + 22 \max_{1\le i\le n} \mathbb{E}(S_i^2) \tag{4.11}$$

and then, condition (2.2) and stationarity implies condition (4.7) with $c_n = \sqrt{n}$. □

5. Appendix

Facts about spectral densities. In the following lemma we combine a few facts about spectral densities, covariances, behavior of variances of sums and their relationships. The first two points are well known. They can be found for instance in Bradley (2007, Vol 1, 0.19–0.21 and Ch.8). The point (iii) was proven in Peligrad and Utev (2006-a).

Lemma 5.1. *Let* $(\xi_i)_{i\in\mathbb{Z}}$ *be a stationary sequence of real-valued variables with* $\mathbb{E}\xi_0 = 0$ *and finite second moment. Let* F *denotes the spectral measure and* f *denotes its* *spectral density (if exists), i.e.,*

$$\mathbb{E}(\xi_0\xi_k) = \int_{-\pi}^{\pi} e^{-ikt} dF(t) = \int_{-\pi}^{\pi} e^{-ikt} f(t)dt.$$

(i) *For any positive integer n and any real numbers $a_1, \dots, a_n$,*

$$\mathbb{E}\Big(\sum_{k=1}^{n} a_k\xi_k\Big)^2 = \int_{-\pi}^{\pi} \Big|\sum_{k=1}^{n} a_k e^{ikt}\Big|^2 f(t)dt \le 2\pi\|f\|_\infty \sum_{k=1}^{n} a_k^2$$
$$\le \Big(\mathbb{E}[\xi_0^2] + 2\sum_{k\ge 1} |\mathbb{E}(\xi_0\xi_k)|\Big) \sum_{k=1}^{n} a_k^2.$$

(ii) *Assume $\sum_{k=1}^{\infty} |\mathbb{E}(\xi_0\xi_k)| < \infty$. Then, f is continuous.*

(iii) *Assume that the spectral density f is continuous, and let $(d_{n,j})_{j\in\mathbb{Z}}$ be a double array of real numbers with $d_n^2 = \Sigma_{j\in\mathbb{Z}} d_{n,j}^2 < \infty$ that satisfies the condition*

$$\frac{1}{d_n^2} \sum_{j\in\mathbb{Z}} |d_{n,j} - d_{n,j-1}|^2 \to 0. \tag{5.1}$$

Then,

$$\lim_{n\to\infty} \frac{1}{d_n^2} \mathbb{E}\Big(\sum_{j\in\mathbb{Z}} d_{n,j}\xi_j\Big)^2 = 2\pi f(0). \tag{5.2}$$

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Part III

Stochastic Processes

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First Exit of Brownian Motion from a One-sided Moving Boundary

Frank Aurzada and Tanja Kramm

Abstract. We revisit a result of Uchiyama (1980): given that a certain integral test is satisfied, the rate of the probability that Brownian motion remains below the moving boundary f is asymptotically the same as for the constant boundary. The integral test for f is also necessary in some sense.

After Uchiyama's result, a number of different proofs appeared simplifying the original arguments, which strongly rely on some known identities for Brownian motion. In particular, Novikov (1996) gives an elementary proof in the case of an increasing boundary. Here, we provide an elementary, half-page proof for the case of a decreasing boundary. Further, we identify that the integral test is related to a repulsion effect of the three-dimensional Bessel process. Our proof gives some hope to be generalized to other processes such as FBM.

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Keywords. Brownian motion; Bessel process; moving boundary; first exit time; one-sided exit problem.

1. Introduction

This note is concerned with the first exit time distribution of Brownian motion from a so-called moving boundary:

$$\mathbb{P}\left[B_t \le f(t), t \le T\right], \qquad \text{as } T \to \infty,$$

where B is a Brownian motion and $f : [0, \infty) \to \mathbb{R}$ is the "moving boundary". The question we treat here is follows: for which functions f does the above probability have the same asymptotic rate as in the case $f \equiv 1$? This problem was considered by a number of authors [1–3, 5, 6, 8] and, besides being a classical problem for Brownian motion, has some implications for the so-called KPP equation (see, e.g., [2]), for branching Brownian motion (see, e.g., [1]), and for other questions.

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The solution of the problem was given by Uchiyama [8], Gärtner [2], and Novikov [5] independently and can be re-phrased as follows.

Theorem 1.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a continuously differentiable function with $f(0) > 0$ and*

$$\int_1^\infty |f(t)|\, t^{-3/2}\, \mathrm{d}t < \infty. \tag{1.1}$$

Then

$$\mathbb{P}\left[B_t \le f(t), t \le T\right] \approx T^{-1/2}, \qquad \text{as } T \to \infty. \tag{1.2}$$

If f is either convex or concave and the integral test (1.1) *fails, $T^{-1/2}$ is not the right order in* (1.2).

Here and in the following, we denote $a(t) \approx b(t)$ if $c_1 a(t) \le b(t) \le c_2 a(t)$ for some constants c_1, c_2 and all t sufficiently large.

Even though the above-mentioned problem has been solved by Uchiyama, there have been various attempts to simplify the proof of this result and to give an interpretation for the integral test (1.1). It is the purpose of this note (a) to give a simplified proof of the theorem for the case of a decreasing boundary. Our proof (b) also allows to interpret the integral test as coming from a repulsion effect of the three-dimensional Bessel process and (c) gives hope to be generalized to other processes, contrary to the existing proofs, which all make use of very specific known identities for Brownian motion.

Let us assume for a moment that f is monotone. Note that the sufficiency part of the theorem can be decomposed into two parts: if $f' \ge 0$ one needs an upper bound of the probability in question, while if $f' \le 0$ one needs a lower bound. The first case is much better studied; in particular, Novikov [6] gives a relatively simple proof of the theorem in this case. To the contrary, in case of a decreasing boundary he wonders that "it would be interesting to find an elementary proof of this bound" ([6], p. 723). We shall provide such an elementary proof here.

The remainder of this note is structured as follows. Section 2 contains the proof of the theorem, which now fits on half a page. We also outline the relation to the Bessel process. In Section 3, we list some additional remarks.

2. Proof

We give a proof of the following theorem, which concerns the part of Theorem 1.1 related to the decreasing boundary.

Theorem 2.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a twice continuously differentiable function with $f(0) > 0$.*

- *Then for some absolute constant* $0 < c < \infty$ *we have*

$$\begin{aligned}\mathbb{P}\left[B_t \le f(t), 0 \le t \le T\right] \\ \ge \mathbb{P}\left[B_t \le f(0), 0 \le t \le T\right] \\ \cdot \exp\left(-\frac{1}{2}\int_0^T f'(s)^2 \mathrm{d}s - c\int_0^T |f''(s)|\sqrt{s}\,\mathrm{d}s - c\sqrt{T}|f'(T)|\right).\end{aligned}$$

- *In particular, if* (1.1) *holds and* $f' \le 0$, $f'' \ge 0$, *for large enough arguments, then we have*

$$\mathbb{P}\left[B_t \le f(t), t \le T\right] \approx T^{-1/2}, \qquad \text{as } T \to \infty. \tag{2.1}$$

Proof. The Cameron-Martin-Girsanov theorem implies that

$$\begin{aligned}\mathbb{P}\left[B_t \le f(t), 0 \le t \le T\right] &= \mathbb{P}\left[B_t - \int_0^t f'(s)\mathrm{d}s \le f(0), 0 \le t \le T\right] \\ &= \mathbb{E}\left[e^{-\int_0^T f'(s)\mathrm{d}B_s}\, \mathbb{1}_{\{B_t \le f(0), 0 \le t \le T\}}\right] e^{-\frac{1}{2}\int_0^T f'(s)^2 \mathrm{d}s}.\end{aligned} \tag{2.2}$$

Further,

$$\int_0^T f'(s)\mathrm{d}B_s = B_T f'(T) - \int_0^T B_u f''(u)\mathrm{d}u,$$

so that the expectation in (2.2) equals

$$\begin{aligned}&\frac{\mathbb{E}[e^{\int_0^T B_u f''(u)\mathrm{d}u - B_T f'(T)}\, \mathbb{1}_{\{B_t \le f(0), 0 \le t \le T\}}]}{\mathbb{P}\left[B_t \le f(0), 0 \le t \le T\right]} \cdot \mathbb{P}\left[B_t \le f(0), 0 \le t \le T\right] \\ &= \mathbb{E}\left[e^{\int_0^T B_u f''(u)\mathrm{d}u - B_T f'(T)} \,\middle|\, \sup_{0 \le t \le T} B_t \le f(0)\right] \cdot \mathbb{P}\left[B_t \le f(0), 0 \le t \le T\right].\end{aligned}$$

By Jensen's inequality, the first term can be estimated from below by

$$\exp\left(\int_0^T \mathbb{E}\left[Y_u\right] f''(u)\mathrm{d}u + \mathbb{E}\left[Y_T\right](-f'(T))\right), \tag{2.3}$$

where we denote by Y the law of B conditioned on $\sup_{0 \le t \le T} B_t \le f(0)$. Since $\mathbb{E}[Y_u] \le 0$ the functions $f''(u)$ and $-f'(T)$ in (2.3) can be estimated from above by the absolute value; and hence the theorem is proved by applying Lemma 2.2 below. □

Lemma 2.2. *Let* B *be a Brownian motion and* $f(0) > 0$ *be some constant. Then there is a constant* $c > 0$ *such that*

$$\mathbb{E}\left[B_u \,\middle|\, \sup_{0 \le t \le T} B_t \le f(0)\right] \ge -c\sqrt{u}, \qquad \forall 0 \le u \le T.$$

In order to show this lemma one can use the joint distribution of maximum over an interval and terminal value of Brownian motion, which is explicitly known (see, e.g., [4], Prop. 2.8.1). However, we do not include this proof here. Let us rather mention that the lemma can also be seen through a relation to the three-dimensional Bessel process, as detailed now.

Recall that a (three-dimensional) Bessel process has three representations: it can be defined firstly as Brownian motion conditioned to be positive for all times, secondly as the solution of a certain stochastic differential equation (which gives rise to Bessel processes of other dimensions), and thirdly as the modulus of a three-dimensional Brownian motion, see, e.g., [4], Chapter 3.3.C. Let us denote by Y the law of a Brownian motion B under the condition $\sup_{0\leq t\leq T} B_t \leq f(0)$. Then, from the first representation, it is intuitively clear that one can find a Bessel process $-X$ such that $Y \geq X$. Now, taking expectations and using the third representation of $-X$ (and Brownian motion scaling) it is clear that $\mathbb{E}Y_s \geq \mathbb{E}X_s = -c\sqrt{s}$. Thus, the integral test is related to the repulsion of Brownian motion by the conditioning.

3. Further remarks

Remark 3.1. Clearly the value of f in a finite time horizon $[0, t_0]$ does not matter for the outcome of the problem, as we are interested in asymptotic results. Any finite time horizon can be cut off with the help of Slepian's inequality [7]:

$$\mathbb{P}\left[B_t \leq f(t), 0 \leq t \leq T\right] \geq \mathbb{P}\left[B_t \leq f(t), 0 \leq t \leq t_0\right] \cdot \mathbb{P}\left[B_t \leq f(t), t_0 \leq t \leq T\right].$$

Remark 3.2. Let us comment on the regularity assumptions: it is clear that these are of technical matter and of no importance to the question. Note that one can easily modify a regular function f such that either (1.1) fails or (1.2) does not hold. The only way to avoid pathologies and to prove a general result is to assume regularity. Note that the theorem is obviously true if we replace f by an irregular function $g \notin C^2(0,\infty)$ with $f \leq g$. The same can be said about the monotonicity/convexity assumption in the second part of Theorem 2.1.

Remark 3.3. It is easy to see that the integral test (1.1) implies

$$\int_0^\infty f''(s)s^{1/2}\,\mathrm{d}s < \infty \qquad \text{and} \qquad \int_0^\infty f'(s)^2\,\mathrm{d}s < \infty$$

under the assumption of monotonicity and concavity.

Remark 3.4. Thanks to [6], Theorem 2, if (1.1) holds one does not only obtain (2.1) but also the strong asymptotic order

$$\lim_{T\to\infty} T^{1/2}\mathbb{P}\left[B_t \leq f(t), t \leq T\right] = \sqrt{\frac{2}{\pi}}\,\mathbb{E}B_\tau,$$

where $0 < \mathbb{E}B_\tau = \mathbb{E}f(\tau) < \infty$ with $\tau := \inf\{t > 0 : B_t = f(t)\}$.

Remark 3.5. Note that the technique of the main proof (Jensen's inequality, Girsanov's theorem) does carry over to other processes. The crucial point is determining the repulsion effect of the conditioning in Lemma 2.2. The authors do not see at the moment how a similar lemma can be established for processes other than Brownian motion, e.g., FBM.

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On Lévy's Equivalence Theorem in Skorohod Space

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Abstract. A new and simple proof of Lévy's Equivalence Theorem in Skorohod space is given. This result and its consequences complement and complete the recent work of the authors [1].

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1. Introduction

Lévy's Equivalence Theorem is a beautiful result in the classical probability theory. It says that the three types of convergence, in distribution, in probability, and almost surely, for partial sums of independent random variables are equivalent. This theorem has a long history and many generalizations. In particular, Itô and Nisio [5] established Lévy's Equivalence Theorem for random variables taking values in a separable Banach space, and added a new and powerful statement allowing to deduce the almost sure convergence in the norm from the convergence of one-dimensional projections. This condition was then applied to show the uniform convergence in series decompositions of a Brownian motion and other continuous Gaussian processes.

In order to investigate series decompositions of jump processes, such as càdlàg Volterra processes driven by Lévy processes, it is natural to consider random series in the space $D[0,1]$. The convergence in the Skorohod as well as in the uniform topologies are of interest. However, $D[0,1]$ is not separable under the uniform norm and it is known that the Itô-Nisio Theorem does not hold in many non-separable Banach spaces, see [1, Remark 2.4]. Nevertheless, for $D([0,1];E)$, the space of càdlàg functions from $[0,1]$ into a separable Banach space E endowed with the uniform topology, Basse-O'Connor and Rosiński [1, Theorem 2.1] showed that the following version of the Itô-Nisio Theorem holds. In the following a random element in $D([0,1];E)$ is a random function taking values in $D([0,1];E)$ measurable for the cylindrical σ-algebra.

Theorem 1 (Basse-O'Connor, Rosiński). *Let $S_n = \sum_{j=1}^{n} X_j$, $n \in \mathbb{N}$, where X_j are independent random elements in $D([0,1];E)$. Suppose there exist a random element Y in $D([0,1];E)$ and a dense subset T of $[0,1]$ such that $1 \in T$ and for any $t_1, \dots, t_k \in T$*

$$(S_n(t_1), \dots, S_n(t_k)) \xrightarrow{d} (Y(t_1), \dots, Y(t_k)) \quad \text{as } n \to \infty. \tag{1}$$

Then there exists a random element S in $D([0,1];E)$ with the same distribution as Y such that

(i) *$S_n \to S$ a.s. uniformly on $[0,1]$, provided X_j are symmetric.*

(ii) *If X_j are not symmetric, then*

$$S_n + y_n \to S \qquad \text{a.s. uniformly on } [0,1]. \tag{2}$$

for some $y_n \in D([0,1];E)$ such that $\lim_{n\to\infty} y_n(t) = 0$ for every $t \in T$.

(iii) *Moreover, if the family $\{|S(t)|_E : t \in T\}$ is uniformly integrable and the functions $t \mapsto \mathbb{E}(X_n(t))$ belong to $D([0,1];E)$, then one can take in (2) y_n given by*

$$y_n(t) = \mathbb{E}(S(t) - S_n(t)).$$

The question whether Lévy's Equivalence Theorem is valid in $D([0,1];E)$ under the Skorohod topology is not addressed by this theorem (in the non-symmetric case). It was answered affirmatively, and somewhat unexpectedly, by Kallenberg [7, Theorem 1] in the case $E = \mathbb{R}$. However, Kallenberg's proof is difficult to follow; it is based on a deep and very convoluted analysis of jumps. The goal of the present note is to give a simpler alternative proof of Lévy's Equivalence Theorem for $D([0,1];E)$, under the Skorohod topology, as a consequence of our Theorem 1. This result and its consequences complement and complete the recent work of the authors [1].

Finally, notice that the validity of Lévy's Equivalence Theorem in the Skorohod space is far from being obvious. Typical methods used to prove such results are based on Lévy-Ottaviani's inequalities, which utilize convexity arguments, and on a centering, see, e.g., [9]. The following two examples are discouraging to this direction of a proof.

Example 2. Let $X(t) = \mathbf{1}_{[U,1]}(t)$, where the random variable U has a continuous distribution on $[0,1]$. Then, for every *convex* compact set $K \subset D[0,1]$, $\mathbb{P}(X \in K) = 0$.

Indeed, let K be a convex compact subset of $D[0,1]$ relative to Skorohod's J_1-topology. According to [3, Theorem 6], for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and $t_1, \dots, t_n \in [0,1]$ such that for each $x \in K$ and $t \in [0,1] \setminus \{t_1, \dots, t_n\}$, we have $|\Delta x(t)| = |x(t) - x(t-)| \le \epsilon$. Taking $\epsilon = 1/2$ we get

$$\mathbb{P}(X \in K) = \mathbb{P}(\mathbf{1}_{[U,1]} \in K) \le \mathbb{P}(U \in \{t_1, \dots, t_n\}) = 0.$$

Another example addresses the discontinuity of addition in $D[0,1]$, which affects centering arguments.

Example 3. Let $f_n = \mathbf{1}_{[2^{-1}-n^{-1},1]}$ and $f = \mathbf{1}_{[2^{-1},1]}$. Then $f_n \to f$ in $D[0,1]$ in the Skorohod topology, but $f_n - f \not\to 0$. In fact, $f_n - f$ does not converge in $D[0,1]$.

1.1. Definitions and notations

$(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $(E, |\cdot|_E)$ is a separable Banach space, and $D([0,1];E)$ is the space of càdlàg functions from $[0,1]$ into E. (Càdlàg means right-continuous with left-hand limits.) The uniform norm of $x \in D([0,1];E)$ is denoted $\|x\| = \sup_{t\in[0,1]} |x(t)|_E$ and put $\Delta x(t) = x(t) - x(t-)$ for the size of jump of x at t. Skorohod's J_1-topology on $D([0,1];E)$ is given by the following metric:

$$d(x,y) = \inf_{\lambda\in\Lambda} \max\Big\{ \sup_{t\in[0,1]} |x(t) - y\circ\lambda(t)|_E,\ \sup_{t\in[0,1]} |\lambda(t) - t| \Big\},$$

where Λ is the class of strictly increasing, continuous mappings of $[0,1]$ onto itself (see, e.g., [2, page 124]). The functionals w and w', given below, are important to characterize compact sets in $D([0,1];E)$. For all $x \in D([0,1];E)$ and $\delta > 0$

$$w(x,\delta) = \sup_{u,t\in[0,1],\,|u-t|\le\delta} |x(u) - x(t)|_E,$$

$$w'(x,\delta) = \inf_{(t_i)_{i=0}^k} \max_{1\le i\le k} \sup_{u,t\in[t_{i-1},t_i)} |x(u) - x(t)|_E,$$

where the infimum is taken over all $(t_i)_{i=0}^k$ with $k \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $\delta \le t_i - t_{i-1}$ for all $i = 1, \ldots, k$. For fixed δ, $w(x,\delta)$ and $w'(x,\delta)$ are upper-semicontinuous in x for the Skorohod topology, thus measurable, see [2, Sec. 12, Lemma 4]. It will be convenient for us to use the following characterization of precompact sets in $D([0,1];E)$. The closure of $A \subset D([0,1];E)$ is compact relative to Skorohod's J_1-topology if and only if

(a) there exists (equivalently, for every) dense set $T \subset [0,1]$, with $1 \in T$, such that, for each $t \in T$, the set $\{x(t) : x \in A\}$ is precompact in E,

(b) $\lim_{\delta\to 0} \sup_{x\in A} w'(x,\delta) = 0$.

In the case $E = \mathbb{R}$, this characterization is proved in the corollary that follows Theorem 13.2 in [2]. In general, a proof of this criterion is a straightforward adaption of arguments from [4, Ch. 3, Theorem 6.3], where a similar characterization is given for $D([0,\infty);E)$. Other useful criteria for precompactness in general Skorohod spaces are given in [6]. For comprehensive information on $D([0,1];E)$ we refer to [2], and [8]. Integrals of E-valued functions are defined in the Bochner sense and $\stackrel{d}{\to}$ and $\stackrel{d}{=}$ denote, respectively, convergence and equality in distribution.

2. Lévy's Equivalence Theorem for $D([0,1];E)$

The theorem reads as follows:

Theorem 4. *Let E be a separable Banach space. Let $S_n = \sum_{j=1}^n X_j$, $n \in \mathbb{N}$, where X_j are independent random elements in $D([0,1];E)$. Then the following condi-*

tions are equivalent for the convergence in $D([0,1];E)$ equipped with the Skorohod topology:

(i) $\{S_n\}_{n\in\mathbb{N}}$ *converges in distribution,*
(ii) $\{S_n\}_{n\in\mathbb{N}}$ *converges in probability,*
(iii) $\{S_n\}_{n\in\mathbb{N}}$ *converges almost surely.*

Proof. We only need to prove that (i) implies (iii). Assume that $\{S_n\}$ converges in distribution to some probability measure μ on $D([0,1];E)$ and let

$$T_\mu := \{t \in (0,1) : \mu(x : \Delta x(t) = 0) = 1\} \cup \{0,1\}.$$

The set $\{t \in [0,1] : t \notin T_\mu\}$ is at most countable, cf. [2, p. 139]. Since for each $t \in T_\mu$, $\lim_n S_n(t)$ exists in distribution, $\{S_n(t)\}$ converges a.s. in E by Lévy's Equivalence Theorem. Therefore, to prove that $\{S_n\}$ converges a.s. in $D([0,1];E)$ it is enough to show that there exists a set $\Omega_0 \subset \Omega$ of probability one such that for each $\omega \in \Omega_0$, the set $\{S_n(\cdot,\omega)\}_{n\in\mathbb{N}}$ is precompact in $D([0,1];E)$. In view of (a)–(b) of Section 1.1, it suffices to show that for some dense set $T \subset [0,1]$, with $1 \in T$, the following two conditions hold a.s.

$$\text{for each } t \in T \text{ the set } \{S_n(t)\}_{n\in\mathbb{N}} \text{ is precompact in } E, \tag{3}$$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} w'(S_n,\delta) = 0. \tag{4}$$

Choose a countable set $T \subset T_\mu$, dense in $[0,1]$ such that $1 \in T$. Since $\{S_n(t)\}$ converges a.s. for each $t \in T$, condition (3) is obviously satisfied. It remains to prove (4).

By Theorem 1(ii) there exist $\{y_n\} \subseteq D([0,1];E)$ and a random element S in $D([0,1];E)$ such that $S_n + y_n \to S$ a.s. in $\|\cdot\|$ and $\lim_n y_n(t) = 0$ for all $t \in T$. Let $\epsilon > 0$ be a fixed positive number and for all processes X in $D([0,1];E)$ define the process X^*, depending on S and ϵ, by

$$X^*(t) = X(t) - \sum_{v\le t:\, |\Delta S(v)|_E > \epsilon} \Delta X(v), \quad t \in [0,1].$$

For $\delta = \delta(\omega) > 0$ small enough we may choose random numbers $0 = t_0 < t_1(\omega) < t_2(\omega) < \cdots < t_{k(\omega)} = 1$ such that $\{t \in [0,1] : |\Delta S(t)|_E > \epsilon\} \subseteq \{t_1,\dots,t_k\}$ and $\delta \le t_i - t_{i-1} \le 2\delta$ for all $i = 1,\dots,k$. For all $n \in \mathbb{N}$ we have

$$\begin{aligned} w'(S_n,\delta) &\le \max_{1\le i\le k} \sup_{u,t\in[t_{i-1},t_i)} |S_n(u) - S_n(t)|_E \\ &= \max_{1\le i\le k} \sup_{u,t\in[t_{i-1},t_i)} |S_n^*(u) - S_n^*(t)|_E \le w(S_n^*, 2\delta) \\ &\le w(S_n^* + y_n^*, 2\delta) + w(y_n^*, 2\delta) \le w(S_n^* + y_n^*, 2\delta) + 2\|y_n^*\|. \end{aligned}$$

Notice that y_n^* are stochastic processes while y_n are non-random. By the uniform convergence, $S_n^* + y_n^* \to S^*$ a.s. in $\|\cdot\|$ and hence

$$\limsup_{n\to\infty} w'(S_n,\delta) \le w(S^*, 2\delta) + 2\limsup_{n\to\infty} \|y_n^*\| \qquad \text{a.s.} \tag{5}$$

We infer that

$$\limsup_{\delta\to 0} w(S^*, 2\delta) \le \|\Delta S^*\| \le \epsilon \qquad \text{a.s.} \tag{6}$$

where the second inequality follows by definition of S^* and the first inequality follows from the inequality $w(x,\delta) \le 2w'(x,\delta) + \|\Delta x\|$ for all $x \in D([0,1];E)$, see [2, eq. (12.9)], and that $\lim_{\delta\to 0} w'(x,\delta) = 0$ for all $x \in D([0,1];E)$, see [2, Sec. 12, Lemma 1].

To show (4) it is, according to (5)–(6), enough to show that

$$\limsup_{n\to\infty} \|y_n^*\| \le \epsilon \quad \text{a.s.} \tag{7}$$

To this end we note that $W_n := (S_n, S_n + y_n)$ converges in distribution to (S,S) in $D([0,1];E)^2$ equipped with the product topology; this follows by tightness of $\{W_n\}_{n\in\mathbb{N}}$ and since for all $t \in T$

$$\lim_{n\to\infty} S_n(t) = \lim_{n\to\infty} \big[S_n(t) + y_n(t)\big] = S(t) \qquad \text{a.s.}$$

By Skorohod's Representation Theorem, see [8, Theorem 6.7], there exist random elements $\{Z_n\}$ and Z in $D([0,1];E)^2$ defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ such that $Z_n \overset{d}{=} W_n$ for $n \in \mathbb{N}$, $Z \overset{d}{=} S$ and $\lim_n Z_n = Z$ a.s. By measurability of addition, $Z_n = (U_n, U_n + y_n)$ and $Z = (U,U)$ for some random elements $U_n \overset{d}{=} S_n$ and $U \overset{d}{=} S$ in $D([0,1];E)$. That is, $\lim_n U_n = U$ a.s. and $\lim_n [U_n + y_n] = U$ a.s. in $D([0,1];E)$. By definition of the Skorohod topology we may choose two sequences $\{\lambda_n^1(\cdot,\omega)\}_{n\in\mathbb{N}}$ and $\{\lambda_n^2(\cdot,\omega)\}_{n\in\mathbb{N}}$ in Λ (defined in Section 1.1) such that

$$\|U_n - U\circ\lambda_n^1\| + \|U_n + y_n - U\circ\lambda_n^2\| + \|\lambda_n^1 - I\| + \|\lambda_n^2 - I\| \to 0 \quad \text{a.s.}$$

where $I(t) = t$ for all $t \in [0,1]$, the first two $\|\cdot\|$ are the sup-norm on E and the last two $\|\cdot\|$ are the sup-norm on $\mathbb{R}$. This implies that

$$\|U\circ\lambda_n^1 - U\circ\lambda_n^2 + y_n\| \to 0 \quad \text{a.s.}$$

For all processes X in $D([0,1];E)$ defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ let X' be the process, depending on U and ϵ, given by

$$X'(t) = X(t) - \sum_{v\le t:\, |\Delta U(v)|_E > \epsilon} \Delta X(v).$$

Then

$$\|U'\circ\lambda_n^1 - U'\circ\lambda_n^2 + y_n'\| \to 0 \quad \text{a.s.}$$

which implies that with probability one

$$\limsup_{n\to\infty} \|y_n'\| \le \limsup_{n\to\infty} \|U'\circ\lambda_n^1 - U'\circ\lambda_n^2\| \le \limsup_{n\to\infty} w(U', \|\lambda_n^1 - \lambda_n^2\|) \le \epsilon$$

in the last inequality we have used that $\|\Delta U'\| \le \epsilon$ and $\|\lambda_n^1 - \lambda_n^2\| \to 0$ a.s. This shows (7) since $\{y_n'\}_{n\in\mathbb{N}}$ and $\{y_n^*\}_{n\in\mathbb{N}}$ has the same finite dimensional distributions. Since (4) follows from (7), the proof is complete. □

Corollary 5. *Under the above notation, suppose also that X_j are symmetric. Then the following conditions are equivalent to* (i)–(iii) *of Theorem* 4.

(iv) $\{S_n\}_{n\in\mathbb{N}}$ *is tight,*
(v) $\{S_n\}_{n\in\mathbb{N}}$ *converges uniformly a.s.*

Proof. Assumption (iv) implies that for all $t \in [0,1]\cap\mathbb{Q}$, $\{S_n(t)\}_{n\in\mathbb{N}}$ is tight in E, see [4, Ch. 3, Theorem 7.2], so by symmetry we have that $\lim_n S_n(t)$ exists a.s. in E cf. Itô and Nisio [5, Theorem 4.1]. This shows that $\{S_n\}$ has at most one cluster point and proves the implication (iv)⇒(i). To prove (i)⇒(v) assume that $\{S_n\}$ converges in distribution to some probability measure μ on $D([0,1];E)$. Condition (1) of Theorem 1 is satisfied for $T = \{t \in (0,1) : \mu(x : \Delta x(t) = 0) = 1\} \cup \{0,1\}$ which by Theorem 1(i) shows (v). □

The next corollary gives an alternative and simpler proof of Theorem 3 in [7] and of Corollary 2.2 in [1]. The proof combines Theorems 1 and 4.

Corollary 6. *If* $S_n \xrightarrow{d} Y$ *in the Skorohod* J_1*-topology and* Y *does not have a jump of non-random size and location, then* S_n *converges a.s. uniformly on* $[0,1]$.

Proof. By Theorem 4, $S_n \to S$ a.s in the Skorohod J_1-topology, so that we may choose a sequence $\{\lambda_n(\cdot,\omega)\}_{n\in N'}$ in Λ such that as $n\to\infty$,

$$\sup_{s\in[0,1]} |S_n(s) - S(\lambda_n(s))|_E + \sup_{s\in[0,1]} |\lambda_n(s) - s| \to 0 \qquad \text{a.s.} \tag{8}$$

Since condition (1) of Theorem 1 holds for

$$T = \{t \in (0,1) :\ \mathbb{P}(\Delta S(t) = 0) = 1\} \cup \{0,1\},$$

by part (ii) of that theorem there exists $\{y_n\} \subseteq D([0,1];E)$ such that $\|S_n + y_n - S\| \to 0$ a.s. Moreover, $\lim_{n\to\infty} y_n(t) = 0$ for every $t \in T$. We want to show that $\|y_n\| \to 0$.

Assume to the contrary that $\limsup_{n\to\infty} \|y_n\| > \epsilon > 0$. Then there exist a subsequence $N' \subseteq \mathbb{N}$, and a monotone sequence $\{t_n\}_{n\in N'} \subset [0,1]$ with $t_n \to t$ such that $|y_n(t_n)|_E \geq \epsilon$ for all $n \in N'$. Assume that $t_n \uparrow t$ (the case $t_n \downarrow t$ follows similarly). From the uniform convergence we have that $S_n(t_n) + y_n(t_n) \to S(t-)$ a.s. ($n\to\infty$, $n \in N'$).

Therefore, using (8),

$$\begin{aligned} &|S(\lambda_n(t_n)) - S(t-) + y_n(t_n)|_E \\ &\quad \leq |S(\lambda_n(t_n)) - S_n(t_n)|_E + |S_n(t_n) + y_n(t_n) - S(t-)|_E \to 0 \quad \text{a.s.} \end{aligned} \tag{9}$$

Since $\lambda_n(t_n) \to t$ a.s. as $n\to\infty$, $n \in N'$, the sequence $\{S(\lambda_n(t_n))\}_{n\in N'}$ is relatively compact in E with at most two cluster points, $S(t)$ or $S(t-)$. By (9), the cluster points for $\{y_n(t_n)\}_{n\in N'}$ are $-\Delta S(t)$ or 0 and since $|y_n(t_n)|_E \geq \epsilon$ we have that $y_n(t_n) \to -\Delta S(t)$ a.s., $n \in N'$. This shows that $\Delta S(t) = c$ a.s. for some non-random $c \in E\setminus\{0\}$, which contradicts our assumption. □

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Continuity Conditions for a Class of Second-order Permanental Chaoses

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Abstract. Just as permanental processes are generalizations of stochastic processes that are the square of Gaussian processes we define permanental fields as a generalization of certain second-order Gaussian chaos processes. A sufficient condition for the continuity of permanental fields is obtained that generalizes an earlier result for second-order Gaussian chaoses.

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1. Introduction

An α-permanental process $\theta := \{\theta_x, x \in S\}$, is a real-valued positive stochastic process that is determined by a real-valued kernel $\Gamma = \{\Gamma(x,y), x, y \in T\}$, in the sense that its finite joint distributions are given by

$$E\left(\exp\left(-\sum_{i=1}^{n} \lambda_i \theta_{x_i}\right)\right) = \frac{1}{|I + \Lambda\Gamma|^{\alpha}}, \tag{1.1}$$

where I is the $n \times n$ identity matrix, Λ is the $n \times n$ diagonal matrix with entries $(\lambda_1, \dots, \lambda_n)$, $\Gamma = \{\Gamma(x_i, x_j)\}_{i,j=1}^{n}$ is an $n \times n$ matrix, and $\alpha > 0$. It is shown in [8, Proposition 4.2] that if $\theta := \{\theta_x, x \in S\}$ is an α-permanental process with kernel Γ, then for any $x_1, \dots, x_n \in S$

$$E\left(\prod_{j=1}^{n} \theta_{x_j}\right) = \sum_{\pi \in \mathcal{P}} \alpha^{c(\pi)} \prod_{j=1}^{n} \Gamma(x_j, x_{\pi(j)}), \tag{1.2}$$

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where $\mathcal{P}$ is the set of permutations π of [1,n], and $c(\pi)$ is the number of cycles in the permutation π.

If Γ is symmetric and positive definite and $\alpha = 1/2$, then $\theta = G^2/2$, where $G = \{G_x, x \in S\}$ is a mean zero Gaussian process with covariance Γ. However, in the definition (1.1), Γ need not be symmetric or positive definite.

Under this definition

$$E(\theta_x \theta_y) = \alpha\Gamma(x,y)\Gamma(y,x) + \alpha^2\Gamma(x,x)\Gamma(y,y). \tag{1.3}$$

We can simplify things if we consider the normalized, mean zero permanental process

$$H_x = \theta_x - E\theta_x, \qquad x \in S. \tag{1.4}$$

We then have

$$E\left(\prod_{j=1}^n H_{x_j}\right) = \sum_{\pi\in\mathcal{P}'} \alpha^{c(\pi)} \prod_{j=1}^n \Gamma(x_j, x_{\pi(j)}), \tag{1.5}$$

where $\mathcal{P}'$ is the set of permutations π of [1,n] such that $\pi(j) \neq j$ for any j, and, as above, $c(\pi)$ is the number of cycles in the permutation π. We now have

$$E(H_x H_y) = \frac{1}{2}\Gamma(x,y)\Gamma(y,x). \tag{1.6}$$

Consider the stochastic process

$$\psi(\mu_j) = \int H_x \, d\mu_j(x) \tag{1.7}$$

for some family of finite measures $\{\mu_j\}$ on S. The moments of $\{\psi(\mu_j)\}$ are given in (1.11) below. We notice that, depending on the measure, the kernel Γ does not have to be finite on its diagonal in order for the right-hand side of (1.11) to be finite.

Let S be a locally compact metric space with countable base. Let $\mathcal{B}(S)$ denote the Borel σ-algebra, and let $\mathcal{M}(S)$ be the set of finite signed Radon measures on $\mathcal{B}(S)$. When $\{\theta_x, x \in S\}$ is the square of a Gaussian process and the kernel Γ, necessarily symmetric and positive definite, and the measures μ in $\mathcal{M}(S)$, are such that

$$\iint \Gamma^2(x,y)\, d\mu(x)d\mu(y) < \infty, \tag{1.8}$$

the integral in (1.7) is a second-order Gaussian chaos and well-known results give good sufficient conditions for $\{\psi(\mu), \mu \in \mathcal{M}(S)\}$ to be continuous on $\mathcal{M}(S)$ with respect to the metric

$$\widetilde{d}(\mu,\nu) = \left(\iint \Gamma^2(x,y)\, d(\mu(x)-\mu(y))d(\nu(x)-\nu(y))\right)^{1/2}. \tag{1.9}$$

This work is done in [5] in which we show that even when $\Gamma(x,x) = \infty$, so that G_x is not defined, as long as

$$\iint \Gamma^2(x,y)\, d\mu(x)d\mu(x) < \infty \qquad \forall\, \mu \in \mathcal{M}(S), \tag{1.10}$$

a second-order Gaussian chaos process $\{\mathcal{G}(\mu), \mu \in \mathcal{M}(S)\}$ exists, and the same well-known results give good sufficient conditions for it to be continuous on $\mathcal{M}(S)$ with respect to the metric in (1.9).

In this paper we consider the existence and continuity of a more general class of processes than second-order Gaussian chaoses that arise when we generalize (1.10) so that the kernel Γ, which may have $\Gamma(x,x) = \infty$, need not be symmetric or positive definite. Since these are not Gaussian chaos processes we don't have a convenient theory available to analyze them or even define them. Therefore, we define them by giving their moments.

Definition 1.1. A map ψ from a subset $\mathcal{V} \subseteq \mathcal{M}(S)$ to $\mathcal{F}$ measurable functions on a probability space $(\Omega, \mathcal{F}, P)$ is called an α-permanental field with kernel Γ if for all $\nu \in \mathcal{V}$, $E\psi(\nu) = 0$ and for all integers $n \geq 2$ and $\nu_1, \dots, \nu_n \in \mathcal{V}$

$$E\Big(\prod\nolimits_{j=1}^{n} \psi(\nu_j)\Big) = \sum\nolimits_{\pi \in \mathcal{P}'} \alpha^{c(\pi)} \int \prod\nolimits_{j=1}^{n} \Gamma(x_j, x_{\pi(j)}) \prod\nolimits_{j=1}^{n} d\nu_j(x_j), \quad (1.11)$$

where $\mathcal{P}'$ is the set of permutations π of $[1,n]$ such that $\pi(j) \neq j$ for any j, and $c(\pi)$ is the number of cycles in the permutation π.

In the course of our proofs we will see that random variables $\psi(\nu)$, $\nu \in \mathcal{V}$ satisfying (1.11) are exponentially integrable for a large class of $\mathcal{V}$ and Γ, hence their finite joint distributions are determined by their moments.

Because Definition 1.1 is satisfied by a second-order Gaussian chaos process when Γ is symmetric and positive definite we refer to processes on $\mathcal{V}$ with finite joint distributions given by (1.11) as second-order permanental chaoses.

We obtain a sufficient condition for the continuity of $\{\psi(\nu); \nu \in \mathcal{V}\}$ that generalizes a very well-known result for the second-order Gaussian chaos, the case when $u(x,y)$ is positive definite and symmetric. The result for the second-order Gaussian chaos depends on the fact that

$$\begin{aligned} \|\psi(\mu) - \psi(\nu)\|_\rho &\leq C\left(E(\psi(\mu) - \psi(\nu))^2\right)^{1/2} \qquad (1.12)\\ &= \frac{C}{2}\left(\iint u^2(x,y)\, d(\mu(x) - \nu(x)) d(\mu(y) - \nu(y))\right)^{1/2}, \end{aligned}$$

where $\|\cdot\|_\rho$ is the norm of the Orlicz space corresponding to $\exp|x| - 1$.

What we do in this paper is find metrics $\tau(\mu,\nu)$ that dominate $\|\psi(\mu) - \psi(\nu)\|_\rho$ when u is not symmetric. Then the same majorizing measure condition for the continuity of a second-order Gaussian chaos given in terms of $\left(E(\psi(\mu) - \psi(\nu))^2\right)^{1/2}$ holds for the second-order permanental chaos with this norm replaced by $\tau(\mu,\nu)$. (Details are given in Section 2.)

Here is a summary of our results:

Theorem 1.1. *Let $\{\psi(\mu), \mu \in \mathcal{V}\}$ be an α-permanental process with kernel u. Then under any of the various conditions on u listed in* 1.–5.

$$\|\psi(\mu) - \psi(\nu)\|_\rho \leq \|\mu - \nu\|_i, \qquad i = a,b,c,d,e \qquad (1.13)$$

for all $\mu, \nu \in \mathcal{V}$.

1. *The kernel $u(x,y)$ is defined on $R^m \times R^m$ and is a function of $y-x$.*

$$\|\mu-\nu\|_a = C_1 \left(\iint \left(\int e^{i(x-y)q} |\hat{u}(q)|\, dq \right)^2 d((\mu-\nu)(x))\, d((\mu-\nu)(y)) \right)^{1/2}$$
$$= C_1' \left(\int |\hat{\mu}(y) - \hat{\nu}(y)|^2 \widetilde{\gamma}(x)\, dx \right)^{1/2}, \tag{1.14}$$

where C_1 is a constant that depends only on α. (Similarly for C_i, $i = 2, \ldots, 5$ and C_3'.)

$$\widetilde{\gamma}(x) = \int |\hat{u}(x-y)|\, |\hat{u}(y)|\, dy \tag{1.15}$$

and $\hat{u}$ and $\hat{\nu}$ denote the Fourier transforms of u and ν.

2. *The kernel u is an α-potential density of a Markov process, i.e.,*

$$u(x,y) = \int_0^\infty e^{-\alpha s} p_s(x,y)\, ds, \qquad \alpha \ge 0 \tag{1.16}$$

where p_s is a transition probability density.

$$\|\mu-\nu\|_b = \left(\iint \Phi(x,y)\, d(\mu(x)-\nu(x))\, d(\mu(y)-\nu(y)) \right)^{1/2}, \tag{1.17}$$

where

$$\Phi(x,y) = \Theta_l(x,y)\Theta_r(x,y) \tag{1.18}$$

and

$$\begin{aligned}\Theta_l(x,y) &= \int_0^\infty e^{-\alpha s} \int p_{s/2}(x,u) p_{s/2}(y,u)\, du\, ds,\\ \Theta_r(x,y) &= \int_0^\infty e^{-\alpha s} \int p_{s/2}(u,x) p_{s/2}(u,y)\, du\, ds.\end{aligned} \tag{1.19}$$

3. *The kernel*

$$u(x,y) = \int \frac{(1-e^{i\lambda x})(1-e^{-i\lambda y})}{\phi(\lambda)}\, d\lambda, \tag{1.20}$$

where ϕ is a Fourier transform.

$$\|\mu-\nu\|_c \le \left(\iint \frac{1}{|\phi(r)\phi(q)|} \left| \int (1-e^{-irz})(1-e^{iqz})\, d(\mu-\nu)(z) \right|^2 dr\, dq \right)^{1/2}$$
$$= \left(\iint \left(\int \frac{(1-e^{i\lambda x})(1-e^{-i\lambda y})}{|\phi(\lambda)|}\, d\lambda \right)^2 d(\mu-\nu)(x)\, d(\mu-\nu)(y) \right)^{1/2} \tag{1.21}$$

4. *The kernel u satisfies the conditions in 1 and is symmetric and $\hat{u}$ is positive, in 2 and the transition probabilities are symmetric, or in 3 and ϕ is real and*

positive.

$$\begin{aligned}\|\mu-\nu\|_i &= C_4\left(\int\!\!\int u^2(x,y)\,d(\mu(x)-\nu(x))\,d(\mu(y)-\nu(y))\right)^{1/2}\\ &= C_4'\left(E(\psi(\mu)-\psi(\nu))^2\right)^{1/2},\end{aligned} \tag{1.22}$$

where $i=a$, b, or c depending on the case considered.

5. *No condition on the kernel $u(x,y)$, which is a function on $S\times S$.*

$$\|\mu-\nu\|_e = \left(\int\!\!\int u^2(x,y)\,d(|\mu-\nu|(x))\,d(|\mu-\nu|(y))\right)^{1/2} \tag{1.23}$$

where $|\nu|$ is the total variation of the measure ν.

Using Theorems 1.1 and 2.1, below, we obtain continuity conditions for α-permanental fields. However, the question remains, how do we know that there are any α-permanental fields which may have $\Gamma(x,x)=\infty$, other than second-order Gaussian chaoses, the case when the kernel u is symmetric and positive definite and $\alpha=1/2$. In [3], using loop soups, we show that permanental fields with kernels u exist. They can be associated with continuous additive functionals of Markov processes with 0-potential densities u. In that work we develop an Isomorphism Theorem, generalizing that of Dynkin, [5], connecting the continuous additive functionals $L=\{L_t^\nu,(\nu,t)\in\mathcal{V}\times R_+^1\}$ of a Markov process X with the associated permanental field $\Psi=\{\psi(\nu),\nu\in\mathcal{V}\}$. In particular, the Isomorphism Theorem can be used to show that if Ψ is almost surely continuous then so is L. This was our motivation for finding sufficient conditions for the continuity of $\{\psi(\nu);\nu\in\mathcal{V}\}$.

The Markov processes we consider are transient Borel right processes with state space S. These processes have jointly measurable transition densities $p_t(x,y)$ with respect to some σ-finite measure m on S. An example is given by Lévy processes on R^m.

Let $Y=\{Y_t,t\in R_+\}$ be a Lévy process in R^m with characteristic function

$$Ee^{i\lambda Y(t)}=e^{-\phi(\lambda)t}. \tag{1.24}$$

We refer to ϕ as the characteristic exponent of Y. Consider a transient Markov process $X=\{X_t,t\in R_+\}$ that is Y killed at ξ_β, an independent exponential time with mean $1/\beta>0$. Then

$$u^\beta(x,y)=\int_0^\infty e^{-\beta t}p_t(x,y)\,dt=\int\frac{e^{i\lambda(x-y)}}{\beta+\phi(\lambda)}\,d\lambda=:u^\beta(x-y) \tag{1.25}$$

is the zero potential density of X. It is kernels such as these that we consider in Theorem 1.1, 1.

In Theorem 1.1, 2, we consider the 0-potential densities u of Markov processes, when u is not a function of $x-y$.

When u is positive and symmetric and the kernel in Theorem 1.1, 3 satisfies (1.10) it defines a second-order Gaussian chaos. If the kernel is finite it is the covariance of a Gaussian process on R^m with stationary increments, [6, page 236] and the 0-potential density of a transient Markov process, [4, Lemma 5.1]. In these cases the α permanental processes exist and when $\alpha \neq 1/2$ they are not the squares of Gaussian processes. We do not know whether the kernel in Theorem 1.1, 3 defines α-permanental fields that are not second-order Gaussian chaoses when the kernel is not finite on the diagonal. It would be interesting to see if they do, since we then would have a strong condition for the fields to be continuous.

In Section 2 we state the well-know sufficient condition for continuity that we refer to above. The bulk of the paper, Section 3, is devoted to proving Theorem 1.1. An example is given in Corollary 3.2.

Lastly, in Section 4, we give conditions under which

$$\|\mu - \nu\|_a \leq C \left(E(\psi(\mu) - \psi(\nu))^2 \right)^{1/2}, \tag{1.26}$$

when u is not symmetric, a result that holds for second-order Gaussian chaoses.

2. Continuity

We mention the well-known sufficient condition for continuity of stochastic process that can be used with the metrics given in Theorem 1.1; see, e.g., [7, Section 3].

Let $\rho(x) = \exp(x) - 1$ and $L^\rho(\Omega, \mathcal{F}, P)$ denote the set of random variables $\xi : \Omega \to R^1$ such that $E\rho\left(|\xi|/c\right) < \infty$ for some $c > 0$. $L^\rho(\Omega, \mathcal{F}, P)$ is a Banach space with norm given by

$$\|\xi\|_\rho = \inf\left\{c > 0 : E\rho\left(|\xi|/c\right) \leq 1\right\}. \tag{2.1}$$

Let (T, τ) be a metric or pseudometric space. Let $B_\tau(t, u)$ denote the closed ball in (T, τ) with radius u and center t. For any probability measure μ on (T, τ) we define

$$J_{T,\tau,\mu}(a) = \sup_{t \in T} \int_0^a \log \frac{1}{\mu(B_\tau(t, u))}\, du. \tag{2.2}$$

The following basic continuity theorem gives sufficient conditions for continuity of permanental fields.

Theorem 2.1. *Let $X = \{X(t) : t \in T\}$ be a stochastic process such that $X(t, \omega) : T \times \Omega \mapsto [-\infty, \infty]$ is $\mathcal{A} \times \mathcal{F}$ measurable for some σ-algebra $\mathcal{A}$ on T. Suppose $X(t) \in L^\rho(\Omega, \mathcal{F}, P)$ and there exists a metric τ on T such that*

$$\|X(s) - X(t)\|_\rho \leq \tau(s, t). \tag{2.3}$$

Suppose furthermore that (T, τ) has finite diameter D, and that there exists a probability measure μ on $(T, \mathcal{A})$ such that

$$J_{T,\tau,\mu}(D) < \infty. \tag{2.4}$$

Then there exists a version $X' = \{X'(t), t \in T\}$ of X such that

$$E \sup_{t \in T} X'(t) \leq C\, J_{T,\tau,\mu}(D), \tag{2.5}$$

for some $C < \infty$. Furthermore for all $0 < \delta \le D$,

$$\sup_{\substack{s,t\in T\\ \tau(s,t)\le\delta}} |X'(s,\omega) - X'(t,\omega)| \le 2Z(\omega)\, J_{T,\tau,\mu}(\delta), \tag{2.6}$$

almost surely, where

$$Z(\omega) := \inf\left\{\alpha > 0 : \int_T \rho(\alpha^{-1}|X(t,\omega)|)\,\mu(dt) \le 1\right\} \tag{2.7}$$

and $\|Z\|_\rho \le K$, where K is a constant.

In particular, if

$$\lim_{\delta\to 0} J_{T,\tau,\mu}(\delta) = 0, \tag{2.8}$$

X' is uniformly continuous on (T,τ) almost surely.

3. Proof of Theorem 1.1

We prove Theorem 1.1 and also present some related material.

The following immediate consequence of Definition 1.1 enables us to simplify the notation.

Lemma 3.1. *Let $\{\psi(\nu); \nu \in \mathcal{M}(S)\}$ be a permanental field. Then*

$$E(\psi^n(\mu-\theta)) = E\left(\psi(\mu) - \psi(\theta)\right)^n. \tag{3.1}$$

Therefore to estimate $E(\psi^n(\mu-\theta))$ it suffices to consider $E(\psi^n(\nu))$ keeping in mind that the measure ν is generally not a positive measure.

We have the following general upper bounds for the terms in Definition 1.1, which we use in several contexts in this section:

Lemma 3.2. *Suppose that the kernel $u(x,y)$ has the from*

$$u(x,y) = \int f(x,\lambda)g(y,\lambda)h(\lambda)\,d\lambda, \tag{3.2}$$

where $\lambda \in R^n$, for some $n \ge 1$. (Recall that $x \in R^m$.) Let

$$I_n(u,\{\nu_j\}_{j=1}^n) = \left|\int u(y_1,y_2)\cdots u(y_{n-1},y_n)u(y_n,y_1)\prod_{j=1}^n d\nu_j(y_j)\right|. \tag{3.3}$$

and set

$$H_j(q,r) = \int f(z,q)g(z,r)\,d\nu_j(z). \tag{3.4}$$

Then

$$\begin{aligned} I_n(u,\{\nu_j\}_{j=1}^n) &\le \prod_{j=1}^n \left(\iint |H_j(r,q)|^2|h(r)|\,dr|h(q)|\,dq\right)^{1/2} \\ &= \prod_{j=1}^n \left(\iint R(z_1,z_2)\,T(z_1,z_2)\,d\nu_j(z_1)\,d\nu_j(z_2)\right)^{1/2}, \end{aligned} \tag{3.5}$$

where

$$R(z_1,z_2)=\int f(z_1,q)\overline{f(z_2,q)}|h(q)|\,dq,\quad T(z_1,z_2)=\int g(z_1,r)\overline{g(z_2,r)}|h(r)|\,dr. \tag{3.6}$$

Proof. We have

$$\prod_{j=1}^n u(z_j,z_{j+1})=\int\dots\int\prod_{j=1}^n f(z_j,\lambda_j)g(z_{j+1},\lambda_j)\prod_{j=1}^n h(\lambda_j)\,d\lambda_j, \tag{3.7}$$

in which we set $z_{n+1}=z_1$. We write

$$\prod_{j=1}^n f(z_j,\lambda_j)g(z_{j+1},\lambda_j)=\prod_{j=1}^n f(z_j,\lambda_j)g(z_j,\lambda_{j-1}) \tag{3.8}$$

where $\lambda_0=\lambda_n$. Using this we see that

$$\begin{aligned} I_n(u,\{\nu_j\}_{j=1}^n) &= \left|\int\left(\prod_{j=1}^n\int f(z_j,\lambda_j)g(z_j,\lambda_{j-1})\,d\nu_j(z_j)\right)\prod_{j=1}^n h(\lambda_j)\,d\lambda_j\right| \\ &= \left|\int\prod_{j=1}^n H_j(\lambda_j,\lambda_{j-1})\prod_{j=1}^n h(\lambda_j)\,d\lambda_j\right| \\ &\le \prod_{j=1}^n\left(\int\int|H_j(s,t)|^2|h(s)|\,ds|h(t)|\,dt\right)^{1/2}. \end{aligned} \tag{3.9}$$

The last inequality in (3.9) in a consequence of multiple applications of the Cauchy-Schwarz inequality as follows:

$$\begin{aligned} &\left|\int\prod_{j=1}^n H_j(\lambda_j,\lambda_{j-1})\prod_{j=1}^n h(\lambda_j)\,d\lambda_j\right| \\ &\le\int\left(\int|H_2(\lambda_2,\lambda_1)|^2|h(\lambda_1)|\,d\lambda_1\right)^{1/2}\left(\int|H_1(\lambda_1,\lambda_n)|^2|h(\lambda_1)|\,d\lambda_1\right)^{1/2} \\ &\qquad |H_3(\lambda_3,\lambda_2)|\cdots|H_n(\lambda_n,\lambda_{n-1})|\prod_{j=2}^n|h(\lambda_j)|\,d\lambda_j. \end{aligned} \tag{3.10}$$

We rearrange the terms so that the right-hand side of (3.10) is equal to

$$\begin{aligned} &\int\left(\int\left(\int|H_2(\lambda_2,\lambda_1)|^2|h(\lambda_1)|\,d\lambda_1\right)^{1/2}|H_3(\lambda_3,\lambda_2)||h(\lambda_2)|\,d\lambda_2\right) \\ &\qquad\left(\int|H_1(\lambda_1,\lambda_n)|^2|h(\lambda_1)|\,d\lambda_1\right)^{1/2}|H_4(\lambda_4,\lambda_3)| \\ &\qquad\qquad\cdots|H_n(\lambda_n,\lambda_{n-1})|\prod_{j=3}^n|h(\lambda_j)|\,d\lambda_j. \end{aligned}$$

Applying the Cauchy-Schwarz inequality again we see that this term is

$$\leq \left(\int\int |H_2(\lambda_2,\lambda_1)|^2|h(\lambda_1)|\,d\lambda_1|h(\lambda_2)|\,d\lambda_2\right)^{1/2}$$
$$\int\left(\int |H_3(\lambda_3,\lambda_2)|^2|h(\lambda_2)|\,d\lambda_2\right)^{1/2}\left(\int |H_1(\lambda_1,\lambda_n|^2|h(\lambda_1)|\,d\lambda_1\right)^{1/2}$$
$$|H_4(\lambda_4,\lambda_3)|\cdots|H_n(\lambda_n,\lambda_{n-1})|\prod_{j=3}^{n}|h(\lambda_j)|\,d\lambda_j.$$

Therefore the left-hand side of (3.10)

$$\leq \prod_{j=1}^{n}\left(\int\int |H_j(\lambda_j,\lambda_{j-1})|^2|h(\lambda_j)|\,d\lambda_j|h(\lambda_{j-1})|\,d\lambda_{j-1}\right)^{1/2}, \tag{3.11}$$

in which, as above, $\lambda_0=\lambda_n$. This is the same as the first line of (3.5).

To get the second line of (3.5) we interchange the order of integration in (3.11). We have

$$|H_j(\lambda_j,\lambda_{j-1})|^2 = \int f(z_1,\lambda_j)g(z_1,\lambda_{j-1})\,d\nu_j(z_1)\int \overline{f(z_2,\lambda_j)g(z_2,\lambda_{j-1})}\,d\nu_j(z_2). \tag{3.12}$$

Interchanging the order of integration, and integrating first with respect to λ_j and λ_{j-1}, we see that

$$\int\int |H_j(\lambda_j,\lambda_{j-1})|^2|h(\lambda_j)|\,d\lambda_j|h(\lambda_{j-1})|\,d\lambda_{j-1} \tag{3.13}$$
$$= \int\int R(z_1,z_2)T(z_1,z_2)\,d\nu_j(z_1)\,d\nu_j(z_2).$$

□

Corollary 3.1. *Let $\{\psi(\nu);\nu\in\mathcal{M}(S)\}$ be a permanental field with kernel u that can be represented as in* (3.2)*. Let*

$$\tau_u(\mu,\nu) := \left(\int\int R(z_1,z_2)T(z_1,z_2)\,d(\mu(z_1)-\nu(z_1))\,d(\mu(z_2)-\nu(z_2))\right)^{1/2}$$
$$= \left(\int\int |H(r,q;\mu,\nu)|^2\,|h(r)|\,dr\,|h(q)|\,dq\right)^{1/2} \tag{3.14}$$

for R and T as given in Lemma 3.2 *and*

$$H(q,r;\mu,\nu) = \int f(z,q)g(z,r)\,d(\mu(z)-\nu(z)). \tag{3.15}$$

Then

$$\|\psi(\mu)-\psi(\nu)\|_\rho \leq 2(\alpha\vee 1)\,\tau_u(\mu,\nu). \tag{3.16}$$

Moreover, if R and T are real valued $\tau_u(\mu,\nu)$ is a metric on $\mathcal{M}(S)$.

Proof. Consider

$$\alpha^{c(\pi)} \int \prod_{j=1}^{n} u(x_j, x_{\pi(j)}) \prod_{j=1}^{n} d\nu(x_j) \tag{3.17}$$

for a fixed permutation $\pi \in \mathcal{P}'$. Suppose it has p cycles of lengths $l_1, \ldots, l_p$. Label the elements in cycle q as $q_1, \ldots, q_{l_q}$. Consider

$$\int \prod_{j=1}^{l_q} u(x_j, x_{j+1}) \prod_{j=1}^{l_q} d\nu(x_j), \tag{3.18}$$

where $l_q + 1 = 1$. By Lemma 3.2,

$$\int \prod_{j=1}^{l_q} u(x_j, x_{j+1}) \prod_{j=1}^{l_q} d\nu(x_j) \leq \prod_{j=1}^{l_q} \left(\int \int R(x,y)T(x,y)\, d\nu(x)\, d\nu(y) \right)^{l_q/2}. \tag{3.19}$$

Therefore (3.17)

$$\leq (\alpha \vee 1)^n \prod_{j=1}^{n} \left(\int \int R(x,y)T(x,y)\, d\nu(x)\, d\nu(y) \right)^{n/2}, \tag{3.20}$$

since $\sum_{j=1}^{l} l_q = n$.

It now follows from Definition 1.1 that

$$E(\psi(\mu) - \psi(\nu))^n \leq (n-1)!\, (\alpha \vee 1)^n\, (\tau_u(\mu,\nu))^{n/2}. \tag{3.21}$$

When n is even the left-hand side of (3.21) is equal to $E|\psi(\mu) - \psi(\nu)|^n$. When n is odd by Hölders inequality $E|\psi(\mu) - \psi(\nu)|^{n-1} \leq (E|\psi(\mu) - \psi(\nu)|^n)^{n/(n-1)}$. Therefore the left-hand side of (3.21) can be replaced by $E|\psi(\mu) - \psi(\nu)|^n$ for all n. Consequently

$$E \left(\frac{|\psi(\mu) - \psi(\nu)|}{2(\alpha \vee 1)\, (\tau_u(\mu,\nu))^{1/2}} \right)^n \leq \frac{1}{2^n}, \tag{3.22}$$

which implies (3.16).

