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Uniqueness Property for C^*-algebras Given by Relations with Circular Symmetry

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Abstract. A general method of investigation of the uniqueness property for C^*-algebra equipped with a circle gauge action is discussed. It unifies isomorphism theorems for various crossed products and Cuntz-Krieger uniqueness theorem for Cuntz-Krieger algebras.

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1. Introduction

The origins of C^*-theory and particularly the theory of universal C^*-algebras generated by operators that satisfy prescribed relations go back to the work of W. Heisenberg, M. Bohr and P. Jordan on matrix formulation of quantum mechanics, and among the most stimulating examples are algebras generated by anti-commutation relations and canonical commutation relations (in the Weyl form). The great advantage of relations of CAR and CCR type is *uniqueness of representation.* Namely, due to the celebrated Slawny's theorem, see, e.g., [1], the C^*-algebras generated by such relations are defined uniquely up to isomorphisms preserving the generators and relations. This *uniqueness property* is not only a strong mathematical tool but also has a significant physical meaning – if we had no such uniqueness, *different representations would yield different physics.*

The aim of the present note is to advertise a program of developing a general approach to investigation of uniqueness property and related problems based on exploring the symmetries of relations. We focus here, as a first attempt, on circular symmetries and propose a two-step method of investigation universal C^*-algebra $C^*(\mathcal{G},\mathcal{R})$ generated by a set of generators $\mathcal{G}$ subject to relations $\mathcal{R}$ which could be schematically presented as follows:

$$\underset{\text{relations, circle action}}{(\mathcal{G},\mathcal{R},\{\gamma_\lambda\}_{\lambda\in\mathbb{T}})} \xrightarrow{\text{step 1}} \underset{\substack{\text{Hilbert bimodule}\\ \text{(reversible dynamics)}}}{(\mathcal{B}_0,\mathcal{B}_1)} \xrightarrow{\text{step 2}} \underset{\text{universal } C^*\text{-algebra}}{C^*(\mathcal{G},\mathcal{R}) = \mathcal{B}_0 \rtimes_{\mathcal{B}_1} \mathbb{Z}}$$

– we fix a circle gauge action $\gamma = \{\gamma\}_{\lambda\in\mathbb{T}}$ on $C^*(\mathcal{G},\mathcal{R})$ which is induced by a circular symmetry in $(\mathcal{G},\mathcal{R})$; in the first step we associate to γ a non-commutative reversible dynamical system which is realized via a Hilbert bimodule $(\mathcal{B}_0,\mathcal{B}_1)$, and in the second step we use this system to determine the uniqueness property for $C^*(\mathcal{G},\mathcal{R})$.

2. Uniqueness property, universal C^*-algebras and gauge actions

Suppose we are given an abstract set of generators $\mathcal{G}$ and a set of *-algebraic relations $\mathcal{R}$ that we want to impose on $\mathcal{G}$. Formally $\mathcal{G}$ is a set and $\mathcal{R}$ is a set consisting of certain *-algebraic relations in a free non-unital *-algebra $\mathbb{F}$ generated by $\mathcal{G}$. By a *representation* of the pair $(\mathcal{G},\mathcal{R})$ we mean a set of bounded operators $\pi = \{\pi(g)\}_{g\in\mathcal{G}} \subset L(H)$ on a Hilbert space H satisfying the relations $\mathcal{R}$, and denote by $C^*(\pi)$ the C^*-algebra generated by $\pi(g)$, $g \in \mathcal{G}$. At this very beginning one faces the following two fundamental problems:

1. (*non-degeneracy problem*) Do there exists a *faithful representation* of $(\mathcal{G},\mathcal{R})$, i.e., a representation $\{\pi(g)\}_{g\in\mathcal{G}}$ of $(\mathcal{G},\mathcal{R})$ such that $\pi(g) \neq 0$ for all $g \in \mathcal{G}$?
2. (*uniqueness problem*) If one has two different faithful representation of $(\mathcal{G},\mathcal{R})$, do they generate isomorphic C^*-algebras? More precisely, does for any two faithful representations π_1, π_2 of $(\mathcal{G},\mathcal{R})$ the mapping

$$\pi_1(g) \longmapsto \pi_2(g), \qquad g \in \mathcal{G},$$

   extends to the (necessarily unique) isomorphism $C^*(\pi_1) \cong C^*(\pi_2)$?

