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Duality and the Abel Map for Complex Supercurves

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Abstract. Supercurves are a generalization to supergeometry of Riemann surfaces or algebraic curves. They naturally appear in pairs related by a duality. The super Riemann surfaces appearing as worldsheets in perturbative superstring theory are precisely the self-dual supercurves. I will review known results and open problems in the geometry of supercurves, with a focus on Abel's Theorem.

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1. Introduction

A supercurve is a generalization to supergeometry of the classical notion of an algebraic curve or Riemann surface. In the smooth case, it is a complex supermanifold of dimension $1|1$. Supercurves naturally occur in pairs connected by a duality generalizing in some sense the Serre duality of line bundles on a Riemann surface. The self-dual supercurves are just the super Riemann surfaces studied extensively during the 1980s in connection with superconformal field theories and string theory. General supercurves have additional applications, for example to supersymmetric integrable systems [1].

In this article I review the definitions and basic examples of supercurves, explain how they generalize both Riemann surfaces and super Riemann surfaces, and describe some work in progress on the "super" analogues of classical results about Riemann surfaces. Section 2 gives the definition and two classes of examples: split supercurves, and super elliptic curves. Section 3 introduces divisors and the duality they lead to: supercurves naturally occur in pairs such that the points of one are the irreducible divisors of the other. Section 4 explains contour integration of differentials on supercurves, and the resulting theory of periods, Jacobians and the Abel map. Section 5 is a sketch of work in progress with Mitchell Rothstein, on Abel's Theorem and the Jacobi Inversion Theorem for supercurves. Section 6 mentions some open problems, such as a theory of theta functions for supercurves.

2. Definitions and examples

I will assume general familiarity with both supermanifolds ([2, 3]) and the classical theory of Riemann surfaces ([4, 5]). Fix a complex Grassmann algebra $\Lambda = \mathbb{C}[\beta_1, \beta_2, \ldots, \beta_n]$, to be thought of as the supercommutative "ring of constants" over which we are working. For us, a (smooth) supercurve X will be a family of $1|1$-dimensional complex supermanifolds over $\operatorname{Spec}\Lambda = (\mathrm{pt}, \Lambda)$. (More general families are possible, but this already displays the characteristic "super" phenomena and is consistent with the viewpoint of physicists.) That is, X is a Riemann surface X_{red} with a sheaf $\mathcal{O}$ of functions locally isomorphic to $\mathcal{O}_{\mathrm{red}} \otimes \Lambda[\theta]$, where θ is an additional odd generator. More explicitly, the holomorphic functions on an open set U, $\mathcal{O}(U)$, have the form $F(z, \theta) = f(z) + \theta\phi(z)$. Here we show explicitly the dependence on the coordinates z, θ while hiding that on the parameters β_i. This is in keeping with the viewpoint of physicists that z, θ are true (even and odd) variables while the β_i are merely "anticommuting constants".

The global structure of X is described by invertible parity-preserving transition functions on chart overlaps, having the form $\tilde{z} = F(z, \theta)$, $\tilde{\theta} = \Psi(z, \theta)$. Here the reduced part, or "body", of $F(z, \theta)$, namely $f_{\mathrm{red}}(z)$, is the transition function for X_{red} on the same overlap. There is *no* requirement that the transition functions be "superconformal" as there would be for a super Riemann surface.

We view the transition functions as giving the transformation law for Λ-valued points of X. A Λ-valued point in some chart U is a parity-preserving Λ-algebra homomorphism that evaluates functions on U to give elements of Λ. The "constants" β_i must of course evaluate to themselves. Since z and θ are themselves local functions, we give such a homomorphism by first specifying the elements of Λ to which they evaluate, say z_0 and θ_0. The reduced part of z_0 is the coordinate of the underlying reduced point of X_{red}. A general function $G(z, \theta)$ must then evaluate to $G(z_0, \theta_0)$, so a Λ-valued point may indeed be identified with a pair of Λ-valued coordinates (z_0, θ_0) in each chart. When charts overlap, their Λ-valued points are identified if they give the same evaluation of every function on the overlap. This defines a transformation rule of their coordinates (z_0, θ_0), coinciding with the transition functions. Physicists tend to think of supermanifolds in the familiar terms of their Λ-valued points.