When $R(z_1, z_2)$ and $T(z_1, z_2)$ are real valued they are symmetric. If they are finite on their diagonals they define Gaussian processes, $\mathcal{R} = \{\mathcal{R}(z), z \in S\}$ and $\mathcal{T} = \{\mathcal{T}(z), z \in S\}$. Take $\mathcal{R}$ and $\mathcal{T}$ to be independent and for $\nu \in \mathcal{M}(S)$ consider the second-order Gaussian chaos

$$\mathcal{G}(\nu) = \int \mathcal{R}(z)\mathcal{T}(z)\, d\nu(z). \tag{3.23}$$

We have

$$\left(E(\mathcal{G}(\mu) - \mathcal{G}(\nu))^2 \right)^{1/2} = \tau_u(\mu,\nu). \tag{3.24}$$

Therefore $\tau_u(\mu,\nu)$ is a metric on $\mathcal{M}(S)$. However, even when $R(z_1, z_2)$ and $T(z_1, z_2)$ are infinite on their diagonals, following the argument in the beginning of Section 2, [5] we can construct a second-order Gaussian chaos for which (3.24) holds. Clearly, $\tau_u(\mu,\nu)$ is a metric on $\mathcal{M}(S)$. □

Remark 3.1. Obviously if (2.4) holds with the metric τ_u the permanental field ψ is continuous. Therefore, for a permanental field with kernel u that is not symmetric, in some sense, (3.23), when $R(z_1,z_2)$ and $T(z_1,z_2)$ are real valued, is a dominating second-order Gaussian chaos.

This second-order Gaussian chaos is not the type that is described by (1.11). However, by the Cauchy-Schwartz inequality

$$\begin{aligned}\tau_u^2(\mu,\nu) \le &\left(\iint R^2(z_1,z_2)\,d(\mu(z_1)-\nu(z_1))\,d(\mu(z_2)-\nu(z_2))\right)^{1/2}\\ &\cdot\left(\iint T^2(z_1,z_2)\,d(\mu(z_1)-\nu(z_1))\,d(\mu(z_2)-\nu(z_2))\right)^{1/2}.\end{aligned}\tag{3.25}$$

The two terms on the right in (3.25) are the L^2 metrics of second-order Gaussian chaoses with kernels $R(z_1,z_2)$ and $T(z_1,z_2)$. Call these metrics $d_1(\mu,\nu)$ and $d_2(\mu,\nu)$. We then have that

$$\tau_u(\mu,\nu) \le \frac{d_1(\mu,\nu)+d_2(\mu,\nu)}{2}.\tag{3.26}$$

We now use Lemma 3.2 and Corollary 3.1 to prove Theorem 1.1.

Proof of Theorem 1.1. We begin with 1. and assume that $u\in L^2$ and denote by $\hat{u}$ its Fourier transform. We write

$$\begin{aligned}u(x-y) &= \frac{1}{(2\pi)^m}\int e^{-i(y-x)\lambda}\hat{u}(\lambda)\,d\lambda\\ &= \frac{1}{(2\pi)^m}\int e^{ix\lambda}e^{-iy\lambda}\hat{u}(\lambda)\,d\lambda\end{aligned}\tag{3.27}$$

and use Lemma 3.2 with

$$f(x,\lambda) = \frac{1}{(2\pi)^{m/2}}e^{ix\lambda},\quad g(y,\lambda) = \frac{1}{(2\pi)^{m/2}}e^{-iy\lambda},\quad h(\lambda)=\hat{u}(\lambda).\tag{3.28}$$

Consequently,

$$H(q,r) = \frac{1}{(2\pi)^m}\int e^{izq}e^{-izr}\,d\nu(z) = \frac{1}{(2\pi)^m}\hat{\nu}(q-r)\tag{3.29}$$

and

$$\begin{aligned}\int |H_j(q,r)|^2|h(r)|\,dr\,|h(q)|\,dq &= \frac{1}{(2\pi)^{2m}}\int |\hat{\nu}(q-r)|^2|\hat{u}(q)||\hat{u}(r)|\,dr\,dq\\ &= \frac{1}{(2\pi)^m}\iint |\hat{\nu}(q)|^2|\hat{u}(q+r)||\hat{u}(r)|\,dr\,dq\\ &= \frac{1}{(2\pi)^m}\iint |\hat{\nu}(q)|^2|\hat{u}(q-r)||\hat{u}(-r)|\,dr\,dq\\ &= \frac{1}{(2\pi)^m}\iint |\hat{\nu}(q)|^2|\hat{u}(q-r)||\hat{u}(r)|\,dr\,dq.\end{aligned}\tag{3.30}$$

Replacing the measure ν by $\mu-\nu$, the second line of (1.14) follows from the second line of (3.14).

The first line of (1.14) is the first line of (3.14), since in this case

$$R(z_1,z_2)=\frac{1}{(2\pi)^m}\int e^{i(z_1-z_2)q}|\hat{u}(q)|\,dq \tag{3.31}$$

and $T(z_1,z_2) = \overline{R(z_1,z_2)}$. Moreover, since $R(z_1,z_2)$ is real we actually have $T(z_1,z_2)=R(z_1,z_2)$. Therefore we can take the square in (1.14).

To prove 2. we write

$$u(x,y)=\int_0^\infty\int e^{-\alpha t}p_{t/2}(x,u)p_{t/2}(u,y)\,du\,dt. \tag{3.32}$$

Considering the notation in (3.2) we take

$$f(x,\lambda)=e^{-\alpha t/2}p_{t/2}(x,u),\quad g(y,\lambda)=e^{-\alpha t/2}p_{t/2}(u,y),\quad h(\lambda)=1, \tag{3.33}$$

in which $\lambda=(u_1,\dots,u_n,t)$. Relabeling the variables we have

$$f(z,q)=e^{-\alpha t/2}p_{t/2}(z,u)\quad\text{and}\quad g(y,r)=e^{-\alpha s/2}p_{s/2}(v,z), \tag{3.34}$$

where $q=(u_1,\dots,u_n,t)$ and $r=(v_1,\dots,v_n,s)$. Therefore,

$$\begin{aligned}R(z_1,z_2)&=\int_0^\infty e^{-\alpha s}\int p_{s/2}(z_1,u)p_{s/2}(z_2,u)\,du\,ds,\\ T(z_1,z_2)&=\int_0^\infty e^{-\alpha s}\int p_{s/2}(z_1,u)p_{s/2}(z_2,u)\,du\,ds.\end{aligned} \tag{3.35}$$

The proof now follows from Lemma 3.2 and Corollary 3.1.

To prove 3. we use Lemma 3.2 with

$$f(x,\lambda)=\frac{1-e^{ix\lambda}}{(2\pi)^{m/2}},\qquad g(y,\lambda)=\frac{1-e^{-iy\lambda}}{(2\pi)^{m/2}},\quad h(\lambda)=\frac{1}{\phi(\lambda)} \tag{3.36}$$

and write

$$H(q,r)=\frac{1}{(2\pi)^m}\int(1-e^{izq})(1-e^{-izr})\,d\nu(z). \tag{3.37}$$

Clearly

$$\begin{aligned}&\int\frac{|H(q,r)|^2}{|\phi(q)||\phi(r)|}\,dr\,dq\\ &\quad=\frac{1}{(2\pi)^{2m}}\iint\frac{1}{|\phi(q)||\phi(r)|}\left|\int(1-e^{izq})(1-e^{-izr})\,d\nu(z)\right|^2dr\,dq.\end{aligned} \tag{3.38}$$

Therefore, the second line of (1.21) follows from Lemma 3.2.

Similarly, the third line of (1.21) follows from Lemma 3.2. (Note that

$$\int\frac{(1-e^{i\lambda x})(1-e^{-i\lambda y})}{|\phi(\lambda)|}\,d\lambda=\int\frac{1-\cos\lambda x-\cos\lambda y+\cos\lambda(x-y)}{|\phi(\lambda)|}\,d\lambda, \tag{3.39}$$

since the numerator in the right-hand side of (3.39) is the real part of the numerator in left-hand side and the imaginary part of the integral is zero, because $|\phi(\lambda)|$ is even.

To prove 5. we use the following lemma:

Lemma 3.3. *When the measures* $\{\nu_j\}_{j=1}^n$ *are positive and* $u(x,y)$ *is positive definite*

$$I_n(u,\{\nu_j\}_{j=1}^n \le \prod_{j=1}^n \left(4\int\int u^2(x,y)\,d\nu_j(x)\,d\nu_j(y)\right)^{1/2}. \tag{3.40}$$

Proof. The proof is essentially the same as the one beginning at (3.10). Using the fact that the kernel u is greater than or equal to zero, we have

$$I_n(u,\{\nu_j\}_{j=1}^n) \le \int \left(\int u^2(y_1,y_2)\,d\nu_1(y_1)\right)^{1/2} \left(\int u^2(y_n,y_1)\,d\nu_1(y_1)\right)^{1/2} u(y_2,y_3)\cdots u(y_{n-1},y_n)\prod_{j=2}^n d\nu_j(y_j). \tag{3.41}$$

This is equal to

$$\int \left(\int \left(\int u^2(y_1,y_2)\,d\nu_1(y_1)\right)^{1/2} u(y_2,y_3)\,d\nu_2\right) \left(\int u^2(y_n,y_1)\,d\nu_1(y_1)\right)^{1/2} u(y_3,y_4)\cdots u(y_{n-1},y_n)\prod_{j=3}^n d\nu_j(y_j), \tag{3.42}$$

which is equal to

$$\left(\int\int u^2(y_1,y_2)\,d\nu_1(y_1)\,d\nu_2(y_2)\right)^{1/2} \int \left(\int u^2(y_2,y_3)\,d\nu_2(y_2)\right)^{1/2} \left(\int u^2(y_n,y_1)\,d\nu_1(y_1)\right)^{1/2} u(y_3,y_4)\cdots u(y_{n-1},y_n)\prod_{j=3}^n d\nu_j(y_j). \tag{3.43}$$

Continuing this procedure we see that

$$I_n(u,\{\nu_j\}_{j=1}^n) \le \prod_{j=1}^n \left(\int\int u^2(y_j,y_{j+1})\,d\nu_j(y_j)\,d\nu_{j+1}(y_{j+1})\right)^{1/2}, \tag{3.44}$$

in which $y_{n+1}=y_1$, which we can simplify to

$$I_n(u,\{\nu_j\}_{j=1}^n) \le \prod_{j=1}^n \left(\int\int u^2(x,y)\,d\nu_j(x)\,d\nu_{j+1}(y)\right)^{1/2}. \tag{3.45}$$

We now use the fact that $u(x,y)$ is positive definite. This implies that $(u(x,y)+u(y,x))^2$ is symmetric and positive definite. Therefore by the Cauchy-

Schwarz inequality

$$\int\int u^2(x,y)\,d\nu_j(x)\,d\nu_{j+1}(y) \le \int\int (u(x,y)+u(y,x))^2\,d\nu_j(x)\,d\nu_{j+1}(y) \quad (3.46)$$

$$\le \left(\int (u(x,y)+u(y,x))^2\,d\nu_j(x)\,d\nu_j(y)\right)^{1/2}$$
$$\left(\int (u(x,y)+u(y,x))^2\,d\nu_{j+1}(x)\,d\nu_{j+1}(y)\right)^{1/2}$$
$$\le 4\left(\int u^2(x,y)\,d\nu_j(x)\,d\nu_j(y)\right)^{1/2}$$
$$\left(\int u^2(x,y)\,d\nu_{j+1}(x)\,d\nu_{j+1}(y)\right)^{1/2}. \quad (3.47)$$

Using this in (3.45) we get (3.40). □

Proof of Theorem 1.1 *continued.* In Lemma 3.3 we need the condition that the kernel is positive definite to deal with the fact that ν_j and ν_{j+1} are not necessarily the same measures. However in 5. they are. When all the $\{\nu_j\}$ are the same positive measure we get (3.40) with no hypotheses on the kernel u. Therefore, 5. follows from Lemma 3.2.

The proof of 4. is given in the following remark. □

Remark 3.2. Let $\{\psi(\nu);\nu\in\mathcal{M}(S)\}$ be an α-permanental field. It follows from Definition 1.1 that

$$\begin{aligned}&\left(\frac{1}{2\alpha}E\left(\psi(\mu)-\psi(\nu)\right)^2\right)^{1/2}\\ &\quad=\left(\int\int u(x,y)u(y,x)\,d(\mu(x)-\nu(x))d(\mu(y)-\nu(y))\right)^{1/2}.\end{aligned} \quad (3.48)$$

So, obviously, when $u(x,y)$ is symmetric,

$$\begin{aligned}&\left(\frac{1}{2\alpha}E\left(\psi(\mu)-\psi(\nu)\right)^2\right)^{1/2}\\ &\quad=\left(\int\int u^2(x,y)\,d(\mu(x)-\nu(x))d(\mu(y)-\nu(y))\right)^{1/2}.\end{aligned} \quad (3.49)$$

Consider the first equality in (1.14). When u is symmetric, $\hat{u}(q)$ is real. If it is also positive

$$\int e^{i(x-y)q}|\hat{u}(q)|\,dq = Cu(y-x) = Cu(x,y). \quad (3.50)$$

We get (1.22) when u satisfies the conditions in 1.

When u satisfies the conditions in 2. and the transition probabilities are symmetric we have

$$\Theta_l(x,y) = \int_0^\infty e^{-\alpha s} \int p_{s/2}(x,u)p_{s/2}(u,y)\,du\,ds = u(x,y) \tag{3.51}$$

and similarly for Θ_l. Consequently (1.22) follows from (1.17).

The proof for 3. when ϕ is real and positive is trivial.

For use in Section 4 we note that when u is a function of $x-y$ the Fourier transform of the square of (3.48) is

$$\begin{aligned} &\int\!\!\int |\hat\mu(\lambda_1-\lambda_2)-\hat\nu(\lambda_1-\lambda_2)|^2\hat u(\lambda_1)\hat u(\lambda_2)\,d\lambda_1\,d\lambda_2 \\ &\quad = \int\!\!\int |\hat\mu(x)-\hat\nu(x)|^2\hat u(x+\lambda_2)\hat u(\lambda_2)\,dx\,d\lambda_2 \\ &\quad = \int\!\!\int |\hat\mu(x)-\hat\nu(x)|^2\hat u(x-y)\overline{\hat u(y)}\,dx\,dy. \end{aligned} \tag{3.52}$$

We can give more concrete results when $u(x,y)$ is a function of $x-y$ and when the measures we consider are translates of a fixed measure ν, which we denote by $\mathcal{V}_K=\{\nu_h, h\in K\}$ where K is some compact symmetric subset of R^m that includes 0. We denote ν_0 by ν. In this case

$$|\hat\nu_h(x)-\hat\nu(x)| = |1-e^{ixh}|^2||\hat\nu(x)|^2 \tag{3.53}$$

so we can write

$$\begin{aligned} \widetilde\tau_u(\nu_{h_1+h},\nu_{h_1}) = \widetilde\tau_u(\nu_h,\nu) &= \alpha\left(\frac{1}{(2\pi)^m}\int\!\!\int |1-e^{ixh}|^2||\hat\nu(x)|^2\widetilde\gamma(x)\,dx\right)^{1/2} \\ &\le \alpha\left(\frac{3}{2(2\pi)^m}\int\!\!\int ((|x||h|)^2\wedge 1)||\hat\nu(x)|^2\widetilde\gamma(x)\,dx\right)^{1/2}. \end{aligned} \tag{3.54}$$

Corollary 3.2. *A sufficient condition for the continuity of the α permanental process $\{\psi(\nu),\nu\in\mathcal{V}_K\}$ is that*

$$\int_2^\infty \frac{\left(\int_{|x|\ge u}|\hat\nu(x)|^2\widetilde\gamma(x)\,dx\right)^{1/2}}{u}\,du<\infty. \tag{3.55}$$

Proof. We use the bound on $\widetilde\tau_u(\nu_h,\nu)$ in (3.54) and follow the proof of Theorem 1.6 in [5]. □

4. Domination by the second moment

Let $\psi=\{\psi(\nu);\nu\in\mathcal{M}(S)\}$ be an α-permanental field with kernel u. We are interested in the situation in which

$$\|\psi(\mu)-\psi(\nu)\|_\rho \le C\left(E(\psi(\mu)-\psi(\nu))^2\right)^{1/2} = C\|\psi(\mu)-\psi(\nu)\|_2, \tag{4.1}$$

for some constant C. This holds when $\alpha=1/2$ and u is symmetric, because in this case ψ is a second-order Gaussian chaos. In Theorem 1.1 we give examples in which this holds for all $\alpha > 0$ when u is symmetric. We now give examples in which this holds when u is not symmetric.

Theorem 4.1. *Let $\psi = \{\psi(\nu), \nu \in \mathcal{M}\}$ be an α-permanental field, with kernel $u^\beta(x-y)$ as given in (1.25). Suppose that $\phi(x)$ satisfies the sectorial condition*

$$|Im\ \phi(x)| \le C\ Re\ (\beta + \phi(x)), \qquad \forall x \in R^m, \tag{4.2}$$

for some constant $C < 1$. Then

$$\|\psi(\nu)\|_\rho \le C' \left(E\psi^2(\nu)\right)^{1/2}, \tag{4.3}$$

for some constant C' that depends on C.

Proof. By (3.52)

$$\left(\frac{1}{2\alpha} E\left(\psi(\mu) - \psi(\nu)\right)^2\right)^{1/2} = C\left(\int |\hat{\mu}(x) - \hat{\nu}(x)|^2 \gamma(x)\, dx\right)^{1/2}$$

where

$$\gamma(x) = \int \hat{u}^\beta(x-y)\overline{\hat{u}^\beta(y)}\, dy. \tag{4.4}$$

Therefore, by Theorem 1.1, 1, to prove this theorem we need only show that (4.2) implies that

$$\int |\hat{u}^\beta(x-y)||\hat{u}^\beta(y)|\, dy \le C'' \int \hat{u}^\beta(x-y)\overline{\hat{u}^\beta(y)}\, dy \qquad \forall x \in R^m \tag{4.5}$$

for some constant $C'' > 0$. Set

$$\hat{u}^\beta(x) = \frac{1}{\beta + \phi(x)} =: \frac{1}{v(x) + iw(x)}. \tag{4.6}$$

Note that v is positive since the real part of a characteristic exponent of a Lévy process, (see (1.24)), is positive.

Since $u^\beta(x)u^\beta(-x)$ is symmetric, its Fourier transform, which is given by the left-hand side of (4.7) immediately below, is real. Consequently

$$\int \hat{u}^\beta(x-y)\overline{\hat{u}^\beta(y)}\, dy = \int \frac{v(x-y)v(x) + w(x-y)w(x)}{|v^2(x-y) + w^2(x-y)||v^2(y) + w^2(y)|}\, dy. \tag{4.7}$$

Therefore, to obtain (4.5) we need only show that

$$\frac{v(x-y)v(x) + w(x-y)w(x)}{|v^2(x-y) + w^2(x-y)|^{1/2}|v^2(y) + w^2(y)|^{1/2}} \ge c, \tag{4.8}$$

for some constant $c > 0$.

By (4.2) and using the fact that v is positive, we see that

$$v(x-y)v(x) + w(x-y)w(x) \ge (1-C)v(x-y)v(x) \tag{4.9}$$

and

$$|v^2(x) + w^2(x)| \le 2v^2(x). \tag{4.10}$$

Therefore, the denominator in (4.8) is less than or equal to $2v(x-y)v(x)$, so we can take $c = (1-C)/2$. □

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Part IV

Random Matrices and Applications

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On the Operator Norm of Random Rectangular Toeplitz Matrices

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Abstract. We consider rectangular $N \times n$ Toeplitz matrices generated by sequences of centered independent random variables and provide bounds on their operator norm under the assumption of finiteness of pth moments ($p > 2$). We also show that if $N \gg n \log n$ then with high probability such matrices preserve the Euclidean norm up to an arbitrarily small error.

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1. Introduction

Generalities on random Toeplitz matrices. In recent years, following a question raised in [3] a considerable amount of work has been devoted to the study of random Toeplitz matrices, i.e., Toeplitz matrices determined by sequences of independent random variables. In particular in [7, 9] the convergence of the spectral measure for random symmetric Toeplitz matrices has been established, while [5] provides a corresponding result for the spectral measure of XX^T, where X is a nonsymmetric random Toeplitz matrix. In both cases the limiting spectral distribution has unbounded support.

A natural further question is the behaviour of the spectral norm of the matrix. In [14] it has been shown that if the underlying random variables are sub-Gaussian and of mean zero, then the operator norm of an $n \times n$ matrix is of the order $\sqrt{n \log n}$. This result has been extended to matrices with bounded variance coefficients in [1], where also a strong law of large numbers with the normalization by expectation has been established. Although both papers consider symmetric matrices, their methods easily generalize to the non-symmetric square ones. Recently in [16], precise asymptotics of the operator norm have been found in the symmetric case.

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It turns out that if T_n is an $n \times n$ random symmetric Toeplitz matrix with mean zero, variance one coefficients and bounded pth moments ($p > 2$), then

$$\frac{\|T_n\|_{\ell_2^n \to \ell_2^n}}{\sqrt{2n\log n}} \xrightarrow{L_p} \|S(x,y)\|_{2\to 4}^2,$$

where $S(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}$ is the sine kernel and $\|S(x,y)\|_{2\to 4}$ denotes the norm of the integral operator associated with it, acting from $L_2(\mathbb{R})$ to $L_4(\mathbb{R})$.

In this article we present results on the behaviour of the operator norm of a rectangular $N \times n$ random Toeplitz matrix with independent coefficients in terms of the matrix size. When n and N are of the same order of magnitude, this question can be easily reduced to the square case, however for general matrices there seems to be no corresponding estimates in the literature. We remark that some results can be obtained from Theorem III.4 in [15], where a more general problem of estimating singular values of submatrices of a square random Toeplitz matrix is considered. This estimate however, when specialized to our problem is not optimal in the whole range of parameters (we discuss it briefly in the sequel).

Our main result gives estimates on the operator norm with optimal dependence on n and N. Additionally, in the case of tall matrices we provide conditions under which a properly scaled Toeplitz matrix preserves the Euclidean norm up to a small error.

Notation and the main result. Throughout the article we will consider a random Toeplitz $N \times n$ matrix

$$T = [T_{ij}]_{1\le i\le N, 1\le j\le n} = [X_{i-j}]_{1\le i\le N, 1\le j\le n},$$

where $X_{1-n}, X_{2-n}, \ldots, X_{N-1}$ is a sequence of independent random variables.

We will denote absolute constants by C, and constants depending on some parameters (say a) by C_a. In both cases the value of a constant may differ between distinct occurrences.

We write ℓ_2^k for $\mathbb{R}^k$ equipped with the standard Euclidean structure (the corresponding inner product will be denoted by $\langle\cdot,\cdot\rangle$). For an $N \times n$ matrix A, by $\|A\|_{\ell_2^n\to\ell_2^N}$ we denote the operator norm of A acting between the spaces ℓ_2^n and ℓ_2^N, i.e., $\|A\|_{\ell_2^n\to\ell_2^N} = \sup_{x\in S^{n-1}}\sup_{y\in S^{N-1}}\langle Ax, y\rangle$.

Having desribed the notation, we are now ready to state our main result which is

Theorem 1.1. *Let $(X_i)_{1-n\le i\le N-1}$ be independent random variables such that $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$ and $\|X_i\|_p \le L$ for some $p > 2$. Then*

$$\mathbb{E}\|T\|_{\ell_2^n\to\ell_2^N} \le C_p L(\sqrt{N\vee n} + \sqrt{(N\wedge n)\log(N\wedge n)}). \tag{1}$$

Moreover, for any $\delta, \varepsilon \in (0,1)$ if $N > C_{L,p,\delta,\varepsilon} n\log n$, then with probability at least $1-\delta$, for all $x \in \mathbb{R}^n$,

$$(1-\varepsilon)|x| \le |\frac{1}{\sqrt{N}}Tx| \le (1+\varepsilon)|x|. \tag{2}$$

We postpone the proof of the above theorem to Section 2.

A brief discussion of optimality. One can easily see that the estimates of Theorem 1.1 are of the right order. Indeed, if X_i are independent Rademacher variables then the Euclidean length of the first column of the matrix is $\sqrt{N}$, while the Euclidean length of the first row is $\sqrt{n}$, which gives $\|T\|_{\ell_2^n \to \ell_2^N} \geq \sqrt{N \vee n}$. Moreover the matrix T contains a square Toeplitz submatrix with independent coefficients of size $(N \wedge n) \times (N \wedge n)$. By a straightforward modification of the argument presented in [14] for the symmetric case (see Theorem 3 therein), one can see that the operator norm of this submatrix is bounded from below by $c\sqrt{(N \wedge n)\log(N \wedge n)}$ for some absolute constant c (in fact instead of mimicking the proof one can also easily reduce the problem to the symmetric case). Standard symmetrization arguments allow to extend such estimates to other sequences of independent random variables satisfying a uniform lower bound on the absolute first moment (cf. the proof of Theorem 6 in [1]).

Further remarks. The constants C_p obtained in our proof of Theorem 1.1 explode when $p \to 2$, contrary to known inequalities on the operator norm of symmetric Toeplitz matrices. We present here a simple proposition, whose proof is based on general methods of probability in Banach spaces, which gives an estimate weaker than that of Theorem 1.1, but under the assumption of finiteness of the second moment of X_i's only. It's proof is deferred to Section 3.

Proposition 1.2. *Let $(X_i)_{1-n \leq i \leq N-1}$ be independent random variables with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$. Then*

$$\mathbb{E}\|T\|_{\ell_2^n \to \ell_2^N} \leq C(\sqrt{N \vee n} + \sqrt[4]{(N \wedge n)(N \vee n)}\sqrt{\log(N \wedge n)}).$$

Restricting our attention to the case $N \geq n$, we see that the above proposition gives an estimate of the same order as Theorem 1.1 (up to constants independent of n and N) if $N \leq C'n$ or $n \log^2 n \leq C'N$. In the former case the operator norm behaves like in the square case, i.e., is of the order $\sqrt{n \log n}$, whereas in the latter one it is of the order $\sqrt{N}$, the same as the Euclidean length of a single column of the matrix (with the implicit constants depending on C'). In the intermediate regime, i.e., when $n \ll N \ll n \log^2 n$ one loses a logarithmic factor.

It is natural to conjecture that whenever $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$, the operator norm is of the order $\sqrt{N} + \sqrt{n \log n}$ for all $N \geq n$, however we do not know how to prove it without additional assumptions on higher moments of X_i's. As for the property (2), clearly it cannot hold just under the assumptions of the above proposition without some stronger integrability assumptions, since assuming just $\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = 1$ still does not exclude the possibility that with probability close to one $X_i = 0$ for all i. Let us also remark that an estimate of the same order as in Proposition 1.2 can be obtained for matrices generated by Rademacher or Gaussian sequences using inequalities presented in [15] (as already mentioned in the introduction). In fact it can be also obtained by a modification of the proof of Theorem 1.1, however the argument presented in Section 3 is more concise.

2. Proof of Theorem 1.1

Without loss of generality we may assume that $N \geq n \geq 2$. In the main part of the proof we will not work with the original Toeplitz matrix, but with its modification, which will be more convenient for the calculations. Consider thus the matrix

$$\Gamma = [\Gamma_{ij}]_{1\leq i\leq N, 1\leq j\leq n}, \tag{3}$$

where $\Gamma_{ij} = T_{ij} = X_{i-j}$ if $j \leq i \leq N-n+j$ and $\Gamma_{ij} = 0$ otherwise. Let us note that T and Γ differ just by two "corners" of Toeplitz type and thus $\mathbb{E}\|T-\Gamma\|_{\ell_2^n\to\ell_2^N}$ can be estimated by means of results for square Toeplitz matrices. More precisely, by using Proposition 4.1 from the Appendix, we obtain

$$\mathbb{E}\|T-\Gamma\|_{\ell_2^n\to\ell_2^N} \leq C\Big(\sum_{i\leq -1 \text{ or } i\geq N-n+1} \mathbb{E}X_i^2\Big)^{1/2}\sqrt{\log n} \leq C\sqrt{n\log n}.$$

Therefore for both assertions made in Theorem 1.1, the contribution from the corners is negligible (for the first part it is a direct consequence of the above inequality, whereas for the second part it follows easily by the above estimate and Chebyshev's inequality).

Denote the standard basis of ℓ_2^n and ℓ_2^N by $(e_j)_{j=1}^n$ and $(E_j)_{j=1}^N$ respectively and let $A_i\colon \ell_2^n \to \ell_2^N$, $i = 0,\ldots,N-n$ be the linear operator such that for all $1 \leq j \leq n$, $A_i e_j = E_{i+j}$ (in the sequel we will identify operators with their matrices in standard basis). Then $\Gamma = \sum_{i=0}^{N-n} X_i A_i$ and so

$$\Gamma^T\Gamma = \sum_{0\leq i,j\leq N-n} X_iX_jA_i^TA_j. \tag{4}$$

Note that $A_i^TE_k = 0$ if $k < i+1$ or $k > i+n$ and $A_i^TE_k = e_{k-i}$ if $i+1 \leq k \leq n+i$. Therefore

$$\langle \Gamma^T\Gamma e_l, e_k\rangle = \sum_{i=0\vee(l-k)}^{(N-n-(k-l))\wedge(N-n)} X_iX_{k-l+i}. \tag{5}$$

In particular $\Gamma^T\Gamma$ is a symmetric Toeplitz matrix.

We will now state the main technical proposition, which will allow us to use standard symmetrization techniques in the proof of Theorem 1.1.

Proposition 2.1. *Let $N \geq n$ be two positive integers, $a_0,\ldots,a_{N-n}$ be real numbers and $g_0,\ldots,g_{N-n}$ be independent standard Gaussian variables. Define a symmetric $n\times n$ Toeplitz matrix $M = [M_{kl}]_{k,l=1}^n$, where $M_{kk} = 0$ and for $k \neq l$,*

$$M_{kl} = Y_{|k-l|} := \sum_{i=0\vee(l-k)}^{(N-n-(k-l))\wedge(N-n)} a_ia_{k-l+i}g_ig_{k-l+i}.$$

Then

$$\mathbb{E}\|M\|_{\ell_2^n\to\ell_2^n} \le C\Big(\sum_{0\le i,j\le N-n} a_i^2a_j^2\mathbf{1}_{\{1\le|i-j|\le n-1\}}\Big)^{1/2}\sqrt{\log n}$$
$$+ C\max_{0\le k\le\lceil(N-n+1)/n\rceil-1}\Big(\sum_{\substack{i\neq j\\ kn\le i,j\le((k+2)n-1)\wedge(N-n)}} a_i^2a_j^2\Big)^{1/2}\log n.$$

Proof of Proposition 2.1. Note first that without loss of generality we can assume that $N-n+1\ge 2n$ and $N-n+1$ is divisible by n (we may simply enlarge N and put zeros as the new a_i's).

Since M is a symmetric Toeplitz matrix, to estimate the operator norm we may use the same strategy as in [14], i.e., relate the operator norm of M to the supremum of a random trigonometric polynomial for which we will use the entropy method. The main difference between our case and [14] is the fact that the coefficients of the polynomial will not be independent and the related supremum will be a chaos of degree 2, which will result in an additional term appearing in the entropy integral. Similarly as in [14] by extending M to an infinite Laurent matrix $[Y_{|k-l|}\mathbf{1}_{\{1\le|k-l|\le n-1\}}]_{k,l\in\mathbb{Z}}$ and then noting that it corresponds to a multiplier on the circle we obtain that

$$\begin{aligned}\|M\|_{\ell_2^n\to\ell_2^n} &\le 2\sup_{0\le x\le 1}\Big|\sum_{j=1}^{n-1}Y_j\cos(2\pi jx)\Big|\\ &= 2\sup_{0\le x\le 1}\Big|\sum_{i=0}^{N-n-1}\sum_{j=1}^{(N-n-i)\wedge(n-1)} a_ia_{i+j}g_ig_{i+j}\cos(2\pi jx)\Big|\\ &= \sup_{0\le x\le 1}\Big|\sum_{0\le i,j\le N-n}B^x_{ij}g_ig_j\Big| =: \sup_{0\le x\le 1}|S_x|,\end{aligned}$$

where for $x\in[0,1]$, the matrix $B^x=[B^x_{ij}]_{i,j=0}^{N-n}$ is defined by

$$B^x_{ij} = a_ia_j\cos(2\pi|i-j|x)\mathbf{1}_{\{1\le|i-j|\le n-1\}}.$$

By Proposition 4.2 in the Appendix we obtain that

$$\mathbb{P}(|S_x-S_y|\ge t)\le 2\exp\Big(-\frac{1}{C}\min\Big(\frac{t^2}{\|B^x-B^y\|_{HS}^2},\frac{t}{\|B^x-B^y\|_{\ell_2^{N-n+1}\to\ell_2^{N-n+1}}}\Big)\Big)$$

and so, by Proposition 4.3, we get

$$\|M\|_{\ell_2^n\to\ell_2^n}\le C\Big(\mathbb{E}|S_0| + \int_0^\infty\sqrt{\log\mathcal{N}([0,1],d_1,\varepsilon)}d\varepsilon + \int_0^\infty\log\mathcal{N}([0,1],d_2,\varepsilon)d\varepsilon\Big), \tag{6}$$

where $d_1(x,y)=\|B^x-B^y\|_{HS}$, $d_2(x,y)=\|B^x-B^y\|_{\ell_2^{N-n+1}\to\ell_2^{N-n+1}}$ and for a metric space $(\mathcal{X},d)$, $\mathcal{N}(\mathcal{X},d,\varepsilon)$ denotes the minimum number of closed balls with radius ε covering $\mathcal{X}$.

Note that $\operatorname{diam}([0,1],d_1) \le 2\sqrt{\sum_{0\le i,j\le N-n} a_i^2 a_j^2 \mathbf{1}_{\{1\le|i-j|\le n-1\}}} =: D_1$. Also, using the Lipschitz property of the cosine function, we get that

$$d_1(x,y)^2 \le 4\pi^2 \sum_{0\le i,j\le N-n} a_i^2 a_j^2 (i-j)^2 \mathbf{1}_{\{1\le|i-j|\le n-1\}} |x-y|^2,$$

which gives $\mathcal{N}([0,1],d_1,\varepsilon) \le C\Delta_1/\varepsilon$ for $\varepsilon \le D_1$, where

$$\Delta_1^2 = \sum_{0\le i,j\le N-n} a_i^2 a_j^2 (i-j)^2 \mathbf{1}_{\{1\le|i-j|\le n-1\}}.$$

We thus obtain

$$\begin{aligned}\int_0^\infty \sqrt{\log([0,1],d_1,\varepsilon)}d\varepsilon &\le \int_0^{D_1} \sqrt{\log\Big(\frac{C\Delta_1}{\varepsilon}\Big)}d\varepsilon \\ &= \frac{C\Delta_1}{\sqrt{2}} \int_{\sqrt{2\log(C\Delta_1/D_1)}}^\infty t^2 e^{-t^2/2} dt \\ &\le D_1\sqrt{\log(C\Delta_1/D_1)} + \sqrt{\pi} D_1 \\ &\le C D_1 \sqrt{\log n}, \end{aligned} \tag{7}$$

where in the last inequality we used the estimate $\Delta_1 \le nD_1$.

Let us now estimate the other integral on the right-hand side of (6). Note that B^x's are band matrices and they may be decomposed as $B^x = B_1^x + B_2^x + B_3^x$, where B_1^x is the block diagonal part of B^x with blocks of size $n\times n$, whereas B_2^x and B_3^x correspond respectively to the part of B^x below and above the block diagonal. More formally,

$$\begin{aligned} B_1^x &= [B_{ij}^x \mathbf{1}_{\{\lfloor i/n\rfloor = \lfloor j/n\rfloor\}}]_{i,j=0}^{N-n}, \\ B_2^x &= [B_{ij}^x \mathbf{1}_{\{\lfloor i/n\rfloor = \lfloor j/n\rfloor + 1\}}]_{i,j=0}^{N-n}, \\ B_3^x &= [B_{ij}^x \mathbf{1}_{\{\lfloor i/n\rfloor + 1 = \lfloor j/n\rfloor\}}]_{i,j=0}^{N-n}. \end{aligned}$$

The matrix $B_1^x - B_1^y$ consists of $(N-n+1)/n$ blocks and the Hilbert-Schmidt norm of the kth block ($k = 1,\ldots,(N-n+1)/n$) is bounded by

$$\begin{aligned} &\Big(\sum_{i,j=(k-1)n}^{kn-1} a_i^2 a_j^2 (\cos(2\pi|i-j|x) - \cos(2\pi|i-j|y))^2 \mathbf{1}_{\{1\le|i-j|\le n-1\}}\Big)^{1/2} \\ &\quad \le 2\pi \Big(\sum_{(k-1)n\le i,j\le kn-1} a_i^2 a_j^2 (i-j)^2\Big)^{1/2} |x-y|. \end{aligned}$$

Thus for $x,y\in[0,1]$,

$$\begin{aligned} &\|B_1^x - B_1^y\|_{\ell_2^{N-n+1}\to\ell_2^{N-n+1}} \\ &\quad \le 2\pi|x-y| \max_{1\le k\le (N-n+1)/n} \Big(\sum_{(k-1)n\le i,j\le kn-1} a_i^2 a_j^2 (i-j)^2\Big)^{1/2}. \end{aligned}$$

By a similar estimate for all $x \in [0,1]$,

$$\|B_1^x - B_1^y\|_{\ell_2^{N-n+1} \to \ell_2^{N-n+1}} \le 2 \max_{1 \le k \le (N-n+1)/n} \Bigg(\sum_{\substack{i \neq j \\ (k-1)n \le i,j \le kn-1}} a_i^2 a_j^2 \Bigg)^{1/2}.$$

Bounds on B_2^x and B_3^x can be obtained in an analogous way, by exploring their block-diagonal structure (the blocks are not on the main diagonal, but still the operator norm of the whole matrix is the maximum of operator norms of individual blocks). Therefore we obtain

$$\begin{aligned}
\operatorname{diam}([0,1], d_2) \le & \max_{1 \le k \le (N-n+1)/n} \Bigg(\sum_{\substack{i \neq j \\ (k-1)n \le i,j \le kn-1}} a_i^2 a_j^2 \Bigg)^{1/2} \\
& + \max_{1 \le k \le (N-n+1)/n-1} \Bigg(\sum_{\substack{(k-1)n \le j \le kn-1 \\ kn \le i \le (k+1)n-1}} a_i^2 a_j^2 \Bigg)^{1/2} \\
& + \max_{1 \le k \le (N-n+1)/n-1} \Bigg(\sum_{\substack{kn \le i \le (k+1)n-1 \\ (k-1)n \le j \le kn-1}} a_i^2 a_j^2 \Bigg)^{1/2} \\
& \le 3 \max_{0 \le k \le ((N-n+1)/n)-2} \Bigg(\sum_{\substack{i \neq j \\ kn \le i,j \le (k+2)n-1}} a_i^2 a_j^2 \Bigg)^{1/2} =: D_2
\end{aligned}$$

and

$$\begin{aligned}
d_2(x,y) \le C & \max_{0 \le k \le ((N-n+1)/n)-2} \Bigg(\sum_{kn \le i,j \le (k+2)n-1} a_i^2 a_j^2 (i-j)^2 \Bigg)^{1/2} |x-y| \\
& =: \Delta_2 |x-y|,
\end{aligned}$$

which allows us to write

$$\mathcal{N}([0,1], d_2, \varepsilon) \le \frac{\Delta_2}{\varepsilon}$$

for $\varepsilon \le D_2$. Thus

$$\begin{aligned}
\int_0^\infty \log \mathcal{N}([0,1], d_2, \varepsilon) d\varepsilon & \le \int_0^{D_2} \log(\Delta_2 \varepsilon^{-1}) d\varepsilon \\
& = D_2 \log(\Delta_2) - D_2 \log D_2 + D_2 \le C D_2 \log n, \qquad (8)
\end{aligned}$$

where in the last inequality we used the estimate $\Delta_2 \le C n D_2$.

Let us now note that $S_0 = \sum_{0 \le i,j \le N-n} a_i a_j g_i g_j \mathbf{1}_{\{1 \le |i-j| \le n-1\}}$ and so by independence of g_i's,

$$\mathbb{E}|S_0| \le \sqrt{\mathbb{E}|S_0|^2} = \sqrt{2 \sum_{0 \le i,j \le N-n} a_i^2 a_j^2 \mathbf{1}_{\{1 \le |i-j| \le n-1\}}},$$

which together with (6), (7) and (8) ends the proof of the proposition. □

Conclusion of the proof of Theorem 1.1. As explained at the beginning of this section, it suffices to prove the corresponding statements for $N \geq n$ and the matrix Γ defined by (3) instead of T.

We have

$$\Gamma^T\Gamma = \Big(\Gamma^T\Gamma - \Big(\sum_{i=0}^{N-n} X_i^2\Big)\mathrm{Id}_n\Big) + \Big(\sum_{i=0}^{N-n} X_i^2\Big)\mathrm{Id}_n. \tag{9}$$

Denote the first term on the right-hand side above by $\tilde{M} = [\tilde{M}_{kl}]_{k,l\leq n}$. From (5) it follows that $\tilde{M}_{kk} = 0$ and for $k \neq l$,

$$\tilde{M}_{kl} = \sum_{i=0\vee(l-k)}^{(N-n-(k-l))\wedge(N-n)} X_i X_{k-l+i},$$

thus $\tilde{M}$ is a tetrahedral chaos of order two (with matrix coefficients).

Let $g_0, \ldots, g_{N-n}$ be i.i.d. standard Gaussian variables independent of the sequence (X_i). By Proposition 4.4 from the Appendix we obtain

$$\mathbb{E}\|\tilde{M}\|_{\ell_2^n\to\ell_2^n} \leq C\mathbb{E}\|M\|_{\ell_2^n\to\ell_2^n},$$

where the matrix M is defined as in Proposition 2.1 with $a_i = X_i$. Therefore, applying this proposition conditionally on (X_i) we get

$$\begin{aligned}\mathbb{E}\|\tilde{M}\|_{\ell_2^n\to\ell_2^n} \leq{}& C\mathbb{E}\Big(\sum_{0\leq i,j\leq N-n} X_i^2X_j^2\mathbf{1}_{\{1\leq|i-j|\leq n-1\}}\Big)^{1/2}\sqrt{\log n} \\ &+ C\mathbb{E}\max_{0\leq k\leq\lceil(N-n+1)/n\rceil-1}\Big(\sum_{kn\leq i\leq((k+2)n-1)\wedge(N-n)} X_i^2\Big)\log n\end{aligned} \tag{10}$$

(note that we have enlarged the second summand of the estimate given in Proposition 2.1 by adding the diagonal terms).

Bounding the first summand on the right-hand side of the above inequality is easy. By Jensen's inequality, independence and the assumption $\mathbb{E}X_i^2 = 1$ we get

$$\mathbb{E}\Big(\sum_{0\leq i,j\leq N-n} X_i^2X_j^2\mathbf{1}_{\{1\leq|i-j|\leq n-1\}}\Big)^{1/2}\sqrt{\log n} \leq \sqrt{2Nn\log n}. \tag{11}$$

Let us now take care of the second term. Denote

$$Z_k = \sum_{kn\leq i\leq((k+2)n-1)\wedge(N-n)} X_i^2,\ \bar{Z}_k = Z_k - \mathbb{E}Z_k,$$

$k = 0, 1, \ldots, \lceil(N-n+1)/n\rceil - 1$.

Set $q = (p/2)\wedge 2 \leq 2$. Let also $\varepsilon_0, \ldots, \varepsilon_{N-n}$ be independent Rademacher variables, independent of the sequence (X_i). By standard symmetrization techniques and Hölder's inequality we get

$$\begin{aligned}\mathbb{E}|\bar{Z}_k|^q &\le 2^q \mathbb{E}_X \mathbb{E}_\varepsilon \Big| \sum_{kn \le i \le ((k+2)n-1)\wedge(N-n)} \varepsilon_i X_i^2 \Big|^q \\ &\le 2^q \mathbb{E}\Big(\sum_{kn \le i \le ((k+2)n-1)\wedge(N-n)} X_i^4 \Big)^{q/2} \\ &\le 2^q \mathbb{E}\Big(\sum_{kn \le i \le ((k+2)n-1)\wedge(N-n)} X_i^{2q} \Big) \\ &\le 2^q \Big(\sum_{kn \le i \le ((k+2)n-1)\wedge(N-n)} L^{2q} \Big) \le 2^{q+2} n L^{2q},\end{aligned}$$

where in the second and third inequality we used the fact $q \le 2$ and in the fourth one, $2q \le p$ and the definition of L. We thus get

$$\begin{aligned}&\mathbb{E} \max_{0 \le k \le \lceil (N-n+1)/n \rceil - 1} \Big(\sum_{kn \le i \le ((k+2)n-1)\wedge(N-n)} X_i^2 \Big) \\ &\le \max_{0 \le k \le \lceil (N-n+1)/n \rceil - 1} \Big(\sum_{kn \le i \le ((k+2)n-1)\wedge(N-n)} \mathbb{E} X_i^2 \Big) \\ &\quad + \mathbb{E} \max_{0 \le k \le \lceil (N-n+1)/n \rceil - 1} |\bar{Z}_k| \\ &\le 2n + \Big(\sum_{0 \le k \le \lceil (N-n+1)/n \rceil - 1} \mathbb{E}|\bar{Z}_k|^q \Big)^{1/q} \\ &\le 2n + \Big(C \frac{N}{n} n L^{2q} \Big)^{1/q} \le 2n + C L^2 N^{1/q},\end{aligned}$$

which together with (10) and (11) gives

$$\mathbb{E}\|\tilde{M}\|_{\ell_2^n \to \ell_2^n} \le C(\sqrt{Nn \log n} + n \log n + L^2 N^{1/q} \log n) \le C_p L^2 (N + n \log n), \tag{12}$$

where we used that $q > 1$. Going back to (9), we see that it remains to estimate the second term on the right-hand side. Clearly $\mathbb{E}\sum_{i=0}^{N-n} X_i^2 = N - n + 1$, which together with (12) gives $\mathbb{E}\|\Gamma\|^2_{\ell_2^n \to \ell_2^N} = \mathbb{E}\|\Gamma^T\Gamma\|_{\ell_2^n \to \ell_2^n} \le C_p L^2 (N + n \log n)$ (recall that $L \ge 1$) and proves the first assertion of the theorem.

To prove the second part, note that in the same way as for $\bar{Z}_k$ above, we get

$$\mathbb{E}\Big| \sum_{i=0}^{N-n} (X_i^2 - 1) \Big| \le C N^{1/q} L^2.$$

Thus by the first inequality of (12), one obtains

$$\mathbb{E}\|\Gamma^T\Gamma - (N-n+1)\mathrm{Id}_n\|_{\ell_2^n \to \ell_2^n} \le C\left(\sqrt{N \log n} + n \log n + L^2 N^{1/q} \log n \right),$$

which gives

$$\mathbb{E}\Big\|\frac{1}{N}\Gamma^T\Gamma - \mathrm{Id}_n\Big\|_{\ell_2^n\to\ell_2^n} \le \varepsilon\delta$$

for $N \ge C_{L,p,\delta,\varepsilon} n\log n$. By Markov's inequality this yields

$$\Big\|\frac{1}{N}\Gamma^T\Gamma - \mathrm{Id}_n\Big\|_{\ell_2^n\to\ell_2^n} \le \varepsilon,$$

with probability at least $1-\delta$, which (after a suitable change of ε) easily implies the second part of the theorem. □

Remark. In the proof above we did not try to obtain explicit dependence of the constant $C_{L,p,\delta,\varepsilon}$ on the parameters. Certain suboptimal estimates can be clearly read from the proof. Moreover, once the expectations of the variables involved are estimated, one can use general concentration inequalities for sums of independent random variables and suprema of polynomial chaoses to get a better estimate on the constants (depending on integrability properties of the underlying sequence of random variables). We do not pursue this direction here.

3. Proof of Proposition 1.2

We can again assume that $N \ge n \ge 2$ and prove the corresponding estimate for the matrix Γ. Going back to the equality (4) we obtain

$$\mathbb{E}\|\Gamma^T\Gamma\|_{\ell_2^n\to\ell_2^n} \le \mathbb{E}\Big\|\sum_{0\le i\le N-n} X_i^2 A_i^T A_i\Big\|_{\ell_2^n\to\ell_2^n} + \mathbb{E}\Big\|\sum_{0\le i\ne j\le N-n} X_iX_jA_i^TA_j\Big\|_{\ell_2^n\to\ell_2^n}.$$

Since $A_i^TA_i = \mathrm{Id}_n$ and $\mathbb{E}X_i^2 = 1$, the first term on the right-hand side equals $N-n+1 \le N$. To bound the second term we use the fact that the space of $n\times n$ matrices equipped with the operator norm has type 2 constant bounded by $C\sqrt{\log n}$ (see Proposition 4.6 in the Appendix). Thus we can use Proposition 4.5 from the Appendix and get

$$\Big(\mathbb{E}\Big\|\sum_{0\le i\ne j\le N-n} X_iX_jA_i^TA_j\Big\|_{\ell_2^n\to\ell_2^n}\Big)^2 \le C\Big(\sum_{0\le i\ne j\le N-n}\|A_i^TA_j\|^2_{\ell_2^n\to\ell_2^n}\Big)\log^2 n.$$

Note that if $|i-j|\ge n$ then $A_i^TA_j = 0$, moreover for all i,j, $\|A_i^TA_j\|_{\ell_2^n\to\ell_2^n}\le 1$, which together with the above inequality gives

$$\mathbb{E}\Big\|\sum_{0\le i\ne j\le N-n} X_iX_jA_i^TA_j\Big\|_{\ell_2^n\to\ell_2^n} \le C\sqrt{Nn}\log n.$$

Combining this with the previous estimates we get

$$\mathbb{E}\|\Gamma\|^2_{\ell_2^n\to\ell_2^N} = \mathbb{E}\|\Gamma^T\Gamma\|_{\ell_2^n\to\ell_2^n} \le C\left(N + \sqrt{nN}\log n\right),$$

which ends the proof. □

4. Appendix

For reader's convenience we gather here several by now standard results which have been used in the proofs above. For most of them we provide detailed references, however in some cases we haven't been able to find the formulation we need in the literature, so we briefly describe how they follow from available references.

The first proposition gives estimates on the operator norm of a square random Toeplitz matrix. It was proved in [1] for symmetric random Toeplitz matrices. A simple modification of the proof gives the result in the non-symmetric case, however it can be also easily obtained by exploring the type 2 property of the space of symmetric matrices (see Proposition 4.6 below) or, e.g., by noncommutative Bernstein inequalities, since a random square Toeplitz matrix can be written as a linear combination with random coefficients of norm one matrices.

Proposition 4.1. *If $N = n$ and $(X_i)_{1-n\le i\le n-1}$ are independent, centered random variables, then*

$$\mathbb{E}\|T\|_{\ell_2^n\to\ell_2^n} \le C\Big(\sum_{i=1-n}^{n-1}\mathbb{E}X_i^2\Big)^{1/2}\sqrt{\log n}.$$

We will now state concentration results for Gaussian chaoses of order 2. In a weaker form they can be traced to [10]. The present formulation can be deduced from results on a Banach space-valued case in [2, 4] and appears explicitly in [11] and [12] (where a generalization to chaoses of higher degree has been obtained).

Proposition 4.2. *Let $g_1, g_2, \ldots, g_n$ be independent standard Gaussian random variables and let $A = (a_{ij})_{1\le i,j\le n}$ be an array of real numbers such that for all $1 \le i \le n$, $a_{ii} = 0$. Then for any $t \ge 0$,*

$$\mathbb{P}\Big(\Big|\sum_{1\le i,j\le n} a_{ij}g_ig_j\Big| \ge t\Big) \le 2\exp\Big(-\frac{1}{C}\min\Big(\frac{t^2}{\|A\|_{HS}^2}, \frac{t}{\|A\|_{\ell_2^n\to\ell_2^n}}\Big)\Big).$$

The next proposition is a consequence of Theorem 1.2.7 in [17] and a comparison between γ_p functionals and entropy integrals.

Proposition 4.3. *Consider a set T equipped with two distances d_1 and d_2 and a stochastic process $(X_t)_{t\in T}$ such that $\mathbb{E}X_t = 0$ for all $t \in T$ and for all $s,t \in T$ and $u > 0$,*

$$\mathbb{P}(|X_s - X_t| \ge u) \le 2\exp\Big(-\min\Big(\frac{u^2}{d_1(s,t)^2}, \frac{u}{d_2(s,t)}\Big)\Big).$$

Then

$$\mathbb{E}\sup_{s,t\in T}|X_s - X_t| \le C\Big(\int_0^\infty \sqrt{\log\mathcal{N}(T,d_1,\varepsilon)}d\varepsilon + \int_0^\infty \log\mathcal{N}(T,d_2,\varepsilon)d\varepsilon\Big).$$

Let us now state a simple proposition which combines standard symmetrization techniques with comparison between Gaussian and Rademacher averages.

Proposition 4.4. *Let $X_1, \ldots, X_n$ be independent centered random variables and let $(a_{ij})_{1\le i\neq j\le n}$ be coefficients from a normed space $(F, \|\cdot\|)$. Finally let $g_1, \ldots, g_n$ be i.i.d. standard Gaussian variables independent of the sequence $X_1, \ldots, X_n$. Then*

$$\mathbb{E}\Big\|\sum_{1\le i\neq j\le n} a_{ij}X_iX_j\Big\| \le 2\pi\mathbb{E}\Big\|\sum_{1\le i\neq j\le n} a_{ij}g_ig_jX_iX_j\Big\|.$$

Proof. Let $\varepsilon_1, \ldots, \varepsilon_n$ be independent Rademacher variables, independent of the sequences (X_i), (g_i). Using repetitively (and conditionally) the fact that for any convex function $\varphi\colon \mathbb{R}\to\mathbb{R}$ we have $\mathbb{E}\varphi(X_i) \le \mathbb{E}\varphi(2\varepsilon_iX_i)$, we get

$$\mathbb{E}\Big\|\sum_{1\le i\neq j\le n} a_{ij}X_iX_j\Big\| \le 4\mathbb{E}\Big\|\sum_{1\le i\neq j\le n} a_{ij}\varepsilon_i\varepsilon_jX_iX_j\Big\|.$$

Now, by symmetry of g_i and Jensen's inequality,

$$\begin{aligned}\frac{2}{\pi}\mathbb{E}\Big\|\sum_{1\le i\neq j\le n} a_{ij}\varepsilon_i\varepsilon_jX_iX_j\Big\| &= \mathbb{E}\Big\|\sum_{1\le i\neq j\le n} a_{ij}\varepsilon_i\varepsilon_j\mathbb{E}_g|g_ig_j|X_iX_j\Big\| \\ &\le \mathbb{E}\Big\|\sum_{1\le i\neq j\le n} a_{ij}\varepsilon_ig_i\varepsilon_jg_jX_iX_j\Big\| \\ &= \mathbb{E}\Big\|\sum_{1\le i\neq j\le n} a_{ij}g_ig_jX_iX_j\Big\|,\end{aligned}$$

which ends the proof. □

Recall that a Banach space $(F, \|\cdot\|)$ is of type 2 if there exists a finite constant T_F, such that for all $a_1, \ldots, a_n \in F$ and independent Rademacher variables $\varepsilon_1, \ldots, \varepsilon_n$, we have

$$\mathbb{E}\|\varepsilon_1a_1 + \cdots + \varepsilon_na_n\|^2 \le T_F^2\sum_{i=1}^n \|a_i\|^2. \tag{13}$$

The next proposition concerns basic properties of polynomial chaoses in spaces of type 2. It is well known, so we skip the proof and remark only that it consists of the three following steps: 1) decoupling inequalities for chaoses (see, e.g., Theorem 3.1.1. in [8]), 2) an iterative application of symmetrization inequalities, 3) iterative application of (13) conditionally on $(X_i)_{i=1}^n$.

Proposition 4.5. *Let $X_1, \ldots, X_n$ be independent centered, variance one random variables and let $(a_{ij})_{1\le i\neq j\le n}$ be coefficients from a normed space $(F, \|\cdot\|)$ with type 2 constant T_F. Then*

$$\mathbb{E}\|\sum_{1\le i\neq j\le n} a_{ij}X_iX_j\|^2 \le CT_F^4\sum_{1\le i\neq j\le n}\|a_{ij}\|^2.$$

Finally, the last proposition gives an estimate of the type 2 constant for the space of symmetric $n\times n$ matrices equipped with the operator norm.

Proposition 4.6. *The space $F = S_\infty^n$ of $n \times n$ symmetric matrices equipped with the operator norm has type 2 with constant $T_F \leq C\sqrt{\log n}$.*

This proposition follows easily from estimates of type 2 constants for Schatten classes S_p^n, which follow from the proof of Theorem 3.1. in [18] ($T_2(S_p^n) \leq C\sqrt{p}$), and the fact that the Banach-Mazur distance between S_∞^n and S_p^n is bounded by $n^{1/p}$ (it is enough to take $p = \log n$). We refer the reader, e.g., to [19] for details on the Banach-Mazur distance and geometry of Banach spaces.

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Edge Fluctuations of Eigenvalues of Wigner Matrices

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Abstract. We establish a moderate deviation principle (MDP) for the number of eigenvalues of a Wigner matrix in an interval close to the edge of the spectrum. Moreover we prove a MDP for the ith largest eigenvalue close to the edge. The proof relies on fine asymptotics of the variance of the eigenvalue counting function of GUE matrices due to Gustavsson. The extension to large families of Wigner matrices is based on the Tao and Vu Four Moment Theorem. Possible extensions to other random matrix ensembles are commented.

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1. Introduction

Recently, in [5] and [4] the Central Limit Theorem (CLT) for the eigenvalue counting function of *Wigner matrices*, that is the number of eigenvalues falling in an interval, was established. This *universality result* relies on fine asymptotics of the variance of the eigenvalue counting function, on the Fourth Moment Theorem due to Tao and Vu as well as on recent localization results due to Erdös, Yau and Yin. There are many random matrix ensembles of interest, but to focus our discussion and to clear the exposition we shall restrict ourselves to the most famous model class of ensembles, the Wigner Hermitian matrix ensembles. For an integer $n \geq 1$ consider an $n \times n$ Wigner Hermitian matrix $M_n = (Z_{ij})_{1 \leq i,j \leq n}$: Consider a family of jointly independent complex-valued random variables $(Z_{ij})_{1 \leq i,j \leq n}$ with $Z_{ji} = \bar{Z}_{ij}$, in particular the Z_{ii} are real valued. For $1 \leq i < j \leq n$ require that the random variables have mean zero and variance one and the $Z_{ij} \equiv Z$ are identically

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distributed, and for $1 \leq i = j \leq n$ require that $Z_{ii} \equiv Z'$ are also identically distributed with mean zero and variance one. The distributions of Z and Z' are called atom distributions. An important example of a Wigner Hermitian matrix M_n is the case where the entries are Gaussian, that is Z_{ij} is distributed according to a complex standard Gaussian $N(0,1)_{\mathbb{C}}$ for $i \neq j$ and Z_{ii} is distributed according to a real standard Gaussian $N(0,1)_{\mathbb{R}}$, giving rise to the so-called Gaussian Unitary Ensembles (GUE). GUE matrices will be denoted by M'_n. In this case, the joint law of the eigenvalues is known, allowing a good description of their limiting behavior both in the global and local regimes (see [1]). In the Gaussian case, the distribution of the matrix is invariant by the action of the group $SU(n)$. The eigenvalues of the matrix M_n are independent of the eigenvectors which are Haar distributed. If $(Z_{i,j})_{1 \leq i<j}$ are real valued the *symmetric Wigner matrix* is defined analogously and the case of Gaussian variables with $\mathbb{E}Z_{ii}^2 = 2$ is of particular importance, since their law is invariant under the action of the orthogonal group $SO(n)$, known as Gaussian Orthogonal Ensembles (GOE).

