A Computer-Assisted Uniqueness Proof for a Semilinear Elliptic Boundary Value Problem

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Dedicated to the memory of Wolfgang Walter

Abstract A wide variety of articles, starting with the famous paper (Gidas, Ni and Nirenberg in Commun. Math. Phys. 68, 209–243 (1979)), is devoted to the uniqueness question for the semilinear elliptic boundary value problem $-\Delta u = \lambda u + u^p$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where λ ranges between 0 and the first Dirichlet Laplacian eigenvalue. So far, this question was settled in the case of Ω being a ball and, for more general domains, in the case $\lambda = 0$. In (McKenna et al. in J. Differ. Equ. 247, 2140–2162 (2009)), we proposed a computer-assisted approach to this uniqueness question, which indeed provided a proof in the case $\Omega = (0, 1)^2$, and $p = 2$. Due to the high numerical complexity, we were not able in (McKenna et al. in J. Differ. Equ. 247, 2140–2162 (2009)) to treat higher values of p. Here, by a significant reduction of the complexity, we will prove uniqueness for the case $p = 3$.

Keywords Semilinear elliptic boundary value problem · Uniqueness · Computer-assisted proof

Mathematics Subject Classification 35J25 · 35J60 · 65N15

1 Introduction

The semilinear elliptic boundary value problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

has attracted a lot of attention since the 19th century. Questions of existence and multiplicity have been (and are still being) extensively studied by means of variational methods, fixed-point methods, sub- and supersolutions, index and degree theory, and more.

In this chapter, we will address the question of *uniqueness* of solutions for the more special problem

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$$\begin{cases} -\Delta u = \lambda u + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where λ ranges between 0 and $\lambda_1(\Omega)$, the first eigenvalue of the Dirichlet Laplacian. It has been shown in a series of papers [1, 2, 19, 28, 29] that the solution of (2) is indeed unique when Ω is a ball, or when Ω is more general but $\lambda = 0$ [9, 10, 12, 30].

We will concentrate on the case where $\Omega = (0, 1)^2$ and $p = 3$, and prove that uniqueness holds for the full range $[0, \lambda_1(\Omega))$ of λ. Thus, our paper constitutes the first uniqueness result for this situation. More precisely we prove

Theorem 1 *Let Ω be the unit square in $\mathbb{R}^2$, $\Omega = (0, 1)^2$. Then the problem*

$$\begin{cases} -\Delta u = \lambda u + u^3 & \textit{in } \Omega, \\ u > 0 & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega \end{cases} \tag{3}$$

admits only one solution for any $\lambda \in [0, \lambda_1(\Omega))$.

Remark 1

a) A simple scaling argument shows that our uniqueness result carries over to all squares $\Omega_l := (0, l)^2$ (and thus, to all squares in $\mathbb{R}^2$): If u is a positive solution of $-\Delta u = \tilde{\lambda} u + u^3$ in Ω_l, $u = 0$ on $\partial\Omega_l$, for some $\tilde{\lambda} \in [0, \lambda_1(\Omega_l))$, then $v(x, y) := lu(lx, ly)$ is a solution of (3) for $\lambda = \tilde{\lambda} l^2 \in [0, \lambda_1(\Omega))$.
b) Since we also show that the unique solution in the square is nondegenerate, by a result of [10] we deduce that the solution is unique also in domains "close to" a square.
c) Finally we observe that having shown in [16] (case $p = 2$) and in this paper (case $p = 3$) that the unique solution is nondegenerate then uniqueness follows also for other nonlinearities of the type $\lambda u + u^p$ for p close to 2 and 3. Indeed, by standard arguments (see for example [9]) nonuniqueness of positive solutions in correspondence to sequences of exponents converging to 3 (resp. to 2) would imply degeneracy of the solution for $p = 3$ (resp. $p = 2$).

Our proof heavily relies on *computer-assistance*. Such computer-assisted proofs are receiving an increasing attention in the recent years since such methods provided results which apparently could not be obtained by purely analytical means (see [5, 6, 17, 18, 24]).

We compute a branch of approximate solutions and prove existence of a true solution branch close to it, using fixed point techniques. By eigenvalue enclosure methods, and an additional analytical argument for λ close to $\lambda_1(\Omega)$ we deduce the non-degeneracy of all solutions along this branch, whence uniqueness follows from the known bifurcation structure of the problem.

In [16] we give a general description of these computer-assisted means and use them to obtain the desired uniqueness result for the case $\Omega = (0, 1)^2$, $p = 2$. To

make the present paper dealing with the case $p = 3$ more self-contained, we recall parts of the content of [16] here. We remark that the numerical tools used in [16] turned out not to be sufficient to treat the case $p = 3$. Now, by some new trick to reduce the numerical complexity, we are able to handle this case.

2 Preliminaries

In the following, let $\Omega = (0, 1)^2$. We remark that the results of this section can be carried over to the more general case of a "doubly symmetric" domain; see [16] for details.

First, note that problem (2) can equivalently be reformulated as finding a *non-trivial* solution of

$$\begin{cases} -\Delta u = \lambda u + |u|^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

since, for $\lambda < \lambda_1(\Omega)$, by the strong maximum principle (for $-\Delta - \lambda$) every non-trivial solution of (4) is positive in Ω. In fact, this formulation is better suited for our computer-assisted approach than (2).

As a consequence of the classical bifurcation theorem of [25] and of the results of [9] the following result was obtained in [20]:

Theorem 2 *All solutions u_λ of (2) lie on a simple continuous curve Γ in $[0, \lambda_1(\Omega)) \times C^{1,\alpha}(\bar{\Omega})$ joining $(\lambda_1(\Omega), 0)$ with $(0, u_0)$, where u_0 is the unique solution of (2) for $\lambda = 0$.*

We recall that the uniqueness of the solution of (2) for $\lambda = 0$ was proved in [10] and [9]. As a consequence of the previous theorem we have

Corollary 1 *If all solutions on the curve Γ are nondegenerate then problem (2) admits only one solution for every $\lambda \in [0, \lambda_1(\Omega))$.*

Proof The nondegeneracy of the solutions implies, by the Implicit Function Theorem, that neither turning points nor secondary bifurcations can exist along Γ. Then, for every $\lambda \in [0, \lambda_1(\Omega))$ there exists only one solution of (2) on Γ. By Theorem 2 all solutions are on Γ, hence uniqueness follows. □

Theorem 2 and Corollary 1 indicate that to prove the uniqueness of the solution of problem (2) for every $\lambda \in [0, \lambda_1(\Omega))$ it is enough to construct a branch of nondegenerate solutions which connects $(0, u_0)$ to $(\lambda_1(\Omega), 0)$. This is what we will do numerically in the next sections with a rigorous computer-assisted proof.

However, establishing the nondegeneracy of solutions u_λ for λ close to $\lambda_1(\Omega)$ numerically can be difficult, due to the fact that the only solution at $\lambda = \lambda_1(\Omega)$, which is the identically zero solution, is obviously degenerate because its linearized

operator is $L_0 = -\Delta - \lambda_1$ which has the first eigenvalue equal to zero. The next proposition shows that there exists a computable number $\bar{\lambda}(\Omega) \in (0, \lambda_1(\Omega))$ such that for any $\lambda \in [\bar{\lambda}(\Omega), \lambda_1(\Omega))$ problem (2) has only one solution which is also nondegenerate. Of course, from the well-known results of Crandall and Rabinowitz, [7, 8], one can establish that for λ "close to" λ_1, all solutions u_λ are nondegenerate. However, in order to complete our program, we need to calculate a precise and *explicit* estimate of how close they need to be. This allows us to carry out the numerical computation only in the interval $[0, \bar{\lambda}(\Omega)]$ as we will do later.

Let us denote by $\lambda_1 = \lambda_1(\Omega)$ and $\lambda_2 = \lambda_2(\Omega)$ the first and second eigenvalue of the operator $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions. We have

Proposition 1 *If there exists $\bar{\lambda} \in (0, \lambda_1)$ and a solution $u_{\bar{\lambda}}$ of (2) with $\lambda = \bar{\lambda}$ such that*

$$\|u_{\bar{\lambda}}\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{\lambda_1}\right)^{\frac{1}{p-1}} \tag{5}$$

then

$$\|u_\lambda\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}}, \tag{6}$$

and u_λ is non-degenerate, for all solutions u_λ of (2) belonging to the branch $\Gamma_2 \subset \Gamma$ which connects $(\bar{\lambda}, u_{\bar{\lambda}})$ to $(\lambda_1, 0)$.

