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Pencils of Conics as a Classification Code

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Abstract. We collect several subjects of the modern Mathematical Physics like integrable quad-graphs, discriminantly separable polynomials, the Petrov classification, the algebro-geometric approach to the Yang-Baxter equation and quadrirational maps since they all lead to the same geometric background. The geometry is related to pencils of conics, and the classification code follows the types of pencils.

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Keywords. Pencil of conics, Petrov classification, integrable quad-graphs, discrminantly separable polynomials, Yang-Baxter equation, quadrirational maps.

1. Pencils of conics

Given two conics in the plane, the set of all conics sharing the same intersection with the two, forms a pencil of conics. We will denote general pencils of conics having four simple common points of intersection as $(1, 1, 1, 1)$, or of type [A]. The case with two simple points of intersection and one double with a common tangent at that point is denoted $(1, 1, 2)$ or [B]. The case with two double points of intersection and with a common tangent in each of them is $(2, 2)$, or [C]. The case $(1, 3)$, denoted also as [D] is defined by one simple and one triple point of intersection. Finally (4), the case of one quadruple point is denoted as [E]. The following Figures 1–5 illustrate these possible configurations of pencils.

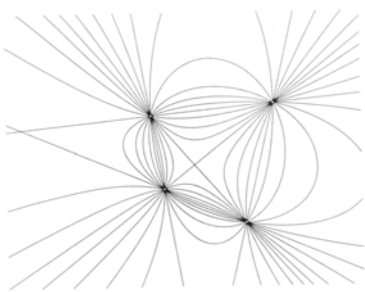


Figure 1. Pencil of type A

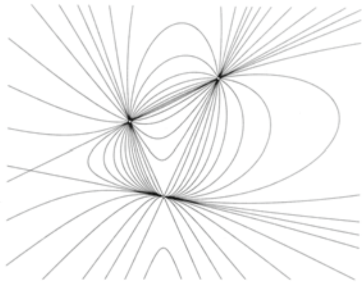


Figure 2. Pencil of type B

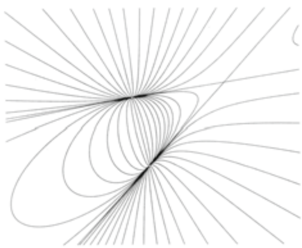


Figure 3. Type C

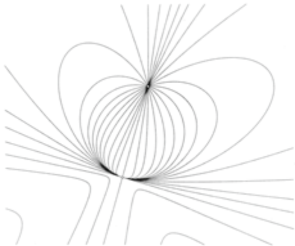


Figure 4. Type D

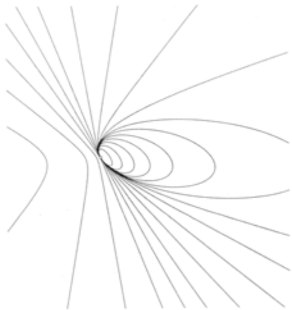
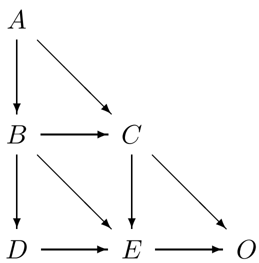


Figure 5. Type E

The transition from a more general pencil to a more special one is represented by the diagram, which is usually associated with Penrose:



(1)

We will need a classical notion of the Darboux coordinates in a projective plane. We fix a conic C in the plane, with a rational parametrization. For a given point P in the plane, there are two tangents from P to the conic C. Let the two values of the rational parameter of the two points of tangency of the tangent lines with the conic C be (x_1, x_2). Then, the pair (x_1, x_2) gives *the Darboux coordinates* of the point P associated with the parametrized conic C.

2. Petrov classification

We will start with historically the first of the stories. The Petrov 1954 classification describes the algebraic symmetries of the Weyl tensor at a point in a Lorentzian manifold (see [1], [2]). It is well known due to its applications to the theory of relativity, in the study of the exact solutions of the Einstein field equations.

