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THE PROBLEM OF LAGRANGE WITH DIFFERENTIAL INEQUALITIES AS ADDED SIDE CONDITIONS

BY

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THE PROBLEM OF LAGRANGE WITH DIFFERENTIAL INEQUALITIES AS ADDED SIDE CONDITIONS

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1. Introduction. The problem of the calculus of variations to be considered here consists in finding in a class of admissible arcs $y_i(x)$ joining two fixed points and satisfying a set of differential equations and inequalities of the form

$$\psi_\alpha(x, y, y') = 0, \qquad \phi_\beta(x, y, y') \geqq 0,$$

that one which minimizes the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx.$$

The problem considered is for a space of n + 1 dimensions. A geometric illustration of a three-dimensional problem was suggested by Zermelo.[1] This problem required the finding of the shortest distance between two points on a surface subject to the condition that the direction of the tangent line at any point of the curve make an angle with the perpendicular which is never greater than a given constant. Bolza in a paper[2] issued in 1914 obtained a first necessary condition for a minimum and several corollaries. However he made no sufficiency proofs.

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[1] E. Zermelo, Jahresberichte der Deutschen Mathematiker-vereinigung, B 11,(1902).

[2] O. Bolza, Über Variationsprobleme mit Ungleichungen als Nebenbedingungen, Mathematische Abhandlungen, H. A. Schwarz, (1914), seite 1.

An equivalent problem is introduced in section 2 of this paper by considering functions $z_\beta(x)$ such that the equations

$$\phi_\beta(x, y, y') = z_\beta'^2$$

hold. This equivalent problem yields a multiplier rule and necessary conditions analogous to those of Weierstrass and Clebsch. These are given in section 3. However as the equivalent problem may become singular, as it does for a composite arc, this method does not provide a complete treatment.

Two sufficiency proofs are made for a composite arc. Such an arc is one without corners composed of two subarcs such that all but one of the functions $\phi_\beta(x, y, y')$ are greater than zero on one subarc, whereas all the functions mentioned are greater than zero on the remaining subarc. An imbedding theorem and a necessary condition analogous to that of Mayer are proved in sections 4, 5 and 6. The first sufficiency proof is made in section 6 and is made with the assumption of normality on sub-intervals. The second sufficiency proof is made without the above assumption and in part depends upon a necessary condition analogous to that of Hestenes for the problem of Bolza.

It should be noted that although the sufficiency proofs are made for a composite arc, any other subcase which might arise could be handled in a similar manner. It is not due to the fact that other subcases present special difficulties that all of them are not treated, but rather to the fact that each subcase has to be handled separately. The case of the composite arc was treated since it represents a fair sample of the variety of cases which do exist. The treatment applied to the composite arcs will in general apply to all other cases. The singularity of the equivalent problem requires the separate treatment of the various subcases. The case considered affords a fairly complete treatment of the plane and 3-dimensional problems.

The following section describes the analytic setting of the problem and introduces the mechanism by means of which all the necessary conditions, save the analogue of the condition of Mayer, may be obtained.

2. Formulation of the problem. In the following pages the set $(x, y_1, \ldots, y_n, y_1', \ldots, y_n')$ will be denoted by (x, y, y'). The functions $y_i(x)$, $(i = 1, \ldots, n)$, defining the minimizing arc E_{12} and the functions

$$(2{:}1) \qquad \begin{array}{lll} f(x, y, y'), & \phi_\beta(x, y, y') & (\beta = 1, \ldots, m), \\ \psi_\alpha(x, y, y') & & (\alpha = m+1, \ldots, m+p < n) \end{array}$$

are required to satisfy the following hypotheses:

(1) The functions $y_i(x)$ are continuous on the interval x_1x_2 and have continuous derivatives on this interval except possibly at a finite number of corners.

(2) In a neighborhood N of the set of values (x, y, y') belonging to the arc E_{12} the functions (2:1) have continuous derivatives up to and including those of the third order.

(3) At every element (x, y, y') of the arc E_{12} the $n \times (m + p)$-dimensional matrix

$$\left\| \begin{array}{c} \psi_{\alpha y_i'}(x, y, y') \\ \phi_{\beta y_i'}(x, y, y') \end{array} \right\| \qquad \begin{array}{r} (i = 1, \ldots, n) \\ (\beta = 1, \ldots, m) \\ (\alpha = m+1, \ldots, m+p) \end{array}$$

has rank $m + p$.

Henceforth the subscripts i, β, and α shall have the ranges specified in hypothesis (3). Moreover a repeated index in a term will indicate summation with respect to that index, unless otherwise stated.

An admissible arc is one with the continuity properties (1) and one whose elements (x, y, y') lie in the region N specified in hypothesis (2).

The problem to be treated here consists in finding in the class of admissible arcs $y_i(x)$, joining two fixed points with coordinates (x_1, y_1) and (x_2, y_2), and satisfying the conditions $\phi_\beta \geqq 0$ and $\psi_\alpha = 0$, that one which minimizes the integral

$$J = \int_{x_1}^{x_2} f(x, y, y')dx. \tag{2:2}$$

A problem of Bolza with variable end-points which is equivalent to the problem just formulated may be obtained by setting

$$\phi_\beta = z_\beta'^2(x), \tag{2:3}$$

where the functions $z_\beta(x)$ will obviously have the same continuity properties as the functions $y_i(x)$ in the above problem. <u>The equivalent problem</u> is stated as follows:

To find in the class of admissible arcs

$$y_i = y_i(x), \qquad z_\beta = z_\beta(x)$$

satisfying the differential equations

$$\begin{aligned} \phi_\beta(x, y, y') - z_\beta'^2(x) &= 0, \\ \psi_\alpha(x, y, y') &= 0, \end{aligned}$$

and satisfying the end-conditions

$$\begin{aligned} x_1 &= a_1, \qquad y(x_1) = y_1, \\ x_2 &= a_2, \qquad y(x_2) = y_2, \end{aligned}$$

that one which minimizes the integral (2:2).

In view of hypotheses (1) to (3) it follows that the corresponding hypotheses for this equivalent problem are also satisfied. Moreover the above end-conditions are independent. Hence one may apply the theory of the problem of Bolza to this problem so as to obtain a number of necessary conditions. However as the equivalent problem may be singular it does not afford a complete attack. As will be seen later, other methods will be necessary in some cases to complete the theory. The equivalent problem is used primarily in sections 3 and 8.

3. First necessary conditions. From the theory for the problem of Bolza it follows that for every minimizing arc E_{12} there must exist constants c_i, d_β and a function

$$G = \lambda_0 f + \lambda_\alpha(x)\psi_\alpha + \lambda_\beta(x)(\phi_\beta - z_\beta'^2)$$

such that the equations

$$G_{y_i'} = \int_{x_1}^{x} G_{y_i}dx + c_i, \qquad \lambda_\beta z_\beta' = d_\beta$$

are satisfied along E_{12}. In the last m equations the repeated index β does not denote summation. Moreover from the transversality conditions in the problem of Bolza it follows that at the end points of E_{12} the expressions

$$(G - y_i'G_{y_i'} - z_\beta'G_{z_\beta'})dx_s + G_{y_i'}dy_{is} + e_s dx_s$$
$$+ b_s dy_{is} - 2\lambda_\beta z_\beta' dz_{\beta s} \qquad (s = 1, 2)$$

must be identically zero in dx_s, dy_{is} and $dz_{\beta s}$. As a consequence the m conditions

$$\lambda_\beta z_\beta'|^{x_2} = \lambda_\beta z_\beta'|^{x_1} = 0, \qquad (\beta \text{ not summed}),$$

must hold. Hence the functions $\lambda_\beta z_\beta'$ must be identically zero along the arc E_{12}. Therefore one obtains the

FIRST NECESSARY CONDITION I. *For every minimizing arc E_{12} joining the fixed points 1 and 2, there must exist constants C_i and a function*

$$F = \lambda_0 f + \lambda_\alpha(x)\psi_\alpha + \lambda_\beta(x)\phi_\beta \tag{3:1}$$

such that the equations

$$\begin{gathered} F_{y_i'} = \int_{x_1}^{x} F_{y_i}\,dx + C_i, \\ \phi_\beta(x, y, y') \geqq 0, \qquad \psi_\alpha(x, y, y') = 0 \end{gathered} \tag{3:2}$$

hold at every point of E_{12}. The constant λ_0 and the functions $\lambda_\alpha(x)$ and $\lambda_\beta(x)$ cannot vanish simultaneously at any point of E_{12}, and are continuous except possibly at values of x defining corners of E_{12}. Moreover the m functions

$$\lambda_\beta\phi_\beta \qquad (\beta \text{ not summed})$$

vanish at all points of E_{12}.

The following corollary may be obtained as an immediate consequence of the preceding sentence.

COROLLARY 3:1. *If all the functions ϕ_β are greater than zero at every point of E_{12}, the minimizing arc is that one which minimizes the integral (2:2) in the class of admissible arcs satisfying the differential equations*

$$\psi_\alpha(x, y, y') = 0.$$

For this case the function F in expression (3:2) reduces to

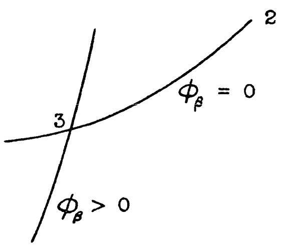
$$F_1 = \lambda_0 f + \lambda_\alpha\psi_\alpha .$$

Since this case is an ordinary problem of Lagrange, a fairly complete treatment of it is known.