(Recall that Γ is the unique continuous branch of solutions given by Theorem 2.)

Proof See [16]. □

Corollary 2 *If on the branch Γ there exists a solution $u_{\bar{\lambda}}$, $\bar{\lambda} \in (0, \lambda_1)$ such that:*

i) *on the sub-branch Γ_1 connecting $(0, u_0)$ with $(\bar{\lambda}, u_{\bar{\lambda}})$ all solutions are nondegenerate*

and

ii)
$$\|u_{\bar{\lambda}}\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{\lambda_1}\right)^{\frac{1}{p-1}}, \tag{7}$$

then all solutions of (2) are nondegenerate, for all $\lambda \in (0, \lambda_1)$, and therefore problem (2) admits only one solution for every $\lambda \in [0, \lambda_1(\Omega))$.

Proof We set $\Gamma = \Gamma_1 \cup \Gamma_2$ with Γ_1 connecting $(0, u_0)$ to $(\bar{\lambda}, u_{\bar{\lambda}})$. On Γ_1 we have that all solutions are nondegenerate by i). On the other hand the hypothesis ii) allows us to apply Proposition 1 which shows nondegeneracy of all solutions on Γ_2. Hence there is nondegeneracy all along Γ so the assertion follows from Corollary 1. □

The last corollary suggests the method of proving the uniqueness through computer assistance: first we construct a branch of nondegenerate "true" solutions near approximate ones in a certain interval $[0, \bar{\lambda}]$ and then verify ii) for the solution $u_{\bar{\lambda}}$. Note that the estimate (7) depends only on p and on the eigenvalues λ_1 and λ_2 of the operator $-\Delta$ in the domain Ω. So the constant on the right-hand side is easily computable. When Ω is the unit square which is the case analyzed in the next sections, the estimate (7) becomes:

$$\|u_{\bar{\lambda}}\|_\infty < \left(\frac{3\pi^2}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{2\pi^2}\right)^{\frac{1}{p-1}} = \left(\frac{3\bar{\lambda}}{2p}\right)^{\frac{1}{p-1}}$$

because $\lambda_1 = 2\pi^2$ and $\lambda_2 = 5\pi^2$.

Fixing $p = 3$ we finally get the condition

$$\|u_{\bar{\lambda}}\|_\infty < \sqrt{\frac{\bar{\lambda}}{2}}. \tag{8}$$

3 The Basic Existence and Enclosure Theorem

We start the computer-assisted part of our proof with a basic theorem on existence, local uniqueness, and non-degeneracy of solutions to problem (4), assuming $p = 3$ now for simplicity of presentation. In this section, the parameter $\lambda \in [0, \lambda_1(\Omega))$ is fixed.

Let $H_0^1(\Omega)$ be endowed with the inner product $\langle u, v\rangle_{H_0^1} := \langle \nabla u, \nabla v\rangle_{L^2} + \sigma\langle u, v\rangle_{L^2}$; actually we choose $\sigma = 1$ in this paper, but different (usually positive) choices of σ are advantageous or even mandatory in other applications, whence we keep σ as a parameter in the following. Let $H^{-1}(\Omega)$ denote the (topological) dual of $H_0^1(\Omega)$, endowed with the usual operator sup-norm.

Suppose that an *approximate* solution $\omega_\lambda \in H_0^1(\Omega)$ of problem (4) has been computed by numerical means, and that a bound $\delta_\lambda > 0$ for its *defect* is known, i.e.

$$\left\|-\Delta\omega_\lambda - \lambda\omega_\lambda - |\omega_\lambda|^3\right\|_{H^{-1}} \le \delta_\lambda, \tag{9}$$

as well as a constant K_λ such that

$$\|v\|_{H_0^1} \le K_\lambda \left\|L_{(\lambda,\omega_\lambda)}[v]\right\|_{H^{-1}} \quad \text{for all } v \in H_0^1(\Omega). \tag{10}$$

Here, $L_{(\lambda,\omega_\lambda)}$ denotes the operator *linearizing* problem (4) at ω_λ; more generally, for $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$, let the linear operator $L_{(\lambda,u)} : H_0^1(\Omega) \to H^{-1}(\Omega)$ be defined by

$$L_{(\lambda,u)}[v] := -\Delta v - \lambda v - 3|u|uv \quad \left(v \in H_0^1(\Omega)\right). \tag{11}$$

The practical computation of bounds δ_λ and K_λ will be addressed in Sects. 6, 7 and 8.

Let C_4 denote a norm bound (embedding constant) for the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, which is bounded since $\Omega \subset \mathbb{R}^2$. C_4 can be calculated e.g. according to the explicit formula given in [23, Lemma 2]. Finally, let

$$\gamma := 3C_4^3.$$

In our example case where $\Omega = (0,1)^2$, the above-mentioned explicit formula gives (with the choice $\sigma := 1$)

$$\gamma = \frac{3\sqrt{2}}{4(\pi^2+1)^{3/4}} \left(< \frac{1}{5} \right).$$

Theorem 3 *Suppose that some $\alpha_\lambda > 0$ exists such that*

$$\delta_\lambda \le \frac{\alpha_\lambda}{K_\lambda} - \gamma \alpha_\lambda^2 \big(\|\omega_\lambda\|_{L^4} + C_4 \alpha_\lambda \big) \tag{12}$$

and

$$2K_\lambda \gamma \alpha_\lambda \big(\|\omega_\lambda\|_{L^4} + C_4 \alpha_\lambda \big) < 1. \tag{13}$$

Then, the following statements hold true:

a) (*existence*) *There exists a solution $u_\lambda \in H_0^1(\Omega)$ of problem* (4) *such that*

$$\|u_\lambda - \omega_\lambda\|_{H_0^1} \le \alpha_\lambda. \tag{14}$$

b) (*local uniqueness*) *Let $\eta > 0$ be chosen such that* (13) *holds with $\alpha_\lambda + \eta$ instead of α_λ. Then,*

$$\left.\begin{array}{l} u \in H_0^1(\Omega) \quad \textit{solution of } (4) \\ \|u - \omega_\lambda\|_{H_0^1} \le \alpha_\lambda + \eta \end{array}\right\} \quad \Longrightarrow \quad u = u_\lambda. \tag{15}$$

c) (*nondegeneracy*)

$$\left.\begin{array}{l} u \in H_0^1(\Omega) \\ \|u - \omega_\lambda\|_{H_0^1} \le \alpha_\lambda \end{array}\right\} \quad \Longrightarrow \quad L_{(\lambda,u)} : H_0^1(\Omega) \to H^{-1}(\Omega) \quad \textit{is bijective}, \tag{16}$$

whence in particular $L_{(\lambda,u_\lambda)}$ is bijective (*by* (14)).

For a proof, see [16].

Corollary 3 *Suppose that* (12) *and* (13) *hold, and in addition that $\|\omega_\lambda\|_{H_0^1} > \alpha_\lambda$. Then, the solution u_λ given by Theorem 3 is non-trivial* (*and hence positive*).

Remark 2

a) The function $\psi(\alpha) := \frac{\alpha}{K_\lambda} - \gamma\alpha^2(\|\omega_\lambda\|_{L^4} + C_4\alpha)$ has obviously a positive maximum at $\bar{\alpha} = \frac{1}{3C_4}(\sqrt{\|\omega_\lambda\|_{L^4}^2 + \frac{3C_4}{K_\lambda\gamma}} - \|\omega_\lambda\|_{L^4})$, and the crucial condition (12) requires that

$$\delta_\lambda \le \psi(\bar{\alpha}) = \frac{4C_4 + \gamma K_\lambda \|\omega_\lambda\|_{L^4}^2}{K_\lambda(\sqrt{\gamma K_\lambda(\gamma K_\lambda \|\omega_\lambda\|_{L^4}^2 + 3C_4)} + \gamma K_\lambda \|\omega_\lambda\|_{L^4})} \cdot \frac{1}{(\sqrt{\gamma K_\lambda(\gamma K_\lambda \|\omega_\lambda\|_{L^4}^2 + 3C_4)} + \gamma K_\lambda \|\omega_\lambda\|_{L^4} + 6C_4)}, \tag{17}$$

i.e. δ_λ has to be sufficiently small. According to (9), this means that ω_λ must be computed with sufficient accuracy, which leaves the "hard work" to the computer!