The first problem is important and interesting in its own rights, see [2], [3], however here we would like to focus on the second problem and thus throughout we assume that all the pairs $(\mathcal{G},\mathcal{R})$ under consideration are non-degenerate. We say that $(\mathcal{G},\mathcal{R})$ possess *uniqueness property* if the answer to question 2 is positive.

Any representation π of $(\mathcal{G},\mathcal{R})$ extends uniquely to a *-homomorphism, also denoted by π, from $\mathbb{F}$ into $L(H)$. The pair $(\mathcal{G},\mathcal{R})$ is said to be *admissible* if the function $|||\cdot||| : \mathbb{F} \to [0,\infty]$ given by

$$|||w||| = \sup\{\|\pi(w)\| : \pi \text{ is a representation of } (\mathcal{G},\mathcal{R})\}$$

is finite. Plainly, admissibility is a necessary condition for uniqueness property and therefore we make it our another standing assumption. Then the function $|||\cdot||| : \mathbb{F} \to [0,\infty)$ is a C^*-seminorm on $\mathbb{F}$ and its kernel

$$\mathbb{I} := \{w \in \mathbb{F} : |||w||| = 0\}$$

is a self-adjoint ideal in $\mathbb{F}$ – it is the smallest self-adjoint ideal in $\mathbb{F}$ such that the relations $\mathcal{R}$ become valid in the quotient $\mathbb{F}/\mathbb{I}$. We put

$$C^*(\mathcal{G},\mathcal{R}) := \overline{\mathbb{F}/\mathbb{I}}^{|||\cdot|||}$$

and call it a *universal C^*-algebra* generated by $\mathcal{G}$ subject to relations $\mathcal{R}$, cf. [4]. C^*-algebra $C^*(\mathcal{G},\mathcal{R})$ is characterized by the property that any representation of $(\mathcal{G},\mathcal{R})$ extends uniquely to a representation of $C^*(\mathcal{G},\mathcal{R})$ and all representations of $C^*(\mathcal{G},\mathcal{R})$ arise in that manner. In particular, $(\mathcal{G},\mathcal{R})$ possess uniqueness property if and only if any faithful representation of $(\mathcal{G},\mathcal{R})$ extends to a faithful representation of $C^*(\mathcal{G},\mathcal{R})$.

3. Gauge actions – exploring the symmetries in the relations

We would like to identify the uniqueness property of $(\mathcal{G},\mathcal{R})$ by looking at the symmetries in $(\mathcal{G},\mathcal{R})$. In order to formalize this we use a natural torus action $\{\gamma_\lambda\}_{\lambda\in\mathbb{T}^\mathcal{G}}$ on $\mathbb{F}$ determined by the formula

$$\gamma_\lambda(g) = \lambda_g\, g, \qquad \text{for} \;\; g\in\mathcal{G} \;\; \text{and} \;\; \lambda = \{\lambda_h\}_{h\in\mathcal{G}}\in\mathbb{T}^\mathcal{G}$$

where $\mathbb{T} = \{z\in\mathbb{C} : |z| = 1\}$ is a unit circle. A closed subgroup $G\subset\mathbb{T}^\mathcal{G}$ may be considered as a *group of symmetries in the pair* $(\mathcal{G},\mathcal{R})$ if the restricted action $\gamma = \{\gamma_\lambda\}_{\lambda\in G}$ leaves invariant the ideal $\mathbb{I}$. Any such group gives rise to a point-wisely continuous group action on $C^*(\mathcal{G},\mathcal{R})$ and actions that arise in that manner are called *gauge actions.*

