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Quantum Configuration Spaces of Extended Objects, Diffeomorphism Group Representations and Exotic Statistics

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and dedicated also to my colleagues
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Abstract. A fundamental approach to quantum mechanics is based on the unitary representations of the group of diffeomorphisms of physical space (and correspondingly, self-adjoint representations of a local current algebra). From these, various classes of quantum configuration spaces arise naturally, as well as the usual exchange statistics for point particles in spatial dimensions $d \geq 3$, induced by representations of the symmetric group. For $d = 2$, this approach led to an early prediction of intermediate or "anyon" statistics induced by unitary representations of the braid group. I review these ideas, and discuss briefly some analogous possibilities for infinite-dimensional configuration spaces, including anyonic statistics for extended objects in three-dimensional space.

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1. Introduction

It is remarkable how slowly physicists gained insight into exotic possibilities for the statistics of quantum particles. Bose-Einstein and Fermi-Dirac statistics, cor-

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responding respectively to the trivial and alternating one-dimensional representations of the symmetric group S_N, have been known of course since the 1920s. During the 1950s and 1960s, quantum theories were studied obeying "parastatistics," associated with various families of higher-dimensional representations of S_N [1, 2]. During this period, Aharonov and Bohm drew attention to what can now be understand as topological effects in quantum mechanics, associated with charged particles circling (but not entering) regions of magnetic flux [3]. In 1971, Laidlaw and DeWitt explicitly connected the topology of N-particle configuration spaces in $\mathbf{R}^3$ with the familiar possibilities of Bose and Fermi statistics [4]. But the first clear suggestion of the possibility of intermediate statistics for indistinguishable particles in $\mathbf{R}^2$ did not come until a 1977 paper by Leinaas and Myrheim [5], fully half a century after the exchange statistics of bosons and fermions had become standard in quantum mechanics – even though the idea can be obtained and expressed in elementary ways.

An early, independent prediction of such intermediate statistics in the plane came from the study by Menikoff, Sharp, and myself of representations of a certain local current algebra for quantum mechanics, and the associated infinite-dimensional group [6, 7]. This group is the natural semidirect product of the additive group $\mathcal{D} = C_0^\infty(M)$ of compactly-supported, real-valued C^∞ scalar functions on the spatial manifold M, with the group $\mathcal{K} = \mathrm{Diff}_0(M)$ of compactly-supported C^∞ diffeomorphisms of M under composition (where, in the case at hand, $M = \mathbf{R}^2$). Particles satisfying intermediate statistics were subsequently termed "anyons" by Wilczek [8, 9], as wave functions can be multiplied by a fixed complex number of modulus one – $\exp i\theta$, for "any" phase $0 \leq \theta < 2\pi$ – as a consequence of the exchange of indistinguishable particles through a single counterclockwise winding in the plane.

Anyons are associated with the equivariance of wave functions under one-dimensional representations of the braid group B_N [10, 11]. Their description fits nicely into the framework of braided tensor products developed by Majid, and when $\exp i\theta$ is a root of unity, generalized exclusion principles occur [12]. Higher-dimensional braid group representations likewise describe possible quantum particle systems in two-dimensional space [13]; such particles or excitations have been termed "nonabelian anyons" or "plektons."

These ideas have found numerous applications in physics, ranging from the theory of the quantum Hall effect to high-T_c superconductivity to quantum computing; for a recent, extensive discussion focusing on the latter, see Nayak *et al.* [14].

