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EXTREMUM PROBLEMS WITH INEQUALITIES AS SUBSIDIARY CONDITIONS

Fritz John

This paper deals with an extension of Lagrange's multiplier rule to the case, where the subsidiary conditions are inequalities instead of equations. Only extrema of differentiable functions of a finite number of variables will be considered. There may however be an infinite number of inequalities prescribed. Lagrange's rule for the situation considered here differs from the ordinary one, in that the multipliers may always be assumed to be positive. This makes it possible to obtain sufficient conditions for the occurence of a minimum in terms of the first derivatives only.

Two geometric applications will be discussed here. From the point of view of applications it would seem desirable to extend the method used here to cases, where the functions involved are not necessarily differentiable, or where they do not depend on a finite number of independent variables.

1. Necessary conditions for a minimum.

Let R be a set of points x in a space E, and F(x) a real-valued function defined in R. We consider a subset R' of R, which is described by a system of inequalities with parameter y:

(1) $$G(x,y) \geqq 0,$$

where G is a function defined for all x in R and all "values" of the parameter y. There may be a finite or infinite number of these inequalities. To gain sufficient generality we assume that the "values" of the parameter y vary over a set of points S in a space H. Then $G(x,y)$ is defined in the set $R \times S$. We are interested in conditions a point x^o of R' has to satisfy in order that

$$M = F(x^o) = \underset{x \subset R'}{\text{Minimum}}\, F(x) \tag{2}$$

In what follows we restrict ourselves to the case, where the space E containing the set R is the n-dimensional euclidean space E_n, and where the set S of parameter values y is a compact set in a metric space H. We make the further assumptions that $F(x)$ and $G(x,y)$ have first derivatives F_i and G_i with respect to the coordinates x_i of the point $(x_1,\ldots,x_n) = x$, and that $F(x)$, $G(x,y)$, $F_i(x)$, $G_i(x,y)$ are continuous functions of (x,y) in $R \times S$.[1)]

Given a function $\phi(x)$ with continuous derivatives $\phi_i(x)$ in R, we denote by

$$\phi'(x,z) = \sum_{i=1}^{n} \phi_i(x) z_i \tag{3}$$

the differential of the function. $\phi'(x,z)$ is then defined for all $x \subset R$ and $z = (z_1,\ldots,z_n) \subset E_n$, and is linear in z.

Theorem I.

Let x^o be an interior point of R, and belong to the set R' of all points x of R, which satisfy (1) for all $y \subset S$. Let $F(x^o) = \underset{x \subset R'}{\text{Minimum}}\, F(x)$.

Then there exists a finite set of points $y^1,\ldots,y^s$ in S and numbers $\lambda_o, \lambda_1,\ldots,\lambda_s$, which do not all vanish, such that

$$G(x^o,y^r) = o \text{ for } r = 1,\ldots,s \tag{4a}$$

1) Here continuity in $R \times S$ is defined so as to agree with the following definition of convergence in $R \times S$: $\lim_{r\to\infty}(x^r,y^r) = (x,y)$, if $\lim_{r\to\infty} x^r = x$ and $\lim_{r\to\infty} y^r = y$.

(4b) $\lambda_o \geqq 0, \quad \lambda_1 > 0, \ldots, \quad \lambda_s > 0$

(4c) $0 \leqq s \leqq n$

(4d) the function

$$\Phi(x) = \lambda_o F(x) - \sum_{r=1}^{s} \lambda_r G(x, y^r)$$

has a critical point at x^o i.e.

$$\Phi_i(x^o) = o \quad \text{for } i = 1, \ldots, n .$$

Proof:

Let S' denote the subset of points y of S, for which

$$G(x^o, y) = 0.$$

We shall first show that the system of inequalities

(5a) $F'(x^o, z) < 0$

(5b) $G'(x^o, z, y) > 0 \quad$ for all $y \subset S'$

can have no solution $z = (z_1, \ldots, z_n)$.

For let (5a,b) be satisfied for a certain z. Denote by S'_ε the set of all points of S having a distance $\leqq \varepsilon$ from sone point of S', and by X^o_ε the set of all points of R having a distance $\leqq \varepsilon$ from x^o. Then there exist positive numbers δ, ε such that

(6) $F'(x,z) < -\delta, \quad G'(x,z,y) > \delta \quad$ for all $x \subset X^o_\varepsilon$, $y \subset S'_\varepsilon$.