Let us from now on consider the case when $G\cong\mathbb{T}$, that is we have a circle gauge action $\gamma = \{\gamma_\lambda\}_{\lambda\in\mathbb{T}}$ on $C^*(\mathcal{G},\mathcal{R})$. Then for each $n\in\mathbb{Z}$ the formula

$$\mathcal{E}_n(b) := \int_\mathbb{T} \gamma_\lambda(b)\lambda^{-n}\, d\lambda$$

defines a projection $\mathcal{E}_n : C^*(\mathcal{G},\mathcal{R})\to C^*(\mathcal{G},\mathcal{R})$, called *nth spectral projection*, onto the subspace

$$\mathcal{B}_n := \{b\in C^*(\mathcal{G},\mathcal{R}) : \gamma_\lambda(b) = \lambda^n b\}$$

called *nth spectral subspace* for γ, cf., e.g., [5]. Spectral subspaces specify a $\mathbb{Z}$-gradation on $C^*(\mathcal{G},\mathcal{R})$. Namely, $\bigoplus_{n\in\mathbb{Z}}\mathcal{B}_n$ is dense in $C^*(\mathcal{G},\mathcal{R})$, and

$$\mathcal{B}_n\mathcal{B}_m\subset\mathcal{B}_{n+m}, \;\; \mathcal{B}_n^* = \mathcal{B}_{-n} \;\; \text{for all} \;\; n,m\in\mathbb{Z}. \tag{1}$$

In particular, $\mathcal{B}_0$ is a C^*-algebra – the fixed point algebra for γ, and $\mathcal{E}_0:\mathcal{B}\to\mathcal{B}_0$ is a conditional expectation. A circle action on a C^*-algebra $\mathcal{B}$ is called *semi-saturated* [5] if $\mathcal{B}$ is generated as a C^*-algebra by its first and zeroth spectral subspaces. We note that every continuous group endomorphism of $\mathbb{T}$ is of the form $\lambda\mapsto\lambda^n$, for certain $n\in\mathbb{Z}$, and hence it follows that $\mathcal{G}\subset\cup_{n\in\mathbb{Z}}\mathcal{B}_n$. In particular, we have

Lemma 1. *The circle gauge action $\gamma = \{\gamma_\lambda\}_{\lambda\in\mathbb{T}}$ on $C^*(\mathcal{G},\mathcal{R})$ is semi-saturated, that is $C^*(\mathcal{G},\mathcal{R}) = C^*(\mathcal{B}_0,\mathcal{B}_1)$ if and only if $\mathcal{G} = \mathcal{G}_0\cup\mathcal{G}_1$ for some disjoint sets $\mathcal{G}_0$, $\mathcal{G}_1$ and $\gamma_\lambda(g_0) = g_0$, $\gamma_\lambda(g_1) = \lambda g_1$, for all $g_i\in\mathcal{G}_i$.*

We introduce an important necessary condition for $(\mathcal{G},\mathcal{R})$ to possess uniqueness property.

Proposition 2. *The following conditions are equivalent:*

i) *each faithful representation of $(\mathcal{G},\mathcal{R})$ give rise to a faithful representation of the fixed-point algebra $\mathcal{B}_0$.*

ii) *each faithful representation π of $(\mathcal{G},\mathcal{R})$ give rise to a faithful representation of $C^*(\mathcal{G},\mathcal{R})$ if and only if there is a circle action β on $C^*(\pi)$ such that*