Furthermore, a "small" defect bound δ_λ allows (via (12)) a "small" error bound α_λ, if K_λ is not too large.

b) If moreover we choose the *minimal* α_λ satisfying (12), then the additional condition (13) follows automatically, which can be seen as follows: the minimal choice of α_λ shows that $\alpha_\lambda \le \bar{\alpha}$. We have

$$\begin{aligned} &2K_\lambda\gamma\bar{\alpha}\big(\|\omega_\lambda\|_{L^4} + C_4\bar{\alpha}\big) \\ &\quad = 1 - \frac{C_4}{3C_4 + 2\gamma K_\lambda \|\omega_\lambda\|_{L^4}^2 + 2\sqrt{\gamma K_\lambda(\gamma K_\lambda \|\omega_\lambda\|_{L^4}^2 + 3C_4)}\|\omega_\lambda\|_{L^4}} \\ &\quad < 1 \end{aligned} \tag{18}$$

and thus condition (13) is satisfied.

Since we will anyway try to find α_λ (satisfying (12)) close to the minimal one, condition (13) is "practically" always satisfied if (12) holds. (Nevertheless, it must of course be checked.)

4 The Branch (u_λ)

Fixing some $\bar{\lambda} \in (0, \lambda_1(\Omega))$ (the actual choice of which is made on the basis of Proposition 1; see also Sect. 5), we assume now that for *every* $\lambda \in [0, \bar{\lambda}]$ an approximate solution $\omega_\lambda \in H_0^1(\Omega)$ is at hand, as well as a defect bound δ_λ satisfying (9), and a bound K_λ satisfying (10). Furthermore, we assume now that, for every $\lambda \in [0, \bar{\lambda}]$, some $\alpha_\lambda > 0$ satisfies (12) and (13), and the additional non-triviality condition $\|\omega_\lambda\|_{H_0^1} > \alpha_\lambda$ (see Corollary 3). We suppose that some *uniform* (λ-independent) $\eta > 0$ can be chosen such that (13) holds with $\alpha_\lambda + \eta$ instead of α_λ (compare Theorem 3b)). Hence Theorem 3 gives a positive solution $u_\lambda \in H_0^1(\Omega)$ of problem (4) with the properties (14), (15), (16), for every $\lambda \in [0, \bar{\lambda}]$.

Finally, we assume that the approximate solution branch $([0,\bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto \omega_\lambda)$ is continuous, and that $([0,\bar{\lambda}] \to \mathbb{R}, \lambda \mapsto \alpha_\lambda)$ is lower semi-continuous.

In Sects. 6, 7 and 8, we will address the actual computation of such branches (ω_λ), (δ_λ), (K_λ), (α_λ).

So far we know nothing about continuity or smoothness of $([0,\bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto u_\lambda)$, which however we will need to conclude that $(u_\lambda)_{\lambda\in[0,\bar{\lambda}]}$ coincides with the sub-branch Γ_1 introduced in Corollary 2.

Theorem 4 *The solution branch*

$$\left\{\begin{array}{c} [0,\bar{\lambda}] \to H_0^1(\Omega) \\ \lambda \mapsto u_\lambda \end{array}\right\}$$

is continuously differentiable.

Proof The mapping

$$\mathcal{F}: \left\{\begin{array}{c} \mathbb{R} \times H_0^1(\Omega) \to H^{-1}(\Omega) \\ (\lambda, u) \mapsto -\Delta u - \lambda u - |u|^3 \end{array}\right\}$$

is continuously differentiable, with $(\partial\mathcal{F}/\partial u)(\lambda, u) = L_{(\lambda,u)}$ (see (11)), and $\mathcal{F}(\lambda, u_\lambda) = 0$ for all $\lambda \in [0,\bar{\lambda}]$. Using the Mean Value Theorem one can show that $L_{(\lambda,u)}$ depends indeed continuously on (λ, u); see [16, Lemma 3.1] for details.

It suffices to prove the asserted smoothness locally. Thus, fix $\lambda_0 \in [0,\bar{\lambda}]$. Since $L_{(\lambda_0,u_{\lambda_0})}$ is bijective by Theorem 3c), the Implicit Function Theorem gives a C^1-smooth solution branch

$$\left\{\begin{array}{c} (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to H_0^1(\Omega) \\ \lambda \mapsto \hat{u}_\lambda \end{array}\right\}$$

to problem (4), with $\hat{u}_{\lambda_0} = u_{\lambda_0}$. By (14),

$$\|\hat{u}_{\lambda_0} - \omega_{\lambda_0}\|_{H_0^1} \le \alpha_{\lambda_0}. \tag{19}$$

Since $\hat{u}_\lambda$ and ω_λ depend continuously on λ, and α_λ lower semi-continuously, (19) implies

$$\|\hat{u}_\lambda - \omega_\lambda\|_{H_0^1} \le \alpha_\lambda + \eta \quad \big(\lambda \in [0,\bar{\lambda}] \cap (\lambda_0 - \tilde{\varepsilon}, \lambda_0 + \tilde{\varepsilon})\big)$$

for some $\tilde{\varepsilon} \in (0, \varepsilon)$. Hence Theorem 3b) provides

$$\hat{u}_\lambda = u_\lambda \quad \big(\lambda \in [0,\bar{\lambda}] \cap (\lambda_0 - \tilde{\varepsilon}, \lambda_0 + \tilde{\varepsilon})\big),$$

implying the desired smoothness in some neighborhood of λ_0 (which of course is one-sided if $\lambda_0 = 0$ or $\lambda_0 = \bar{\lambda}$). □

As a consequence of Theorem 4, $(u_\lambda)_{\lambda\in[0,\bar{\lambda}]}$ is a continuous solution curve connecting the point $(0, u_0)$ with $(\bar{\lambda}, u_{\bar{\lambda}})$, and thus must coincide with the sub-branch

Γ_1, connecting these two points, of the unique simple continuous curve Γ given by Theorem 2. Using Theorem 3c), we obtain

Corollary 4 *On the sub-branch* Γ_1 *of* Γ *which connects* $(0, u_0)$ *with* $(\bar{\lambda}, u_{\bar{\lambda}})$, *all solutions are nondegenerate.*

Thus, if we can choose $\bar{\lambda}$ such that condition (7) holds true, Corollary 2 will give the desired uniqueness result.

5 Choice of $\bar{\lambda}$

We have to choose $\bar{\lambda}$ such that condition (7) is satisfied. For this purpose, we use computer-assistance again. With x_M denoting the intersection of the symmetry axes of the (doubly symmetric) domain Ω, i.e. $x_M = (\frac{1}{2}, \frac{1}{2})$ for $\Omega = (0, 1)^2$, we choose $\bar{\lambda} \in (0, \lambda_1(\Omega))$, not too close to $\lambda_1(\Omega)$, such that our *approximate* solution $\omega_{\bar{\lambda}}$ satisfies

$$\omega_{\bar{\lambda}}(x_M) < \left(\frac{\lambda_2(\Omega) - \lambda_1(\Omega)}{3}\right)^{\frac{1}{2}} \cdot \left(\frac{\bar{\lambda}}{\lambda_1(\Omega)}\right)^{\frac{1}{2}}, \tag{20}$$

with "not too small" difference between right- and left-hand side. Such a $\bar{\lambda}$ can be found within a few numerical trials.

