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1. Introduction.

In the following, X and Y will be vectors with components X_i, Y_j. By $X \geq 0$ will be meant $X_i \geq 0$ for all i. Let $g(X)$, $f_j(X)$ $(j = 1, \cdots, m)$ be functions with suitable differentiability properties, where $f_j(X) \geq 0$ for all X, and define

(1) $$F(X, Y) = g(X) + \sum_{j=1}^{m} Y_j \left\{ 1 - \left[f_j(X) \right]^{1+\gamma} \right\}.$$

Let $(\bar{X}, \bar{Y})$ be a saddle-point of (1) subject to the conditions $X \geq 0$, $Y \geq 0$; assume it unique in X. The function $F(X, \bar{Y})$ attains its maximum for variation in X subject to the condition $X \geq 0$ at the point $X = \bar{X}$. Since F is a maximum for variation in each component X_i separately, it follows that

(2) $\bar{F}_{X_i} \leq 0$ for all i, and

(3) $\bar{X}_i = 0$ if $\bar{F}_{X_i} < 0$.

We will refer to those subscripts for which (3) holds as <u>corner</u> indices and the remainder as <u>interior</u> indices. Let X^1 be the vector of components of X with corner indices, and X^2 the vector of interior components. Since F is also a maximum for variations in X^2 alone (holding X^1 at 0 and Y at $\bar{Y}$), and the first-order terms vanish by (2) and (3), it follows, under the usual differentiability assumptions, that the matrix,

(4) $\bar{F}_{X^2X^2}$ is negative semi-definite,

where $\overline{F}_{X^2X^2}$ is the matrix of elements $\partial^2 F / \partial X_i \partial X_j$, with i and j ranging over interior indices, evaluated at $(\overline{X}, \overline{Y})$.

It is shown in another paper, now in preparation, that for all γ sufficiently large, $\overline{X}$ maximizes $g(X)$ subject to the restraints $f_j(X) \leq 1$, $X \geq 0$, and

(5) $\overline{F}_{X^2X^2}$ is negative definite.

Hence the determination of the constrained maximum is equivalent to finding the saddle-point of a function $F(X, Y)$ which is linear in Y and satisfies (5). We seek here a convergent process for approximating such a saddle-point. The intuitively natural method, in terms of the motivations of the two players (interpreting F as the pay-off of a game in which player I chooses X and player II chooses Y), is for the player who chooses X to move "uphill" with regard to variation in that variable, while the other player moves against the gradient with respect to Y. Such processes have been investigated by Brown and von Neumann [1][1] for the case where F is linear in both X and Y. In that case, the "naive" gradient method just described leads to an oscillatory behavior (see Samuelson [2], pp. 17-22) and must be modified. In the present case, even if the functions g, f_j were linear to begin with, the introduction of the power γ creates a nonlinear system satisfying (5); as will be seen, this implies that the naive gradient method will be at least locally stable.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

2. Description of the Gradient Method.

It must be recalled that the variables X and Y are constrained to be non-negative, so that the movements of the players with and against the gradients of X and Y, respectively, cannot carry the variables into areas of negativity. The gradient method for finding a saddle-point then is the following system of differential equations:

(1) $\dot{X}_i = 0$ if $F_{X_i} < 0$ and $X_i = 0$,

(2) $= F_{X_i}$ otherwise;

(3) $\dot{Y}_j = 0$ if $F_{Y_j} > 0$ and $Y_j = 0$,

(4) $= - F_{Y_j}$ otherwise;

the dot denotes differentiation with respect to time. In this system the derivatives are discontinuous functions of the variables. The usual existence theorems for nonlinear differential equations assume continuity (see [3], Chapter II). If it could be shown that equations (2.1-4) have a unique solution for any initial position continuous with respect to variations in the starting-point, considerably stronger statements could be made about the convergence of the system.

3. Theorem.

Let F(X, Y) be linear in Y, possess a saddle-point $(\bar{X}, \bar{Y})$ under the constraint $X \geq 0$, $Y \geq 0$, and be analytic in some neighborhood of $(\bar{X}, \bar{Y})$. Suppose further that (a) condition (1.5) holds and (b) $\bar{X}_i > 0$ and $\bar{Y}_j > 0$ for every interior index i or j.[2]

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[2]Analogously to (1.3), j is a corner index for Y if $\bar{Y}_j = 0$, $\bar{F}_{Y_j} < 0$; an interior index for Y is any subscript which is not a corner index.