The matrix $W_n := \frac{1}{\sqrt{n}} M_n$ is called the coarse-scale normalized Wigner Hermitian matrix, and $A_n := \sqrt{n} M_n$ is called the fine-scale normalized Wigner Hermitian matrix. For any $n \times n$ Hermitian matrix A we denote by $\lambda_1(A), \dots, \lambda_n(A)$ the real eigenvalues of A. We introduce the *eigenvalue counting function*

$$N_I(A) := \big| \{ 1 \leq i \leq n : \lambda_i(A) \in I \} \big|$$

for any interval $I \subset \mathbb{R}$. We will consider $N_I(W_n)$ as well as $N_I(A_n)$. Remark that $N_I(W_n) = N_{nI}(A_n)$. The global *Wigner theorem* states that the empirical measure $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$ on the eigenvalues λ_i of the coarse-scale normalized Wigner Hermitian matrix W_n converges weakly almost surely as $n \to \infty$ to the semicircle law

$$d\varrho_{sc}(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\, 1_{[-2,2]}(x)\, dx,$$

(see [1, Theorem 2.1.21, Theorem 2.2.1]). Consequently, for any interval $I \subset \mathbb{R}$,

$$\lim_{n \to \infty} \frac{1}{n} N_I(W_n) = \varrho_{sc}(I) := \int_I \varrho_{sc}(y)\, dy$$

almost surely. At the fluctuation level, it is well known that for the GUE, $W'_n := \frac{1}{\sqrt{n}} M'_n$ satisfies a CLT (see [17]): Let I_n be an interval in $\mathbb{R}$. If $\mathbb{V}(N_{I_n}(W'_n)) \to \infty$ as $n \to \infty$, then

$$\frac{N_{I_n}(W'_n) - \mathbb{E}[N_{I_n}(W'_n)]}{\sqrt{\mathbb{V}(N_{I_n}(W'_n))}} \to N(0,1)_{\mathbb{R}}$$

as $n \to \infty$ in distribution. In [14] the asymptotic behavior of the expectation and the variance of the counting function $N_{I_n}(W'_n)$ for intervals $I_n = [y(n), \infty)$ with $y(n) = G^{-1}(k/n)$ (where $k = k(n)$ is such that $k/n \to a \in (0,1)$ – *strictly in the bulk* –, and G denotes the distribution function of the semicircle law) was established:

$$\mathbb{E}[N_{I_n}(W_n')] = n - k(n) + O\left(\frac{\log n}{n}\right) \text{ and } \mathbb{V}(N_{I_n}(W_n')) = \left(\frac{1}{2\pi^2} + o(1)\right) \log n. \tag{1.1}$$

The proof applied strong asymptotics for orthogonal polynomials with respect to exponential weights, see [6]. These conclusions were extended to non-Gaussian Wigner Hermitian matrices in [5].

In this article we focus on the behaviour of the eigenvalue counting function $N_{I_n}(W_n)$ evaluated *at the edge* of the spectrum. Fine asymptotics of expectation and variance as well as the CLT at the edge of the spectrum of GUE matrices are known, see [14]. Let $I_n = [y_n, \infty)$ where $y_n \to 2^-$ for $n \to \infty$ in a certain speed. We prove a global moderate deviation principle (MDP) for $Z_n := \frac{N_{I_n}(W_n') - \frac{2}{3\pi} n(2-y_n)^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2} \log(n(2-y_n)^{3/2})}}$, that means: For any sequence $(a_n)_n$ with $1 \ll a_n \ll \sqrt{\log n}$, we have

$$P\left(\frac{N_{I_n}(W_n') - \frac{2}{3\pi} n(2-y_n)^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2} \log(n(2-y_n)^{3/2})}} \sim x\right) \approx e^{-a_n^2 I(x)},$$

see Theorem 2.1 for a precise statement. Using the equivalence

$$N_{[y,\infty)}(W_n) \le n - i \text{ if and only if } \lambda_i(W_n) \le y$$

in Theorem 3.1 we prove a local MDP for an eigenvalue near the edge of the spectrum of W_n'. Applying the Four Moment Theorem due to Tao and Vu, the local MDP can be generalized to a large class of Hermitian Wigner matrices. Finally this yields the main theorem of the present article: a universal global MDP of the eigenvalue counting function $N_{[y,\infty)}(W_n)$ for y near the edge of the spectrum. We state it in Theorem 5.1. The last section indicates how to achieve the previous moderate deviation results for symmetric Wigner matrices.

2. Global moderate deviations at the edge of the spectrum

Certain deviations results and concentration properties for Wigner matrices were considered. Our aim is to establish moderate deviation principles. Recall that a sequence of laws $(P_n)_{n\ge 0}$ on a Polish space Σ satisfies a large deviation principle (LDP) with good rate function $I : \Sigma \to \mathbb{R}_+$ and speed s_n going to infinity with n if and only if the level sets $\{x : I(x) \le M\}$, $0 \le M < \infty$, of I are compact and for all closed sets F

$$\limsup_{n\to\infty} s_n^{-1} \log P_n(F) \le -\inf_{x\in F} I(x)$$

whereas for all open sets O

$$\liminf_{n\to\infty} s_n^{-1} \log P_n(O) \ge -\inf_{x\in O} I(x).$$

We say that a sequence of random variables satisfies the LDP when the sequence of measures induced by these variables satisfies the LDP. Formally a moderate deviation principle is nothing else but the LDP. However, we speak about a moderate deviation principle (MDP) for a sequence of random variables, whenever the scaling of the corresponding random variables is between that of an ordinary Law of Large Numbers (LLN) and that of a CLT.

Large deviation results for the empirical measures of Wigner matrices are still only known for the Gaussian ensembles since their proof is based on the explicit joint law of the eigenvalues, see [2] and [1]. A moderate deviation principle for the empirical measure of the GUE or GOE is also known, see [7]. This moderate deviations result does not have yet a fully universal version for Wigner matrices. It has been generalised to Gaussian divisible matrices with a deterministic self-adjoint matrix added with converging empirical measure [7] and to Bernoulli matrices [9]. Recently we proved in [11] a MDP for the number of eigenvalues *in the bulk* of the spectrum of a GUE matrix. If M_n' is a GUE matrix and $W_n' := \frac{1}{\sqrt{n}} M_n'$ and I_n be an interval in $\mathbb{R}$. If $\mathbb{V}(N_{I_n}(W_n')) \to \infty$ for $n \to \infty$, then, for any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll \sqrt{\mathbb{V}(N_{I_n}(W_n'))}$, the sequence $(Z_n)_n$ with

$$Z_n = \frac{N_{I_n}(W_n') - \mathbb{E}[N_{I_n}(W_n')]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n'))}}$$

satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$. Moreover let $I = [y, \infty)$ with $y \in (-2, 2)$ strictly in the bulk, then the sequence $(\hat{Z}_n)_n$ with $\hat{Z}_n = \frac{N_I(W_n') - n\varrho_{sc}(I)}{a_n \sqrt{\frac{1}{2\pi^2} \log n}}$ satisfies the MDP with the same speed, the same rate function, and in the regime $1 \ll a_n \ll \sqrt{\log n}$ (called the MDP with numerics; see Theorem 1.1 in [11]). It follows applying (1.1). In [11], these conclusions were extended to non-Gaussian Wigner Hermitian matrices. In [10] we proved a further universal MDP for the logarithm of the determinants of Wigner matrices.

The first observation in this paper is, that the MDP for $(Z_n)_n$ and $(\hat{Z}_n)_n$, respectively, is not restricted to the bulk of the spectrum. To state the result, let $\delta > 0$ and assume that $y_n \in [-2 + \delta, 2)$ and $n(2 - y_n)^{3/2} \to \infty$ when $n \to \infty$. Then with [14, Lemma 2.3] the variance of the number of eigenvalues of W_n' in $I_n := [y_n, \infty)$ satisfies

$$\mathbb{V}(N_{I_n}(W_n')) = \frac{1}{2\pi^2} \log\big(n(2 - y_n)^{3/2}\big) \, (1 + \eta(n)), \tag{2.1}$$

where $\eta(n) \to 0$ as $n \to \infty$. Moreover the expected number of eigenvalues of W_n' in I_n, when $y_n \to 2^-$, is given by [14, Lemma 2.2]:

$$\mathbb{E}(N_{I_n}(W_n')) = \frac{2}{3\pi} n(2 - y_n)^{3/2} + O(1). \tag{2.2}$$

Hence applying Theorem 1.1 in [11] we immediately obtain:

Theorem 2.1. *Let M_n' be a GUE matrix and $W_n' = \frac{1}{\sqrt{n}} M_n$. Let $I_n = [y_n, \infty)$ where $y_n \to 2^-$ for $n \to \infty$. Assume that $y_n \in [-2 + \delta, 2)$ and $n(2 - y_n)^{3/2} \to \infty$ when $n \to \infty$. Then, for any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll$*

$\sqrt{\mathbb{V}(N_{I_n}(W_n'))}$, *the sequence* $\frac{N_{I_n}(W_n')-\mathbb{E}[N_{I_n}(W_n')]}{a_n\sqrt{\mathbb{V}(N_{I_n}(W_n'))}}$ *satisfies a MDP with speed* a_n^2 *and rate function* $I(x)=\frac{x^2}{2}$. *Moreover the sequence*

$$Z_n := \frac{N_{I_n}(W_n') - \frac{2}{3\pi}n(2-y_n)^{3/2}}{a_n\sqrt{\frac{1}{2\pi^2}\log(n(2-y_n)^{3/2})}}$$

satisfies the MDP with the same speed, the same rate function, and in the regime $1 \ll a_n \ll \sqrt{\log(n(2-y_n)^{3/2})}$ (*called the MDP with numerics*).

For symmetry reasons an analogous result could be formulated for the counting function $N_{I_n}(W_n')$ near the left edge of the spectrum.

3. Local moderate deviations at the edge of the spectrum

Under certain conditions on i it was proved in [14] that the ith eigenvalue λ_i of the GUE W_n' satisfies a CLT. Consider $t(x)\in[-2,2]$ defined for $x\in[0,1]$ by

$$x=\int_{-2}^{t(x)}\varrho_{sc}(t)\,dt=\frac{1}{2\pi}\int_{-2}^{t(x)}\sqrt{4-x^2}\,dx.$$

Then for $i=i(n)$ such that $i/n\to a\in(0,2)$ as $n\to\infty$ (i.e., λ_i is eigenvalue in the bulk), $\lambda_i(W_n')$ satisfies a CLT:

$$X_n := \sqrt{\frac{4-t(i/n)^2}{2}}\,\frac{\lambda_i(W_n')-t(i/n)}{\frac{\sqrt{\log n}}{n}} \to N(0,1) \tag{3.1}$$

for $n\to\infty$. Remark that $t(i/n)$ is sometimes called the *classical or expected location* of the ith eigenvalue. The standard deviation is $\frac{\sqrt{\log n}}{\pi\sqrt{2}}\frac{1}{n\varrho_{sc}(t(i/n))}$. Note that from the semicircular law, the factor $\frac{1}{n\varrho_{sc}(t(i/n))}$ is the mean eigenvalue spacing. Informally, (3.1) asserts in the GUE case, that each eigenvalue $\lambda_i(W_n')$ typically deviates by $O\big(\sqrt{\log n}/(n\varrho(t(i/n)))\big)$ around its classical location. This result can be compared with the so-called *eigenvalue rigidity property* $\lambda_i(W_n') = t(i/n)+O(n^{-1+\varepsilon})$ established in [12], which has a slightly worse bound on the deviation but which holds with overwhelming probability and for general Wigner ensembles. See also discussions in [20, Section 3]. We proved in [11, Theorem 4.1] a MDP for $\big(\frac{1}{a_n}X_n\big)_n$ with X_n in (3.1), for any $1\ll a_n\ll\sqrt{\log n}$, with speed a_n^2 and rate $x^2/2$. Moreover in [11, Theorem 4.2], these conclusions were extended to non-Gaussian Wigner Hermitian matrices. The proofs are achieved by the tight relation between eigenvalues and the counting function expressed by the elementary equivalence, for $I(y)=[y,\infty)$, $y\in\mathbb{R}$,

$$N_{I(y)}(W_n)\le n-i \text{ if and only if } \lambda_i(W_n)\le y. \tag{3.2}$$

This relation is true for any eigenvalue $\lambda_i(W_n)$, independent of sitting being in the bulk of the spectrum or very close to the right edge of the spectrum. Hence the next goal is to transport the MDP for the counting function of eigenvalues close to the (right) edge, Theorem 2.1, to a MDP for any singular eigenvalue close to the right edge of the spectrum. Consider $i = i(n)$ with $i \to \infty$ but $i/n \to 0$ as $n \to \infty$ and define $\lambda_{n-i}(W_n')$ as eigenvalue number $n-i$ in the GUE. An example is $i(n) = n - \log n$. In [14, Theorem 1.2] a CLT was proven, which is

$$Z_{n,i} := \frac{\lambda_{n-i}(W_n') - \big(2 - \big(\frac{3\pi}{2}\frac{i}{n}\big)^{2/3}\big)}{\text{const}\big(\frac{\log i}{i^{2/3}n^{4/3}}\big)^{1/2}} \to N(0,1)_{\mathbb{R}} \tag{3.3}$$

in distribution with const $= \big((3\pi)^{2/3}2^{1/3}\big)^{-1/2}$. Remark that the formulation in [14, Theorem 1.2] is different, since first of all the GUE in [14] was defined such that the limiting semicircular law has support $[-1,1]$ and, second the CLT in [14] is formulated for $\lambda_{n-i}(M_n')$ instead of $\lambda_{n-i}(W_n')$. The choice of the asymptotic expectation and variance in (3.3) can be explained as follows. Let $g(y_n)$ be the expected number of eigenvalues in $I_n = [y_n, \infty)$. Then with (3.2)

$$\begin{aligned} P\big(\lambda_{n-i}(W_n') \le y_n\big) &= P\big(N_{I_n}(W_n') \le i\big) \\ &= P\bigg(\frac{N_{I_n}(W_n') - g(y_n)}{\mathbb{V}(N_{I_n}(W_n'))^{1/2}} \le \frac{i - g(y_n)}{\mathbb{V}(N_{I_n}(W_n'))^{1/2}}\bigg). \end{aligned}$$

Trying to apply the CLT for N_{I_n} is choosing y_n such that $\frac{i-g(y_n)}{\sqrt{\mathbb{V}(N_{I_n}(W_n'))}} \to x$ for $n \to \infty$, because this would imply $P\big(\lambda_{n-i}(W_n') \le y_n\big) \to \int_{-\infty}^x \varphi_{0,1}(t)\,dt$, where $\varphi_{0,1}(\cdot)$ denotes the density of the standard normal distribution. The candidate for y_n can be found as in the proof of [14, Theorem 1.2], with $g(y_n) = \frac{2}{3\pi}n(2-y_n)^{3/2} + O(1)$ and $h(y_n) = \mathbb{V}(N_{I_n}(W_n'))^{1/2} = \frac{1}{\sqrt{2\pi}} \log^{1/2}(n(2-y_n)^{3/2}) + o(\log^{1/2}(n(2-y_n)^{3/2}))$. Applying the same heuristic as on page 157 in [14], we obtain

$$y_n \approx 2 - \left(\frac{3\pi}{2}\frac{i}{n}\right)^{2/3} + x\left(\big((3\pi)^{2/3}2^{1/3}\big)^{-1/2}\frac{\log i}{i^{2/3}n^{4/3}}\right)^{1/2}.$$

With respect to the statement in Theorem 2.1 one might expect a MDP for $\big(\frac{1}{a_n}Z_{n,i}\big)_n$ for certain growing sequences $(a_n)_n$. We have

$$\begin{aligned} P\big(Z_{n,i}/a_n \le x\big) &= P\big(\lambda_{n-i}(W_n') \le y_n(a_n)\big) = P\big(N_{I_n}(W_n') \le i\big) \\ &= P\bigg(\frac{N_{I_n}(W_n') - \mathbb{E}(N_{I_n}(W_n'))}{a_n\mathbb{V}(N_{I_n}(W_n'))^{1/2}} \le \frac{i - \mathbb{E}(N_{I_n}(W_n'))}{a_n\mathbb{V}(N_{I_n}(W_n'))^{1/2}}\bigg) \end{aligned}$$

with

$$y_n(a_n) := 2 - \left(\frac{3\pi}{2}\frac{i}{n}\right)^{2/3} + a_n\,x\left(\big((3\pi)^{2/3}2^{1/3}\big)^{-1/2}\Big(\frac{\log i}{i^{2/3}n^{4/3}}\Big)\right)^{1/2} \tag{3.4}$$

and $I_n = [y_n(a_n), \infty)$. Since $i \to \infty$ and $i/n \to 0$ for $n \to \infty$, we have that $y_n(a_n) \to 2^-$ for every a_n such that $a_n \ll \big(\log i\big)^{1/2}$. Hence we can apply (2.2), that is $\mathbb{E}(N_{I_n}(W_n')) = \frac{2}{3\pi}n(2-y_n(a_n))^{3/2} + O(1)$. With

$$2 - y_n(a_n) = \left(\frac{3\pi}{2}\frac{i}{n}\right)^{2/3}\left(1 - \frac{a_n\,x\,\log^{1/2} i}{(3\pi/\sqrt{2})i}\right)$$

by Taylor's expansion we obtain

$$\frac{2}{3\pi} n(2 - y_n(a_n))^{3/2} = i - \frac{1}{\sqrt{2}\pi} a_n\, x \, \log^{1/2} i + o\big(a_n\, x\, \log^{1/2} i\big), \tag{3.5}$$

and therefore $i - \mathbb{E}(N_{I_n}(W_n')) = \frac{1}{\sqrt{2}\pi} a_n\, x\, \log^{1/2} i + o\big(a_n\, x\, \log^{1/2} i\big) + O(1)$. From (3.5) we obtain that $n(2 - y_n(a_n))^{3/2} \to \infty$ for $n \to \infty$ for every $a_n \ll \big(\log i\big)^{1/2}$. Hence we can apply (2.1), that is $\mathbb{V}(N_{I_n}(W_n')) = \frac{1}{2\pi^2} \log\big(n(2 - y_n(a_n))^{3/2}\big)\,(1 + o(1))$. With (3.5) the variance $\mathbb{V}(N_{I_n}(W_n'))$ equals

$$\left(\frac{1}{2\pi^2} \log\Big(\frac{3\pi}{2} i\Big) + \frac{1}{2\pi^2} \log\Big(1 - \frac{a_n\, x\, (\log i)^{1/2}}{\sqrt{2}\pi i} + o\Big(\frac{a_n\, x\, (\log i)^{1/2}}{i}\Big)\Big)\right)(1 + o(1)).$$

Summarizing we have proven that for any growing sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll (\log i)^{1/2}$

$$\frac{i - \mathbb{E}(N_{I_n}(W_n'))}{a_n \mathbb{V}(N_{I_n}(W_n'))^{1/2}} = x + o(1).$$

By Theorem 2.1 we obtain for every $x < 0$ that $\lim_{n\to\infty} \frac{1}{a_n^2} \log P\big(Z_{n,i}/a_n \le x\big) = -\frac{x^2}{2}$. With $P\big(Z_{n,i}/a_n \ge x\big) = P\big(N_{I_n}(W_n') \ge i - 1\big)$ the same calculations lead, for every $x > 0$, to

$$\lim_{n\to\infty} \frac{1}{a_n^2} \log P\big(Z_{n,i}/a_n \ge x\big) = -\frac{x^2}{2}. \tag{3.6}$$

Next we choose all open intervals (a, b), where at least one of the endpoints is finite and where none of the endpoints is zero. Denote the family of such intervals by $\mathcal{U}$. Now it follows for each $U = (a, b) \in \mathcal{U}$,

$$\mathcal{L}_U := \lim_{n\to\infty} \frac{1}{a_n^2} \log P\big(Z_{n,i}/a_n \in U\big) = \begin{cases} b^2/2 & : \quad a < b < 0 \\ 0 & : \quad a < 0 < b \\ a^2/2 & : \quad 0 < a < b \end{cases}$$

By [8, Theorem 4.1.11], $(Z_{n,i}/a_n)_n$ satisfies a weak MDP (see definition in [8, Section 1.2]) with speed a_n^2 and rate function $t \mapsto \sup_{U\in\mathcal{U}; t\in U} \mathcal{L}_U = \frac{t^2}{2}$. With (3.6), it follows that $(Z_{n,i}/a_n)_n$ is exponentially tight (see definition in [8, Section 1.2]), hence by Lemma 1.2.18 in [8], $(Z_{n,i}/a_n)_n$ satisfies the MDP with the same speed and the same good rate function. Hence we have proven:

Theorem 3.1. *Let M_n' be a GUE matrix and $W_n' = \frac{1}{\sqrt{n}} M_n'$. Consider $i = i(n)$ such that $i \to \infty$ but $i/n \to 0$ as $n \to \infty$. If λ_{n-i} denotes the eigenvalue number $n - i$ in the GUE matrix W_n' it holds that for any sequence $(a_n)_n$ of real numbers such that $1 \ll a_n \ll (\log i)^{1/2}$, the sequence $\big(\frac{1}{a_n} Z_{n,i}\big)_n$ with $Z_{n,i}$ given by (3.3) satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$.*

4. Universal local moderate deviations near the edge

Our next goal is to check whether the precise distribution of the atom variables Z_{ij} of a Hermitian random matrix M_n are relevant for the conclusion of the MDP stated in Theorems 2.1 and 3.1, so long as they are normalized to have mean zero and variance one, and are jointly independent on the upper-triangular portion of M_n. It is a remarkable feature of our MDP results that they are *universal*, hence the distribution of the atom variables are irrelevant in some sense. The arguments used above relied heavily on the special structure of the GUE ensemble, in particular on the determinantal structure of the joint probability distribution (see [11, Theorem 1.1 and 1.3]) and on the fine asymptotics of the expectation and the variance of the eigenvalue counting function of GUE presented in [14]. We apply the swapping method due to Tao and Vu, in which one replaces the entries of one Wigner Hermitian matrix M_n with another matrix M_n' which are close in the sense of matching moments. This goes back to Lindeberg's exchange strategy for proving the classical CLT, [15], and applied to Wigner matrices, e.g., in [3]. The precise statement of the so-called Four Moment Theorem needs some preparation. We will use the notation as in [20].

We say that two complex random variables η_1 and η_2 *match to order* k with $k \in \mathbb{N}$ if

$$\mathbb{E}\big[\mathrm{Re}(\eta_1)^m \,\mathrm{Im}(\eta_1)^l\big] = \mathbb{E}\big[\mathrm{Re}(\eta_2)^m \,\mathrm{Im}(\eta_2)^l\big]$$

for all $m, l \geq 0$ such that $m + l \leq k$. We will consider the case when the real and the imaginary parts of η_1 or of η_2 are independent, then the matching moment condition simplifies to the assertion that $E\big[\mathrm{Re}(\eta_1)^m\big] = E\big[\mathrm{Re}(\eta_2)^m\big]$ and $E\big[\mathrm{Im}(\eta_1)^l\big] = E\big[\mathrm{Im}(\eta_2)^l\big]$ for all $m, l \geq 0$ such that $m + l \leq k$.

We say that the Wigner Hermitian matrix M_n obeys Condition (**C0**) if we have the exponential decay condition

$$P\big(|Z_{ij}| \geq t^C\big) \leq e^{-t}$$

for all $1 \leq i, j \leq n$ and $t \geq C'$, and some constants C, C' independent of i, j, n. We say that the Wigner Hermitian matrix M_n obeys Condition (**C1**) with constant C_0 if one has

$$\mathbb{E}|Z_{ij}|^{C_0} \leq C$$

for some constant C independent of n. Of course, Condition (**C0**) implies Condition (**C1**) for any C_0, but not conversely. The statement of the Four Moment Theorem for eigenvalues is:

Theorem 4.1 (Four Moment Theorem due to Tao and Vu). *Let $c_0 > 0$ be a sufficiently small constant. Let $M_n = (Z_{ij})$ and $M_n' = (Z_{ij}')$ be two $n \times n$ Wigner Hermitian matrices satisfying Condition* (**C1**) *for some sufficiently large constant C_0. Assume furthermore that for any $1 \leq i < j \leq n$, Z_{ij} and Z_{ij}' match to order 4 and for any $1 \leq i \leq n$, and Z_{ii} and Z_{ii}' match to order 2. Set $A_n := \sqrt{n} M_n$ and $A_n' := \sqrt{n} M_n'$, let $1 \leq k \leq n^{c_0}$ be an integer, and let $G : \mathbb{R}^k \to \mathbb{R}$ be a smooth function obeying the derivative bounds $|\nabla^j G(x)| \leq n^{c_0}$ for all $0 \leq j \leq 5$ and $x \in \mathbb{R}^k$.*

Then for any $1 \le i_1 < i_2 < \cdots < i_k \le n$, *and for* n *sufficiently large we have*

$$|\mathbb{E}\big(G(\lambda_{i_1}(A_n),\ldots,\lambda_{i_k}(A_n))\big) - \mathbb{E}\big(G(\lambda_{i_1}(A_n'),\ldots,\lambda_{i_k}(A_n'))\big)| \le n^{-c_0}. \qquad (4.1)$$

The preliminary version of this Theorem was first established in the case of bulk eigenvalues and assuming Condition (**C0**), [19]. Later the restriction to the bulk was removed and the Condition (**C0**) was relaxed to Condition (**C1**) for a sufficiently large value of C_0, [18]. Moreover, a natural question is whether the requirement of four matching moments is necessary. As far as the distribution of individual eigenvalues $\lambda_i(A_n)$ are concerned, the answer is essentially yes. For this see the discussions in [20].

Applying this Theorem for the special case when M_n' is GUE, we obtain the following MDP:

Theorem 4.2. *The MDP for* $\big(\frac{1}{a_n} Z_{n,i}\big)_n$, *Theorem* 3.1, *hold for Wigner Hermitian matrices obeying Condition* (**C1**) *for a sufficiently large* C_0, *and whose atom distributions match that of GUE to second order on the diagonal and fourth order off the diagonal. Given* $i = i(n)$ *such that* $i \to \infty$ *and* $i/n \to 0$ *as* $n \to \infty$ *we have: The sequence* $\big(\frac{1}{a_n} Z_{n,i}\big)_n$ *with*

$$Z_{n,i} := \frac{\lambda_{n-i}(W_n) - \Big(2 - \big(\frac{3\pi}{2}\frac{i}{n}\big)^{2/3}\Big)}{\mathrm{const}\big(\frac{\log i}{i^{2/3} n^{4/3}}\big)^{1/2}} \qquad (4.2)$$

satisfies the MDP for any sequence $(a_n)_n$ *of real numbers such that* $1 \ll a_n \ll (\log i)^{1/2}$ *with speed* a_n^2 *and rate function* $I(x) = \frac{x^2}{2}$.

Proof. Let M_n be a Wigner Hermitian matrix whose entries satisfy Condition (**C1**) and match the corresponding entries of GUE up to order 4. Let i be as in the statement of the Theorem, and let c_0 be as in Theorem 4.1. Then [19, (18)] says that

$$P\big(\lambda_i(A_n') \in I_-\big) - n^{-c_0} \le P\big(\lambda_i(A_n) \in I\big) \le P\big(\lambda_i(A_n') \in I_+\big) + n^{-c_0} \qquad (4.3)$$

for all intervals $I = [b,c]$, and n sufficiently large, where $I_+ := [b - n^{-c_0/10}, c + n^{-c_0/10}]$ and $I_- := [b + n^{-c_0/10}, c - n^{-c_0/10}]$. We present the argument of proof of (4.3) just to make the presentation more self-contained. One can find a smooth bump function $G : \mathbb{R} \to \mathbb{R}_+$ which is equal to one on the smaller interval I and vanishes outside the larger interval I_+. It follows that $P\big(\lambda_i(A_n) \in I\big) \le \mathbb{E}G(\lambda_i(A_n))$ and $\mathbb{E}G(\lambda_i(A_n')) \le P\big(\lambda_i(A_n') \in I_+\big)$. One can choose G to obey the condition $|\nabla^j G(x)| \le n^{c_0}$ for $j = 0, \ldots, 5$ and hence by Theorem 4.1 one gets

$$|\mathbb{E}G(\lambda_i(A_n)) - \mathbb{E}G(\lambda_i(A_n'))| \le n^{-c_0}.$$

Therefore the second inequality in (4.3) follows from the triangle inequality. The first inequality is proven similarly.

Now for n sufficiently large we consider the interval $I_n := [b_n, c_n]$ with

$$b_n := b\, a_n\, n\, \mathrm{const}\Big(\frac{\log i}{i^{2/3} n^{4/3}}\Big)^{1/2} + n\bigg(2 - \Big(\frac{3\pi}{2}\frac{i}{n}\Big)^{2/3}\bigg),$$

$$c_n := c\, a_n\, n \text{ const}\Big(\frac{\log i}{i^{2/3} n^{4/3}}\Big)^{1/2} + n\bigg(2 - \Big(\frac{3\pi}{2}\frac{i}{n}\Big)^{2/3}\bigg)$$

with $b, c \in \mathbb{R}$, $b \le c$ and $\text{const} = \big((3\pi)^{2/3} 2^{1/3}\big)^{-1/2}$. Then for $\frac{1}{a_n} Z_{n,i}$ defined as in the statement of the Theorem we have $P\big(Z_{n,i}/a_n \in [b,c]\big) = P\big(\lambda_{n-i}(A_n) \in I_n\big)$. With (4.3) and [8, Lemma 1.2.15] we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{a_n^2} \log P\big(Z_{n,i}/a_n \in [b,c]\big) \\ &\quad \le \max\bigg(\limsup_{n\to\infty} \frac{1}{a_n^2} \log P\big(\lambda_{n-i}(A_n') \in (I_n)_+\big); \limsup_{n\to\infty} \frac{1}{a_n^2} \log n^{-c_0}\bigg). \end{aligned}$$

For the first object we have

$$\begin{aligned} &P\big(\lambda_{n-i}(A_n') \in (I_n)_+\big) \\ &\quad = P\Bigg(\frac{\lambda_{n-i}(A_n') - n\big(2 - \big(\frac{3\pi}{2}\frac{i}{n}\big)^{2/3}\big)}{a_n\, n \text{ const}\big(\frac{\log i}{i^{2/3}n^{4/3}}\big)^{1/2}} \in [b - \eta(n), c + \eta(n)]\Bigg) \end{aligned}$$

with $\eta(n) = n^{-c_0/10}\big(a_n\, n \text{ const}\big(\frac{\log i}{i^{2/3}n^{4/3}}\big)^{1/2}\big)^{-1} \to 0$ as $n \to \infty$. Since $c_0 > 0$ and $\log n / a_n^2 \to \infty$ for $n \to \infty$ by assumption, applying Theorem 3.1 we have

$$\limsup_{n\to\infty} \frac{1}{a_n^2} \log P\big(Z_{n,i}/a_n \in [b,c]\big) \le - \inf_{x \in [b,c]} \frac{x^2}{2}.$$

Applying the first inequality in (4.3) in the same manner we also obtain the upper bound

$$\limsup_{n\to\infty} \frac{1}{a_n^2} \log P\big(Z_{n,i}/a_n \in [b,c]\big) \ge - \inf_{x \in [b,c]} \frac{x^2}{2}.$$

Finally the argument in the last part of the proof of Theorem 3.1 can be repeated to obtain the MDP for $(Z_{n,i}/a_n)_n$. □

5. Universal global moderate deviations near the edge

Next we show the MDP for the eigenvalue counting function near the edge of the spectrum for Wigner Hermitian matrices matching moments with GUE up to order 4:

Theorem 5.1. *The MDP for $(Z_n)_n$, Theorem 2.1, hold for Wigner Hermitian matrices M_n obeying Condition (**C1**) for a sufficiently large C_0, and whose atom distributions match that of GUE to second order on the diagonal and fourth order off the diagonal. Let $W_n = \frac{1}{\sqrt{n}} M_n$, let $I_n = [y_n, \infty)$ where $y_n \to 2^-$ for $n \to \infty$. Assume that $y_n \in [-2+\delta, 2)$ and $n(2-y_n)^{3/2} \to \infty$ when $n \to \infty$. Then the sequence*

$$Z_n = \frac{N_{I_n}(W_n) - \frac{2}{3\pi} n(2-y_n)^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2}\log(n(2-y_n)^{3/2})}}$$

satisfies the MDP with speed a_n^2*, rate function* $x^2/2$ *and in the regime* $1 \ll a_n \ll \sqrt{\log(n(2-y_n)^{3/2})}$.

Proof. For every $\xi \in \mathbb{R}$ and k_n defined by

$$k_n := \xi\, a_n \sqrt{\frac{1}{2\pi^2}\log(n(2-y_n)^{3/2})} + \frac{2}{3\pi} n(2-y_n)^{3/2}$$

we obtain that $P\big(Z_n \leq \xi\big) = P\big(N_{I_n}(W_n) \leq k_n\big)$. Hence using (3.2) it follows

$$P\big(Z_n \leq \xi\big) = P\big(\lambda_{n-k_n}(W_n) \leq y_n\big) = P\big(\lambda_{n-k_n}(A_n) \leq n\, y_n\big).$$

With (4.3) we have

$$P\big(\lambda_{n-k_n}(A_n) \leq n\, y_n\big) \leq P\big(\lambda_{n-k_n}(A_n') \leq n\, y_n + n^{-c_0/10}\big) + n^{-c_0}$$

and

$$\begin{aligned} P\big(\lambda_{n-k_n}(A_n') \leq n\, y_n + n^{-c_0/10}\big) &= P\big(\lambda_{n-k_n}(W_n') \leq y_n + n^{-1-c_0/10}\big) \\ &= P\big(N_{J_n}(W_n') \leq k_n\big), \end{aligned}$$

where $J_n = [y_n + n^{-1-c_0/10}, \infty)$. With $y_n' := y_n + n^{-1-c_0/10}$ we consider

$$Z_n' = \frac{N_{J_n}(W_n') - \frac{2}{3\pi} n(2-y_n')^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2}\log(n(2-y_n')^{3/2})}}.$$

In order to apply Theorem 2.1 for $(Z_n')_n$, we have to check if $y_n' \to 2^-$ and $n(2-y_n')^{3/2} \to \infty$ when $n \to \infty$. For a proof see [4, Section 2]. We present the arguments just to make the presentation more self-contained. By assumption we take $y_n \in [-2+\delta, 2)$ with $y_n \to 2^-$. Suppose that $y_n' > 2$ for some n, then $y_n - 2 + n^{-1-c_0/10} > 0$, hence $2 - y_n < n^{-1-c_0/10}$, which implies $n(2-y_n)^{3/2} < n\, n^{-3/2-3c_0/20}$, but the left-hand side is growing by assumption, a contradiction. We have proven $y_n' \to 2^-$. Moreover we have

$$\begin{aligned} (2-y_n')^{3/2} &= (2-y_n)^{3/2}\left(1 - \frac{n^{-1-c_0/10}}{2-y_n}\right)^{3/2} \\ &= (2-y_n)^{3/2}\left(1 - \frac{3}{2}\frac{n^{-1-c_0/10}}{2-y_n} + o\left(\frac{n^{-1-c_0/10}}{2-y_n}\right)\right). \end{aligned}$$

Notice that $\frac{n^{-1-c_0/10}}{2-y_n} = \frac{n^{-1/3-c_0/10}}{(n(2-y_n)^{3/2})^{2/3}} \to 0$ and $n(2-y_n)^{3/2} \to \infty$ when $n \to \infty$ by assumption. Hence we can apply Theorem 2.1, which is the MDP for $(Z_n')_n$. Summarizing we have

$$P\big(Z_n \leq \xi\big) \leq P\big(Z_n' \leq \xi_n\big) + n^{-c_0}$$

with

$$\xi_n = \frac{k_n - \frac{2}{3\pi} n(2-y_n')^{3/2}}{a_n \sqrt{\frac{1}{2\pi^2}\log(n(2-y_n')^{3/2})}}$$
$$= \frac{\frac{2}{3\pi} n\big((2-y_n)^{3/2} - (2-y_n')^{3/2}\big)}{a_n \sqrt{\frac{1}{2\pi^2}\log(n(2-y_n')^{3/2})}} + \xi \left(\frac{\log(n(2-y_n)^{3/2})}{\log(n(2-y_n')^{3/2})} \right)^{1/2}.$$

We will prove that $\xi_n = \xi + o(1)$. Using the preceding representation we have

$$n\big((2-y_n)^{3/2} - (2-y_n')^{3/2}\big) = \frac{3}{2} n^{-c_0/10} (2-y_n)^{1/2} + o(n^{-c_0/10}) \to 0$$

and $a_n \sqrt{\frac{1}{2\pi^2}\log(n(2-y_n')^{3/2})} \to \infty$ when $n \to \infty$. Moreover

$$\frac{\log(n(2-y_n)^{3/2})}{\log(n(2-y_n')^{3/2})} = \frac{\log(n(2-y_n)^{3/2})}{\log(n(2-y_n)^{3/2}) + \frac{3}{2}\log\big(1 - \frac{n^{-1-c_0/10}}{2-y_n}\big)} \to 1.$$

Applying Theorem 2.1, it follows that $\lim_{n\to\infty} \frac{1}{a_n^2} \log P\big(Z_n \le \xi\big) = -\frac{\xi^2}{2}$ for all $\xi < 0$. Similarly we obtain for any $\xi > 0$ that $\lim_{n\to\infty} \frac{1}{a_n^2} \log P\big(Z_n \ge \xi\big) = -\frac{\xi^2}{2}$ and the MDP for $(Z_n)_n$ follows along the lines of the proof of Theorem 2.1. □

Remark 5.2. In a next step one could ask whether the statement of Theorem 5.1 is true also for the sequence

$$\frac{N_{I_n}(W_n) - \mathbb{E}[N_{I_n}(W_n)]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n))}}.$$

Hence the question is whether the asymptotic behavior of the expectation and the variance of $N_{I_n}(W_n)$ is identical to the one for GUE matrices, given in (2.1) and (2.2). The answer is yes, but only for Wigner matrices obeying Condition (**C0**). The reason for this is that the Four Moment Theorem 4.1 deals with a finite number of eigenvalues, whereas $N_{I_n}(W_n)$ involves all the eigenvalues of the Wigner matrix M_n. Theorem 4.1 does not give the asymptotics (2.1) and (2.2) for Wigner matrices. A recent result of Erdös, Yau and Yin [12] describe strong localization of the eigenvalues of Wigner matrices and this result provides the additional step necessary to obtain (2.1) and (2.2) for Wigner matrices M_n obeying Condition (**C0**). The result in [12] is that for M_n being a Wigner Hermitian matrix obeying Condition (**C0**), there is a constant $C > 0$ such that for any $i \in \{1, \ldots, n\}$

$$P\big(|\lambda_i(W_n) - t(i/n)| \ge (\log n)^{C \log\log n} \min(i, n-i+1)^{-1/3} n^{-2/3}\big) \le n^{-3}.$$

Along the lines of the proof of [5, Lemma 5] one obtains (2.1) and (2.2). We will not present the details.

6. Further random matrix ensembles

In this section, we indicate how the preceding results for Wigner Hermitian matrices can be stated and proved for *real Wigner symmetric* matrices. Real Wigner matrices are random symmetric matrices M_n of size n such that, for $i<j$, $(M_n)_{ij}$ are i.i.d. with mean zero and variance one, $(M_n)_{ii}$ are i.i.d. with mean zero and variance 2. The case where the entries are Gaussian is the GOE mentioned in the introduction. As in the Hermitian case, the main issue is to establish our conclusions for the GOE. On the level of CLT, this was developed in [16] by means of the famous *interlacing formulas* due to Forrester and Rains, [13], that relates the eigenvalues of different matrix ensembles.

Theorem 6.1 (Forrester, Rains, 2001). *The following relation holds between GUE and GOE matrix ensembles:*

$$\mathrm{GUE}_n = \mathrm{even}\big(\mathrm{GOE_n} \cup \mathrm{GOE_{n+1}}\big). \tag{6.1}$$

The statement is: Take two independent (!) matrices from the GOE: one of size $n \times n$ and one of size $(n+1)\times(n+1)$. Superimpose the $2n+1$ eigenvalues on the real line and then take the n even ones. They have the same distribution as the eigenvalues of a $n\times n$ matrix from the GUE. Let $M_n^{\mathbb{R}}$ denote a GOE matrix and let $W_n^{\mathbb{R}} := \frac{1}{\sqrt{n}} M_n^{\mathbb{R}}$. In [11, Theorem 4.2] we have proved a MDP for

$$Z_n^{\mathbb{R}} := \frac{N_{I_n}(W_n^{\mathbb{R}}) - \mathbb{E}[N_{I_n}(W_n^{\mathbb{R}})]}{a_n \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))}} \tag{6.2}$$

for any $1 \ll a_n \ll \sqrt{\mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))}$, I_n an interval in $\mathbb{R}$, with speed a_n^2 and rate $x^2/2$. If $M_n^{\mathbb{C}}$ denotes a GUE matrix and $W_n^{\mathbb{C}}$ the corresponding normalized matrix, the nice consequences of (6.1) were already suitably developed in [16]: applying Cauchy's interlacing theorem one can write

$$N_{I_n}(W_n^{\mathbb{C}}) = \frac{1}{2}\left[N_{I_n}(W_n^{\mathbb{R}}) + N_{I_n}(\widehat{W}_n^{\mathbb{R}}) + \eta_n'(I_n)\right], \tag{6.3}$$

where one obtains GOE_n' in $N_{I_n}(\widehat{W}_n^{\mathbb{R}})$ from GOE_{n+1} by considering the principle sub-matrix of GOE_{n+1} and $\eta_n'(I_n)$ takes values in $\{-2,-1,0,1,2\}$. Note that $N_{I_n}(W_n^{\mathbb{R}})$ and $N_{I_n}(\widehat{W}_n^{\mathbb{R}})$ are independent because GOE_{n+1} and GOE_n denote independent matrices from the GOE. Now the same arguments as in [11, Section 4] and Theorem 2.1 lead to the MDP for $(Z_n^{\mathbb{R}})_n$, if we consider intervals $I_n = [y_n, \infty)$ where $y_n \to 2^-$ for $n \to \infty$. Remark that the interlacing formula (6.3) leads to $2\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) + O(1) = \mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))$ if $\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) \to \infty$. Next the proof of Theorem 3.1 can be adapted to obtain an MDP for $\lambda_{n-i}(W_n^{\mathbb{R}})$: Consider

$$Z_{n,i}^{\mathbb{R}} := \frac{\lambda_{n-i}(W_n^{\mathbb{R}}) - \Big(2 - \big(\frac{3\pi}{2}\frac{i}{n}\big)^{2/3}\Big)}{\mathrm{const}\big(\frac{2\log i}{i^{2/3}n^{4/3}}\big)^{1/2}}.$$

With

$$\mathbb{E}[N_{I_n}(W_n^{\mathbb{R}})] = \mathbb{E}[N_{I_n}(W_n^{\mathbb{C}})] + O(1) \quad \text{and} \quad 2\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) + O(1) = \mathbb{V}(N_{I_n}(W_n^{\mathbb{R}}))$$

if $\mathbb{V}(N_{I_n}(W_n^{\mathbb{C}})) \to \infty$ we get a MDP along the lines of the proof of Theorem 3.1. We omit the details. The Four Moment Theorem also applies for real symmetric matrices. The proof of the next Theorem is nearly identical to the proofs of Theorem 4.2 and Theorem 5.1.

Theorem 6.2. *Consider a real symmetric Wigner matrix $W_n = \frac{1}{\sqrt{n}} M_n$ whose entries satisfy Condition* (**C1**) *and match the corresponding entries of GOE up to order 4. Consider $i = i(n)$ such that $i \to \infty$ and $i/n \to 0$ as $n \to \infty$. Denote the ith eigenvalue of W_n by $\lambda_i(W_n)$. Let $(a_n)_n$ be a sequence of real numbers such that $1 \ll a_n \ll \sqrt{\log i}$. Then the sequence $(Z_{n,i})_n$ with*

$$Z_{n,i} = \frac{\lambda_{n-i}(W_n) - \left(2 - \left(\frac{3\pi}{2}\frac{i}{n}\right)^{2/3}\right)}{\mathrm{const}\left(\frac{2\log i}{i^{2/3} n^{4/3}}\right)^{1/2}}$$

universally satisfies a MDP with speed a_n^2 and rate function $I(x) = \frac{x^2}{2}$. Moreover the statement of Theorem 5.1 *can be adapted and proved analogously.*

Remark that one could consider the Gaussian Symplectic Ensemble (GSE). Quaternion self-dual Wigner Hermitian matrices have not been studied. Due to Forrester and Rains, the following relation holds between matrix ensembles: $\mathrm{GSE}_n = \mathrm{even}\big(\mathrm{GOE}_{2n+1}\big)\frac{1}{\sqrt{2}}$. The multiplication by $\frac{1}{\sqrt{2}}$ denotes scaling the $(2n+1) \times (2n+1)$ GOE matrix by the factor $\frac{1}{\sqrt{2}}$. Let $x_1 < x_2 < \cdots < x_n$ denote the ordered eigenvalues of an $n \times n$ matrix from the GSE and let $y_1 < y_2 < \cdots < y_{2n+1}$ denote the ordered eigenvalues of an $(2n+1) \times (2n+1)$ matrix from the GOE. Then it follows that $x_i = y_{2i}/\sqrt{2}$ in distribution. Hence the MDP for the ith eigenvalue of the GSE follows from the MDP in the GOE case. We omit formulating the result.

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On the Limiting Shape of Young Diagrams Associated with Inhomogeneous Random Words

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Abstract. The limiting shape of the random Young diagrams associated with an inhomogeneous random word is identified as a multidimensional Brownian functional. This functional is identical in law to the spectrum of a Gaussian random matrix. Since the length of the top row of the Young diagrams is also the length of the longest (weakly) increasing subsequence of the random word, the corresponding limiting law follows. The Poissonized word problem is also briefly studied, and the asymptotic behavior of the shape analyzed.

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1. Introduction

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of random variables taking values in an ordered alphabet. The length of the longest (weakly) increasing subsequence of $X_1, X_2, \ldots, X_n$, denoted by LI_n, is the maximal $1 \le k \le n$ such that there exists an increasing sequence of integers $1 \le i_1 < i_2 < \cdots < i_k \le n$ with $X_{i_1} \le X_{i_2} \le \cdots \le X_{i_k}$, i.e.,

$$LI_n = \max\{k : \exists\ 1 \le i_1 < i_2 < \cdots < i_k \le n,\ with\ X_{i_1} \le X_{i_2} \le \cdots \le X_{i_k}\}.$$

When the $X_i s$ take their values independently and uniformly in an m-letter ordered alphabet, through a careful analysis of the exponential generating function of LI_n, Tracy and Widom [27] gave the limiting distribution of LI_n (properly centered and normalized) as that of the largest eigenvalue of a matrix drawn from the $m \times m$ traceless Gaussian Unitary Ensemble (GUE). This result, motivated by

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the celebrated random permutation result of Baik, Deift and Johansson [2], was further extended to the non-uniform case by Its, Tracy and Widom ([18], [19]). In that last setting, the corresponding limiting law is the maximal eigenvalue of a direct sum of mutually independent GUEs subject to an overall trace constraint.

A method to study the asymptotic behavior of the length of longest increasing subsequences is through Young diagrams ([10], [24]). Recall that a Young diagram of size n is a collection of n boxes arranged in left-justified rows, with a weakly decreasing number of boxes from row to row. The shape of a Young diagram is the vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and for each i, λ_i is the number of boxes in the ith row while k is the total number of rows of the diagram (and so $\lambda_1 + \cdots + \lambda_k = n$). Recall also that a (semi-standard) Young tableau is a Young diagram, with a filling of a positive integer in each box, in such a way that the integers are weakly increasing along the rows and strictly increasing down the columns. A standard Young tableau of size n is a Young tableau in which the fillings are the integers from 1 to n.

Let now $[m] := \{1, 2, \dots, m\}$ be an m-letter ordered alphabet. A *word* of length n is a mapping W from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, m\}$, and let $[m]^n$ denotes the set of words of length n with letters taken from the alphabet $\{1, 2, \dots, m\}$. A word is a *permutation* if $m = n$, and W is onto. The Robinson-Schensted correspondence is a bijection between the set of words $[m]^n$ and the set of pairs of Young tableaux $\{(P, Q)\}$, where P is semi-standard with entries from $\{1, 2, \dots, m\}$, while Q is standard with entries from $\{1, 2, \dots, n\}$. Moreover P and Q share the same shape which is a partition of n, and so, we do not distinguish between shape and partition. If the word is a permutation, then P is also standard. A word W in $[m]^n$ can be represented uniquely as an $m \times n$ matrix $\mathbf{X}_W$ with entries

$$(\mathbf{X}_W)_{i,j} = \mathbf{1}_{W(j)=i}. \tag{1.1}$$

The Robinson-Schensted correspondence actually gives a one-to-one correspondence between the set of pairs of Young tableaux and the set of matrices whose entries are either 0 or 1 and with exactly a unique 1 in each column. This was generalized by Knuth to the set of $m \times n$ matrices with nonnegative integer entries. Let $\mathcal{M}(m, n)$ be the set of $m \times n$ matrices with nonnegative integer entries. Let $\mathcal{P}(P, Q)$ be the set of pairs of semi-standard Young tableaux (P, Q) sharing the same shape and whose size is the sum of all the entries, where P has elements in $\{1, \dots, m\}$ and Q has elements in $\{1, \dots, n\}$. The Robinson-Schensted-Knuth (RSK) correspondence is a one-to-one mapping between $\mathcal{M}(m, n)$ and $\mathcal{P}(P, Q)$. If the matrix corresponds to a word in $[m]^n$, then Q is standard.

Johansson [20], using orthogonal polynomial methods, proved that the limiting shape of the Young diagrams, associated with homogeneous words, i.e., the i.i.d. uniform m-letter framework, through the RSK correspondence, is the spectrum of the traceless $m \times m$ GUE. Since LI_n is also equal to the length of the top row of the associated Young diagrams, these results recover those of [27]. The permutation result is also obtained by Johansson [20], Okounkov [22] and Borodin, Okounkov and Olshanki [5]. More recently, for inhomogeneous words and via simple probabilistic tools, the limiting law of LI_n is given, in [15], as a Brownian functional. Via the results of Baryshnikov [3] or of Gravner, Tracy and Widom [12] this functional can then be identified as a maximal eigenvalue of a certain matrix ensemble. For the shape of the associated Young diagrams, the corresponding open problem is resolved below.

Let us now describe the content of the present paper. In Section 2, we list some simple properties of a matrix ensemble, which we call generalized traceless GUE; and relate various properties of the GUE to this generalized one. In Section 3, we obtain the limiting shape, of the RSK Young diagrams associated with an inhomogeneous random word, as a multivariate Brownian functional. In turn, this functional is identified as the spectrum of an $m \times m$ element of the generalized traceless GUE. Therefore, the limiting law of LI_n is the largest eigenvalue of the block of the $m \times m$ generalized traceless GUE corresponding to the most probable letters. Finally, the corresponding Poissonized word problem is studied in Section 4.

2. Generalized traceless GUE

In this section, we list, without proofs, some elementary properties of the generalized traceless GUE. Proofs are omitted since simple consequences of known GUE results as exposed, for example, in [21] or [1], except for the proof of Proposition 2.7 which relies on simple arguments presented in the Appendix.

Recall that an element of the $m \times m$ GUE is an $m \times m$ Hermitian random matrix $\mathbf{G} = (G_{i,j})_{1 \le i,j \le m}$, whose entries are such that: $G_{i,i} \sim N(0,1)$, for $1 \le i \le m$, $\mathrm{Re}\,(G_{i,j}) \sim N(0,1/2)$ and $\mathrm{Im}\,(G_{i,j}) \sim N(0,1/2)$, for $1 \le i < j \le m$, and $G_{i,i}$, $\mathrm{Re}\,(G_{i,j})$, $\mathrm{Im}\,(G_{i,j})$ are mutually independent for $1 \le i \le j \le m$. Now, for $m \ge 1$, $k = 1, \dots, K$ and $d_1, \dots, d_K$ such that $\sum_{k=1}^{K} d_k = m$, let $\mathcal{G}_m\,(d_1, \dots, d_K)$ be the set of random matrices $\mathbf{X}$ which are direct sums of mutually independent elements of the $d_k \times d_k$ GUE, $k = 1, \dots, K$ (i.e., $\mathbf{X}$ is an $m \times m$ block diagonal matrix whose K blocks are mutually independent elements of the $d_k \times d_k$ GUE, $k = 1, \dots, K$). Let $p_1, \dots, p_m > 0$, $\sum_{j=1}^{m} p_j = 1$, be such that the multiplicities of the K distinct probabilities $p^{(1)}, \dots, p^{(K)}$ are respectively $d_1, \dots, d_K$, i.e., let $m_1 = 0$ and for $k = 2, \dots, K$, let $m_k = \sum_{j=1}^{k-1} d_j$, and so $p_{m_k+1} = \cdots = p_{m_k+d_k} = p^{(k)}$, $k = 1, \dots, K$. The generalized $m \times m$ traceless GUE associated with the probabilities $p_1, \dots, p_m$ is the set, denoted by $\mathcal{G}^0\,(p_1, \dots, p_m)$, of $m \times m$ matrices $\mathbf{X}^0$, of the form

$$\mathbf{X}^0_{i,j} = \begin{cases} \mathbf{X}_{i,i} - \sqrt{p_i} \sum_{l=1}^{m} \sqrt{p_l} \mathbf{X}_{l,l}, & \text{if } i = j; \\ \mathbf{X}_{i,j}, & \text{if } i \ne j, \end{cases} \tag{2.1}$$

where $\mathbf{X} \in \mathcal{G}_m\,(d_1, \dots, d_K)$. Clearly, from (2.1), $\sum_{i=1}^{m} \sqrt{p_i} \mathbf{X}^0_{i,i} = 0$. Note also that the case $K = 1$ (for which $d_1 = m$) recovers the traceless GUE, whose elements are of the form $\mathbf{X} - \mathrm{tr}(\mathbf{X}) \mathbf{I}_m / m$, with $\mathbf{X}$ an element of the GUE and $\mathbf{I}_m$ the $m \times m$ identity matrix.

Here is an equivalent way of defining the generalized traceless GUE: let $\mathbf{X}^{(k)}$ be the $m \times m$ diagonal matrix such that

$$\mathbf{X}^{(k)}_{i,i} = \begin{cases} \sqrt{p^{(k)}}\sum_{l=1}^{m}\sqrt{p_l}\mathbf{X}_{l,l}, & \text{if } m_k < i \le m_k + d_k; \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

and let $\mathbf{X} \in \mathcal{G}_m(d_1, \ldots, d_K)$. Then, $\mathbf{X}^0 := \mathbf{X} - \sum_{k=1}^{K}\mathbf{X}^{(k)} \in \mathcal{G}^0(p_1, \ldots, p_m)$.

Equivalently, there is an "ensemble" description of $\mathcal{G}^0(p_1, \ldots, p_m)$.

Proposition 2.1. $\mathbf{X}^0 \in \mathcal{G}^0(p_1, \ldots, p_m)$ *if and only if* $\mathbf{X}^0$ *is distributed according to the probability distribution*

$$\mathbb{P}\left(d\mathbf{X}^0\right) = C\gamma\left(d\mathbf{X}^0_{1,1}, \ldots, d\mathbf{X}^0_{m,m}\right)\prod_{k=1}^{K}\left(e^{-\sum_{m_k<i<j\le m_k+d_k}\left|\mathbf{X}^0_{i,j}\right|^2}\right.$$
$$\left.\prod_{m_k<i<j\le m_k+d_k} d\,\mathrm{Re}\left(\mathbf{X}^0_{i,j}\right)d\,\mathrm{Im}\left(\mathbf{X}^0_{i,j}\right)\right), \tag{2.3}$$

on the space of $m \times m$ *Hermitian matrices, which are direct sum of* $d_k \times d_k$ *Hermitian matrices,* $k = 1, \ldots, K$, $\sum_{k=1}^{K} d_k = m$, *and where* $m_1 = 0$, $m_k = \sum_{j=1}^{k-1} d_j$, $k = 2, \ldots, K$. *Above,* $C = \pi^{-\sum_{k=1}^{K} d_k(d_k-1)/2}$ *and* $\gamma\left(d\mathbf{X}^0_{1,1}, \ldots, d\mathbf{X}^0_{m,m}\right)$ *is the distribution of an* m*-dimensional centered (degenerate) multivariate Gaussian law with covariance matrix*

$$\mathbf{\Sigma}^0 = \begin{pmatrix} 1-p_1 & -\sqrt{p_1p_2} & \cdots & -\sqrt{p_1p_m} \\ -\sqrt{p_2p_1} & 1-p_2 & \cdots & -\sqrt{p_2p_m} \\ \vdots & \ddots & \ddots & \vdots \\ -\sqrt{p_mp_1} & \cdots & -\sqrt{p_mp_{m-1}} & 1-p_m \end{pmatrix}.$$

We provide next a relation between the spectra of $\mathbf{X}$ and $\mathbf{X}^0$.

Proposition 2.2. *Let* $\mathbf{X} \in \mathcal{G}_m(d_1, \ldots, d_K)$, *and let* $\mathbf{X}^0 \in \mathcal{G}^0(p_1, \ldots, p_m)$. *Let* $\xi_1, \ldots, \xi_m$ *be the eigenvalues of* $\mathbf{X}$, *where for each* $k = 1, \ldots, K$, $\xi_{m_k+1}, \ldots, \xi_{m_k+d_k}$ *are the eigenvalues of the* k*th diagonal block (an element of the* $d_k \times d_k$ *GUE). Then, the eigenvalues of* $\mathbf{X}^0$ *are given by:*

$$\xi^0_i = \xi_i - \sqrt{p_i}\sum_{l=1}^{m}\sqrt{p_l}\mathbf{X}_{l,l} = \xi_i - \sqrt{p_i}\sum_{l=1}^{m}\sqrt{p_l}\xi_l, \quad i = 1, \ldots, m.$$

Let $\xi_1^{GUE,m}, \xi_2^{GUE,m}, \ldots, \xi_m^{GUE,m}$ be the eigenvalues of an element of the $m \times m$ GUE. It is well known that the empirical distribution of the eigenvalues $\left(\xi_i^{GUE,m}/\sqrt{m}\right)_{1\le i\le m}$ converges almost surely to the semicircle law ν with density $\sqrt{4-x^2}/2\pi$, $-2 \le x \le 2$. Equivalently, the semicircle law is also the almost sure limit of the empirical spectral measure for the kth block of the generalized traceless GUE, provided $d_k \to \infty$, $k = 1, \ldots, K$. This is, for example, the case of the uniform alphabet, where $K = 1$, $d_1 = m$ and $p^{(1)} = 1/m$.

Proposition 2.3. *Let $\xi_1^0, \xi_2^0, \ldots, \xi_m^0$ be the eigenvalues of an element of the $m \times m$ generalized traceless GUE, such that $\xi_{m_k+1}^0, \ldots, \xi_{m_k+d_k}^0$ are the eigenvalues of the kth diagonal block, for each $k = 1, \ldots, K$. For any $k = 1, \ldots, K$, the empirical distribution of the eigenvalues $\left(\xi_i^0/\sqrt{d_k}\right)_{m_k < i \le m_k+d_k}$ converges almost surely to the semicircle law ν with density $\sqrt{4-x^2}/2\pi$, $-2 \le x \le 2$, whenever $d_k \to \infty$.*

Now for $p_1, \ldots, p_m$ considered, so far, i.e., such that the multiplicities of the K distinct probabilities $p^{(1)}, \ldots, p^{(K)}$ are respectively $d_1, \ldots, d_K$ and $p_{m_k+1} = \cdots = p_{m_k+d_k} = p^{(k)}$, $k = 1, \ldots, K$, let

$$\mathcal{L}^{p_1,\ldots,p_m} := \Bigg\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_{m_k+1} \ge \cdots \ge x_{m_k+d_k},\ k = 1, \ldots, K; \sum_{j=1}^m \sqrt{p_j} x_j = 0 \Bigg\}. \tag{2.4}$$

In other words, $\mathcal{L}^{p_1,\ldots,p_m}$ is a subset of the hyperplane $\sum_{j=1}^m \sqrt{p_j} x_j = 0$, where within each block of size d_k, $k = 1, \ldots, K$, the coordinates $x_{m_k+1}, \ldots, x_{m_k+d_k}$, are ordered. For any $s_1, \ldots, s_m \in \mathbb{R}$, let also

$$\mathcal{L}^{p_1,\ldots,p_m}_{(s_1,\ldots,s_m)} := \mathcal{L}^{p_1,\ldots,p_m} \cap \Big\{ (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \le s_i,\ i = 1, \ldots, m \Big\}. \tag{2.5}$$

The distribution function of the eigenvalues, written in non-increasing order within each $d_k \times d_k$ GUE, of an element of $\mathcal{G}^0(p_1, \ldots, p_m)$ is given now.

