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Some Non-standard Examples of Coherent States and Quantization

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Abstract. We look at certain non-standard constructions of coherent states, viz., over matrix domains, on quaternionic Hilbert spaces and C*-Hilbert modules and their possible use in quantization. In particular we look at families of coherent states built over Cuntz algebras and suggest applications to non-commutative spaces. The present considerations might also suggest an extension of Berezin-Toeplitz and coherent state quantization to quaternionic Hilbert spaces and Hilbert modules.

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1. Standard coherent states

Coherent states are a much used concept, both physically and mathematically. Generically, they are obtained from a reproducing kernel subspace (see, for example, [1]) of an L^2-space,

$$\mathfrak{H}_K \subset \mathfrak{H} = L^2(X, \mu),$$

where μ is a finite measure on the Borel σ-field of a locally compact topological space X. If

$$\Phi_0, \Phi_1, \ldots, \Phi_n, \ldots$$

is any orthonormal basis of $\mathfrak{H}_K$, then the reproducing kernel is given by

$$K(x, y) = \sum_k \Phi_k(x)\overline{\Phi_k(y)}\ . \tag{1}$$

Using this fact and taking another Hilbert space $\mathfrak{K}$ of the same dimension as that of $\mathfrak{H}_K$, the non-normalized coherent states are defined as

$$\mid x\rangle = \sum_k \psi_k\ \overline{\Phi_k(x)}, \tag{2}$$

where $\psi_1, \psi_2, \ldots, \psi_n, \ldots$ is an orthonormal basis of $\mathfrak{K}$.

It is then easy to verify that

$$\langle x \mid y\rangle = K(x,y) \quad \text{and} \quad \int_X \mid x\rangle\langle x \mid \; d\mu(x) = I_{\mathfrak{K}} \;, \tag{3}$$

the integral converging in the weak operator topology. If, furthermore,

$$K(x,x) = \sum_k |\Phi_k(x)|^2 := \mathcal{N}(x) > 0,$$

for all $x \in X$, normalized CS can be defined as:

$$\widehat{\mid x\rangle} = \mathcal{N}(x)^{-\frac{1}{2}} \mid x\rangle \;,$$

which then satisfy the conditions,

$$\|\widehat{\mid x\rangle}\| = 1 \quad \text{and} \quad \int_X \widehat{\mid x\rangle}\widehat{\langle x \mid} \,\mathcal{N}(x)\; d\mu(x) = I_{\mathfrak{K}} \;.$$

These are the physical coherent states.

Berezin-Toeplitz quantization or, coherent state quantization, of functions f on the space X is given by the operator association (see, for example, [2] and references cited therein),

$$f \longmapsto \hat{f} = \int_X f(x)|x\rangle\langle x| \; d\mu(x) \;, \tag{4}$$

provided the integral exists in some appropriate sense.

In view of their usefulness and interest in various areas of physics and mathematics, it is natural to look for generalizations of the above concept of coherent states.

One such possibility is to construct analogous objects on a Hilbert C^*-module, which is analogous to a Hilbert space, but has an inner product taking values in a C^*-algebra. We shall call the resulting vectors module-valued coherent states (MVCS). In simple terms, we shall replace both the set of functions $\Phi_k(x)$ and the vectors ψ_k, in the definition of coherent states in (2) by elements of Hilbert modules. Another possibility for generalization could be to construct coherent states on quaternionic Hilbert spaces.

Since the field of complex numbers $\mathbb{C}$ is trivially a C^*-algebra, coherent states on Hilbert spaces are special cases of MVCS.

2. Module-valued coherent states

The discussion of this section is based mainly on [3]. Consider two unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and a Hilbert C^*-correspondence $\mathbf{E}$ from $\mathcal{A}$ to $\mathcal{B}$. This means that $\mathbf{E}$ is a Hilbert C^*-module over $\mathcal{B}$, with a left action from $\mathcal{A}$, i.e., there is a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{L}(\mathbf{E})$, the bounded adjointable operators on $\mathbf{E}$. Let (X, μ) be a finite measure space and consider the set of functions,

$$\mathbb{F} = \{F : X \longmapsto \mathbf{E} \mid F \;\text{ is a strongly measurable function}\} \;.$$

Then clearly, for any two F, G in $\mathbb{F}$, $x \longmapsto \langle F(x) \mid G(x)\rangle_{\mathbf{E}}$ is a strongly measurable function. Let

$$\mathfrak{H} = \{F \in \mathbb{F} \mid \text{ the function} \langle F(x) \mid F(x)\rangle \text{ is Bochner integrable}\} \,. \tag{5}$$