$$\beta_z(\pi(g)) = \pi(\gamma_z(g)), \qquad g \in \mathcal{G}.$$

Proof. i) $\Longrightarrow$ ii). It suffices to apply the gauge invariance uniqueness for circle actions, see, e.g., [5, 2.9] or [6, 4.2]. ii) $\Longrightarrow$ i). Assume that π is a faithful representation of $(\mathcal{G},\mathcal{R})$ such that its extension is not faithful on $\mathcal{B}_0$. The spaces $\{\pi(\mathcal{B}_n)\}_{n\in\mathbb{Z}}$ form a $\mathbb{Z}$-graded C^*-algebra and thus by [6, 4.2], there is a (unique) C^*-norm $\|\cdot\|_\beta$ on $\bigoplus_{n\in\mathbb{Z}}\pi(\mathcal{B}_n)$ such that the circle action β on $\bigoplus_{n\in\mathbb{Z}}\pi(\mathcal{B}_n)$ established by gradation extends onto the C^*-algebra $\mathcal{B} = \overline{\bigoplus_{n\in\mathbb{Z}}\pi(\mathcal{B}_n)}^{||\cdot||_\beta}$. Composing π with the embedding $\bigoplus_{n\in\mathbb{Z}}\pi(\mathcal{B}_n) \subset \mathcal{B}$ one gets a faithful representation π' of $(\mathcal{G},\mathcal{R})$ which is gauge-invariant but not faithful on $C^*(\mathcal{G},\mathcal{R})$. □

In the literature the statements showing that the condition ii) in Proposition 2 holds are often called *gauge-invariance uniqueness theorems* and therefore we shall say that the triple $(\mathcal{G},\mathcal{R},\gamma)$ has the *gauge-invariance uniqueness property* if each faithful representation of $(\mathcal{G},\mathcal{R})$ give rise to a faithful representation of the fixed-point algebra $\mathcal{B}_0$. In particular, this always holds for triples $(\mathcal{G},\mathcal{R},\gamma)$ such that $C^*(\mathcal{G},\mathcal{R})$ can be modeled as relative Cuntz-Pimsner algebra, see [3, Sect. 9] and sources cited there.

4. From relations to Hilbert bimodules

Let us fix a pair $(\mathcal{G},\mathcal{R})$ with a circle gauge action $\gamma = \{\gamma_\lambda\}_{\lambda\in\mathbb{T}}$. It follows from (1) that $\mathcal{B}_1$ can be naturally viewed as a *Hilbert bimodule* over $\mathcal{B}_0$, in the sense introduced in [7, 1.8]. Namely, $\mathcal{B}_1$ is a $\mathcal{B}_0$-bimodule with bimodule operations inherited from $C^*(\mathcal{G},\mathcal{R})$ and additionally is equipped with two $\mathcal{B}_0$-valued inner products

$$\langle a,b\rangle_R := a^*b, \qquad {}_L\langle a,b\rangle := ab^*$$

that satisfy the so-called imprimitivtiy condition: $a\cdot\langle b,c\rangle_R = {}_L\langle a,b\rangle\cdot c = ab^*c$, for all $a,b,c\in\mathcal{B}_1$. Thus we can consider *crossed product* $\mathcal{B}_1 \rtimes_{\mathcal{B}_0}\mathbb{Z}$ of $\mathcal{B}_0$ by *the Hilbert bimodule* $\mathcal{B}_1$ constructed in [8], which could be alternatively defined as the universal C^*-algebra:

$$\mathcal{B}_1 \rtimes_{\mathcal{B}_0}\mathbb{Z} = C^*(\mathcal{G}_\gamma,\mathcal{R}_\gamma)$$

where $\mathcal{G}_\gamma = \mathcal{B}_0\cup\mathcal{B}_1$ and $\mathcal{R}_\gamma$ consists of all algebraic relations in the Hilbert bimodule $(\mathcal{B}_0,\mathcal{B}_1)$.