Then for every initial position in a sufficiently small neighborhood of $(\overline{X}, \overline{Y})$, there is a unique solution $X(t)$, $Y(t)$ of the equations (2.1-4), such that $\lim_{t \to \infty} X(t) = \overline{X}$ and, for every limit-point Y^* of $Y(t)$, $(\overline{X}, Y^*)$ is a saddle-point of $F(X, Y)$.

4. Proof.

If (X^*, Y^*) were another saddle-point of $F(X, Y)$, then $(X^*, \overline{Y})$ would be still another. That is, X^* would maximize $F(X, \overline{Y})$ for variation in X. Then (1.5) implies that if X^* is in a sufficiently small neighborhood of $\overline{X}$, then $\overline{X} = X^*$, so that $\overline{X}$ is at least locally unique.

In what follows, let $x = X - \overline{X}$, $y = Y - \overline{Y}$, expanding the derivatives of F into power series. Then

$$(1) \qquad F_{X^1} = \overline{F}_{X^1} + a(x, y),$$

where $a(x, y)$ is a continuous vector function with $a(0, 0) = 0$.

In the expansion of F_{X^2}, there are no constant terms by definition. Divide the terms of the expansion into four types: those containing components of x^1; the terms linear in x^2; the terms linear in y; and the terms in x^2 and y of degree higher than the first. Define the (variable) matrix A as follows: for any interior index i and corner index j, let $A_{ij}\, x_j$ be the sum of all terms in the expansion of F_{X_i} which have x_j as a factor but do not have x_k as a factor for any corner index $k < j$. Then, clearly, $\sum_{j1} A_{ij}\, x_j$ is the sum of all terms in the expansion of F_{X_i} which contain corner components of x, the summation extending only over corner indices. The matrix A is a function of x and y. Now consider the fourth type of term in the expansion of F_{X^2}, the non-linear terms involving x^2 and y only. Since F, and therefore F_{X^2}, is linear in y, each such term must involve a component of x^2. Define the matrix B so that, for every pair of interior indices i and j, $B_{ij}\, x_j$ is the sum of all non-linear terms x in the expansion of F_{X_i} which have x_j for a factor but do not have x_k as a factor for any corner index k or for any interior index $k < j$. Then, $\sum_{j^2} B_{ij}\, x_j$ is the sum of all non-linear terms in the expansion of F_{X_i} which contain no corner components of x as factors. B therefore is a function of x^2 and y; further, since each component of Bx^2 is non-linear, B vanishes if both x^2 and y do.

(2) $$F_{X^2} = A(x, y)\, x^1 + \bar{F}_{X^2X^2}\, x^2 + \bar{F}_{X^2Y}\, y + B(x^2, y)\, x^2,$$

where A and B are continuous matrix functions, and $B(0, 0) = 0$.

Since F is linear in y, F_Y is independent of y. By a discussion similar to the preceding, it follows that

(3) $$F_Y = \bar{F}_Y + C(x)\, x^1 + \bar{F}'_{X^2Y}\, x^2 + b(x^2),$$

where C is a continuous matrix function and the vector $b(x^2)$ is of the second order with respect to components of x^2.

Now define

(4) $$D = (1/2)\, (x'x + y'y).$$

D is proportional to the distance in the (X, Y) space to the saddle-point $(\bar{X}, \bar{Y})$. Differentiate (4) with respect to time.

(5) $$DD = x'x + y'y.$$

First suppose that for each i either $X_i > 0$ or $F_{X_i} \geq 0$ and that for each j either $Y_j > 0$ or $F_{Y_j} \leq 0$. Then from (2.2), (2.4) and (5),

(6) $$DD = x'F_X - y'F_Y.$$

Substitute from (1-3) into (6)

(7) $$DD = (x^1)'F_{X^1} + (x^2)'F_{X^2} - y'F_Y$$

$$= (x^1)'\overline{F}_{X^1} + (x^1)'a(x, y) + (x^2)'A(x, y)(x^1) + (x^2)'\overline{F}_{X^2X^2}\, x^2 + (x^2)'\overline{F}_{X^2Y}\, y$$

$$+ (x^2)'B(x^2, y)\, x^2 - y'\overline{F}_Y - y'C(x)\, x^1 - y'\overline{F}'_{X^2Y}\, x^2 - y'\, b(x^2).$$

The last term is homogeneous linear in y and of the second order in x^2. Hence, it can be written in the form,

(8) $$y'\, b(x^2) = (x^2)'\, E(x^2, y)\, (x^2),$$

where E is a continuous matrix function and $E(x^2, 0) = 0$.

