The Right Square

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Abstract It is shown that there is a way of interpreting the traditional Square of Opposition that overcomes the main historic problem that led to the present supremacy of modern predicate logic over Aristotelian Syllogistic: handing multiply quantificational, relational expressions like 'All boys love some girls'. The interpretation has other advantages, is plausibly Aristotle's own, original view, and was certainly known by several significant Aristotelian logicians in the late nineteenth century, and the early part of the twentieth century, when Syllogistic was falling from favour. That leads to a puzzle about why the proposed interpretation was not seen to overcome the problem of multiple generality at the time, and some points are made showing what might need to change before the interpretation is more widely accepted.

Keywords Square of Opposition · Conditional probability · Logically proper names · Epsilon terms

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1 A Historical Puzzle

The recent resurgence of interest in Aristotle's Square of Opposition is surprising in view of the disregard with which it has been held in modern logic for nearly a century. On the reading of the statement forms in the Square of Opposition given in most logic texts of today, the only traditional relations that hold are the contradictoriness of the A and O forms, and the contradictoriness of the E and I forms. The contrariety of A and E, the subcontrariety of I and O, and the implications of I by A, and of O by E, have gained no support in the tradition that has grown up since Frege's and Russell's day. But not only that makes the present resurgence of interest in the Square surprising, since, also, little if any defense of the traditional relations has been common, let alone agreed, in the present resurgence of interest. Yet, as we shall see in this paper, a much more viable one is easily obtained through a reading of Aristotle's initial views.

But that leaves us with a considerable historical puzzle. For, given the defense available, what is most surprising is that the appropriate analysis was lost sight of by the generality of logicians in the early twentieth century, when concentration on merely universal quantifiers and their negations took over public attention in modern logic. Syllogistic Logic reigned from Aristotle's day for many centuries, and it was only in the latter part of the nineteenth century that it started to lose its place. Extensions with negative and complex terms were developed around then, but it could, in the end, provide no competition

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to the complexities and rigour that modern logic has derived from the work of Frege and Peirce. By the middle of the twentieth century Syllogistic was no longer respected. One supposed difficulty with Aristotelian Syllogistic, for instance, was its seeming restriction to monadic predicate logic. In this area the polyadic logics developed by Peirce and Frege were taken to win out. How could Aristotelian Syllogistic handle relational expressions like ‘All boys love some girls’, for example? That was, and continues to be, thought a great stumbling block. But not only can the appropriate interpretation easily overcome this problem, it also solves a substantial problem that contemporary predicate logic has made no attempt to address.

2 Aristotle’s Understanding

Let us look at some central aspects of the problem of Existential Import, as it was understood in the early twentieth century.

An extended analysis of the possibilities of defending the traditional relations in the Square of Opposition by means of a rendering in the language of modern logic is to be found in P.F. Strawson *Introduction to Logical Theory* [6]. This book appeared just about the time when Aristotelian Logic was losing its traditional pre-eminent public place. On page 167 Strawson gives the standard modern view, but then he considers two further possibilities, on pages 169 and 173, the latter of which does verify all the traditional relations—though at a severe cost. The standard rendering has the A form as a universal quantification ‘$(x)(Sx \supset Px)$’, the E form as a universal quantification ‘$(x)(Sx \supset \neg Px)$’, the I form as an existential quantification ‘$(\exists x)(Sx \,\&\, Px)$’, and the O form as an existential quantification ‘$(\exists x)(Sx \,\&\, \neg Px)$’. This reproduces just the contradictories, as before. By adding existential import just to the universal forms, i.e. by conjoining each of them with ‘$(\exists x)Sx$’, Strawson then gets his second interpretation:

$$\begin{array}{l} \text{A } (x)(Sx \supset Px) \,\&\, (\exists x)Sx \\ \text{E } (x)(Sx \supset \neg Px) \,\&\, (\exists x)Sx \\ \text{I } (\exists x)(Sx \,\&\, Px) \\ \text{O } (\exists x)(Sx \,\&\, \neg Px). \end{array}$$

