Microlocal Analysis of FIOs with Singularities

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Overview In this talk we describe the composition calculus of Fourier Integral Operators (FIOs) with fold and cusp singularities. Such operators appear in many inverse scattering problems, where the composition calculus can be used as a tool for recovering images. In these problems, caustics occur and create artifacts which make the reconstruction more complicated and challenging. The goal is to understand these artifacts, find their strength and try to remove them.

Motivation In the seismic problems [13, 14], acoustic waves are generated at the surface of the earth, scatter the heterogeneities of the subsurface and return to the surface. The pressure field is measured at the surface and is used to reconstruct an image of the subsurface. We consider the linearized operator F which maps singular perturbations of a smooth background sound speed in the subsurface, to perturbations of the pressure field. We study different cases where the waves are generated from a fixed single source and from a moving single source. In reality caustics appear. We define a caustic in the following way: a ray departing from a source s in the direction α reaches at time t a point denoted $x(t,\alpha)$ in the subsurface. If the projection map $(t,\alpha) \to x(t,\alpha)$ is singular then $x(t,\alpha)$ lies on a caustic. We consider only fold and cusp caustics which occur when this map exhibits fold or cusp singularities. In order to reconstruct the image we apply the operator F^* to the data. In the case when no caustics occur, F^*F is a pseudodifferential operator which can be inverted [1]. The focus is to understand the reconstruction operator F^*F when caustics are present.

Background Let $I^m(X,Y;C)$ be the class of Fourier integral operators, F : $E'(Y) \to D'(X)$ associated to a canonical relation $C \subset (T^*X \times T^*Y)\setminus\{0\}$. Under certain geometric conditions like transverse and clean intersection conditions, this class is closed under composition [2, 11]. When these conditions fail to be satisfied,

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the geometry of each canonical relation plays an important role in establishing a composition calculus. Let π_L and π_R be the projections to the left and right from C to $T^*X \setminus 0$ and $T^*Y \setminus 0$.

Singularities My interest is in the case when both projections are singular in specific ways: π_L and π_R have both fold singularities (when C is called a two sided fold) or π_R is a submersion with folds and π_L has a cross cap singularity (when C is called a folded cross cap) or π_R and π_L have both cusp singularities (when C is called a two sided cusp). In the above examples, the operator F^*F is not an FIO anymore but it belongs to a new class of operators associated to a pair of intersecting Lagrangeans (Δ, C) where Δ is the diagonal in $T^*X \times T^*Y$ and C is the artifact. In the first two cases, C is a smooth canonical relation and intersects Δ cleanly and we show that F^*F belongs to $I^{p,l}(\Delta, C)$, a class of distributions studied in [8–10, 12] while in the last case, C is a singular Lagrangean called an open umbrella [5, 7].

Fixed source, fold caustics In the case when a single source generates acoustics waves in the presence of fold caustics only, Nolan showed that F is associated to a two sided fold [13]. We proved that $F^*F \in I^{2m,0}(\Delta, \hat{C})$ where $\hat{C}$ is another two sided fold [3, 13]. Using the properties of $I^{p,l}$ classes, this implies that $F^*F \in I^{2m}(\Delta \setminus \hat{C})$ and $F^*F \in I^{2m}(\hat{C} \setminus \Delta)$. Thus the artifact $\hat{C}$ has the same strength as the pseudodifferential part and it cannot be removed.

Fixed source, cusp caustics In the single source case, in the presence of cusp caustics, F is associated to a two sided cusp and $F^*F \in I^{2m}(\Delta, \Lambda)$ [5], where Λ is an open umbrella. We also obtain that away from $\Delta \cap \Lambda$, F^*F is of order $2m$ on both Δ and Λ which means that the artifact is as strong as the initial location of the singularities.

Moving source, fold caustics We prove that under the fold caustic assumption, F is an FIO associated to a folded cross cap canonical relation C, and that $F^*F \in I^{2m-\frac{1}{2},\frac{1}{2}}(\Delta, \tilde{C})$ where $\tilde{C}$ is a two sided fold [4]. In this case $F^*F \in I^{2m}(\Delta \setminus \tilde{C})$ and $F^*F \in I^{2m-\frac{1}{2}}(\tilde{C} \setminus \Delta)$, hence the artifact is $\frac{1}{2}$ smoother and there is hope for the image recovering. So far we obtained H^s estimates for operators belonging to $I^{p,l}(\Delta, \hat{C})$ and $I^{p,l}(\Delta, \tilde{C})$ [6].

Open problems We would like to find Sobolev estimates for the operators in the class $I^{2m}(\Delta, \Lambda)$ and to invert the operators from $I^{2m-\frac{1}{2},\frac{1}{2}}(\Delta, \tilde{C})$.

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