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# Inequalities and Applications 2010

## Dedicated to the Memory of Wolfgang Walter

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# Preface[1]

Inequalities are an essential component occurring in various mathematical areas. On the one hand, they form a highly important collection of tools e.g. for proving analytic or stochastic theorems or for deriving error estimates in numerical mathematics, and on the other hand they also constitute a fascinating and challenging research field of their own. Inequalities also appear directly in mathematical models for many kinds of applications e.g. from science, engineering, and economics. This volume reflects all these aspects of the area. Classical inequalities related to means or to convexity are addressed as well as inequalities arising in the field of ordinary and partial differential equations, like Sobolev or Hardy-type inequalities, and inequalities occurring in geometrical contexts.

Within the last five decades, great contributions to the field of inequalities have been made by late Wolfgang Walter. His book on differential and integral inequalities was a real breakthrough in the 1970's and has generated a vast variety of further research in this field. He also organized six of the seven "General Inequalities" Conferences held at Oberwolfach between 1976 and 1995, and co-edited their proceedings volumes. He participated as an honorary member of the Scientific Committee in the "General Inequalities 8" conference in Hungary. As a recognition of his great achievements, this volume is dedicated to Wolfgang Walter's memory.

The "General Inequalities" meetings found their continuation in the "Conferences on Inequalities and Applications" which, so far, have been held twice in Hungary. This volume contains some contributions of the participants of the second of these conferences which took place in Hajdúszoboszló in September 2010, as well as additional articles written upon invitation. These contributions reflect many theoretical and practical aspects in the field of inequalities, and will be useful for

[1]The editors are thankful to the members of the staff of the publisher Birkhäuser, in particular to Dr. Barbara Hellriegel, for their kind help during the whole publication process of the book. The publication was partially supported by the Hungarian Scientific Research Fund (OTKA) Grant NK-81402 and by TÁMOP 4.2.1./B-09/1/KONV-2010-0007/IK/IT project.

researchers and lecturers, as well as for students who want to familiarize themselves with the area.

The Editors

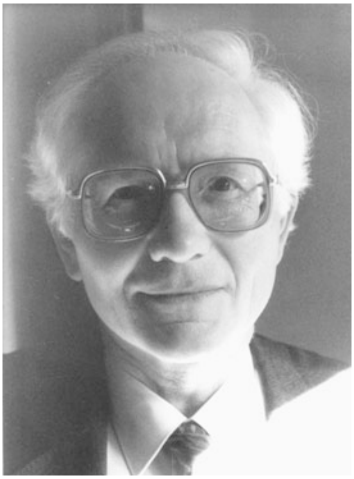
# In Memoriam Wolfgang Walter (1927–2010)[2]

**Wolfgang Reichel**

**Zusammenfassung** Am 26. Juni 2010 verstarb Wolfgang Walter im Alter von 83 Jahren in Karlsruhe. Durch seine Beiträge zur Theorie der Differentialungleichungen wurde er international bekannt. Seine Lehrbücher sind weit verbreitet und werden von Studierenden und Lehrenden gerne benutzt. Als Hochschullehrer, Wissenschaftler, Buchautor, Mitherausgeber von Zeitschriften und Organisator von Konferenzen hat er die Mathematik nachhaltig bereichert und über viele Jahre mitgestaltet.

**Schlüsselwörter** Differentialungleichungen · Quasimonotonie · parabolische Systeme · Hängebrücken · wandernde Wellen · nichtlineare Oszillationen

**Mathematics Subject Classification** Primary 01A70 · 35K40 · Secondary 74J30 · 35L75



Bildarchiv des Mathematischen Forschungsinstituts Oberwolfach

Professor emeritus Dr. Wolfgang Walter verstarb am 26. Juni 2010 im Alter von 83 Jahren in Karlsruhe. Mit ihm verlor die Gemeinschaft der Mathematikerinnen und Mathematiker einen engagierten Hochschullehrer, einen begeisterten Forscher, einen angesehenen Kollegen. Die Familie Walter verlor mit ihm den Ehemann, Bruder, Vater, Schwiegervater und Großvater.

Um den Lesern die Gelegenheit zu geben, die Stationen in Wolfgang Walters Leben mitzuverfolgen, ist dieser Nachruf fast durchgehend in der Gegenwartsform geschrieben.

## 1 Studium und Promotion in Tübingen (1947–1956)

Wolfgang Walter wird am 2. Mai 1927 in Schwäbisch Gmünd geboren. Seine Schulzeit wird 1943 abrupt unterbrochen durch seine Einberufung zunächst als Flakhelfer und später als Soldat an der Ostfront, gefolgt von Verwundung und amerikanischer Kriegsgefangenschaft. Nach seiner Entlassung Ende 1946 schließt er seine Schulausbildung ab und studiert von 1947 bis 1952 Mathematik und Physik an der Universität Tübingen. Er legt 1952 die erste Dienstprüfung und 1955 nach eineinhalbjähriger Referendarzeit die zweite Dienstprüfung für das Lehramt an Höheren Schulen ab. Es folgt seine Promotionszeit bei Erich Kamke mit dem Abschluss der Promotion im Jahr 1956 über das Thema „Mittelwertsätze und ihre Verwendung zur Lösung von Randwertaufgaben". Kamkes Einfluss auf Wolfgang Walters wissenschaftliche Entwicklung ist Zeit seines Lebens deutlich spürbar. Stets spricht Walter mit größtem Respekt von seinem wissenschaftlichen Lehrer und Förderer. Er legt großen Wert auf die Würdigung von Kamkes Verdiensten bei der Neubegründung der Theorie gewöhnlicher und partieller Differentialgleichungen, vgl. [21]. Aber vor allem thematisch und methodisch hat Kamke enormen Einfluss auf Wolfgang Walters wissenschaftliches Werk. Kamkes Bücher über Differentialgleichungen sind Standardwerke der Dreißiger bis Fünfziger Jahre. Sein Bemühen um äußerste Klarheit, Strenge und Präzision im Umgang mit Differentialgleichungen findet deutlichen Widerhall in Wolfgang Walters Publikationen. Kamkes Formulierung der Eindeutigkeitsbedingungen für Lösungen von Anfangswertproblemen bei gewöhnlichen Differentialgleichungen, seine Formulierung der Monotoniesätze, die Verwendung der Methode der sukzessiven Approximationen sowie Prüfers Polarkoordinatenmethode zur Lösung Sturm-Liouvillescher Randwertaufgaben finden Eingang und Würdigung in Walters Lehrbuch über gewöhnliche Differentialgleichungen [27] und werden dort in eleganter Form, prägnanter Darstellung und konsequenter begrifflicher Erweiterung zum Standardrepertoire für Vorlesungen über gewöhnliche Differentialgleichungen.

## 2 Familiengründung und erster USA-Aufenthalt (1957–1962)

Im Jahr 1957 heiraten Wolfgang Walter und Irmgard Scheu, verlassen im selben Jahr Tübingen und beginnen gemeinsam einen neuen Lebensabschnitt in Karlsruhe. Zunächst arbeitet Wolfgang Walter an der Universität Karlsruhe als Assistent bei Johannes Weissinger. Es folgt 1958–1959 an der University of Maryland, College Park, der erste von zahlreichen USA-Aufenthalten. Das Ehepaar Walter fährt per Schiffspassage von Le Havre über den Atlantik und erlebt bei der Ankunft im Hafen von New York einen äußerst emotionalen Moment. Auch noch Jahrzehnte später lässt Wolfgang Walter diesen Augenblick, in dem er und seine Frau zum ersten Mal die Freiheitsstatue sehen und hoffnungsvoll einer gemeinsamen Zukunft entgegenblicken, in seinen Erinnerungen wiederaufleben und teilt ihn mit Kollegen und

Freunden. Kurz nach der Ankunft in den Vereinigten Staaten wird er zum ersten Mal Vater. Es ist die Zeit des kalten Krieges unmittelbar nach dem Sputnik-Schock, in der er die Dynamik des wissenschaftlichen Aufbruchs in den USA fasziniert miterlebt.

Nach seiner Rückkehr nach Karlsruhe habilitiert er sich 1960 mit einer Arbeit über Existenz- und Eindeutigkeitssätze für eine spezielle Klasse von partiellen Differentialgleichungen. Seine Habilitationsschrift wird mit dem Dozentenpreis der Karl-Freudenberg-Stiftung ausgezeichnet. Noch im selben Jahr wird er Dozent und ein Jahr später wissenschaftlicher Rat.

Im gleichen Zeitraum wächst durch die Geburt der Kinder seine Familie; nach Wolfgang (1959 in den USA) werden Susanne (1962) und Katrin (1963) geboren.

## 3 Professor in Karlsruhe (1963–1995)

Nach einem abgelehnten Ruf an die Universität Wien im Jahr 1962 erfolgt 1963 seine Berufung auf ein neu eingerichtetes Ordinariat für Mathematik an der Universität Karlsruhe, das er bis zu seiner Emeritierung im Jahre 1995 innehat. Die lange Zeitspanne von 32 Jahren, in denen er als Professor im aktiven Dienst der Universität Karlsruhe steht, stellt eine äußerst produktive Phase seiner akademischen Karriere dar.

1965 erhält Wolfgang Walter gleich drei Rufe auf Ordinariate an den Universitäten Hamburg, Erlangen-Nürnberg und an die University of Notre Dame, Indiana. Es folgen im Jahr 1971 drei weitere Rufe auf Positionen als full professor an die University of Delaware, an die State University of New York (SUNY) und an die Michigan State University. Eine besondere Auszeichnung ist der Ruf an die University of Delaware auf einen für ihn neu geschaffenen Unidel-Stiftungslehrstuhl. Trotz insgesamt sieben Rufen bleibt Wolfgang Walter an der Universität Karlsruhe. Der Aufbau der Fakultät für Mathematik in den Jahrzehnten nach dem zweiten Weltkrieg und die Wiederaufnahme der internationalen wissenschaftlichen Beziehungen, insbesondere zu Instituten in den USA, sind dabei sicherlich eine starke Motivation für ihn. In Karlsruhe findet er passende Strukturen vor, die ihm seine erfolgreiche Arbeit in Forschung und Lehre ermöglichen und seine administrativen Pflichten in der akademischen Selbstverwaltung erleichtern. Die höchst effiziente Zusammenarbeit mit seiner langjährigen Sekretärin Irene Jendrasik bietet eine sehr gute Rahmenbedingung für seine wissenschaftliche Produktivität. Gemeinsam setzen Irene Jendrasik und Wolfgang Walter bei der Editierung wissenschaftlicher Texte konsequent auf Textverarbeitungssysteme und gehören zu den ersten Benutzern von LaTeX an der Karlsruher Fakultät.

Von 1975 bis 1977 leitet er die Geschicke der Fakultät für Mathematik als Dekan. Seine gut vorbereiteten und stets kurz und knapp gehaltenen Sitzungen finden den Beifall des Kollegiums. Seine persönliche Integrität, seine herzliche, freundliche und humorvolle Art des Umgangs bringt ihm die Wertschätzung seiner Karlsruher Kollegen ein.

Wolfgang Walter besitzt eine starke Affinität zu den akademischen Institutionen in den USA und zu den Vereinigten Staaten selbst. Sie ist begründet einerseits im Vertrauen gegenüber den US-amerikanischen Befreiern, das während seiner Zeit in Gefangenschaft herangewachsen war, und andererseits in der weltöffnenden Erfahrung seines ersten USA-Aufenthaltes in den Jahren 1958 bis 1959. Zudem lebt seine Schwester seit den Fünfziger Jahren in den USA. Mitte der Sechziger bzw. Anfang der Siebziger Jahre, als die Rufe aus den USA kommen, zieht die Familie Walter ernsthaft eine Übersiedlung in die USA in Erwägung. Durch mehrere längere Aufenthalte in Begleitung seiner Familie ist Wolfgang Walter mit dem US-amerikanischen Hochschulsystem vertraut und macht gute Erfahrungen mit dem Leben in den USA. Er kennt die Vorzüge und Nachteile des Lebens und Forschens auf beiden Seiten des Atlantiks genau, als er sich schließlich für den Verbleib in Karlsruhe entscheidet. Auch danach findet er weiterhin die meisten seiner wissenschaftlichen Kontakte in den USA. Während seiner gesamten wissenschaftlichen Tätigkeit ist er mindestens elf Mal zu Aufenthalten, die mehrere Monate bis hin zu einem ganzen akademischen Jahr dauern, als Gastprofessor an nordamerikanischen Universitäten. Gerne berichten Irmgard und Wolfgang Walter auch noch Jahre später davon, wie wohl sich ihre Familie bei diesen Aufenthalten gefühlt hat.

## 4 Buchautor und akademischer Lehrer

Sein erstes, 1964 erschienenes Buch über Differential- und Integralungleichungen [20] sowie dessen 1970 erschienene englischsprachige Erweiterung [22] haben Wolfgang Walter unter Fachkollegen bekannt gemacht. In Karlsruhe verfolgt er das Ziel, die Ausbildung der Studierenden in der Analysis auf eine neue Basis zu stellen. Anfänglich gibt er zu seinen Vorlesungen eigene Skripten heraus, aus denen im Laufe der Jahre schließlich Lehrbücher werden.

Sein erstes Lehrbuch über Distributionentheorie [25] erscheint 1970, sein zweites über Potentialtheorie [23] 1971. Beiden Büchern gemeinsam ist der knappe, klare Stil, in dem sich sowohl Leser als auch Autor auf die wesentlichen Elemente der Theorie und ihren Anwendungen konzentrieren. In erfrischend kurzer Darstellung wird der Leser der Distributionentheorie vom einfachen Kalkül der Distributionen bis hin zum Satz von Paley-Wiener geführt. Mit einem auf Heinz König zurückgehenden einfachen Beweis des Satzes von Malgrange-Ehrenpreis zur Existenz von Grundlösungen für partielle Differentialgleichungen mit konstanten Koeffizienten und einem Kapitel über Sobolevräume beendet Wolfgang Walter seine Distributionentheorie und verdeutlicht, dass ihm vor allem die Anwendungen auf partielle Differentialgleichungen am Herzen liegen. In seiner Potentialtheorie weht ebenfalls ein frischer Wind. Nach den Kapiteln über harmonische Funktionen und Einfach- und Doppelschichtpotentiale finden sich gleich drei methodisch unterschiedliche Beweise für die Existenz von Lösungen des Dirichletschen Randwertproblems: ein Beweis mittels Fredholmscher Integralgleichungen, ein zweiter basierend auf der Perronschen Methode von Ober- bzw. Unterfunktionen und ein dritter Beweis mit Hilfe von Differenzenverfahren auf Gittern inklusive Konvergenzbetrachtung beim Grenzübergang der Gitterweite gegen Null.

1972 erscheint Wolfgang Walters beliebtestes Lehrbuch über Gewöhnliche Differentialgleichungen [27] mit insgesamt sieben Auflagen, einer später erschienenen Übersetzung ins Englische [26] sowie lizenzierten und sehr erfolgreichen Nachdrucken in China. Mit steigender Auflage und steter Erprobung des Lehrstoffes im Hörsaal entsteht ein Gesamtwerk, in dem Wolfgang Walter abwechselt zwischen konsequenter Verwendung des Banachschen Fixpunktsatzes, funktionalanalytischen Argumenten, Methoden der Differentialungleichungen, Phasenebenenargumenten, Floquet-Theorie, Attraktoren, Lyapunov-Funktionen und Stabilitätsbegriffen. Stets visiert er den mit den vorhandenen Hilfsmitteln bestmöglichen Satz an. Dieses Buch ist am stärksten von Wolfgang Walters Streben nach einem ebenso knappen wie lesbaren Stil geprägt.

Zwischen 1985 und 2002 erscheinen mehrere Auflagen von Wolfgang Walters Analysis 1 [28] und Analysis 2 [29] im Springer Verlag in der Reihe „Grundwissen Mathematik“. In dieser Reihe wird der neuartige Ansatz verfolgt, die Grundbegriffe der Analysis in ihrem historischen Entwicklungsprozess darzustellen. Dabei nehmen Autor und Leser eine Perspektive ein, in der sie das Ringen um die Begriffe der modernen Analysis wie Stetigkeit, Grenzwert, Funktion, Konvergenz miterleben und gleichzeitig ihre Bedeutung bei der Lösung wichtiger Probleme erkennen, z.B. beim Nachweis der Keplerschen Gesetze aus dem Newtonschen Gravitationsgesetz. Durch viele historisch interessante Details und Anmerkungen, die Wolfgang Walter mit viel Liebe recherchiert, wird das Lesen zum Vergnügen. Gleichzeitig bleibt er seinem Stil, die wichtigen Sätze und Beweise knapp und prägnant darzustellen, treu. Im zweiten Band des Analysis-Lehrbuches treten die historischen Erläuterungen zugunsten der modernen Darstellung in den Hintergrund. Dafür finden sich in der Darstellung der Theorie des Lebesgueschen Integrals, im Beweis des Transformationssatzes für Lebesgueintegrale mit Hilfe des Sardschen Lemmas und in der Konvergenztheorie der Fourierreihen die von Lehrenden und Studierenden gleichermaßen geschätzten Höhepunkte seiner Lehrbücher.

Auch in seinen Vorlesungen ist das Streben nach Effizienz greifbar. Er hält mit Freude seine Vorlesungen und versteht es, Studierende für Themen der angewandten Analysis zu begeistern. Aus seinen Vorlesungen und Seminaren gehen zahlreiche Diplomanden hervor. Elf Doktoranden erlangen mit Hilfe seiner Betreuung den Doktorgrad und fünf Wissenschaftler seiner engeren Arbeitsgruppe habilitieren sich. Mit seinen Kollegen und Mitarbeitern diskutiert er gerne und ausgiebig an der Tafel in seinem Büro oder auf einer Papierserviette beim gemeinsamen Mittagessen. Seine Denk- und Schlussweisen trägt er in bewundernswerter Klarheit vor. Oft sind sie neuartig und überraschend und bereichern diejenigen, die von ihm Mathematik lernen und mit ihm über Mathematik diskutieren.

*Liste der Doktoranden*

Herbert Weigel, 1968
Klaus Deimling, 1969
Gerhard Schleinkofer, 1969
Alexander Voigt, 1971
Roland Lemmert, 1974
Gerhard Lamott, 1976
Jörg Heuß, 1979
Dietrich Wendland, 1981
Reinhard Redlinger, 1982
Volkmar Weckesser, 1993
Wolfgang Reichel, 1996

*Liste der Habilitationen*

Klaus Ritter, 1968
Klaus Deimling, 1971
Peter Volkmann, 1975
Roland Lemmert, 1979
Reinhard Redlinger, 1988

## 5 Wissenschaftliches Werk

Neben seinen 8 Büchern verfasst Wolfgang Walter über 130 wissenschaftliche Publikationen, die inzwischen mehr als 600-mal zitiert worden sind. Seine Themengebiete sind vielfältig, insbesondere interessieren ihn gewöhnliche und partielle Differentialgleichungen sowie angewandte und numerische Mathematik. Auf internationalen Kongressen erfahren seine wissenschaftlichen Vorträge Anerkennung. Seine Beiträge auf dem Gebiet der Differentialungleichungen sind bahnbrechend und grundlegend für eine Vielzahl weiterer Untersuchungen. Bis heute entfalten seine prägnant geschriebenen Publikationen ihre inspirierende Wirkung und zeigen, wie stark die Theorie der Differentialgleichungen von der Idee der Ungleichungen profitiert. Wolfgang Walter sieht die Trennung von gewöhnlichen und partiellen Differentialgleichungen undogmatisch und widmet sich beiden Feldern mit großem Interesse.

Eine besondere Stellung unter seinen Koautoren nehmen Ray Redheffer, UCLA (21 gemeinsame Arbeiten) und Joe McKenna, Univ. of Connecticut (8 gemeinsame Arbeiten) ein. Mit beiden verbindet ihn nicht nur eine fruchtbare mathematische Kooperation sondern auch eine private Freundschaft, die sich auf die Familien Walter, Redheffer und McKenna erstreckt.

Anlässlich seines 66. Geburtstages wird Wolfgang Walter Band 3 der World Scientific Series in Applicable Analysis [1] gewidmet. Auf knapp 600 Seiten sind Beiträge von Wissenschaftlerinnen und Wissenschaftlern enthalten über das Thema Ungleichungen und ihre Anwendungen in den Gebieten Analysis, Wirtschaftswissenschaften, Differential- und Funktionalgleichungen. Im ersten Beitrag dieses Bandes findet sich „R.M. Redheffer's 66th birthday tribute to Wolfgang Walter" [15]. Diesem lesenswerten Beitrag über Wolfgang Walters wissenschaftliches Werk kommt eine besondere Rolle zu, da die Würdigung zu seinen Lebzeiten stattfindet und von ihm als ehrenvolle Auszeichnung betrachtet wird. Aus diesem Grund soll hier keine Wiederholung oder Kopie vorgenommen werden. Statt dessen findet sich am Ende dieses Nachrufes in Abschnitt 9 eine exemplarische Würdigung von Wolfgang Walters wissenschaftlichen Beiträgen in Form einiger detaillierter Auszüge, die sinngemäß, aber nicht wörtlich aus seinen Arbeiten stammen.

## 6 Zeitschriften, Tagungen, Ämter

Nicht nur durch seine eigenen wissenschaftlichen Beiträge bereichert Wolfgang Walter die Mathematik. Auch als Mitherausgeber der Zeitschriften *Applicable Analysis* ab 1971, *Journal of Nonlinear Analysis – TMA* ab 1976, *Journal of Dynamic Systems and Applications* ab 1992 und *Journal of Inequalities and Applications* ab 1997 ist er einer großen Zahl von Mathematikern bekannt. Ebenfalls gibt er die Springer Reihe *Grundwissen Mathematik* und die *Scientific Series in Applicable Analysis* (WSSAA, World Sci. Publ., River Edge, NJ) mit heraus.

Als 1976 die erste „General Inequalities“ - Tagung in Oberwolfach stattfindet, ist Wolfgang Walter von Anfang an dabei. In seiner wissenschaftlichen Karriere lässt er sich seit langem vom Thema Ungleichungen leiten. Integralungleichungen, Normabschätzungen, Ober- und Unterfunktionen interessieren ihn ebenso wie später die verifizierte Einschließung von Lösungen gewöhnlicher Differentialgleichungen mittels computerunterstützter Methoden der Intervallarithmetik. Ab 1978 gehört er dem Leitungsgremium der Tagungsreihe an und übernimmt ab 1983 die Herausgabe und Editierung der Tagungsbände „General Inequalities 3–7“. Die Tagungen haben einen internationalen Charakter und sind von der Idee durchdrungen, die Mathematik einmal nicht nach Disziplinen einzuteilen und zu separieren, sondern vielmehr einen vereinigenden Gedanken in den Vordergrund zu stellen. Dazu eignet sich das Thema Ungleichungen bestens – nicht zuletzt aufgrund der herausragenden Leistungen von Hardy, Littlewood und Pólya sowie Beckenbach und Bellman. Bei vielen „General Inequalities“ - Tagungen werden in den Nächten Ungleichungen bewiesen oder widerlegt, die tags zuvor zur Diskussion gestellt worden sind. Mit den „General Inequalities“ - Tagungen gelingen Wolfgang Walter wichtige Beiträge zur Internationalisierung der Mathematik und zur Verbreitung und Weitergabe wissenschaftlicher Forschungsresultate.

Wolfgang Walter hat ein engagiertes Interesse am wissenschaftlichen Fortschritt, an der Unterstützung seiner Kollegen und an der Förderung junger Nachwuchswissenschaftler. Er ist Mitglied der DMV und der GAMM sowie der AMS, MAA und SIAM. Die unnatürliche Einteilung in „Reine“ und „Angewandte“ Mathematik ist ihm fremd. Als einen wichtigen Teil seines mathematischen Lebenswerks betrachtet er den Fortschritt in der angewandten Analysis und der numerischen Mathematik. Daher fühlt er sich der GAMM besonders verbunden und fördert sie in vielerlei Hinsicht. Er ist von 1986 bis 1989 Präsident und von 1989 bis 1992 Vizepräsident der GAMM. Als 1987 die erste ICIAM Konferenz (International Conference on Industrial and Applied Mathematics) in Paris stattfindet, ist Wolfgang Walter von Beginn an involviert und fördert nach Kräften diese Konferenz, die 2011 zum siebten Mal stattfinden wird. Er ist Mitbegründer des Richard-von-Mises-Preises der GAMM, ist Mitglied des Preiskomitees und unterstützt den GAMM-Vorstand als beratendes Mitglied bis weit nach seiner Emeritierung.

## 7 Persönliche Begegnungen

In Wolfgang Walters Leben spielen zahlreiche Begegnungen mit Kollegen, Koautoren und Freunden eine wichtige Rolle. Ohne Anspruch auf Vollständigkeit seien mit George Knightly, Jean Mawhin, Djairo de Figueiredo, Alexander Weinstein, Hans Weinberger, Bill Ames, Ivo Babuška, Norrie Everitt, Catherine Bandle, Bernd Kawohl, Vangipuram Lakshmikantham, László Losonczi, Russell Thompson einige Persönlichkeiten aus dem Umfeld seiner mathematischen Tätigkeit genannt, zu denen er über viele Jahre kollegiale Verbindungen bis hin zu guten Freundschaften pflegt.

Die besonders intensiven Beziehungen zu seinen langjährigen Freunden und Koautoren Ray Redheffer und Joe McKenna wurden bereits erwähnt und werden in Abschnitt 9 nochmals zur Sprache kommen. Am besten wird dies durch die nachfolgenden Erinnerungen von Joe McKenna selbst beschrieben.

**Memories of Wolfgang Walter. By Joe McKenna, Univ. of Connecticut**

I first met Wolfgang Walter in Texas, at a conference in 1980, almost exactly thirty years to the day before his death. At the time, I was a young associate professor and he struck me as very old and distinguished. (He was younger than I am now!)

He also impressed me with a beautiful lecture on differential inequalities. We talked and our conversations evolved to the point where he was to spend a semester of an upcoming sabbatical in Gainesville in the autumn of 1982. My hope was to use the visit to learn about the field of differential inequalities. During that time, we worked on a competing species problem with Dirichlet boundary conditions and got some partial results. The problem remains open.

Later, I visited him in Karlsruhe several times, in the summers of 1984 and 1986. We still worked on differential inequalities for finite difference equations, but also on results using degree theory and nonlinear functional analysis. Later, he visited me in Storrs, in 1988. There we worked on a travelling wave problem for a suspension bridge equation. This involved nothing more advanced than calculus and we found explicit solutions of the nonlinear equation. The area started by this paper is still quite active today. Later, in the nineties, I visited Karlsruhe again, and (with his then student Wolfgang Reichel), we worked on radially symmetric solutions of semilinear equations with boundary blowup. This involved ordinary differential equation techniques.

Over the years, my family and his became very close and my children have happy memories of visits to the house on Breslauerstrasse. Looking back, I am struck mainly by the variety of different problems Wolfgang Walter would tackle with gusto. He loved all mathematics and would tackle any problem, regardless of what was involved in the solution. A truly natural mathematician.

Die Musik spielt in Wolfgang Walters Leben eine wichtige Rolle. Er singt gerne, spielt gut Klavier und hat bereits als Student in Tübingen Vorlesungen über Musiktheorie besucht. Im Hause Walter wird gerne und oft gesungen und musiziert. Mit seinem Freund und Koautor Ray Redheffer verbindet ihn neben der Liebe zur Mathematik auch die Freude am Klavierspielen. Bei zahlreichen Oberwolfach-Tagungen nutzt Wolfgang Walter die Gelegenheit, um mit gleichgesinnten Kolleginnen und Kollegen Musik zu spielen. In diesem Zusammenhang ergibt sich zu seinen Kollegen Ulrich Kulisch (Univ. Karlsruhe) und Klaus Kirchgässner (Univ. Stuttgart) eine enge freundschaftliche Beziehung. Die folgenden Erinnerungen von Ulrich Kulisch zeichnen den Beginn dieser Freundschaft nach und zeigen, wie die Musik zu einem Leitthema dieser Freundschaft wurde.

**Erinnerungen an Wolfgang Walter. Von Ulrich Kulisch, Univ. Karlsruhe**

Anfang 1969 erhielten sowohl Wolfgang Walter als auch ich ein Angebot für einen Forschungsaufenthalt am Mathematics Research Center (MRC) der University of Wisconsin. Die Karlsruher Fakultät genehmigte beide Forschungsaufenthalte für das Wintersemester 1969/70. Am 1. August 1969 reiste ich mit Familie dorthin ab. Wir hatten damals zwei Töchter im Alter von zwei Jahren und drei Monaten.

Am MRC wurde mir ein Zimmer zugewiesen. Im Nachbarzimmer saß Klaus Kirchgässner, ein Zimmer weiter George Knightly. Beide waren uns bei der Überwindung der Anfangsschwierigkeiten (Beschaffung von Auto, Wohnung, Möbeln, Kinderbetten usw.) behilflich. Sie machten mir auch klar, dass es hier üblich sei, mittags zum Joggen zu gehen.

Am 1. September traf dann Familie Walter ein. Sie hatten drei Schulkinder im Alter bis zu zehn Jahren und mussten sich daher in den University Houses einmieten. Aber sie hatten ja schon Amerika-Erfahrung. Sie wussten, dass man zunächst einmal die ganze Wohnung durchputzen und in Ordnung bringen musste. Wolfgang griff zu Pinsel und Farbe und zimmerte angekaufte Möbel zurecht. Auch Familie Knightly hatte Schulkinder und wohnte in den University Houses. Als die Anfangsschwierigkeiten überwunden waren, wurde auch Wolfgang Walter davon überzeugt, dass man mittags zum Joggen geht.

In Wisconsin gab es einen bereits damals berühmten amerikanischen Architekten namens Frank Lloyd Wright. Man erzählte uns, dass es in der Nähe des Ortes Spring Green am Wisconsin River etwa 50 Meilen nordöstlich von Madison ein von ihm erbautes, architektonisch interessantes Restaurant gibt. Nachdem alle einigermaßen eingerichtet waren, hatte Familie Walter die Idee, am nächsten Sonntag dorthin zum Mittagessen zu fahren. Wir hätten eine so weite Reise in eine unbekannte Gegend mit unserer kleinen Tochter damals wohl nicht gewagt, aber in Begleitung von Frau Walter als Ärztin willigten wir ein. Es lief alles sehr harmonisch ab und auch unsere kleine Tochter benahm sich zufriedenstellend. Gegen Ende unseres Ausfluges boten Irmgard und Wolfgang Walter uns das „Du" an. Dies war der Anfang unserer inzwischen über 40-jährigen Freundschaft.

Von da ab traf man sich in Madison ziemlich regelmäßig bei uns, bei Familie Walter, bei Familie Knightly oder bei anderen Kollegen oder Freunden. Klaus Kirchgässner reiste bereits im November wieder ab. Bei Treffen im Hause Walter wurde immer musiziert. Sie hatten selbstverständlich ein Klavier gemietet und verfügten als Familie bereits damals über ein beachtliches Repertoire im Gesang, das mit den Jahren beständig erweitert wurde. Irmgard und Wolfgang Walter hatten sich in einem Singkreis an der Universität Tübingen, den er gegen Ende seines Studiums selbst leitete, kennen gelernt.

Im April 1970 reiste auch Familie Walter wieder ab. Ich habe sie mit schwankendem Auto nach Chicago zum Flughafen gefahren. Dies war die Zeit, als unsere kleine Tochter anfing zu sprechen. „Mein Wolfgang" gehörte zu ihren ersten Worten, die sie sagen konnte, wenn er sie auf den Arm nahm.

Als alle wieder in Karlsruhe waren, ging es mit gemeinsamen Wochenendausflügen weiter. Irgendwann stellte Irmgard fest, dass ich einmal Cello gespielt hatte. Ich hatte ja bis zum Abitur die Lehrerbildungsanstalt in Freising besucht. Entweder Geige oder Klavier war für jeden Schüler Pflicht. Ein zweites Instrument war erwünscht. Dies war bei mir das Cello. So wurde sofort ein Klaviertrio ins Leben gerufen mit Irmgard und Wolfgang Walter am Klavier, Klaus Kirchgässner als Geiger und ich mit dem Cello. Von da ab trafen wir uns 30 Jahre lang reihum etwa vier Mal im Jahr und spielten Klaviertrios. Wir hatten alle viel Freude daran.

## 8 Professor Emeritus (1995–2010)

Am Ende des Sommersemesters 1995 läßt sich Wolfgang Walter von seinen Lehrverpflichtungen entbinden und im Herbst desselben Jahres nimmt Michael Plum als sein Nachfolger den Ruf nach Karlsruhe an. Mit seiner Emeritierung ändert sich der typische Arbeitstag von Wolfgang Walter nur wenig. Spätestens nach Abschluss der letzten von ihm betreuten Doktorarbeit im Januar 1996 sind die regelmäßigen Verpflichtungen zwar entfallen, aber seine Publikationstätigkeit bleibt in den Jahren nach seiner Emeritierung auf hohem Niveau. Allein im Zeitraum 1995–2003 erscheinen 17 Beiträge in wissenschaftlichen Zeitschriften. Er wird als Vortragender auf internationale Konferenzen eingeladen und reist 1999 für eine Reihe von Vorlesungen zu einer internationalen Sommerschule nach Chile. Er nimmt bis Ende der Neunziger Jahre regelmäßig an wissenschaftlichen Tagungen teil, spricht in Seminaren und Kolloquien und steht in engem Kontakt zu seinen Koautoren und Freunden Joe McKenna und Ray Redheffer. Ray Redheffer, dessen Ehefrau Heddy 1994 starb, heiratet 1997 Wolfgang Walters langjährige Sekretärin Irene Jendrasik. Vom Tod seines Freundes Ray Redheffer im Jahr 2005 ist Wolfgang Walter tief betroffen.

Seine Lehrbücher werden weiterhin von ihm intensiv gepflegt und regelmäßig erscheinen Neuauflagen. Die aufwändige Übersetzung und Umstrukturierung der Gewöhnlichen Differentialgleichungen ins Englische [26] erscheint im Jahr 1998. Die Analysis 1 [28] geht 2002 in die siebte, die Analysis 2 [29] 2002 in die fünfte Auflage und die Gewöhnlichen Differentialgleichungen [27] gehen 2000 ebenfalls in die siebte Auflage.

Auch in den ersten Jahren des neuen Jahrtausends bleibt Wolfgang Walter wissenschaftlich aktiv und ist als Mitherausgeber mehrerer wissenschaftlicher Zeitschriften weiterhin an der Front der Forschung und am Puls der Zeit. Er nimmt aktiv am universitären Leben teil, seine mathematische Expertise und sein persönlicher Rat werden an der Fakultät für Mathematik geschätzt und es vergeht kaum ein Tag, an dem er nicht sowohl vormittags als auch nachmittags sein Büro am Institut für Analysis aufsucht und arbeitet.

Auch privat erlebt Wolfgang Walter wie in den Jahrzehnten zuvor glückliche Jahre. Seine Kinder haben im Leben Fuß gefasst und er ist mehrfach Großvater geworden. Alle mir bekannten Zeitzeugen stimmen in der Einschätzung überein, dass Wolfgang Walters wissenschaftliche Karriere und sein großes familiäres Glück sich gegenseitig bedingen.

Leider ist es Wolfgang Walter aufgrund einer Erkrankung nicht vergönnt, die letzten Jahre seines Lebens so vital und aktiv wie zuvor erleben zu können. Obwohl die Zeichen einer schweren und unheilbaren Krankheit sichtbarer werden, erlebt Wolfgang Walter seinen 80. Geburtstag im Kreis seiner Kollegen und seiner Familie bei einem zu seinen Ehren veranstalteten Festkolloquium an der Universität Karlsruhe. Auch an seinen 81. Geburtstag erinnere ich mich gerne. Bei meinem Besuch im Haus der Familie Walter anlässlich dieses Geburtstages lebt die alte Vertrautheit zwischen ihm als Lehrer und mir als seinem Schüler noch einmal auf. Bei unserer letzten, herzlichen Verabschiedung planen wir ein Wiedersehen, das leider nicht mehr stattfindet.

Seine Familie gibt ihm Kraft und Unterstützung in den beiden letzten Jahren seines Lebens und nur wenige, sehr enge Freunde können ihn regelmäßig besuchen. Wolfgang Walter stirbt am 26. Juni 2010 im Alter von 83 Jahren in Karlsruhe. Seine Kolleginnen und Kollegen, insbesondere in Karlsruhe, seine Mitarbeiterinnen und Mitarbeiter, seine Schüler, Freunde und Koautoren vermissen ihn. Seine exakte Arbeitsweise, seine Liebe zum Detail, sein immerwährender Drang nach Verbesserung, seine Fähigkeit, Fragen zu stellen und neue Einsichten zu gewinnen, haben ihn ausgezeichnet. Durch sein Werk bleibt die Erinnerung an ihn erhalten, die von ihm gewonnenen Erkenntnisse werden zukünftige Forschung inspirieren und sein mathematisches Erbe wird weiteren Generationen den Weg weisen.

## 9 Ausschnitte aus Wolfgang Walters wissenschaftlichem Werk

### *9.1 Systeme parabolischer Differentialungleichungen*

Sein am häufigsten zitiertes wissenschaftliches Werk ist das 1970 erschienene Buch „Differential and Integral Inequalities“ [22], welches die englische Übersetzung und Erweiterung seines 1964 erschienenen Buches über „Differential- und Integralungleichungen“ [20] darstellt. Die englische Ausgabe von 1970 hat seine Karriere stark gefördert, ihn international bekannt gemacht und wird auch weiterhin häufig zitiert (über 700 Zitate in google scholar). Aus dem Umfeld dieses Werkes möchte ich einige Ergebnisse über Systeme parabolischer Differentialgleichungen erläutern.

Es sei $D \subset \mathbb{R}^N$ eine beschränkte, offene Menge, $T > 0$ und $G = D \times (0, T]$ der zugeordnete parabolische Zylinder. Auf $G$ betrachtet man das folgende System parabolischer Differentialgleichungen für die vektorwertige Funktion $u = (u^1, \dots, u^n) : \overline{G} \to \mathbb{R}^n$

$$u_t^k = f^k\big(x, t, u, u_x^k, u_{xx}^k\big) \quad \text{in } G, \quad k = 1, \dots, n, \tag{1}$$

bzw. in Kurzschreibweise

$$u_t = f(x, t, u, u_x, u_{xx}) \quad \text{in } G. \tag{2}$$

Dabei steht $u_t^k$ für die partielle Ableitung nach $t$, $u_x^k$ für den Gradienten, und $u_{xx}^k$ für die Hesse-Matrix der Funktion $u^k$. Das System (1) wird zum parabolischen System indem vorausgesetzt wird, dass die Funktionen $f^k(x, t, z, q, r)$ wachsend in der Variablen $r \in S^N$ sind, d.h. falls für zwei symmetrische $N \times N$-Matrizen $r, s \in S^N$ gilt: aus $r \geq s$ (im Sinne von $r - s$ positiv semi-definit) folgt $f^k(x, t, z, q, r) \geq f^k(x, t, z, q, s)$ für alle Werte $(x, t, z, q) \in G \times \mathbb{R}^n \times \mathbb{R}^N$. Oftmals wird das System (1) ergänzt durch Anfangsbedingungen bei $t = 0$ und Randbedingungen auf $\partial D$, d.h. es werden die Werte von $u$ auf dem parabolischen Rand $\Gamma = \overline{G} \setminus G = (\overline{D} \times \{0\}) \cup (\partial D \times (0, T])$ vorgeschrieben.

Als einfacher Fall sei das semilineare Beispiel $f^k(x,t,z,q,r) = \operatorname{spur} r + g^k(z) + h^k(z^k)$, $k = 1,\ldots,n$, genannt. Da $\operatorname{spur} u^k_{xx} = \sum_{\lambda=1}^N \partial^2_{x_\lambda x_\lambda} u^k$ gerade der Laplace-Operator $\Delta$ angewandt auf $u^k$ ist, reduziert sich das parabolische System (1) in diesem Fall auf

$$u^k_t = \Delta u^k + g^k(u) + h^k(u^k) \quad \text{in } G, \quad k = 1,\ldots,n.$$

Eine grundsätzliche Frage, die man für das parabolische System (1) stellen kann, ist die Folgende: führen geordnete Anfangs- und Randdaten zu geordneten Lösungen, d.h. folgt für zwei Lösungen $u, \tilde{u}$ mit $u \le \tilde{u}$ auf $\Gamma$ die Beziehung $u \le \tilde{u}$ in $G$? In vielen physikalischen, chemischen oder biologischen Anwendungen, in denen durch (1) die Konzentration von Stoffen, Chemikalien oder Populationen modelliert wird, ist ein solches qualitatives Verhalten eine sehr wichtige Information. Die Ordnung $u \le \tilde{u}$ wird dabei komponentenweise verstanden, d.h. zwei Vektoren $y = (y^1,\ldots,y^n), z = (z^1,\ldots,z^n) \in \mathbb{R}^n$ erfüllen die Relation

$$y \le z \quad \text{falls gilt} \quad y^i \le z^i \quad \text{für } i = 1,\ldots,n$$

und

$$y \ll z \quad \text{falls gilt} \quad y^i < z^i \quad \text{für } i = 1,\ldots,n.$$

Ein Satz, der es erlaubt, aus der Ordnung zweier Lösungen von (1) auf dem parabolischen Rand $\Gamma$ auf die Ordnung der Lösungen im parabolischen Zylinder $G$ zu schließen, wird als „Monotoniesatz" oder „Vergleichssatz" bezeichnet.

Im skalaren Fall $n = 1$ gelten Monotoniesätze unter geeigneten Annahmen an die Lipschitz-Stetigkeit der rechten Seite bzgl. der Variablen $z$ für parabolische Gleichungen ebenso wie auch für gewöhnliche Differentialgleichungen; zu erwähnen sind dabei die Arbeiten von Nagumo [13, 14] und Westphal [30]. Mit den Arbeiten von Müller [11, 12] und Kamke [5] wurde eine strukturelle Bedingung bei Systemen von gewöhnlichen Differentialgleichungen bekannt, die für Monotoniesätze sorgte. Dieser Bedingung gibt Walter den Namen „Quasimonotonie", vgl. [20, 22], und sie lautet im Kontext des parabolischen Systems (1) wie folgt:

**Definition 1** (Quasimonotonie[3]) Die in (2) auftretende Funktion $f = (f^1,\ldots,f^n)$ mit Komponentenfunktionen $f^k(x,t,z,q,r)$, $k = 1,\ldots,n$ heißt quasimonoton wachsend in $z$, falls für alle $(x,t,q,r) \in G \times \mathbb{R}^N \times S^N$, alle $y, z \in \mathbb{R}^n$ und alle $k \in \{1,\ldots,n\}$ gilt:

$$y \le z, \qquad y^k = z^k \Rightarrow f^k(x,t,y,q,r) \le f^k(x,t,z,q,r).$$

Hier wurde zur Vereinfachung angenommen, dass die Funktionen $f^k$ für alle $z \in \mathbb{R}^n$ definiert sind. In diesem Fall bedeutet „Quasimonotonie" von $f(x,t,z,q,r)$

[3] Anstelle der Begriffe „quasimonoton wachsend", „quasimonoton fallend" findet man in der Literatur gelegentlich die Ausdrücke „kooperativ", „kompetitiv" (engl. „cooperative", „competitive").

bzgl. der Variablen $z$ kurz gesprochen, dass $f^k(x,t,z,q,r)$ als Funktion von $z$ monoton wachsend ist bezüglich jeder Komponente $z^i$, $i \neq k$. Betrachtet man das semilineare Beispiel $f^k(x,t,z,q,r) = \operatorname{spur} r + g^k(z) + h^k(z^k)$, so ist Quasimonotonie von $f = (f^1, \dots, f^n)$ in $z$ gleichbedeutend mit Quasimonotonie von $g = (g^1, \dots, g^n)$ in $z$, d.h. es werden an die Funktionen $h^k$ keinerlei Bedingungen gestellt. Sind die Funktionen $g^k(z) = \sum_{i=1}^N g_i^k z^i$ linear mit $g_i^k \in \mathbb{R}$, so ist $g$ quasimonoton wachsend in $z$ genau dann, wenn die Einträge der Matrix $(g_i^k)_{k,i=1}^N$ außerhalb der Diagonale nichtnegativ sind.

Im Folgenden sei für die Funktion $f = (f^1, \dots, f^n)$ stets vorausgesetzt, dass $f^k(x,t,z,q,r)$ wachsend in $r \in S^N$ ist. Außerdem sollen die auftretenden Funktionen $u, v, w : \overline{G} \to \mathbb{R}^n$ stetig auf $\overline{G}$ sein und in $G$ stetige Ableitungen $u_t$, $u_x$, $u_{xx}$, $v_t$, $v_x$, $v_{xx}$, $w_t$, $w_x$, $w_{xx}$ besitzen.

Ausgehend von dem von Walter geprägten Begriff der „Quasimonotonie" lassen sich die folgenden beiden, auf Mlak [9] zurückgehenden Vergleichssätze (Satz 1, Satz 2) formulieren. Die Beweise sind nicht wörtlich, aber doch sinngemäß und im Stil von Wolfgang Walter übernommen und folgen seinem Credo [24]:

> The main results on differential inequalities are simple and elementary and should be proved accordingly. Heavier machinery, in particular existence theory, should only be used when necessary.

**Satz 1** (Vergleichsprinzip (I)) *Die Funktion $f = f(x,t,z,q,r)$ sei quasimonoton wachsend in $z$. Für zwei Funktionen $v, w : \overline{G} \to \mathbb{R}^n$ gelte*

$$v_t - f(x,t,v,v_x,v_{xx}) \ll w_t - f(x,t,w,w_x,w_{xx}) \quad \text{in } G. \tag{3}$$

*Dann folgt aus $v \ll w$ auf $\Gamma$ die Beziehung $v \ll w$ in $G$.*

*Beweis* Falls die Aussage falsch ist, so existiert ein Punkt $(\bar{x}, \bar{t}) \in \overline{G}$, so dass für die Funktion $u = w - v$ gilt

$$\begin{aligned} &u(\bar{x},\bar{t}) \geq 0, \qquad u^k(\bar{x},\bar{t}) = 0 \quad \text{für ein } k \in \{1,\dots,n\}, \\ &u(x,t) \gg 0 \quad \forall x \in G,\ 0 \leq t < \bar{t}. \end{aligned}$$

Aufgrund der Annahmen über $v$, $w$ auf $\Gamma$ ist $\bar{x} \in G$ und $\bar{t} > 0$. Es folgt daher

$$u_x^k(\bar{x},\bar{t}) = 0, \qquad u_{xx}^k(\bar{x},\bar{t}) \geq 0 \quad \text{und} \quad u_t^k(\bar{x},\bar{t}) \leq 0.$$

Aus (3) folgt im Punkt $(\bar{x},\bar{t})$ der Widerspruch

$$\begin{aligned} u_t^k &> f^k\big(\bar{x},\bar{t},w,w_x^k,w_{xx}^k\big) - f^k\big(\bar{x},\bar{t},v,v_x^k,v_{xx}^k\big) \\ &\geq f^k\big(\bar{x},\bar{t},w,v_x^k,v_{xx}^k\big) - f^k\big(\bar{x},\bar{t},v,v_x^k,v_{xx}^k\big) \\ &\geq 0. \end{aligned}$$

Im Übergang von der ersten zur zweiten Zeile wurde die Monotonie von $f^k(x,t,z,q,r)$ bzgl. $r \in S^N$ benutzt und im Übergang von der zweiten zur dritten Zeile die Quasimonotonie von $f$. □

Der Beweis dieses ersten Vergleichsprinzips ist sehr elementar und benötigt neben der Quasimonotonie fast keine Voraussetzungen an $f$. Lässt man in (3) und bei $v \ll w$ auf $\Gamma$ die schwachen Ungleichheitszeichen $\leq$ zu, so gilt ebenfalls ein Vergleichsprinzip – allerdings benötigt man dann zusätzlich eine Form von einseitiger Lipschitzbedingung (5) bzw. (5′) für $f(x,t,z,q,r)$ bzgl. $z$.

**Satz 2** (Vergleichsprinzip (II)) *Die Funktion* $f = f(x,t,z,q,r)$ *sei quasimonoton wachsend in* $z$. *Für zwei Funktionen* $v, w : \overline{G} \to \mathbb{R}^n$ *gelte*

$$v_t - f(x,t,v,v_x,v_{xx}) \leq w_t - f(x,t,w,w_x,w_{xx}) \quad \text{in } G. \tag{4}$$

*Ferner sei* $\mathbf{e} = (1,\ldots,1) \in \mathbb{R}^n$ *und es gebe Konstanten* $K, \Lambda > 0$, *so dass für* $k = 1,\ldots,n$ *und alle* $(x,t) \in G$, $\lambda \in (0,\Lambda)$ *gilt*

$$f^k(x,t,w+\lambda\mathbf{e},w_x,w_{xx}) - f^k(x,t,w,w_x,w_{xx}) \leq K\lambda. \tag{5}$$

*Dann folgt aus* $v \leq w$ *auf* $\Gamma$ *die Beziehung* $v \leq w$ *in* $G$.

*Beweis* Sei $w_\epsilon := w + \epsilon e^{\overline{K}t}\mathbf{e}$ für $\overline{K} > K$ und $\epsilon > 0$ hinreichend klein. Dann gilt aufgrund der einseitigen Lipschitzbedingung (5)

$$\begin{aligned} w^k_{\epsilon,t} - f^k(x,t,w_\epsilon,w_{\epsilon,x},w_{\epsilon,xx}) &= w^k_t + \overline{K}\epsilon e^{\overline{K}t} - f^k(x,t,w_\epsilon,w_x,w_{xx}) \\ &> w^k_t - f^k(x,t,w,w_x,w_{xx}), \end{aligned}$$

und Satz 1 angewandt auf das Funktionenpaar $v, w_\epsilon$ liefert $v \ll w_\epsilon$ in $G$, woraus für $\epsilon \to 0$ die Behauptung folgt. □

In Satz 2 kann man anstelle von (5) auch die Bedingung

$$f^k(x,t,v,v_x,v_{xx}) - f^k(x,t,v-\lambda\mathbf{e},v_x,v_{xx}) \leq K\lambda \tag{5′}$$

verwenden. Im semilinearen Beispiel $f^k(x,t,z,q,r) = \operatorname{spur} r + g^k(z) + h^k(z^k)$ sind (5), (5′) erfüllt, wenn $g^k \in C^1(\mathbb{R}^n)$ und $h^k$ monoton fallend auf $\mathbb{R}$ ist.

Satz 2 lässt sich insbesondere dann auf Funktionen $u, v, w$ anwenden, wenn (5), (5′) gilt sowie

$$\begin{aligned} &v_t \leq f(x,t,v,v_x,v_{xx}), \qquad u_t = f(x,t,u,u_x,u_{xx}), \\ &w_t \geq f(x,t,w,w_x,w_{xx}) \quad \text{in } G \end{aligned}$$

und $v \leq u \leq w$ auf $\Gamma$. Die Folgerung $v \leq u \leq w$ in $G$ liefert damit Einschließungen von Lösungen $u$ des parabolischen Systems (2).

Der von Walter eingeführte Begriff der „Quasimonotonie" erweist sich als sehr erfolgreich, denn mit Hilfe dieses Begriffes können zahlreiche Ergebnisse für parabolische Systeme bewiesen werden. Von Volkmann [19] wird der Begriff „Quasimonotonie" verallgemeinert für Funktionen, die ihre Werte in topologischen

Vektorräumen annehmen; dies zieht auf dem Gebiet der gewöhnlichen und partiellen Differentialgleichungen in Banachräumen sowie in der nichtlinearen Funktionalanalysis eine Vielzahl von Ergebnissen und Publikationen nach sich.

In Walters Buch [22] finden sich wesentlich allgemeinere Versionen der obigen beiden Sätze. Unter anderem interessiert er sich auch sehr für den Fall, dass das zugrundeliegende parabolische System keine Quasimonotonie-Eigenschaft hat. In diesem Zusammenhang findet Walter folgenden Satz, der im Falle gewöhnlicher Differentialgleichungssysteme erster Ordnung auf Müller [12] zurückgeht. Dabei ist die verwendete einseitige Lipschitzbedingung (5′), (5′) an $f(x,t,z,q,r)$ bzgl. $z$ stärker als diejenige von Satz 2. Für das semilineare Beispiel $f^k(x,t,z,q,r) = \operatorname{spur} r + g^k(z) + h^k(z^k)$ ist (5′), (5′) erfüllt, wenn $g^k \in C^1(\mathbb{R}^n)$ und $h^k$ monoton fallend auf $\mathbb{R}$ ist.

**Satz 3** (Vergleichsprinzip (III)) *Für zwei Funktionen $v, w : \overline{G} \to \mathbb{R}^n$ mit $v \le w$ in $\overline{G}$ und für $k = 1, \ldots, n$ gelte*

$$v_t^k \le f^k(x,t,z,v_x^k,v_{xx}^k) \quad \textit{in } G$$
$$\textit{für alle } z \in \mathbb{R}^n \textit{ mit } v(x,t) \le z \le w(x,t),\ v^k(x,t) = z^k,$$
$$w_t^k \ge f^k(x,t,z,w_x^k,w_{xx}^k) \quad \textit{in } G$$
$$\textit{für alle } z \in \mathbb{R}^n \textit{ mit } v(x,t) \le z \le w(x,t),\ w^k(x,t) = z^k.$$

*Ferner gebe es Konstanten $K, \Lambda > 0$, so dass für $k = 1, \ldots, n$ und alle $(x,t) \in G$, $z, \tilde{z} \in \mathbb{R}^n$ mit $v(x,t) \le z \le w(x,t)$, $\|z - \tilde{z}\|_\infty \le \Lambda$ gilt*

$$f^k(x,t,z,v_x,v_{xx}) - f^k(x,t,\tilde{z},v_x,v_{xx}) \le K\|z-\tilde{z}\|_\infty \quad \textit{für } z^k \ge \tilde{z}^k \qquad (5')$$

*und*

$$f^k(x,t,\tilde{z},w_x,w_{xx}) - f^k(x,t,z,w_x,w_{xx}) \le K\|z-\tilde{z}\|_\infty \quad \textit{für } \tilde{z}^k \ge z^k. \qquad (5')$$

*Ist $u$ Lösung von (2), dann folgt aus $v \le u \le w$ auf $\Gamma$ die Einschließung $v \le u \le w$ in $G$.*

*Beweis* Zuerst wird die entsprechende Aussage bewiesen, falls in den Differentialungleichungen für $v, w$ die Relationen $\le, \ge$ durch strikte Ungleichungen $<, >$ ersetzt werden und ebenso die Voraussetzung $v \le u \le w$ auf $\Gamma$ durch $v \ll u \ll w$ auf $\Gamma$ ersetzt wird. Falls etwa die Behauptung $v \ll u$ in $\overline{G}$ nicht gilt, so existiert wie im Beweis von Satz 1 ein Punkt $(\bar{x},\bar{t}) \in G$ mit

$$v(\bar{x},\bar{t}) \le u(\bar{x},\bar{t}), \qquad v^k(\bar{x},\bar{t}) = u^k(\bar{x},\bar{t}) \quad \text{für ein } k \in \{1,\ldots,n\},$$
$$v(x,t) \ll u(x,t) \quad \forall x \in G,\ 0 \le t < \bar{t}$$

sowie

$$v_x^k(\bar{x},\bar{t}) = u_x^k(\bar{x},\bar{t}), \qquad v_{xx}^k(\bar{x},\bar{t}) \le u_{xx}^k(\bar{x},\bar{t}) \quad \text{und} \quad v_t^k(\bar{x},\bar{t}) \ge u_t^k(\bar{x},\bar{t}).$$

Mit Hilfe der Differentialungleichung für $v$, in der $z = u(\bar{x}, \bar{t})$ zulässig ist, folgt im Punkt $(\bar{x}, \bar{t})$ der Widerspruch

$$v_t^k < f^k\big(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), v_x^k, v_{xx}^k\big) \le f^k\big(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), u_x^k, u_{xx}^k\big) = u_t^k.$$

Analog beweist man die Ungleichung $u \ll w$ in $\overline{G}$. Die Behauptung mit den schwachen Ungleichheitszeichen wird bewiesen, indem man $v, w$ durch $v_\epsilon = v - \epsilon e^{\widetilde{K}t}\mathbf{e}$, $w_\epsilon = w + \epsilon e^{\widetilde{K}t}\mathbf{e}$ mit $\epsilon > 0$, $\widetilde{K} > K$ ersetzt, die Lipschitzbedingung (5′), (5′) ausnutzt, um den Satz mit strikten Ungleichheitszeichen anwenden zu können, und schließlich $\epsilon \searrow 0$ streben lässt. □

Satz 3 nimmt eine sehr prägnante Form an im Fall $n = 2$, $v(x,t) = \text{const.} = (\alpha, \gamma) \in \mathbb{R}^2$, $w(x,t) = \text{const.} = (\beta, \delta) \in \mathbb{R}^2$. In diesem Fall reduzieren sich die Voraussetzungen von Satz 3 an $f = (f^1, f^2)$ neben der einseitigen Lipschitzbedingung (5′), (5′) auf die beiden Bedingungen

$$f^1\big(x, t, (\alpha, z^2), 0, 0\big) \ge 0 \ge f^1\big(x, t, (\beta, z^2), 0, 0\big) \quad \text{in } G \text{ für alle } z^2 \in [\gamma, \delta],$$
$$f^2\big(x, t, (z^1, \gamma), 0, 0\big) \ge 0 \ge f^2\big(x, t, (z^1, \delta), 0, 0\big) \quad \text{in } G \text{ für alle } z^1 \in [\alpha, \beta].$$

Betrachtet man das von $v$ und $w$ aufgespannte Rechteck $R = [\alpha, \beta] \times [\gamma, \delta] \subset \mathbb{R}^2$, so besagt Satz 3, dass für Lösungen $u$ von (2) aus $u(\Gamma) \subset R$ die Aussage $u(G) \subset R$ folgt. In diesem Zusammenhang spricht man davon, dass $R$ ein invariantes Rechteck ist. Die Bedingungen an $f$ lassen sich in diesem Fall so verstehen, dass gilt

$$f(x, t, z, 0, 0) \cdot \nu(z) \le 0 \quad \text{für alle } z \in \partial R,$$

wobei $\nu(z)$ den äußeren Normalenvektor von $R$ in $z$ bezeichnet. Diese Bedingung an $f$ kann man so interpretieren, dass der parabolische Fluss, der durch das Vektorfeld $f$ repräsentiert wird, in die Menge $R$ hineinzeigt.

