The Adiabatic Limit of the Laplacian on Thin Fibre Bundles

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We consider the time dependent Schrödinger equation on a fibre bundle in the adiabatic limit. We allow the fibres to have a boundary, in which case we impose Dirichlet conditions. In particular this allows us to understand, in a general setting, results obtained in the context of quantum waveguides (see the review [2]), in which case the bundle is usually a solid cylinder or a square. In order to extract such results from our effective operator one has to embed the total space into $\mathbb{R}^n$ and expand the induced metric. The leading order term will be a Riemannian submersion. When calculating the effective operator the additional corrections due to the metric need to be taken into account. The resulting operator can then be analysed to obtain e.g. expansions for the eigenvalues.
One of the virtues of the approach is a scaling that gives results for energies that are infinite in many of the settings treated in the literature. More precisely, Theorem 1 gives an approximation for finite energies when the energies of the fibre dynamics are independent of the scaling parameter. In comparison the often considered scaling in which these energies go to infinty as the fibres get thin only yields results for small energies above the ground state (after substracting the increasing energy of the fibre ground state).

We use methods of adiabatic perturbation theory (see [3] for a comprehensive presentation). The basic idea is that the separation of scales between the base and the fibre leads to a decoupling of the corresponding dynamics. This means that for special initial conditions (one may think of eigenstates of the fibre dynamics) the dynamics in the fibre direction will be trivial for long times. One can thus decompose the total problem into a set of simpler problems, one for every such initial condition. Each of these has dynamics only in directions on the base and is governed by an effective equation.

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These methods were applied in a differential geometric setting to the problem of constraints in quantum mechanics [4].

Let $F \to M \overset{\pi}{\to} B$ be a smooth fibre bundle with a Riemannian submersion metric $g = g_{F_x} \oplus \pi^* h$ (see [1, Chap. 2] for a discussion of basic properties). By the adiabatic limit we mean the asymptotic limit as $\varepsilon \to 0$ for the scaled family of metrics $g_\varepsilon = g_{F_x} \oplus \varepsilon^{-2}\pi^* h$. We study this limit for the time dependent Schrödinger equation

$$\mathrm{i}\partial_t \psi = -\Delta_{g_\varepsilon}\psi. \tag{1}$$

Δ_{g_ε} is self adjoint on the Dirichlet domain $H^2(M) \cap H^1_0(M) (= H^2(M)$ if $\partial M = \emptyset)$.

It can easily be seen (for example using the quadratic form) that the Laplacian decomposes into

$$\Delta_{g_\varepsilon} = \mathrm{tr}_{TF}\nabla^2 + \varepsilon^2 \left(\mathrm{tr}_{NF}\nabla^2 - \eta\right) =: \Delta_{F_x} + \varepsilon^2 \Delta_h \tag{2}$$

where η is the mean curvature vector of the fibres (in the metric $g_{\varepsilon=1}$). We interpret these terms in the following way:

- For every x the Laplacian of the fibre Δ_{F_x} is a bounded linear operator from its domain to $L^2(F_x)$. It can thus be viewed as a section of a bundle over B: Let $L^2(F;\pi)$ and D be vector bundles induced by π (with fibres $L^2(F)$ and $H^2(F_x) \cap H^1_0(F_x)$ respectively). The fibre Laplacian is precisely a section of the bundle of continuous linear maps $\mathcal{L}\left(D, L^2(F;\pi)\right)$ between both.
- Derivation in the horizontal direction formally defines a connection ∇^h on $L^2(F;\pi)$. $-\Delta_h$ can be identified as the Laplacian $\nabla^{h*}\nabla^h$ of this connection.

From this point of view the operator $\Delta_{g_\varepsilon} = \Delta_{F_x} + \varepsilon^2\Delta_h$ fits nicely into the framework of adiabatic perturbation theory, where one might call it "fibred over B". Now we proceed by noting that for every $x \in B$ the spectrum of Δ_{F_x} is discrete, of finite multiplicity and independent of ε. As it depends continuously on x it has a band structure i.e. one can choose continuous functions $E(x)$ that are eigenvalues for every x.

Let E be such a band and $P(x)$ its spectral projection, i.e.

$$\forall x: \quad -\Delta_{F_x} P(x) = E(x)P(x). \tag{3}$$

Since $P(x)$ is a bounded linear map on $L^2(F_x)$, P is a section of $\mathcal{L}(L^2(F;\pi))$. If E is separated from the rest of the spectrum then the dimension of ran P is constant and $PL^2(F;\pi)$ is a finite-rank subbundle that is locally spanned by eigenfunctions of Δ_{F_x}.