Given a strongly measurable function F, a necessary and sufficient condition for $\langle F(x) \mid F(x)\rangle$ to be Bochner integrable is that

$$\int_X \|\langle F(x) \mid F(x)\rangle_{\mathbf{E}}\|_{\mathcal{B}}\, d\mu(x) < \infty \,.$$

This immediately shows that $\mathfrak{H}$ is a complex vector space. Also, $\mathfrak{H}$ is an inner product module over $\mathcal{B}$, where the right multiplication and the inner product respectively are

$$(F \cdot b)(x) = F(x)b \text{ for all } b \in \mathcal{B}, \;\; \langle F \mid G\rangle_{\mathfrak{H}} = \int_X \langle F(x) \mid G(x)\rangle_{\mathbf{E}}\, d\mu(x).$$

Its completion in the resulting norm $\|F\|_{\mathfrak{H}} = \|\langle F \mid \; F\rangle_{\mathfrak{H}}\|_{\mathcal{B}}^{\frac{1}{2}}$ is a Hilbert C^*-module over $\mathcal{B}$ and can be identified with $L^2(X) \otimes \mathbf{E}$. There is a natural left action of $\mathcal{A}$ on $\mathfrak{H}$ because $\mathbf{E}$ is an $\mathcal{A} - \mathcal{B}$ correspondence.

For $e \in \mathbf{E}$, we define the map $\langle e| : \mathbf{E} \longrightarrow \mathcal{B}$, by

$$\langle e|(f) = \langle e \mid f\rangle_{\mathbf{E}} \,, \quad f \in \mathbf{E} \,.$$

This is an adjointable map. We shall denote its adjoint by $|e\rangle$. Then $|e\rangle : \mathcal{B} \longrightarrow \mathbf{E}$ has the action

$$|e\rangle(b) = eb \,, \quad b \in \mathcal{B} \,,$$

so that for $e_1, e_2 \in \mathbf{E}$,

$$|e_1\rangle\langle e_2|(f) = e_1\langle e_2 \mid f\rangle_{\mathbf{E}} \,. \tag{6}$$

Thus formally, one may use the standard bra-ket notation for Hilbert modules as one does for Hilbert spaces.

Let us choose a set of vectors

$$F_0, F_1, \ldots, F_n, \ldots \,,$$

(finite or infinite) in the function space $\mathfrak{H}$, which are pointwise defined (for all $x \in X$) and which satisfy the orthogonality relations,

$$\int_X \mid F_k(x)\rangle\langle F_\ell(x) \mid \; d\mu(x) = I_{\mathbf{E}}\; \delta_{k\ell} \,. \tag{7}$$

We now introduce module-valued coherent states for two separate situations, highlighting the fact that a Hilbert C^*-module is a generalization of both a Hilbert space and a C^*-algebra. The resulting MVCS depend on an auxiliary object $\mathbf{G}$, which in the first instance is a Hilbert space and in the second, the Cuntz algebras $\mathcal{O}_n$ or $\mathcal{O}_\infty$.

To proceed with the first construction of MVCS let $\mathbf{G}$ be a Hilbert space of the same dimension as the cardinality of the F_k. In $\mathbf{G}$ we choose an orthonormal basis, $\phi_0, \phi_1, \ldots, \phi_n, \ldots$. Let $\mathbf{H} = \mathbf{E} \otimes \mathbf{G}$ denote the exterior tensor product of $\mathbf{E}$ and $\mathbf{G}$, which is then itself a Hilbert module over $\mathcal{B}$.

For each $x \in X$ and co-isometry $a \in \mathcal{A}$ (i.e., $aa^* = \mathrm{id}_{\mathcal{A}}$), we define the vectors,

$$| \, x, a\rangle = \sum_k aF_k(x) \otimes \phi_k \in \mathbf{H} \, , \tag{8}$$

assuming of course that the sum converges in the norm of $\mathbf{H}$. We call these vectors (non-normalized) module-valued coherent states (MVCS).

Proposition 2.1. *The MVCS in* (8) *satisfy the resolution of the identity,*

$$\int_X | \, x, a\rangle\langle x, a \, | \;\; d\mu(x) = I_{\mathbf{H}}, \tag{9}$$

the integral converging in the sense that for any two $h_1, h_2 \in \mathbf{H}$,

$$\int_X \langle h_1 \mid x, a\rangle_{\mathbf{H}} \langle x, a \mid h_2\rangle_{\mathbf{H}} \, d\mu(x) = \langle h_1 \mid h_2\rangle_{\mathbf{H}} \, ,$$

as a Bochner integral.