Es stellt sich die allgemeinere Frage, unter welchen Bedingungen eine Teilmenge $S \subset \mathbb{R}^n$ die Eigenschaft der Invarianz besitzt, d.h. unter welchen Bedingungen an $S$ folgt für Lösungen $u$ von (2) aus $u(\Gamma) \subset S$ die Aussage $u(G) \subset S$? Gemeinsam mit Ray Redheffer geht Wolfgang Walter dieser Frage in mehreren Arbeiten, vgl. [16, 17], nach und es gelingt ihnen, die Frage zu beantworten für semilineare parabolische Systeme der Form

$$u_t^k = Lu^k + g^k(x, t, u, u_x) \quad \text{in } G, \quad k = 1, \dots, n \tag{5′}$$

mit einem Operator $L = \sum_{\lambda,\mu=1}^N a_{\lambda\mu}(x,t)\partial^2_{x_\lambda x_\mu} + \sum_{\lambda=1}^N b_\lambda(x,t)\partial_{x_\lambda}$, der für alle $n$ Gleichungen des Systems derselbe ist und für dessen Koeffizienten nur die positive Semidefinitheit der Matrix $(a_{\lambda\mu})_{\lambda,\mu=1}^N$ vorausgesetzt werden muss. Unter Lipschitzbedingungen an die Funktionen $g^k(x, t, z, q)$ bzgl. der Variablen $z \in \mathbb{R}^n$ lautet die Antwort, dass die Bedingung

$$g(x, t, z, q) \cdot \nu(z) \le 0 \quad \text{für alle } z \in \partial S$$
$$\text{und alle } q = (q^1, \dots, q^N) \in \mathbb{R}^{Nn},\ q^\lambda \in \mathbb{R}^n \text{ mit } q^\lambda \cdot \nu(z) = 0,\ \lambda = 1, \dots, N,$$

wobei $\nu(z)$ eine äußere Normale an $S$ in $z$ bezeichnet, und die Konvexität der abgeschlossenen Menge $S \subset \mathbb{R}^n$ eine hinreichende Bedingung für ihre Invarianz unter (5′) ist. Invarianzaussagen dieser Form spielen eine wesentliche Rolle z.B. bei der Untersuchung des Langzeitverhaltens der Lösungen von Systemen von Reaktions-Diffusionsgleichungen.

## *9.2 Hängebrückenmodelle*

1987 veröffentlichen Lazer und McKenna eine Arbeit [6] über ein Modell zur Beschreibung von Hängebrücken. In diesem Modell wird das Brückenbett als elastischer Balken und die Kabel der Hängebrücke als Federn modelliert, deren Rückstellkraft bei Streckung gemäß dem Hookeschen Gesetz angenommen wird, während sie (in Abweichung vom Hookeschen Gesetz) keine Rückstellkraft bei Kompression aufweisen. Für die Auslenkung $u(x,t)$ eines solchen Brückenbetts schlagen Lazer und McKenna die Wellengleichung

$$u_{tt} + K_1 u_{xxxx} + K_2 u^+ = W(x) + \epsilon f(x,t)$$

vor. Dabei sind $K_1, K_2 > 0$ positive Konstanten, $u > 0$ entspricht einer Auslenkung nach unten, $u^+ = \max\{u, 0\}$, $W(x)$ ist die Massendichte der eindimensional modellierten Brücke und $\epsilon f(x,t)$ modelliert eine kleine äußere Kraft (z.B. Wind). Unter geeigneter Reskalierung der Variablen und unter der Annahme homogener Massenverteilung lässt sich dieses Modell reduzieren zu

$$u_{tt} + u_{xxxx} + bu^+ = 1 + \epsilon h(x,t) \qquad (5')$$

mit $b > 0$, $\epsilon \in \mathbb{R}$ klein. Als Walter über McKenna von diesem Modell erfährt, ist er davon sofort angetan und beginnt mit McKenna an zwei Aspekten zu arbeiten: periodische, stehende Wellen [7] und wandernde Wellen [8].

Bei der Untersuchung periodischer, stehender Wellen betrachten McKenna und Walter (5′) auf $Q = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$, ergänzen (5′) mit Randbedingungen („hinged boundary conditions“)

$$u\left(-\frac{\pi}{2}, t\right) = u\left(\frac{\pi}{2}, t\right) = u_{xx}\left(-\frac{\pi}{2}, t\right) = u_{xx}\left(\frac{\pi}{2}, t\right) = 0 \qquad (5')$$

und beweisen folgendes Ergebnis, vgl. [7].

**Satz 4** *Sei $h \in L^2(Q)$ gerade in $x$, $t$ mit $\|h\|_{L^2(Q)} = 1$ und $3 < b < 15$. Dann existiert $\epsilon_0 > 0$ so, dass für $|\epsilon| < \epsilon_0$ die Gleichung (5′) auf $Q$ mit Randbedingungen (5′) mindestens zwei zeitlich $\pi$-periodische Lösungen besitzt, die gerade in $x, t$ sind.*

Der Operator $L = \frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}$ mit Randbedingungen (5′) und $\pi$-Periodizität in $t$ ist selbstadjungiert auf dem Hilbertraum $H$ der in $x$ und $t$ geraden $L^2$-integrierbaren

Funktionen auf $Q$. Sein Spektrum ist daher reell, aber nach oben und unten unbeschränkt. Die Eigenwerte von $L$ in der Nähe von 0 sind gegeben durch $\lambda_{-2} = -15$, $\lambda_{-1} = -3$, $\lambda_1 = 1$, $\lambda_2 = 17$. Die Bedingung an $b$ in Satz 4 bedeutet also, dass $-b$ zwischen zwei aufeinanderfolgenden Eigenwerten von $L$ liegt. Würde man anstelle von ($5'$) die zugehörige lineare Gleichung $u_{tt} + u_{xxxx} + bu = 1 + \epsilon h(x,t)$ betrachten, so gäbe es nur eine eindeutige Lösung mit Randbedingung ($5'$), die gerade in $x$, $t$ und $\pi$-periodisch in $t$ ist. Satz 4 belegt also, dass die Nichtlinearität $u^+$ in ($5'$) neue Lösungen hervorbringt, die es im linearen Modell nicht gibt. Als McKenna und Walter Satz 4 beweisen, können sie auf ihre reichhaltige Erfahrung im Beweis von Existenzsätzen für nichtlineare Randwertprobleme zurückgreifen, bei denen die auftretende Nichtlinearität in Resonanz mit dem Spektrum des linearisierten Problems steht.

Die im Folgenden geschilderte Beweisidee von McKenna und Walter ist ein schönes Beispiel für die Ausnutzung von a-priori Schranken im Wechselspiel mit dem auf Leray und Schauder zurückgehenden topologischen Abbildungsgrad. Der Abbildungsgrad ist eine ganzzahlige Abbildung $\mathrm{d}_{LS}$, die einer beschränkten, offenen, nichtleeren Menge $\Omega \subset X$ eines Banachraumes $X$, einer Abbildung $\mathrm{Id} - F : \overline{\Omega} \to X$ sowie einem Element $z \in X$, $z \notin (\mathrm{Id} - F)(\partial\Omega)$ eine ganze Zahl zuordnet. Dabei ist z.B. vorauszusetzen, dass $F : \overline{\Omega} \to X$ stetig ist und beschränkte, abgeschlossene Mengen auf kompakte Mengen abbildet. Falls $\mathrm{d}_{LS}(\mathrm{Id} - F, \Omega, z) \neq 0$ ist, so besitzt die Gleichung $x - F(x) = z$ eine Lösung $x \in \Omega$. Anstatt $\mathrm{d}_{LS}(\mathrm{Id} - F, \Omega, z)$ schreibt man auch $\mathrm{d}_{LS}(u - F(u), \Omega, z)$.

*Beweisskizze* (a) Schwache Lösungen von ($5'$), ($5'$) erhält man als Lösungen der Gleichung

$$u - L^{-1}\big(1 - bu^+ + \epsilon h\big) = 0$$

im Hilbertraum $(H, \|\cdot\|)$ der in $x$ und $t$ geraden $L^2$-integrierbaren Funktionen auf $Q$. Dabei steht $L^{-1}$ für den Lösungsoperator zum linearen Problem $Lw = f$ mit Randbedingungen ($5'$) und $\pi$-Periodizität in $t$ für die Lösung $w$.

(b) Fixiert man eine kleine positive Zahl $\alpha > 0$, betrachtet $b \in [-1 + \alpha, 15 - \alpha]$ sowie $\epsilon \in [-1, 1]$ und berücksichtigt die Normierung $\|h\| = 1$, so existiert ein $R_0 > 0$ derart, dass für jede Lösung $u$ von ($5'$), ($5'$) die a-priori Schranke $\|u\| < R_0$ gilt. Denn falls es keine solche Schranke gäbe, dann würde eine Folge $u_k$ von Lösungen existieren mit $\|u_k\| \to \infty$ und $w_k := u_k/\|u_k\| \to w_0$ für $k \to \infty$, wobei $w_0$ eine nicht-triviale Lösung von $Lw_0 + bw_0^+ = 0$ mit „hinged boundary conditions" wäre. Ein Widerspruch – denn die Existenz einer solchen nicht-trivialen Lösung hatten McKenna und Walter bereits zuvor ausschließen können.

(c) Mit Hilfe der a-priori Schranke und der Invarianz des Leray-Schauder-Abbildungsgrades bzgl. des Homotopieparameters $b \in [-1 + \alpha, 15 - \alpha]$ folgt für alle Radien $R \geq R_0$

$$\mathrm{d}_{LS}\big(u - L^{-1}\big(1 - bu^+ + \epsilon h\big), B_R(0), 0\big) = \mathrm{d}_{LS}\big(u - L^{-1}(1 + \epsilon h), B_R(0), 0\big) = 1. \tag{5'}$$

(d) Als nächstes zeigen McKenna und Walter, dass das elliptische Randwertproblem

$$y^{(iv)} + by^{+} = 1 \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ mit } y\left(\pm\frac{\pi}{2}\right) = y_{xx}\left(\pm\frac{\pi}{2}\right) = 0$$

genau eine Lösung besitzt. Diese Lösung $y \in H$ ist auch eine stationäre Lösung von (5′), (5′) im Fall $\epsilon = 0$. Betrachtet man eine Kugel $B_\gamma(y)$ mit hinreichend kleinem Radius $\gamma > 0$ so zeigen McKenna, Walter, dass das folgende Problem mit $u^- = -\min\{u, 0\}$

$$Lu + bu = 1 + \lambda\big(\epsilon h - bu^-\big) \quad \text{in } Q \text{ mit Randbedingungen (5′)}$$

für $\lambda \in [0, 1]$ und $|\epsilon| < \epsilon_0$ keine Lösung auf $\partial B_\gamma(y)$ hat. Daher ist folgende Berechnung des Abbildungsgrades unter Ausnutzung der Homotopieinvarianz in $\lambda \in [0, 1]$ gerechtfertigt:

$$\begin{aligned} &\mathrm{d}_{LS}\big(u - L^{-1}\big(1 - bu^+ + \epsilon h\big), B_\gamma(y), 0\big) \\ &= \mathrm{d}_{LS}\big(u - L^{-1}\big(1 - bu + \lambda\big(\epsilon h - bu^-\big)\big), B_\gamma(y), 0\big) \\ &= \mathrm{d}_{LS}\big(u - L^{-1}(1 - bu), B_\gamma(y), 0\big) \\ &= \mathrm{d}_{LS}\big(u + L^{-1}(bu), B_\gamma(0), 0\big) \\ &= -1, \end{aligned} \tag{5′}$$

wobei für die letzte Auswertung des Abbildungsgrades entscheidend ist, dass unter der Bedingung $3 < b < 15$ der Operator $\mathrm{Id} + bL^{-1}$ auf $H$ nur einen negativen Eigenwert besitzt. Schließlich folgt aus (5′), (5′) und der Ausschöpfungseigenschaft des Abbildungsgrades, dass (5′), (5′) mindestens zwei in $x, t$ gerade und in der Zeit $\pi$-periodische Lösungen besitzt: eine in $B_\gamma(y)$ und eine in $B_R(0) \setminus \overline{B_\gamma(y)}$ für $R \geq R_0$. □

Sucht man für die Gleichung (5′) Lösungen in Form wandernder anstatt stehender Wellen, so betrachtet man (5′) für $(x, t) \in \mathbb{R}^2$. Außerdem steht in diesem Fall die äußere Kraft $h(x, t)$ nicht im Vordergrund, so dass man $\epsilon = 0$ wählt. Mit Hilfe des Ansatzes

$$u(x, t) = \frac{1}{b} y\big(b^{-1/4}x - b^{-1/2}ct\big)$$

reduziert sich (5′) auf

$$y^{(iv)} + c^2 y'' + y^+ = 1 \tag{5′}$$

bzw. indem man $z = y - 1$ setzt auf

$$z^{(iv)} + c^2 z'' + z = 0 \quad \text{falls } z(x) \geq -1, \tag{5′}$$

$$z^{(iv)} + c^2 z'' - 1 = 0 \quad \text{falls } z(x) \leq -1. \tag{5′}$$

Gesucht werden „homokline“ Lösungen mit $z(x) \to 0$ (exponentiell) für $|x| \to \infty$. In ihrer zweiten Arbeit [8] konstruieren McKenna und Walter explizite Lösungen, indem sie auf einem Intervall $[-r, r]$ eine gerade Lösung von (5′) an der Stelle $t = \pm r$ mit einer exponentiell gegen 0 fallenden Lösung von (5′) zusammenpassen. Die vier Übergangsbedingungen, die entstehen um eine glatte Lösung zu erhalten, führen auf transzendente Gleichungen für die freien Parameter der Lösungsscharen und für die Wellengeschwindigkeit $c$. Die Suche nach Lösungen der transzendenten Gleichungen ist in geschlossener Form nicht möglich. Daher benutzen McKenna und Walter Standardsoftware zur numerischen Berechnung der Nullstellen der transzendenten Gleichungen. Sobald die Parameter und die Wellengeschwindigkeit $c$ numerisch bestimmt sind, ist anschließend noch zu überprüfen, dass die Lösungen tatsächlich $\geq -1$ auf $(-\infty, -r)$, $(r, \infty)$ bzw. $\leq -1$ auf $[-r, r]$ sind. Es stellt sich heraus, dass die Wellengeschwindigkeit $c$ in einem Intervall $[c_1, c_2]$ liegt mit $0 < c_1 < c_2 < \sqrt{2}$. Obwohl hierbei nur relativ einfache Rechnungen notwendig sind, finden McKenna und Walter in [8] auf diese Weise einige sehr interessante „travelling wave“ – Lösungen von (5′), (5′), die den in Abb. 1 dargestellten sehr ähnlich sind.[4]

Die beiden Arbeiten [7, 8] von McKenna und Walter, die bisher 28 bzw. 24 Mal zitiert wurden, haben weitere interessante Forschungsarbeiten nach sich gezogen, über die hier in einer kurzen Übersicht berichtet wird:

(1) Chen und McKenna [3] gaben 1995 einen Beweis der Existenz von Lösungen von (5′) für alle Wellengeschwindigkeiten $c \in (0, \sqrt{2})$ mit Hilfe variationeller Methoden. Gleichzeitig benutzten sie einen von Choi und McKenna [4] entwickelten numerischen Mountain Pass Algorithmus, um Approximationen der wandernden Wellen konkret zu bestimmen. Ebenso wurden hier erstmalig die Stabilitäts- und Interaktionseigenschaften dieser wandernden Wellen numerisch untersucht.

(2) Chen und McKenna erkannten, dass die Nichtlinearität $(z + 1)^+ - 1$ für numerische Rechnungen wenig geeignet war, und schlugen vor, sie durch $e^z - 1$ zu ersetzen. Diese Nichtlinearität hat für $z \ll -1$ und für $z > 0$ (aber nicht zu groß) praktisch denselben Effekt wie die zuvor betrachtete.

(3) Smets und van den Berg [18] bewiesen 2002 die Existenz von mindestens einer abklingenden Lösung der Gleichung $z^{(iv)} + c^2 z'' + e^z - 1 = 0$ für fast alle Wellengeschwindigkeiten $c \in (0, \sqrt{2})$ ebenfalls unter Verwendung variationeller Methoden.

(4) 2006 untersuchten Breuer, Horák, McKenna und Plum [2] die Gleichung $z^{(iv)} + c^2 z'' + e^z - 1 = 0$ hinsichtlich Existenz abklingender Lösungen. Mit Hilfe computerunterstützter Methoden gelang ihnen der analytische Nachweis und die rigorose Einschließung von 36 unterschiedlichen Lösungen bei der Wellengeschwindigkeit $c = 1.3$, vgl. Abb. 1. Dieses Resultat steht in bemerkenswertem Kontrast zum Ergebnis von Smets und van den Berg: die Aus-

[4] Die Graphiken aus [8] sind nicht erhalten. Die Graphiken aus Abb. 1 stammen aus dem Besitz von J. Horák.

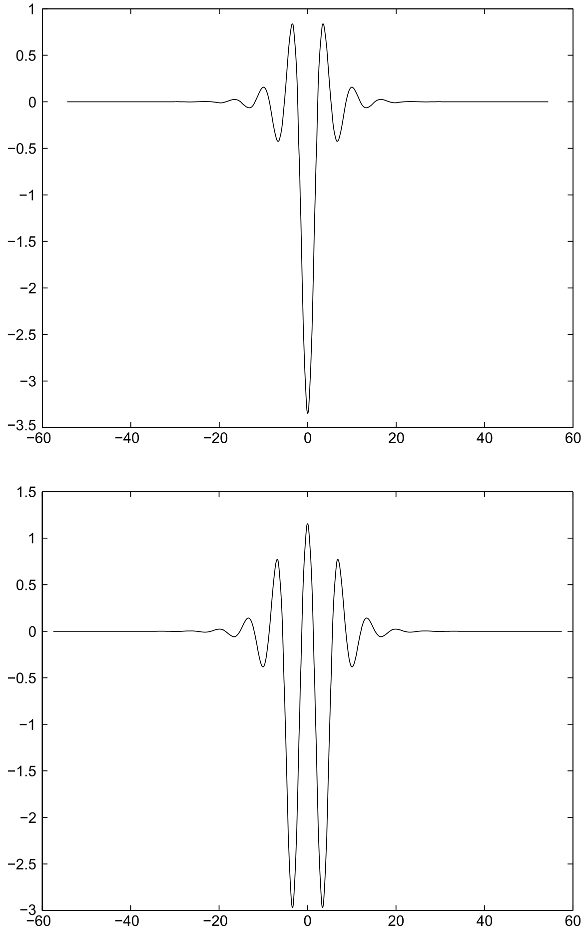



**Abbildung 1** Oben „Travelling wave" wie in McKenna, Walter [8]. Unten „Travelling wave" wie in Breuer, Horák, McKenna und Plum [2]

sage *mindestens eine Lösung für fast alle c* steht der Aussage *mindestens 36 Lösungen bei festem* $c = 1.3$ gegenüber.

(5) 2002 publizierte Moore eine Arbeit [10], in der neben longitudinalen Oszillationen auch Torsionsoszillationen quer zum Brückenbett modelliert wurden. Mit Hilfe von Abbildungsgradtheorie konnte Moore die Existenz periodischer Schwingungen nachweisen. Auffällig war bei den numerischen Experimenten, dass große longitudinale Oszillationen, die durch kleine Torsionsoszillationen gestört werden, beinahe ansatzlos in starke Torsionsoszillationen übergehen können. Vergleichbare Beobachtungen wurden 1940 von Augenzeugen beim Einsturz der Tacoma Narrows Bridge gemacht, siehe z.B. http://de.wikipedia.org/wiki/Tacoma-Narrows-Brucke.

Wolfgang Walter hat die Weiterentwicklung der Arbeiten zu Hängebrücken aufmerksam und mit großem Interesse verfolgt. Die Benutzung unterschiedlicher Methoden aus der Analysis partieller Differentialgleichungen, der Numerik und der Modellierung, die bei diesem Problem nötig und von gleichrangiger Bedeutung sind, entspricht sehr gut seiner Vorstellung von Mathematik.

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# Conference on Inequalities and Applications ’10

The *Conference on Inequalities and Applications ’10* meeting was a resumption of the General Inequalities symposia with the following brief history:

| | | |
|---|---|---|
| General Inequalities 1 | 1976 | Oberwolfach, Germany |
| General Inequalities 2 | 1978 | Oberwolfach, Germany |
| General Inequalities 3 | 1981 | Oberwolfach, Germany |
| General Inequalities 4 | 1983 | Oberwolfach, Germany |
| General Inequalities 5 | 1986 | Oberwolfach, Germany |
| General Inequalities 6 | 1990 | Oberwolfach, Germany |
| General Inequalities 7 | 1995 | Oberwolfach, Germany |
| General Inequalities 8 | 2002 | Noszvaj, Hungary |
| Conference on Inequalities and Applications ’07 | 2007 | Noszvaj, Hungary |

The meeting was dedicated to the memory of Wolfgang Walter.

The meeting took place from September 19 to 25, 2010, at In Hotel in Hajdúszoboszló, Hungary, and was organized by the Institute of Mathematics of the University of Debrecen with the financial supports of the Hungarian Scientific Research Fund Grant OTKA NK-81402.

The members of the Scientific Committee were Professors Catherine Bandle (Basel), W. Norrie Everitt (Birmingham, honorary member), László Losonczi (Debrecen), Zsolt Páles (Debrecen) and Michael Plum (Karlsruhe).

The Local Organizing Committee consisted of Professors Zoltán Daróczy (honorary chairman), Zoltán Boros (co-chairman), Attila Gilányi (co-chairman), Gyula Maksa and Mihály Bessenyei (scientific secretary). The Committee Members were ably assisted by Szabolcs Baják, Eszter Gselmann, Judit Makó and Fruzsina Mészáros. There were 47 participants from 8 countries.

Professor Plum opened the Symposium in the name of the Scientific Committee and welcomed the participants in the name of the Local Organizing Committee.

The talks at the symposium focused on the following topics: convexity and its generalizations; mean values and functional inequalities; matrix and operator inequalities; inequalities for ordinary and partial differential operators; integral and differential inequalities; variational inequalities; numerical methods.

A number of sessions were, as usual, devoted to problems and remarks.

Of course, there were some social and cultural events during the conference, such as visiting the Thermal Bath of Hajdúszoboszló on Wednesday afternoon and the Banquet on Thursday evening.

The scientific sessions were followed in the evening of Thursday by a festive banquet in the De La Motte Castle. The conference was closed on Friday by Professor Zsolt Páles.

Abstracts of the talks are in alphabetical order of the authors. These are followed by the problems and remarks (in approximate chronological order), and finally, the list of participants.

## 1 Abstracts of Talks

**Abramovich, Shoshana** *On convex, superquadratic and superterzatic functions.* (Joint work with Slavića Ivelić and Josip Pečarić.)

This talk is mainly about inequalities satisfied by functions called superterzatic and their relations to convex and to superquadratic functions.

In analogy to inequalities satisfied by convex functions and by superquadratic functions that are reduced to equalities when $f(x) = x$, $f(x) = x^2$, $x \geq 0$ respectively, the inequalities satisfied by superterzatic functions reduce to equalities when $f(x) = x^3$, $x \geq 0$.

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**Baják, Szabolcs** *A non-linear form of the Hahn–Banach separation theorem.* (Joint work with Zs. Páles.)

A generalization of the Hahn–Banach separation theorem—due to Dubovitskii and Milyutin—is a classical result of the linear functional analysis. In this presentation an even more general setting is considered: the local disjointness at a given point of non-linear inverse images of convex sets are investigated. Thus, by using the notion of approximation of a point and the notion of the so-called tangent maps, it is possible to develop first- and higher-order necessary conditions for various non-smooth optimization problems.

**Bandle, Catherine** *Variation of domains and Pohozaev's identity.* (Joint work with Alfred Wagner.)

The main goal of this talk is to present a connection between domain derivatives of energy functionals and Pohozaev's identity. These derivatives can then be applied to discuss nonexistence and integral estimates of quasilinear boundary value problems, and to find candidates for optimal domains for energy functionals. Similar and very general results are found in the Lecture Note of W. Reichel [1] via the method of transformation groups.

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**Batra, Prashant** *Half-plane mappings and coefficient inequalities and moment problems in the class L-P.*

Coefficient inequalities for functions with exclusively real zeros were already investigated by Newton and Euler. The well-known Laguerre-Turán inequalities for functions in L-P lead to function inequalities for orthogonal polynomials on the interval of orthogonality as well as to necessary coefficient conditions for Hurwitz-stable functions and functions in L-P. We show how to derive Grommer's characterization of logarithmic derivatives of functions in L-P from mapping properties. Showing that certain mapping properties for logarithmic derivatives $f'/f$ correspond to specific integral representations for $f$ in L-P, we obtain the connection between structured determinants and reality of zeros. We give two examples of our new approach: For f in L-P, we embed the Laguerre–Turán inequalities together with a recent inequality by Dimitrov into a new infinite family of inequalities. For a real Hurwitz-stable polynomial we show how to obtain quantifiably stronger inequalities if more than three coefficients are considered.

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**Behnke, Henning** *Curve Veering for the Parameter-Dependent Clamped Plate.*

The computation of vibrations of a thin rectangular clamped plate results in an eigenvalue problem with a partial differential equation of fourth order.

$$\frac{\partial^4}{\partial x^4}\varphi + P\frac{\partial^4}{\partial x^2 \partial y^2}\varphi + Q\frac{\partial^4}{\partial y^4}\varphi = \lambda\varphi \quad \text{in } \Omega,$$

$$\varphi = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

for $P, Q \in \mathbb{R}$, $P > 0$, $Q > 0$, and $\Omega = (-\frac{a}{2}, \frac{a}{2}) \times (-\frac{b}{2}, \frac{b}{2}) \subseteq \mathbb{R}^2$.

If we change the geometry of the plate for fixed area, this results in a parameter-dependent eigenvalue problem. For certain parameters, the eigenvalue curves seem to cross. We give a numerically rigorous proof of curve veering, which is based on the Lehmann-Goerisch inclusion theorems and the Rayleigh–Ritz procedure.

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**Bessenyei, Mihály** *Separation by certain nonlinear interpolation families.*

By standard separation theorems, if a convex function is above a concave one, then there exists an affine function between them. In more general, characterizations of the existence of an affine separation between two functions, in different settings, are also known. The references below contains some contributions for the topic. Motivated by these results, the talk presents a characterization for the existence of separation by members of such nonlinear interpolation families that are closed under convex combinations. The proof is based on the classical Helly theorem and some geometric properties of Beckenbach families.

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**Boros, Zoltán** *Strong dyadic derivatives.*

Let $I$ denote an open interval in the real line, and let us consider a function $f : I \to \mathbb{R}$. For $x \in I$ and $h \in \mathbb{R}$, we define the lower and upper strong dyadic derivatives of $f$ by

$$\underline{D}_h^\diamond f(x) = \liminf_{\substack{y\to x\\ n\to\infty}} 2^n\left(f\left(y+2^{-n}h\right) - f(y)\right)$$

and

$$\overline{D}_h^\diamond f(x) = \limsup_{\substack{y\to x\\ n\to\infty}} 2^n\left(f\left(y+2^{-n}h\right) - f(y)\right),$$

respectively. We call $f$ strongly dyadically differentiable if

$$\underline{D}_h^\diamond f(x) = \overline{D}_h^\diamond f(x) \in \mathbb{R}$$

holds for every $x \in I$ and $h \in \mathbb{R}$. We say that $f$ has increasing strong dyadic derivatives if

$$-\infty < \overline{D}_h^{\diamond} f(x) \leq \underline{D}_h^{\diamond} f(y) < +\infty$$

holds for every $h > 0$ and $x, y \in I$ such that $x < y$. These properties are characterized by the following decomposition theorems:

**Theorem 1** *The function $f$ is strongly dyadically differentiable if, and only if, there exist a continuously differentiable function $g : I \to \mathbb{R}$ and an additive mapping $\varphi : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) + \varphi(x)$ for every $x \in I$.*

**Theorem 2** *The function $f$ has increasing strong dyadic derivatives if, and only if, there exist a convex function $g : I \to \mathbb{R}$ and an additive mapping $\varphi : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) + \varphi(x)$ for every $x \in I$.*

Applying these results, we characterize affine (respectively, Wright-convex) functions as locally approximately affine (respectively, locally approximately Wright-convex) functions in a specific sense. In particular, we obtain a localization principle for these classes of functions.

**Burai, Pál** *Regularity and convexity results on Orlicz- and Breckner-convex functions*. (Joint work with Attila Házy.)

It is an accustomed thing in the research of convex functions to derive a better property from a weaker one with the aid of a convexity type inequality. Probably the best known result of this kind is the theorem of Bernstein and Doetsch. They proved that the locally boundedness from above of a Jensen-convex function implies its continuity and convexity as well. The main goal of this talk to present some similar results on Orlicz- and Breckner-convex functions.

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**Chudziak, Jacek** *Quotient stability of some composite functional equations.*
Recently, J. Brzdęk [1, 2], dealing with the quotient stability of the Gołąb-Schinzel type functional equations, has stated several open questions. In our talk we present the answers to some of them.

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**Cristescu, Gabriela** *Jordan type representation of functions with generalized high order bounded variation.*
The aim of this work is to identify few classes of functions with generalized type of bounded variation for which a decomposition theorem of Jordan type holds. We refer especially to functions with $n$th order bounded variation with respect to a Tchebycheff system. The particular case of trigonometric Tchebycheff systems bring interesting results.

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**Čuljak, Vera** *Schur-convexity of the means.* (Joint work with J. Pečarić.)
The property of Schur-covexity (Schur-concavity) of means is considered and compared with recent results in the literature [1–4]. A new proof for convexity (concavity) and Schur-covexity (Schur-concavity) of the integral arithmetic mean is presented. We established the sufficient conditions such that the generalized quasi-

arithmetic mean

$$M_f(k;x,y) = f^{-1}\left(\frac{1}{y-x}\int_x^y f\big(k(t)\big)\,dt\right), \quad (x,y) \in I^2$$

and the generalized weighted integral quasi-arithmetic mean

$$M_f(p,k;x,y) = f^{-1}\left(\frac{1}{\int_x^y p(t)\,dt}\int_x^y p(t) f\big(k(t)\big)\,dt\right)$$

are Schur-convex (or concave) with respect to $(x,y)$. The applications for the extended mean values $E(r,s;x,y)$ and weighted power integral mean $M^{[r]}(p,k;x,y)$ are pointed out.

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**Dascăl, Judita** *The equality problem of conjugate means*. (Joint work with Zoltán Daróczy.)

Let $I \subset \mathbb{R}$ be a nonvoid open interval and let $n \geq 3$ be a fixed natural number. The question is which conjugate means of n variables generated by the weighted arithmetic mean $M : I^n \to I$ are weighted quasi-arithmetic means of $n$ variables at the same time? This question is a functional equation problem:

Characterize the real valued continuous and strictly monotone functions $\varphi$, $\psi$ defined on $I$ and the parameters $p_1, \ldots, p_n, q_1, \ldots, q_n$ for which the equation

$$\varphi^{-1}\left(\sum_{i=1}^n p_i\varphi(x_i) + \left(1 - \sum_{i=1}^n p_i\right)\varphi\big(M(x_1,\ldots,x_n)\big)\right) = \psi^{-1}\left(\sum_{i=1}^n q_i\psi(x_i)\right)$$

holds for all $x_1, \ldots, x_n \in I$, where

$$q_i > 0 \quad (i = 1,\ldots,n), \quad \sum_{i=1}^n q_i = 1,$$

$$p_j \geq 0 \quad \text{and} \quad \sum_{i=1}^n p_i - p_j \leq 1 \quad (j = 1,\ldots,n).$$

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**Fechner, Włodzimierz** *Functional inequalities and equivalences of some estimates.*

Let $A$, $G$ and $L$ denote the arithmetic, geometric ans logarithmic mean, respectively. It is well known that $G \le L \le A$ (see F. Burk [3]) and also $G^{\frac{2}{3}} \cdot A^{\frac{1}{3}} \le L \le \frac{2}{3}G + \frac{1}{3}A$ (see B.C. Carleson [4], F. Burk [3], E.B. Leach and M.C. Sholander [7]). This estimates motivate the study of several functional inequalities (e.g. C. Alsina and J.L. Garcia-Roig [1], C. Alsina and R. Ger [2] and our papers [5, 6]). An inspection of this results shows the possibility of introducing the following notion of equivalence of inequalities for real numbers. Namely, one may say that a given inequality (A) is sharper than another given inequality (B) or equivalent to it if one can prove the inclusion or the equality between the sets of all solutions of respective functional inequalities (FA) and (FB) formed from (A) and (B). In the talk we discuss this concept in details and we study some related estimates and the corresponding functional inequalities.

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**Gilányi, Attila** *Regularity of weakly subquadratic functions*. (Joint work with Katarzyna Troczka-Pawelec.)

In the recent years, subquadratic and weakly subquadratic functions have been considered by several authors (cf., e.g., [1, 2] and [3]).

In the present talk, related to and motivated by some regularity results on convex and subadditive functions, we investigate regularity properties of weakly subquadratic functions, that is, solutions of the inequality

$$f(x+y)+f(x-y)\le 2f(x)+2f(y) \quad (x,y\in G),$$

in the case when $f$ is a real valued function defined on a group $G=(G,+)$.

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**Glavosits, Tamás** *Examples and counter-examples concerning the uniqueness of Hahn-Banach extension.* (Joint work with Csaba Gábor Kézi.)

According to the classical Hahn-Banach dominated extension theorem if $X$ is a real linear space, $V$ is a subspace of $X$, $p : X \to \mathbb{R}$ is a positively homogeneous sub-additive function and $\phi : V \to \mathbb{R}$ is a linear function which is dominated by $p$, then there exists at least one function $\psi : X \to \mathbb{R}$ such that it is also linear, it is also dominated by $p$ and extends $\phi$. Such a function $\psi$ is called Hahn-Banach extension (of $\phi$ from $V$ to $X$). In all cases investigated by us the linear space will be always the $n$-dimensional real linear space $\mathbb{R}^n$, the linear subspace will be always

$$V=\Big\{(x,x,\dots,x)|x\in\mathbb{R}\Big\}$$

and the linear function $\phi : V \to \mathbb{R}$ will be always defined in the following way

$$\phi(x,x\dots,x)=x \quad (x\in\mathbb{R}).$$

We will give all the Hahn-Banach extensions of $\phi$ (from $V$ to $X$) with respect several remarkable dominating norms. For this we will frequently use the technique of infimal convolution which will be also presented.

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**Goldberg, Moshe** *Homotonic Algebras.*

An algebra $\mathcal{A}$ of real or complex valued functions defined on a set $\mathbf{S}$ shall be called *homotonic* if $\mathcal{A}$ is closed under forming of absolute values, and for all $f$ and $g$ in $\mathcal{A}$, the product $f \times g$ satisfies $|f \times g| \leq |f| \times |g|$. Our purpose in this talk is to provide a simple inequality which characterizes sub-multiplicativity for weighted sup norms on homotonic algebras.

**Gselmann, Eszter** *Approximate $n$-Jordan homomorphisms.*

The study of additive mappings from a ring into another ring which preserve squares was initiated by G. Ancochea in [1] in connection with problems arising in projective geometry. Later, these results were strengthened by (among others) Kaplansky [4] and Jacobson–Rickart [3].

Let $n \geq 2$ be a fixed integer, a function $\varphi$ from a ring $R$ into another ring $R'$ is called an *$n$-Jordan homomorphism*, if

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi\left(a^n\right) = \varphi(a)^n \quad (a, b \in R)$$

hold. 2-Jordan homomorphisms will be simply called Jordan homomorphisms. Furthermore, an additive function $\varphi : R \to R'$ is an *$n$-homomorphism*, if

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n) \quad (a_1, \ldots a_n \in R).$$

In Herstein [2] the following statement was proved: *Let $\varphi$ be an $n$-Jordan homomorphism from a ring $R$ onto a prime ring $R'$ of characteristic larger than $n$. Suppose further that $R$ has a unit element. Then $\varphi = \varepsilon\tau$ where $\tau$ is either a homomorphism or an anti–homomorphism and $\varepsilon$ is an $(n-1)$st root of unity lying in the center of $R'$.*

The aim of this talk it to characterize $n$-Jordan homomorphisms and to investigate their connection with $n$-homomorphisms as well as homomorphisms. Furthermore, the following implication is also verified: if a function $\varphi$ is additive and (for a fixed integer $n \geq 2$) the mapping

$$x \longmapsto \varphi\left(x^n\right) - \varphi(x)^n$$

satisfies some mild regularity assumption (e.g., boundedness, continuity at a point, measurability), then the function $\varphi$ is an $n$-homomorphism or it is a continuous additive function.

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**Házy, Attila** *On $h$-convex and approximately $h$-convex functions.* (Joint work with Pál Burai.)

In our talk we introduce a class of $h$-convex functions (which is a common generalization of the convexity, the Breckner $s$-convexity, the Godunova-Levin functions and the $P$-functions) and a class of approximately $h$-convex functions.

Bernstein and Doetsch in 1915 proved that the local upper boundedness of a Jensen-convex function yields its local boundedness and continuity as well on the whole domain, which implies the convexity of the function. We show some Bernstein-Doetsch type results for $h$-convex and approximately $h$-convex functions.

**Horváth, László** *A refinement of the classical Jensen's inequality.*

In this talk we consider some integral inequalities in probability spaces, which go back to some discrete variants of the Jensen's inequality. Especially, we refine the classical Jensen's inequality. Convergence results corresponding to the inequalities are also studied.

**Jabłoński, Wojciech** *Stability of $\varphi$-Homogeneity and Completeness of a target space.*

Many authors examined stability of different functional equations since the time when S.M. Ulam [3] had posed his famous problem concerning the stability of an equation of homomorphism. Among others the stability of a $\varphi$-homogeneity equation

$$f(\alpha x) = \varphi(\alpha) f(x)$$

under different assumptions and in various spaces was considered. In all cases, where the homogeneity equation is stable, but not superstable, it has been assumed, that the target space of the considered function $f$ is complete.

In 1989 G.L. Forti and J. Schwaiger shown (see [1]) that a normed space $Y$ has to be a Banach one provided for some abelian group $A$ containing an element of infinite order and for all functions $f : A \to Y$ such that the Cauchy difference $f(x+y) - f(x) - f(y)$ is bounded, there is some additive function $h$ such that $f - h$ is bounded. The similar result is also true for stability of a homogeneity (see [2]).

Following the idea of the proof of Theorem from [2] we show that the statement remains true with weakened and simplified assumptions.

Assume that $(A, \cdot)$ is a group admitting a homomorphism $\phi : A \to \mathbb{K}^* =: \mathbb{K} \setminus \{0\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, which is not a character. Let moreover $\cdot : A \times X \to X$ be an action of $A$ on some set $X$ such that the stabilizer $A_{x_0} := \{\alpha \in A : \alpha x_0 = x_0\}$ is trivial for some $x_0 \in X$. Finally, let $Y$ be a normed space.

**Theorem** *Let $\varphi : A \to \mathbb{K}^\star$ be a homomorphisms which is not a character on $A$. Assume that for all functions $f : X \to Y$ such that for some positive $\varepsilon, \delta$*

$$\left\| f(\alpha x) - \varphi(\alpha) f(x) \right\| \leq \varepsilon \left| \varphi(\alpha) \right| + \delta \quad \text{for } \alpha \in A, x \in X,$$

*holds, there exists a $\varphi$-homogeneous function $h : X \to Y$ such that $f - h$ is bounded. Then $Y$ is a Banach space, i.e., a complete normed space.*

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**Kézi, Csaba Gábor** *Functional equations and group substitutions*. (Joint work with Mihály Bessenyei.)

Motivated by some investigation of Babbage and a method of solving certain functional equations arising in competition problems, the functional equation

$$F(f \circ g_1, \ldots, f \circ g_n, \mathrm{id}) = 0, \tag{1}$$

is studied. Here id denotes the identity of $\mathbb{R}$, the functions $F$, $g_1, \ldots, g_n$ are given (with appropriate range and domain) such that $g_1, \ldots, g_n$ form a group and $f$ is to determined. Under some further analytical assumptions on the involved known functions, a local existence and uniqueness theorem is proved for (1). In the proof, the Implicit Function Theorem and the Global Existence and Uniqueness Theorem are applied.

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**Klaričić Bakula, Milica** *Converse Jensen-Steffensen inequality*. (Joint work with S. Ivelić and J. Pečarić.)

Let $I$ be any open interval in $\mathbb{R}$ and $[a, b] \subset I$, $a < b$. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a monotonic $n$-tuple in $[a, b]^n$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ a real $n$-tuple satisfying

$$w_i \neq 0, \quad i = 1, \ldots, n,$$

$$0 \leq W_j \leq W_n = 1, \quad j = 1, \ldots, n,$$

where $W_j = \sum_{i=1}^{j} w_i$. We prove that for any convex function $f : I \to \mathbb{R}$ the inequalities

$$\begin{aligned}\sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right) &\\ \le f(a) + f(b) - f\left(\sum_{i=1}^{n} w_i x_i\right) - f\left(a + b - \sum_{i=1}^{n} w_i x_i\right) &\\ \le f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) & \qquad (1)\end{aligned}$$

hold. Furthermore, we use *exp-convex method* to obtain inequalities of the form

$$\big(\Omega(s)\big)^{t-r} \le \big(\Omega(r)\big)^{t-s} \big(\Omega(t)\big)^{s-r},$$

where $r < s < t$ and $\Omega$ is some specially chosen exp-convex function related to (1). The obtained inequalities are used to prove monotonicity of some Cauchy type means.

**Kobayashi, Kenta** *On the interpolation constant over triangular elements.*
Let $T$ be the triangle in $\mathbb{R}^2$ and $V^{1,1}(T)$, $V^{1,2}(T)$, $V^2(T)$ be the function spaces defined by

$$\begin{aligned} V^{1,1}(T) &= \left\{\varphi \in H^1(T) \;\middle|\; \int_T \varphi \, dx\, dy = 0\right\}, \\ V^{1,2}(T) &= \left\{\varphi \in H^1(T) \;\middle|\; \int_{\gamma_k} \varphi \, ds = 0,\ k = 1,2,3\right\}, \\ V^2(T) &= \left\{\varphi \in H^2(T) \;\middle|\; \varphi(p_k) = 0,\ k = 1,2,3\right\}, \end{aligned}$$

where $p_1$, $p_2$, $p_3$ and $\gamma_1$, $\gamma_2$, $\gamma_3$ are the vertices and sides of $T$ respectively. Then, it is known that the following constants $C_1(T)$, $C_2(T)$, $C_3(T)$, $C_4(T)$ exist:

$$C_1(T) = \sup_{\varphi \in V^{1,1}(T)\setminus 0} \frac{\|\varphi\|_{L^2(T)}}{\|\nabla\varphi\|_{L^2(T)}}, \qquad C_2(T) = \sup_{\varphi \in V^{1,2}(T)\setminus 0} \frac{\|\varphi\|_{L^2(T)}}{\|\nabla\varphi\|_{L^2(T)}},$$

$$C_3(T) = \sup_{\varphi \in V^{2}(T)\setminus 0} \frac{\|\varphi\|_{L^2(T)}}{|\varphi|_{H^2(T)}}, \qquad C_4(T) = \sup_{\varphi \in V^{2}(T)\setminus 0} \frac{\|\nabla\varphi\|_{L^2(T)}}{|\varphi|_{H^2(T)}},$$

where $|\varphi|_{H^2(T)}$ is a $H^2$ semi-norm of $\varphi$ defined by

$$|\varphi|_{H^2(T)}^2 = \|\varphi_{xx}\|_{L^2(T)}^2 + 2\|\varphi_{xy}\|_{L^2(T)}^2 + \|\varphi_{yy}\|_{L^2(T)}^2.$$

For these constants, we have obtained the formulas which give sharp upper bound of $C_j(T)$ as

$$C_j(T) < K_j(T), \quad j = 1,2,3,4.$$

Concrete form of the formulas $K_j(T)$ will be shown in the talk.

**Fig. 1** $\widetilde{C}_4(T)$

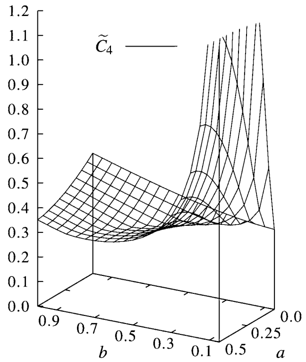
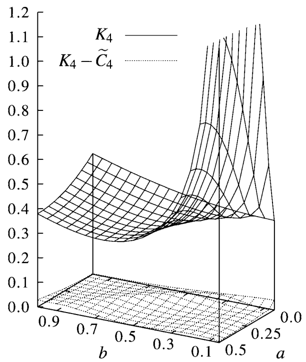



**Fig. 2** $K_4$ and $K_4(T) - \widetilde{C}_4(T)$




Since we don't have enough space here, we will only show the graph of $\widetilde{C}_4(T)$, $K_4$ and $K_4(T) - \widetilde{C}_4(T)$ in Figs. 1 and 2, where $\widetilde{C}_4(T)$ is an approximate solution of $C_4(T)$ and $T$ is a triangle whose vertices are $(0,0)$, $(1,0)$ and $(a,b)$.

**Kolumbán, József** *Parametric equilibrium problems*. (Joint work with Marcel Bogdan.)

Let $(X,\sigma)$ be a Hausdorff topological space. For $n \in \mathbf{N}$ we consider the following equilibrium problem:

Find an element $a_n \in X$ such that

$$f_n(a_n, b) \geq \Phi_n(a_n, b) - \Phi_n(a_n, a_n), \quad \forall b \in X, \tag{1}$$

where $f_n : X \times X \to \mathbf{R}$ and $\Phi_n : X \times X \to \mathbf{R} \cup \{+\infty\}$ are given functions.

Along with the problem (1) we consider the so-called "limit" problem:

Find an element $a \in X$ such that

$$f(a, b) \geq \Phi(a, b) - \Phi(a, a), \quad \forall b \in X. \tag{2}$$

Supposing that $a_n$ is a solution of problem (1) and $a_n \stackrel{\sigma}{\to} a$, we give sufficient conditions for $a$ to be a solution of problem (2).

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**Losonczi, László** *Minkowski-type inequalities for means generated by two functions and a measure*. (Joint work with Zsolt Páles.)

Given two continuous functions $f, g : I \to \mathbb{R}$ such that $g$ is positive and $f/g$ is strictly monotone, and a probability measure $\mu$ on the Borel subsets of [0, 1], the two variable mean $M_{f,g;\mu} : I^2 \to I$ is defined by

$$M_{f,g;\mu}(x, y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f(tx + (1-t)y)\, d\mu(t)}{\int_0^1 g(tx + (1-t)y)\, d\mu(t)}\right) \quad (x, y \in I).$$

We study Minkowski-type inequalities for these means, i.e., try to find conditions for the generating functions $f_0, g_0 : I_0 \to \mathbb{R}$, $f_1, g_1 : I_1 \to \mathbb{R}, \ldots, f_n, g_n : I_n \to \mathbb{R}$, and for the measure $\mu$ such that

$$M_{f_0,g_0;\mu}(x_1 + \cdots + x_n, y_1 + \cdots + y_n) \underset{[\geq]}{\leq} M_{f_1,g_1;\mu}(x_1, y_1) + \cdots + M_{f_n,g_n;\mu}(x_n, y_n)$$

holds for all $x_1, y_1 \in I_1, \ldots, x_n, y_n \in I_n$ with $x_1 + \cdots + x_n, y_1 + \cdots + y_n \in I_0$. The particular case when the generating functions are power functions, i.e., when the means are generalized Gini means is also investigated.

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**Makó, Judit** *Approximate convexity of Takagi type functions*. (Joint work with Zsolt Páles.)

Given a nonnegative function $\phi : [0, \frac{1}{2}] \to \mathbb{R}_+$, we define the Takagi type function $S_\phi : \mathbb{R} \to \mathbb{R}$ by

$$S_\phi(x) := \sum_{n=0}^{\infty} 2\phi\left(\frac{1}{2^{n+1}}\right) d_Z\big(2^n x\big),$$

where $d_\mathbb{Z}(x) := \inf\{|x - k| : k \in \mathbb{Z}\}$. Our main result states that if $\phi(0) = 0$ and the mapping $x \mapsto \phi(x)/x$ is concave, then the Takagi type function $S_\phi$ is approximately

Jensen convex in the following sense

$$S_\phi\left(\frac{x+y}{2}\right) \le \frac{S_\phi(x)+S_\phi(y)}{2} + \phi \circ d_{\mathbb{Z}}\left(\frac{x-y}{2}\right) \quad (x, y \in \mathbb{R}).$$

Applications to the theory of approximately convex functions are also given.

**Maksa, Gyula** *The equality case in some recent convexity inequalities*. (Joint work with Zsolt Páles.)

In a recent paper [7] by Varošanec, a common generalization of convex and $s$-convex functions, Godunova–Levin functions, and $\mathcal{P}$-functions is introduced in the following way: Let $I$ be a nonvoid subinterval of $\mathbb{R}$ (the set of all real numbers), $h : [0, 1] \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be real-valued functions satisfying the inequality

$$f\big(tx + (1-t)y\big) \le h(t)f(x) + h(1-t)f(y) \tag{1}$$

for all $x, y \in I$ and $t \in ]0, 1[$. An even more general notion, the so-called $h$-convexity ($(H, h)$-convexity), can be found in Házy [5]: Let $X$ be a real or complex normed space, $D \subset X$ be a nonempty convex set, $\emptyset \ne H \subset [0, 1]$, and $h : H \to \mathbb{R}$ be a function. A function $f : D \to \mathbb{R}$ is $(H, h)$-convex if (1) holds for all $x, y \in D$ and $t \in H$. It is clear that this generalizes the concepts of convexity ($h(t) = t$, $t \in [0, 1]$, [7, 2]), the Breckner–convexity ($h(t) = t^s$, $t \in ]0, 1[$, for some $s \in \mathbb{R}$, [3, 4]), the Godunova–Levin functions ($h(t) = t^{-1}$, $t \in ]0, 1[$, [1]), and $\mathcal{P}$-functions ($h(t) = 1$, $t \in [0, 1]$, [6]).

In this talk, we present some results on the functional equation

$$f\big(\alpha(t)x + \beta(t)y\big) = g(t)f(x) + h(t)f(y) \quad (x, y \in D,\ t \in H)$$

related to (1), where $D$ is a nonempty open subset of a real or complex topological vector space, $H \subset \mathbb{R}$ is a nonempty set, $\alpha, \beta, g, h : H \to \mathbb{R}$ are given functions, and $f : D \to \mathbb{R}$ is the unknown function. Furthermore, we suppose on $D$ that $\alpha(t)x + \beta(t)y \in D$ whenever $x, y \in D$ and $t \in H$.

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**Matković, Anita** *Some Variants of the Jensen-Mercer Inequality and their Applications*. (Joint work with J. Pečarić.)

In 2003. A.McD. Mercer proved the following variant of Jensen's inequality

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \le f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i),$$

for a convex function $f : [a,b] \to \mathbb{R}$, real numbers $x_1, \ldots, x_n \in [a,b]$ and positive real numbers $w_1, \ldots, w_n$, where $W_n = \sum_{i=1}^{n} w_i$.

We call it Jensen-Mercer inequality and we present some of its generalizations in various spaces with adequate orders, and for several types real valued functions. As their applications we establish the monotonicity property of the weighted means of Mercer's type and an upper bound for the normalized Jensen-Mercer functional.

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**Mészáros, Fruzsina** *Density function solutions of functional equations by the help of interval expansions*. (Joint work with Károly Lajkó.)

To prove that the so-called density function solutions of some functional equations are positive almost everywhere on their domain, we use interval expansions and the following generalization of Steinhaus' theorem (see [1]):

Let $U$ be an open subset of $\mathbb{R}^2$ and $F : U \to \mathbb{R}$ be a continuously differentiable function with nonvanishing partial derivatives, moreover let $A, B \subset \mathbb{R}$ $(A \times B \subset U)$ be measurable sets with positive Lebesgue measure, then the set $F(A, B)$ has an interior point, i.e. $F(A, B)$ contains a nonvoid open interval.

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**Minuţă, Flavia-Corina** *An Extension of Young's Inequality*.

Young's inequality asserts that every strictly increasing continuous function $f$ : $[0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$ verifies an inequality of the following form,

$$ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy,$$

whenever $a$ and $b$ are nonnegative real numbers. The equality occurs if and only if $f(a) = b$.

In our extension $f$ will denote a nondecreasing function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$ and $K(x, y)$ is a Lebesgue locally integrable function. We will attach to $f$ a *pseudo-inverse* $f_{\sup}^{-1}$. Then for every pair of nonnegative numbers $a < b$, and every number $c \geq f(a)$ we have

$$\int_a^b \int_{f(a)}^c K(x, y)\,dy\,dx \leq \int_a^b \left( \int_{f(a)}^{f(x)} K(x, y)\,dy \right) dx + \int_{f(a)}^c \left( \int_a^{f_{\sup}^{-1}(y)} K(x, y)\,dx \right) dy.$$

If in addition $K$ is strictly positive almost everywhere, then the equality occurs if and only if $c \in [f(b-), f(b+)]$.

The following corollary incorporates the Legendre duality:

Let $f : [0, \infty) \to [0, \infty)$ be a continuous nondecreasing function and $\Phi : [0, \infty) \to \mathbb{R}$ a convex function whose conjugate is also defined on $[0, \infty)$. Then for all $b > a \geq 0$, $c \geq f(a)$, and $\varepsilon > 0$ we have

$$\int_a^b \Phi\left( \varepsilon \int_{f(a)}^{f(x)} K(x, y)\,dy \right) dx + \int_{f(a)}^c \Phi^*\left( \frac{1}{\varepsilon} \int_a^{f_{\sup}^{-1}(y)} K(x, y)\,dx \right) dx \geq \int_a^b \int_{f(a)}^c K(x, y)\,dy\,dx - \big(c - f(a)\big)\Phi(\varepsilon) - (b - a)\Phi^*(1/\varepsilon).$$

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**Mitrea, Alexandru Ioan** *Two-sided inequalities for the norm of some linear operators in Approximation Theory.* (Joint work with Delia Mitrea.)

The estimate of the norm of some operators which appear in various approximation procedures is essential in order to establish their convergence. Our goal is to give estimates for classes of operators involved in the following approximation procedures:

(i) Chebyshev best approximation on finite point sets,
(ii) product-quadrature formulas.

Based on these estimates, some theorems concerning the convergence, the error estimate or the topological structure of the unbounded divergence set for the corresponding approximation formulas will be derived.

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**Mitrea, Paulina** *On the approximation error of EGO-Algorithms for obtaining energy-minimizing surfaces*. (Joint work with Octavian M. Gurzău and Alexandru I. Mitrea.)

A 3D deformable model (i.e. a deformable surface) of variational type is defined, usually, as a triple $(\mathcal{A}, I, E)$, where $\mathcal{A} \subseteq C^2(D, \mathbb{R}^3)$, $D = [0, 1] \times [0, 1]$, is the set of admissible deformations, $I \in C^2(\mathbb{R}^3)$ is the image-intensity function and $E : \mathcal{A} \to \mathbb{R}$ is the energy-functional, given by a double integral on $D$ involving the partial derivatives of first and second order of $v \in \mathcal{A}$ and the gradient of $I$.

The basic problem of a deformable model is to minimize its energy-functional, i.e. to obtain an energy-minimizing surface, by solving the associated Euler-Gauss-Ostrogradski (EGO) Equation of Calculus of Variations.

The main goals of this paper are to derive the corresponding EGO-Algorithms, based on discretization methods, to obtain estimates for the approximation-error of these algorithms and, consequently, to establish their convergence, under given restrictions.

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**Molnár, Lajos** *Order automorphisms on positive definite operators and a few applications*.

We determine the order automorphisms of the set of all positive definite operators with respect to the usual order and to the so-called chaotic order. We then apply those results to the following problems: 1) description of all bijective transformations on the space of nonsingular density operators (quantum states) which preserve the Umegaki or the Belavkin-Staszewski relative entropy; 2) characterization of the logarithmic product as the essentially unique binary operation on the set of positive definite operators that makes it an ordered commutative group with respect to the chaotic order.

**Nagatou, Kaori** *Eigenvalue excluding for perturbed-periodic 1D Schrödinger operators*. (Joint work with Michael Plum and Mitsuhiro T. Nakao.)