Each term in (7) is a scalar and therefore equal to its transpose. In particular, $(x^2)'\overline{F}_{X^2Y}\, y = y'\overline{F}'_{X^2Y}\, x^2$. Let

(9) $$c(x, y) = a(x, y) + A'(x, y)\, x^2 - C'(x)\, y,$$

(10) $$G(x^2, y) = B(x^2, y) - E(x^2, y).$$

In view of (8-10) and the preceding remarks, (7) can be simplified to the following expression:

(11) $$DD = (x^1)'\overline{F}_{X^1} + (x^1)'\, c(x, y) + (x^2)'\overline{F}_{X^2X^2}\, x^2 + (x^2)'\, G(x^2, y)\, (x^2) - y'\overline{F}_Y.$$

From (1) and (9),

(12) $$c(0, 0) = 0.$$

Let m_1 be the minimum of $\left|\bar{F}_{X_i}\right|$ over the corner indices i; by definition, $m_1 > 0$. By (12), we can choose ϵ_1 so that every component of c is less than m_1 whenever $D < \epsilon_1$. Let $\sum_1$ denote summation over the corner indices only; then, since $x_i \geq 0$ for all corner indices,

$$(x^1)' c(x, y) < m_1 \sum_{i^1} |x_i| \leq - (x^1)' \bar{F}_{X^1}, \quad \text{if } x^1 \neq 0 \text{ and } D < \epsilon_1, \text{ or}$$

$$(x^1)' \bar{F}_{X^1} + (x^1)' c(x, y) < 0 \quad \text{if } x^1 \neq 0 \text{ and } D < \epsilon_1. \tag{13}$$

From (1.5),

$$(x^2)' \bar{F}_{X^2X^2} \; x^2 < 0 \quad \text{unless } x^2 = 0.$$

Let m_2 be the maximum of $(x^2)' \bar{F}_{X^2X^2} \, x^2$ subject to the condition $D = 1$, m_3 the maximum of $\sum_{i^2} \left| x_i \right|$ subject to the same condition. Then,

$$m_2 < 0, \; (x^2)' \bar{F}_{X^2X^2} \, x^2 \leq m_2 D^2, \; \sum_{i^2} |x_i| \leq m_3 D. \tag{14}$$

From (2), (8) and (10), $G(0, 0) = 0$. If D is sufficiently small, (x^2, y) will be sufficiently close to $(0, 0)$ to insure that the largest of the components g_{ij} of G is less than $- m_2/(m_3)^2$ in absolute value. Then, from (14),

$$(x^2)' \, G(x^2, y) \; x^2 \leq \left| \sum_{i^2} \sum_{j^2} g_{ij} \; x_i x_j \right| \leq \sum_{i^2} \sum_{j^2} \left| g_{ij} \right| \left| x_i \right| \left| x_j \right|$$

$$< (-m_2/m_3^{\,2}) \Big(\sum_{i^2} \left| x_i \right| \Big)^2 \leq - m_2 \; D^2 \leq - (x^2)' \bar{F}_{X^2X^2} \; x^2,$$

the strict inequality holding provided that $\sum_{i^2} \left| x_i \right| > 0$, which is equivalent to $x^2 \neq 0$.

$$(x^2)' \bar{F}_{X^2X^2} \, x^2 + (x^2)' \, G(x^2, y) \; x^2 < 0 \quad \text{if } D < \epsilon_2, \; x^2 \neq 0. \tag{15}$$

For each corner index j, $y_j \geq 0$ always, while $\bar{F}_{Y_j} > 0$; for interior indices j, $\bar{F}_{Y_j} = 0$. Hence,

(16) $$y'\bar{F}_Y \geq 0.$$

By (11), (13), (15) and (16),

(17) $$D\dot{D} < 0 \text{ if } D < \epsilon,\ x \neq 0;\quad D\dot{D} \leq 0 \text{ if } D < \epsilon,$$

where ϵ is chosen smaller than ϵ_1 or ϵ_2, and also sufficiently small so that,

(18) $$F_{X_i} < 0,\ F_{Y_j} > 0 \text{ when } D \leq \epsilon, \text{ for all corner indices } i \text{ and } j;$$

(19) $$\epsilon < \min_i{}_2\, \bar{X}_i,\ \epsilon < \min_j{}_2\, \bar{Y}_j,$$ the minima being taken over all interior indices;

(20) $F_{X^2X^2}$ is negative definite when $X = \bar{X}$ and $y'y/2 < \epsilon$.