The simplest examples of supercurves are the *split* supercurves. To construct one, choose a Riemann surface to serve as X_{red}. Fix some "soul" line bundle $\mathcal{S}$ on X_{red} and define X by transition functions

$$\tilde{z} = f(z), \quad \tilde{\theta} = \theta g(z),$$

where $f(z)$ are transition functions for X_{red} and $g(z)$ are transition functions for $\mathcal{S}$. In effect, X becomes the total space of the dual bundle, with θ as (odd) fiber coordinate. For example, if X_{red} is the complex plane $\mathbb{C}$ and $\mathcal{S}$ is the trivial line bundle, then X is the affine superspace $\mathbb{C}^{1|1}$.

A set of nonsplit examples is provided by super elliptic curves. Fix an even element $\tau \in \Lambda$ with $\operatorname{Im}\tau_{\mathrm{red}} > 0$, and two odd elements $\epsilon, \delta \in \Lambda$. X will be $\mathbb{C}^{1|1}/G$, where the group $G \cong \mathbb{Z} \times \mathbb{Z}$ has generators A, B acting on $\mathbb{C}^{1|1}$ by

$$A(z,\theta) = (z+1,\theta), \quad B(z,\theta) = (z+\tau+\theta\epsilon, \theta+\delta). \tag{1}$$

Then X_{red} is the torus with lattice generated by 1 and τ_{red}. Associated to a supercurve X there is always a split supercurve $X/(\beta_1, \beta_2, \ldots, \beta_n)$, obtained by "setting the β_i equal to zero", and in this case it is the torus with the trivial line bundle on it.

We use these examples to highlight some differences in the behavior of cohomology for ordinary curves and supercurves. For a split supercurve, it is easy to see that the global functions are $H^0(X, \mathcal{O}) = (\mathbb{C}|\Gamma(\mathcal{S})) \otimes \Lambda$. This notation indicates the even and odd subspaces of a super vector space over Λ. That is, the "even functions" of the form $f(z)$ are the even constants from Λ as expected, but there are also "odd" global holomorphic functions $\theta s(z)$ coming from the global sections $s(z)$ of $\mathcal{S}$, if any. Of course, one can take Λ-linear combinations of these, respecting parity, as well. The presence of nonconstant global functions is a counterintuitive but important feature of supergeometry.

For a super elliptic curve, it is not hard to see that global functions are either constants a or of the form $\theta\alpha$ with α constant, but not all of the latter are G-invariant, because of the action $\theta \mapsto \theta + \delta$ of the generator B. In this way one computes that

$$H^0(X, \mathcal{O}) = \{a + \theta\alpha : \alpha\delta = 0\}. \tag{2}$$

Because of the restriction on α, the cohomology is not freely generated as a Λ-module. This is typical for nonsplit supercurves and is a major complication in dealing with them. It means, for example, that there is no simple result like the Riemann-Roch theorem that characterizes cohomology modules by computing their ranks.

Fortunately Serre duality does work for supercurves: $H^1(X, \mathcal{O}) \cong H^0(X, \mathrm{Ber})^*$ as Λ-modules, as shown in [6]. Here the dual space consists of the Λ-linear functionals on $H^0(X, \mathrm{Ber})$. Earlier work had established Serre duality in the sense of $\mathbb{C}$-linear functionals on individual supermanifolds rather than families [7, 8]

Here the dualizing Berezinian or "canonical" sheaf Ber is the line bundle (see Section 4) on X with transition functions

$$\mathrm{ber} \begin{bmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{bmatrix} = \frac{\partial_z F - \partial_z \Psi (\partial_\theta \Psi)^{-1} \partial_\theta F}{\partial_\theta \Psi}. \tag{3}$$

Serre duality is parity-reversing: even elements of $H^1(X, \mathcal{O})$ correspond to odd linear functionals.