Here, we impose the additional requirement

$$\omega_{\bar{\lambda}} \in H^2(\Omega) \cap H_0^1(\Omega), \tag{21}$$

which is in fact a condition on the numerical method used to compute $\omega_{\bar{\lambda}}$. (Actually, condition (21) could be avoided if we were willing to accept additional technical effort.) Moreover, exceeding (9), we will now need an L^2-bound $\hat{\delta}_{\bar{\lambda}}$ for the defect:

$$\left\| -\Delta\omega_{\bar{\lambda}} - \bar{\lambda}\omega_{\bar{\lambda}} - |\omega_{\bar{\lambda}}|^3 \right\|_{L^2} \le \hat{\delta}_{\bar{\lambda}}. \tag{22}$$

Finally, we note that Ω is *convex*, and hence in particular H^2-regular, whence every solution $u \in H_0^1(\Omega)$ of problem (4) is in $H^2(\Omega)$.

Using the method described in Sect. 3, we obtain, by Theorem 3a), a positive solution $u_{\bar{\lambda}} \in H^2(\Omega) \cap H_0^1(\Omega)$ of problem (4) satisfying

$$\|u_{\bar{\lambda}} - \omega_{\bar{\lambda}}\|_{H_0^1} \le \alpha_{\bar{\lambda}}, \tag{23}$$

provided that (12) and (13) hold, and that $\|\omega_{\bar{\lambda}}\|_{H_0^1} > \alpha_{\bar{\lambda}}$.

Now we make use of the explicit version of the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ given in [21]. There, explicit constants $\hat{C}_0$, $\hat{C}_1$, $\hat{C}_2$ are computed such that

$$\|u\|_\infty \le \hat{C}_0\|u\|_{L^2} + \hat{C}_1\|\nabla u\|_{L^2} + \hat{C}_2\|u_{xx}\|_{L^2} \quad \text{for all } u \in H^2(\Omega),$$

with $\|u_{xx}\|_{L^2}$ denoting the L^2-Frobenius norm of the Hessian matrix u_{xx}. E.g. for $\Omega = (0,1)^2$, [21] gives

$$\hat{C}_0 = 1, \qquad \hat{C}_1 = 1.1548 \cdot \sqrt{\frac{2}{3}} \le 0.9429, \qquad \hat{C}_2 = 0.22361 \cdot \sqrt{\frac{28}{45}} \le 0.1764.$$

Moreover, $\|u_{xx}\|_{L^2} \le \|\Delta u\|_{L^2}$ for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ since Ω is convex (see e.g. [14]). Consequently,

$$\|u_{\bar\lambda} - \omega_{\bar\lambda}\|_\infty \le \hat{C}_0 \|u_{\bar\lambda} - \omega_{\bar\lambda}\|_{L^2} + \hat{C}_1 \|u_{\bar\lambda} - \omega_{\bar\lambda}\|_{H_0^1} + \hat{C}_2 \|\Delta u_{\bar\lambda} - \Delta\omega_{\bar\lambda}\|_{L^2}. \tag{24}$$

To bound the last term on the right-hand side, we first note that

$$\begin{aligned}
\big\| |u_{\bar\lambda}|^3 - |\omega_{\bar\lambda}|^3 \big\|_{L^2} &= \left\| 3 \int_0^1 \big|\omega_{\bar\lambda} + t(u_{\bar\lambda} - \omega_{\bar\lambda})\big| \big(\omega_{\bar\lambda} + t(u_{\bar\lambda} - \omega_{\bar\lambda})\big)\, dt \cdot (u_{\bar\lambda} - \omega_{\bar\lambda}) \right\|_{L^2} \\
&\le 3 \int_0^1 \big\| \big|\omega_{\bar\lambda} + t(u_{\bar\lambda} - \omega_{\bar\lambda})\big|^2 \cdot |u_{\bar\lambda} - \omega_{\bar\lambda}| \big\|_{L^2}\, dt \\
&\le 3 \int_0^1 \big\|\omega_{\bar\lambda} + t(u_{\bar\lambda} - \omega_{\bar\lambda})\big\|_{L^6}^2 \|u_{\bar\lambda} - \omega_{\bar\lambda}\|_{L^6}\, dt \\
&\le 3 \int_0^1 \big(\|\omega_{\bar\lambda}\|_{L^6} + t C_6 \alpha_{\bar\lambda}\big)^2\, dt \cdot C_6 \alpha_{\bar\lambda} \qquad (25) \\
&= 3C_6 \left(\|\omega_\lambda\|_{L^6}^2 + C_6 \|\omega_\lambda\|_{L^6} \alpha_{\bar\lambda} + \frac{1}{3} C_6^2 \alpha_{\bar\lambda}^2 \right) \alpha_{\bar\lambda}, \qquad (26)
\end{aligned}$$

using (23) and an embedding constant C_6 for the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ in the last but one line; see e.g. [23, Lemma 2] for its computation. Moreover, by (4) and (22),

$$\|\Delta u_{\bar\lambda} - \Delta\omega_{\bar\lambda}\|_{L^2} \le \hat{\delta}_{\bar\lambda} + \bar\lambda \|u_{\bar\lambda} - \omega_{\bar\lambda}\|_{L^2} + \big\| |u_{\bar\lambda}|^3 - |\omega_{\bar\lambda}|^3 \big\|_{L^2}. \tag{27}$$

Using (23)–(27), and the Poincaré inequality

$$\|u\|_{L^2} \le \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}} \|u\|_{H_0^1} \quad \big(u \in H_0^1(\Omega)\big), \tag{28}$$

we finally obtain

$$\begin{aligned}
\|u_{\bar\lambda} - \omega_{\bar\lambda}\|_\infty \le \Bigg[&\frac{\hat{C}_0 + \bar\lambda \hat{C}_2}{\sqrt{\lambda_1(\Omega) + \sigma}} + \hat{C}_1 \\
&+ 3C_6 \hat{C}_2 \left(\|\omega_\lambda\|_{L^6}^2 + C_6 \|\omega_\lambda\|_{L^6} \alpha_{\bar\lambda} + \frac{1}{3} C_6^2 \alpha_{\bar\lambda}^2 \right) \Bigg] \cdot \alpha_{\bar\lambda} + \hat{C}_2 \hat{\delta}_{\bar\lambda}, \qquad (29)
\end{aligned}$$

and the right-hand side is "small" if $\alpha_{\bar{\lambda}}$ and $\hat{\delta}_{\bar{\lambda}}$ are "small", which can (again) be achieved by sufficiently accurate numerical computations.

Finally, since

$$u_{\bar{\lambda}}(x_M) \le \omega_{\bar{\lambda}}(x_M) + \|u_{\bar{\lambda}} - \omega_{\bar{\lambda}}\|_\infty,$$

(29) yields an upper bound for $u_{\bar{\lambda}}(x_M)$ which is "not too much" larger than $\omega_{\bar{\lambda}}(x_M)$. Hence, since $u_{\bar{\lambda}}(x_M) = \|u_{\bar{\lambda}}\|_\infty$ by [11], condition (7) can easily be checked, and (20) (with "not too small" difference between right- and left-hand side) implies a good chance that this check will be successful; otherwise, $\bar{\lambda}$ has to be chosen a bit larger.

6 Computation of ω_λ and δ_λ for Fixed λ

In this section we report on the computation of an approximate solution $\omega_\lambda \in H^2(\Omega) \cap H_0^1(\Omega)$ to problem (4), and of bounds δ_λ and K_λ satisfying (9) and (10), where $\lambda \in [0, \lambda_1(\Omega))$ is *fixed* (or one of *finitely* many values). We will again restrict ourselves to the unit square $\Omega = (0, 1)^2$.

An approximation ω_λ is computed by a *Newton iteration* applied to problem (4), where the linear boundary value problems

$$L_{(\lambda, \omega_\lambda^{(n)})}[v_n] = \Delta\omega_\lambda^{(n)} + \lambda\omega_\lambda^{(n)} + \left|\omega_\lambda^{(n)}\right|^3 \tag{30}$$

occurring in the single iteration steps are solved approximately by an ansatz

$$v_n(x_1, x_2) = \sum_{i,j=1}^{N} \alpha_{ij}^{(n)} \sin(i\pi x_1)\sin(j\pi x_2) \tag{31}$$

and a Ritz-Galerkin method (with the basis functions in (31)) applied to problem (30). The update $\omega_\lambda^{(n+1)} := \omega_\lambda^{(n)} + v_n$ concludes the iteration step.

