NONLINEAR PROGRAMMING

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1. Introduction

Linear programming deals with problems such as (see [4], [5]): to maximize a linear function $g(x) \equiv \sum c_i x_i$ of n real variables $x_1, \ldots, x_n$ (forming a vector x) constrained by $m + n$ linear *inequalities*,

$$f_h(x) \equiv b_h - \sum a_{hi} x_i \geqq 0,\ x_i \geqq 0, \qquad h = 1, \ldots, m;\ i = 1, \ldots, n.$$

This problem can be transformed as follows into an equivalent saddle value (minimax) problem by an adaptation of the calculus method customarily applied to constraining *equations* [3, pp. 199–201]. Form the Lagrangian function

$$\phi(x, u) \equiv g(x) + \sum u_h f_h(x).$$

Then, a particular vector x^0 maximizes $g(x)$ subject to the $m + n$ constraints if, and only if, there is some vector u^0 with nonnegative components such that

$$\phi(x, u^0) \leqq \phi(x^0, u^0) \leqq \phi(x^0, u) \quad \text{for all nonnegative } x, u.$$

Such a saddle point (x^0, u^0) provides a solution for a related zero sum two person game [8], [9], [12]. The bilinear symmetry of $\phi(x, u)$ in x and u yields the characteristic duality of linear programming (see section 5, below).

This paper formulates necessary and sufficient conditions for a saddle value of any differentiable function $\phi(x, u)$ of nonnegative arguments (in section 2) and applies them, through a Lagrangian $\phi(x, u)$, to a maximum for a differentiable function $g(x)$ constrained by inequalities involving differentiable functions $f_h(x)$ mildly qualified (in section 3). Then, it is shown (in section 4) that the above equivalence between an inequality constrained maximum for $g(x)$ and a saddle value for the Lagrangian $\phi(x, u)$ holds when $g(x)$ and the $f_h(x)$ are merely required to be concave (differentiable) functions for nonnegative x. (A function is *concave* if linear interpolation between its values at any two points of definition yields a value not greater than its actual value at the point of interpolation; such a function is the negative of a *convex* function—which would appear in a corresponding minimum problem.) For example, $g(x)$ and the $f_h(x)$ can be quadratic polynomials in which the pure quadratic terms are negative semidefinite (as described in section 5).

In terms of *activity analysis* [11], x can be interpreted as an activity vector, $g(x)$ as the resulting output of a desired commodity, and the $f_h(x)$ as unused balances of primary commodities. Then the Lagrange multipliers u can be interpreted as a price vector [13, chap. 8] corresponding to a unit price for the desired commodity, and the Lagrangian function $\phi(x, u)$ as the combined worth of the output of the desired commodity and the unused balances of the primary commodities. These

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price interpretations seem to relate closely to the price theory in the contemporary paper of K. J. Arrow [1].

A "vector" maximum—of T. C. Koopmans' *efficient point* type [11]—for several concave functions $g_1(x), \ldots, g_p(x)$ can be transformed into a "scalar" maximum for $g(x) \equiv \sum v_k^0 g_k(x)$ by suitable choice of positive constants v_k^0 (as described in section 6). These positive constants can be interpreted as prices to be assigned (for efficient production) to several desired commodities with outputs $g_k(x)$ produced by the activity vector x.

Likewise, a maximum for min $[g_1(x), \ldots, g_p(x)]$ can be transformed into a maximum for $g(x) \equiv \sum v_k^0 g_k(x)$ by suitable choice of nonnegative constants v_k with unit sum (as described in section 7). Such a maximum of a minimum component, example, is the objective of the first player in a zero sum two person game [12].

Modifications resulting from changes in the $m + n$ basic constraints are also considered (in section 8).

Throughout this paper it is assumed that the functions occurring are differentiable. But it seems to be an interesting consequence of the directional derivative properties of general convex (or concave) functions [2, pp. 18–21] that the equivalence between an inequality constrained maximum for $g(x)$ and a saddle value for the Lagrangian $\phi(x, u)$ still holds when the assumption of differentiability is dropped. Then proofs would involve the properties (of linear sum, intersection, and polar) of general closed convex "cones" rather than those of the polyhedral convex "cones" [7], [14] that occur implicitly in this paper through homogeneous linear differential inequalities. However, to assure finite directional derivatives at boundary points of the orthant of nonnegative x, one needs some mild requirement. For this purpose, it is certainly sufficient to assume that the functions are convex (or concave) in some open region containing the orthant of nonnegative x.

NOTATION. Vectors, denoted usually by lower case roman letters, will be treated as one column matrices, unless transposed by an accent $'$ into one row matrices. Vector inequalities or equations stand for systems of such inequalities or equations, one for each component. Thus $x \geqq 0$ means that all the components of the vector x are nonnegative. Rectangular matrices and mapping operators will be denoted by capital letters.