For otherwise there would exist sequences of points x^r in R, y^r in S, η^r in S', such that

$$\lim_{r\to\infty} x^r = x^o, \quad \lim_{r\to\infty} (\text{distance of } y^r \text{ and } \eta^r) = 0$$

and either

$$\liminf_{r\to\infty} F'(x^r, z) \geqq 0$$

or

$$\limsup_{r\to\infty} G'(x^r, z, y^r) \leqq 0.$$

As S is compact and G is continuous, S' is compact as well. We can then form a suitable subsequence of the r, such that y^r and η^r converge towards a point y of S'. As F' and G' are continuous, it would follow that either $F'(x^0,z) \geqq 0$ or $G'(x^0,z,y) \leqq 0$, contrary to (5a,b).

Hence (6) holds for suitable positive ε, δ. On the other hand there exists a positive constant $\mu = \mu(\varepsilon)$ such that

(7) $$G(x^0,y) > \mu$$

for all y of S outside S'. For $G(x^0,y)$ is non-negative in S (as $x^0 \subset R'$), vanishes only on S', and is continuous on the compact set S.

As x^0 is an interior point of R, we have for sufficiently small positive t

$$F(x^0 + tz) = F(x^0) + tF'(x^0 + \theta tz, z)$$

$$G(x^0 + tz,y) = G(x^0,y) + tG'(x^0 + \theta tz,z,y)$$

where θ stands for any quantity between 0 and 1. If here t is chosen so small that

$$t\sqrt{\sum_i z_i^2} < \varepsilon, \qquad t\cdot\underset{\substack{y \subset S \\ x \subset X^0_\varepsilon}}{\text{Maximum}} |G'(x,z,y)| < \mu ,$$

we can apply (6), (7) and find that

$$F(x^0 + tz) \leqq F(x^0) - t\delta < F(x^0)$$

$$G(x^0 + tz,y) \geqq G(x^0,y) + t\delta > 0 \quad \text{for all } y \subset S'_\varepsilon$$

$$G(x^0 + tz,y) \geqq \mu - t|G'(x^0+\theta tz,z,y)| > 0 \text{ for all } y \text{ of } S \text{ outside } S'_\varepsilon$$

This would however contradict the assumed minimum property of x^0. Consequently, there can be no z satisfying (5a,b).

The non-existence of a solution z of the system of linear homogeneous inequalities (5a,b) can be seen to be equivalent to the existence of non-negative solutions of a certain system of equations.[2] For this purpose we introduce the "representative" points corresponding to (5a,b), i.e. the points in n-space given respectively by

$$(8)\qquad \begin{aligned} q &= (-F_1(x^o),\ldots,-F_n(x^o)) \\ p_y &= (G_1(x^o,y),\ldots,G_n(x^o,y)) \quad \text{for } y \subset S'. \end{aligned}$$

The non-existence of a solution z of (5a,b) implies that the set Σ consisting of q and all p_y does not lie in an open half-space bounded by a hyper-plane through the origin. Then the origin is a point of the convex hull of Σ. As in addition, as a consequence of our assumptions, Σ is closed and bounded, it follows that the origin belongs to a simplex with vertices in Σ, where the point q may be chosen as one of the vertices.[3] Then the origin is center of mass of $s+1$ non-negative masses ($s \leq n$), located in q and s other points of Σ. Equations (4a,b,c,d) are the analytic expression for this fact.

2. Sufficient conditions for a minimum. Equivalence with finite systems of inequalities.

Theorem II.

Let x^o be an interior point of R and belong to the set R' of all points x of R, which satisfy

$$G(x,y) \geqq 0 \quad \text{for all } y \subset S.$$

Let there exist a function $\Phi(x)$ of the form

$$\Phi(x) = \lambda_o F(x) - \sum_{r=1}^{s} \lambda_r G(x,y^r)$$

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2) See L. L. Dines: "Linear inequalities," Bull. Am. Math. Soc. vol. 42 (1936), pp. 353-365. R. W. Stokes: "A geometric theory of linear inequalities," Trans. Am. Math. Soc. vol. 33 (1931), pp. 782-805.