The Newton iteration is terminated when the coefficients $\alpha_{ij}^{(n)}$ in (31) are "small enough", i.e. their modulus is below some pre-assigned tolerance.

To *start* the Newton iteration, i.e. to find an appropriate $\omega_\lambda^{(0)}$ of the form (31), we first consider some λ close to $\lambda_1(\Omega)$, and choose $\omega_\lambda^{(0)}(x_1, x_2) = \alpha \sin(\pi x_1)\sin(\pi x_2)$; with an appropriate choice of $\alpha > 0$ (to be determined in a few numerical trials), the Newton iteration will "converge" to a non-trivial approximation $\omega^{(\lambda)}$. Then, starting at this value, we diminish λ in small steps until we arrive at $\lambda = 0$, while in each of these steps the approximation $\omega^{(\lambda)}$ computed in the *previous* step is taken as a start of the Newton iteration. In this way, we find approximations ω_λ to problem (4) for "many" values of λ. Note that all approximations ω_λ obtained in this way are of the form (31).

The computation of an L^2-defect bound $\hat{\delta}_\lambda$ satisfying

$$\left\| -\Delta\omega_\lambda - \lambda\omega_\lambda - |\omega_\lambda|^3 \right\|_{L^2} \le \hat{\delta}_\lambda \tag{32}$$

amounts to the computation of an integral over Ω.

Due to [11] every solution of (4) is symmetric with respect to reflection at the axes $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$. Therefore it is useful to look for approximate solutions of the form

$$\omega_\lambda(x_1, x_2) = \sum_{\substack{i,j=1 \\ i,j \text{ odd}}}^{N} \alpha_{ij} \sin(i\pi x_1) \sin(j\pi x_2). \tag{33}$$

Using sum formulas for sin and cos one obtains for all $n \in \mathbb{N}_0$, $x \in \mathbb{R}$

$$\sin\big((2n+1)\pi x\big) = \left(2\sum_{k=1}^{n} \cos(2k\pi x) + 1\right) \sin(\pi x)$$

and thus ω_λ can be written as follows:

$$\begin{aligned} \omega_\lambda(x_1, x_2) = {} & \alpha_{11} \sin(\pi x_1) \sin(\pi x_2) \\ & + \sum_{k,l=1}^{\lfloor \frac{N-1}{2} \rfloor} \alpha_{2k+1,2l+1} \left(2\sum_{i=1}^{k} \cos(2i\pi x_1) + 1\right)\left(2\sum_{j=1}^{l} \cos(2j\pi x_2) + 1\right) \\ & \cdot \sin(\pi x_1) \sin(\pi x_2). \end{aligned} \tag{34}$$

Since $\cos(x)$ ranges in $[-1, 1]$ and $\sin(\pi x_1)\sin(\pi x_2)$ is positive for $(x_1, x_2) \in \Omega = (0, 1)^2$, ω_λ will be positive if

$$\alpha_{11} + \sum_{k,l=1}^{\lfloor \frac{N-1}{2} \rfloor} \alpha_{2k+1,2l+1} \big([-2k+1, 2k+1]\big)\big([-2l+1, 2l+1]\big) \subset (0, \infty). \tag{35}$$

Condition (35) can easily be checked using interval arithmetic and is indeed always satisfied for our approximate solutions, since α_{11} turns out to be "dominant" and the higher coefficients decay quickly. Hence ω_λ is positive and one can omit the modulus in the computations. Therefore the integral in (32) can be computed in closed form, since only products of trigonometric functions occur in the integrand. After calculating them, various sums $\sum_{i=1}^{N}$ remain to be evaluated. In order to obtain a *rigorous* bound $\hat{\delta}_\lambda$, these computations (in contrast to those for obtaining ω_λ as described above) need to be carried out in *interval* arithmetic [13, 27], to take rounding errors into account.

Note that the complexity in the evaluation of the defect integral in (32), without any further modifications, is $O(N^{12})$ due to the term ω_λ^3. Using some trick, it is however possible to reduce the complexity to $O(N^6)$:

Applying the sum formulas $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ and $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$ one obtains:

$$\begin{aligned}
&\sin(i_1\pi x)\sin(i_2\pi x)\sin(i_3\pi x)\sin(i_4\pi x)\sin(i_5\pi x)\sin(i_6\pi x)\\
&\quad = -\frac{1}{32}\sum_{\substack{\sigma_2,\sigma_3,\sigma_4,\\ \sigma_5,\sigma_6\in\{-1,1\}}} \sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\cos\big((i_1+\sigma_2 i_2+\sigma_3 i_3+\sigma_4 i_4+\sigma_5 i_5+\sigma_6 i_6)\pi x\big).
\end{aligned}$$

Since $\int_0^1 \cos(n\pi x)\,dx = \left\{\begin{matrix} 1 & \text{for } n=0 \\ 0 & \text{for } n\in\mathbb{Z}\backslash\{0\}\end{matrix}\right\} =: \delta_n$, we get

$$\begin{aligned}
&\int_\Omega \omega_\lambda(x_1,x_2)^6\,d(x_1,x_2)\\
&\quad = \frac{1}{1024}\sum_{\sigma_2,\dots,\sigma_6\in\{-1,1\}}\ \sum_{\rho_2,\dots,\rho_6\in\{-1,1\}} \sigma_2\cdot\ldots\cdot\sigma_6\cdot\rho_2\cdot\ldots\cdot\rho_6\\
&\qquad\cdot\sum_{i_1,\dots,i_6=1}^{N}\ \sum_{j_1,\dots,j_6=1}^{N} \delta_{i_1+\sigma_2 i_2+\cdots+\sigma_6 i_6}\delta_{j_1+\rho_2 j_2+\cdots+\rho_6 j_6}\alpha_{i_1 j_1}\cdot\ldots\cdot\alpha_{i_6 j_6}.
\end{aligned}$$

Setting $\alpha_{ij} := 0$ for $(i,j)\in\mathbb{Z}^2\backslash\{1,\dots,N\}^2$ the previous sum can be rewritten as

$$\begin{aligned}
&\frac{1}{1024}\sum_{\substack{\sigma_2,\dots,\sigma_6,\\ \rho_2\dots,\rho_6\in\{-1,1\}}} \sigma_2\cdot\ldots\cdot\sigma_6\cdot\rho_2\cdot\ldots\cdot\rho_6\\
&\quad\cdot\sum_{k=-2N+1}^{3N}\ \sum_{l=-2N+1}^{3N}\Bigg(\sum_{\substack{i_1+\sigma_2 i_2\\ +\sigma_3 i_3=k}}\ \sum_{\substack{j_1+\rho_2 j_2\\ +\rho_3 j_3=l}} \alpha_{i_1 j_1}\alpha_{i_2 j_2}\alpha_{i_3 j_3}\Bigg)\\
&\quad\cdot\Bigg(\sum_{\substack{\sigma_4 i_4+\sigma_5 i_5\\ +\sigma_6 i_6=-k}}\ \sum_{\substack{\rho_4 j_4+\rho_5 j_5\\ +\rho_6 j_6=-l}} \alpha_{i_4 j_4}\alpha_{i_5 j_5}\alpha_{i_6 j_6}\Bigg).
\end{aligned}$$

For fixed σ_i, ρ_i, k and l each of the two double-sums in parentheses is $O(N^4)$. Since they are independent, the product is still $O(N^4)$. The sums over k and l then give $O(N^6)$, whereas the sums over σ_i and ρ_i do not change the complexity.

Moreover the sum $\sum_{k=-2N+1}^{3N}$ is only

$$\begin{cases}
\sum_{k=3}^{3N} & \text{if } \sigma_2=1, \sigma_3=1,\\
\sum_{k=2-N}^{2N-1} & \text{if } \sigma_2\cdot\sigma_3=-1,\\
\sum_{k=-2N+1}^{N-2} & \text{if } \sigma_2=-1, \sigma_3=-1.
\end{cases}$$

Similarly, also certain constellations of σ_4, σ_5, σ_6 reduce the k-sum, and of course analogous reductions are possible for the l-sum. Since $\alpha_{ij} = 0$ if i or j is even, the result does not change if the sum is only taken over odd values of i_n, j_n, k and l.