Recently attention has been drawn to possibilities for exotic statistics associated with configurations of extended objects. For example, Niemi discusses anyonic statistics that can occur for "leapfrogging" vortex rings, deriving this possibility in an elementary way that suggests to Niemi that it is generic [15], and providing inspiration for the present discussion. Here, I hope to indicate how such possibilities for the exotic statistics of extended objects arise naturally from the diffeomorphsim group approach to quantum mechanics. With relatively few equations, I shall survey some of the key ideas in this approach, unifying in a way the discussion of extended configuration spaces with that of exotic statistics. More detail about some of these ideas may be found in the references [16, 17, 18],

Section 2 offers a general description of representations of the semidirect product group $\mathcal{D} \times \mathcal{K}$ modeled on various classes of configuration spaces. Section 3 highlights induced representations and corresponding 1-cocycles in the N-particle case. Finally Section 4 indicates briefly how these ideas are generalized to extended objects, including configurations of loops and tori. Possible applications are to those domains of quantum physics where topologically nontrivial objects are fundamental, such as loops, ribbons, or rings of vorticity, configurations of magnetic flux, quantized strings, geons, and so forth.

2. Diffeomorphism group representations and quantum configuration spaces

Let M be the manifold of physical space (assumed to be smooth, connected, separable, locally compact, and σ-compact), and let $\mathbf{x}$ denote a general point in M. The support of a diffeomorphism $\phi : M \to M$ is defined to be the intersection of all closed sets outside of which $\phi(\mathbf{x}) \equiv \mathbf{x}$. The set of compactly supported diffeomorphisms $\mathcal{K}$ of M forms a group under composition: to be precise, we define $\phi_1\phi_2 = \phi_2 \circ \phi_1$, where $\circ$ denotes composition. Then $\mathcal{K}$ is an infinite-dimensional topological group in the topology of uniform convergence in all derivatives on compact sets. Similarly, $\mathcal{D}$ is an infinite-dimensional topological group under addition, endowed with the topology of uniform convergence in all derivatives on compact sets. The semidirect product $G = \mathcal{D} \times \mathcal{K}$ is defined by the group law

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + f_2 \circ \phi_1, \phi_2 \circ \phi_1)\,. \tag{1}$$

In an important sense, G may be considered a fundamental symmetry group of physical space for the purpose of defining the kinematics of quantum mechanics. It is a *local* symmetry group, in that given any fixed compact region $K \subset M$, we have a closed subgroup $\mathcal{D}_K \subset \mathcal{D}$ of functions supported in K (i.e., vanishing outside K), a closed subgroup $\mathcal{K}_K$ of diffeomorphisms having support in K, and the semidirect product $G_K = \mathcal{D}_K \times \mathcal{K}_K$ which is a closed subgroup of $G = \mathcal{D} \times \mathcal{K}$.

The group G is obtained by exponentiating the singular local current algebra proposed in 1968 by Dashen and Sharp [19], interpreted as a Lie algebra of operator-valued distributions [20]. This algebra, in turn, can be obtained formally from canonical creation and annihilation fields. The inequivalent, continuous unitary representations of G then correspond to distinct quantum systems, infinite as well as finite, so that their classification and interpretation becomes of central physical interest [21, 22]. A consequence is that one can describe – and, in fact, predict – exotic particle statistics as well as topological quantum effects, in a mathematically satisfying way. Let us see briefly how this works.

Let $f \in \mathcal{D}$ and $\phi \in \mathcal{K}$, for a particular spatial manifold M. A very general unitary representation of the semidirect product is given by the equations,

$$
\begin{aligned}
(\gamma) &= \exp i\langle \gamma, f\rangle \Psi(\gamma) \quad \text{a.e.}\,(\mu)\,,\\
[V(\phi)\Psi](\gamma) &= \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)} \quad \text{a.e.}\,(\mu)\,,
\end{aligned} \tag{2}
$$

which we shall now spend a little time interpreting and discussing.