Proposition 3. *We have a natural embedding $\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z} \hookrightarrow C^*(\mathcal{G},\mathcal{R})$ which is an isomorphism if and only if γ is semi-saturated. Moreover, if γ is semi-saturated, then the following conditions are equivalent:*

i) $(\mathcal{G},\mathcal{R})$ *possess uniqueness property*
ii) $(\mathcal{G},\mathcal{R},\gamma)$ *has gauge-invariance uniqueness property and* $(\mathcal{G}_\gamma,\mathcal{R}_\gamma)$ *possess uniqueness property*

Proof. Since the homomorphism $\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z} \mapsto C^*(\mathcal{G},\mathcal{R})$ is gauge-invariant and injective on $\mathcal{B}_0$ it is injective onto the whole $\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z}$ by [5, 2.9]. The rest, in view of Proposition 2, is clear. □

The Hilbert bimodule $(\mathcal{B}_0,\mathcal{B}_1)$ is an imprimitivity bimodule (called also Morita-Rieffel equivalence bimodule), see [9], if and only if $\overline{\mathcal{B}_1^*\mathcal{B}_1} = \mathcal{B}_0$ and $\overline{\mathcal{B}_1\mathcal{B}_1^*} = \mathcal{B}_0$. In general, $\overline{\mathcal{B}_1^*\mathcal{B}_1}$ and $\overline{\mathcal{B}_1\mathcal{B}_1^*}$ are non-trivial ideals in $\mathcal{B}_0$ and we may treat $\mathcal{B}_1$ as a $\overline{\mathcal{B}_1\mathcal{B}_1^*} - \overline{\mathcal{B}_1^*\mathcal{B}_1}$-imprimitivity bimodule. This means, cf. [9, Cor. 3.33], that the induced representation functor

$$\widehat{h} = \mathcal{B}_1\text{-Ind}$$

is a homeomorphism $\widehat{h} : \widehat{\overline{\mathcal{B}_1^*\mathcal{B}_1}} \to \widehat{\overline{\mathcal{B}_1\mathcal{B}_1^*}}$ between the spectra of $\overline{\mathcal{B}_1^*\mathcal{B}_1}$ and $\overline{\mathcal{B}_1\mathcal{B}_1^*}$. Treating these spectra as open subsets of the spectrum $\widehat{\mathcal{B}_0}$ of $\mathcal{B}_0$ we may treat $\widehat{h}$ as a partial homeomorphism of $\widehat{\mathcal{B}_0}$. We shall say that $(\widehat{\mathcal{B}},\widehat{h})$ is a *partial dynamical system dual to the bimodule* $(\mathcal{B}_0,\mathcal{B}_1)$. Partial homeomorphism $\widehat{h}$ is said to be *topologically free* if for each $n \in N$ the set of points in $\widehat{\mathcal{B}_0}$ for which the equality $\widehat{h}^n(x) = x$ (makes sense and) holds has empty interior.

Theorem 4 (main result). *Suppose that the partial homeomorphism $\widehat{h} = B_1\text{-Ind}$ is topologically free. Then the pair $(\mathcal{G}_\gamma,\mathcal{R}_\gamma)$ possess uniqueness property and in particular*

i) *if $(\mathcal{G},\mathcal{R},\gamma)$ possess gauge-invariance uniqueness property, then any faithful representation of $(\mathcal{G},\mathcal{R})$ extends to the faithful representation of $\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z} \subset C^*(\mathcal{G},\mathcal{R})$.*
ii) *if γ is semi-saturated and $(\mathcal{G},\mathcal{R},\gamma)$ possess gauge-invariance uniqueness property, then $(\mathcal{G},\mathcal{R})$ possess uniqueness property.*

Proof. Apply the main result of [10] and Proposition 3. □

5. Applications to crossed products and Cuntz-Krieger algebras

We show that our main result is a generalization of the so-called isomorphisms theorem for crossed products by automorphisms (see, for instance, [11, pp. 225, 226] for a brief survey of such results) by applying it to a crossed product by an endomorphisms which is considered to be one of the most successful constructions of this sort, see [12] and sources cited there. In particular, we shall use this crossed product to identify the uniqueness property for Cuntz-Krieger algebras.