Remark 3

a) Computing trigonometric sums in an efficient way is an object of investigation since a very long time, but up to our knowledge the above complexity reduction has not been published before.
b) As an alternative to the closed form integration described above, we also tried quadrature for computing the defect integral, but due to the necessity of computing a safe remainder term bound in this case, we ended up in a very high numerical effort, since a large number of quadrature points had to be chosen. So *practically* closed-form integration turned out to be more efficient, although its complexity (as $N \to \infty$) is higher than the quadrature complexity.

Once an L^2-defect bound $\hat{\delta}_\lambda$ (satisfying (32)) has been computed, an H^{-1}-defect bound δ_λ (satisfying (9)) is easily obtained via the embedding

$$\|u\|_{H^{-1}} \leq \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}} \|u\|_{L^2} \quad \left(u \in L^2(\Omega)\right) \tag{36}$$

which is a result of the corresponding dual embedding (28). Indeed, (32) and (36) imply that

$$\delta_\lambda := \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}} \hat{\delta}_\lambda$$

satisfies (9).

The estimate (36) is suboptimal but, under practical aspects, seems to be the most suitable way for obtaining an H^{-1}-bound for the defect. At this point we also wish to remark that, as an alternative to the weak solutions approach used in this paper, we could also have aimed at a computer-assisted proof for *strong* solutions (see [23]), leading to H^2-and C^0-error bounds; in this case an L^2-bound is needed directly (rather than an H^{-1}-bound).

7 Computation of K_λ for Fixed λ

For computing a constant K_λ satisfying (10), we use the isometric isomorphism

$$\Phi : \left\{ \begin{array}{r} H_0^1(\Omega) \to H^{-1}(\Omega) \\ u \mapsto -\Delta u + \sigma u \end{array} \right\}, \tag{37}$$

and note that $\Phi^{-1} L_{(\lambda,\omega_\lambda)} : H_0^1(\Omega) \to H_0^1(\Omega)$ is $\langle \cdot, \cdot \rangle_{H_0^1}$-symmetric since

$$\left\langle \Phi^{-1} L_{(\lambda,\omega_\lambda)}[u], v \right\rangle_{H_0^1} = \int_\Omega \left[\nabla u \cdot \nabla v - \lambda u v - 3|\omega_\lambda| \omega_\lambda u v \right] dx, \tag{38}$$

and hence selfadjoint. Since $\|L_{(\lambda,\omega_\lambda)}[u]\|_{H^{-1}} = \|\Phi^{-1}L_{(\lambda,\omega_\lambda)}[u]\|_{H_0^1}$, (10) thus holds for any

$$K_\lambda \geq \left[\min\left\{|\mu| : \mu \text{ is in the spectrum of } \Phi^{-1}L_{(\lambda,\omega_\lambda)}\right\}\right]^{-1}, \tag{39}$$

provided the min is positive.

A particular consequence of (38) is that

$$\left\langle\left(I - \Phi^{-1}L_{(\lambda,\omega_\lambda)}\right)[u], u\right\rangle_{H_0^1} = \int_\Omega W_\lambda u^2\,dx \quad \left(u \in H_0^1(\Omega)\right) \tag{40}$$

where

$$W_\lambda(x) := \sigma + \lambda + 3\left|\omega_\lambda(x)\right|\omega_\lambda(x) \quad (x \in \Omega). \tag{41}$$

Note that, due to the positivity of our approximate solutions ω_λ established in Sect. 6, the modulus can be omitted here, which again facilitates numerical computations. Choosing a *positive* parameter σ in the H_0^1-product (recall that we actually chose $\sigma := 1$), we obtain $W_\lambda > 0$ on $\bar{\Omega}$. Thus, (40) shows that all eigenvalues μ of $\Phi^{-1}L_{(\lambda,\omega_\lambda)}$ are less than 1, and that its essential spectrum consists of the single point 1. Therefore, (39) requires the computation of *eigenvalue bounds* for the eigenvalue(s) μ neighboring 0.

Using the transformation $\kappa = 1/(1-\mu)$, the eigenvalue problem $\Phi^{-1}L_{(\lambda,\omega_\lambda)}[u] = \mu u$ is easily seen to be equivalent to

$$-\Delta u + \sigma u = \kappa W_\lambda u,$$

or, in weak formulation,

$$\langle u, v\rangle_{H_0^1} = \kappa \int_\Omega W_\lambda u v\,dx \quad \left(v \in H_0^1(\Omega)\right), \tag{42}$$

and we are interested in bounds to the eigenvalue(s) κ neighboring 1. It is therefore sufficient to compute two-sided bounds to the first m eigenvalues $\kappa_1 \leq \cdots \leq \kappa_m$ of problem (42), where m is (at least) such that $\kappa_m > 1$. In all our practical examples, the computed enclosures $\kappa_i \in [\underline{\kappa}_i, \bar{\kappa}_i]$ are such that $\bar{\kappa}_1 < 1 < \underline{\kappa}_2$, whence by (39) and $\kappa = 1/(1-\mu)$ we can choose

$$K_\lambda := \max\left\{\frac{\bar{\kappa}_1}{1-\bar{\kappa}_1}, \frac{\underline{\kappa}_2}{\underline{\kappa}_2 - 1}\right\}. \tag{43}$$

Remark 4 By [11] and the fact that ω_λ is symmetric with respect to reflection at the axes $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$, all occurring function spaces can be replaced by their intersection with the class of reflection symmetric functions. This has the advantage that some eigenvalues κ_i drop out, which possibly reduces the constant K_λ.

The desired *eigenvalue bounds* for problem (42) can be obtained by computer-assisted means of their own. For example, *upper* bounds to $\kappa_1, \ldots, \kappa_m$ (with $m \in \mathbb{N}$ given) are easily and efficiently computed by the *Rayleigh-Ritz* method [26]:

Let $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m \in H_0^1(\Omega)$ denote linearly independent trial functions, for example approximate eigenfunctions obtained by numerical means, and form the matrices

$$A_1 := \left(\langle \tilde{\varphi}_i, \tilde{\varphi}_j \rangle_{H_0^1}\right)_{i,j=1,\ldots,m}, \qquad A_0 := \left(\int_\Omega W_\lambda \tilde{\varphi}_i \tilde{\varphi}_j \, dx\right)_{i,j=1,\ldots,m}.$$

Then, with $\Lambda_1 \le \cdots \le \Lambda_m$ denoting the eigenvalues of the matrix eigenvalue problem

$$A_1 x = \Lambda A_0 x$$

(which can be enclosed by means of verifying numerical linear algebra; see [3]), the Rayleigh-Ritz method gives

$$\kappa_i \le \Lambda_i \quad \text{for } i = 1, \ldots, m.$$

However, also *lower* eigenvalue bounds are needed, which constitute a more complicated task than upper bounds. The most accurate method for this purpose has been proposed by Lehmann [15], and improved by Goerisch concerning its range of applicability [4]. Its numerical core is again (as in the Rayleigh-Ritz method) a matrix eigenvalue problem, but the accompanying analysis is more involved. In particular, in order to compute lower bounds to the first m eigenvalues, a *rough* lower bound to the $(m+1)$st eigenvalue must be known already. This a priori information can usually be obtained via a *homotopy method* connecting a simple "base problem" with known eigenvalues to the given eigenvalue problem, such that all eigenvalues increase (index-wise) along the homotopy; see [22] or [5] for details on this method, a detailed description of which would be beyond the scope of this chapter. In fact, [5] contains the newest version of the homotopy method, where only very small (2×2 or even 1×1) matrix eigenvalue problems need to be treated rigorously in the course of the homotopy.