Proposition 2.4. *The joint distribution function of the eigenvalues, written in non-increasing order within each $d_k \times d_k$ GUE, of an element of $\mathcal{G}^0(p_1, \ldots, p_m)$ is given, for any $s_1, \ldots, s_m \in \mathbb{R}$, by*

$$\mathbb{P}\Big(\xi_1^0 \le s_1, \xi_2^0 \le s_2, \ldots, \xi_m^0 \le s_m\Big) = \int_{\mathcal{L}^{p_1,\ldots,p_m}_{(s_1,\ldots,s_m)}} f(x) dx_1 \cdots dx_{m-1}, \tag{2.6}$$

where for $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$,

$$f(x) := c_m \prod_{k=1}^K \Delta_k(x)^2 e^{-\sum_{i=1}^m x_i^2/2} \mathbf{1}_{\mathcal{L}^{p_1,\ldots,p_m}}(x), \tag{2.7}$$

with $c_m = (2\pi)^{-(m-1)/2} \prod_{k=1}^K (0!1! \cdots (d_k-1)!)^{-1}$ and where $\Delta_k(x)$ is the Vandermonde determinant associated with those x_i for which $p_i = p^{(k)}$, i.e.,

$$\Delta_k(x) = \prod_{m_k+1 \le i < j \le m_k+d_k} (x_i - x_j).$$

Remark 2.5. When the eigenvalues are not ordered within each $d_k \times d_k$ GUE, the identity (2.6) remains valid, multiplying c_m, above, by $\prod_{k=1}^K (d_k!)^{-1}$, and also by omitting the ordering constraints $x_{m_k+1} \ge \cdots \ge x_{m_k+d_k}$, $k = 1, \ldots, K$, in the definition of $\mathcal{L}^{p_1,\ldots,p_m}$.

The next proposition gives a relation in law between the spectra of elements of $\mathcal{G}_m(d_1, \ldots, d_K)$ and of $\mathcal{G}^0(p_1, \ldots, p_m)$.

Proposition 2.6. *For any $m \geq 2$, let $\mathbf{X} \in \mathcal{G}_m(d_1, \ldots, d_K)$ and let $\mathbf{X}^0 \in \mathcal{G}^0(p_1, \ldots, p_m)$. Let $\xi_1, \ldots, \xi_m$ be the eigenvalues of $\mathbf{X}$, and let $\xi_1^0, \ldots, \xi_m^0$ be the eigenvalues of $\mathbf{X}^0$ as given in Proposition 2.2. Then,*

$$(\xi_1, \ldots, \xi_m) \stackrel{d}{=} \left(\xi_1^0, \ldots, \xi_m^0\right) + (Z_1, \ldots, Z_m),$$

where $(Z_1, \ldots, Z_m)$ is a centered (degenerate) multivariate Gaussian vector with covariance matrix $\left(\sqrt{p_i p_j}\right)_{1\leq i,j\leq m}$. Moreover, $(Z_1, \ldots, Z_m)$ can be chosen to be independent of $\left(\xi_1^0, \ldots, \xi_m^0\right)$.

The asymptotic behavior of the maximal eigenvalues, within each block, of $\mathbf{X}^0 \in \mathcal{G}^0(p_1, \ldots, p_m)$ is well known and well understood (see also Proposition 5.2 and Proposition 5.4 of the Appendix for elementary arguments leading to the result below).

Proposition 2.7. *For $k = 1, \ldots, K$, let $\max_{m_k < i \leq m_k + d_k} \xi_i^0$ be the largest eigenvalue of the $d_k \times d_k$ block of $\mathbf{X}^0 \in \mathcal{G}^0(p_1, \ldots, p_m)$, then*

$$\lim_{d_k \to \infty} \frac{\max_{m_k < i \leq m_k + d_k} \xi_i^0}{\sqrt{d_k}} = 2,$$

both almost surely and in the mean.

3. Young diagrams and inhomogeneous random words

Throughout the rest of this paper, let $W = X_1 X_2 \cdots X_n$ be a random word, where $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with $\mathbb{P}(X_1 = j) = p_j$, where $j = 1, \ldots, m$, $p_j > 0$, and $\sum_{j=1}^m p_j = 1$. Let τ be a permutation of $\{1, \ldots, m\}$ corresponding to a non-increasing ordering of $p_1, p_2, \ldots, p_m$, i.e., $p_{\tau(1)} \geq \cdots \geq p_{\tau(m)}$. Assume also there are K distinct probabilities in $\{p_1, p_2, \ldots, p_m\}$ which are reordered as $p^{(1)} > \cdots > p^{(K)}$, in such a way that the multiplicity of each $p^{(k)}$ is d_k, $k = 1, \ldots, K$. In our notation, $K = 1$ corresponds to the uniform case, where $d_1 = m$. Let $m_1 = 0$ and for any $k = 2, \ldots, K$, let $m_k = \sum_{j=1}^{k-1} d_j$ and so the multiplicity of each $p_{\tau(j)}$ is d_k if $m_k < \tau(j) \leq m_k + d_k$, $j = 1, \ldots, m$. Finally, let $\mathbf{X}_W$ be as in (1.1) the matrix corresponding to such a random word W of length n.

Its, Tracy and Widom ([18], [19]) have obtained the limiting law of the length of the longest increasing subsequence of such a random word. To recall their result, let $(\xi_1, \ldots, \xi_m)$ be the eigenvalues of an element of $\mathcal{G}^0\left(p_{\tau(1)}, \ldots, p_{\tau(m)}\right)$, written in such a way that $(\xi_1, \ldots, \xi_m) = \left(\xi_1^{d_1}, \ldots, \xi_{d_1}^{d_1}, \ldots, \xi_1^{d_K}, \ldots, \xi_{d_K}^{d_K}\right)$, i.e., $\xi_1^{d_k}, \ldots, \xi_{d_k}^{d_k}$ are the eigenvalues of the kth block, $k = 1, \ldots, K$. Then (see [19]), the limiting law of the length of the longest increasing subsequence, properly centered and normalized, is the law of $\max_{1\leq i\leq d_1} \xi_i^{d_1}$. A representation of this limiting law, as a Brownian functional is given in [15]. A multidimensional Brownian functional representation of the whole shape of the diagrams associated with a Markov random word is further given in [17] (see also Chistyakov and Götze [7] or [16] for the binary case). Below, we obtain the convergence of the whole shape of the diagrams, in the i.i.d. non-uniform case via a different set of techniques which is related to the work of Glynn and Whitt [11], Baryshnikov [3], Gravner, Tracy and Widom [12] and Doumerc [9].

Let $\left(\hat{B}^1(t), \hat{B}^2(t), \ldots, \hat{B}^m(t)\right)$ be the m-dimensional Brownian motion having covariance matrix

$$\mathbf{\Sigma}_t := \begin{pmatrix} p_{\tau(1)}\left(1-p_{\tau(1)}\right) & -p_{\tau(1)}p_{\tau(2)} & \cdots & -p_{\tau(1)}p_{\tau(m)} \\ -p_{\tau(2)}p_{\tau(1)} & p_{\tau(2)}\left(1-p_{\tau(2)}\right) & \cdots & -p_{\tau(2)}p_{\tau(m)} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{\tau(m)}p_{\tau(1)} & -p_{\tau(m)}p_{\tau(2)} & \cdots & p_{\tau(m)}\left(1-p_{\tau(m)}\right) \end{pmatrix} t. \quad (3.1)$$

For each $l = 1, \ldots, m$, there is a unique $1 \le k \le K$ such that $p_{\tau(l)} = p^{(k)}$, and let

$$\hat{L}^l_m = \sum_{j=1}^{m_k} \hat{B}^j(1) + \sup_{J(l-m_k, d_k)} \sum_{j=m_k+1}^{m_k+d_k} \sum_{i=1}^{l-m_k} \left(\hat{B}^j(t^i_{j-i+1}) - \hat{B}^j(t^i_{j-i})\right), \quad (3.2)$$

where the first sum is understood to vanish when $m_k = 0$ and where the set $J(l-m_k, d_k)$ consists of all the subdivisions (t^i_j) of $[0,1]$, $1 \le i \le l-m_k$, $j \in \mathbb{N}$, of the form:

$$t^i_j \in [0,1];\ t^{i+1}_j \le t^i_j \le t^i_{j+1};\ t^i_j = 0 \ \mathit{for}\ j \le m_k \\ \mathit{and}\ t^i_j = 1 \ \mathit{for}\ j \ge m_{k+1} - (l-m_k) + 1. \quad (3.3)$$

With these preliminaries, we have:

Theorem 3.1. *Let $\lambda(RSK(\mathbf{X}_W)) = (\lambda_1, \ldots, \lambda_m)$ be the common shape of the Young diagrams associated with W through the RSK correspondence. Then, as $n \to \infty$,*

$$\left(\frac{\lambda_1 - np_{\tau(1)}}{\sqrt{n}}, \ldots, \frac{\lambda_m - np_{\tau(m)}}{\sqrt{n}}\right) \Longrightarrow \left(\hat{L}^1_m, \hat{L}^2_m - \hat{L}^1_m, \ldots, \hat{L}^m_m - \hat{L}^{m-1}_m\right). \quad (3.4)$$

Proof. Let $(\mathbf{e_j})_{j=1,\ldots,m}$ be the canonical basis of $\mathbb{R}^m$, and let $\mathbf{V} = (V_1, \ldots, V_m)$ be the random vector such that

$$\mathbb{P}\left(\mathbf{V} = \mathbf{e_j}\right) = p_j, \quad j = 1, \ldots, m.$$

Clearly, for each $1 \le j \le m$,

$$\mathbb{E}\left(V_j\right) = p_j,\ \mathrm{Var}(V_j) = p_j\left(1-p_j\right),$$

and for $j_1 \ne j_2$, $\mathrm{Cov}(V_{j_1}, V_{j_2}) = -p_{j_1}p_{j_2}$. Hence the covariance matrix of $\mathbf{V}$ is

$$\mathbf{\Sigma} = \begin{pmatrix} p_1\left(1-p_1\right) & -p_1p_2 & \cdots & -p_1p_m \\ -p_2p_1 & p_2\left(1-p_2\right) & \cdots & -p_2p_m \\ \vdots & \vdots & \ddots & \vdots \\ -p_mp_1 & -p_mp_2 & \cdots & p_m\left(1-p_m\right) \end{pmatrix}. \quad (3.5)$$

Let $\mathbf{V_1}, \mathbf{V_2}, \ldots, \mathbf{V_n}$ be independent copies of $\mathbf{V}$, where $\mathbf{V_i} = (V_{i,1}, V_{i,2}, \ldots, V_{i,m})$, $i = 1, \ldots, n$. Then $\mathbf{X}_W$ has the same law as the matrix formed by all the $V_{i,j}$ on the lattice $\{1, \ldots, n\} \times \{1, \ldots, m\}$.

It is a well-known combinatorial fact (see Section 3.2 in [10]) that, for all $1 \le l \le m$,

$$\lambda_1 + \cdots + \lambda_l = G^l(m,n) := \max\Bigg\{ \sum_{(i,j)\in\pi_1\cup\cdots\cup\pi_l} V_{i,j} : \pi_1, \ldots, \pi_l \in \mathcal{P}(m,n), \text{ and } \pi_1, \ldots, \pi_l \text{ are all disjoint}\Bigg\}, \tag{3.6}$$

where $\mathcal{P}(m,n)$ is the set of all paths π taking only unit steps up or to the right in the rectangle $\{1, \ldots, n\} \times \{1, \ldots, m\}$ and where, by disjoint, it is meant that any two paths do not share a common point in $\{1, \ldots, n\} \times \{1, \ldots, m\}$ when $V_{i,j} = 1$. We prove next that, for any $l = 1, \ldots, m$,

$$\frac{G^l(m,n) - ns_l}{\sqrt{n}} \overset{n\to\infty}{\Longrightarrow} \hat{L}^l_m, \tag{3.7}$$

where $s_l = \sum_{j=1}^{l} p_{\tau(j)}$. For $l = 1$,

$$G^1(m,n) = \max\Bigg\{ \sum_{(i,j)\in\pi} V_{i,j} \ ; \pi \in \mathcal{P}(m,n)\Bigg\}. \tag{3.8}$$

Moreover, each path π is uniquely determined by the weakly increasing sequence of its $m-1$ jumps, namely $0 = t_0 \le t_1 \le \cdots \le t_{m-1} \le 1$, such that π is horizontal on $[\lfloor t_{j-1}n \rfloor, \lfloor t_j n \rfloor] \times \{j\}$ and vertical on $\{\lfloor t_j n \rfloor\} \times [j, j+1]$. Hence

$$G^1(m,n) = \sup_{0=t_0\le t_1\le\cdots\le t_{m-1}\le t_m=1} \sum_{j=1}^{m} \sum_{i=\lfloor t_{j-1}n \rfloor}^{\lfloor t_j n \rfloor} V_{i,j}.$$

Let $p_{\max} = \max_{1\le j\le m} p_j$, $J(m) = \{j : p_j = p_{\max}\} \subset \{1, \ldots, m\}$ and so $d_1 = \operatorname{card}(J(m))$ ($J(m)$ is the set of all the most probable letters). As shown in [17, Section 3 and 4], the distribution of $G^1(m,n)$ is very close, for large n, to that of a very similar expression which involves only those $V_{i,j}$ for which $j \in J(m)$. To recall this result, if

$$\hat{G}^1(m,n) = \sup_{\substack{0=t_0\le t_1\le\cdots\le t_{m-1}\le t_m=1 \\ t_{j-1}=t_j \ for \ j\notin J(m)}} \sum_{j=1}^{m} \sum_{i=\lfloor t_{j-1}n \rfloor}^{\lfloor t_j n \rfloor} V_{i,j},$$

then, as $n \to \infty$,

$$\frac{G^1(m,n)}{\sqrt{n}} - \frac{\hat{G}^1(m,n)}{\sqrt{n}} \overset{\mathbb{P}}{\longrightarrow} 0, \tag{3.9}$$

i.e., as $n \to \infty$, the distribution of the maximum (over all the northeast paths) in (3.8) is approximately the distribution of the maximum over the northeast paths going eastbound only along the rows corresponding to the most probable letters. Now,

$$\frac{\hat{G}^1(m,n) - np_{\max}}{\sqrt{n}} = \sup_{\substack{0=t_0 \le t_1 \le \cdots \\ \le t_{m-1} \le t_m = 1 \\ t_{j-1}=t_j \ for \ j \notin J(m)}} \sum_{j=1}^{m} \frac{\sum_{i=\lfloor t_{j-1}n \rfloor}^{\lfloor t_j n \rfloor} V_{i,j} - (t_j - t_{j-1}) n p_{\max}}{\sqrt{n}}. \tag{3.10}$$

We next claim that, as $n \to \infty$, for any $t > 0$,

$$\left(\frac{\sum_{i=1}^{\lfloor tn \rfloor} V_{i,j} - tnp_j}{\sqrt{n}} \right)_{1 \le j \le m} \Longrightarrow \left(\tilde{B}^j(t) \right)_{1 \le j \le m},$$

where $\left(\tilde{B}^j(t) \right)_{1 \le j \le m}$ is an m-dimensional Brownian motion with covariance matrix $\mathbf{\Sigma} t$. Indeed, for any $t > 0$, since $\mathbf{V_1}, \mathbf{V_2}, \ldots$ are independent, each with mean vector $\mathbf{p} = (p_1, \ldots, p_m)$, and covariance matrix $\mathbf{\Sigma}$,

$$\frac{\sum_{i=1}^{\lfloor tn \rfloor} \mathbf{V_i} - tn\mathbf{p}}{\sqrt{n}} \Longrightarrow \left(\tilde{B}^j(t) \right)_{1 \le j \le m},$$

by the central limit theorem for i.i.d. random vectors and Slutsky's lemma. Next, for any $t > s > 0$, and from the independence of the $\mathbf{V_i}s$,

$$\begin{aligned} &\left(\frac{\sum_{i=\lfloor sn \rfloor + 1}^{\lfloor tn \rfloor} \mathbf{V_i} - \lfloor (t-s)n \rfloor \mathbf{p}}{\sqrt{n}}, \frac{\sum_{i=1}^{\lfloor sn \rfloor} \mathbf{V_i} - \lfloor sn \rfloor \mathbf{p}}{\sqrt{n}} \right) \\ &\Longrightarrow \left(\left(\tilde{B}^j(t-s) \right)_{1 \le j \le m}, \left(\tilde{B}^j(s) \right)_{1 \le j \le m} \right). \end{aligned} \tag{3.11}$$

The continuous mapping theorem allows to conclude that

$$\begin{aligned} &\left(\frac{\sum_{i=1}^{\lfloor tn \rfloor} \mathbf{V_i} - tn\mathbf{p}}{\sqrt{n}}, \frac{\sum_{i=1}^{\lfloor sn \rfloor} \mathbf{V_i} - sn\mathbf{p}}{\sqrt{n}} \right) \\ &\Longrightarrow \left(\left(\tilde{B}^j(t) \right)_{1 \le j \le m}, \left(\tilde{B}^j(s) \right)_{1 \le j \le m} \right). \end{aligned} \tag{3.12}$$

The convergence for the time points $t_1 > t_2 > \cdots > t_n > 0$ can be treated in a similar fashion. Thus the finite dimensional distributions converge to that of $\left(\tilde{B}^j(t) \right)_{1 \le j \le m}$. Since tightness in $C([0,1]^m)$ is as in the proof of Donsker's invariance principle (e.g., see [4]), we are just left with identifying the covariance structure of the limiting Brownian motion $\left(\tilde{B}^j(t)\right)_{1\le j\le m}$. But,

$$\begin{aligned}\operatorname{Cov}\left(\tilde{B}^{j_1}(t),\tilde{B}^{j_2}(t)\right) &= \lim_{n\to\infty}\operatorname{Cov}\left(\frac{\sum_{i=1}^{\lfloor tn\rfloor}V_{i,j_1}}{\sqrt{n}},\frac{\sum_{i=1}^{\lfloor tn\rfloor}V_{i,j_2}}{\sqrt{n}}\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{\lfloor tn\rfloor}\operatorname{Cov}\left(V_{1,j_1},V_{1,j_2}\right) \qquad (3.13)\\ &= \operatorname{Cov}\left(V_{1,j_1},V_{1,j_2}\right)t.\end{aligned}$$

Hence the m-dimensional Brownian motion $\left(\tilde{B}^j(t)\right)_{1\le j\le m}$ has covariance matrix $\mathbf{\Sigma}t$ with $\mathbf{\Sigma}$ given in (3.5). In particular, as $n\to\infty$, for any $t>0$,

$$\left(\frac{\sum_{i=1}^{\lfloor tn\rfloor}V_{i,j}-tnp_{\max}}{\sqrt{n}}\right)_{1\le j\le m,\ j\in J(m)} \Longrightarrow \left(\hat{B}^j(t)\right)_{1\le j\le m,\ j\in J(m)}.$$

It is also straightforward to see that the covariance matrix of $\left(\hat{B}^j(t)\right)_{j\in J(m)}$ is the $d_1\times d_1$ matrix

$$\begin{pmatrix} p_{\max}(1-p_{\max}) & -p_{\max}^2 & \cdots & -p_{\max}^2\\ -p_{\max}^2 & p_{\max}(1-p_{\max}) & \cdots & -p_{\max}^2\\ \vdots & \vdots & \ddots & \vdots\\ -p_{\max}^2 & -p_{\max}^2 & \cdots & p_{\max}(1-p_{\max})\end{pmatrix}t. \qquad (3.14)$$

By the continuous mapping theorem,

$$\frac{\hat{G}^1(m,n)-np_{\max}}{\sqrt{n}} \overset{n\to\infty}{\Longrightarrow} \sup_{J(1,d_1)}\sum_{j=1}^{d_1}\left(\hat{B}^j(t_j)-\hat{B}^j(t_{j-1})\right), \qquad (3.15)$$

and the right-hand side of (3.15) is exactly $\hat{L}_m^1$, then (3.9), leads to

$$\frac{G^1(m,n)-np_{\max}}{\sqrt{n}} \overset{n\to\infty}{\Longrightarrow} \hat{L}_m^1. \qquad (3.16)$$

Now, for $l\ge 2$, $G^l(m,n)$ is the maximum, of the sums of the $V_{i,j}$, over l disjoint paths. Still by the argument in [17], $\left(G^l(m,n)-\hat{G}^l(m,n)\right)/\sqrt{n}\xrightarrow{\mathbb{P}}0$, as $n\to\infty$, where $\hat{G}^l(m,n)$ is the maximal sums of the $V_{i,j}$ over l disjoint paths we now describe. Let $1\le k\le K$ be the unique integer such that $p_{\tau(l)}=p^{(k)}$. Denote by $\alpha_{j(1)},\dots,\alpha_{j(m_k)}$ the letters corresponding to the m_k probabilities that are strictly larger than $p_{\tau(l)}$. For each $1\le s\le m_k$, the horizontal path from $(1,j(s))$ to $(n,j(s))$ is included, and thus so are these m_k paths. The remaining $l-m_k$ disjoint paths only go eastbound along the rows corresponding to the d_k letters having probability $p_{\tau(l)}$. The set of these $l-m_k$ paths is in a one-to-one correspondence with the set of subdivisions of $[0,1]$ given in (3.3). Therefore

$$\hat{G}^l(m,n) = \sum_{j=1}^{m_k}\sum_{i=1}^{n} V_{i,\tau(j)} + \sup_{J(l-m_k,d_k)} \sum_{j=m_k+1}^{m_k+d_k} \sum_{i=1}^{l-m_k} \sum_{r=\lfloor t^i_{j-i} n\rfloor}^{\lfloor t^i_{j-i+1} n\rfloor} V_{r,\tau(j)}. \tag{3.17}$$

Now,

$$\frac{\hat{G}^l(m,n) - ns_l}{\sqrt{n}} = \sum_{j=1}^{m_k} \frac{\sum_{i=1}^{n} V_{i,\tau(j)} - np_{\tau(j)}}{\sqrt{n}} + \sup_{J(l-m_k,d_k)} \sum_{j=m_k+1}^{m_k+d_k} \sum_{i=1}^{l-m_k} \frac{\sum_{r=\lfloor t^i_{j-i} n\rfloor}^{\lfloor t^i_{j-i+1} n\rfloor} V_{r,\tau(j)} - \left(t^i_{j-i+1} - t^i_{j-i}\right) np^{(k)}}{\sqrt{n}}. \tag{3.18}$$

Since the column vectors $\mathbf{V_1}, \mathbf{V_2}, \ldots, \mathbf{V_n}$ are i.i.d., again, as $n \to \infty$, for any $t > 0$,

$$\left(\frac{\sum_{r=1}^{\lfloor tn\rfloor} V_{r,\tau(j)} - tnp_{\tau(j)}}{\sqrt{n}}\right)_{1\le j\le m} \Longrightarrow \left(\hat{B}^j(t)\right)_{1\le j\le m},$$

where $\left(\hat{B}^j(t)\right)_{1\le j\le m}$ is an m-dimensional Brownian motion with covariance matrix given in (3.1). Hence, (3.18) and standard arguments give

$$\frac{G^l(m,n) - ns_l}{\sqrt{n}} \overset{n\to\infty}{\Longrightarrow} \hat{L}^l_m.$$

Finally, by the Cramér-Wold theorem, as $n \to \infty$,

$$\left(\frac{\lambda_1 - ns_1}{\sqrt{n}}, \frac{\sum_{j=1}^{2}\lambda_j - ns_2}{\sqrt{n}}, \ldots, \frac{\sum_{j=1}^{m}\lambda_j - ns_m}{\sqrt{n}}\right) \Longrightarrow \left(\hat{L}^1_m, \hat{L}^2_m, \ldots, \hat{L}^m_m\right), \tag{3.19}$$

therefore, as $n \to \infty$, by the continuous mapping theorem,

$$\begin{aligned}
&\left(\frac{\lambda_1 - np_{\tau(1)}}{\sqrt{n}}, \frac{\lambda_2 - np_{\tau(2)}}{\sqrt{n}}, \ldots, \frac{\lambda_m - np_{\tau(m)}}{\sqrt{n}}\right) \\
&= \left(\frac{G^1 - ns_1}{\sqrt{n}}, \frac{(G^2 - ns_2) - (G^1 - ns_1)}{\sqrt{n}}, \ldots, \frac{(G^m - ns_m) - (G^{m-1} - ns_{m-1})}{\sqrt{n}}\right) \\
&\Longrightarrow \left(\hat{L}^1_m, \hat{L}^2_m - \hat{L}^1_m, \ldots, \hat{L}^m_m - \hat{L}^{m-1}_m\right).
\end{aligned} \tag{3.20}$$

The proof is now complete. □

Remark 3.2. (i) In Theorem 3.2 of [17], the limiting shape of the Young diagrams generated by an irreducible, aperiodic, homogeneous Markov word with finite state space is obtained as a multivariate Brownian functional similar to the one obtained above. The arguments there are based on a careful analysis of the reconfiguration of disjoint subsequences. Specifically, the smallest letter appearing in the disjoint subsequences is then solely in the first subsequence, the second smallest letter, not included in the first subsequence, is completely in the second subsequence, etc. With this new configuration of the disjoint subsequences, a subdivision of the interval $[0,1]$ can be described and a Brownian functional representation is then available. Our approach takes advantage of the lattice with zeros and ones entries (exactly a unique one in each column), and the fact that each subsequence corresponds to a north-east path on the lattice, and that the length of the subsequence is identical to the sum of all the entries on that path. Moreover, for $1 \le l \le m$, and $1 \le i \le l$, the ith lowest path can be chosen to be from $(1,i)$ to $(N, M-l+i)$. Then the subdivision of $[0,1]$ is naturally determined by describing the jumps of all the paths involved.

(ii) Let $\left(\xi_1^0, \dots, \xi_m^0\right)$ represent the vector of the eigenvalues of an element of $\mathcal{G}^0\left(p_{\tau(1)}, \dots, p_{\tau(m)}\right)$, written in such a way that $\xi_{m_k+1}^0 \ge \cdots \ge \xi_{m_k+d_k}^0$ for $k = 1, \dots, K$. Its, Tracy and Widom [18] have shown that the limiting density of $\left(\left(\lambda_1 - np_{\tau(1)}\right)/\sqrt{np_{\tau(1)}}, \dots, \left(\lambda_m - np_{\tau(m)}\right)/\sqrt{np_{\tau(m)}}\right)$, as $n \to \infty$, is that of the eigenvalues of an element of $\mathcal{G}^0\left(p_{\tau(1)}, \dots, p_{\tau(m)}\right)$, given by (2.7). By a simple Riemann integral approximation argument, it follows that

$$\left(\frac{\lambda_1 - np_{\tau(1)}}{\sqrt{np_{\tau(1)}}}, \dots, \frac{\lambda_m - np_{\tau(m)}}{\sqrt{np_{\tau(m)}}}\right) \Longrightarrow \left(\xi_1^0, \dots, \xi_m^0\right).$$

Thus, from Theorem 3.1,

$$\left(\frac{\hat{L}_m^1}{\sqrt{p_{\tau(1)}}}, \frac{\hat{L}_m^2 - \hat{L}_m^1}{\sqrt{p_{\tau(2)}}}, \dots, \frac{\hat{L}_m^m - \hat{L}_m^{m-1}}{\sqrt{p_{\tau(m)}}}\right) \stackrel{d}{=} \left(\xi_1^0, \dots, \xi_m^0\right). \tag{3.21}$$

(iii) Let $\left(B^1(t), B^2(t), \dots, B^m(t)\right)$ be a standard m-dimensional Brownian motion. For $k = 1, \dots, m$, let

$$D_m^k = \sup \sum_{i=1}^{m} \sum_{p=1}^{k} \left(B^i(t_{i-p+1}^p) - B^i(t_{i-p}^p)\right),$$

where the sup is taken over all the subdivisions (t_i^p) of $[0,1]$ described in (3.3). The very approach to prove Theorem 3.1 can be used to obtain a Brownian functional representation of the spectrum of the $m \times m$ GUE, namely,

$$\left(D_m^1, D_m^2 - D_m^1, \dots, D_m^m - D_m^{m-1}\right) \stackrel{d}{=} \left(\xi_1^{GUE,m}, \xi_2^{GUE,m}, \dots, \xi_m^{GUE,m}\right). \tag{3.22}$$

From the observation that the supremum in the definition of $G^k(m,n)$ is attained on a particular set of k disjoint northeast paths for each $k = 1, \dots, m$, Doumerc ([9]) found Brownian functional representations for $\sum_{i=1}^k \xi_i^{GUE,m}$. These functionals are similar to the D_m^k except that the supremum is taken over a different set of subdivisions of $[0,1]$. In fact, we believe that the subdivisions given in (3.3) should be the ones present in [9] (we believe the conditions $t_1 \le s_2, t_2 \le s_3, \dots$, present at the top of page 7 of [9], should not be there). With a similar consideration of k disjoint increasing subsequences, a specific expression for the sum of the first k

rows of the Young diagram associated with a Markov random word is obtained, in [17], in terms of the number of occurrences of the letters among the sequence (see also Chistyakov and Götze [7] or [16] for the binary case). The multidimensional convergence of the whole diagram towards a corresponding multidimensional Brownian functional is also obtained there.

In contrast to the approach in [9], our potential proof of (3.22) does not require passing through the matrix central limit theorem. To briefly describe the approach in [9], let the $V_{i,j}$ in (3.6) be i.i.d. geometric random variables, i.e., for $r = 0, 1, \ldots$, let $\mathbb{P}\left(V_{i,j} = r\right) = q(1-q)^r$. With such $V_{i,j}$, the probability of a given matrix realization only depends on the sum of the matrix entries, which is also the sum of the entries in the shape of the associate Young diagrams. The joint probability mass function of the shape of the associate Young diagrams through the RSK correspondence can then be expressed through the well-known number of Young diagrams sharing this given shape. Next, by setting $q = 1 - L^{-1}$, and letting $L \to \infty$, the random variables on the lattice converge to i.i.d. exponential random variables with parameter one, while the corresponding shape of the associated Young diagrams converges to the spectrum of the $m \times n$ Laguerre Unitary Ensemble. As $n \to \infty$, for any $k = 1, \ldots, m$, the corresponding $G^k(m,n)$, properly normalized, converge in distribution to D_m^k. With the same normalization, it is proved in [9] that the spectrum of the $m \times n$ Laguerre Unitary Ensemble converges to the spectrum of the $m \times m$ GUE. Hence, the continuous mapping theorem, gives $\sum_{j=1}^k \xi_j^{GUE,m} \stackrel{d}{=} D_m^k$. Via the large n asymptotics of the corresponding numbers of Young diagrams, we are able to directly show that the limiting joint probability mass function of the shape of the diagrams converges to the joint probability density function of the eigenvalues of an element of the GUE. Thus, $\sum_{j=1}^k \xi_j^{GUE,m} \stackrel{d}{=} D_m^k$, and (3.22) follows from the Cramér-Wold theorem. Similar ideas are already developed by Johansson (Theorem 1.1 in [20]) to prove that the Poissonized Plancherel measure can be obtained as a limit of the Meixner measure. Johansson also proves the convergence of the whole diagram corresponding to a random word for uniform alphabets, and obtains the joint density of the limiting law.

(iv) The functionals $\left(\hat{L}_m^1, \ldots, \hat{L}_m^m\right)$ can also be represented via m-dimensional standard Brownian motion $\left(B^1, \ldots, B^m\right)$. Indeed, for $i = 1, \ldots, m$, let

$$\begin{aligned}\hat{B}^i(t) &= \sqrt{p_{\tau(i)}}\left(1 - p_{\tau(i)}\right) B^i(t) - p_{\tau(i)} \sum_{q=1,\ q\neq i}^{m} \sqrt{p_{\tau(q)}} B^q(t) \\ &= \sqrt{p_{\tau(i)}} B^i(t) - p_{\tau(i)} \sum_{q=1}^{m} \sqrt{p_{\tau(q)}} B^q(t). \end{aligned} \tag{3.23}$$

Then, it is easy to check that the multidimensional Brownian motion obtained via this linear transformation has covariance matrix $\mathbf{\Sigma}_t$ as in (3.1). Next, recall that for each $l = 1, \ldots, m$, there is a unique $1 \le k \le K$ such that $p_{\tau(l)} = p^{(k)}$, each $p^{(k)}$ having multiplicity d_k. Recall also that we set $m_1 = 0$, and for $k = 2, \ldots, K$,

$m_k = \sum_{j=1}^{k-1} d_j$, so that the multiplicity of each $p_{\tau(j)}$ is d_k if $m_k + 1 \le j \le m_k + d_k$, $k = 1, \dots, m$. With these notations and using the transformation (3.23), and for any $l = 1, \dots, m$, with $m_k + 1 \le l \le m_k + d_k$, (3.2) becomes:

$$\begin{aligned}
\hat{L}_m^l &= \sum_{j=1}^{m_k} \sqrt{p_{\tau(j)}} B^j(1) - \sum_{j=1}^{m_k} p_{\tau(j)} \sum_{q=1}^{m} \sqrt{p_{\tau(q)}} B^q(1) \\
&\quad + \sup_{J(l-m_k,d_k)} \sum_{j=m_k+1}^{m_k+d_k} \sum_{i=1}^{l-m_k} \Big\{ \sqrt{p_{\tau(j)}} \big(B^j(t_{j-i+1}^i) - B^j(t_{j-i}^i)\big) \\
&\qquad\qquad - p_{\tau(j)} \sum_{q=1}^{m} \sqrt{p_{\tau(q)}} \big(B^q(t_{j-i+1}^i) - B^q(t_{j-i}^i)\big) \Big\} \\
&= \sum_{j=1}^{m_k} \sqrt{p_{\tau(j)}} B^j(1) - \left(\sum_{j=1}^{m_k} p_{\tau(j)} + (l - m_k) p_{\tau(m_k+1)} \right) \sum_{q=1}^{m} \sqrt{p_{\tau(q)}} B^q(1) \\
&\quad + \sqrt{p_{\tau(m_k+1)}} \sup_{J(l-m_k,d_k)} \sum_{j=m_k+1}^{m_k+d_k} \sum_{i=1}^{l-m_k} \big(B^j(t_{j-i+1}^i) - B^j(t_{j-i}^i)\big) \qquad (3.24) \\
&= \sum_{j=1}^{m_k} \sqrt{p_{\tau(j)}} B^j(1) - \sum_{j=1}^{l} p_{\tau(j)} \sum_{q=1}^{m} \sqrt{p_{\tau(q)}} B^q(1) \\
&\quad + \sqrt{p_{\tau(m_k+1)}} \sup_{J(l-m_k,d_k)} \sum_{j=m_k+1}^{m_k+d_k} \sum_{i=1}^{l-m_k} \big(B^j(t_{j-i+1}^i) - B^j(t_{j-i}^i)\big), \qquad (3.25)
\end{aligned}$$

where all the sums $\sum_{j=1}^{m_k}$ are understood to vanish when $m_k = 0$. In particular, for $l = 1$, $m_1 = 0$ and (3.24) becomes:

$$\hat{L}_m^1 = -p_{\max} \sum_{q=1}^{m} \sqrt{p_{\tau(q)}} B^q(1) + \sqrt{p_{\max}} \sup_{J(1,d_1)} \sum_{j=1}^{d_1} \big(B^j(t_j^1) - B^j(t_{j-1}^1)\big). \qquad (3.26)$$

4. The Poissonized Word Problem

"Poissonization" is another useful tool in dealing with length asymptotics for longest increasing subsequence problems. It was introduced by Hammersley in [13] in order to show the existence of $\lim_{n\to\infty} \mathbb{E} L\sigma_n/\sqrt{n}$, for σ_n a random permutation of $\{1, 2, \dots, n\}$. Since then, this technique has been widely used and we use it below in connection with the inhomogeneous word problem.

Johansson [20] studied the Poissonized measure on the set of shapes of Young diagrams associated with the homogeneous random word, while, Its Tracy and Widom [19] also studied the Poissonization of LI_n for inhomogeneous random words. They showed that the Poissonized distribution of the length of the longest increasing subsequence, as a function of $p_1, \dots, p_m$, can be identified as the solution of a certain integrable system of nonlinear PDEs. Below, we show that the

Poissonized distribution of the shape of the whole Young diagrams associated with an inhomogeneous random word converges to the spectrum of the corresponding direct sum of GUEs. Next, using this result, together with "de-Poissonization", we obtain the asymptotic behavior of the shape of the diagrams.

Let $W = X_1X_2\cdots X_n$ be a random word of length n, with each letter independently drawn and with $\mathbb{P}_m(X_i = j) = p_j$, $i = 1,\ldots,n$, where $p_j > 0$ and $\sum_{j=1}^m p_j = 1$, i.e., the random word is distributed according to $\mathbb{P}_{W,m,n} = \mathbb{P}_m \times \cdots \times \mathbb{P}_m$ on the set of words $[m]^n$. Using the terminology of [20], with $\mathbb{N} = \{0,1,2,\ldots\}$, let

$$\mathcal{P}_m^{(n)} := \left\{\lambda = (\lambda_1,\ldots,\lambda_m) \in \mathbb{N}^m : \lambda_1 \geq \cdots \geq \lambda_m,\ \sum_{i=1}^m \lambda_i = n\right\},$$

denote the set of partitions of n, of length at most m. The RSK correspondence defines a bijection from $[m]^n$ to the set of pairs of Young diagrams (P,Q) of common shape $\lambda \in \mathcal{P}_m^{(n)}$, where P is semi-standard with elements in $\{1,\ldots,m\}$ and Q is standard with elements in $\{1,\ldots,n\}$.

For any $W \in [m]^n$, let $S(W)$ be the common shape of the Young diagrams associated with W by the RSK correspondence. Then S is a mapping from $[m]^n$ to $\mathcal{P}_m^{(n)}$, which, moreover, is a surjection. The image (or push-forward) of $\mathbb{P}_{W,m,n}$ by S is the measure $\mathbb{P}_{m,n}$ given, for any $\lambda_0 \in \mathcal{P}_m^{(n)}$, by

$$\mathbb{P}_{m,n}(\lambda_0) := \mathbb{P}_{W,m,n}(\lambda(RSK(\mathbf{X}_W)) = \lambda_0).$$

Next, let

$$\mathcal{P}_m := \{\lambda = (\lambda_1,\ldots,\lambda_m) \in \mathbb{N}^m : \lambda_1 \geq \cdots \geq \lambda_m\},$$

be the set of partitions, of elements of $\mathbb{N}$, of length at most m. The set $\mathcal{P}_m$ consists of the shapes of the Young diagrams associated with the random words of any finite length made up from the m letter alphabet.

For $\alpha > 0$, the Poissonized measure of $\mathbb{P}_{m,n}$ on the set $\mathcal{P}_m$ is then defined as

$$\mathbb{P}_m^\alpha(\lambda_0) := e^{-\alpha}\sum_{n=0}^{\infty} \mathbb{P}_{m,n}(\lambda_0)\frac{\alpha^n}{n!}. \tag{4.1}$$

The Poissonized measure $\mathbb{P}_m^\alpha$ coincides with the distribution of the shape of the Young diagrams associated with a random word whose length is a Poisson random variable with mean α. Such a random word is called *Poissonized*, and LI_α denote the length of its longest increasing subsequence.

The Charlier ensemble is closely related to the Poissonized word problem. It is used by Johansson [20] to investigate the asymptotics of LI_n for finite uniform alphabets. For the non-uniform alphabets we consider, let us define the generalized Charlier ensemble to be:

$$\mathbb{P}_{Ch,m}^\alpha(\lambda^0) = \prod_{1\leq i<j\leq m}(\lambda_i^0 - \lambda_j^0 + j - i)\prod_{j=1}^m \frac{1}{(\lambda_j^0 + m - j)!} s_{\lambda^0}(p) e^{-\alpha}\prod_{i=1}^m \alpha^{\lambda_i^0}, \tag{4.2}$$

for all $\lambda^0 = (\lambda^0_1, \lambda^0_2, \ldots, \lambda^0_m) \in \mathcal{P}_m$, and where $s_{\lambda^0}(p)$ is the Schur function of shape λ^0 in the variable $p = \big(p_{\tau(1)}, \ldots, p_{\tau(m)}\big)$ which we describe next. Let $\mathcal{A}_1, \ldots, \mathcal{A}_K$ be the decomposition of $\{1, \ldots, m\}$ such that $p_{\tau(i)} = p_{\tau(j)} = p^{(k)}$ if and only if $i, j \in \mathcal{A}_k$, for some $1 \le k \le K$. Clearly, $d_k = \text{card}\,(\mathcal{A}_k)$. Then,

$$s_{\lambda^0}(p) = \frac{\sum\limits_{\sigma \in \mathcal{S}_m} (-1)^\sigma \prod_{k=1}^K \prod_{i \in \mathcal{A}_k} \Big(p_{\tau(i)}^{m-\sigma(i)-m_k-d_k+\tau(i)} h_{\sigma(i)}^{m_k+d_k-\tau(i)}\Big)}{\prod_{k=1}^K (0!1!\cdots(d_k-1)!) \prod_{k<l} \big(p^{(k)} - p^{(l)}\big)^{d_k d_l}}, \tag{4.3}$$

where $\mathcal{S}_m$ is the set of all the permutations of $\{1, \ldots, m\}$ and where $h_i = \lambda^0_i + m - i$ for $i = 1, \ldots, m$.

The next theorem gives, for inhomogeneous random words, both $\mathbb{P}_{m,n}(\lambda_0)$ and the distribution of LI_α. The first statement is due to Its, Tracy and Widom ([18], [19]), while the second follows directly from the fact that the length of the longest increasing subsequence is equal to the length of the first row of the corresponding Young diagrams.

Theorem 4.1.

(i) *On $[m]^n$, the image (or push-forward) of $\mathbb{P}_{W,m,n}$ by the mapping $S : [m]^n \to \mathcal{P}^{(n)}_m$ is, for any $\lambda^0 = (\lambda^0_1, \lambda^0_2, \ldots, \lambda^0_m) \in \mathcal{P}^{(n)}_m$, given by*

$$\mathbb{P}_{m,n}(\lambda^0) = s_{\lambda^0}(p) f^{\lambda^0}. \tag{4.4}$$

Above, f^{λ^0} is the number of Young diagrams of shape λ^0 with elements in $\{1, \ldots, n\}$:

$$f^{\lambda^0} = n! \prod_{1 \le i < j \le m} (\lambda^0_i - \lambda^0_j + j - i) \prod_{j=1}^m \frac{1}{(\lambda^0_j + m - j)!},$$

and $s_{\lambda^0}(p)$ is the Schur function of shape λ^0 in the variable $p = \big(p_{\tau(1)}, \ldots, p_{\tau(m)}\big)$ given in (4.3), with τ a permutation of $\{1, \ldots, m\}$ corresponding to a non-increasing ordering of $p_1, p_2, \ldots, p_m$.

(ii) *The Poissonization of $\mathbb{P}_{m,n}$ is the generalized Charlier ensemble $\mathbb{P}^\alpha_{Ch,m}$ defined in (4.2). In particular, for the Poissonized word problem,*

$$\mathbb{P}^\alpha_{W,m}\,(LI_\alpha \le t) := e^{-\alpha} \sum_{n=0}^\infty \mathbb{P}_{m,n}\,(\lambda_1 \le t)\,\frac{\alpha^n}{n!} = \mathbb{P}^\alpha_{Ch,m}\,(\lambda_1 \le t)\,. \tag{4.5}$$

For uniform alphabet, Johansson [20] obtained the convergence, as $\alpha \to \infty$, of the Poissonized measure on $\mathcal{P}_m$ to the joint law of the ordered eigenvalues of the GUE. Next, following his lead and techniques, we generalize this result to the non-uniform case, where the convergence is towards the joint law of the eigenvalues $(\xi_1, \ldots, \xi_m)$, ordered within each block, of an element of $\mathcal{G}_m\,(d_1, \ldots, d_K)$. The density of $(\xi_1, \ldots, \xi_m)$ is, for any $x \in \mathbb{R}^m$, given by

$$f_{\xi_1,\ldots,\xi_m}(x) = \frac{1}{\sqrt{2\pi}} c_m \prod_{k=1}^K \Delta_k(x)^2 e^{-\sum_{i=1}^m x_i^2/2}, \tag{4.6}$$

where $c_m = (2\pi)^{-(m-1)/2} \prod_{k=1}^{K} (0!1! \cdots (d_k-1)!)^{-1}$, and where

$$\Delta_k(x) = \prod_{m_k+1 \le i < j \le m_k+d_k} (x_i - x_j)\,.$$

Theorem 4.2. *Let $\lambda(RSK(\mathbf{X}_W)) = (\lambda_1, \dots, \lambda_m)$ be the common shape of the Young diagrams associated with W through the RSK correspondence. Let $(\xi_1, \dots, \xi_m)$ be the eigenvalues of an element of $\mathcal{G}_m(d_1, \dots, d_K)$, written in such a way that $\xi_{m_k+1} \ge \cdots \ge \xi_{m_k+d_k}$ for $k = 1, \dots, K$, and let $f_{\xi_1,\dots,\xi_m}$ be its density given by* (4.6)*. Then, for any continuous function g on $\mathbb{R}^m$,*

$$\lim_{\alpha\to\infty} \mathbb{E}_m^\alpha \left(g\left(\frac{\lambda_1 - \alpha p_{\tau(1)}}{\sqrt{\alpha p_{\tau(1)}}}, \dots, \frac{\lambda_m - \alpha p_{\tau(m)}}{\sqrt{\alpha p_{\tau(m)}}} \right)\right) = \int_{\mathbb{R}^m} g(x) f_{\xi_1,\dots,\xi_m}(x)dx. \quad (4.7)$$

Proof. By Theorem 4.1, for any partition $\lambda^0 = (\lambda_1^0, \lambda_2^0, \dots, \lambda_m^0)$ of $n \in \mathbb{N}$,

$$\mathbb{P}_{m,n}(\lambda(RSK(\mathbf{X}_W)) = \lambda^0) = s_{\lambda^0}(p) f^{\lambda^0},$$

where

$$f^{\lambda^0} = n! \prod_{1\le i<j\le m} (\lambda_i^0 - \lambda_j^0 + j - i) \prod_{j=1}^{m} \frac{1}{(\lambda_j^0 + m - j)!},$$

and where $s_{\lambda^0}(p)$ is the Schur function of shape λ^0 in the variable $p = \big(p_{\tau(1)}, \dots, p_{\tau(m)}\big)$ as given in (4.3). Hence the Poissonized measure is

$$\mathbb{P}_m^\alpha\left(\lambda^0\right) = e^{-\alpha} \sum_{n=0}^{\infty} n! \prod_{1\le i<j\le m} (\lambda_i^0 - \lambda_j^0 + j - i) \prod_{j=1}^{m} \frac{1}{(\lambda_j^0 + m - j)!} s_{\lambda^0}(p) \frac{\alpha^n}{n!}.$$

Next, for $i = 1, \dots, m$, let

$$x_i = \frac{\lambda_i^0 - \alpha p_{\tau(i)}}{\sqrt{\alpha p_{\tau(i)}}},$$

then, as $\alpha \to \infty$,

$$\prod_{j=1}^{m} \frac{1}{(\lambda_j^0 + m - j)!} \sim (2\pi)^{-m/2} \frac{e^\alpha}{\alpha^n} \alpha^{-m(m-1)/2} \left(\prod_{i=1}^{m} p_{\tau(i)}^{\tau(i)-m} \right) e^{-\sum_{i=1}^m x_i^2/2}, \quad (4.8)$$

and

$$\prod_{1\le i<j\le m} (\lambda_i^0 - \lambda_j^0 + j - i) \quad (4.9)$$

$$\sim \alpha^{m(m-1)/2 - \sum_{k=1}^K d_k(d_k-1)/4} \prod_{k=1}^{K} \left(\left(p^{(k)}\right)^{d_k(d_k-1)/4} \Delta_k(x) \right) \prod_{k<l} \left(p^{(k)} - p^{(l)}\right)^{d_k d_l}.$$

Together with

$$\sum_{\sigma\in\mathcal{S}_m}(-1)^\sigma\prod_{k=1}^K\prod_{i\in\mathcal{A}_k}\left(p_{\tau(i)}^{m-\sigma(i)-m_k-d_k+\tau(i)}h_{\sigma(i)}^{m_k+d_k-\tau(i)}\right) \tag{4.10}$$
$$\sim\prod_{i=1}^m p_{\tau(i)}^{m-\tau(i)}\prod_{k=1}^K\left(p^{(k)}\right)^{-d_k(d_k-1)/2}\alpha^{\sum_{k=1}^K d_k(d_k-1)/4}\prod_{k=1}^K\left(\left(p^{(k)}\right)^{d_k(d_k-1)/4}\Delta_k(x)\right),$$

the limiting density of $\left(\left(\lambda_1-\alpha p_{\tau(1)}\right)/\sqrt{\alpha p_{\tau(1)}},\dots,\left(\lambda_m-\alpha p_{\tau(m)}\right)/\sqrt{\alpha p_{\tau(m)}}\right)$, as $\alpha\to\infty$, is

$$\sqrt{2\pi}c_m\prod_{k=1}^K\Delta_k(x)^2e^{-\sum_{i=1}^m x_i^2/2},\quad x=(x_1,\dots,x_m)\in\mathbb{R}^m,$$

which is just the joint density of the eigenvalues, ordered within each block, of an element of $\mathcal{G}_m\left(d_1,\dots,d_K\right)$. The statement then follows from a Riemann sums approximation argument as in [20]. □

The next result is concerned with "de-Poissonization", and again is the non-uniform version (with a similar proof) of a result of Johansson.

Proposition 4.3. *Let $\alpha_n=n+3\sqrt{n\log n}$ and $\beta_n=n-3\sqrt{n\log n}$. Then there is a constant C such that, for sufficiently large n, and for any $0\le n_i\le n$, $i=1,\dots,m$,*

$$\begin{aligned}\mathbb{P}_m^{\alpha_n}\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right)-\frac{C}{n^2}&\le\mathbb{P}_{m,n}\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right)\\&\le\mathbb{P}_m^{\beta_n}\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right)+\frac{C}{n^2}.\end{aligned}\tag{4.11}$$

Proof. The proof is analogous to the proof of the corresponding uniform alphabet result, given in [20] (see also Lemma 4.7 in [5]). First, a simple consequence of the description of the RSK correspondence ensures that $\mathbb{P}_{m,n}\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right)$ is non-increasing in n, i.e.,

$$\mathbb{P}_{m,n+1}\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right)\le\mathbb{P}_{m,n}\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right).\tag{4.12}$$

Next,

$$\mathbb{P}_m^\alpha\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right)=\sum_{n=0}^\infty e^{-\alpha}\frac{\alpha^n}{n!}\mathbb{P}_{m,n}\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right),$$

and then, proceeding as in [20],

$$\begin{aligned}\left|\mathbb{P}_m^\alpha\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right)-\sum_{|n-\alpha|\le\sqrt{8\alpha\log\alpha}}e^{-\alpha}\frac{\alpha^n}{n!}\mathbb{P}_{m,n}\left(\lambda_1\le n_1,\dots,\lambda_m\le n_m\right)\right|\\\le\frac{C}{\alpha^2},\end{aligned}\tag{4.13}$$

for some constant C, α sufficiently large and all $1\le n_i\le n$, $i=1,\dots,m$. Replacing α by respectively $n+3\sqrt{n\log n}$ and $n-3\sqrt{n\log n}$ completes the proof. □

We are now ready to obtain asymptotics for the shape of the Young diagrams associated with a random word $W \in [m]^n$, when m and n go to infinity. Before stating our result, let us recall the well-known, large m, asymptotic behavior of the spectrum of the $m \times m$ GUE ([25], [26], [20]):

Let $\xi_j^{GUE,m}$ be the jth largest eigenvalue of an element of the $m\times m$ GUE. For each $r \geq 1$, there is a distribution function F_r on $\mathbb{R}^r$, such that, for all $(t_1,\ldots,t_r) \in \mathbb{R}^r$,

$$\lim_{m\to\infty} \mathbb{P}_{GUE,m}\left(\xi_j^{GUE,m} \leq 2\sqrt{m} + t_j/m^{1/6}, j=1,\ldots,r\right) = F_r(t_1,\ldots,t_r).$$

The multivariate distribution function F_r originates in [25] and [26], another expression for it is also given in [20] (see (3.48) there) and its one-dimensional marginals are Tracy-Widom distributions.

Once more, our next theorem is already present, for uniform alphabets, in Johansson [20].

Theorem 4.4. *Let $r\geq 1$. Let $d_1 \to +\infty$, as $m\to +\infty$. Then, for all $(t_1,\ldots,t_r) \in \mathbb{R}^r$,*

$$\lim_{m\to\infty}\lim_{\alpha\to\infty} \mathbb{P}_m^\alpha\left(\lambda_j \leq \alpha p_{\max} + 2\sqrt{d_1\alpha p_{\max}} + t_j d_1^{-1/6}\sqrt{\alpha p_{\max}}, j=1,\ldots,r\right) = F_r(t_1,\ldots,t_r), \tag{4.14}$$

and,

$$\lim_{d_1\to\infty}\lim_{n\to\infty} \mathbb{P}_{m,n}\left(\lambda_j \leq n p_{\max} + 2\sqrt{d_1 n p_{\max}} + t_j d_1^{-1/6}\sqrt{n p_{\max}}, j=1,\ldots,r\right) = F_r(t_1,\ldots,t_r). \tag{4.15}$$

Proof. By Theorem 4.2, for each $r \geq 1$, and for all $(s_1,\ldots,s_r) \in \mathbb{R}^r$,

$$\lim_{\alpha\to\infty} \mathbb{P}_{W,m}^\alpha\left(\frac{\lambda_j - \alpha p_{\max}}{\sqrt{\alpha p_{\max}}} \leq s_j,\ j=1,\ldots,r\right) = \mathbb{P}_{GUE,d_1}\left(\xi_j \leq s_j,\ j=1,\ldots,r\right), \tag{4.16}$$

where ξ_j is the jth largest eigenvalue of the $d_1 \times d_1$ GUE.

Hence, for any $(t_1,\ldots,t_r) \in \mathbb{R}^r$,

$$\begin{aligned}
&\lim_{\alpha\to\infty} \mathbb{P}_m^\alpha\left(\lambda_j \leq \alpha p_{\max} + 2\sqrt{d_1\alpha p_{\max}} + t_j d_1^{-1/6}\sqrt{\alpha p_{\max}}, j=1,\ldots,r\right)\\
&\quad= \lim_{\alpha\to\infty} \mathbb{P}_m^\alpha\left(\frac{\lambda_j - \alpha p_{\max}}{\sqrt{\alpha p_{\max}}} \leq 2\sqrt{d_1} + t_j d_1^{-1/6}, j=1,\ldots,r\right)\\
&\quad= \mathbb{P}\left(\xi_j \leq 2\sqrt{d_1} + t_j d_1^{-1/6}, j=1,\ldots,r\right).
\end{aligned} \tag{4.17}$$

As $d_1 \to \infty$, the result of Tracy-Widom on the convergence of the spectrum of the GUE gives the first conclusion, proving (4.14). Next, by Proposition 4.3, with $\alpha_n = n + 3\sqrt{n\log n}$ and $\beta_n = n - 3\sqrt{n\log n}$, there is a constant C such that, for sufficiently large n, and for any $0 \le s_j \le n$, $j = 1, \ldots, r$,

$$\begin{aligned} \mathbb{P}_m^{\alpha_n}\left(\lambda_j \le s_j, j = 1, \ldots, r\right) - \frac{C}{n^2} &\le \mathbb{P}_{m,n}\left(\lambda_j \le s_j, j = 1, \ldots, r\right) \\ &\le \mathbb{P}_m^{\beta_n}\left(\lambda_j \le s_j, j = 1, \ldots, r\right) + \frac{C}{n^2}. \end{aligned} \tag{4.18}$$

But,

$$n = (1 - \varepsilon_\alpha)\,\alpha_n, \quad \text{with} \quad \varepsilon_\alpha = 3\sqrt{n \log n}/\left(n + 3\sqrt{n \log n}\right),$$

whereas

$$n = (1 + \varepsilon_\beta)\,\beta_n \quad \text{with} \quad \varepsilon_\beta = 3\sqrt{n \log n}/\left(n - 3\sqrt{n \log n}\right).$$

Since $\varepsilon_\alpha, \varepsilon_\beta \to 0$, as $n \to \infty$, it follows from (4.18), by setting $s_j = np_{\max} + 2\sqrt{d_1 n p_{\max}} + t_j d_1^{-1/6}\sqrt{np_{\max}}$, that

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{P}_m^{\alpha_n}\left(\lambda_j \le \alpha_n p_{\max} + 2\sqrt{d_1 \alpha_n p_{\max}} + t_j d_1^{-1/6}\sqrt{\alpha_n p_{\max}}, j = 1, \ldots, r\right) \\ &\le \lim_{n\to\infty} \mathbb{P}_{m,n}\left(\lambda_j \le np_{\max} + 2\sqrt{d_1 n p_{\max}} + t_j d_1^{-1/6}\sqrt{np_{\max}}, j = 1, \ldots, r\right) \\ &\le \lim_{n\to\infty} \mathbb{P}_m^{\beta_n}\left(\lambda_j \le \beta_n p_{\max} + 2\sqrt{d_1 \beta_n p_{\max}} + t_j d_1^{-1/6}\sqrt{\beta_n p_{\max}}, j = 1, \ldots, r\right). \end{aligned} \tag{4.19}$$

Now, (4.17) holds true with α replaced by α_n or β_n. Finally, (4.15) follows from (4.19) by letting $d_1 \to \infty$. □

Remark 4.5. The convergence results in Theorem 4.4 are obtained by taking successive limits, i.e., first in n and then in m. For uniform finite alphabets, in which case $d_1 = m$, Johansson [20] obtained the simultaneous convergence, for the length of the longest increasing subsequence, via a careful analysis of corresponding kernels and methods of orthogonal polynomials. These results demand: $(\log n)^{3/2}/m \to 0$ and $\sqrt{n}/m \to \infty$. Also in the uniform case, under the assumption $m = o\left(n^{3/10}(\log n)^{-3/5}\right)$, the simultaneous convergence result (4.15) is obtained, via Gaussian approximation, in [6] where non-uniform results are also given.

5. Appendix

Let $\xi_{\max,0}^{GUE,m}$ (resp. $\xi_{\max}^{GUE,m}$) be the maximal eigenvalue of an element of the $m \times m$ traceless GUE (resp. GUE). Below, we give simple proofs of the convergence of $\xi_{\max,0}^{GUE,m}/\sqrt{m}$ (or equivalently of $\xi_{\max}^{GUE,m}$) towards 2. These proofs are based on the "tridiagonalization" technique originating in Trotter [28] (see also Silverstein [23] where similar ideas are used). Our first result is the well-known Householder representation of Hermitian matrices.

Lemma 5.1. *Let* $\mathbf{G} = (G_{i,j})_{1\le i,j\le m}$ *be an element of the GUE. Then, there exists a unitary matrix* $\mathbf{U}$*, such that*

$$\mathbf{T} := \mathbf{U}\mathbf{G}\mathbf{U}^* = \begin{pmatrix} A_{1,1} & \chi^2_{m-1} & 0 & \cdots & 0 \\ \chi^2_{m-1} & A_{2,2} & \chi^2_{m-2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \chi^2_2 & A_{m-1,m-1} & \chi^2_1 \\ 0 & \cdots & 0 & \chi^2_1 & A_{m,m} \end{pmatrix}, \tag{5.1}$$

where $A_{1,1}, \dots, A_{m,m}$ *are independent standard normal random variables, and for each* $1 \le k \le m-1$, χ^2_{m-k} *has a chi-squared distribution, with* $m-k$ *degrees of freedom. Moreover, for each* $k = 1, \dots, m-1$, $A_{k,k}$ *is independent of* $\chi^2_{m-k}, \dots, \chi^2_1$.