The Weyl tensor, is a $(2,2)$-tensor, evaluated at some point, and it acts on the space of bivectors at that point as a linear operator:

$$W : Y^{\alpha\beta} \mapsto \frac{1}{2} W^{\alpha\beta}_{pq} Y^{pq}. \tag{2}$$

The equation

$$W^{\alpha\beta}_{pq} Y^{pq} = 2\lambda Y^{\alpha\beta}$$

defines the eigenvalues and the eigenbivectors. In the case of a space-time of dimension four, the space of antisymmetric bivectors at a point is of dimension six, and, due to the symmetries of the Weyl tensor, the eigenbivectors lie in a subspace of dimension four. Thus, the Weyl tensor at each point has at most four linearly independent eigenbivectors. The eigenbivectors of the Weyl tensor can occur with multiplicities, indicating a kind of algebraic symmetry of the tensor at the point. The multiplicities reflect the structure of zeros of a certain polynomial of degree four. The eigenbivectors are associated with null vectors in the original space-time, the principal null directions at point. According to the Petrov classification theorem, there are six possible types of algebraic symmetry, the six Petrov types:

[I] four simple principal null directions;
[II] two simple principal null directions and one double;
[D] two double principal null directions;
[III] one simple and one triple principal null direction;
[N] one quadruple principal null direction.
[O] the case where the Weyl tensor vanishes.

A relationship between the Petrov classification and the pencils of conics has been elaborated in [3]. It has been represented by a diagram of type (1) by Penrose, see [4], with the following correspondence

$$(A, B, C, D, E, 0) \to (\mathrm{I}, \mathrm{II}, \mathrm{D}, \mathrm{III}, \mathrm{N}, 0).$$

3. Integrable quad-graphs

Let us denote by $\mathcal{P}^n_d$ the set of polynomials in d variables of degree at most n in each.

Recall that the basic building blocks of systems on quad-graphs from works of Adler, Bobenko, Suris [5] are the equations on quadrilaterals of the form

$$Q(x_1, x_2, x_3, x_4) = 0 \tag{3}$$

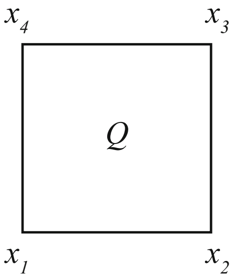


Figure 6. Quad-equation $Q(x_1, x_2, x_3, x_4) = 0$.

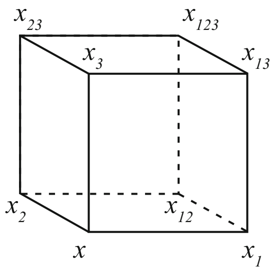


Figure 7. A 3D consistency.

where $Q \in \mathcal{P}_4^1$. Equations of type (3) are called *quad-equations*. The field variables x_i are assigned to four vertices of a quadrilateral as in Figure 6.

Following [5] we consider the idea of integrability as consistency, see Figure 7. We assign six quad-equations to the faces of coordinate cube. The system is said to be *3D-consistent* if three values for x_{123} obtained from equations on right, back and top faces coincide for arbitrary initial data x, x_1, x_2, x_3. Then, applying discriminant-like operators introduced in [5] $\delta_{x,y} : \mathcal{P}_4^1 \to \mathcal{P}_2^2, \quad \delta_x : \mathcal{P}_2^2 \to \mathcal{P}_1^4$ by formulae

$$h(z, w) := \delta_{x,y}(Q) = Q_x Q_y - Q Q_{xy}, \quad P(z) := \delta_w(h) = h_w^2 - 2h h_{ww}, \tag{4}$$

there is a descent from the faces to the edges and then to the vertices of the cube: from a polynomial $Q(x_1, x_2, x_3, x_4) \in \mathcal{P}_4^1$ to a biquadratic polynomial $h \in \mathcal{P}_2^2$ and further, to a polynomial $P \in \mathcal{P}_1^4$ of one variable of degree 4.