2. Necessary and sufficient conditions for a saddle value

Let $\phi(x, u)$ be a differentiable function of an n-vector x with components $x_i \geqq 0$ and an m-vector u with components $u_h \geqq 0$. Taking partial derivatives, evaluated at a particular point x^0, u^0, let

$$\phi_x^0 = \left[\frac{\partial \phi}{\partial x_i}\right]^0, \qquad \phi_u^0 = \left[\frac{\partial \phi}{\partial u_h}\right]^0.$$

Here ϕ_x^0 is an n-vector and ϕ_u^0 an m-vector.

SADDLE VALUE PROBLEM. *To find nonnegative vectors x^0 and u^0 such that*

$$\phi(x, u^0) \leqq \phi(x^0, u^0) \leqq \phi(x^0, u) \textit{ for all } x \geqq 0,\ u \geqq 0.$$

LEMMA 1. *The conditions*

$$(1) \qquad \phi_x^0 \leqq 0, \qquad \phi_x^{0\prime} x^0 = 0, \qquad x^0 \geqq 0$$

(2) $$\phi_u^0 \geqq 0, \qquad \phi_u^{0\prime} u^0 = 0, \qquad u^0 \geqq 0$$

are necessary that x^0, u^0 *provide a solution for the saddle value problem.*

PROOF. The components of ϕ_x^0 and ϕ_u^0 must vanish except possibly when the corresponding components of x^0 and u^0 vanish, in which case they must be nonpositive and nonnegative, respectively. Hence (1) and (2) must hold.

LEMMA 2. *Conditions* (1), (2) *and*

(3) $$\phi(x, u^0) \leqq \phi(x^0, u^0) + \phi_x^{0\prime}(x - x^0)$$

(4) $$\phi(x^0, u) \geqq \phi(x^0, u^0) + \phi_u^{0\prime}(u - u^0)$$

for all $x \geqq 0$, $u \geqq 0$, *are sufficient that* x^0, u^0 *provide a solution for the saddle value problem.*

PROOF. Applying (3), (1), (2), (4) in turn, one has

$$\begin{aligned}\phi(x, u^0) &\leqq \phi(x^0, u^0) + \phi_x^{0\prime}(x - x^0)\\ &\leqq \phi(x^0, u^0)\\ &\leqq \phi(x^0, u^0) + \phi_u^{0\prime}(u - u^0)\\ &\leqq \phi(x^0, u)\end{aligned}$$

for all $x \geqq 0$, $u \geqq 0$.

Conditions (3) and (4) are not as artificial as may appear at first sight. They are satisfied if $\phi(x, u^0)$ is a concave function of x and $\phi(x^0, u)$ is a convex function of u (see section 4).

3. Lagrange multipliers for an inequality constrained maximum

Let $x \to u = F(x)$ be a differentiable mapping of nonnegative n-vectors x into m-vectors u. That is, $F(x)$ is an m-vector whose components $f_1(x), \ldots, f_m(x)$ are differentiable functions of x defined for $x \geqq 0$. Let $g(x)$ be a differentiable function of x defined for $x \geqq 0$. Taking partial derivatives, evaluated at x^0, let

$$F^0 = [\partial f_h/\partial x_i]^0, \qquad g^0 = [\partial g/\partial x_i]^0.$$

Here F^0 is an m by n matrix and g^0 an n-vector.

MAXIMUM PROBLEM. *To find an* x^0 *that maximizes* $g(x)$ *constrained by* $F(x) \geqq 0$, $x \geqq 0$.

CONSTRAINT QUALIFICATION. Let x^0 belong to the boundary of the *constraint set* of points x satisfying $Fx \geqq 0$, $x \geqq 0$. Let the inequalities $F(x^0) \geqq 0$, $Ix^0 \geqq 0$ (where I is the identity matrix of order n) be separated into

$$F_1 x^0 = 0, \qquad I_1 x^0 = 0 \quad \text{and} \quad F_2 x^0 > 0, \qquad I_2 x^0 > 0.$$

It will be assumed for each x^0 of the boundary of the constraint set that any vector differential dx satisfying the homogeneous linear inequalities

(5) $$F_1^0 dx \geqq 0, \qquad I_1 dx \geqq 0$$

is tangent to an arc contained in the constraint set; that is, to any dx satisfying (5) there corresponds a differentiable arc $x = a(\theta)$, $0 \leqq \theta \leqq 1$, contained in the constraint set, with $x^0 = a(0)$, and some positive scalar λ such that $[da/d\theta]^0 = \lambda dx$. This assumption is designed to rule out singularities on the boundary of the constraint set, such as an outward pointing "cusp." For example, the constraint set in two dimensions determined by

$$(1 - x_1)^3 - x_2 \geqq 0\,, \qquad x_1 \geqq 0\,, \qquad x_2 \geqq 0$$

does not satisfy the constraint qualification at the boundary point $x_1^0 = 1$, $x_2^0 = 0$, since it does not contain an arc leading from this point in the direction $dx_1 = 1$, $dx_2 = 0$. At such a singular point condition (1) in theorem 1, below, may fail to hold for any u^0—as would be the case for $g(x) \equiv x_1$ subject to the above constraints.