By adding further expressions of existential import to the universal forms, and then the negatives of all such additions to the particular forms, he gets his third interpretation:

$$\begin{array}{l} \text{A } (x)(Sx \supset Px) \,\&\, (\exists x)Sx \,\&\, (\exists x)\neg Px \\ \text{E } (x)(Sx \supset \neg Px) \,\&\, (\exists x)Sx \,\&\, (\exists x)Px \\ \text{I } (\exists x)(Sx \,\&\, Px) \text{ v } (x)\neg Sx \text{ v } (x)\neg Px \\ \text{O } (\exists x)(Sx \,\&\, \neg Px) \text{ v } (x)\neg Sx \text{ v } (x)Px. \end{array}$$

The second interpretation saves the contrariety of A and E, and the implication relations between A and I, and between E and O, but loses the contradictoriness of A and O, and E and I, and also the subcontrariety of I and O. The third interpretation fares better formally, because, as before, it does render all the traditional relations within the Square valid. But it loses out completely on natural language plausibility, since, for instance, Strawson has this to say about it: ‘It is quite implausible to suggest that if someone says “Some students of English will get Firsts this year”, it is a sufficient condition of his having made a true statement, that no one at all should get a First. But this would be a consequence of accepting the above interpretation for I’ [6, 173].

Strawson took all this to be a part of his motivation for introducing his pre-suppositional approach, for which he is famous. But that move, whatever its other merits, is not justified on the basis of the other analyses Strawson presented, since there is further analysis available that delivers all the traditional relations within the Square, and which does not suffer any natural language implausibility. What might have persuaded Strawson to neglect this further analysis is the fact that, in addition to considering representations of the Square of Opposition he also was concerned to assess the validity of other Laws of the Syllogism on the various interpretations. So he was concerned to assess laws such as Conversion, Conversion *per Accidens*, Obversion, and Contraposition, as well as the 24 valid Syllogistic moods themselves. Only the last of his proffered interpretations justifies the totality of all of these laws, and that is quite implausible as we have seen. Even the alternative interpretation we shall now consider will not render valid all of the traditional ‘Laws of the Syllogism,’ so if the search was to find a way to do that, then this fourth formal analysis would fare no better in total than the two interpretations Strawson first considered. Nevertheless there is a very good reason why this further interpretation is to be preferred.

For, in direct response to Strawson’s story, Manley Thompson, in the *Philosophical Review* for 1953 [7], put forward a further account of Aristotle’s four forms. This was Aristotle’s view of the situation according to Thompson, although others have had some doubts (see [2, 169], for instance). Certainly the account was well backed up by the passages from Aristotle himself that Thompson presented, and indeed this further interpretation was recommended by several notable Aristotelian logicians in the immediately preceding period such as Keynes, Johnson, and Popper [2, 169 n1].

One Aristotelian logician who kept to the further interpretation even wrote a whole text book using a version of it. That was Lewis Carroll, author of the ‘Alice’ books. In professional life Carroll was Charles Dodgson, a logician at the University of Oxford, and he produced several significant articles on logic, as well as a substantial book. On Carroll’s analysis it is the positive forms A and I that carry existential import while their contradictories do not. Thus he says [1, 74]:

> A Proposition of Relation, beginning with “All” asserts (as we already know) that “*All* Members of the Subject are members of the Predicate.” This evidently contains, as a *part* of what it tells us, the smaller Proposition “*Some* Members of the Subject are Members of the Predicate.” [Thus, the Proposition “*All* bankers are rich men” evidently contains the smaller Proposition “*Some* bankers are rich men.”] The question now arises “What is the *rest* of the information which this Proposition gives us?”

After some investigation he concludes:

> Hence a Proposition of Relation, beginning with “All” is a *Double* Proposition, and is *equivalent* to (i.e. gives the same information as) the *two* propositions *Some* Members of the Subject are Members of the Predicate; *No* Members of the Subject are members of the Class whose Differentia is *contradictory* to that of the Predicate. [Thus, the Proposition “*All* bankers are rich men” is a *Double* Proposition, and is equivalent to the *two* Propositions (1) “Some bankers are rich men”; (2) “No bankers are poor men.”]