Proposition 5.2. *Let* $\xi^{GUE,m}_{\max,0}$ *(resp.* $\xi^{GUE,m}_{\max}$*) be the maximal eigenvalue of an element of the* $m \times m$ *traceless GUE (resp. GUE), then as* $m \to \infty$,

$$\frac{\xi^{GUE,m}_{\max,0}}{\sqrt{m}} \to 2, \quad \left(\textit{resp. } \frac{\xi^{GUE,m}_{\max}}{\sqrt{m}} \to 2\right) \textit{ almost surely.}$$

Proof. An elementary proof is obtained along the following lines: First, by Lemma 5.1, $\mathbf{G}$ and $\mathbf{T}$ share the same eigenvalues. Next, by the Gerŝgorin circle theorem (see [14]), for any eigenvalue ξ_i of $\mathbf{G}$, letting also $\chi^2_0 = \chi^2_m = 0$,

$$\xi_i \in \bigcup_{k=1,\dots,m} \left[A_{k,k} - \chi^2_{m-k+1} - \chi^2_{m-k}, A_{k,k} + \chi^2_{m-k+1} + \chi^2_{m-k}\right].$$

Hence

$$\frac{\xi^{GUE,m}_{\max}}{\sqrt{m}} \le \max_{k=1,\dots,m} \left(\frac{A_{k,k}}{\sqrt{m}} + \frac{\chi^2_{m-k+1}}{\sqrt{m}} + \frac{\chi^2_{m-k}}{\sqrt{m}}\right). \tag{5.2}$$

For each $1 \le k \le m$, $A_{k,k} \sim N(0,1)$, and thus very classically $\max_{k=1,\dots,m} A_{k,k}/\sqrt{m} \overset{\text{a.s.}}{\to} 0$. Next, for any fixed $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{P}\left(\left|\max_{k=1,\dots,m} \frac{\chi^2_{m-k+1}}{m} - 1\right| > \varepsilon\right) \\ &\quad \le \mathbb{P}\left(\chi^2_m < m(1-\varepsilon)\right) + m\mathbb{P}\left(\chi^2_m > m(1+\varepsilon)\right), \end{aligned} \tag{5.3}$$

and the tail behavior of χ^2_m ensures that $\sum_{m=1}^\infty m\mathbb{P}\left(\chi^2_m > m(1+\varepsilon)\right) < +\infty$, and that $\sum_{m=2}^\infty \mathbb{P}\left(\chi^2_m < m(1-\varepsilon)\right) < +\infty$. Therefore, $\max_{k=1,\dots,m} \chi^2_{m-k+1}/m \overset{\text{a.s.}}{\to} 1$, and almost surely,

$$\limsup_{m\to\infty} \frac{\xi^{GUE,m}_{\max}}{\sqrt{m}} \le 2. \tag{5.4}$$

Next, since the empirical distribution of the eigenvalues $\left(\xi_i^{GUE,m}/\sqrt{m}\right)_{1\le i\le m}$ converges almost surely to the semicircle law ν with density $\sqrt{4-x^2}/2\pi$, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\liminf_{m\to\infty} \frac{\xi_{\max}^{GUE,m}}{\sqrt{m}} > 2-\varepsilon\right) = 1. \tag{5.5}$$

Letting $\varepsilon \to 0$ in (5.5) yields,

$$\liminf_{m\to\infty} \frac{\xi_{\max}^{GUE,m}}{\sqrt{m}} \ge 2 \quad \text{a.s.} \tag{5.6}$$

Combining (5.4) and (5.6), $\xi_{\max}^{GUE,m}/\sqrt{m} \to 2$ almost surely, and a similar result also follows for $\xi_{\max,0}^{GUE,m}/\sqrt{m}$. □

To prove our next convergence result, we first need a simple lemma.

Lemma 5.3. *For each $k = 1, 2, \ldots,$ let χ_k^2 be a chi-square random variable with k degrees of freedom. Then,*

$$\lim_{m\to\infty} \mathbb{E}\left(\frac{\max_{k=1,\ldots,m} \chi_k^2}{m}\right) = 1. \tag{5.7}$$

Proof. First,

$$\mathbb{E}\Big(\max_{k=1,\ldots,m} \chi_k^2\Big) \ge \mathbb{E}\left(\chi_m^2\right) = m.$$

Next, by the concavity of the logarithm, for any $0 < t < 1/2$,

$$\begin{aligned} t\mathbb{E}\left(\frac{\max_{k=1,\ldots,m} \chi_k^2}{m}\right) &\le \frac{1}{m}\ln\left(\sum_{k=1}^m \mathbb{E}e^{t\chi_k^2}\right) \\ &\le \frac{1}{m}\ln\left(m\frac{1}{(1-2t)^{m/2}}\right) \\ &= \frac{\ln m}{m} - \frac{1}{2}\ln(1-2t). \end{aligned}$$

Hence,

$$t\limsup_{m\to\infty} \mathbb{E}\left(\frac{\max_{k=1,\ldots,m} \chi_k^2}{m}\right) \le -\frac{1}{2}\ln(1-2t),$$

and letting $t \to 0$,

$$\limsup_{m\to\infty} \mathbb{E}\left(\frac{\max_{k=1,\ldots,m} \chi_k^2}{m}\right) \le \lim_{t\to 0} -\frac{\ln(1-2t)}{2t} = 1.$$

$\big($Since $-\ln(1-2t) \le 2t + 4t^2$, for $0 \le t \le 1/3$, taking $t = \sqrt{\ln m/2m}$ in (5.8), will give $\mathbb{E}\big(\max_{k=1,\ldots,m} \chi_k^2/m\big) \le 1 + 2\sqrt{2\ln m/m}$, for $m > 10$.$\big)$ □

Again, in the uniform finite alphabet case, where $p_1 = \cdots = p_m = 1/m$, we have $K = 1$, $d_1 = m$. For $k = 1, \ldots, m$, and to keep up with the notation of [15], denote by $\tilde{H}_m^k$ the particular version of $\hat{L}_m^k$, as in (3.2). Let $\left(\tilde{B}^1(t), \tilde{B}^2(t), \ldots, \tilde{B}^m(t)\right)$ be the m-dimensional Brownian motion having covariance matrix

$$\begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix} t, \tag{5.8}$$

with $\rho = -1/(m-1)$. Then, for $k = 1, \ldots, m$ (see also [15], [9]),

$$\tilde{H}_m^k = \sqrt{\frac{m-1}{m}} \sup \sum_{i=1}^{m} \sum_{p=1}^{k} \left(\tilde{B}^i(t_{i-p+1}^p) - \tilde{B}^i(t_{i-p}^p)\right),$$

where the sup is taken over all the subdivisions (t_i^p) of $[0,1]$ as in (3.3). As a corollary to Theorem 3.1 (see also [15]), for each $m \geq 2$,

$$\left(\tilde{H}_m^1, \tilde{H}_m^2 - \tilde{H}_m^1, \ldots, \tilde{H}_m^m - \tilde{H}_m^{m-1}\right) \stackrel{d}{=} \left(\xi_{1,0}^{GUE,m}, \xi_{2,0}^{GUE,m}, \ldots, \xi_{m,0}^{GUE,m}\right). \tag{5.9}$$

Moreover, convergence in L^1 also holds.

Proposition 5.4. *As $m \to \infty$,*

$$\frac{\xi_{\max,0}^{GUE,m}}{\sqrt{m}} \to 2, \quad in \;\; L^1.$$

Equivalently,

$$\frac{\xi_{\max}^{GUE,m}}{\sqrt{m}} \to 2, \quad in \;\; L^1.$$

Equivalently,

$$\frac{\tilde{H}_m^1}{\sqrt{m}} \to 2, \quad in \;\; L^1.$$

Proof. Note that when $p_1 = \cdots = p_m = 1/m$, $\mathcal{L}_{(s_1,\ldots,s_m)}^{p_1,\ldots,p_m}$, given by (2.4) is the empty set when $s_1 < 0$. Hence $\xi_{\max,0}^{GUE,m}$ is nonnegative (this is actually clear from the traceless requirement). By Theorem 3.1, $\tilde{H}_m^1$ and $\xi_{\max,0}^{GUE,m}$ are equal in distribution, and so it suffices to prove that, as $m \to \infty$,

$$\frac{\mathbb{E}\left(\xi_{\max,0}^{GUE,m}\right)}{\sqrt{m}} \to 2. \tag{5.10}$$

Next, by Proposition 2.6, $\mathbb{E}\left(\xi_{\max,0}^{GUE,m}\right) = \mathbb{E}\left(\xi_{\max}^{GUE,m}\right)$. Moreover, taking expectations on both sides of (5.2) gives:

$$\mathbb{E}\left(\xi_{\max}^{GUE,m}\right) \leq \mathbb{E}\left(\max_{k=1,\ldots,m} A_{k,k}\right) + \mathbb{E}\left(\max_{k=1,\ldots,m} \chi_{m-k+1}^2\right) + \mathbb{E}\left(\max_{k=1,\ldots,m} \chi_{m-k}^2\right).$$

It is well known that,

$$\mathbb{E}\left(\max_{k=1,\dots,m} A_{k,k}\right) \le \sqrt{2\ln m},$$

while, by Lemma 5.3,

$$\limsup_{m\to\infty} \mathbb{E}\left(\max_{k=1,\dots,m} \frac{\chi_k^2}{\sqrt{m}}\right) = 1,$$

leading to

$$\limsup_{m\to\infty} \mathbb{E}\left(\frac{\xi_{\max,0}^{GUE,m}}{\sqrt{m}}\right) \le 2.$$

Now, $\xi_{\max,0}^{GUE,m}$ is nonnegative and by Proposition 5.2, $\xi_{\max,0}^{GUE,m}/\sqrt{m} \to 2$, almost surely. Thus, by Fatou's Lemma,

$$\liminf_{m\to\infty} \mathbb{E}\left(\frac{\xi_{\max,0}^{GUE,m}}{\sqrt{m}}\right) \ge \mathbb{E}\left(\liminf_{m\to\infty} \frac{\xi_{\max,0}^{GUE,m}}{\sqrt{m}}\right) = 2,$$

and so, $\lim_{m\to\infty} \mathbb{E}\left(\xi_{\max,0}^{GUE,m}/\sqrt{m}\right) = 2$. Using once more the fact that $\xi_{\max,0}^{GUE,m}$ is nonnegative, we conclude that $\lim_{m\to\infty} \mathbb{E}\left|\xi_{\max,0}^{GUE,m}/\sqrt{m} - 2\right| = 0$, and by the weak law of large number, $\lim_{m\to\infty} \mathbb{E}\left|\xi_{\max}^{GUE,m}/\sqrt{m} - 2\right| = 0$. □

Remark 5.5. A small and elementary tightening of the arguments of Davidson and Szarek [8] will also provide an alternative proof of Proposition 5.4.

Proof of Proposition 2.7. By Proposition 2.2,

$$\max_{m_k < i \le m_k + d_k} \xi_i^0 = \max_{m_k < i \le m_k + d_k} \xi_i - \sqrt{p^{(k)}} \sum_{l=1}^{m} \sqrt{p_l} \mathbf{X}_{l,l}.$$

Since $\max_{m_k < i \le m_k + d_k} \xi_i$ is the maximal eigenvalue of an element of the $d_k \times d_k$ GUE, with probability one or in the mean, $\lim_{d_k\to\infty} \max_{m_k < i \le m_k + d_k} \xi_i/\sqrt{d_k} = 2$. Moreover, $\sum_{l=1}^m \sqrt{p_l}\mathbf{X}_{l,l}$ is a centered Gaussian random variable with variance $\mathrm{Var}\left(\sum_{l=1}^m \sqrt{p_l}\mathbf{X}_{l,l}\right) = \sum_{l=1}^m p_l = 1$. Hence, with probability one or in the mean, $\lim_{d_k\to\infty} \sqrt{p^{(k)}} \sum_{l=1}^m \sqrt{p_l}\mathbf{X}_{l,l}/\sqrt{d_k} = 0$. □

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Part V

High Dimensional Statistics

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Low Rank Estimation of Similarities on Graphs

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Abstract. Let (V, E) be a graph with vertex set V and edge set E. Let $(X, X', Y) \in V \times V \times \{-1, 1\}$ be a random triple, where X, X' are independent uniformly distributed vertices and Y is a label indicating whether X, X' are "similar" ($Y = +1$), or not ($Y = -1$). Our goal is to estimate the regression function

$$S_*(u, v) = \mathbb{E}(Y|X = u, X' = v), u, v \in V$$

based on training data consisting of n i.i.d. copies of (X, X', Y). We are interested in this problem in the case when S_* is a symmetric low rank kernel and, in addition to this, it is assumed that S_* is "smooth" on the graph. We study estimators based on a modified least squares method with complexity penalization involving both the nuclear norm and Sobolev type norms of symmetric kernels on the graph and prove upper bounds on L_2-type errors of such estimators with explicit dependence both on the rank of S_* and on the degree of its smoothness.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E, $\text{card}(V) = m$. Let $A := (a(u, v))_{u,v \in V}$ be the adjacency matrix of G, that is, $a(u, v) = 1$ if u and v are connected with an edge and $a(u, v) = 0$ otherwise. Let $\Delta := D - A$ be the Laplacian of G, D being the diagonal matrix with the degrees of vertices on the diagonal. Let $(X, X', Y) \in V \times V \times \{-1, 1\}$ be a random triple with X, X' being independent

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vertices sampled at random from the uniform distribution Π on V and Y being an "indicator" of a symmetric binary relationship between X, X' called in what follows a "similarity". More precisely, $Y = +1$ indicates that the vertices X, X' are similar and $Y = -1$ indicates that they are not. The conditional distribution of Y given X, X' is completely characterized by the regression function

$$S_*(u,v) := \mathbb{E}(Y|X = u, X' = v), u, v \in V$$

that is assumed to be a symmetric kernel on $V \times V$ and will be called the *similarity kernel.* It is well known that $\mathrm{sign}(S_*(X, X'))$ is the Bayes classifier, that is, the best possible predictor of Y based on an observation of X, X' in the sense that it minimizes the generalization error $\mathbb{P}\{Y \neq g(X, X')\}$ over all possible predictors $g : V \times V \mapsto \{-1, 1\}$. Our goal is to estimate S_* based on the training data $(X_1, X_1', Y_1), \ldots, (X_n, X_n', Y_n)$ consisting of n i.i.d. copies of (X, X', Y). We are especially interested in the class of problems such that, on the one hand, S_* is a matrix (kernel) of relatively small rank and, on the other hand, S_* possesses certain degree of smoothness on the graph.

Throughout the paper, $\mathcal{S}_V$ denotes the linear space of *symmetric kernels* $S : V \times V \mapsto \mathbb{R}$, $S(u,v) = S(v,u), u, v \in V$, that can be also viewed as real-valued symmetric $m \times m$ matrices. For $S \in \mathcal{S}_V$, let $\mathrm{rank}(S)$ denote the *rank* of S and $\mathrm{tr}(S)$ denote the *trace* of S. The *spectral representation* of S has the form $S = \sum_{j=1}^r \sigma_j(\psi_j \otimes \psi_j)$, where $r = \mathrm{rank}(S)$, $\sigma_1 \leq \cdots \leq \sigma_r$ are non-zero eigenvalues of S (repeated with their multiplicities) and $\psi_1, \ldots, \psi_r$ are the corresponding orthonormal eigenfunctions (there is a multiple choice of ψ_js in the case of repeated eigenvalues). We also use the notation $\mathrm{sign}(S) := \sum_{j=1}^r \mathrm{sign}(\sigma_j)(\psi_j \otimes \psi_j)$ and we define the *support* of S, denoted by $\mathrm{supp}(S)$, as the linear span of $\{\psi_1, \ldots, \psi_r\}$ in $\mathbb{R}^V$.

For $1 \leq p < \infty$, the Schatten p-norm of $S \in \mathcal{S}_V$ is defined as

$$\|S\|_p := (\mathrm{tr}(|S|^p))^{1/p} = \left(\sum_{j=1}^r |\sigma_j|^p\right)^{1/p},$$

where $|S| := \sqrt{S^2}$. For $p = 1$, $\|\cdot\|_1$ is called the *nuclear norm*, while, for $p = 2$, $\|\cdot\|_2$ is the *Hilbert–Schmidt* or *Frobenius norm*, that is, the norm induced by the Hilbert–Schmidt inner product which will be denoted by $\langle\cdot,\cdot\rangle$. The *operator* or *spectral norm* is defined as $\|S\| := \max_j |\sigma_j|$.

Let us also denote by $\Pi^2 := \Pi \times \Pi$ the distribution of random couple (X, X') in $V \times V$ and let $\|S\|_{L_2(\Pi^2)}$ be the $L_2(\Pi^2)$-norm of kernel S :

$$\|S\|_{L_2(\Pi^2)}^2 = \int_{V\times V} |S(u,v)|^2 \Pi^2(du, dv) = \mathbb{E}|S(X, X')|^2.$$

The corresponding inner product is denoted by $\langle\cdot,\cdot\rangle_{L_2(\Pi^2)}$. Clearly, under the assumption that the distribution Π is uniform in V, we have $\|S\|_{L_2(\Pi^2)}^2 = m^{-2}\|S\|_2^2$ and $\langle S_1, S_2\rangle_{L_2(\Pi^2)} = m^{-2}\langle S_1, S_2\rangle$.

The smoothness of a symmetric kernel $S : V \times V \mapsto \mathbb{R}$ can be characterized in terms of Sobolev type norms $\|\Delta^{p/2}S\|_2^2$ for some $p > 0$. Note that if S is a kernel of rank r with spectral representation $S = \sum_{k=1}^{r} \mu_k(\psi_k \otimes \psi_k)$, then[1]

$$\|\Delta^{p/2}S\|_2^2 = \mathrm{tr}(\Delta^{p/2}S^2\Delta^{p/2}) = \mathrm{tr}(\Delta^p S^2)$$
$$= \sum_{k=1}^{m} \mu_k^2 \langle \Delta^p \psi_k, \psi_k \rangle = \sum_{k=1}^{m} \mu_k^2 \|\Delta^{p/2}\psi_k\|^2,$$

so, essentially, the smoothness of the kernel S depends on the smoothness of its eigenfunctions ψ_k on the graph. In particular, for $p = 1$, we have

$$\|\Delta^{1/2}S\|_2^2 = \sum_{k=1}^{m} \mu_k^2 \sum_{u \sim v} |\psi_k(u) - \psi_k(v)|^2,$$

where the sum is over the pairs of vertices connected with an edge.

Given a kernel S, let $L_n(S)$ denote the following penalized empirical risk:

$$\begin{aligned} L_n(S) &:= \|S\|_{L_2(\Pi^2)}^2 - \frac{2}{n}\sum_{j=1}^{n} Y_j S(X_j, X_j') + \varepsilon\|S\|_1 + \bar{\varepsilon}\|W^{1/2}S\|_{L_2(\Pi^2)}^2 \\ &= \|S\|_{L_2(\Pi^2)}^2 - \frac{2}{n}\sum_{j=1}^{n} Y_j S(X_j, X_j') + \varepsilon\|S\|_1 + \varepsilon_1\|W^{1/2}S\|_2^2 \end{aligned} \tag{1.1}$$

where $W = d\Delta^p$ for some constants $d > 0$ and $p > 0$, $\varepsilon, \bar{\varepsilon} > 0$ are regularization parameters and $\varepsilon_1 = \frac{\bar{\varepsilon}}{m^2}$. We will study the following estimation method:

$$\hat{S} := \mathrm{argmin}_{S \in \mathbb{D}} L_n(S), \tag{1.2}$$

where $\mathbb{D}$ is a closed convex subset of the linear space $\mathcal{S}_V$ of all symmetric kernels. Note that there are two complexity penalties involved in the definition of penalized empirical risk (1.1). The first penalty is based on the nuclear norm $\|S\|_1$ and it is used to "promote" low rank solutions. The second penalty is based on a "Sobolev type norm" $\|W^{1/2}S\|_2^2$. It is used to "promote" the smoothness of the solution on the graph. In principle, W in the definition of $L_n(S)$ could be an arbitrary symmetric nonnegatively definite matrix. Therefore, alternative interpretations of the problem under consideration are possible (such as, for instance, learning similarities on weighted graphs).

We will derive an upper bound on the error $\|\hat{S} - S_*\|_{L_2(\Pi^2)}^2 = m^{-2}\|\hat{S} - S_*\|_2^2$ of estimator $\hat{S}$ in terms of spectral characteristics of the target similarity matrix S_* and matrix W. Before stating the main results, let us recall recent advances on low rank matrix completion problems in which the approach based on nuclear norm penalization has been crucial.

Suppose first that a symmetric kernel $S_* \in \mathcal{S}_V$ is observed at random points $(X_j, X_j'), j = 1, \ldots, n$, where $X_j, X_j', j = 1, \ldots, n$ are independent and sampled

[1]Below $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^V$; there is a little abuse of notation here since we also denote the operator norm by $\|\cdot\|$.

from the uniform distribution Π in V. In this case, V is an arbitrary finite set of cardinality m and the set of edges E is not specified. It is assumed that $Y_j = S_*(X_j, X_j')$, so, there are no errors in the observations. In such a noiseless case, the following method is used to recover S_* based on the observations $(X_1, X_1', Y_1), \dots, (X_n, X_n', Y_n)$:

$$\check{S} := \mathrm{argmin}\{\|S\|_1 : S \in \mathcal{S}_V, S(X_j, X_j') = Y_j, j = 1, \dots, n\}.$$

Such methods of recovery of low rank target matrices S_* have been extensively studied in the recent literature (see Candes and Recht (2009), Recht, Fazel and Parrilo (2010), Candes and Tao (2010), Gross (2011) and references therein). It is easy to see that there are low rank matrices S_* that can not be recovered based on a random sample of n entries unless n is very large (comparable with the total number of entries of the matrix). Indeed, consider S_* such that, for given $u, v \in V$, $S_*(u, v) = S_*(v, u) = 1$ and $S_*(u', v') = 0$ otherwise. For this rank 2 matrix, the probability that the two "informative" entries are not present in the sample is $(1 - \frac{2}{m^2})^n$, which is close to 1 if $n = o(m^2)$. Such sparse low rank matrices should be excluded to make it possible to recover the target low rank matrix based on relatively small samples of entries. This is done by introducing so-called *low coherence* assumptions. Let $\{e_v : v \in V\}$ be the canonical orthonormal basis of $\mathbb{R}^V$ equipped with the standard Euclidean inner product. Given a linear subspace $L \subset \mathbb{R}^V$, denote by $L^\perp$ the orthogonal complement of L and by P_L the projector onto the subspace L. Let $L := \mathrm{supp}(S_*)$, $r = \mathrm{rank}(S_*)$ and suppose there exists a constant $\nu \geq 1$ (*coherence coefficient*) such that

$$\|P_L e_v\|^2 \leq \frac{\nu r}{m},\ v \in V \ \text{ and } \ |\langle \mathrm{sign}(S_*) e_u, e_v\rangle|^2 \leq \frac{\nu r}{m^2}, u, v \in V. \tag{1.3}$$

The following result is due to Candes and Tao (2010) and Gross (2011) (we state here a version of Gross that is an improvement of an earlier result of Candes and Tao with significant simplification of the proof).

Theorem 1. *Suppose conditions* (1.3) *hold for some* $\nu \geq 1$. *Then, there exists a numerical constant* $C > 0$ *such that, for all* $n \geq C\nu r m \log^2 m$, $\check{S} = S_*$ *with probability at least* $1 - m^{-2}$.

Thus, if, for the target matrix S_*, the coherence coefficient $\nu \geq 1$ is relatively small, the nuclear norm minimization algorithm (1.2) does provide the exact recovery of S_* as soon as the number of observed entries n is of the order mr (up to a log factor).

In the case when Y_j are noisy observations of $S_*(X_j, X_j')$ with

$$\mathbb{E}(Y_j | X_j = u, X_j' = v) = S_*(u, v),$$

one can use the following estimation method based on penalized empirical risk minimization with quadratic loss and with nuclear norm penalty:

$$\check{S} := \mathrm{argmin}_{S \in \mathcal{S}_V} \left[n^{-1} \sum_{j=1}^n (Y_j - S(X_j, X_j'))^2 + \varepsilon \|S\|_1 \right]. \tag{1.4}$$

This method has been also extensively studied for the recent years, in particular, by Candes and Plan (2011), Rohde and Tsybakov (2011), Negahban and Wainwright (2010), Koltchinskii, Lounici and Tsybakov (2011), Koltchinskii (2011b). It was also pointed out by Koltchinskii, Lounici and Tsybakov (2011) that in the case of known design distribution Π (which is the case in our paper) one can use instead of (1.4) the following modified method:[2]

$$\check{S} := \mathrm{argmin}_{S \in \mathcal{S}_V} \Big[\|S\|^2_{L_2(\Pi^2)} - \frac{2}{n} \sum_{j=1}^n Y_j S(X_j, X_j') + \varepsilon \|S\|_1 \Big]. \tag{1.5}$$

Clearly, (1.5) is equivalent to method (1.2) defined above for $\bar{\varepsilon} = 0$.

When the observations $|Y_j| \le 1, j = 1, \dots, n$ (for instance, when $Y_j \in \{-1, 1\}$, which is the case studied in the paper), the next result follows from Theorem 4 in Koltchinskii, Lounici and Tsybakov (2011).

Theorem 2. *For* $t > 0$, *suppose that*

$$\varepsilon \ge 4 \bigg(\sqrt{\frac{t + \log(2m)}{nm}} \bigvee \frac{2(t + \log(2m))}{n} \bigg).$$

Then with probability at least $1 - e^{-t}$

$$\|\check{S} - S_*\|^2_{L_2(\Pi)} \le \Big(\frac{1 + \sqrt{2}}{2} \Big)^2 m^2 \varepsilon^2 \mathrm{rank}(S_*).$$

Our main goal is to show that this bound can be improved in the case when the target kernel S_*, in addition to having relatively small rank, is also smooth on the graph and when the estimation method (1.2) is used with a proper choice of regularization parameters $\varepsilon, \bar{\varepsilon}$.

2. Main results

Suppose that W has the following spectral representation: $W = \sum_{k=1}^m \lambda_k (\phi_k \otimes \phi_k)$, where $0 \le \lambda_1 \le \dots \le \lambda_m$ are the eigenvalues of W (repeated with their multiplicities) and $\phi_1, \dots, \phi_m$ are the corresponding orthonormal eigenfunctions (of course, there is a multiple choice of ϕ_k in the case of repeated eigenvalues). Let k_0 be the smallest k such that $\lambda_k > 0$. We will assume that for some (arbitrarily large) $\zeta \ge 1$, $\lambda_m \le m^{\zeta}$ and $\lambda_{k_0} \ge m^{-\zeta}$. In addition, it is assumed that, for some constant $c > 1$ and for all $k = k_0, \dots, m-1$, $\lambda_{k+1} \le c\lambda_k$. The following spectral function characterizes the distribution of the eigenvalues:

$$F(\lambda) := \sum_{j=1}^m I(\lambda_j \le \lambda), \lambda \ge 0.$$

[2]Note that, if the norm $\|S\|_{L_2(\Pi^2)}$ in the definition below is replaced by the $L_2(\Pi_n)$-norm, where Π_n is the empirical distribution based on $(X_1, X_1'), \dots, (X_n, X_n')$, then the resulting estimator coincides with (1.4).

We will also use an upper bound $\bar{F}(\lambda) \geq F(\lambda), \lambda \geq 0$ that possesses some "regularity" in the sense that $\frac{\bar{F}(\lambda)}{\lambda}, \lambda \geq 0$ is a nonincreasing function and, for some $\gamma \in (0,1)$,

$$\int_\lambda^\infty \frac{\bar{F}(t)}{t^2} dt \leq \frac{1}{\gamma}\frac{\bar{F}(\lambda)}{\lambda}, \lambda > 0.$$

It is easy to see that the last two conditions are satisfied if $\frac{\bar{F}(\lambda)}{\lambda^{1-\gamma}}, \lambda \geq 0$ is a nonincreasing function and that the smallest upper bound on F with this property is

$$\bar{F}(\lambda) = \sup_{s \leq \lambda} s^{1-\gamma} \sup_{t \geq s} \frac{F(t)}{t^{1-\gamma}}, \lambda \geq 0.$$

We also can assume that, for all $\lambda \geq m$, $\bar{F}(\lambda) = m$ (otherwise, $\bar{F}$ can be replaced by the function $\bar{F} \wedge m$).

Suppose now that the spectral representation of S_* is $S_* = \sum_{k=1}^r \mu_k(\psi_k \otimes \psi_k)$, where $r = \mathrm{rank}(S_*) \geq 1$, μ_k are non-zero eigenvalues of S_* (possibly repeated) and ψ_k are the corresponding orthonormal eigenfuctions. Denote $L := \mathrm{supp}(S_*)$. Let φ be an arbitrary nondecreasing function such that $k \mapsto \frac{\varphi(k)}{\bar{F}(\lambda_k)}$ is nonincreasing and

$$\sum_{j=1}^k \|P_L \phi_j\|^2 \leq \varphi(k), \quad k = 0, 1, \ldots, m.$$

It will be convenient to set $\varphi(k) = \varphi(m)$ for all $k \geq m$. We will denote by $\Psi = \Psi_{S_*,W}$ the class of all the functions satisfying these properties.

The following *coherence function* will be crucial in our analysis:

$$\bar{\varphi}(k) := \bar{\varphi}(S_*, k) := \max_{l \leq k} \bar{F}(\lambda_l) \max_{j \geq l} \frac{1}{\bar{F}(\lambda_j)} \sum_{i=1}^j \|P_L \phi_i\|^2,$$
$$k = 1, \ldots, m, \quad \bar{\varphi}(0) = 0.$$

It is straightforward to check that $\bar{\varphi} \in \Psi$ and, for all $\varphi \in \Psi$, $\bar{\varphi}(k) \leq \varphi(k)$, $k = 0, \ldots, m$. Thus, $\bar{\varphi}$ is the smallest function $\varphi \in \Psi$.

Also, $\bar{\varphi}(m) = r$ since $\sum_{j=1}^m \|P_L \phi_j\|^2 = \|P_L\|_2^2 = r$. Moreover, since $\frac{\bar{\varphi}(k)}{\bar{F}(\lambda_k)}$ is nonincreasing, we have

$$\bar{\varphi}(k) \geq \frac{r\bar{F}(\lambda_k)}{m}, \quad k = 0, \ldots, m.$$

Given $t > 0$, let $t_{n,m} := t + \log(2m \log_2(16 n^\zeta m^{(3/2)\zeta}))$. We will assume in what follows that $m t_{n,m} \leq n$. If $t \asymp \log m$, which is a typical choice of t, this assumption means that n should be larger than m times a log factor. The following value of regularization parameter ε in (1.1) will be used:

$$\varepsilon := 4\sqrt{\frac{t + \log(2m)}{nm}}.$$

Theorem 3. *There exists constants C, C_1 depending only on c such that, for all $s \in \{k_0+1, \dots, m+1\}$ and all $\bar{\varepsilon} \in [\lambda_s^{-1}, \lambda_{s-1}^{-1}]$,*[3] *with probability at least $1 - e^{-t}$,*

$$\|\hat{S} - S_*\|^2_{L_2(\Pi^2)} \le C\frac{\bar{\varphi}(S_*; s)mt_{n,m}}{n} + \bar{\varepsilon}\|W^{1/2}S_*\|^2_{L_2(\Pi^2)} + C_1 \max_{v\in V}\|P_L e_v\|^2\Big(\frac{mt_{n,m}}{n}\Big)^2. \tag{2.1}$$

Remarks. Note that $\max_{v\in V}\|P_L e_v\|^2 \le 1$. Thus, the last term in the right-hand side of bound (2.1) is smaller than the first term, provided that

$$\bar{\varphi}(S_*; s) \ge \frac{mt_{n,m}}{n}.$$

Moreover, this term is much smaller under a low coherence condition

$$\max_{v\in V}\|P_L e_v\|^2 \le \frac{\nu r}{m}$$

for some $\nu \ge 1$ (see conditions (1.3)). In this case,

$$\max_{v\in V}\|P_L e_v\|^2\Big(\frac{mt_{n,m}}{n}\Big)^2 \le \frac{\nu r m t_{n,m}^2}{n^2} \le \frac{\nu r t_{n,m}}{n}.$$

Note also that Theorem 3 holds in the case when $\bar{\varepsilon} = 0$. In this case, $s = m$ and $\bar{\varphi}(S_*, m) = r$, so the bound of Theorem 3 becomes

$$\|\hat{S} - S_*\|^2_{L_2(\Pi^2)} \le C\frac{rmt_{n,m}}{n}, \tag{2.2}$$

which also follows from the result of Koltchinskii, Lounici and Tsybakov (2011) (see Theorem 2 in Section 1).

The function $\bar{\varphi}$ involved in the statement of the theorem has some connection to the low coherence assumptions frequently used in the literature on low rank matrix completion. To be specific, suppose that, for some $\nu \ge 1$,

$$\sum_{j=1}^{k}\|P_L\phi_j\|^2 \le \frac{\nu r\bar{F}(\lambda_k)}{m}, k = 1, \dots, m. \tag{2.3}$$

Then

$$\bar{\varphi}(k) \le \frac{\nu r\bar{F}(\lambda_k)}{m}, k = 1, \dots, m.$$

A part of standard low coherence assumptions on matrix S_* with respect to the orthonormal basis $\{\phi_k\}$ is (see (1.3))

$$\|P_L\phi_k\|^2 \le \frac{\nu r}{m}, k = 1, \dots, m$$

and it implies condition (2.3) that can be viewed as a weak version of low coherence. Under condition (2.3), the following corollary of Theorem 3 holds.

[3] Here and in what follows, we use a convention that $\lambda_{m+1} = +\infty$ and $\lambda_{m+1}^{-1} = 0$.

Corollary 1. *Suppose that condition* (2.3) *holds. Then, there exists a constant* $C > 0$ *depending only on* ζ *such that, for all* $s \in \{k_0 + 1, \dots, m + 1\}$ *and all* $\bar{\varepsilon} \in (\lambda_s^{-1}, \lambda_{s-1}^{-1}]$, *with probability at least* $1 - e^{-t}$,

$$\|\hat{S} - S_*\|^2_{L_2(\Pi^2)} \le C \frac{\nu r \bar{F}(\lambda_s) t_{n,m}}{n} + \bar{\varepsilon} \|W^{1/2} S_*\|^2_{L_2(\Pi^2)} + C_1 \max_{v \in V} \|P_L e_v\|^2 \Big(\frac{m t_{n,m}}{n} \Big)^2.$$

Note that, if $\lambda_k \asymp k^{2\beta}$ for some $\beta > 1/2$, then it is easy to see that one can choose $\bar{F}(\lambda) \asymp \lambda^{1/2\beta}$ and, with this choice, $\bar{F}(\lambda_s) \asymp s$. Thus, the value of s that minimizes the bound of Corollary 1 is

$$s \asymp \left(\frac{n}{\nu r t_{n,m}} \right)^{1/(2\beta+1)} \|W^{1/2} S_*\|^{2/(2\beta+1)}_{L_2(\Pi)},$$

which, under a low coherence assumption $\max_{v \in V} \|P_L e_v\|^2 \le \frac{\nu r}{m}$, yields the bound

$$\|\hat{S} - S_*\|^2_{L_2(\Pi^2)} \le C \left(\frac{\nu r t_{n,m}}{n} \right)^{2\beta/(2\beta+1)} \|W^{1/2} S_*\|^{2/(2\beta+1)}_{L_2(\Pi)}. \tag{2.4}$$

As a simple example, one can consider a "cycle" with m vertices, that is, a graph with vertex set $V = \mathbb{Z}_m = \{0, 1, \dots, m-1\}$, $u, v \in V$ being connected with an edge iff $u - v \equiv 1 \pmod{m}$ or $v - u \equiv 1 \pmod{m}$. In this case, the spectrum of the Laplacian Δ consists of the following eigenvalues (repeated with their multiplicities): $4 \sin^2 \frac{\pi k}{m}, k = 0, \dots, m-1$. Let $W := m^{2p} \Delta^p$ for some $p > 1/2$. Then, it is easy to check that $\bar{F}(\lambda) \asymp \lambda^{1/2p}$, so, bound (2.4) holds with $\beta = p$.

The advantage of (2.4) comparing with (2.2) (that holds for $\bar{\varepsilon} = 0$ and does not rely on any smoothness assumption on the kernel S_*) is due to the fact that there is no factor m in the numerator in the right-hand side of (2.4). Due to this fact, when m is large enough and ν is not too large, bound (2.4) becomes sharper than (2.2).

3. Proofs

Proof of Theorem 3. Bound (2.1) will be proved for an arbitrary function $\varphi \in \Psi_{S_*, W}$ with $\varphi(k) = r, k \ge m$ instead of $\bar{\varphi}$. It then can be applied to the function $\bar{\varphi}$ (which is the smallest function in $\Psi_{S_*, W}$). We will also assume throughout the proof that $s \in \{k_0, \dots, m\}$ and $\bar{\varepsilon} \in [\lambda_{s+1}^{-1}, \lambda_s^{-1}]$ (at the end of the proof, we replace $s + 1 \mapsto s$).

Denote $\mathcal{P}_L(A) := A - P_{L^\perp} A P_{L^\perp}$, $\mathcal{P}_L^\perp(A) = P_{L^\perp} A P_{L^\perp}, A \in \mathcal{S}_V$. Clearly, this defines orthogonal projectors $\mathcal{P}_L, \mathcal{P}_L^\perp$ in the space $\mathcal{S}_V$ with Hilbert–Schmidt inner product. We will use the following well-known representation of subdifferential of convex function $S \mapsto \|S\|_1$:

$$\partial \|S\|_1 = \left\{ \mathrm{sign}(S) + \mathcal{P}_L^\perp(M) : M \in \mathcal{S}_V, \|M\| \le 1 \right\},$$

where $L = \mathrm{supp}(S)$ (see Koltchinskii (2011b), Appendix A.4 and references therein). An arbitrary matrix $A \in \partial L_n(\hat{S})$ can be represented as follows:

$$A = \frac{2}{m^2}\hat{S} - \frac{2}{n}\sum_{i=1}^{n} Y_i E_{X_i,X_i'} + \varepsilon\hat{V} + 2\varepsilon_1 W\hat{S}, \tag{3.1}$$

where $\hat{V} \in \partial\|\hat{S}\|_1$ and $E_{u,v} = E_{v,u} = \frac{1}{2}(e_u \otimes e_v + e_v \otimes e_u)$. Since $\hat{S}$ is a minimizer of $L_n(S)$, there exists a matrix $A \in \partial L_n(\hat{S})$ such that $-A$ belongs to the normal cone of $\mathbb{D}$ at the point $\hat{S}$ (see Aubin and Ekeland (1984), Chap. 2, Corollary 6). This implies that $\langle A, \hat{S} - S_*\rangle \le 0$ and, in view of (3.1),

$$2\langle \hat{S}, \hat{S} - S_*\rangle_{L_2(\Pi^2)} - \left\langle \frac{2}{n}\sum_{i=1}^{n} Y_i E_{X_i,X_i'}, \hat{S} - S_* \right\rangle + \varepsilon\langle \hat{V}, \hat{S} - S_*\rangle + 2\varepsilon_1\langle W\hat{S}, \hat{S} - S_*\rangle \le 0$$

It follows by a simple algebra that

$$\begin{aligned} 2\|\hat{S} - S_*\|^2_{L_2(\Pi^2)} + 2\varepsilon_1\|W^{1/2}(\hat{S} - S_*)\|_2^2 + \varepsilon\langle \hat{V}, \hat{S} - S_*\rangle \\ \le -2\varepsilon_1\langle S_*, W(\hat{S} - S_*)\rangle + 2\langle \Xi, \hat{S} - S_*\rangle, \end{aligned} \tag{3.2}$$

where

$$\Xi := \frac{1}{n}\sum_{j=1}^{n} Y_j E_{X_j,X_j'} - \mathbb{E}YE_{X,X'}.$$

Note that $\langle \Xi, S\rangle = \frac{1}{n}\sum_{j=1}^{n}\big(Y_j S(X_j, X_j') - \mathbb{E}YS(X, X')\big)$.

On the other hand, let $V_* \in \partial\|S_*\|_1$. Therefore, the representation $V_* = \mathrm{sign}(S_*) + \mathcal{P}_L^{\perp}(M)$ holds, where M is a matrix with $\|M\| \le 1$. It follows from the trace duality property that there exists an M with $\|M\| \le 1$ such that

$$\langle \mathcal{P}_L^{\perp}(M), \hat{S} - S_*\rangle = \langle M, \mathcal{P}_L^{\perp}(\hat{S} - S_*)\rangle = \langle M, \mathcal{P}_L^{\perp}(\hat{S})\rangle = \|\mathcal{P}_L^{\perp}(\hat{S})\|_1$$

where in the first equality we used that $\mathcal{P}_L^{\perp}$ is a self-adjoint operator and in the second equality we used that S_* has support L. Using this equation and monotonicity of subdifferentials of convex functions, we get

$$\langle \mathrm{sign}(S_*), \hat{S} - S_*\rangle + \|\mathcal{P}_L^{\perp}(\hat{S})\|_1 = \langle V_*, \hat{S} - S\rangle \le \langle \hat{V}, \hat{S} - S_*\rangle$$

Substituting this in (3.2), it is easy to get

$$\begin{aligned} 2\|\hat{S} - S_*\|^2_{L_2(\Pi^2)} + \varepsilon\|\mathcal{P}_L^{\perp}(\hat{S})\|_1 + 2\varepsilon_1\|W^{1/2}(\hat{S} - S_*)\|_2^2 \\ \le -\varepsilon\langle \mathrm{sign}(S_*), \hat{S} - S_*\rangle - 2\varepsilon_1\langle W^{1/2}S_*, W^{1/2}(\hat{S} - S_*)\rangle + 2\langle \Xi, \hat{S} - S_*\rangle \end{aligned} \tag{3.3}$$

We will bound separately each term in the right-hand side. First note that

$$\begin{aligned} \varepsilon|\langle \mathrm{sign}(S_*), \hat{S} - S_*\rangle| &\le \varepsilon\|\mathrm{sign}(S_*)\|_2\|\hat{S} - S_*\|_2 \\ &= \varepsilon\sqrt{r}m\|\hat{S} - S_*\|_{L_2(\Pi^2)} \le \frac{1}{2}rm^2\varepsilon^2 + \frac{1}{2}\|\hat{S} - S_*\|^2_{L_2(\Pi^2)}. \end{aligned} \tag{3.4}$$

We will also need a more subtle bound on $\langle \mathrm{sign}(S_*), \hat{S} - S_* \rangle$, expressed in terms of function φ. Note that, for all $k_0 \le s \le m$,

$$\begin{aligned}\langle \mathrm{sign}(S_*), \hat{S} - S_* \rangle &= \sum_{k=1}^{m} \langle \mathrm{sign}(S_*)\phi_k, (\hat{S} - S_*)\phi_k \rangle \\ &= \sum_{k=1}^{s} \langle \mathrm{sign}(S_*)\phi_k, (\hat{S} - S_*)\phi_k \rangle \\ &\quad + \sum_{k=s+1}^{m} \left\langle \frac{\mathrm{sign}(S_*)\phi_k}{\sqrt{\lambda_k}}, \sqrt{\lambda_k}(\hat{S} - S_*)\phi_k \right\rangle,\end{aligned}$$

which easily implies

$$\begin{aligned}|\langle \mathrm{sign}(S_*), \hat{S} - S_* \rangle| &\le \left(\sum_{k=1}^{s} \|\mathrm{sign}(S_*)\phi_k\|^2\right)^{1/2} \left(\sum_{k=1}^{s} \|(\hat{S} - S_*)\phi_k\|^2\right)^{1/2} \\ &\quad + \left(\sum_{k=s+1}^{m} \frac{\|\mathrm{sign}(S_*)\phi_k\|^2}{\lambda_k}\right)^{1/2} \left(\sum_{k=s+1}^{m} \lambda_k \|(\hat{S} - S_*)\phi_k\|^2\right)^{1/2} \\ &\le \left(\sum_{k=1}^{s} \|P_L\phi_k\|^2\right)^{1/2} \|\hat{S} - S_*\|_2 \\ &\quad + \left(\sum_{k=s+1}^{m} \frac{\|P_L\phi_k\|^2}{\lambda_k}\right)^{1/2} \|W^{1/2}(\hat{S} - S_*)\|_2. \qquad (3.5)\end{aligned}$$

We will now use the following elementary lemma.

Lemma 1. *Let c, γ be the constants involved in the conditions on the spectrum of W and in the definition of $\bar{F}$. For all $s \ge k_0 - 1$,*

$$\sum_{k=s+1}^{m} \frac{\|P_L\phi_k\|^2}{\lambda_k} \le c_\gamma \frac{\varphi(s+1)}{\lambda_{s+1}} \quad \textit{and} \quad \sum_{k=s+1}^{m} \frac{1}{\lambda_k} \le c_\gamma \frac{\bar{F}(\lambda_{s+1})}{\lambda_{s+1}},$$

where $c_\gamma := \frac{c}{\gamma} + 1$.

Proof. Denote $F_s := \sum_{k=1}^{s} \|P_L\phi_k\|^2, s = 1, \dots, m$. Then, using the properties of functions $\varphi \in \Psi$ and $\bar{F}$, and of the spectrum of W, we get

$$\begin{aligned}\sum_{k=s+1}^{m} \frac{\|P_L\phi_k\|^2}{\lambda_k} &= \sum_{k=s+1}^{m-1} F_k \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right) + \frac{F_m}{\lambda_m} - \frac{F_s}{\lambda_{s+1}} \\ &\le \sum_{k=s+1}^{m-1} \varphi(k) \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right) + \frac{\varphi(m)}{\lambda_m} \\ &\le \frac{\varphi(s+1)}{\bar{F}(\lambda_{s+1})} \left[\sum_{k=s+1}^{m-1} \frac{\bar{F}(\lambda_k)}{\lambda_k \lambda_{k+1}} (\lambda_{k+1} - \lambda_k) + \frac{\bar{F}(\lambda_m)}{\lambda_m}\right]\end{aligned}$$

$$\le c\frac{\varphi(s+1)}{\bar F(\lambda_{s+1})}\sum_{k=s+1}^{m-1}\frac{\bar F(\lambda_{k+1})}{\lambda_{k+1}^2}(\lambda_{k+1}-\lambda_k)+\frac{\varphi(s+1)}{\bar F(\lambda_{s+1})}\frac{\bar F(\lambda_{s+1})}{\lambda_{s+1}}$$
$$\le c\frac{\varphi(s+1)}{\bar F(\lambda_{s+1})}\int_{\lambda_{s+1}}^{\infty}\frac{\bar F(t)}{t^2}dt+\frac{\varphi(s+1)}{\lambda_{s+1}}$$
$$\le \frac{c}{\gamma}\frac{\varphi(s+1)}{\bar F(\lambda_{s+1})}\frac{\bar F(\lambda_{s+1})}{\lambda_{s+1}}+\frac{\varphi(s+1)}{\lambda_{s+1}}=c_\gamma\frac{\varphi(s+1)}{\lambda_{s+1}}. \tag{3.6}$$

The proof of the second bound is similar (with some simplifications). □

It follows from (3.5) and the bound of Lemma 1 that

$$\begin{aligned}|\langle \mathrm{sign}(S_*),\hat S-S_*\rangle| &\le \sqrt{\varphi(s)}\|\hat S-S_*\|_2+\sqrt{c_\gamma\frac{\varphi(s+1)}{\lambda_{s+1}}}\|W^{1/2}(\hat S-S_*)\|_2\\ &= m\sqrt{\varphi(s)}\|\hat S-S_*\|_{L_2(\Pi^2)}+m\sqrt{c_\gamma\frac{\varphi(s+1)}{\lambda_{s+1}}}\|W^{1/2}(\hat S-S_*)\|_{L_2(\Pi^2)}.\end{aligned} \tag{3.7}$$

This implies the following bound:

$$\begin{aligned}\varepsilon|\langle \mathrm{sign}(S_*),\hat S-S_*\rangle| &\le \varphi(s)m^2\varepsilon^2+\frac14\|\hat S-S_*\|^2_{L_2(\Pi^2)}\\ &\quad +c_\gamma\frac{\varphi(s+1)}{\lambda_{s+1}}\frac{m^2\varepsilon^2}{\bar\varepsilon}+\frac{\bar\varepsilon}{4}\|W^{1/2}(\hat S-S_*)\|^2_{L_2(\Pi^2)},\end{aligned} \tag{3.8}$$

where we used twice an elementary inequality $ab\le a^2+\frac14 b^2, a,b>0$. Since, under the assumptions of the theorem, $\bar\varepsilon\lambda_{s+1}\ge 1$, (3.8) yields the following bound:

$$\begin{aligned}&\varepsilon|\langle \mathrm{sign}(S_*),\hat S-S_*\rangle|\\ &\le (c_\gamma+1)\varphi(s+1)m^2\varepsilon^2+\frac14\|\hat S-S_*\|^2_{L_2(\Pi^2)}+\frac{\bar\varepsilon}{4}\|W^{1/2}(\hat S-S_*)\|^2_{L_2(\Pi^2)}.\end{aligned} \tag{3.9}$$

To bound the second term in the right-hand side of (3.3), note that

$$|\langle W^{1/2}S_*,W^{1/2}(\hat S-S_*)\rangle|\le \|W^{1/2}S_*\|_2\|W^{1/2}(\hat S-S_*)\|_2, \tag{3.10}$$

which implies

$$\begin{aligned}\varepsilon_1|\langle W^{1/2}S_*,W^{1/2}(\hat S-S_*)\rangle| &\le \varepsilon_1\|W^{1/2}S_*\|_2^2+\frac{\varepsilon_1}{4}\|W^{1/2}(\hat S-S_*)\|_2^2 \\ &= \bar\varepsilon\|W^{1/2}S_*\|^2_{L_2(\Pi^2)}+\frac{\bar\varepsilon}{4}\|W^{1/2}(\hat S-S_*)\|^2_{L_2(\Pi^2)}.\end{aligned} \tag{3.11}$$

Finally, we bound $\langle \Xi,\hat S-S_*\rangle$:

$$\begin{aligned}|\langle \Xi,\hat S-S_*\rangle| &\le |\langle \Xi,\mathcal P_L(\hat S-S_*)\rangle|+|\langle \Xi,\mathcal P_L^\perp(\hat S)\rangle|\\ &\le |\langle \mathcal P_L\Xi,\hat S-S_*\rangle|+\|\Xi\|\|\mathcal P_L^\perp(\hat S)\|_1.\end{aligned} \tag{3.12}$$

To bound $\|\Xi\|$, we use a version of the noncommutative Bernstein inequality. Such inequalities go back to Ahlswede and Winter (2002). The version stated below follows from Tropp (2010) (see also Tropp (2010), Koltchinskii (2011a, 2011b, 2011c) for other versions of such inequalities).

Lemma 2. *Let Z be a bounded random symmetric matrix with $\mathbb{E}Z = 0$, $\sigma_Z^2 := \|\mathbb{E}Z^2\|$ and $\|Z\| \le U$ for some $U > 0$. Let $Z_1, \dots, Z_n$ be n i.i.d. copies of Z. Then for all $t > 0$, with probability at least $1 - e^{-t}$*

$$\left\| \frac{1}{n} \sum_{i=1}^n Z_i \right\| \le 2 \left(\sigma_Z \sqrt{\frac{t + \log(2m)}{n}} \bigvee U \frac{t + \log(2m)}{n} \right)$$

It is applied to i.i.d. random matrices $Z_i := Y_i E_{X_i, X_i'} - \mathbb{E}(Y_i E_{X_i X_i'}), i = 1, \dots, n$. Since $\|Z_i\| \le 2$ and, by a simple computation, $\sigma_{Z_i}^2 := \|\mathbb{E}Z_i^2\| \le 1/m$ (see, e.g., Koltchinskii (2011b), Section 9.4), Lemma 2 implies that with probability at least $1 - e^{-t}$

$$\|\Xi\| = \left\| \frac{1}{n} \sum_{i=1}^n Z_i \right\| \le 2 \left[\sqrt{\frac{t + \log(2m)}{nm}} \bigvee \frac{2(t + \log(2m))}{n} \right].$$

Under the assumption that

$$\varepsilon \ge 4 \left[\sqrt{\frac{t + \log(2m)}{nm}} \bigvee \frac{2(t + \log(2m))}{n} \right],$$

this yields $\|\Xi\| \le \varepsilon/2$ and

$$|\langle \Xi, \hat{S} - S_* \rangle| \le |\langle \mathcal{P}_L \Xi, \hat{S} - S_* \rangle| + \frac{\varepsilon}{2} \|\mathcal{P}_L^\perp(\hat{S})\|_1. \tag{3.13}$$

For simplicity, it is assumed that $n \ge 2m(t + \log(2m))$. In this case, one can take $\varepsilon = 4\sqrt{\frac{t+\log(2m)}{nm}}$, as it has been done in the statement of the theorem.

We have to bound $|\langle \mathcal{P}_L \Xi, \hat{S} - S_* \rangle|$ and we start with the following simple bound:

$$\begin{aligned} |\langle \mathcal{P}_L \Xi, \hat{S} - S_* \rangle| &\le m \|\mathcal{P}_L \Xi\|_2 \|\hat{S} - S_*\|_{L_2(\Pi^2)} \\ &\le m\sqrt{2r} \|\Xi\| \|\hat{S} - S_*\|_{L^2(\Pi^2)} \\ &\le \frac{1}{2} m \varepsilon \sqrt{2r} \|\hat{S} - S_*\|_{L^2(\Pi^2)} \\ &\le \frac{1}{2} m^2 \varepsilon^2 r + \frac{1}{4} \|\hat{S} - S_*\|_{L_2(\Pi^2)}^2, \end{aligned} \tag{3.14}$$

where we use the fact that $\text{rank}(\mathcal{P}_L \Xi) \le 2r$. Substituting (3.4), (3.11), (3.13) and (3.14) in (3.3), we easily get that

$$\|\hat{S} - S_*\|_{L_2(\Pi^2)}^2 \le \frac{3}{2} r \varepsilon^2 m^2 + 2\bar{\varepsilon} \|W^{1/2} S_*\|_{L_2(\Pi^2)}^2. \tag{3.15}$$

For $\bar{\varepsilon} = 0$, this bound follows from the results of Koltchinskii, Lounici and Tsybakov (2011). However, we need a more subtle bound expressed in terms of function φ, which is akin to bound (3.9). To this end, we will use the following lemma.

Lemma 3. *For $\delta > 0$, let $k(\delta) := F(\delta^{-2})$ (that is, $k(\delta)$ is the largest value of $k \leq m$ such that $\lambda_k^{-1} \geq \delta^2$). For all $t > 0$, with probability at least $1 - e^{-t}$,*

$$\sup_{\|M\|_2 \leq \delta, \|W^{1/2}M\|_2 \leq 1} |\langle \mathcal{P}_L \Xi, M \rangle| \leq 2\sqrt{4(c_\gamma + 1)}\sqrt{\frac{t}{nm}}\delta\sqrt{\varphi(k(\delta)+1)} + 2\sqrt{2}\delta \max_{v \in V} \|P_L e_v\| \frac{t}{n},$$

provided that $k(\delta) < m$, and

$$|\langle \mathcal{P}_L \Xi, M \rangle| \leq 4\sqrt{2}\delta\sqrt{\frac{rt}{nm}} + 2\sqrt{2}\delta \max_{v \in V} \|P_L e_v\| \frac{t}{n},$$

provided that $k(\delta) \geq m$.

Proof. The proof is somewhat akin to the derivation of the bounds on Rademacher processes in terms of Mendelson's complexities used in learning theory (see, e.g., Proposition 3.3 in Koltchinskii (2011b)).

Note that, for all symmetric $m \times m$ matrices M,

$$\langle \mathcal{P}_L \Xi, M \rangle = \sum_{k,j=1}^{m} \langle \mathcal{P}_L \Xi, \phi_k \otimes \phi_j \rangle \langle M, \phi_k \otimes \phi_j \rangle.$$

Suppose that

$$\|M\|_2^2 = \sum_{k,j=1}^{m} |\langle M, \phi_k \otimes \phi_j \rangle|^2 \leq \delta^2$$

and

$$\|W^{1/2}M\|_2^2 = \sum_{k,j=1}^{m} \lambda_k |\langle M, \phi_k \otimes \phi_j \rangle|^2 \leq 1.$$

Then, it easily follows that

$$\sum_{k,j=1}^{m} \frac{|\langle M, \phi_k \otimes \phi_j \rangle|^2}{\lambda_k^{-1} \wedge \delta^2} \leq 2,$$

which implies

$$\begin{aligned} |\langle \mathcal{P}_L \Xi, M \rangle|^2 &\leq \sum_{k,j=1}^{m} (\lambda_k^{-1} \wedge \delta^2) |\langle \mathcal{P}_L \Xi, \phi_k \otimes \phi_j \rangle|^2 \sum_{k,j=1}^{m} \frac{|\langle M, \phi_k \otimes \phi_j \rangle|^2}{\lambda_k^{-1} \wedge \delta^2} \\ &\leq 2 \sum_{k,j=1}^{m} (\lambda_k^{-1} \wedge \delta^2) |\langle \mathcal{P}_L \Xi, \phi_k \otimes \phi_j \rangle|^2 \qquad (3.16) \end{aligned}$$

Define now the following inner product:

$$\langle M_1, M_2 \rangle_w := \sum_{k,j=1}^{m} (\lambda_k^{-1} \wedge \delta^2) \langle M_1, \phi_k \otimes \phi_j \rangle \langle M_2, \phi_k \otimes \phi_j \rangle$$

and let $\|\cdot\|_w$ be the corresponding norm. We will provide an upper bound on

$$\|\mathcal{P}_L\Xi\|_w = \left(\sum_{k,j=1}^{m} (\lambda_k^{-1} \wedge \delta^2)|\langle \mathcal{P}_L\Xi, \phi_k \otimes \phi_j\rangle|^2\right)^{1/2}.$$

To this end, we use a standard Bernstein type inequality for random variables in a Hilbert space. It is given in the following lemma (which follows, for instance, from Theorem 3.3.4(b) in Yurinsky (1995)).