Subject of investigation in this talk is a 1D-Schrödinger equation, where the potential is a sum of a periodic function and a perturbation decaying at $\pm\infty$. It is well known that the essential spectrum consists of spectral bands, and that there may or may not be additional eigenvalues below the lowest band or in the gaps between the bands. While *enclosures* for gap eigenvalues can comparably easily be obtained from numerical approximation, e.g. by D. Weinstein's bounds, there seems to be no method available so far which is able to *exclude* eigenvalues in spectral gaps, i.e. which identifies sub-regions (of a gap) which contain no eigenvalues. Here, we propose such a method. It makes heavy use of computer assistance; nevertheless, the results are completely rigorous in the strict mathematical sense, since all computational errors are taken into account.

**Nagy, Károly** *On the Walsh-Marcinkiewicz means and kernels.*

For the two-dimensional Walsh-Fourier series Weisz [1] proved that the maximal operator

$$\mathcal{M}^{\star} f := \sup_{n\in\mathbb{P}} \frac{1}{n} \left| \sum_{j=0}^{n-1} S_{j,j}(f) \right|$$

is bounded from the two-dimensional dyadic martingale Hardy space $H_p$ to the space $L_p$ for $p > 2/3$ and is of weak type (1,1). Goginava [2] proved that the assumption $p > 2/3$ is essential for the boundedness of the maximal operator $\mathcal{M}^{\star}$ from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$. Namely, in the endpoint case $p = 2/3$ he gave a counterexample for which the boundedness does not hold. In the endpoint case $p = 2/3$, Goginava [3] proved that the maximal operator $\mathcal{M}^{\star}$ of the Walsh-Marcinkiewicz means of double Fourier series is bounded from the Hardy space $H_{2/3}$ to the space weak-$L_{2/3}$.

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**Ohwada, Tomoyoshi** *On a continuous mapping and sharp triangle inequalities.*

The triangle inequality is one of the most fundamental inequalities in analysis and have been treated by several authors. Kato, Saito and Tamura [1] showed sharp triangle inequality and its reverse inequality with $n$ elements in a Banach space. After this, Mitani, Saito, Kato and Tamura [2] extended it.

In this talk, we shall present a non-negative valued continuous mapping on a Normed space and, as an intermediate values of it, we will show the two kinds of sharp triangle inequalities in [1] and [2].

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**Páles, Zsolt** *Connection between the Ingham-Jessen and the Kedlaya inequalities.*

The classical Kedlaya inequality (which was conjectured by F. Holland) establishes the relation

$$A\big(G(x_1), G(x_1, x_2), \ldots, G(x_1, \ldots, x_n)\big) \le G\big(A(x_1), A(x_1, x_2), \ldots, A(x_1, \ldots, x_n)\big),$$

where $x_1, \ldots, x_n$ are positive numbers and $A$ and $G$ denote the arithmetic and geometric means, respectively.

Using a combinatorial argument, this inequality can be deduced from the classical Ingham-Jessen inequality. This argument will be proved to obtain a similar implication in terms of quasi-arithmetic and more general means.

**Plum, Michael** *A uniqueness result for a semilinear elliptic problem: A computer-assisted proof.* (Joint work with P.J. McKenna, F. Pacella, and D. Roth.)

Starting with the famous article [A. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209–243], many papers have been devoted to the uniqueness question for positive solutions of $-\Delta u = \lambda u + u^p$ in $\Omega$, $u = 0$ on $\partial\Omega$, where $p > 1$ and $\lambda$ ranges between 0 and the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of $-\Delta$. For the case when $\Omega$ is a ball, uniqueness could be proved, mainly by ODE techniques. But very little is known when $\Omega$ is not a ball, and then only for $\lambda = 0$. In this talk, we prove uniqueness, for all $\lambda \in [0, \lambda_1(\Omega))$, in the case $\Omega = (0, 1)^2$ and $p = 2$. This constitutes the first positive answer to the uniqueness question in a domain different from a ball. Our proof makes heavy use of computer assistance: we compute a branch of approximate solutions and prove existence of a true solution branch close to it, using fixed point techniques. By eigenvalue enclosure methods, and an additional analytical argument for $\lambda$ close to $\lambda_1(\Omega)$, we deduce the non-degeneracy of all solutions along this branch, whence uniqueness follows from the known bifurcation structure of the problem.

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**Popa, Dorian** *Hyers-Ulam stability of the first order linear partial differential equation.* (Joint work with Nicolaie Lungu.)

We prove that the linear partial differential equation of order one $a_1(x_1)\frac{\partial u}{\partial x_1} + \cdots + a_n(x_n)\frac{\partial u}{\partial x_n} = a(x_1)u + f(x_1, x_2, \ldots, x_n)$ is stable in Hyers-Ulam sense under suitable conditions.

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**Reichel, Wolfgang** *Symmetry of solutions for quasimonotone second-order elliptic systems in ordered Banach spaces.* (Joint work with Gerd Herzog.)

We consider symmetry properties of solutions to nonlinear elliptic boundary value problems defined on bounded symmetric domains of $\mathbb{R}^n$. The solutions take values in ordered Banach spaces $E$, e.g. $E = \mathbb{R}^N$ ordered by a suitable cone. The nonlinearity is supposed to be quasimonotone increasing. By considering cones which are different from the standard cone of componentwise nonnegative elements we can prove symmetry of solutions to nonlinear elliptic systems which are not covered by previous results. We use methods based on maximum principles (the method of moving planes) suitably adapted to cover the case of solutions of nonlinear elliptic problems with values in ordered Banach spaces.

**Saito, Kichi-Suke** *The sharp triangle inequalities and its applications to the geometry of Banach spaces.*

Let $X$ be a Banach space (or a normed space). The triangle inequality

$$\|x + y\| \le \|x\| + \|y\| \quad (x, y \in X)$$

plays a fundamental role in studying various analytic and geometric properties of such space. Several authors have been treating its generalizations and reverse inequalities.

In 2005, Kato-Saito-Tamura in [1] announced the sharp triangle inequality and its reverse inequality with $n$ elements for a Banach space, as follows:

$$\begin{aligned}
&\left\| \sum_{j=1}^{n} x_j \right\| + \left( n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \le j \le n} \|x_j\| \\
&\quad \le \sum_{j=1}^{n} \|x_j\| \\
&\quad \le \left\| \sum_{j=1}^{n} x_j \right\| + \left( n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1 \le j \le n} \|x_j\|
\end{aligned}$$

for nonzero elements $x_1, \ldots, x_n$ in $X$. These inequalities are useful to study the geometrical structure of Banach spaces, such as uniformly non-$\ell_1^n$-ness [2]. After that, we have the series paper about the sharp triangle inequality. Moreover, Mitani-Saito-Kato-Tamura [3] and Mitani-Saito [4] presented the refinement of sharp triangle inequality and its reverse inequality.

In this talk, we present the recent results about sharp triangle inequalities of a Banach space given in [2–4]. In particular, we give the equality condition of these inequalities. As an application, we give a characterization of geometrical properties of Banach spaces, such as strict convexity, uniform convexity and uniform non-squareness.

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**Saitoh, Saburou** *Fundamental Error Estimate Inequalities for the Tikhonov regularization using reproducing kernels.*

First of all, as two news, I would like to introduce simply the two general inequalities: For a real-valued absolutely continuous function on [0, 1] satisfying $f(0) = 0$ and $\int_0^1 f'(x)^2 dx < 1$, we have, by using the theory of reproducing kernels

$$\int_0^1 \left( \frac{f(x)}{1 - f(x)} \right)^{\prime 2} (1 - x)^2 dx \leq \frac{\int_0^1 f'^2(x)\, dx}{1 - \int_0^1 f'^2(x)\, dx},$$

[6]. Yamada [9] gave a direct proof for this inequality with a generalization and as its application, he unified the famous Opial's inequality [5] and its generalizations.

Meanwhile, we gave some explicit representations of the solutions of nonlinear simultaneous equations [8] and of the explicit functions [1] in the implicit function theory by using singular integrals, and we [7] derived the estimate inequalities for the regularizations: for example, for the singularity $\frac{1}{(|x-y|)^\alpha}$, for an small $\delta$, $\frac{1}{(|x-y|+\delta)^\alpha}$.

Our main purpose in this talk is to introduce our method [2, 4] constructing approximate and numerical solutions of bounded linear operator equations on Hilbert spaces using the Tikhonov regularizations and reproducing kernels; there, for the error estimates for the solutions, we will need the inequalities for the approximate solutions; as a typical example, we shall present our new numerical and real inversion formulas [3] of the Laplace transform whose problems are famous as typical ill-posed and difficult ones; for this software realizing the formulas in computers, we are requesting international patents. Here, we will be able to see a great computer power of H. Fujiwara with infinite precision algorithms in connection with the error estimates.

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**Tabor, Jacek** *Strongly quasiconvex functions behave like convex ones*. (Joint work with Jó. Tabor and M. Żołdak.)

Let $f : I \to \mathbb{R}$, when $I$ is a subinterval of $\mathbb{R}$. We say that $f$ is *$\varepsilon$-strongly quasiconvex* if

$$f\big(tx+(1-t)y\big) \le \max\big(f(x), f(y)\big) - \varepsilon \min(t, 1-t)|x-y| \quad \text{for } x, y \in I, t \in [0, 1].$$

Dually, $f$ is *$\varepsilon$-strongly quasiconcave* if

$$f\big(tx+(1-t)y\big) \ge \max\big(f(x), f(y)\big) - \varepsilon \max(t, 1-t)|x-y| \quad \text{for } x, y \in I, t \in [0, 1].$$

We show that strongly quasiconvex and strongly quasiconcave functions build an interesting family which behaves in many ways like classical convex and concave functions.

**Tabor, Józef** *How to estimate the module of uniform convexity*. (Joint work with Ja. Tabor.)

Our main results give a convenient tool to compute the module of uniform convexity of a given function. By $\mathbb{R}_+$ we denote the interval $[0, \infty)$. Let $X$ be a normed space and $V$ a convex subset of $X$. For a convex function $f : V \to \mathbb{R}$ by $\omega_f : \mathbb{R}_+ \to [0, \infty]$ we understand the *module of uniform convexity of $f$*, that is

$$\omega_f(r) := \inf\left\{ \frac{tf(x) + (1-t)f(y) - f(tx + (1-t)y)}{t(1-t)} \,\middle|\, t \in (0, 1), \right.$$
$$\left. x, y \in V, \|x - y\| = r \right\}.$$

We say that $f$ is *uniformly convex* if $\omega_f(r) > 0$ for $r > 0$. Our results read as follows.

(1) Let $f : [a, \infty) \to \mathbb{R}$ be a convex function of class $C^1$, such that $f'$ is convex. Then

$$\omega_f(r) = f(a+r) - f(a) - f'(a)r.$$

(2) Let $f : [a, b] \to \mathbb{R}$ be a convex function. Let $x_0 \in (a, b)$ and let $f_1 = f|_{[a,x_0]}$ and $f_2 := f|_{[x_0,b]}$. Then

$$\omega_f(r) \geq \inf_{s+t=r, s,t>0} \left[ \omega_{f_1}(s) + \omega_{f_2}(t) + 2st \cdot \min\left( \frac{\omega_{f_1}(s)}{s^2}, \frac{\omega_{f_2}(t)}{t^2} \right) \right] \quad \text{for } r > 0.$$

**Wagner, Alfred** *On some rescaled shape optimization problems*. (Joint work with G. Buttazzo.)

We consider Cheeger-like shape optimization problems of the form

$$\min\left\{ |\Omega|^\alpha J(\Omega) : \Omega \subset D \right\}$$

where $D$ is a given bounded domain and $\alpha$ is above the natural scaling. We show the existence of a solution and analyze as $J(\Omega)$ the particular cases of the compliance functional $C(\Omega)$ and of the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet Laplacian. We prove that optimal sets are open and we obtain some necessary conditions of optimality.

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**Wąsowicz, Szymon** *Probabilistic characterization of strong convexity*. (Joint work with Teresa Rajba.)

The function $\varphi : \mathcal{I} \to \mathbb{R}$ (where $\mathcal{I} \subset \mathbb{R}$ is an interval) is called *strongly convex with modulus* $c > 0$, if

$$\varphi\big(tx + (1-t)y\big) \leq t\varphi(x) + (1-t)\varphi(y) - ct(1-t)(x-y)^2$$

for any $x, y \in \mathcal{I}$, $t \in [0, 1]$. We state the following

**Theorem** *The function $\varphi : \mathcal{I} \to \mathbb{R}$ is strongly convex with modulus $c$ if and only if*

$$\varphi\big(\mathbb{E}[X]\big) \leq \mathbb{E}\big[\varphi(X)\big] - c\mathbb{D}^2[X]$$

*for any integrable random variable taking values in $\mathcal{I}$.*

Some inequalities of Jensen-type (known from [1]) will be derived. The geometrical interpretations of the above theorem will be also given.

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**Witkowski, Alfred** *Gini and Stolarsky means in geometric problems.*

The note [1] shows how different means appear as lengths of horizontal sections of a trapezoid. Replacing a trapezoid with an $n$-frustum (i.e. a truncated cone with $n$-dimensional balls as its bases) we see that the volumes of some horizontal sections can be expressed by the Stolarsky or Gini means of the volumes of its bases.

For example, if the $n$-volumes of bases of a frustum equal $x$ and $y$, then the $n$-volume of the base of a cylinder with the same height and the same $(n+1)$-volume equals

$$E(1+1/n, 1/n; x, y),$$

where $E$ is the Stolarsky mean.

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**Volkmann, Peter** *The continuous solutions* $f : \mathbb{R} \to \mathbb{R}$ *of the functional equation* $\min\{f(x+y), f(x-y)\} = |f(x) - f(y)|$[1].

They are given by $f(x) = c|x|$ $(x \in \mathbb{R})$, where $c \geq 0$, and by

$$f(x) = c|x| \quad \big(|x| \leq p/2\big), \qquad f(x+p) = f(x) \quad (x \in \mathbb{R}), \tag{1}$$

where $c \geq 0$, $p > 0$.

Background information:

(i) Jointly with Alice Simon (Aeq. Math. 47, 60–68 (1994)) the functions $f(x) = |a(x)|$ $(x \in G)$ had been studied, $a : G \to \mathbb{R}$ being additive on an abelian group $G$. They solve

$$\max\big\{f(x+y), f(x-y)\big\} = f(x) + f(y) \quad (x, y \in G) \tag{2}$$

as well as

$$\min\big\{f(x+y), f(x-y)\big\} = \big|f(x) - f(y)\big| \quad (x, y \in G). \tag{3}$$

Equation (1) characterizes these functions $f : G \to \mathbb{R}$, but for (2) in case $G = \mathbb{R}$ also (1) (with $c = p = 1$) had been found as solution.

(ii) Jointly with Raymond M. Redheffer the equation

$$f(x) + g(y) = \max\big\{h(x+y), h(x-y)\big\} \quad (x, y \in G)$$

for functions $f, g, h : G \to \mathbb{R}$ had been solved (General Inequalities 7, 311–318 (1997)).

[1]This talk was presented during the Conference on Inequalities and Applications '07, Noszvaj, Hungary, September 9–15, 2007. However, due to technical reasons, it did not appear in the book Inequalities and Applications (Eds. C. Bandle, A. Gilányi, L. Losonczi, Zs. Páles, M. Plum), Birkhäuser Verlag, Basel, Boston, Berlin, 2009, therefore, we publish it here.

(iii) Jointly with Karol Baron the complex case had been considered (Sem. LV, No. 28, 10pp. (2006); http://www.mathematik.uni-karlsruhe.de/~semlv):

The functions $f(x) = |\varphi(x)|$ $(x \in V)$ with $\varphi : V \to \mathbb{C}$ being linear on a complex vector space $V$ are characterized by each of the equations $\sup_{t\in\mathbb{R}} f(x + e^{it}y) = f(x) + f(y)$ $(x, y \in V)$ and $\inf_{t\in\mathbb{R}} f(x + e^{it}y) = |f(x) - f(y)|$ $(x, y \in V)$ (where $f : V \to \mathbb{R}$).

# 2 Problems and Remarks

## *2.1 Problem (The Problem of Convex Separation)*

Let $H \subset \mathbb{R}$ be a set of at least $n$ elements, and $\omega_1, \dots, \omega_n : H \to \mathbb{R}$ be given functions. We say that $\boldsymbol{\omega} := (\omega_1, \dots, \omega_n)$ is a (positive) *Chebyshev system* over $H$, if, for all elements $x_1 < \dots < x_n$ of $H$, the following inequality holds:

$$\begin{vmatrix} \boldsymbol{\omega}(x_1) & \cdots & \boldsymbol{\omega}(x_n) \end{vmatrix} := \begin{vmatrix} \omega_1(x_1) & \cdots & \omega_1(x_n) \\ \vdots & \ddots & \vdots \\ \omega_n(x_1) & \cdots & \omega_n(x_n) \end{vmatrix} > 0.$$

Given a positive Chebyshev system $\omega$ on $H$, a function $f : H \to \mathbb{R}$ is said to be *generalized convex with respect to* $\boldsymbol{\omega}$ (or briefly: $\boldsymbol{\omega}$*-convex*) if, for all elements $x_0 \le \dots \le x_n$ of $H$, we have the inequality

$$\begin{vmatrix} \boldsymbol{\omega}(x_0) & \cdots & \boldsymbol{\omega}(x_n) \\ f(x_0) & \cdots & f(x_n) \end{vmatrix} := \begin{vmatrix} \omega_1(x_0) & \cdots & \omega_1(x_n) \\ \vdots & \ddots & \vdots \\ \omega_n(x_0) & \cdots & \omega_n(x_n) \\ f(x_0) & \cdots & f(x_n) \end{vmatrix} \ge 0.$$

Clearly, the notion of convexity induced by Chebyshev systems involves the case of classical convexity. The well-known Sandwich Theorem states that if a convex function lies "above" a concave one, then there exists an affine function separating them. It turns out that the existence of an affine separator can be characterized even in the general case:

**Theorem** *Let $H \subset \mathbb{R}$ be a set of at least n elements, let $\boldsymbol{\omega}$ be a positive Chebyshev system over $H$, and $f, g : H \to \mathbb{R}$. Denote the linear hull of the components of $\boldsymbol{\omega}$ by $\Omega_n(H)$. Then, the following statements are equivalent:*

(i) *There exists an element $\omega$ of $\Omega_n(H)$ such that $f \le \omega \le g$;*
(ii) *there exists an $\boldsymbol{\omega}$-concave function $\phi : H \to \mathbb{R}$ and an $\boldsymbol{\omega}$-convex function $\psi : H \to \mathbb{R}$ satisfying the inequalities $f \le \phi \le g$ and $f \le \psi \le g$;*

(iii) *for all elements* $x_0 \leq \cdots \leq x_n$ *of* $H$ *the next inequalities are satisfied*:

$$\begin{vmatrix} \cdots & \boldsymbol{\omega}(x_{n-3}) & \boldsymbol{\omega}(x_{n-2}) & \boldsymbol{\omega}(x_{n-1}) & \boldsymbol{\omega}(x_n) \\ \cdots & g(x_{n-3}) & f(x_{n-2}) & g(x_{n-1}) & f(x_n) \end{vmatrix} \leq 0;$$

$$\begin{vmatrix} \cdots & \boldsymbol{\omega}(x_{n-3}) & \boldsymbol{\omega}(x_{n-2}) & \boldsymbol{\omega}(x_{n-1}) & \boldsymbol{\omega}(x_n) \\ \cdots & f(x_{n-3}) & g(x_{n-2}) & f(x_{n-1}) & g(x_n) \end{vmatrix} \geq 0.$$

In the particular setting when the underlying Chebyshev system is $\boldsymbol{\omega} := (1, \mathrm{id})$, the previous theorem reduces to the main result of Nikodem–Wąsowicz [5]. In fact, Wąsowicz obtained similar results both for the polynomial [6] and for the Chebyshev setting [1] using some selection principles an algebraic manipulations.

On the other hand, there is a characterization for the existence of convex (respectively, concave) separation due to Baron–Matkowski–Nikodem [2], which corresponds to the first (respectively, second) inequality of the presented result. (For the nonlinear correspondence, consult the paper of Nikodem–Páles [4].) Therefore the question arises, quite evidently, whether the separated inequalities mentioned characterize the existence of convex (respectively, concave) separation in the general case.

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M. Bessenyei and Zs. Páles

## *2.2 Problem*

Suppose $M(x, y)$ is a continuous, strict mean and $\lambda \in (0, 1)$ with binary representation $\lambda = 0.\lambda_1\lambda_2 \ldots$. For $x < y$ we built a decreasing (in terms of inclusion) sequence of intervals $[x_0, y_0], [x_1, y_1], \ldots, [x_n, y_n], \ldots$ as follows:

$$[x_0, y_0] = [x, y],$$

$$[x_{n+1}, y_{n+1}] = \begin{cases} [M(x_n, y_n), y_n] & \lambda_{n+1} = 0, \\ [x_n, M(x_n, y_n)] & \lambda_{n+1} = 1. \end{cases}$$

In other words: the point $M(x_n, y_n)$ divides interval $[x_n, y_n]$ into two parts, and the $(n+1)^{th}$ digit of $t$ decides which one to choose. The assumptions on $M$ guarantee that their intersection consists of one point—$M(x, y; \lambda)$

$$\widehat{M}(x, y) = \int_0^1 M(x, y; \lambda)\, d\lambda.$$

*Question: what are the properties of $\widehat{M}$ and which means can be obtained this way?* Some of means that can be obtained are:

$$\widehat{A}(x, y) = A(x, y),$$
$$\widehat{G}(x, y) = L(x, y),$$
$$\widehat{M_r}(x, y) = E(r+1, r; x, y).$$

And generally if $M_f(x, y) = f^{-1}(\frac{f(x)+f(y)}{2})$ is a pseudoarithmetic mean then

$$\widehat{M_f}(x, y) = \frac{\int_{f(x)}^{f(y)} f^{-1}(t)\, dt}{f(y) - f(x)}.$$

If we use a more general approach by defining

$$\widehat{M}^g = g^{-1}\left(\int_0^1 g\big(M(x, y, \lambda)\big)\, d\lambda\right)$$

them we can obtain all Stolarsky means by taking a power means as $M$ and a power or logarithmic function as $g$.

A. Witkowski

## 2.3 Remark

Let $\mathcal{I} \subset \mathbb{R}$ be an interval. The function $f : \mathcal{I} \to \mathbb{R}$ is called

- *Mercer–convex*, if

$$f\left(a + b - \sum_{i=1}^n t_i x_i\right) \le f(a) + f(b) - \sum_{i=1}^n t_i f(x_i) \tag{1}$$

  for any $a, b \in \mathcal{I}$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathrm{co}(\{a, b\})$ and $t_1, \ldots, t_n \ge 0$ with $t_1 + \cdots + t_n = 1$;
- *Wright–convex*, if

$$f\big(tx + (1-t)y\big) + f\big(ty + (1-t)x\big) \le f(x) + f(y) \tag{2}$$

  for any $x, y \in \mathcal{I}$, $t \in [0, 1]$;

- *Jensen–convex*, if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for any $x, y \in \mathcal{I}$.

Of course, every Mercer–convex function is Wright–convex (use (1) for $n = 1$, which, in fact, is equivalent to (2)) and any Wright–convex function is Jensen–convex (use (2) for $t = 1/2$). Next, not every Jensen–convex function is Wright–convex. The easy example is the absolute value of a discontinuous additive function, the graph of which is not dense in the whole plane, while Wright–convex functions possess this property because of a decomposition into the sum of an additive component and a convex one.

Then the natural question arises, whether every Wright–convex function is Mercer–convex. However, the answer is negative. Indeed, using (1) for $n = 2$, $x_1 = a$, $x_2 = b$, we easily obtain that Mercer convexity is equivalent to the ordinary convexity. Then, for instance, any discontinuous additive function is Wright–convex and it is not convex (i.e. Mercer–convex) at all.

A. Witkowski and Sz. Wąsowicz

## *2.4 Remark (On Two Different Concepts of Subquadraticity)*

A function $f : [0,\infty[\to \mathbb{R}$ is called strongly subquadratic if, for all $x \geq 0$, there exists a constant $c_x \in \mathbb{R}$ such that

$$f(y) - f(x) \leq c_x(y-x) + f\big(|y-x|\big) \quad (y \geq 0). \tag{3}$$

Functions of this type have been investigated, among others, by S. Abramovich, S. Banić, J. Barić, G. Jameson, M. Matić, J.A. Oguntuase L.E. Persson, J. Pečarić, G. Sinnamon, S. Varošanec.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be weakly subquadratic, if

$$f(x+y) + f(x-y) \leq 2f(x) + 2f(y) \quad (x, y \in \mathbb{R}). \tag{4}$$

Functions satisfying this inequality have been considered by Z. Kominek, K. Troczka-Pawelec, W. Smajdor. (Cf. also R.A. Rosenbaum (1950) and M. Kuczma (1985, 2009).)

**Theorem** *Let $f : [0,\infty[\to \mathbb{R}$ be a function. If $f$ is strongly subquadratic then its even extension $\bar{f} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \geq 0, \\ f(-x), & \text{if } x < 0, \end{cases}$$

*is weakly subquadratic.*

**Theorem** *A weakly subquadratic function* $f\colon [0,\infty[\to \mathbb{R}$ *is not necessarily strongly subquadratic.*

**Example** Let $f\colon [0,\infty[\to \mathbb{R}$ defined by

$$f(x)=\begin{cases}3, & \text{if } x=0,\\ 1, & \text{if } x\neq 0.\end{cases} \tag{5}$$

Then $f$ is weakly subquadratic but not strongly subquadratic.

**Theorem** *Let* $f : [0,\infty[\to \mathbb{R}$. *The function* $f$ *fulfils the inequality*

$$f(x+y)+f(x-y)\le 2f(x)+2f(y) \quad (x\ge y\ge 0) \tag{6}$$

*if and only if*

$$f(x+y)+f\big(|x-y|\big)\le 2f(x)+2f(y) \quad (x,y\ge 0). \tag{7}$$

**Theorem** *If* $f\colon [0,\infty]\to \mathbb{R}$ *is a strongly subquadratic function then*

$$f(x+y)+f\big(|x-y|\big)\le 2f(x)+2f(y) \quad (x,y\ge 0). \tag{8}$$

The converse of the previous theorem is not true. Namely, the function given in (5) fulfils inequality (8) but it is not strongly subquadratic.

**Theorem** *Suppose that* $f : [0,\infty[\to \mathbb{R}$ *satisfies* (8), *that is,*

$$f(x+y)+f\big(|x-y|\big)\le 2f(x)+2f(y) \quad (x,y\ge 0).$$

*Then the even extension* $\bar{f} : \mathbb{R}\to\mathbb{R}$ *of* $f$ *defined by fulfils*

$$f(x+y)+f\big(|x-y|\big)\le 2f(x)+2f(y) \quad (x,y\in \mathbb{R}). \tag{9}$$

**Theorem** *Let* $f : \mathbb{R}\to\mathbb{R}$. *If* $f$ *is weakly subquadratic then it solves*

$$f(x+y)+f\big(|x-y|\big)\le 2f(x)+2f(y) \quad (x,y\in \mathbb{R}). \tag{10}$$

The converse of the theorem above is not true: the function $f : \mathbb{R}\to\mathbb{R}$

$$\bar{f}(x) := \begin{cases}1 & \text{if } x\ge 0,\\ 4 & \text{if } x<0,\end{cases}$$

satisfies (10) but it is not weakly subquadratic.

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## *2.5 Problem*

Let us start from the well-known inequality between geometric and logarithmic mean. One may apply it for $e^x$ and $e^y$ to reach the following estimate:

$$e^{\frac{x+y}{2}} \leq \frac{e^y - e^x}{y - x} \tag{11}$$

for all $x, y \in \mathbb{R}$. This estimate motivates the study of the following functional inequality:

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(y)-f(x)}{y-x} \tag{12}$$

for all $x$, $y$ belonging to a nonempty open interval $I$. This inequality appeared first in 1998 in a paper of C. Alsina and R. Ger [1]. In particular they proved the following statements:

**Theorem** *A nonnegative function $f\colon I \to \mathbb{R}$ satisfies (12) if and only if there exists a nondecreasing nonnegative function $i\colon I \to \mathbb{R}$ such that $f(x) = i(x)e^x$ for $x \in I$.* [1, *Lemma* 3]

**Theorem** *If a nonpositive and nonincreasing function $f\colon I \to \mathbb{R}$ satisfies (12) then there exists a nondecreasing function $i\colon I \to \mathbb{R}$ such that $f(x) = i(x)e^x$ for $x \in I$.* [1, *Lemma* 4]

**Theorem** *A nonpositive, nonincreasing and Jensen-convex function $f\colon I \to \mathbb{R}$ satisfies (12) if and only if there exists a nonpositive and nondecreasing function $i\colon I \to \mathbb{R}$ such that $x \to i(x)e^x$ is convex and $f(x) = i(x)e^x$ for $x \in I$.* [1, *Lemma* 5]

The authors asked if there is possible to drop or weaken the additional assumptions imposed upon $f$. In 2008 we partially answered this question. More precisely, we proved the following result.

**Theorem** *If $f\colon I \to \mathbb{R}$ satisfies (12) and*

$$\limsup_{h\to 0+} f(x+h) \geq f(x), \quad x \in I, \tag{13}$$

*then there exists a nondecreasing function $i\colon I \to \mathbb{R}$ such that $f(x) = i(x)e^x$ for $x \in I$.* [2, *Theorem* 1]

However, it is easy to notice that the converse implication in the last theorem is not true (take $i = -1$). Therefore, the following two open problems are well-motivated:

**Problem** What should be added to the assertion of the theorem above to get an "if and only if" result?

**Problem** Is it possible to drop or weaken the assumption (13)?

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W. Fechner

## 2.6 Remark (The Continuous Case for Generalized Jensen Functional)

Professor László Horváth has given the talk entitled *A refinement of the classical Jensen's inequality.* Professor Michael Plum asked about the integral case related to the talk of Professor László Horváth. Here are few results related tho this question; the proofs have been omitted, the interested reader can find them in the paper [1].

Assume that we have $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [a,b]^n$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ such that $p_i > 0$, $\sum_{i=1}^n p_i = 1$ and $\mathbf{q} = (q_1, q_2, \ldots, q_k)$ such that $q_i > 0$, $\sum_{i=1}^k q_i = 1$ $(1 \leq k \leq n)$. We define the normalized Jensen functional

$$\mathcal{T}_k(f, \mathbf{p}, \mathbf{q}, \mathbf{x}) = \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1} \ldots p_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right).$$

**Theorem** *Assume $f$ is a convex function and let $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ be such that $r_i > 0$, $\sum_{i=1}^n r_i = 1$. Then*

$$\min_{1 \leq i_1,\ldots,i_k \leq n} \left\{\frac{p_{i_1} \ldots p_{i_k}}{r_{i_1} \ldots r_{i_k}}\right\} \mathcal{T}_k(f, \mathbf{r}, \mathbf{q}, \mathbf{x})$$
$$\leq \mathcal{T}_k(f, \mathbf{p}, \mathbf{q}, \mathbf{x}) \leq \max_{1 \leq i_1,\ldots,i_k \leq n} \left\{\frac{p_{i_1} \ldots p_{i_k}}{r_{i_1} \ldots r_{i_k}}\right\} \mathcal{T}_k(f, \mathbf{r}, \mathbf{q}, \mathbf{x}).$$

For $\mu$ a Steffensen-Popoviciu measure on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ a convex function then we have

$$f\left(\frac{1}{\mu([a,b])} \int_a^b x \, d\mu(x)\right) \leq \frac{1}{\mu([a,b])} \int_a^b f(x) \, d\mu(x).$$

(See C.P. Niculescu and L.-E. Persson [2, Chap. 4] for more results concerning the Steffensen-Popoviciu measures.) We consider $p : [a,b] \to \mathbb{R}$ such that $p(x)\,dx$ is an absolutely continuous measure and

$$0 < \int_a^b p(x)\,dx, \qquad 0 \leq \int_a^t p(x)\,dx \leq \int_a^b p(x)\,dx \tag{SP}$$

for all $t \in [a,b]$. Then $p(x)\,dx$ is a Steffensen-Popoviciu measure and $f$ verifies

$$f\left(\frac{\int_a^b x p(x)\,dx}{\int_a^b p(x)\,dx}\right) \leq \frac{1}{\int_a^b p(x)\,dx} \int_a^b f(x) p(x)\,dx. \tag{SP}$$

**Lemma** *Let $p(x)\,dx$ be an absolutely continuous measure, $p\ [a,b] \to (0,\infty)$ increasing such that $\int_a^b p(x)\,dx = 1$. We consider $\mathbf{q} = (q_1, q_2, \ldots, q_k)$ such that*

$q_i > 0$, $\sum_{i=1}^{k} q_i = 1$ $(1 \le k)$. *If $f$ is convex then*

$$f\left(\int_a^b xp(x)\,dx\right) \le \int_{[a,b]^k} f\left(\sum_{i=1}^{k} q_i x_i\right) \prod_{i=1}^{k} \big(p(x_i)\,dx_i\big)$$
$$\le \int_a^b f(x)p(x)\,dx$$

*for all positive integers $k$.*

Let $p(x)\,dx$ be an absolutely continuous measure, where $p : [a,b] \to (0,\infty)$ is increasing such that $\int_a^b p(x)\,dx = 1$. We define the normalized Jensen functional

$$\mathcal{T}_k(f,p,q) := \int_{[a,b]^k} f\left(\sum_{i=1}^{k} q_i x_i\right) \prod_{i=1}^{k} \big(p(x_i)\,dx_i\big)$$
$$- f\left(\left(\int_a^b xp(x)\,dx\right)\right)(\ge 0).$$

The precision of the first inequality of the above lemma is estimated in the following theorem:

**Theorem** *Let $r(x)\,dx$ be an absolutely continuous measure, where $r : [a,b] \to (0,\infty)$ is increasing such that $\int_a^b r(x)\,dx = 1$. If $f$ is convex then*

$$\inf_{t,s\in[a,b],s\neq t} \left\{ \frac{\int_{[t,s]^k} \prod_{i=1}^{k} (p(x_i)\,dx_i)}{\int_{[t,s]^k} \prod_{i=1}^{k} (r(x_i)\,dx_i)} \right\} \mathcal{T}_k(f,r,q)$$
$$\le \mathcal{T}_k(f,p,q) \le \max_{t,s\in[a,b],s\neq t} \left\{ \frac{\int_{[t,s]^k} \prod_{i=1}^{k} (p(x_i)\,dx_i)}{\int_{[t,s]^k} \prod_{i=1}^{k} (r(x_i)\,dx_i)} \right\} \mathcal{T}_k(f,r,q).$$

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(Compiled by **Mihály Bessenyei**)

# Contents

# Part 1
# Boundary Value Problems

# Domain Derivatives for Energy Functionals with Boundary Integrals

**Catherine Bandle and Alfred Wagner**

**Abstract** This paper deals with domain derivatives of energy functionals related to elliptic boundary value problems. Emphasis is put on boundary conditions of mixed type which give rise to a boundary integral in the energy. A formal computation for rather general functionals is given. It turns out that in the radial case the first derivative vanishes provided the perturbations are volume preserving. In the simplest case of a torsion problem with Robin boundary conditions, the sign of the first variation shows that the energy is monotone with respect to domain inclusion for nearly circular domains. In this case also the second variation is derived.

**Keywords** Domain derivative · Optimality

**Mathematics Subject Classification** 49Q10

## 1 Introduction

In this paper we are concerned with *energy* functionals $\mathcal{E} : \Omega_t \to \mathbb{R}$ where $\Omega_t \subset \mathbb{R}^N$, $t \in [0, \tau]$, are small perturbations of a domain $\Omega$. Important tools in shape optimization are variational formulas exhibiting the domain dependence. Under sufficient smoothness assumptions $\mathcal{E}(t)$ can be expanded into powers of $t$,

$$\mathcal{E}(t) = \dot{\mathcal{E}}(0)t + \ddot{\mathcal{E}}(0)t^2 + o\big(t^2\big) \quad \text{as } t \to 0.$$

The terms $\dot{\mathcal{E}}(0)$ and $\ddot{\mathcal{E}}(0)$ are called the *first variation*, resp. *second variation* of $\mathcal{E}(t)$. They depend on $\Omega$ and on the particular perturbations. The simplest example we have in mind are problems of the type

$$\mathcal{E}(t) = \inf_{W^{1,2}(\Omega_t)} \left\{ \int_{\Omega_t} \left( \frac{1}{2}|\nabla u|^2 - u \right) dx + \frac{\alpha}{2} \oint_{\partial\Omega_t} u^2 \, ds, \quad \alpha \in \mathbb{R}^+ \right\}. \tag{1}$$

It is well-known that a minimizer exists and that it satisfies Euler-Lagrange equation

$$\Delta u + 1 = 0 \quad \text{in } \Omega_t, \qquad \frac{\partial u}{\partial n_t} = 0 \quad \text{on } \partial\Omega. \tag{2}$$

C. Bandle et al. (eds.), *Inequalities and Applications 2010*,
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Here $n_t$, $(n)$ stands for the outer normal of $\Omega_t$, $(\Omega)$. Then

$$\partial\Omega_t = \{x + tg(x)n(x) : x \in \partial\Omega\},$$

where $tg(x)$ is the normal displacement of each boundary point $x \in \partial\Omega$. In the case of Dirichlet boundary conditions $u = 0$ on $\partial\Omega_t$

$$\mathcal{E}^D(t) = \inf_{W_0^{1,2}(\Omega_t)} \int_{\Omega_t} \left(\frac{1}{2}|\nabla u|^2 - u\right) dx.$$

Its minimizer solves $\Delta u + 1 = 0$ in $\Omega_t$ and vanishes on the boundary. Its first variation assumes the simple form

$$\dot{\mathcal{E}^D}(0) = -\frac{1}{2}\oint_{\partial\Omega} |\nabla u|^2 g\, ds.$$

From this expression and the positivity of $u$ it follows immediately that $\mathcal{E}^D$ is a decreasing functional of the domain. Moreover if $\Omega$ is a ball and $|\Omega_t| = |\Omega|$, i.e. $\oint_{\partial\Omega} g\, ds = 0$ then $\dot{\mathcal{E}}^D(0) = 0$. The first statement follows directly from the variational characterization of $\mathcal{E}^D(t)$. In fact if $u$ is extended by zero outside $\Omega$ it remains an admissible function for the energy in $\Omega_t$. In addition it does not change the energy and its minimum therefore decreases. The second assertion is a consequence of Pólya's theorem on the maximal torsional rigidity [5]. By means of Schwarz symmetrization it is easily proved that among all domains of given volume the sphere has the minimal energy $\mathcal{E}^D(t)$.

For Robin boundary conditions it is not known whether such results are true. No global tools seem to be available to discuss question such as:

1. For what kind of deformations does $\mathcal{E}(t)$ decrease?
2. Does the ball yield the minimum of $\mathcal{E}(t)$, among all domains $\Omega_t$ of prescribed volume?

In this paper we give an answer to the first question for nearly circular domains. Concerning the second question we have only been able to show that for balls $\dot{\mathcal{E}}(0) = 0$. We have computed $\ddot{\mathcal{E}}(0)$ for the ball, its sign however does not seem clear.

The paper is organized as follows. We first derive the first variational formula for general energies. Such formulas are already known in the literature [3, 4, 6]. Since we are dealing with slightly more general energy functionals containing boundary integrals we include the formal computation for the reader's convenience. We then apply the first variation to radial problems and show that it vanishes for the ball. We then study the first and second variations of the torsion problem with Robin boundary conditions in the case of a ball. A study of the second variation for a different optimization problem is found in [2]. At the end some open problems related to these investigations are listed.

# 2 Variation Formulas

## 2.1 Domain Variation

Let $\Omega_t \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and let $\theta(t) : \Omega \to \Omega_t$, $t \in [0, \tau]$ be a family of diffeomorphisms such that

$$\Omega_t = \theta(t, \Omega) \quad \text{and} \quad \Omega = \theta(0, \Omega).$$

Since we will be interested in small perturbations of $\Omega$ we shall assume that

$$\theta(t, x) = x + tv(x), \tag{3}$$

where $v : \Omega \to \mathbb{R}^N$ is a smooth vector field and $t$ is a small parameter. We shall use the notation

$$D_v := \left(\frac{\partial v_i}{\partial x_j}\right), \quad D_v^2 = \left(\frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_j}\right), \quad i, j = 1, \ldots, N,$$

$$D_{\theta(t,x)} : \quad \text{Jacobian matrix,}$$

$$J(t) = \det D_{\theta(t,x)} : \quad \text{Jacobian determinant.}$$

Here and in the sequel repeated indices are understood to be summed from 1 to $N$. If $\theta$ is of the form (3) then

$$J(t) = 1 + t(\operatorname{trace} D_v) + \frac{t^2}{2}\big((\operatorname{trace} D_v)^2 - \operatorname{trace} D_v^2\big) + o\big(t^2\big), \tag{4}$$

$$\text{where } \operatorname{trace} D_v = \frac{\partial v_i}{\partial x_i}.$$

Observe that

$$\left(\frac{\partial x_k}{\partial \theta_i}\right) = D_\theta^{-1} = (I + tD_v)^{-1}.$$

For small $t$ we have

$$D_\theta^{-1} = I - tD_v + t^2 D_v^2 + o\big(t^2\big).$$

Hence

$$\frac{\partial}{\partial \theta_i} = \frac{\partial x_k}{\partial \theta_i}\frac{\partial}{\partial x_k} = \left(\delta_{ik} - t\frac{\partial v_k}{\partial x_i} + t^2\frac{\partial v_k}{\partial x_s}\frac{\partial v_s}{\partial x_i}\right)\frac{\partial}{\partial x_k} + o\big(t^2\big). \tag{5}$$

Our aim is to study the dependence of integrals involving $u : \Omega_t \to \mathbb{R}$ on domain deformations under the assumption that $u$ is sufficiently regular in $t$.

## 2.2 Variation of Volume Integrals

Consider a function[1] $L(y, \tilde{u}, p) : \Omega_t \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ which is continuously differentiable in all its argument and denote by $\nabla_\theta \tilde{u}$ the gradient $(u_{\theta_i})$. Define

$$\mathcal{L}(\tilde{u}, \Omega_t) := \int_{\Omega_t} L(y, \tilde{u}, \nabla_y \tilde{u})\, dy.$$

After the change of variable $y = \theta(t, x)$ we obtain

$$\mathcal{L}(\tilde{u}, \Omega_t) := \int_\Omega L\left(\theta, u(x,t), u_{x_k} \frac{\partial x_k}{\partial \theta_i}\right) J(t)\, dx, \quad i = 1, \dots, N.$$

Here we have written $u(x,t)$ for $\tilde{u}(\theta, t)$. Differentiation with respect to $t$ yields

$$\frac{\partial L}{\partial t} = L_{\theta_i} \frac{\partial \theta_i}{\partial t} + L_u \frac{\partial u}{\partial t} + L_{p_i}\left(\frac{\partial u_{x_k}}{\partial t} \frac{\partial x_k}{\partial \theta_i} + u_{x_k} \frac{\partial^2 x_k}{\partial t \partial \theta_i}\right).$$

For the particular diffeomorphism (3)

$$\begin{aligned}
\frac{\partial \theta_i}{\partial t} &= v_i, \\
\frac{\partial x_k}{\partial \theta_i} &= \delta_{ik} - t \frac{\partial v_k}{\partial x_i} + o(t), \\
\frac{\partial^2 x_k}{\partial t \partial \theta_i} &= -\frac{\partial v_k}{\partial x_i} + 2t \frac{\partial v_k}{\partial x_l} \frac{\partial v_l}{\partial x_i} + o(t).
\end{aligned}$$

Formal differentiation of $\mathcal{L}$ with respect to $t$ yields,

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} = &\int_\Omega \left\{ L_{\theta_i} v_i + L_u \frac{\partial u}{\partial t} + L_{p_i}\left(\frac{\partial u_{x_i}}{\partial t} - u_{x_k} \frac{\partial v_k}{\partial x_i}\right)\right\} J(t)\, dx \\
&+ \int_\Omega L \frac{\partial v_s}{\partial x_s}\, dx + O(t),
\end{aligned} \tag{6}$$

where (4) was used in the last integral.

## 2.3 Variation of Boundary Integrals

Suppose that $\partial\Omega = \Gamma^0 \cup \Gamma^1$ such that $\Gamma^0 \cap \Gamma^1 = \emptyset$ and let $\Gamma_t^k = \{x + tv : x \in \Gamma^k\}$ $(k = 0, 1)$. Consider integrals of the form

$$\mathcal{B}(\tilde{u}, \Gamma_t^1) := \int_{\Gamma_t^1} b(y, \tilde{u}(y,t))\, ds_y,$$

[1] This function will be called the Lagrangian following the usage in the calculus of variations.

where $b(y,\tilde{u}) : \Gamma_t^1 \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable in $y$ and $\tilde{u}$. Let $x(\xi)$, $\xi \in \mathcal{U} \subset \mathbb{R}^{N-1}$ be local coordinates of $\Gamma^1$. Then $\Gamma_t^1$ is represented locally by $\{y(\xi) := x(\xi) + tv(x(\xi)) : \xi \in \mathcal{U}\}$. Throughout this paper $(x, y)$ stands for the Euclidean scalar product of two vectors $x$ and $y$ in $\mathbb{R}^N$ and $|x| = (x,x)^{1/2}$. We have, setting $g_{ij} := (x_{\xi_i}, x_{\xi_j})$, $\tilde{v}(\xi) := v(x(\xi))$, $c_{ij} := (x_{\xi_i}, D_v x_{\xi_j}) = (x_{\xi_i}, \tilde{v}_{\xi_j})$, $a_{ij} = \frac{1}{2}(c_{ij} + c_{ji})$ and $b_{ij} = (\tilde{v}_{\xi_i}, \tilde{v}_{\xi_j})$,

$$|dy|^2 = \left(g_{ij} + 2ta_{ij} + t^2 b_{ij}\right) d\xi_i \, d\xi_j =: g_{ij}^t \, d\xi_i \, d\xi_j.$$

Write for short $G = (g_{ij})$, $G^{-1} = (g^{ij})$, $A = (a_{ij})$, $B = (b_{ij})$ and correspondingly $G^t = (g_{ij}^t)$. Then

$$ds_y = \left(\det G^t\right)^{1/2} d\xi.$$

Clearly

$$\sqrt{\det G^t} = \sqrt{\det G} \{ \underbrace{\det\left(I + 2tG^{-1}A + t^2 G^{-1} B\right)}_{k(\xi,t)} \}^{1/2}.$$

Set

$$\sigma_A = \operatorname{trace} G^{-1}A, \qquad \sigma_B = \operatorname{trace} G^{-1}B \quad \text{and} \quad \sigma_{A^2} = \operatorname{trace}\left(G^{-1}A\right)^2.$$

The Taylor expansion yields

$$k(\xi,t) = 1 + 2t\sigma_A + t^2\left(\sigma_B + 2\sigma_A^2 - 2\sigma_{A^2}\right) + o\left(t^2\right).$$

For small $t$ we have

$$\sqrt{k(\xi,t)} = 1 + t\sigma_A + t^2\left(\frac{\sigma_B}{2} - \sigma_{A^2} + \frac{\sigma_A^2}{2}\right) + o\left(t^2\right) := 1 + t\sigma_A + t^2\frac{\nu}{2} + o\left(t^2\right). \quad (7)$$

As before we set $u(x,t) = \tilde{u}(\theta(x,t),t)$. Then, since $ds = \sqrt{\det G}\, d\xi$, it follows that

$$\mathcal{B}(t) := \mathcal{B}\left(\tilde{u}, \Gamma_t^1\right) = \int_{\Gamma^1} b(\theta,u)\left\{1 + t\sigma_A + \frac{t^2}{2}\nu + o\left(t^2\right)\right\} ds.$$

Consequently

$$\begin{aligned}\frac{d\mathcal{B}}{dt}(t) = &\int_{\Gamma^1}\left\{b\sigma_A + b_{\theta_i} v_i + b_u \frac{\partial u}{\partial t}\right\} ds \\ &+ t\int_{\Gamma^1}\left\{\sigma_A\left(b_{\theta_i} v_i + b_u \frac{\partial u}{\partial t}\right) + b\nu + o(1)\right\} ds, \end{aligned} \quad (8)$$

and

$$\frac{d\mathcal{B}}{dt}(0) = \int_{\Gamma^1}\left\{b\sigma_A + b_{x_i} v_i + b_u \frac{\partial u}{\partial t}\right\} ds. \quad (9)$$

### 2.3.1 Discussion of $\sigma_A$, $\sigma_{A^2}$ and $\sigma_B$

In order to have a better understanding of the term $\sigma_A$ let us decompose the vector field $v$ on $\Gamma^1$ in the following way

$$\tilde{v}(\xi) := v\big(x(\xi)\big) = \underbrace{\big(v\big(x(\xi)\big), n(\xi)\big)n(\xi)}_{\tilde{v}^n} + \underbrace{\sum_{k=1}^{N-1} \big(v\big(x(\xi)\big), x_{\xi_k}\big)x_{\xi_k}}_{\tilde{v}^*}. \tag{10}$$

We set

$$\eta^k := \big(v\big(x(\xi)\big), x_{\xi_k}\big), \quad k = 1, \dots, N-1,$$
$$\eta^N := \big(v\big(x(\xi)\big), n(\xi)\big).$$

Clearly $\tilde{v}^n \perp \tilde{v}^*$. In the language of differential geometry we have

$$\tilde{v}^*_{\xi_j} = \eta^k_{,j} x_{\xi_k} = \left[\frac{\partial \eta^k}{\partial \xi_j} + \Gamma^k_{ij}\eta^i\right] x_{\xi_k}$$

where $\Gamma^k_{ij}$ denotes the Christoffel symbol and $\eta^k_{,j}$ is the covariant derivative with respect to $g_{ij}$. Using this decomposition we can compute $G^{-1}A$ and $G^{-1}B$ explicitely.

$$\begin{aligned}\big(G^{-1}B\big)_{ik} = g^{ij}b_{jk} = g^{ij}\big(\eta^N_{\xi_j} n(\xi) + \eta^N n(\xi)_{\xi_j}, \eta^N_{\xi_k} n(\xi) + \eta^N n(\xi)_{\xi_k}\big)\\ + 2g^{ij}\big(\eta^N_{\xi_j} n(\xi) + \eta^N n(\xi)_{\xi_j}, \eta^l_{,k} x_{\xi_l}\big) g^{ij}\big(\eta^m_{,j} x_{\xi_m}, \eta^l_{,k} x_{\xi_l}\big)\end{aligned}$$

where $l = 1, \dots, N-1$. We observe that $(n(\xi), n(\xi)) = 1$, $(n(\xi), n(\xi)_{\xi_i}) = 0$, $(n(\xi), x_{\xi_l}) = 0$ and we assume that $(x_{\xi_m}, x_{\xi_l}) = \delta_{kl}$ for $m, l = 1, \dots, N-1$. Thus

$$\big(G^{-1}B\big)_{ik} = g^{ij}b_{jk} = g^{ij}\eta^N_{\xi_j}\eta^N_{\xi_k} + \big(\eta^N\big)^2 g^{ij}\big(\eta^N_{\xi_j}, \eta^N_{\xi_k}\big) + g^{ij}\eta^l_{,j}\eta^l_{,k}.$$

For the trace $\sigma_B$ we compute

$$\sigma_B = \big(1 + \big(\eta^N\big)^2\big) g^{ij}\big(\eta^N_{\xi_j}, \eta^N_{\xi_i}\big) + g^{ij}\eta^l_{,j}\eta^l_{,i}.$$

Moreover

$$c_{ij} = (x_{\xi_i}, \tilde{v}_{\xi_j}) = \eta^N(\xi)(x_{\xi_i}, n_{\xi_j}) + \eta^k_{,j}(\xi)(x_{\xi_i}, x_{\xi_k}), \quad k = 1, \dots, N-1.$$

Thus

$$\begin{aligned}\big(G^{-1}A\big)_{ij} &= g^{ik}a_{kj} = \frac{1}{2}g^{ik}(c_{kj} + c_{jk})\\ &= \frac{1}{2}g^{ik}\big(\eta^N(\xi)(x_{\xi_k}, n_{\xi_j}) + \eta^l_{,k}(\xi)(x_{\xi_j}, x_{\xi_l}) + \eta^N(\xi)(x_{\xi_j}, n_{\xi_k}\big)\end{aligned}$$

$$+ \eta^l_{,j}(\xi)(x_{\xi_k}, x_{\xi_l})\big)$$
$$= \frac{1}{2} g^{ik}\big(\eta^N(\xi)(x_{\xi_k}, n_{\xi_j}) + \eta^l_{,k}(\xi) g_{jl} + \eta^N(\xi)(x_{\xi_j}, n_{\xi_k}) + \eta^l_{,j}(\xi) g_{lk}\big).$$

Analogously for the trace $\sigma_A$ we compute

$$\sigma_A = \frac{1}{2} g^{ik}\big(\eta^N(\xi)(x_{\xi_k}, n_{\xi_i}) + \eta^l_{,k}(\xi) g_{il} + \eta^N(\xi)(x_{\xi_i}, n_{\xi_k}) + \eta^l_{,i}(\xi) g_{lk}\big)$$
$$= \eta^N(\xi) g^{ik}(x_{\xi_k}, n_{\xi_i}) + \eta^i_{,i}(\xi).$$

Observe that $\tau^i_{,i} =: \operatorname{div}^* \tilde{v}^*$ is the surface divergence of $\Gamma^1$. Furthermore $(n_{\xi_i}, x_{\xi_s}) = -(n, x_{\xi_s \xi_i}) = L_{is}$ is the second fundamental form.[2] Let $\kappa_i$, $i = 1, 2, \dots, N-1$, denote the principle curvatures of $\Gamma^1$. Then

$$g^{is} L_{is} = \sum_{i=1}^{N-1} \kappa_i =: (N-1)H, \quad H \text{ mean curvature of } \Gamma^1.$$

In conclusion we have

$$\sigma_A = g(N-1)H + \operatorname{div}^* \tilde{v}^*. \tag{11}$$

Finally we give an explicit expression for $\sigma_{A^2}$. We use the following notation:

$$h_{ij} = \frac{1}{2}(L_{ij} + L_{ji}) \quad \text{and} \quad H_{ij} := g^{ik} h_{kj}.$$

Then a lengthly computation gives

$$\sigma_{A^2} = \big(\eta^N(\xi)\big)^2 \operatorname{trace} H^2 + \eta^N(\xi)\Bigg(h_{ij}\eta^j_{,k} g^{ki} + H_{ij}\eta^i_{,j} + \frac{1}{2}\big(\eta^i_{,j}\eta^j_{,i} + g^{ij}\eta^k_{,i}\eta^l_{,j} g_{kl}\big)\Bigg).$$

## 2.4 Domain Variation of Critical Points

Consider the following energy functional

$$\mathcal{E}(t) = \mathcal{L}(\Omega_t, \tilde{u}) + \mathcal{B}\big(\tilde{u}, \Gamma^1_t\big).$$

Suppose that for all $t$, $\tilde{u}(y,t)$ is a critical point of the energy functional $\mathcal{E}$—in the sense that the Fréchet of $\mathcal{E}(\Omega_t, \cdot)$ derivative vanishes at this point. Thus $\tilde{u}$ solves in $\Omega_t$ the Euler-Lagrange equation

$$\frac{\partial L_{p_i}(y, \tilde{u}, \nabla\tilde{u})}{\partial y_i} = L_{\tilde{u}}(y, \tilde{u}, \nabla\tilde{u}) \quad \text{in } \Omega_t, \tag{12}$$

[2] Notice that the minus sign is due to the fact that $n$ is the outer normal.

and boundary conditions

$$\begin{aligned} &\tilde{u}=0 \quad \text{on } \Gamma_t^0: \quad \text{Dirichlet boundary conditions,}\\ &L_{p_i}(y,\tilde{u},\nabla\tilde{u})n_i + b_{\tilde{u}}(y,\tilde{u}) = 0 \quad \text{on } \Gamma_t^1: \quad \text{Robin boundary conditions.} \end{aligned} \tag{13}$$

Observe that if $b=0$ the Robin condition becomes a Neumann boundary condition

$$L_{p_i}(y,u,\nabla u)n_i = 0.$$

In the $x$-coordinates the Euler-Lagrange equation for $u$ assumes the form

$$L_u J = \frac{\partial}{\partial x_k}\left(L_{p_i} J \frac{\partial x_k}{\partial \theta_i}\right) \quad \text{in } \Omega. \tag{14}$$

The boundary conditions are

$$\begin{aligned} u(x,t) &= 0 \quad \text{on } \Gamma^0,\\ L_{p_i} J \frac{\partial x_k}{\partial \theta_i} n_k + b_u \sqrt{k(x,t)} &= 0 \quad \text{on } \Gamma^1. \end{aligned} \tag{15}$$

Introducing (14) into (6) and letting $t \to 0$ we find

$$\frac{d\mathcal{L}}{dt}\bigg|_{t=0} = \int_\Omega \left\{ L_{x_i} v_i - L_{p_i} u_{x_k} \frac{\partial v_k}{\partial x_i} + L \frac{\partial v_s}{\partial x_s} \right\} dx + \oint_{\partial\Omega} L_{p_k} \frac{\partial u}{\partial t} n_k \, ds.$$

Taking into account the boundary conditions we conclude that $\frac{\partial u}{\partial t} = 0$ on $\Gamma^0$ and $L_{p_i} n_i = -b_u$ on $\Gamma^1$. Thus

$$\oint_{\partial\Omega} L_{p_k} \frac{\partial u}{\partial t} n_k \, ds = -\int_{\Gamma^1} b_u \frac{\partial u}{\partial t}\, ds.$$

This together with (9) implies

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t}\bigg|_{t=0} = &\int_\Omega \left\{ L_{x_i} v_i - L_{p_i} u_{x_k} \frac{\partial v_k}{\partial x_i} + L \frac{\partial v_s}{\partial x_s} \right\} dx \\ &+ \int_{\Gamma^1} \left\{ b(x,u)\sigma_A + b_{x_i} v_i \right\} ds. \end{aligned} \tag{16}$$

The volume integral can be transformed into a boundary integral. In fact if $u$ is a solution of (12) in $\Omega$ then

$$\frac{\partial}{\partial x_i}(L v_i - L_{p_i} u_{x_j} v_j) = L \frac{\partial v_i}{\partial x_i} + v_i L_{x_i} - L_{p_i} u_{x_j} \frac{\partial v_j}{\partial x_i},$$

and hence

$$\frac{\partial \mathcal{E}}{\partial t}\bigg|_{t=0} = \oint_{\partial\Omega} \left\{ L(v,n) - L_{p_i} n_i (\nabla u, v) \right\} ds$$

$$+\int_{\Gamma^1}\{b(x,u)\sigma_A + b_{x_i}v_i\}\,ds. \tag{17}$$

## 3 Applications

### 3.1 Optimality of Radial Problems

Suppose that $\Omega$ is a ball of radius $R$ and that $L = L(r, u(r), u'(r))$ and $b = b(r, u(r))$, $r = |x|$ are radially symmetric. Then on $\partial\Omega$ we have

$$L = \text{const.} \quad \text{and} \quad L_{p_i}n_i(\nabla u, v) = L_{u'}u'(v, n).$$

Thus

$$\oint_{\partial\Omega}\{L(v,n) - L_{p_i}n_i(\nabla u, v)\}\,ds = \left(L - L_{u'}u'\right)\oint_{\partial\Omega}(n, v)\,ds.$$

By (11), $\sigma_A = (v,n)(N-1)/R + \operatorname{div}^* \tilde{v}^*$ and

$$\int_{\Gamma^1}\{b(r,u)\sigma_A + b_{x_i}v_i\}\,ds = \left(\frac{b(N-1)}{R} + b_r\right)\oint_{\partial\Omega}(v,n)\,ds.$$

Finally we get

$$\left.\frac{d\mathcal{E}}{dt}\right|_{t=0} = \left(L - L_{u'}u' + \frac{b(N-1)}{R} + b_r\right)\oint_{\partial\Omega}(v,n)\,ds.$$

From the divergence theorem and (4) we get

$$\oint_{\partial\Omega}(v,n)\,ds = \int_{\Omega}\operatorname{trace} D_v\,dx = \frac{1}{t}\left(\int_{\Omega_t}dx - \int_{\Omega}dx + o(t)\right).$$

Hence $\oint_{\partial\Omega}(v,n)\,ds = 0$ if $|\Omega_t| = |\Omega|$.