3) That one of the vertices can be chosen arbitrarily in Σ, is evident from the proof of the fundamental theorem that any point of the convex hull of Σ belongs to a simplex with vertices in Σ. See Bonnesen-Fenchel: "Theorie der konvexen Körper," p. 9.

where $y^r \subset S$, such that (4a,b,d) hold. Let in addition the matrix

$$A = \begin{pmatrix} \lambda_o F_1(x^o) & G_1(x^o,y^1) & G_1(x^o,y^2) \ldots\ldots G_1(x^o,y^s) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \lambda_o F_n(x^o) & G_n(x^o,y^1) & G_n(x^o,y^2) \ldots\ldots G_n(x^o,y^s) \end{pmatrix}$$

have rank n.

Then F(x) has a relative minimum at x^o in the set defined by the finite number of inequalities

$$(9) \qquad G(x,y^r) \geqq 0 \qquad \text{for } r = 1,\ldots,s$$

and has, a fortiori, a relative minimum at x^o for the set R'.

Proof:

If F(x) did not have a relative minimum at x^o for the set (9), we could find a sequence of positive numbers t_r and a set of points $z^r \subset E_n$, such that

$$\lim_{r\to\infty} t_r \quad 0, \qquad \sum_{i=1}^{n} (z^r_i)^2 = 1$$

$$F(x^o + t_r z^r) < F(x^o)$$

$$G(x^o + t_r z^r, y^k) \geqq 0 \quad \text{for } k = 1,\ldots,s.$$

Then with suitable θ between 0 and 1

$$F'(x^o + \theta t_r z^r, z^r) \leqq 0$$

$$G'(x + \theta t_r z^r, z^r, y^k) \geqq 0 \quad \text{for } k = 1,\ldots,s.$$

For a suitable subsequence the z^r converge towards a vector $z \neq 0$, for which then

$$F'(x^o,z) \leqq 0$$

$$G'(x^o,z,y^k) \geqq 0 \quad \text{for } k = 1,\ldots,s.$$

But, as $\phi(x)$ is stationary at x^o, we have

$$0 = \phi'(x^o,z) = \lambda_o F'(x^o,z) - \sum_{k=1}^{s} \lambda_k G'(x^o,z,y^k).$$

Hence, making use of (4b), we see that z satisfies the system of linear homogeneous equations

$$\lambda_o F'(x^o,z) = 0, \qquad G'(x^o,z,y^k) = 0 \quad \text{for } k = 1,\ldots,s.$$

The existence of a solution $z \neq 0$ of this system contradicts however the assumption made on the rank of A. This completes the proof of theorem II.

Theorem II shows that under suitable conditions a relative minimum of F(x) for the set R' determined by infinitely many inequalities is at the same time a relative minimum for the set determined by a suitable finite number of these inequalities. An example will show that this is not the case for every minimum problem. Consider the problem of finding the minimum of the function

$$F(x) = -x^2$$

in the set of all x satisfying the inequalities

(10) $G(x,y) = y^2 - yx^2 \geqq 0$ for all y with $0 \leqq y \leqq 1$.

The set R' of all x satisfying (10) consists of the point $x = 0$. That point then is also a relative minimum point of F for the set. If, on the other hand, x is only subjected to a finite number of inequalities

$$G(x,y^k) \geqq 0 \quad \text{for } k = 1,\ldots,s,$$

where $0 \leqq y^k \leqq 1$, then all points of a neighbourhood of $x = 0$ are admitted, and F(x) has no relative minimum in the resulting set at 0.

3. Application to minimum sphere containing a set.[4]

Let S be a bounded set in E_m. A sphere in E_m may be described

[4] See H. W. E. Jung: "Ueber die kleinste Kugel, die eine räumliche Figur einschliesst," Journal für die reine und angewandte Mathematik, vol. 123 (1901), pp. 241-257. For a historical account of this well knwon problem see the paper by L. M. Blumenthal and G. E. Wahlin: "On the spherical surface of smallest radius enclosing a bounded subset of n-dimensional euclidean space," Bull. Am.

by $x = (x_1, \ldots, x_{m+1})$, where $x_1, \ldots, x_m$ are the coordinates of its center and x_{m+1} the square of its radius. Let x^0 denote the sphere of least radius enclosing S. The existence of a sphere of least positive radius enclosing S is evident, if the assumption is made that S contains at least two distinct points.

Then

(11a) $$F(x) = x_{m+1}$$

has a minimum for $x = x^0$ in the set of all x satisfying the inequalities

(11b) $$G(x,y) = x_{m+1} - \sum_{i=1}^{m} (x_i - y_i)^2 \geqq 0 \quad \text{for all } y \subset S.$$

As every sphere containing S also contains the closure S of S, we can replace S by $\bar{S}$ in (11b).