In the split case, $\mathrm{Ber} = K\mathcal{S}^{-1}|K$ (we omit the $\otimes\Lambda$ by abuse of notation). That is, the sections of Ber are generated by even sections $f(z)$ of $K\mathcal{S}^{-1}$, where K is the canonical bundle of differentials on X_{red}, and odd sections having the form $\theta s(z)$ with $s(z)$ itself a differential on X_{red}.

In general, $H^0(X, \mathcal{O})$, respectively $H^1(X, \mathcal{O})$, is always a submodule, respectively a quotient, of a free Λ-module. The free modules in question are isomorphic to the cohomologies of the associated split supercurve, and their ranks can be found from the Riemann-Roch theorem applied to $X_{\rm red}$ and $\mathcal{S}$.

The validity of Serre duality for supercurves can be traced to the fact that the Grassmann algebra Λ is a self-injective, or Gorenstein, ring [9, 10]. This means that linear functionals behave almost as nicely as they do on a vector space: any Λ-linear functional on an ideal $I \subset \Lambda$ is given by multiplication by an element of Λ, modulo those elements that annihilate the ideal.

3. Divisors and the dual curve

We use the standard basis for vector fields on a supercurve, $\partial = \partial_z, \quad D = \partial_\theta + \theta\partial_z$, and observe that $D^2 = \frac{1}{2}[D, D] = \partial$. A divisor on X is a subvariety of dimension $0|1$, locally given by an even equation $G(z, \theta) = 0$ with $G_{\rm red}$ not identically zero. For example, $z - z_0 - \theta\theta_0 = 0$ locally defines a divisor. In general, near a simple zero of $G_{\rm red}$, $G(z, \theta)$ contains a factor $z - z_0 - \theta\theta_0$ with the parameters z_0, θ_0 determined by the conditions

$$G(z_0, \theta_0) = DG(z_0, \theta_0) = 0. \tag{4}$$

This follows from the Taylor series expansion in the form

$$G(z, \theta) = \sum_{j=0}^{\infty} \frac{1}{j!}(z - z_0 - \theta\theta_0)^j [\partial^j G(z_0, \theta_0) + (\theta - \theta_0) D\partial^j G(z_0, \theta_0)]. \tag{5}$$

Although irreducible divisors depend on two parameters (z_0, θ_0) just like Λ-valued points, a crucial observation is that they are *not* points. To see this, we ask how the parameters of the same divisor are related in two overlapping charts. This is easily computed by using the transition functions to write

$$\tilde{z} - \tilde{z}_0 - \tilde{\theta}\tilde{\theta}_0 = F(z, \theta) - \tilde{z}_0 - \Psi(z, \theta)\tilde{\theta}_0, \tag{6}$$

and applying the conditions (4) to this function G to obtain

$$\tilde{z}_0 = F(z_0, \theta_0) + \frac{DF(z_0, \theta_0)}{D\Psi(z_0, \theta_0)}\Psi(z_0, \theta_0), \quad \tilde{\theta}_0 = \frac{DF(z_0, \theta_0)}{D\Psi(z_0, \theta_0)}. \tag{7}$$

Thus the parameters of a divisor have their own transformation rule distinct from that of points. It is automatic that these new transition functions satisfy a cocycle condition and thus they define a new supercurve denoted $\hat{X}$ and called the dual to X. It has the same reduced curve, and due to the symmetry of the function $z - z_0 - \theta\theta_0$ between (z, θ) and (z_0, θ_0), the dual of $\hat{X}$ is necessarily X again. Thus, supercurves naturally occur in pairs, with the points of each representing the irreducible divisors of the other [11]. Not only does either supercurve determine the other, but a chosen atlas on one determines an associated atlas with the same collection of charts on the other.

We easily determine the duals of our basic examples of supercurves. For split X, we find $\hat{X} = (X_{\rm red}, K\mathcal{S}^{-1})$. That is, this duality simply acts as Serre duality on the line bundle characterizing X. The dual of the super elliptic curve X with parameters τ, ϵ, δ is again a super elliptic curve, with parameters $\tau + \epsilon\delta, \delta, \epsilon$. Note in particular the interchange $\epsilon \leftrightarrow \delta$.

