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On Maximal $\mathbb{R}$-split Tori Invariant under an Involution

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Abstract. Symmetric k-varieties have been a topic of interest in several fields of mathematics and physics since the 1980's. For $k = \mathbb{R}$, symmetric $\mathbb{R}$-varieties are commonly called real symmetric spaces; however, the generalization over other fields play a role in the study of arithmetic subgroups, geometry, singularity theory, Harish Chandra modules and most importantly representation theory of Lie groups.

The preliminary study of the rationality properties of these spaces over various base fields was published by Helminck and Wang [1]. In order to study the representations associated with these symmetric k-varieties one needs a thorough understanding of the orbits of parabolic k-subgroups, P_k, acting on the symmetric k-varieties, G_k/H_k. This paper's contribution is the classification of the orbits of $P \setminus G/H$ which are determined by the H-conjugacy classes of σ-stable maximal quasi k-split tori.

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1. Introduction and notation

Symmetric k-varieties are the homogeneous spaces defined G_k/H_k where G_k and H_k are the k-points of a reductive group G and H, the fixed point group of some involution. They play a role in geometry, singularity theory, and the cohomology of arithmetic groups. However, they are probably best known for their role in representation theory. The first breakthrough was made when Harish-Chandra commenced his study of general semisimple Lie groups, which finally led to the Plancherel formula. The next step was to study the representation theory of the general semisimple symmetric spaces which has been considered by Brylinski, Delorme, Carmona, Matsuki, Oshima, Schlichtkrull, van der Ban and many others.

The orbits of parabolic k-subgroups acting on a symmetric k-variety are of fundamental importance in the study of induced representations. The characterization of these orbits involves conjugacy classes of σ-stable maximal k-split tori and for each of these σ-stable maximal k-split tori a quotient of Weyl groups.

There are descriptions of some of these orbit decompositions in [1], the focus is on the orbits of parabolic k-subgroups acting on a variety, $P_k \setminus G_k/H_k$. Such a decomposition can be characterized as the P_k-orbits action on G_k/H_k, the H_k-orbits on $P_k \setminus G_k$ or the orbits of $P_k \times H_k$ on G. While these orbits are characterized for any field k the actually classification requires first the classification of orbit decompositions of the related $P \setminus G/H$. There exists a map between the orbits of $P_k \setminus G_k/H_k$ onto orbits of $P \setminus G/H$. After classifying the orbits of the latter one determines the fibers of the representatives and find the classification of the former. This paper's will discuss the classification of the orbits of $P \setminus G/H$ which are determined by the H-conjugacy classes of σ-stable maximal quasi k-split tori; however, there are 171 cases to consider and the! classification is quite long. Please see [2] for the full classification.

Helminck and Wang described the double cosets as follows:

Theorem 1 ([1, Proposition 6.10]). *Let $\{A_i \mid i \in I\}$ be representatives of the H_k-conjugacy classes of σ-stable maximal k-split tori in G. Then*

$$P_k \setminus G_k/H_k \cong \bigcup_{i\in I} W_{G_k}(A_i)/W_{H_k}(A_i).$$

The goal will be to explicitly determine the set I for $k = \mathbb{R}$ in order to calculate the Weyl groups, $W_{G_k}(A_i)$ and $W_{H_k}(A_i)$.

1.1. Notation

Definition 1. A torus, T, is called σ-stable if $\sigma(T) = T$. Then $T = T_\sigma^+ T_\sigma^-$, where

$$T_\sigma^+ = (T \cap H)^0 \text{ and } T_\sigma^- = \{x \in T | \sigma(x) = x^{-1}\}^0$$

A torus, A, is called σ-split if $\sigma(a) = a^{-1}$ for all $a \in A$. A quasi k-split torus is a torus that is G-conjugate to a k-split torus. Last, a torus, S, is called σ-fixed if $\sigma(s) = s$ for all $s \in S$. Note, a (σ, k)-split torus is both σ-split and k-split. Let $\mathfrak{A}_k^{(\theta,\sigma)}$ be the set of all (θ, σ)-stable maximal k-split tori. $\mathfrak{A}^{(\theta,\sigma)}$ be the set of (θ, σ)-stable maximal quasi k-split tori. Also, $\mathfrak{A}_0^{(\theta,\sigma)}$ be the set of quasi k-split tori that are H-conjugate with a k-split torus.