The following corollaries and further necessary conditions, with the exception of the necessary condition of Mayer, are obtained for the general problem stated above. In the case of the Mayer condition the problem considered is the one in which all but one of the functions ϕ_β are greater than zero on the interval x_1x_2, whereas the remaining function is zero on certain subintervals of x_1x_2 and greater than zero on the remaining subintervals. It will be no restriction to label this last function by ϕ_1. For this problem the function F occurring in the expression (3:1) has the form

$$F = \lambda_0 f + \lambda_\alpha \psi_\alpha + \lambda_1 \phi_1 .$$

If a minimizing arc E_{12} is composed of two subarcs E_{13} and E_{32}, the functions ϕ_β being greater than zero on E_{13}, and zero on E_{32}, it follows that the functions λ_β are zero on E_{13}. Hence the arc E_{12} is normal if the arc E_{13} is normal. The arc E_{13} is defined by equations (3:2) in which F has been replaced by the function F_1 occurring in corollary (3:1).



The following corollaries are an immediate consequence of the first necessary condition.

COROLLARY 3:2. On every subarc between corners of a minimizing arc E_{12} the differential equations and inequalities

$$dF_{y_i'}/dx = F_{y_i}, \qquad \phi_\beta(x, y, y') \geqq 0, \qquad \psi_\alpha(x, y, y') = 0$$

must be satisfied, where F is the function (3:1).

COROLLARY 3:3. *At every corner of a minimizing arc* E_{12} *the conditions*

$$F_{y_i'}(x, y, y'(x-0), \lambda(x-0)) = F_{y_i'}(x, y, y'(x+0), \lambda(x+0))$$

must be satisfied.

The analogue of the Weierstrass necessary condition for the *equivalent problem* yields the result that at each element $(x, y, z, y', z', \lambda)$ of a minimizing arc which is normal, the inequality

$$\begin{aligned}\mathcal{E} = G(x, y, z, Y', Z', \lambda) - G(x, y, z, y' z', \lambda) \\ - (Y_i' - y_i')G_{y_i'} - (Z_\beta' - z_\beta')G_{z_\beta'} \qquad \geqq 0\end{aligned}$$

must be satisfied for every admissible set $(x, y, z, Y', Z') \neq (x, y, z, y', z')$, satisfying the equations

$$\phi_\beta(x, y, Y') - Z_\beta'^2 = 0 \qquad \psi_\alpha(x, y, Y') = 0.$$

Since the functions $\lambda_\beta z_\beta'$ are identically zero on E_{12} one obtains immediately the

SECOND NECESSARY CONDITION II. *At each element* (x, y, y', λ) *of a minimizing arc* E_{12} *which is normal the inequality*

$$(3:3) \qquad E(x, y, y', Y', \lambda_\alpha, \lambda_\beta) - \lambda_\beta\phi_\beta(x, y, Y') \geqq 0$$

must hold for all sets $(x, y, Y') \neq (x, y, y')$, *and satisfying the differential equations and inequalities*

$$\phi_\beta(x, y, Y') \geqq 0, \qquad \psi_\alpha(x, y, Y') = 0,$$

where $E(x, y, y', Y', \lambda_\alpha, \lambda_\beta)$ *is the function*

$$F(x, y, Y', \lambda_\alpha, \lambda_\beta) - F(x, y, y', \lambda_\alpha, \lambda_\beta) - (Y_i' - y_i')F_{y_i'}.$$

In a similar manner the analogue of the Clebsch condition for the equivalent problem gives the following condition.

THIRD NECESSARY CONDITION III. At every element (x, y, y', λ) of a minimizing arc E_{12} which is normal the inequality

$$F_{y_i'y_k'}\pi_i\pi_k - 2\lambda_\beta \varkappa_\beta^2 \geqq 0 \tag{3:4}$$

must be satisfied for every set $[\pi_1, \ldots, \pi_n, \varkappa_1, \ldots, \varkappa_m] \neq [0, \ldots, 0, 0, \ldots, 0]$ and satisfying the equations

$$\psi_{\alpha y_i'}\pi_i = 0, \qquad \phi_{\beta y_i'}\pi_i - 2z_\beta'\varkappa_\beta = 0.$$

At any point of E_{12} where any one of the functions z_β', say z_1', is zero, choose $[\pi_i] = [0]$, and all the $\varkappa_\beta$ except $\varkappa_1$ zero. Hence at such a point of E_{12} the condition $\lambda_1 \leqq 0$ must hold. Where $z_1' \neq 0$, it follows from the first necessary condition that $\lambda_1 = 0$. Hence one obtains the following corollary.

COROLLARY 3:4. At every element (x, y, y') of a minimizing arc E_{12} it is necessary that the inequalities

$$\lambda_\beta \leqq 0$$

be satisfied.

As a consequence of the paragraph preceding the above corollary the condition III yields the following result.

COROLLARY 3:5. At every element of a minimizing arc E_{12} which is normal the inequality

$$F_{y_i'y_k'}\pi_i\pi_k \geqq 0 \tag{3:5}$$

must be satisfied for every set $[\pi_1, \ldots, \pi_n] \neq [0, \ldots, 0]$ and satisfying the equations

$$\psi_{\alpha y_i'}\pi_i = 0, \qquad \phi_{\beta y_i'}\pi_i = 0.$$

The equivalent problem was used to obtain the preceding necessary conditions. In the following sections 4 to 7 special methods are used to obtain the necessary condition of Mayer and a sufficiency proof.

4. Imbedding theorem. In the following section an imbedding theorem is established for the case in which all but one of the functions ϕ_β are greater than zero on E_{12}. The remaining function, which will be denoted by ϕ_1, is to be greater than zero on one subarc E_{13} of E_{12} and zero on the remaining subarc E_{32}. Let R_1 and R_2 represent the determinants

$$(4:1) \qquad R_1 = \begin{vmatrix} F_{y_i' y_k'} & \psi_{\alpha y_i'} \\ \psi_{\delta y_k'} & 0 \end{vmatrix}, \qquad R_2 = \begin{vmatrix} F_{y_i' y_k'} & \psi_{\alpha y_i'} & \phi_{1 y_i'} \\ \psi_{\delta y_k'} & 0 & 0 \\ \phi_{1 y_k'} & 0 & 0 \end{vmatrix}$$

where (α, $\delta = m + 1, \ldots, m + p$) and ($i, k = 1, \ldots, n$). Let F_1 and F_2 denote the functions

$$(4:2) \qquad \begin{aligned} F_1 &= \lambda_0 f + \lambda_\alpha \psi_\alpha , \\ F_2 &= \lambda_0 f + \lambda_\alpha \psi_\alpha + \lambda_1 \phi_1 . \end{aligned}$$

The symbol F_1 represents the function F which occurs in the first necessary condition for the problem in which E_{13} is an extremal; similarly F_2 denotes the corresponding function for the problem in which E_{32} is an extremal. The class of arcs defined by equations (3:2) with the function F replaced by F_1 will be denoted by A; whereas the class of arcs which are defined by equations (3:2) with F replaced by F_2, and along which the equation $\phi_1 = 0$ is satisfied, will be represented by B. A composite arc is defined to be one composed of two subarcs, one subarc belonging to A, and the second belonging to B, such that the functions $y_i(x)$ defining *the entire arc and their derivatives* $y_i'(x)$ *are continuous.* From the first necessary condition it follows that the multipliers $\lambda_\alpha(x)$ and $\lambda_1(x)$ are continuous on a composite arc. With this definition in mind, one may prove the following theorem.

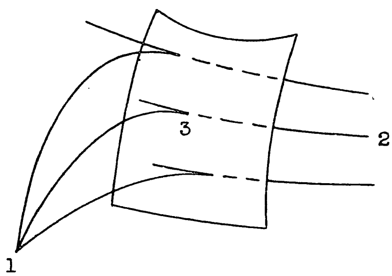
IMBEDDING THEOREM. *Consider a composite arc* $E_{12} = E_{13} + E_{32}$ *satisfying the conditions that* R_1 *and* R_2 *be different from zero on* E_{13} *and* E_{32} *respectively, and that* $\phi_1' \neq 0$ *on* E_{13} *at* 3. *Such an arc is a member of an n parameter family of composite arcs defined by the equations*

$$\begin{aligned} y_i &= y_i(x, a_1, \ldots, a_n) \\ \lambda_\alpha &= \lambda_\alpha(x, a_1, \ldots, a_n) \qquad [x_1 \leqq x \leqq x_3(a)], \\ \lambda_1 &= \lambda_1(x, a_1, \ldots, a_n) \end{aligned}$$

$$\begin{aligned} y_i &= Y_i(x, a_1, \ldots, a_n) \\ \lambda_\alpha &= \Lambda_\alpha(x, a_1, \ldots, a_n) \qquad [x_3(a) \leqq x \leqq x_2] \\ \lambda_1 &= \Lambda_1(x, a_1, \ldots, a_n) \end{aligned}$$

for the special values a_0 *of the parameters.*

Proof: Henceforth the letter a will stand for the set $(a_1, \ldots, a_n)$. Consider a composite extremal arc $E_{12} = E_{13} + E_{32}$. Since $R_1 \neq 0$ on the arc E_{13} it follows from the theory of the problem of Lagrange that E_{13} can be imbedded in an n-parameter family of extremals belonging to A, passing through the point 1 or through a point 0 on the extension of E_{13}. Similarly it



is known that if $R_2 \neq 0$ on E_{32}, then E_{32} may be imbedded in a

2n-parameter family of extremals of class[1] B. Denote the n-parameter family of extremals passing through 1 and containing E_{13} by