Finding a base problem for problem (42), and a suitable homotopy connecting them, is rather simple here since Ω is a bounded rectangle, whence the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$ are known: We choose a constant upper bound c_0 for $|\omega_\lambda|\omega_\lambda = \omega_\lambda^2$ on Ω, and the coefficient homotopy

$$W_\lambda^{(s)}(x) := \sigma + \lambda + 3\big[(1-s)c_0 + s\omega_\lambda(x)^2\big] \quad (x \in \Omega, 0 \le s \le 1).$$

Then, the family of eigenvalue problems

$$-\Delta u + \sigma u = \kappa^{(s)} W_\lambda^{(s)} u$$

connects the explicitly solvable constant-coefficient base problem ($s = 0$) to problem (42) ($s = 1$), and the eigenvalues increase in s, since the Rayleigh quotient does, by Poincaré's min-max principle.

8 Computation of Branches (ω_λ), (δ_λ), (K_λ), (α_λ)

In the previous section we described how to compute approximations ω_λ for a grid of finitely many values of λ within $[0, \lambda_1(\Omega))$. After selecting $\bar{\lambda}$ (among these) according to Sect. 5, we are left with a grid

$$0 = \lambda^0 < \lambda^1 < \cdots < \lambda^M = \bar{\lambda}$$

and approximate solutions $\omega^i = \omega_{\lambda^i} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ $(i = 0, \ldots, M)$. Furthermore, according to the methods described in the previous sections, we can compute bounds $\delta^i = \delta_{\lambda^i}$ and $K^i = K_{\lambda^i}$ such that (9) and (10) hold at $\lambda = \lambda^i$.

Now we define a piecewise linear (and hence continuous) approximate solution branch $([0, \bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto \omega_\lambda)$ by

$$\omega_\lambda := \frac{\lambda^i - \lambda}{\lambda^i - \lambda^{i-1}}\omega^{i-1} + \frac{\lambda - \lambda^{i-1}}{\lambda^i - \lambda^{i-1}}\omega^i \quad \big(\lambda^{i-1} < \lambda < \lambda^i, i = 1, \ldots, M\big). \tag{44}$$

To compute corresponding defect bounds δ_λ, we fix $i \in \{1, \ldots, M\}$ and $\lambda \in [\lambda^{i-1}, \lambda^i]$, and let $t := (\lambda - \lambda^{i-1})/(\lambda^i - \lambda^{i-1}) \in [0, 1]$, whence

$$\lambda = (1-t)\lambda^{i-1} + t\lambda^i, \qquad \omega_\lambda = (1-t)\omega^{i-1} + t\omega^i. \tag{45}$$

Using the classical linear interpolation error bound we obtain, for fixed $x \in \Omega$,

$$\begin{aligned}
&\big|\omega_\lambda(x)^3 - \big[(1-t)\omega^{i-1}(x)^3 + t\omega^i(x)^3\big]\big| \\
&\quad \le \frac{1}{2}\max_{s\in[0,1]}\left|\frac{d^2}{ds^2}\big[(1-s)\omega^{i-1}(x) + s\omega^i(x)\big]^3\right| \cdot t(1-t) \\
&\quad \le \frac{3}{4}\max_{s\in[0,1]}\big[(1-s)\omega^{i-1}(x) + s\omega^i(x)\big] \cdot \big(\omega^i(x) - \omega^{i-1}(x)\big)^2 \\
&\quad \le \frac{3}{4}\max\big\{\|\omega^{i-1}\|_\infty, \|\omega^i\|_\infty\big\}\|\omega^i - \omega^{i-1}\|_\infty^2,
\end{aligned} \tag{46}$$

$$\begin{aligned}
&\big|\lambda\omega_\lambda(x) - \big[(1-t)\lambda^{i-1}\omega^{i-1}(x) + t\lambda^i\omega^i(x)\big]\big| \\
&\quad \le \frac{1}{2}\max_{s\in[0,1]}\left|\frac{d^2}{ds^2}\big[\big((1-s)\lambda^{i-1} + s\lambda^i\big)\big((1-s)\omega^{i-1}(x) + s\omega^i(x)\big)\big]\right| \cdot t(1-t) \\
&\quad \le \frac{1}{4}\big(\lambda^i - \lambda^{i-1}\big)\|\omega^i - \omega^{i-1}\|_\infty.
\end{aligned} \tag{47}$$

Since $\|u\|_{H^{-1}} \le C_1\|u\|_\infty$ for all $u \in L^\infty(\Omega)$, with C_1 denoting an embedding constant for the embedding $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$ (e.g. $C_1 = \sqrt{|\Omega|}C_2$), (46) and (47) imply

$$\begin{aligned}
&\left\|\omega_\lambda^3 - \left[(1-t)\left(\omega^{i-1}\right)^3 + t\left(\omega^i\right)^3\right]\right\|_{H^{-1}} \\
&\quad \le \frac{3}{4} C_1 \max\left\{\left\|\omega^{i-1}\right\|_\infty, \left\|\omega^i\right\|_\infty\right\} \left\|\omega^i - \omega^{i-1}\right\|_\infty^2 =: \rho_i, \qquad (48)\\
&\left\|\lambda\omega_\lambda - \left[(1-t)\lambda^{i-1}\omega^{i-1} + t\lambda^i\omega^i\right]\right\|_{H^{-1}} \\
&\quad \le \frac{1}{4} C_1 \left(\lambda^i - \lambda^{i-1}\right) \left\|\omega^i - \omega^{i-1}\right\|_\infty =: \tau_i. \qquad (49)
\end{aligned}$$

Now (45), (48), (49) give

$$\begin{aligned}
&\left\|-\Delta\omega_\lambda - \lambda\omega_\lambda - \omega_\lambda^3\right\|_{H^{-1}} \\
&\quad \le (1-t)\left\|-\Delta\omega^{i-1} - \lambda^{i-1}\omega^{i-1} - \left(\omega^{i-1}\right)^3\right\|_{H^{-1}} \\
&\qquad + t\left\|-\Delta\omega^i - \lambda^i\omega^i - \left(\omega^i\right)^3\right\|_{H^{-1}} + \tau_i + \rho_i \\
&\quad \le \max\left\{\delta^{i-1}, \delta^i\right\} + \tau_i + \rho_i =: \delta_\lambda. \qquad (50)
\end{aligned}$$

Thus, we obtain a branch $(\delta_\lambda)_{\lambda\in[0,\bar\lambda]}$ of defect bounds which is constant on each subinterval $[\lambda^{i-1}, \lambda^i]$. In the points $\lambda^1, \ldots, \lambda^{M-1}$, δ_λ is possibly doubly defined by (50), in which case we choose the smaller of the two values. Hence, $([0, \bar\lambda] \to \mathbb{R}, \lambda \mapsto \delta_\lambda)$ is lower semi-continuous.

Note that δ_λ given by (50) is "small" if δ^{i-1} and δ^i are small (i.e. if the approximations ω^{i-1} and ω^i have been computed with sufficient accuracy; see Remark 2a)) *and* if ρ_i, τ_i are small (i.e. if the grid is chosen sufficiently fine; see (48), (49)).

In order to compute bounds K_λ satisfying (10) for $\lambda \in [0, \bar\lambda]$, with ω_λ given by (44), we fix $i \in \{1, \ldots, M-1\}$ and $\lambda \in [\frac{1}{2}(\lambda^{i-1} + \lambda^i), \frac{1}{2}(\lambda^i + \lambda^{i+1})]$. Then,

$$\begin{aligned}
\left|\lambda - \lambda^i\right| &\le \frac{1}{2} \max\left\{\lambda^i - \lambda^{i-1}, \lambda^{i+1} - \lambda^i\right\} =: \mu_i, \\
\left\|\omega_\lambda - \omega^i\right\|_{H_0^1} &\le \frac{1}{2} \max\left\{\left\|\omega^i - \omega^{i-1}\right\|_{H_0^1}, \left\|\omega^{i+1} - \omega^i\right\|_{H_0^1}\right\} =: \nu_i,
\end{aligned} \qquad (51)$$

whence a coefficient perturbation result given in [16, Lemma 3.2] implies: If

$$\zeta_i := K^i \left[\frac{1}{\lambda_1(\Omega) + \sigma}\mu_i + 2\gamma\left(\left\|\omega^i\right\|_{L^4} + C_4\nu_i\right)\nu_i\right] < 1, \qquad (52)$$

then (10) holds for

$$K_\lambda := \frac{K^i}{1 - \zeta_i}. \qquad (53)$$

Note that (52) is indeed satisfied if the grid is chosen sufficiently fine, since then μ_i and ν_i are "small" by (51).