That means that all the relations in the Square of Opposition are validated, along with the 24 Syllogistic moods, although, in tune with many of Aristotle's remarks, not all the laws of Obversion, or the laws of Contraposition, for instance, hold, on the following account:

$$\begin{aligned}
&\text{A } (x)(Sx \supset Px) \,\&\, (\exists x)(Sx \,\&\, Px)\\
&\text{E } (x)(Sx \supset \neg Px)\\
&\text{I } (\exists x)(Sx \,\&\, Px)\\
&\text{O } (\exists x)(Sx \,\&\, \neg Px) \vee (x)(Sx \supset \neg Px).
\end{aligned}$$

Thus SEP does not imply SA¬P, and SOP does not imply SI¬P; and neither does SAP imply ¬PA¬S, or SOP imply ¬PO¬S (see [7, 262–265]). The lack of implication in the latter case, for instance, is because the O form is now read with an external negation: 'Not all Ss are Ps', in place of 'Some Ss are not Ps', which has an internal or predicate negation. 'Not all Ss are Ps', as can be seen, is the disjunction of 'Some Ss are not Ps' and 'No Ss are Ps' (or 'There are no Ps'). What has come down to us from antiquity is a mix of this account together with additions probably provided by Boethius amongst others (see [2, 126f]), and it is the totality of this whole tradition that Strawson was trying to justify, without success. Indeed it is not justifiable at all, once one takes a more rigorous look at it than the medievalists evidently took.

What additionally substantiates the above as Aristotle's reading, according to Thompson, is that he applied the principle of it to other cases. For Aristotle applied the distinction between external and internal negations to singular statements, saying that both 'Socrates is well' and 'Socrates is ill' would be false if 'Socrates does not exist' was true [7, 254-5]. Taking 'Socrates' to be a 'disguised description', in the manner of Russell (cf. [3]), this shows that Russell's analysis of definite descriptions followed both Aristotle's and Carroll's line of analysis. In fact it is well known that Russell thought highly of Lewis Carroll's work, so it could well have been one inspiration for Russell's Theory of Descriptions. Thus, for Russell, 'The king of France is bald' entails 'Some king of France is bald'. But 'It is not the case that the king of France is bald' does not entail 'The king of France is not bald'. The former contains an external negation, and in it 'the king of France' has a 'secondary occurrence' as a result. The latter contains an internal, or predicate negation, and in it 'the king of France' has a 'primary occurrence' as a result.

3 Multiple Generality

So how does this account overcome the standard objection that Aristotelian Syllogistic cannot be extended to relational expressions? It is because it parallels a *probabilistic* account. For a probabilistic analysis of the Aristotelian forms also supports the variant reading. Thus $pr(Yx/Xx) = 1$ (i.e. 'All Xs are Ys') entails $Pr(Yx/Xx) > 0$ (i.e. 'Some Xs are Ys'). But $pr(Yx/Xx) \neq 1$ (i.e. 'Not all Xs are Ys') does not entail $Pr(\neg Yx/Xx) > 0$ (i.e. 'Some Xs are not Ys'), since it is possible that $pr(Xx) = 0$ (i.e. 'there are no Xs') in which case the probability is not defined. That means that the pr(Yx/Xx) is certainly not 1, although it is not greater than 0, since it cannot be given a value at all. The probabilistic analysis is applicable also, of course, to many other quantifiers. Thus

Most Xs are Ys

can be represented as

$$\mathrm{pr}(\mathrm{Yx}/\mathrm{Xx}) > 1/2,$$

and

Few Xs are not Ys

(where by ‘Few Xs’ I do not mean ‘Few, if any, Xs’) can be represented as

$$\mathrm{pr}(\neg\mathrm{Yx}/\mathrm{Xx}) < 1/2.$$

This latter facility is a feat that standard predicate logic has no means of tackling, and yet it is supported by everyday facts about many other quantifiers. Indeed there is very strong supporting evidence for the probabilistic interpretation of the A and O forms, when one considers other quantifiers. For ‘Almost all Xs are Ys’, ‘Most Xs are Ys’, and ‘A lot of Xs are Ys’ surely all entail ‘Some Xs are Ys’. Also a negative form like ‘Not a lot of Xs are Ys’, for instance, unlike the related positive form ‘A few Xs are not Ys’, allows it to be possible that no Xs are Ys (or that there are no Xs at all). So, unlike with the form where there is an internal negation (‘A few Xs are not Ys’), there is no entailment from the form with the external negation (‘Not a lot of Xs are Ys’), to ‘Some Xs are not Ys’. All of this parallels the behaviour of ‘all’ and ‘some not’ on the probabilistic interpretation.