This construction may easily be modified to obtain normalized MVCS under certain conditions. For that, we fix a notation for a certain positive element of $\mathcal{B}$. Let

$$\mathcal{N}(x, a) := \langle x, a \mid x, a\rangle_{\mathbf{H}} = \sum_k \langle F_k(x) \mid a^* a F_k(x)\rangle_{\mathbf{E}} \, . \tag{10}$$

Proposition 2.2. *If $\phi_1, \phi_2, \ldots$ is an orthonormal basis for $\mathbf{G}$ and a is a unitary element of $\mathcal{A}$ and $\mathcal{N}(x, id_{\mathcal{A}})$ is invertible, then the MVCS constructed above can be normalized, i.e., we can construct MVCS $\widehat{| \, x, a\rangle} = | \, x, a\rangle \otimes \mathcal{N}(x, id_{\mathcal{A}})^{-\frac{1}{2}}$ which along with* (7) *also satisfy*

$$\langle \widehat{x, a} \mid \widehat{x, a}\rangle = id_{\mathcal{B}} \otimes id_{\mathcal{C}} \, . \tag{11}$$

The well-known vector coherent states [4, 5] (or multi-component coherent states), used in nuclear and atomic physics, can all be obtained from module-valued coherent states using the above construction. Furthermore, one can define adjointable operators on the Hilbert module $\mathbf{H}$ following a Berezin-Toeplitz type prescription as in (4):

$$f \longrightarrow \widehat{f} = \int_X f(x) |x, \mathrm{id}_{\mathcal{A}}\rangle\langle x, \mathrm{id}_{\mathcal{A}}| \; d\mu(x) \, ,$$

and study the resulting quantization problem.

3. MVCS from certain Cuntz algebras

We now construct MVCS using the notion of Cuntz algebras [6] (see also [7]). Let $S_1, S_2, \ldots$ be isometries on a complex separable Hilbert space $\mathcal{K}$ (necessarily infinite-dimensional) such that

$$\sum_{j=1}^{\infty} S_j S_j^* = I_{\mathcal{K}}$$

where the sum converges in the strong operator topology of $\mathcal{B}(\mathcal{K})$. Multiplying both sides by S_i^*, we get

$$S_i^* + S_i^* \sum_{j\neq i} S_j S_j^* = S_i^*$$

so that

$$S_i^* \sum_{j\neq i} S_j S_j^* = 0 \ .$$

But $\sum_{j\neq i} S_j S_j^*$ is the projection onto the closure of the span of the ranges of S_j for $j \neq i$. So the range of S_i is orthogonal to the range of S_j for all $j \neq i$. This is a representation of the Cuntz algebra $\mathcal{O}_\infty$ with infinitely many generators.

We take $\mathbf{G}$ to be the C^*-algebra generated by the isometries $S_1, S_2, \ldots$. The coherent states are defined as

$$|x, a\rangle = \left(\sum_{k=1}^{\infty} a \cdot F_k(x) \otimes S_k\right)(\boldsymbol{\mathcal{N}}(x)^{-1/2} \otimes I). \tag{12}$$

An explicit example of a Cuntz algebra is as follows. Let

$$\omega : \mathbb{N}^{>0} \longrightarrow \mathbb{N}^{>0} \times \mathbb{N}^{>0}$$

be a bijection ($\mathbb{N}^{>0}$ denoting the set of positive integers). Consider a Hilbert space $\mathfrak{H}$ and let $\{\phi_n\}_{n\in\mathbb{N}^{>0}}$ be an orthonormal basis of it. Writing $\omega(n) = (k, \ell)$ we define a re-transcription of this basis in the manner

$$\psi_{k\ell} := \phi_n = \psi_{\omega(n)} \ , \qquad k, n, \ell \in \mathbb{N}^{>0} \ . \tag{13}$$

The C^*-algebra $\mathcal{O}_\infty$, generated by these isometries, is then a Cuntz algebra.

The MVCS obtained using these S_k in (12) have an immediate physical application. We consider the non-normalized version (with a set to the unit element of $\mathcal{A}$),

$$|x\rangle = \sum_{k=1}^{\infty} F_k(x) \otimes S_k.$$

Let

$$X = \mathbb{C} \qquad \text{and} \qquad \mathbf{E} = L^2\left(\mathbb{C},\ \frac{e^{-|z|^2}}{2\pi}\, dx\, dy\right), \quad z = \frac{1}{\sqrt{2}}(x + iy) \ ,$$