Treating the vector u as a set of m nonnegative Lagrange multipliers [10], form the function

$$\phi\,(x, u) \equiv g\,(x) + u'Fx\,.$$

Then

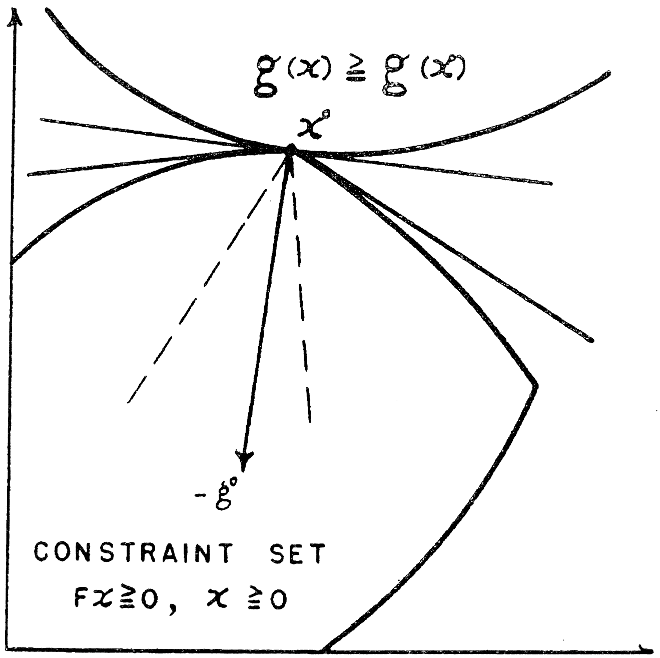
$$\phi_x^0 = g^0 + F^{0\prime}u^0\,, \qquad \phi_u^0 = Fx^0\,.$$

THEOREM 1. *In order that x^0 be a solution of the maximum problem, it is necessary that x^0 and some u^0 satisfy conditions* (1) *and* (2) *for* $\phi(x, u) \equiv g(x) + u'Fx$.)

PROOF. Let x^0 maximize $g(x)$ constrained by $Fx \geqq 0$, $x \geqq 0$ (subject to the above constraint qualification). Then, the inequality $g^{0\prime}dx \leqq 0$ must hold for all vector differentials dx satisfying (5). But, it is a fundamental property of homogeneous linear inequalities (indicated by H. Minkowski and proved by J. Farkas at the turn of the century) that an inequality $b'x \geqq 0$ holds for all n-vectors x satisfying a system of m inequalities $Ax \geqq 0$ only if $b = A't$ for some m-vector $t \geqq 0$ [6, pp. 5–7], [7, corollary to theorem 2], [9, lemma 1] and [14, theorem 3]. Hence

$$-g^0 = F_1^{0\prime}u_1^0 + I_1'w_1^0 \qquad \text{for some } u_1^0 \geqq 0,\ w_1^0 \geqq 0\,.$$

This equation expresses the intuitively evident geometric fact that at the point x^0 the outward normal $-g^0$ to the set of points x for which $g(x) \geqq g(x^0)$ must belong



to the convex polyhedral "cone" of inward normals to the constraint set. Of course, if x^0 is an interior point of the latter set, then F_1^0 and I_1 are both vacuous. In this case x^0 maximizes $g(x)$ independent of the constraints, so $g^0 = 0$ and conditions (1), (2) hold for $u^0 = 0$.

The above equation may be rewritten as

$$-g^0 = F^{0\prime}u^0 + w^0 \qquad \text{for some } u^0 \geqq 0\,,\ w^0 \geqq 0$$

by adding zeros as components to u_1^0 and w_1^0 to form u^0 and w^0. Consequently,

$$\phi_x^0 = g^0 + F^{0\prime}u^0 \leqq g^0 + F^{0\prime}u^0 + w^0 = 0\,.$$

At the same time, since $w^{0\prime}x^0 = w_1^{0\prime}I_1x^0 = 0$,

$$\phi_x^{0\prime}x^0 = g^{0\prime}x^0 + u^{0\prime}F^0x^0 = 0\,.$$

Moreover,

$$\phi_u^0 = Fx^0 \geqq 0 \quad \text{and} \quad \phi_u^{0\prime}u^0 = u^{0\prime}Fx^0 = u_1^{0\prime}F_1x^0 = 0\,.$$

This completes the proof of theorem 1.