A biquadratic polynomial $h(x, y) \in \mathcal{P}_2^2$ is said to be *non degenerate* if no polynomial in its equivalence class with respect to fractional linear transformations is divisible by a factor of the form $x - c$ or $y - c$, with $c = \text{const}$. A multiaffine function $Q(x_1, x_2, x_3, x_4) \in \mathcal{P}_4^1$ is said to be of *type* Q if all four of its accompanying biquadratic polynomials h^{jk} are non degenerate. Otherwise, it is of *type* H. Previous notions were introduced in [5], where the classification list of mulitiaffine polynomials of type Q has been obtained, based on the structure of zeros of the associated nonzero polynomial P of degree four. There are five cases, [A], [B], [C], [D], [E]. For example, in the case $[B] = (1, 1, 2)$:

$$\begin{aligned} Q_B = (\alpha - \alpha^{-1})(x_1 x_2 + x_3 x_4) &+ (\beta - \beta^{-1})(x_1 x_4 + x_2 x_3) \\ &- (\alpha\beta - \alpha^{-1}\beta^{-1})(x_1 x_3 + x_2 x_4) \\ &+ \frac{\delta}{4}(\alpha - \alpha^{-1})(\beta - \beta^{-1})(\alpha\beta - \alpha^{-1}\beta^{-1}) \end{aligned}$$

for $\delta \neq 0$. In the case $[C] = (2, 2)$ Q_C is obtained from Q_B with $\delta = 0$.

4. Discriminantly separable polynomials

The notion of discriminantly separable polynomials has been introduced in [6]. A family of such polynomials has been constructed there as pencil equations from the theory of conics $\mathcal{F}(w, x_1, x_2) = 0$, where w, x_1, x_2 are the pencil parameter and the Darboux coordinates respectively. The key algebraic property of the pencil equation, as quadratic equation in each of three variables w, x_1, x_2 is: *all three of its discriminants are expressed as products of two polynomials in one variable each*:

$$\mathcal{D}_w(\mathcal{F}) = P(x_1)P(x_2),\ \mathcal{D}_{x_1}(\mathcal{F}) = J(w)P(x_2)\, \mathcal{D}_{x_2}(\mathcal{F}) = P(x_1)J(w), \qquad (5)$$

where J, P are polynomials of degree up to 4, and the elliptic curves $\Gamma_1 : y^2 = P(x)$, $\Gamma_2 : y^2 = J(s)$ are isomorphic (see Proposition 1 of [6]).

A classification of strongly discriminantly separable polynomials

$$\mathcal{F}(x_1, x_2, x_3) \in \mathcal{P}_3^2,$$

which are those satisfying the above relations 5 with $P = J$, has been performed modulo a gauge group of the following fractional-linear transformations $x_i \mapsto (ax_i + b)/(cx_i + d), \quad i = 1, 2, 3$ in [7], where more details can be found.

The classification of such polynomials, following [7], goes along the study of structure of zeros of a nonzero polynomial $P \in \mathcal{P}_1^4$. There are five cases: [A] with four simple zeros; [B] with a double zero and two simple zeros; [C] corresponds to polynomials with two double zeros; [D] is the case of one triple and one simple zero; finally, [E] is the case of one zero of degree four. The corresponding families of polynomials $\mathcal{F}_A$, $\mathcal{F}_B$, $\mathcal{F}_{C1}$, $\mathcal{F}_{C2}$, $\mathcal{F}_D$, $\mathcal{F}_{E1}$, $\mathcal{F}_{E2}$, $\mathcal{F}_{E3}$, $\mathcal{F}_{E4}$ are listed in Theorem 4 of [7]. Here, we are giving an example.

[B] $(1, 1, 2)$: two simple zeros and one double zero, for a canonical form of the polynomial $P(x) = x^2 - \epsilon^2$, the corresponding discriminantly separable polynomial is $\mathcal{F}_B = x_1x_2x_3 + (\epsilon/2)(x_1^2 + x_2^2 + x_3^2 - \epsilon^2)$.