and let $F_k : \mathbb{C} \longrightarrow \mathbb{C}$ be the functions,

$$F_k(z) = \frac{z^{k-1}}{\sqrt{(k-1)!}} \ , \qquad k = 1, 2, 3, \ldots \ .$$

Next let $\psi_{k\ell}$ be the complex Hermite polynomials,

$$\psi_{k\ell}(\overline{z}, z) = \frac{(-1)^{n+k-2}}{\sqrt{(\ell-1)!(k-1)!}}\ e^{|z|^2} \partial_{\overline{z}}^{\ell-1} \partial_z^{k-1}\ e^{-|z|^2} \ , \qquad k, \ell = 1, 2, 3, \ldots \ , \tag{14}$$

which form an orthonormal basis of $L^2\big(\mathbb{C},\ \frac{e^{-|z|^2}}{2\pi}\ dx\ dy\big)$. The module-valued coherent states now become

$$|z\rangle = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} S_k \ . \qquad (15)$$

Let ϕ_n be as in (13), consider the vectors

$$\xi_{\overline{z}',\, n} = \frac{\overline{z}'^{n-1}}{\sqrt{(n-1)!}} \phi_n \ .$$

Then the vectors (in $L^2(\mathbb{C},\ \frac{e^{-|z|^2}}{2\pi}\ dx\ dy)$),

$$|z, \overline{z}', n\rangle = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} S_k \xi_{\overline{z}',\, n} = \overline{z}'^{n-1} \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!\ (n-1)!}}\ \psi_{kn}\ , \qquad (16)$$

($\ell = 1, 2, 3, \ldots, \infty,$) are just the non-normalized versions of the infinite component vector CS found in [5] and associated to the energy levels (the so-called Landau levels) of an electron in a constant magnetic field.

4. Matrix-valued and quaternionic MVCS

In [4] analytic vector coherent states, built using powers of matrices from $\mathcal{M}_N(\mathbb{C})$, were defined:

$$|\,\mathfrak{Z}, i\rangle = \sum_k \frac{\mathfrak{Z}^k}{\sqrt{c_k}} \chi^i \otimes \mathbf{\Phi}_k\ , \qquad \mathfrak{Z} \in \mathcal{M}_N(\mathbb{C})\ , \qquad (17)$$

where the c_k are the numbers,

$$c_k = \frac{1}{(k+1)(k+2)} \left[\prod_{j=1}^{k+1} (N+j) - \prod_{j=1}^{k+1} (N-j) \right], \qquad k = 0, 1, 2, \ldots,$$

Let $z_{ij},\ i, j = 1, 2, \ldots, N$ be the matrix elements of $\mathfrak{Z}$. Then, writing

$$F_k(\mathfrak{Z}) = \frac{\mathfrak{Z}^k}{\sqrt{c_k}} \quad \text{and} \quad z_{ij} = x_{ij} + i y_{ij}\ ,$$

it can be shown that,

$$\int_{\mathcal{M}_N(\mathbb{C})} F_k(\mathfrak{Z}) F_\ell(\mathfrak{Z})^*\ d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = \delta_{k\ell} \mathbb{I}_N\ ,$$

where

$$d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = \frac{e^{-\mathrm{Tr}[\mathfrak{Z}^*\mathfrak{Z}]}}{(2\pi)^N} \prod_{i,j=1}^{N} dx_{ij}\ dy_{ij}\ .$$

Using this fact, it is easy to prove the resolution of identity,

$$\sum_{i=1}^{N} \int_{\mathcal{M}_N(\mathbb{C})} |\,\mathfrak{Z}, i\rangle\langle \mathfrak{Z}, i\,|\ \ d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = \mathbb{I}_N \otimes I_{\mathfrak{H}_\mathbf{K}}\ .$$

To construct the related MVCS, we take $\mathbf{E} = \mathcal{B} = \mathcal{M}_N(\mathbb{C})$. The module $\mathfrak{H}$, containing the functions F_k, then consists of functions from $\mathcal{M}_N(\mathbb{C})$ to itself. Considering $\mathfrak{H}_{\mathbf{K}}$ as a module over $\mathbb{C}$, we may define MVCS in $\mathbf{H} = \mathcal{M}_N(\mathbb{C}) \otimes \mathfrak{H}_{\mathbf{K}}$ as

$$| \mathfrak{z}, a\rangle = \sum_k aF_k(\mathfrak{z}) \otimes \mathbf{\Phi}_k = \sum_k a \frac{\mathfrak{z}^k}{\sqrt{c_k}} \otimes \mathbf{\Phi}_k \ , \tag{18}$$

where a is a unitary element in $\mathcal{M}_N(\mathbb{C})$. These then satisfy the resolution of the identity,

$$\int_{\mathcal{M}_N(\mathbb{C})} | \mathfrak{z}, a\rangle\langle \mathfrak{z}, a \mid \ d\mu(\mathfrak{z}, \mathfrak{z}^*) = I_{\mathbf{H}} \ . \tag{19}$$

In the particular case when $N = 2$ the set $\mathcal{M}_N(\mathbb{C})$, of all complex 2×2 matrices, can be identified with the space of complex quaternions. The resulting MVCS may then be called complex quaternionic MVCS.