One can show that $PL^2(M)$ is left invariant by the dynamics up to errors of order ε by estimating $[\Delta_{g_\varepsilon}, P] = [\varepsilon^2\Delta_h, P]$ using the following:

Lemma 1. *Let M, B and π be of bounded geometry.*[1] *Let E be bounded and uniformly separated.*
Then $\left\| [\varepsilon\nabla^h_X, P]P \right\|_{\mathcal{L}(L^2(M))} \le C\varepsilon$ *for every bounded X.*

This allows us to describe the leading order dynamics (with initial conditions in $PL^2(M)$) by the effective operator $P\Delta_{g_\varepsilon}P$ on the L^2-sections of our finite-rank bundle $PL^2(F;\pi)$. However we aim for more precision because:

1. Horizontal distances increase as $\varepsilon \to 0$, so with kinetic energies of order one it takes times of order $1/\varepsilon$ for global effects to occur, and
2. The spacing of the eigenvalues of Δ_{g_ε} decreases and an error of order ε might not be enough to distinguish them.

Therefore we need better approximations to understand the dynamics and completely resolve the spectrum. For this purpose we construct a corrected projection $P^\varepsilon = P + \mathcal{O}(\varepsilon)$ (that is no longer a fibrewise operator), for which $[\Delta_{g_\varepsilon}, P]P = \mathcal{O}(\varepsilon^N)$ holds, and an intertwining unitary $U^\varepsilon : P^\varepsilon L^2(M) \to PL^2(M)$ that allows us to define the effective operator on sections of $PL^2(F;\pi)$ (see [4] for a detailed exposition of the technique).

Theorem 1. *Under the assumptions of the lemma there exist*

- *A projection $P^\varepsilon \in \mathcal{L}\left(L^2(M)\right)$,*
- *A unitary operator $U^\varepsilon \in \mathcal{L}\left(L^2(M)\right)$ that maps $P^\varepsilon L^2(M) \to PL^2(M)$,*
- *A self-adjoint operator $(H_{\text{eff}}, D_{\text{eff}})$ on $L^2(PL^2(F;\pi))$,*

such that

$$\left\| \left(\mathrm{e}^{-\mathrm{i}H^\varepsilon t} - U^{\varepsilon *}\mathrm{e}^{-\mathrm{i}H_{\text{eff}}t}U^\varepsilon\right) P^\varepsilon \chi_{(-\infty, E_{\max})}(\Delta_{g_\varepsilon}) \right\|_{\mathcal{L}(L^2(M))} \le C\varepsilon^N t \tag{4}$$

for all $t \ge 0$ and $E_{\max} < \infty$;
If λ is an eigenvalue of H_{eff} then there is a ball B of radius $C\varepsilon^N$ around λ such that $B \cap \sigma\left(\Delta_{g_\varepsilon}\right) \neq \emptyset$.

The effective operator is given by

$$H_{\text{eff}} = PU^\varepsilon \Delta_{g_\varepsilon} U^{\varepsilon *} P = -\varepsilon^2 \Delta_B + E(x) + \mathcal{O}(\varepsilon^2)\,, \tag{5}$$

where $-\Delta_B$ is the Laplacian of the induced connection $\nabla^B = P\nabla^h P$. The higher order corrections can be computed explicitly from the construction of P^ε.

[1] When there is no boundary we require that M, B be of bounded geometry and π have bounded derivatives. In the case with boundary there is a similar condition. Both are trivially satisfied if M is compact.

References

1. Peter B. Gilkey, John V. Leahy, and Jeonghyeong Park, *Spectral geometry, Riemannian submersions, and the Gromov-Lawson conjecture*, Studies in Advanced Mathematics, Chapman & Hall/CRC, 1999.
2. D. Grieser, *Thin tubes in mathematical physics, global analysis and spectral geometry*, Analysis on Graphs and its Applications, Proceedings of Symposia in Pure Mathematics, vol. 77, 2008.
3. Stefan Teufel, *Adiabatic perturbation theory in quantum dynamics*, Lecture Notes in Mathematics, Springer, 2003.
4. Jakob Wachsmuth and Stefan Teufel, *Effective Hamiltonians for constrained quantum systems*, arXiv:0907.0351 [math-ph], 2009.