Analogous estimates give K_λ also on the two remaining half-intervals $[0, \frac{1}{2}\lambda^1]$ and $[\frac{1}{2}(\lambda^{M-1} + \lambda^M), \lambda^M]$.

Choosing again the smaller of the two values at the points $\frac{1}{2}(\lambda^{i-1}+\lambda^i)$ ($i = 1,\dots,M$) where K_λ is possibly doubly defined by (53), we obtain a lower semi-continuous, piecewise constant branch ($[0,\bar\lambda]\to\mathbb{R}, \lambda\mapsto K_\lambda$).

According to the above construction, both $\lambda\mapsto\delta_\lambda$ and $\lambda\mapsto K_\lambda$ are constant on the $2M$ half-intervals. Moreover, (44) implies that, for $i = 1,\dots,M$,

$$\|\omega_\lambda\|_{L^4} \le \left\{\begin{array}{l} \max\{\|\omega^{i-1}\|_{L^4}, \frac{1}{2}(\|\omega^{i-1}\|_{L^4}+\|\omega^i\|_{L^4})\} \ \text{ for } \lambda\in[\lambda^{i-1},\frac{1}{2}(\lambda^{i-1}+\lambda^i)] \\ \max\{\frac{1}{2}(\|\omega^{i-1}\|_{L^4}+\|\omega^i\|_{L^4}), \|\omega^i\|_{L^4})\} \ \text{ for } \lambda\in[\frac{1}{2}(\lambda^{i-1}+\lambda^i),\lambda^i] \end{array}\right\}$$

and again we choose the smaller of the two values at the points of double definition.

Using these bounds, the crucial inequalities (12) and (13) (which have to be satisfied for all $\lambda\in[0,\bar\lambda]$) result in *finitely* many inequalities which can be fulfilled with "small" and piecewise constant α_λ if δ_λ is sufficiently small, i.e. if $\omega^0,\dots,\omega^M$ have been computed with sufficient accuracy (see Remark 2a)) and if the grid has been chosen sufficiently fine (see (48)–(50)). Moreover, since $\lambda\mapsto\delta_\lambda$, $\lambda\mapsto K_\lambda$ and the above piecewise constant upper bound for $\|\omega_\lambda\|_{L^4}$ are lower semi-continuous, the structure of the inequalities (12) and (13) clearly shows that also $\lambda\mapsto\alpha_\lambda$ can be chosen to be lower semi-continuous, as required in Sect. 4. Finally, since (13) now consists in fact of *finitely* many strict inequalities, a uniform (λ-independent) $\eta>0$ can be chosen in Theorem 3b), as needed for Theorem 4.

9 Numerical Results

All computations have been performed on an AMD Athlon Dual Core 4800+ (2.4 GHz) processor, using MATLAB (version R2010a) and the interval toolbox INTLAB [27]. For some of the time consuming nested sums occurring in the computations, we used moreover mexfunctions to outsource these calculations to C++. For these parts of the program we used C-XSC [13] to verify the results. Our source code can be found on our webpage.[1]

In the following, we report on some more detailed numerical results.

Using $\bar\lambda = 18.5$ (which is not the minimally possible choice; e.g. $\bar\lambda = 15.7$ could have been chosen) and $M+1 = 94$ values $0=\lambda^0<\lambda^1<\dots<\lambda^{93}=18.5$ (with $\lambda^1 = 0.1$, $\lambda^2 = 0.3$ and the remaining gridpoints equally spaced with distance 0.2) we computed approximations $\omega^0,\dots,\omega^{93}$ with $N = 16$ in (31), as well as defect bounds $\delta^0,\dots,\delta^{93}$ and constants $K^0,\dots,K^{93}$, by the methods described in Sects. 6 and 7.

Figure 1 shows an approximate branch $[0,2\pi^2)\to\mathbb{R}, \lambda\mapsto\|\omega_\lambda\|_\infty$. The continuous plot has been created by interpolation of the above grid points λ^j, plus some more grid points between 18.5 and $2\pi^2$, where we computed additional approximations.

[1] http://www.math.kit.edu/iana2/~roth/page/publ/en

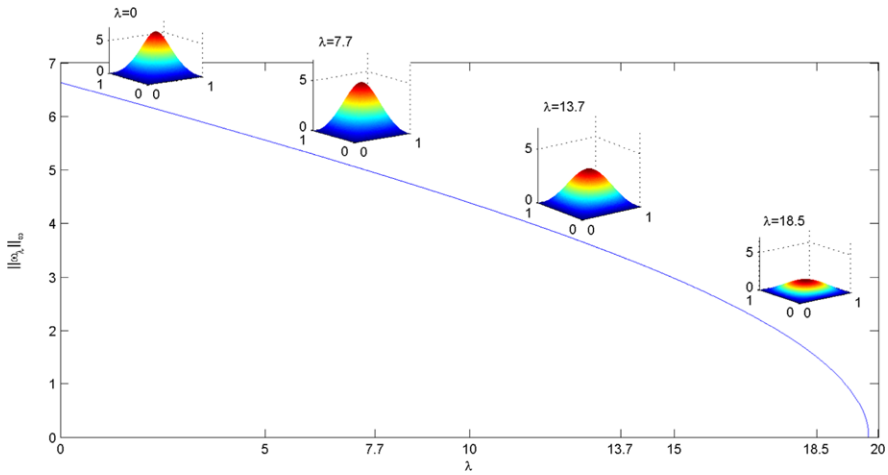

Fig. 1 Curve $(\lambda, \|\omega_\lambda\|_\infty)$ with samples of ω_λ in the case $p = 3$

Table 1 Eigenvalue enclosures for the first two eigenvalues

	κ_1	κ_2
ω_0	$0.34350814513^{840}_{229}$	2.492570^{712}_{450}
$\omega_{2.7}$	$0.37521912233^{850}_{290}$	2.6221837^{653}_{393}
$\omega_{6.7}$	0.4373273950^{411}_{355}	2.87378161^{409}_{204}
$\omega_{10.7}$	$0.52752354636^{621}_{169}$	$3.2234170 42^{515}_{185}$
$\omega_{14.7}$	0.6676848259^{417}_{379}	3.725209290^{988}_{830}
$\omega_{18.5}$	$0.89237445994^{742}_{555}$	4.46288110^{102}_{093}

For some selected values of λ, Table 1 shows, with an obvious sub- and superscript notation for enclosing intervals, the computed eigenvalue bounds for problem (42) (giving K_λ by (43)). These were obtained using the Rayleigh-Ritz and the Lehmann-Goerisch method, and the homotopy method briefly mentioned at the end of Sect. 7 (exploiting also the symmetry considerations addressed in Remark 4). The integer m, needed for these procedures, has been chosen different (between 3 and 10) for different values of λ, according to the outcome of the homotopy. This resulted in a slightly different quality of the eigenvalue enclosures.

Table 2 contains, for some selected of the 186 λ-half-intervals,

a) the defect bounds δ_λ obtained by (50) from the grid-point defect bounds δ^{i-1}, δ^i, and from the grid-width characteristics ρ_i, τ_i defined in (48), (49),
b) the constants K_λ obtained by (53) from the grid-point constants K^i and the grid-width parameters ν_i defined in (51) (note that $\mu_i = 0.1$ for all i),
c) the error bounds α_λ computed according to (12), (13).

Table 2

λ-interval	δ_λ	K_λ	α_λ
[0, 0.05)	0.0005943	1.7443526	0.0010378
(2, 2.1)	0.0023344	1.7707941	0.0041521
(6, 6.1)	0.0022937	1.6669879	0.0038369
(10, 10.1)	0.0023644	1.5677657	0.0037168
(14, 14.1)	0.0026980	1.9582604	0.0053028
(16, 16.1)	0.0031531	3.2267762	0.0102701
(18.4, 18.5]	0.0050056	13.8930543	0.0882899

Thus, Corollary 1, together with all the considerations in the previous sections, proves Theorem 1.

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