By assumption (b) of the theorem, (19) is possible with positive ϵ. By (1.5), $F_{X^2X^2}$ is negative definite when $X = \bar{X}$ and $y = 0$; since F is certainly continuous in Y, (20) can hold for sufficiently small ϵ.

We will now show that there does in fact exist a unique solution of (2.1-4) continuous in the initial position and in time if, at the initial position $[X(0), Y(0)]$, $D < \epsilon$. Let S_0 be the set of all indices for which $X_i(0) = 0$, By (19), any index in S_0 must be a corner index for X. Similarly, let T_0 be the set of all indices for which $Y_j(0) = 0$. By (18), $F_{X_i} < 0$, $F_{Y_j} > 0$ for all indices in S_0 and T_0, respectively. By the differential equation system (S_0, T_0), we shall mean

(21) $$\dot{X}_i = F_{X_i} \text{ for } i \text{ not in } S_0,\quad \dot{Y}_j = -F_{Y_j} \text{ for } j \text{ not in } T_0,$$
$$X_i = Y_j = 0 \text{ for } i \text{ in } S_0,\ j \text{ in } T_0.$$

In this system, the derivatives are continuous functions of the variables. By the Cauchy-Lipschitz Existence Theorem (see [3], Theorem (4.1), p. 23), the system (S_0, T_0) has a solution uniquely defined by the initial conditions. Let $Z = (X, Y)$, and let a given solution be $Z[t, Z(0)]$, where $Z(0)$ is the initial position. Then it is further known ([3], (7.3), p.30) that

(22) $\quad Z[t, Z(0)]$ is a continuous function of $Z(0)$ and of t.

For every i not in S_0, $X_i[t, Z(0)] > 0$ in some interval of time; similarly, $Y_j[t, Z(0)] > 0$ in some interval for every j not in T_0. Since F_{X_i} and F_{Y_j} are continuous functions of Z, which is in turn continuous in t there is an interval in which $F_{X_i} < 0$, $F_{Y_j} > 0$ for i in S_0, and j in T_0. The solution to system (S_0, T_0) is then a solution to the system (2.1-4), and further it is clearly the only one.

Since S_0 and T_0 contain only corner indices, $x_i = y_j = 0$ for all i in S_0 and j in T_0. If we fix these variables at 0, F, considered as a function of the remaining variables, has the same properties as assumed to begin with. Hence, (17) is valid; since $\dot{D} \leq 0$, $D[t, Z(0)]$ (the value of D for the point $Z[t, Z(0)]$) is non-increasing. Since $D[0, Z(0)] < \epsilon$, $D[t, Z(0)] < \epsilon$ for all t. Hence, $F_{X_i} < 0$, $F_{Y_j} > 0$ for all i in S_0 and j in T_0 for all points of the solution $Z[t, Z(0)]$. The solution for (S_0, T_0) therefore ceases to be a solution for (2.1-4) only when $X_i[t, Z(0)] = 0$ for some i not in S_0 or $Y_j[t, Z(0)] = 0$ for some j not in T_0. Let this occur at time t_0. Since $D[t_0, Z(0)] < \epsilon$, $X_i[t_0, Z(0)] > 0$, $Y_j[t_0, Z(0)] > 0$ for all interior indices by (19); hence, i or j must be a corner index by (18). Let S_1 be now the set of all indices for which $X_i[t_0, Z(0)] = 0$, T_1 the set of all indices for which $Y_j[t_0, Z(0)] = 0$. Clearly, S_1 includes S_0, T_1 includes T_0. Again, the solution of the system (S_1, T_1) is the unique solution of (2.1-4) in some interval of time beginning with t_0. The argument can be repeated; since the sets S_i, T_i are increasing and there are only a finite number of indices, only a finite number of systems are involved. It then follows easily that the system (2.1-4) has a unique solution $Z[t, Z(0)]$ continuous in t and in $Z(0)$.