Since we will be looking at the H_k^+ or H-conjugacy classes of these various sets, we will denote these classes by: $\mathfrak{A}_k^{(\sigma,\theta)}/H_k^+$, $\mathfrak{A}^{(\theta,\sigma)}/H$, and $\mathfrak{A}_0^{(\theta,\sigma)}/H$, respectively.

We will call $\Phi(A) = \Phi_\theta$ the root system of a torus θ-split torus A with associated Weyl group $W(A)$. In general, the Weyl group of a torus, T, will be $W(T, L_k) = W_{L_k}(T) = N_{L_k}(T)/Z_{L_k}(T)$, where

$$N_{L_k}(T) = \{x \in L_k \mid xTx^{-1} \subset T\},$$
$$Z_{L_k}(T) = \{x \in L_k \mid xt = tx \text{ for all } t \in T\}.$$

We will also be looking at $\Phi_{\theta,\sigma} = \Phi(A, A_\sigma^-) = \Phi(A) \cap \Phi(A_\sigma^-)$. For $w \in W(A)$, $\Phi(w) = \{\alpha \in \Phi(A) \mid w(\alpha) = -\alpha\}$.

The following sections will highlight important portions of the final classification. The goal is to determine the H_k-conjugacy classes of maximal $\mathbb{R}$-split tori for the orbit decomposition $P_\mathbb{R} \setminus G_\mathbb{R}/H_\mathbb{R}$. The following steps will be discussed.

1. A Cartan involution, θ, commuting with σ will convert the problem into a pair, (θ, σ), of commuting involutions over $\mathbb{C}$ while simplifying the $\mathbb{R}$-split requirement. One involution over $\mathbb{R}$ becomes a pair of commuting involutions over $\mathbb{C}$.
2. All tori can be put into standard position and each torus can be associated with a Weyl group element.
3. Classify the H-conjugacy classes of σ-stable maximal quasi $\mathbb{R}$-split tori on route to the $H_\mathbb{R}$-conjugacy classes of σ-stable maximal $\mathbb{R}$-split tori.
4. Employ the use of the associated pair $(\theta, \sigma\theta)$ and classify the $H_\mathbb{R}$-conjugacy classes of σ-stable maximal $\mathbb{R}$-split tori.

This paper will demonstrate 1. through 3. and end with a description of associated pairs and the role played to determine 4. My current research is to complete the $H_\mathbb{R}$-conjugacy classes of σ-stable maximal $\mathbb{R}$-split tori.

2. Cartan involutions

Definition 2. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the decomposition into the $+1$ and -1-eigenspaces of θ. Then $\theta \in \mathrm{Aut}(\mathfrak{g}_0)$ is called a Cartan involution if $\mathfrak{k}_0$ is a maximal compact subalgebra of $\mathfrak{g}_0$. A subalgebra be called compact if the Killing form restricted to $\mathfrak{k}_0$ is negative definite.

The Cartan involution plays an important role, when $k = \mathbb{R}$, in the classification of the representatives of the $H_\mathbb{R}$-conjugacy classes of σ-stable maximal $\mathbb{R}$-split tori. A Cartan involution, θ, commuting with σ will simplify the into a pair, (θ, σ), of commuting involutions over $\mathbb{C}$ while simplifying the $\mathbb{R}$-split requirement. This changes the problem from one involution to commuting involutions over $\mathbb{C}$.

In our discussion, we have a fixed involution σ and can find a Cartan involution that will commute with σ.

Theorem 2 ([3, Lemma 10.2]). *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, θ a Cartan involution, and σ any involution. Then there exists $\phi \in \mathrm{Int}(\mathfrak{g}_0)$ such that $\phi\theta\phi^{-1}$ commutes with σ.*

Theorem 3 ([4, Theorem 10.6]). *The inner isomorphism classes of semisimple locally symmetric pairs $(\mathfrak{g}_0, \mathfrak{h})$ correspond bijectively to the inner isomorphism classes of ordered pairs of commuting involutions (θ, σ) of $\mathfrak{g}$ or $\mathrm{Aut}(\mathfrak{g})^0$. The outer isomorphism classes correspond bijectively as well.*

For $k = \mathbb{R}$ one studies the structure of real reductive algebraic groups in the complex case with a pair of commuting involutions (where one is a Cartan involution) instead of one involution of a real reductive algebraic group.