According to theorem I we can find s points $(s \leqq m+1)$ $y^1, \ldots, y^s$ of S and numbers $\lambda_0, \ldots, \lambda_s$ such that

(12a) $$\lambda_0 = \sum_{r=1}^{s} \lambda_r$$

(12b) $$\sum_{r=1}^{s} \lambda_r (x_i^r - y_i^r) = 0 \quad \text{for } i = 1, \ldots, m$$

(12c) $$x_{m+1}^0 - \sum_{i=1}^{m} (x_i^0 - y_i^r)^2 = 0 \quad \text{for } r = 1, \ldots, s$$

(12d) $$\lambda_0 \geqq 0, \quad \lambda_1 > 0, \ldots, \lambda_s > 0.$$

It follows from (12a,d) that $\lambda_0 > 0$. From (12) we get for any $x = (x_1, \ldots, x_{m+1})$

$$\sum_{r=1}^{s} \lambda_r \left(x_{m+1} - \sum_{i=1}^{m} (x_i - y_i^r)^2 \right) = \lambda_0 \left(x_{m+1} - x_{m+1}^0 - \sum_{i=1}^{m} (x_i - x_i^0)^2 \right)$$

Math. Soc., vol. 47 (1941), pp. 771-777.

This identity shows that any sphere containing the points $y^1,\dots,$ y^s has a radius $\geqq \sqrt{x^o_{m+1}}$, where the = sign only holds, if its center is also at $(x^o_1,\dots,x^o_m)$. Hence the smallest sphere containing S is uniquely determined, and is at the same time the smallest sphere containing the points $y^1,\dots,y^s$ of the closure of S.

If D_{rt} denotes the distance of the points y^r and y^t $(r,t = 1,\dots,s)$, we have from (12)

$$\sum_{r\neq t} \lambda_r \lambda_t D^2_{rt} = \sum_{r,t,i} \lambda_r\lambda_t\left[(x^o_i-y^r_i)-(x^o_i-y^t_i)\right]^2 = 2\lambda^2_o x^o_{m+1}$$

On the other hand

$$\sum_{r\neq t} \lambda_r\lambda_t = \frac{s-1}{s}\left(\sum_r \lambda_r\right)^2 - \frac{1}{2s}\sum_{r,t}(\lambda_r-\lambda_t)^2 \leqq \frac{s-1}{s}\lambda^2_o .$$

Dividing the last two inequalities by each other and observing that the $\lambda_r\,\lambda_t$ are positive, it follows that

$$\text{(13)}\qquad D = \text{diameter of } S \geqq \underset{r,t}{\text{Maximum}}\; D_{rt} \geqq \sqrt{\frac{2s}{s-1}\,x^o_{m+1}}$$

As $s \leqq m+1$, this leads to "Jung's inequality"

$$\text{(14)}\qquad D \geqq \sqrt{\frac{2(m+1)}{m}}\; R$$

between the diameter D of a set S in E_m and the radius R of the smallest sphere containing the set.[5)]

This result can be extended in various directions. Following L. A. Santaló,[6)] we can consider a set S, which lies on the surface K of the unit-sphere in E_m and is contained entirely in a closed subset interior to a hemi-sphere of K.

We consider the set $y^1,\dots,y^s$ belonging to S through (12).

5) See Jung, l.c., note 4.
6) "Convex regions on the n-dimensional spherical surface," Annals of Mathematics, vol. 47 (1946), pp. 448-459.

If $y^1,\ldots,y^s$ do not lie in a hyper-plane of E_m, then K is the smallest sphere containing S, for the y^r lie on the smallest sphere and lie only on one sphere. This however is impossible, as S lies in a closed subset interior to a hemisphere of K, and hence is certainly contained in spheres of radius < 1.

