The Identification Problems in SPECT: Uniqueness, Non-uniqueness and Stability

Plamen Stefanov

We study an inverse problem arising in Single Photon Emission Computerized Tomography (SPECT): recover an unknown source distribution with a unknown attenuation. The mathematical model is the attenuated X-ray transform

$$X_a f(x,\theta) = \int e^{-Ba(x+t\theta,\theta)} f(x+t\theta)\, \underset{.}{t}, \quad x \in \mathbf{R}^2,\ \theta \in S^1, \tag{1}$$

in the plane with a source f and an attenuation a that we want to recover. We denote by

$$Ba(x,\theta) = \int_0^\infty a(x+t\theta)\, dt \tag{2}$$

the "beam transform" of a, usually denoted by Da. The functions a and f are assumed to be compactly supported. We analyze whether one can recover both a and f, and if so; whether this can be done in a stable way.

The linearization $\delta X_{a,f}$ of $X_a f$ with respect to (a, f) turns out to be a sum of two weighted X-ray transforms:

$$\delta X_{a,f}(\delta a, \delta f) = I_w \delta a + X_a \delta f,$$

where

$$I_w f(x,\theta) = \int w(x+t\theta,\theta) f(x+t\theta)\, dt, \quad x \in \mathbf{R}^2,\ \theta \in S^1,$$

P. Stefanov (✉)
Department of Mathematics, Purdue University, west lafayette, IN, USA
e-mail: stefanov@math.purdue.edu

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and

$$w = -e^{-Ba} u,$$

with

$$u(x,\theta) = \int_{-\infty}^{0} e^{-\int_t^0 a(x+s\theta)\,ds} f(x+t\theta)\,dt.$$

This motivates the study of the inversion of the more general transform $I_{w_1} g_1 + I_{w_2} g_2$.

The data $\delta X_{a,f}(\delta a, \delta f)$ contains an integral over each line twice—once in each direction. On the other hand, the integral over each such line (and some neighborhood of it) contains information about singularities conormal to it. So microlocally, we have a 2×2 system. To get a pseudo-differential system, instead of an FIO one, we take the Fourier transform with respect to the initial point $z \in \theta^\perp$ of each line, for each direction θ. The determinant of the so transformed $\delta X_{a,f}$, up to elliptic factors, is

$$p_0(x,\xi) = u(x,\theta) - u(x,-\theta)|_{\theta=\xi^\perp/|\xi|}.$$

Since p_0 is an odd function of θ, it has zeros over any point x. Therefore, elliptic methods would not work. The Hamiltonian flow of p_0 then plays an important role. We call the projections of the zero bicharacteristics of p_0 to the x-space *rays*. By the Duistermaat-Hörmaner propagation of singularities theorem, on any bicharacteristics, either each point is a singularity, or none is. If we a priori know that δa, δf are supported in a non-trapping set K (no ray lies entirely in K), then recovery of the singularities of δa, δf is possible with a loss of one derivative. Then we also have an a priori regularity estimate. A more careful analysis actually reveals that we need to assume the non-trapping condition for δa only.

Using this, we prove that under the non-trapping assumption the linearization $\delta X_{a,f}$ is invertible in a stable way, if we know that it is injective. We provide conditions for injectivity: either $(\delta a, \delta f)$ are supported in a small enough set, or (a, f) satisfy the following analyticity conditions: Ba and u are analytic in $K \times S^1$. For the non-linear identification problem we prove local uniqueness near such a, f, in particular assuming the non-trapping condition for δa, and a conditional Hölder stability estimate.

As an example, we consider radial a and f. First, we consider $a = 0$ and f being equal to the characteristic function of the unit disk. The rays then are the concentric circles $|x| = R$, $0 \le R < 1$. K is non-trapping, if and only if no entire circle of that family lies in K. We also study the case of general radial a and f. There is no uniqueness in that case, and in fact, given any C_0^∞ radial a, f, one can find a radial f_0, so that $X_a f = X_0 f_0$. This is a trapping case (without non-trapping support restrictions, except that all functions are supported in the unit disk) and demonstrates that in the trapping case we may lose the well-posedness of the problem.

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