Let θ be a Cartan involution of G over k and σ a k-involution with $\sigma\theta = \theta\sigma$. Consider the following propositions from [1]:

1. (Proposition 11.18) Given any σ-stable maximal k-split torus A of G, there is a $h \in H_k$ such that hAh^{-1} is θ-stable.
2. Any θ-stable k-split torus is θ-split.
3. (Lemma 11.5) Any maximal θ-split k torus of G is maximal (θ, k)-split.

Therefore, any σ-stable maximal $\mathbb{R}$-split torus of G can be viewed as a (σ, θ)-stable maximal $\mathbb{R}$-split torus (or θ-split torus) of G. An important corollary follows from Theorem 1 when using this relation.

Corollary 4 ([1, Corollary 12.11]). *Let K be the fixed point group of θ, H a k-open subgroup of the fixed point group of σ and $H^+ = H \cap K$. Then*

$$P_k \backslash G_k / H_k \cong \bigcup_{i \in I} W_{G_k}(A_i)/W_{H_k^+}(A_i)$$

where $\{A_i \mid i \in I\}$ are the representatives of the H_k^+-conjugacy classes of (σ, θ)-stable maximal k-split tori in G.

In fact, pairs of commuting involutions over complex groups were classified in [4]. The notation from that paper will used to represent involutions through this section and next. Each involution has a Cartan type and each type has a diagram representation. From these diagrams, which were created using an ordered basis, one determines the type of the maximal $\mathbb{R}$-split (θ-split) torus (Φ_θ with basis Δ_θ) and the σ-split torus in the maximal $\mathbb{R}$-split ($\Phi_{\sigma,\theta}$ with basis $\Delta_{\sigma,\theta}$) for each pair of commuting involutions. There are 171 irreducible pairs to consider. Knowing the type and dimension of the maximal $(\sigma, \mathbb{R})$-split torus is necessary for the classification.

3. Characterizing standard involutions

As seen in the previous section, we can find the type and dimension of the maximal $(\sigma, \mathbb{R})$-split torus in the set $\mathfrak{A}_k^{(\theta,\sigma)}$.

3.1. Standard position

Definition 3. For A_1, $A_2 \in \mathfrak{A}_k^{(\theta,\sigma)}$, the pair (A_1, A_2) is called standard if $A_1^- \subset A_2^-$ and $A_1^+ \supset A_2^+$. We say that A_1 is standard with respect to A_2.

Theorem 5 ([5, Theorem 3.6]). *Let (A_1, A_2) be a standard pair of (θ, σ)-stable $\mathbb{R}$-split (or quasi $\mathbb{R}$-split) tori of G. Then the following hold:*

1. *There exists $g \in Z(A_1^- A_2^+)$ such that $gA_1g^{-1} = A_2$.*
2. *If $n_1 = g^{-1}\sigma(g)$ and $n_2 = \sigma(g)g^{-1}$, then $n_1 \in N(A_1)$ and $n_2 \in N(A_2)$.*
3. *Let w_1 and w_2 be the images of n_1 and n_2 in $W(A_1)$ and $W(A_2)$ respectively. Then $w_1^2 = e, w_2^2 = e$, and $(A_1)_{w_1}^+ = (A_2)_{w_2}^+ = A_1^- A_2^+$ which characterizes w_1 and w_2.*

Corollary 6. *Fix an element $A \in \mathfrak{A}_k^{(\theta,\sigma)}$ such that A_σ^- is maximal. Let A_1 be put in standard position with A where A^- is a maximal $(\sigma,\mathbb{R})$-split torus of G Then the following hold:*

1. *There exists $g \in Z(A_1^- A^+)$ such that $gA_1g^{-1} = A$.*
2. *If $n = \sigma(g)g^{-1}$, then $n \in N(A)$.*
3. *Let w be the image of n in $W(A)$. Then $w^2 = e$, and $(A)_w^+ = A_1^- A^+$ which characterizes w.*

For any tori $A_1, A_2 \in \mathfrak{A}_k^{(\theta,\sigma)}$ put in standard position with A, there is an associated element in $W(A)$. Each has an element g which is associated with an $n \in N(A)$ whose image in $W(A)$ is w. These images w_1 and w_2 are called the A_1-standard and A_2-standard involutions, respectively.