Riemann surfaces are special among algebraic varieties in that their irreducible divisors coincide with their points. We have seen that general supercurves do not share this property. The super-analog of a Riemann surface would thus be a self-dual supercurve. These are the "super Riemann surfaces" (also known as superconformal manifolds or SUSY curves) introduced in connection with string theory in the 1980s. From (7) we find that the transition functions of a super Riemann surface are "superconformal", meaning that $DF = \Psi D\Psi$. For split X this means $\mathcal{S}^2 = K$, so that the Serre self-dual line bundle $\mathcal{S}$ defines a spin structure on X_{red}. For super elliptic curves self-duality means $\epsilon = \delta$.

4. Differentials, integration, line bundles

The fundamental exact sequence underlying contour integration theory for supercurves is

$$0 \to \Lambda \to \mathcal{O} \xrightarrow{D} \hat{\text{Ber}} \to 0. \tag{8}$$

It is the analog of the sequence

$$0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{d} \Omega^1 \to 0 \tag{9}$$

on a Riemann surface. That is, given representatives $F(z, \theta)$ of a function in some local charts on X, one can check that the derivatives $DF(z, \theta)$ transform as local sections of the canonical bundle $\hat{\text{Ber}}$ of the dual curve $\hat{X}$ [following the cosmetic replacement of the arguments (z, θ) by $(\hat{z}, \hat{\theta})$]. Sections $\hat{\omega}$ of $\hat{\text{Ber}}$ should be viewed as "holomorphic differentials" on $\hat{X}$, and locally have antiderivatives with respect to D, which are functions on X determined up to a constant. An antiderivative of $f(\hat{z})+\hat{\theta}\phi(\hat{z})$ is $\theta f(z)+\int^z \phi$. Note that integration is parity-reversing, in addition to mapping between a curve and its dual. Once we have local antiderivatives, contour integrals of the form $\int_P^Q \hat{\omega}$ make sense, as follows. If the points P and Q of X lie in a single (contractible) chart, and F is an antiderivative of $\hat{\omega}$ in this chart, then the integral is defined to be $F(Q) - F(P)$. More generally, we define a super contour C as the pair of points P, Q together with a contour from P_{red} to Q_{red} on X_{red}, and we choose a sequence of points $P = P_1, P_2, \ldots, P_k = Q$ along this contour such that each consecutive pair lies in a common chart. Then the contour integral is defined to be

$$\int_C \hat{\omega} = \sum_{i=1}^{k-1} \int_{P_i}^{P_{i+1}} \hat{\omega}. \tag{10}$$

As for Riemann surfaces, this is independent of the choice of intermediate points.

Similarly, periods and residues of a meromorphic differential make sense: the former is the integral around a nontrivial homology cycle (for example, one of the basis A and B cycles) and the latter is the integral around a closed contour encircling a pole. Among the classical facts about Riemann surfaces which generalize to this context, I point out the Riemann bilinear period relation for holomorphic differentials, which here takes the form

$$\sum_{i=1}^{g}[A_i(\omega)B_i(\hat{\omega}) - B_i(\omega)A_i(\hat{\omega})] = 0. \tag{11}$$

Here g is the genus of the (reduced) curve, ω and $\hat{\omega}$ are arbitrary and independent holomorphic differentials on X and $\hat{X}$ respectively, and the notation $A_i(\omega)$ denotes the period of ω around the cycle A_i. On a Riemann surface, this relation is responsible for the symmetry of the period matrix.