The relationship between the discriminantly separable polynomials of degree two in each of three variables, and integrable quad-graphs of Adler, Bobenko and Suris has been established in [7]. The key point is the following formula, which defines an h, a biquadratic ingredient of quad-graph integrability, starting form a discriminantly separable polynomial $\mathcal{F}$: $\hat{h}(x_1, x_2, \alpha) = \mathcal{F}(x_1, x_2, \alpha)/\sqrt{P(\alpha)}$.

5. Quantum Yang-Baxter equation

The next subject is devoted to *the Yang–Baxter equation*

$$R^{12}(t_1 - t_2, h)R^{13}(t_1, h)R^{23}(t_2, h) = R^{23}(t_2, h)(R^{13}(t_1, h)R^{12}(t_1 - t_2, h). \qquad (6)$$

Here t is so-called *spectral parameter* and h is *Planck constant.* Here we assume that $R(t, h)$ is a linear operator from $V \otimes V$ to $V \otimes V$ and $R^{ij} : V \otimes V \otimes V \to V \otimes V \otimes V$ is an operator acting on the ith and jth components as $R(t, h)$ and as identity on

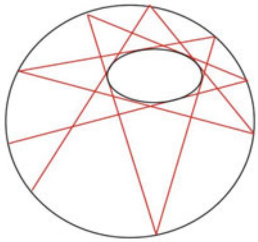


FIGURE 8. The Euler-Chasles correspondence

the third component. For example $R^{12}(t,h) = R \otimes Id$. In the first nontrivial case, matrix $R(t,h)$ is 4×4 and the space V is two-dimensional.

Krichever's approach is based on the *vacuum vector representation* of a 4×4 matrix L, understood as a 2×2 matrix with blocks of 2×2 matrices. In other words, $L = L^{i\alpha}_{j\beta}$ is a linear operator in the tensor product $C^2 \otimes C^2$. *The vacuum vectors* X, Y, U, V satisfy, by definition, the relation

$$LX \otimes U = hY \otimes V. \tag{7}$$

The vacuum vectors are parametrized by *the vacuum curve* Γ_L. In [8] Krichever proved that in the case of general position, the vacuum curve is elliptic, and rank one solutions are equivalent to the Baxter R-matrix. In [9], [10] the cases of rational vacuum curves have been studied.

The geometric background of the above algebro-geometric classification is connected with pencils of conics. It is based on the fact that the vacuum curve is a biquadratic, or the Euler-Chasles 2-2 correspondence (see [11]) of the form

$$E : ax^2y^2 + b(x^2y + xy^2) + c(x^2 + y^2) + 2dxy + e(x + y) + f = 0. \tag{8}$$

Using the Darboux coordinates, we visualize the Euler-Chasles correspondence (8) by Figure 8 and a relationship with pencils of conics becomes obvious. Thus, again, the classification follows the Penrose diagram (1) where the case [A] corresponds to the Baxter R-matrix, [B] to the Cherednik R-matrix, and [C] to the six-vertex R-matrix of Yang.

6. Quadrirational maps

The last section is devoted to quadrirational maps on $\mathbb{CP}^1$ which are introduced and classified in [12]. Following Adler, Bobenko and Suris, we consider a rational map $F : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ and its graph as an algebraic variety $\Gamma_F \subset (\mathbb{CP}^1)^4$. Such a map is called *quadrirational* if for any fixed pair $(X, Y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ (modulo some closed subvariety of co-dimension at least one) the graph Γ_F intersects each of the sets $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \{X\} \times \{Y\}$, $\{X\} \times \{Y\} \times \mathbb{CP}^1 \times \mathbb{CP}^1$,