In (2), the variable γ ranges over elements of a *quantum configuration space* Δ that one has defined (see below). The first equation requires that we have identified a sense in which γ also acts as a *continuous real-valued linear functional* on $\mathcal{D}$ (i.e., as a distribution over $\mathcal{D}$). The value of the distribution γ at $f \in \mathcal{D}$ is denoted $\langle \gamma, f\rangle \in \mathbf{R}$. That is, the elements of Δ are somehow (see below) identified with some of the elements of the dual space $\mathcal{D}'$. The second equation presupposes a natural, μ-measurable group action of the diffeomorphism group $\mathcal{K} = \mathrm{Diff}_0(M)$ on Δ, denoted by $(\phi, \gamma) \to \phi\gamma$, where μ is a measure on Δ having the important technical property of *quasiinvariance* under this group action. To be precise, this is actually a *right* group action, so that $[\phi_1\phi_2]\gamma = \phi_2(\phi_1\gamma)$. Quasiinvariance means that for all $\phi \in G$, the transformed measure μ_ϕ is absolutely continuous with respect to μ. This implies the existence of the Radon-Nikodym derivative $[d\mu_\phi/d\mu](\gamma)$ *almost everywhere* (a.e.) – i.e., outside of μ-measure zero sets.

Of course, to have such a measure μ, Δ must be a measurable space, endowed with a σ-algebra $\mathcal{B}_\Delta$ of "measurable" subsets which is closed under countable unions, countable intersections, and complements. We shall also need $\langle \gamma, f\rangle$ to be a measurable function of γ, for all $f \in \mathcal{D}$.

Now, in both equations (2), Ψ belongs to a Hilbert space $\mathcal{H}$, denoted $L^2_{d\mu}(\Delta, \mathcal{W})$, and defined to be the space of μ-measurable functions $\Psi(\gamma)$ on Δ, square-integrable with respect to μ, taking values in a complex inner product space $\mathcal{W}$. The inner product in $\mathcal{H}$ is given by

$$
(\Phi, \Psi) = \int_\Delta \langle \Phi(\gamma), \Psi(\gamma)\rangle_{\mathcal{W}}\, d\mu(\gamma)\,, \tag{3}
$$

where $\langle\ ,\ \rangle_{\mathcal{W}}$ denotes the inner product in $\mathcal{W}$. When $\mathcal{W} = \mathbb{C}$ (the complex numbers), equation (3) becomes simply $(\Phi, \Psi) = \int_\Delta \overline{\Phi(\gamma)}\Psi(\gamma)\, d\mu(\gamma)$; but when $\mathcal{W}$ is a higher-dimensional space, Ψ may have (finitely or infinitely many) components.

Finally, χ is a measurable, unitary 1-cocycle. This means that (for each $\phi \in \mathcal{K}$) χ_ϕ is a measurable function of $\gamma \in \Delta$ taking values in the group of unitary operators on $\mathcal{W}$; and, furthermore, satisfying for each $\phi_1, \phi_2 \in \mathcal{K}$ the cocycle equation,

$$
\chi_{\phi_1\phi_2}(\gamma) \;=\; \chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma) \;\; \text{a.e.}\,(\mu)\,. \tag{4}
$$

Note that the system of Radon-Nikodym derivatives $\alpha_\phi(\gamma) = [d\mu_\phi/d\mu](\gamma)$ is a measurable, positive real-valued cocycle, as is also $\alpha^{1/2}$. Let us remark that the failure sets for cocycle equations here may actually depend on ϕ_1 and ϕ_2 in such fashion that there is no measure zero set outside of which the equation holds for every pair of diffeomorphisms. The factor $\alpha^{1/2}$ in equation (2) is precisely what is needed to ensure that the representation is unitary; indeed, the fact that $V(\phi)$ preserves the inner product in $\mathcal{H}$ is demonstrated simply by making a change of variable in calculating $(V(\phi)\Phi, V(\phi)\Psi)$ using equations (2) and (3). Furthermore the action of the cocycle χ_ϕ in equation(2), being unitary in $\mathcal{W}$, does not alter the value of this inner product.

Unitarily inequivalent representations of G are now to be associated with inequivalent measures μ, and (for equivalent measures) with inequivalent (noncohomologous) cocycles χ.

The representation theory of the diffeomorphism group specified by the second equation in (2), viewed in this way, thus incorporates and unifies two features: (1) the class of possible quantum configuration spaces Δ equipped with quasiinvariant measures, describing the kinds of configurations for which there exists a consistent quantum theory on M (i.e., a consistent quantization of some classical motion in M), and (2) the 1-cocycles with respect to the action of the group $\mathrm{Diff}_0(M)$ on Δ, describing the possible quantum statistics of such configurations (in the generalized sense of statistics that includes exotic statistics).