5.1. Crossed products by monomorphisms with hereditary range

Let $\alpha : \mathcal{A} \to \mathcal{A}$ be a monomorphism of a unital C^*-algebra $\mathcal{A}$. Let $\mathcal{G} = \mathcal{A} \cup \{S\}$ and let $\mathcal{R}$ consists of all *-algebraic relations in $\mathcal{A}$ plus the covariance relations

$$SaS^* = \alpha(a), \qquad S^*S = 1, \qquad a \in \mathcal{A}. \tag{2}$$

Then $C^*(\mathcal{G},\mathcal{R}) \cong \mathcal{A} \rtimes_\alpha \mathbb{N}$ is the crossed product of $\mathcal{A}$ by α, which is equipped with a semi-saturated circle gauge action: $\gamma_\lambda(a) = a$, $\gamma_\lambda(S) = \lambda S$, $a \in \mathcal{A}$. Let us additionally assume that $\alpha(\mathcal{A})$ is a hereditary subalgebra of $\mathcal{A}$. This is equivalent to $\alpha(\mathcal{A}) = \alpha(1)\mathcal{A}\alpha(1)$. Then we have $S^*\mathcal{A}S \subset \mathcal{A}$ since for any $a \in \mathcal{A}$ there is $b \in \mathcal{A}$ such that $\alpha(b) = \alpha(1)a\alpha(1)$ and therefore

$$S^*aS = S^*\alpha(1)a\alpha(1)S = S^*\alpha(b)S = S^*SbS^*S = b \in \mathcal{A}.$$

Hence on one hand $\mathcal{A} = \mathcal{B}_0$ is the fixed point algebra for γ and $\mathcal{B}_1 = \mathcal{B}_0 S$ is the first spectral subspace. On the other hand the spectrum of the hereditary subalgebra $\alpha(\mathcal{A})$ may be naturally identified with an open subset of $\widehat{\mathcal{A}}$, see, e.g., [13, Thm. 5.5.5], and then the dual $\widehat{\alpha} : \widehat{\alpha(\mathcal{A})} \to \widehat{\mathcal{A}}$ to the isomorphism $\alpha : \mathcal{A} \to \alpha(\mathcal{A})$ becomes a partial homeomorphism of $\widehat{\mathcal{A}}$. Under this identification one gets

$$\widehat{\alpha} = \mathcal{B}_1\text{-Ind}$$

and hence if the partial system $(\widehat{\mathcal{A}}, \widehat{\alpha})$ dual to $(\mathcal{A}, \alpha)$ is topologically free, then $(\mathcal{G},\mathcal{R})$ possess uniqueness property.

5.2. Cuntz-Krieger algebras

Let $\mathcal{G} = \{S_i : i = 1,\dots,n\}$, where $n \geq 2$, and let $\mathcal{R}$ consists of the Cuntz-Krieger relations

$$S_i^*S_i = \sum_{j=1}^{n} A(i,j)S_jS_j^*, \qquad S_i^*S_k = \delta_{i,k}S_i^*S_i, \qquad i,k = 1,\dots,n, \tag{3}$$

where $\{A(i,j)\}$ is a given $n \times n$ zero-one matrix such that every row and every column of A is non-zero, and $\delta_{i,j}$ is Kronecker symbol. Then $C^*(\mathcal{G},\mathcal{R})$ is the Cuntz-Krieger algebra $\mathcal{O}_A$ and the celebrated Cuntz-Krieger uniqueness theorem, cf. [14, Thm. 2.13], states that the pair $(\mathcal{G},\mathcal{R})$ possess uniqueness property if and only if the so-called *condition* (I) holds:

- **(I)** the space $X_A := \{(x_1,x_2,\dots) \in \{1,\dots,n\}^{\mathbb{N}} : A(x_k,x_{k+1}) = 1\}$ has no isolated points (considered with the product topology)