Let w_1 and w_2 be the A_1-standard and A_2-standard involutions, respectively, in $W(A)$. Now, we can discuss the tori based on these elements of the finite Weyl group.

Proposition 7 ([1, Proposition 12.6]). *Assume that $A_1, A_2 \in \mathfrak{A}_k^{(\theta,\sigma)}$ are both standard with respect to A. Let w_1 and w_2 be the A_1-standard and A_2-standard involutions, respectively, in $W(A)$. Then A_1 and A_2 are H_k^+-conjugate if and only if w_1 and w_2 are conjugate under $W(A, H_k^+)$*

Corollary 8. *Assume that A_1', $A_2' \in \mathfrak{A}^{(\theta,\sigma)}$ are both standard with respect to A. Let w_1' and w_2' be the A_1'-standard and A_2'-standard involutions, respectively, in $W(A)$. Then A_1' and A_2' are H-conjugate if and only if w_1 and w_2 are conjugate under $W(A, H)$.*

3.2. Singular involutions

What remains is to determine which involutions in $W(A)$ are A_i-standard involutions for some $A_i \in \mathfrak{A}^{(\theta,\sigma)}$ or $\mathfrak{A}_k^{(\theta,\sigma)}$.

Definition 4. Let $A \in \mathfrak{A}^{(\sigma,\theta)}$, $w \in W(A)$ and $G_w = Z(A_w^+)$. w is called σ-singular when following properties hold.

1. $w^2 = e$.
2. $\sigma w = w\sigma$.
3. $\sigma|[G_w, G_w]$ is k-split.

w is called (θ,σ)-singular if w if σ-singular and $\sigma\theta|[G_w, G_w]$ is k-split. A root $\alpha \in \Phi(A)$ is called σ-singular ((θ,σ)-singular) if the corresponding reflection $s_\alpha \in W(A)$ is σ-singular ((θ,σ)-singular).

Proposition 9. *An involution $w \in W(A)$ is a σ-singular ((σ,θ)-singular) involution iff w is an A_i-standard involution for some $A_i \in \mathfrak{A}^{(\theta,\sigma)}$ ($\mathfrak{A}_k^{(\sigma,\theta)}$).*

Proposition 10. *Let $A \in \mathfrak{A}^{(\theta,\sigma)}$ ($\mathfrak{A}_k^{(\theta,\sigma)}$) with A_σ^- maximal. Then there is a one-to-one correspondence between the $W(A,H)$-($W(A, H_k^+)$)-conjugacy classes of A_i-standard involutions in $W(A)$ and the $W(A,H)$-($W(A, H_k^+)$)-conjugacy classes of σ-singular ((θ,σ)-singular) involutions in $W(A)$.*

Now the goal is to classify the singular involutions in $W(A)$. A complete discussion of conjugacy classes of elements in the Weyl group can be found in [5]. In summary, let $\Phi(A)$ be irreducible and $w \in W(A)$ an involution, then $\Phi(w)$ is of type $r \cdot A_1 + X_\ell$, where either $X_\ell = \emptyset$ or one of $B_\ell(\ell \geq 1)$, $C_\ell(\ell \geq 1)$, $D_\ell(\ell \geq 1)$, E_7, E_8, F_4, or G_2, where $r \cdot A_1 = A_1 + A_1 + \cdots + A_1$ r times.

Let $\mathfrak{W}$ be the set of all W-conjugacy classes of involutions in W. If we define an order $>$ on $\mathfrak{W}$ then for $[w_1], [w_2] \in \mathfrak{W}$ we have $[w_1] > [w_2]$ if and only if $\Delta(w_1) \subset \Delta(w_2)$ for some representatives w_i of $[w_i](i = 1, 2)$.

One builds diagrams of these conjugacy classes as seen in [6]. Once the A_i-standard involutions in $W(A)$ are identified the diagram describes the conjugacy classes and types of tori. If $w_1, w_2 \in W(A)$ are A_1 and A_2-standard involutions of A_1 and A_2 then

$$A_1^- \subset A_2^- \iff A_{w_1}^- \supset A_{w_2}^-.$$

Hence,

$$[A_1] < [A_2] \iff [w_1] < [w_2].$$

Example. Suppose Φ_θ is of type B_3, let $\Delta_\theta = \{\alpha_1, \alpha_2, \alpha_3\}$ be a basis for Φ_θ. Then $\Phi(w)$ is some subset of Φ_θ. The following list describes possible types of the basis for $\Phi(w)$, $\Delta(w)$. We use the notation B_1 to designate the unique shortest root of type A_1.

