On the Closure of Elliptic Wedge Operators

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We present a semi-Fredholm theorem for the minimal extension of an elliptic differential operator on a manifold with wedge singularities and give, under suitable assumptions, a full asymptotic expansion of the trace of the resolvent.

1 Wedge Operators

Let $\mathcal{M}$ be a smooth compact manifold with boundary. Assume that the boundary is the total space of a locally trivial fiber bundle $\wp : \partial\mathcal{M} \to \mathcal{Y}$ with typical fiber $\mathcal{Z}$, where $\mathcal{Y}$ and $\mathcal{Z}$ are smooth compact manifolds. Let $E, F \to \mathcal{M}$ be smooth vector bundles. We are interested in the space $x^{-m}\operatorname{Diff}_e^m(\mathcal{M}; E, F)$ of differential *wedge operators* of order m, where $\operatorname{Diff}_e^m(\mathcal{M}; E, F)$ denotes the space of differential edge operators, as introduced in [3], and $x : \mathcal{M} \to \mathbb{R}$ is any smooth defining function for $\partial\mathcal{M}$.

Locally, near a point $p \in \partial\mathcal{M}$, a wedge operator $A \in x^{-m}\operatorname{Diff}_e^m(\mathcal{M}; E, F)$ can be represented as

$$A = x^{-m} \sum_{k+|\alpha|+|\beta|\le m} a_{k,\alpha,\beta}(x, y, z)(xD_x)^k (xD_y)^\alpha D_z^\beta \tag{1}$$

with coefficients $a_{k,\alpha,\beta}$ smooth up to $x = 0$.

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For example, if g_w is a Riemannian metric on $\mathcal{M}$ that near $\partial\mathcal{M}$ takes the form $g_w = dx^2 + x^2 g_{\mathcal{Z}} + g_{\mathcal{Y}}$ (wedge metric), then the Laplacian associated with g_w is a wedge operator of order 2 and has the local representation

$$x^{-2}\big((xD_x)^2 - i(\dim\mathcal{Z} - 1)(xD_x) + \Delta_{\mathcal{Z}} + x^2\Delta_{\mathcal{Y}}\big).$$

Note that if $\mathcal{Y} = \{\text{pt}\}$, the space $x^{-m}\,\text{Diff}_e^m(\mathcal{M}; E, F)$ reduces to the class of cone operators, and if $\mathcal{Z} = \{\text{pt}\}$, then $x^{-m}\,\text{Diff}_e^m(\mathcal{M}; E, F)$ contains all regular differential operators on $\mathcal{M}$.

Let A be a wedge operator or order m, locally near the boundary represented as in (1). The following principal symbols are intrinsically associated with A.

The w-principal symbol. There is a natural structure bundle ${}^wT^*\mathcal{M} \to \mathcal{M}$ associated with wedge geometry. The principal symbol of a wedge operator A extends from the interior of $\mathcal{M}$ to ${}^wT^*\mathcal{M} \setminus 0$. Locally near $\partial\mathcal{M}$, the w-principal symbol of A can be represented as

$${}^w\boldsymbol{\sigma}(A) = \sum_{k+|\alpha|+|\beta|=m} a_{k,\alpha,\beta}(x,y,z)\xi^k\eta^\alpha\zeta^\beta.$$

The operator A is said to be w-elliptic if ${}^w\boldsymbol{\sigma}(A)$ is invertible.

The conormal symbol (indicial family). The indicial family restricts to the fibers of $\wp : \partial\mathcal{M} \to \mathcal{Y}$. We have

$$\hat{A}(y,\sigma) = \sum_{k+|\beta|\le m} a_{k,0,\beta}(0,y,z)\sigma^k D_z^\beta$$

for $y \in \mathcal{Y}$ and $\sigma \in \mathbb{C}$, and $\hat{A}(y,\sigma)$ acts on $C^\infty(\mathcal{Z})$. The set

$$\text{spec}_e(A) = \big\{(y,\sigma) \in \mathcal{Y} \times \mathbb{C} : \hat{A}(y,\sigma) \text{ is not invertible}\big\}$$

is called the boundary spectrum of A.

The principal edge symbol (normal family). The choice of defining function trivializes the inward pointing half $\mathcal{N}_+(\partial\mathcal{M})$ of the normal bundle of $\partial\mathcal{M}$ in $\mathcal{M}$. We get an induced fibration $\wp_\wedge : \mathcal{N}_+(\partial\mathcal{M}) \to \mathcal{Y}$ with typical fiber $\mathcal{Z}^\wedge = \overline{\mathbb{R}}_+ \times \mathcal{Z}$. Locally, the normal family takes the form

$$A_\wedge(y,\eta) = x^{-m} \sum_{k+|\alpha|+|\beta|\le m} a_{k,\alpha,\beta}(0,y,z)(xD_x)^k(x\eta)^\alpha D_z^\beta,$$

where $(y,\eta) \in T^*\mathcal{Y} \setminus 0$, and $A_\wedge(y,\eta)$ acts in the canonically induced conic L^2-space on the fiber $\mathcal{Z}^\wedge$.

Under suitable conditions on the normal family and the boundary spectrum of a w-elliptic wedge operator, we present the following results, cf. [2].

2 Main Results

For simplicity of the exposition, we assume that the operators are scalar.