$\mathbb{CP}^1 \times \{Y\} \times \{X\} \times \mathbb{CP}^1$ exactly once. In that case Γ_F defines four rational maps $F, F^{-1}, \bar{F}, \bar{F}^{-1} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$. It has been proven in [12] that for a quadrirational map, its graph is defined by polynomial equations $f(x, y, u) = 0$ and $h(y, x, v) = 0$, where the degrees of f in x and of h in y are one or two. We will consider further only the case when both of the degrees are equal to two, denoted in [12] as $[2 : 2]$. Then, the following classification takes place:

Theorem (Adler, Bobenko, Suris 2004). *Any quadrirational map of type* $[2 : 2]$ *is, up to Möbius gauge transformations on variables, equivalent to one and only one of the five maps:*

$$[A] \qquad F_A : u = ayP, \qquad v = bxP, \qquad P = \frac{(1-b)x + b - a + (a-1)y}{b(1-a)x + (a-b)yx + a(b-1)y};$$

$$[B] \qquad F_B : u = \frac{y}{a}P, \qquad v = \frac{x}{b}P, \qquad P = \frac{ax - by + b - a}{x - y};$$

$$[C] \qquad F_C : u = \frac{y}{a}P, \qquad v = \frac{x}{b}P, \qquad P = \frac{ax - by}{x - y};$$

$$[D] \qquad F_D : u = yP, \qquad v = xP, \qquad P = \frac{x - y + b - a}{x - y};$$

$$[E] \qquad F_E : u = y + P, \quad v = x + P, \quad P = \frac{b - a}{x - y};$$

where a, b *are given constants.*

The mappings F_A, F_B, F_C, F_D, F_E are related with pencils of conics of types A, B, C, D, E respectively, in the following way: given two conics C_1, C_2 of a pencil, with fixed rational parametrizations. For a pair of points $x \in C_1, y \in C_2, x \neq y$, the line they define intersects conics C_1 and C_2 in other two points u, v. Then, as it has been shown in [12], $F(x, y) = (u, v)$ is a quadrirational mapping, with the formula given above.

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References

[1] A.Z. Petrov, *The classification of spaces definig gravitational fields* Sci. Notices of Kazan State University, Vol. 114, 1954.

[2] A.Z. Petrov, *Einstein spaces*, Pergamon press, 1969.

[3] M. Cahen, R. Debever, L. Defrise, *A Complex Vectorial Formalism in General Relativity*, Journal of Mathematics and Mechanics, Vol. 16, no. 7 (1967).

[4] R. Penrose, W. Rindler, *Spinors and space-time*, Vol. 2, Cambridge University Press, 1986.

[5] E.V. Adler, A.I. Bobenko, Y.B. Suris, *Discrete nonlinear hiperbolic equations. Classification of integrable cases*, Funct. Anal. Appl **43** (2009) 3–21.

[6] V. Dragović, *Generalization and geometrization of the Kowalevski top*, Communications in Math. Phys. 298 (2010), no. 1, 37–64.

[7] V. Dragović, K. Kukić, *Integrable Kowalevski type systems, discriminantly separable polynomials and quad graphs* 2011 arXiv: 1106.5770.

[8] I. M. Krichever, *Baxter's equation and algebraic geometry*, Func. Anal. Appl. **15** (1981), 92–103 (in Russian).

[9] V. I. Dragovich, *Solutions to the Yang equation with rational spectral curves*, St. Petersb. Math. J. **4**:5 (1993), 921–931.

[10] V. I. Dragovich, *Solutions to the Yang equation with rational irreducible spectrual curves*, Russ. Acad. Sci., Izv., Math. **42**:1 (1994), 51–65.

[11] V. Dragović, M. Radnović, *Poncelet porisms and beyond*, Springer Basel AG, 2011.

[12] E.V. Adler, A.I. Bobenko, Y.B. Suris, *Geometry of Yang-Baxter Maps: pencils of conics and quadrirational mappings*, Comm. Analysis and Geometry, Vol. 15, No. 5, (2004) 967–1007.

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