Let us close this section by mentioning briefly some of the approaches to constructing configuration spaces that are pertinent to this description. More discussion of some of these may be found in earlier papers and the references therein [18, 23].

1. Systems of N *indistinguishable point particles* in M correspond to configuration spaces $\Gamma^{(N)}$ of finite (N-element) subsets of M. When M is noncompact, systems of *infinitely many* such point particles are described by configurations which are countably infinite but locally finite subsets of M, defining the space $\Gamma^{(\infty)}$. When $M = \mathbf{R}^d$, this is the usual configuration space for statistical mechanics [24, 25, 26, 27]. Of course, diffeomorphisms of M act on subsets of M in the obvious way; they do not create or destroy particles, but move them around in M.

2. General configuration spaces may be defined as orbits or unions of orbits (under the diffeomorphism group action) in the space $\mathcal{D}'$ of distributions over M (for $M = \mathbf{R}^d$, one also has the possibility of considering tempered distributions). Particle configurations, in particular, are associated with linear combinations of evaluation functionals (δ-functions) in this space. Coefficients of δ-functions may be interpreted as particle masses, allowing configurations of distinguishable as well as indistinguishable particles to be described in this way. Here diffeomorphisms of M act on $\mathcal{D}$ as specified by the semidirect product law in G, and on distributions by the dual action [20].

3. Letting N be a manifold (typically of lower dimension than M), a class of configuration spaces may be constructed as spaces of (not necessarily infinitely differentiable) embeddings (or, more generally, immersions) of N in M; let us write such a configuration as $\beta : N \to M$. For example, with $N = S^1$, we have configuration spaces of loops in M.

Such embeddings or immersions may be parametrized (so that the map β itself is the configuration), or unparametrized (so that the image set $[\beta]$ of β is the configuration; then $\beta_1 \sim \beta_2$ if they are related by a diffeomorphism of N). For $N = S^1$, we thus have the possibility of parametrized or unparametrized loops. If M is three-dimensional, we also have distinct configuration spaces for different kinds of knots. A prerequisite for the existence of measures on such spaces that are quasiinvariant under (C^∞) diffeomorphisms of M seems to be that the continuity class of β be fixed at a finite value. To the best of my knowledge, this theory is still incomplete.

4. General configuration spaces may be defined as spaces of *closed* subsets of M, as proposed and developed by Ismagilov; see [28] and references therein. Note that unparametrized embeddings or immersions of N in M are special cases of such closed subsets, while parametrized embeddings or immersions are not.
5. Still more general configuration spaces may be defined as spaces of *countable* subsets of M (without imposing the condition of local finiteness). This generalizes $\Gamma^{(\infty)}$, in that it allows for infinite-point configurations with accumulation points. It also generalizes Ismagilov's approach, in that (M being separable) a closed subset can be recovered as the closure of many distinct countable subsets (see [29] and references therein). Parametrized configurations require consideration of ordered countable subsets.
6. Consideration of the *coadjoint representation* of $\mathcal{K}$, or of the semidirect product group $G = \mathcal{D} \times \mathcal{K}$, suggests that one construct configuration spaces from the dual space to the corresponding (infinite-dimensional) Lie algebra – i.e., the dual space to the current algebra of compactly-supported scalar functions and vector fields on M. Then one needs to introduce a "polarization" (in the spirit of geometric quantization) in the corresponding coadjoint orbit or class of orbits, which amounts to selecting certain coordinates as "position-like" and others as "momentum-like" – with the former defining the quantum configurations. The additional (symplectic) structure on coadjoint orbits provides a systematic way to obtain cocycles in this context.
7. Finite or countably infinite subsets of *bundles* over M provide another approach to configuration spaces. For example, returning to configuration spaces in $\mathcal{D}'$, derivatives of δ-functions (including higher derivatives) are perfectly satisfactory configurations, and lead to quantum theories of point-like dipoles, quadrupoles, etc. [30]. However, these configurations belong not to M itself, but to the jet bundle over M, to which the action of diffeomorphisms on M lifts naturally.
8. Finally, in the spirit of the approach via bundles over M, there is a physically important generalization to what has been termed "marked configuration spaces." Here one identifies a compact manifold S describing the "internal degrees of freedom" of a particle, and a compact Lie group L that acts on S. One then associates to each point in an ordinary configuration a value or