Although a Hilbert space over the quaternions is not a Hilbert module, we may still build coherent states in such a space using the above construction on Hilbert modules. Such coherent states also have interesting physical applications [8]. Suppose that $\mathfrak{H}_{\text{quat}}$ is a Hilbert space over the quaternions. (Multiplication by elements of $\mathbb{H}$ from the right is assumed, i.e., if $\mathbf{\Phi} \in \mathfrak{H}_{\text{quat}}$ and $\mathfrak{q} \in \mathbb{H}$, then $\mathbf{\Phi}\mathfrak{q} \in \mathfrak{H}_{\text{quat}}$). The well-known canonical coherent states [1] may then be readily generalized to quaternionic coherent states over $\mathfrak{H}_{\text{quat}}$. Indeed take an orthonormal basis $\{\mathbf{\Psi}_n^{\text{quat}}\}_{n=0}^{\infty}$ in $\mathfrak{H}_{\text{quat}}$ and define the vectors

$$| \mathfrak{q}\rangle = e^{-\frac{r^2}{2}} \sum_{n=0}^{\infty} \mathbf{\Psi}_n^{\text{quat}} \frac{\mathfrak{q}^n}{\sqrt{n!}} \in \mathfrak{H}_{\text{quat}}, \qquad \mathfrak{q} \in \mathbb{H}, \qquad \langle \mathfrak{q} \mid \mathfrak{q}\rangle_{\mathfrak{H}_{\text{quat}}} = \mathbb{I}_2 \ . \tag{20}$$

They satisfy the resolution of the identity,

$$\int_{\mathbb{H}} | \mathfrak{q}\rangle\langle \mathfrak{q} \mid \ d\nu(\mathfrak{q}, \mathfrak{q}^\dagger) = I_{\mathfrak{H}_{\text{quat}}} \ , \qquad d\nu(\mathfrak{q}, \mathfrak{q}^\dagger) = \frac{1}{4\pi^2}\, r dr \ d\xi \ \sin\theta d\theta \ d\phi \ . \tag{21}$$

In [8] these coherent states were obtained using a group theoretical argument. Here they appear as a special case of our more general construction.

5. Some possible applications

We end this discussion by mentioning some possible applications of the above general constructions of non-standard families of coherent states.

- Coherent states are naturally associated to positive definite kernels [1], coming from the reproducing kernel Hilbert spaces used to build them. It would be interesting to study such kernels for the MVCS and the coherent states on quaternionic Hilbert spaces. Then there would also be related positive operator-valued measures and a Naimark type dilation theorem. One could also study subnormal operators in this context.
- As already mentioned, a Berezin-Toeplitz type quantization on Hilbert modules would be a natural problem to study.

- Module-valued coherent states have been used to define localization on non-commutative spaces [3], which is another direction for further investigation. Indeed, it is in this direction, where standard quantum mechanics might not be readily applicable, that we see greater possibility of application of this general concept.

References

[1] S.T. Ali, J.-P. Antoine and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*, Springer-Verlag, New York 1999.

[2] S.T. Ali and M. Engliš *Quantization methods: A guide for physicists and analysts,* Rev. Math. Phys. **17** (2005), 301–490.

[3] S.T. Ali, T. Bhattacharayya and S.S. Roy, *Coherent states on Hilbert modules*, J. Phys. A: Math. Theor., **44** (2011), 205202 (16pp).

[4] S.T. Ali, M. Engliš and J.-P. Gazeau, *Vector coherent states from Plancherel's theorem, Clifford algebras and matrix domains*, J. Phys. **A37** (2004), 6067–6089.

[5] S.T. Ali and F. Bagarello, *Some physical appearances of vector coherent states and coherent states related to degenerate Hamiltonians*, J. Math. Phys. **46** (2005), 053518 (28pp).

[6] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.

[7] E.C. Lance, *Hilbert C^*-Modules*, A toolkit for operator algebraists, Lond. Math. Soc. Lec. Notes Series. 210, Cambridge University Press, Cambridge 1995.

[8] S.L. Adler and A.C. Millard, *Coherent states in quaternionic quantum mechanics*, J. Math. Phys. **38** (1996), 2117–2126.

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