THEOREM 2. *In order that x^0 be a solution of the maximum problem, it is sufficient that x^0 and some u^0 satisfy conditions* (1), (2), *and* (3) *for* $\phi(x, u) \equiv g(x) + u'Fx$.

PROOF. From (3), (1), and (2) one has that

$$\begin{aligned} g(x) + u^{0\prime}Fx = \phi(x, u^0) &\leqq \phi(x^0, u^0) + \phi_x^{0\prime}(x - x^0) \\ &\leqq \phi(x^0, u^0) = g(x^0) + u^{0\prime}Fx^0 = g(x^0) \ \text{ for all } x \geqq 0\,. \end{aligned}$$

But $u^{0\prime}Fx \geqq 0$ for all x satisfying $Fx \geqq 0$. Hence $g(x) \leqq g(x^0)$ for all x satisfying the constraints $Fx \geqq 0$, $x \geqq 0$. This proves theorem 2.

One notes in theorem 2 that (3) need only hold for $Fx \geqq 0$, $x \geqq 0$.

4. Convexity-concavity properties and the equivalence theorem

In this section restrictions are placed on Fx and $g(x)$ which will insure the equivalence of solutions of the maximum problem and the saddle value problem for $\phi(x, u) \equiv g(x) + u'Fx$.

DEFINITIONS. *A function $f(x)$ is* convex *if*

$$(1-\theta)f(x^0) + \theta f(x) \geqq f\{(1-\theta)x^0 + \theta x\}$$

for $0 \leqq \theta \leqq 1$ and all x^0 and x in the (convex) region of definition of $f(x)$. A function $f(x)$ is concave *if $-f(x)$ is convex* (that is, if the interpolation inequality holds with $\leqq$ instead of $\geqq$).

LEMMA 3. *If $f(x)$ is convex and differentiable, then*

$$f(x) \geqq f(x^0) + f^{0\prime}(x - x^0) \qquad \left(\textit{where } f^0 = \left[\frac{\partial f}{\partial x_i}\right]^0\right)$$

for all x^0 and x in the region of definition. [With $f(x)$ concave, the inequality is reversed.]

PROOF. From the above definition of convexity one has, for $0 < \theta \leqq 1$,

$$f(x) - f(x^0) \geqq \frac{f\{x^0 + \theta(x - x^0)\} - f(x^0)}{\theta}$$

Hence, in the limit,

$$f(x) - f(x^0) \geqq f^{0\prime}(x - x^0) .$$

THEOREM 3 (Equivalence theorem). *Let the functions* $f_1(x), \ldots, f_m(x), g(x)$ *be concave as well as differentiable for* $x \geqq 0$. *Then,* x^0 *is a solution of the maximum problem if, and only if,* x^0 *and some* u^0 *give a solution of the saddle value problem for* $\phi(x, u) \equiv g(x) + u'Fx$.

PROOF. By lemma 3 (for concavity)

$$Fx \leqq Fx^0 + F^0(x - x^0)$$
$$g(x) \leqq g(x^0) + g^{0\prime}(x - x^0)$$

for all $x^0 \geqq 0$ and $x \geqq 0$. Hence, for any $u^0 \geqq 0$,

$$\begin{aligned}\phi(x, u^0) &= g(x) + u^{0\prime}Fx\\ &\leqq g(x^0) + u^{0\prime}Fx^0 + (g^{0\prime} + u^{0\prime}F^0)(x - x^0)\\ &= \phi(x^0, u^0) + \phi_x^{0\prime}(x - x^0) .\end{aligned}$$

That is, condition (3) holds for all $x^0 \geqq 0$ and $x \geqq 0$. Under these circumstances, theorems 1 and 2 combine to make conditions (1) and (2) both necessary and sufficient that x^0 provide a solution for the maximum problem.

Condition (4) holds automatically, since the linearity of $\phi(x, u)$ with respect to u implies that

$$\phi(x^0, u) = \phi(x^0, u^0) + \phi_u^{0\prime}(u - u^0)$$

identically. So lemmas 1 and 2 combine to make conditions (1) and (2) both necessary and sufficient that x^0 and u^0 provide a solution for the saddle value problem. This completes the proof of theorem 3.

5. Quadratic and linear problems

LEMMA 4. *A quadratic form*

$$x'Qx = \sum \sum q_{ij}x_i x_j$$

is a convex function for all x, *if* $x'Qx \geqq 0$ *for all* x (*that is, if the form is positive semi-definite*).