Lemma 4. *Let ξ be a bounded random variable with values in a Hilbert space $\mathcal{H}$. Suppose that $\mathbb{E}\xi = 0$, $\mathbb{E}\|\xi\|_{\mathcal{H}}^2 = \sigma^2$ and $\|\xi\|_{\mathcal{H}} \le U$. Let $\xi_1, \dots, \xi_n$ be n i.i.d. copies of ξ_i. Then for all $t > 0$, with probability at least $1 - e^{-t}$*

$$\left\|\frac{1}{n}\sum_{i=1}^{n} \xi_i\right\|_{\mathcal{H}} \le 2\left[\sigma\sqrt{\frac{t}{n}} \bigvee U\frac{t}{n}\right]$$

Applying Lemma 4 to the random variable $\xi = Y\mathcal{P}_L(E_{X,X'}) - \mathbb{E}Y\mathcal{P}_L(E_{X,X'})$, we get that for all $t > 0$, with probability at least $1 - e^{-t}$,

$$\begin{aligned}\|\mathcal{P}_L\Xi\|_w &= \left\|\frac{1}{n}\sum_{j=1}^{n} Y_j\mathcal{P}_L(E_{X_j,X_j'}) - \mathbb{E}Y\mathcal{P}_L(E_{X,X'})\right\|_w \\ &\le 2\left[\mathbb{E}^{1/2}\|Y\mathcal{P}_L(E_{X,X'})\|_w^2\sqrt{\frac{t}{n}} + \Big\|\|Y\mathcal{P}_L(E_{X,X'})\|_w\Big\|_{L_\infty}\frac{t}{n}\right].\end{aligned} \tag{3.17}$$

Using the fact that $Y \in \{-1, 1\}$, we get

$$\begin{aligned}\mathbb{E}\|Y\mathcal{P}_L(E_{X,X'})\|_w^2 &= \mathbb{E}\|\mathcal{P}_L(E_{X,X'})\|_w^2 \\ &= \mathbb{E}\sum_{k,j=1}^{m} (\lambda_k^{-1} \wedge \delta^2)|\langle \mathcal{P}_L(E_{X,X'}), \phi_k \otimes \phi_j\rangle|^2 \\ &= \sum_{k,j=1}^{m} (\lambda_k^{-1} \wedge \delta^2)\mathbb{E}|\langle E_{X,X'}, \mathcal{P}_L(\phi_k \otimes \phi_j)\rangle|^2 \\ &= \sum_{k,j=1}^{m} (\lambda_k^{-1} \wedge \delta^2)m^{-2}\sum_{u,v\in V} |\langle E_{u,v}, \mathcal{P}_L(\phi_k \otimes \phi_j)\rangle|^2 \\ &\le m^{-2}\sum_{k,j=1}^{m} (\lambda_k^{-1} \wedge \delta^2)\|\mathcal{P}_L(\phi_k \otimes \phi_j)\|_2^2 \\ &\le 2m^{-2}\sum_{k,j=1}^{m} (\lambda_k^{-1} \wedge \delta^2)(\|P_L\phi_k\|^2 + \|P_L\phi_j\|^2) \\ &= 2m^{-1}\sum_{k=1}^{m}(\lambda_k^{-1} \wedge \delta^2)\|P_L\phi_k\|^2 + 2m^{-2}\sum_{k=1}^{m}(\lambda_k^{-1} \wedge \delta^2)\sum_{j=1}^{m} \|P_L\phi_j\|^2\end{aligned}$$

$$= 2m^{-1}\sum_{k=1}^{m}(\lambda_k^{-1}\wedge\delta^2)\|P_L\phi_k\|^2 + 2m^{-2}\sum_{k=1}^{m}(\lambda_k^{-1}\wedge\delta^2)\|P_L\|_2^2$$

$$= 2m^{-1}\sum_{k=1}^{m}(\lambda_k^{-1}\wedge\delta^2)\|P_L\phi_k\|^2 + 2m^{-2}r\sum_{k=1}^{m}(\lambda_k^{-1}\wedge\delta^2). \tag{3.18}$$

To bound $\mathbb{E}\|Y\mathcal{P}_L(E_{X,X'})\|_w^2$ further, note that

$$\sum_{k=1}^{m}(\lambda_k^{-1}\wedge\delta^2)\|P_L\phi_k\|^2 \le \delta^2\sum_{k\le k(\delta)}\|P_L\phi_k\|^2 + \sum_{k>k(\delta)}\lambda_k^{-1}\|P_L\phi_k\|^2. \tag{3.19}$$

Assuming that $1 \le k(\delta) \le m-1$, using the first bound of Lemma 1, the fact that $\lambda_{k(\delta)+1}^{-1} < \delta^2$ and the monotonicity of function φ, we get from (3.19) that

$$\sum_{k=1}^{m}(\lambda_k^{-1}\wedge\delta^2)\|P_L\phi_k\|^2 \le \delta^2\varphi(k(\delta)) + c_\gamma\frac{\varphi(k(\delta)+1)}{\lambda_{k(\delta)+1}} \tag{3.20}$$

$$\le \delta^2\varphi(k(\delta)) + c_\gamma\delta^2\varphi(k(\delta)+1) \le (c_\gamma+1)\delta^2\varphi(k(\delta)+1).$$

It is easy to check that (3.20) holds also for $k(\delta) = 0$ and $k(\delta) = m$ (in the last case, $\varphi(k(\delta)+1) = r$). We also have

$$\sum_{k=1}^{m}(\lambda_k^{-1}\wedge\delta^2) \le \sum_{k\le k(\delta)}\delta^2 + \sum_{k>k(\delta)}\lambda_k^{-1},$$

which, in view of the second bound of Lemma 1 and the properties of function φ, implies

$$\sum_{k=1}^{m}(\lambda_k^{-1}\wedge\delta^2) \le \delta^2 k(\delta) + c_\gamma\frac{\bar{F}(\lambda_{k(\delta)+1})}{\lambda_{k(\delta)+1}} \tag{3.21}$$

$$\le (c_\gamma+1)\delta^2\bar{F}(\lambda_{k(\delta)+1}) \le (c_\gamma+1)\frac{m}{r}\delta^2\varphi(k(\delta)+1).$$

Using bounds (3.18), (3.20) and (3.21), we get, under the condition that $k(\delta) < m$,

$$\mathbb{E}\|Y\mathcal{P}_L(E_{X,X'})\|_w^2 \le 2m^{-1}(c_\gamma+1)\delta^2\varphi(k(\delta)+1) + 2m^{-2}r(c_\gamma+1)\frac{m}{r}\delta^2\varphi(k(\delta)+1)$$

$$\le 4(c_\gamma+1)m^{-1}\delta^2\varphi(k(\delta)+1). \tag{3.22}$$

In the case when $k(\delta) \ge m$, it is easy to show that

$$\mathbb{E}\|Y\mathcal{P}_L(E_{X,X'})\|_w^2 \le 4m^{-1}\delta^2 r. \tag{3.23}$$

We can also bound $\Big\|\|Y\mathcal{P}_L(E_{X,X'})\|_w\Big\|_{L_\infty}^2$ as follows:

$$\begin{aligned}\Big\|\|Y\mathcal{P}_L(E_{X,X'})\|_w\Big\|_{L_\infty}^2 &= \Big\|\|\mathcal{P}_L(E_{X,X'})\|_w\Big\|_{L_\infty}^2 \\ &= \Big\|\sum_{k,j=1}^m (\lambda_k^{-1}\wedge\delta^2)|\langle \mathcal{P}_L(E_{X,X'}),\phi_k\otimes\phi_j\rangle|^2\Big\|_{L_\infty} \\ &\le \max_{1\le k\le m}(\lambda_k^{-1}\wedge\delta^2)\max_{u,v\in V}\sum_{k,j=1}^m |\langle \mathcal{P}_L E_{u,v},\phi_k\otimes\phi_j\rangle|^2 \\ &\le \max_{1\le k\le m}(\lambda_k^{-1}\wedge\delta^2)\max_{u,v\in V}\|\mathcal{P}_L E_{u,v}\|_2^2 \\ &\le \delta^2\max_{u,v\in V}\|\mathcal{P}_L(e_u\otimes e_v)\|_2^2 \le 2\delta^2\max_{v\in V}\|P_L e_v\|^2.\end{aligned} \tag{3.24}$$

If $k(\delta) < m$, it follows from (3.16), (3.17), (3.22) and (3.24) that with probability at least $1-e^{-t}$, for all symmetric matrices M with $\|M\|_2 \le \delta$ and $\|W^{1/2}M\|_2 \le 1$,

$$|\langle \mathcal{P}_L\Xi, M\rangle| \le 2\sqrt{4(c_\gamma+1)}\sqrt{\frac{t}{nm}}\delta\sqrt{\varphi(k(\delta)+1)} + 2\sqrt{2}\delta\max_{v\in V}\|P_L e_v\|\frac{t}{n}.$$

Alternatively, if $k(\delta)\ge m$, we use (3.23) to get

$$|\langle \mathcal{P}_L\Xi, M\rangle| \le 4\delta\sqrt{\frac{rt}{nm}} + 2\sqrt{2}\delta\max_{v\in V}\|P_L e_v\|\frac{t}{n}.$$

This completes the proof of Lemma 3. □

It follows from Lemma 3 that, for all $\delta>0$, the following bound holds with probability at least $1-e^{-t}$

$$\begin{aligned}&\sup_{\|M\|_2\le\delta,\|W^{1/2}M\|_2\le 1} |\langle \mathcal{P}_L\Xi, M\rangle| \\ &\le 4\sqrt{c_\gamma+1}\sqrt{\frac{t}{nm}}\delta\sqrt{\varphi(k(\delta)+1)} + 2\sqrt{2}\delta\max_{v\in V}\|P_L e_v\|\frac{t}{n}\end{aligned} \tag{3.25}$$

(recall that $\varphi(k)=r$ for $k\ge m$, so, the second bound of the lemma can be included in the first bound). Moreover, the bound can be easily made uniform in $\delta\in[\delta_-,\delta_+]$ for arbitrary $\delta_- < \delta_+$. To this end, take $\delta_j := \delta_+ 2^{-j}, j=0,1,\dots,[\log_2(\delta_+/\delta_-)]+1$ and use (3.25) for each $\delta=\delta_j$ with $\bar t := t+\log([\log_2(\delta_+/\delta_-)]+2)$ instead of t. An application of the union bound and monotonicity of the left-hand side and the right-hand side of (3.25) with respect to δ then implies that with probability at least $1-e^{-t}$ for all $\delta\in[\delta_-,\delta_+]$

$$\sup_{\|M\|_2\le\delta,\|W^{1/2}M\|_2\le 1} |\langle \mathcal{P}_L\Xi, M\rangle| \le C\sqrt{\frac{\bar t}{nm}}\delta\sqrt{\varphi(k(\delta)+1)} + 4\sqrt{2}\delta\max_{v\in V}\|P_L e_v\|\frac{\bar t}{n}. \tag{3.26}$$

where $C > 0$ is a constant depending only on c. Indeed, by the union bound, (3.25) holds with probability at least

$$1 - ([\log_2(\delta_+/\delta_-)] + 2)e^{-\bar{t}} = 1 - e^{-t}$$

for all $\delta = \delta_j, j = 0, \dots, [\log_2(\delta_+/\delta_-)] + 1$.

Therefore, for all $j = 0, \dots, [\log_2(\delta_+/\delta_-)] + 1$ and all $\delta \in (\delta_{j+1}, \delta_j]$

$$\begin{aligned}&\sup_{\|M\|_2 \le \delta, \|W^{1/2}M\|_2 \le 1} |\langle \mathcal{P}_L \Xi, M\rangle| \\ &\quad \le 4\sqrt{c_\gamma + 1}\sqrt{\frac{\bar{t}}{nm}}\delta_j\sqrt{\varphi(k(\delta_j) + 1)} + 2\sqrt{2}\delta_j \max_{v \in V} \|P_L e_v\| \frac{\bar{t}}{n}\end{aligned} \tag{3.27}$$

(by monotonicity of the left-hand side). Note that $k(\delta_j) \le k(\delta) \le k(\delta_{j+1})$. We can now use the fact that $\frac{\varphi(k)}{\lambda_k} = \frac{\varphi(k)}{\bar{F}(\lambda_k)}\frac{\bar{F}(\lambda_k)}{\lambda_k}$ is a nonincreasing function and the condition $\lambda_{k+1}/\lambda_k \le c$ to show that

$$\begin{aligned}\sqrt{\frac{\bar{t}}{nm}}\delta_j\sqrt{\varphi(k(\delta_j) + 1)} + &\le 2\sqrt{\frac{\bar{t}}{nm}}\delta_{j+1}\sqrt{\varphi(k(\delta_{j+1}) + 1)} \\ &\le 2\sqrt{\frac{\bar{t}}{nm}}\sqrt{\frac{\varphi(k(\delta_{j+1}) + 1)}{\lambda_{k(\delta_{j+1})}}} \le 2\sqrt{c}\sqrt{\frac{\bar{t}}{nm}}\sqrt{\frac{\varphi(k(\delta_{j+1}) + 1)}{\lambda_{k(\delta_{j+1})+1}}} \\ &\le 2\sqrt{c}\sqrt{\frac{\bar{t}}{nm}}\sqrt{\frac{\varphi(k(\delta) + 1)}{\lambda_{k(\delta)+1}}} \le 2\sqrt{c}\sqrt{\frac{\bar{t}}{nm}}\delta\sqrt{\varphi(k(\delta) + 1)}.\end{aligned}$$

This and bound (3.27) imply that

$$\begin{aligned}&\sup_{\|M\|_2 \le \delta, \|W^{1/2}M\|_2 \le 1} |\langle \mathcal{P}_L \Xi, M\rangle| \\ &\quad \le 8\sqrt{c(c_\gamma + 1)}\sqrt{\frac{\bar{t}}{nm}}\delta\sqrt{\varphi(k(\delta) + 1)} + 4\sqrt{2}\delta \max_{v \in V} \|P_L e_v\| \frac{\bar{t}}{n},\end{aligned} \tag{3.28}$$

which proves bound (3.26).

Set δ as

$$\delta := \frac{\|\hat{S} - S_*\|_2}{\|W^{1/2}(\hat{S} - S_*)\|_2} = \frac{\|\hat{S} - S_*\|_{L_2(\Pi^2)}}{\|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)}}$$

and assume for now that $\delta \in [\delta_-, \delta_+]$.

For a particular choice of $M := \frac{\hat{S} - S_*}{\|W^{1/2}(\hat{S} - S_*)\|_2}$, we get from (3.26) that

$$\begin{aligned}|\langle \mathcal{P}_L \Xi, \hat{S} - S_*\rangle| &\le C\sqrt{\frac{\bar{t}}{nm}}\|\hat{S} - S_*\|_2\sqrt{\varphi(k(\delta) + 1)} \\ &\quad + 4\sqrt{2}\max_{v \in V} \|P_L e_v\| \frac{\bar{t}}{n}\|\hat{S} - S_*\|_2.\end{aligned} \tag{3.29}$$

Suppose now that $\delta^2 \geq \bar{\varepsilon}$. Since, under assumptions of the theorem, $\bar{\varepsilon} \in (\lambda_{s+1}^{-1}, \lambda_s^{-1}]$, this implies that $k(\delta) \leq k(\sqrt{\bar{\varepsilon}}) = s$ and

$$\begin{aligned}
|\langle \mathcal{P}_L \Xi, \hat{S} - S_* \rangle| &\leq C\sqrt{\frac{\bar{t}}{nm}} \|\hat{S} - S_*\|_2 \sqrt{\varphi(s+1)} \\
&\quad + 4\sqrt{2} \max_{v \in V} \|P_L e_v\| \frac{\bar{t}}{n} \|\hat{S} - S_*\|_2 \\
&= C\sqrt{\frac{m\bar{t}}{n}} \|\hat{S} - S_*\|_{L_2(\Pi^2)} \sqrt{\varphi(s+1)} \\
&\quad + 4\sqrt{2} \max_{v \in V} \|P_L e_v\| \frac{m\bar{t}}{n} \|\hat{S} - S_*\|_{L_2(\Pi)} \\
&\leq 2C^2 \frac{\varphi(s+1) m\bar{t}}{n} + 64 \max_{v \in V} \|P_L e_v\|^2 \Big(\frac{m\bar{t}}{n}\Big)^2 + \frac{1}{4}\|\hat{S} - S_*\|^2_{L_2(\Pi^2)}.
\end{aligned} \tag{3.30}$$

In the case when $\delta^2 < \bar{\varepsilon}$, we have $k(\delta) \geq k(\sqrt{\bar{\varepsilon}}) = s$. In this case, we again use the fact that $\frac{\varphi(k)}{\lambda_k}$ is a nonincreasing function and the condition $\lambda_{k+1}/\lambda_k \leq c$ to show that

$$\begin{aligned}
\sqrt{\frac{\bar{t}}{nm}} \|\hat{S} - S_*\|_2 \sqrt{\varphi(k(\delta)+1)} &= \sqrt{\frac{m\bar{t}}{n}} \|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)} \sqrt{\delta^2 \varphi(k(\delta)+1)} \\
&\leq \sqrt{\frac{m\bar{t}}{n}} \|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)} \sqrt{\frac{\varphi(k(\delta)+1)}{\lambda_{k(\delta)}}} \\
&\leq \sqrt{c}\sqrt{\frac{m\bar{t}}{n}} \|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)} \sqrt{\frac{\varphi(k(\delta)+1)}{\lambda_{k(\delta)+1}}} \\
&\leq \sqrt{c}\sqrt{\frac{m\bar{t}}{n}} \|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)} \sqrt{\frac{\varphi(s+1)}{\lambda_{s+1}}} \\
&\leq \sqrt{c}\sqrt{\frac{m\bar{t}}{n}} \sqrt{\bar{\varepsilon}} \|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)} \sqrt{\varphi(s+1)}.
\end{aligned}$$

This allows us to deduce from (3.29) that

$$\begin{aligned}
|\langle \mathcal{P}_L \Xi, \hat{S} - S_* \rangle| &\leq \sqrt{c} C \sqrt{\frac{m\bar{t}}{n}} \sqrt{\bar{\varepsilon}} \|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)} \sqrt{\varphi(s+1)} \\
&\quad + 4\sqrt{2} \max_{v \in V} \|P_L e_v\| \frac{m\bar{t}}{n} \|\hat{S} - S_*\|_{L_2(\Pi)} \\
&\leq cC^2 \frac{\varphi(s+1) m\bar{t}}{n} + \frac{1}{4}\bar{\varepsilon} \|W^{1/2}(\hat{S} - S_*)\|^2_{L_2(\Pi^2)} \\
&\quad + 32 \max_{v \in V} \|P_L e_v\|^2 \Big(\frac{m\bar{t}}{n}\Big)^2 + \frac{1}{4}\|\hat{S} - S_*\|^2_{L_2(\Pi^2)}.
\end{aligned} \tag{3.31}$$

It follows from bounds (3.30) and (3.31) that with probability at least $1 - e^{-t}$,

$$\begin{aligned} |\langle \mathcal{P}_L \Xi, \hat{S} - S_* \rangle| \leq (2 \vee c) C^2 \frac{\varphi(s+1) m \bar{t}}{n} + 64 \max_{v \in V} \|P_L e_v\|^2 \Big(\frac{m\bar{t}}{n}\Big)^2 \\ + \frac{1}{4} \|\hat{S} - S_*\|^2_{L_2(\Pi^2)} + \frac{1}{4} \bar{\varepsilon} \|W^{1/2}(\hat{S} - S_*)\|^2_{L_2(\Pi^2)}, \end{aligned} \tag{3.32}$$

provided that

$$\delta = \frac{\|\hat{S} - S_*\|_2}{\|W^{1/2}(\hat{S} - S_*)\|_2} = \frac{\|\hat{S} - S_*\|_{L_2(\Pi^2)}}{\|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)}} \in [\delta_-, \delta_+]. \tag{3.33}$$

It remains now to substitute bounds (3.9), (3.11), (3.13) and (3.32) in bound (3.3) to get that with some constants $C > 0, C_1 > 0$ depending only on c and with probability at least $1 - 2e^{-t}$

$$\begin{aligned} \|\hat{S} - S_*\|^2_{L_2(\Pi^2)} \leq C \frac{\varphi(s+1) m (\bar{t} + t_m)}{n} \\ + \bar{\varepsilon} \|W^{1/2} S_*\|^2_{L_2(\Pi^2)} + C_1 \max_{v \in V} \|P_L e_v\|^2 \Big(\frac{m\bar{t}}{n}\Big)^2, \end{aligned} \tag{3.34}$$

where $t_m := t + \log(2m)$.

We still have to choose the values of δ_-, δ_+ and to handle the case when

$$\delta = \frac{\|\hat{S} - S_*\|_2}{\|W^{1/2}(\hat{S} - S_*)\|_2} = \frac{\|\hat{S} - S_*\|_{L_2(\Pi^2)}}{\|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)}} \notin [\delta_-, \delta_+]. \tag{3.35}$$

First note that, since the largest eigenvalue of W is λ_m and it is bounded from above by m^ζ, we have

$$\|W^{1/2}(\hat{S} - S_*)\|_2 \leq \sqrt{\lambda_m} \|\hat{S} - S_*\|_2 \leq m^{\zeta/2} \|\hat{S} - S_*\|_2.$$

Thus, $\delta \geq m^{-\zeta/2}$. Next note that

$$\|W^{1/2} S_*\|^2_{L_2(\Pi^2)} \leq m^{-2} m^\zeta \|S_*\|_2^2 \leq m^\zeta,$$

where we also took into account that the absolute values of the entries of S_* are bounded by 1. It now follows from (3.15) that, under the assumption $\frac{2mt_m}{n} \leq 1$,

$$\begin{aligned} \|\hat{S} - S_*\|^2_{L_2(\Pi^2)} \leq \frac{3}{2} r m^2 \varepsilon^2 + 2 \bar{\varepsilon} m^\zeta \\ \leq 24 r m^2 \frac{t + \log(2m)}{nm} + 2 \frac{m^\zeta}{\lambda_s} \leq 12m + 2m^{2\zeta} \leq 14 m^{2\zeta}, \end{aligned}$$

which holds with probability at least $1 - e^{-t}$. Therefore, as soon as $\|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)} \geq n^{-\zeta}$, we have $\delta \leq 4 n^\zeta m^\zeta$.

We will now take $\delta_- := m^{-\zeta/2}, \delta_+ := 4 n^\zeta m^\zeta$. Then, the only case when (3.35) can possibly hold is if $\|W^{1/2}(\hat{S} - S_*)\|_{L_2(\Pi^2)} \leq n^{-\zeta}$. In this case, we can set

$$\delta := n^\zeta \|\hat{S} - S_*\|_{L_2(\Pi^2)} \in [\delta_-, \delta_+]$$

and follow the proof of bound (3.32) replacing throughout the argument $\|W^{1/2}(\hat{S}-S_*)\|_{L_2(\Pi^2)}$ with $n^{-\zeta}$. This yields

$$|\langle \mathcal{P}_L \Xi, \hat{S} - S_* \rangle| \leq (2 \vee c) C^2 \frac{\varphi(s+1) m\bar{t}}{n} \tag{3.36}$$
$$+ 64 \max_{v \in V} \|P_L e_v\|^2 \Big(\frac{m\bar{t}}{n}\Big)^2 + \frac{1}{4}\|\hat{S} - S_*\|^2_{L_2(\Pi^2)} + \frac{1}{4}\bar{\varepsilon} n^{-2\zeta}.$$

Bound (3.36) can be now used instead of (3.32) to prove that

$$\|\hat{S} - S_*\|^2_{L_2(\Pi^2)} \leq C \frac{\varphi(s+1) m(\bar{t} + t_m)}{n} \tag{3.37}$$
$$+ \bar{\varepsilon}\|W^{1/2} S_*\|^2_{L_2(\Pi^2)} + C_1 \max_{v \in V} \|P_L e_v\|^2 \Big(\frac{m\bar{t}}{n}\Big)^2 + \bar{\varepsilon} n^{-2\zeta}$$

with some constants $C, C_1 > 0$ depending only on c.

Clearly, we can assume that $C_1 \geq 1$ and $\bar{t} \geq 1$. Since $m \leq n^2$ (recall that we even assumed that $m t_{n,m} \leq 1$), $\zeta \geq 1$, $\max_{v \in V} \|P_L e_v\|^2 \geq \frac{r}{m}$[4] and $\bar{\varepsilon} \leq \lambda_{k_0}^{-1} \leq m^{\zeta}$, it is easy to check that

$$C_1 \max_{v \in V} \|P_L e_v\|^2 \Big(\frac{m\bar{t}}{n}\Big)^2 \geq \frac{m}{n^2} \geq \frac{m^{\zeta}}{n^{2\zeta}} \geq \bar{\varepsilon} n^{-2\zeta}.$$

Thus, the last term of bound (3.37) can be dropped (with a proper adjustment of constant C_1).

Note also that with our choice of δ_-, δ_+

$$\bar{t} = t + \log(\log_2(\delta_+/\delta_-) + 2) \leq t + \log\log_2(16 n^{\zeta} m^{(3/2)\zeta})$$

and $\bar{t} + t_m \leq 2 t_{n,m}$. It is now easy to conclude that, with some constants C, C_1 depending only on c and with probability at least $1 - 3e^{-t}$

$$\|\hat{S} - S_*\|^2_{L_2(\Pi^2)} \leq C \frac{\varphi(s+1) m t_{n,m}}{n} + \bar{\varepsilon}\|W^{1/2} S_*\|^2_{L_2(\Pi^2)} + C_1 \max_{v \in V} \|P_L e_v\|^2 \Big(\frac{m\bar{t}}{n}\Big)^2. \tag{3.38}$$

The probability bound $1 - 3e^{-t}$ can be rewritten as $1 - e^{-t}$ by changing the value of constants C, C_1. Also, by changing the notation $s + 1 \mapsto s$, bound (3.38) yields (2.1). This completes the proof of the theorem. □

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Sparse Principal Component Analysis with Missing Observations

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Abstract. In this paper, we study the problem of sparse Principal Component Analysis (PCA) in the high dimensional setting with missing observations. Our goal is to estimate the first principal component when we only have access to partial observations. Existing estimation techniques are usually derived for fully observed data sets and require a prior knowledge of the sparsity of the first principal component in order to achieve good statistical guarantees. Our contributions is essentially theoretical in nature. First, we establish the first information-theoretic lower bound for the sparse PCA problem with missing observations. Second, we study the properties of a BIC type estimator that does not require any prior knowledge on the sparsity of the unknown first principal component or any imputation of the missing observations and adapts to the unknown sparsity of the first principal component. Third, if the covariance matrix of interest admits a sparse first principal component and is in addition approximately low-rank, then we can derive a completely data-driven choice of the regularization parameter and the resulting BIC estimator will also enjoy optimal statistical performances (up to a logarithmic factor).

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1. Introduction

Let $X, X_1, \dots, X_n \in \mathbb{R}^p$ be i.i.d. zero mean vectors with unknown covariance matrix $\Sigma = \mathbb{E}X \otimes X$ of the form

$$\Sigma = \sigma_1 \theta_1 \theta_1^\top + \sigma_2 \Upsilon, \tag{1.1}$$

where $\sigma_1 > \sigma_2 \geq 0$, $\theta_1 \in \mathcal{S}^p$ (the l_2 unit sphere in $\mathbb{R}^p$) and Υ is a $p \times p$ symmetric positive semi-definite matrix with spectral norm $\|\Upsilon\|_\infty \leq 1$ and such that $\Upsilon\theta_1 = 0$.

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The eigenvector θ_1 is called the first principal component of Σ. Our objective is to estimate the first principal component θ_1 when the vectors $X_1, \ldots, X_n$ are partially observed. More precisely, we consider the following framework. Denote by $X_i^{(j)}$ the jth component of the vector X_i. We assume that each component $X_i^{(j)}$ is observed independently of the others with probability $\delta \in (0,1]$. Note that δ can be easily estimated by the proportion of observed entries. Therefore, we will assume in this paper that δ is known. Note also that the case $\delta = 1$ corresponds to the standard case of fully observed vectors. Let $(\delta_{i,j})_{1\le i\le n, 1\le j\le p}$ be a sequence of i.i.d. Bernoulli random variables with parameter δ and independent from $X_1, \ldots, X_n$. We observe n i.i.d. random vectors $Y_1, \ldots, Y_n \in \mathbb{R}^p$ whose components satisfy

$$Y_i^{(j)} = \delta_{i,j} X_i^{(j)}, \quad 1 \le i \le n,\, 1 \le j \le p. \tag{1.2}$$

We can think of the $\delta_{i,j}$ as masked variables. If $\delta_{i,j} = 0$, then we cannot observe the jth component of X_i and the default value 0 is assigned to $Y_i^{(j)}$. Our goal is then to estimate θ_1 given the partial observations $Y_1, \ldots, Y_n$.

Principal Component Analysis (PCA) is a popular technique to reduce the dimension of a data set that has been used for many years in a variety of different fields including image processing, engineering, genetics, meteorology, chemistry and many others. In most of these fields, data are now high dimensional, that is the number of parameters p is much larger than the sample size n, and contain missing observations. This is especially true in genomic with gene expression microarray data where PCA is used to detect the genes responsible for a given biological process. Indeed, despite the recent improvements in gene expression techniques, microarray data can contain up to 10% missing observations affecting up to 95% of the genes. Unfortunately, it is a known fact that PCA is very sensitive even to small perturbations of the data including in particular missing observations. Therefore, several strategies have been developed to deal with missing values. The simple strategy that consists in eliminating from the PCA study any gene with at least one missing observation is not acceptable in this context since up to 95% of the genes can be eliminated from the study. An alternative strategy consists in inferring the missing values prior to the PCA using complex imputation schemes [3, 6]. These schemes usually assume that the genes interactions follow some specified model and involve intensive computational preprocessing to impute the missing observations. We propose in this paper a different strategy. Instead of building an imputation technique based on assumptions describing the genome structure (about which we usually have no prior information), we propose a technique based on the analysis of the perturbations process. In other words, if we understand the process generating the missing observations, then we can efficiently correct the data prior to the PCA analysis. This strategy was first introduced in [8] to estimate the spectrum of low-rank covariance matrices. One of our goal is to show that this approach can be successfully applied to perform accurate PCA with missing observations.

Standard PCA in the full observation framework ($\delta = 1$) consists in extracting the first principal components of Σ (that is the eigenvector θ_1 associated to the largest eigenvalue) based on the i.i.d. observations $X_1, \ldots, X_n$:

$$\hat{\theta} = \text{argmax}_{\theta^\top \theta = 1} \left(\theta^\top \Sigma_n \theta \right), \tag{1.3}$$

where $\Sigma_n = \frac{1}{n} \sum_{i=1}^n X_i X_i^\top$. The standard PCA presents two majors drawbacks. First, it is not consistent in high dimension [4, 11, 12]. Second, the solution $\hat{\theta}$ is usually a dense vector whereas sparse solutions are preferred in most applications in order to obtain simple interpretable structures. For instance, in microarray data, we typically observe that only a few among the thousands of screened genes are involved in a given biological process. In order to improve interpretability, several approaches have been proposed to perform sparse PCA, that is to enforce sparsity of the PCA outcome. See for instance [13, 14, 18] for SVD based iterative thresholding approaches. [19] reformulated the sparse PCA problem as a sparse regression problem and then used the LASSO estimator. See also [10] for greedy methods. Note also the recent paper [1] which studies the testing problem for the presence of a sparse principal component. We consider now the approach by [5] which consists in computing a solution of (1.3) under the additional l_1-norm constraint $|\theta|_1 \leq \bar{s}$ for some fixed integer $\bar{s} \geq 1$ in order to enforce sparsity of the solution. The same approach with the l_1-norm constraint replaced by the l_0-norm gives the following estimator

$$\hat{\theta}_{\text{o}} = \text{argmax}_{\theta \in \mathcal{S}^p \,:\, |\theta|_0 \leq \bar{s}} \left(\theta^\top \Sigma_n \theta \right), \tag{1.4}$$

where $|\theta|_0$ denotes the number of nonzero components of θ and $\bar{s} \geq 1$ is a parameter of this estimator. In the recent paper [17], the following oracle inequality was established. If $|\theta_1|_0 \leq \bar{s}$, then we have

$$\left(\mathbb{E} \| \hat{\theta}_{\text{o}} \hat{\theta}_{\text{o}}^\top - \theta_1 \theta_1^\top \|_2 \right)^2 \leq C \left(\frac{\sigma_1}{\sigma_1 - \sigma_2} \right)^2 \bar{s} \frac{\log(p/\bar{s})}{n},$$

for some absolute constant $C > 0$. Note that this estimator requires the knowledge of an upper bound $\bar{s} \geq |\theta_1|_0$. In practice, we generally do not have access to any prior information on the sparsity of θ_1. Consequently, if the parameter $\bar{s}$ we use in the estimator is too small, then the above upper bound does not hold, and if $\bar{s}$ is too large, then the above upper bound, even though valid, is sub-optimal. In other words, the estimator (1.4) with $\bar{s} = |\theta_1|_0$ can be seen as an oracle and our goal is to propose an estimator that performs as well as this oracle without any prior information on $|\theta_1|_0$.

In order to circumvent the fact that $|\theta_1|_0$ is unknown, we consider the following estimator proposed by [2]

$$\hat{\theta}_1 = \text{argmax}_{\theta \in \mathcal{S}^p} \left(\theta^\top \Sigma_n \theta - \lambda |\theta|_0 \right), \tag{1.5}$$

where $\lambda > 0$ is a regularization parameter to be tuned properly. This optimization problem is unfortunately hard to solve in practice. [2] proposed to solve instead a convex relaxation for the above problem. [7] proposed a gradient procedure computationally tractable even in high dimension to solve (1.5). However, a weakness of their procedure is that it can trapped into a local maximizer. Although the computational interest of (1.5) is not clearly established yet, we believe nevertheless that the investigation of the statistical performances of this estimator is important to get a better theoretical understanding of the sparse PCA problem. Indeed, the BIC estimator in high dimensional regression is not computationally tractable but enjoys optimal statistical properties. This BIC estimator is then used as a theoretical benchmark to assess the statistical performances of other computationally tractable procedures like the Lasso estimator. Similarly, we propose to show in this paper that the statistical properties of (1.5) are optimal in the sparse PCA problem. In particular, (1.5) is minimax rate optimal (up to a logarithmic factor) and adapts to the unknown sparsity of θ_1 provided the regularization parameter is tuned properly.

When the data contains incomplete observations ($\delta < 1$), we do not have access to the empirical covariance matrix Σ_n. Given the observations $Y_1, \dots, Y_n$, we can build the following empirical covariance matrix

$$\Sigma_n^{(\delta)} = \frac{1}{n} \sum_{i=1}^{n} Y_i Y_i^\top .$$

As noted in [8], $\Sigma_n^{(\delta)}$ is not an unbiased estimator of Σ, Consequently, we need to consider the following correction in order to get sharp estimation results:

$$\tilde{\Sigma}_n = (\delta^{-1} - \delta^{-2}) \mathrm{diag}\left(\Sigma_n^{(\delta)}\right) + \delta^{-2} \Sigma_n^{(\delta)}, \tag{1.6}$$

where for any $p \times p$ matrix A, $\mathrm{diag}(A)$ is the $p \times p$ matrix obtained by keeping only the diagonal elements of A and putting all the non-diagonal elements of A equal to 0. Indeed, we can check by elementary algebra that $\tilde{\Sigma}_n$ is an unbiased estimator of Σ in the missing observation framework $\delta \in (0, 1]$. Therefore, we consider the following estimator in the missing observation framework

$$\hat{\theta}_1 = \mathrm{argmax}_{\theta \in \mathcal{S}^p \,:\, |\theta|_0 \le \bar{s}} \left(\theta^\top \tilde{\Sigma}_n \theta - \lambda |\theta|_0 \right), \tag{1.7}$$

where $\lambda > 0$ is a regularization parameter to be tuned properly and $\bar{s}$ is a mild constraint on $|\theta_1|_0$. More precisely, $\bar{s}$ can be chosen as large as $\frac{\delta^2 n}{\log(ep)}$ when no prior information on $|\theta_1|_0$ is available. We will prove in particular that the estimator (1.7) adapts to the unknown sparsity of θ_1 provided that $|\theta_1|_0 \le \bar{s}$. We also investigate the case where Σ is in addition approximately low-rank. In that case, we can remove the restriction $|\theta|_0 \le \bar{s}$ (taking $\bar{s} = p$) in the procedure (1.7) and propose a data-driven choice of the regularization parameter λ. Finally, we establish information theoretic lower bounds for the sparse PCA problem in the missing observation framework $\delta \in (0, 1]$, showing in particular that the dependence of our bounds on the parameter δ is sharp. Note that our results are nonasymptotic in nature and hold for any setting of n, p including in particular the high dimensional setting $p > n$.

The rest of the paper is organized as follows. In Section 2, we recall some tools and definitions that will be useful for our statistical analysis. Section 3 contains our main theoretical results. Finally, Section 4 contains the proofs of our results.

2. Tools and definitions

In this section, we introduce various notations and definitions and we recall some known facts that we will use to establish our results.

The l_q-norms of a vector $x = \left(x^{(1)}, \dots, x^{(p)}\right)^\top \in \mathbb{R}^p$ is given by

$$|x|_q = \left(\sum_{j=1}^{p} |x^{(j)}|^q \right)^{1/q}, \text{ for } 1 \le q < \infty, \quad \text{and} \quad |x|_\infty = \max_{1 \le j \le p} |x^{(j)}|.$$

The support of a vector $x = \left(x^{(1)}, \dots, x^{(p)}\right)^\top \in \mathbb{R}^p$ is defined as follows

$$J(x) = \left\{ j \,:\, x^{(j)} \neq 0 \right\}.$$

We denote the number of nonzero components of x by $|x|_0$. Note that $|x|_0 = |J(x)|$. Set $\mathcal{S}^p = \{x \in \mathbb{R}^p : |x|_2 = 1\}$ and $[p] = 2^{\{1,\dots,p\}}$. For any $J \in [p]$, we define $\mathcal{S}^p(J) = \{x \in \mathcal{S}^p \,:\, J(x) = J\}$. For any integer $1 \le s \le p$, we define $\mathcal{S}^p_s = \{x \in \mathcal{S}^p \,:\, |x|_0 = s\}$. Note that $\mathcal{S}^p_s = \cup_{J \in [p] \,:\, |J| = s} \mathcal{S}^p(J)$.

For any $p \times p$ symmetric matrix A with eigenvalues $\sigma_1(A), \dots, \sigma_p(A)$, we define the Schatten q-norm of A by

$$\|A\|_q = \left(\sum_{j=1}^{p} |\sigma_j(A)|^q \right)^{1/q}, \ \forall 1 \le q < \infty, \quad \text{and} \quad \|A\|_\infty = \max_{1 \le j \le p} \left\{ |\sigma_j(A)| \right\}.$$

Define the usual matrix scalar product $\langle A, B \rangle = \mathrm{tr}(A^\top B)$ for any $A, B \in R^{p \times p}$. Note that $\|A\|_2 = \sqrt{\langle A, A \rangle}$ for any $A \in \mathbb{R}^{p \times p}$. Recall the trace duality property

$$|\langle A, B \rangle| \le \|A\|_\infty \|B\|_1, \quad \forall A, B \in \mathbb{R}^{p \times p}.$$

We recall now some basic facts about ϵ-nets (See for instance Section 5.2.2 in [16]).

Definition 1. Let (A, d) be a metric space and let $\epsilon > 0$. A subset $\mathcal{N}_\epsilon$ of A is called an ϵ-net of A if for every point $a \in A$, there exists a point $b \in \mathcal{N}_\epsilon$ so that $d(a, b) \le \epsilon$.

We recall now an approximation result of the spectral norm on an ϵ-net.

Lemma 1. *Let A be a $k \times k$ symmetric matrix for some $k \ge 1$. For any $\epsilon \in (0, 1/2)$, there exists an ϵ-net $\mathcal{N}_\epsilon \subset \mathcal{S}^k$ (the unit sphere in $\mathbb{R}^k$) such that*

$$|\mathcal{N}_\epsilon| \le \left(1 + \frac{2}{\epsilon}\right)^k, \quad \textit{and} \quad \sup_{\theta \in \mathcal{S}^k} |\langle Ax, x \rangle| \le \frac{1}{1 - 2\epsilon} \sup_{\theta \in \mathcal{N}_\epsilon} |\langle Ax, x \rangle|.$$

See for instance Lemma 5.2 and Lemma 5.3 in [16] for a proof.

We recall now the definition and some basic properties of sub-exponential random vectors.

Definition 2. The ψ_α-norms of a real-valued random variable V are defined by

$$\|V\|_{\psi_\alpha} = \inf\left\{u > 0 : \mathbb{E}\exp\left(|V|^\alpha/u^\alpha\right) \leq 2\right\}, \quad \alpha \geq 1.$$

We say that a random variable V with values in $\mathbb{R}$ is sub-exponential if $\|V\|_{\psi_\alpha} < \infty$ for some $\alpha \geq 1$. If $\alpha = 2$, we say that V is sub-Gaussian.

We recall some well-known properties of sub-exponential random variables:

1. For any real-valued random variable V such that $\|V\|_{\psi_\alpha} < \infty$ for some $\alpha \geq 1$, we have
$$\mathbb{E}|V|^m \leq 2\frac{m}{\alpha}\Gamma\left(\frac{m}{\alpha}\right)\|V\|_{\psi_\alpha}^m, \quad \forall m \geq 1 \tag{2.1}$$
where $\Gamma(\cdot)$ is the Gamma function.
2. If a real-valued random variable V is sub-Gaussian, then V^2 is sub-exponential. Indeed, we have
$$\|V^2\|_{\psi_1} \leq 2\|V\|_{\psi_2}^2. \tag{2.2}$$

Definition 3. A random vector $X \in \mathbb{R}^p$ is sub-exponential if $\langle X, x\rangle$ are sub-exponential random variables for all $x \in \mathbb{R}^p$. The ψ_α-norms of a random vector X are defined by

$$\|X\|_{\psi_\alpha} = \sup_{x\in\mathcal{S}^p}\|\langle X, x\rangle\|_{\psi_\alpha}, \quad \alpha \geq 1.$$

We recall a version of Bernstein's inequality for unbounded real-valued random variables.

Proposition 1. *Let $Y_1, \dots, Y_n$ be independent real-valued random variables with zero mean. Let there exist constants σ, σ' and K such that for any $m \geq 2$*

$$\frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[|Y_i|^m\right] \leq \frac{m!}{2}K^{m-2}\sigma^2. \tag{2.3}$$

Then for every $t \geq 0$, we have with probability at least $1 - 2e^{-t}$

$$\left|\frac{1}{n}\sum_{i=1}^n Y_i\right| \leq \sigma\sqrt{\frac{2t}{n}} + K\frac{t}{n}.$$

3. Main results for sparse PCA with missing observations

In this section, we state our main statistical results concerning the procedure (1.7). We will establish these results under the following condition on the distribution of X.

Assumption 1 (Sub-Gaussian observations). *The random vector $X \in \mathbb{R}^p$ is sub-Gaussian, that is $\|X\|_{\psi_2} < \infty$. In addition, there exist a numerical constant $c_1 > 0$ such that*

$$\mathbb{E}(\langle X, u\rangle)^2 \geq c_1 \|\langle X, u\rangle\|_{\psi_2}^2,\ \forall u \in \mathbb{R}^p. \tag{3.1}$$

3.1. Oracle inequalities for sparse PCA

We first establish a preliminary results on the stochastic deviation of the following empirical process

$$\mathbf{Z}_n(s) = \max_{\theta \in \mathcal{S}_s^p} \left\{ \left| \theta^\top (\tilde{\Sigma}_n - \Sigma)\theta \right| \right\}, \quad \forall 1 \leq s \leq p.$$

We consider first the full observation case ($\delta = 1$). To this end, we introduce the following quantity

$$\zeta_n(s,p) := \max\left\{ \sqrt{\frac{s\log(ep)}{n}}, \frac{s\log(ep)}{n} \right\}, \tag{3.2}$$

Proposition 2. *Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $\mathbb{R}^p$ with covariance matrix* (1.2)*. Let Assumption* 1 *be satisfied. Then, we have*

$$\mathbb{P}\left(\bigcap_{s=1}^{p} \left\{ \mathbf{Z}_n(s) \leq c\frac{\sigma_1}{c_1 \wedge 1} \zeta_n(s,p) \right\} \right) \geq 1 - \frac{1}{p} \tag{3.3}$$

where $c > 0$ is an absolute constant.

We now treat the missing observations case ($\delta < 1$). For the sake of simplicity, we will assume in addition that $|X|_2 \leq \sqrt{K}$, almost surely, for some constant $K > 0$. We can deduce from this case similar results for general distributions satisfying Assumption 1 by taking $K = \mathrm{trace}(\Sigma)\log(ep)$ and a simple union bound argument. See [8] for more details. We introduce the following quantity

$$\zeta_n(s,p,\delta) := \max\left\{ \sqrt{\frac{s\log(ep)}{\delta^2 n}}, \frac{K}{\sigma_1}\frac{s\log(ep)}{\delta^2 n} \right\}, \tag{3.4}$$

Proposition 3. *Let Assumption* 1 *be satisfied. Assume in addition that there exists a constant $K > 0$ such that $|X|_2 \leq \sqrt{K}$, almost surely. Let $Y_1, \ldots, Y_n$ be defined in* (1.2) *with $\delta \in (0,1]$. Then, we have*

$$\mathbb{P}\left(\bigcap_{s=1}^{p} \left\{ \mathbf{Z}_n(s) \leq c\frac{\sigma_1}{c_1 \wedge 1} \zeta_n(s,p,\delta) \right\} \right) \geq 1 - \frac{1}{p} \tag{3.5}$$

where $c > 0$ is an absolute constant.

We can now state our main results. We start with the full observation framework ($\delta = 1$).

Theorem 1. *Let $X_1,\dots,X_n$ be i.i.d. random vectors in $\mathbb{R}^p$ with covariance matrix* (1.2)*. Let Assumption* 1 *be satisfied. Assume that $n \geq \bar{2}s\log(ep)$ where $\bar{s}$ is a parameter of the estimator* (1.7)*. Take*

$$\lambda = C\frac{\sigma_1^2}{\sigma_1-\sigma_2}\frac{\log(ep)}{n}, \tag{3.6}$$

where $C>0$ is a large enough numerical constant. If $|\theta_1|_0 \leq \bar{s}$, then the estimator (1.7) *satisfies with probability at least $1-\frac{1}{p}$*

$$\|\hat{\theta}_1\hat{\theta}_1^\top - \theta_1\theta_1^\top\|_2^2 \leq C'|\theta_1|_0\tilde{\sigma}^2\frac{\log(ep)}{n}.$$

where $\tilde{\sigma} = \frac{\sigma_1}{\sigma_1-\sigma_2}$ and $C'>0$ is a numerical constant that can depend only on c_1.

We consider now the missing observation framework ($\delta < 1$).

Theorem 2. *Let the assumptions of Proposition* 3 *be satisfied. In addition, assume that $n \geq 2\delta^{-2}\frac{K^2}{\sigma_1^2}\bar{s}\log(ep)$ where $\bar{s}$ is a parameter of the estimator* (1.7)*. Take*

$$\lambda = C\frac{\sigma_1^2}{\sigma_1-\sigma_2}\frac{\log(ep)}{\delta^2 n}, \tag{3.7}$$

where $C>0$ is a large enough numerical constant. If $|\theta_1|_0 \leq \bar{s}$, then the estimator (1.7) *satisfies with probability at least $1-\frac{1}{p}$*

$$\|\hat{\theta}_1\hat{\theta}_1^\top - \theta_1\theta_1^\top\|_2^2 \leq C'|\theta_1|_0\tilde{\sigma}^2\frac{\log(ep)}{\delta^2 n}.$$

where $\tilde{\sigma} = \frac{\sigma_1}{\sigma_1-\sigma_2}$ and $C'>0$ is a numerical constant that can depend only on c_1.

1. We observe that the estimation bound increases as the difference $\sigma_1-\sigma_2$ decreases. The problem of estimation of the first principal component is statistically more difficult when the largest and second largest eigenvalues are close. We also observe that the optimal choice of the regularization parameters in (3.6) and (3.7) depend on the eigenvalues σ_1,σ_2 of Σ. Unfortunately, these quantities are typically unknown in practice. In order to circumvent this difficulty, we propose in Section 3.3 a data-driven choice of λ with optimal statistical performances (up to a logarithmic factor) provided that Σ is approximately low-rank.
2. Let now consider the full observation framework ($\delta = 1$). In Theorem 1, we established the following upper bound with probability at least $1-\frac{1}{p}$

   $$\|\hat{\theta}_1\hat{\theta}_1^\top - \theta_1\theta_1^\top\|_2^2 \leq C'|\theta_1|_0\tilde{\sigma}^2\frac{\log(ep)}{n}.$$

   We can compare this result with that obtained for the procedure (1.4) in [17]

   $$\left(\mathbb{E}\|\hat{\theta}_o\hat{\theta}_o^\top - \theta_1\theta_1^\top\|_2\right)^2 \leq C'\bar{s}\tilde{\sigma}^2\frac{\log(ep/\bar{s})}{n}.$$

   We see that in order to achieve the rate $|\theta_1|_0\tilde{\sigma}^2\log(ep/|\theta_1|_0)$ with the procedure (1.4), we need to know the sparsity of θ_1 in advance, whereas our procedure adapts to the unknown sparsity of θ_1 and achieves the minimax optimal rate up to a logarithmic factor provided that $|\theta_1|_0 \le \bar{s}$ (see Section 3.4 for the lower bounds). This logarithmic factor is the price we pay for adaptation to the sparsity of θ_1. Note also that we can formulate a version of (1.4) when observations are missing ($\delta < 1$) by replacing Σ_n with $\tilde{\Sigma}_n$. In that case, our techniques of proof will give with probability at least $1 - \frac{1}{p}$

$$\|\hat{\theta}_o\hat{\theta}_o^\top - \theta_1\theta_1^\top\|_2^2 \le C'\bar{s}\tilde{\sigma}^2\frac{\log(ep/\bar{s})}{\delta^2 n}.$$

3. In the case where observations are missing ($\delta < 1$), Theorem 2 guarantees that recovery of the first principal component is still possible using the procedure (1.7). We observe the additional factor δ^{-2}. Consequently, the estimation accuracy of the procedure (1.7) will decrease as the proportion of observed entries δ decreases. We show in Section 3.4 below that the dependence of our bounds on δ^{-2} is sharp. In other words, there exists no statistical procedure that achieves an upper bound without the factor δ^{-2}. Thus, we can conclude that the factor δ^{-2} is the statistical price to pay to deal with missing observations in the principal component estimation problem. If we consider for instance microarray datasets where typically about 10% of the observations are missing (that is $\delta = 0.9$), then the optimal bound achieved for the first principal component estimation increases by a factor 1.24 as compared to the full observation framework ($\delta = 1$).

3.2. Study of approximately low-rank covariance matrices

We concentrate from now on the full observation case ($\delta = 1$). We now assume that Σ defined in (1.1) is also approximately low-rank and study the different implications of this additional condition. We recall that the effective rank of Σ is defined by $\mathbf{r}(\Sigma) = \mathrm{trace}(\Sigma)/\|\Sigma\|_\infty$. We say that Σ is approximately low-rank when $\mathbf{r}(\Sigma) \ll p$. Note also that the effective rank of an approximately low-rank covariance matrix can be estimated efficiently in our context by $\mathbf{r}(\tilde{\Sigma}_n)$. See [8] for more details.

First, we can propose a different control of the stochastic quantities $\mathbf{Z}_n(s)$. Note indeed that $\mathbf{Z}_n(s) \le \|\tilde{\Sigma}_n - \Sigma\|_\infty$ for any $1 \le s \le p$. We apply now Proposition 3 in [8] and get the following control on $\mathbf{Z}_n(s)$. Under the assumptions of Proposition 3, we have with probability at least $1 - e^{-t}$ that

$$\|\tilde{\Sigma}_n - \Sigma\|_\infty \le C\frac{\sigma_1}{c_1}\max\left\{\sqrt{\frac{\mathbf{r}(\Sigma)\,(t+\log(2p))}{n}}, \frac{\mathbf{r}(\Sigma)\,(t+\log(2p))}{n}\,(c_1 + t + \log n)\right\}, \tag{3.8}$$

where $C > 0$ is an absolute constant. We concentrate now on the high dimensional setting $p > n$. Assume that

$$n \ge c\mathbf{r}(\Sigma)\log^2(ep), \tag{3.9}$$

for some sufficiently large numerical constant $c > 0$. Taking $t = \log(ep)$, we get from the two above displays, with probability at least $1 - \frac{1}{ep}$ that

$$\|\tilde{\Sigma}_n - \Sigma\|_\infty \le c'\sigma_1\sqrt{\frac{\mathbf{r}(\Sigma)\log(ep)}{n}}, \tag{3.10}$$

where $c' > 0$ can depends only on c_1. Combining the previous display with Proposition 2 and a union bound argument, we immediately obtain the following control on $\mathbf{Z}_n(s)$.

Proposition 4. *Let the conditions of Proposition* 2 *be satisfied. In addition, let* (3.9) *be satisfied. Then we have*

$$\mathbb{P}\left(\bigcap_{s=1}^{p}\left\{\mathbf{Z}_n(s) \le c\sigma_1\sqrt{\min\{\mathbf{r}(\Sigma),\, s\}}\sqrt{\frac{\log(ep)}{n}}\right\}\right) \ge 1 - \frac{1}{p},$$

where $c > 0$ is a numerical constant that can depend only on c_1.

The motivation behind the new bound in Proposition 4 is the following. If (3.9) is satisfied, then we can remove the restriction $|\theta|_0 \le \bar{s}$ in (1.7) and we obtain exactly the estimator (1.5) that was the initial object of our interest. we can now investigate its statistical performances. Following the proof of Theorem 1, we establish the following result for (1.5).

Theorem 3. *Let the assumptions of Theorem* 1 *be satisfied. In addition, let* (3.9) *be satisfied. Take*

$$\lambda = C\frac{\sigma_1^2}{\sigma_1 - \sigma_2}\frac{\log(ep)}{n}, \tag{3.11}$$

where $C > 0$ is a large enough numerical constant. Then the estimator (1.5) *satisfies, with probability at least $1 - \frac{1}{p}$,*

$$\|\hat{\theta}_1\hat{\theta}_1^\top - \theta_1\theta_1^\top\|_2^2 \le C'|\theta_1|_0\tilde{\sigma}^2\frac{\log(ep)}{n}.$$

where $\tilde{\sigma} = \frac{\sigma_1}{\sigma_1 - \sigma_2}$ and $C' > 0$ is a numerical constant that can depend only on c_1.

Note that this result holds without any condition on the sparsity of θ_1. Of course, as we already commented for Theorem 1, the result is of statistical interest only when θ_1 is sparse: $|\theta_1|_0 \le \frac{n}{\tilde{\sigma}^2\log(ep)}$.

3.3. Data-driven choice of $\boldsymbol{\lambda}$

As we see in Theorem 3, the optimal choice of the regularization parameter depends on the largest and second largest eigenvalues of Σ. These quantities are typically unknown in practice. To circumvent this difficulty, we propose the following data-driven choice for the regularization parameter λ

$$\lambda_D = C\frac{\hat{\sigma}_1^2}{\hat{\sigma}_1 - \hat{\sigma}_2}\frac{\log(ep)}{n}, \tag{3.12}$$

where $C > 0$ is a numerical constant and $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are the two largest eigenvalues of $\tilde{\Sigma}_n$. If (3.9) is satisfied, then as a consequence of Proposition 3 in [8], $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are good estimators of σ_1 and σ_2. In order to guarantee that λ_D is a suitable choice, we will need a more restrictive condition on the number of measurements n than (3.9). This new condition involves in addition the "variance" $\tilde{\sigma}^2 = \frac{\sigma_1^2}{(\sigma_1-\sigma_2)^2}$:

$$n \geq c\tilde{\sigma}^2 \mathbf{r}(\Sigma)\log^2(ep), \tag{3.13}$$

where $c > 0$ is a sufficiently large numerical constant. As compared to (3.9), we observe the additional factor $\tilde{\sigma}^2$ in the above condition. We already noted that matrices Σ for which the difference $\sigma_1 - \sigma_2$ is small are statistically more difficult to estimate. We observe that the number of measurements needed to construct a suitable data-driven estimator also increases as the difference $\sigma_1 - \sigma_2$ decreases to 0.

We have the following result.

Lemma 2. *Let the conditions of Proposition* 3 *be satisfied, Assume in addition that* (3.13) *is satisfied. Let* λ_D *be defined in* (3.12) *with a sufficiently large numerical constant* $C > 0$. *Then, we have with probability at least* $1 - \frac{1}{p}$ *that*

$$\mathbf{Z}_n^2(s) \leq (\sigma_1 - \sigma_2)\lambda_D\, s, \quad \forall 1 \leq s \leq p,$$

and

$$\lambda_D \leq C' \frac{\sigma_1^2}{\sigma_1 - \sigma_2}\frac{\log(ep)}{n},$$

for some numerical constant $C' > 0$.

Consequently, the conclusion of Theorem 3 holds true for the estimator (1.5) with $\lambda = \lambda_D$ provided that (3.13) is satisfied.

3.4. Information theoretic lower bounds

We derive now minimax lower bounds for the estimation of the first principal component θ_1 in the missing observation framework.

Let $s_1 \geq 1$. We denote by $\mathcal{C} = \mathcal{C}_{s_1}(\sigma_1, \sigma_2)$ the class of covariance matrices Σ satisfying (1.1) with $\sigma_1 > \sigma_2$, $\theta_1 \in \mathcal{S}^p$ with $|\theta|_0 \leq s_1$ and Υ is a $p \times p$ symmetric positive semi-definite matrix with spectral norm $\|\Upsilon\|_\infty \leq 1$ and such that $\Upsilon\theta_1 = 0$. We prove now that the dependence of our estimation bounds on $\sigma_1 - \sigma_2, \delta, s_1, n, p$ in Theorem 2 is sharp in the minimax sense. Set $\bar{\sigma}^2 = \frac{\sigma_1\sigma_2}{(\sigma_1-\sigma_2)^2}$.

Theorem 4. *Fix* $\delta \in (0, 1]$ *and* $s_1 \geq 3$. *Let the integers* $n, p \geq 3$ *satisfy*

$$2\bar{\sigma}^2 s_1 \log(ep/s_1) \leq \delta^2 n. \tag{3.14}$$

Let $X_1, \ldots, X_n$ *be i.i.d. random vectors in* $\mathbb{R}^p$ *with covariance matrix* $\Sigma \in \mathcal{C}$. *We observe* n *i.i.d. random vectors* $Y_1, \ldots, Y_n \in \mathbb{R}^p$ *such that*

$$Y_i^j = \delta_{i,j} X_i^{(j)},\ 1 \leq i \leq n,\ 1 \leq j \leq p,$$

where $(\delta_{i,j})_{1\leq i\leq n,\, 1\leq j\leq p}$ *is an i.i.d. sequence of Bernoulli* $B(\delta)$ *random variables independent of* $X_1, \ldots, X_n$.

Then, there exist absolute constants $\beta \in (0,1)$ and $c > 0$ such that

$$\inf_{\hat{\theta}_1} \sup_{\Sigma \in \mathcal{C}} \mathbb{P}_\Sigma \left(\|\hat{\theta}_1 \hat{\theta}_1^\top - \theta_1 \theta_1^\top\|_2^2 > c\bar{\sigma}^2 \frac{s_1}{\delta^2 n} \log\left(\frac{ep}{s_1}\right)\right) \geq \beta, \tag{3.15}$$

where $\inf_{\hat{\theta}_1}$ denotes the infimum over all possible estimators $\hat{\theta}_1$ of θ_1 based on $Y_1, \dots, Y_n$.

Remark 1. For $s_1 = 1$, we can prove a similar lower bound with the factor δ^{-2} replaced by δ^{-1}. This is actually the right dependence on δ for 1-sparse vectors. We can indeed derive an upper bound of the same order for the selector $e_{\hat{\jmath}} = \operatorname{argmax}_{1 \leq j \leq p} \left(e_j^\top \tilde{\Sigma}_n e_j\right)$ where $e_1, \dots, e_p$ are the canonical vectors of $\mathbb{R}^p$.

For $s_1 = 2$, we can prove a lower bound of the form (3.15) without the logarithmic factor by comparing for instance the hypothesis $\theta_0 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $\theta_1 = \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$. Getting a lower bound for $s_1 = 2$ with the logarithmic factor remains an open question.

4. Proofs

4.1. Proof of Propositions 2 and 3

We start with the proof of Proposition 3.

Proof. For any $s \geq 1$, we have

$$\mathbf{Z}_n(s) \leq \delta^{-1}\mathbf{Z}_n^{(1)}(s) + \delta^{-2}\mathbf{Z}_n^{(2)}(s) \tag{4.1}$$

where

$$\mathbf{Z}_n^{(1)}(s) = \max_{\theta \in \mathcal{S}_s^p} \left\{ \left| \theta^\top \operatorname{diag}\left(\Sigma_n^{(\delta)} - \Sigma^{(\delta)}\right)\theta \right| \right\},$$
$$\mathbf{Z}_n^{(2)}(s) = \max_{\theta \in \mathcal{S}_s^p} \left\{ \left| \theta^\top \left(A_n^{(\delta)} - A^{(\delta)}\right)\theta \right| \right\}$$

with $A_n^{(\delta)} = \Sigma_n^{(\delta)} - \operatorname{diag}(\Sigma_n^{(\delta)})$, $A^{(\delta)} = \Sigma^{(\delta)} - \operatorname{diag}(\Sigma^{(\delta)})$ and $\Sigma^{(\delta)} = \delta^2[\Sigma - \operatorname{diag}(\Sigma)] + \delta \operatorname{diag}(\Sigma)$.

Before we proceed with the study of the empirical processes $\mathbf{Z}_n^{(1)}(s)$ and $\mathbf{Z}_n^{(2)}(s)$, we need to introduce some additional notations. Define

$$Y = (\delta_1 X^{(1)}, \dots, \delta_p X^{(p)})^\top,$$

where $\delta_1, \dots, \delta_p$ are i.i.d. Bernoulli random variables with parameter δ and independent from X. Denote by $\mathbb{E}_\delta$ and $\mathbb{E}_X$ the expectations w.r.t. $(\delta_1, \dots, \delta_p)$ and X respectively.

We now proceed with the study of $\mathbf{Z}_n^{(2)}(s)$. For any $s \geq 1$ and any fixed $\theta \in \mathcal{S}_s^p$, we have

$$\theta^\top \left(A_n^{(\delta)} - A^{(\delta)}\right)\theta = \frac{1}{n}\sum_{i=1}^n \left[\theta^\top \left(Y_i Y_i^\top - \operatorname{diag}(Y_i Y_i^\top)\right)\theta - \delta^2 \theta^\top \left(\Sigma - \operatorname{diag}(\Sigma)\right)\theta\right].$$

Set

$$Z_i = \left[\left(Y_iY_i^\top - \mathrm{diag}(Y_iY_i^\top)\right) - \delta^2\left(\Sigma - \mathrm{diag}(\Sigma)\right)\right], \quad 1 \le i \le n,$$

and

$$Z = \left[\left(YY^\top - \mathrm{diag}(YY^\top)\right) - \delta^2\left(\Sigma - \mathrm{diag}(\Sigma)\right)\right].$$

We note that $Z, Z_1, \ldots, Z_n$ are i.i.d with zero mean.