Let $A \in x^{-m}\,\mathrm{Diff}_e^m(\mathcal{M})$ be w-elliptic, considered as an unbounded operator

$$A : C_c^\infty(\mathring{\mathcal{M}}) \subset x^{-\gamma} L_b^2(\mathcal{M}) \to x^{-\gamma} L_b^2(\mathcal{M})$$

for some fixed $\gamma \in \mathbb{R}$. Here $L_b^2(\mathcal{M})$ denotes the L^2 space defined using a fixed density of the form $x^{-1}\mathfrak{m}$ for some smooth positive density $\mathfrak{m}$ on $\mathcal{M}$.

We let $H_e^m(\mathcal{M})$ denote the corresponding Sobolev space defined using edge differential operators of order $\leq m$. Let $\mathcal{D}_{\min}$ be the closure of $C_c^\infty(\mathring{\mathcal{M}})$ with respect to the graph norm of A, and let $\mathcal{D}_{\wedge,\min}(y)$ be the closure of $C_c^\infty(\mathcal{Z}^\wedge)$ in $x^{-\gamma} L_b^2(\mathcal{Z}^\wedge)$ with respect to the one of $A_\wedge(y,\eta)$.

Our first result concerns the minimal domain and the semi-Fredholm property of the minimal extension of A.

Theorem 1. *Let A be as above. If $A_\wedge(y,\eta) : \mathcal{D}_{\wedge,\min}(y) \to x^{-\gamma} L_b^2$ is injective on $T^*\mathcal{Y}\backslash 0$, and if $\pi_{\mathbb{C}}\,\mathrm{spec}_e(A) \cap \{\Im\sigma = \gamma - m\} = \emptyset$, then $\mathcal{D}_{\min}(A) = x^{-\gamma+m} H_e^m(\mathcal{M})$ and $A : \mathcal{D}_{\min} \to x^{-\gamma} L_b^2(\mathcal{M})$ is a semi-Fredholm operator with finite-dimensional kernel and closed range.*

For our next result, let Λ be a closed sector properly contained in $\mathbb{C}$. Such a sector is called a *sector of minimal growth* for $A_{\mathcal{D}} : \mathcal{D} \subset x^{-\gamma} L_b^2 \to x^{-\gamma} L_b^2$, if $A_{\mathcal{D}} - \lambda$ is invertible for $|\lambda|$ large, and $\|\big(A_{\mathcal{D}} - \lambda\big)^{-1}\|_{\mathscr{L}(x^{-\gamma} L_b^2)} = O(|\lambda|^{-1})$ as $|\lambda| \to \infty$.

Theorem 2 (Resolvent expansion). *Let $m > 0$, let $A \in x^{-m}\,\mathrm{Diff}_e^m(\mathcal{M})$ be such that* $\mathrm{spec}({}^w\!\sigma(A)) \cap \Lambda = \emptyset$ *on ${}^wT^*\mathcal{M} \setminus 0$. If $\pi_{\mathbb{C}}\,\mathrm{spec}_e(A) \cap \{\Im\sigma = \gamma - m\} = \emptyset$, and if $A_\wedge(y,\eta) - \lambda : \mathcal{D}_{\wedge,\min}(y) \to x^{-\gamma} L_b^2$ is bijective on $(T^*\mathcal{Y} \times \Lambda) \setminus 0$, then Λ is a sector of minimal growth for $A_{\mathcal{D}_{\min}}$, and for every $\ell \in \mathbb{N}$ with $\ell > \frac{\dim \mathcal{M}}{m}$,*

$$\big(A_{\mathcal{D}_{\min}} - \lambda\big)^{-\ell} : x^{-\gamma} L_b^2(\mathcal{M}) \to x^{-\gamma} L_b^2(\mathcal{M})$$

is of trace class. For every $\varphi \in C^\infty(\mathcal{M})$, we have an expansion

$$\mathrm{Tr}\big(\varphi\big(A_{\mathcal{D}_{\min}} - \lambda\big)^{-\ell}\big) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} \alpha_{jk} \lambda^{\frac{\dim \mathcal{M} - j}{m} - \ell} \log^k(\lambda) \text{ as } |\lambda| \to \infty. \tag{2}$$

Here $m_j \leq 1$ for all j, and $m_j = 0$ for $j \leq \dim \mathcal{Z}$.

By standard methods, this asymptotic expansion leads to short time asymptotics of the heat trace when $A_{\mathcal{D}_{\min}}$ is sectorial, and to results concerning the meromorphic structure of the ζ-function when $A_{\mathcal{D}_{\min}}$ is positive.

The above theorems rely on the construction of suitable parametrices within the class of wedge pseudodifferential operators. Our approach makes substantial use of pseudodifferential methods developed by Schulze, see e.g. [4].

The asymptotic expansion in (2) is of course consistent with what is known in the special cases of boundary value problems ($\dim \mathcal{Z} = 0$) and of elliptic cone operators ($\dim \mathcal{Y} = 0$). For closed extensions other than the minimal extension, one should generally expect a more intricate asymptotic structure of the resolvent. In fact, in the case when $\dim \mathcal{Y} = 0$, the corresponding expansion of the resolvent sometimes involves rational functions in $\log \lambda$ and complex powers of λ, see [1].

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