Pseudodifferential Operators on Manifolds with Foliated Boundaries

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1 Manifolds with Fibred Boundaries

Let X be a compact manifold with boundary ∂X endowed with a fibration

$$\begin{array}{ccc} Z & \longrightarrow & \partial X \\ & & \downarrow{\scriptstyle \Phi} \\ & & Y. \end{array}$$

Let $x \in \mathcal{C}^\infty(X)$ be a boundary defining function and let g_Φ be a complete Riemannian metric on $X \setminus \partial X$ which in a collar neighborhood of ∂X is of the form

$$g_\Phi = \frac{dx^2}{x^4} + \frac{\Phi^* h}{x^2} + \kappa, \tag{1}$$

where κ is a symmetric 2-tensor restricting to give a metric on each fibre of Φ and h is a Riemannian metric on Y. To study geometric operators (Laplacian, Dirac operators) associated to such metrics, Mazzeo and Melrose introduced a calculus of

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pseudodifferential operators: the Φ-calculus. The starting point is the Lie algebra of Φ-vector fields:

$$\mathcal{V}_{\Phi}(X) = \{\xi \in \Gamma(TX) \mid \xi x \in x^2\mathcal{C}^{\infty}(X),\ \Phi_*(\xi|_{\partial X}) = 0\}.$$

In local coordinates, such a vector $\xi \in \mathcal{V}_{\Phi}(X)$ takes the form:

$$\xi = ax^2\frac{\partial}{\partial x} + \sum_i b^i x\frac{\partial}{\partial y^i} + \sum_j c^j\frac{\partial}{\partial z^j}, \quad a, b^i, c^j \in \mathcal{C}^{\infty}(X).$$

Since it is a Lie algebra, we can consider its universal enveloping algebra to define Φ-differential operators. Mazzeo and Melrose defined more generally Φ-pseudodifferential operators. They are useful to study mapping properties, for instance to determine when a Φ-differential operator is Fredholm.

2 Manifolds with Foliated Boundaries

Question 1. What can we do when the fibration Φ is replaced by a smooth foliation $\mathcal{F}$ on ∂X?

The notion of $\mathcal{F}$-vector fields is easy to define:

$$\mathcal{V}_{\mathcal{F}}(X) = \{\xi \in \Gamma(TX) \mid \xi x \in x^2\mathcal{C}^{\infty}(X),\ \xi|_{\partial X} \in \Gamma(T\mathcal{F})\}.$$

This is still a Lie algebra, so we can define $\mathcal{F}$-differential operators. However, since pseudodifferential operators are not local, we expect global aspects of the foliation $\mathcal{F}$ to come into play. One approach consists in using groupoid theory, namely, since $\mathcal{V}_{\mathcal{F}}(X)$ is in fact a Lie algebroid, we can integrate it to get a Lie groupoid $\mathcal{G}$. We can then use the general approach of Nistor-Weinstein-Xu to construct a pseudodifferential calculus. We will instead proceed differently by assuming the foliation can be 'resolved' by a fibration. This restricts the class of foliations that can be considered, but will allow us to develop further the underlying analysis.

We will assume the foliation arises as follows:

1. $\partial X = \partial\widetilde{X}/\Gamma$, where Γ is a discrete group acting freely and properly discontinuously on $\partial\widetilde{X}$, a possibly non-compact manifold;
2. There is a fibration $\Phi : \partial\widetilde{X} \to Y$ with Y a compact manifold;
3. The group Γ acts Y in a locally free manner (that is, if $\gamma \in \Gamma$ and $\mathcal{U} \subset Y$ an open set are such that $y \cdot \gamma = y$ for all $y \in \mathcal{U}$, then γ is the identity element) and so that $\Phi(p \cdot \gamma) = \Phi(p) \cdot \gamma$ for all $p \in \partial\widetilde{X}$ and $\gamma \in \Gamma$;
4. The images of the fibres of Φ under the quotient map $q : \partial\widetilde{X} \to \partial X$ give the leaves of the foliation $\mathcal{F}$.

Example 1. The Kronecker foliation on the 2-torus with lines of irrational slope θ arise in this way. One takes $\partial\widetilde{X} = \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ with the fibration Φ given by the projection on the right factor $Y = \mathbb{R}/\mathbb{Z}$, and the group Γ to be the integers with action given by

$$(x, [y]) \cdot k = (x + k, [y - k\theta]), \; [y] \cdot k = [y - k\theta], \; k \in \mathbb{Z}.$$

The identification with the standard definition of the Kronecker foliation is then given by the map

$$\begin{aligned} \Psi : (\mathbb{R} \times \mathbb{R}/\mathbb{Z})/\mathbb{Z} &\to \mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \\ [x, [y]] \quad &\mapsto \quad ([x], [y + \theta x]). \end{aligned}$$

Example 2. Seifert fibrations (circle foliations on a compact 3-manifold) typically arise in this way, except when the space of leaves is a bad orbifold.