Consequently $y^1,\ldots,y^s$ must lie in an (m-1)-dimensional linear space. Then however the inequality (18) between diameter of the set of the y^r and the radius of the least sphere containing the y^r applies with m replaced by m-1. As the least sphere containing the y^r is identical with the one containing S, we have for S

(15) $$D \geqq \sqrt{\frac{2m}{m-1}}\, R .$$

We can introduce the "spherical diameter" Δ of S as the least upper bound of the lengths of the greatcircle arcs on K joining any two points of S. Then obviously

$$D = 2 \sin \frac{\Delta}{2} .$$

Similarly we can introduce the "spherical radius" ρ of the least "spherical" (m-1)-dimensional sphere on K containing S. Obviously

$$R = \sin \rho .$$

We then obtain form (15), as analogue of (14) in (m-1)-dimensional spherical space of curvature 1, the inequality

$$2 \sin \frac{\Delta}{2} \geqq \sqrt{\frac{2m}{m-1}}\, \sin \rho$$

or

(16) $$\cos \Delta \leqq \frac{m \cos^2 \rho - 1}{m-1} .$$

This inequality is the best possible one between ρ and Δ as is seen from the example of a set S on K consisting of the vertices of an m-dimensional regular simplex.[7)]

In a different direction an obvious extension of (14) to Hilbert space suggests itself for $m \to \infty$:

If S is a set in Hilbert-space with the property that any two points of S have a distance $\leq D$, then there exists a point in Hilbert space, from which all points of S have a distance $\leq \frac{1}{\sqrt{2}} D$.

For a proof of this statement one forms the projection S_n of S on the $x_1 \ldots x_n$-coordinate plane. It is easily seen that the center

$$(x_1^n, x_2^n, \ldots, x_n^n, 0, 0, \ldots)$$

of the smallest sphere containing S_n converges for $n \to \infty$ towards a point of Hilbert space with the desired properties.

The constant $\frac{1}{\sqrt{2}}$ is again the best possible one in this connestion, as is shown by the example of the set consisting of the points (1,0,0,...),(0,1,0,....),(0,0,1,0,....), etc.

4. Application to the ellipsoid of least volume containing a set S in E_m.[8)]

A solid ellipsoid in running coordinates $y_1, \ldots, y_m$ may be described by a relation

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7) Santaló l.c. obtains an inequality, which in appearance is stronger than (16) for $\Delta > \pi/2$. The explanation of this discrepancy must lie in the fact that he uses a different definition of "spherical diameter" from the one used here. (No definition of that term is given in his paper.) For sets of spherical diameter $> \pi/2$ (as used here) the diameter of the set need not be the same, as that of its "spherical convex hull," whereas they seem to be the same in Santalo's use of the term.

8) Related questions have been considered for $m = 2$ by F. Behrend: "Ueber einige Affininvarianten konvexer Bereiche", Math. Ann., vol. 113 (1937), pp. 713-747; "Ueber die kleinste umbeschriebene und die grösste einbeschriebene Ellipse eines konvexen Bereiches", ibid., vol. 115 (1938), pp. 379-411; F. John: "Moments of inertia of convex regions", Duke Math. J., vol. 2 (1936), pp. 447-452;

for $m = 3$ by O. B. Ader: "An affine invariant of convex regions", Duke Math. J. vol. 4 (1938) pp. 291-299; for general m by F. John: "An inequality for convex bodies", U. of Kentucky Research Club

(17) $$\sum_{i,k=1}^{m} x_{ik}(y_i-x_i)(y_i-x_i) \leqq 1$$

where

(18) $$x_{ik} = x_{ki}$$

and the x_{ik} are coefficients of a positive definite quadratic form. The volume of the ellipsoid is given by

$$V = \frac{\omega_m}{\sqrt{d}} ,$$

where ω_m denotes the volume of the unit-sphere in E_m and

$$d = \det(x_{ik}).$$

If the assumption is made that S is not contained in any hyper-plane, the existence of an ellipsoid of least volume containing S can be seen as follows. There is a sphere of radius $r > 0$ contained in the convex hull of S, and hence contained in any ellipsoid, which contains S. Thus any ellipsoid containing S contains the sphere of radius r about the center of the ellipsoid. We have then for any x_{ik}, x_i satisfying (17) for all $y \subset S$

$$\underset{\sum_i u_i^2 = r^2}{\text{Maximum}} \sum_{i,k} x_{ik} u_i u_k \leqq 1$$

As the x_{ik} are also coefficients of a definite form, it follows that

$$|x_{ik}| \leqq \frac{1}{r^2} .$$

Thus the x_{ik} satisfying (17) for all $y \subset S$ form a bounded set. Moreover we have for those x_{ik}, x_i

$$\lim V = \infty \qquad \text{for } (x_1, \ldots, x_m) \to \infty ,$$

as the ellipsoid contains the convex hull of S and the point $(x_1,$

Bull. 8 (1942), pp. 8-11.