PROOF. From the hypothesis, one has

$$\theta(x - x^0)'Q(x - x^0) \geqq \theta^2(x - x^0)'Q(x - x^0)$$

for all $0 \leqq \theta \leqq 1$ and all x, x^0. Hence

$$\begin{aligned}&(1 - \theta)x^{0\prime}Qx^0 + \theta x'Qx\\ &\quad = x^{0\prime}Qx^0 + \theta x^{0\prime}Q(x - x^0) + \theta(x - x^0)'Qx^0 + \theta(x - x^0)'Q(x - x^0)\\ &\quad \geqq x^{0\prime}Qx^0 + \theta x^{0\prime}Q(x - x^0) + \theta(x - x^0)'Qx^0 + \theta^2(x - x^0)'Q(x - x^0)\\ &\quad = \{x^0 + \theta(x - x^0)\}'Q\{x^0 + \theta(x - x^0)\}\\ &\quad = \{(1 - \theta)x^0 + \theta x\}'Q\{(1 - \theta)x^0 + \theta x\}\end{aligned}$$

for all $0 \leqq \theta \leqq 1$ and all x, x^0.

QUADRATIC MAXIMUM PROBLEM. *To find an x^0 that maximizes*

$$g(x) \equiv \sum c_i x_i - \sum \sum c_{ij} x_i x_j$$

constrained by the $m + n$ inequalities

$$f_h(x) \equiv b_h - \sum a_{hi} x_i - \sum \sum a_{hij} x_i x_j \geqq 0 \quad \text{and} \quad x_i \geqq 0 .$$

It is assumed that the quadratic forms in the above double sums (including the preceding sign) are nonpositive for all x (that is, negative semidefinite).

From lemma 4 it follows that these quadratic functions $f_h(x)$ and $g(x)$ are concave for all x, since their linear parts are concave and convex both. Hence, by theorem 3, solution of the quadratic maximum problem is equivalent to solution of the saddle value problem for

$$\phi(x, u) \equiv \sum c_i x_i - \sum \sum c_{ij} x_i x_j + \sum b_h u_h - \sum \sum a_{hi} u_h x_i - \sum \sum \sum a_{hij} u_h x_i x_j .$$

When all of the quadratic terms vanish (an extreme but legitimate special case of semidefiniteness), the quadratic maximum problem reduces to the following problem of *linear programming.*

LINEAR MAXIMUM PROBLEM. *To find an x^0 that maximizes $\sum c_i x_i$ constrained by the $m + n$ linear inequalities*

$$\sum a_{hi} x_i \leqq b_h , \qquad x_i \geqq 0 .$$

Now the equivalent saddle point problem concerns the *bilinear* function

$$\phi(x, u) \equiv \sum c_i x_i + \sum b_h u_h - \sum \sum a_{hi} u_h x_i .$$

The minimum maximum rôles of x and u can be interchanged by replacing $\phi(x, u)$ by $-\phi(x, u)$. Hence, solution of the following *dual problem* of linear programming is equivalent to solution of the saddle point problem for the bilinear function $\phi(x, u)$.

LINEAR MINIMUM PROBLEM. *To find a u^0 that minimizes $\sum b_h u_h$ constrained by the $n + m$ inequalities*

$$\sum a_{hi} u_h \geqq c_i , \qquad u_h \geqq 0 .$$

6. Extension to a vector maximum problem

This section extends the previous results to a maximum problem for a vector function Gx constrained by $Fx \geqq 0$, $x \geqq 0$. Here the concept of maximum—like T. C. Koopmans' *efficient point* [11]—depends on a partial ordering of vectors by the relation $\geq$, where $v \geq v^0$ means that $v \geqq v^0$ but $v \neq v^0$.

Let $x \to v = Gx$ be a differentiable mapping of nonnegative n-vectors x into

p-vectors v. That is, Gx is a p-vector whose components $g_1(x), \ldots, g_p(x)$ are differentiable functions of x defined for $x \geqq 0$. Taking partial derivatives, evaluated at a particular x^0, let

$$G^0 = \left[\frac{\partial g_k}{\partial x_i}\right]^0 .$$

Here G^0 is a p by n matrix. Let g_k^0 denote the n-vector whose components form the k-th row of G^0. Let Fx have the meaning assigned in section 3.

VECTOR MAXIMUM PROBLEM. *To find an x^0 that maximizes the vector function Gx constrained by $Fx \geqq 0$, $x \geqq 0$—that is, to find an x^0 satisfying the constraints and such that $Gx \geq Gx^0$ for no x satisfying the constraints.*

RESTRICTION. Attention will be restricted to solutions x^0 of the vector maximum problem that are *proper* in the sense that $G^0dx \geq 0$ for no vector differential dx if x^0 is interior to the constraint set determined by $Fx \geqq 0$, $x \geqq 0$, and for no dx satisfying

$$(5) \qquad F_1^0 dx \geqq 0 , \qquad I_1 dx \geqq 0$$

if x^0 belongs to the boundary of the constraint set (as qualified in section 3).