$$(4:3) \qquad y_i = y_i(x, a), \qquad \lambda_\alpha = \lambda_\alpha(x, a), \qquad \lambda_1 = 0,$$

and the 2n-parameter family containing E_{32} by

$$y_i = Y_i(x, c), \qquad \lambda_\alpha = \Lambda_\alpha(x, c), \qquad \lambda_1 = \Lambda_1(x, c),$$

where $(c = c_1, \ldots, c_{2n})$, and E_{32} is defined for the value of the parameters $c = c_0$. It is also known that at the special values (x_3, c_0) the condition

$$D = \begin{vmatrix} Y_{ic_k} \\ u_{ic_k} \end{vmatrix} \neq 0 \qquad (k = 1, \ldots, 2n)$$

holds, where

$$u_i = F_{y_i'}(x, Y, Y', \Lambda).$$

The necessary conditions

$$(4:4) \qquad \begin{aligned} & y_i(x_4, a) - Y_i(x_4, c) = 0, \\ & F_{1y_i'}[x_4, y(x_4,a), y'(x_4,a), \lambda(x_4,a)] \\ & \quad - F_{2y_i'}[x_4, Y(x_4,c), Y'(x_4,c), \Lambda(x_4,c)] = 0, \\ & \Phi_1[x_4, y(x_4,a), y'(x_4,a)] = 0 \end{aligned}$$

must hold at the point 3, that is for the values $x_4 = x_3$, $a = a_0$, and $c = c_0$. The functional determinant of these equations (4:4) with respect to x_4 and c is

$$(4:5) \qquad \begin{vmatrix} 0 & -Y_{ic_k} \\ F_{1y_i'}' - F_{2y_i'}' & -u_{ic_k} \\ \Phi_1' & 0 \end{vmatrix} = -\Phi_1' D.$$

---

[1] G. A. Bliss, Problem of Lagrange in the calculus of variations, American Journal of Mathematics, vol. 52 (1930), p. 687.

The above determinant will be different from zero at 3 if the function ϕ_1' at $x = x_3$ is different from zero. In the theorem it was assumed that $\phi_1' \neq 0$ holds at $x = x_3$. From the theory of implicit functions it follows that one may solve equations (4:4) for x_4 and c as functions of a. Denote these solutions by

$$(4:6) \qquad x_4 = x_4(a), \qquad c = c(a).$$

There remains to show that for values of a sufficiently close to a_0, the subarcs defined by the equations

$$(4:7) \qquad \begin{aligned} y_i &= y_i(x, a) & [x_1 \leqq x \leqq x_4(a)], \\ y_i &= Y_i(x, c(a)) & [x_4(a) \leqq x \leqq x_2], \end{aligned}$$

are tangent along the n-space defined by the first n equations of (4:3) and by (4:6). To show this consider the equations

$$(4:8) \qquad \begin{aligned} u_i &= F_{2y_i'}(x, Y, Y', \Lambda), \\ 0 &= \psi_\alpha(x, Y, Y'), \\ 0 &= \phi_1(x, Y, Y'). \end{aligned}$$

Since R_2 as defined in expression (4:1) is different from zero, equations (4:8) have a unique solution for Y', Λ, Λ_1. Moreover since $Y = y$ is a solution of (4:8) with

$$u_i = F_{1y_i'}[x_4, y(x_4,a), y'(x_4,a), \lambda(x_4,a)]$$

it is plain that $Y' = y'$, and $\Lambda = \lambda$ at $x = x_4(a)$. Hence the arcs defined by equations (4:7) are composite arcs. Thus there exists an n-parameter family of composite arcs imbedding the composite arc $E_{12} = E_{13} + E_{32}$.

5. The Mayer condition for a composite minimizing arc. In developing this condition a geometric argument will be given first. In section 8 another proof is given by means of the accessory minimum problem associated with the second variation.

Consider an n-parameter family of composite extremals through the point 1 defined by the equations

$$
(5{:}1)\qquad
\begin{aligned}
y_i &= y_i(x,\ a) && [x_1 \leqq x \leqq x_4(a)],\\
y_i &= Y_i(x,\ a) && [x_4(a) \leqq x \leqq x_2],\\
y_i(x_3,\ a) &= Y_i(x_3,\ a). &&
\end{aligned}
$$

Also consider a one-parameter family of these arcs having an envelope D obtained by letting $a = a(t)$. Let the equation of D be

$$x = x(t), \qquad y_i = g_i(t).$$

The fact that D is tangent at each of its points to an extremal

$$y_i = Y_i[x,\ a(t)]$$

may be expressed by the equations

$$x'(t) = k, \qquad Y_i'x' + Y_{ia_j}a_j' = g_{it} = kY_i'.$$

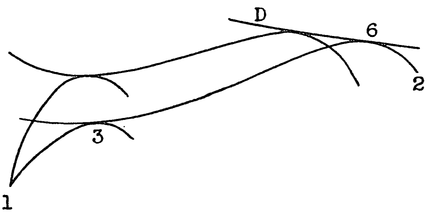
These equations have the solution $(a_j') \neq (0)$ if and only if the determinant

$$\Delta(x,\ a) \equiv |Y_{ia_j}|$$

is identically zero in t, when x and a are replaced by $x(t)$ and $a(t)$.

DEFINITION. A value x_6 is said to define a point conjugate to the point 1 if it is a root of the determinant $\Delta(x,\ a)$ belonging to an n-parameter family of composite arcs (5:1).

To prove that if Δ vanishes at x_6, the equations (5:1) do have an envelope to which E_{32} is tangent at $x = x_6$, let E_{12} be contained in an n-parameter family of composite extremals with equations of the form (5:1) for values of the parameters $a = a_0$. All the extremals satisfy the equations



$$y_i'(x_4, a) = Y_i'(x_4, a)$$

where $x_4(a)$ is defined by the first of equations (4:6). Let x_6 define a conjugate point to 3 on E_{32}. We assume for purposes of the proof that $\Delta_x(x_6, a_0) \neq 0$. Hence at least some one n - 1 rowed minor of the determinant $|Y_{ia_j}|$ is different from zero. Suppose for example that the determinant

$$|Y_{ka_t}| \qquad (k, t = 1, \ldots, n - 1)$$

is different from zero. Then the first n differential equations of the set

$$\begin{aligned} \Delta_x dx + \Delta_{a_j} da_j &= 0 \\ Y_{ia_j} da_j &= 0 \end{aligned} \qquad (j = 1, \ldots, n),$$

can be solved for dx/da_n, da_t/da_n. They determine uniquely a solution

$$x = x_5(a_n), \qquad a_t = a_t(a_n) \qquad (t = 1, \ldots, n - 1)$$

through the initial point (x_6, a_0). The determinant $\Delta(x, a)$ is identically zero on this solution since it vanishes at (x_6, a_0) and since its total derivative with respect to a_n is identically zero. Hence the last equation is also satisfied. A similar argument can be made for any other n - 1 rowed minor which may be different from zero. One thus determines

$$x = x_5(t), \qquad a_s = a_s(t) \qquad (s = 1, \ldots, n),$$

t being a properly selected one of the parameters a.

On the one-parameter family of extremals

$$y_i = Y_i(x, a(t)) = Y_i(x, t) \tag{5:2}$$

the curve D is define by the equations

$$x = x_5(t), \qquad y_i = Y_i[x_5(t), a(t)] = y_i(t),$$

and satisfies the equations

$$Y_i'x' + Y_{ia_j}a_j' = ky_i'$$

since

$$Y_{ia_j}a_j' = 0.$$

Hence the family (5:2) is a one-parameter family of composite extremals with an envelope D, touching the extremal E_{32} at the conjugate point 6.

FOURTH NECESSARY CONDITION IV. <u>Let</u> $E_{12} = E_{13} + E_{32}$ <u>be a composite arc which is normal on every subinterval of</u> x_1x_2 <u>and which is imbedded in an n-parameter family of composite arcs. Moreover suppose that</u> R_1 <u>and</u> R_2 <u>are different from zero on</u> E_{13} <u>and</u> E_{32} <u>respectively. Then if</u> E_{12} <u>is a minimizing arc there can exist no conjugate point to 1 on the arc</u> E_{12}.

In the following proof it is assumed that the envelope D of the one-parameter family of arcs (5:1) has a branch projecting backward from 6 to the point 1, as shown in the figure below. It is also assumed that the envelope D is not tangent anywhere to the n-space of tangency defined by the first n equations of (4:3) and by (4:6). That there can exist no conjugate point to 1 on E_{13} between 1 and 3 follows from the theory of the problem of Lagrange which applies to extremals E_{13}. To prove that there can exist no point conjugate to 1 on E_{32} between 3 and 2 consider the integral

$$I(E_{14} + E_{45} + D_{56}) = \int_{x_1}^{x_4(t)} f[x,\ y(x,t),\ y'(x,t)]dx$$

$$+ \int_{x_4(t)}^{x_5(t)} f[x,\ Y(x,\ t),\ Y'(x,\ t)]dx$$

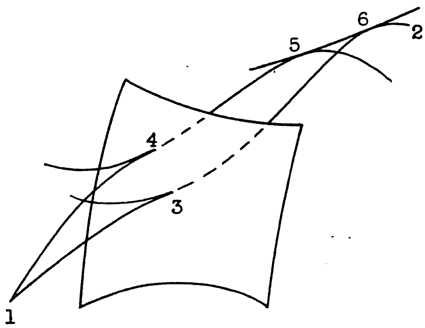
$$+ \int_{t}^{t_0} f\big[x(u),\ Y[x(u),u],\ Y'[x(u),u]\big]x'(u)\,du.$$

In this expression the equations

$$y_i = y_i(x,\ t) \qquad (x_1 \leqq x \leqq x_4),$$
$$y_i = Y_i(x,\ t) \qquad (x_4 \leqq x \leqq x_2),$$

define the one-parameter family of composite arcs, having an envelope D which has the equations

$$x = x_5(t), \qquad y_i = Y_i[x_5(t),\ t] = g_i(t).$$



Add $\Lambda_1 \phi_1(x,\ Y,\ Y') + \Lambda_\alpha \psi_\alpha(x,\ Y,\ Y')$ to the integrands of the second and third integrals in the above expression for $I(E_{14} + E_{45} + D_{56})$, and add $\lambda_\alpha \psi_\alpha$ to the integrand of the first integral. Then the derivative of I with respect to x_5 is

$$\frac{dI}{dx_5} = \frac{dI}{dt}\frac{dt}{dx_5} = -E(x, Y, Y', g', \lambda_\alpha, \lambda_1) + \lambda_1\phi_1(x, g, g')|^5$$

where g' is the slope of D. But since

$$Y_i'[x, t] = g_i'$$

at every point of D, it follows that

$$dI/dx_5 \equiv 0$$

in t. Consequently one obtains the result

$$I(E_{14} + E_{45} + E_{32}) = I(E_{32}).$$

Hence by the usual argument[1] $I(E_{12})$ cannot be a minimum.