By (18), for each corner index i, there is a number $m_i < 0$ such that $F_{X_i} \leq m_i$ whenever $D \leq \epsilon$. As $\dot{D}[t, Z(0)] \leq 0$ for all t, $D[t, Z(0)] < \epsilon$. So long as $X_i[t, Z(0)] > 0$, $\dot{X}_i[t, Z(0)] \leq m_i$, so that $X_i[t, Z(0)]$ reaches 0 in finite time. Since $F_{X_i} < 0$ for all t, $X_i[t, Z(0)] = 0$ for all t from then on. The same argument holds for corner indices of Y.

(23) $$X^1[t, Z(0)] = Y^1[t, Z(0)] = 0 \text{ for all } t \text{ sufficiently large.}$$

As $\dot{D}[t, Z(0)] \leq 0$ for all t, $D[t, Z(0)]$ converges to a limit. Let

(24) $$\lim_{t \to \infty} D[t, Z(0)] = D^*.$$

Let $Z^* = (X^*, Y^*)$ be any limit point of $Z[t, Z(0)]$. There is a sequence $\{t_n\}$ such that

(25) $$\lim_{n \to \infty} t_n = \infty, \lim_{n \to \infty} Z[t_n, Z(0)] = Z^*.$$

Let $Z_n = Z[t_n, Z(0)]$. Then, by (22),

(26) $$Z(t, Z^*) = \lim_{n \to \infty} Z(t, Z_n) = \lim_{n \to \infty} Z[t + t_n, Z(0)].$$

Since D is a continuous function of Z, it follows from (26) and (24) that

(27) $D(t, Z*) = \lim_{n \to \infty} D[t + t_n, Z(0)] = D*,$

a constant. That is, $\dot{D}(t, Z*) = 0$ for all t. By (17), $x(t, Z*) = 0$ for all t, or

(28) $X(t, Z*) = \overline{X}$ for all t.

In particular, $X(0, Z*) = X* = \overline{X}$. Since $Z*$ was any limit-point of $Z[t, Z(0)]$,

(29) $\lim_{t \to \infty} X[t, Z(0)] = \overline{X}.$

Let an asterisk denote evaluation at $Z* = (\overline{X}, Y*)$. By (23) and (18),

(30) $\overline{X}^1 = 0, F^*_{X^1} < 0,$

(31) $Y*^1 = 0, F^*_{Y^1} > 0.$

By (28), $\dot{X}^2(t, Z*) = 0$; since $\overline{X}^2 > 0$ by hypothesis, it follows from (2.2) that

(32) $F^*_{X^2} = 0.$

By (20), $F^*_{X^2X^2}$ is negative definite. In conjunction with (30) and (32), this shows that

(33) $F(X, Y*)$ has a maximum at $\overline{X}$ for variation in X subject to $X \geq 0$.

Since F is linear in Y, F_Y is independent of Y, so that $F^*_{Y^2} = \overline{F}_{Y^2} = 0$. That is, $F(\overline{X}, Y)$ is independent of Y^2. From (31), then,

(34) $F(\bar{X}, Y)$ has a minimum at Y^* for variation in Y subject to $Y \geq 0$.

(28), (33) and (34) complete the proof of the theorem.

5. A Remark on the Hypotheses of the Theorem.

Condition (b) of the theorem, that no component of $\bar{X}$ or $\bar{Y}$ is at the boundary of the domain of variation unless it is actually a corner extremum in the proper sense, is inserted to avoid the possibility that at some point $X_i = 0$ and $F_{X_i} = 0$ for some i. We have been unable to show, in this situation, either that there exists a solution of (2.1-4) with such an initial position or that, if it exists, it is unique. Some experiments with simple systems suggest that in fact there is a unique solution beginning at such a point; if so, condition (b) could be dropped.