"mark" in S [31, 32]. The local symmetry group itself can be correspondingly enlarged to include compactly supported C^∞ mappings from M to L under the pointwise Lie group operation, and/or to include bundle diffeomorphisms of $M \times S$.

Each of these methods of characterizing quantum configuration spaces has some significant literature that develops it, and in some instances is associated with a point of view about quantization or about quantum mechanics. The diffeomorphism group approach helps us understand these distinct but overlapping methods as techniques for the construction of classes of unitary group representations embodying the local symmetry of physical space in the quantum kinematics.

3. Induced representations and particle statistics

Next let us consider briefly the examples of Bose statistics, Fermi statistics, and parastatistics for N indistinguishable particles in $\mathbf{R}^d$, $d \geq 2$, and of anyonic statistics for N (distinguishable or) indistinguishable particles in $\mathbf{R}^2$.

The configuration space $\Gamma^{(N)}$ is the set of N-point subsets of $\mathbf{R}^d$; we write $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \in \Gamma^{(N)}$. The space $\Gamma^{(N)}$ is sometimes written in the more complicated way $[\mathbf{R}^{dN}\text{-}D]/S_N$; where $\mathbf{R}^{dN}$ is the set of ordered N-tuples $(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ of points in $\mathbf{R}^d$, D is the "diagonal" set of N-tuples for which $\mathbf{x}_i = \mathbf{x}_j$ for some $i \neq j$, and S_N is the symmetric group for N objects. Thus $\Gamma^{(N)}$ is the set of ordered N-tuples without repeated points, modulo all permutations of the values of the points. A diffeomorphism ϕ acts on $\Gamma^{(N)}$ by (the right action) $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \to \phi\gamma = \{\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N)\}$.

Note that for $d \geq 2$, $\Gamma^{(N)}$ is *multiply connected* – indeed, any continuous path in $\Gamma^{(N)}$ that begins at a configuration γ_0 and non trivially *permutes* the locations of the points in γ_0 forms a closed loop in the configuration space, based at γ_0, that cannot be continuously contracted to γ_0.

First let us consider $d \geq 3$. The fundamental group $\pi_1(\Gamma^{(N)})$, which is the group of distinct homotopy classes of such loops (under composition), is then just isomorphic to S_N, according to the particular permutation of the locations of the points in γ_0 implemented by a loop based there. The *universal covering space* $\tilde{\Gamma}^{(N)}$ is then the space of *ordered* N-tuples without repeating points; i.e., $\tilde{\Gamma}^{(N)} = [\mathbf{R}^{dN}\text{-}D]$, and we shall write $\tilde{\gamma} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \tilde{\Gamma}^{(N)}$. Then we have the projection $p : \tilde{\Gamma}^{(N)} \to \Gamma^{(N)}$ from the universal covering space to the base space, given by $p(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$; i.e., p tells us to "forget the ordering." There are, of course, $N!$ sheets in $\tilde{\Gamma}^{(N)}$ (for $d \geq 3$), corresponding to the $N!$ elements of the fundamental group S_N.