Example. To maximize $g_1(x) \equiv x$, $g_2(x) \equiv 2x - x^2$, x being a real variable (one dimensional vector) constrained only by $x \geqq 0$. Here, $Gx \geq Gx^0$ for no x if $x^0 \geqq 1$, and $G^0dx \geq 0$ for no dx except at $x^0 = 1$, where $G^0dx \geq 0$ for $dx > 0$. So, any $x > 1$ is a proper solution of this particular vector maximum problem, but $x^0 = 1$ is a solution that is not proper. An argument against admitting $x^0 = 1$ as a "proper" solution is that it would usually be natural to accept a second order loss in $g_2(x) \equiv 2x - x^2$ to achieve a first order gain in $g_1(x) \equiv x$. (The anomaly indicated by $x^0 = 1$ in this example was noticed by C. B. Tompkins. A rather similar anomaly occurs in the paper [1] of K. J. Arrow.)

THEOREM 4. *In order that x^0 be a proper solution of the vector maximum problem, it is necessary that there be some $v^0 > 0$ such that x^0 and some u^0 satisfy conditions (1) and (2) for $\phi(x, u) \equiv v^{0\prime}Gx + u'Fx$.*

PROOF. Let x^0 be a proper solution of the vector maximum problem. Then, for each $k = 1, \ldots, p$, one must have $g^{0\prime}dx \leqq 0$ for all dx satisfying

$$F_1^0 dx \geqq 0 , \qquad I_1 dx \geqq 0 , \qquad G^0 dx \geqq 0$$

(where F_1^0 and I_1 may be vacuous). Hence, by the fundamental property of homogeneous linear inequalities used in the proof of theorem 1,

$$-g_k^0 = F_1^{0\prime}u_1^k + I_1'w_1^k + G^{0\prime}v^k \quad \text{for some} \quad u_1^k \geqq 0 , \quad w_1^k \geqq 0 , \quad v^k \geqq 0 .$$

Now, summing for $k = 1, \ldots, p$, and transferring the G^0 terms to the left side, one has

$$-G^{0\prime}v^0 = F_1^{0\prime}u_1^0 + I_1'w_1^0 ,$$

where $u_1^0 = \sum u_1^k \geqq 0$, $w_1^0 = \sum w_1^k \geqq 0$, and $v^0 = e + \sum v^k > 0$, e being a p-vector whose components are all 1's.

Let $g(x) \equiv v^{0\prime}Gx$. Then

$$-g^0 = -G^{0\prime}v^0 = F_1^{0\prime}u_1^0 + I_1'w_1^0 .$$

From this point on the proof of theorem 4 is completed by following the remaining steps of theorem 1.

THEOREM 5. *In order that x^0 be a proper solution of the vector maximum problem, it is sufficient that there be some $v^0 > 0$ such that x^0 and some u^0 satisfy conditions* (1), (2), *and* (3) *for* $\phi(x, u) \equiv v^{0\prime}Gx + u'Fx$.

PROOF. From the proof of theorem 2, with $g(x) \equiv v^{0\prime}Gx$, it follows that

$$v^{0\prime}Gx \leqq v^{0\prime}Gx^0$$

for all x satisfying the constraints $Fx \geqq 0$, $x \geqq 0$. But $v^0 > 0$, so $Gx \geq Gx^0$ for no x satisfying the constraints.

If x^0 is interior to the constraint set, then $G^{0\prime}v^0 = 0$ by (1), since $x^0 > 0$, $Fx^0 > 0$, and $u^0 = 0$. So $G^0dx \geq 0$ for no dx. If x^0 belongs to the boundary of the constraint set, then (1) implies that

$$-G^{0\prime}v^0 - F^{0\prime}u^0 = I_1'w_1^0 \quad \text{for some} \quad w_1^0 \geqq 0\,.$$

Through (2) this can be written

$$-G^{0\prime}v^0 = F_1^{0\prime}u_1^0 + I_1'w_1^0 \quad \text{for} \quad u_1^0 \geqq 0\,.$$

Hence $G^0dx \geq 0$ for no dx satisfying

$$F_1^0dx \geqq 0\,, \qquad I_1dx \geqq 0\,. \tag{5}$$

This completes the proof of theorem 5.

THEOREM 6 (Equivalence theorem). *Let the functions $f_1(x), \ldots, f_m(x)$, $g_1(x), \ldots, g_p(x)$ be concave as well as differentiable for $x \geqq 0$. Then, x^0 is a proper solution of the vector maximum problem if, and only if, there is some $v^0 > 0$ such that x^0 and some u^0 give a solution of the saddle value problem for $\phi(x, u) \equiv v^{0\prime}Gx + u'Fx$.*

PROOF. Clearly $g(x) \equiv v^{0\prime}Gx$ is concave, since $v^0 > 0$. So the proof of theorem 3 can be duplicated, using theorems 4 and 5 in place of theorems 1 and 2.