This together with the previous observations implies

**Theorem 1** *Let $\Omega$ be a ball of radius $R$ in $\mathbb{R}^N$ and let $\Omega_t$ be a small, volume preserving perturbation in the sense of Sect. 2. Let $u(r)$ be a solution of*

$$\frac{dL_{u'}(r,u(r),u'(r))}{dr} = L_u\big(r,u(r),u'(r)\big) \quad \text{in } (0,R).$$

*Then the energy $\mathcal{E}(t)$ given by $\int_{\Omega_t}L(r,u,u')\,dx + \oint_{\partial\Omega_t}b(r,u)\,ds$ is stationary in $t = 0$, i.e., $\dot{\mathcal{E}}(0) = 0$.*

## 3.2 Torsion Problem with Robin Boundary Conditions

### 3.2.1 First Variation

In this section we discuss the problem

$$\mathcal{E}(t) = \int_{\Omega_t} \left( \frac{|\nabla u|^2}{2} - u \right) dx + \frac{\alpha}{2} \oint_{\partial\Omega_t} u^2 \, ds, \tag{18}$$

where $u$ is a solution of the corresponding Euler-Lagrange equation

$$\Delta u + 1 = 0 \quad \text{in } \Omega_t, \qquad \frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \partial\Omega_t.$$

The first variation is according to (17)

$$\dot{\mathcal{E}}(0) = \oint_{\partial\Omega} \left\{ \left(|\nabla u|^2/2 - u\right)(v, n) + \alpha u (\nabla u, v) + \alpha u^2 \sigma_A/2 \right\} ds.$$

For the ball $\Omega = B_R$ the solution can be computed explicitly. In this case we have $u(r) = \frac{R}{N}(\frac{R}{2} + \frac{1}{\alpha}) - \frac{r^2}{2N}$, $L(r, u, u') = u'^2/2 - u$,

$$\mathcal{E}(0) = -|\partial B_1| \left( \frac{R^{N+2}}{2N^2(N+2)} + \frac{R^{N+1}}{2\alpha N^2} \right),$$

and

$$\frac{\partial \mathcal{E}(0)}{\partial t} = -\left[ \frac{R^2}{2N^2} + \frac{(N+1)R}{2\alpha N^2} \right] \oint_{\partial B_R} (v, n) \, ds.$$

It follows immediately that for volume preserving perturbations $\dot{\mathcal{E}}(0) = 0$, in accordance with Theorem 1. The monotonicity of $\mathcal{E}(t)$ with respect to nearly circular domain changes if $\alpha \geq -(N+1)/R$ or if $\alpha \leq -(N+1)/R$.

Next we want to find out if for volume preserving perturbations the ball is a local maximum or minimum. For this we need the second variation.

### 3.2.2 Second Variation for Balls and Divergence Free Vector Fields

In order to make the computation more transparent we introduce some abbreviations.

$$\dot{w} =: \frac{\partial w}{\partial t}, \qquad \operatorname{div} y(x) := \frac{\partial y_k}{\partial x_k}(x),$$

$$\nabla u \cdot D_v = \frac{\partial u}{\partial x_i} \frac{\partial v_k}{\partial x_i} \quad \text{thus} \quad \nabla u \cdot D_v \cdot X = \frac{\partial u}{\partial x_i} \frac{\partial v_k}{\partial x_i} X_k \quad \forall X \in \mathbb{R}^N,$$

$$\nabla u \cdot D_v^2 = \frac{\partial u}{\partial x_i}\frac{\partial v_k}{\partial x_i}\frac{\partial v_j}{\partial x_k} \quad \text{thus} \quad \nabla u \cdot D_v^2 \cdot X = \frac{\partial u}{\partial x_i}\frac{\partial v_k}{\partial x_i}\frac{\partial v_j}{\partial x_k} X_j \quad \forall X \in \mathbb{R}^N,$$

$$\text{trace } D_v^2 =: \sigma_{D_v^2}.$$

Observe that the definition of $\sigma_{D_v^2}$ differs slightly from those of $\sigma_A$ and $\sigma_B$. From the Euler-Lagrange equation we deduce that, taking into account that $\dot{J}(0) = 0$,

$$\ddot{\mathcal{E}}(0) = -\int_{B_1} \big(\ddot{u} + u\ddot{J}(0)\big)\, dx.$$

In order to evaluate this integral we need an equation for $\dot{u}$ and $\ddot{u}$. For that we differentiate (14) and (15) with respect to $t$. After each differentiation we set $t = 0$. This gives

$$\dot{L_u} J(0) + L_u \dot{J}(0) = \frac{\partial}{\partial x_k}\left(\dot{L_{p_i}} J(0)\frac{\partial x_k}{\partial \theta_i} + L_{p_i}\dot{J}(0)\frac{\partial x_k}{\partial \theta_i} + L_{p_i} J(0)\frac{\partial \dot{x}_k}{\partial \theta_i}\right)$$

and

$$\begin{aligned}
&\ddot{L_u} J(0) + 2\dot{L_u}\dot{J}(0) + L_u \ddot{J}(0) \\
&\quad = \frac{\partial}{\partial x_k}\bigg(\ddot{L_{p_i}} J(0)\frac{\partial x_k}{\partial \theta_i} + 2\dot{L_{p_i}}\dot{J}(0)\frac{\partial x_k}{\partial \theta_i} + 2\dot{L_{p_i}} J(0)\frac{\partial \dot{x}_k}{\partial \theta_i} \\
&\qquad + L_{p_i}\ddot{J}(0)\frac{\partial x_k}{\partial \theta_i} + 2L_{p_i}\dot{J}(0)\frac{\partial \dot{x}_k}{\partial \theta_i} + L_{p_i} J(0)\frac{\partial \ddot{x}_k}{\partial \theta_i}\bigg)
\end{aligned}$$

in $B_1$. For $t = 0$ and divergence free vector fields we have (see also (4) and (5))

$$\begin{aligned}
&J(0) = 1, \qquad \dot{J}(0) = \operatorname{div} v = 0, \qquad \ddot{J}(0) = -\sigma_{D_v^2}, \\
&\frac{\partial x_k}{\partial \theta_i} = \delta_{ik}, \qquad \frac{\partial \dot{x}_k}{\partial \theta_i} = -\frac{\partial v_k}{\partial x_i}, \qquad \frac{\partial \ddot{x}_k}{\partial \theta_i} = 2D_v^2.
\end{aligned}$$

Moreover

$$L = \frac{1}{2}|\nabla u|^2 - u, \qquad L_u = -1, \qquad L_{p_i} = p_i.$$

Thus we obtain an equation for $\dot{u}$ and $\ddot{u}$ in $B_1$.

$$0 = \operatorname{div}(\nabla \dot{u} - \nabla u \cdot D_v), \tag{19}$$

$$\sigma_{D_v^2} = \operatorname{div}\big(\nabla \ddot{u} - 2\nabla \dot{u} \cdot D_v - \sigma_{D_v^2}\nabla u + 2\nabla u \cdot D_v^2\big). \tag{20}$$

For the boundary conditions we work similarly. For the case of Robin condition on $\partial B_1$, we consider the second equation in (15) on $\partial B_1$. After differentiation in $t = 0$ and taking $\dot{J}(0) = 0$ into account, we get

$$\dot{L_{p_i}} J(0)\frac{\partial x_k}{\partial \theta_i} n_k + L_{p_i} J(0)\frac{\partial \dot{x}_k}{\partial \theta_i} n_k + \dot{b_u}\sqrt{k} + b_u\dot{\sqrt{k}} = 0 \quad \text{in } \partial B_1,$$

and

$$\ddot{L}_{p_i} J(0) \frac{\partial x_k}{\partial \theta_i} n_k + 2\dot{L}_{p_i} J(0) \frac{\partial \dot{x}_k}{\partial \theta_i} n_k + L_{p_i} \ddot{J}(0) \frac{\partial x_k}{\partial \theta_i} n_k + L_{p_i} J(0) \frac{\partial \ddot{x}_k}{\partial \theta_i} n_k$$

$$+ \ddot{b}_u \sqrt{k} + 2\dot{b}_u \dot{\sqrt{k}} + b_u \ddot{\sqrt{k}} = 0 \quad \text{in } \partial B_1.$$

From (7) we have in $t = 0$

$$\sqrt{k} = 1, \qquad \dot{\sqrt{k}} = \sigma_A, \qquad \ddot{\sqrt{k}} = \nu = \sigma_B - 2\sigma_{A^2} + \sigma_A^2.$$

Moreover

$$b(u) = \frac{\alpha}{2} u^2, \qquad b_u = \alpha u.$$

From that we obtain the following Robin boundary conditions for $\dot{u}$ and $\ddot{u}$ on $\partial B_1$.

$$\frac{\partial \dot{u}}{\partial n} + \alpha \dot{u} = \nabla u \cdot D_v \cdot n - \alpha \sigma_A u, \tag{21}$$

$$\frac{\partial \ddot{u}}{\partial n} + \alpha \ddot{u} = 2\nabla \dot{u} \cdot D_v \cdot n + \sigma_{D_v^2} \frac{\partial u}{\partial n} - 2\nabla u \cdot D_v^2 \cdot n - 2\alpha \sigma_A \dot{u} - \alpha \nu u. \tag{22}$$

We first consider the equation for $\ddot{u}$ in $B_1$. We multiply it with $u$ and integrate over $B_1$. After integration by parts this gives

$$\begin{aligned} \int_{B_1} u \sigma_{D^2}\, dx = & \oint_{\partial B_1} \left\{ u \frac{\partial \ddot{u}}{\partial n} - \ddot{u} \frac{\partial u}{\partial n} \right\} ds - \int_{B_1} \ddot{u}\, dx \\ & - 2 \oint_{\partial B_1} u \nabla \dot{u} \cdot D_v \cdot n\, ds + 2 \int_{B_1} \nabla \dot{u} \cdot D_v \cdot \nabla u\, dx \\ & - \oint_{\partial B_1} u \frac{\partial u}{\partial n} \sigma_{D_v^2}\, ds + \int_{B_1} |\nabla u|^2 \sigma_{D_v^2}\, dx \\ & + 2 \oint_{\partial B_1} u \nabla u \cdot D_v^2 \cdot n\, ds - 2 \int_{B_1} \nabla u \cdot D_v^2 \cdot \nabla u\, dx. \end{aligned} \tag{23}$$

Next we make use of the boundary condition (21) for $\ddot{u}$ and obtain

$$\begin{aligned} \int_{B_1} u \sigma_{D_v^2}\, dx = & \oint_{\partial B_1} \left\{ u(-\alpha \ddot{u} - 2\alpha \sigma_A \dot{u} - \alpha \nu u) - \ddot{u} \frac{\partial u}{\partial n} \right\} ds - \int_{B_1} \ddot{u}\, dx \\ & + 2 \int_{B_1} \nabla \dot{u} \cdot D_v \cdot \nabla u\, dx + \int_{B_1} |\nabla u|^2 \sigma_{D_v^2}\, dx - 2 \int_{B_1} \nabla u \cdot D_v^2 \cdot \nabla u\, dx. \end{aligned}$$

We can simplify, since $\frac{\partial u}{\partial n} = -\alpha u$ on $\partial B_1$.

$$\int_{B_1} u \sigma_{D_v^2}\, dx = -\alpha \oint_{\partial B_1} u(2\sigma_A \dot{u} + \nu u)\, ds - \int_{B_1} \ddot{u}\, dx + 2 \int_{B_1} \nabla \dot{u} \cdot D_v \cdot \nabla u\, dx$$

$$+\int_{B_1} |\nabla u|^2 \sigma_{D_v^2}\,dx - 2\int_{B_1} \nabla u \cdot D_v^2 \cdot \nabla u\,dx.$$

After rearranging terms we obtain a formula for $\ddot{\mathcal{E}}(0)$ which does not depend on $\ddot{u}$ anymore (recall $\ddot{J}(0) = -\sigma_{D_v^2}$).

$$\ddot{\mathcal{E}}(0) = -\int_{B_1} \big(|\nabla u|^2 - 2u\big)\sigma_{D_v^2}\,dx + \alpha \oint_{\partial B_1} u(2\sigma_A \dot{u} + \nu u)\,ds \\ - 2\int_{B_1} \nabla \dot{u} \cdot D_v \cdot \nabla u\,dx + 2\int_{B_1} \nabla u \cdot D_v^2 \cdot \nabla u\,dx. \tag{24}$$

At this point it is convenient to use the explicitly known solution of the torsion problem with Robin boundary conditions on $B_1$. We have

$$u = \frac{1}{N}\left(\frac{1}{2} + \frac{1}{\alpha}\right) - \frac{r^2}{2N}. \tag{25}$$

Consequently

$$\nabla u = -\frac{x}{N}, \qquad \frac{\partial^2 u}{\partial x_i \partial x_k} = -\frac{\delta_{ik}}{N}. \tag{26}$$

In particular we can use this information in (19) and obtain $\Delta \dot{u} = 0$ in $B_1$. Then the third integral in (24) can be simplified. Partial integration gives

$$\begin{aligned} &2\int_{B_1} \nabla \dot{u} \cdot D_v \cdot \nabla u\,dx \\ &= 2\oint_{\partial B_1} (\nabla u, v)\frac{\partial \dot{u}}{\partial n}\,ds - 2\int_{B_1} \left\{\frac{\partial^2 u}{\partial x_i \partial x_k} v_k \dot{u}_{x_i} + (\nabla u, v)\Delta \dot{u}\right\} dx \\ &= -\frac{2}{N}\oint_{\partial B_1} (v, n)\frac{\partial \dot{u}}{\partial n}\,ds + \frac{2}{N}\int_{B_1} v \cdot \nabla \dot{u}\,dx \\ &= -\frac{2}{N}\oint_{\partial B_1} (v, n)\left\{\frac{\partial \dot{u}}{\partial n} - \dot{u}\right\} ds. \end{aligned}$$

Introducing this expression into (24) we obtain

$$\ddot{\mathcal{E}}(0) = -\int_{B_1} \big(|\nabla u|^2 - 2u\big)\sigma_{D_v^2}\,dx + \alpha \oint_{\partial B_1} u(2\sigma_A \dot{u} + \nu u)\,ds \\ + \frac{2}{N}\oint_{\partial B_1} (v, n)\left\{\frac{\partial \dot{u}}{\partial n} - \dot{u}\right\} ds + 2\int_{B_1} \nabla u \cdot D_v^2 \cdot \nabla u\,dx. \tag{27}$$

If we replace $u$ and $\nabla u$ by (25) and (26) and use the abbreviation $g = (v, n)|_{\partial B_1}$ (cf. (10)) we find

$$\ddot{\mathcal{E}}(0) = N^{-1}\left(1 + \frac{2}{\alpha}\right)\int_{B_1} \sigma_{D_v^2}\,dx - \frac{N-1}{N^2}\int_{B_1} |x|^2\sigma_{D_v^2}\,dx + \frac{2}{N^2}\int_{B_1} xD_v^2 x\,dx$$
$$+ \frac{1}{\alpha N^2}\oint_{\partial B_1} \nu\,ds + \frac{2}{N}\oint_{\partial B_1}\left[\sigma_A\dot{u} - g\dot{u} + g\frac{\partial \dot{u}}{\partial n}\right]ds. \tag{28}$$

The explicit formulas for $\sigma_A$ and $\nu$ are given in Sect. 2.3.1 and $\sigma_{D_v^2} = \text{trace}\, D_v^2$. In view if (21) the term $\frac{\partial \dot{u}}{\partial n}$ on $\partial B_1$ can be substituted by

$$\frac{\partial \dot{u}}{\partial n} = -\frac{x}{N}D_v x - \frac{\sigma_A}{N} - \alpha\dot{u}.$$

From this computation it is not clear if $\ddot{\mathcal{E}}(0)$ has constant sign. The normal displacement $g : \partial B_1 \to \mathbb{R}$ necessarily needs to satisfy the compatibility condition

$$\oint_{\partial B_1} g(\xi)\,ds = 0.$$

Moreover, for simply connected domains, it is not restrictive to set

$$v(x) = \nabla\phi(x), \quad x \in B_1.$$

Necessarily

$$\Delta\phi = 0 \quad \text{in } B_1, \qquad \frac{\partial\phi}{\partial n} = g \quad \text{in } \partial B_1.$$

In this case we have $\sigma_{D_v^2} = \phi_{x_j x_i}\phi_{x_j x_i} > 0$. Thus the contribution of the volume integrals in (28) is positive.

# 4 Open Problems

**Problem 1** Let $B \subset \Omega$. Prove or disprove that for the torsion problem with Robin boundary conditions $\mathcal{E}(\Omega) \le \mathcal{E}(B)$?

**Problem 2** Let $\Omega$ be convex and $\Omega_t \supset \Omega$. Prove or disprove that $\dot{\mathcal{E}}(0) \le 0$.

**Problem 3** Prove the existence of an optimal domain with given volume for an energy with a boundary integral. Once the existence is established a symmetry argument leads to the conjecture.

**Problem 4** (Conjecture) Among all Lipschitz domains of given volume the ball yields the minimum of $\mathcal{E}$ given in (1) and (2). This conjecture is supported by the

Faber-Krahn inequality for the first membrane eigenvalue with Robin boundary conditions [1].

**Problem 5** Give conditions on the data which justify the formal computations. More precisely under what conditions are the solutions of the Euler-Lagrange (14) with the boundary conditions (15) differentiable in $t$?

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# The Asymptotic Shape of a Boundary Layer of Symmetric Willmore Surfaces of Revolution

**Hans-Christoph Grunau**

*Dedicated to the memory of Prof. Wolfgang Walter, who uncovered so many deep insights into Analysis*

**Abstract** We consider the Willmore boundary value problem for surfaces of revolution over the interval $[-1, 1]$ where, as Dirichlet boundary conditions, any symmetric set of position $\alpha$ and angle $\arctan\beta$ may be prescribed. Energy minimising solutions $u_{\alpha,\beta}$ have been previously constructed and for fixed $\beta \in \mathbb{R}$, the limit $\lim_{\alpha\searrow 0} u_{\alpha,\beta}(x) = \sqrt{1-x^2}$ has been proved locally uniformly in $(-1, 1)$, irrespective of the boundary angle. Subject of the present note is to study the asymptotic behaviour for fixed $\beta \in \mathbb{R}$ and $\alpha \searrow 0$ in a boundary layer of width $k\alpha$, $k > 0$ fixed, close to $\pm 1$. After rescaling $x \mapsto \frac{1}{\alpha} u_{\alpha,\beta}(\alpha(x-1)+1)$ one has convergence to a suitably chosen cosh on $[1-k, 1]$.

**Keywords** Dirichlet boundary conditions · Willmore surfaces of revolution · Asymptotic shape · Boundary layer

**Mathematics Subject Classification** 49Q10 · 53C42 · 35J65 · 34L30

## 1 Introduction

Recently, the Willmore functional has attracted a lot of attention. For a smooth surface $\Gamma \subset \mathbb{R}^3$ we define it by

$$\mathcal{W}(\Gamma) := \int_\Gamma \left(H^2 - K\right) dS = \frac{1}{4}\int_\Gamma (\kappa_1 - \kappa_2)^2 \, dS,$$

where $\kappa_1, \kappa_2$ denote the principal curvatures, $H = (\kappa_1 + \kappa_2)/2$ the mean curvature and $K$ the Gaussian curvature of $\Gamma$. Apart from being of geometric interest [18, 19], the functional $\mathcal{W}$ and its variants are models for the elastic energy of thin shells [13] or biological membranes [8, 14]. Furthermore, they are used in image processing for

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problems of surface restoration and image inpainting [4]. In these applications one is usually concerned with minima, or more generally with critical points of the Willmore functional. It is well-known since Thomsen's work [18] that the corresponding surface $\Gamma$ has to satisfy the Willmore equation

$$\Delta_\Gamma H + 2H\big(H^2 - K\big) = 0 \quad \text{on } \Gamma, \tag{1}$$

where $\Delta_\Gamma$ denotes the Laplace–Beltrami operator on $\Gamma$ with respect to the induced metric. A solution of (1) is called a Willmore surface. Moreover, from the geometric point of view an essential property of the Willmore functional is its conformal invariance. This means that $\mathcal{W}(\Gamma) = \mathcal{W}(\Phi \circ \Gamma)$ for any Möbius transform $\Phi$ of $\mathbb{R}^3$ being regular on $\Gamma$. For an easily accessible derivation of these facts one may also see [7].

A particular difficulty in the analytical investigation of (1) arises from the fact that $\Delta_\Gamma$ depends on the unknown surface so that the equation is highly nonlinear. Moreover, it is of fourth order where many of the established techniques do not apply. Existence of closed Willmore surfaces of prescribed genus has been proved by Simon [17] and by Bauer and Kuwert [1]. Recently, Rivière [15] has developed a different approach which seems to open opportunities to address many further questions. For more detailed information and further references we refer to [6].

If one is interested in surfaces with boundaries, then appropriate boundary conditions have to be added to (1). Since this equation is of fourth order one requires two sets of conditions; a discussion of possible choices can be found in [13]. Of particular interest is the Dirichlet problem where at its boundary, the position and the direction of the unknown Willmore surface are prescribed. Existence results for the Dirichlet problem, which are not subject to unnatural smallness conditions, can be found e.g. in [5, 6, 16]. The result by Schätzle [16] is put into a very general context and so does not provide very detailed information about the topological and geometrical shape of the solutions. In [5, 6] the authors proceed just the other way round: They confine themselves to symmetric surfaces of revolution but at the same time they obtain rather precise information on the geometric shape of their energy minimising solutions.

More precisely, they look at surfaces of revolution, which are obtained by rotating a graph over the $x = x_1$-axis in $\mathbb{R}^3$ around the $x_1$-axis. These are described by a sufficiently smooth function

$$u : [-1, 1] \to (0, \infty),$$

which is moreover restricted to be even about $x = 0$, and are parametrised as follows:

$$(x, \varphi) \mapsto \big(x, u(x)\cos\varphi, u(x)\sin\varphi\big), \quad x \in [-1, 1],\ \varphi \in [0, 2\pi].$$

In the present chapter we consider the Willmore problem under Dirichlet boundary conditions, where the height $u(\pm 1) = \alpha > 0$ and the slope $u'(-1) = -u'(1) = \beta \in \mathbb{R}$ are symmetrically prescribed at the boundary. The focus will be on the asymptotic behaviour of energy minimising solutions as $\alpha \searrow 0$. However, in order to explain this one needs to recall first a bit of the underlying existence theory.

## *1.1 Some Basics*

We consider the Willmore energy of the surface of revolution $\Gamma(u)$ generated by the graph of the smooth positive function $u : [-1, 1] \to (0, \infty)$

$$\begin{aligned}\mathcal{W}(u) &= \int_{\Gamma(u)} \big(H^2 - K\big)\, dS \\ &= \frac{\pi}{2} \int_{-1}^{1} \left( \frac{u''(x)}{(1+u'(x)^2)^{3/2}} - \frac{1}{u(x)\sqrt{1+u'(x)^2}} \right)^2 u(x)\sqrt{1+u'(x)^2}\, dx \\ &\quad + 2\pi \int_{-1}^{1} \frac{u''(x)}{(1+u'(x)^2)^{3/2}}\, dx.\end{aligned}$$

**Definition 1** For $\alpha > 0$ and $\beta \in \mathbb{R}$ we introduce the function space

$$\begin{aligned}N_{\alpha,\beta} := \big\{ u \in C^{1,1}\big([-1, 1], (0, \infty)\big) \,:\, &u \text{ positive, symmetric,} \\ &u(1) = \alpha \text{ and } u'(-1) = \beta \big\}\end{aligned}$$

as well as

$$M_{\alpha,\beta} := \inf \big\{ \mathcal{W}(u) \,:\, u \in N_{\alpha,\beta} \big\}.$$

This notation here should not be mixed with that in [6]. We also need the following number

$$\alpha^* = \min \left\{ \frac{\cosh(b)}{b} : b > 0 \right\} = 1.5088795\ldots.$$

For $\alpha$ below $\alpha^*$ there is no catenary satisfying this boundary condition, irrespective of the prescribed slope at the boundary. In this regime—for $\beta < 0$—the existence proof and also the qualitative properties of solutions are different.

We recall as a special case from [6] the following existence result: For each $\alpha \in (0, \alpha^*)$ and each $\beta \in \mathbb{R}$ we find $u_{\alpha,\beta} \in N_{\alpha,\beta}$ satisfying

$$\mathcal{W}(u_{\alpha,\beta}) = M_{\alpha,\beta}.$$

The corresponding surface of revolution $\Gamma(u_{\alpha,\beta}) \subset \mathbb{R}^3$ is smooth and solves the Dirichlet problem for the Willmore equation

$$\begin{cases} \Delta_\Gamma H + 2H\big(H^2 - K\big) = 0 & \text{in } (-1, 1), \\ u_{\alpha,\beta}(-1) = u_{\alpha,\beta}(+1) = \alpha, & u'_{\alpha,\beta}(-1) = -u'_{\alpha,\beta}(+1) = \beta. \end{cases}$$

In [5, 6] the authors took advantage of looking at the Willmore energy of surfaces of revolution from a different point of view. It was observed by Bryant, Griffiths, and Pinkall (see [2, 3, 9]) and intensively exploited among others by Langer and Singer [11, 12] that the Willmore energy and the elastic energy of the profile curve considered in the hyperbolic half plane coincide up to a factor. The half-plane $\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is equipped with the hyperbolic metric $ds_h^2 := \frac{1}{y^2}(dx^2 + dy^2)$. As explained in detail in [6, Sect. 2.2] one finds for the hyperbolic curvature of the curve $x \mapsto (x, u(x))$

$$\begin{aligned}\kappa_h(x) &= -\frac{u(x)^2}{u'(x)}\frac{d}{dx}\left(\frac{1}{u(x)\sqrt{1+u'(x)^2}}\right) = \frac{u(x)u''(x)}{(1+u'(x)^2)^{3/2}} + \frac{1}{\sqrt{1+u'(x)^2}}\\ &= \pm u(x)\big(\kappa_1(x) - \kappa_2(x)\big).\end{aligned}$$

The hyperbolic Willmore energy is defined in the following natural way and one observes that $\kappa_h^2 = 4u^2(H^2 - K)$ and obtains so the following simple relation with the original energy:

$$\mathcal{W}_h(u) := \int_\gamma \kappa_h^2\, ds_h := \int_{-1}^{1} \kappa_h^2 \frac{\sqrt{1+u'^2}}{u}\, dx = \frac{2}{\pi}\mathcal{W}(u). \tag{2}$$

## 1.2 The Asymptotic Result

The previous work [6] also contains some asymptotic considerations. It should be mentioned that the numerically calculated pictures displayed there give the clear idea that for $\alpha \searrow 0$, the central part of any Willmore minimiser $u_{\alpha,\beta}$ looks pretty much like a sphere while close to the boundary the graph resembles a catenary. Combinations of these prototype functions were not only employed as initial data for the numerical flow method but were also used as comparison functions for precise estimates of the optimal Willmore energy, see [6, Sect. 5.1, Theorem 5.4]. Moreover, in [6, Theorem 5.8] it was proved for fixed $\beta \in \mathbb{R}$ and $\alpha \searrow 0$ that $u_{\alpha,\beta}(x) \to \sqrt{1-x^2}$ in $C^m([-1+\delta, 1-\delta])$ for any $m \in \mathbb{N}_0$ and $\delta > 0$. A related result under so called natural boundary conditions was proved by Jachalski [10].

It remains to study the asymptotic behaviour in boundary layers close to $x = \pm 1$. To this end it will be crucial to have the following comparison function which generates a minimal surface of revolution:

$$v_{\alpha,\beta}(x) := \frac{\alpha}{\sqrt{1+\beta^2}}\cosh\left(\frac{\sqrt{1+\beta^2}}{\alpha}(1-x) + \operatorname{arsinh}(\beta)\right).$$

We prove the following result:

**Theorem 1** *Fix some $\beta \in \mathbb{R}$ and $k > 0$. For $\alpha > 0$ small enough let $u_{\alpha,\beta} \in N_{\alpha,\beta}$ minimise the Willmore energy in this class, i.e. $\mathcal{W}(u_{\alpha,\beta}) = M_{\alpha,\beta}$. Then we have uniform smooth convergence*

$$\lim_{\alpha \searrow 0} \frac{1}{\alpha} u_{\alpha,\beta}\big(\alpha(x-1)+1\big) = v_{1,\beta}(x).$$

*on* $[1-k, 1]$.

This means that in this sense

$$u_{\alpha,\beta}(x) \approx v_{\alpha,\beta}(x) \quad \text{for } x \in [1-k\alpha, 1]$$

for $\alpha \searrow 0$ while a careful analysis of the proof in [6] indicates that for any $\varepsilon > 0$ one may expect that

$$u_{\alpha,\beta}(x) \approx \sqrt{1-x^2} \quad \text{for } |x| \in \left[0, 1-\alpha^{1-\varepsilon}\right].$$

## 2 Rescaled Convergence to a Suitable cosh for $\alpha \searrow 0$

In this section, we choose any $\beta \in \mathbb{R}$, keep it fixed and study the singular limit $\alpha \searrow 0$, where the "holes" $\{\pm 1\} \times B_\alpha(0)$ in the cylindrical surfaces of revolution disappear.

### 2.1 Known Properties of Minimisers

We first recall for $\alpha$ small from [6, Sect. 5] the following properties of any minimiser $u_{\alpha,\beta} \in N_{\alpha,\beta}$ of $\mathcal{W}$, i.e. $\mathcal{W}(u_{\alpha,\beta}) = M_{\alpha,\beta}$.

**Lemma 1** *We assume that* $\alpha < \min\{\alpha^*, 1/|\beta|\}$. *Let* $u \in N_{\alpha,\beta}$ *be such that* $\mathcal{W}(u) = M_{\alpha,\beta}$. *Then,* $u \in C^\infty([-1,1],(0,\infty))$ *and* $u$ *has the following additional properties*:

1. *If* $\beta \geq 0$, *then* $u' < 0$ *in* $(0,1)$ *and*

$$\alpha \leq u(x) \leq \sqrt{1+\alpha^2-x^2} \quad \textit{in } [-1,1], \qquad x + u(x)u'(x) > 0 \quad \textit{in } (0,1).$$

2. *If* $\beta < 0$, *then* $u$ *has at most one critical point in* $(0,1)$, *i.e. there exists* $x_0 \in [0,1)$ *such that* $u' > 0$ *in* $(x_0, 1]$, $u'(x_0) = 0$ *and* $u' < 0$ *in* $(0, x_0)$. *Moreover,*

$$x + u(x)u'(x) > 0 \quad \textit{in } (0,1], \qquad u'(x) \leq \gamma := \max\left\{-\beta, \alpha^*\right\} \quad \textit{in } [x_0, 1]$$

$$\textit{and} \quad u(x) \geq \min\left\{\frac{\alpha}{2\sqrt{1+\beta^2}}, \frac{\gamma}{2(e^C-1)}\right\} \quad \textit{in } [-1,1],$$

*with* $C = 6\gamma\sqrt{1+\gamma^2} > 0$. *Moreover,*

$$\lim_{\alpha \searrow 0} x_0 = \lim_{\alpha \searrow 0} x_0(\alpha) = 1.$$

**Lemma 2** *Keep some* $\beta \in \mathbb{R}$ *fixed. For* $\alpha > 0$ *small enough let* $\delta_\alpha > 0$ *be such that* $-u'_{\alpha,\beta}(1-\delta_\alpha)$ *is maximal. Then we know that*

$$\lim_{\alpha \searrow 0} \delta_\alpha = 0, \tag{3}$$

$$\lim_{\alpha\searrow 0}\left(-u'_{\alpha,\beta}(1-\delta_\alpha)\right)=\infty, \tag{4}$$

$$\lim_{\alpha\searrow 0} M_{\alpha,\beta}=\lim_{\alpha\searrow 0}\mathcal{W}(u_{\alpha,\beta})=4\pi-4\pi\frac{\beta}{\sqrt{1+\beta^2}}, \tag{5}$$

$$\lim_{\alpha\searrow 0}\int_0^{1-\delta_\alpha}\kappa_h[u_{\alpha,\beta}]^2\,ds_h[u_{\alpha,\beta}]=0. \tag{6}$$

*Proof* Statements (3) and (4) follow from [6, Lemma 5.3, Theorem 5.8]. For (6), see the proof of [6, Corollary 5.5]. According to [6, Theorem 5.4] and (2) we finally have for $\alpha\searrow 0$

$$8-\frac{8\beta}{\sqrt{1+\beta^2}}+o(1)=\mathcal{W}_h(u_{\alpha,\beta})=\frac{2}{\pi}\mathcal{W}(u_{\alpha,\beta});$$

statement (5) follows. □

## 2.2 Further Comparison Results

In order to guarantee compactness in our limit process we need some further uniform bounds.

We study first the simpler case $\beta\geq 0$.

**Lemma 3** *Fix some* $\beta\geq 0$. *For* $0<\alpha<\min\{\alpha^*,1/|\beta|\}$ *we have for any Willmore minimiser* $u_{\alpha,\beta}\in N_{\alpha,\beta}$ *that*

$$u_{\alpha,\beta}(x)<v_{\alpha,\beta}(x)\quad \textit{for } x\in[0,1).$$

*Proof* Since both $u_{\alpha,\beta}$ and $v_{\alpha,\beta}$ are strictly decreasing on $[0,1]$ they may be considered as graphs over the angular variable. This means that for each $x\in[0,1]$ we find uniquely determined $\varphi,\psi\in[0,\pi/2]$ and $r_1(\varphi)$, $r_2(\psi)$ such that

$$\left(x,u_{\alpha,\beta}(x)\right)=r_1(\varphi)(\cos\varphi,\sin\varphi),\qquad \left(x,v_{\alpha,\beta}(x)\right)=r_2(\psi)(\cos\psi,\sin\psi).$$

Considering the curves $\varphi\mapsto r_j(\varphi)(\cos\varphi,\sin\varphi)$ let $T_j(\varphi)=(t_j^1(\varphi),t_j^2(\varphi))$ denote the corresponding unit tangent vectors with $t_j^2(\varphi)\leq 0$.

Let us assume by contradiction that $u_{\alpha,\beta}>v_{\alpha,\beta}$ on some subinterval of $[0,1]$. The case where the graphs touch tangentially in some point is simpler and can be treated similarly. Then we find $0<\varphi_1<\varphi_2$ such that

$$0>\frac{t_2^2(\varphi_1)}{t_2^1(\varphi_1)}>\frac{t_1^2(\varphi_1)}{t_1^1(\varphi_1)}\quad\text{and}\quad 0>\frac{t_1^2(\varphi_2)}{t_1^1(\varphi_2)}>\frac{t_2^2(\varphi_2)}{t_2^1(\varphi_2)}.$$

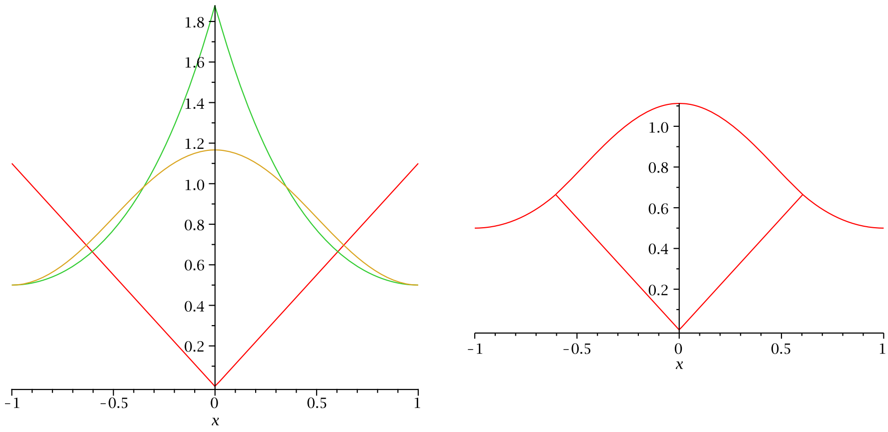


**Fig. 1** *Left*: Assume that the minimiser is somewhere above the cosh. *Right*: Rescale minimiser inside the wedge and fit it into the cosh. This results in a decrease of Willmore energy

By the intermediate value theorem there exists a $\varphi_0 \in (\varphi_1, \varphi_2)$ satisfying

$$\frac{t_2^2(\varphi_0)}{t_2^1(\varphi_0)} = \frac{t_1^2(\varphi_0)}{t_1^1(\varphi_0)}.$$

Hence $T_1(\varphi_0) = T_2(\varphi_0)$, the tangents on the ray with angle $\varphi_0$ from the $x$-axis coincide.

We may now construct a new even function $\hat{u}_{\alpha,\beta} \in N_{\alpha,\beta}$ which coincides with the catenary $v_{\alpha,\beta}$ on $[r_2(\varphi_0)\cos\varphi_0, 1]$ and with

$$x \mapsto \frac{r_2(\varphi_0)}{r_1(\varphi_0)} u_{\alpha,\beta}\left(\frac{r_1(\varphi_0)}{r_2(\varphi_0)} x\right)$$

on $[0, r_2(\varphi_0)\cos\varphi_0]$. We emphasise that the Willmore energy is scaling invariant. Since $u_{\alpha,\beta}$ is nowhere locally equal to a cosh, we would end up with $\mathcal{W}(\hat{u}_{\alpha,\beta}) < \mathcal{W}(u_{\alpha,\beta})$, a contradiction.

See Fig. 1. □

**Lemma 4** *Fix some* $\beta \geq 0$, $k \in \mathbb{N}$. *For* $0 < \alpha < \min\{\alpha^*, 1/|\beta|, 1/k\}$ *we have for any Willmore minimiser* $u_{\alpha,\beta} \in N_{\alpha,\beta}$ *that*

$$|u'_{\alpha,\beta}(x)| \leq \sinh\bigl(k\sqrt{1+\beta^2} + \operatorname{arsinh}(\beta)\bigr) \quad \textit{for } x \in [1 - k\alpha, 1].$$

*Proof* We proceed similarly as in the proof of Lemma 3 and consider rays which intersect $[0, 1] \ni x \mapsto (x, u_{\alpha,\beta}(x))$ and $[0, 1] \ni x \mapsto (x, v_{\alpha,\beta}(x))$. Using the same argument as before—cf. Fig. 2—we see that on each ray, the slope of $u_{\alpha,\beta}$ is less negative than the slope of $v_{\alpha,\beta}$. Since $v_{\alpha,\beta}(x) \geq u_{\alpha,\beta}(x)$, we find that the rays, which

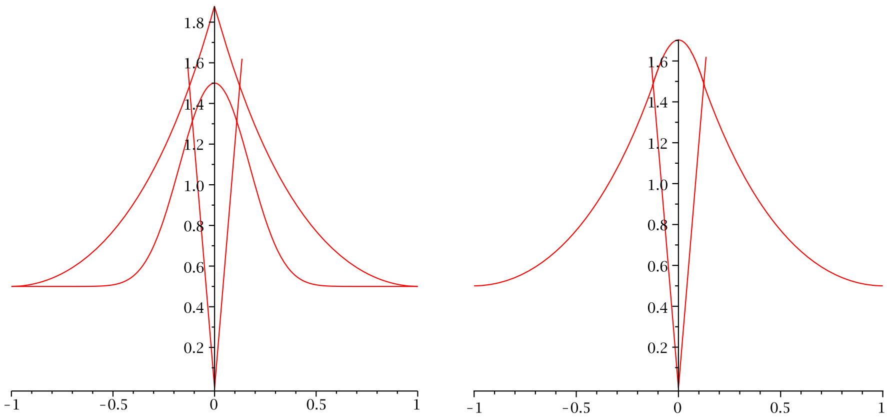


**Fig. 2** *Left*: Assume that the minimiser is on some ray steeper than the cosh. *Right*: Rescale minimiser inside the wedge and fit it into the cosh. This results in a decrease of Willmore energy

cover $(x, u_{\alpha,\beta}(x))$ for $x \in [1-k\alpha, 1]$, cover $(x, v_{\alpha,\beta}(x))$ with $x$ in a subinterval of $[1-k\alpha, 1]$:

$$\max_{x\in[1-k\alpha,1]}\left|u'_{\alpha,\beta}(x)\right| \le \max_{x\in[1-k\alpha,1]}\left|v'_{\alpha,\beta}(x)\right| \le \sinh\bigl(k\sqrt{1+\beta^2}+\operatorname{arsinh}(\beta)\bigr). \qquad \square$$

Combining the previous results with the statements from Lemma 1 we can also treat the case $\beta < 0$.

**Lemma 5** *Fix some* $\beta < 0$, *then there exists a constant* $C = C(\beta) > 0$ *such that for all* $0 < \alpha < \min\{\alpha^*, 1/|\beta|\}$ *we have for any Willmore minimiser* $u_{\alpha,\beta} \in N_{\alpha,\beta}$ *that*

$$\frac{\alpha}{C} \le u_{\alpha,\beta}(x) < \alpha\cosh\left(\frac{C}{\alpha}(1-x)\right) \quad \text{for } x \in [0,1).$$

*Proof* Let $x_0 \in [0,1)$ be as mentioned in Lemma 1, i.e. $u_{\alpha,\beta}(x_0) = \min\{u_{\alpha,\beta}(x) : x \in [-1,1]\} =: u_{\min,\alpha}$. We take further from this lemma that there exists a constant $C = C(\beta) > 0$ such that for all $0 < \alpha < \min\{\alpha^*, 1/|\beta|\}$

$$u_{\min,\alpha} \ge \frac{\alpha}{C}.$$

Then, according to Lemma 3 we have for $x \in [0, x_0)$:

$$\begin{aligned} u_{\alpha,\beta}(x) &< u_{\min,\alpha}\cosh\left(\frac{1}{u_{\min,\alpha}}(x_0-x)\right) \\ &\le \alpha\cosh\left(\frac{1}{u_{\min,\alpha}}(1-x)\right) \le \alpha\cosh\left(\frac{C}{\alpha}(1-x)\right). \end{aligned}$$

Since $u_{\alpha,\beta}(x) < \alpha \le \alpha\cosh(\frac{C}{\alpha}(1-x))$ on $[x_0, 1)$, the proof is complete. $\square$

**Lemma 6** *Fix some $\beta < 0$, $k \in \mathbb{N}$. There exists a bound $C = C(k, \beta)$ such that for all $\alpha > 0$ small enough we have for any Willmore minimiser $u_{\alpha,\beta} \in N_{\alpha,\beta}$ that*

$$\left|u'_{\alpha,\beta}(x)\right| \leq C \quad \text{for } x \in [1 - k\alpha, 1].$$

*Proof* We proceed similarly as in the previous Lemma 5. Let $x_0$, $u_{\min,\alpha}$ be as there and let $C_1 = C_1(\beta) > 0$ be the constant used there. In particular we use that $u_{\min,\alpha} \geq \frac{\alpha}{C_1}$ From Lemma 4 we see that there exists a constant $C_2 = C_2(k, \beta) > 0$ such that for $\alpha > 0$ small enough

$$\left|u'_{\alpha,\beta}(x)\right| \leq C_2 \quad \text{on } [x_0 - kC_1 u_{\min,\alpha}, x_0] \supset [x_0 - k\alpha, x_0].$$

One should observe that $\lim_{\alpha \searrow 0} x_0(\alpha) = 1$. According to Lemma 1 we have that

$$0 \leq u'_{\alpha,\beta}(x) \leq \max\left\{-\beta, \alpha^*\right\} \quad \text{in } [x_0, 1].$$

Putting all together proves the claim. □

**Corollary 1** *For any $k \in \mathbb{N}$ we have that for $\alpha > 0$ small enough*

$$\delta_\alpha > k\alpha.$$

## *2.3 Concentration of the Willmore Energy*

According to Lemma 2, for $\alpha \searrow 0$, the hyperbolic Willmore energy concentrates close to $\pm 1$. The following lemma shows the reverse result for $\int H^2\, dS$.

**Lemma 7** *Fix some $\beta \in \mathbb{R}$, $k \in \mathbb{N}$. Let $u_{\alpha,\beta} \in N_{\alpha,\beta}$ be any Willmore minimiser and let $H_{\alpha,\beta}$ denote its mean curvature. Then*

$$\lim_{\alpha \searrow 0} \int_{1-k\alpha}^{1} H^2_{\alpha,\beta} u_{\alpha,\beta} \sqrt{1 + \left(u'_{\alpha,\beta}\right)^2}\, dx = 0.$$

*Proof* According to Lemma 2, we have for $\alpha \searrow 0$:

$$\begin{aligned}
4\pi + o(1) &= \mathcal{W}(u_{\alpha,\beta}) + 4\pi \frac{\beta}{\sqrt{1+\beta^2}} \\
&= \mathcal{W}(u_{\alpha,\beta}) + 2\pi \int_{-1}^{1} K[u_{\alpha,\beta}] u_{\alpha,\beta} \sqrt{1 + \left(u'_{\alpha,\beta}\right)^2}\, dx \\
&= 4\pi \int_{1-\delta_\alpha}^{1} H^2_{\alpha,\beta} u_{\alpha,\beta} \sqrt{1 + \left(u'_{\alpha,\beta}\right)^2}\, dx \\
&\quad + 4\pi \int_{0}^{1-\delta_\alpha} H^2_{\alpha,\beta} u_{\alpha,\beta} \sqrt{1 + \left(u'_{\alpha,\beta}\right)^2}\, dx
\end{aligned}$$

$$= 4\pi \int_{1-\delta_\alpha}^{1} H_{\alpha,\beta}^2 u_{\alpha,\beta} \sqrt{1+\left(u_{\alpha,\beta}'\right)^2}\, dx$$

$$+ \pi \int_0^{1-\delta_\alpha} \kappa_h[u_{\alpha,\beta}]^2\, ds_h[u_{\alpha,\beta}] + 4\pi \frac{|u_{\alpha,\beta}'(1-\delta_\alpha)|}{\sqrt{1+u_{\alpha,\beta}'(1-\delta_\alpha)^2}}$$

$$= 4\pi \int_{1-\delta_\alpha}^{1} H_{\alpha,\beta}^2 u_{\alpha,\beta} \sqrt{1+\left(u_{\alpha,\beta}'\right)^2}\, dx + o(1) + 4\pi + o(1).$$

This yields

$$\int_{1-\delta_\alpha}^{1} H_{\alpha,\beta}^2 u_{\alpha,\beta} \sqrt{1+\left(u_{\alpha,\beta}'\right)^2}\, dx = o(1)$$

which in view of Corollary 1 proves the claim. □

### *2.4 Limit of the Rescaled Solutions, Proof of Theorem 1*

We introduce the rescaled solutions

$$\hat{u}_{\alpha,\beta} := \frac{1}{\alpha} u_{\alpha,\beta}\big(\alpha(x-1)+1\big)$$

and keep some $k \in \mathbb{N}$ fixed in what follows. Lemmas 3–6 show that $(\hat{u}_{\alpha,\beta})_{\alpha\searrow 0}$ is uniformly bounded in $C^1([1-k, 1])$ and uniformly bounded from below on $[1-k, 1]$ while Lemma 7 proves that its mean curvature converges to 0 in $L^2([1-k, 1])$. By standard arguments (cf. [6, Proof of Theorem 5.8]) we find a strong $C^1$- and weak $H^2$-limit $u : [1-k, 1] \to (0, \infty)$ satisfying

$$u(1) = 1, \qquad u'(1) = -\beta, \qquad H[u](x) \equiv 0.$$

By direct integration this gives

$$u(x) = v_{1,\beta}(x) = \frac{1}{\sqrt{1+\beta^2}} \cosh\big(\sqrt{1+\beta^2}(1-x) + \operatorname{arsinh}(\beta)\big)$$

and so, the proof of Theorem 1. As for convergence in higher order norms one may see the proof of [6, Theorem 5.8].

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# A Computer-Assisted Uniqueness Proof for a Semilinear Elliptic Boundary Value Problem

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*Dedicated to the memory of Wolfgang Walter*

**Abstract** A wide variety of articles, starting with the famous paper (Gidas, Ni and Nirenberg in Commun. Math. Phys. 68, 209–243 (1979)), is devoted to the uniqueness question for the semilinear elliptic boundary value problem $-\Delta u = \lambda u + u^p$ in $\Omega$, $u > 0$ in $\Omega$, $u = 0$ on $\partial\Omega$, where $\lambda$ ranges between 0 and the first Dirichlet Laplacian eigenvalue. So far, this question was settled in the case of $\Omega$ being a ball and, for more general domains, in the case $\lambda = 0$. In (McKenna et al. in J. Differ. Equ. 247, 2140–2162 (2009)), we proposed a computer-assisted approach to this uniqueness question, which indeed provided a proof in the case $\Omega = (0, 1)^2$, and $p = 2$. Due to the high numerical complexity, we were not able in (McKenna et al. in J. Differ. Equ. 247, 2140–2162 (2009)) to treat higher values of $p$. Here, by a significant reduction of the complexity, we will prove uniqueness for the case $p = 3$.

**Keywords** Semilinear elliptic boundary value problem · Uniqueness · Computer-assisted proof

**Mathematics Subject Classification** 35J25 · 35J60 · 65N15

## 1 Introduction

The semilinear elliptic boundary value problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

has attracted a lot of attention since the 19th century. Questions of existence and multiplicity have been (and are still being) extensively studied by means of variational methods, fixed-point methods, sub- and supersolutions, index and degree theory, and more.

In this chapter, we will address the question of *uniqueness* of solutions for the more special problem

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$$\begin{cases} -\Delta u = \lambda u + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\lambda$ ranges between 0 and $\lambda_1(\Omega)$, the first eigenvalue of the Dirichlet Laplacian. It has been shown in a series of papers [1, 2, 19, 28, 29] that the solution of (2) is indeed unique when $\Omega$ is a ball, or when $\Omega$ is more general but $\lambda = 0$ [9, 10, 12, 30].

We will concentrate on the case where $\Omega = (0,1)^2$ and $p = 3$, and prove that uniqueness holds for the full range $[0, \lambda_1(\Omega))$ of $\lambda$. Thus, our paper constitutes the first uniqueness result for this situation. More precisely we prove

**Theorem 1** *Let $\Omega$ be the unit square in $\mathbb{R}^2$, $\Omega = (0,1)^2$. Then the problem*

$$\begin{cases} -\Delta u = \lambda u + u^3 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

*admits only one solution for any $\lambda \in [0, \lambda_1(\Omega))$.*

*Remark 1*

a) A simple scaling argument shows that our uniqueness result carries over to all squares $\Omega_l := (0,l)^2$ (and thus, to all squares in $\mathbb{R}^2$): If $u$ is a positive solution of $-\Delta u = \tilde{\lambda} u + u^3$ in $\Omega_l$, $u = 0$ on $\partial\Omega_l$, for some $\tilde{\lambda} \in [0, \lambda_1(\Omega_l))$, then $v(x,y) := lu(lx, ly)$ is a solution of (3) for $\lambda = \tilde{\lambda} l^2 \in [0, \lambda_1(\Omega))$.
b) Since we also show that the unique solution in the square is nondegenerate, by a result of [10] we deduce that the solution is unique also in domains “close to” a square.
c) Finally we observe that having shown in [16] (case $p = 2$) and in this paper (case $p = 3$) that the unique solution is nondegenerate then uniqueness follows also for other nonlinearities of the type $\lambda u + u^p$ for $p$ close to 2 and 3. Indeed, by standard arguments (see for example [9]) nonuniqueness of positive solutions in correspondence to sequences of exponents converging to 3 (resp. to 2) would imply degeneracy of the solution for $p = 3$ (resp. $p = 2$).

Our proof heavily relies on *computer-assistance*. Such computer-assisted proofs are receiving an increasing attention in the recent years since such methods provided results which apparently could not be obtained by purely analytical means (see [5, 6, 17, 18, 24]).

We compute a branch of approximate solutions and prove existence of a true solution branch close to it, using fixed point techniques. By eigenvalue enclosure methods, and an additional analytical argument for $\lambda$ close to $\lambda_1(\Omega)$ we deduce the non-degeneracy of all solutions along this branch, whence uniqueness follows from the known bifurcation structure of the problem.

In [16] we give a general description of these computer-assisted means and use them to obtain the desired uniqueness result for the case $\Omega = (0,1)^2$, $p = 2$. To make the present paper dealing with the case $p = 3$ more self-contained, we recall parts of the content of [16] here. We remark that the numerical tools used in [16] turned out not to be sufficient to treat the case $p = 3$. Now, by some new trick to reduce the numerical complexity, we are able to handle this case.

## 2 Preliminaries

In the following, let $\Omega = (0, 1)^2$. We remark that the results of this section can be carried over to the more general case of a "doubly symmetric" domain; see [16] for details.

First, note that problem (2) can equivalently be reformulated as finding a *non-trivial* solution of

$$\begin{cases} -\Delta u = \lambda u + |u|^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

since, for $\lambda < \lambda_1(\Omega)$, by the strong maximum principle (for $-\Delta - \lambda$) every non-trivial solution of (4) is positive in $\Omega$. In fact, this formulation is better suited for our computer-assisted approach than (2).

As a consequence of the classical bifurcation theorem of [25] and of the results of [9] the following result was obtained in [20]:

**Theorem 2** *All solutions $u_\lambda$ of* (2) *lie on a simple continuous curve $\Gamma$ in $[0, \lambda_1(\Omega)) \times C^{1,\alpha}(\bar{\Omega})$ joining $(\lambda_1(\Omega), 0)$ with $(0, u_0)$, where $u_0$ is the unique solution of* (2) *for $\lambda = 0$.*

We recall that the uniqueness of the solution of (2) for $\lambda = 0$ was proved in [10] and [9]. As a consequence of the previous theorem we have

**Corollary 1** *If all solutions on the curve $\Gamma$ are nondegenerate then problem* (2) *admits only one solution for every $\lambda \in [0, \lambda_1(\Omega))$.*

*Proof* The nondegeneracy of the solutions implies, by the Implicit Function Theorem, that neither turning points nor secondary bifurcations can exist along $\Gamma$. Then, for every $\lambda \in [0, \lambda_1(\Omega))$ there exists only one solution of (2) on $\Gamma$. By Theorem 2 all solutions are on $\Gamma$, hence uniqueness follows. □

Theorem 2 and Corollary 1 indicate that to prove the uniqueness of the solution of problem (2) for every $\lambda \in [0, \lambda_1(\Omega))$ it is enough to construct a branch of nondegenerate solutions which connects $(0, u_0)$ to $(\lambda_1(\Omega), 0)$. This is what we will do numerically in the next sections with a rigorous computer-assisted proof.