- Type $\Delta(w) =$ empty.
- Type $\Delta(w) = A_1$.
- Type $\Delta(w) = B_1$.
- Type $\Delta(w) = 2 \cdot A_1$.
- Type $\Delta(w) = B_2$.
- Type $\Delta(w) = B_3$.

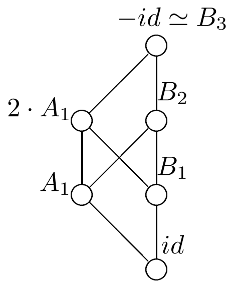


Figure 1. Conjugacy classes of involutions in Weyl group of Φ type B_3

4. H-conjugacy classes of $\mathfrak{A}^{(\theta,\sigma)}$

Proposition 11. *$\alpha \in \Phi(A)$ is a σ-singular root if and only if $\alpha \in \Phi(A) \cap \Phi(A_\sigma^-)$.*

Lemma 12 ([5, Theorem 4.6]). *Let A be a (θ,σ)-stable $\mathbb{R}$-split torus of G with A_σ^- a maximal $(\sigma,\mathbb{R})$-split torus of G and $w \in W(A)$, $w^2 = e$. Then the following are equivalent:*

1. *w is σ-singular.*
2. *$A_w^- \subset A_\sigma^-$.*

Proof. ($\Longrightarrow$) α is a σ-singular root then by Lemma 12, $A_{s_\alpha} \subset A_\sigma^-$. Therefore, $\alpha \in \Phi(A_\sigma^-)$. Since $\alpha \in \Phi(A)$ then $\alpha \in \Phi(A) \cap \Phi(A_\sigma^-)$.

($\Longleftarrow$) $\alpha \in \Phi(A) \cap \Phi(A_\sigma^-)$, then $\alpha \in \Phi(A)$ and $w = s_\alpha \in W(A)$ so $w^2 = e$. Since $\alpha \in \Phi(A_\sigma^-)$, $A_{s_\alpha}^- \subset A_\sigma^-$. By Lemma 12, s_α is σ-singular and α is a σ-singular root. □

Theorem 13. *Let $A \in \mathfrak{A}^{(\theta,\sigma)}$ with A_σ^- maximal. Then there is a one-to-one correspondence between the $W(A)$-conjugacy classes of σ-singular involutions in $W(A)$ and the $W(A)$-conjugacy classes of elements in $W(A, A_\sigma^-)$ where $W(A, A_\sigma^-)$ is the Weyl group of $\Phi(A, A_\sigma^-) = \Phi(A) \cap \Phi(A_\sigma^-)$.*

Example.

Ex.	Type (θ,σ)	Type Φ_θ $\Phi(A)$	Type $\Phi_{\sigma,\theta} \cap \Phi_\theta$ $\Phi(A, A_\sigma^-)$	max. involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$
(1)	$A_{2\ell+1}^{2\ell+1,\ell}(\mathrm{I},\mathrm{II})$	$A_{2\ell+1}$	$\emptyset$	id
(2)	$A_{4\ell-1}^{2\ell,2\ell-1,\epsilon_0}(\mathrm{III_b},\mathrm{II})$	$C_{2\ell}$	$\ell \cdot A_1$	$\ell \cdot A_1$
$\ell = 2$	$A_7^{4,3}(\mathrm{III_b},\mathrm{II},\epsilon_0)$	C_4	$2 \cdot A_1$	$2 \cdot A_1$
(3)	$B_\ell^{q,p}(\mathrm{I_a},\mathrm{I_a},\epsilon_\mathrm{i})$	B_q	B_p	B_p
$\ell = 5$	$B_5^{4,3}(\mathrm{I_a},\mathrm{I_a},\epsilon_\mathrm{i})$	B_4	B_3	B_3

TABLE 1

In Ex. (1), $\Phi(A, A_\sigma^-) = \emptyset$ and $W(A, A_\sigma^-) = \mathrm{id}$. There is only one $W(A)$-conjugacy class of σ-singular roots; therefore, there is only one H-conjugacy class (σ,θ)-stable maximal quasi $\mathbb{R}$-split tori. In Ex. (2), seen in Figure 2, there is only