As usual, a line bundle on X is defined by even, invertible transition functions $g_{ij}(z, \theta)$ in chart overlaps $U_i \cap U_j$, satisfying a cocycle condition, and line bundles are therefore classified by $H^1(X, \mathcal{O}_{\text{ev}}^{\times})$. The usual exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\text{ev}} \stackrel{\exp 2\pi i\cdot}{\longrightarrow} \mathcal{O}_{\text{ev}}^{\times} \to 0 \tag{12}$$

holds, and shows that degree-zero bundles are classified by the component of the Picard group $\text{Pic}^0(X) = H^1(X, \mathcal{O}_{\text{ev}})/H^1(X, \mathbb{Z})$. By means of Serre and Poincaré duality, this is isomorphic to the Jacobian

$$\text{Jac}(X) = H^0(X, \text{Ber})^*_{\text{odd}}/H_1(X, \mathbb{Z}).$$

This isomorphism is given explicitly by the *Abel map*: a degree-zero bundle on X can be described by the divisor $\sum_a n_a \hat{P}_a$ of a meromorphic section, and corresponds to the odd linear functional on holomorphic differentials (on X) given by

$$\sum_a n_a \int_{\hat{P}_0}^{\hat{P}_a}$$

modulo periods. Here $\sum_a n_a = 0$, and $\hat{P}_0$ is an arbitrary basepoint on $\hat{X}$. Abel's Theorem is due to [12] in the (free) super Riemann surface case, and to [6] in general.

5. Abel's theorem and Jacobi inversion

The classical Abel's Theorem characterizes those divisors of degree zero which are the divisor of some meromorphic function on a Riemann surface. The analog for supercurves was proved in [6] and states that a degree-zero divisor $\Delta = \sum_a n_a \hat{P}_a$ is the divisor of a meromorphic function F if and only if the associated linear functional $\sum_a n_a \int_{\hat{P}_0}^{\hat{P}_a}$ acting on $H^0(X, \text{Ber})$ vanishes modulo periods. That is, the value of this linear functional on any holomorphic differential is equal to the period of the differential around some fixed cycle which is the same for all differentials. Among many classical proofs of Abel's Theorem, that in [5] is based on criteria for the existence of meromorphic differentials with specified poles and residues on X. In order to better understand such criteria in the super case, M. Rothstein and I (work in progress) are adapting this proof to supercurves. A key ingredient is the Riemann reciprocity law generalizing the above bilinear relation:

$$\sum_{i=1}^{g}[A_i(\omega)B_i(\hat{\eta}) - B_i(\omega)A_i(\hat{\eta})] = 2\pi i \sum_a \operatorname{res}_{\hat{P}_a}(\hat{\eta}) \int_{\hat{P}_0}^{\hat{P}_a} \omega. \tag{13}$$

Here ω is a holomorphic differential on X, $\hat{\eta}$ is a meromorphic differential on $\hat{X}$, and the equation holds on the simply-connected interior of the $2g$-sided polygon obtained by cutting X open along the cycles A_i, B_i.

Here is a sketch of the proof of Abel's Theorem in the case of split X, which is technically simplest. The "easy" direction assumes that the divisor Δ is that of a meromorphic function F, in which case we set $2\pi i\hat{\eta} = D\log F$ and apply (13). The right side becomes the Abel map associated to the divisor, and the left side is an integer combination of periods of ω.

For the "hard" direction we have a divisor Δ whose associated linear functional is zero mod periods, and we must construct a meromorphic F with this divisor, which we do by first constructing the differential $2\pi i\hat{\eta}$ which would be $D\log F$. Recall that the sum of residues of a meromorphic differential at all poles vanishes. If $\hat{\eta}$ is such a differential on $\hat{X}$ then so is $\hat{G}\hat{\eta}$ for any holomorphic function $\hat{G}$. The new ingredient in the super case is that $h^0(\hat{\mathcal{S}})$ such nonconstant holomorphic functions do generally exist. Thus, the residues of $\hat{\eta}$ must satisfy $1|h^0(\hat{\mathcal{S}})$ vanishing conditions, which turn out to be sufficient as well as necessary for the existence of such a differential. These conditions can be shown to hold for the differential we seek, because the divisor has degree zero (1 condition) and because the Abel linear functional is assumed to vanish on the holomorphic differentials $D\hat{G}$ ($h^0(\hat{\mathcal{S}})$ conditions). Now that we have a differential with appropriate residues to be $(D\log F)/2\pi i$, its periods can be adjusted to be integers by adding a suitable combination of holomorphic differentials from $H^0(X,\mathrm{Ber})$; we then reconstruct F by integration and exponentiation. This is all as in the classical proof.