Consider next the action of $\mathcal{K} = \mathrm{Diff}_0(\mathbf{R}^d)$ on $\Gamma^{(N)}$. The stability subgroup $\mathcal{K}_\gamma \subset \mathcal{K}$ consists of those compactly-supported diffeomorphisms which leave γ fixed; i.e., just those which permute the points in γ. Thus $\mathcal{K}_\gamma$ contains $N!$ disconnected components, and we obtain a natural homomorphism h_γ from $\mathcal{K}_\gamma$ to S_N. Referring back to equations (2) and (4), observe that when ϕ_1 and ϕ_2 belong to

$\mathcal{K}_\gamma$, the cocycle equation at γ becomes a unitary representation in $\mathcal{W}$ of $\mathcal{K}_\gamma$. Thus we have an association between cocycles describing quantum theories modeled on $\mathbf{R}^d$ ($d \geq 3$) and unitary representations of $\mathcal{K}_\gamma$. Note too that any unitary representation π of S_N in the inner product space $\mathcal{W}$ now gives us a continuous unitary representation $\pi \circ h_\gamma$ of $\mathcal{K}_\gamma$ in $\mathcal{W}$. Cocycles describing quantum theories of Bose statistics, Fermi statistics, and parastatistics correspond in this way to inequivalent representations of S_N: the trivial (Bose) and alternating (Fermi) one-dimensional representations (for $\mathcal{W} = \mathbb{C}$), and additional (para) higher-dimensional representations described by Young tableaux (with $\mathcal{W} = \mathbb{C}^n$).

The corresponding unitary representations of $\mathcal{D} \times \mathcal{K}$ can be obtained in a different way, making use of a generalization of Mackey's theory of induced representations. The action of $\phi \in \mathcal{K}$ on $\Gamma^{(N)}$ lifts naturally to an action $\tilde{\phi}$ on the universal covering space $\tilde{\Gamma}^{(N)}$, so that $\phi(p\tilde{\gamma}) = p\tilde{\phi}(\tilde{\gamma})$. Diffeomorphisms belonging to $\mathcal{K}_\gamma$, in their action on $\tilde{\Gamma}^{(N)}$, now *permute* the elements of $p^{-1}\gamma$. In the induced representation approach, the Hilbert space consists of wave functions on $\tilde{\Gamma}^{(N)}$ that are *equivariant* with respect to the given unitary representation of the fundamental group S_N, and thus with respect to the corresponding unitary representation of $\mathcal{K}_\gamma$ in its action on $\tilde{\Gamma}^{(N)}$.

In short, for $d \geq 3$, we see how the topology of the N-particle configuration spaces in $\mathbf{R}^d$ gives rise to the possible exchange statistics of indistinguishable particles in the representation theory of the group of diffeomorphisms of $\mathbf{R}^d$. Corresponding to the unitary representations of the fundamental group of Δ are inequivalent cocycles for the action of $\mathrm{Diff}_0(M)$ on Δ, and different equivariance conditions for wave functions written on the universal covering space of Δ.

Finally, consider the case $d = 2$. The fundamental group $\pi_1(\Gamma^{(N)}(\mathbf{R}^2))$ is larger than S_N, because loops based at a configuration γ_0 that implement (let us say) a clockwise exchange of two points of γ_0 in $\mathbf{R}^2$ are not homotopically equivalent to loops that implement a counterclockwise exchange of the same two points. Here, the fundamental group is Artin's braid group B_N, an infinite discrete group which for $N > 2$ is nonabelian. One may think of the braid group element b_j, for $j = 1, \ldots, N-1$, as exchanging the pair of points $\mathbf{x}_j, \mathbf{x}_{j+1}$ (which are *adjacent* with respect to some coordinatization of the plane), in a counterclockwise direction; the element b_j^{-1} then exchanges the same pair of points in a clockwise direction. The group B_N is the free group generated by these elements, *modulo* the equivalence relation $b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1}$.

Now the space of ordered N-tuples of points in the plane is a covering space of $\Gamma^{(N)}(\mathbf{R}^2)$, but it is no longer the *universal* covering space; the latter has infinitely many sheets. Ultimately wave functions on the universal covering space, equivariant with respect to a unitary representation of the braid group, define the Hilbert space for the desired representation of G.

We omit further details, but close this section by focusing on a key step in this induced representation construction for anyons, which we shall then indicate how to generalize to configurations of extended objects. This step is the association of the connected components of the stability subgroup $\mathcal{K}_\gamma$ (i.e., the subgroup of compactly supported diffeomorphisms of the plane that leave fixed the subset of points $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$) with elements of the fundamental group B_N, by means of a homomorphism h_γ.