7. Another extension

Let Fx and Gx be differentiable mappings, as previously defined (with the constraint qualification on $Fx \geqq 0$, $x \geqq 0$ still in effect). Let min $[Gx]$ denote the (scalar) function whose value for each $x \geqq 0$ is the least among the p values $g_1(x), \ldots, g_p(x)$ of the components of the vector Gx.

MINIMUM COMPONENT MAXIMUM PROBLEM. *To find an x^0 that maximizes min $[Gx]$ constrained by $Fx \geqq 0$, $x \geqq 0$.*

THEOREM 7. *In order that x^0 be a solution of the minimum component maximum problem, it is necessary that there be some nonnegative v^0 with unit component sum satisfying*

$$v^{0\prime}Gx^0 = \min\,[Gx^0] \tag{6}$$

and such that x^0 and some u^0 satisfy conditions (1) *and* (2) *for* $\phi(x, u) \equiv v^{0\prime}Gx + u'Fx$.

PROOF. Let F_1x^0 and I_1x^0 have the meanings assigned them in section 3. Further, let Gx^0 be separated into $G_1x^0 = \min\,[Gx^0]$ and $G_2x^0 > \min\,[Gx^0]$ (see note preceding theorem 10, below). Then, since x^0 is assumed to maximize min $[Gx]$ constrained by $Fx \geqq 0$, $x \geqq 0$, one must have that $G_1^0dx > 0$ for no vector differential

dx satisfying

$$F_1^0 dx \geqq 0\,, \qquad I_1 dx \geqq 0$$

(or for no dx at all, if F_1^0 and I_1 are vacuous). That is, for each k belonging to a certain nonvacuous subset of the set of indices corresponding to the rows of G^0 that belong to G_1^0 one must have that $g_k^{0\prime}dx \leqq 0$ for all dx satisfying

$$F_1^0 dx \geqq 0\,, \qquad I_1 dx \geqq 0\,, \qquad G_1^0 dx \geqq 0\,.$$

Hence, by the fundamental property of homogeneous linear inequalities used in the proof of theorem 1,

$$-g_k^0 = F_1^{0\prime} u_1^k + I_1' w_1^k + G_1^{0\prime} v_1^k \quad \text{for some} \quad u_1^k \geqq 0\,, \qquad w_1^k \geqq 0\,, \qquad v_1^k \geqq 0\,.$$

Now, summing for k over the nonvacuous subset and transferring the G_1^0 terms to the left side, one has

$$-G_1^{0\prime} v_1^0 = F_1^{0\prime} u_1^0 + I_1' w_1^0\,,$$

where $u_1^0 = \sum u_1^k \geqq 0$, $w_1^0 = \sum w_1^k \geqq 0$, and $v_1^0 = e_1 + \sum v_1^k \geqq 0$, e_1 being a vector whose components are 0's or 1's—with at least one 1. Here it can be assumed that the sum of the components of v_1^0 is one, since the above vector equation is homogeneous and the sum of the components of v_1^0 is positive. Form v^0 from v_1^0 by adding zeros as components. Then

$$v^{0\prime} G x^0 = v_1^{0\prime} G_1 x^0 = \min\,[Gx^0]\,.$$

By setting $g(x) \equiv v^{0\prime}Gx$, the above vector equation can be rewritten as

$$-g^0 \doteq -G^{0\prime} v_1^0 = -G_1^{0\prime} v_1^0 = F_1^{0\prime} u_1^0 + I_1' w_1^0\,.$$

From this point on the proof of theorem 7 is completed by following the remaining steps of theorem 1.

THEOREM 8. *In order that x^0 be a solution of the minimum component maximum problem, it is sufficient that there be some nonnegative v^0 with unit component sum satisfying condition* (6) *and such that x^0 and some u^0 satisfy conditions* (1), (2), *and* (3) *for $\phi(x, u) \equiv v^{0\prime}Gx + u'Fx$.*

PROOF. From the proof of theorem 2 with $g(x) \equiv v^{0\prime}Gx$, it follows that

$$v^{0\prime}Gx \leqq v^{0\prime}Gx^0$$

for all x satisfying the constraints $Fx \geqq 0$, $x \geqq 0$. But v^0 is nonnegative with unit component sum and satisfies condition (6). Hence

$$\min\,[Gx] \leqq v^{0\prime}Gx \leqq v^{0\prime}Gx^0 = \min\,[Gx^0]$$

for all x satisfying the constraints. This proves theorem 8.