6. Sufficiency proof. The four necessary conditions have been denoted by I, II, III, IV, the order being the same as they occur in this paper. The notation II' will be used to designate the condition II when the equality sign in expression (3:3) is omitted. The condition III' is defined for a composite arc as follows: The normal composite arc will be denoted by $E_{12} = E_{13} + E_{32}$, where E_{13} belongs to A and E_{32} belongs to B. For every element (x, y, y', λ) of E_{13} the inequality

$$F_{1y_i'y_k'}\pi_i\pi_k > 0$$

holds for every set $(\pi_i) \neq (0)$ satisfying the equations

$$\psi_{\alpha y_i'}\pi_i = 0,$$

whereas for every element (x, y, y', λ) of E_{32} the inequality

$$F_{2y_i'y_k'}\pi_i\pi_k > 0$$

is satisfied for every set $(\pi_i) \neq (0)$ satisfying the equations

$$\psi_{\alpha y_i'}\pi_i = 0, \qquad \phi_{1y_i'}\pi_i = 0.$$

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[1]See Bliss, loc. cit., p. 722.

Condition IV' excludes the point 2 as well as the interior points of E_{12} from being a conjugate point to 1. An arc E_{12} satisfies condition II_N' if the inequality

$$E(x, y, y', Y', \lambda) - \lambda_1 \phi_1(x, y, Y') > 0$$

holds for all sets (x, y, y', Y', λ) for which the sets (x, y, y', λ) are in a neighborhood of similar sets belonging to E_{12}, and $(x, y, Y') \neq (x, y, y')$ satisfies

$$\phi_1(x, y, Y') \geqq 0, \qquad \psi_\alpha(x, y, Y') = 0.$$

We can now state the following theorem.

SUFFICIENCY THEOREM FOR A STRONG RELATIVE MINIMUM. _If an admissible composite arc_ $E_{12} = E_{13} + E_{32}$ _with an extension normal on every subinterval satisfies the conditions_ II_N', III', IV', _then there exists a neighborhood_ M _of the points_ (x, y) of E_{12} _such that the inequality_ $I(C_{12}) > I(E_{12})$ _holds for every admissible arc_ C_{12} _satisfying_

$$\phi_1 \geqq 0, \qquad \psi_\alpha = 0,$$

which is in M, _and which is not identical with_ E_{12}.

In the first place since E_{12} is normal and satisfies I it is true that there exists a unique set of multipliers $\lambda_0 = 1$, λ_α, λ_1 and constants c_i which with the equations of E_{12} satisfy equations (3:2). In order to complete the proof, the following lemma is established.

LEMMA 6:1. _The condition_ III' _for a composite arc_ $E_{12} = E_{13} + E_{32}$ _implies that the determinants_ R_1 _and_ R_2 _defined by_ (4:1) _are different from zero on_ E_{13} _and_ E_{32} _respectively._

The fact that $R_1 \neq 0$ on E_{13} follows from the theory of the Lagrange problem which applies to E_{13}. However along the extremal E_{32} if the determinant $R_2 = 0$, the equations

$$(6:1) \qquad F_{y_i' y_k'} \pi_k + \lambda_\alpha \psi_{\alpha y_k'} + \lambda_1 \phi_{1 y_k'} = 0,$$

$$(6:2) \qquad \psi_{\alpha y_k'} \pi_k = 0, \qquad \phi_{1 y_k'} \pi_k = 0,$$

would have solutions $(\pi_i, \lambda_\alpha, \lambda_1) \neq (0, 0, 0)$ with π_k not all zero since by hypothesis the matrix

$$\begin{Vmatrix} \psi_{\alpha y_k'} \\ \phi_{1 y_k'} \end{Vmatrix}$$

must have rank p + 1. By multiplying equations (6:1) by $(\pi_1, \ldots, \pi_n)$ respectively, and adding the result, one obtains

$$F_{y_i' y_k'} \pi_i \pi_k = 0$$

on account of equations (6:2). But this contradicts the latter part of condition III' which states that

$$F_{2 y_i' y_k'} \pi_i \pi_k > 0$$

is satisfied for every set $(\pi_i) \neq (0)$ satisfying the equations

$$\psi_{\alpha y_i'} \pi_i = 0, \qquad \phi_{1 y_i'} \pi_i = 0.$$

Thus condition III' implies that $R_2 \neq 0$ on E_{32}.

According to the imbedding theorem in section 4 a point O can be chosen on the normal extension of E_{13}, so that E_{12} can be imbedded in an n-parameter family of composite extremals passing through O. From the first n of equations (4:4) it follows that along the n-space of tangency of this composite family we have

$$y_{ia}(x, a) = Y_{ia}(x, a)$$

for values of a close to a_0 which defines E_{12}. Hence $\Delta(x, a)$

defined in section 5 is continuous in x. On account of condition IV' this n-parameter family simply covers a region containing $E_{12} = E_{13} + E_{32}$. For $\Delta(x, a) \neq 0$ on x_1x_2 implies from implicit function theory that there exists a neighborhood M of the points (x, y) on E_{12} in which the equations

$$(6{:}3)\qquad \begin{aligned} y_i &= y_i(x, a) && (x_1 \leqq x \leqq x_3),\\ y_i &= Y_i(x, a) && (x_3 \leqq x \leqq x_2),\\ y_i(x_3, a) &= Y_i(x_3, a), && \end{aligned}$$

have solutions

$$a_i = a_i(x, y).$$

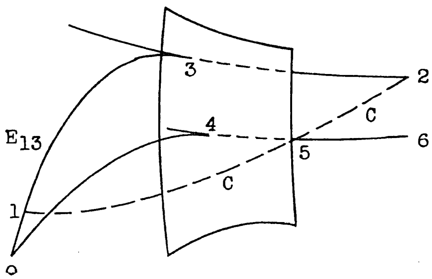
If the region M is taken sufficiently small the values (x, y, p, λ) belonging to M will remain in so small a neighborhood of the sets (x, y, y', λ) of E_{13} and E_{32} that according to II_N' the inequality

$$E(x, y, p, y', \lambda) - \lambda_1\phi_1(x, y, y') > 0$$

will be satisfied for all sets $(x, y, y') \neq (x, y, p)$ in M, where $p_i(x, y) = y_{ix}$ or Y_{ix} according as the notation refers to an arc of class A or B. Hence one may show that $I(E_{12})$ is a minimum in M as follows.

Let any admissible curve C in M satisfying the conditions $\phi_\beta(x, y, y') \geqq 0$ and $\psi_\alpha(x, y, y') = 0$ be defined by the equations

$$y_i = g_i(x).$$



The integral $I(x_5)$ is de-

$$I(x_5) = \int_{x_1}^{x_4} f[x,\ y(x,a),\ y'(x,a)]dx$$
$$+ \int_{x_4}^{x_5} f[x,\ Y(x,a),\ Y'(x,a)]dx$$
$$+ \int_{x_5}^{x_2} f[x,\ g(x,a),\ g'(x,a)]dx,$$

where $y(x, a)$ and $Y(x, a)$ define the unique composite arc $E_{06} = E_{04} + E_{46}$ joining an arbitrary point 5 on C_{12} to the point 0. If the point 5 lies between the point 0 and the point of tangency 4 on E_{04}, then $I(x_5)$ has the derivative

$$I'(x_5) = -E(x,\ y,\ y',\ g',\ \lambda_\alpha,\ \lambda_1),$$

whereas if 5 lies between 4 and 6 on E_{46}, then $I(x_5)$ has the derivative

$$I'(x_5) = -E(x,\ Y,\ Y',\ g',\ \lambda_\alpha,\ \lambda_1) + \lambda_1\phi_1(x,\ g,\ g').$$

In either case $I'(x_5)$ is less than or equal to zero, since $\lambda_1 = 0$ along arcs of class A. Moreover we have the equations

$$I(x_0) = I(E_{01}) + I(C_{12}),$$
$$I(x_2) = I(E_{01}) + I(E_{12}).$$

The condition II_N' now implies that $I(E_{12})$ is a minimum.

