Adiabatic Limits and Related Lattice Point Problems

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1 Preliminaries on Adiabatic Limits

Let $(M, \mathcal{F})$ be a closed foliated manifold endowed with a Riemannian metric g. Then we have a direct sum decomposition $TM = F \oplus H$ of the tangent bundle TM of M, where $F = T\mathcal{F}$ is the tangent bundle of $\mathcal{F}$ and $H = F^{\perp}$ is the orthogonal complement of F, and the corresponding decomposition of the metric: $g = g_F + g_H$. Consider the one-parameter family of Riemannian metrics on M,

$$g_\varepsilon = g_F + \varepsilon^{-2} g_H, \quad \varepsilon > 0,$$

and the corresponding Laplace-Beltrami operator Δ_ε. We are interested in the asymptotic behavior of the trace of the operator $f(\Delta_\varepsilon)$ for sufficiently nice functions f on $\mathbb{R}$, in particular, of the eigenvalue distribution function $N_\varepsilon(\lambda)$ of Δ_ε, as $\varepsilon \to 0$ (in the adiabatic limit).

In [4] (see also [2, 3, 5]), the first author proved an asymptotic formula for $\operatorname{tr} f(\Delta_\varepsilon)$ in the case when the foliation $\mathcal{F}$ is Riemannian and the metric g is bundle-like. For particular examples of non-Riemannian foliations, such an asymptotic formula was proved by the second author in [11, 12] (see also a survey paper [6] for some historic remarks and references).

As the simplest example, one can consider the linear foliation $\mathcal{F}$ on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ given by the leaves $L_x = x + F \mod \mathbb{Z}^n$, $x \in \mathbb{T}^n$, where F is a linear subspace of $\mathbb{R}^n$. Let g be the standard Euclidean metric on $\mathbb{T}^n$. The foliation $\mathcal{F}$ is Riemannian, and the metric g is bundle-like. In this case, the eigenvalue distribution function $N_\varepsilon(\lambda)$ of Δ_ε equals the number of integer points in

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the ellipsoid $\{\xi \in \mathbb{R}^n : \sum_{j,\ell=1}^n g_\varepsilon^{j\ell}\xi_j\xi_\ell < \lambda/(2\pi)^2\}$. So we arrive at the following lattice point problem.

2 Lattice Point Problems

Let F be a p-dimensional linear subspace of $\mathbb{R}^n$ and $H = F^\perp$ the orthogonal complement of F with respect to the standard Euclidean inner product $(\cdot,\cdot)$ in $\mathbb{R}^n$, $p+q=n$. For any $\varepsilon > 0$, consider the linear transformation $T_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T_\varepsilon(x) = x, \text{ if } x \in F, \quad T_\varepsilon(x) = \varepsilon^{-1}x, \text{ if } x \in H.$$

Let S be a bounded open set with smooth boundary in $\mathbb{R}^n$. Put

$$n_\varepsilon(S) = \#(T_\varepsilon(S) \cap \mathbb{Z}^n), \quad \varepsilon > 0.$$

The problem is to study the asymptotic behavior of $n_\varepsilon(S)$ as $\varepsilon \to 0$. It appears that, in the general case, the leading term in the asymptotic formula for $n_\varepsilon(S)$ as $\varepsilon \to 0$ was unknown. In a slightly different context, this problem was studied in considerable detail in [9, 10] (see also the references therein).

Let $\Gamma = \mathbb{Z}^n \cap F$. Γ is a free abelian group. Denote by $r = \operatorname{rank}\Gamma \leq p$ the rank of Γ. Let V be the r-dimensional subspace of $\mathbb{R}^n$ spanned by the elements of Γ. Observe that Γ is a lattice in V. Let $\Gamma^* \subset V$ denote the dual lattice to Γ: $\Gamma^* = \{\gamma^* \in V : (\gamma^*, \Gamma) \subset \mathbb{Z}\}$. For any $x \in V$, denote by P_x the $(n-r)$-dimensional affine subspace of $\mathbb{R}^n$, passing through x orthogonal to V.

Theorem 1 ([7]). *Under the current assumptions, we have*

$$n_\varepsilon(S) = \frac{\varepsilon^{-q}}{\operatorname{vol}(V/\Gamma)} \sum_{\gamma^* \in \Gamma^*} \operatorname{vol}_{n-r}(P_{\gamma^*} \cap S) + R_\varepsilon(S),$$

where the remainder $R_\varepsilon(S)$ satisfies the estimate

$$R_\varepsilon(S) = O(\varepsilon^{\frac{1}{p-r+1}-q}), \quad \varepsilon \to 0.$$

Theorem 2 ([7, 8]).