For any $\theta = (\theta^{(1)}, \ldots, \theta^{(p)})^\top \in \mathbb{R}^p$ and $\delta_1, \ldots, \delta_p \in \{0,1\}^p$, we set $\theta_\delta = (\delta_1\theta^{(1)}, \ldots, \delta_p\theta^{(p)})^\top$. Next, for any $\theta \in \mathcal{S}^p$ and $\delta_1, \ldots, \delta_p$, we have by assumption on X that

$$\begin{aligned}
|\theta^\top Z\theta| &\le |\theta_\delta^\top[XX^\top - \mathrm{diag}(XX^\top)]\theta_\delta| + \delta^2 \max_{\theta\in\mathcal{S}^p}\left\{\left|\theta^\top[\Sigma - \mathrm{diag}(\Sigma)]\theta\right|\right\} \\
&\le \max\left\{(\theta_\delta^\top X)^2, \sum_{j=1}^p(\delta_j\theta^{(j)}X^{(j)})^2\right\} + \delta^2 \max_{\theta\in\mathcal{S}^p}\left\{\max\{\theta^\top\Sigma\theta, \theta^\top\mathrm{diag}(\Sigma))\theta\}\right\} \\
&\le K + \delta^2\sigma_1 \le 2(K\vee\sigma_1), \quad \text{a.s.}
\end{aligned}$$

For any $\theta \in \mathcal{S}_s^p$, we establish in Section 4.9 below that

$$\mathbb{E}\left[(\theta^\top Z\theta)^2\right] \le c\frac{1}{c_1^2}\delta^2\sigma_{\max}^2(s),$$

for some numerical constant $c > 0$. Combining the last two displays, we deduce, for any $m \ge 2$ and any $\theta \in \mathcal{S}_s^p$, that

$$\mathbb{E}\left[|\theta^\top Z\theta|^m\right] \le \mathbb{E}\left[(\theta^\top Z\theta)^2\right][2(K\vee\sigma_1)]^{m-2} \le \frac{c}{c_1^2}\delta^2\sigma_{\max}^2(s)[2(K\vee\sigma_1)]^{m-2}.$$

Thus, for any fixed $\theta \in \mathcal{S}_s^p$, Bernstein's inequality gives for any $t' > 0$ that

$$\mathbb{P}\left(\left|\theta^\top(A_n^{(\delta)} - A^{(\delta)})\theta\right| > C\max\left\{\frac{\delta\sigma_{\max}(s)}{c_1}\sqrt{\frac{t'}{n}}, (K\vee\sigma_1)\frac{t'}{n}\right\}\right) \le 2e^{-t'},$$

where $C > 0$ is an absolute constant. Note now that

$$\mathbf{Z}_n^{(2)}(s) = \max_{\theta\in\mathcal{S}_s^p}\left\{\left|\theta^\top(A_n^{(\delta)} - A^{(\delta)})\theta\right|\right\} = \max_{J\in[p]\,:\,|J|=s}\ \max_{\theta\in\mathcal{S}^p(J)}\left\{\left|\theta^\top(A_n^{(\delta)} - A^{(\delta)})\theta\right|\right\}.$$

For any fixed $J \in [p]$ such that $|J| = s$, Lemma 1 guarantees the existence of a $\frac{1}{4}$-net $\mathcal{N}(J)$ such that $|\mathcal{N}(J)| \le 9^s$ and

$$\max_{\theta\in\mathcal{S}^p(J)}\left\{\left|\theta^\top(A_n^{(\delta)} - A^{(\delta)})\theta\right|\right\} \le 2\max_{\theta\in\mathcal{N}(J)}\left\{\left|\theta^\top(A_n^{(\delta)} - A^{(\delta)})\theta\right|\right\}.$$

Combining the last three displays with a union bound argument, we get for $t' = t + s\log(9) + s\log\left(\frac{ep}{s}\right)$ and $t > 0$ that

$$\mathbb{P}\left(\mathbf{Z}_n^{(2)}(s) > \zeta_n^{(2)}(s,t)\right) \le 2e^{-t}, \tag{4.2}$$

with

$$\zeta_n^{(2)}(s,t) = C\max\Bigg\{\frac{\delta\sigma_{\max}(s)}{c_1}\sqrt{\frac{t+s\log(9)+s\log\left(\frac{ep}{s}\right)}{n}}, \\ (K\vee\sigma_1)\frac{t+s\log(9)+s\log\left(\frac{ep}{s}\right)}{n}\Bigg\}.$$

We proceed similarly to treat the quantity $\mathbf{Z}_n^{(1)}(s)$. We first note that

$$\theta^\top\left(\mathrm{diag}(\Sigma_n^{(\delta)}-\Sigma^{(\delta)})\right)\theta = \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^p\left(\left[\theta^{(j)}Y_i^{(j)}\right]^2 - \delta\Sigma_{j,j}\left(\theta^{(j)}\right)^2\right).$$

Next, we have for any $m\geq 2$ and any $\theta\in\mathcal{S}_s^p$

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg(\sum_{j=1}^p\left(\theta^{(j)}Y^{(j)}\right)^2 - \delta\Sigma_{j,j}\left(\theta^{(j)}\right)^2\Bigg)^m\Bigg] \\
&\quad\leq \sum_{j=1}^p\left(\theta^{(j)}\right)^2\left(2^{m-1}\mathbb{E}\left[\left(Y^{(j)}\right)^{2m}\right]+2^{m-1}\delta^m\Sigma_{j,j}^m\right) \\
&\quad\leq \sum_{j=1}^p(\theta^{(j)})^2\left[\delta 2^{m-1}\mathbb{E}_X\left[\left(X^{(j)}\right)^{2m}\right]+\delta^m 2^{m-1}\Sigma_{j,j}^m\right] \\
&\quad\leq \sum_{j=1}^p(\theta^{(j)})^2\left[2\delta m!\left(4\|X^{(j)}\|_{\psi_2}^2\right)^m+\delta^m 2^{m-1}\Sigma_{j,j}^m\right] \\
&\quad\leq \sum_{j=1}^p(\theta^{(j)})^2\left[2\delta m!\left(\frac{4}{c_1}\Sigma_{j,j}\right)^m+\delta^m 2^{m-1}\Sigma_{j,j}^m\right] \\
&\quad\leq \frac{m!}{2}\left(\frac{C}{c_1\wedge 1}\sqrt{\delta}\sigma_{\max}(1)\right)^2\left(\frac{C\sigma_{\max}(1)}{c_1\wedge 1}\right)^{m-2},
\end{aligned}$$

for some numerical constant $C>0$. Then, for any $\theta\in\mathcal{S}_s^p$, Bernstein's inequality (Proposition 2.9 in [9]) gives for any $t'>0$ that

$$\mathbb{P}\left(\left|\theta^\top\left(\mathrm{diag}(\Sigma_n^{(\delta)}-\Sigma^{(\delta)})\right)\theta\right| > C\frac{\sigma_{\max}(1)}{c_1\wedge 1}\max\left\{\sqrt{\frac{\delta t'}{n}},\frac{t'}{n}\right\}\right)\leq 2e^{-t'},$$

for some numerical constant $C>0$. Next, a similar union bound argument as we used above for $\mathbf{Z}_n^{(2)}(s)$ gives

$$\mathbb{P}\left(\mathbf{Z}_n^{(1)}(s) > \zeta_n^{(1)}(s,t)\right)\leq 2e^{-t}, \tag{4.3}$$

with

$$\zeta_n^{(1)}(s,t) = C\frac{\sigma_{\max}(1)}{c_1\wedge 1}\max\left\{\sqrt{\frac{\delta(t+s\log\left(\frac{ep}{s}\right))}{n}},\frac{t+s\log\left(\frac{ep}{s}\right)}{n}\right\},$$

for some numerical constant $C > 0$. Next, easy computations give $\frac{1}{\delta}\zeta_n^{(1)}(s,t) + \frac{1}{\delta^2}\zeta_n^{(2)}(s,t) \leq \bar{\zeta}_n(s,t)$ where

$$\bar{\zeta}_n(s,t) = C\frac{\sigma_1}{c_1 \wedge 1}\max\left\{\sqrt{\frac{t + s\log\left(\frac{ep}{s}\right)}{\delta^2 n}}, \left(\frac{K}{\sigma_1} \vee 1\right)\frac{t + s\log\left(\frac{ep}{s}\right)}{\delta^2 n}\right\},$$

for some numerical constant $C > 0$. Combining (4.1), (4.2) and (4.3) with a union bound argument, we get, for any $s = 1, \ldots, p$, that

$$\mathbb{P}\left(\mathbf{Z}_n(s) > \bar{\zeta}_n(s,t)\right) \leq 4e^{-t}.$$

Finally, using again a union bound argument, we get from the previous display that

$$\mathbb{P}\left(\bigcap_{s=1}^{p}\left\{\mathbf{Z}_n(s) > \bar{\zeta}_n(s,t)\right\}\right) \leq 4pe^{-t}.$$

Replacing t by $t + \log(ep)$ and up to a rescaling of the constants, we get that

$$\mathbb{P}\left(\bigcap_{s=1}^{p}\left\{\mathbf{Z}_n(s) > \bar{\zeta}_n(s,t)\right\}\right) \leq e^{-t}, \tag{4.4}$$

for some numerical constant $C > 0$. Finally, taking $t = \log(ep)$ yields the result. □

The proof of Proposition 2 is essentially the same as that of Proposition 3. The only difference is a sharper control of the moment $\mathbb{E}[|\theta^\top Z\theta|^m]$, which yields a smaller bound in the large deviation regime. Indeed, we have for any $m \geq 2$ and any $\theta \in \mathcal{S}_s^p$ that

$$\begin{aligned}\mathbb{E}[|\theta^\top Z\theta|^m] &\leq 2^{m-1}\mathbb{E}[(\theta^\top X)^{2m}] + 2^{m-1}\sum\nolimits_{j=1}^{p}(\theta^{(j)})^2\mathbb{E}[(X^{(j)})^{2m}]\\ &\leq \frac{m!}{2}\left(C\frac{\sigma_{\max}(s)}{c_1}\right)^m,\end{aligned}$$

for some sufficiently large numerical constant $C > 0$, where we have (2.1) and Assumption 1. Therefore, we can apply Bernstein's inequality with $\sigma = K = \frac{\sigma_{\max}(s)}{c_1}$. The rest of the proof is identical to that of Proposition 3.

4.2. Proof of Theorems 1 and 2

We start with the proof of Theorem 2. We will use the following lemma in order to prove our results.

Lemma 3. *Let $\theta \in \mathcal{S}^p$. Let $\Sigma \in \mathbb{R}^{p\times p}$ be a symmetric positive semi-definite matrix with largest eigenvalue σ_1 of multiplicity 1 and second largest eigenvalue σ_2. Then, for any $\theta \in \mathcal{S}^p$, we have*

$$\frac{1}{2}(\sigma_1 - \sigma_2)\|\theta\theta^\top - \theta_1\theta_1^\top\|_2^2 \leq \langle\Sigma, \theta_1\theta_1^\top - \theta\theta^\top\rangle.$$

See Lemma 3.2.1 in [17] for a proof of this result.

Proof. We have by definition of $\hat{\theta}_1$ and in view of Lemma 3 that

$$\begin{aligned}
\frac{\sigma_1-\sigma_2}{2}\|\hat{\theta}_1\hat{\theta}_1^\top-\theta_1\theta_1^\top\|_2^2 &\le \left\langle \Sigma, \theta_1\theta_1^\top-\hat{\theta}_1\hat{\theta}_1^\top\right\rangle \\
&\le \left\langle \Sigma-\tilde{\Sigma}_n, \theta_1\theta_1^\top-\hat{\theta}_1\hat{\theta}_1^\top\right\rangle + \left\langle \tilde{\Sigma}_n, \theta_1\theta_1^\top-\hat{\theta}_1\hat{\theta}_1^\top\right\rangle \\
&\le \left\langle \Sigma-\tilde{\Sigma}_n, \theta_1\theta_1^\top-\hat{\theta}_1\hat{\theta}_1^\top\right\rangle + \left[\theta_1^\top\tilde{\Sigma}_n\theta_1-\lambda|\theta_1|_0\right] \\
&\quad - \left[\hat{\theta}_1^\top\tilde{\Sigma}_n\hat{\theta}_1-\lambda|\hat{\theta}_1|_0\right]+\lambda|\theta_1|_0-\lambda|\hat{\theta}_1|_0 \\
&\le \left\langle \Sigma-\tilde{\Sigma}_n, \theta_1\theta_1^\top-\hat{\theta}_1\hat{\theta}_1^\top\right\rangle +\lambda|\theta_1|_0-\lambda|\hat{\theta}_1|_0 \\
&\le \|\Pi_{\hat{J}\cup J_1}(\Sigma-\tilde{\Sigma}_n)\Pi_{\hat{J}\cup J_1}\|_\infty\sqrt{2}\|\theta_1\theta_1^\top-\hat{\theta}_1\hat{\theta}_1^\top\|_2 \\
&\quad +\lambda|\theta_1|_0-\lambda|\hat{\theta}_1|_0,
\end{aligned}$$

where $\Pi_{\hat{J}\cup J_1}$ is the orthogonal projection onto l.s.$(e_j,\, j\in\hat{J}\cup J_1)$, $\hat{J}=J(\hat{\theta}_1)$ and $J_1=J(\theta_1)$.

Thus we get

$$\begin{aligned}
\|\hat{\theta}_1\hat{\theta}_1^\top-\theta_1\theta_1^\top\|_2^2 &\le \frac{2\sqrt{2}}{\sigma_1-\sigma_2}\|\Pi_{\hat{J}\cup J_1}(\Sigma-\tilde{\Sigma}_n)\Pi_{\hat{J}\cup J_1}\|_\infty\|\theta_1\theta_1^\top-\hat{\theta}_1\hat{\theta}_1^\top\|_2 \\
&\quad + \frac{2}{\sigma_1-\sigma_2}\lambda\left(|\theta_1|_0-|\hat{\theta}_1|_0\right).
\end{aligned}$$

Set

$$A=\|\hat{\theta}_1\hat{\theta}_1^\top-\theta_1\theta_1^\top\|_2,\quad \beta=\frac{2\sqrt{2}}{\sigma_1-\sigma_2}\|\Pi_{\hat{J}\cup J_1}(\Sigma-\tilde{\Sigma}_n)\Pi_{\hat{J}\cup J_1}\|_\infty$$

and

$$\gamma=\frac{2}{\sigma_1-\sigma_2}\lambda\left(|\theta_1|_0-|\hat{\theta}_1|_0\right).$$

The above display becomes

$$A^2-\beta A-\gamma\le 0.$$

Next, basic computations on second order polynoms yield the following necessary condition on A

$$A\le\frac{\beta+\sqrt{\beta^2+4\gamma}}{2}\le\sqrt{\frac{2\beta^2+4\gamma}{2}}=\sqrt{\beta^2+2\gamma},$$

where we have used concavity of $x\to\sqrt{x}$.

Set $\hat{s}_1=|\hat{\theta}_1|_0$ and $s_1=|\theta_1|_0$. Note that $|\hat{J}\cup J_1|\le\hat{s}_1+s_1\le 2\bar{s}$. Next, we have in view of Proposition 3 and under the condition $2\frac{K^2}{\sigma_1^2}\bar{s}\log^2(ep)\le\delta^2 n$, with probability at least $1-\frac{1}{p}$ that

$$\|\Pi_{\hat{J}\cup J_1}\left(\tilde{\Sigma}_n-\Sigma\right)\Pi_{\hat{J}\cup J_1}\|_\infty^2\le c^2\frac{\sigma_1^2}{c_1^2\wedge 1}\frac{\log(ep)}{\delta^2 n}(\hat{s}_1+s_1).$$

Thus, we get, with probability at least $1 - \frac{1}{p}$ that

$$\beta^2 + 2\gamma \le \frac{8}{(\sigma_1-\sigma_2)^2}\left(c^2\frac{\sigma_1^2}{c_1^2\wedge 1}\frac{\log(ep)}{\delta^2 n} - \frac{\sigma_1-\sigma_2}{8}\lambda\right)\hat{s}_1 + \frac{8}{(\sigma_1-\sigma_2)^2}\left[c^2\frac{\sigma_1^2}{c_1^2\wedge 1}\frac{\log(ep)}{\delta^2 n} + (\sigma_1-\sigma_2)\frac{\lambda}{4}\right]s_1.$$

Next, we note that

$$c^2\frac{\sigma_1^2}{c_1^2\wedge 1}\frac{\log(ep)}{\delta^2 n} \le \frac{\sigma_1-\sigma_2}{8}\lambda,$$

provided that λ satisfies (3.7) with a large enough numerical constant $C > 0$. Combining the last two displays, we get for λ taken as in (3.7), with probability at least $1 - \frac{1}{p}$ that

$$\beta^2 + 2\gamma \le C'\frac{\sigma_1^2}{(\sigma_1-\sigma_2)^2}\frac{\log(ep)}{\delta^2 n}s_1,$$

where $C' > 0$ can depend only on c_1. □

The proof of Theorem 1 is virtually the same and is left to the reader.

4.3. Proof of Lemma 2

Proof. A standard matrix perturbation argument gives $|\hat{\sigma}_j - \sigma_j| \le \|\tilde{\Sigma}_n - \Sigma\|_\infty$, $\forall 1 \le j \le p$. Consequently, we get

$$\begin{aligned}\sigma_1 - \|\tilde{\Sigma}_n - \Sigma\|_\infty \le \hat{\sigma}_1 &\le \sigma_1 + \|\tilde{\Sigma}_n - \Sigma\|_\infty,\\ \hat{\sigma}_1 - \hat{\sigma}_2 &= \hat{\sigma}_1 - \sigma_1 + \sigma_1 - \sigma_2 + \sigma_2 - \hat{\sigma}_2\\ &\ge \sigma_1 - \sigma_2 - (|\hat{\sigma}_1 - \sigma_1| + |\hat{\sigma}_2 - \sigma_2|)\\ &\ge \sigma_1 - \sigma_2 - 2\|\tilde{\Sigma}_n - \Sigma\|_\infty\end{aligned}$$

and similarly

$$\hat{\sigma}_1 - \hat{\sigma}_2 \le \sigma_1 - \sigma_2 + 2\|\tilde{\Sigma}_n - \Sigma\|_\infty.$$

Combining now (3.8) with (3.13) with a sufficiently large constant $c > 0$, we get with probability at least $1 - \frac{1}{p}$ that

$$\frac{1}{2}\sigma_1 \le \hat{\sigma}_1 \le 2\sigma_1, \quad \frac{1}{2}(\sigma_1 - \sigma_2) \le \hat{\sigma}_1 - \hat{\sigma}_2 \le 2(\sigma_1 - \sigma_1),$$

and

$$\frac{\sigma_1^2}{8(\sigma_1-\sigma_2)} \le \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1-\hat{\sigma}_2} \le \frac{8\sigma_1^2}{\sigma_1-\sigma_2}. \tag{4.5}$$

Next, Proposition 4 gives, with probability at least $1-\frac{1}{p}$, for any $1 \le s \le p$ that

$$\begin{aligned}\mathbf{Z}_n^2(s) &\le c^2\sigma_1^2\frac{s\log(ep)}{n}\\ &\le c^2(\sigma_1-\sigma_2)\frac{\sigma_1^2}{\sigma_1-\sigma_2}\frac{\log(ep)}{n}s.\end{aligned}$$

Combining the last two displays with a union bound argument, we get with probability at least $1-\frac{2}{p}$ that

$$\mathbf{Z}_n^2(s) \le 8\frac{c^2}{C}(\sigma_1-\sigma_2)\lambda_D s, \quad \forall 1 \le s \le p.$$

Choosing the numerical constant $C>0$ large enough in (3.12), we get from the previous display with probability at least $1-\frac{2}{p}$ that

$$\mathbf{Z}_n^2(s) \le (\sigma_1-\sigma_2)\lambda_D s, \quad \forall 1 \le s \le p.$$

Up to a rescaling of the constants, we can assume that the above inequality holds true with probability at least $1-\frac{1}{p}$. This gives the first inequality in Lemma 2.

The second inequality is immediate in view of (3.12) and (4.5). □

4.4. Proof of Theorem 4

This proof uses standard tools of the minimax theory (see for instance [15]). The proof is more technical in the missing observation case ($\delta < 1$) in order the get the sharp dependence δ^{-2} factor. In order to improve readability, we will decompose the proof into several technical facts and proceed first with the main arguments. Then, we give the proofs for the technical facts.

Proof. We consider the following class $\tilde{\mathcal{C}}$ of $p \times p$ covariance matrices

$$\begin{aligned}\tilde{\mathcal{C}} = \big\{\Sigma_\theta = \Sigma(\theta,\sigma_1,\sigma_2) = \sigma_1\theta\theta^\top + \sigma_2(I_p-\theta\theta^\top),\ \forall\theta\in\mathcal{S}^p\ :\ |\theta|_0 \le s_1,\\ \forall\sigma_1 \ge (1+\eta)\sigma_2 > 0\big\},\end{aligned} \tag{4.6}$$

where I_p is the $p\times p$ identity matrix and $\eta > 0$ is some absolute constant.

Note that the set $\tilde{\mathcal{C}}$ contains only full rank matrices with the same determinant and whose first principal component θ is s_1-sparse. Note also that $\tilde{\mathcal{C}} \subset \mathcal{C}$. Indeed, it is easy to see that σ_1 is the largest eigenvalue of Σ with multiplicity 1 and associated eigenvector θ with less than s_1 nonzero components, $\|I_p-\theta\theta^\top\|_\infty = 1$ and $(I_p-\theta\theta^\top)\theta = 0$.

Next, we define $\omega_0 = (1,1,0,\dots,0)\in\{0,1\}^p$ and

$$\Omega = \Big\{\omega = (\omega^{(1)},\dots,\omega^{(p)})\in\{0,1\}^p\ :\ \omega^{(1)}=\omega^{(2)}=1,\ |\omega|_0 = s_1\Big\}\cup\{\omega_0\}.$$

A Varshamov-Gilbert's type bound (see for instance Lemma 4.10 in [9]) guarantees the existence of a subset $\mathcal{N}\subset\Omega$ with cardinality $\log(\mathrm{Card}(\tilde{\mathcal{N}})) \ge C_1(s_1 -$

2) $\log(e(p-2)/(s_1-2))$ containing ω_0 such that, for any two distinct elements ω and ω' of $\mathcal{N}$, we have

$$|\omega - \omega'|_0 \geq \frac{s_1}{8}$$

where $C_1 > 0$ is an absolute constant.

Set $\epsilon = a\sqrt{\frac{\bar{\sigma}^2 s_1 \log(ep/s_1)}{\delta^2 n}}$ for some numerical constant $a \in (0, 1/\sqrt{2})$. Note that we have $\epsilon < 1/2$ under Condition (3.14). Consider now the following set of normalized vectors

$$\Theta = \left\{ \theta(\omega) = \left(\sqrt{\frac{1-\epsilon^2}{2}}, \sqrt{\frac{1-\epsilon^2}{2}}, \frac{\omega^{(3)}\epsilon}{\sqrt{s_1-2}}, \ldots, \frac{\omega^{(p)}\epsilon}{\sqrt{s_1-2}} \right)^\top \,:\, \omega \in \mathcal{N} \setminus \{\omega_0\} \right\}$$
$$\cup \left\{ \theta_0 = \frac{1}{\sqrt{2}} \omega_0^\top \right\}. \tag{4.7}$$

Note that $|\Theta| = |\mathcal{N}|$ and $|\theta|_0 \leq s_1$ for any $\theta \in \Theta$.

Lemma 4. *For any $a > 0$ and any distinct $\theta_1, \theta_2 \in \Theta$, we have*

$$\|\theta_1\theta_1^\top - \theta_2\theta_2^\top\|_2^2 \geq \frac{a^2}{8}\bar{\sigma}^2 \frac{s_1 \log(ep/s_1)}{\delta^2 n}. \tag{4.8}$$

Clearly, for any $\theta \in \Theta$, we have $\Sigma_\theta \in \tilde{\mathcal{C}}$. We introduce now the class

$$\mathcal{C}(\Theta) = \left\{ \Sigma_\theta \in \tilde{\mathcal{C}} \,:\, \theta \in \Theta \right\}.$$

Denote by $\mathbb{P}_\Sigma$ the distribution of $(Y_1, \ldots, Y_n)$. For any $\theta, \theta' \in \mathcal{S}^p$, the Kullback-Leibler divergences $K\big(\mathbb{P}_{\Sigma_{\theta'}}, \mathbb{P}_{\Sigma_\theta}\big)$ between $\mathbb{P}_{\Sigma_{\theta'}}$ and $\mathbb{P}_{\Sigma_\theta}$ is defined by

$$K\left(\mathbb{P}_{\Sigma_{\theta'}}, \mathbb{P}_{\Sigma_\theta}\right) = \mathbb{E}_{\Sigma_{\theta'}} \log\left(\frac{d\mathbb{P}_{\Sigma_{\theta'}}}{d\mathbb{P}_{\Sigma_\theta}}\right).$$

We have the following result

Lemma 5. *Let $X_1, \ldots, X_n \in \mathbb{R}^p$ be i.i.d. $N(0, \Sigma)$ with $\Sigma = \Sigma_\theta \in \mathcal{C}(\Theta)$. Assume that $\frac{\sigma_1}{\sigma_2} \geq 1 + \eta$ for some absolute $\eta > 0$. Taking $a > 0$ sufficiently small, we have for any $\theta' \in \mathcal{S}^p$, that*

$$K\big(\mathbb{P}_{\Sigma_{\theta'}}, \mathbb{P}_{\Sigma_{\theta_0}}\big) \leq \frac{a^2}{2} s_1 \log\left(\frac{ep}{s_1}\right).$$

Thus, we have that

$$\frac{1}{\mathrm{Card}(\Theta) - 1} \sum_{\theta \in \Theta \setminus \{\theta_0\}} K(\mathbb{P}_{\Sigma_\theta}\mathbb{P}_{\Sigma_{\theta_0}}) \;\leq\; \alpha \log\big(\mathrm{Card}(\Theta) - 1\big) \tag{4.9}$$

is satisfied for any $\alpha > 0$ if $a > 0$ is chosen as a sufficiently small numerical constant depending on α. In view of (4.8) and (4.9), (3.15) now follows by application of Theorem 2.5 in [15]. □

4.5. Proof of Lemma 4

Proof. For any distinct $\theta_1, \theta_2 \in \Theta$, we have

$$|\theta_1 - \theta_2|_2^2 \geq \frac{1}{8}\epsilon^2 = \frac{a^2}{8}\bar{\sigma}^2 \frac{s_1 \log(ep/s_1)}{\delta^2 n}.$$

Next, we need to compare $\|\theta_1\theta_1^\top - \theta_2\theta_2^\top\|_2$ to $|\theta_1 - \theta_2|_2$.

For any $\theta_1, \theta_2 \in \Theta$, we have

$$\begin{aligned}\|\theta_1\theta_1^\top - \theta_2\theta_2^\top\|_2^2 &= 2 - 2(\theta_1^\top\theta_2)^2 \\ &= |\theta_1|_2^2 + |\theta_2|_2^2 - 2(\theta_1^\top\theta_2)^2 \\ &= |\theta_1 - \theta_2|_2^2 + 2[(\theta_1^\top\theta_2) - (\theta_1^\top\theta_2)^2].\end{aligned}$$

We immediately get from the previous display that $\|\theta_1\theta_1^\top - \theta_2\theta_2^\top\|_2 \geq |\theta_1 - \theta_2|_2$ for any $\theta_1, \theta_2 \in \Theta$ since $\theta_1^\top\theta_2 > 0$ for any $\theta_1, \theta_2 \in \Theta$ when $\epsilon < \frac{1}{2}$. □

4.6. Proof of Lemma 5

Recall that $X_1, \ldots, X_n \in \mathbb{R}^p$ are i.i.d. $N(0, \Sigma)$ with $\Sigma = \Sigma_\theta \in \mathcal{C}(\Theta)$. For any $1 \leq i \leq n$, set $\delta_i = (\delta_{i,1}, \ldots, \delta_{i,p})^\top \in \mathbb{R}^p$. We note that $\delta_1, \ldots, \delta_n$ are random vectors in $\mathbb{R}^p$ with i.i.d. entries $\delta_{i,j} \sim B(\delta)$ and independent from $(X_1, \ldots, X_n)$. Recall that the observations $Y_1, \ldots, Y_n$ satisfies $Y_i^{(j)} = \delta_{i,j}X_i^{(j)}$. Denote by $\mathbb{P}_\Sigma$ the distribution of $(Y_1, \ldots, Y_n)$ and by $\mathbb{P}_\Sigma^{(\delta)}$ the conditional distribution of $(Y_1, \ldots, Y_n)$ given $(\delta_1, \ldots, \delta_n)$. Next, we note that for any $1 \leq i \leq n$ the conditional random variables $Y_i \mid (\delta_1, \ldots, \delta_n)$ are independent Gaussian vectors $N(0, \Sigma_\theta^{(\delta_i)})$, where

$$(\Sigma_\theta^{(\delta_i)})_{j,k} = \begin{cases} \delta_{i,j}\delta_{i,k}\Sigma_{j,k} & \text{if } j \neq k, \\ \delta_{i,j}\Sigma_{j,j} & \text{otherwise.} \end{cases}$$

Thus, we have $\mathbb{P}_{\Sigma_\theta}^{(\delta)} = \otimes_{i=1}^n \mathbb{P}_{\Sigma_\theta^{(\delta_i)}}$. Denote respectively by $\mathbb{P}_\delta$ and $\mathbb{E}_\delta$ the probability distribution of $(\delta_1, \ldots, \delta_n)$ and the associated expectation. We also denote by $\mathbb{E}_{\Sigma_\theta}$ and $\mathbb{E}_{\Sigma_\theta}^{(\delta)}$ the expectation and conditional expectation associated respectively with $\mathbb{P}_{\Sigma_\theta}$ and $\mathbb{P}_{\Sigma_\theta}^{(\delta)}$.

Next, for any $\theta, \theta' \in \mathcal{S}^p$, the Kullback-Leibler divergences $K(\mathbb{P}_{\Sigma_{\theta'}}, \mathbb{P}_{\Sigma_\theta})$ between $\mathbb{P}_{\Sigma_{\theta'}}$ and $\mathbb{P}_{\Sigma_\theta}$ satisfies

$$\begin{aligned}K\left(\mathbb{P}_{\Sigma_{\theta'}}, \mathbb{P}_{\Sigma_\theta}\right) &= \mathbb{E}_{\Sigma_{\theta'}} \log\left(\frac{d\mathbb{P}_{\Sigma_{\theta'}}}{d\mathbb{P}_{\Sigma_\theta}}\right) = \mathbb{E}_{\Sigma_{\theta'}} \log\left(\frac{d(\mathbb{P}_\delta \otimes \mathbb{P}_{\Sigma_{\theta'}}^{(\delta)})}{d(\mathbb{P}_\delta \otimes \mathbb{P}_{\Sigma_\theta}^{(\delta)})}\right) \\ &= \mathbb{E}_\delta \mathbb{E}_{\Sigma_{\theta'}}^{(\delta)} \log\left(\frac{d\mathbb{P}_{\Sigma_{\theta'}}^{(\delta)}}{d\mathbb{P}_{\Sigma_\theta}^{(\delta)}}\right) = \mathbb{E}_\delta K\left(\mathbb{P}_{\Sigma_{\theta'}}^{(\delta)}, \mathbb{P}_{\Sigma_\theta}^{(\delta)}\right) \\ &= \sum_{i=1}^n \mathbb{E}_{\delta_i} K\left(\mathbb{P}_{\Sigma_{\theta'}^{(\delta_i)}}, \mathbb{P}_{\Sigma_\theta^{(\delta_i)}}\right). \qquad (4.10)\end{aligned}$$

Set $\theta_{\delta_i} = (\delta_{i,1}\theta^{(1)}, \dots, \delta_{i,p}\theta^{(p)})^\top$. In view of (4.6), we have

$$\Sigma_\theta^{(\delta_i)} = \left[(\sigma_1 - \sigma_2)|\theta_{\delta_i}|_2^2 + \sigma_2\right] \Pi_{\theta,\delta_i} + \sigma_2 \left(I_p^{(\delta_i)} - \Pi_{\theta,\delta_i}\right), \tag{4.11}$$

and Π_{θ,δ_i} is the orthogonal projection onto $l.s.(\theta_{\delta_i})$ (Note indeed that we have in general $|\theta_{\delta_i}|_2 \leq 1$, therefore $\Pi_{\theta,\delta_i} = |\theta_{\delta_i}|_2^{-2}\theta_{\delta_i}\theta_{\delta_i}^\top$.) For any $\theta \in \Theta$, we set $\sigma_1(\theta) = (\sigma_1 - \sigma_2)|\theta_{\delta_i}|_2^2 + \sigma_2$.

- **Fact 1:** For any $1 \leq i \leq n$, any $\theta, \theta' \in \mathcal{S}^p$ and any realization of $\delta_i \in \{0,1\}^p$, we have

$$K\big(\mathbb{P}_{\Sigma_{\theta'}^{(\delta_i)}}, \mathbb{P}_{\Sigma_\theta^{(\delta_i)}}\big) = \frac{1}{2}\left(\frac{\sigma_2}{\sigma_1(\theta)} + \frac{\sigma_1(\theta')}{\sigma_2} - 2\right) + \frac{1}{2}\log\left(\frac{\sigma_1(\theta)}{\sigma_1(\theta')}\right) + \frac{1}{2}\mathrm{tr}\left(\Pi_{\theta,\delta_i}\Pi_{\theta',\delta_i}\right)\left[\frac{\sigma_1(\theta')}{\sigma_1(\theta)} + 1 - \frac{\sigma_2}{\sigma_1(\theta)} - \frac{\sigma_1(\theta')}{\sigma_2}\right].$$

We apply Fact 1 with $\theta = \theta_0 = \frac{1}{\sqrt{2}}\omega_0^\top$ and take the expectation w.r.t. δ_i for any $i = 1, \dots, n$. Thus, we get the following.

- **Fact 2:** Assume that $\frac{\sigma_1}{\sigma_2} \geq 1 + \eta$ for some absolute $\eta > 0$. Taking $a > 0$ sufficiently small (that can depend only on η), we have for any $i = 1, \dots, n$, any $\theta' \in \mathcal{S}^p$, that

$$\mathbb{E}_{\delta_i}\left[K\big(\mathbb{P}_{\Sigma_{\theta'}^{(\delta_i)}}, \mathbb{P}_{\Sigma_{\theta_0}^{(\delta_i)}}\big)\right] \leq \frac{\delta^2}{2\bar{\sigma}^2}\epsilon^2.$$

We immediately get from Fact 2 for any $\theta' \in \Theta$ that

$$K\big(\mathbb{P}_{\Sigma_{\theta'}}, \mathbb{P}_{\Sigma_{\theta_0}}\big) = \sum_{i=1}^n \mathbb{E}_{\delta_i}\left[K\big(\mathbb{P}_{\Sigma_{\theta'}^{(\delta_i)}}, \mathbb{P}_{\Sigma_{\theta_0}^{(\delta_i)}}\big)\right] \leq \frac{\delta^2 n}{2\bar{\sigma}^2}\epsilon^2 = \frac{a^2}{2}s_1\log(ep/s_1).$$

4.7. Proof of Fact 1

In view of (4.11), we get for any $1 \leq i \leq n$, any $\theta, \theta' \in \mathcal{S}^p$ and any realization $\delta_i \in \{0,1\}^p$ that $\mathbb{P}_{\Sigma_\theta^{(\delta_i)}} \ll \mathbb{P}_{\Sigma_{\theta'}^{(\delta_i)}}$ and hence $K\big(\mathbb{P}_{\Sigma_{\theta'}^{(\delta_i)}}, \mathbb{P}_{\Sigma_\theta^{(\delta_i)}}\big) < \infty$.

Define $J_i = \{j : \delta_{i,j} = 1, 1 \leq j \leq r\}$ and $d_i = |J_i|$. Define the mapping $P_i : \mathbb{R}^p \to \mathbb{R}^{d_i}$ as follows $P_i(x) = x(J_i)$ where for any $x = (x^{(1)}, \dots, x^{(p)})^\top \in \mathbb{R}^p$, $x(J_i) \in \mathbb{R}^{d_i}$ is obtained by keeping only the components $x^{(k)}$ with their index $k \in J_i$. We denote by $P_i^* : \mathbb{R}^{d_i} \to \mathbb{R}^p$ the right inverse application of P_i. We note that

$$P_i\Sigma_\theta^{(\delta_i)}P_i^* = \sigma_1(\theta)\Pi_{\theta(J_i),\delta_i} + \sigma_2\left[I_{d_i} - \Pi_{\theta(J_i),\delta_i}\right],$$

where $\Pi_{\theta(J_i),\delta_i}$ denotes the orthogonal projection onto the subspace l.s.$(\theta_{\delta_i}(J_i))$ of $\mathbb{R}^{d_i}$. Note also that $P_i\Sigma_\theta^{(\delta_i)}P_i^*$ admits an inverse for any $\theta \in \mathcal{S}^p$ provided that δ_i is not the null vector in $\mathbb{R}^p$ and we have

$$(P_i\Sigma_\theta^{(\delta_i)}P_i^*)^{-1} = \frac{1}{\sigma_1(\theta)}\Pi_{\theta(J_i),\delta_i} + \frac{1}{\sigma_2}\left[I_{d_i} - \Pi_{\theta(J_i),\delta_i}\right].$$

Thus, we get for any $\theta, \theta' \in \mathcal{S}^p$ that

$$\begin{aligned}
K\left(\mathbb{P}_{\Sigma_{\theta'}^{(\delta_i)}}, \mathbb{P}_{\Sigma_{\theta}^{(\delta_i)}}\right) &= K\left(\mathbb{P}_{P_i\Sigma_{\theta'}^{(\delta_i)}P_i^*}, \mathbb{P}_{P_i(\Sigma_{\theta}^{(\delta_i)})P_i^*}\right) \\
&= \frac{1}{2}\mathrm{tr}\left((P_i\Sigma_{\theta}^{(\delta_i)}P_i^*)^{-1}P_i(\Sigma_{\theta'}^{(\delta_i)})P_i^*\right) + \frac{1}{2}\log\left(\frac{\det\left(P_i\Sigma_{\theta}^{(\delta_i)}P_i^*\right)}{\det\left(P_i\Sigma_{\theta'}^{(\delta_i)}P_i^*\right)}\right) - \frac{d_i}{2} \\
&= \frac{1}{2}\mathrm{tr}\Bigg(\left[\frac{1}{\sigma_1(\theta)}\Pi_{\theta(J_i),\delta_i} + \frac{1}{\sigma_2}\left(I_{d_i} - \Pi_{\theta(J_i),\delta_i}\right)\right] \\
&\qquad \times \left[\sigma_1(\theta')\Pi_{\theta'(J_i),\delta_i} + \sigma_2\left(I_{d_i} - \Pi_{\theta'(J_i),\delta_i}\right)\right]\Bigg) \\
&\quad + \frac{1}{2}\log\left(\frac{\det\left(P_i\Sigma_{\theta}^{(\delta_i)}P_i^*\right)}{\det\left(P_i\Sigma_{\theta'}^{(\delta_i)}P_i^*\right)}\right) - \frac{d_i}{2} \\
&= \frac{1}{2}\left(\frac{\sigma_2}{\sigma_1(\theta)} + \frac{\sigma_1(\theta')}{\sigma_2} - 2\right) \\
&\quad + \frac{1}{2}\mathrm{tr}\left(\Pi_{\theta(J_i),\delta_i}\Pi_{\theta'(J_i),\delta_i}\right)\left[\frac{\sigma_1(\theta')}{\sigma_1(\theta)} + 1 - \frac{\sigma_2}{\sigma_1(\theta)} - \frac{\sigma_1(\theta')}{\sigma_2}\right] \\
&\quad + \frac{1}{2}\log\left(\frac{\det\left(P_i\Sigma_{\theta}^{(\delta_i)}P_i^*\right)}{\det\left(P_i\Sigma_{\theta'}^{(\delta_i)}P_i^*\right)}\right) \\
&= \frac{1}{2}\left(\frac{\sigma_2}{\sigma_1(\theta)} + \frac{\sigma_1(\theta')}{\sigma_2} - 2\right) + \frac{1}{2}\mathrm{tr}\left(\Pi_{\theta,\delta_i}\Pi_{\theta',\delta_i}\right)\left[\frac{\sigma_1(\theta')}{\sigma_1(\theta)} + 1 - \frac{\sigma_2}{\sigma_1(\theta)} - \frac{\sigma_1(\theta')}{\sigma_2}\right] \\
&\quad + \frac{1}{2}\log\left(\frac{\sigma_1(\theta)}{\sigma_1(\theta')}\right),
\end{aligned}$$

where we have used $\sigma_1(\theta)$ and σ_2 are eigenvalues of $P_i\Sigma_{\theta}^{(\delta_i)}P_i^*$ with respective multiplicity 1 and $d_1 - 1$ for any $\theta \in \Theta$, and also that $\mathrm{tr}\left(\Pi_{\theta(J_i),\delta_i}\Pi_{\theta'(J_i),\delta_i}\right) = \mathrm{tr}\left(\Pi_{\theta,\delta_i}\Pi_{\theta',\delta_i}\right)$ for any $\theta, \theta' \in \Theta$. □

4.8. Proof of Fact 2

For any $i = 1, \ldots, n$, we have that $\sigma_1(\theta_0) = (\sigma_1 - \sigma_2)\frac{\delta_{i,1}+\delta_{i,2}}{2} + \sigma_2$ and

$$\sigma_1(\theta') = (\sigma_1 - \sigma_2)\left[\left(\frac{1-\epsilon^2}{2}\right)(\delta_{i,1} + \delta_{i,2}) + \epsilon^2\sum_{j=3}^{p}\frac{\omega^{(j)}}{s_1 - 2}\delta_{i,j}\right] + \sigma_2.$$

Note that the random quantities in the last three displays depend on $\delta_{i,1}, \delta_{i,2}$ only through the sum $Z_i := \delta_{i,1} + \delta_{i,2} \sim \mathrm{Bin}(2, \delta)$ and that Z_i is independent of $(\delta_{i,3}, \ldots, \delta_{i,p})$.

Thus, we get

$$\mathbb{E}_{\delta_i}\left[\frac{\sigma_2}{2\sigma_1(\theta_0)}\right] = \mathbb{E}_{\delta_i}\left[\frac{\sigma_2}{(\sigma_1-\sigma_2)(\delta_{i,1}+\delta_{i,2})+2\sigma_2}\right]$$
$$= \frac{\delta^2\sigma_2}{2\sigma_1} + \frac{2\delta(1-\delta)\sigma_2}{\sigma_1+\sigma_2} + \frac{(1-\delta)^2}{2}.$$

Similarly, we obtain

$$\mathbb{E}_{\delta_i}\left[\frac{\sigma_1(\theta')}{2\sigma_2}\right] = \frac{\delta(\sigma_1-\sigma_2)|\theta'|_2^2+\sigma_2}{2\sigma_2} = \frac{\delta\sigma_1}{2\sigma_2} + \frac{1-\delta}{2}.$$

Combining the last two displays, we get

$$\mathbb{E}_{\delta_i}\left[\frac{\sigma_2}{2\sigma_1(\theta_0)} + \frac{\sigma_1(\theta')}{2\sigma_2} - 1\right] = \frac{\delta(\sigma_1-\sigma_2)^2(\sigma_1+\delta\sigma_2)}{2\sigma_1\sigma_2(\sigma_1+\sigma_2)}. \tag{4.12}$$

We study now the following quantity

$$\frac{1}{2}\mathrm{tr}\left(\Pi_{\theta_0,\delta_i}\Pi_{\theta',\delta_i}\right)\left[\frac{\sigma_1(\theta')}{\sigma_1(\theta_0)} + 1 - \frac{\sigma_2}{\sigma_1(\theta_0)} - \frac{\sigma_1(\theta')}{\sigma_2}\right].$$

We note first that

$$\Pi_{\theta_0,\delta_i} = \frac{1}{\delta_{i,1}+\delta_{i,2}}\left(\begin{array}{cc|c} \delta_{i,1} & \delta_{i,1}\delta_{i,2} & O \\ \delta_{i,1}\delta_{i,2} & \delta_{i,2} & O \\ \hline O & O & O \end{array}\right)$$

and

$$\Pi_{\theta',\delta_i} = \frac{1-\epsilon^2}{2|\theta'_{\delta_i}|_2^2}\left(\begin{array}{cc|c} \delta_{i,1} & \delta_{i,1}\delta_{i,2} & * \\ \delta_{i,1}\delta_{i,2} & \delta_{i,2} & * \\ \hline * & * & * \end{array}\right).$$

Thus, we get that

$$\frac{1}{2}\mathrm{tr}\left(\Pi_{\theta_0,\delta_i}\Pi_{\theta',\delta_i}\right) = \left(1-\epsilon^2\right)\frac{\delta_{i,1}^2+2\delta_{i,1}\delta_{i,2}+\delta_{i,2}^2}{4|\theta'_{\delta_i}|_2^2(\delta_{i,1}+\delta_{i,2})} = \left(1-\epsilon^2\right)\frac{\delta_{i,1}+\delta_{i,2}}{4|\theta'_{\delta_i}|_2^2}.$$

Next, we set

$$\tilde{\sigma}(\theta_0,\theta') = \frac{\sigma_2\sigma_1(\theta')+\sigma_2\sigma_1(\theta_0)-\sigma_2^2-\sigma_1(\theta_0)\sigma_1(\theta')}{\sigma_2\sigma_1(\theta_0)}.$$

If $Z_i = 1$, then $\sigma_1(\theta_0) = (\sigma_1+\sigma_2)/2$ and

$$\tilde{\sigma}(\theta_0,\theta') = \frac{2\sigma_2\sigma_1(\theta')+\sigma_2(\sigma_1+\sigma_2)-2\sigma_2^2-(\sigma_1+\sigma_2)\sigma_1(\theta')}{\sigma_2(\sigma_1+\sigma_2)}$$
$$= -\frac{(\sigma_1-\sigma_2)^2|\theta'_{\delta_i}|_2^2}{\sigma_2(\sigma_1+\sigma_2)}.$$

If $Z_i = 2$, then $\sigma_1(\theta_0) = \sigma_1$ and

$$\tilde{\sigma}(\theta_0,\theta') = -\frac{(\sigma_1-\sigma_2)^2|\theta'_{\delta_i}|_2^2}{\sigma_1\sigma_2}.$$

We now freeze $(\delta_{i,3},\dots,\delta_{i,p})$ and compute the following expectation w.r.t Z_i

$$\mathbb{E}_{Z_i}\left[\frac{1}{2}\mathrm{tr}\left(\Pi_{\theta_0,\delta_i}\Pi_{\theta',\delta_i}\right)\tilde{\sigma}(\theta_0,\theta')\right] = -\delta^2\left(1-\epsilon^2\right)\frac{(\sigma_1-\sigma_2)^2}{2\sigma_1\sigma_2} - \delta(1-\delta)\left(1-\epsilon^2\right)\frac{(\sigma_1-\sigma_2)^2}{2\sigma_2(\sigma_1+\sigma_2)}.$$

We note that the above display does not depend on $(\delta_{i,3},\dots,\delta_{i,p})$. Thus, we get

$$\mathbb{E}_{\delta}\left[\frac{1}{2}\mathrm{tr}\left(\Pi_{\theta_0,\delta_i}\Pi_{\theta',\delta_i}\right)\tilde{\sigma}(\theta_0,\theta')\right] = -\delta^2\left(1-\epsilon^2\right)\frac{(\sigma_1-\sigma_2)^2}{2\sigma_1\sigma_2} - \delta(1-\delta)\left(1-\epsilon^2\right)\frac{(\sigma_1-\sigma_2)^2}{2\sigma_2(\sigma_1+\sigma_2)}.$$

Combining the above display with (4.12), we get

$$\begin{aligned}\Delta_1 &:= \frac{\delta(\sigma_1-\sigma_2)^2(\sigma_1+\delta\sigma_2)}{2\sigma_1\sigma_2(\sigma_1+\sigma_2)} - \delta^2\left(1-\epsilon^2\right)\frac{(\sigma_1-\sigma_2)^2}{2\sigma_1\sigma_2} \\ &\quad - \delta(1-\delta)\left(1-\epsilon^2\right)\frac{(\sigma_1-\sigma_2)^2}{2\sigma_2(\sigma_1+\sigma_2)} \\ &= -\frac{\delta}{2}\frac{(\sigma_1-\sigma_2)^2}{\sigma_1\sigma_2(\sigma_1+\sigma_2)}\left((1-\delta)(1-2\epsilon^2)\sigma_1 - \delta\epsilon^2(\sigma_1+\sigma_2)\right).\end{aligned}$$

We study now the logarithm factor

$$\Delta_2 := \frac{1}{2}\mathbb{E}_{\delta}\log\left(\frac{\sigma_1(\theta_0)}{\sigma_1(\theta')}\right).$$

Recall that $\sigma_1(\theta_0) = (\sigma_1-\sigma_2)\frac{Z_i}{2}+\sigma_2$ with $Z_i = \delta_{i,1}+\delta_{i,2} \sim \mathrm{Bin}(2,\delta)$ and

$$\sigma_1(\theta') = (\sigma_1-\sigma_2)|\theta_{\delta_i}|_2^2+\sigma_2 = (\sigma_1-\sigma_2)\left[\frac{Z_i}{2}+\frac{\epsilon^2}{s_1-2}\tilde{Z}_i\right]+\sigma_2.$$

with $\tilde{Z}_i = \sum_{j=3}^{p}\omega^{(j)}\delta_{i,j} \sim \mathrm{Bin}(s_1-2,\delta)$ and is independent of $(\delta_{i,1},\delta_{i,2})$.

We now freeze $\tilde{Z}_i$ and take the expectation w.r.t. Z_i. Thus, we get

$$\begin{aligned}\mathbb{E}_{Z_i}\left[\frac{1}{2}\log\left(\frac{\sigma_1(\theta_0)}{\sigma_1(\theta')}\right)\right] &= -\frac{(1-\delta)^2}{2}\log\left(\frac{(\sigma_1-\sigma_2)\frac{\epsilon^2}{s_1-2}\tilde{Z}_i+\sigma_2}{\sigma_2}\right) \\ &\quad - \delta(1-\delta)\log\left(\frac{(\sigma_1-\sigma_2)[(1-\epsilon^2)+\frac{2\epsilon^2}{s_1-2}\tilde{Z}_i]+2\sigma_2}{\sigma_1+\sigma_2}\right) \\ &\quad - \frac{\delta^2}{2}\log\left(\frac{(\sigma_1-\sigma_2)[(1-\epsilon^2)+\frac{\epsilon^2}{s_1-2}\tilde{Z}_i]+\sigma_2}{\sigma_1}\right).\end{aligned}$$

We study now the first term in the right-hand side of the above display. We have

$$\frac{(\sigma_1-\sigma_2)\frac{\epsilon^2}{s_1-2}\tilde{Z}_i+\sigma_2}{\sigma_2} = 1-\epsilon^2+\epsilon^2\left[\left(\frac{\sigma_1}{\sigma_2}-1\right)\frac{\tilde{Z}_i}{s_1-2}+1\right]$$
$$= 1-\epsilon^2+\epsilon^2\left[\sum_{j=3}^{p}\left[\left(\frac{\sigma_1}{\sigma_2}-1\right)\delta_{i,j}+1\right]\frac{\omega^{(j)}}{s_1-2}\right],$$

since $\sum_{j=3}^{p}\omega^{(j)} = s_1-2$ by construction. Next, we notice that $-\log$ is convex. Thus applying Jensen's inequality twice gives that

$$\begin{aligned}
&\mathbb{E}_{\tilde{Z}_i}\left[-\log\left(\frac{(\sigma_1-\sigma_2)\frac{\epsilon^2}{s_1-2}\tilde{Z}_i+\sigma_2}{\sigma_2}\right)\right]\\
&\quad\leq \epsilon^2\mathbb{E}_{\tilde{Z}_i}\left[-\log\left(\sum_{j=3}^{p}\left[\left(\frac{\sigma_1}{\sigma_2}-1\right)\delta_{i,j}+1\right]\frac{\omega^{(j)}}{s_1-2}\right)\right]\\
&\quad\leq -\epsilon^2\sum_{j=3}^{p}\frac{\omega^{(j)}}{s_1-2}\mathbb{E}_{\delta_{i,j}}\log\left[\left(\frac{\sigma_1}{\sigma_2}-1\right)\delta_{i,j}+1\right]\\
&\quad\leq -\delta\epsilon^2\log\left(\frac{\sigma_1}{\sigma_2}\right).
\end{aligned}$$

We proceed similarly and obtain

$$\begin{aligned}
&\frac{(\sigma_1-\sigma_2)[(1-\epsilon^2)+\frac{2\epsilon^2}{s_1-2}\tilde{Z}_i]+2\sigma_2}{\sigma_1+\sigma_2}\\
&\quad= (1-\epsilon^2)+\epsilon^2\left(\frac{2\sigma_2}{\sigma_1+\sigma_2}+\frac{2(\sigma_1-\sigma_2)\tilde{Z}_i}{(s_1-2)(\sigma_1+\sigma_2)}\right)\\
&\quad= (1-\epsilon^2)+\epsilon^2\sum_{j=3}^{p}\frac{\omega^{(j)}}{s_1-2}\left(\frac{2\sigma_2+\delta_{i,j}(\sigma_1-\sigma_2)}{\sigma_1+\sigma_2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{\tilde{Z}_i}\left[-\log\left(\frac{(\sigma_1-\sigma_2)[(1-\epsilon^2)+\frac{2\epsilon^2}{s_1-2}\tilde{Z}_i]+2\sigma_2}{\sigma_1+\sigma_2}\right)\right]\\
&\quad\leq -\epsilon^2\sum_{j=3}^{p}\frac{\omega^{(j)}}{s_1-2}\mathbb{E}_{\delta_{i,j}}\log\left[\frac{2\sigma_2+\delta_{i,j}(\sigma_1-\sigma_2)}{\sigma_1+\sigma_2}\right]\leq -(1-\delta)\epsilon^2\log\left(\frac{2\sigma_2}{\sigma_1+\sigma_2}\right).
\end{aligned}$$

We obtain similarly

$$\frac{(\sigma_1-\sigma_2)[(1-\epsilon^2)+\frac{\epsilon^2}{s_1-2}\tilde{Z}_i]+\sigma_2}{\sigma_1} = (1-\epsilon^2)+\epsilon^2\left(\frac{\tilde{Z}_i(\sigma_1-\sigma_2)}{(s_1-2)\sigma_1}+\frac{\sigma_2}{\sigma_1}\right)$$
$$= (1-\epsilon^2)+\epsilon^2\sum_{j=3}^{p}\frac{\omega^{(j)}}{s_1-2}\left(\frac{\sigma_2+\delta_{i,j}(\sigma_1-\sigma_2)}{\sigma_1}\right),$$

and

$$\mathbb{E}_{\tilde{Z}_i}\left[-\log\left(\frac{(\sigma_1-\sigma_2)[(1-\epsilon^2)+\frac{\epsilon^2}{s_1-2}\tilde{Z}_i]+\sigma_2}{\sigma_1}\right)\right]$$
$$\le -\epsilon^2\sum_{j=3}^{p}\frac{\omega^{(j)}}{s_1-2}\mathbb{E}_{\delta_{i,j}}\log\left[\frac{\sigma_2+\delta_{i,j}(\sigma_1-\sigma_2)}{\sigma_1}\right]$$
$$\le -(1-\delta)\epsilon^2\log\left(\frac{\sigma_2}{\sigma_1}\right).$$

Thus, we get

$$\begin{aligned}\Delta_2 = &-\frac{(1-\delta)^2\delta}{2}\epsilon^2\log\left(\frac{\sigma_1}{\sigma_2}\right)-\delta(1-\delta)^2\epsilon^2\log\left(\frac{2\sigma_2}{\sigma_1+\sigma_2}\right)\\ &-\frac{(1-\delta)\delta^2}{2}\epsilon^2\log\left(\frac{\sigma_2}{\sigma_1}\right)\\ = &-\frac{(1-\delta)\delta}{2}\epsilon^2\left[(1-\delta)\log\left(\frac{\sigma_1}{\sigma_2}\right)+2\log\left(\frac{2\sigma_2}{\sigma_1+\sigma_2}\right)+\delta\log\left(\frac{\sigma_2}{\sigma_1}\right)\right]\\ = &\frac{(1-\delta)\delta}{2}\epsilon^2\left[\log\left(\frac{(\sigma_1+\sigma_2)^2}{2\sigma_1\sigma_2}\right)-\log(2)+2\delta\log\left(\frac{\sigma_1}{\sigma_2}\right)\right].\end{aligned}$$

Set $\Delta := \Delta_1+\Delta_2$. We have

$$\begin{aligned}\Delta = &-\frac{\delta}{2}\frac{(\sigma_1-\sigma_2)^2}{\sigma_1\sigma_2(\sigma_1+\sigma_2)}\left((1-\delta)(1-2\epsilon^2)\sigma_1-\delta\epsilon^2(\sigma_1+\sigma_2)\right)\\ &+\frac{(1-\delta)\delta}{2}\epsilon^2\left[\log\left(\frac{(\sigma_1+\sigma_2)^2}{4\sigma_1\sigma_2}\right)+2\delta\log\left(\frac{\sigma_1}{\sigma_2}\right)\right]\\ \le &\frac{\delta^2}{2\bar{\sigma}^2}\epsilon^2-\frac{\delta(1-\delta)}{2}\left[\frac{(\sigma_1-\sigma_2)^2}{\sigma_2(\sigma_1+\sigma_2)}\left((1-2\epsilon^2)\right)\right.\\ &\left.-\epsilon^2\left[\log\left(\frac{(\sigma_1+\sigma_2)^2}{4\sigma_1\sigma_2}\right)+2\log\left(\frac{\sigma_1}{\sigma_2}\right)\right]\right]\\ \le &\frac{\delta^2}{2\bar{\sigma}^2}\epsilon^2-\frac{\delta(1-\delta)}{2}\left[\frac{(\sigma_1-\sigma_2)^2}{\sigma_2(\sigma_1+\sigma_2)}\left((1-2\epsilon^2)\right)-\epsilon^2\log\left(\frac{\sigma_1(\sigma_1+\sigma_2)^2}{4\sigma_2^3}\right)\right].\end{aligned}$$

We now show that if the absolute constant $a>0$ (recall that $\epsilon = a\bar{\sigma}\sqrt{\frac{s_1\log(ep/s_1)}{\delta^2 n}}$) is taken sufficiently small, then we have

$$\frac{(\sigma_1-\sigma_2)^2}{\sigma_2(\sigma_1+\sigma_2)}\left((1-2\epsilon^2)\right)-\epsilon^2\log\left(\frac{\sigma_1(\sigma_1+\sigma_2)^2}{4\sigma_2^3}\right)\ge 0,\quad \forall\sigma_1>(1+\eta)\sigma_2.$$

We set $u = \sigma_1 - \sigma_2$ and $x = u/(2\sigma_2)$. Then, we have

$$\begin{aligned}
&\frac{(\sigma_1-\sigma_2)^2}{\sigma_2(\sigma_1+\sigma_2)}\left((1-2\epsilon^2)\right) - \epsilon^2 \log\left(\frac{\sigma_1(\sigma_1+\sigma_2)^2}{4\sigma_2^3}\right)\\
&\quad = \frac{u^2}{\sigma_2(2\sigma_2+u)}(1-2\epsilon^2) - \epsilon^2\log\left(\frac{(\sigma_2+u)(2\sigma_2+u)^2}{4\sigma_2^3}\right)\\
&\quad = \frac{u^2}{2\sigma_2^2(1+u/(2\sigma_2))}(1-2\epsilon^2) - \epsilon^2\log\left(\left[1+\frac{u}{\sigma_2}\right]\left[1+\frac{u}{2\sigma_2}\right]^2\right)\\
&\quad \geq \frac{u^2}{2\sigma_2^2(1+u/(2\sigma_2))}(1-2\epsilon^2) - 3\epsilon^2\log\left(1+\frac{u}{\sigma_2}\right)\\
&\quad \geq 2(1-2\epsilon^2)\frac{x^2}{1+x} - 3\epsilon^2\log(1+2x) \geq 0, \quad \forall x \geq \frac{\eta}{2},
\end{aligned}$$

provided that the numerical constant $a > 0$ is taken sufficiently small (and this choice can depend only on $\eta > 0$). □

4.9. Bounding of the moment $\mathbb{E}[(\boldsymbol{\theta}^\top \boldsymbol{Z}\boldsymbol{\theta})^2]$

Set $\bar{Z} = XX^\top - \mathrm{diag}(XX^\top)$. For any $\theta = (\theta^{(1)}, \ldots, \theta^{(p)})^\top \in \mathbb{R}^p$ and $\delta_1, \ldots, \delta_p \in \{0,1\}^p$, we set $\theta_\delta = (\delta_1\theta^{(1)}, \ldots, \delta_p\theta^{(p)})^\top$. Note that

$$\theta^\top[YY^\top - \mathrm{diag}(YY^\top)]\theta = \theta_\delta^\top \bar{Z}\theta_\delta.$$

We have for any $\theta \in \mathcal{S}_s^p$ that

$$(\theta^\top Z\theta)^2 \leq 2(\theta_\delta^\top \bar{Z}\theta_\delta)^2 + 2\delta^4(\theta^\top[\Sigma - \mathrm{diag}(\Sigma)]\theta)^2. \tag{4.13}$$

It is easy to see that $(\theta^\top[\Sigma - \mathrm{diag}(\Sigma)]\theta)^2 \leq \sigma_{\max}^2(s)$ for any $\theta \in \mathcal{S}_s^p$. Next, we concentrate on $(\theta_\delta^\top \bar{Z}\theta_\delta)^2$.