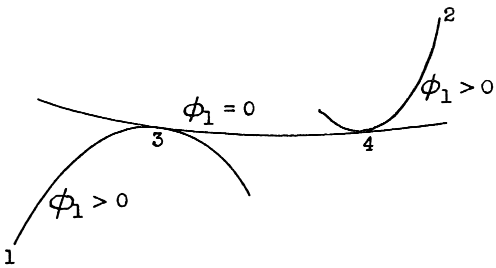
7. Generalizations to more complex arcs. In sections 4, 5, and 6 the composite arc as defined consisted of only two subarcs. The proofs made in these sections for the imbedding theorem, the Mayer condition, and for the sufficiency proof may be extended to apply to arcs without corners composed of n subarcs, ϕ_1 being zero on some of these subarcs, and greater than zero on the remaining subarcs. An imbedding theorem is here established for an arc without corners consisting of three sub-

arcs. It will then be obvious how to construct an imbedding theorem for an arc composed of four or more subarcs.

Consider an extremal $E_{12} = E_{13} + E_{34} + E_{42}$ without corners along which all the functions ϕ_β save one are greater than zero. The function ϕ_1 is to be greater than zero on the subarcs E_{13} and E_{42}, but zero on the arc E_{34}. Suppose that R_1 is different from zero on E_{13} and E_{42}, whereas R_2 is different from zero on E_{34}. From the imbedding theorem for composite arcs, we know that the arc $E_{14} = E_{13} + E_{34}$ may be imbedded in an n-parameter family of composite arcs of the same form. Denote this n-parameter family by

$$\begin{aligned} y_i &= h_i(x, a) && (x_1 \leqq x \leqq x_3), \\ y_i &= Y_i(x, a) && (x_3 \leqq x \leqq x_4), \\ \lambda_\alpha &= \lambda_\alpha(x, a), \\ \lambda_1 &= \lambda_1(x, a). \end{aligned}$$

Moreover it is known that under the above hypotheses E_{42} may be imbedded in a 2n-parameter family of arcs of class A. Let these extremals be defined by the equations



$$\begin{aligned} y_i &= y_i(x, b) \\ \lambda_\alpha &= \lambda_\alpha(x, b) && (x_4 \leqq x \leqq x_2). \\ \lambda_1 &= \lambda_1(x, b) \end{aligned}$$

The following conditions

$$(7:1)\qquad \begin{aligned} &Y_i(x, a) - y_i(x, b) = 0, \\ &F_{2y_i'}[x, Y(x,a), Y'(x,a), \lambda(x,a)] \\ &\quad - F_{1y_i'}[x, y(x,b), y'(x,b), \lambda(x,b)] = 0, \\ &\phi_1[x, y(x,b), y'(x,b)] = 0 \end{aligned}$$

hold at the point 4 on E_{12}. Moreover it is known that the determinant D satisfies the condition

$$D = \begin{vmatrix} y_{ib_k} \\ u_{ib_k} \end{vmatrix} \neq 0, \qquad u_i = F_{1y_i'}$$

at the point 4 on E_{12}. The functional determinant of the above expression with respect to x and b is

$$C = \begin{vmatrix} Y_i' - y_i' & -y_{ib_k} \\ F_{1y_i'}' - F_{2y_i'}' & -u_{ib_k} \\ \phi_1' & \phi_{1b_k} \end{vmatrix} .$$

Since at the point 4 we know that

$$\begin{aligned} y_i(x_4, b) &= Y_i(x_4, a), \\ y_i'(x_4, b) &= Y_i'(x_4, a), \end{aligned}$$

holds, it follows from Corollary 3:2 that the determinant C has the value

$$C = \phi_1' \begin{vmatrix} y_{ib_k} \\ u_{ib_k} \end{vmatrix} .$$

For purposes of the proof we assume that $\phi_1' \neq 0$ holds at the point 4. Hence it is true that $C \neq 0$ at the point 4 on E_{12}. Thus one may solve equations (7:1) for x and b as functions of a. Consequently under the above hypotheses the arc $E_{12} = E_{13} + E_{34} + E_{42}$ can be imbedded in an n-parameter family of arcs of the same kind, that is, consisting of three subarcs, two belonging to class A, and one belonging to class B. A proof similar to that given in section 4 shows that the members of this family have no corners. An imbedding theorem for an arc E_{12} composed of n subarcs, ϕ_1 being greater than zero on every other subarc, and zero on the remaining subarcs, can now be made by alternately repeating the processes described in section 6 and in this section.

The proof of the Mayer condition and the sufficiency proof for the arcs considered in this section are so similar to those given for a composite arc that they will need no repetition.

8. The analogue of the Mayer condition and the second variation. The following section establishes the condition IV, formulated geometrically in section 6, by means of the second variation. The equivalent problem stated in section 2 will now be used again.

For a normal extremal E_{12} of the equivalent problem it is known that if $\eta_i(x)$, $\zeta_\beta(x)$ is a set of admissible variations satisfying the equations

$$\begin{aligned} &\Psi_\alpha(x, \eta, \eta') = \psi_{\alpha y_i}\eta_i + \psi_{\alpha y_i'}\eta_i' = 0, \\ (8{:}1) \quad &\Phi_\beta(x, \eta, \eta') - 2z_\beta' \zeta_\beta' = \phi_{\beta y_i}\eta_i + \phi_{\beta y_i'}\eta_i' - 2z_\beta' \zeta_\beta', \\ &\eta_i(x_1) = \eta_i(x_2) = 0, \end{aligned}$$

then there exists a one-parameter family of admissible arcs

$$y_i = y_i(x, b), \qquad z_\beta = z_\beta(x, b)$$

containing E_{12} for $b = b_0$, and having the set $\eta_i(x)$, $\zeta_\beta(x)$ as its variations along E_{12}. In this section the second variation is to be calculated for an admissible arc E_{12} without corners satisfying the equations $\psi_\alpha = 0$ and $\phi_\beta(x, y, y') - z_\beta'^2 = 0$, and also satisfying the multiplier rule with multipliers $\lambda_0 = 1$, $\lambda_\alpha(x)$ and $\lambda_1(x)$.

When the members of the equations

$$\begin{aligned} I(b) &= \int_{x_1}^{x_2} f[x,\ y(x,b),\ y'(x,b)]dx, \\ 0 &= \Psi_\alpha[x,\ y(x,b),\ y'(x,b)], \\ 0 &= \Phi_\beta[x,\ y(x,b),\ y'(x,b)] - z_\beta'^2(x,\ b), \end{aligned}$$

are differentiated twice with respect to b, one may obtain[1] the equation

$$I''(b_0) = \int_{x_1}^{x_2} [2\omega(x,\ \eta,\ \eta') - 2\,\zeta_\beta'^2\lambda_\beta]dx, \tag{8:2}$$

where

$$2\omega = F_{y_i\,y_k}\,\eta_i\eta_k + 2F_{y_i y_k'}\eta_i\eta_k' + F_{y_i' y_k'}\,\eta_i'\eta_k'.$$

The accessory minimum problem for this problem consists in finding in the class of arcs $\eta_i(x)$, $\zeta_\beta(x)$ satisfying the equations (8:1) that one which minimizes the second variation (8:2). The case to be considered here is the one in which the minimizing arc is a composite one $E_{12} = E_{13} + E_{32}$. The extremals for the accessory minimum problem for this case must satisfy the differential equations

$$\begin{aligned} &\frac{d}{dx}\Omega_{\eta_i'} = \Omega_{\eta_i}, && \zeta_1'\lambda_1 + \mu_1 z_1' = d_1 \\ &\Psi_\alpha(x,\ \eta,\ \eta') = 0, && \Phi_1(x,\ \eta,\ \eta') - 2z_1'\zeta_1' = 0 \end{aligned} \tag{8:3}$$

where

$$\Omega = \mu_0(\omega - \zeta_1'^2\lambda_1) + \mu_\alpha\Psi_\alpha + \mu_1(\Phi_1 - 2z_1'\zeta_1')$$

and d_1 is a constant. From the transversality condition one finds that

$$\zeta_1'\lambda_1 + \mu_1 z_1'\,|^1 = \zeta_1'\lambda_1 + \mu_1 z_1'\,|^2 = 0$$

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[1]See Bliss, loc. cit., p. 723.

holds. Hence it is true that

$$\zeta_1'\lambda_1 + \mu_1 z_1' \equiv 0$$

on x_1x_2. Since λ_1 is zero on E_{13} and z_1' is zero on E_{32}, it follows that

$$\zeta_1'\lambda_1 \equiv \mu_1 z_1' \equiv 0 \qquad (x_1 \leqq x \leqq x_2)$$

holds. The functions $\eta(x)$, $\zeta_1(x)$ which define the minimizing arc for the accessory minimum problem are determined by the equations

$$\Omega_{\eta_i'} = \int_{x_1}^{x} \Omega_{\eta_i} dx + c_i,$$
$$\Psi_\alpha = 0, \qquad \Phi_1 - 2z_1'\zeta_1' = 0.$$

It follows that η_i', μ_α, μ_1 are continuous at x_3 as well as at all other points on x_1x_2 since $\eta_i(x)$ are continuous, and thus all three terms not involving η_i', μ_α, and μ_1 are continuous since the determinant of coefficients of η_i', μ_α, μ_1 is R_2 or R_1 which are different from zero on x_3x_2 and x_1x_3 respectively.