We have not completed the proof in the general case, but believe that it presents only technical obstacles. The major complication is that $H^0(X,\mathrm{Ber})$ is not freely generated; in particular it does not have a basis ω_j normalized as in the classical case to have A-periods $A_i(\omega_j) = \delta_{ij}$. One must show that nevertheless there are enough holomorphic differentials to adjust the periods of $\hat{\eta}$ as required in the last step of the proof.

More information about the Abel map is provided by the classical Jacobi Inversion Theorem, which is also the subject of work in progress. The naive super analog would say that every point in the Jacobian of X is the image under the Abel map of a "g-point divisor" having the form $\Delta = \sum_{a=1}^{g}(\hat{P}_a - \hat{P}_0)$. This is not quite true as stated; again we can only sketch the situation in the split case thus far.

Let the points $\hat{P}_a$ have coordinates $(\hat{z}_a, \hat{\theta}_a)$ in some chart. The divisor Δ corresponds to the linear functional that sends the odd holomorphic differentials $\theta\omega_j$ to $\sum_a \int_{\hat{z}_0}^{\hat{z}_a} \omega_j$, and the even differentials s_j to $\sum_a \hat{\theta} s_j(\hat{z})|_{\hat{P}_0}^{\hat{P}_a}$. Given the images of all these differentials, the Jacobi Inversion Problem is to determine the g points

$\hat{P}_a$. In the split case, their even and odd coordinates can be found separately. The classical Jacobi Inversion Theorem determines the $\hat{z}_a$ from the values of $\sum_a \int_{\hat{z}_0}^{\hat{z}_a} \omega_j$. Knowing these, the prescribed values of $\sum_a \hat{\theta} s_j(\hat{z})|_{\hat{P}_0}^{\hat{P}_a}$ give a system of $h^0(\hat{\mathcal{S}})$ linear equations in g unknowns for the $\hat{\theta}_a$. Thus, the divisor is determined uniquely if $h^0(\hat{\mathcal{S}}) = g$ and the coefficient matrix $s_j(\hat{z}_a)$ has maximal rank. The solution is nonunique, and the Abel map has a nontrivial fiber, if $h^0(\hat{\mathcal{S}}) < g$. Finally, if $h^0(\hat{\mathcal{S}}) > g$ one generally needs to allow for more than g points in the divisor Δ.

6. Open problems

Most of the classical theory of Riemann surfaces was extended to super Riemann surfaces during the 1980s, at least under the simplifying assumption that relevant cohomology groups were free modules. Much has now been further extended to general supercurves, and without restriction on the cohomology, but many interesting questions remain open. For lack of space I mention just two.

Can the duality between X and $\hat{X}$ be described explicitly in terms of classical algebraic geometry? That is, if X is given explicitly as the solution set of some polynomial equations in a projective superspace, can the equations of $\hat{X}$ be constructed?

Theta functions for supercurves need to be better understood. Such theta functions exist when the Jacobian is free, and are related to the super tau functions associated to supersymmetric integrable systems [6, 13]. They can also be constructed on super elliptic curves, for example

$$H(z,\theta) = \sum_{n\in\mathbb{Z}} \exp \pi i \left(2nz + n^2\tau + n\theta\epsilon + n^2\theta\epsilon + \frac{1}{3}n^3\delta\epsilon \right) \tag{14}$$

is such a theta function. By this I mean that it is invariant under the A transformation but acquires a phase linear in the coordinates under B:

$$H(z+\tau+\theta\epsilon, \theta+\delta) = H(z,\theta) \exp -\pi i \left(2z + \tau + 2\theta\epsilon + \frac{1}{3}\delta\epsilon \right). \tag{15}$$

One can define a theta subvariety of the Jacobian as the image by the Abel map of $(g-1)$-point divisors. Assuming free cohomology, it would be expected to have codimension $1|0$, making it a true theta divisor, if $h^1(X_{\mathrm{red}}, \mathcal{S}) = g-1$. Its properties are completely unexplored.

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