But not only can probabilistic analyses handle cases involving ‘all’ and ‘some not’ appropriately, they also can be easily generalised to *relations* involving these as well as many other quantifiers, which is a feat quite out of the question using just the devices in modern logic. Thus not only is

All boys love some girls,

re-expressible as

$$\mathrm{pr}\big(\big[\mathrm{pr}(\mathrm{Lxy}/\mathrm{Gy}) > 0\big]/\mathrm{Bx}\big) = 1,$$

but

Most boys love few girls

is

$$\mathrm{pr}\big(\big[\mathrm{pr}(\mathrm{Lxy}/\mathrm{Gy}) < 1/2\big]/\mathrm{Bx}\big) > 1/2.$$

(If probabilities are given in terms of proportions of existent cases, there is no difficulty with these nested probabilities.) So, whatever it was that made Carroll, Keynes, Johnson, Popper, and many others, miss these generalisations, clearly on the right interpretation of the Square there is a distinct possibility of Aristotelian Syllogistic competing much more strongly with the current paradigm, and maybe even regaining the crown that it lost.

4 Proper Names

But it can only do that if the current paradigm gives ground on the logic of individual statements. For, despite Russell, proper names like ‘Socrates’ are not treated as disguised descriptions in standard texts on modern logic, which leads to the internal relation between ‘Socrates is well’ and ‘Socrates is ill’ being unrepresentable. Only each of these expressions separately, together with their contradictories is part of the modern system, becoming, for instance, 'Ws', 'Is', '$\neg$Ws', '$\neg$Is'. If 'Socrates' is taken to be a disguised description, on the other hand, we can exhibit the four Aristotelian interrelated forms thus:

Socrates is well: $(\exists x)(Sx \,\&\, Wx)$, Socrates is ill: $(\exists x)(Sx \,\&\, \neg Wx)$,
Socrates is not well: $(x)(Sx \supset \neg Wx)$, Socrates is not ill: $(x)(Sx \supset Wx)$,

all under the condition that there is at most one S, i.e. that $(y)(z)((Sy \,\&\, Sz) \supset y = z)$.

In fact, from the standpoint of Russell, the use of ordinary proper names in place of variables is logically impossible, since the only name that could replace a variable would be a 'logically proper name' in Russell's terminology, i.e. a name for something whose existence cannot be questioned. But the text-book tradition seems to have forgotten this more strict understanding of the matter. And Russell cannot escape blame for the mix-up. For in *Principia Mathematica* [4], not only were logically proper names not provided, but also descriptive terms were allowed into what otherwise would be the place for logically proper names, in such expressions as '$(\exists y)(y = \iota x Kx)$'. So what else were Russell's followers to use as substitutes for variables but ordinary proper names and the like?

The reprehensibility of the text-book tradition, and especially its leaders, is heightened once it is realised that there is in fact no difficulty in finding the required logically proper names in classical logic. I have pointed this out in a large number of publications starting in 1986 (for instance, most recently [5]). For one classical theorem is:

$$(\exists x)\big((\exists y)Ky \supset Kx\big),$$

so the existence of certain objects is guaranteed in classical logic, and in the epsilon calculus they are given names: epsilon terms. Thus

$$(\exists y)Ky \supset K\epsilon x Kx,$$

is a theorem in the epsilon calculus, and

$$(\exists y)(y = \epsilon x Kx),$$

containing what Russell called a 'complete' term for an individual [3], is also a theorem there, while

$$(\exists y)(y = \iota x Kx),$$

containing Russell's 'incomplete' iota term still isn't. Certainly it is contingent whether anyone is *a sole king of France* (i.e. $(\exists y)(y = \iota x Kx)$), but what is not contingent is that there is something that is *a sole king of France if anything is* (i.e. $(\exists y)(y = \epsilon x Kx)$).

It is because of the ready availability of 'logically proper names' in this straightforward way that it is particularly unfortunate that Russell allowed his iota expressions to be thought of as individual terms in his system. For from this it too easily seems that contingently descriptive terms—and even ordinary names, and indeed any other singular term or phrase that is only contingently applicable—can take the place of variables. It is this, therefore, that primarily needs to be corrected in current predicate logic if there is to be a recovery of the Aristotelian system. For it is the common use of ordinary proper names as substitutes for individual variables in modern logic that has led to the abandonment of the Aristotelian four fold logic for contingent individuals.

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