The functions η, ζ_1 are defined for the intervals x_1x_3 and x_3x_2 by the following equations,

$$\frac{d}{dx}\Omega_{2\eta_i'} = \Omega_{2\eta_i}, \qquad \Psi_\alpha = 0, \qquad \Phi_1 - 2z_1'\zeta_1' = 0, \qquad (x_3 \leqq x \leqq x_2), \tag{8:4}$$

$$\frac{d}{dx}\Omega_{1\eta_i'} = \Omega_{1\eta_i}, \qquad \Psi_\alpha = 0, \qquad (x_1 \leqq x \leqq x_3), \tag{8:5}$$

where

$$\Omega_1 = \mu_0\omega + \mu_\alpha\Psi_\alpha,$$
$$\Omega_2 = \mu_0\omega + \mu_\alpha\Psi_\alpha + \mu_1\Phi_1.$$

On the interval x_1x_3 the function $\zeta_1(x)$ is defined by the equation $\Phi_1 - 2z_1'\zeta_1' = 0$. The function $\zeta_1(x)$ is admissible, since $z_1' \neq 0$ for $(x_1 \leqq x < x_3)$ and the equations

$$\Phi_1[x_3, \eta(x_3-0), \eta'(x_3-0)] = \Phi_1[x_3, \eta(x_3+0), \eta'(x_3+0)] = 0,$$
$$z_1'(x_3) = 0,$$

hold at the point 3. Also the function $\zeta_1'(x)$ is zero at $x = x_3$ since the limit

$$\lim_{x=x_3-0} \Phi_1/z_1' \tag{8:6}$$

exists at $x = x_3$ and is zero. For if the numerator and denominator of the function

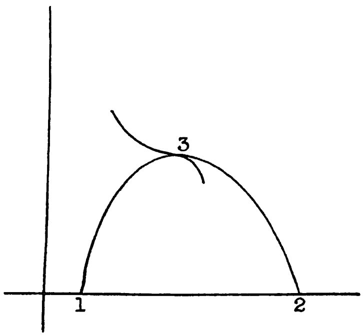
$$\Phi_1{}^2/z_1'{}^2$$

are differentiated separately, one gets

$$2\Phi_1\Phi_1'(x, \eta, \eta')/\phi_1'(x, y, y').$$

Since it has already been assumed that $\phi_1'(x_3, y, y') \neq 0$, it is true that the limit (8:6) is zero at 3. Thus the minimizing arcs for the accessory minimum problem are defined by equations (8:5) and (8:4).

In the following argument the determinants R_1 and R_2 defined by expression (4:1) are assumed to be different from zero on E_{13} and E_{32} respectively.



Definition. A value x_4 is said to be conjugate to x_1 on the arc $E_{13} + E_{32}$

if there exists an extremal of the accessory minimum problem of the form

$$\eta_i(x) = u_i(x), \qquad \mu_\alpha = \rho_\alpha(x) \qquad (x_1 \leqq x \leqq x_3),$$

$$\eta_i(x) = v_i(x), \qquad \begin{matrix} \mu_\alpha = \rho_\alpha(x) \\ \mu_1 = \rho_1(x) \end{matrix} \qquad (x_3 \leqq x \leqq x_2),$$

continuous and having continuous derivatives on x_1x_2 and satisfying

$$\eta_i(x_1) = \eta_i(x_4) = 0,$$

but not identically zero on x_1x_2.

ANALOGUE OF THE MAYER CONDITION. Suppose $E_{12} = E_{13} + E_{32}$ is a composite arc which is normal on every subinterval, and which is such that R_1 and R_2 are different from zero on E_{13} and E_{32} respectively. If E_{12} is a minimizing arc there can exist no point conjugate to 1 between the points 1 and 2.

To prove this statement consider the special solution

$$\left.\begin{matrix} \eta_i(x) \equiv u_i(x), & \mu_\alpha(x) \equiv \rho_\alpha(x), & (x_1 \leqq x \leqq x_4) \\ \eta_i(x) \equiv 0, & & (x_4 \leqq x \leqq x_2) \end{matrix}\right. \text{ when } x_4 \leqq x_3,$$

$$\left.\begin{matrix} \eta_i(x) \equiv u_i(x), & \mu_\alpha(x) \equiv \rho_\alpha(x), & (x_1 \leqq x \leqq x_3) \\ \eta_i(x) \equiv v_i(x), & \mu_\alpha(x) \equiv \rho_\alpha(x), & \\ & \mu_1(x) \equiv \rho_1(x), & (x_3 \leqq x \leqq x_4) \\ \eta_i(x) \equiv 0, & & (x_4 \leqq x \leqq x_2) \end{matrix}\right\} \text{ when } x_3 < x_4.$$

For this choice of $\eta_i(x)$ the second variation has the value

$$I''(b_0) = \int_{x_1}^{x_3} 2\omega(x, u, u')dx + \int_{x_3}^{x_4} 2\omega(x, v, v')dx,$$

which has the form

$$I''(b_0) = \int_{x_1}^{x_3}(u_i\Omega_{1u_i} + u_i'\Omega_{1u_i'} + \rho_\alpha\Omega_{1\rho_\alpha} + \rho_1\Omega_{1\rho_1})dx$$
$$+ \int_{x_3}^{x_4}(v_i\Omega_{2v_i} + v_i'\Omega_{2v_i'} + \rho_\alpha\Omega_{2\rho_\alpha} + \rho_1\Omega_{1\rho_1})dx.$$

Upon using equations (8:5) and (8:4) this integral may be evaluated to be

$$I''(b_0) = u_i\Omega_{1u_i'}\Big|_1^3 + v_i\Omega_{2v_i'}\Big|_3^4.$$

But since the relation

$$u_i\Omega_{1u_i'} = v_i\Omega_{2v_i'}$$

holds at the point 3, $I''(b_0)$ has the value

$$I''(b_0) = v_i\Omega_{2v_i'}\Big|^4 - u_i\Omega_{1u_i}\Big|^1,$$

or

$$I''(b_0) = v_i\Omega_{2v_i} = 0.$$

Since for a minimizing arc $\eta_i(x)$ the corner conditions

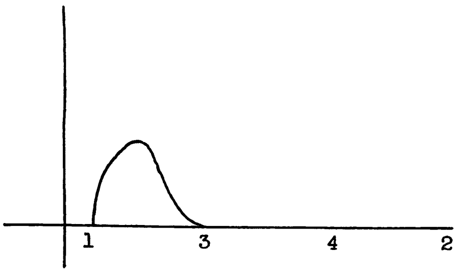
$$\Omega_{\eta_i'}[x_4, \eta, \eta'(x_4-0), \mu(x_4-0)]$$
$$-\Omega_{\eta_i'}[x_4, \eta, \eta'(x_4+0), \mu(x_4+0)] = 0,$$

hold,[1] and since

$$\Psi(x, \eta, \eta') = 0, \qquad \Phi_1(x, \eta, \eta') = 0, \qquad R_2 \neq 0,$$

hold, it is true that

$$v_i \equiv v_i' \equiv \rho_\alpha \equiv 0 \qquad (x_3 \leqq x \leqq x_4).$$



Similarly it follows that

$$u_i \equiv u_i' \equiv \rho_\alpha \equiv 0$$

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[1]See Bliss, loc. cit., p. 725-6.

for the interval $(x_1 \leqq x \leqq x_3)$. Thus the Mayer condition has been established.

9. Analogue of the necessary condition of Hestenes. In sections 10 and 11 a sufficiency proof is made for a composite arc without the assumption of normality. In order to lead up to this proof another necessary condition, analogous to the necessary condition IV_1, given by Hestenes for the problem of Bolza, is derived.

As shown in section 8 the minimizing arc $\eta_i(x)$ for the accessory minimum problem, when $E_{12} = E_{13} + E_{32}$ is a composite arc, is defined by (8:5) for $(x_1 \leqq x \leqq x_3)$ and by (8:4) for $(x_3 \leqq x \leqq x_2)$. The functional determinant of equations (8:5) with respect to η_i' and μ_α is R_1, whereas the functional determinant of equations (8:4) with respect to η_i', μ_α and μ_1 is R_2, R_1 and R_2 being defined by (4:1). Since we suppose that R_1 and R_2 are different from zero on E_{13} and E_{32} respectively, the equations

$$(9:1) \qquad \begin{aligned} t_i &= \Omega_{1\eta_i'}(x, \eta, \eta', \mu) \\ 0 &= \Psi_\alpha \end{aligned} \qquad (x_1 \leqq x \leqq x_3),$$

have the solutions

$$\begin{aligned} \eta_i' &= \Pi_i(x, \eta, t) \\ \mu_\alpha &= M_\alpha(x, \eta, t) \end{aligned} \qquad (x_1 \leqq x \leqq x_3),$$

and the equations

$$(9:2) \qquad \begin{aligned} t_i &= \Omega_{2\eta_i'}(x, \eta, \eta', \mu) \\ 0 &= \Psi_\alpha \\ 0 &= \Phi_1 \end{aligned} \qquad (x_3 \leqq x \leqq x_2),$$

have the solutions

$$\begin{aligned} \eta_i' &= K_i(x, \eta, t) \\ \mu_\alpha &= N_\alpha(x, \eta, t) \\ \mu_1 &= N_1(x, \eta, t) \end{aligned} \qquad (x_3 \leqq x \leqq x_2).$$

Equations (8:5) and (8:4) may be put into the usual canonical forms by introducing the Hamiltonian functions[1] H_1 and H_2. Equations (8:5) will then be equivalent to

$$\begin{aligned} \eta_i' &= H_{1t_i} \\ t_i' &= -H_{1\eta_i} \end{aligned} \qquad (x_1 \leqq x \leqq x_3), \tag{9:3}$$

and equations (8:4) will be equivalent to

$$\begin{aligned} \eta_i' &= H_{2t_i} \\ t_i' &= -H_{2\eta_i} \end{aligned} \qquad (x_3 \leqq x \leqq x_2). \tag{9:4}$$

For an arbitrary pair of solutions of (9:3), (η_i, t_i) and (η_i^*, t_i^*), it is known that

$$\eta_i t_i^* - \eta_i^* t_i = \text{constant} = c. \tag{9:5}$$

The same relation holds for an arbitrary pair of solutions of (9:4

DEFINITION. <u>The solution</u> (η_i^*, t_i^*) <u>is said to be conjugate to the solution</u> (η_i, t_i) <u>if equation</u> (9:5) <u>holds with</u> $c = 0$.