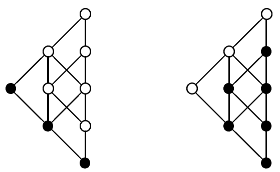


FIGURE 2. $\mathfrak{A}^{(\theta,\sigma)}/H$ for Ex. (2) & Ex. (3)

one $W(A)$-conjugacy class of σ-singular roots at each dimension; therefore, there is only one H-conjugacy class (σ,θ)-stable maximal quasi $\mathbb{R}$-split tori for each dimension. Last, in Ex. (3), as seen in Figure 2, one sees the $W(A)$-conjugacy class of σ-singular roots at each dimension from the diagram. Also, ones count the H-conjugacy classes (σ,θ)-stable maximal quasi $\mathbb{R}$-split tori for each dimension.

The complete classification of $\mathfrak{A}^{(\theta,\sigma)}/H$ in all 171 cases is quite long and can be found in [2]. This classification will help to determine the $H_{\mathbb{R}}$-conjugacy classes of (θ,σ)-stable maximal $\mathbb{R}$-split tori.

5. H_k-conjugacy classes of (θ,σ)-stable maximal k-split tori

The classification of the $H_{\mathbb{R}}$-conjugacy classes of σ-stable maximal tori can be simplified into a classification of objects in the Weyl group. However, determining the (θ,σ)-singular involutions and the appropriate conjugacy classes requires deeper investigation.

5.1. Associated Pairs

$$\begin{array}{ccccc}
\begin{matrix}(\mathfrak{g},\mathfrak{h})\\(\theta,\sigma)\end{matrix} & \leftarrow \text{associated} \rightarrow & \begin{matrix}(\mathfrak{g},\mathfrak{h}^a)\\(\theta,\sigma\theta)\end{matrix} & \leftarrow \text{dual} \rightarrow & \begin{matrix}(\mathfrak{g}^{ad},\mathfrak{h}^d)\\(\sigma\theta,\theta)\end{matrix} \\
\uparrow & & & & \uparrow \\
\text{dual} & & & & \text{associated} \\
\downarrow & & & & \downarrow \\
\begin{matrix}(\mathfrak{g}^d,\mathfrak{h}^d)\\(\sigma,\theta)\end{matrix} & \leftarrow \text{associated} \rightarrow & \begin{matrix}(\mathfrak{g}^d,\mathfrak{h}^a)\\(\sigma,\sigma\theta)\end{matrix} & \leftarrow \text{dual} \rightarrow & \begin{matrix}(\mathfrak{g}^{ad},\mathfrak{h})\\(\sigma\theta,\sigma)\end{matrix}
\end{array}$$

Previously, we used the action of σ on Φ_θ to determine the σ-split portion inside the θ-split torus. Similarly, we look at the action of $\sigma\theta$ on Φ_θ to find the $\sigma\theta$-split portion inside the θ-split torus. Let the maximal $\mathbb{R}$-split torus for (θ,σ) be A (as usual) and the maximal $\mathbb{R}$-split torus for $(\theta,\sigma\theta)$, S. So $S^-_{\sigma\theta}$ is maximal $\sigma\theta$-split and θ-split which is equivalent to S^+_σ which is a maximal in the fixed point group.

Definition 5. Let A and S be as above. The singular rank is the difference in rank of the (σ,θ)-stable maximal $(\sigma,\mathbb{R})$-split torus and the (σ,θ)-stable maximal $\mathbb{R}$-split, σ-fixed torus. The singular rank is calculated as follows:

$$\text{singular rank} = \dim(A^-_\sigma) + \dim(S^-_{\sigma\theta}) - \dim(A).$$

The singular rank helps to determine the maximal singular involution. From there we determine the appropriate structure of the remaining classes between the maximal σ-split and the maximal σ-fixed ($\theta\sigma$-split). It has been shown that under certain conditions of H_k (namely H_k pseudo-connected), one uses representatives of the same conjugacy classes in the H_k or $G_{\sigma\theta}$ (the fixed point group of $\sigma\theta$).