However, establishing the nondegeneracy of solutions $u_\lambda$ for $\lambda$ close to $\lambda_1(\Omega)$ numerically can be difficult, due to the fact that the only solution at $\lambda = \lambda_1(\Omega)$, which is the identically zero solution, is obviously degenerate because its linearized operator is $L_0 = -\Delta - \lambda_1$ which has the first eigenvalue equal to zero. The next proposition shows that there exists a computable number $\bar{\lambda}(\Omega) \in (0, \lambda_1(\Omega))$ such that for any $\lambda \in [\bar{\lambda}(\Omega), \lambda_1(\Omega))$ problem (2) has only one solution which is also non-degenerate. Of course, from the well-known results of Crandall and Rabinowitz, [7, 8], one can establish that for $\lambda$ "close to" $\lambda_1$, all solutions $u_\lambda$ are nondegenerate. However, in order to complete our program, we need to calculate a precise and *explicit* estimate of how close they need to be. This allows us to carry out the numerical computation only in the interval $[0, \bar{\lambda}(\Omega)]$ as we will do later.

Let us denote by $\lambda_1 = \lambda_1(\Omega)$ and $\lambda_2 = \lambda_2(\Omega)$ the first and second eigenvalue of the operator $-\Delta$ in $\Omega$ with homogeneous Dirichlet boundary conditions. We have

**Proposition 1** *If there exists* $\bar{\lambda} \in (0, \lambda_1)$ *and a solution* $u_{\bar{\lambda}}$ *of* (2) *with* $\lambda = \bar{\lambda}$ *such that*

$$\|u_{\bar{\lambda}}\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{\lambda_1}\right)^{\frac{1}{p-1}} \tag{5}$$

*then*

$$\|u_\lambda\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}}, \tag{6}$$

*and* $u_\lambda$ *is non-degenerate, for all solutions* $u_\lambda$ *of* (2) *belonging to the branch* $\Gamma_2 \subset \Gamma$ *which connects* $(\bar{\lambda}, u_{\bar{\lambda}})$ *to* $(\lambda_1, 0)$.

(Recall that $\Gamma$ is the unique continuous branch of solutions given by Theorem 2.)

*Proof* See [16]. □

**Corollary 2** *If on the branch* $\Gamma$ *there exists a solution* $u_{\bar{\lambda}}$, $\bar{\lambda} \in (0, \lambda_1)$ *such that*:

i) *on the sub-branch* $\Gamma_1$ *connecting* $(0, u_0)$ *with* $(\bar{\lambda}, u_{\bar{\lambda}})$ *all solutions are nondegenerate*

*and*

ii)
$$\|u_{\bar{\lambda}}\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{\lambda_1}\right)^{\frac{1}{p-1}}, \tag{7}$$

*then all solutions of* (2) *are nondegenerate, for all* $\lambda \in (0, \lambda_1)$, *and therefore problem* (2) *admits only one solution for every* $\lambda \in [0, \lambda_1(\Omega))$.

*Proof* We set $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1$ connecting $(0, u_0)$ to $(\bar{\lambda}, u_{\bar{\lambda}})$. On $\Gamma_1$ we have that all solutions are nondegenerate by i). On the other hand the hypothesis ii) allows us to apply Proposition 1 which shows nondegeneracy of all solutions on $\Gamma_2$. Hence there is nondegeneracy all along $\Gamma$ so the assertion follows from Corollary 1. □

The last corollary suggests the method of proving the uniqueness through computer assistance: first we construct a branch of nondegenerate "true" solutions near approximate ones in a certain interval $[0, \bar{\lambda}]$ and then verify ii) for the solution $u_{\bar{\lambda}}$. Note that the estimate (7) depends only on $p$ and on the eigenvalues $\lambda_1$ and $\lambda_2$ of the operator $-\Delta$ in the domain $\Omega$. So the constant on the right-hand side is easily computable. When $\Omega$ is the unit square which is the case analyzed in the next sections, the estimate (7) becomes:

$$\|u_{\bar{\lambda}}\|_\infty < \left(\frac{3\pi^2}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{2\pi^2}\right)^{\frac{1}{p-1}} = \left(\frac{3\bar{\lambda}}{2p}\right)^{\frac{1}{p-1}}$$

because $\lambda_1 = 2\pi^2$ and $\lambda_2 = 5\pi^2$.

Fixing $p = 3$ we finally get the condition

$$\|u_{\bar{\lambda}}\|_\infty < \sqrt{\frac{\bar{\lambda}}{2}}. \tag{8}$$

## 3 The Basic Existence and Enclosure Theorem

We start the computer-assisted part of our proof with a basic theorem on existence, local uniqueness, and non-degeneracy of solutions to problem (4), assuming $p = 3$ now for simplicity of presentation. In this section, the parameter $\lambda \in [0, \lambda_1(\Omega))$ is fixed.

Let $H_0^1(\Omega)$ be endowed with the inner product $\langle u, v\rangle_{H_0^1} := \langle \nabla u, \nabla v\rangle_{L^2} + \sigma\langle u, v\rangle_{L^2}$; actually we choose $\sigma = 1$ in this paper, but different (usually positive) choices of $\sigma$ are advantageous or even mandatory in other applications, whence we keep $\sigma$ as a parameter in the following. Let $H^{-1}(\Omega)$ denote the (topological) dual of $H_0^1(\Omega)$, endowed with the usual operator sup-norm.

Suppose that an *approximate* solution $\omega_\lambda \in H_0^1(\Omega)$ of problem (4) has been computed by numerical means, and that a bound $\delta_\lambda > 0$ for its *defect* is known, i.e.

$$\left\| -\Delta\omega_\lambda - \lambda\omega_\lambda - |\omega_\lambda|^3 \right\|_{H^{-1}} \le \delta_\lambda, \tag{9}$$

as well as a constant $K_\lambda$ such that

$$\|v\|_{H_0^1} \le K_\lambda \left\| L_{(\lambda,\omega_\lambda)}[v] \right\|_{H^{-1}} \quad \text{for all } v \in H_0^1(\Omega). \tag{10}$$

Here, $L_{(\lambda,\omega_\lambda)}$ denotes the operator *linearizing* problem (4) at $\omega_\lambda$; more generally, for $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$, let the linear operator $L_{(\lambda,u)} : H_0^1(\Omega) \to H^{-1}(\Omega)$ be defined by

$$L_{(\lambda,u)}[v] := -\Delta v - \lambda v - 3|u|uv \quad \big(v \in H_0^1(\Omega)\big). \tag{11}$$

The practical computation of bounds $\delta_\lambda$ and $K_\lambda$ will be addressed in Sects. 6, 7 and 8.

Let $C_4$ denote a norm bound (embedding constant) for the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, which is bounded since $\Omega \subset \mathbb{R}^2$. $C_4$ can be calculated e.g. according to the explicit formula given in [23, Lemma 2]. Finally, let

$$\gamma := 3C_4^3.$$

In our example case where $\Omega = (0, 1)^2$, the above-mentioned explicit formula gives (with the choice $\sigma := 1$)

$$\gamma = \frac{3\sqrt{2}}{4(\pi^2+1)^{3/4}} \left( < \frac{1}{5} \right).$$

**Theorem 3** *Suppose that some* $\alpha_\lambda > 0$ *exists such that*

$$\delta_\lambda \le \frac{\alpha_\lambda}{K_\lambda} - \gamma\alpha_\lambda^2\big(\|\omega_\lambda\|_{L^4} + C_4\alpha_\lambda\big) \tag{12}$$

*and*

$$2K_\lambda\gamma\alpha_\lambda\big(\|\omega_\lambda\|_{L^4} + C_4\alpha_\lambda\big) < 1. \tag{13}$$

*Then, the following statements hold true*:

a) (*existence*) *There exists a solution* $u_\lambda \in H_0^1(\Omega)$ *of problem* (4) *such that*

$$\|u_\lambda - \omega_\lambda\|_{H_0^1} \le \alpha_\lambda. \tag{14}$$

b) (*local uniqueness*) *Let* $\eta > 0$ *be chosen such that* (13) *holds with* $\alpha_\lambda + \eta$ *instead of* $\alpha_\lambda$. *Then,*

$$\left.\begin{array}{ll} u \in H_0^1(\Omega) & \textit{solution of } (4) \\ \|u - \omega_\lambda\|_{H_0^1} \le \alpha_\lambda + \eta & \end{array}\right\} \implies u = u_\lambda. \tag{15}$$

c) (*nondegeneracy*)

$$\left.\begin{array}{l} u \in H_0^1(\Omega) \\ \|u - \omega_\lambda\|_{H_0^1} \le \alpha_\lambda \end{array}\right\} \implies L_{(\lambda,u)} : H_0^1(\Omega) \to H^{-1}(\Omega) \quad \textit{is bijective}, \tag{16}$$

*whence in particular* $L_{(\lambda,u_\lambda)}$ *is bijective* (*by* (14)).

For a proof, see [16].

**Corollary 3** *Suppose that* (12) *and* (13) *hold, and in addition that* $\|\omega_\lambda\|_{H_0^1} > \alpha_\lambda$. *Then, the solution* $u_\lambda$ *given by Theorem* 3 *is non-trivial* (*and hence positive*).

*Remark 2*

a) The function $\psi(\alpha) := \frac{\alpha}{K_\lambda} - \gamma\alpha^2(\|\omega_\lambda\|_{L^4} + C_4\alpha)$ has obviously a positive maximum at $\bar{\alpha} = \frac{1}{3C_4}(\sqrt{\|\omega_\lambda\|_{L^4}^2 + \frac{3C_4}{K_\lambda\gamma}} - \|\omega_\lambda\|_{L^4})$, and the crucial condition (12) requires that

$$\delta_\lambda \leq \psi(\bar{\alpha}) = \frac{4C_4 + \gamma K_\lambda\|\omega_\lambda\|_{L^4}^2}{K_\lambda(\sqrt{\gamma K_\lambda(\gamma K_\lambda\|\omega_\lambda\|_{L^4}^2 + 3C_4)} + \gamma K_\lambda\|\omega_\lambda\|_{L^4})} \cdot \frac{1}{(\sqrt{\gamma K_\lambda(\gamma K_\lambda\|\omega_\lambda\|_{L^4}^2 + 3C_4)} + \gamma K_\lambda\|\omega_\lambda\|_{L^4} + 6C_4)}, \tag{17}$$

i.e. $\delta_\lambda$ has to be sufficiently small. According to (9), this means that $\omega_\lambda$ must be computed with sufficient accuracy, which leaves the "hard work" to the computer!

Furthermore, a "small" defect bound $\delta_\lambda$ allows (via (12)) a "small" error bound $\alpha_\lambda$, if $K_\lambda$ is not too large.

b) If moreover we choose the *minimal* $\alpha_\lambda$ satisfying (12), then the additional condition (13) follows automatically, which can be seen as follows: the minimal choice of $\alpha_\lambda$ shows that $\alpha_\lambda \leq \bar{\alpha}$. We have

$$\begin{aligned} &2K_\lambda\gamma\bar{\alpha}\big(\|\omega_\lambda\|_{L^4} + C_4\bar{\alpha}\big) \\ &= 1 - \frac{C_4}{3C_4 + 2\gamma K_\lambda\|\omega_\lambda\|_{L^4}^2 + 2\sqrt{\gamma K_\lambda(\gamma K_\lambda\|\omega_\lambda\|_{L^4}^2 + 3C_4)}\|\omega_\lambda\|_{L^4}} \\ &< 1 \end{aligned} \tag{18}$$

and thus condition (13) is satisfied.

Since we will anyway try to find $\alpha_\lambda$ (satisfying (12)) close to the minimal one, condition (13) is "practically" always satisfied if (12) holds. (Nevertheless, it must of course be checked.)

## 4 The Branch $(u_\lambda)$

Fixing some $\bar{\lambda} \in (0, \lambda_1(\Omega))$ (the actual choice of which is made on the basis of Proposition 1; see also Sect. 5), we assume now that for *every* $\lambda \in [0, \bar{\lambda}]$ an approximate solution $\omega_\lambda \in H_0^1(\Omega)$ is at hand, as well as a defect bound $\delta_\lambda$ satisfying (9), and a bound $K_\lambda$ satisfying (10). Furthermore, we assume now that, for every $\lambda \in [0, \bar{\lambda}]$, some $\alpha_\lambda > 0$ satisfies (12) and (13), and the additional non-triviality condition $\|\omega_\lambda\|_{H_0^1} > \alpha_\lambda$ (see Corollary 3). We suppose that some *uniform* ($\lambda$-independent) $\eta > 0$ can be chosen such that (13) holds with $\alpha_\lambda + \eta$ instead of $\alpha_\lambda$ (compare Theorem 3b)). Hence Theorem 3 gives a positive solution $u_\lambda \in H_0^1(\Omega)$ of problem (4) with the properties (14), (15), (16), for every $\lambda \in [0, \bar{\lambda}]$.

Finally, we assume that the approximate solution branch $([0,\bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto \omega_\lambda)$ is continuous, and that $([0,\bar{\lambda}] \to \mathbb{R}, \lambda \mapsto \alpha_\lambda)$ is lower semi-continuous.

In Sects. 6, 7 and 8, we will address the actual computation of such branches $(\omega_\lambda)$, $(\delta_\lambda)$, $(K_\lambda)$, $(\alpha_\lambda)$.

So far we know nothing about continuity or smoothness of $([0,\bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto u_\lambda)$, which however we will need to conclude that $(u_\lambda)_{\lambda\in[0,\bar{\lambda}]}$ coincides with the sub-branch $\Gamma_1$ introduced in Corollary 2.

**Theorem 4** *The solution branch*

$$\left\{\begin{array}{c}[0,\bar{\lambda}] \to H_0^1(\Omega)\\ \lambda \mapsto u_\lambda\end{array}\right\}$$

*is continuously differentiable.*

*Proof* The mapping

$$\mathcal{F}: \left\{\begin{array}{r}\mathbb{R}\times H_0^1(\Omega) \to H^{-1}(\Omega)\\ (\lambda,u) \mapsto -\Delta u - \lambda u - |u|^3\end{array}\right\}$$

is continuously differentiable, with $(\partial\mathcal{F}/\partial u)(\lambda,u) = L_{(\lambda,u)}$ (see (11)), and $\mathcal{F}(\lambda,u_\lambda)=0$ for all $\lambda\in[0,\bar{\lambda}]$. Using the Mean Value Theorem one can show that $L_{(\lambda,u)}$ depends indeed continuously on $(\lambda,u)$; see [16, Lemma 3.1] for details.

It suffices to prove the asserted smoothness locally. Thus, fix $\lambda_0\in[0,\bar{\lambda}]$. Since $L_{(\lambda_0,u_{\lambda_0})}$ is bijective by Theorem 3c), the Implicit Function Theorem gives a $C^1$-smooth solution branch

$$\left\{\begin{array}{r}(\lambda_0-\varepsilon,\lambda_0+\varepsilon) \to H_0^1(\Omega)\\ \lambda \mapsto \hat{u}_\lambda\end{array}\right\}$$

to problem (4), with $\hat{u}_{\lambda_0} = u_{\lambda_0}$. By (14),

$$\|\hat{u}_{\lambda_0} - \omega_{\lambda_0}\|_{H_0^1} \le \alpha_{\lambda_0}. \tag{19}$$

Since $\hat{u}_\lambda$ and $\omega_\lambda$ depend continuously on $\lambda$, and $\alpha_\lambda$ lower semi-continuously, (19) implies

$$\|\hat{u}_\lambda - \omega_\lambda\|_{H_0^1} \le \alpha_\lambda + \eta \quad \big(\lambda\in[0,\bar{\lambda}]\cap(\lambda_0-\tilde{\varepsilon},\lambda_0+\tilde{\varepsilon})\big)$$

for some $\tilde{\varepsilon}\in(0,\varepsilon)$. Hence Theorem 3b) provides

$$\hat{u}_\lambda = u_\lambda \quad \big(\lambda\in[0,\bar{\lambda}]\cap(\lambda_0-\tilde{\varepsilon},\lambda_0+\tilde{\varepsilon})\big),$$

implying the desired smoothness in some neighborhood of $\lambda_0$ (which of course is one-sided if $\lambda_0=0$ or $\lambda_0=\bar{\lambda}$). □

As a consequence of Theorem 4, $(u_\lambda)_{\lambda\in[0,\bar{\lambda}]}$ is a continuous solution curve connecting the point $(0,u_0)$ with $(\bar{\lambda},u_{\bar{\lambda}})$, and thus must coincide with the sub-branch

$\Gamma_1$, connecting these two points, of the unique simple continuous curve $\Gamma$ given by Theorem 2. Using Theorem 3c), we obtain

**Corollary 4** *On the sub-branch $\Gamma_1$ of $\Gamma$ which connects $(0, u_0)$ with $(\bar{\lambda}, u_{\bar{\lambda}})$, all solutions are nondegenerate.*

Thus, if we can choose $\bar{\lambda}$ such that condition (7) holds true, Corollary 2 will give the desired uniqueness result.

## 5 Choice of $\bar{\lambda}$

We have to choose $\bar{\lambda}$ such that condition (7) is satisfied. For this purpose, we use computer-assistance again. With $x_M$ denoting the intersection of the symmetry axes of the (doubly symmetric) domain $\Omega$, i.e. $x_M = (\frac{1}{2}, \frac{1}{2})$ for $\Omega = (0, 1)^2$, we choose $\bar{\lambda} \in (0, \lambda_1(\Omega))$, not too close to $\lambda_1(\Omega)$, such that our *approximate* solution $\omega_{\bar{\lambda}}$ satisfies

$$\omega_{\bar{\lambda}}(x_M) < \left(\frac{\lambda_2(\Omega) - \lambda_1(\Omega)}{3}\right)^{\frac{1}{2}} \cdot \left(\frac{\bar{\lambda}}{\lambda_1(\Omega)}\right)^{\frac{1}{2}}, \tag{20}$$

with "not too small" difference between right- and left-hand side. Such a $\bar{\lambda}$ can be found within a few numerical trials.

Here, we impose the additional requirement

$$\omega_{\bar{\lambda}} \in H^2(\Omega) \cap H_0^1(\Omega), \tag{21}$$

which is in fact a condition on the numerical method used to compute $\omega_{\bar{\lambda}}$. (Actually, condition (21) could be avoided if we were willing to accept additional technical effort.) Moreover, exceeding (9), we will now need an $L^2$-bound $\hat{\delta}_{\bar{\lambda}}$ for the defect:

$$\left\| -\Delta\omega_{\bar{\lambda}} - \bar{\lambda}\omega_{\bar{\lambda}} - |\omega_{\bar{\lambda}}|^3 \right\|_{L^2} \le \hat{\delta}_{\bar{\lambda}}. \tag{22}$$

Finally, we note that $\Omega$ is *convex*, and hence in particular $H^2$-regular, whence every solution $u \in H_0^1(\Omega)$ of problem (4) is in $H^2(\Omega)$.

Using the method described in Sect. 3, we obtain, by Theorem 3a), a positive solution $u_{\bar{\lambda}} \in H^2(\Omega) \cap H_0^1(\Omega)$ of problem (4) satisfying

$$\|u_{\bar{\lambda}} - \omega_{\bar{\lambda}}\|_{H_0^1} \le \alpha_{\bar{\lambda}}, \tag{23}$$

provided that (12) and (13) hold, and that $\|\omega_{\bar{\lambda}}\|_{H_0^1} > \alpha_{\bar{\lambda}}$.

Now we make use of the explicit version of the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ given in [21]. There, explicit constants $\hat{C}_0, \hat{C}_1, \hat{C}_2$ are computed such that

$$\|u\|_\infty \le \hat{C}_0\|u\|_{L^2} + \hat{C}_1\|\nabla u\|_{L^2} + \hat{C}_2\|u_{xx}\|_{L^2} \quad \text{for all } u \in H^2(\Omega),$$

with $\|u_{xx}\|_{L^2}$ denoting the $L^2$-Frobenius norm of the Hessian matrix $u_{xx}$. E.g. for $\Omega=(0,1)^2$, [21] gives

$$\hat{C}_0=1, \qquad \hat{C}_1=1.1548\cdot\sqrt{\frac{2}{3}}\le 0.9429, \qquad \hat{C}_2=0.22361\cdot\sqrt{\frac{28}{45}}\le 0.1764.$$

Moreover, $\|u_{xx}\|_{L^2}\le\|\Delta u\|_{L^2}$ for $u\in H^2(\Omega)\cap H_0^1(\Omega)$ since $\Omega$ is convex (see e.g. [14]). Consequently,

$$\|u_{\bar\lambda}-\omega_{\bar\lambda}\|_\infty\le\hat{C}_0\|u_{\bar\lambda}-\omega_{\bar\lambda}\|_{L^2}+\hat{C}_1\|u_{\bar\lambda}-\omega_{\bar\lambda}\|_{H_0^1}+\hat{C}_2\|\Delta u_{\bar\lambda}-\Delta\omega_{\bar\lambda}\|_{L^2}. \quad (24)$$

To bound the last term on the right-hand side, we first note that

$$\begin{aligned}\big\||u_{\bar\lambda}|^3-|\omega_{\bar\lambda}|^3\big\|_{L^2}&=\left\|3\int_0^1\big|\omega_{\bar\lambda}+t(u_{\bar\lambda}-\omega_{\bar\lambda})\big|\big(\omega_{\bar\lambda}+t(u_{\bar\lambda}-\omega_{\bar\lambda})\big)\,dt\cdot(u_{\bar\lambda}-\omega_{\bar\lambda})\right\|_{L^2}\\&\le 3\int_0^1\big\|\big|\omega_{\bar\lambda}+t(u_{\bar\lambda}-\omega_{\bar\lambda})\big|^2\cdot|u_{\bar\lambda}-\omega_{\bar\lambda}|\big\|_{L^2}\,dt\\&\le 3\int_0^1\|\omega_{\bar\lambda}+t(u_{\bar\lambda}-\omega_{\bar\lambda})\|_{L^6}^2\|u_{\bar\lambda}-\omega_{\bar\lambda}\|_{L^6}\,dt\\&\le 3\int_0^1\big(\|\omega_{\bar\lambda}\|_{L^6}+tC_6\alpha_{\bar\lambda}\big)^2\,dt\cdot C_6\alpha_{\bar\lambda} \qquad (25)\\&=3C_6\left(\|\omega_\lambda\|_{L^6}^2+C_6\|\omega_\lambda\|_{L^6}\alpha_{\bar\lambda}+\frac{1}{3}C_6^2\alpha_{\bar\lambda}^2\right)\alpha_{\bar\lambda}, \qquad (26)\end{aligned}$$

using (23) and an embedding constant $C_6$ for the embedding $H_0^1(\Omega)\hookrightarrow L^6(\Omega)$ in the last but one line; see e.g. [23, Lemma 2] for its computation. Moreover, by (4) and (22),

$$\|\Delta u_{\bar\lambda}-\Delta\omega_{\bar\lambda}\|_{L^2}\le\hat{\delta}_{\bar\lambda}+\bar\lambda\|u_{\bar\lambda}-\omega_{\bar\lambda}\|_{L^2}+\big\||u_{\bar\lambda}|^3-|\omega_{\bar\lambda}|^3\big\|_{L^2}. \quad (27)$$

Using (23)–(27), and the Poincaré inequality

$$\|u\|_{L^2}\le\frac{1}{\sqrt{\lambda_1(\Omega)+\sigma}}\|u\|_{H_0^1}\quad\big(u\in H_0^1(\Omega)\big), \quad (28)$$

we finally obtain

$$\begin{aligned}\|u_{\bar\lambda}-\omega_{\bar\lambda}\|_\infty\le\Bigg[&\frac{\hat{C}_0+\bar\lambda\hat{C}_2}{\sqrt{\lambda_1(\Omega)+\sigma}}+\hat{C}_1\\&+3C_6\hat{C}_2\left(\|\omega_\lambda\|_{L^6}^2+C_6\|\omega_\lambda\|_{L^6}\alpha_{\bar\lambda}+\frac{1}{3}C_6^2\alpha_{\bar\lambda}^2\right)\Bigg]\cdot\alpha_{\bar\lambda}+\hat{C}_2\hat{\delta}_{\bar\lambda}, \quad (29)\end{aligned}$$

and the right-hand side is "small" if $\alpha_{\bar{\lambda}}$ and $\hat{\delta}_{\bar{\lambda}}$ are "small", which can (again) be achieved by sufficiently accurate numerical computations.

Finally, since

$$u_{\bar{\lambda}}(x_M) \leq \omega_{\bar{\lambda}}(x_M) + \|u_{\bar{\lambda}} - \omega_{\bar{\lambda}}\|_\infty,$$

(29) yields an upper bound for $u_{\bar{\lambda}}(x_M)$ which is "not too much" larger than $\omega_{\bar{\lambda}}(x_M)$. Hence, since $u_{\bar{\lambda}}(x_M) = \|u_{\bar{\lambda}}\|_\infty$ by [11], condition (7) can easily be checked, and (20) (with "not too small" difference between right- and left-hand side) implies a good chance that this check will be successful; otherwise, $\bar{\lambda}$ has to be chosen a bit larger.

## 6 Computation of $\omega_\lambda$ and $\delta_\lambda$ for Fixed $\lambda$

In this section we report on the computation of an approximate solution $\omega_\lambda \in H^2(\Omega) \cap H_0^1(\Omega)$ to problem (4), and of bounds $\delta_\lambda$ and $K_\lambda$ satisfying (9) and (10), where $\lambda \in [0, \lambda_1(\Omega))$ is *fixed* (or one of *finitely* many values). We will again restrict ourselves to the unit square $\Omega = (0, 1)^2$.

An approximation $\omega_\lambda$ is computed by a *Newton iteration* applied to problem (4), where the linear boundary value problems

$$L_{(\lambda,\omega_\lambda^{(n)})}[v_n] = \Delta\omega_\lambda^{(n)} + \lambda\omega_\lambda^{(n)} + \left|\omega_\lambda^{(n)}\right|^3 \tag{30}$$

occurring in the single iteration steps are solved approximately by an ansatz

$$v_n(x_1, x_2) = \sum_{i,j=1}^{N} \alpha_{ij}^{(n)} \sin(i\pi x_1)\sin(j\pi x_2) \tag{31}$$

and a Ritz-Galerkin method (with the basis functions in (31)) applied to problem (30). The update $\omega_\lambda^{(n+1)} := \omega_\lambda^{(n)} + v_n$ concludes the iteration step.

The Newton iteration is terminated when the coefficients $\alpha_{ij}^{(n)}$ in (31) are "small enough", i.e. their modulus is below some pre-assigned tolerance.

To *start* the Newton iteration, i.e. to find an appropriate $\omega_\lambda^{(0)}$ of the form (31), we first consider some $\lambda$ close to $\lambda_1(\Omega)$, and choose $\omega_\lambda^{(0)}(x_1, x_2) = \alpha\sin(\pi x_1)\sin(\pi x_2)$; with an appropriate choice of $\alpha > 0$ (to be determined in a few numerical trials), the Newton iteration will "converge" to a non-trivial approximation $\omega^{(\lambda)}$. Then, starting at this value, we diminish $\lambda$ in small steps until we arrive at $\lambda = 0$, while in each of these steps the approximation $\omega^{(\lambda)}$ computed in the *previous* step is taken as a start of the Newton iteration. In this way, we find approximations $\omega_\lambda$ to problem (4) for "many" values of $\lambda$. Note that all approximations $\omega_\lambda$ obtained in this way are of the form (31).

The computation of an $L^2$-defect bound $\hat{\delta}_\lambda$ satisfying

$$\left\| -\Delta\omega_\lambda - \lambda\omega_\lambda - |\omega_\lambda|^3 \right\|_{L^2} \leq \hat{\delta}_\lambda \tag{32}$$

amounts to the computation of an integral over $\Omega$.

Due to [11] every solution of (4) is symmetric with respect to reflection at the axes $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$. Therefore it is useful to look for approximate solutions of the form

$$\omega_\lambda(x_1, x_2) = \sum_{\substack{i,j=1 \\ i,j \text{ odd}}}^{N} \alpha_{ij} \sin(i\pi x_1)\sin(j\pi x_2). \tag{33}$$

Using sum formulas for sin and cos one obtains for all $n \in \mathbb{N}_0$, $x \in \mathbb{R}$

$$\sin\big((2n+1)\pi x\big) = \left(2\sum_{k=1}^{n}\cos(2k\pi x) + 1\right)\sin(\pi x)$$

and thus $\omega_\lambda$ can be written as follows:

$$\begin{aligned}\omega_\lambda(x_1, x_2) &= \alpha_{11}\sin(\pi x_1)\sin(\pi x_2) \\ &\quad + \sum_{k,l=1}^{\lfloor\frac{N-1}{2}\rfloor} \alpha_{2k+1,2l+1}\left(2\sum_{i=1}^{k}\cos(2i\pi x_1) + 1\right)\left(2\sum_{j=1}^{l}\cos(2j\pi x_2) + 1\right) \\ &\quad \cdot \sin(\pi x_1)\sin(\pi x_2).\end{aligned} \tag{34}$$

Since $\cos(x)$ ranges in $[-1, 1]$ and $\sin(\pi x_1)\sin(\pi x_2)$ is positive for $(x_1, x_2) \in \Omega = (0, 1)^2$, $\omega_\lambda$ will be positive if

$$\alpha_{11} + \sum_{k,l=1}^{\lfloor\frac{N-1}{2}\rfloor} \alpha_{2k+1,2l+1}\big([-2k+1, 2k+1]\big)\big([-2l+1, 2l+1]\big) \subset (0, \infty). \tag{35}$$

Condition (35) can easily be checked using interval arithmetic and is indeed always satisfied for our approximate solutions, since $\alpha_{11}$ turns out to be "dominant" and the higher coefficients decay quickly. Hence $\omega_\lambda$ is positive and one can omit the modulus in the computations. Therefore the integral in (32) can be computed in closed form, since only products of trigonometric functions occur in the integrand. After calculating them, various sums $\sum_{i=1}^{N}$ remain to be evaluated. In order to obtain a *rigorous* bound $\hat{\delta}_\lambda$, these computations (in contrast to those for obtaining $\omega_\lambda$ as described above) need to be carried out in *interval* arithmetic [13, 27], to take rounding errors into account.

Note that the complexity in the evaluation of the defect integral in (32), without any further modifications, is $O(N^{12})$ due to the term $\omega_\lambda^3$. Using some trick, it is however possible to reduce the complexity to $O(N^6)$:

Applying the sum formulas $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ and $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$ one obtains:

$$\begin{aligned}&\sin(i_1\pi x)\sin(i_2\pi x)\sin(i_3\pi x)\sin(i_4\pi x)\sin(i_5\pi x)\sin(i_6\pi x)\\&= -\frac{1}{32}\sum_{\substack{\sigma_2,\sigma_3,\sigma_4,\\ \sigma_5,\sigma_6\in\{-1,1\}}} \sigma_2\sigma_3\sigma_4\sigma_5\sigma_6 \cos\big((i_1+\sigma_2 i_2+\sigma_3 i_3+\sigma_4 i_4+\sigma_5 i_5+\sigma_6 i_6)\pi x\big).\end{aligned}$$

Since $\int_0^1 \cos(n\pi x)\,dx = \left\{\begin{matrix} 1 & \text{for } n=0 \\ 0 & \text{for } n\in\mathbb{Z}\backslash\{0\}\end{matrix}\right\} =: \delta_n$, we get

$$\begin{aligned}&\int_\Omega \omega_\lambda(x_1,x_2)^6\,d(x_1,x_2)\\&= \frac{1}{1024}\sum_{\sigma_2,\dots,\sigma_6\in\{-1,1\}}\ \sum_{\rho_2,\dots,\rho_6\in\{-1,1\}} \sigma_2\cdot\ldots\cdot\sigma_6\cdot\rho_2\cdot\ldots\cdot\rho_6\\&\quad\cdot\sum_{i_1,\dots,i_6=1}^{N}\ \sum_{j_1,\dots,j_6=1}^{N} \delta_{i_1+\sigma_2 i_2+\dots+\sigma_6 i_6}\,\delta_{j_1+\rho_2 j_2+\dots+\rho_6 j_6}\,\alpha_{i_1 j_1}\cdot\ldots\cdot\alpha_{i_6 j_6}.\end{aligned}$$

Setting $\alpha_{ij} := 0$ for $(i,j)\in\mathbb{Z}^2\backslash\{1,\dots,N\}^2$ the previous sum can be rewritten as

$$\begin{aligned}&\frac{1}{1024}\sum_{\substack{\sigma_2,\dots,\sigma_6,\\ \rho_2\dots,\rho_6\in\{-1,1\}}} \sigma_2\cdot\ldots\cdot\sigma_6\cdot\rho_2\cdot\ldots\cdot\rho_6\\&\cdot\sum_{k=-2N+1}^{3N}\ \sum_{l=-2N+1}^{3N}\Bigg(\sum_{\substack{i_1+\sigma_2 i_2\\ +\sigma_3 i_3=k}}\ \sum_{\substack{j_1+\rho_2 j_2\\ +\rho_3 j_3=l}} \alpha_{i_1 j_1}\alpha_{i_2 j_2}\alpha_{i_3 j_3}\Bigg)\\&\cdot\Bigg(\sum_{\substack{\sigma_4 i_4+\sigma_5 i_5\\ +\sigma_6 i_6=-k}}\ \sum_{\substack{\rho_4 j_4+\rho_5 j_5\\ +\rho_6 j_6=-l}} \alpha_{i_4 j_4}\alpha_{i_5 j_5}\alpha_{i_6 j_6}\Bigg).\end{aligned}$$

For fixed $\sigma_i$, $\rho_i$, $k$ and $l$ each of the two double-sums in parentheses is $O(N^4)$. Since they are independent, the product is still $O(N^4)$. The sums over $k$ and $l$ then give $O(N^6)$, whereas the sums over $\sigma_i$ and $\rho_i$ do not change the complexity.

Moreover the sum $\sum_{k=-2N+1}^{3N}$ is only

$$\begin{cases}\sum_{k=3}^{3N} & \text{if } \sigma_2=1, \sigma_3=1,\\ \sum_{k=2-N}^{2N-1} & \text{if } \sigma_2\cdot\sigma_3=-1,\\ \sum_{k=-2N+1}^{N-2} & \text{if } \sigma_2=-1, \sigma_3=-1.\end{cases}$$

Similarly, also certain constellations of $\sigma_4$, $\sigma_5$, $\sigma_6$ reduce the $k$-sum, and of course analogous reductions are possible for the $l$-sum. Since $\alpha_{ij} = 0$ if $i$ or $j$ is even, the result does not change if the sum is only taken over odd values of $i_n$, $j_n$, $k$ and $l$.

*Remark 3*

a) Computing trigonometric sums in an efficient way is an object of investigation since a very long time, but up to our knowledge the above complexity reduction has not been published before.
b) As an alternative to the closed form integration described above, we also tried quadrature for computing the defect integral, but due to the necessity of computing a safe remainder term bound in this case, we ended up in a very high numerical effort, since a large number of quadrature points had to be chosen. So *practically* closed-form integration turned out to be more efficient, although its complexity (as $N \to \infty$) is higher than the quadrature complexity.

Once an $L^2$-defect bound $\hat{\delta}_\lambda$ (satisfying (32)) has been computed, an $H^{-1}$-defect bound $\delta_\lambda$ (satisfying (9)) is easily obtained via the embedding

$$\|u\|_{H^{-1}} \leq \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}} \|u\|_{L^2} \quad \bigl(u \in L^2(\Omega)\bigr) \tag{36}$$

which is a result of the corresponding dual embedding (28). Indeed, (32) and (36) imply that

$$\delta_\lambda := \frac{1}{\sqrt{\lambda_1(\Omega) + \sigma}} \hat{\delta}_\lambda$$

satisfies (9).

The estimate (36) is suboptimal but, under practical aspects, seems to be the most suitable way for obtaining an $H^{-1}$-bound for the defect. At this point we also wish to remark that, as an alternative to the weak solutions approach used in this paper, we could also have aimed at a computer-assisted proof for *strong* solutions (see [23]), leading to $H^2$-and $C^0$-error bounds; in this case an $L^2$-bound is needed directly (rather than an $H^{-1}$-bound).

## 7 Computation of $K_\lambda$ for Fixed $\lambda$

For computing a constant $K_\lambda$ satisfying (10), we use the isometric isomorphism

$$\Phi : \left\{ \begin{array}{r} H_0^1(\Omega) \to H^{-1}(\Omega) \\ u \mapsto -\Delta u + \sigma u \end{array} \right\}, \tag{37}$$

and note that $\Phi^{-1} L_{(\lambda,\omega_\lambda)} : H_0^1(\Omega) \to H_0^1(\Omega)$ is $\langle \cdot, \cdot \rangle_{H_0^1}$-symmetric since

$$\bigl\langle \Phi^{-1} L_{(\lambda,\omega_\lambda)}[u], v \bigr\rangle_{H_0^1} = \int_\Omega \bigl[\nabla u \cdot \nabla v - \lambda u v - 3|\omega_\lambda|\omega_\lambda u v\bigr]\, dx, \tag{38}$$

and hence selfadjoint. Since $\|L_{(\lambda,\omega_\lambda)}[u]\|_{H^{-1}} = \|\Phi^{-1}L_{(\lambda,\omega_\lambda)}[u]\|_{H_0^1}$, (10) thus holds for any

$$K_\lambda \geq \left[\min\left\{|\mu| : \mu \text{ is in the spectrum of } \Phi^{-1}L_{(\lambda,\omega_\lambda)}\right\}\right]^{-1}, \tag{39}$$

provided the min is positive.

A particular consequence of (38) is that

$$\left\langle \left(I - \Phi^{-1}L_{(\lambda,\omega_\lambda)}\right)[u], u\right\rangle_{H_0^1} = \int_\Omega W_\lambda u^2\, dx \quad \left(u \in H_0^1(\Omega)\right) \tag{40}$$

where

$$W_\lambda(x) := \sigma + \lambda + 3\big|\omega_\lambda(x)\big|\omega_\lambda(x) \quad (x \in \Omega). \tag{41}$$

Note that, due to the positivity of our approximate solutions $\omega_\lambda$ established in Sect. 6, the modulus can be omitted here, which again facilitates numerical computations. Choosing a *positive* parameter $\sigma$ in the $H_0^1$-product (recall that we actually chose $\sigma := 1$), we obtain $W_\lambda > 0$ on $\bar{\Omega}$. Thus, (40) shows that all eigenvalues $\mu$ of $\Phi^{-1}L_{(\lambda,\omega_\lambda)}$ are less than 1, and that its essential spectrum consists of the single point 1. Therefore, (39) requires the computation of *eigenvalue bounds* for the eigenvalue(s) $\mu$ neighboring 0.

Using the transformation $\kappa = 1/(1-\mu)$, the eigenvalue problem $\Phi^{-1}L_{(\lambda,\omega_\lambda)}[u] = \mu u$ is easily seen to be equivalent to

$$-\Delta u + \sigma u = \kappa W_\lambda u,$$

or, in weak formulation,

$$\langle u, v\rangle_{H_0^1} = \kappa \int_\Omega W_\lambda u v\, dx \quad \left(v \in H_0^1(\Omega)\right), \tag{42}$$

and we are interested in bounds to the eigenvalue(s) $\kappa$ neighboring 1. It is therefore sufficient to compute two-sided bounds to the first $m$ eigenvalues $\kappa_1 \leq \cdots \leq \kappa_m$ of problem (42), where $m$ is (at least) such that $\kappa_m > 1$. In all our practical examples, the computed enclosures $\kappa_i \in [\underline{\kappa}_i, \bar{\kappa}_i]$ are such that $\bar{\kappa}_1 < 1 < \underline{\kappa}_2$, whence by (39) and $\kappa = 1/(1-\mu)$ we can choose

$$K_\lambda := \max\left\{\frac{\bar{\kappa}_1}{1-\bar{\kappa}_1}, \frac{\underline{\kappa}_2}{\underline{\kappa}_2 - 1}\right\}. \tag{43}$$

*Remark 4* By [11] and the fact that $\omega_\lambda$ is symmetric with respect to reflection at the axes $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$, all occurring function spaces can be replaced by their intersection with the class of reflection symmetric functions. This has the advantage that some eigenvalues $\kappa_i$ drop out, which possibly reduces the constant $K_\lambda$.

The desired *eigenvalue bounds* for problem (42) can be obtained by computer-assisted means of their own. For example, *upper* bounds to $\kappa_1, \ldots, \kappa_m$ (with $m \in \mathbb{N}$ given) are easily and efficiently computed by the *Rayleigh-Ritz* method [26]:

Let $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m \in H_0^1(\Omega)$ denote linearly independent trial functions, for example approximate eigenfunctions obtained by numerical means, and form the matrices

$$A_1 := \left(\langle \tilde{\varphi}_i, \tilde{\varphi}_j \rangle_{H_0^1}\right)_{i,j=1,\ldots,m}, \qquad A_0 := \left(\int_\Omega W_\lambda \tilde{\varphi}_i \tilde{\varphi}_j \, dx\right)_{i,j=1,\ldots,m}.$$

Then, with $\Lambda_1 \leq \cdots \leq \Lambda_m$ denoting the eigenvalues of the matrix eigenvalue problem

$$A_1 x = \Lambda A_0 x$$

(which can be enclosed by means of verifying numerical linear algebra; see [3]), the Rayleigh-Ritz method gives

$$\kappa_i \leq \Lambda_i \quad \text{for } i = 1, \ldots, m.$$

However, also *lower* eigenvalue bounds are needed, which constitute a more complicated task than upper bounds. The most accurate method for this purpose has been proposed by Lehmann [15], and improved by Goerisch concerning its range of applicability [4]. Its numerical core is again (as in the Rayleigh-Ritz method) a matrix eigenvalue problem, but the accompanying analysis is more involved. In particular, in order to compute lower bounds to the first $m$ eigenvalues, a *rough* lower bound to the $(m+1)$st eigenvalue must be known already. This a priori information can usually be obtained via a *homotopy method* connecting a simple "base problem" with known eigenvalues to the given eigenvalue problem, such that all eigenvalues increase (index-wise) along the homotopy; see [22] or [5] for details on this method, a detailed description of which would be beyond the scope of this chapter. In fact, [5] contains the newest version of the homotopy method, where only very small ($2 \times 2$ or even $1 \times 1$) matrix eigenvalue problems need to be treated rigorously in the course of the homotopy.

Finding a base problem for problem (42), and a suitable homotopy connecting them, is rather simple here since $\Omega$ is a bounded rectangle, whence the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$ are known: We choose a constant upper bound $c_0$ for $|\omega_\lambda|\omega_\lambda = \omega_\lambda^2$ on $\Omega$, and the coefficient homotopy

$$W_\lambda^{(s)}(x) := \sigma + \lambda + 3\left[(1-s)c_0 + s\omega_\lambda(x)^2\right] \quad (x \in \Omega, 0 \leq s \leq 1).$$

Then, the family of eigenvalue problems

$$-\Delta u + \sigma u = \kappa^{(s)} W_\lambda^{(s)} u$$

connects the explicitly solvable constant-coefficient base problem ($s = 0$) to problem (42) ($s = 1$), and the eigenvalues increase in $s$, since the Rayleigh quotient does, by Poincaré's min-max principle.

## 8 Computation of Branches $(\omega_\lambda)$, $(\delta_\lambda)$, $(K_\lambda)$, $(\alpha_\lambda)$

In the previous section we described how to compute approximations $\omega_\lambda$ for a grid of finitely many values of $\lambda$ within $[0, \lambda_1(\Omega))$. After selecting $\bar{\lambda}$ (among these) according to Sect. 5, we are left with a grid

$$0 = \lambda^0 < \lambda^1 < \cdots < \lambda^M = \bar{\lambda}$$

and approximate solutions $\omega^i = \omega_{\lambda^i} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ $(i = 0, \ldots, M)$. Furthermore, according to the methods described in the previous sections, we can compute bounds $\delta^i = \delta_{\lambda^i}$ and $K^i = K_{\lambda^i}$ such that (9) and (10) hold at $\lambda = \lambda^i$.

Now we define a piecewise linear (and hence continuous) approximate solution branch $([0, \bar{\lambda}] \to H_0^1(\Omega), \lambda \mapsto \omega_\lambda)$ by

$$\omega_\lambda := \frac{\lambda^i - \lambda}{\lambda^i - \lambda^{i-1}}\omega^{i-1} + \frac{\lambda - \lambda^{i-1}}{\lambda^i - \lambda^{i-1}}\omega^i \quad \big(\lambda^{i-1} < \lambda < \lambda^i, i = 1, \ldots, M\big). \tag{44}$$

To compute corresponding defect bounds $\delta_\lambda$, we fix $i \in \{1, \ldots, M\}$ and $\lambda \in [\lambda^{i-1}, \lambda^i]$, and let $t := (\lambda - \lambda^{i-1})/(\lambda^i - \lambda^{i-1}) \in [0, 1]$, whence

$$\lambda = (1-t)\lambda^{i-1} + t\lambda^i, \qquad \omega_\lambda = (1-t)\omega^{i-1} + t\omega^i. \tag{45}$$

Using the classical linear interpolation error bound we obtain, for fixed $x \in \Omega$,

$$\begin{aligned}
&\big|\omega_\lambda(x)^3 - \big[(1-t)\omega^{i-1}(x)^3 + t\omega^i(x)^3\big]\big| \\
&\le \frac{1}{2}\max_{s\in[0,1]}\left|\frac{d^2}{ds^2}\big[(1-s)\omega^{i-1}(x) + s\omega^i(x)\big]^3\right| \cdot t(1-t) \\
&\le \frac{3}{4}\max_{s\in[0,1]}\big[(1-s)\omega^{i-1}(x) + s\omega^i(x)\big] \cdot \big(\omega^i(x) - \omega^{i-1}(x)\big)^2 \\
&\le \frac{3}{4}\max\big\{\|\omega^{i-1}\|_\infty, \|\omega^i\|_\infty\big\}\|\omega^i - \omega^{i-1}\|_\infty^2,
\end{aligned} \tag{46}$$

$$\begin{aligned}
&\big|\lambda\omega_\lambda(x) - \big[(1-t)\lambda^{i-1}\omega^{i-1}(x) + t\lambda^i\omega^i(x)\big]\big| \\
&\le \frac{1}{2}\max_{s\in[0,1]}\left|\frac{d^2}{ds^2}\big[\big((1-s)\lambda^{i-1} + s\lambda^i\big)\big((1-s)\omega^{i-1}(x) + s\omega^i(x)\big)\big]\right| \cdot t(1-t) \\
&\le \frac{1}{4}\big(\lambda^i - \lambda^{i-1}\big)\|\omega^i - \omega^{i-1}\|_\infty.
\end{aligned} \tag{47}$$

Since $\|u\|_{H^{-1}} \le C_1\|u\|_\infty$ for all $u \in L^\infty(\Omega)$, with $C_1$ denoting an embedding constant for the embedding $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$ (e.g. $C_1 = \sqrt{|\Omega|}C_2$), (46) and (47) imply

$$\begin{aligned}&\left\|\omega_\lambda^3-\left[(1-t)\left(\omega^{i-1}\right)^3+t\left(\omega^i\right)^3\right]\right\|_{H^{-1}}\\&\quad\le\frac{3}{4}C_1\max\left\{\left\|\omega^{i-1}\right\|_\infty,\left\|\omega^i\right\|_\infty\right\}\left\|\omega^i-\omega^{i-1}\right\|_\infty^2=:\rho_i,\end{aligned}\tag{48}$$

$$\begin{aligned}&\left\|\lambda\omega_\lambda-\left[(1-t)\lambda^{i-1}\omega^{i-1}+t\lambda^i\omega^i\right]\right\|_{H^{-1}}\\&\quad\le\frac{1}{4}C_1\left(\lambda^i-\lambda^{i-1}\right)\left\|\omega^i-\omega^{i-1}\right\|_\infty=:\tau_i.\end{aligned}\tag{49}$$

Now (45), (48), (49) give

$$\begin{aligned}&\left\|-\Delta\omega_\lambda-\lambda\omega_\lambda-\omega_\lambda^3\right\|_{H^{-1}}\\&\quad\le(1-t)\left\|-\Delta\omega^{i-1}-\lambda^{i-1}\omega^{i-1}-\left(\omega^{i-1}\right)^3\right\|_{H^{-1}}\\&\qquad+t\left\|-\Delta\omega^i-\lambda^i\omega^i-\left(\omega^i\right)^3\right\|_{H^{-1}}+\tau_i+\rho_i\\&\quad\le\max\left\{\delta^{i-1},\delta^i\right\}+\tau_i+\rho_i=:\delta_\lambda.\end{aligned}\tag{50}$$

Thus, we obtain a branch $(\delta_\lambda)_{\lambda\in[0,\bar\lambda]}$ of defect bounds which is constant on each subinterval $[\lambda^{i-1},\lambda^i]$. In the points $\lambda^1,\dots,\lambda^{M-1}$, $\delta_\lambda$ is possibly doubly defined by (50), in which case we choose the smaller of the two values. Hence, $([0,\bar\lambda]\to\mathbb{R},\lambda\mapsto\delta_\lambda)$ is lower semi-continuous.

Note that $\delta_\lambda$ given by (50) is "small" if $\delta^{i-1}$ and $\delta^i$ are small (i.e. if the approximations $\omega^{i-1}$ and $\omega^i$ have been computed with sufficient accuracy; see Remark 2a)) *and* if $\rho_i,\tau_i$ are small (i.e. if the grid is chosen sufficiently fine; see (48), (49)).

In order to compute bounds $K_\lambda$ satisfying (10) for $\lambda\in[0,\bar\lambda]$, with $\omega_\lambda$ given by (44), we fix $i\in\{1,\dots,M-1\}$ and $\lambda\in[\frac{1}{2}(\lambda^{i-1}+\lambda^i),\frac{1}{2}(\lambda^i+\lambda^{i+1})]$. Then,

$$\begin{aligned}\left|\lambda-\lambda^i\right|&\le\frac{1}{2}\max\left\{\lambda^i-\lambda^{i-1},\lambda^{i+1}-\lambda^i\right\}=:\mu_i,\\\left\|\omega_\lambda-\omega^i\right\|_{H_0^1}&\le\frac{1}{2}\max\left\{\left\|\omega^i-\omega^{i-1}\right\|_{H_0^1},\left\|\omega^{i+1}-\omega^i\right\|_{H_0^1}\right\}=:\nu_i,\end{aligned}\tag{51}$$

whence a coefficient perturbation result given in [16, Lemma 3.2] implies: If

$$\zeta_i:=K^i\left[\frac{1}{\lambda_1(\Omega)+\sigma}\mu_i+2\gamma\left(\left\|\omega^i\right\|_{L^4}+C_4\nu_i\right)\nu_i\right]<1,\tag{52}$$

then (10) holds for

$$K_\lambda:=\frac{K^i}{1-\zeta_i}.\tag{53}$$

Note that (52) is indeed satisfied if the grid is chosen sufficiently fine, since then $\mu_i$ and $\nu_i$ are "small" by (51).

Analogous estimates give $K_\lambda$ also on the two remaining half-intervals $[0,\frac{1}{2}\lambda^1]$ and $[\frac{1}{2}(\lambda^{M-1}+\lambda^M),\lambda^M]$.

Choosing again the smaller of the two values at the points $\frac{1}{2}(\lambda^{i-1} + \lambda^i)$ ($i = 1, \dots, M$) where $K_\lambda$ is possibly doubly defined by (53), we obtain a lower semi-continuous, piecewise constant branch $([0, \bar{\lambda}] \to \mathbb{R}, \lambda \mapsto K_\lambda)$.

According to the above construction, both $\lambda \mapsto \delta_\lambda$ and $\lambda \mapsto K_\lambda$ are constant on the $2M$ half-intervals. Moreover, (44) implies that, for $i = 1, \dots, M$,

$$\|\omega_\lambda\|_{L^4} \leq \left\{ \begin{array}{ll} \max\{\|\omega^{i-1}\|_{L^4}, \frac{1}{2}(\|\omega^{i-1}\|_{L^4} + \|\omega^i\|_{L^4})\} & \text{for } \lambda \in [\lambda^{i-1}, \frac{1}{2}(\lambda^{i-1} + \lambda^i)] \\ \max\{\frac{1}{2}(\|\omega^{i-1}\|_{L^4} + \|\omega^i\|_{L^4}), \|\omega^i\|_{L^4})\} & \text{for } \lambda \in [\frac{1}{2}(\lambda^{i-1} + \lambda^i), \lambda^i] \end{array} \right\}$$

and again we choose the smaller of the two values at the points of double definition.

Using these bounds, the crucial inequalities (12) and (13) (which have to be satisfied for all $\lambda \in [0, \bar{\lambda}]$) result in *finitely* many inequalities which can be fulfilled with "small" and piecewise constant $\alpha_\lambda$ if $\delta_\lambda$ is sufficiently small, i.e. if $\omega^0, \dots, \omega^M$ have been computed with sufficient accuracy (see Remark 2a)) and if the grid has been chosen sufficiently fine (see (48)–(50)). Moreover, since $\lambda \mapsto \delta_\lambda$, $\lambda \mapsto K_\lambda$ and the above piecewise constant upper bound for $\|\omega_\lambda\|_{L^4}$ are lower semi-continuous, the structure of the inequalities (12) and (13) clearly shows that also $\lambda \mapsto \alpha_\lambda$ can be chosen to be lower semi-continuous, as required in Sect. 4. Finally, since (13) now consists in fact of *finitely* many strict inequalities, a uniform ($\lambda$-independent) $\eta > 0$ can be chosen in Theorem 3b), as needed for Theorem 4.

## 9 Numerical Results

All computations have been performed on an AMD Athlon Dual Core 4800+ (2.4 GHz) processor, using MATLAB (version R2010a) and the interval toolbox INTLAB [27]. For some of the time consuming nested sums occurring in the computations, we used moreover mexfunctions to outsource these calculations to C++. For these parts of the program we used C-XSC [13] to verify the results. Our source code can be found on our webpage.[1]

In the following, we report on some more detailed numerical results.

Using $\bar{\lambda} = 18.5$ (which is not the minimally possible choice; e.g. $\bar{\lambda} = 15.7$ could have been chosen) and $M + 1 = 94$ values $0 = \lambda^0 < \lambda^1 < \cdots < \lambda^{93} = 18.5$ (with $\lambda^1 = 0.1$, $\lambda^2 = 0.3$ and the remaining gridpoints equally spaced with distance 0.2) we computed approximations $\omega^0, \dots, \omega^{93}$ with $N = 16$ in (31), as well as defect bounds $\delta^0, \dots, \delta^{93}$ and constants $K^0, \dots, K^{93}$, by the methods described in Sects. 6 and 7.

Figure 1 shows an approximate branch $[0, 2\pi^2) \to \mathbb{R}$, $\lambda \mapsto \|\omega_\lambda\|_\infty$. The continuous plot has been created by interpolation of the above grid points $\lambda^j$, plus some more grid points between 18.5 and $2\pi^2$, where we computed additional approximations.

[1] http://www.math.kit.edu/iana2/~roth/page/publ/en

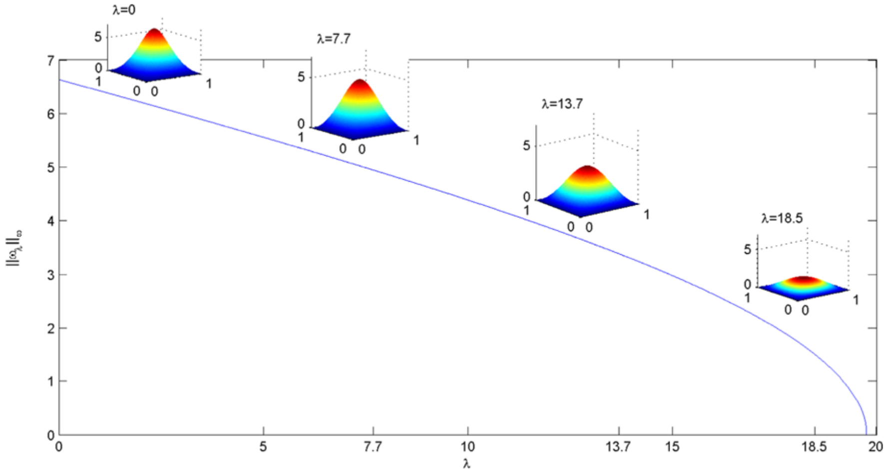


**Fig. 1** Curve $(\lambda, \|\omega_\lambda\|_\infty)$ with samples of $\omega_\lambda$ in the case $p = 3$

**Table 1** Eigenvalue enclosures for the first two eigenvalues

| | $\kappa_1$ | $\kappa_2$ |
|---|---|---|
| $\omega_0$ | $0.34350814513_{229}^{840}$ | $2.492570_{450}^{712}$ |
| $\omega_{2.7}$ | $0.37521912233_{290}^{850}$ | $2.6221837_{393}^{653}$ |
| $\omega_{6.7}$ | $0.4373273950_{355}^{411}$ | $2.87378161_{204}^{409}$ |
| $\omega_{10.7}$ | $0.52752354636_{169}^{621}$ | $3.223417042_{185}^{515}$ |
| $\omega_{14.7}$ | $0.6676848259_{379}^{417}$ | $3.725209290_{830}^{988}$ |
| $\omega_{18.5}$ | $0.89237445994_{555}^{742}$ | $4.46288110_{093}^{102}$ |

For some selected values of $\lambda$, Table 1 shows, with an obvious sub- and superscript notation for enclosing intervals, the computed eigenvalue bounds for problem (42) (giving $K_\lambda$ by (43)). These were obtained using the Rayleigh-Ritz and the Lehmann-Goerisch method, and the homotopy method briefly mentioned at the end of Sect. 7 (exploiting also the symmetry considerations addressed in Remark 4). The integer $m$, needed for these procedures, has been chosen different (between 3 and 10) for different values of $\lambda$, according to the outcome of the homotopy. This resulted in a slightly different quality of the eigenvalue enclosures.