6. Economic Interpretation.

Let X_i $(i = 1, \cdots, n)$ be activity levels of the n different possible production activities (measured, e. g., by the outputs of one of the products). Let $g(X)$ be the social utility derived from activities, $F_{ij}(X_i)$ the quantity of input j needed to carry on activity i at level X_i, α_j the stock of input j available to begin with, and $f_j(X) = \sum_i f_{ij}(X_i) - \alpha_j + 1$. The unit of measurement of commodity j should be chosen sufficiently large that $f_j(X)$ (which is excess demand plus one) will be positive throughout the adjustment process. Note that production of an output j by means of process i would be represented by a negative value for the function f_{ij}; also, $\alpha_j = 0$ for intermediate products. Hence, it is desired to choose a set of activity levels $\bar{X}$ which will maximize $g(X)$ subject to the constraints that the excess demand of the productive system for any input does not exceed the initial supply, i.e., $\sum_i f_{ij}(X) \leq \alpha_j$, or, $f_j(X) \leq 1$ for all j. By definition, $X_i \geq 0$ for all i. As noted in section 1, if $\overline{X}$ is the optimum set of activity levels, then, there is some $\overline{Y}$ such that $(\overline{X}, \overline{Y})$ is the saddle-point of the function $F(X, Y)$ defined in (1.1).

It follows then, by the Theorem, that the X-components of the solution of the system of differential equations (2.1-4) will approach $\overline{X}$. The equations (2.1-2) can be written,

(1) $$\dot{X}_i = (\partial g/\partial X_i) - \sum_j Y_j (1 + \eta)(f_j)^{\eta} (df_{ij}/dX_i),$$

unless the right-hand side is negative when $X_i = 0$, in which case the right-hand side is replaced by 0. Let

(2) $$q_i = \partial g/\partial X_i,$$

(3) $$p_j = Y_j (1 + \eta)(f_j)^{\eta}.$$

Then, from (1-3),

(4) $$\dot{X}_i = q_i - \sum_j p_j (df_{ij}/dX_i) \quad \text{if} \quad X_i > 0,$$

$$= \max \left[0, q_i - \sum_j p_j (df_{ij}/dX_i)\right] \quad \text{if} \quad X_i = 0.$$

By (2.3-4), p_j is determined by (3) in conjunction with the equations,

(5) $$\dot{Y}_j = f_j^{1+\eta} - 1 \quad \text{if} \quad Y_j > 0,$$

$$= \max \left[f_j^{1+\eta} - 1, 0\right] \quad \text{if} \quad Y_j = 0.$$

Note that $\dot{Y}_j > 0$ if $f > 1$, i. e., if there is excess demand, and $\dot{Y}_j < 0$ if there is excess supply (except for free goods).

Institutionally, the process can be visualized as follows: there is a central board which evaluates the social worth of a given constellation of activity levels, and therefore the marginal social valuation q_i of each; for each activity, there is a plant manager who determines the activity level X_i; for each primary or intermediate product, there is a price-fixing authority who determines p_j. The central board announces the marginal social valuations q_i, and each price-fixing authority announces a price p_j. then, each plant manager expands or contracts at a rate equal to the difference between the marginal social valuation of the activity, q_i, and the marginal cost of increasing the activity, $\sum_j p_j \ (df_{ij}/dX_i)$ (apart from the corner case of unused activities). At the same time, the price-fixing authority adjusts Y_j in accordance with the excess demand, as given in (5) and then arrives at p_j.

It is important to observe that these rules of decision-making are highly decentralized. Once the prices are announced, the individual activity managers need know only their own technologies to determine their rate of expansion. Similarly, the price-fixers need know only the excess demands on their own markets.

Even the decisions of the central board in regard to the marginal social valuations of the commodities can be simplified. Actually, the social valuation depends on the outputs of final products. Let $g_{ik}(X_i)$ be the output of final product k if activity i is operated at level X_i, $g_k(X) = \sum_i g_{ik} \ (X_i)$ be the total output of final product k, and $U(g_1, \cdots, g_m)$ the social utility derived from having total outputs $g_1, \cdots, g_m$ of the final products $1, \cdots, m$, respectively. Then $g(X) = U[g_1(X), \cdots, g_m(X)]$, so that

(6) $$\partial g / \partial X_i = \sum_k (\partial U / \partial g_k)(dg_{ik}/dX_i).$$

If the central board announces merely the marginal social valuations of the various final products, $r_k = \partial U / \partial g_k$, the firm can compute its marginal social valuation,

(7) $$q_i = \sum_k r_k (dg_{ik}/dX_i),$$

by the knowledge of its own technology.

Bibliography

1. G. W. Brown and J. von Neumann, "Solutions of Games by Differential Equations," P-142, 19 April 1950, RAND.

2. P. A. Samuelson, "Market Mechanisms and Maximizations," P-69, 28 March 1949, RAND.

3. S. Lefschetz, Lectures on Differential Equations, Princeton: Princeton University Press, 1946, 209 pp.

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