$\ldots, x_m$). Consequently there exists a set x^o_{ik}, x^o_i, for which V is a minimum, among all x_{ik}, which satisfy (18) and (17) for all $y \subset S$, and for which the x_{ik} are coefficients of a positive definite form.[9)]

We are here again more interested in deriving significant properties of the minimum ellipsoid than in actually "determining" it in terms of S.

As V and -d take their least value simultaneously, we can conclude from theorem I that there exist non-negative constants λ_o, $\ldots$, λ_s, which do not all vanish, such that the function

$$(18) \qquad \Phi(x) = \lambda_o d + \sum_{r=1}^{s} \lambda_r \Big[1 - \sum_{i,k} x_{ik}(y^r_i - x_i)(y^r_k - x_k)\Big]$$

of the $n = \frac{m(m+3)}{2}$ independent variables

$$x_i \qquad (i = 1, \ldots, m)$$

$$x_{ik} \qquad (i,k = 1, \ldots, m; \quad i \leqq k)$$

has a critical value at x^o. Here $y^1, \ldots, y^s$ are points on the boundary of the convex hull of S, for which

$$\sum_{i,k} x^o_{ik}\,(y^r_i - x^o)(y^r_k - x^o) = 1.$$

As $\Phi(x)$ is symmetric in x_{ik} and x_{ki}, the first derivatives of Φ with respect to the x_i $(i = 1, \ldots, m)$ and all x_{ik} $(i,k = 1, \ldots, m)$ must vanish at the critical point. We may apply an affine transformation to E_m, so that the minimum ellipsoid becomes the unit sphere about the origin:

$$x^o_{ik} = \delta_{ik}\,, \qquad x^o_i = 0.$$

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9) For a minimizing sequence the x_{ik} cannot tend towards the coefficients of a non-definte form, as the determinant d of the x_{ik} has to become a maximum, and hence is bounded away from 0.

As

$$\left(\frac{\partial d}{\partial x_{rt}}\right)_{x_{ik} = \delta_{ik}} = \delta_{rt} ,$$

we obtain the following relations:

(19a) $$\lambda_o \delta_{ik} = \sum_{r=1}^{s} \lambda_r y_i^r y_k^r \quad \text{for } i,k = 1,\ldots,m$$

(19b) $$0 = \sum_{r=1}^{s} \lambda_r y_i^r \quad \text{for } i = 1,\ldots,m$$

(19c) $$\lambda_o \geqq 0, \ \lambda_1 > 0, \ldots, \lambda_m > 0$$

(19d) $$\sum_{i=1}^{m} (y_i^r)^2 = 1 \quad \text{for } r = 1,\ldots,s$$

Summing (19a) over all $i = k$, we obtain from (19d) the relation

(19e) $$m\lambda_o = \sum_{r=1}^{s} \lambda_r ,$$

which shows that λ_o is positive. It follows from (19) for any ellipsoid containing the points y^r

$$m\lambda_o = \sum_r \lambda_r \geqq \sum_r \lambda_r \left(\sum_{i,k} x_{ik}(y_i^r - x_i)(y_k^r - x_k)\right)$$

$$= \lambda_o \sum_i x_{ii} + m \sum_{i,k} x_{ik} x_i x_k \geqq \lambda_o \sum_i x_{ii}$$

$$\geqq m\lambda_o (\det x_{ik})^{1/m} . \quad 10)$$

Consequently the volume of any ellipsoid containing S is at least

10) For a deinite form the expression $\sum_i x_{ii}$ is the sum of the Eigen-values, $d = \det(x_{ik})$ is the product. Hence by the well known inequality between arithmetic and geometric means it follows that

$$\sum_i x_{ii} \geqq md^{1/m}$$

equal to that of the unit-sphere. This shows that the ellipsoid of least volume containing S is at the same time the ellipsoid of least volume containing the points $y^1, \ldots, y^s$ of the boundary of the convex hull of S, where $s \leq \frac{m(m+3)}{2}$.