Table 2 contains, for some selected of the 186 $\lambda$-half-intervals,

a) the defect bounds $\delta_\lambda$ obtained by (50) from the grid-point defect bounds $\delta^{i-1}$, $\delta^i$, and from the grid-width characteristics $\rho_i$, $\tau_i$ defined in (48), (49),
b) the constants $K_\lambda$ obtained by (53) from the grid-point constants $K^i$ and the grid-width parameters $\nu_i$ defined in (51) (note that $\mu_i = 0.1$ for all $i$),
c) the error bounds $\alpha_\lambda$ computed according to (12), (13).

**Table 2**

| $\lambda$-interval | $\delta_\lambda$ | $K_\lambda$ | $\alpha_\lambda$ |
|---|---|---|---|
| $[0, 0.05)$ | 0.0005943 | 1.7443526 | 0.0010378 |
| $(2, 2.1)$ | 0.0023344 | 1.7707941 | 0.0041521 |
| $(6, 6.1)$ | 0.0022937 | 1.6669879 | 0.0038369 |
| $(10, 10.1)$ | 0.0023644 | 1.5677657 | 0.0037168 |
| $(14, 14.1)$ | 0.0026980 | 1.9582604 | 0.0053028 |
| $(16, 16.1)$ | 0.0031531 | 3.2267762 | 0.0102701 |
| $(18.4, 18.5]$ | 0.0050056 | 13.8930543 | 0.0882899 |

Thus, Corollary 1, together with all the considerations in the previous sections, proves Theorem 1.

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# Green Function Estimates Lead to Neumann Function Estimates

**Guido Sweers**

*To the memory of Wolfgang Walter*

**Abstract** It is shown that the Neumann function for $-\Delta u + u = f$ on a bounded domain $\Omega \subset \mathbb{R}^n$ can be estimated pointwise from below in a uniform way. The proof is based on known uniform estimates from below for the Green function.

**Keywords** Neumann function · Fundamental solution · Comparison principle

**Mathematics Subject Classification** 35J25 · 35B51 · 35J08

Since the contributions by Krasovskii [5, 6], estimates from above for the Green function of elliptic boundary value problems of arbitrary order, say $2m$, on bounded domains $\Omega$ are known:

$$\begin{aligned} G(x,y) &\le C|x-y|^{2m-n} && \text{if } n > 2m, \\ G(x,y) &\le C\log\left(1+\frac{1}{|x-y|}\right) && \text{if } n = 2m. \end{aligned}$$

Estimates from below are much harder to obtain and crucially depend on the boundary conditions. For the second order Laplace equation under homogeneous Dirichlet boundary conditions, Zhongxin Zhao in [9, 10] (see also [7]) was the first to obtain the sharp two-sided result. For biharmonic Green functions optimal estimates were recently proven in [4]. Surprisingly enough, optimal results for homogeneous Neumann boundary conditions do not seem to be known, at least I have not been able to trace them. In this manuscript we will show that the estimates for the Dirichlet Green function lead to optimal estimates for the Green function under Neumann boundary conditions.

Of course we cannot use the Laplace equation since 0 lies in the spectrum if one considers Neumann boundary conditions, but we shall show that on a bounded domain (connected open subset) $\Omega \subset \mathbb{R}^n$ with $n \ge 3$ and a smooth boundary $\partial\Omega$, the kernel function $N_\Omega(x,y)$ for the boundary value problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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International Series of Numerical Mathematics 161, DOI 10.1007/978-3-0348-0249-9_4,

satisfies for some positive constant $c_\Omega$ :

$$N_{\Omega,1}(x,y) \geq c_\Omega |x-y|^{2-n} \quad \text{for all } x,y \in \Omega.$$

In (1) $\nu$ stands for the (exterior) normal.

For $\Omega$ a bounded domain in $\mathbb{R}^2$ one may show that

$$N_{\Omega,1}(x,y) \geq c_\Omega \log\left(1+\frac{1}{|x-y|}\right) \quad \text{for all } x,y \in \Omega.$$

For the proof we will use the maximum principle directly, elementary estimates for Green functions and the fundamental solution, and the fact that for $f \gneqq 0$ the solution of (1) satisfies $u > 0$.

## 1 Elementary Estimates

Let $N_{\Omega,1}(x,y)$ denote the kernel function for (1), that is, the solution of (1) can be written as

$$u(x) = \int_\Omega N_{\Omega,1}(x,y) f(y)\, dy.$$

By the strong maximum principle it holds that $N_{\Omega,1}(x,y) > 0$ on $\overline{\Omega} \times \overline{\Omega}$. Indeed, for $f \geq 0$ the maximum principle implies that the solution of (1) cannot have a negative interior minimum. Hopf's boundary point lemma implies $\partial_\nu u(x_0) < 0$ if $u$ attains a negative minimum at a boundary point $x_0$. So $N_{\Omega,1}(x,y) \geq 0$ and the strong maximum principle together with Hopf's boundary point lemma implies $N_{\Omega,1}(x,y) > 0$.

We will also exploit estimates for the Green function on balls. Indeed, let $B_1(0)$ be the unit ball in $\mathbb{R}^n$ and consider for $c \geq 0$ the boundary value problem

$$\begin{cases} -\Delta u + cu = f & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0). \end{cases} \tag{2}$$

Then there exists a unique Green function $G_{B_1(0),c}(x,y)$ such that the solution of (2) satisfies

$$u(x) = \int_{B_1(0)} G_{B_1(0),c}(x,y) f(y)\, dy.$$

The typical behavior of $y \mapsto N_{\Omega,1}(x,y)$ and $y \mapsto G_{\Omega,1}(x,y)$ is illustrated in Fig. 1.

Let us start by recalling some properties of these Green functions and the fundamental solution. First we recall the radially symmetric solutions of

$$(-\Delta + c)F = 0 \quad \text{on } \mathbb{R}^n \backslash \{0\}. \tag{3}$$

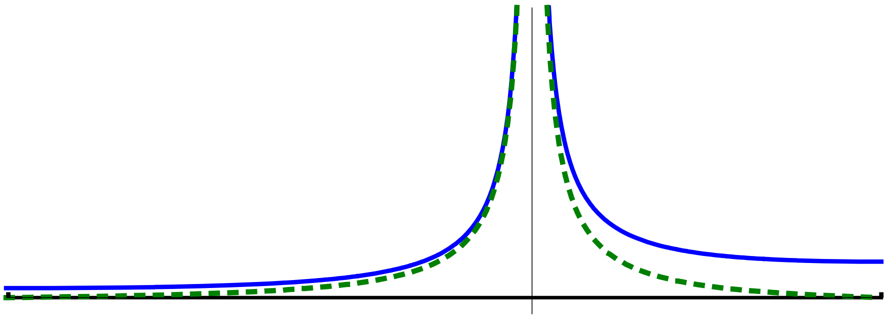

**Fig. 1** Typical behavior of Neumann and Green (*dashed*) function

**Lemma 1** *Let $c \in \mathbb{R}^+$ and $n \geq 2$. Every radially symmetric solution of* (3) *can be written by using modified Bessel functions of the first and second kind*:

$$F(x) = c_1 |x|^{\frac{2-n}{2}} K_{\frac{2-n}{2}}\left(\sqrt{c}|x|\right) + c_2 |x|^{\frac{2-n}{2}} I_{\frac{n-2}{2}}\left(\sqrt{c}|x|\right). \tag{4}$$

*Remark 1.1* The modified Bessel functions of the first kind are defined by:

$$I_\nu(r) = \frac{r^\nu}{2^\nu} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}r)^{2m}}{m!\,\Gamma(\nu + m + 1)}.$$

For $\nu \in \mathbb{Z}$ one finds $I_\nu(r) = I_{-\nu}(r)$. For $\nu \notin \mathbb{Z}$ the functions $I_\nu$ and $I_{-\nu}$ are independent, but even then it will be convenient to introduce the modified Bessel functions of the second kind, which are defined by:

$$K_\nu(r) = \begin{cases} \pi \dfrac{I_{-\nu}(r) - I_\nu(r)}{2\sin(\nu\pi)} & \text{for } \nu \notin \mathbb{Z}, \\ \lim\limits_{\mu \to \nu} \pi \dfrac{I_{-\mu}(r) - I_\mu(r)}{2\sin(\mu\pi)} & \text{for } \nu \in \mathbb{Z}. \end{cases}$$

Next we consider fundamental solutions for $-\Delta + c$, by which we mean a radially symmetric solution $F$ of

$$\begin{cases} -\Delta F + cF = \delta_0 & \text{in distributional sense}, \\ \lim\limits_{|x| \to \infty} F(x) = 0. \end{cases} \tag{5}$$

**Lemma 2** *Let $c \in \mathbb{R}^+$ and $n \geq 2$. Then there exists a unique fundamental solution $F_{n,c} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ of* (5), *namely*

$$\begin{cases} F_{2,c}(x) = \dfrac{1}{2\pi} K_0\left(\sqrt{c}|x|\right) & \text{if } n = 2, \\ F_{n,c}(x) = \dfrac{1}{(n-2)\omega_n} |x|^{2-n} f_n\left(\sqrt{c}|x|\right) & \text{if } n > 2, \end{cases} \tag{6}$$

*with $\omega_n$ the surface area of the unit sphere in $\mathbb{R}^n$, and with*

$$f_n(r) = \frac{2}{\Gamma(\frac{n-2}{2})}\left(\frac{1}{2}r\right)^{\frac{n-2}{2}} K_{\frac{2-n}{2}}(r). \tag{7}$$

*Moreover, $F_{n,c}(x)$ is strictly positive on $\mathbb{R}^n \setminus \{0\}$ and*

$$\text{Strictly decreasing:} \quad \frac{\partial}{\partial|x|}F_{n,c}(x) < 0 \quad \text{for } |x| > 0. \tag{8}$$

*Remark 2.1* For $n \in \{1, 2\}$ and $c = 0$ there is no radially symmetric solution of (3) that goes to zero at infinity. When $c > 0$, there exists for any $n \in \mathbb{N}^+$ a fundamental solution.

*Proof* The radially solutions of $(-\Delta + c)F = 0$ on $\mathbb{R}^n \backslash\{0\}$ are given by (4). From [8, Formula (1) and (2) in 7·23.] one finds that $\lim_{|x|\to\infty} F(x) = 0$ if and only if $c_2 = 0$. Calculating the appropriate constant to find $-\Delta F + cF = \delta_0$, leads to the constant in (7).

In radial coordinates one finds $-\Delta + c = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + c$ and an ode-argument shows that any solution to

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + c\right)F(r) = 0$$

cannot have a positive local maximum, nor can it have a negative local minimum. Since $\lim_{|x|\downarrow 0} F_{n,c}(x) = +\infty$ and $\lim_{|x|\to\infty} F_{n,c}(x) = 0$ one finds as a consequence that (8) holds and that $F_{n,c}$ is positive. □

Let us recall some properties of the functions $f_n$ defined in (7) from [8, Formula (1) in 7·23.]. For $n \geq 3$ the function $f_n$

- is bounded in 0, namely $f_n(0) = 1$,
- goes exponentially to zero at $\infty$, more precisely, with $c_n = \sqrt{\pi}\, 2^{\frac{3-n}{2}}\, \Gamma(\frac{n-2}{2})^{-1}$:

$$r^{2-n} f_n(r) = r^{\frac{1-n}{2}}\, e^{-r}\left(c_n + \mathcal{O}\left(\frac{1}{r}\right)\right) \quad \text{for } r \to \infty.$$

**Lemma 3** *Let $c \in \mathbb{R}^+$ and $n \geq 2$. The Green function for (2) on $B_1(0) \subset \mathbb{R}^n$ has the following properties*:

- *for $y = 0$*:

$$\begin{cases} x \mapsto G_{B_1(0),c}(x,0) & \text{is radially symmetric and} \\ \text{strictly decreasing:} & \frac{\partial}{\partial|x|}G_{B_1(0),c}(x,0) < 0 \quad \text{for } 0 < |x| \leq 1; \end{cases} \tag{9}$$

- *for all* $y \in B_1(0)$ *and* $x \in B_1(0)\backslash\{y\}$:

$$G_{B_1(0),c}(x,y) \geq \begin{cases} C_{n,c}\min\left(1, \dfrac{(1-|x|)(1-|y|)}{|x-y|^2}\right)|x-y|^{2-n} & \text{if } n \geq 3, \\ C_{2,c}\log\left(1+\dfrac{(1-|x|)(1-|y|)}{|x-y|^2}\right) & \text{if } n = 2, \end{cases} \tag{10}$$

*where* $C_{n,c}$ *is a positive constant.*

*Proof* Since the Green function is unique, $x \mapsto G_{B_1(0),c}(x,0)$ is radially symmetric. Hence $G_{B_1(0),c}(x,0)$ can be written as in (4). Then similar arguments as for the fundamental solution imply (9).

The second statement is the standard estimate from below for the Green function. The estimate for the Green function for $-\Delta$ is due to Zhao. The fact that lower order perturbations of the Green function (as long as one does not pass the first eigenvalue) allow the same estimate, goes back to Ancona [1]. This result may also be found in the book by Chung and Zhao [2]. See also [7]. □

## 2 On Starshaped and Convex Domains

**Theorem 4** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded smooth domain which is starshaped with respect to* $y$. *Then there is a positive constant* $c_\Omega$ *such that*

$$N_{\Omega,1}(x,y) \geq \left\{\begin{matrix} c_\Omega|x-y|^{2-n} & \text{if } n \geq 3 \\ c_\Omega\log\left(1+\dfrac{1}{|x-y|}\right) & \text{if } n = 2 \end{matrix}\right\} \quad \text{for all } x \in \Omega. \tag{11}$$

*In fact, if* $F_{n,1}$ *is the fundamental solution for* $-\Delta + 1$ *on* $\mathbb{R}^n$, *then*

$$N_{\Omega,1}(x,y) \geq F_{n,1}(x-y) \quad \text{for all } x \in \Omega. \tag{12}$$

*Proof* Consider

$$u(x) = N_{\Omega,1}(x,y) - F_{n,1}(x-y).$$

So $-\Delta u + u = 0$ in $\Omega$. Since $\Omega$ is starshaped, one finds that the outside normal $\nu_x$ in $x \in \partial\Omega$ satisfies

$$(x-y)\cdot\nu_x \geq 0. \tag{13}$$

Due to (9) and (13) the function $u$ satisfies

$$\frac{\partial}{\partial\nu}u(x) = -\frac{\partial}{\partial\nu}F_{n,1}(x-y) \geq 0 \quad \text{for all } x \in \partial\Omega. \tag{14}$$

By the maximum principle, a solution of $-\Delta u + u = 0$ in $\Omega$ cannot have a negative interior minimum. So if $u$ is somewhere negative, then the minimum of $u$ is assumed at a boundary point, say at $\tilde{x} \in \partial\Omega$ and by Hopf's boundary point lemma $\frac{\partial}{\partial \nu} u(\tilde{x}) < 0$, which contradicts (14). So $u \geq 0$ in $\Omega$ and (12) has been proven. The estimate in (11) follows from the behavior of the fundamental solution. □

**Corollary 5** *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain which is convex. Then there is a positive constant $c_\Omega$ such that*

$$N_{\Omega,1}(x,y) \geq \left\{ \begin{matrix} c_\Omega |x-y|^{2-n} & \text{if } n \geq 3 \\ c_\Omega \log\left(1 + \frac{1}{|x-y|}\right) & \text{if } n = 2 \end{matrix} \right\} \quad \text{for all } x, y \in \Omega. \tag{15}$$

*In fact, if $F_{n,1}$ is the fundamental solution for $-\Delta + 1$ on $\mathbb{R}^n$, then*

$$N_{\Omega,1}(x,y) \geq F_{n,1}(x-y) \quad \text{for all } x, y \in \Omega. \tag{16}$$

*Proof* Since $\Omega$ is convex, it is starshaped with respect to any $y \in \Omega$ and one may use Theorem 4. □

One may localize the results above, in the sense that when the boundary is locally convex, similar results can be obtained by considering Green functions instead of the fundamental solution.

**Theorem 6** *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let $d$ be such that $\Omega \cap B_{2d}(y)$ is starshaped with respect to $y$. Then there is a positive constant $c_{\Omega,d}$ such that*

$$N_{\Omega,1}(x,y) \geq \left\{ \begin{matrix} c_{\Omega,d} |x-y|^{2-n} & \text{if } n \geq 3 \\ c_{\Omega,d} \log\left(1 + \frac{1}{|x-y|}\right) & \text{if } n = 2 \end{matrix} \right\} \quad \text{for all } x \in \Omega \cap B_d(y). \tag{17}$$

*In fact, if $G_{B_{2d}(y),1}$ is the Green function for $-\Delta + 1$ on $B_{2d}(y)$, then*

$$N_{\Omega,1}(x,y) \geq G_{B_{2d}(y),1}(x,y) \quad \text{for all } x \in \Omega \cap B_{2d}(y). \tag{18}$$

*Remark 6.1* The estimate in (18) allows us to find a uniform estimate from below near convex boundary parts. A straightforward application near nonconvex boundary parts would force us to reduce the radius of $B_{2d}(y)$ and hence would fail to give a uniform estimate. This complication can be overcome if one could locally transform the non-convex part of the domain $\Omega$ to a convex set without changing the differential operator. The Möbius transforms, which are discussed in the next section, offer such a possibility. The corresponding estimates we shall see in the last section.

*Proof* The arguments are similar as in the proof of Theorem 4 except that one uses a Green function instead of the fundamental solution. Consider

$$u(x) = N_{\Omega,1}(x,y) - G_{B_{2d}(y),1}(x,y)$$

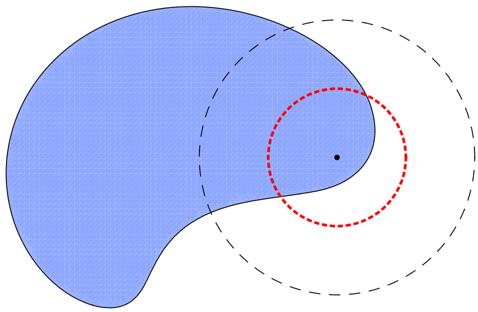

**Fig. 2** A sketch of $\Omega$ and disks $B_{2d}(y)$ and $B_d(y)$ such that $\Omega \cap B_{2d}(y)$ is starshaped with respect to $y$. Then we find $G_{B_{2d}(y),1}(x,y) < N_{\Omega,1}(x,y)$ for $x \in \Omega \cap B_{2d}(y)$, which supplies us with the appropriate estimate for $x \in \Omega \cap B_d(y)$

and one finds

$$\begin{cases} u(x) = N_{\Omega,1}(x,y) > 0 & \text{for } x \in \Omega \cap \partial B_{2d}(y), \\ -\dfrac{\partial}{\partial \nu} u(x) = \dfrac{\partial}{\partial \nu} G_{B_{2d}(y),1}(x,y) \le 0 & \text{for } x \in \partial\Omega \cap B_{2d}(y). \end{cases}$$

By the strong maximum principle it follows that $u > 0$ for $x \in \Omega \cap B_{2d}(y)$, that is (18), and hence also (17). See Fig. 2. □

**Corollary 7** *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain such that $\Omega \cap B_{3d}(z)$ is convex. Then there is a positive constant $c_{\Omega,d}$ such that*

$$N_{\Omega,1}(x,y) \ge \left\{ \begin{matrix} c_{\Omega,d}|x-y|^{2-n} & \text{if } n \ge 3 \\ c_{\Omega,d} \log\left(1 + \dfrac{1}{|x-y|}\right) & \text{if } n = 2 \end{matrix} \right\} \quad \text{for all } x, y \in \Omega \cap B_d(z). \tag{19}$$

*Proof* For any $y \in B_d(z)$ one finds that $B_{2d}(y) \subset B_{3d}(z)$ and that $\Omega \cap B_{2d}(y)$ is starshaped with respect to $y$. By using Theorem 6 one finds (18) and hence (19). □

## 3 Recalling Möbius Transformations

In 2 dimensions every simply connected domain can be mapped on the unit disk by a conformal mapping. Conformal mappings almost preserve the Laplace operator. This allows one to compare Green and Neumann functions on arbitrary domains to explicitly known ones on the disk. In higher dimensions only very few nontrivial conformal mappings exist. These are the so-called Möbius transformations.

**Definition 8** A mapping $h : \mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\}$ is called a Möbius transformation, if

$$h = \phi_1 \circ j_0 \circ \phi_2$$

with $\phi_i$ $(i = 1, 2)$ a similarity:

$$\phi_i(x) = a_i + c_i M_i x \quad \text{for } M_i \text{ an orthogonal } n \times n \text{ matrix, } c_i \in \mathbb{R}^+ \text{ and } a_i \in \mathbb{R}^n;$$

and $j_0$ the standard inversion:

$$j_0(x) = \frac{x}{\|x\|^2} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}.$$

*Remark 8.1* A similarity transformation by itself trivially preserves the Laplace operator except for a scaling constant.

*Remark 8.2* To exploit the relation with holomorphic functions in 2 dimensions, one usually considers similarities $\phi(z) = \alpha + \beta z$ for $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$, and as similarity $j_0(z) = z^{-1} = \frac{\bar{z}}{|z|^2}$ where $z = x_1 + i x_2$.

It is known, see e.g. Corollary 2 of [3], that for a Möbius transformation $h$ it holds that

$$\Delta\big(J_h^{\frac{1}{2}-\frac{1}{n}} u \circ h\big) = J_h^{\frac{1}{2}+\frac{1}{n}} (\Delta u) \circ h,$$

where $J_h = |\det(\frac{\partial h_i}{\partial x_j})|$. So

$$(-\Delta + 1)\big(J_h^{\frac{1}{2}-\frac{1}{n}} u \circ h\big) = J_h^{\frac{1}{2}+\frac{1}{n}} \big(\big(-\Delta + J_h^{-\frac{2}{n}} \circ h^{-1}\big) u\big) \circ h.$$

Setting

$$\tilde{f} = (-\Delta + 1)\big(J_h^{\frac{1}{2}-\frac{1}{n}} u \circ h\big) : \Omega \to \mathbb{R},$$

$$f = \big(-\Delta + J_h^{-\frac{2}{n}} \circ h^{-1}\big) u : h(\Omega) \to \mathbb{R},$$

one finds

$$\tilde{f} = J_h^{\frac{1}{2}+\frac{1}{n}} \, f \circ h$$

and that the Green, i.e. $Bu = u$, and Neumann, $Bu = \frac{\partial}{\partial \nu} u$, functions for

$$\begin{cases} -\Delta \tilde{u} + \tilde{u} = \tilde{f} & \text{in } \Omega, \\ B\tilde{u} = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u + \big(J_h^{-\frac{2}{n}} \circ h^{-1}\big) u = f & \text{in } h(\Omega), \\ Bu = 0 & \text{on } \partial h(\Omega), \end{cases}$$

are related through

$$\begin{aligned} \big(J_h^{\frac{1}{2}-\frac{1}{n}} u \circ h\big)(x) = \tilde{u}(x) &= \int_\Omega G_{\Omega,1}(x, y) \tilde{f}(y)\, dy \\ &= \int_\Omega G_{\Omega,1}(x, y) J_h^{\frac{1}{2}+\frac{1}{n}}(y) f\big(h(y)\big)\, dy \end{aligned}$$

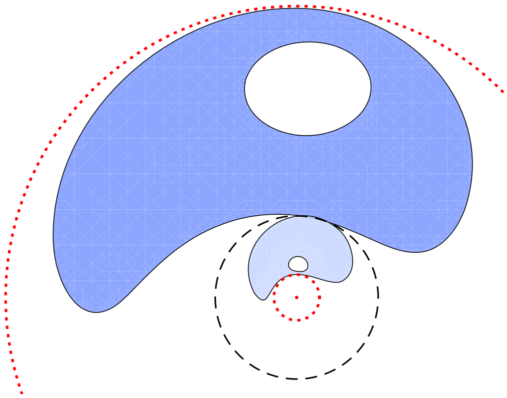

**Fig. 3** The original domain is *dark*; the Möbius transformed domain is *lighter*. The *dashed black circle* is used for the reflection. Since $\Omega$ and $h(\Omega)$ lie between the two *dotted red circles* the factor $J_h$ is bounded where it matters

and

$$J_h^{\frac{1}{2}-\frac{1}{n}}(x)u\big(h(x)\big) = J_h^{\frac{1}{2}-\frac{1}{n}}(x)\int_{h(\Omega)} G_{h(\Omega),J_h^{-2/n}}\big(h(x),z\big)f(z)\,dz$$
$$= J_h^{\frac{1}{2}-\frac{1}{n}}(x)\int_{h(\Omega)} G_{h(\Omega),J_h^{-2/n}}\big(h(x),h(y)\big)f\big(h(y)\big)J_h(y)\,dy.$$

That is

$$J_h^{\frac{1}{n}-\frac{1}{2}}(x)G_{\Omega,1}(x,y)J_h^{\frac{1}{n}-\frac{1}{2}}(y) = G_{h(\Omega),J_h^{-2/n}}\big(h(x),h(y)\big).$$

Similar formulae hold when the Green function is replaced by the Neumann function. The fact that $h$ is conform implies that

$$\frac{\partial}{\partial\nu_x}(u\circ h)(x) = 0 \quad\Leftrightarrow\quad \left(\frac{\partial}{\partial\nu_y}u\right)\big(h(x)\big) = 0.$$

The function $J_h(y)$ is bounded from above and bounded away from 0 on compact sets that do not contain the center of the reflection circle. Note that a Möbius mapping which is an inversion (circular reflection) with respect to a circle outside of $\Omega$ that intersects the boundary at one point, maps the boundary locally into a convex boundary and the image domain stays away from the center. See Fig. 3.

## 4 Smooth Domains

**Theorem 9** *Let $\Omega$ satisfy a uniform exterior sphere condition, then there is a positive constant $c_\Omega$ such that*

$$N_{\Omega,1}(x,y) \ge \left\{\begin{array}{ll} c_\Omega|x-y|^{2-n} & \textit{if } n\ge 3 \\ c_\Omega\log\left(1+\dfrac{1}{|x-y|}\right) & \textit{if } n=2 \end{array}\right\} \quad \textit{for all } x,y\in\Omega.$$

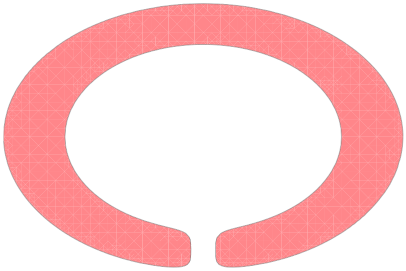

**Fig. 4** Taking $x$ and $y$ near and on opposite sides of the bottom gap, one expects $N_{\Omega,c}(x,y)$ to be much smaller than in the case where $[x,y] \subset \Omega$

*Remark 9.1* For non-convex domains there is no hope to find a result as in (16) as one may guess from Fig. 4.

*Proof* We only have to prove this estimate for $|x-y| < \delta$ for some fixed $\delta > 0$ and for $x$ and $y$ near a part of the boundary which is not convex. All other cases are covered by the results in Sect. 2. For this remaining case we use a Möbius mapping that is a circular reflection in a sphere that touches the boundary. We use that for $f \geq 0$ the solutions of

$$\begin{cases} -\Delta u_\alpha + \alpha u_\alpha = f & \text{in } \Omega, \\ Bu_\alpha = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_\beta + \beta u_\beta = f & \text{in } \Omega, \\ Bu_\beta = 0 & \text{on } \partial\Omega, \end{cases}$$

can be compared:

$$\text{If } \alpha \geq \beta \geq 0 \quad \text{then } u_\alpha \leq u_\beta.$$

Indeed, since $u_\alpha \geq 0$ one finds

$$(-\Delta + \beta)(u_\beta - u_\alpha) = (\alpha - \beta) u_\alpha \geq 0$$

and the maximum principle implies $u_\beta \geq u_\alpha$. After this circular reflection the new domain has become locally convex and we may use the estimates from Sect. 2 to find the result we want. □

**Acknowledgements** This note would not have been written, if Pavol Quittner had not been asking the right questions. I thank him for the inspiration. I also thank the referee for his comments which thoroughly simplified the proof.

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# Part 2
# Numerical Methods

# Deriving Inequalities in the Laguerre-Pólya Class from Properties of Half-Plane Mappings

**Prashant Batra**

**Abstract** Newton, Euler and many after them gave inequalities for real polynomials with only real zeros. We show how to extend classical inequalities ensuring a guaranteed minimal improvement. Our key is the construction of mappings with bounded image domains such that existing coefficient criteria from complex analysis are applicable. Our method carries over to the Laguerre-Pólya class $\mathcal{L}$–$\mathcal{P}$ which contains real polynomials with exclusively real zeros and their uniform limits. The class $\mathcal{L}$–$\mathcal{P}$ covers quasi-polynomials describing delay-differential inequalities as well as infinite convergent products representing entire functions, while it is at present not known whether the Riemann $\xi$-function belongs to this class. For the class $\mathcal{L}$–$\mathcal{P}$ we obtain a new infinite family of inequalities which contains and generalizes the Laguerre-Turán inequalities.

**Keywords** Coefficient inequalities · Reality of zeros · Moment problem · Hankel determinants · Logarithmic derivative

**Mathematics Subject Classification** 26D05 · 30D20 · 33C45

## 1 Introduction

The study of astronomical constellations, control mechanisms or chemical reactions leads to equations with limited locus of the corresponding zeros. A typical property of such equations is that all corresponding zeros belong to the open interior of a half-plane or lie on the real axis, maybe even only on a line-segment. Newton already searched for characterizations of algebraic polynomials with such restricted root-location.

**Theorem 1** (Newton) *Let* $P(x) = \sum_0^n a_i x^i$ *be a real polynomial of degree* $n$ *with exclusively negative roots. Then it holds true for all* $k$ *with* $1 \leq k \leq n-1$ *that*

$$a_k \geq \sqrt{\frac{k+1}{k}\frac{n+1-k}{n-k}}\sqrt{a_{k-1}a_{k+1}}. \tag{1}$$

We want to derive a method which allows to consider more than three consecutive coefficients. Such additional criteria should yield a quantifiable improvement over

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Newton's criterion Theorem 1 to justify the increased effort. We show that for the even and odd parts of stable polynomials quantifiably stronger results hold true. This is profitable for applications such as stability tests for interval polynomials arising from control problems.

Moreover, we show that our computational approach extends easily to entire functions which are uniform limits of real polynomials with real zeros i.e., the computational approach extends to the Laguerre-Pólya class $\mathcal{L}\text{–}\mathcal{P}$. For control theoretic problems this implies that we may apply our necessary stability tests to the even and odd part respectively, of quasi-polynomials describing delay-differential equations.

We are not restricted to essentially even expansions. Varying our arguments, we show how the best-known inequality for $f \in \mathcal{L}\text{–}\mathcal{P}$, the Laguerre-Turán inequality:

$$f(x)f''(x) \leq f'(x)^2, \quad x \in \mathbb{R} \quad \Leftrightarrow \quad \begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} \leq 0, \quad x \in \mathbb{R}, \tag{2}$$

can be embedded into an infinite family. Such embeddings have been sought after for a long time as in [9, 10, 13] and [14]. One reason for this is the fact that the Riemann hypothesis (RH) holds true if and only if the Riemann $\xi$-function, given by

$$\frac{\xi(\frac{x}{2})}{8} := \int_0^\infty \cos(xt) \sum_{n=1}^\infty \big(2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t}\big) e^{-n^2\pi e^{4t}}\, dt = \int_0^\infty \cos(xt)\Phi(t)\, dt,$$

is in $\mathcal{L}\text{–}\mathcal{P}$, for references *viz.* [30] or [33].

**NB** Pólya remarked in [25] on the futility of a numerical proof of the Riemann hypothesis *loc. cit. p. 9*:

> Überhaupt scheint mir (wie den meisten andern in der Frage Interessierten) das Vorhaben, ein derartiges Problem durch numerische Rechnung entscheiden zu wollen, aussichtslos zu sein.

and became more explicit on this point *loc. cit. p. 33*:

> Man kann durch eine numerische Rechnung, die mit beschränkter Genauigkeit geführt werden muss, nie entscheiden, ob eine Zahl irrational ist oder nicht, und ebensowenig ob sie genau $= 1$ ist oder nicht.
> Wenn man aber bei dem erreichbaren Genauigkeitsgrad der numerischen Rechnung etwa 0,9999 nicht von 1,0001 zu unterscheiden vermag, kann man auch die Fälle, in denen $[2\int_0^\infty \Psi(t)\cos(zt)\,dt = 2\sqrt{2\pi e}e^{-\frac{z^2}{2}}(a + \cos(z))]$ nur reelle Nullstellen hat, nicht von denen unterscheiden, in denen es keine hat.

Pólya's remark addresses the *possibility to numerically decide the RH*. Not mentioning a possible falsification, Csordas, Norfolk and Varga inquired into the numerical correctness of the Laguerre-Turán inequalities (2) for the RH. This investigation led to an analytic proof of inequality (1) for a whole class of integral-transforms in $\mathcal{L}\text{–}\mathcal{P}$ *viz.* [11, 12, 33]. With the hope that our new approach together with the obtained inequalities stated will be equally useful for the community of researchers in complex and numerical analysis, we commence.

## 2 Essentially Even Polynomials and Positive Mappings

Given a real polynomial $P$, Hurwitz [18] considered the quotient of the odd part divided by the even part of $P$ i.e., the rational function

$$H(P)(x) := \frac{P(x) - P(-x)}{P(x) + P(-x)}, \quad \text{where } P \in \mathbb{R}[x]. \tag{3}$$

To motivate the introduction of $H(P)$, we may consider the argument growth of $P$ on the boundary of the left open half-plane, namely, the argument growth on the imaginary axis. If the argument growth of $P$ is maximal, the Cauchy index of $H(P)$ is maximal. Hurwitz computes in [18] this Cauchy index via quadratic forms, and cleverly splitting and re-arranging derives his well-known determinantal characterization of 'open left half-plane'-stability i.e., Hurwitz-stability. If $P$ is Hurwitz-stable, then $H(P)$ is a mapping from the right half-plane to itself, and the converse holds true under some mild restrictions cf. [19]. Hurwitz' arguments can be re-arranged and adapted if we consider the mapping property of $H(P)$ instead of the argument growth of $H(P)$, for one such discussion see [19]. We wish to exploit solely the mapping property in the remainder of the paper.

**Definition 2** A function $f$ analytic in the open right half-plane of the complex numbers is called *positive*: $\Leftrightarrow \Re f(s) > 0$ for all $s \in \mathbb{C}$ with $\Re s > 0$.

*Remark* The class of (strictly) positive functions is of great importance for questions involving the existence and design of stabilizing controllers, adaptive control and simultaneous stabilization (see, e.g., [34]), but seemingly was not used to derive stability criteria other than Hurwitz'.

Relying on the Hermite-Biehler theorem (cf., e.g., [17]) which shows that Hurwitz-stability of a *real* polynomial is equivalent to interlacing of the zeros of the even and odd part on the imaginary axis we construct odd, positive functions as follows.

**Lemma 3** *Suppose $P \in \mathbb{R}[x]$ is a Hurwitz-stable polynomial of degree $n > 0$. Let $p$ denote either of the following four functions: the even or the odd part of $P$ defined by $P_e(x) := \frac{1}{2}(P(x) + P(-x))$ or $P_o(x) := \frac{1}{2}(P(x) - P(-x))$, respectively, or the reduced even or odd parts, $P_e(\sqrt{x})$ or $P_o(\sqrt{x})/\sqrt{x}$, respectively. Then*

$$p^{(j)}(x)/p^{(j-1)}(x), \quad j = 1, \ldots, n,$$

*is a positive function.*

*Proof* The Hermite-Biehler theorem guarantees that the even and odd part have all zeros on the imaginary axis, and the same holds true for the derivatives by Rolle's theorem. Hence, it suffices to prove that $p'/p$ is positive. This follows from the representation

$$p'(s)/p(s) = \sum \frac{1}{s - \zeta_i}. \tag{4}$$

□

Positive functions may be expressed via certain Stieltjes integrals. In the case of an odd, real function bounded near zero we use the following auxiliary result (which is mentioned as an exercise in Henrici's classic textbook [17]).

**Theorem 4** *A positive real, analytic function $f$ has an asymptotic expansion near the origin given by*

$$f(s) \sim c_0 s + c_1 s^3 + c_2 s^5 + O\big(s^7\big), \quad s \to 0,\ \Re s > 0, \tag{5}$$

*if and only if either of the following conditions hold true*:

i) *the determinants*

$$\begin{aligned} H_k^{(0)} &:= |c_{i+j}|, \\ H_k^{(1)} &:= (-1)^{k+1}|c_{i+j+1}|, \quad 0 \le i, j \le k, \end{aligned} \tag{6}$$

*are positive for all $k$*;

ii) *the determinants*

$$\begin{aligned} H_k^{(0)} &:= |c_{i+j}|, \\ H_k^{(1)} &:= (-1)^{k+1}|c_{i+j+1}|, \quad 0 \le i, j \le k, \end{aligned} \tag{7}$$

*are positive for $k \le k_0$, and identically zero for $k > k_0$.*

*In the latter case, the function $f$ is a rational function of degree $k_0$.*

*Proof* In the context of interpolation with positive real functions (see, e.g., [38]) the following special Stieltjes integral representation (due to Cauer) for real, positive functions $f$ bounded near the origin is known:

$$f(s) = s \int_0^\infty \frac{1}{1 + \tau^2 s^2}\, d\psi(\tau), \quad \Re s > 0, \tag{8}$$

where $\psi$ is non-decreasing and bounded.

The positive analytic function $f$ given by (8) is represented by the asymptotic expansion (5) near the origin if and only if all moments $\mu_k := \int_0^\infty \tau^{2k}\, d\psi(\tau)$, $k \ge 0$, exist, in which case $c_k = (-1)^k \mu_k$. This follows from the known arguments connecting the existence of moments to the asymptotic expansion, and may be established by [17, Th. 12.9h, p. 582] or [1, Lemma 3.3.6, p. 111].

The existence of moments $\mu_k = \int_0^\infty \tau^{2k}\, d\psi(\tau)$ in the Stieltjes moment problem is equivalent [19] to the mutual exclusive conditions i), ii) above on the Hankel determinants constructed with $c_k = (-1)^k \mu_k$. This yields the theorem. □

We may apply the preceding theorem to a function $r$ of the form $r(x) = \frac{q'(x)}{q(x)}$, where $q$ is a three-term polynomial, to derive a weak version of Newton's inequality for the even or odd part of a polynomial $P$. In splitting the polynomial $P$, we will have to be careful to get the degrees of the resulting polynomials right. An even-degree polynomial $P$ with $\deg(P) = n$, has $n/2 =: N$ odd coefficients and $N + 1$ even coefficients. To index the coefficients $a_{2k+\tau}$ ($\tau \in \{0, 1\}$ indicates the parity of $n$) via $k$ we use $\mu := N - \tau \cdot (2N + 1 - n)$, where $N$ is the integer $\lfloor n/2 \rfloor$. For even $n$ and $\tau = 1$ we have $\mu = N - 1$, whereas in all other cases $\mu = N$.

**Corollary 5** *Let $P(x) = \sum_0^n a_i x^i$ be Hurwitz-stable with positive coefficients. Let us designate by $N$ the integer $\lfloor n/2 \rfloor$. Let $\tau \in \{0, 1\}$. Let $\mu := N - \tau \cdot (2N + 1 - n)$. Then for $k \in \mathbb{N}$ such that $1 \le k \le \mu - 1$ the following holds true*:

$$a_{2k+\tau} \ge \sqrt{\frac{k+1}{k}\frac{\mu - (k-1)}{\mu - k}}\sqrt{a_{2k-2+\tau}a_{2k+2+\tau}}. \tag{9}$$

We show how to derive this straight-forward corollary of Newton's inequalities from the non-negative Hankel determinants associated with a positive rational mapping. (This corollary to Newton's Theorem 1 was recently re-discovered by several authors who were interested into applications *cf.* [7, 36, 37].)

*Proof* Consider the real Hurwitz-stable polynomial $P(x) = \sum_{i=0}^n a_i x^i$ of degree $n$. Put $N := \lfloor n/2 \rfloor$, and let us fix the index shift $\tau \in \{0, 1\}$. If $\tau = 0$, we consider the reduced even part $P_e(\sqrt{x})$, and if $\tau = 1$ we consider the reduced odd part $P_o(\sqrt{x})/\sqrt{x}$, and denote the chosen intermediary polynomial by $R$. The chosen polynomial $R$ has degree $\mu$.

To obtain limits involving $a_{2p+\tau}$, $a_{2p+2+\tau}$ and $a_{2p+4+\tau}$ we compute the $p$th derivative of $R$ as

$$r_1(x) = p!a_{2p+\tau} + (p+1)!a_{2p+2+\tau}x + \frac{(p+2)!}{2}a_{2p+4+\tau}x^2 + \cdots.$$

We take the reciprocal $r_1^*(x) = x^{\mu-p} r_1(1/x)$ of $r_1(x)$, and, with the short-hand

$$\gamma_p^\mu := (\mu - p)!p!,$$

we obtain the $(\mu - p - 2)$nd derivative (of $r_1^*(x)$), called $r_2(x)$, as

$$a_{2p+\tau}\frac{\gamma_p^\mu}{2} + a_{2p+2+\tau}\gamma_{p+1}^\mu x^2 + a_{2p+4+\tau}\frac{\gamma_{p+2}^\mu}{2}x^4.$$

The first three coefficients $c_i$ of the Taylor expansion $r_2'(x)/r_2(x) = \sum c_i x^{2i+1}$ read, using auxiliary terms $s_i$, as

$$s_0 := \frac{(p+1)a_{2p+2+\tau}}{(\mu - p)a_{2p+\tau}},$$

$$s_1 := \frac{(p+2)(p+1)a_{2p+4+\tau}}{(\mu - p)(\mu - p - 1)a_{2p+\tau}},$$

$$c_0 = 4 \cdot s_0, \qquad c_1 = -8s_0^2 + 4s_1, \qquad c_2 = -12s_0 s_1 + 16s_0^3.$$

The non-negativity of the Hankel determinant $H_1^{(0)}$ of order 2 i.e., the fact that $c_0 \cdot c_2 - c_1^2 \geq 0$, yields the inequalities (9) of Corollary 5 after an index shift from $p \geq 0$ to $k = p + 1$. □

The double series (6) and (7) of nonnegative Hankel determinants allow to improve on the lower bound (9) i.e., using more information we improve on the Newton inequalities for essentially even structures.

**Theorem 6** *Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be Hurwitz-stable with positive coefficients. Let us designate by $N$ the integer $\lfloor n/2 \rfloor$. Let $\tau \in \{0, 1\}$, and $\mu := N - \tau \cdot (2N + 1 - n)$. Let $\sigma_k = \sqrt{\frac{k+1}{k}\frac{\mu-k+1}{\mu-k}}$. Then for all $k \in \mathbb{N}$ such that $1 \leq k \leq \mu - 2$ the following holds true.*

1.
$$a_{2k-2+\tau} \leq \frac{3}{\sigma_k^2} \frac{a_{2k+2+\tau} a_{2k+\tau}^2}{4a_{2(k+1)+\tau}^2 - \sigma_{k+1}^2 a_{2(k+2)+\tau} a_{2k+\tau}} \tag{10}$$

$$\leq \frac{a_{2k+\tau}^2}{\sigma_k^2 a_{2(k+1)+\tau}}. \tag{11}$$

2. *If $a_{2k+\tau}a_{2k+2+\tau} = f \cdot (\sigma_k \cdot \sigma_{k+1})^2 a_{2k-2+\tau}a_{2k+4+\tau}$, where $f \geq 1$, we obtain*

$$a_{2k+\tau} \geq L(f) \cdot \sigma_k \cdot \sqrt{a_{2k-2\tau}a_{2k+2+\tau}},$$

*where*

$$L(f) := \sqrt{\frac{9f^2 - 1 + 10f - \sqrt{81f^4 + 82f^2 - 108f^3 + 1 - 20f}}{12f}} \geq 1.$$

3. *Let*

$$f_1 := \frac{a_{2k+2+\tau}^{3/2}}{\sigma_k(\sigma_{k+1})^2\sqrt{a_{2k-2+\tau}}a_{2k+4+\tau}},$$

*then it holds true that*

$$a_{2k+\tau} > L(f_1)\sigma_k\sqrt{a_{2k-2+\tau}a_{2k+2+\tau}}. \tag{12}$$

We established this result earlier (*viz.* [5]) using the quadratic form associated with the asymptotic expansion of $q'/q$ where $q$ is a suitable four-term polynomial. The following proof employs the Maclaurin expansions at Zero, and avoids the artificial perturbation term employed in our work [5].

*Proof* To obtain the inclusion result of Theorem 6, take the even or odd part of the polynomial $P$, reduce it via differentiation (as above), to the four term expression $R(x)$ given by

$$\frac{\gamma^{\mu_p}}{3!}a_{(2p+\tau)} + \frac{\gamma^{\mu}_{p+1}}{2!}a_{(2p+2+\tau)}x^2 + \frac{\gamma^{\mu}_{p+2}}{2!}a_{(2p+4+\tau)}x^4 + \frac{\gamma^{\mu}_{p+3}}{3!}\cdot a_{(2p+6+\tau)}x^6,$$

with derivative $R'(x)$

$$\gamma^{\mu}_{p+1}a_{2p+2+\tau}x + 2\gamma^{\mu}_{p+2}a_{2p+4+\tau}x^3 + \gamma^{\mu}_{p+3}a_{2p+6+\tau}x^5,$$

where $0 \le p \le \mu - 3$ and $\gamma^{\mu}_{p} := (\mu - p)!p!$.

With $s_2 := \frac{(p+3)(p+2)(p+1)a_{2p+6+\tau}}{(\mu-p)(\mu-p-1)(\mu-p-2)a_{2p+\tau}}$ and $s_1$, $s_0$ as in the proof of Corollary 5 the odd, positive function $R'(x)/R(x)$ with expansion $\sum c_i x^{2i+1}$ has the first four coefficients $c_i$

$$c_0 = 6\cdot s_0, \qquad c_1 = 12\cdot s_1 - 18s_0^2,$$

$$c_2 = 6s_2 - 54s_0s_1 + 54s_0^3, \qquad c_3 = -162s_0^4 - 36s_1^2 + 216s_0^2s_1 - 24s_2s_0.$$

Again we have the inequality

$$H_1^{(0)} = c_0\cdot c_1 - c_2^2 \ge 0.$$

This yields an expression involving $a_{2k+\tau},\ldots,a_{2k+6+\tau}$. Multiplying it by $a^4_{2p+\tau}(\mu - p)^3(\mu-p-1)^2(\mu-p-2)/((p+1)^2(p+2))$ obtain the inequality

$$\begin{aligned}
&-144(\mu-p)(\mu-p-2)(p+2)a^2_{2p+\tau}a^2_{2p+4+\tau}\\
&\quad + 108(\mu-p-1)(\mu-p-2)(p+1)a_{2p+\tau}a^2_{2p+2+\tau}a_{2p+4+\tau}\\
&\quad + 36(\mu-p)(\mu-p-1)(p+3)a^2_{2p+\tau}a_{2p+2+\tau}a_{2p+6+\tau} \ge 0.
\end{aligned}$$

Equating the left-hand side to zero, we obtain the estimate (10) after an index shift from $p$ to $k = p+1$. The estimate (10) is not weaker than (11) by Corollary 5.

If the polynomial $P$ is Hurwitz-stable, all Hankel determinants (7) are non-negative according to Theorem 4. Consider especially the inequality

$$H_1^{(1)} = c_1\cdot c_3 - c_2^2 \ge 0. \tag{13}$$

This yields an inequality involving $a_{2p+\tau},\ldots,a_{2p+6+\tau}$.

With $\phi := \frac{a_{2(p+2)+\tau}}{(\sigma_{p+1}\cdot\sigma_{p+2})^2 a_{2p+\tau} a_{2(p+3)+\tau}}$ the three-term inequalities for $p$ and $p+1$ from Corollary 5 yield

$$f_1 = \frac{a_{2(p+2)+\tau}^{3/2}}{\sigma_{p+1}(\sigma_{p+2})^2\sqrt{a_{2p+\tau}}\,a_{2(p+3)+\tau}} \le \phi a_{2(p+1)+\tau} \le f_1^2.$$

Multiplying the two three-term inequalities yields

$$a_{2p+2+\tau}a_{2p+4+\tau} \ge (\sigma_{p+1}\cdot\sigma_{p+2})^2 a_{2p+\tau}a_{2p+6+\tau}. \tag{14}$$

We may express (14) as an equality using a suitable multiplier $f \ge 1$ for the right-hand side of (14), and may re-order to obtain

$$\frac{1}{f}\frac{p+1}{p+3}\frac{\mu-(p+2)}{\mu-p}\frac{a_{2p+2+\tau}a_{2p+4+\tau}}{a_{2p+\tau}} = a_{2p+6+\tau}. \tag{15}$$

We note here that $a_{2(p+1)+\tau}\cdot\phi = f$.

We may obtain an expression involving just $a_{2p+\tau}$, $a_{2p+2+\tau}$ and $a_{2p+4+\tau}$ from substitution of (15) into the inequality (13).

This yields the following non-negative expression (where $0 \le p \le \mu-3$)

$$\begin{aligned}
&-\frac{216}{f}\frac{(p+2)(p+1)^5 a_{2p+4+\tau}}{(\mu-p-1)(\mu-p)^5 a_{2p+\tau}^5}\cdot a_{2p+2+\tau}^4 \\
&+(324+\frac{360}{f}-\frac{36}{f^2})\frac{(p+2)^2(p+1)^4 a_{2p+4+\tau}^2 a_{2p+2+\tau}^2}{(\mu-p-1)^2(\mu-p)^4 a_{2p+\tau}^4} \\
&-\frac{432(p+2)^3(p+1)^3 a_{2p+4+\tau}^3}{(\mu-p)^3(\mu-p-1)^3 a_{2p+\tau}^3}.
\end{aligned}$$

The term $a_{2p+2+\tau}$ appears as a square and a fourth power inside this expression. To ease comparison of results shift the indices from $p$ to $k = p+1$ (which yields the index range $1 \le k \le \mu-2$). Equate the expression to zero, solve the corresponding equation for the middle index, $a_{2k+\tau}$, and obtain the positive solutions

$$U(f)\cdot\sigma_k\sqrt{a_{2k-2+\tau}a_{2k+2+\tau}}, \qquad L(f)\cdot\sigma_k\sqrt{a_{2k-2+\tau}a_{2k+2+\tau}},$$

where the factors $U(f)$ and $L(f)$ (with $U(f) > L(f)$) are given by the following pair of positive roots

$$\left(\frac{9f^2-1+10f \pm ((9f^2-1+10)^2-288f)^{0.5}}{12f}\right)^{0.5}.$$

It is easily verified that $f=1$ yields $U(f)=\sqrt{2}$, $L(f)=1$, and that the quartic becomes negative if $a_{2k+\tau}\to+\infty$. We use the estimate $f_1 \le f = a_{2(k+1)+\tau}\phi \le f_1^2$

in place of the unknown $f > 1$, and may compute $L(f_1)$ with $L(f) \geq L(f_1) \geq 1$. Hence, the real coefficient $a_{2k+\tau}$ is bounded from below as

$$a_{2k+\tau} \geq L(f) \cdot \sigma_k \sqrt{a_{2k-2+\tau} a_{2k+2+\tau}} \geq L(f_1) \cdot \sigma_k \sqrt{a_{2k-2+\tau} a_{2k+2+\tau}}. \qquad \square$$

Theorem 6 gives improved lower bounds compared to (9), and an additional upper bound.

*Example* Consider a real, Hurwitz-stable *interval polynomial*

$$P(x) = \sum_{i=0}^{6} a_i x^i \quad \text{with positive coefficients } a_i \text{ in real intervals } \left[\alpha_i^-, \alpha_i^+\right] \subset \mathbb{R}.$$

The inequalities (9) with $N := \lfloor n/2 \rfloor = 3$ (and $\tau = 0$, $\mu = N$) for the four coefficients of even index yield

$$a_2 \geq \sqrt{\frac{2N}{N-1}} \sqrt{a_0 a_4}, \quad \text{and} \quad a_4 \geq \sqrt{\frac{3}{2} \frac{N-1}{N-2}} \sqrt{a_2 a_6}. \tag{16}$$

Suppose that $a_2$ and $a_0$ are allowed to vary in some finite interval. Let $a_4 = 1600$ and $a_6 = 1$. If $a_2 \leq 6500$, the constraints (16) yield the estimate $a_0 \leq 8802.08$. The new estimate (10) of Theorem 6 yields

$$a_0 \leq 6614.15,$$

while the maximal value retaining stability is slightly smaller than 6609.968.

If the admissible lower bound for $a_2$ is unknown, and $a_0 \geq 5000$, then the inequalities (16) yield $a_2 \geq 4898.97$.

With $f_1 = \frac{a_{2k+2+\tau}^{3/2}}{\sigma_k (\sigma_{k+1})^2 \sqrt{a_{2k-2+\tau}} a_{2(k+2)+\tau}} \sim 174.186$ our new inequality (12) of Theorem 6 yields

$$a_2 \geq 5653.24.$$

This is close to the lowest possible value of approximately 5653.726. Hence, before any test for Hurwitz-stability of an interval family

$$P(x) = x^6 + a_5 x^5 + 1600 x^4 + a_3 x^3 + [5000, 6500] x^2 + a_1 x^1 + [5000, 8000]$$

we may exclude certain coefficient values, and limit the test to

$$P(x) = x^6 + a_5 x^5 + 1600 x^4 + a_3 x^3 + [5653, 6500] x^2 + a_1 x^1 + [5000, 6615].$$

## *2.1 Functions Derived from Quasi-polynomials*

**Definition 7** A finite sum $\sum \sum a_{k,v} z^k e^{\tau_v z} \in \mathbb{R}(z)$, where $a_{k,v} \in \mathbb{R}$, $\tau_v \in \mathbb{R}$, is called a *real quasi-polynomial*.

It is an ill-posed problem to decide stability of quasi-polynomials with commensurate delays i.e., to decide stability for quasi-polynomials with a set of delays $\tau_v$ such that $\tau_w/\tau_v \in \mathbb{Q}$ for all possible indices. Stability of a *real* quasi-polynomial with commensurate delays is equivalent to positivity of the infinity of principal minors of the associated Hurwitz form $H$, while the case of incommensurate delays bounded in intervals is NP-hard by results of Özbay and Toker, (for reference, see [16]).

To obtain necessary, low-order conditions for quasi-polynomials we invoke the following result of Schwengeler (for reference, see [16]) which shows that the genus of quasi-polynomials is at most one while the *genus of the zeros* is zero (for terminology and basic results in the theory of entire functions *viz.* [6] or [30]).

**Lemma 8** (Schwengeler) *A real quasi-polynomial G may be written as*

$$G(z) = z^m e^{\alpha \cdot z + \beta} \prod \left(1 - \frac{z}{z_n}\right), \quad \alpha, \beta, \in \mathbb{R},\ z_n \in \mathbb{C} \setminus \{0\}.$$

Suppose the real quasi-polynomial $G$ has a non-trivial product part. If $G$ is Hurwitz-stable, then $F(z) := G(\sqrt{-1} \cdot z) = F_R(z) + \sqrt{-1} \cdot F_I(z)$ has all zeros on the upper half-plane. The real and imaginary part, $F_R$ and $F_I$ respectively, have all zeros on the real axis according to a well-known generalization of the Hermite-Biehler theorem (attributed by Levin [23] to Tschebotareff and Meiman). As these functions are real and of order less than two, a result of Laguerre's [6] shows that all derivatives $F_R^{(k)}$ and $F_I^{(k)}$ for $k \in \mathbb{N}_0$ have all zeros on the real axis.

Fix now $f = f(z)$ as one of $F_R^{(k)}(z)$ or $[F_I(z)/z]^{(k)}$. Use Hadamard's representation of the logarithmic derivative of entire functions

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + g'(z) + \sum_{n=1}^{\infty} \left[\frac{1}{z - z_n}\right], \quad m \in \mathbb{N} \cup 0,\ g(z) = \alpha \cdot z + \beta, \tag{17}$$

to establish that $-f'/f$ maps the upper half-plane to itself, and hence that

$$\sqrt{-1} \cdot \frac{f'(\sqrt{-1} \cdot z)}{f(\sqrt{-1} \cdot z)}$$

is an odd, positive function real on the real axis. This odd, positive mapping gives rise to the following new necessary stability condition.