The sets $[\eta_{ik}, t_{ik}]$ form a conjugate system if any pair of them are conjugate to each other.

A conjugate system of solutions (η_{ik}, t_{ik}) of equations (9:3) and (9:4) may be found such that (η_{ik}, t_{ik}) are continuous. Suppose $\eta_{ik} = \sigma_{ik}$ and $t_{ik} = s_{ik}$ form a conjugate system of solutions of (9:4) on $(x_3 \leqq x \leqq x_2)$ where Ω has been replaced by Ω_2. The solutions $\eta_{ik} = u_{ik}$, $t_{ik} = r_{ik}$ of equations (9:3) with the end conditions

$$\begin{aligned} u_{ik}(x_3) &= \sigma_{ik}(x_3), \\ r_{ik}(x_3) &= s_{ik}(x_3), \end{aligned}$$

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[1] G. A. Bliss, <u>Problem of Bolza in the Calculus of Variations</u>, Lecture notes at the University of Chicago, Winter 1935, p. 74.

on the interval $(x_1 \leqq x \leqq x_3)$ are well defined. The system of solutions (η_{ik}, t_{ik}) thus obtained is continuous on the entire interval $(x_1 \leqq x \leqq x_2)$ and is a conjugate system.

ANALOGUE OF THE CONDITION OF HESTENES, IV_1. Suppose the arc $E_{12} = E_{13} + E_{32}$ satisfies the hypotheses assumed for the calculation of the second variation. The arc is said to satisfy condition IV_1 if the inequality

$$(\zeta_{ij}u_{ik} - \eta_{ij}v_{ik})a_j b_k \geqq 0 \tag{9:6}$$

is satisfied on $(x_1 \leqq x \leqq x_2)$, where the constants a_j and b_j satisfy the equations

$$\eta_{ij}a_j = u_{ik}b_k, \tag{9:7}$$

and where the set (η_{ij}, ζ_{ij}) is a conjugate system of solutions of equations (9:3), and (u_{ij}, v_{ij}) is a conjugate system of solutions of equations (9:4). The first set (η_{ij}, ζ_{ij}) is defined by the transversality and end-conditions for the point 1, whereas the second set (u_{ij}, v_{ij}) is defined by the corresponding conditions for the point 2. Every normal composite minimizing arc $E_{12} = E_{13} + E_{32}$, for which R_1 and R_2 are different from zero on E_{13} and E_{32} respectively, must satisfy the condition IV_1.

We will first prove the necessity of this condition on E_{32}. Let the set (η_{ij}, ζ_{ij}) be defined as follows

$$\begin{aligned} \eta_{ij} &= \tau_{ij}(x) \\ \zeta_{ij} &= r_{ij}(x) \end{aligned} \qquad (x_1 \leqq x \leqq x_3),$$

and

$$\begin{aligned} \eta_{ij} &= \sigma_{ij}(x) \\ \zeta_{ij} &= s_{ij}(x) \end{aligned} \qquad (x_3 \leqq x \leqq x_2).$$

Moreover let

$$u_{ik} \equiv m_{ik}(x), \quad v_{ik} \equiv n_{ik}(x) \qquad (x_1 \leqq x \leqq x_3),$$

and

$$u_{ik} \equiv p_{ik}(x), \quad v_{ik} \equiv q_{ik}(x) \qquad (x_3 \leqq x \leqq x_2).$$

Consider a solution a_j, b_k of (9:7) for a value x_4 between x_3 and x_2, and let the sets (τ_i, r_i), (σ_i, s_i), (m_i, n_i) and (p_i, q_i) represent

$$\tau_i = \tau_{ij}a_j,\quad r_i = r_{ij}a_j,\quad m_i = m_{ik}b_k,\quad n_i = n_{ik}b_k \qquad (x_1 \leqq x \leqq x_3),$$

$$\sigma_i = \sigma_{ij}a_j,\quad s_i = s_{ij}a_j,\quad p_i = p_{ik}b_k,\quad q_i = q_{ik}b_k \qquad (x_3 \leqq x \leqq x_2).$$

The arc defined by $\tau_i(x)$ on $(x_1 \leqq x \leqq x_3)$, by $\sigma_i(x)$ on $(x_3 \leqq x \leqq x_4)$, and by $p_i(x)$ on $(x_4 \leqq x \leqq x_2)$ is continuous by (9:7), and satisfies the equations $\Psi_\alpha = 0$ on x_1x_2 and $\Phi_1 = 0$ on x_3x_2. This arc gives to the second variation (8:2) the value

$$I''(b_0) = \int_{x_1}^{x_3} 2\omega(x, \tau, \tau')dx + \int_{x_3}^{x_4} 2\omega(x, \sigma, \sigma')dx + \int_{x_4}^{x_2} 2\omega(x, p, p')dx.$$

If $\mu_\alpha\Psi_\alpha$ is added to the integrand of the first integral, and if $\mu_\alpha\Psi_\alpha + \mu_1\Phi_1$ is added to the integrands of the second and third integrals, $I''(b_0)$ will have the form

$$I''(b_0) = \int_{x_1}^{x_3} 2\Omega_1(x, \tau, \tau')dx + \int_{x_3}^{x_4} 2\Omega_2(x, \sigma, \sigma')dx + \int_{x_4}^{x_2} 2\Omega_2(x, p, p')dx.$$

By the use of the homogeneity property of quadratic forms,[1] one may find the value of $I''(b_0)$ to be

$$I''(b_0) = \tau_i\Omega_{1\tau_i'}\Big|_1^3 + \sigma_i\Omega_{2\sigma_i'}\Big|_3^4 + p_i\Omega_{2p_i'}\Big|_4^2,$$

which reduces to

$$I''(b_0) = \tau_i r_i\Big|_1^3 + \sigma_i s_i\Big|_3^4 + p_i q_i\Big|_4^2.$$

Since the equations

$$\tau_i(x_3) = \sigma_i(x_3), \qquad r_i(x_3) = s_i(x_3),$$
$$\tau_i(x_1) = p_i(x_2) = 0, \qquad \sigma_i(x_4) = p_i(x_4),$$

hold, it follows that

$$I''(0) = s_i(x_4)p_i(x_4) - \sigma_i(x_4)q_i(x_4),$$

and this last expression for $I''(0)$ is

$$(\zeta_{ij}u_{ik} - \eta_{ij}v_{ik})\, a_j b_k \qquad (x_3 \leqq x \leqq x_2).$$

A similar proof can be made when the point 4 lies between the points 1 and 3. In event the point 4 is taken at the point 3, the second variation $I''(b_0)$ will have the form

$$I''(b_0) = \int_{x_1}^{x_3} 2\Omega_1(x, \tau, \tau')dx + \int_{x_3}^{x_2} 2\Omega_2(x, p, p')dx,$$

where $\tau_i(x)$ and $p_i(x)$ are defined above. The completion of the proof for this case is then easily made. Thus the condition IV_1 has been established.

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[1] Bliss, Problem of Bolza, p. 87.

10. Sufficiency proof without the assumption of normality. One may now prove the following theorem with the aid of the preceding section and some auxiliary lemmas.

THEOREM 10:1. *Let $E_{12} = E_{13} + E_{32}$ be an admissible composite arc, satisfying the conditions II'_N, III', IV_1', with a set of multipliers $\lambda_0 = 1$, $\lambda_\alpha(x)$, $\lambda_1(x)$. Then there exists a neighborhood F of $E_{12} = E_{13} + E_{32}$ such that $J(C_{12}) > J(E_{12})$ for every admissible arc C_{12} in F joining the points 1 and 2, satisfying*

$$\psi_\alpha = 0, \qquad \phi_1 \geqq 0,$$

and distinct from E_{12}.