We may recover this result applying our method to the standard circle gauge action on $\mathcal{O}_A$ determined by equations $\gamma_\lambda(S_i) = \lambda S_i$, $i = 1,\dots,n$. Indeed, the fixed point C^*-algebra for γ coincides with the so-called AF-core

$$\mathcal{F}_A = \overline{\text{span}}\{S_\mu S_\nu^* : |\mu| = |\nu| = k,\ k = 1,\dots\}$$

where for a multiindex $\mu = (i_1,\dots,i_k)$, with $i_j \in 1,\dots,n$, we denote by $|\mu|$ the length k of μ and write $S_\mu = S_{i_1}S_{i_2}\cdots S_{i_k}$. Moreover, any faithful representation of the Cuntz-Krieger relations (3) yields a faithful representation of $\mathcal{F}_A$, that is $(\mathcal{G}, \mathcal{R}, \gamma)$ possess gauge-invariance uniqueness property. Following [12] we put

$$S := \sum_{i,j} \frac{1}{\sqrt{n_j}} S_i P_j$$

where $n_j = \sum_{i=1}^n A(i,j)$ and $P_j = S_j S_j^*$, $j = 1, \dots, n$. A routine computation shows that $S\mathcal{F}_A S^* \subset \mathcal{F}_A$, $S^*\mathcal{F}_A S \subset \mathcal{F}_A$ and $S^*S = 1$ (S is an isometry). Hence the mapping $\mathcal{F}_A \ni a \mapsto \alpha(a) := SaS^* \in \mathcal{F}_A$ is a monomorphism with a hereditary range. It is uniquely determined by the formula

$$\alpha\Big(S_{i_2\mu} S^*_{j_2\nu}\Big) = \frac{1}{\sqrt{n_{i_2} n_{j_2}}} \sum_{i,j=1}^{n} S_{i\, i_2\mu} S^*_{j\, j_2\nu}. \tag{4}$$

From the construction any representation of relations (3) yields a representation of $(\mathcal{F}_A, \alpha)$ as introduced in the previous subsection. Conversely, if S satisfies (2) where $\mathcal{A} = \mathcal{F}_A$, then one gets representation of (3) by putting $S_i := \sum_{j=1}^n A(i,j)\sqrt{n_j} P_i S P_j$. Thus we have a natural isomorphism

$$\mathcal{O}_A \cong \mathcal{F}_A \rtimes_\alpha \mathbb{N}$$

under which the introduced gauge actions coincide. Hence we may identify the partial dynamical system dual to the Hilbert bimodule $(\mathcal{B}_1, \mathcal{B}_0)$ where $\mathcal{B}_0 = \mathcal{F}_A$ and $\mathcal{B}_1 = \mathcal{F}_A S$ with the partial dynamical system $(\widehat{\mathcal{F}_A}, \widehat{\alpha})$ dual to $(\mathcal{F}_A, \alpha)$, as introduced in the previous subsection.

In order to identify the topological freeness of $\widehat{\alpha}$ we define $\pi_\mu \in \widehat{\mathcal{A}}$ for any infinite path $\mu = (i_1, i_2, \dots)$, $A(i_j, i_{j+1}) = 1$, $j \in \mathbb{N}$, to be the GNS-representation associated to the pure state $\omega_\mu : \mathcal{F}_A \to \mathbb{C}$ determined by the formula

$$\omega_\mu(S_\nu S_\eta^*) = \begin{cases} 1 & \nu = \eta = (\mu_1, \dots, \mu_n) \\ 0 & \text{otherwise} \end{cases} \qquad \text{for} \quad |\nu| = |\eta| = n. \tag{5}$$

Using description of the ideal structure in $\mathcal{F}_A$ in terms of Bratteli diagrams [15], similarly as in [10], one can show that representations π_μ form a dense subset of $\widehat{\mathcal{F}_A}$ and

$$\widehat{\alpha}(\pi_{(\mu_1,\mu_2,\mu_3,\dots)}) = \pi_{(\mu_2,\mu_3,\dots)}, \qquad \text{for any } (\mu_1, \mu_2, \mu_3, \dots).$$

In particular, it follows that *topological freeness of* $\widehat{\alpha}$ *is equivalent to condition* (I). Accordingly

> *our main result, Theorem 4, when applied to Cuntz-Krieger relations is equivalent to the Cuntz-Krieger uniqueness theorem.*

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