For such foliations, we can define $\mathcal{F}$-operators as follows. We let $M = \partial X \times [0, \epsilon)_x \subset X$ be a collar neighborhood of ∂X and consider $\widetilde{M} = \partial\widetilde{X} \times [0, \epsilon)_x$ with Γ acting on $\widetilde{M}$ in obvious way so that $\widetilde{M}/\Gamma = M$. On $\widetilde{M}$, we consider the space of Γ-invariant Φ-operators $\Psi^*_{\Phi,\Gamma}(\widetilde{M})$ with support away from $x = \epsilon$. Given $\widetilde{P} \in \Psi^k_{\Phi,\Gamma}(\widetilde{M})$, we can make it act on $f \in \mathcal{C}^\infty(M)$ by requiring that $\widetilde{P}(q^* f) = q^* \widetilde{P}(f)$, where $q : \widetilde{M} \to M$ is the quotient map. This is meaningful because $\widetilde{P}$ acts on Γ invariant functions to give again Γ-invariant functions. We denote by $q_* \widetilde{P}$ the operator acting on $\mathcal{C}^\infty(M)$ obtained from $\widetilde{P}$ in this way.

Definition 1. An $\mathcal{F}$-pseudodifferential operator $P \in \Psi^m_{\mathcal{F}}(X)$ is an operator of the form

$$P = q_* P_1 + P_2, \quad P_1 \in \Psi^m_{\Phi,\Gamma}(\widetilde{M}), \; P_2 \in \dot{\Psi}^m(X).$$

From the Φ-calculus, we deduce relatively easily that $\mathcal{F}$-operators are closed under composition, that they map smooth functions to smooth functions and that they are bounded when acting on appropriate Sobolev spaces. One can also introduce a notion of principal symbol $\sigma_m(P)$ as well as a notion of normal operator $N_{\mathcal{F}}(P)$ defined by 'restricting' the operator P to the boundary. This leads to a simple criterion to describe Fredholm operators. An operator $P \in \Psi^m_{\mathcal{F}}(X)$ is Fredholm (when acting on suitable Sobolev spaces) if and only if its principal symbol $\sigma_m(P)$ and its normal operator $N_{\mathcal{F}}(P)$ are invertible.

3 An Index Theorem for Some Dirac-Type Operators

Assume now that the the foliation $\mathcal{F}$ is also such that $\partial\widetilde{X}$ is compact and the group Γ is finite. In particular, the leaves of $\mathcal{F}$ must be compact. Let $g_{\mathcal{F}}$ be a metric such that $q^*(g_{\mathcal{F}}|_M)$ takes the form (1) near $\partial\widetilde{X}$. Suppose X is even dimensional and that X, $\widetilde{X}$ and Y are spin manifolds. Let $D_{\mathcal{F}}$ be the induced Dirac operator. Suppose its normal operator $N_{\mathcal{F}}(D_{\mathcal{F}})$ is invertible, which is the case for instance when the induced metric on the leaves of the foliation $\mathcal{F}$ has positive scalar curvature. Under the decomposition $S = S^+ \oplus S^-$ of the spinor bundle, the Dirac operator can be written as

$$D_{\mathcal{F}} = \begin{pmatrix} 0 & D_{\mathcal{F}}^- \\ D_{\mathcal{F}}^+ & 0 \end{pmatrix}.$$

Since $N_{\mathcal{F}}(D_{\mathcal{F}})$ is invertible, the operator $D_{\mathcal{F}}^+$ is Fredholm.

Theorem 1. *The index of $D_{\mathcal{F}}^+$ is given by*

$$\operatorname{ind}(D_{\mathcal{F}}^+) = \int_X \widehat{A}(X, g_{\mathcal{F}}) - \frac{1}{|\Gamma|}\int_Y \widehat{A}(Y,h)\widehat{\eta}(\widetilde{D}_0) + \frac{\rho}{2},$$

where $\widetilde{D}_0$ is a family of Dirac operators on the fibres of $\Phi : \partial\widetilde{X} \to Y$ associated to $q^*(D_{\mathcal{F}}|_M)$ and $\rho = \frac{\eta(\widetilde{D}_\delta)}{|\Gamma|} - \eta(D_\delta)$ is a difference of two eta invariants with $\widetilde{D}_\delta$ the Dirac operator on $(\partial\widetilde{X}, \frac{\Phi^* h}{\delta^2} + \kappa)$ and D_δ the Dirac operator on $(\partial X, q_*(\frac{\Phi^* h}{\delta^2} + \kappa))$. Both $\widetilde{D}_\delta$ and D_δ are invertible for $\delta > 0$ small enough and ρ does not *depend on δ.*

The strategy to prove this theorem is to take an adiabatic limit.