Proposition 14 ([6, Proposition 9.24]). *Let w_1 and w_2 be (σ,θ)-singular involutions and let H_k be pseudo-connected. Then the following are equivalent.*

1. *w_1 and w_2 are conjugate under $W(A, H_k^+)$.*
2. *w_1 and w_2 are conjugate under $W(A_\sigma^-, H_k^+)$.*
3. *w_1 and w_2 are conjugate under $W(A, G_{\sigma\theta})$.*
4. *w_1 and w_2 are conjugate under $W(A_\sigma^-, G_{\theta\sigma})$.*

In some cases, the number of H_k-conjugacy classes is determined quickly because the singular rank is maximal or 0. The final caveat is that while the structure from the $A^{(\theta,\sigma)}/H$ conjugacy classes is useful here when one considers the $H_\mathbb{R}$-conjugacy classes in $W(A)$, then involutions that were previously conjugate split as demonstrated in the following example.

Example. Consider the case for $\ell = 7, p = 2, q = 4, i = 1$ from he general case in Table 2.

- $\Phi_\theta = \Phi(A) = BC_2$ and $\Phi_{\theta,\sigma} = \Phi(A, A_\sigma^-) = BC_2$.
- The rank of the maximal σ-split torus is 2 and the rank of the maximal σ-fixed torus is also 2, but the rank of the maximal ℝ-split (i.e., θ-split) torus is also 2. Then the "top" the maximal ℝ-split torus is a σ-split torus and the "bottom" the torus is a σ-fixed torus.
- Consider the tori that are standard to A where $\dim((A_i)_\sigma^-)$= 2,1, and 0.

Through direction computation on the six roots in $\Phi(A, A_\sigma^-)^+$ ($e_1 \pm e_2$, e_1, $2e_1$, e_2, $2e_2$) the two (θ,σ)-singular roots can be determined. These roots are the unique short roots, usually denoted e_1 and e_2.

In $W(A)$, roots of type A_1 are conjugate. So the conjugacy classes of (σ,θ)-singular roots in $W(A)$ are the blackened dots in the diagram in Figure 3. This classifies the quasi ℝ-split tori that are H-conjugate to a maximal ℝ-split torus.

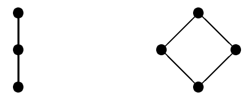
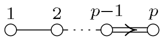
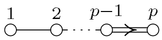


FIGURE 3. $A_0^{(\theta,\sigma)}/H$ & $A_\mathbb{R}^{(\theta,\sigma)}/H_\mathbb{R}$

There is one conjugacy class at each level. So all tori $A_i \in A_0^{(\sigma,\theta)}$ such that $\dim((A_i)_\sigma^-) = 2$ are conjugate. Similarly those with dimension 1 and 0. However, if we consider the conjugacy classes of these singular involutions in $W(A, H_\mathbb{R}^+) = BC_1 + BC_1$, then e_1 and e_2 are both type A_1 but no longer conjugate. So the one-dimensional level will split and there will two $H_\mathbb{R}^+$-conjugacy classes of tori where the rank of the σ-split portion is 1. It should be noted that these calculations are done in the associated Lie algebra and lifted to the group.

My current research is to complete the classification of the $H_\mathbb{R}^+$-conjugacy classes thus completing the classification of orbits of parabolic ℝ-subgroups on the symmetric space $G_\mathbb{R}/H_\mathbb{R}$, $P_\mathbb{R} \setminus G_\mathbb{R}/H_\mathbb{R}$.

Type (θ, σ)	Type $(\theta, \sigma\theta)$
$A_\ell^{p,q}(\mathrm{III_a}, \mathrm{III_a}, \epsilon_{\mathrm{i}})$ $p < q - p + 2i \leq \frac{1}{2}(\ell+1)$ $1 \leq p < q \leq \frac{1}{2}(\ell+1)$ $(0 \leq i \leq p)$	$A_\ell^{p,q-p+2i}(\mathrm{III_a}, \mathrm{III_a}, \epsilon_{\mathrm{p-i}})$

rank $\Phi_{\sigma,\theta}$	$\sigma\|\Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta\|\Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p

max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W-conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank
BC_p	BC_p	$B(p)$	BC_p	p

$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$	$\Phi(\mathfrak{t}_1) + \Phi(\mathfrak{t}_2)$
$su(\ell+1-q-i, p-i) + su(q-p+i, i) + so(2)$	$BC_i + BC_{p-i}$

TABLE 2

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