Consider a one-parameter family of composite arcs

$$\begin{array}{llll} & y_i = y_i(x, a), & \lambda_\alpha = \lambda_\alpha(x, a) & (x_1 \leqq x \leqq x_3), \\ (10:1) & y_i = Y_i(x, a), & \lambda_\alpha = \lambda_\alpha(x, a) & (x_3 \leqq x \leqq x_2), \\ & & \lambda_1 = \lambda_1(x, a), & \end{array}$$

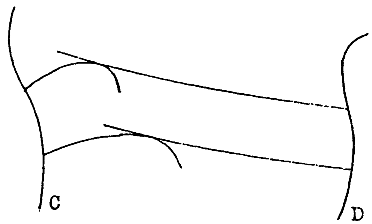
and a set of functions $x_1(t)$, $x_2(t)$, $a(t)$ having continuous derivatives, and such that y_i, Y_i, y_{ix}, Y_{ix}, λ_β and λ_1 have continuous first partial derivatives in a neighborhood of the sets (x, a) defined by

$$x_1(t) \leqq x \leqq x_2(t), \qquad a(t) \qquad (t' \leqq t \leqq t'').$$

The end points 1 and 2 of the curves describe two arcs C and D, the equations of C being

$$x = x_1(t),$$
$$y_i = y_i[x_1(t), a(t)],$$

and the curve D being defined by



$$x = x_2(t), \qquad y_i = Y_i[x_2(t), a(t)].$$

The differentials dx, dy_i along the curves C and D are given by the equations

$$(10{:}2) \qquad \begin{aligned} dx_1 &= x_1'(t)dt, & dy_i &= y_{ix_1}dx_1 + y_{ia}da, \\ dx_2 &= x_2'(t)dt, & dy_i &= Y_{ix_2}dx_2 + Y_{ia}da. \end{aligned}$$

Along the particular composite extremal arc defined by a value t the integral I has the form

$$I(t) = \int_{x_1(t)}^{x_3(t)} F_1[x, y(x,a), y'(x,a), \lambda]dx + \int_{x_3(t)}^{x_2(t)} F_2[x, y(x,a), y'(x,a), \lambda]dx.$$

The derivative of I with respect to t is

$$\frac{dI}{dt} = F_1\frac{dx}{dt}\Big|_1^3 + \int_{x_1}^{x_3}(F_{1y_i}y_{ia}\frac{da}{dt} + F_{1y_i'}y_i'{}_a\frac{da}{dt})dx + F_2\frac{dx}{dt}\Big|_3^2 + \int_{x_3}^{x_2}(F_{2y_i}Y_{ia}\frac{da}{dt} + F_{2y_i'}Y_i'{}_a\frac{da}{dt})dx.$$

Upon integrating by parts and using equations (10:2) one gets

$$(10{:}3) \qquad dI = [F - y_i'F_{y_i'}]dx + F_{y_i'}dy_i\Big|_1^2.$$

The symbol I^* denotes the integral

$$I^* = \int\left[[F - y_i'F_{y_i'}]dx + F_{y_i'}dy_i\right].$$

By integrating (10:3) from t' to t" one obtains the following result.

LEMMA 10:1. _If the composite extremal arcs of the one parameter family (10:1) corresponding to the values_ t' _and_ t" _of the parameter_ t _are_ E_{34} _and_ E_{56} _respectively, then_

$$I(E_{56}) - I(E_{34}) = I^*(D_{46}) - I^*(C_{35}).$$

Definition of a Field. A field is a region $\mathcal{F}$ of (x, y) space with a set of slope-functions and multipliers

$$p_i(x, y), \quad \ell_0 = 1, \quad \ell_1(x, y), \quad \ell_\beta(x, y),$$

having continuous first partial derivatives in $\mathcal{F}$, and such that the sets (x, y, p) are admissible and satisfy $\psi_\alpha = 0$, $\phi_1 \geqq 0$, and make the I^* integral

$$I^* = \int \{[F - p_i F_{y_i'}]dx + F_{y_i'} dy_i\}$$

independent of the path in $\mathcal{F}$.

LEMMA[1] 10:2. Let $E_{12} = E_{13} + E_{32}$ be a composite arc such that $R_1 \neq 0$ on E_{13} and $R_2 \neq 0$ on E_{32}, and having a conjugate system of solutions (U_{ik}, V_{ik}) of the accessory equations (9:3) and (9:4). This solution has the form

$$\begin{aligned} U_{ik} &= u_{ik} \\ V_{ik} &= v_{ik} \end{aligned} \qquad (x_1 \leqq x \leqq x_3),$$

$$\begin{aligned} U_{ik} &= \sigma_{ik} \\ V_{ik} &= s_{ik} \end{aligned} \qquad (x_3 \leqq x \leqq x_2).$$

Moreover suppose $|U_{ik}| \neq 0$ on x_1x_2. Then E_{12} is an extremal of a field $\mathcal{F}$ consisting of an n-parameter family of composite arcs

$$\begin{aligned} y_i &= y_i(x, \alpha_1, \ldots, \alpha_n), & r_i &= F_{1y_i'} & [x_1 \leqq x \leqq x_3(\alpha)], \\ y_i &= Y_i(x, \alpha_1, \ldots, \alpha_n), & R_i &= F_{2y_i'} & [x_3(\alpha) \leqq x \leqq x_2], \end{aligned}$$

and containing E_{12} for values (x, α) satisfying

$$(x_1 \leqq x \leqq x_2), \qquad \alpha_k = 0, \qquad (k = 1, \ldots, n).$$

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[1] Bliss, loc. cit., Problem of Bolza, p. 103.

<u>The functions</u> y_i, Y_i, y_{ix}, Y_{ix}, r_i, R_i <u>have continuous first partial derivatives in a neighborhood of the values</u> (x, α) <u>belonging to</u> E_{12}, <u>and the variations of that family along</u> E_{12} <u>have the values</u>

$$y_{i\alpha_k}(x, 0) = u_{ik}(x), \quad r_{i\alpha_k}(x, 0) = v_{ik}(x) \quad [x_1 \leqq x \leqq x_3(0)],$$
$$Y_{i\alpha_k}(x, 0) = \sigma_{ik}(x), \quad R_{i\alpha_k}(x, 0) = s_{ik}(x) \quad [x_3(0) \leqq x \leqq x_2].$$

The proof of this lemma can be obtained by an extension of a lemma given by Bliss for the problem of Bolza. By a proof whose details are identical with those given for the imbedding theorem in section 4, it may be proved that the composite arc $E_{12} = E_{13} + E_{32}$ may be imbedded in a 2n-parameter family of composite arcs. As shown by Bliss[1] it is true that this 2n-parameter family may have the form

$$\begin{aligned} y_i &= y_i(x, \alpha_1, \ldots, \alpha_n) \\ r_i &= r_i(x, \alpha_1, \ldots, \alpha_n) \end{aligned} \qquad [x_1 \leqq x \leqq x_3(\alpha)],$$

$$\begin{aligned} y_i &= Y_i(x, \alpha_1, \ldots, \alpha_n) \\ R_i &= R_i(x, \alpha_1, \ldots, \alpha_n) \end{aligned} \qquad [x_3(\alpha) \leqq x \leqq x_2],$$

containing E_{12} for $(\alpha_1, \ldots, \alpha_n) = (0, \ldots, 0)$. It follows from the theory given by Bliss and from the imbedding theorem mentioned that the equations

$$y_{i\alpha_k}(x_3) = Y_{i\alpha_k}(x_3) = \sigma_{ik}(x_3) = u_{ik}(x_3)$$

hold. The remainder of the proof of the above lemma is so similar to that made by Bliss that it will not be repeated.

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[1]Bliss, <u>loc. cit.</u>, Problem of Bolza, p. 105.

THEOREM 10:2. A FUNDAMENTAL SUFFICIENCY THEOREM. *If an arc $E_{12} = E_{13} + E_{32}$ is a composite arc in a field $\mathcal{F}$ and satisfies the condition II_N', then $I(E_{12})$ is a minimum as described in Theorem* 10:1.

In view of the assumption that E_{12} satisfies the condition II_N', the field $\mathcal{F}$ may be restricted to a sufficiently small neighborhood of E_{12} so that all the elements $[x, y, p(x,y), \ell(x,y)]$ belonging to $\mathcal{F}$ lie in the neighborhood N. Then at all points of $\mathcal{F}$ the condition

$$E(x, y, p(x,y), Y', \ell) - \ell_1 \phi(x, y, Y') > 0$$

must be satisfied for every set $(x, y, Y') \neq (x, y, y')$ and satisfying $\phi_1(x, y, Y') \geqq 0$, $\psi_\alpha(x, y, Y') = 0$. Since

$$I^*(E_{12}) = I(E_{12})$$

it is true that

$$\begin{aligned} I(C_{12}) - I(E_{12}) &= I(C_{12}) - I^*(E_{12}) \\ &= \int_{x_1}^{x_2} f(x, Y, Y')dx - \int_{x_1}^{x_2} ([F - p_i F_{y_i'}]dx + F_{y_i'} dy_i) \\ &= \int_{x_1}^{x_2} [F - \ell_1 \phi_1(x, y, Y') - F(x, y, p_i) - (p_i - Y_i')F_{y_i'}]dx \\ &= \int_{x_1}^{x_2} [E(x, y, p, Y', \lambda) - \ell_1 \phi_1(x, y, Y')]dx. \end{aligned}$$

Hence the theorem is established.

LEMMA[1] 10:3. *Let $E_{12} = E_{13} + E_{32}$ be an admissible composite arc satisfying conditions III', IV_1' with a set of multipliers $\lambda_0 = 1$, λ_α, λ_1. Then there exists a conjugate system*

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[1]Bliss, *loc. cit.*, Problem of Bolza, p. 112.

of solutions $U_{ik}(x)$, $V_{ik}(x)$ *of the canonical accessory equations* (9:3) *and* (9:4) *with* $|U_{ik}| \neq 0$ *on* x_1x_2.

The proof of this lemma is almost identical with that given by Bliss in his notes on the problem of Bolza, hence it will not be repeated.

Now one is in a position to prove the sufficiency theorem 10:1. According to Lemma 6:1 an admissible arc $E_{12} = E_{13} + E_{32}$ satisfying III' must be such that R_1 and R_2 are different from zero on E_{13} and E_{32} respectively. Condition IV_1' and Lemma 10:3 imply the existence of a conjugate system of solutions $U_{ik}(x)$, $V_{ik}(x)$ of the canonical equations (9:3) and (9:4) with determinant $|U_{ik}(x)| \neq 0$ on x_1x_2. Hence by Lemma 10:2 the composite arc E_{12} is in a field $\mathcal{F}$, contained in an n-parameter family of composite arcs. Thus by these conditions and II_N' it follows that the hypotheses of the sufficiency Theorem 10:2 are fulfilled, and therefore the conclusion of Theorem 10:1 is established.