One way to define this homomorphism, described in Ref. [33], is as follows. Choose an arbitrary direction in the plane M, let us say for specificity the y-direction with respect to Cartesian coordinate axes x and y, such that for the points in the configuration γ no two x-coordinates coincide. Index the points $\mathbf{x}_j$ in order of increasing x-coordinate value. Attach to each point in γ a *strand* which is a straight line extending to infinity in the negative y-direction; the parallel strands in this set of strands do not intersect. Now a compactly-supported diffeomorphism ϕ in the stability subgroup of γ leaves all of the strands fixed at infinity (because of the compact support of ϕ), but can permute their terminal points. Still more generally, the images of the strands of under ϕ constitute a *new* set of non-intersecting strands coming in from $y = -\infty$ and terminating at the points in γ. This set of strands may be homotopically inequivalent to the original set, *even when* $\phi(\mathbf{x}_j) = \mathbf{x}_j$ *for all* j; i.e., even when ϕ implements no permutation of the points.

In fact, such sets of strands fall into distinct homotopy classes, encoding the passages of strands above or below each other (with respect to the coordinate y) as one moves in from $y = -\infty$ to the points of γ. When no such passage occurs, we map ϕ to the identity element of B_N. When the only such passage is that the strand terminating in $\mathbf{x}_{j+1}$ passes once above the strand terminating in $\mathbf{x}_j$, we map the diffeomorphism to b_j. In this way, the stability subgroup $\mathcal{K}_\gamma$ is mapped homomorphically to B_N.

Then a unitary representation of B_N in a space $\mathcal{W}$ immediately implements a continuous unitary representation of $\mathcal{K}_\gamma$, which induces the desired representation of G. In short, all the needed information about braiding in encoded in the compactly supported diffeomorphism belonging to the stability subgroup. The one-dimensional representations of B_N, in which b_j is represented by $\exp i\theta$, describe anyons; while the higher-dimensional representations describe nonabelian anyons.

One draws certain physical inferences immediately from the above construction.

First, it is not a prerequisite for intermediate statistics in the plane that there be a hard core potential excluding two or more particles from occupying the same position in M, any more than such a potential is required for ordinary Bose or Fermi statistics. Diffeomorphisms act transitively on the configuration space $\Gamma^{(N)}$, and cannot bring distinct points into coincidence. Thus configuration spaces from which the diagonal D is not excluded may be written as the union of mutually disjoint orbits under the group action, and the corresponding possible irreducible unitary representations still include those that are anyonic.

Secondly, it is not a prerequisite for exotic statistics of particles in the plane that they be indistinguishable. The configuration space of *ordered* N-tuples of points in the plane, excluding N-tuples with coincident points, is still multiply-connected. Its fundamental group is the group of "colored braids." Correspondingly, given such a configuration, the elements ϕ of $\mathcal{K}$ for which $\phi(\mathbf{x}_j) = \mathbf{x}_j$ for all j form a closed subgroup. Elements of this subgroup map the original set of parallel strands (from $y = -\infty$, terminating at the points $\mathbf{x}_j$) to various non-homotopic sets of strands from $y = -\infty$ terminating at the same points.

4. Exotic statistics for extended configurations

The ideas in the preceding sections generalize to consideration of topologically nontrivial configurations in higher-dimensional physical spaces. Let us consider just a couple of examples.

Suppose that Δ is a configuration space whose elements are unparametrized single oriented loops in (for specificity) $\mathbf{R}^3$; i.e., a configuration $\gamma \in \Delta$ is a continuous embedding $[\beta]$ of S^1 (modulo C^∞ reparametrization) of some smoothness class, for which (let us say) the arc length in the target space is defined. Diffeomorphisms of $\mathbf{R}^3$ act on Δ in the obvious way. We remark that we shall not be able to concentrate a quasiinvariant measure on a single orbit under $\mathcal{K}$, but will need an uncountable family of orbits. Nevertheless, we envision being able to infer exotic statistics by selecting configurations from such a family of orbits in a measurable way, and describing topological invariants across orbits of the way diffeomorphisms act on such sets of loops.