We have for any $\theta \in \mathcal{S}_s^p$ that

$$\begin{aligned}
(\theta_\delta^\top \bar{Z}\theta_\delta)^2 &= \left(\sum_{j\neq k}\delta_j\delta_k\theta^{(j)}\theta^{(k)}X^{(j)}X^{(k)}\right)^2\\
&= \sum_{j_1,j_2:j_1\neq j_2}\delta_{j_1}\delta_{j_2}\left(\theta^{(j_1)}\right)^2\left(\theta^{(j_2)}\right)^2\left(X^{(j_1)}\right)^2\left(X^{(j_2)}\right)^2\\
&\quad + \sum_{j_1,j_2,j_3\text{ distinct}}\delta_{j_1}\delta_{j_2}\delta_{j_3}\left(\theta^{(j_1)}\right)^2\theta^{(j_2)}\theta^{(j_3)}\left(X^{(j_1)}\right)^2X^{(j_2)}X^{(j_3)}\\
&\quad + \sum_{j_1,j_2,j_3j_4\text{ distinct}}\delta_{j_1}\delta_{j_2}\delta_{j_3}\delta_{j_4}\theta^{(j_1)}\theta^{(j_2)}\theta^{(j_3)}\theta^{(j_4)}X^{(j_1)}X^{(j_2)}X^{(j_3)}X^{(j_4)}.
\end{aligned}$$

Taking the expectation w.r.t. to $\delta_1, \ldots, \delta_p$, we get

$$\mathbb{E}_\delta\left[(\theta_\delta^\top \bar{Z}\theta_\delta)^2\right] = \delta^2 \sum_{j_1,j_2:j_1\neq j_2} \left(\theta^{(j_1)}\right)^2 \left(\theta^{(j_2)}\right)^2 \left(X^{(j_1)}\right)^2 \left(X^{(j_2)}\right)^2$$
$$+ \delta^3 \sum_{j_1,j_2,j_3 \text{ distinct}} \left(\theta^{(j_1)}\right)^2 \theta^{(j_2)}\theta^{(j_3)} \left(X^{(j_1)}\right)^2 X^{(j_2)}X^{(j_3)}$$
$$+ \delta^4 \sum_{j_1,j_2,j_3 j_4,\text{distinct}} \theta^{(j_1)}\theta^{(j_2)}\theta^{(j_3)}\theta^{(j_4)}X^{(j_1)}X^{(j_2)}X^{(j_3)}X^{(j_4)}.$$

Set

$$A = \sum_{j_1,j_2:j_1\neq j_2} \left(\theta^{(j_1)}\right)^2 \left(\theta^{(j_2)}\right)^2 \left(X^{(j_1)}\right)^2 \left(X^{(j_2)}\right)^2$$
$$B = \sum_{j_1,j_2,j_3 \text{ distinct}} \left(\theta^{(j_1)}\right)^2 \theta^{(j_2)}\theta^{(j_3)} \left(X^{(j_1)}\right)^2 X^{(j_2)}X^{(j_3)}$$
$$C = \sum_{j_1,j_2,j_3 j_4,\text{distinct}} \theta^{(j_1)}\theta^{(j_2)}\theta^{(j_3)}\theta^{(j_4)}X^{(j_1)}X^{(j_2)}X^{(j_3)}X^{(j_4)}.$$

We have

$$\begin{aligned}
\mathbb{E}_\delta\left[\theta_\delta^\top \bar{Z}\theta_\delta\right] &= \delta^2 A + \delta^3 B + \delta^4 C \\
&= (\delta^2 - \delta^4)A + (\delta^3 - \delta^4)B + \delta^4(A+B+C) \\
&= [(\delta^2 - \delta^4 - (\delta^3 - \delta^4)]A + (\delta^3 - \delta^4)(A+B) + \delta^4(A+B+C) \\
&= \delta^2(1-\delta)A + \delta^3(1-\delta)(A+B) + \delta^4(A+B+C). \qquad (4.14)
\end{aligned}$$

Next, we note that

$$A+B+C = (\theta^\top \bar{Z}\theta)^2 = \left((\theta^\top X)^2 - \sum_j (\theta^{(j)}X^{(j)})^2\right)^2$$
$$\leq 2(\theta^\top X)^4 + 2\left(\sum_j (\theta^{(j)})^2 (X^{(j)})^2\right)^2 \leq 2(\theta^\top X)^4 + 2\sum_j (\theta^{(j)})^2 (X^{(j)})^4.$$

Taking now the expectation w.r.t X, we get for any $\theta \in \mathcal{S}_s^p$ that

$$\begin{aligned}
\mathbb{E}_X[A+B+C] &\leq 2\mathbb{E}_X[(\theta^\top X)^4] + 2\sum_j (\theta^{(j)})^2 \mathbb{E}_X[(X^{(j)})^4] \qquad (4.15) \\
&\leq 8\|\theta^\top X\|_{\psi_2}^4 + 8\sum_j (\theta^{(j)})^2 \|X^{(j)}\|_{\psi_2}^4 \\
&\leq \frac{8}{c_1^2}\left(\mathbb{E}_X[(\theta^\top X)^2]\right)^2 + \frac{8}{c_1^2}\sum_j (\theta^{(j)})^2 \left(\mathbb{E}_X[(X^{(j)})^2]\right)^2 \leq \frac{16}{c_1^2}\sigma_{\max}^2(s),
\end{aligned}$$

where we have used (2.1) and Assumption 1.

We now treat $A+B$ similarly. We have

$$\begin{aligned}A+B &= \sum_{j_1}(\theta^{(j_1)})^2(X^{(j_1)})^2\left(\sum_{j_2,j_3\,:\,j_2\neq j_1,j_3\neq j_1}\theta^{(j_2)}\theta^{(j_3)}X^{(j_2)}X^{(j_3)}\right)\\ &= \sum_{j_1}(\theta^{(j_1)})^2(X^{(j_1)})^2\left(\theta^\top X-\theta^{(j_1)}X^{(j_1)}\right)^2\\ &\le 2\sum_{j_1}(\theta^{(j_1)})^2(X^{(j_1)})^2\left(\theta^\top X\right)^2+2\sum_{j_1}(\theta^{(j_1)})^4(X^{(j_1)})^4.\end{aligned}$$

Next, we note that

$$\mathbb{E}_X[(X^{(j_1)})^2(\theta^\top X)^2]\le\sqrt{\mathbb{E}_X[(X^{(j_1)})^4]}\sqrt{\mathbb{E}_X[(\theta^\top X)^4]}\le\frac{4}{c_1^2}\sigma^2_{\max}(s).$$

Combining the two previous displays, we get

$$\mathbb{E}_X[A+B]\le\frac{16}{c_1^2}\sigma^2_{\max}(s). \tag{4.16}$$

We now deal with A. We have

$$A\le\left(\sum_{j_1}(\theta^{(j_1)})^2(X^{(j_1)})^2\right)^2-\sum_{j_1}(\theta^{(j_1)})^4(X^{(j_1)})^4\le\sum_{j_1}(\theta^{(j_1)})^2(X^{(j_1)})^4.$$

Taking the expectation w.r.t. X, we get

$$\mathbb{E}_X[A]\le\mathbb{E}_X\left(\sum_{j_1}(\theta^{(j_1)})^2(X^{(j_1)})^4\right)\le\frac{4}{c_1^2}\sigma^2_{\max}(s). \tag{4.17}$$

Combining (4.14)–(4.17), we get

$$\mathbb{E}\left[(\theta_\delta^\top\bar{Z}\theta_\delta)^2\right]\le\frac{16}{c_1^2}\delta^2\sigma^2_{\max}(s)\left(1-\delta+\delta(1-\delta)+\delta^2\right)=\frac{16}{c_1^2}\delta^2\sigma^2_{\max}(s).$$

Combining the above display with (4.13), we get that

$$\mathbb{E}\left[(\theta^\top Z\theta)^2\right]\le c\frac{1}{c_1^2}\delta^2\sigma^2_{\max}(s), \tag{4.18}$$

for some numerical constant $c>0$.

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High Dimensional CLT and its Applications

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Abstract. We study the behavior of empirical processes indexed by finite classes of functions but we allow that the cardinality of these classes tend to infinity. We prove general results showing that one can bootstrap these types of processes even if they do not converge. We show that these results can be used to construct novel statistical tests. To this end we offer a new goodness of fit test.

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1. Introduction

Let X_i be a sequence of i.i.d. random variables on a Polish space $\mathbf{S}$, and let $\mathcal{F}$ be a collection of functions $\mathcal{F} = \{f : \mathbf{S} \to \mathbf{R}\}$. We will consider the stochastic process

$$Z_n(f)_{f\in\mathcal{F}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(f(X_i) - Ef(X_i)).$$

Traditionally, this setting is studied under two scenarios. If the collection $\mathcal{F}$ is finite then we are dealing with finite dimensional Central Limit Theorem (FIDI-CLT), while in the case of infinite $\mathcal{F}$ the Empirical Processes Theory applies. In this paper we will address the middle case, that is, we will consider a sequence of classes $\mathcal{F}_n$ where for each n the cardinality of $\mathcal{F}_n$ is finite but tends to infinity. We call this case high dimensional CLT.

The motivation. It is well known (Gine and Zinn [4]) that the necessary and sufficient conditions for the Uniform CLT are FIDI convergence and stochastic equicontinuity.

FIDI:

$$\overrightarrow{Z}_n = (Z_n(f_1), \ldots, Z_n(f_d)) \to \text{converges weakly} \tag{1.1}$$

for any finite collections $\{f_1, \ldots, f_d\} \subset \mathcal{F}$.

Stochastic equicontinuity: for $\varepsilon > 0$

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P\Big(\sup_{d(f,g)<\delta}|Z_n(f-g)| > \varepsilon\Big) = 0. \tag{1.2}$$

These two conditions are used in the following way. For an arbitrary small $\delta > 0$ we choose a covering class $\{f_1, \ldots, f_{d_\delta}\} = \mathcal{F}_\delta \subset \mathcal{F}$ and construct a piecewise constant process that changes the values only at the balls of radius δ which are centered at $f \in \mathcal{F}_\delta$. This process is de facto a finite dimensional vector $\overrightarrow{Z}_{n,\delta} = (Z_n(f_1), \ldots, Z_n(f_{d_\delta}))$. Next we let

$$Z_n = (Z_n - \overrightarrow{Z}_{n,\delta}) + \overrightarrow{Z}_{n,\delta} = I_n + II_n,$$

and argue that part I_n converges to zero by (1.2) while part II_n converges weakly to the appropriate limit by (1.1).

It is important to observe that the distribution of the limiting stochastic process (i.e., $Z_n \to Z_0$) depends on the behavior of FIDI part only (since $I_n \to 0$) and for this reason alone it makes sense to study the FIDI part more carefully. Since the above technique applies only if δ tends to zero, and since this typically implies that the cardinality of $\mathcal{F}_\delta$ tends to infinity (i.e., $d_\delta \to \infty$) we believe that one should investigate the properties of

$$\overrightarrow{Z}_n = (Z_n(f_1), \ldots, Z_n(f_{d_n})) \text{ for } d_n \to \infty. \tag{1.3}$$

By considering only the high dimensional part (1.3) we gain on several fronts. These are much simpler and easier objects to study (i.e., random vectors versus empirical processes), their behavior does not depend on stochastic equicontinuity which considerably relaxes the restrictions commonly encountered in Empirical Processes Theory (see Dudley [2]). This in turn opens the doors for some novel and unusual statistical applications.

For example, the classical approach typically requires that we first prove the weak convergence (i.e., $Z_n \to Z_o$) , and then derive statistical applications by arguing that for appropriately chosen functional H the laws of $H(Z_n)$ and $H(Z_o)$ are similar (i.e., $\mathcal{L}H(Z_n) \approx \mathcal{L}H(Z_o)$). Here we adopt a different strategy. We show that one could use the bootstrap version Z_n^* in order to approximate the distribution of Z_n (i.e., $\mathcal{L}H(Z_n) \approx \mathcal{L}H(Z^*)$) and to do so one does not need the assumption on weak convergence. In fact, in Section 3 we present a novel goodness of fit test which diverges (i.e., $\mathcal{L}H(Z_n) \to \infty$) but is still applicable; since its distribution can be approximated by the bootstrap version of the process.

This approach of using the bootstrap to construct statistical tests, even for statistics that diverge, is not new. To the best of our knowledge the first to demonstrate this possibility were Bickel and Freedman [1]. The authors argued that in some cases the regression, for which the number of parameters increases with n, could be efficiently bootstrapped even though the original statistic H_n does not necessarily converge. A more general (albeit less applicable) result was presented in Radulovic [5] where it was shown that there exists a class of empirical processes for which we can apply bootstrapping techniques without assuming limiting distributions. Recently this result is expanded to more applicable settings in Radulovic [6] and with some applications to copula functions in Fremanian et al. [3].

The main novelty presented here is that we are not dealing with empirical processes but high dimensional vectors. This approach fits nicely with the Bickel and Freedman idea. By treating the vectors we are able to bypass the cumbersome P-Donsker type requirements (see van deer Vaart [7] and Dudley [2]) and consequently we are able to characterize the whole new class of statistics for which this interesting phenomenon (statistical testing without the weak limit) applies.

The paper is organized as follows. In Section 2 we state the main results: Theorem 2.1 provides a general tool for the construction of specific types of statistics for which we could have hypothesis testing without assuming the weak convergence. Corollary 2.1 and Theorem 2.2 expand on this idea by providing more concrete (applicable) framework for such constructions.

In Section 3 we show that a straightforward application of Theorem 2.2 yields a novel goodness of fit (GOF) test for which we provide a small simulation study. We would like to stress here that the main purpose of this paper is not to introduce a novel GOF test, but rather to demonstrate that the unusual and somewhat cumbersome results (Theorem 2.1 and Theorem 2.2) could indeed produce concrete and applicable test statistics. Section 4 is reserved for the proofs.

2. Main results

Definitions. Given two random vectors $X, Y : \Omega \to \mathbf{R}^d$ we define the Bounded Lipschitz distance

$$d_{BL_1}(X,Y) = \sup_{H \in BL(\mathbf{R}^d)} |EH(X) - EH(Y)|,$$

where $BL(\mathbf{R}^d) := \{H : \mathbf{R}^d \to \mathbf{R},\ |H(x) - H(y)| \le \min(||x-y||_\infty, 1)\}$. We will also use d-3 distance

$$d_3(X,Y) = \sup_{H \in C_3(\mathbf{R}^d)} |EH(X) - EH(Y)|$$

where $C_3(\mathbf{R}^d) := \{H : \mathbf{R}^d \to \mathbf{R},\ ||H_{i,j,k}(x)||_\infty \le 1)\}$, and $H_{i,j,k}(x)$ denotes all partial derivatives up to order 3 (we let $H_{0,0,0}(x) \equiv H(x)$).

Next we let $d \le n^m$ for some $m > 0$, and let $\overrightarrow{X}_n, \overrightarrow{Y}_n : \Omega \to R^d$ such that

$$\overrightarrow{X}_n = \sum_{i=1}^n \overrightarrow{X}_{i,n} = \sum_{i=1}^n (X_{1,i,n}, X_{2,i,n}, \ldots, X_{d,i,n})$$

and

$$\overrightarrow{Y}_n = \sum_{i=1}^n \overrightarrow{Y}_{i,n} = \sum_{i=1}^n (Y_{1,i,n}, Y_{2,i,n}, \ldots, Y_{d,i,n}).$$

We will assume that the vectors $\overrightarrow{X}_{i,n}$ $i = 1, \ldots, n$ are independent and that the vectors $\overrightarrow{Y}_{i,n}$ $i = 1, \ldots, n$ are independent but $\overrightarrow{Y}_{i,n}$'s are not necessarily independent from $\overrightarrow{X}_{i,n}$'s. We let E_Y stand for a conditional expectation with respect to sigma algebra generated by $Y_{j,i,n}, i \leq n, j \leq d$. The following theorem is a version of Lemma A in Radulovic [6].

Theorem 2.1. *Let $EX_{j,i,n} = E_Y Y_{j,i,n} = 0$ and suppose that for some constant C and sequence Q_n and R_n, the following holds*

$$\max_{j,i} E|E_Y(Y_{j,i,n}^2) - E(X_{j,i,n}^2)| \leq \frac{C}{Q_n n^{3/2}} \tag{2.1}$$

and

$$\max_{j,i} E\left(|X_{j,i,n}|^3 + E_Y(|Y_{j,i,n}|^3\right) \leq \frac{C}{R_n n^{3/2}}. \tag{2.2}$$

Then for some $\gamma > 0$

$$E(d_{BL_1}(\overrightarrow{X}_n, \overrightarrow{Y}_n | \sigma(X_{j,i,n}, i \leq n, j \leq d)) = O\left(\frac{\ln n^\gamma d^2}{n^{1/2} Q_n} + \frac{\ln n^\gamma d^3}{n^{1/2} R_n}\right).$$

The above theorem does not follow the classical narrative where one shows that a sequence of random variables converges to a fixed distribution (i.e., $\overrightarrow{X}_n \to \overrightarrow{X}_0$). However, it allows for the construction of an alternative sequence designed to approximate the original process (i.e., $\overrightarrow{Y}_n \approx \overrightarrow{X}_n$).

The statement as well as the conditions of Theorem 2.1 is cumbersome but it is fairly easy to adapt them into the setting of Empirical Processes and the bootstrap approximation. For this we turn to the following corollary. Let $\{X_i\}_{i=1}^n$, be i.i.d. P sequences defined on a Polish space $\mathbf{S}$ and let $\{ X_i^* \}_{i=1}^n$ be a bootstrap sample based on X_i, (i.e., $X_i^* = X_j$ with probability $1/n$). Let $d_n \leq n^m$ for some $m > 0$ and let us consider a sequence of indexing functions: $\mathcal{F}_n = \{f_{j,n} : \mathbf{S} \to \mathbf{R}, j = 1, \ldots, d_n\}$. If we let

$$X_{j,i,n} = n^{-1/2}(f_{j,n}(X_i) - Ef_{j,n}(X_i))$$

and

$$Y_{j,i,n} = n^{-1/2}(f_{j,n}(X_i^*) - Ef_{j,n}(X_i^*)),$$

then the vectors $\overrightarrow{X}_n, \overrightarrow{Y}_n : \Omega \to R^{d_n}$ introduced prior Theorem 2.1 could be written using more familiar notation:

$$\overrightarrow{X}_n = \overrightarrow{Z}_n(f)_{f \in \mathcal{F}_n} \quad \text{and} \quad \overrightarrow{Y}_n = \overrightarrow{Z}_n^*(f)_{f \in \mathcal{F}_n},$$

where we use the notation $\overrightarrow{Z}$, just to emphasize that we are dealing with finite dimensional indexing class.

Corollary 2.1. *Suppose that $\overrightarrow{Z}_n(f)_{f \in \mathcal{F}_n}$ and $\overrightarrow{Z}_n^*(f)_{f \in \mathcal{F}_n}$ are defined as above and suppose that*

$$\max_{j,n} E(f_{j,n}(X_1))^4 \lesssim \frac{1}{Q_n} \quad \textit{and} \quad \max_{j,n} E|f_{j,n}(X_1)|^3 \lesssim \frac{1}{R_n} \tag{2.3}$$

then for some $\gamma > 0$

$$Ed_{BL_1}(\overrightarrow{Z}_n, \overrightarrow{Z}^*_n) = O\left(\frac{\ln n^\gamma d_n^2}{n^{1/2}Q_n^{1/2}} + \frac{\ln n^\gamma d_n^3}{n^{1/2}R_n}\right).$$

If $G : R^{d_n} \to R$ *is a Lipschitz under the sup norm on* R^{d_n} *then*

$$Ed_{BL_1}(G(\overrightarrow{Z}_n), G(\overrightarrow{Z}^*_n)) = O\left(\frac{\ln n^\gamma d_n^2}{n^{1/2}Q_n^{1/2}} + \frac{\ln n^\gamma d_n^3}{n^{1/2}R_n}\right). \tag{2.4}$$

Remark 2.1. If d_n stays fix then the above results reduce to usual multidimensional bootstrap CLT. Thus the obvious novelty here is that we could let $d_n \to \infty$. Since in this case it is not even clear how to define a weak convergence of $\overrightarrow{Z}_n$, the above results open the doors for some creative and novel statistical constructions. The second part of Corollary 2.1 provides the additional tool for such constructions. Namely the classical Empirical Processes techniques often use the continuous mapping theorem in order to derive the concrete statistical inference. Here we offer the second part of Corollary 2.1, which is a version of continuous mapping theorem (we need a slightly stronger Lipschitz assumption).

Since the functional $G(a_1, \ldots, a_{d_n}) = \max_{i \le d_n} |a_i|$ is clearly Lipschitz, the above Corollary 2.1 allows us to bootstrap sup norm (i.e., $\sup_{f \in \mathcal{F}_n}(\overrightarrow{Z}_n(f))$ $\approx \sup_{f \in \mathcal{F}_n}(\overrightarrow{Z^*}_n(f))$). But we can do even more. The following result is designed to deal with L_1 and L_2 norms. That is, with statistics:

$$\sum_{i=1}^{d_n} |Z_n(f_{i,n})| \ , \ \sum_{i=1}^{d_n} (Z_n(f_{i,n}))^2 \ \text{ or more general } \ \sum_{i=1}^{d_n} g_n(Z_n(f_{i,n})).$$

To this end we start with some definitions. For $\widetilde{f}_{k,n} \in \mathcal{F}_n$ we define the centered version of the processes

$$W_{n,k} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_{k,n}(X_i) \quad \text{and} \quad W^*_{n,k} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_{k,n}(X^*_i) \tag{2.5}$$

where $f_{k,n}(X_i) = \widetilde{f}_{k,n}(X_i) - E\widetilde{f}_{k,n}(X_i)$ and $f_{k,n}(X^*_i) = \widetilde{f}_{k,n}(X^*_i) - E^*\widetilde{f}_{k,n}(X^*_i)$. For any $g_n : \mathbf{R} \to \mathbf{R}$ we let

$$T_n = \sum_{k=1}^{d_n} g_n(W_{n,k}) \quad \text{and} \quad T^*_n = \sum_{k=1}^{d_n} g_n(W^*_{n,k}).$$

The assumptions

1) There exists a universal constant C and a sequence M_n such that

$$||g'_n||_\infty \le C, \quad ||g''_n||_\infty \le CM_n, \quad ||g'''_n||_\infty \le CM_n^2$$

2) There exist a universal constant C and sequences K_n and R_n such that

$$\max_k E|f_{k,n}(X_1)|^3 \le \frac{1}{R_n} \quad \text{and} \quad \max_k E(f_{k,n}(X_1))^4 \le \frac{1}{K_n}$$

Theorem 2.2. *Under the assumptions* 1 *and* 2 *the following is true*

$$d_3(T_n', T_n^*) := \sup_{H \in C_3(1)} |EH(T_n') - E^* H(T_n^*)|$$
$$= O_P\left(\frac{d^3 + d^2 M_n + d M_n^2}{n^{1/2} R_n} + \frac{d^2 + d M_n}{n^{1/2} K_n^{1/2}}\right)$$

where $C_3(1)$ *denotes a set of functions* $h : R \to R$ *such that* $||h||_\infty \le 1$, $||h'||_\infty \le 1$, $||h''||_\infty \le 1$ and $||h'''||_\infty \le 1$.

Remark 2.2. Unfortunately $|x|$ and x^2 are not three times differentiable with bounded derivatives and consequently we cannot deal with statistics

$$\sum_{i=1}^{d_n} |Z_n(f_{i,n})| \quad \text{and} \quad \sum_{i=1}^{d_n} (Z_n(f_{i,n}))^2$$

using the "plug-in" approach. Nevertheless, the above Theorem 2.2 allows us to construct $g_n(x)$ as an approximation. In Section 3 we show how to use Theorem 2.2, and $g_n(x) \approx |x|$ in order to bootstrap the statistic $\sum_{i=1}^{d_n} |Z_n(f_{i,n})|$ and $\sum_{i=1}^{d_n} (Z_n(f_{i,n}))^2$.

3. Applications

Using the results presented thus far we derive a novel GOF test and we show that on limited set of simulations the suggested test is equal or better than Kolmogorov-Smirnov test. However, we would like to emphasize here that we do not claim nor do we prove that the proposed test is superior. The only reason we include this study is to emphasize that the results presented in Section 2 are not just a theoretical pedantry but that indeed this approach of constructing statistical tests without the weak limit assumption could yield to some relevant and potentially useful statistical applications.

Before we proceed we have to deal with quantile approximation. As we mentioned earlier the main novelty of our approach is the fact that we are not requiring the weak convergence of the process $\overrightarrow{Z}_n$. Instead we use the bootstrap version of the process (i.e., $\overrightarrow{Z}_n^*$) and argue that the two are close in d_{BL_1} or d_3 metric. However, as stated above, it is not clear if the closeness in these metrics allows for quantile approximation. The following simple computation testifies that indeed we can use the aforementioned results in such a way.

Quantile approximation justification

For an arbitrary $\varepsilon \in (0, 1)$ and a function $H_{t,\varepsilon}(x) = 1_{x \le t} + \frac{t+\varepsilon-x}{\varepsilon} 1_{t < x \le t+\varepsilon}$ we have that

$$P(X_n \le t) \le EH_{t,\varepsilon}(X_n) = EH_{t,\varepsilon}(Y_n) + EH_{t,\varepsilon}(X_n) - EH_{t,\varepsilon}(Y_n)$$

(since $\varepsilon H_{t,\varepsilon}(x) \in BL(\mathbf{R})$ and by letting $d_{BL_1} = d_{BL_1}(X_n, Y_n)$)

$$\leq P(Y_n \leq t + \varepsilon) + \frac{d_{BL_1}}{\varepsilon}.$$

Similar computation with $t - \varepsilon$ yields

$$P(Y_n \leq t - \varepsilon) - \frac{d_{BL_1}}{\varepsilon} \leq P(X_n \leq t) \leq P(Y_n \leq t + \varepsilon) + \frac{d_{BL_1}}{\varepsilon}. \tag{3.1}$$

Since $\varepsilon > 0$ is arbitrary and if we assume that $d_{BL_1} \to 0$, one could use statement (3.1) in order to justify the quantile approximation of $P(X_n \leq t)$ by $P(Y_n \leq t)$. In our case Y_n is a bootstrapped variable; thus a discrete random variable with finitely many atoms. Consequently for almost all $t \in R$, and for ε small enough we have that $P(Y_n \leq t - \varepsilon) = P(Y_n \leq t + \varepsilon)$.

One could easily adapt the above argument (by constructing a three times differentiable version of $H_{t,\varepsilon}(x)$) and show that

$$P(Y_n \leq t - \varepsilon^{1/3}) - \frac{d_3}{\varepsilon} \leq P(X_n \leq t) \leq P(Y_n \leq t + \varepsilon^{1/3}) + \frac{d_3}{\varepsilon}.$$

Application

Now we are ready for the applications. Let us assume that a sequence $\{X_i\}_{i=1}^n$ is i.i.d. P, and X_i: $\Omega \to \mathbf{R}^m$ for some integer $m > 0$. We consider the sequence of sets (partitions) of $\mathbf{R}^m$ $\Pi_n = \{B_{1,n}, \ldots, B_{d_n,n}\}$ such that $P(B_{i,n} \cap B_{j,n}) = 0$ and $\max_{k \leq d_n} P(B_{k,n}) \leq \frac{C}{d_n}$ for some fixed constant C. We use the notation

$$Z_n(B_{k,n}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1_{X_i \in B_{k,n}} - P(B_{k,n})) \text{ and } Z_n^*(B_{k,n})$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1_{X_i^* \in B_{k,n}} - P_n(B_{k,n})).$$

Theorem 3.1. *For $d_n \leq n^\gamma$, $\gamma < 1/4$, let a sequence of sets Π_n be defined as above and let $\widetilde{T}_n = \sum_{k=1}^{d_n} |Z_n(B_k)|$ and $\widetilde{T}_n^* = \sum_{k=1}^{d_n} |Z_n^*(B_k)|$,then*

$$d_3(\widetilde{T}_n, \widetilde{T}_n^*) := \sup_{H \in C_3(1)} |EH(\widetilde{T}_n) - E^* H(\widetilde{T}_n^*)| = o_P(1)$$

Remark 3.1. Theorem 3.1 is a direct consequence of Theorem 2.1 and the assumption $\gamma < 1/4$ comes from the fact that we need that $d^2/n^{1/2} \to 0$. Moreover, the rate $n^{-1/2}$ is the well-known rate for the CLT (under fourth moment assumption) while d^2 comes from the number of partial derivatives. For the standard approach dimension d is fixed and the factor d^2 is absorbed into a constant. Here we let $d \to \infty$.

GOF test

Next we describe the application of Theorem 3.1 for the one-dimensional case but the extension to higher dimensions (i.e., $X_i : \Omega \to R^m\ m > 1$) is straightforward. Let us consider the partitions $B_{k,n} = [a_{k,n}, a_{k+1,n}]$, where we let $a_{0,n} = -\infty$ and $a_{d_n,n} = \infty$ and $P(B_{k,n}) = \frac{1}{d_n}$, (one could relax the last assumption to $\max_k P(B_{k,n}) \leq \frac{C}{d_n}$ for some constant C). The test statistic and its bootstrap versions are

$$\widetilde{T}_n = n^{1/2} \sum_{k=1,d_n} |P_n(B_{k,n}) - P(B_{k,n})| \text{ and } \widetilde{T}_n^* = n^{1/2} \sum_{k=1,d_n} |P_n^*(B_{k,n}) - P_n(B_{k,n})|$$

The intuition is clear: as $d_n \to \infty$, we are using finer and finer partitions and statistic $\widetilde{T}_n$ behaves like a *total variation* metric. Namely, under the null hypothesis (if P is continuous measure) the statistics $\widetilde{T}_n$ is stochastically greater than $\frac{1}{2}\sum_{i=1}^{d_n} |G(a_{k+1}) - G(a_k)|$, where $G_P(t)$ is a Brownian Bridge process, and this quantity obviously diverges.

On the other hand one can easily show that under the alternative hypothesis $H_1 : P \neq Q$ we have $\widetilde{T}_n \gtrsim n^{1/2}$ as long as there exists an interval I such that $P(I) > Q(I)$. However, we can control Type 1 error since one can show that under the null hypothesis

$$E\widetilde{T}_n = n^{1/2} \sum_{k=1,d_n} E|P_n(B_{k,n}) - P(B_{k,n})| \leq d_n \max_{k \leq d_n} \left(E\left(1_{X_i \in B_{k,n}}\right)\right)^{1/2} \lesssim d_n^{1/2} \leq n^{1/8}.$$

So, to summarize:

Under null hypothesis $(\mathcal{L}(X_i) = P)$

$$\widetilde{T}_n \to \infty \quad \textit{and} \quad \widetilde{T}_n \lesssim n^{1/8} \ \textit{in probab.}$$

Under alternative $(\mathcal{L}(X_i) = Q \neq P)$

$$\widetilde{T}_n \gtrsim n^{1/2} \ \textit{in probab.}$$

Clearly $n^{1/2} >> n^{1/8}$ which in turns implies that under alternative hypothesis the upper quantile of $\widetilde{T}_n$ will be much larger than the upper quantile under the null, resulting with the rejection.

Statistic $\widetilde{T}_n$ is related to the statistic

$$\widehat{T}_n = \sup_{\text{partition } \Pi_n} n^{1/2} \sum_{B_{k,n} \in \Pi_n} |P_n(B_{k,n}) - P(B_{k,n})|$$

that was considered in Radulovic [6] and Fremanian et al. [3]. The main difference is that here we do not have the supremum over all the partitions $\Pi_n = \{B_1, \ldots, B_{L_n}\}$ and consequently the statistic $\widetilde{T}_n$ might have lower power (i.e., harder time to distinguish P from the alternative). However, statistic $\widehat{T}_n$ comes with some issues as well. To start with, the size of partitions grows logarithmically (i.e., $d_n \approx \ln n$) while here for $\widetilde{T}_n$ we have $d_n \approx n^{1/4}$. More importantly, the computation of $\widetilde{T}_n$ is trivial while for $\widehat{T}_n$ we need to evaluate the supremum, which is often a non-trivial computational problem.

Numerical results

In order to better understand the behavior of $\widetilde{T}_n$ and proposed GOF test we offer the following simulations. We created two tests and for each one we stipulated wrong null hypothesis (i.e., $N(0,1)$), while the data came from a mixture of two Gaussian random variables and Double Exponential andrespectively. (We denote these two casses as $Alt1$ and $Alt2$.) Each of these distributions was designed to be symmetric with variance one.

For a fixed sample size n we computed $\widetilde{T}_n$ as well as 1000 bootstrap resamplings $\{\widetilde{T}^*_{n,m}\}_{m=1}^{1000}$, evaluated at mth bootstrap sample $\{X^*_{i,m}, \ldots, X^*_{n,m}\}$. Both, $\widetilde{T}_n$ and $\widetilde{T}^*_n$ are evaluated at equidistant partition of interval $[-3,3]$ (i.e., $a_k = -3 + \frac{6k}{d_n}$ with $k = 0, \ldots, d_n$) where $d_n = 3 + \lfloor \sqrt[4]{n} \rfloor$. We used the bootstrap sample to estimate the $P - val$ (i.e., $P - Val = \sum_{m=1}^{1000} 1_{\widetilde{T}_n < \widetilde{T}^*_{n,m}} / 1000$). For comparison we also performed the usual KS test and recorded the corresponding P-values. For each n, we repeated this 100 times and averaged the resulting P-values. Results are presented in Table 3.1.

Table 3.1. Performance of GOF test on simulated data

Alt1	KS GOF	Novel GOF		Alt2	KS GOF	Novel GOF	
N	Avg P-Val	Avg P-Val	d_n	N	Avg P-Val	Avg P-Val	d_n
200	0.3527	0.1089	4	200	0.1562	0.0134	4
400	0.2407	0.1185	5	400	0.0348	0.0261	5
600	0.0445	0.0053	5	600	0.0094	0.0004	5
800	0.038	0.0	6	800	0.0031	0.0	6

Comments for Table 3.1

Clearly as n increases it is easier to reject the (wrong) null hypothesis. This was captured by both methods and in both simulations since the average P-value decreases toward zero. However it is evident that in these two examples the proposed test did considerably better than the KS test. For example: In the case of Gaussian mixture (Alt1) the new test needed $n = 600$ before the average P-value dropped below 1%. On the other hand KS test, even for $n = 800$ did not drop below 3% (in fact we needed $n > 1000$ to get below 1%). Double exponential example is interesting too. Both tests had an easier time distinguishing the null from alternative, but here too, the new test did better, since its P-value is consistently lower. We also can observe the effect of increased partition size d_n which nicely follows the heuristic: "with larger data size we should consider more powerful statistic – with finer partition".

4. Proofs

Proof of Theorem 2.1. For a fixed Lipschitz function $H : \mathbf{R}^d \to \mathbf{R}$ we need to estimate

$$|EH(\overrightarrow{X}_n) - E_Y H(\overrightarrow{Y}_n)|.$$

Next we construct vectors $(\overrightarrow{X'}_{i,n})$ based on $X'_{j,i,n}, i \leq n, j \leq d$ that are defined on the same probability space as $X_{j,i,n}$'s (enlarged if necessary) but such that $\mathcal{L}(X'_{j,i,n}, i \leq n, j \leq d) = \mathcal{L}(X_{j,i,n}, i \leq n, j \leq d)$ and such that $X'_{j,i,n}$, are independent from $X_{j,i,n}$. We will use the notation E' to indicate the expectation taken with respect to $X'_{j,i,n}$. Clearly $E'H(\overrightarrow{X'}_n) = EH(\overrightarrow{X}_n)$ and since both are constants we only need to estimate

$$|E'H(\overrightarrow{X'}_n) - E_Y H(\overrightarrow{Y}_n)|.$$

This transformation is useful since now we can interchange the order of integration E_Y and E'.

Next we let $\overrightarrow{G} := (G_1, \ldots, G_d)$ be normal random variable with distribution $N(0, I_d)$, independent from $\overrightarrow{Y}_n$, $\overrightarrow{X}_n$ and $\overrightarrow{X'}_n$; and we let $\delta_n = (\ln n)^{-3}$. By adding and subtracting the appropriate terms:

$$\begin{aligned}
|E'H(\overrightarrow{X'}_n) - E_Y H(\overrightarrow{Y}_n)| &\leq |E'E_G H(\overrightarrow{X'}_n) - E'E_G H(\overrightarrow{X'}_n + \delta_n \overrightarrow{G})| \\
&\quad + |E'E_G H(\overrightarrow{X'}_n + \delta_n \overrightarrow{G}) - E_Y E_G H(\overrightarrow{Y}_n + \delta_n \overrightarrow{G})| \\
&\quad + |E_Y E_G H(\overrightarrow{Y}_n) - E_Y E_G H(\overrightarrow{Y}_n + \delta_n \overrightarrow{G})| \\
&= I + II + III,
\end{aligned}$$

where E_G stands for the expectation with respect to $\overrightarrow{G}$. We first estimate I. Since H is bounded by one and Lipschitz we can let $\tau_n = (\ln n)^{-1}$and obtain the following estimate:

$$\begin{aligned}
I &\leq \tau_n + P(\max_{i \leq d} |\delta_n G_i| > \tau_n) \leq (\ln n)^{-1} + dP(|G_1| > \tau_n/\delta_n) \\
&\leq (\ln n)^{-1} + n^m P(|G_1| > (\ln n)^2) \lesssim (\ln n)^{-1} + n^m e^{-(\ln n)^2} = o(1).
\end{aligned}$$

The estimate for III is exactly the same. Next we observe that for a $\overrightarrow{a} \in \mathbf{R}^d$ we can define

$$\widetilde{H}(\overrightarrow{a}) := E_G H(\overrightarrow{a} + \delta_n \overrightarrow{G}) = \int (2\pi\delta_n^2)^{-d/2} H(\overrightarrow{T}) \exp(-(1/2\delta_n^2))||\overrightarrow{T} - \overrightarrow{a}||^2 d\overrightarrow{T}.$$

This convoluted version of H belongs to $C^\infty(\mathbf{R}^d)$. It is easy to see that there exists a universal constant C such that all the partial derivatives (up to order 3) of $\widetilde{H}$ are bounded by a constant multiple of $\delta_n^{-C} \lesssim (\ln n)^{3C}$. Thus

$$II \leq \sup_{\widetilde{H} \in C_{3,n}} |E'\widetilde{H}(\overrightarrow{X'}_n) - E_Y \widetilde{H}(\overrightarrow{Y}_n)|$$

where $C_{3,n}$ contains all the functions with partial derivatives (up to order 3) bounded by $(\ln n)^{3C}$. Now the proof proceeds as in the classical (Lindeberg's) proof for CLT.

For a fixed $\widetilde{H}$ we let $S_j := \sum_{i=1}^{j-1} \overrightarrow{X'}_{i,n} + \sum_{i=j+1}^{n} \overrightarrow{Y}_{i,n}$, and by virtue of adding and subtracting we get the following estimate:

$$|E'\widetilde{H}(\overrightarrow{X'_n}) - E_Y\widetilde{H}(\overrightarrow{Y}_n)| \leq \sum_{j=1}^{n} |E'E_Y\widetilde{H}(S_j + \overrightarrow{X'}_{j,n}) - E'E_Y\widetilde{H}(S_j + \overrightarrow{Y}_{j,n})|. \quad (4.1)$$

Next we develop $\widetilde{H}$ as a second degree Taylor polynomial. In order to avoid cumbersome notation we present the computation for $d = 1$. The extension to $d > 1$ is straightforward but we need to keep in mind that we are dealing with d^2 and d^3 second- and third-order partial derivatives respectively. For a fixed j we have that

$$\begin{aligned}
&E'E_Y\widetilde{H}(S_j + X'_{j,n}) - E'E_Y\widetilde{H}(S_j + Y_{j,n}) \\
&\quad = E'E_Y\widetilde{H}(S_j) + E'E_Y\left(\widetilde{H'}(S_j)X'_{j,n}\right) + E'E_Y\left(\widetilde{H''}(S_j)X'^2_{j,n}\right)/2! \\
&\quad\quad + E'E_Y\left(\widetilde{H'''}(\xi_j)X'^3_{j,n}\right)/3! - E'E_Y\widetilde{H}(S_j) - E'E_Y\left(\widetilde{H'}(S_j)Y_{j,n}\right) \\
&\quad\quad - E'E_Y\left(\widetilde{H''}(S_j)Y^2_{j,n}\right)/2! - E'E_Y\left(\widetilde{H'''}(\eta_j)Y^3_{j,n}\right)/3!
\end{aligned}$$

for some ξ_j and η_j. The terms with $\widetilde{H}$ cancel. Since $S_j, X'_{j,n}$ as well as $S_j, Y_{j,n}$ are independent and since $E'X'_{j,n} = E_Y Y_{j,n} = 0$ the terms with $\widetilde{H'}$ are equal zero. Next we observe that by the assumption $E'E_Y\widetilde{H}''(S_j) \leq (\ln n)^{3C}$ and consequently

$$E|E'E_Y\widetilde{H}''(S_j)(E'X'^2_{j,n} - E_Y Y^2_{j,n})| \leq \frac{(\ln n)^{3C}}{n^{3/2}Q_n}.$$

This estimate is valid for any second-order partial derivative and it does not depend on j. Since we are dealing with $j = 1, \ldots, n$ and d^2 second-order partial derivatives; the first moment of the total contribution from the terms with second-order partial derivatives is bounded by

$$\frac{d^2(\ln n)^{3C}}{n^{1/2}Q_n}.$$

Similar argument could be applied to estimate the contribution from the reminder terms. Namely for each j there are d^3 reminders and the first moment of their sum is bounded by

$$nd^3(\ln n)^{3C}\max_{j,i}(E'|X'_{j,i,n}|^3 + EE_Y|Y_{j,i,n}|^3) \leq \frac{d^3(\ln n)^{3C}}{n^{1/2}R_n}.$$

This proves Theorem 2.1. □

Proof of Corollary 2.1. We will use the following substitution

$$X_{i,j,n} := n^{-1/2}(f_{j,n}(X_i) - Ef_{j,n}(X_i)) \qquad \text{and}$$

$$Y_{i,j,n} = X^*_{i,j,n} := n^{-1/2}(f_{j,n}(X^*_i) - E^*f_{j,n}(X^*_i)).$$

In order to apply Theorem 2.1 we need to verify (2.1) and (2.2).

$$\max_{i,j} E|X_{i,j,n}|^3 \lesssim n^{-3/2} \max_j E|f_j(X_1)|^3 \lesssim \frac{1}{n^{3/2} R_n}$$

and

$$\max_{i,j} E|Y_{i,j,n}|^3 = \max_{i,j} E^*|X^*_{i,j,n}|^3 \lesssim n^{-3/2} \max_j E|f_j(X_1)|^3 \lesssim \frac{1}{n^{3/2} R_n}.$$

Finally, since $E_Y\ Y_{i,j,n} = E^* X^*_{i,j,n}$ we need to estimate

$$\begin{aligned}
&\max_j E|E^*(X^*_{i,j,n})^2 - EX^2_{i,j,n}| \\
&\lesssim n^{-1} \max_j E\left(\left|\frac{1}{n}\sum_{i=1}^n f_j^2(X_i) - Ef_j^2(X_i)\right| + \left|\frac{1}{n}\sum_{i=1}^n f_j(X_i) - Ef_j(X_i)\right|\right) \\
&\leq n^{-3/2} \max_j \left(Ef_j^4(X_1)\right)^{1/2} \lesssim \frac{1}{n^{3/2} Q_n^{1/2}}.
\end{aligned}$$

With these estimates and Theorem 2.1 the proof follows easily.

Proof of Theorem 2.2. Let a sequence $\{X'_i\}_{i=1}^n$ be i.i.d. P, independent from $\{X_i\}_{i=1}^n$ and defined on same probability space. We let T'_n be defined in a same ways as T_n but based on the sequence $\{X'_i\}_{i=1}^n$ and not the sequence $\{X_i\}_{i=1}^n$. We observe that since $\mathcal{L}(\{X_i\}_{i=1}^n) = \mathcal{L}(\{X'_i\}_{i=1}^n)$ we can replace $E'H(T'_n)$ with $EH(T_n)$. This simple trick is important since now the variables $\{X'_i\}_{i=1}^n$ and $\{X^*_i\}_{i=1}^n$ are independent and we can freely interchange the order of integration (i.e., $E^*E' = E^*E'$), where we indicate E' as integration with respect to X'_i only. By definition of d-3 metric we need to estimate

$$\sup_{H \in C_3(1)} |EHT_n - E^*HT^*_n| = \sup_{H \in C_3(1)} |E'HT_n - E^*HT^*_n|.$$

In what follows we will simplify the notation and use k, d and g instead of k_n, d_n and g_n. Let

$$\overrightarrow{W}_n = (W_{n,1}, \ldots, W_{n,d}) \quad \text{and} \quad \overrightarrow{W}^*_n = (W^*_{n,1}, \ldots, W^*_{n,d}).$$

For a fixed n and $H \in C_3(1)$ we define $G : R^d \to R$ as

$$G(\overrightarrow{a}) := G(a_1, \ldots, a_d) := H\left(\sum_{i=1}^d g_n(a_i)\right),$$

where g_n has the properties as stipulated in the assumptions preceding Theorem 2.2. We will write g instead of g_n and we let

$$G_j(\overrightarrow{a}) := \frac{\partial}{\partial a_j} G(\overrightarrow{a}) = H'\left(\sum_{i=1}^d g(a_i)\right) g'(a_j)$$

We let $G_{j,l}(\overrightarrow{a}) := \frac{\partial}{\partial a_j \partial a_l} G(\overrightarrow{a})$ and observe that

$$G_{j,l}(\overrightarrow{a}) = \begin{cases} H''\left(\sum_{i=1}^{d} g(a_i)\right) g'(a_j) g'(a_l) & \text{if } j \neq l \\ H''\left(\sum_{i=1}^{d} g(a_i)\right) (g'(a_j))^2 + H'\left(\sum_{i=1}^{d} g(a_i)\right) g''(a_j) & \text{if } j = l. \end{cases}$$

We let $G_{j,l,m}(\overrightarrow{a}) := \frac{\partial}{\partial a_j \partial a_l \partial a_m} G(\overrightarrow{a})$ and observe that

$$G_{j,l,m}(\overrightarrow{a}) = \begin{cases} H'''\left(\sum_{i=1}^{d} g(a_i)\right) g'(a_j) g'(a_l) g'(a_m) & \text{if } j \neq l \neq m \\ H'''\left(\sum_{i=1}^{d} g(a_i)\right) (g'(a_j))^2 g'(a_m) & \\ \quad + H''\left(\sum_{i=1}^{d} g(a_i)\right) g'(a_m) g''(a_j) & \text{if } j = l \neq m \end{cases}$$

while in case $j = l = m$ we have

$$G_{j,l,m}(\overrightarrow{a}) = H'''\left(\sum_{i=1}^{d} g(a_i)\right) (g'(a_j))^3 + 2H''\left(\sum_{i=1}^{d} g(a_i)\right) g'(a_j) g''(a_j)$$
$$+ H''\left(\sum_{i=1}^{d} g(a_i)\right) g''(a_j) g'(a_j) + H'\left(\sum_{i=1}^{d} g(a_i)\right) g'''(a_j).$$

Assumption 1 now implies that

$$||G||_\infty \leq 1 \text{ and } ||G_j||_\infty \leq C \text{ , } ||G_{j,l}||_\infty \leq 1_{j \neq l} C^2 + 1_{j=l} C M_n \tag{4.2}$$

and

$$||G_{j,l,m}||_\infty \leq 1_{j \neq l \neq m} C^3 + 1_{j=l \neq m} C M_n + 1_{j=l=m} (C^3 + 3CM_n + CM_n^2). \tag{4.3}$$

Next we adopt Linderberg inclusion-exclusion trick and derive an estimate for $|EH(T_n') - E^* H(T_n^*)|$ which will not depend on the choice of $H \in C_3(1)$. For this purpose we define

$$S_{m,l} := \sum_{j=1}^{m-1} f_l(X_j') n^{-1/2} + \sum_{j=m+1}^{n} f_l(X_j^*) n^{-1/2}$$

and

$$\overrightarrow{S}_m := (S_{m,1}, \ldots, S_{m,d}) \text{ and } \overrightarrow{f}(X_m') := n^{-1/2} (f_1(X_m'), \ldots, f_d(X_m'))).$$

For a fixed $H \in C_3(1)$ we used the earlier defined G and observe that

$$H(T_n) = G(\overrightarrow{S}_{n+1}) \text{ and } H(T_n^*) = G(\overrightarrow{S}_0).$$

Thus by using notation $f_l(X_{n+1}^*) := 0$ and $f_l(X_0^*) := 0$ we get

$$H(T_n') - H(T_n^*) = \sum_{j=0}^{n} \left(G\left(\overrightarrow{S}_{n+1-j} + \overrightarrow{f}(X_{n+1-j}^*)\right) - G\left(\overrightarrow{S}_{n-j} + \overrightarrow{f}(X_{n-j}^*)\right)\right)$$

(since $S_{m,l} + f_l(X_m^*) n^{-1/2} = S_{m-1,l} + f_l(X_{m-1}') n^{-1/2}$)

$$= \sum_{j=0}^{n} G\left(\overrightarrow{S}_{n-j} + \overrightarrow{f}(X_{n-j}')\right) - G\left(\overrightarrow{S}_{n-j} + \overrightarrow{f}(X_{n-j}^*)\right). \tag{4.4}$$

Claim. There exist universal constants C_1 and C_2 (not depending on $k \in \{0, \ldots, n\}$) such that

$$E \sup_{H\in C_3(1)} \left| E'E^* G_H\left(\overrightarrow{S}_k + \overrightarrow{f}(X'_k)n^{-1/2}\right) - E'E^* G_H\left(\overrightarrow{S}_k + \overrightarrow{f}(X^*_k)n^{-1/2}\right)\right|$$
$$\leq C_1 \frac{d^3 + d^2 M_n + dM_n^2}{n^{3/2} R_n} + C_2 \frac{d^2 + dM_n}{n^{3/2} K_n^{1/2}}.$$

We use the notation E', E, E^* in order to indicate the integration with respect to X'_i, X_i and X^*_i respectively. We will simplify the notation and write G instead of G_H. Since G is three times differentiable there exist vectors $\overrightarrow{\xi}$ and $\overrightarrow{\nu}$ such that

$$\begin{aligned} G(\overrightarrow{S}_k + \overrightarrow{f}(X'_k)) &= G(\overrightarrow{S}_k) + \sum\nolimits_{j=1}^d G_j(\overrightarrow{S}_k) f_j(X'_k) n^{-1/2} \\ &\quad + \sum\nolimits_{j,l=1}^d G_{j,l}(\overrightarrow{S}_k) f_j(X'_k) f_l(X'_k) n^{-1} \\ &\quad + \sum\nolimits_{j,l,m=1}^d G_{j,l,m}(\overrightarrow{\xi}) f_j(X'_k) f_l(X'_k) f_m(X'_k) n^{-3/2} \\ &= A_0 + A_1 + A_2 + A_3 \end{aligned}$$

and

$$\begin{aligned} G(\overrightarrow{S}_k + \overrightarrow{f}(X^*_k)) &= G(\overrightarrow{S}_k) + \sum\nolimits_{j=1}^d G_j(\overrightarrow{S}_k) f_j(X^*_k) n^{-1/2} \\ &\quad + \sum\nolimits_{j,l=1}^d G_{j,l}(\overrightarrow{S}_k) f_j(X^*_k) f_l(X^*_k) n^{-1} \\ &\quad + \sum\nolimits_{j,l,m=1}^d G_{j,l,m}(\overrightarrow{\nu}) f_j(X^*_k) f_l(X^*_k) f_m(X^*_k) n^{-3/2} \\ &= A^*_0 + A^*_1 + A^*_2 + A^*_3. \end{aligned}$$

Clearly, $A_0 = A^*_0$ and since under the $E'E^*$ the vector $\overrightarrow{S}_k$ is independent from both X'_k as well as X^*_k we have that $E'E^* A_1 = E'E^* A^*_1 = 0$. Thus

$$E \sup_{H\in C_3(1)} \left| E'E^* G\left(\overrightarrow{S}_k + \overrightarrow{f}(X'_k)\right) - E'E^* G\left(\overrightarrow{S}_k + \overrightarrow{f}(X^*_k)\right)\right|$$
$$\leq E \sup_{H\in C_3(1)} |E'E^* A_2 - E'E^* A^*_2| + E \sup_{H\in C_3(1)} |E'E^* A_3| + E \sup_{H\in C_3(1)} |E'E^* A^*_3|.$$

Let us estimate $E \sup_{H\in C_3(1)} |E'E^* A_3|$ first. Using the estimate (4.3), assumption 2 and Hölder's inequality we have

$$|E'E^* A_3| \leq n^{-3/2} \sum_{j,l,m=1}^d ||G_{j,l,m}||_\infty E' |f_j(X'_k) f_l(X'_k) f_m(X'_k)|$$

(since $\mathcal{L}(X'_i) = \mathcal{L}(X_i)$ and $f_j = \widetilde{f}_j - E\widetilde{f}_j$)

$$\leq 2n^{-3/2}(C^3 d^3 + Cd^2 M_n + dC^3 + 3dCM_n + dCM_n^2) \max\nolimits_j E|\widetilde{f}_j(X_1)|^3$$
$$\leq \widetilde{C} \frac{d^3 + d^2 M_n + dM_n^2}{n^{3/2} R_n},$$

for some universal constant $\widetilde{C}$. The estimate for $E|A_3^*|$ is very similar.

$$E \sup_{H\in C_3(1)} |E'E^*A_3^*| \le n^{-3/2} \sum_{j,l,m=1}^{d} ||G_{j,l,m}||_\infty E \sup_{H\in C_3(1)} E^*|f_j(X_k^*)f_l(X_k^*)f_m(X_k^*)|$$

However,

$$\begin{aligned} EE^*|f_j(X_k^*)f_l(X_k^*)f_m(X_k^*)| &\le 2En^{-1}\sum_{i=1}^{n}|\widetilde{f}_j(X_i)\widetilde{f}_l(X_i)\widetilde{f}_m(X_i)| \\ &= E|\widetilde{f}_j(X_1)\widetilde{f}_l(X_1)\widetilde{f}_m(X_1)| \end{aligned}$$

and consequently

$$\begin{aligned} &E \sup_{H\in C_3(1)} |E'E^*A_3^*| \\ &\quad\le 2n^{-3/2}(C^3d^3 + Cd^2M_n + dC^3 + 3dCM_n + dCM_n^2)\max_j E|\widetilde{f}_j(X_1)|^3. \end{aligned}$$

Finally we need to estimate

$$E \sup_{H\in C_3(1)} |E'E^*A_2 - E'E^*A_2^*|$$

since $\overrightarrow{S}_k$ is independent from X_k' as well as X_k^*

$$\le n^{-1}\sum_{j,l=1}^{d}||G_{j,l}||_\infty E \sup_{H\in C_3(1)} |E'(f_j(X_k')f_l(X_k') - E^*f_j(X_k^*)f_l(X_k^*)|. \tag{4.5}$$

First we estimate

$$E \sup_{H\in C_3(1)} |E'(f_j(X_k')f_l(X_k') - E^*f_j(X_k^*)f_l(X_k^*)|$$

since $\mathcal{L}(\{X_k\}_{k>0}) = \mathcal{L}(\{X_k'\}_{k>0})$ and since f_j's do not depend on H

$$\begin{aligned} &= n^{-1/2}E|n^{-1/2}\sum_{i=1}^{n}(f_j(X_i)f_l(X_i) - E(f_j(X_i)f_l(X_i))| \\ &\le n^{-1/2}\sqrt{E(f_j(X_1)^2f_l(X_1)^2)} \le n^{-1/2}(E(f_j(X_1)^4))^{1/2} \le \frac{1}{n^{1/2}K_n^{1/2}}. \end{aligned}$$

Thus using estimate (4.2) the expression (4.5) is bounded by

$$\widehat{C}\frac{d^2 + dM_n}{n^{3/2}K_n^{1/2}}$$

for some universal constant $\widehat{C}$. This proves the Claim. Finally, since the Linderberg trick (computation (4.4)) yields n parts we have shown that

$$E \sup_{H\in C_3(1)} |EH(T_n) - E^*H(T_n^*)| \le C_1\frac{d^3 + d^2M_n + dM_n^2}{n^{1/2}R_n} + C_2\frac{d^2 + dM_n}{n^{1/2}K_n^{1/2}}$$

which proves Theorem 2.2. □

Proof of Theorem 3.1. For a $B_{k,n} \in \Pi_n$ we define $\widetilde{f}_{k,n}(X_i) = 1_{X_i \in B_{k,n}}$ and let

$$W_{n,k} := Z_n(B_{k,n}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_{k,n}(X_i)$$

and

$$W_{n,k}^* := Z_n^*(B_{k,n}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_{k,n}(X_i^*)$$

with $f_{k,n}(X_i) = \widetilde{f}_{k,n}(X_i) - E\widetilde{f}_{k,n}(X_i)$ and $f_{k,n}(X_i^*) = \widetilde{f}_{k,n}(X_i^*) - E^*\widetilde{f}_{k,n}(X_i^*)$. Next, for $M_n = 1/(d \ln d)$, we let

$$g_n(x) = \left(\frac{1}{16M_n^5} x^6 - \frac{5}{16M_n^3} x^4 + \frac{15}{16M_n} x^2 + \frac{5M_n}{16} \right) 1_{|x| \le M_n} + |x| 1_{|x| > M_n}$$

and observe that function $g_n(x)$ has the following properties:

$$||(g_n(x) - |x|)||_\infty \le M_n^{-1}, \quad ||g_n'||_\infty \le C, \quad ||g_n''||_\infty \le CM_n, \quad ||g_n''||_\infty \le CM_n^2.$$

We define

$$T_n = \sum_{k=1}^{d_n} g_n(W_{n,k}) \quad \text{and} \quad T_n^* = \sum_{k=1}^{d_n} g_n(W_{n,k}^*),$$

and observe that for any $H \in C_3(1)$

$$|EH(T_n) - EH(\widetilde{T}_n)| \lesssim \sum_{k=1}^{d} \| |W_{n,k}| - g_n(W_{n,k}) \|_\infty \le \frac{1}{\ln d} \to 0$$

and

$$|EH(T_n^*) - EH(\widetilde{T}_n^*)| \lesssim \sum_{k=1}^{d} \left\| |W_{n,k}| - g_n(W_{n,k}^*) \right\|_\infty \le \frac{1}{\ln d} \to 0.$$

Thus

$$\sup_{H \in C_3(1)} |EH(\widetilde{T}_n) - E^*H(\widetilde{T}_n^*)| \le \sup_{H \in C_3(1)} |EH(T_n) - E^*H(T_n^*)| + o(1)$$

(by Theorem 2.2)

$$= O_P\left(\frac{d^3 + d^2 M_n + dM_n^2}{n^{1/2} R_n} + \frac{d^2 + dM_n}{n^{1/2} K_n^{1/2}} \right) + o(1)$$

$$= O_P\left(\frac{d^3 \ln^2 d}{n^{1/2} R_n} + \frac{d^2 \ln d}{n^{1/2} K_n^{1/2}} \right) + o(1)$$

$$= O_P\left(\frac{\ln^2 d}{n^{1/2-3\gamma} R_n} + \frac{\ln d}{n^{1/2-2\gamma} K_n^{1/2}} \right) + o(1)$$

where as before K_n and R_n are such that

$$\max_k E|f_k(X_1)|^3 \le \frac{1}{R_n} \quad \text{and} \quad \max_k E(f_k(X_1))^4 \le \frac{1}{K_n}.$$

However, from the definition of $\widetilde{f}$ it follows that

$$E|f_k(X_1)|^3 \lesssim \max_k P(B_k) \lesssim \frac{1}{d_n} \quad \text{and} \quad E|f_k(X_1)|^4 \lesssim \max_k P(B_k) \lesssim \frac{1}{d_n}.$$

Consequently, we can let $K_n = R_n = d = n^\gamma$ and since $\gamma < 1/4$ the above estimate becomes

$$O_P\left(\frac{\ln^2 d}{n^{1/2-2\gamma}} + \frac{\ln d}{n^{1/2-3/2\gamma} K_n}\right) + o(1) = o_P(1).$$

This proves Theorem 3.1.

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