Let $u_1, \ldots, u_m$ be any numbers with $\sum_i u_i^2 = 1$. Introduce

$$P_r = \sum_{i=1}^{m} u_i y_i^r$$

Then, because of (19d)

(20) $$|P_r| \leq 1 \qquad \text{for } r = 1, \ldots, s.$$

On the other hand we have for any t, using (19a, b e)

$$\sum_{r=1}^{s} \lambda_r (P_r + t)^2 = (t^2 + \frac{1}{m}) \sum_{r=1}^{s} \lambda_r .$$

It follows that

(21) $$\underset{r}{\text{Maximum}} (P_r + t)^2 \geqq t^2 + \frac{1}{m} .$$

Hence for any t there exists an r such that

$$P_r^2 + 2tP_r - \frac{1}{m} \geqq 0 .$$

The lefthand side of this inequality is a quadratic function of P_r, whose roots may be α, β. We then see that for any α, β with $\alpha\beta = -\frac{1}{m}$, there is a P_r outside the interval

$$\alpha < x < \beta .$$

If we put

(22) $$M = \underset{r}{\text{Maximum}}\, P_r, \quad -\mu = \underset{r}{\text{Minimum}}\, P_r$$

it follows that

(23) $$M\mu \geqq \frac{1}{m} .$$

Consequently

(24) $$M+\mu \geqq \frac{2}{\sqrt{m}} ,$$

and, because of (20),

$$(25) \qquad M \geqq \frac{1}{m}, \qquad \mu \geqq \frac{1}{m}.$$

As M and μ are the distances of the two planes of support of the set formed by the y^r in the direction u, it follows that the convex hull of the y^r contains the sphere of radius $\frac{1}{m}$ about the origin, and that the distance of any two parallel planes of support of that convex hull is $\geqq \frac{2}{\sqrt{m}}$. The same holds then for the convex hull of S. We have then the following theorem in terms of the original space before the affine transformation:

Theorem III.

If K is the ellipsoid of smallest volume containing a set S in E_m, then the ellipsoid K' which is concentric and homothetic to K at the ratio 1/m is contained in the convex hull of S.

The example of a simplex shows that the constant $\frac{1}{m}$ is the best possible one in this connection.

As the boundary of the convex hull of S may be an arbitrary convex surface, we see that any closed convex surface lies between two concentric homothetic ellipsoids of ratio $= \frac{1}{m}$. We also have from (24):

Any convex body can be transformed by an affine transformation into a body, for which the ratio of diameter and breadth is $\leqq \frac{1}{\sqrt{m}}$.[11)]

A stronger inequality can be derived in the case, where S is symmetric to a point, say the origin. Let K be the ellipsoid of least volume containing S, which has its center at the origin. In this case we again obtain $\lambda_0, \ldots, \lambda_s, y^1, \ldots, y^s$, such that (19a,c,d) are satisfied. We can conclude that (21) holds for t = 0, i.e. that

11) Here "breadth" is defined as minimum distance of any two parallel planes of support.

$$\underset{r}{\text{Maximum}}\; P_r^2 \geq \frac{1}{m} .$$

Then of any two parallel planes of support of the convex hull of S (after suitable affine transformation) at least one has a distance $\frac{1}{\sqrt{m}}$ from the origin. As however S is symmetric to the origin, the same holds then for the other plane of support. Hence:

If S is a set symmetric to the point 0, and K the ellipsoid of least volume containing S and having its center at 0, then the ellipsoid, which is concentric and homothetic to K at the ratio $\frac{1}{\sqrt{m}}$ is contained in the convex hull of S.

Again $\frac{1}{\sqrt{m}}$ is the best possible constant in this connection, as is seen from the example of the m-dimensional "cube" or of the m-dimensional analogon to an octahedron.

If the convex hull of S is represented by its "gage function" ("Distanzfunktion"),[12] we have the following theorem:

For any function $f(x) = f(x_1, \ldots, x_m)$, which satisfies the conditions

$$f(\mu x) = |\mu|\, f(x) \qquad \text{for all numbers } \mu$$

$$f(x) > 0 \quad \text{for } x \neq 0$$

$$f(x + y) \leqq f(x) + f(y)$$

there exists a positive definite quadratic form $Q = Q(x)$, such that

$$\sqrt{\frac{1}{m} Q(x)} \leqq f(x) \leqq \sqrt{Q(x)} \qquad \text{for all } x.$$

It is to be expected that for a convex body S the ratio of minimum circumscribed ellipsoid to the volume of S reaches its largest value for a simplex (respectively for a cube, in case S is symmetric to a point). However the author has been unable to prove

12) See Bonnesen-Fenchel, loc. cit. p. 21.

this statement for general m.[13] If true, the statement must be a consequence of the relations (19), which are characteristic for the circumscribed ellipsoid of least volume.

13) For $m = 2$ this was proved by F. Behrend, loc. cit.