For a particular oriented loop γ, consider the stability subgroup $\mathcal{K}_\gamma$. An element $\phi \in \mathcal{K}_\gamma$ leaves the loop invariant as a set, but not necessarily pointwise. Thus there is a homomorphism h_γ that maps $\mathcal{K}$ to $\mathrm{Diff}(S^1)$, with $h_\gamma(\phi)$ specified straightforwardly by looking at how ϕ transforms γ (parametrized by its own arc length). A unitary representation of $\mathrm{Diff}(S^1)$ may then describe the "internal statistics" of γ. This is, in a sense, analogous to the ordinary statistics of particles – an equivariance condition for wave functions can be written that depends only on γ and $\phi\gamma$.

But ϕ encodes still more information. If we introduce a set of continuous, non-selfintersecting strands that become parallel (say, for specificity, on the surface of a circular cylinder) in some fixed direction at infinity, and that terminate at correspondingly ordered points of γ, we see that because ϕ is compactly supported, its action on these strands keeps track of how many times it has, in effect, *rotated* the loop. The stability subgroup thus maps not just to $\mathrm{Diff}(S^1)$, but to a *covering group* of $\mathrm{Diff}(S^1)$. "Bringing the loop in from infinity" (and watching what $\phi \in \mathcal{K}_\gamma$ does) tells us how many windings ϕ is to be associated with. Diffeomorphisms that leave every point of γ fixed still encode the number of rotations, and we have the possibility of introducing an extra, additional phase for a single directed rotation of γ. The loop thus can have internal "intermediate statistics."

If instead of a loop γ is an embedded torus (the continuous image of $S^1 \times S^1$) of some smoothness class, the same idea allows us to associate a pair of winding numbers with a compactly-supported diffeomorphism that leave the torus pointwise fixed. Thus we infer further possibilities for intermediate statistics, associating distinct phases with each directed rotation.

Next consider a configuration γ that is the union of a point and an embedded, oriented loop. Imagine a non-selfintersecting cylindrical membrane from infinity (in some fixed direction) that is bounded by the loop, and a strand from infinity in the same direction, intersecting neither itself nor the membrane. Let us say, for specificity, that at infinity the strand is inside the cylinder, terminating at the point particle inside the cylinder. Now, consider the image of this strand-cylinder combination under the action of $\phi \in \mathcal{K}_\gamma$. The image of the strand may now pass through and around the loop as the image of the membrane is moved to one side, so that the strand finally reaches the particle from "outside" the membrane. All such topological complexity takes place within the compact support of ϕ; outside of this support, the original strand and cylinder are fixed. We see from the homotopy class of this image that ϕ encodes the net number of times the particle passes *through* the oriented loop, and again we can have an arbitrary associated phase.

Finally, consider a configuration γ that is the union of a pair of oriented loops in $\mathbf{R}^3$; the discussion will readily extend to pairs of closed filaments of vorticity, vortex rings, or tori. Now we envision two non-intersecting and non-selfintersecting membranes extending to infinity in a fixed direction, bounded by the respective loops. Suppose that a compactly-supported diffeomorphism $\phi \in \mathcal{K}_\gamma$ exchanges the loops. The homotopy class of the pair of image membranes is now labeled by the sequence of passages of one loop through the other. The diffeomorphism encodes "leapfrogging" as a sequence of such passages. The condition of equivariance of the wave function on configuration-space with respect to a unitary representation of $\mathcal{K}_\gamma$ can associate (in particular) a phase with each such passage, leading again to anyonic statistics.

In conclusion, the idea of describing quantum systems by means of continuous unitary representations of the infinite-dimensional group $G = \mathcal{D} \times \mathcal{K}$ leads to a unifying kinematical description of interesting quantum configuration spaces and associated possibilities for exotic statistics.

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