(1) If, for any $x \in F$, the intersection $\{x+H\} \cap S$ is strictly convex, then

$$R_\varepsilon(S) = O(\varepsilon^{\frac{2q}{q+1+2(p-r)}-q}), \quad \varepsilon \to 0.$$

(2) If, for any $x \in F$, the intersection $\{x+V^\perp\} \cap S$ is strictly convex, then

$$R_\varepsilon(S) = O(\varepsilon^{\frac{2q}{n-r+1}-q}), \quad \varepsilon \to 0.$$

In [8], we also study the average remainder estimates, where the average is taken over rotations of the domain S by orthogonal transformations in $\mathbb{R}^n$.

3 Applications to Adiabatic Limits

As a straightforward consequence of Theorem 2, we obtain a more precise estimate for the remainder in the asymptotic formula of [4] in the above mentioned case when $\mathcal{F}$ is a linear foliation on $\mathbb{T}^n$ and g is the standard Euclidean metric.

Theorem 3 ([7]). *For $\lambda > 0$, we have, as $\varepsilon \to 0$,*

$$N_\varepsilon(\lambda) = \varepsilon^{-q} \frac{\omega_{n-r}}{\mathrm{vol}(V/\Gamma)} \sum_{\gamma^* \in \Gamma^*} \left(\frac{\lambda}{4\pi^2} - |\gamma^*|^2 \right)^{(n-r)/2} + O(\varepsilon^{\frac{2q}{n-r+1} - q}),$$

where ω_{n-r} is the volume of the unit ball in $\mathbb{R}^{n-r}$.

4 Some Open Problems

1. *Prove an asymptotic formula for* $\operatorname{tr} f(\Delta_\varepsilon)$ *when $\mathcal{F}$ is an arbitrary foliation.*
   The case when $\mathcal{F}$ is given by the fibers of a fibration over a compact manifold and the metric g is not bundle-like, is already quite interesting.
2. *Prove a complete asymptotic expansion for the heat trace* $\operatorname{tr} e^{-t\Delta_\varepsilon}$ *as $\varepsilon \to 0$ (even if the metric g is bundle-like).*
3. *Study the adiabatic limits of more complicated spectral invariants like the eta-invariant, the analytic torsion etc. (even if the metric g is bundle-like).*
   Here the extension of the Mazzeo-Melrose result on small eigenvalues in the adiabatic limit and spectral sequences to Riemannian foliations [1] might be useful.
4. *Study the remainder estimates for $N_\varepsilon(\lambda)$.*
5. *Continue the study of the remainder $R_\varepsilon(S)$, depending on geometry of a domain S and properties of F and H.*

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References

1. J. Álvarez López, Yu. A. Kordyukov, *Adiabatic limits and spectral sequences for Riemannian foliations*, Geom. and Funct. Anal. **10** (2000), 977–1027.

2. Yu. A. Kordyukov, *Quasiclassical asymptotics of the spectrum of elliptic operators in a foliated manifold.* Math. Notes **53** (1993), no. 1–2, 104–105.
3. Yu. A. Kordyukov, *On the quasiclassical asymptotics of the spectrum of hypoelliptic operators on a manifold with foliation.* Funct. Anal. Appl. **29** (1995), no. 3, 211–213.
4. Yu. A. Kordyukov, *Adiabatic limits and spectral geometry of foliations*, Math. Ann. **313** (1999), 763–783.
5. Yu. A. Kordyukov, *Semiclassical spectral asymptotics on foliated manifolds*, Math. Nachr. **245** (2002), 104–128.
6. Yu. A. Kordyukov, A. A. Yakovlev, *Adiabatic limits and the spectrum of the Laplacian on foliated manifolds*, C^*-algebras and elliptic theory. II, Trends in Mathematics, Birkhäuser, Basel, 2008, 123–144.
7. Yu. A. Kordyukov, A. A. Yakovlev, *Integer points in domains and adiabatic limits*, Algebra i Analiz, **23** (2011), no. 6, 80–95; preprint arXiv:1006.4977.
8. Yu. A. Kordyukov, A. A. Yakovlev, *The number of integer points in a family of anisotropically expanding domains*, in preparation.
9. Nikichine, N. A.; Skriganov, M. M., *Nombre de points d'un réseau dans un produit cartésien de domaines convexes*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 6, 671–675.
10. Nikishin, N. A.; Skriganov, M. M.. *On the distribution of algebraic numbers in parallelotopes*, St. Petersburg Math. J. **10** (1999), no. 1, 53–68
11. A. A. Yakovlev, *Adiabatic limits on Riemannian Heisenberg manifolds*, Sb. Math. **199** (2008), no. 1–2, 307–318.
12. A. A. Yakovlev, *Asymptotics of the spectrum of the Laplace operator on Riemannian Sol-manifolds in the adiabatic limit*, Sib. Math. J. **51** (2010), no. 2, 370–382.