THEOREM 9 (Equivalence theorem). *Let the functions $f_1(x), \ldots, f_m(x), g_1(x), \ldots, g_p(x)$ be concave as well as differentiable for $x \geqq 0$. Then, x^0 is a solution of the minimum component maximum problem if, and only if, there is some nonnegative v^0 with unit component sum satisfying condition* (6) *and such that x^0 and some u^0 give a solution of the saddle value problem for $\phi(x, u) \equiv v^{0\prime}Gx + u'Fx$.*

PROOF. Clearly $g(x) \equiv v^{0\prime}Gx$ is concave, since v^0 is nonnegative. The proof of theorem 3 can be duplicated, using theorems 7 and 8 in place of theorems 1 and 2.

The fact that the constraints $Fx \geqq 0$ can be written equivalently as $\min\,[Fx] \geqq 0$

suggests the possibility of interchanging the rôles of Fx and Gx. The following theorem exploits this possibility. As before, constraints are subject to the constraint qualification introduced in section 3. (It is to be noted that a constant, such as min $[Gx^0]$, appearing as a vector in a vector inequality or equation is to be interpreted as a vector all of whose components equal that constant.)

THEOREM 10. *Let the functions* $f_1(x), \ldots, f_m(x), g_1(x), \ldots, g_p(x)$ *be concave as well as differentiable for* $x \geqq 0$. *Then, in order that* x^0 *maximize min* $[Gx]$ *constrained by* $Fx \geqq$ *min* $[Fx^0]$, $x \geqq 0$, *it is sufficient that* x^0 *maximize min* $[Fx]$ *constrained by* $Gx \geqq$ *min* $[Gx^0]$, $x \geqq 0$*—provided* $Fx >$ *min* $[Fx^0]$ *for some* $x \geqq 0$.

PROOF. Let x^0 maximize min $[Fx]$ constrained by $(Gx - \min [Gx^0]) \geqq 0$, $x \geqq 0$, as hypothesized. Then, by theorem 7 applied to this reversed situation, there must be some nonnegative u^0 with unit component sum and some v^0 such that

$$u^{0\prime}Fx^0 = \min [Fx^0] ,$$

$$F^{0\prime}u^0 + G^{0\prime}v^0 \leqq 0 , \quad u^{0\prime}F^0x^0 + v^{0\prime}G^0x^0 = 0 , \quad x^0 \geqq 0$$

$$(Gx^0 - \min [Gx^0]) \geqq 0 , \quad v^{0\prime} (Gx^0 - \min [Gx^0]) = 0 , \quad v^0 \geqq 0 .$$

Assume, if possible, that $v^0 = 0$. Then, using the concavity of the functions forming Fx and the above conditions, one has

$$u^{0\prime}Fx \leqq u^{0\prime}Fx^0 + u^{0\prime}F^0 (x - x^0) \leqq u^{0\prime}Fx^0 \quad \text{for all } x \geqq 0 ,$$

contradicting the proviso that $Fx > \min [Fx^0]$ for some $x \geqq 0$. Therefore the vector $v^0 \geq 0$ and one can assume that it has unit component sum by dropping the same assumption concerning u^0. Under these circumstances

$$(Fx^0 - \min [Fx^0]) \geqq 0 , \quad u^{0\prime} (Fx^0 - \min [Fx^0]) = 0 , \quad u^0 \geqq 0 ,$$
$$\text{and } v^{0\prime}Gx^0 = \min [Gx^0] .$$

While, by the concavity of the functions forming Fx and Gx,

$$v^{0\prime}Gx + u^{0\prime}Fx \leqq v^{0\prime}Gx^0 + u^{0\prime}Fx^0 + (v^{0\prime}G^0 + u^{0\prime}F^0) (x - x^0) \quad \text{for all } x \geqq 0 .$$

Consequently, by theorem 8, x^0 is a solution of the minimum component maximum problem for Gx constrained by $(Fx - \min [Fx^0]) \geqq 0$, $x \geqq 0$. This completes the proof of theorem 10.

8. Other types of constraints

The foregoing results admit simple modifications when the constraints $Fx \geqq 0$, $x \geqq 0$ are changed to:

(1) $Fx \geqq 0$, or (2) $Fx = 0$, $x \geqq 0$, or (3) $Fx = 0$.

These modifications are outlined below.

Case 1: $Fx \geqq 0$.

Here, using $\phi(x, u) \equiv g(x) + u'Fx$ defined for all x and constrained only by $u \geqq 0$, one must replace condition (1) by

(1*) $$\phi_x^0 = 0 .$$

Case 2: $Fx = 0$, $x \geqq 0$.

Here, using $\phi(x, u) \equiv g(x) + u'Fx$ defined for all u and constrained only by $x \geqq 0$, one must replace condition (2) by

$$(2^*) \qquad \phi_u^0 = 0 .$$

Case 3: $Fx = 0$.

Here, using $\phi(x, u) \equiv g(x) + u'Fx$ defined for all x and u without constraints, one must replace conditions (1) and (2) by (1*) and (2*). This corresponds to the customary use of the method of Lagrange multipliers for side *equations* [3].

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