**Proposition 9** *Given a real quasi-polynomial $F$ and a parity index $\tau \in \{0, 1\}$, we choose the even or odd part of $F$ respectively, and denote it by $f$. If the expansion of $f$ near the origin is*

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^{2\nu+\tau},$$

*and if* $a_k \neq 0$, *then* $a_{k+2}$ *is contained in the real interval with center*

$$\frac{(k+1)a_{k+1}^2}{4(k+2)a_k} \tag{18}$$

*and radius*

$$\frac{1}{4(k+2)|a_k|}\sqrt{a_{k+1}^4 \cdot (k+1)^2 + 8a_{k+1}a_{k+3}a_k^2 \cdot \left(k^2+5k+6\right)}. \tag{19}$$

*Proof* Consider $a_k \neq 0$ with $k = 2\nu + \tau$ for some $\nu \in \mathbb{N}_0$ The function $R(z) := f^{(2\nu+\tau+1)}(z)/f^{(2\nu+\tau)}(z)$ is analytic near the origin as $a_k \neq 0$. Compute the first three coefficients $c_i$ of the expansion

$$R(z) = f^{(2\nu+\tau+1)}(z)/f^{(2\nu+\tau)}(z) = c_0 z + c_1 z^3 + c_2 z^5 + c_5 z^7 + \cdots,$$

where

$$f^{(k)}(\sqrt{z}) := k!a_k + (k+1)!a_{k+1}z + \frac{(k+2)!}{2}a_{k+2}z^2 + \frac{(k+3)!}{6}a_{k+3}z^3 + \cdots,$$

as

$$c_0 := (k+1)\frac{a_{k+1}}{a_k}, \tag{20}$$

$$c_1 := (k+1)(k+2)\frac{a_{k+2}}{a_k} - (k+1)^2\frac{a_{k+1}^2}{a_k^2}, \tag{21}$$

$$\begin{aligned} c_2 := {} & 1/2(k+1)(k+2)(k+3)\frac{a_{k+3}}{a_k} + (k+1)^3\frac{a_{k+1}^3}{a_k^3} \\ & + \frac{-3/2 \cdot a_{k+1}a_{k+2}k^3 - 6a_{k+1}a_{k+2}k^2 - 15/2 \cdot a_{k+1}a_{k+2}k - 3a_{k+1}a_{k+2}}{a_k^2}. \end{aligned} \tag{22}$$

The function $R$ maps the upper half-plane to the lower half-plane. After a suitable transformation, we may use Cauer's integral representation (8) used in the proof of Theorem 4 together with its determinantal characterization to obtain non-negativity of the two sequences of Hankel determinants $H_m^{(0)} = |c_{i+j}|$ and $H_m^{(1)} = (-1)^{m+1}|c_{i+j+1}|$, $0 \leq i, j \leq m$.

We multiply the value $c_0c_2 - c_1^2$ of the Hankel determinant $H_2^{(0)}$ by $a_k^4 > 0$ to obtain from $|c_{i+j}| \geq 0$ the equivalent inequality

$$a_k^4\left(c_0c_2 - c_1^2\right) \geq 0.$$

Writing out the $c_j$ in terms of $a_\nu$ we obtain the following inequality constraint for the coefficients:

$$\left(-a_k^2k^4 - 6a_k^2k^3 - 12a_k^2k - 4a_k^2 - 13a_k^2k^2\right)a_{k+2}^2$$

$$+ \big(1/2a_k a_{k+1}^2 k^4 + 5/2a_k a_{k+1}^2 k^3 + 9/2a_k a_{k+1}^2 k^2 + 7/2a_k a_{k+1}^2 \cdot k + a_k a_{k+1}^2\big) a_{k+2}$$
$$+ 3a_{k+1}a_{k+3}a_k^2 + 1/2a_k^2 a_{k+1}a_{k+3}k^4 + 7/2a_{k+1}a_{k+3}a_k^2 k^3$$
$$+ 17/2a_{k+1}a_{k+3}a_k^2 k^2 + 17/2a_{k+1}a_{k+3}a_k^2 k \geq 0.$$

This inequality may not hold true if for fixed $a_k \neq 0$ and real $a_{k+1}$, $a_{k+3}$ the coefficient $a_{k+2} \in \mathbb{R}$ grows without bounds. Solving the corresponding quadratic equation

$$a_k^4 \big(c_0 c_2 - c_1^2\big) = 0$$

for $a_{k+2}$ shows that the real coefficient must be contained in the closed real interval with center

$$\frac{(k+1)a_{k+1}^2}{4(k+2)a_k}$$

and radius

$$\frac{1}{4(k+2)|a_k|}\sqrt{a_{k+1}^4 \cdot (k+1)^2 + 8a_{k+1}a_{k+3}a_k^2 \cdot \big(k^2+5k+6\big)}. \qquad \square$$

*Remark* Our first written proof of Proposition 9 was given in the unpublished thesis [4], although the result was presented somewhat earlier [2, 3].

The above result for essentially even expansions compares to Laguerre's inequalities as follows. Suppose $f$ is chosen as the even or odd part of a real quasi-polynomial $F$, and $\tau$ is chosen to be 0 or 1, respectively, such that $f$ satisfies the assumptions of the preceding proposition. Consider then the even function $\phi(z) := f(z)/z^\tau$. As all roots are real, we may apply Laguerre's inequality for $k$ and $k+1$ so that we have

$$(k+1)a_{k+1}a_{k-1} < ka_k^2, \quad (k+2)a_{k+2}a_k < (k+1)a_{k+1}^2, \quad \text{where } k \geq 1.$$

Shifting indices we obtain

$$\frac{k+2}{k+1}a_{k+2}a_k < a_{k+1}^2, \quad \frac{k+3}{k+2}a_{k+3}a_{k+1} < a_{k+2}^2, \quad \text{for } k \geq 0. \tag{23}$$

If $a_k < 0$, we obtain no coefficient inclusion from Laguerre's inequalities. If $a_k > 0$, combine the inequalities to obtain the four-term expression

$$\frac{k+3}{k+2}a_{k+3}a_{k+1} < a_{k+2}^2 < \left(\frac{k+1}{k+2}\frac{a_{k+1}^2}{a_k}\right)^2 =: L^2.$$

The upper bound $L > a_{k+2}$, combining two consecutive Laguerre inequalities, is strictly weaker than our new upper bound from Proposition 9 with coefficient interval center $c := \frac{(k+1)a_{k+1}^2}{4(k+2)a_k}$ and new radius $\rho$ given by (18) as

$$\rho = \frac{1}{4(k+2)a_k}\sqrt{a_{k+1}^4 \cdot (k+1)^2 + 8a_{k+1}a_{k+3}a_k^2 \cdot \left(k^2+5k+6\right)},$$

because the inequalities (23) show that $\rho < \frac{3}{4}\frac{k+1}{k+2}\frac{a_{k+1}^2}{a_k}$, and thus $c + \rho < L$.

*Example 1* Consider the even function

$$\left(a_0 + a_2 z^2 + a_4 z^4\right)\cos\lambda_1 z + \left(\alpha_0 + \alpha_2 z^2 + \alpha_4 z^4\right)\cos\lambda_2 z.$$

We may test for reality of zeros using Corollary 9.

*Example 2* Consider the function

$$U_1 + U_2 + \sqrt{-1}(V_1 + V_2), \tag{24}$$

where

$$\begin{aligned}
U_1 &:= \frac{2}{\pi}\int_0^1 \left(1-t^2\right)^{-1/2}\cos\lambda_1 zt\,dt \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m(\lambda_1 z/2)^{2m}}{m!\Gamma(m+1)} =: J_0(\lambda_1 z), \\
V_1 &:= \frac{2}{\pi}\int_0^1 \left(1-t^2\right)^{-1/2}\sin\lambda_2 zt\,dt \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m(\lambda_2 z/2)^{2m+1}}{\Gamma(m+3/2)^2} =: H_0(\lambda_2 z), \\
U_2 &:= \int_0^1 t\cos\lambda_3 zt\,dt = \frac{\cos(\lambda_3 z) + (\lambda_3 z)\sin(\lambda_3 z)}{(\lambda_3 z)^2} - \frac{1}{(\lambda_3 z)^2}, \\
V_2 &:= \int_0^1 t\sin\lambda_4 zt\,dt = \frac{\sin(\lambda_4 z) - (\lambda_4 z)\cos(\lambda_4 z)}{(\lambda_4 z)^2} \\
&= \frac{\cos(\lambda_4 z)}{(\lambda_4 z)^2}\left(\tan(\lambda_4 z) - \lambda_4 z\right).
\end{aligned}$$

The function $U_1$ is identifiable as the scaled Bessel-function $J_0$, while $V_1$ is essentially the so-called Struve function $H_0$ (see [35, §10.4, p. 328]).

We may test stability for parameters $\lambda_i$ in a neighborhood of 1 applying our new Proposition 9. The non-trivial fact that the function defined in (24) is stable for the choice of parameters $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$ is a consequence of results obtained by Pólya in [26].

## 3 Coefficient Inequalities for Real Entire Functions

Real entire functions with zeros restricted to a line or half-plane satisfy certain characteristic inequalities if the roots lie sufficiently separated. For a real entire function $f$ with exclusively real zeros $x_k$ satisfying $\sum_{x_k \neq 0} x_k^{-2} < 1$, Laguerre proved (for reference, see, e.g. [9]) that for all $x \in \mathbb{R}$

$$f(x)f''(x) \leq f'(x)^2 \quad \text{which may be formulated as} \quad \begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} \leq 0. \qquad (25)$$

When evaluated at the special point $x = 0$, the resulting coefficient inequalities are known as Turán inequalities [32].[1] The Laguerre-Turán inequalities (25) received special interest in connection with the Riemann hypothesis as well as in the theory of orthogonal polynomials. For orthogonal polynomials, the inequalities (25) may be transformed to prove inequalities on the interval of orthogonality. Szegö [28] derived from (25) Turán's inequality

$$P_{n-1}(x)P_{n+1}(x) \leq P_n(x)^2, \quad x \in [-1, 1],\ n \geq 1,$$

for the ultraspherical as well as other orthogonal polynomials, see [29].

The Riemann hypothesis holds true if and only if (*cf.* [10, 30]) the function (of order 1/2)

$$F_0(z) := \sum_{m=0}^{\infty} \frac{\hat{b}_m z^m}{(2m)!}, \quad \text{where } \hat{b}_k := \int_0^{\infty} t^{2k} \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t}\right) e^{-n^2\pi e^{4t}}\, dt,$$

has only negative zeros. Correctness of the Laguerre-Turán inequalities (25) for $F_0$ for every real $x$, especially for $x = 0$, is a necessary condition for the reality of the zeros of $F_0$. Pólya remarked in 1927 that it was not known whether these latter inequalities i.e., the estimates

$$\hat{b}_k^2 \geq \frac{2k-1}{2k+1} \hat{b}_{k-1} \hat{b}_{k+1}, \quad k \in \mathbb{N}, \qquad (26)$$

held true. Correctness of these inequalities was established by Csordas, Norfolk, and Varga in 1986 [11]. Subsequently, Csordas and Varga proved (26) as a consequence of a general inequality for a family of integral moments $\int_0^{\infty} t^{2k} e^{\lambda t^2} \Phi(t)\, dt$.

Given $f(x) = \sum \frac{\gamma_k}{k!} x^k$, $f$ in the Laguerre-Pólya class (defined in the following section), Turán's inequalities read $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$. These are generalized by the Mařik-Dimitrov inequality [24] (Dimitrov [14] lifted Mařik's original inequality from polynomials to the Laguerre-Pólya class applying Lemma 12 below *viz.* [20]) which connects four coefficients of $f$ via

$$4\left(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1}\right)\left(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}\right) - (\gamma_k\gamma_{k+1} - \gamma_{k-1}\gamma_{k+2})^2 \geq 0, \quad \text{for } k \in \mathbb{N}. \qquad (27)$$

[1] Csordas and Dimitrov pointed out in [10] that the name Euler-Laguerre-Pólya-Schur-Turán inequalities would reflect the major contributors.

A *conjectured* infinite family of equations, called ‘Higher order Turán inequalities’ (in [10]), was considered in [10] as well as in [9] and [8]. The question of the general validity of these inequalities for functions with only non-negative coefficients in the Laguerre-Pólya class was considered in the cited works because an ‘affirmative answer [...] would provide a set of strong necessary conditions for an entire function to have only real negative zeros’ (cf. [10, Problem 3.1]). Thereby motivated, we proceed to deduce our new infinite family of inequalities generalizing the Laguerre-Turán inequalities.

We show how to derive the Mařik-Dimitrov inequality (27) together with Laguerre’s inequality (25) as the first members of a newly established infinite family. Thus, we embed these inequalities in a common framework for the first time here, and give an independent new derivation of Mařik’s inequality.

## 4 New Infinite Family of Inequalities for the Class $\mathcal{L}$–$\mathcal{P}$

Real polynomials with exclusively real zeros together with their uniform limits essentially constitute the Laguerre-Pólya class (*viz.* [10], also for references pertaining to the subsequent basic lemmas).

**Definition 10** A real entire function $f$ belongs to the *Laguerre-Pólya* class (we write: $f \in \mathcal{L}$–$\mathcal{P}$) if

$$f(x) = cx^m e^{-ax^2+bx} \prod_{m=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}}$$

$$\text{with } a \geq 0,\ b \in \mathbb{R},\ c, x_k \in \mathbb{R} \setminus \{0\},\ \text{and } \sum_k x_k^{-2} < \infty.$$

The condition $\sum_k x_k^{-2} < \infty$ on the non-vanishing zeros implies that $f$ is the product of a function of genus at most one with the factor $e^{-ax^2}$, $a \geq 0$.

Differentiation of $f \in \mathcal{L}$–$\mathcal{P}$ is an operation inside the class $\mathcal{L}$–$\mathcal{P}$ (which follows from a classical result of Laguerre’s) as noted by Pólya.

**Lemma 11** *The class $\mathcal{L}$–$\mathcal{P}$ is closed under differentiation.*

We may derive inequalities for a transcendental entire function in the Laguerre-Pólya class via the following connection to polynomials.

**Lemma 12** (Pólya/Schur) *A power series $f$ analytic about* 0 *with expansion*

$$f(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$$

*has exclusively real* (*negative*) *roots if and only if all non-constant* Jensen polynomials

$$\sum_{j=0}^{n} \frac{\gamma_j}{j!(n-j)!} x^j$$

*have only real* (*negative*) *roots*.

Differentiating $F(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{–}\mathcal{P}$ yields $f(x) := F^{(l)}(x) = \sum_{k=0}^{\infty} \frac{\gamma_{k+l}}{k!} x^k$. Using Lemmas 11 and 12, we find that $f$ lies in $\mathcal{L}\text{–}\mathcal{P}$ together with the Jensen polynomials

$$\sum_{j=0}^{n} \frac{\gamma_{j+l}}{j!(n-j)!} x^j .$$

This allows to apply a well-known characterization of functions in $\mathcal{L}\text{–}\mathcal{P}$ to obtain an infinite family of inequalities.

**Proposition 13** *Let* $f(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{–}\mathcal{P}$, *and* $r_n(x) := \sum_{j=0}^{n} \frac{\gamma_j}{j!(n-j)!} x^{n-j}$. *Let* $s_v$, $v \in \mathbb{N}_0$, *be determined by the function* $g := r_n'/r_n = (\log r_n)'$ *via the expansion at Infinity*

$$g(z) = \sum_{v=1}^{\infty} \frac{s_{v-1}}{z^v}, \quad s_v \in \mathbb{R}.$$

*Then we have the inequalities*

$$\begin{vmatrix} s_0 & s_1 & \cdots & s_l \\ s_1 & s_2 & \dots & s_{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_l & s_{l+1} & \cdots & s_{2l} \end{vmatrix} > 0 \quad \textit{for all } l \textit{ with } 0 \le l \le n-1. \tag{28}$$

This result follows from a famous, deep result of Grommer's (*viz.* [15]), partly rederived by Kritikos [22] and later Tchebotareff [31], see also the exercise collection by Pólya and Szegö [27, Problem 43.1, p. 103, and p. 289], or alternatively Krein's crisp review of Grommer's achievements (pointing to a minor, but famous, error) in [21].

*Proof* By assumption we have $f \in \mathcal{L}\text{–}\mathcal{P}$, hence its Jensen polynomial $R_n(x) := \sum_{j=0}^{n} \frac{\gamma_j}{j!(n-j)!} x^j$ and the reciprocal $r_n(x) := x^n R_n(1/x)$ have only real zeros. Thus, we can invoke Grommer's result Th. III, [15], p.156*f.*, which implies positivity of the Hankel determinants (28) connected with $g$. □

We deduce in the following the Laguerre-Turán inequalities and the Mařik-Dimitrov inequalities as special consequences of the above result. Given $f(x) =$

$\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{–}\mathcal{P}$, we obtain for $n = 2$, $n = 3$ and $n = 4$ the following reciprocals of Jensen polynomials:

$$r_2(x) := \frac{\gamma_k}{2} x^2 + \frac{\gamma_{k+1}}{1} x + \frac{\gamma_{k+2}}{2},$$
$$r_3(x) := \frac{\gamma_k}{6} x^3 + \frac{\gamma_{k+1}}{2} x^2 + \frac{\gamma_{k+2}}{2} x + \frac{\gamma_{k+3}}{6},$$
$$r_4(x) := \frac{\gamma_k}{24} x^4 + \frac{\gamma_{k+1}}{6} x^3 + \frac{\gamma_{k+2}}{4} x^2 + \frac{\gamma_{k+3}}{6} x + \frac{\gamma_{k+4}}{24}.$$

We obtain the following expansions at Infinity for the rational functions $r_2'/r_2$ and $r_3'/r_3$ (if $\gamma_k \neq 0$):

$$\frac{r_2'(x)}{r_2(x)} = \frac{2}{x} - \frac{2\gamma_{k+1}}{\gamma_k} \frac{1}{x^2} + \frac{-2\gamma_{k+2}\gamma_k + 4\gamma_{k+1}^2}{\gamma_k^2} \frac{1}{x^3} + \cdots,$$

$$\frac{r_3'(x)}{r_3(x)} = \frac{3}{x} - \frac{3\gamma_{k+1}}{\gamma_k} \frac{1}{x^2} + \frac{-6\gamma_{k+2}\gamma_k + 9\gamma_{k+1}^2}{\gamma_k^2} \frac{1}{x^3}$$
$$+ \frac{-3\gamma_{k+3}\gamma_k^2 + 27\gamma_{k+2}\gamma_{k+1}\gamma_k - 27\gamma_{k+1}^3}{\gamma_k^3} \frac{1}{x^4}$$
$$+ \frac{(12\gamma_{k+1}\gamma_{k+3} + 18\gamma_{k+2}^2)\gamma_k^2 - 108\gamma_{k+2}\gamma_k\gamma_{k+1}^2 + 81\gamma_{k+1}^4}{\gamma_k^4} \frac{1}{x^5} + \cdots.$$

If $\gamma_k \neq 0$, the function $r_2'(x)/r_2(x) = \sum \frac{s_{\upsilon-1}}{x^{\upsilon}}$ gives rise to the positive $2 \times 2$ minor $M_2 := s_0 s_2 - s_1^2$ which yields

$$\gamma_{k+1}^2 > \gamma_k \gamma_{k+2}.$$

This implies the Laguerre-Turán inequalities as $\gamma_{k+1}^2 \geq \gamma_k \gamma_{k+2} = 0$, if $\gamma_k = 0$.

Similarly, if $\gamma_k \neq 0$, we consider $r_3'/r_3$ to obtain the $3 \times 3$ minor $|s_{i+j}|_{\substack{i=0,\ldots,2 \\ j=0,\ldots,2}}$ which is non-negative. After multiplication by $3^3 \cdot \gamma_k^4$, we obtain the inequality

$$-\gamma_{k+3}^2\gamma_k^2 + \left(6\gamma_{k+2}\gamma_{k+1}\gamma_{k+3} - 4\gamma_{k+2}^3\right)\gamma_k - 4\gamma_{k+3}\gamma_{k+1}^3 + 3\gamma_{k+2}^2\gamma_{k+1}^2 \geq 0, \quad k \geq 0.$$

This is a rearrangement (with indices shifted) of the claimed Mařik-Dimitrov inequality (27) (cf. [14, 24]). If $\gamma_k = 0$, Dimitrov's inequality reduces simply to $3\gamma_{k+2}^2\gamma_{k+1}^2 \geq 4\gamma_{k+3}\gamma_{k+1}^3$ which would be trivial if $\gamma_{k+1} = 0$. But if $\gamma_{k+1} \neq 0$ we consider

$$\rho_2(x) := \frac{\gamma_{k+1}}{2} x^2 + \frac{\gamma_{k+2}}{2} x + \frac{\gamma_{k+3}}{6},$$

with logarithmic derivative near Infinity given by

$$\rho_2'(x)/\rho_2(x) = \frac{2}{x} - \frac{\gamma_{k+2}}{\gamma_{k+1}} \frac{1}{x^2} + + \frac{-2/3\gamma_{k+3}\gamma_k + \gamma_{k+2}^2}{\gamma_{k+1}^2} \frac{1}{x^3} + \cdots$$

(which we may have obtained from $r_2'(x)/r_2(x)$ using the formal substitution $\gamma_k \leftarrow \gamma_{k+1}$, $\gamma_{k+1} \leftarrow \gamma_{k+2}/2$, $\gamma_{k+2} \leftarrow \gamma_{k+3}/3$). Computing the $2 \times 2$ minor, we obtain, as above, the claimed inequality: $3\gamma_{k+2}^2\gamma_{k+1}^2 \geq 4\gamma_{k+3}\gamma_{k+1}^3$ (which also follows from Laguerre's inequality using the indicated substitution).

We state the two new five-term inequalities resulting from Proposition 13 for $n = 4$ as a corollary.

**Corollary 14** *Let* $f(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{–}\mathcal{P}$, *then we have the inequalities*

$$\begin{aligned}
&\left(\gamma_{k+4} \cdot \gamma_{k+2} - 3 \cdot \gamma_{k+3}^2\right) \cdot \gamma_k^2 \\
&\quad + \left(-\gamma_{k+4} \cdot \gamma_{k+1}^2 - 9 \cdot \gamma_{k+2}^3 + 14 \cdot \gamma_{k+3} \cdot \gamma_{k+1} \cdot \gamma_{k+2}\right) \cdot \gamma_k \\
&\quad - 8 \cdot \gamma_{k+1}^3 \cdot \gamma_{k+3} + 6 \cdot \gamma_{k+2}^2 \cdot \gamma_{k+1}^2 \geq 0; \\
&\gamma_{k+4}^3 \cdot \gamma_k^3 + \left(54 \cdot \gamma_{k+2} \cdot \gamma_{k+4} \cdot \gamma_{k+3}^2\right. \\
&\quad \left.- 18 \cdot \gamma_{k+2}^2 \cdot \gamma_{k+4}^2 - 12 \cdot \gamma_{k+3} \cdot \gamma_{k+4}^2 \cdot \gamma_{k+1} - 27 \cdot \gamma_{k+3}^4\right) \cdot \gamma_k^2 \\
&\quad + \left(81 \cdot \gamma_{k+2}^4 \cdot \gamma_{k+4} + 54 \cdot \gamma_{k+2} \cdot \gamma_{k+4}^2 \cdot \gamma_{k+1}^2 - 54 \cdot \gamma_{k+2}^3 \cdot \gamma_{k+3}^2\right. \\
&\quad \left.- 6 \cdot \gamma_{k+1}^2 \gamma_{k+4} \gamma_{k+3}^2 + 108 \gamma_{k+2} \gamma_{k+1} \gamma_{k+3}^3 - 180 \gamma_{k+2}^2 \gamma_{k+4} \gamma_{k+3} \gamma_{k+1}\right) \gamma_k \\
&\quad - 64 \gamma_{k+1}^3 \gamma_{k+3}^3 + 108 \gamma_{k+2} \gamma_{k+4} \gamma_{k+1}^3 \gamma_{k+3} - 27 \gamma_{k+1}^4 \gamma_{k+4}^2 \\
&\quad + 36 \gamma_{k+2}^2 \gamma_{k+1}^2 \gamma_{k+3}^2 - 54 \gamma_{k+2}^3 \gamma_{k+4} \gamma_{k+1}^2 \geq 0.
\end{aligned}$$

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# Fundamental Error Estimate Inequalities for the Tikhonov Regularization Using Reproducing Kernels

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*Dedicated to the Memory of Wolfgang Walter*

**Abstract** First of all, we will be concentrated in some particular but very important inequalities. Namely, for a real-valued absolutely continuous function on [0, 1], satisfying $f(0) = 0$ and $\int_0^1 f'(x)^2\,dx < 1$, we have, by using the theory of reproducing kernels

$$\int_0^1 \left(\frac{f(x)}{1-f(x)}\right)'^2 (1-x)^2\,dx \le \frac{\int_0^1 f'^2(x)\,dx}{1-\int_0^1 f'^2(x)\,dx}.$$

A. Yamada gave a direct proof for this inequality with a generalization and, as an application, he unified the famous Opial inequality and its generalizations.

Meanwhile, we gave some explicit representations of the solutions of nonlinear simultaneous equations and of the explicit functions in the implicit function theory by using singular integrals. In addition, we derived estimate inequalities for the consequent regularizations of singular integrals.

Our main purpose in this paper is to introduce our method of constructing approximate and numerical solutions of bounded linear operator equations on reproducing kernel Hilbert spaces by using the Tikhonov regularization. In view of this, for the error estimates of the solutions, we will need the inequalities for the approximate solutions. As a typical example, we shall present our new numerical and real inversion formulas of the Laplace transform whose problems are famous as typical ill-posed and difficult ones. In fact, for this matter, a software realizing the corresponding formulas in computers is now included in a present request for international patent. Here, we will be able to see a great computer power of H. Fujiwara with infinite precision algorithms in connection with the error estimates.

**Keywords** Reproducing kernel · Inequality · Implicit function · Singular integral · Best approximation · Tikhonov regularization · Real inversion of Laplace transform · Infinite precision method

**Mathematics Subject Classification** Primary 30C40 · 46E32 · 44A05 · 44A10 · 35A22 · Secondary 44A15 · 35K05 · 35A22 · 46E22

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# 1 Yamada's Results

Let $H_K$ denote a Hilbert space admitting a reproducing kernel $K$ on a set $E$. For all $f \in H_K$ and for a very general transform $\phi$ of $f$, there exists a naturally determined function $\Phi$ satisfying

$$\|\phi(f)\|^2_{H(\Phi(K))} \leq \Phi\big(\|f\|^2_{H_K}\big). \tag{1}$$

Here, $H(\Phi(K))$ is the reproducing kernel Hilbert space which is determined by the positive definite quadratic function $\Phi(K)$ (cf. [16–19]).

We are considering a very general nonlinear transform $\phi(f)$. As an application of (1), we derived the identification method for the nonlinear system $\phi(f)$ in [23].

As a typical example of (1), in the framework of [17–19] we have that for a real-valued absolutely continuous function on [0, 1], satisfying $f(0) = 0$ and $\int_0^1 f'(x)^2\,dx < 1$, it holds

$$\int_0^1 \left(\frac{f(x)}{1-f(x)}\right)'^2 (1-x)^2\,dx \leq \frac{\int_0^1 f'^2(x)\,dx}{1-\int_0^1 f'^2(x)\,dx},$$

for the nonlinear transform $f + f^2 + f^3 + \cdots$. We would like to call the reader's attention to [15, Appendix] and [18], where some essays on this inequality and mathematics in general can be found.

Meanwhile, we know the Opial inequality [14]: For $f \in \mathrm{AC}[0,a]$ (i.e., an absolutely continuous function on $[0,a]$), with $f(0)=0$, we have

$$\int_0^a \big|f(x)f'(x)\big|\,dx \leq \frac{a}{2}\int_0^a \big|f'(x)\big|^2\,dx.$$

Since this starting result proved in 1960 by Opial, a wide variety of generalizations and extensions was introduced in the last half-century. One of the most natural extensions concerns the weighted Opial inequality

$$\int_a^b \big|f(x)\big|^q f'(x)^r w(x)\,dx \leq C\left(\int_a^b \big|f'(x)\big|^p v(x)\,dx\right)^{(q+r)/p}, \tag{2}$$

with $f(a)=0$. Different characterizations of weights $w$ and $v$ and exponents $p$, $q$, $r$ for which (2) holds true are known. We are particularly interested in the following generalization provided by A. Yamada (see [22]), which he managed to derived by a direct proof.

**Theorem 1** *Let $G$ be a function of class $C^1$ on an interval $(-R,R)$ $(0<R\leq+\infty)$ satisfying the conditions $G(0)=0$, $|G'(x)|\leq G'(|x|)$, for all $x\in(-R,R)$, and if $x^2\leq yz$ $(0<x,y,z<R)$, then $0<G'^2(x)\leq G'(y)G'(z)$.*

*Assume that functions $F$, $f\in \mathrm{AC}[a,b]$ with $F(a)=f(a)=0$ satisfy $F'(x)>0$ a.e. on $[a,b]$, $F(b)\leq R$, and $\int_a^b |f'(t)|^p/F'(t)^{p-1}\,dt < R$ for some $p>1$.*

*Then*,

$$\int_a^b \frac{|(G \circ f)'(x)|^p}{(G \circ F)'(x)^{p-1}}\,dx \le G\left(\int_a^b \frac{|f'(x)|^p}{F'(x)^{p-1}}\,dx\right). \tag{3}$$

*If* $f(x) = C \cdot F(\min\{x, y\})$ $(a < y \le b,\ C = 0, 1)$, *then the equality holds in* (3).

For the proof, from the identity $f(x) = \int_a^x f'(t)\,dt$ and

$$\left|f(x)\right| \le F(x)^{1/q}\left(\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\,dt\right)^{1/p},$$

he used the assumptions (directly) and the Hölder inequality, and so the equality case was also solved, completely. Furthermore, by some specialization of Theorem 1, he was able to give a full generalization (cf. [1–3, 9, 11, 13]) of the Opial inequality with the equality statement:

**Theorem 2** *Let* $s(x), t(x)$ *be nonnegative, measurable functions on* $[a, b]$ *such that* $\int_a^b t(x)^{-1/(p-1)}\,dx < +\infty$ *for some* $p > 1$. *Set* $F(x) = \int_a^x t(\xi)^{-1/(p-1)}\,d\xi$ *and assume that the functions* $G(x)$, $F(x)$ *and* $f(x)$ *satisfy the same conditions as stated in Theorem* 1 *with* $R = +\infty$. *Then, if* $C < +\infty$, *we have*

$$\left\{\int_a^b \left|(G \circ f)'(x)\right|^q s(x)\,dx\right\}^{1/q} \le C \cdot G\left(\int_a^b \left|f'(x)\right|^p t(x)\,dx\right)^{1/p}, \tag{4}$$

*where* $1/p + 1/r = 1/q$, $r > 0$ *and*

$$C = \left\{\int_a^b (G \circ F)'(x)^{r(1-1/p)} s(x)^{r/q}\,dx\right\}^{1/r}.$$

## 2 The Implicit Function Theorem

Let us now turn to the *Implicit Function Theorem*. For a simplification of the statement, we shall assume some global properties: On a smooth bounded domain $U \subset \mathbb{R}^{n+k}$ surrounded by a finite number of $C^1$ class and simple closed surfaces, for $k$ functions

$$f_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}), \quad i = 1, 2, \ldots, k,$$

we assume that for some point on $U$ it holds

$$f_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) = 0$$

and on $U$ we have

$$\det \frac{\partial(f_1, f_2, \ldots, f_k)}{\partial(x_{n+1}, x_{n+2}, \ldots, x_{n+k})}(x) > 0.$$

Then, we assume globally that there exist $k$ functions of $C^1$ class, $g_j(x_1, x_2, \dots, x_n)$ for $j = 1, 2, \dots, k$, on $U \cap \mathbb{R}^n$, satisfying the properties:

$$f_i(x_1, x_2, \dots, x_n, g_1, g_2, \dots, g_k) = 0, \quad i = 1, 2, \dots, k,$$

and

$$x_{n+j} = g_j(x_1, x_2, \dots, x_n), \quad j = 1, 2, \dots, k.$$

We were able to represent the functions $g_j$ explicitly, in terms of the implicit functions $\{f_i\}$, by using singular integrals in the sense of Cauchy's principal value in [4], and by using some explicit representations of the solutions of nonlinear simultaneous equations (cf. [24]). We shall state here the results for the simplest cases.

Let $D \subset \mathbb{R}^2$ be a bounded domain with a finite number of piecewise $C^1$ class boundary components. Let $f$ be a one-to-one $C^1$ class mapping from $\overline{D}$ into $\mathbb{R}^2$ and we assume that its Jacobian $J(x)$ is positive on $D$. We shall represent $f$ in the form

$$\begin{aligned} y_1 &= f_1(x) = f_1(x_1, x_2), \\ y_2 &= f_2(x) = f_2(x_1, x_2) \end{aligned} \tag{5}$$

and the inverse mapping $f^{-1}$ of $f$ as follows:

$$\begin{aligned} x_1 &= \left(f^{-1}\right)_1(y) = \left(f^{-1}\right)_1(y_1, y_2), \\ x_2 &= \left(f^{-1}\right)_2(y) = \left(f^{-1}\right)_2(y_1, y_2). \end{aligned} \tag{6}$$

Then, we would like to represent

$$\begin{pmatrix} (f^{-1})_1(y^*) \\ (f^{-1})_2(y^*) \end{pmatrix} \tag{7}$$

in terms of the direct mapping (5).

Additionally, we are also interested in some numerical and practical solutions of the non-linear simultaneous equations.

**Theorem 3** *For the mappings* (5) *and* (6) *with* (7), *we obtain the representation, for any* $y^* = (y_1^*, y_2^*) \in f(D)$,

$$\begin{aligned} \begin{pmatrix} (f^{-1})_1(y^*) \\ (f^{-1})_2(y^*) \end{pmatrix} &= \frac{1}{2\pi} \oint_{\partial D} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} d \operatorname{Arctan} \frac{f_2(x) - y_2^*}{f_1(x) - y_1^*} \\ &\quad - \frac{1}{2\pi} \int\int_D \frac{1}{|f(x) - y^*|^2} \operatorname{adj} J(x) \begin{pmatrix} f_1(x) - y_1^* \\ f_2(x) - y_2^* \end{pmatrix} dx_1\, dx_2. \end{aligned} \tag{8}$$

*Meanwhile, for an outside point* $y^*$ *of the image* $f(D)$, *the left-hand side of* (8) *is zero.*

**Theorem 4** *For a* $C^1$ *class function* $f(x_1, x_2)$ *on a domain* $U$ *in* $\mathbb{R}^2$, *we assume that for a point* $x^0 = \binom{x_1^0}{x_2^0}$ *it holds*

$$f\left(x_1^0, x_2^0\right) = 0,$$
$$\frac{\partial f}{\partial x_2}\left(x_1^0, x_2^0\right) \neq 0.$$

*Then, there exist a neighborhood* $U_1 \times U_2$ $(\subset U)$ *around the point* $x^0$ *and an explicit function* $g : U_1 \to U_2$ *determined by the implicit function* $f = 0$ *as* $f(x_1, g(x_1)) = 0$ *and, furthermore, it is represented as follows*:

$$g\left(x_1^*\right) = \frac{1}{2\pi}\left(\int_{\partial(U_1 \times U_2)} x_2\, d\theta - \int_{U_1 \times U_2} dx_2 \wedge d\theta\right),$$
$$\theta = \operatorname{Arctan} \frac{f(x_1, x_2)}{x_1 - x_1^*},$$

*for any* $x_1^* \in U_1$.

**Corollary 5** (Representations of the inverse functions) *On an interval* $[a, b]$, *for a* $C^1$ *class function* $f$ *satisfying* $f'(x) > 0$, *its inverse function* $f^{-1}(y^*)$ *on* $[f(a), f(b)]$ *is represented as follows*:

$$f^{-1}\left(y^*\right) = \frac{1}{2\pi}\left(\int_{\partial([a,b]\times[f(a),f(b)])} x\, d\theta_1 - \int_{[a,b]\times[f(a),f(b)]} dx \wedge d\theta_1\right),$$
$$\theta_1 = -\operatorname{Arctan} \frac{y - f(x)}{y - y^*},$$

*for any* $y^* \in [f(a), f(b)]$.

## 3 Singular Integral Estimates

We gave various error estimates for the regularizations for the singular integrals appearing in the representations in Theorems 3 and 4. For example, for the singularity

$$\frac{1}{(|x-y|)^\alpha},$$

we consider the regularization

$$\frac{1}{(|x-y|+\delta)^\alpha}$$

for a small $\delta$ and then analyze their error estimates. For the regularized integrals, their numerical calculations are done easily by using computers.

For example, for

$$\mathcal{K}(x, y) = \frac{1}{(|x-y|+\delta)^{n-\alpha}} - \frac{1}{(|x-y|+\delta')^{n-\alpha}}, \tag{9}$$

we obtained in [21] the following error estimates.

**Theorem 6**

1. *Assume that*

$$1 < \alpha < n+1, \qquad 1 < p < q < \infty, \qquad \frac{1}{q} = \frac{1}{p} - \frac{\alpha - 1}{n}.$$

   *Then, the integral operator $K$ with the kernel $\mathcal{K}$ defined by* (9) *on* $\mathbb{R}^n$, *satisfies* $\|K\|_{L^p \to L^q} \le c|\delta - \delta'|$, *where c is independent of* $\delta$ *and* $\delta'$.
2. *Assume that*

$$1 < \alpha < n+1, \qquad p = \frac{n}{\alpha - 1}.$$

   *Then, the integral operator $K$ with the kernel $\mathcal{K}$ defined by* (9), *satisfies* $\|K\|_{L^p \to BMO} \le c\,|\delta - \delta'|$, *where c is independent of* $\delta$ *and* $\delta'$.
3. *Assume that*

$$1 < \alpha < n+1, \qquad 0 < \beta = \alpha - 1 - \frac{n}{p} < 1.$$

   *Then, the integral operator $K$ with the kernel $\mathcal{K}$ defined by* (9), *satisfies* $\|K\|_{L^p \to \mathrm{Lip}^\beta} \le c|\delta - \delta'|$, *where c is independent of* $\delta$ *and* $\delta'$.

We also considered the integral operator for $p = 1$ and, furthermore, for the integral kernel case,

$$If(x) := \int_\Omega f(y) \log|x-y|\, dy,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$.

Anyhow, for our regularizations, we see that the convergence or error estimates are good.

## 4 Best Approximations

Let $L$ be any bounded linear operator from a reproducing kernel Hilbert space $H_K$ into a Hilbert space $\mathcal{H}$. Then, the following problem is a classical and fundamental problem known as the best approximate mean square norm problem: For any member $\mathbf{d}$ of $\mathcal{H}$, we would like to find

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}}.$$

It is clear that we are considering operator equations, generalized solutions and corresponding generalized inverses within the framework of $f \in H_K$ and $\mathbf{d} \in \mathcal{H}$, having in mind

$$Lf = \mathbf{d}. \tag{10}$$

However, this problem has a complicated structure, specially in the infinite dimension Hilbert spaces case, leading in fact to the consideration of generalized inverses (in the Moore-Penrose sense). Following our theory (cf. [19]), we can realize its complicated structure. Anyway, the problem turns to be well-posed within the reproducing kernels theory framework. That is, the existence, uniqueness and representation of the solutions are mathematically well-formulated as follows.

**Theorem 7** *For any member* $\mathbf{d}$ *of* $\mathcal{H}$, *there exists a function* $\tilde{f}$ *in* $H_K$ *satisfying*

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} = \|L\tilde{f} - \mathbf{d}\|_{\mathcal{H}} \tag{11}$$

*if and only if, for the reproducing kernel Hilbert space* $H_k$, *admitting the kernel defined by* $k(p,q) = (L^*LK(\cdot,q), L^*LK(\cdot,p))_{H_K}$,

$$L^*\mathbf{d} \in H_k. \tag{12}$$

*Furthermore, when there exists a function* $\tilde{f}$ *satisfying the condition* (11), *then there is a uniquely determined function that minimizes the norms in* $H_K$ *among the functions satisfying the equality, and its function* $f_{\mathbf{d}}$ *is represented as follows*:

$$f_{\mathbf{d}}(p) = \big(L^*\mathbf{d}, L^*LK(\cdot,p)\big)_{H_k} \quad \text{on } E. \tag{13}$$

We would like to point out that the adjoint operator $L^*$ of $L$ is represented upon the known data $\mathbf{d}$, $L$, $K(p,q)$ and $\mathcal{H}$, as we may realize from

$$\big(L^*\mathbf{d}\big)(p) = \big(L^*\mathbf{d}, K(\cdot,p)\big)_{H_K} = \big(\mathbf{d}, LK(\cdot,p)\big)_{\mathcal{H}}.$$

However, the result is involved and so *the Moore-Penrose generalized inverse* $f_{\mathbf{d}}$ *is not good*, when the data contain error or noises in some practical cases. So, we shall introduce the idea of the Tikhonov regularization in the present framework.

## 5 The Tikhonov Regularization

We shall give some practical and more concrete representation in the extremal functions involved in the Tikhonov regularization by using the theory of reproducing kernels.

Let $\{E_\lambda\}$ be the spectral family for the self-adjoint operator $L^*L$. If $L^*L$ has a continuous inverse, then we have

$$\big(L^*L\big)^{-1} = \int \frac{1}{\lambda}\, dE_\lambda.$$

Then, the Moore-Penrose generalized inverse for (10) is represented by

$$f_{\mathbf{d}}(p) = \int \frac{1}{\lambda} \, dE_\lambda \, L^* \mathbf{d}.$$

When $\mathcal{R}(L)$ is not closed and $\mathbf{d} \notin \mathcal{D}(L^\dagger)$, that is, $Lf = \mathbf{d}$ does not have the Moore-Penrose generalized inverse, by taking $\alpha > 0$, we define

$$f_{\mathbf{d},\alpha}(p) = \int \frac{1}{\lambda + \alpha} \, dE_\lambda \, L^* \mathbf{d}.$$

If the Moore-Penrose generalized inverse does exist, when $\alpha$ tends to zero, then the function $f_{\mathbf{d},\alpha}(p)$ converges to the Moore-Penrose generalized inverse in a good way and the convergence property is examined in a detailed way. It is also interesting to realize that even when the Moore-Penrose generalized inverse does not exist, and furthermore, even when the function spaces do not belong to our function spaces above, we are able to see—as numerical experiments—some good approximate solutions for some suitable parameter $\alpha$.

**Theorem 8** *Let* $\mathbf{d} \in \mathrm{D}(L^\dagger)$. *Then, we have*

$$\lim_{\alpha \downarrow 0} f_{\mathbf{d},\alpha}(p) = f_{\mathbf{d}}$$

*in the topology of* $H_K$.

**Theorem 9** *Under the same notation, we have*

$$\|Lf_{\mathbf{d},\alpha} : \mathcal{H}\| \leq \|\mathbf{d} : \mathcal{H}\|$$

*and*

$$\|f_{\mathbf{d},\alpha} : H_K\| \leq \frac{\|\mathbf{d} : \mathcal{H}\|}{2\sqrt{\alpha}}.$$

The function $f_{\mathbf{d},\alpha}$ is characterized as the following extremal function that makes a minimum in the Tikhonov functional:

**Theorem 10** *Let* $\alpha > 0$. *Then the following minimizing problem admits a unique solution*

$$\min_{f \in H_K} \left\{ \alpha \|f : H_K\|^2 + \|\mathbf{d} - Lf : \mathcal{H}\|^2 \right\}.$$

*Furthermore, the minimum is attained by*

$$f_{\mathbf{d},\alpha} = \left( \int_{\mathbb{R}} \frac{1}{\lambda + \alpha} \, dE_\lambda \right) L^* \mathbf{d}.$$

The following theorem gives an approximate solution for the operator equation (10) with error or noises:

**Theorem 11** *Suppose that* $\alpha : (0, 1) \to (0, \infty)$ *is a function of* $\delta$ *such that*

$$\lim_{\delta \downarrow 0}\left(\alpha(\delta) + \frac{\delta^2}{\alpha(\delta)}\right) = 0.$$

*Let* $D : (0, 1) \to \mathcal{H}$ *be a function such that*

$$\left\| D(\delta) - \mathbf{d} \right\|_{\mathcal{H}} \le \delta$$

*for all* $\delta \in (0, 1)$. *If* $\mathbf{d} \in \mathrm{D}(L^\dagger)$, *then we have*

$$\lim_{\delta \downarrow 0} f_{D(\delta),\alpha(\delta)} = f_{\mathbf{d}} = L^\dagger \mathbf{d}.$$

**Theorem 12** *Let* $L : H_K \to \mathcal{H}$ *be a bounded linear operator, and define the inner product*

$$\langle f_1, f_2 \rangle_{H_{K_\lambda}} = \alpha \langle f_1, f_2 \rangle_{H_K} + \langle L f_1, L f_2 \rangle_{\mathcal{H}}$$

*for* $f_1, f_2 \in H_K$. *Then* $(H_K, \langle \cdot, \cdot \rangle_{H_{K_\alpha}})$ *is a reproducing kernel Hilbert space whose reproducing kernel is given by*

$$K_\alpha(p, q) = \left[\left(\alpha + L^* L\right)^{-1} K_q\right](p).$$

*Here,* $K_\alpha(p, q)$ *is the solution* $\tilde{K}_\alpha(p, q)$ *of the functional equation*

$$\tilde{K}_\alpha(p, q) + \frac{1}{\alpha}(L\tilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\alpha} K(p, q), \tag{14}$$

*that is corresponding to the Fredholm integral equation of the second kind for many concrete cases. Moreover, we are using*

$$\tilde{K}_q = \tilde{K}_\alpha(\cdot, q) \in H_K \quad \text{for } q \in E, \qquad K_p = K(\cdot, p) \quad \text{for } p \in E.$$

Now we wish to represent the Tikhonov extremal function $f_{\mathbf{d},\alpha(\delta)}$ in terms of a reproducing kernel in order to turn the computation possible in practice. We shall, furthermore, need error estimates, when $\mathbf{d}$ contains error or noises. For this fundamental problem, we obtain the following conclusion.

**Theorem 13** *Under the same assumption as Theorem 11,*

$$f \in H_K \mapsto \left\{\alpha \| f : H_K \|^2 + \| Lf - \mathbf{d} : \mathcal{H} \|^2\right\} \in \mathbb{R}$$

*attains the minimum and the minimum is attained only at* $f_{\mathbf{d},\alpha} \in H_K$ *such that*

$$(f_{\mathbf{d},\alpha})(p) = \left\langle \mathbf{d}, LK_\alpha(\cdot, p)\right\rangle_{\mathcal{H}}.$$

*Furthermore,* $(f_{\mathbf{d},\alpha})(p)$ *satisfies*

$$\left|(f_{\mathbf{d},\alpha})(p)\right| \le \sqrt{\frac{K(p, p)}{2\alpha}} \|\mathbf{d}\|_{\mathcal{H}}. \tag{15}$$

Note that in (15), the factor 2 is missing in the result presented in [12] (and in other works); that is, the estimate (15) is improved in here. As the example of $\mathcal{H} = H_K$ and $L = I$ shows, the equality in (15) is attained.

This theorem means that in order to obtain good approximate solutions, we must take a sufficiently small $\alpha$, however, here we have restrictions for them, as we see, when $\mathbf{d}$ moves to $\mathbf{d}'$, by considering $f_{\mathbf{d},\alpha}(p) - f_{\mathbf{d}',\alpha}(p)$ in connection with the relation of the difference $\|\mathbf{d} - \mathbf{d}'\|_{\mathcal{H}}$. This fact is a very natural one, because we cannot obtain good solutions from the data containing errors. Here we wish to know how to take a small $\alpha$ a prior and what is the bound for it. These problems are very important practically and delicate ones, and we have many methods.

The basic idea may be given as follows. We examine for various $\alpha$ tending to zero, the corresponding extremal functions. By examining the sequence of the extremal functions, when it converges to some function numerically and after then when the sequence diverges numerically, it will give the bound for $\alpha$ numerically (see [5, 7, 8, 10, 12]).

Meanwhile, the inequality (15) will be also interesting in the following viewpoints: (i) firstly, from

$$f_{\mathbf{d},\alpha}(p) = \big(f_{\mathbf{d},\alpha}(\cdot), K(\cdot, p)\big)_{H_K},$$

we obtain the best possible inequality

$$\big|f_{\mathbf{d},\alpha}(p)\big| \le \|f_{\mathbf{d},\alpha}\|_{H_K}\sqrt{K(p, p)};$$

(ii) secondly, the inequality is independent of the linear mappings $L$ for $H_K$ into $\mathcal{H}$.

As concrete examples, many particular cases we have been discussed. Typical famous problems are the inversion for the heat conduction (see [12]) and the real inversion of the Laplace transform that are famous ill-posed and difficult problems, historically. We were able to obtain good results for these problems. We shall introduce the results simply for the Laplace transform in the next section.

## 6 Real and Numerical Inversion Formula of the Laplace Transform

We shall consider the inversion formula of the Laplace transform

$$(\mathcal{L}F)(p) = f(p) = \int_0^\infty e^{-pt} F(t)\,dt, \quad p > 0$$

for some natural function spaces. For more general functions, we shall apply their transforms suitably in order to apply the results (cf. [20]). We shall consider, in general, the complex inversion formulas, because the images of the Laplace transform are analytic functions. However, we are requested to use only real and discrete data to obtain the inversion formula. This is the *real inversion formula* of the Laplace transform, and we must represent the analytic function of the image in terms of the data on the positive real line. This problem is a very famous difficult one.

In order to consider a bounded linear operator by the Laplace transform, we shall recall a natural function space in [20].

On the positive real line $\mathbb{R}^+$, we shall consider the norm

$$\|f\|_{H_K} = \left\{ \int_0^\infty \left|F'(t)\right|^2 \frac{1}{t} e^t \, dt \right\}^{1/2}$$

for absolutely continuous functions $F$ satisfying $F(0) = 0$. This space $H_K$ admits the reproducing kernel

$$K(t,t') = \int_0^{\min(t,t')} \xi e^{-\xi} \, d\xi. \tag{16}$$

Then, we have

$$\int_0^\infty \left|(\mathcal{L}F)(p)p\right|^2 dp \leq \frac{1}{2}\|F\|_{H_K}^2; \tag{17}$$

that is, $(\mathcal{L}F)(p)p$ is a bounded linear operator from $H_K$ into $L_2(\mathbb{R}^+, dp) = L_2(\mathbb{R}^+)$. Recently, considering a H. Fujiwara conjecture, Y. Sawano proved that this operator is compact. By using this reproducing kernel Hilbert space, we obtain, following our general method the following consequences.

**Theorem 14** *For any $g \in L_2(\mathbb{R}^+)$ and for any $\alpha > 0$, in the sense*

$$\inf_{F \in H_K} \left\{ \alpha \int_0^\infty \left|F'(t)\right|^2 \frac{1}{t} e^t dt + \left\|(\mathcal{L}F)(p)p - g\right\|_{L_2(\mathbb{R}^+)}^2 \right\}$$
$$= \alpha \int_0^\infty \left|F_{\alpha,g}^{*\prime}(t)\right|^2 \frac{1}{t} e^t \, dt + \left\|(\mathcal{L}F_{\alpha,g}^*)(p)p - g\right\|_{L_2(\mathbb{R}^+)}^2 \tag{18}$$

*there exists a uniquely determined best approximate function $F_{\alpha,g}^*$ and it is represented by*

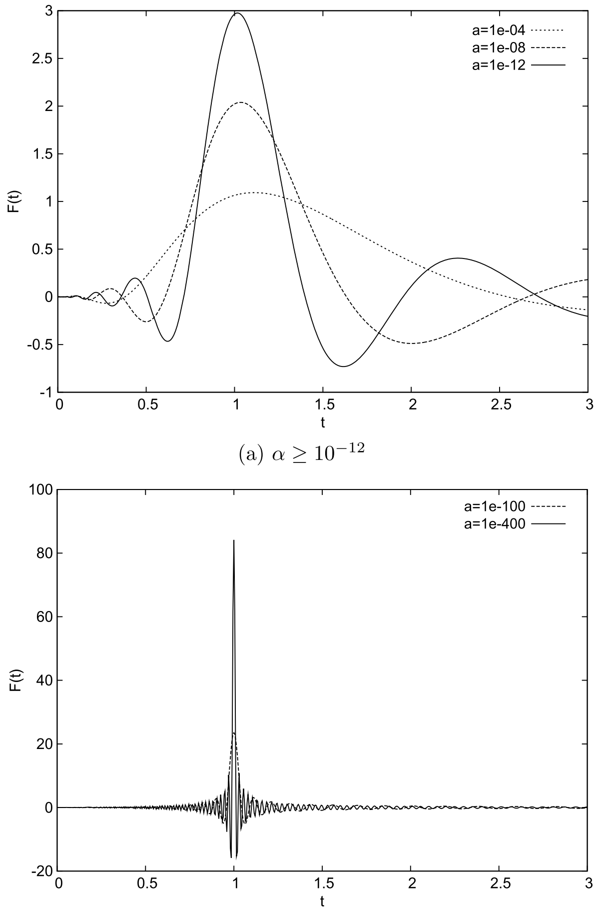
$$F_{\alpha,g}^*(t) = \int_0^\infty g(\xi)\big(\mathcal{L}K_\alpha(\cdot,t)\big)(\xi)\xi \, d\xi. \tag{19}$$

*Here, $K_\alpha(\cdot,t)$ is determined by the functional equation, for $K_{\alpha,t'} = K_\alpha(\cdot,t')$, $K_t = K(\cdot,t)$,*

$$K_\alpha(t,t') = \frac{1}{\alpha}K(t,t') - \frac{1}{\alpha}\big((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}K_t)(p)p\big)_{L_2(\mathbb{R}^+)}. \tag{20}$$

We calculate the approximate inverse $F_{\alpha,g}^*(t)$ by using (19). By taking the Laplace transform of (20) with respect to $t$, and by changing the variables $t$ and $t'$, it holds

**Fig. 1** Numerical results for the delta function $\delta_1$




(a) $\alpha \geq 10^{-12}$

(b) $\alpha = 10^{-100}, 10^{-400}$

$$\big(\mathcal{L}K_\alpha(\cdot,t)\big)(\xi) = \frac{1}{\alpha}\big(\mathcal{L}K(\cdot,t)\big)(\xi) - \frac{1}{\alpha}\Big((\mathcal{L}K_{\alpha,t})(p)p, \big(\mathcal{L}(\mathcal{L}K.)(p)p\big)(\xi)\Big)_{L_2(\mathbb{R}^+)}. \tag{21}$$

Here,

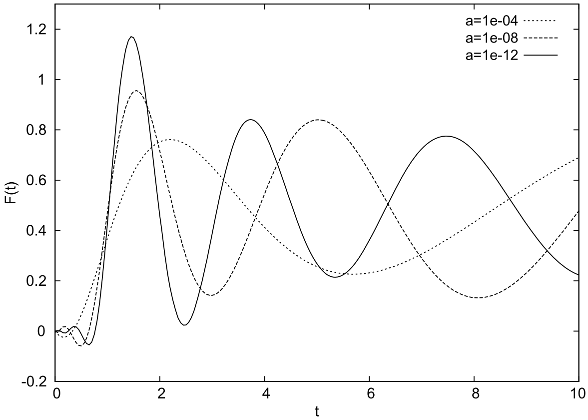
$$K(t,t') = \begin{cases} -te^{-t} - e^{-t} + 1 & \text{for } t \leq t', \\ -t'e^{-t'} - e^{-t'} + 1 & \text{for } t \geq t'. \end{cases}$$

Additionally,

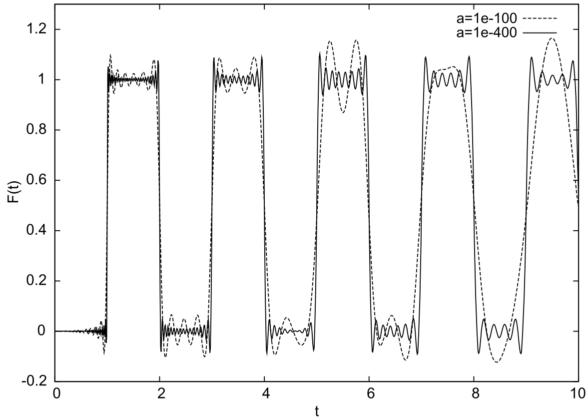
$$\big(\mathcal{L}K(\cdot,t')\big)(p) = e^{-t'p}e^{-t'}\left[\frac{-t'}{p(p+1)} + \frac{-1}{p(p+1)^2}\right] + \frac{1}{p(p+1)^2},$$

$$\int_0^\infty e^{-qt'}\big(\mathcal{L}K(\cdot,t')\big)(p)\,dt' = \frac{1}{pq(p+q+1)^2}.$$

**Fig. 2** Numerical results for a square wave function




(a) $\alpha \geq 10^{-12}$




(b) $\alpha = 10^{-100}, 10^{-400}$

Therefore, by setting as $(\mathcal{L}K_\alpha(\cdot,t))(\xi)\xi = H_\alpha(\xi,t)$, we obtain the following *Fredholm integral equation of the second kind*:

$$\alpha H_\alpha(\xi,t) + \int_0^\infty \frac{H_\alpha(p,t)}{(p+\xi+1)^2}dp = -\frac{e^{-t\xi}e^{-t}}{\xi+1}\left(t+\frac{1}{\xi+1}\right)+\frac{1}{(\xi+1)^2}. \quad (22)$$

By solving this integral equation, H. Fujiwara (cf. [6–8]) derived a very reasonable numerical inversion formula for the integral transform and he expanded very good algorithms for numerical and real inversion formulas of the Laplace transform. Figure 1 is an example for $\mathcal{L}F(p) = \exp(-p)$ for which $F(t) = \delta_1(t)$ in the distribution sense, and Fig. 2 is for

$$\mathcal{L}F(p) = \frac{e^{-p}}{p(1+e^{-p})}$$

for which $F(t)$ is a square wave function.

In both figures, (a) is computed with large regularization parameters $\alpha \geq 10^{-12}$, and (b) is computed with small regularization parameters $\alpha = 10^{-100}, 10^{-400}$. At this moment, theoretically we shall use the whole data of the output - in fact, 6000 data. Surprisingly enough, Fujiwara gave the solutions with $\alpha = 10^{-400}$ and *600 digits precision*. The core of the above mentioned and corresponding patent is *10 GB* data for the solutions.

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# On the Approximation-Error of Some Numerical Methods for Obtaining the Optimal Deformable Model

**Paulina Mitrea, Octavian Mircia Gurzău, and Alexandru Ioan Mitrea**

**Abstract** This paper deals with two approximation methods for obtaining the optimal deformable model: a discretization scheme by finite differences that generates an algorithm providing the approximating solution for energy-minimizing surfaces and a reconstruction method based on the Chebyshev discrete best approximation which approximates a plane deformable model represented by a finite set of points. Estimates for the approximation-error of these methods and results concerning their convergence or the topological structure of the set of unbounded divergence are presented, too.

**Keywords** Deformable model · Error approximation · Superdense set

**Mathematics Subject Classification** 41A10 · 41A50 · 65N06

## 1 Introduction

The theory of deformable models originates from the general theory of continuous multidimensional deformable models in a Lagrangian dynamics setting of D. Terzopoulos (1986) [13] based on deformation energies in the form of generalized splines [7]. The deformable curves (2D models) and the deformable surfaces (3D models) gained popularity after their use in computer vision by D. Terzopoulos, M. Kaas and A. Witkin [8] and in computer graphics, by D. Terzopoulos and K. Fleischer (1988) [14]. The theory of deformable models joins methods, results and techniques of various mathematical fields, physics and mechanics. The mathematical foundation of this theory represents the confluence of Functional Analysis, Approximation Theory, Differential Equations, Differential Geometry, Calculus of Variations, Numerical Analysis.

Two general types of deformable models have been developed: firstly, the variational models, which originate from the papers of D. Terzopoulos, M. Kaas and A. Witkin [8] and are based on the minimization of the energy-functional associated to the model, and secondly, the geometric models, which were introduced independently by V. Caselles, F. Catle, T. Coll, F. Dibos (1993) [1] and R. Maladi, J. Sethian, B. Vermuri (1995) [9], and are based on the front propagation theory (1988) [10]; on this subject, see also [6].

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International Series of Numerical Mathematics 161, DOI 10.1007/978-3-0348-0249-9_7,

Two approaches were emphasized in order to obtain the optimal deformable variational model. The first one starts from an initial estimate of the model, based on image data, restricted to the domain of evolving estimate; then, by using the Euler-Gauss-Ostrogradski (EGO) Equation of Calculus of Variations, this initial model undergoes a deformation until reaching a local minimum of the energy-functional. The second one (the classical approach) consists of using reconstruction methods, such as the interpolation of the sparse data extracted from the image, in order to obtain a representation of the original data.

The paper is organized as follows. The next section is devoted to present the 3D deformable models, both in static and dynamic forms. In Sect. 3, after presenting a method for reducing the 3D problem to a 2D modeling problem (see [4]), we derive the EGO-Algorithm for obtaining the energy-minimizing surfaces and we estimate its error-approximation. The last section deals with a reconstruction method, based on the Chebyshev discrete best approximation; more exactly, we provide convergence-type results and we emphasize the phenomenon of condensation of singularities for pointwise approximation formulas on equidistant nodes, related to this approximation method. To this goal, we need the following *principle of condensation of singularities*:

**Theorem 1** [3] *If $X$ is a Banach space, $Y$ a normed space and $(A_n)_{n\geq 1}$ is a sequence of continuous linear operators from $X$ into $Y$ so that the set of the norms $\{\|A_n\| : n \geq 1\}$ is unbounded then the set of singularities of the family $\{A_n\}_{n\geq 1}$, namely $S = \{x \in X : \limsup_{n\to\infty} \|A_n x\| = \infty\}$ is superdense in $X$.*

We recall that a subset $S$ of a topological space $T$ is said to be *superdense* in $T$ if it is residual (i.e. its complement is of first Baire category), uncountable and dense in $T$.

In this paper the notations $m$, $M$, $M_k$, $k \geq 1$ stand for some generic positive constants which do not depend on $n$. If $(a_n)$ and $(b_n)$ are sequences of real numbers (see [12]), with $b_n \neq 0$, we write $a_n \sim b_n$ if $0 < m \leq |a_n/b_n| \leq M$, for all $n \geq 1$.

## 2 Deformable Variational Models

In this section, we describe, according to [4], a 3D deformable variational model, both in their static (initial) and dynamic forms, together its EGO-Equation, which leads to the optimal surface; then we present a method for reducing the problem of its optimization to a 2D modeling problem.

Denoting by $D = [0, 1] \times [0, 1]$ the unit square of $\mathbb{R}^2$; let us consider a surface of vectorial equation

$$(S): \quad v = v(s, r), \quad (s, r) \in D \tag{1}$$

where $v \in C^2(D, \mathbb{R}^3)$, $v = (x, y, z)^T$; in what follows we set $|v|^2 = x^2 + y^2 + z^2$, $v_s = \frac{\partial v}{\partial s}$, $v_{ss} = \frac{\partial^2 v}{\partial s^2}$, $v_{sr} = \frac{\partial^2 v}{\partial s \partial r}$, $v_{rr} = \frac{\partial^2 v}{\partial r^2}$. Given the functions $g \in C^2(\partial D, \mathbb{R}^3)$

and $h \in C^1(\partial D, \mathbb{R}^3)$, where $\partial D$ is the boundary of $D$, let $\mathcal{A}$ be the set of *admissible deformations*, which consists of all functions $v \in C^2(D, \mathbb{R}^3)$ satisfying the boundary conditions $v(s,r) = g(s,r)$ and $\frac{\partial v}{\partial n}(s,r) = h(s,r)$ on $\partial D$, where $n$ is the normal vector with respect to the surface ($S$) defined by (1). Further, let us consider the following functions: *the image intensity function* $I \in C^2(\mathbb{R}^3)$; *the potential function associated* to the external forces $P(v) = -\lambda|\nabla I(v)|^2$, $\lambda > 0$; *the control functions* corresponding to the internal forces acting on the shape of the surface, namely the elasticity functions $w_{10}(s;r)$ and $w_{01}(s;r)$; the *rigidity functions* $w_{20}(s;r)$ and $w_{02}(s;r)$, and *the twist resistance function* $w_{11}(s;r)$. The energy functional $E : \mathcal{A} \to \mathbb{R}$, associated to these data, is defined as follows:

$$E(v) = \iint_D F(v, v_s, v_r, v_{ss}, v_{sr}, v_{rr})\, ds\, dr \tag{2}$$

where:

$$\begin{aligned} F(v, v_s, v_r, v_{ss}, v_{sr}, v_{rr}) = w_{10}|v_s|^2 + w_{01}|v_r|^2 + w_{20}|v_{ss}|^2 + 2w_{11}|v_{sr}|^2 \\ + w_{02}|v_{rr}|^2 + f(v, v_s, v_r), \end{aligned} \tag{3}$$

$$f(v, v_s, v_r) = P(v) + \det(c_0 v, v_s, v_r). \tag{4}$$

We notice that $E(v)$ represents the sum of *the internal energy* (the terms of (2) excepting $f(v, v_s, v_r)$), *the external energy* (defined by the terms containing $P(v)$) and *the balloon energy*, which is added, optionally, by the users (the term including $\det(c_0 v, v_s, v_r)$).

The triple $(\mathcal{A}, I, E)$ is said to be a *3D deformable model*, sometimes a *deformable surface*. The basic problem of the deformable model is to minimize its energy-functional, namely to obtain the optimal deformable surface. To this purpose, the Euler-Gauss-Ostrogradski (EGO) Equation of Calculus of Variations, i.e.

$$\begin{aligned} \frac{\partial F}{\partial v} - \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial v_s}\right) - \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial v_r}\right) + \frac{\partial^2}{\partial s^2}\left(\frac{\partial F}{\partial v_{ss}}\right) \\ + \frac{\partial^2}{\partial v \partial r}\left(\frac{\partial F}{\partial v_{sr}}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial F}{\partial v_{rr}}\right) = 0 \end{aligned} \tag{5}$$

is used.

By simple calculation we obtain from (3), (4) and (5):

$$\begin{aligned} &\frac{\partial^2}{\partial s^2}(w_{20} v_{ss}) + \frac{\partial^2}{\partial r^2}(w_{02} v_{rr}) + 2\frac{\partial^2}{\partial s \partial r}(w_{11} v_{sr}) \\ &\quad - \frac{\partial}{\partial s}(w_{10} v_s) - \frac{\partial}{\partial r}(w_{01} v_r) + \frac{1}{2}\left(\nabla f - \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial v_s}\right) - \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial v_r}\right)\right) = 0. \end{aligned} \tag{6}$$

Generally, the energy-functional may has many local minima, i.e. there may exists many local minimum energy-surfaces. Because the goal of the user (in medical imaging, for example) is to find a good 3D contour in a given area, we can suppose that a rough prior estimate of surface is accessible, namely:

$$\left(S^0\right): \quad v = v_0(s, r), \quad (s, r) \in D. \tag{7}$$

Further, this surface is refined step by step, according to (EGO)-Equation; so, a sequence of surfaces, which leads to the energy-minimizing surface, is provided. More exactly, let

$$\left(S^t\right): \quad v = v(t, s, r); \quad t \geq 0, \ (s, r) \in D \tag{8}$$

be a family of surfaces, where the parameter $t$ describes the evolution in time of the model. We associate to the previous static model $(\mathcal{A}, I, E)$ the *evolution equation*

$$\frac{\partial v}{\partial t} + G(v, v_s, v_r, v_{ss}, v_{sr}, v_{rr}) = 0 \tag{9}$$

where $G(v, v_s, v_r, v_{ss}, v_{sr}, v_{rr})$ is the left-hand member of (6), together with the *initial estimate* (*condition*)

$$v(0, s, r) = v_0(s, r), \quad (s, r) \in D \tag{10}$$

and the boundary dynamic conditions

$$\begin{cases} v(t, s, r) = v_0(s, r); & (s, r) \in \partial D, \ t \geq 0, \\ \dfrac{\partial v(t, s, r)}{\partial n} = \dfrac{\partial v_0(s, r)}{\partial n}; & (s, r) \in \partial D, \ t \geq 0. \end{cases} \tag{11}$$

A solution of the “static” problem described by (6) is achieved, when the solution $v(t, s, r)$ becomes stable with respect to the time-parameter, i.e. $\lim_{t\to\infty} \frac{\partial v}{\partial t}(t, s, r) = 0$, uniformly, with respect to $(s, r) \in D$; in this case, the evolution equation (9) provides a solution of the static problem (6).

## 3 Approximation Error of a Discretization Scheme

The problem of obtaining directly energy-minimizing surfaces (i.e. solution of (6)) is not practically possible, due to the complicated form of (6). On the other hand, by using discretization schemes for solving (6), we get a system of algebraic equations, with a high computational level. In order to eliminate these drawbacks, the problem of finding the energy-minimizing surface will be reduced to a 2D modeling problem, [4]. In this approach, the surface that we seek is obtained as a sequence of plane curves (slices) indexed by a parameter $r$ so that each given value of $r$ provides a closed curve, lying in a slice of the 3D image. Consequently, let

$$(\gamma_r): \quad v(s) = \big(x(s), y(s)\big), \quad s \in [0, 1] \tag{12}$$

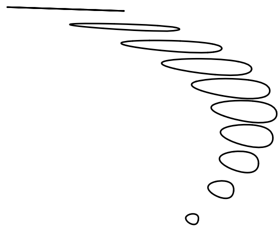

**Fig. 1** The slices

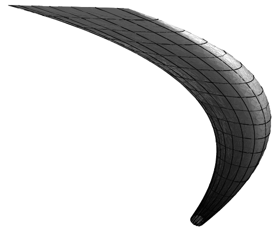

**Fig. 2** The surface

be the plane curve obtain by this procedure, for a given $r$. Throughout this section we suppose that the control functions $w_{10}$ and $w_{20}$ are positive constants. With the notations $\alpha = w_{10}$ and $\beta = w_{20}$, (6) which corresponds to $(\gamma_r)$ of (12) is

$$2\beta \frac{d^4 v}{ds^4} - 2\alpha \frac{d^2 v}{ds^2} - 2c_0 J_2 \frac{dv}{ds} + \nabla P = 0, \tag{13}$$

where $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

*Example 1* If we consider in (13) $c_0 = 0$, $\alpha = 5$, $\beta = 0.05$, $P = r(x^2 + y^2)$ and $r = 0.1, 0.2, \ldots, 1$ with boundary conditions $x(0) = x(1) = 1 + r(1-r)/25$, $y(0) = y(1) = r(1-r)/25$, $y'(0) = y'(1) = -r(1-r)/5$, we obtain the graphs of slices (Fig. 1) and a 3D reconstruction of the surface (Fig. 2).

Further, let us consider the dynamic 2D model corresponding to the dynamic 3D model described by the relations (7)–(11), whose evolution equation has the form:

$$\frac{\partial v}{\partial t} + \beta \frac{\partial^4 v}{\partial s^4} - \alpha \frac{\partial^2 v}{\partial s^2} - c_0 J_2 \frac{\partial v}{\partial s} + \frac{1}{2} \nabla P = 0. \tag{14}$$

Passing to the discretization of (14) by the method of finite differences, denote by $\delta$ and $h$ the time and the space discretization steps, respectively, and let $\mathcal{R} = \{(t_k, s_i), k \geq 0, 0 \leq i \leq N\}$ be the plane net of discretization, with $N \in \mathbb{N}^*$, $Nh = 1$, $t_k = k\delta$ and $s_i = ih$. The following notations will be used, too: $v_i^k = v(t_k, s_i)$, $v^k = (x^k, y^k)^T$, $k \geq 0$; $g^k = (g_1^k, g_2^k)$, with $g_1^k = -\frac{1}{2}(\frac{\partial P}{\partial x}(v^k))$, ${g_2}^k = -\frac{1}{2}(\frac{\partial P}{\partial y}(v^k))$, $a_1 = 2\frac{\alpha}{h^2} + 6\frac{\beta}{h^4}$, $a_2 = -\frac{\alpha}{h^2} - 4\frac{\beta}{h^4}$, $a_3 = \frac{\beta}{h^4}$. Further, the $N$th order square matrices $K$ (known as *stiffness-matrix*) and $L$ are defined as circular matrices with first row $(a_1, a_2, a_3, 0, \ldots, 0, a_3, a_2)$ and $(1, -1, 0, 0, \ldots, 0)$, respectively. We notice that $v^k = (v_0^k, v_1^k, \ldots, v_{N-1}^k)^T$ with $v_i^k = v(t_k, s_i)$ and similarly for $x^k$, $y^k$, $g_1^k$, $g_2^k$, as example

$$g_1^k = -\left(\frac{\partial P}{\partial x}\left(v_0^k\right), \frac{\partial P}{\partial x}\left(v_1^k\right), \ldots, \frac{\partial P}{\partial x}\left(v_{N-1}^k\right)\right)^T;$$

since $(\gamma_r)$ is a closed curve it results $v_i^k = v_{i+N}^k$, $i \in \mathbb{Z}$. Thus the differential equation (14) leads to the following algebraic system:

$$\frac{v^k - v^{k-1}}{\delta} + K v^{k-1} + \gamma_0 L\left(J_2 v^{k-1}\right) = g^{k-1}, \quad k \geq 1,\ \gamma_0 = c_0/h, \tag{15}$$

whose solutions approximate the values of $v(t, s)$ at the nodes of the plane net $\mathcal{R}$.

Denote by $V^k = (X^k, Y^k)^T$, $k \geq 1$ the solutions of the system (15). After some calculations we obtain the following EGO-Algorithm:

$$\begin{cases} X^k = (I_N - \delta K)X^{k-1} - \gamma_0 \delta L Y^{k-1} + \delta g_1^{k-1}, \\ Y^k = (I_N - \delta K)Y^{k-1} + \gamma_0 \delta L X^{k-1} + \delta g_2^{k-1}, \end{cases} \tag{16}$$

where $I_N$ is the identity matrix of order $N$. The vectorial form of (16) is:

$$V^k = (I_N - \delta K)V^{k-1} - \gamma_0 \delta L\left(J_2 V^{k-1}\right) + \delta g^{k-1}, \quad k \geq 1. \tag{17}$$

With the notation $r_1 = \alpha\delta/h^2$, $r_2 = \beta\delta/h^4$, $r_3 = c_0\delta/h = \gamma_0\delta$ we derive from (17) the following equation:

$$\begin{aligned} V_i^k &= -r_2 V_{i+2}^{k-1} + \left((r_1 + 4r_2)I_2 + r_3 J_2\right)V_{i+1}^{k-1} + \left((1 - 6r_2 - 2r_1)I_2 - r_3 J_2\right)V_i^{k-1} \\ &\quad + (r_1 + 4r_2)V_{i-1}^{k-1} - r_2 V_{i-2}^{k-1} + g_i^{k-1}, \quad k \geq 1,\ 0 \leq i \leq N-1, \end{aligned} \tag{18}$$

with $V_j^k = (X_j^k, Y_j^k)^T$, $V_j^k = V_{j+N}^k$, $j \in \mathbb{Z}$ and $g_i^k = (g_{1i}^k, g_{2i}^k)^T$, $g_{1i}^k = -\frac{1}{2}\frac{\partial P}{\partial x}(v_i^k)$, $g_{2i}^k = -\frac{1}{2}\frac{\partial P}{\partial y}(v_i^k)$.

The main aim of this section is to give an estimate regarding the approximation-error of the (EGO)-Algorithm (18) and to establish its convergence.

Denote by

$$\varepsilon_i^k = v_i^k - V_i^k, \quad k \geq 0,\ 0 \leq i \leq N-1 \tag{19}$$

the difference between the value of the solution $v = v(t, s)$ of (EGO)-Equation (14) and its approximation $V_i^k$ at the points of $\mathcal{R}$, for a given $k \geq 1$. Using the method of Taylor expansion, we obtain from (18) and (19):

$$\begin{aligned} \varepsilon_i^k &= \left((1 - 6r_2 - 2r_1)I_2 - r_3 J_2\right)\varepsilon_i^{k-1} + \left((r_1 + 4r_2)I_2 + r_3 J_2\right)\varepsilon_{i+1}^{k-1} \\ &\quad + (r_1 + 4r_2)\varepsilon_{i+1}^{k-1} - r_2\left(\varepsilon_{i+2}^{k-1} + \varepsilon_{i-2}^{k-1}\right) + \delta R v_i^{k-1}, \end{aligned} \tag{20}$$

where $Rv_i^k$ is *the residue* of the algorithm, see [15], given by:

$$\begin{aligned} Rv_i^k &= \delta\left(\frac{1}{2}\frac{\partial^2 v}{\partial t^2} + \frac{1}{6}\delta\frac{\partial^3 v}{\partial t^3} + \cdots\right)(t_k, s_i) \\ &\quad + h^2\beta\left(\frac{1}{6}\frac{\partial^6 v}{\partial s^6} + \frac{127}{5040}h^2\frac{\partial^8 v}{\partial s^8} + \cdots\right)(t_k, s_i) \\ &\quad - \alpha h^2\left(\frac{1}{12}\frac{\partial^4 v}{\partial s^4} + \frac{1}{60}h^2\frac{\partial^6 v}{\partial s^6} + \cdots\right)(t_k, s_i) \\ &\quad - c_0 J_2 h\left(\frac{1}{2}\frac{\partial^2 v}{\partial s^2} + \frac{h}{24}\frac{\partial^4 v}{\partial s^4} + \cdots\right)(t_k, s_i). \end{aligned} \tag{21}$$

Denoting by $E^k = \max\{|\varepsilon_{i-2}^k|, |\varepsilon_{i-1}^k|, |\varepsilon_i^k|, |\varepsilon_{i+1}^k|, |\varepsilon_{i+2}^k| : i \geq 0\}$, $k \geq 0$, the approximation error at the $k$th iteration of the EGO-Algorithm, the relation (20) leads to (for $k \geq 1$):

$$\begin{aligned} E^k &\leq \delta|Rv_i| \\ &\quad + \left(\sqrt{(4r_2 + r_1)^2 + r_3^2} + \sqrt{(1 - 6r_2 - 2r_1)^2 + r_3^2} + 4r_2 + r_1\right)E^{k-1}. \end{aligned} \tag{22}$$

Under the assumption that the partial derivatives in (21) are uniformly bounded, we derive from (21) and (22)

$$\begin{aligned} E^k &\leq \left(10r_2 + 2r_1 + 2r_3 + |1 - 6r_2 - 2r_1|\right)E^{k-1} \\ &\quad + M_1\delta^2 + M_2|2\beta - \alpha|\delta h^2 + M_3 c_0 \delta h. \end{aligned} \tag{23}$$

We admit $1 - 6r_2 - 2r_1 > 0$ (this condition is satisfied in medical imaging applications, [4]), therefore (23) yields:

$$E^k \leq qE^{k-1} + M_1\delta^2 + M_2|\alpha - 2\beta|\delta h^2 + M_3 c_0\delta h; \quad k \geq 1 \tag{24}$$

with

$$q = 1 + 4r_2 + 2r_3 \quad \text{and} \quad E_0 = 0.$$

Writing the estimate (24) for $k, k-1, \ldots, 1$ we get:

$$E^k \leq \left(1 + q + q^2 + \cdots + q^{k-1}\right)A(h,\delta) \leq \frac{q^k}{q-1}A(h,\delta) \tag{25}$$

with

$$A(h,\delta) = M_1\delta^2 + M_2|\alpha - 2\beta|\delta h^2 + M_3 c_0\delta h. \tag{26}$$

The inequality $1 - 6r_2 - 2r_1 > 0$ implies $\frac{\delta}{h^4} \leq \frac{1}{6\beta}$, which gives $q = 1 + 4\frac{\beta\delta}{h^4} + 2r_3 \leq 2(1 + r_3)$, so we obtain from (25) and (26):

$$E^k \leq \frac{2^{k-1}h^4(1+r_3)^k}{2\beta + c_0h^3}\left(M_1\delta + M_2|\alpha - \beta|h^2 + M_3c_0h\right), \quad k \geq 1.$$

This estimate proves the convergence of the EGO-Algorithm, if $\delta \to 0$ and $h \to 0$, under the hypothesis $1 - 6r_2 - 2r_1 > 0$.

## 4 A Reconstruction Method in the Theory of Deformable Models

Let us consider a 2D-deformable model, which is given, usually, by a parametric curve: $x = x(s)$, $y = y(s)$, $s \in [-1, 1]$. In many practical situations, a plane deformable model is represented by a discrete set of points $v_n = (x_n, y_n)$, $n \geq 1$, named *snaxels*.

A classic problem in the theory of deformable models is the so-called *best fitting*, given the position of a curve at a set of points $v_n = (x(s_n), y(s_n))$, $n \geq 1$. We can interpret physically this problem as a spring which connects a point $g(s_n)$ of the *reconstruction* $g(s)$ and the a given point $A_n(x(s_n), y(s_n))$, [4]. Denoting by $\Phi(v, g)$ a function which measures the distance between the *reconstruction* $g(s_n)$ and the given data $v(s)$, our goal is to find the function $g^*$ that minimizes $\Phi(v, g)$, according to a given criterion.

In this section, we refer to a method of *discrete best approximation type* in order to obtain the reconstruction. Denote by $C$ the Banach space of all real continuous functions defined on the real interval $[-1, 1]$, endowed with the supremum norm and suppose that the plane curve $(\gamma)$ has an explicit representation $y = f(x)$, $f \in C$, $-1 \leq x \leq 1$, so the snaxels will be denoted by $v_n^k = (x_n^k, f(x_n^k))$, $1 \leq k \leq n$, $n \geq 1$. More exactly, given an integer $m = m(n) \geq n + 1$, $n \in \mathbb{N}^*$, a node matrix $\mathcal{M} = \{x_m^k : 1 \leq k \leq m, m \geq 1\}$ with $-1 \leq x_m^1 < x_m^2 < \cdots < x_m^m \leq 1$ and a function $f$ in $C$, we search a polynomial $g^* = U_n(f) \in P_n$ satisfying the condition:

$$\begin{aligned} &\max\{|U_n(f)(x_m^k) - f(x_m^k)| : 1 \leq k \leq m\} \\ &\quad = \min\{\max\{|P(x_m^k) - f(x_m^k)| : 1 \leq k \leq m\} : P \in \mathcal{P}_n\}. \end{aligned} \tag{27}$$

The polynomial $U_n(f) = U_n(f; \mathcal{M}) \in \mathcal{P}_n$, which provides the best approximation of $f$ in the Chebyshev sense, in respect to the finite point set $J_n = \{x_m^k : 1 \leq k \leq m\}$, will be refereed as the *M-projection of $f$ on the space $\mathcal{P}_n$*; similarly, the operator $U_n = U_n(\cdot, \mathcal{M})$ will be named the *$\mathcal{M}$-projection operator* associated to $\mathcal{P}_n$. If $m = n + 1$, the solution is given by the Lagrange projections, i.e.

$$U_n f = L_n f = \sum_{k=1}^{n+1} f(x_{n+1}^k)\; l_{n+1}^k, \tag{28}$$

where $l_{n+1}^k$, $n \geq 1$, $1 \leq k \leq n + 1$ are the fundamental polynomials of Lagrange interpolation with respect to the node matrix $\mathcal{M}$. If $m = n + 2$, $n \geq 0$, let us denote by $a_{n+1}(f) = a_{n+1}(M; f)$ the leading coefficient of Lagrange polynomial $L_{n+1} f$, which interpolates $f$ at the points $x_{n+2}^k$, $1 \leq k \leq n + 2$ and consider a function

$\sigma_{n+2} \in C$ satisfying the relations $\sigma_{n+2}(x_{n+2}^k) = (-1)^k$, $1 \le k \le n+2$. From the theorem of Charles de la Vallée-Poussin, [2] we have, according to [5]:

$$U_n f = L_{n+1} f - \frac{a_{n+1}(f)}{a_{n+1}(\sigma_{n+2})} L_{n+1}(\sigma_{n+2}),$$

i.e.

$$(U_n f)(x) = \sum_{k=1}^{n+2} \left( f\left(x_{n+2}^k\right) + \frac{a_{n+1}(f)}{a_{n+1}(\sigma_{n+2})} (-1)^{k+1} \right) l_{n+2}^k(x), \quad x \in [-1, 1]. \quad (29)$$

Let $\Lambda_{n+2}(x) = \sum_{k=1}^{n+2} |l_{n+2}^k(x)|$, $x \in [-1, 1]$ be the Lebesgue functions, associated to $\mathcal{M}$. In what follows we examine the pointwise convergence at origin, related the problem of Chebyshev best approximation, with respect to the equidistant node matrix of $[-1, 1]$, i.e.

$$\mathcal{M}_E = \left\{ x_{2n}^k = -1 + \frac{2(k-1)}{2n-1} = \frac{2k-2n-1}{2n-1}, 1 \le k \le 2n \right\}.$$

We remark the relations:

$$x_{2n}^k = -x_{2n}^{2n-k+1}; \quad 1 \le k \le 2n. \quad (30)$$

We define the functionals $T_n : C \to \mathbb{R}$,

$$T_{2n}(f) = (U_{2n} f)(0), \quad n \ge 1 \quad (31)$$

and consider the pointwise approximation formulas:

$$f(0) = T_{2n}(f) + R_{2n}(f), \quad n \ge 1 \quad (32)$$

where $R_{2n} f$ are known as *approximation errors* of (32). Taking into account the relation $U_{2n} P = P, \forall P \in \mathcal{P}_{2n}$, it follows from (32):

$$|R_{2n} f| = \left| R_{2n}(f - P) \right| = \left| f(0) - P(0) \right| + \left| T_{2n}(f - P) \right|, \quad P \in \mathcal{P}_{2n}. \quad (33)$$

On the other hand, we derive from the definition of $\sigma_{n+2}$ the inequality $|a_{n+1}(f)| \le |a_{n+1}(\sigma_{n+2})| \|f\|$, which combined with (30) gives:

$$|T_{2n} f| \le 2\Lambda_{2n+2}(0) \|f\|. \quad (34)$$

Further, let $f \in C^r$, $r \ge 0$ a function which possesses a $r$th continuous derivative on $[-1, 1]$. It follows from the inequality of Gopenganz (see [11]) the existence of a polynomial $\tilde{p} \in \mathcal{P}_{2n}$ so that

$$\|f - \tilde{p}\| \le M_7 n^{-r} \omega\left( f^{(r)}; \frac{1}{n} \right), \quad n \ge 1, \quad (35)$$

where $\omega(g;\cdot)$ is the modulus of continuity of $g \in C$. The relations (33), (34) and (35) yield:

$$\left|R_{2n}(f)\right| \le M_7 n^{-r}\big(1+2\Lambda_{2n+2}(0)\big)\omega\left(f^{(r)};\frac{1}{n}\right); \quad n \ge 1,\ f \in C^r. \tag{36}$$

Now let us estimate $\Lambda_{2n}(0)$. Taking into account that $l_{2n}^{n+k}(0) = l_{2n}^{n-k+1}(0) = \frac{(-1)^{k+1}((2n-1)!!)^2}{2^{2n-1}(2k-1)(n-k)!(n+k-1)!}$ for $1 \le k \le n$ we obtain:

$$\Lambda_{2n}(0) = 2\sum_{k=1}^{n}\left|l_{2n}^{n+k}(0)\right| = \frac{((2n)!)^2}{2^{4n-2}(n!)^2}\sum_{k=1}^{n}\frac{1}{(2k-1)(n-k)!(n+k-1)!}. \tag{37}$$

The Stirling formula $n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{a_n}$, $\frac{1}{2n+1} < a_n < \frac{1}{2n}$, $n \in \mathbb{N}^*$ together with (37) leads to the estimate:

$$\Lambda_{2n}(0) \sim n^{2n}\sum_{k=1}^{n-1}\frac{n+k}{(2k-1)(n+k)^{n+k}(n-k)^{n-k}\sqrt{n^2-k^2}}. \tag{38}$$

Now we are in a position to prove the following statement:

**Theorem 2** *Let consider the pointwise approximation formulas with respect to the origin, described by the formulas* (31) *and* (32).

1° *These formulas are convergent on* $C^r$, *for all integers* $r \ge 1$, *i.e.*

$$\lim_{n\to\infty} T_{2n}(f) = f(0), \quad f \in C^r.$$

2° *The set of all functions* $f \in C$ *for which these formulas unboundedly diverge, namely* $\limsup_{n\to\infty}|T_{2n}(f)| = \infty$, *is superdense in the Banach space* $(C, \|\cdot\|)$.

*Proof* 1° Let us establish an upper estimate for $\Lambda_{2n}(0)$. We derive from (38):

$$\Lambda_{2n}(0) \le M_8\sum_{k=1}^{n-1}\frac{1}{\sqrt{k}}\frac{k}{2k-1}\sqrt{\frac{n+k}{k(n-k)}}\left(\frac{n}{n+k}\right)^{n+k}\left(\frac{n}{n-k}\right)^{n-k}. \tag{39}$$

The function $g_n : [1, n-1] \to \mathbb{R}$, $g_n(x) = \frac{x+n}{x(n-x)}$ satisfy the relations:

$$g_n(x) \le g_n(n-1) = \frac{2n-1}{2n} \sim 2, \quad x \in [1, n-1]. \tag{40}$$

On the other hand it is simple exercise to prove the inequality

$$\left(\frac{n}{n+k}\right)^{n+k}\left(\frac{n}{n-k}\right)^{n-k} \le 1. \tag{41}$$

Now (39), (40) and (41) yield:

$$\Lambda_{2n}(0) \le M_9 \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}} \le M_{10}\sqrt{n}; \quad n \ge 2. \tag{42}$$

A combination of (36) and (42) leads to the conclusion of 1°.

2° In order to apply Theorem 1 we estimate the norm of $T_{2n}$. Let us define $f_{2n} \in C$ by:

$$f_{2n}(x) = \begin{cases} \operatorname{sign} l_{2n}^k(0), & \text{if } x = x_{2n}^k,\ 1 \le k \le 2n, \\ \text{linear}, & \text{otherwise.} \end{cases}$$

It follows from (29) and (31)

$$T_{4n-2}(f_{4n}) = \sum_{k=1}^{4n} \left(1 + (-1)^k \frac{a_{4n-1}(f_{4n})}{a_{4n-1}(\sigma_{4n})} \operatorname{sign}\big(l_{4n}^k(0)\big)\right) \big|l_{4n}^k(0)\big|. \tag{43}$$

It is clear that $f_{4n}$ is an even function, since $\operatorname{sign}(l_{4n}^k(0)) = (-1)^k$, $1 \le k \le 2n$ and $\operatorname{sign}(l_{4n}^k(0)) = (-1)^{k+1}$, $2n+1 \le k \le 4n$, so we obtain, via (30):

$$\begin{aligned} a_{4n-1}(f_{4n}) &= \sum_{k=1}^{4n} \tau_{4n}^k f_{4n}\big(x_{4n}^k\big) = \sum_{k=1}^{4n} \tau_{4n}^{4n-k+1} f_{4n}\big(x_{4n}^{4n-k+1}\big) \\ &= \sum_{k=1}^{4n} \frac{f_{4n}(-x_{4n}^k)}{u_{4n}'(-x_{4n}^k)} = -\sum_{k=1}^{4n} \frac{f_{4n}(x_{4n}^k)}{u_{4n}'(x_{4n}^k)} = -a_{4n-1}(f_{4n}), \end{aligned}$$

where $u_{4n}(x) = \prod_{k=1}^{4n}(x - x_{4n}^k)$ and $\tau_{4n}^k = \frac{1}{u_{4n}'(x_{4n}^k)}$, $1 \le k \le 4n$. Therefore:

$$a_{4n-1}(f_{4n}) = 0. \tag{44}$$

Now, the relations (43) and (44) lead to:

$$T_{4n-2}(f_{4n}) = \Lambda_{4n}(0). \tag{45}$$

Further, we deduce from (38):

$$\begin{aligned} \Lambda_{2n}(0) &\ge M_{12} \sum_{k=1}^{n-1} \frac{1}{2k-1} \sqrt{\frac{n+k}{n-k}} \left(\frac{n^2}{n^2-k^2}\right)^{n-k} \left(\frac{n}{n+k}\right)^{2k} \\ &\ge M_{12} \sum_{k=1}^{n-1} \frac{1}{2k-1} \left(\frac{n}{n+k}\right)^{2k}. \end{aligned} \tag{46}$$

Taking into account (45) and the definition of $\|T_{2n}\|$ we have:

$$\|T_{4n-2}\| \ge \big|T_{4n-2}(f_{4n})\big| = \Lambda_{4n}(0),$$

which combined with (46) gives:

$$\|T_{4n-2}\| \geq M_{12} \sum_{k=1}^{2n-2} \frac{1}{2k-1} \left( \frac{2n-1}{2n-1+k} \right)^{2k}. \tag{47}$$

The inequality $(1-x)^m \geq 1 - mx$ for $x \in [0, 1]$ and $m \geq 1$ gives:

$$\left( \frac{2n-1}{2n-1+k} \right)^{2k} = \left( 1 - \frac{k}{2n-1+k} \right)^{2k} \geq 1 - \frac{2k^2}{2n-1+k} \geq \frac{1}{2}, \tag{48}$$

if $4k^2 - k - (2n-1) \leq 0$; the last inequality is satisfied for

$$k \leq \frac{\sqrt{2n-1}}{2}. \tag{49}$$

By choosing $n = 2m^2 + 1$, we infer from (47), (48), and (49):

$$\|T_{8m^2}\| \geq M_{13} \sum_{k=1}^{m} \frac{1}{2k-1} \geq M_{14} \ln m, \quad m \geq 1. \tag{50}$$

Finally, let us prove that the set of norms $\{\|T_{2n}\| : n \geq 1\}$ is unbounded. Indeed, by means of (50) we get:

$$\sup\left\{\|T_{2n}\|; n \geq 1\right\} \geq \sup\left\{\|T_{8m^2}\|; m \geq 1\right\} \geq M_{14} \sup\{\ln m; m \geq 1\} = \infty. \tag{51}$$

Now, apply Theorem 1 with $X = C$, $Y = \mathbb{R}$ and $A_n = T_{2n}$ and take into account (51), which completes the proof. □

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# Part 3
# Geometric and Norm Inequalities

# The Longest Shortest Piercing

**Bernd Kawohl and Vasilii V. Kurta**

**Abstract** We generalize an old problem and its partial solution of Pólya (Pólya in Elem. Math. 13, 40–41 (1958)) to the $n$-dimensional setting. Given a plane domain $\Omega \subset \mathbb{R}^2$, Pólya asked in 1958 for the shortest bisector of $\Omega$, that is for the shortest line segment $l(\Omega)$ which divides $\Omega$ into two subsets of equal area. He claimed that among all centrosymmetric domains of given area $l(\Omega)$ becomes longest for a disk. His proof, however, does not seem to be valid for domains that are not starshaped with respect to the center of $\Omega$. In the present note we provide two proofs that it suffices to restrict attention to starshaped sets. Moreover we state and prove a related inequality in $\mathbb{R}^n$. Given the volume of a measurable set $\Omega$ with finite Lebesgue measure, only a ball centered at zero maximizes the length of the shortest line segments running through the origin. In this sense the ball has the longest shortest piercing.

**Keywords** Starshaped rearrangement · Geometric inequality · Bisector · Piercing

**Mathematics Subject Classification** 52A40

## 1 Motivation and Result

Let $\Omega$ be a measurable (possibly unbounded) set with finite volume in $\mathbb{R}^n$, $n \geq 2$. For every $z$ on the unit sphere $\mathbb{S}^n$ and ray $L_z$ emanating from the origin and passing through $z$ we can measure $\ell_z(\Omega)$, the length of $L_z \cap \Omega$, which can be given by

$$\ell_z(\Omega) = \int_0^\infty \chi_\Omega(z, r)\, dr, \tag{1}$$

where $\chi_\Omega(z, r)$ is the characteristic function of the set $L_z \cap \Omega$. Note that $\ell_z(\Omega)$ can be infinite for some $z \in \mathbb{S}^n$, but since the volume of $\Omega$ is finite, this can happen only on a nullset of $\mathbb{S}^n$. We define the piercing length of $\Omega$ as

$$\ell(\Omega) := \inf_{z \in \mathbb{S}^n} \big(\ell_z(\Omega) + \ell_{-z}(\Omega)\big) \tag{2}$$

and prove the following result:

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**Theorem 1** *Let $\Omega$ be a measurable (possibly unbounded) set with finite volume in $\mathbb{R}^n$, $n \geq 2$. Then the inequality*

$$\ell(\Omega) \leq \ell\big(\Omega^*\big), \tag{3}$$

*holds, where $\Omega^*$ denotes the ball centered at the origin and of the same volume as $\Omega$. Moreover, equality holds in* (3) *if and only if $\Omega = \Omega^*$ modulo a set of measure zero.*

For the case $n = 2$ this theorem was implicitly stated in [7], but the proof given there had starshaped centrosymmetric domains in mind. We call a set $\Omega$ *centrosymmetric* (with respect to the origin) iff $x \in \Omega$ implies $-x \in \Omega$ and *starshaped* (with respect to the origin) iff $x \in \Omega$ and $t \in [0, 1]$ imply $tx \in \Omega$. Later Cianchi gave an independent proof of (3) for convex centrosymmetric plane domains, see Theorem 4 in [2].

## 2 Proof

First we prove the theorem for starshaped centrosymmetric sets. In a second step we show that the maximum of $\ell(\Omega)$ over all starshaped sets is assumed among centrosymmetric sets. In a third step we show that the maximum of $\ell(\Omega)$ over all measurable sets with finite $n$-dimensional Lebesgue measure is necessarily attained among starshaped sets.

*Step 1* Suppose $\Omega$ is an arbitrary centrosymmetric starshaped (possibly unbounded) set with finite volume in $\mathbb{R}^n$ but not a ball (modulo a nullset). Then there must exist a boundary point $x$ of $\Omega$ which lies in the interior of $\Omega^*$, and so does $-x$. Therefore, the line segment connecting $x$ with $-x$ is strictly shorter than the diameter of $\Omega^*$, that is $\ell(\Omega) < \ell(\Omega^*)$. For $n = 2$ this is Pólya's proof, but it extends without changes to general $n \geq 2$.

*Step 2* We will prove that *if $\Omega$ is starshaped but not centrosymmetric, then it can be replaced by a centrosymmetric starshaped set $\tilde{\Omega}$ of same volume as $\Omega$ such that* $\ell(\Omega) \leq \ell(\tilde{\Omega})$. In fact, if we replace the representation of $\Omega$ in polar coordinates $\ell_z(\Omega)$ by $\tilde{\ell}_z(\Omega)$ with

$$\left(\frac{1}{n}\big(\ell_z^n + \ell_{-z}^n\big)\right)^{1/n} = \left(\frac{2}{n}\tilde{\ell}_z^n\right)^{1/n}, \tag{4}$$

then the set $\tilde{\Omega}$ whose boundary is described by $\ell_z(\tilde{\Omega}) := \tilde{\ell}_z(\Omega)$ is of same volume as $\Omega$. Its piercing length, however, has not decreased, because the convexity of the mapping $t \mapsto t^n$ implies

$$\big(\tilde{\ell}_z(\Omega)\big)^n = \frac{1}{2}\big(\ell_z^n + \ell_{-z}^n\big) \geq \left(\frac{\ell_z(\Omega) + \ell_{-z}(\Omega)}{2}\right)^n \geq \big(\ell(\Omega)\big)^n,$$

and after infimizing over $z \in \mathbb{S}^n$ we arrive at

$$\ell(\tilde{\Omega}) \geq \ell(\Omega)$$

as claimed.

*Step 3* We will prove the following *claim*: *If $\Omega$ is not starshaped, then it can be replaced by a starshaped set $\Omega^\#$ of same volume as $\Omega$ such that $\ell(\Omega) \leq \ell(\Omega^\#)$*. This claim and Steps 1 and 2 result in a proof of Theorem 1.

For an arbitrary (possibly unbounded) set $\Omega$ with finite volume in $\mathbb{R}^n$ its volume is given by

$$|\Omega| = \int_{\mathbb{S}^n} \int_0^\infty \chi_\Omega(z, r) r^{n-1} \, dr \, dz$$

in polar coordinates. Let us consider the function

$$R_\Omega^\#(z) = \left[ n \int_0^\infty \chi_\Omega(z, r) r^{n-1} \, dr \right]^{\frac{1}{n}} \tag{5}$$

which can be infinite for some $z \in \mathbb{S}^n$, but since the volume of $\Omega$ is finite, only on a nullset of $\mathbb{S}^n$. If $\Omega^\#$ is defined (modulo a nullset) as the starshaped set bounded in polar coordinates by $R_\Omega^\#(z)$, then $|\Omega| = |\Omega^\#|$, that is $\Omega$ has been rearranged by starshaped rearrangement into an equimeasurable starshaped set, see [6]. What happens to $\ell(\Omega)$ after this operation? First, it is clear that the inequality

$$\ell(\Omega) \leq \ell_z(\Omega) + \ell_{-z}(\Omega) \tag{6}$$

holds for all $z \in \mathbb{S}^n$. Second, due to monotonicity of the function $\rho(r) = r^{n-1}$, $n \geq 2$, we have the inequality

$$\int_0^{\ell_z(\Omega)} r^{n-1} \, dr \leq \int_0^\infty \chi_\Omega(z, r) r^{n-1} \, dr$$

and thus the inequality

$$\frac{1}{n} \big(\ell_z(\Omega)\big)^n \leq \frac{1}{n} \big(R_\Omega^\#(z)\big)^n, \tag{7}$$

which holds for all $z \in \mathbb{S}^n$. In turn, (6) and (7) yield the inequalities

$$\ell(\Omega) \leq \ell_z(\Omega) + \ell_{-z}(\Omega) \leq R_\Omega^\#(z) + R_\Omega^\#(-z) = \ell_z\big(\Omega^\#\big) + \ell_{-z}\big(\Omega^\#\big) \tag{8}$$

for all $z \in \mathbb{S}^n$. Finally, after infimizing (8) over all $z \in \mathbb{S}^n$ we obtain

$$\ell(\Omega) \leq \ell\big(\Omega^\#\big)$$

which shows that the piercing length of $\Omega$ does not decrease in passing from $\Omega$ to $\Omega^{\#}$. Since we try to maximize the domain functional $\ell(\Omega)$, it suffices to study starshaped sets.

For a *second* proof of Step 3 we can also follow the idea in [3], there for the case $n = 2$, and recall a Hardy-Littlewood inequality that seems to be mathematical folklore. If $u$ and $v$ are two nonnegative functions defined on $\mathbb{R}_+$, and if $u^*$ denotes the decreasing and $v_*$ the increasing rearrangement of $u$ and $v$, then

$$\int_0^\infty u(r)v(r)\,dr \geq \int_0^\infty u^*(r)v_*(r)\,dr.$$

For the benefit of the reader let us remark in passing that its proof goes along the lines of Lemma 2.1 in [6] by reduction to the product of two nonnegative finite sequences. This product becomes minimal when the sequences are oppositely ordered, see Theorem 368 in [5].

Identifying $u$ with $\chi_\Omega(z,r)$ and $v$ with $r^{n-1}$ gives now

$$|\Omega| = \int_{\mathbb{S}^n}\int_0^\infty \chi_\Omega(z,r)r^{n-1}\,dr\,dz \geq \int_{\mathbb{S}^n}\int_0^\infty \chi_\Omega^*(z,r)r^{n-1}\,dr\,dz = |\tilde{\Omega}|,$$

i.e. the volume of a starshaped set $\tilde{\Omega}$ whose characteristic function is given by $\chi_\Omega^*(z,r)$. While the piercing length $\ell(\Omega)$ remains invariant under this rearrangement, in fact $\ell(\Omega) = \ell(\tilde{\Omega})$ by construction, the volume decreases, unless $\Omega$ was already starshaped. Now we define $\Omega^{\#}$ to be a rescaled (enlarged) version of $\tilde{\Omega}$, so that $|\Omega^{\#}| = |\Omega|$. Then again $\ell(\Omega) \leq \ell(\Omega^{\#})$ as claimed.

## 3 Related Questions

In this section we address related questions.

A) We have learned from F. Brock that he and M. Willem have considered planes which cut centrosymmetric $n$-dimensional bodies into two halves of equal volume. If $A_{n-1}(\Omega)$ denotes a cut through $\Omega$ which minimizes $(n-1)$-dimensional area, they were able to show that $A_{n-1}(\Omega) \leq A_{n-1}(\Omega^*)$.

B) If one wants to trade the assumption of centrosymmetry against convexity, already in 2 dimensions the question of the longest shortest cut that bisects area poses a major challenge. If one allows only straight lines to cut a convex plane set $\Omega$ into two parts of equal area (and any straight line can be shifted to do so), then among all (convex plane) sets of given area, the length $A_1$ of the bisecting line segment becomes maximal *not for the disk* $\Omega^*$, but for the so-called Auerbach triangle $T$, see [3]. The Auerbach triangle belongs to a class of so-called Zindler sets. By definition a Zindler set $Z$ has the remarkable property that every line-segment which bisects the area of $Z$ has the same length. To be precise [3] contains a proof that

$$A_1(\Omega) \leq A_1(T) \tag{9}$$

for every plane convex set $\Omega$ of given area, while [4] proves (9) for the smaller class of plane convex Zindler sets.

On the other hand, the shortest curve that bisects the area of the Auerbach triangle, is a circular arc and its length is shorter than the diameter of the disc of equal area. In fact, it has long been conjectured that the among all plane convex sets of given area, the disk maximizes length of the shortest curve that bisects the area. In [3] this conjecture is confirmed by a long and rather technical proof.

C) The result of Brock and Willem described above under A) as well as our Theorem 1 can be generalized to the $k$-dimensional setting for all $k = 1, \dots, n-1$. Then it reads as follows: If $A_k(\Omega)$ denotes a $k$-dimensional cut (or generalized piercing) through $\Omega$ which minimizes $k$-dimensional area, then $A_k(\Omega) \le A_k(\Omega^*)$. To prove these results by induction with respect to $k$ one can follow Steps 1–3: First, using an analytic version of Pólya's proof and an iteration from $n-k+1$ to $n-k$ for any $k = 1, \dots, n-1$, one can obtain the corresponding $(n-k)$-dimensional results for starshaped centrosymmetric sets. Next, using Step 2, one can show that the maximum of $(\ell(\Omega))^k$ is assumed among centrosymmetric sets. Finally, one can show that the maximum of $A_k(\Omega)$ over all measurable sets with finite $n$-dimensional Lebesgue measure is necessarily attained among starshaped sets by using the following observation which can be of independent interest. To formulate the corresponding result, for any $z \in \mathbb{S}^n$ and any $1 \le p \le n$ let us consider the function

$$R_\Omega(z, p) = \left[ p \int_0^\infty \chi_\Omega(z, r) r^{p-1}\, dr \right]^{\frac{1}{p}}. \tag{10}$$

**Lemma 1** *Let $\Omega$ be a measurable (possibly unbounded) domain with finite volume in $\mathbb{R}^n$, $n \ge 2$. Then the inequality*

$$R_\Omega(z, p) \le R_\Omega(z, q) \tag{11}$$

*holds for any $z \in \mathbb{S}^n$ and any $1 \le p < q \le n$. Moreover, equality in (11) holds if and only if $\Omega$ is a starshaped set.*

*Proof* Let $r = \rho^{1/q}$. Then by (10),

$$\left[R_\Omega(z, p)\right]^p = p \int_0^\infty \chi_\Omega(z, r) r^{p-1}\, dr = \frac{p}{q} \int_0^\infty \chi_\Omega\big(z, \rho^{1/q}\big) \rho^{\frac{p}{q}-1}\, d\rho. \tag{12}$$

It is clear that in the integral on the right-hand side of (12) one integrates over the set $\mathcal{L}_z \subset L_z$ of length

$$|\mathcal{L}_z| = \int_0^\infty \chi_\Omega\big(z, \rho^{1/q}\big)\, d\rho.$$

Changing in (12) the variable of integration to $\rho = r^q$, we have the relations

$$|\mathcal{L}_z| = \int_0^\infty \chi_\Omega\big(z, \rho^{1/q}\big)\, d\rho = q \int_0^\infty \chi_\Omega(z, r) r^{q-1}\, dr = \left[R_\Omega(z, q)\right]^q. \tag{13}$$

Further, due to the fact that the function $f(\rho) = \rho^{\frac{p}{q}-1}$ decreases monotonically on $\mathbb{R}_+$, we conclude from (12) by (13) that

$$\frac{p}{q}\int_0^\infty \chi_\Omega\big(z, \rho^{1/q}\big)\rho^{\frac{p}{q}-1}\,d\rho \le \frac{p}{q}\int_0^{(R_\Omega(z,q))^q} \rho^{\frac{p}{q}-1}\,d\rho. \tag{14}$$

Integrating on the right-hand side of (14) we obtain the inequality

$$\frac{p}{q}\int_0^\infty \chi_\Omega\big(z, \rho^{1/q}\big)\rho^{\frac{p}{q}-1}\,d\rho \le \big(R_\Omega(z,q)\big)^p$$

which, together with (12), yields (11). Equality in (11) holds iff one has equality in (14), which in turn implies

$$R_\Omega(z,q) = \left[q\int_0^{\ell_z(\Omega)} r^{q-1}\,dr\right]^{\frac{1}{q}},$$

i.e., iff the set $\Omega$ under consideration is already starshaped. Finally let us remark that a discrete version of this lemma can be found in [6, p. 64]. □

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# On a Continuous Mapping and Sharp Triangle Inequalities

**Tomoyoshi Ohwada**

**Abstract** This is a survey on some recent results concerning the sharp triangle inequalities. Our results refine and generalize the corresponding ones in (Kato et al. in Math. Inequal. Appl. 10(2), 451–460 (2007)) and (Mitani et al. in J. Math. Anal. Appl. 10(2), 451–460 (2007)).

**Keywords** Triangle inequality · Banach space

**Mathematics Subject Classification** Primary 46B20 · Secondary 46B99

## 1 Introduction

In this note we want to survey some of the most recent results concerning the sharp triangle inequalities on a Banach (or normed linear) space.

There are many results concerning norm inequalities under various settings. The typical one is as follows.

**Problem 1** *Let* $(X, \|\cdot\|)$ *be a normed linear space. Suppose that, for* $A, B \in X$, *a norm inequality* $\|A\| \leqq \|B\|$ *holds. Construct a positive value* $C$ *with respect to* $A$ *and* $B$ *satisfying* $\|A\| + C \leqq \|B\|$.

Note that Problem 1 is the same to find the intermediate value between 0 and $\|B\| - \|A\|$. We are interested in this problem to the triangle inequality. The (generalized) triangle inequality, namely

$$\left\| \sum_{j=1}^{n} x_j \right\| \leqq \sum_{j=1}^{n} \|x_j\|,$$

where $x_1, x_2, \ldots, x_n$, are elements in a normed linear space $(X, \|\cdot\|)$, is one of the most fundamental norm inequalities in analysis. This inequality has attracted the

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attention of a number of authors, and many interesting refinements and reverse inequalities of it have been obtained (cf. [1, 2, 10, 14, 15]). For the triangle inequality, we consider the following problem.

**Problem 2** *Characterize all the intermediate values C which satisfy*

$$0 \leqq C \leqq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|$$

*by using $x_1, x_2, \dots, x_n$ in $X$.*

In 1992, Hudzik and Landes [6] proved the following inequality which gives the solution of Problem 2 in the case of $n = 2$.

**Theorem 3** [6, Lemma 1] *For all nonzero elements $x$, $y$ in $X$,*

$$0 \leqq \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|\right) \min\bigl\{\|x\|, \|y\|\bigr\} \leqq \|x\| + \|y\| - \|x + y\|.$$

In 2005, Kato, Saito and Tamura [7] extended this result for an arbitrary number of finitely many nonzero elements $x_1, x_2, \dots, x_n$ in $X$ to treat the uniform non-$\ell_1^n$-ness of Banach spaces as follows (see [8, 9]):

**Theorem 4** [7, Lemma 2] *For all nonzero elements $x_1, x_2, \dots, x_n$ in $X$,*

$$0 \leqq \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\|\right) \min_{1 \leqq j \leqq n} \|x_j\| \leqq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|.$$

After that, several authors improved and generalized these inequalities (cf. [3–5]). Mitani, Saito, Kato and Tamura [13] succeeded in the further extension of Theorem 4 as follows:

**Theorem 5** [13, Theorem 1] *For all nonzero elements $x_1, x_2, \dots, x_n$ in $X$,*

$$0 \leqq \sum_{k=2}^{n} \left(k - \left\| \sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|} \right\|\right) \bigl(\|x_k^*\| - \|x_{k+1}^*\|\bigr) \leqq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|,$$

*where $x_1^*, x_2^*, \dots, x_n^*$ are the rearrangement of $x_1, x_2, \dots, x_n$ satisfying $\|x_1^*\| \geqq \|x_2^*\| \geqq \cdots \geqq \|x_n^*\|$ and $x_{n+1}^* = 0$.*

Since the intermediate value in Theorem 5 can be calculated as follows:

$$\sum_{k=2}^{n} \left(k - \left\| \sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|} \right\|\right) \bigl(\|x_k^*\| - \|x_{k+1}^*\|\bigr)$$

$$= \left(n - \left\|\sum_{j=1}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right)\|x_n^*\| + \sum_{k=2}^{n-1}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|}\right\|\right)\left(\|x_k^*\| - \|x_{k+1}^*\|\right)$$

$$\geqq \left(n - \left\|\sum_{j=1}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right)\min_{1\leqq j\leqq n}\|x_j\| \geqq 0,$$

if we put

$$(\mathrm{KST}) = \left(n - \left\|\sum_{j=1}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right)\min_{1\leqq j\leqq n}\|x_j\|$$

and

$$(\mathrm{MSKT}) = \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|}\right\|\right)\left(\|x_k^*\| - \|x_{k+1}^*\|\right),$$

then we see that

$$0 \leqq (\mathrm{KST}) \leqq (\mathrm{MSKT}) \leqq \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|.$$

Hence we know that (KST) and (MSKT) give one solution of Problem 2. However, other intermediate values are still unknown. So, in Sect. 2, we shall characterize the all the intermediate values in the triangle inequality, and two kinds of inequalities in Theorems 4 and 5 are concretely expressed as the intermediate values of it.

## 2 Intermediate Values

First, we shall give a norm inequality related to the triangle inequality. For a positive integer $n \geqq 2$, let $M_n([0, 1])$ be the set of all $n \times n$ matrices whose all elements belong to the interval $[0, 1]$ and $L_n$ denote the set of all lower triangular matrices of $M_n([0, 1])$; i.e.,

$$L_n = \left\{a = (a_{ij}) \in M_n\big([0, 1]\big) \mid a_{ij} = 0 \ (i < j)\right\}.$$

Let $1 \leqq m \leqq n$. For each $a = (a_{ij})$ in $L_n$, we set

$$\ell_{1j}^a(m) = a_{1j} \quad \text{and} \quad \ell_{mj}^a(m) = a_{mj} \quad (1 \leqq j \leqq m)$$

and if $3 \leqq n$, then, for each $m$ with $3 \leqq m \leqq n$, we put

$$\ell_{ij}^a(m) = a_{ij}\prod_{k=i+1}^{m}(1 - a_{kj}) \quad (2 \leqq i \leqq m-1, 1 \leqq j \leqq m).$$

Note that the $n \times n$ matrix $(\ell^a_{ij}(n))$ also belong to $L_n$. Take any $a = (a_{ij}) \in L_n$ and fix it. Considering $(\ell^a_{ij}(n))$ as the matrix acting on the Banach space $\underbrace{X \oplus X \oplus \cdots \oplus X}_{n \text{ times}}$, we have

$$\begin{pmatrix} \ell^a_{11}(n) & & & \\ \ell^a_{21}(n) & \ell^a_{22}(n) & 0 & \\ \vdots & \vdots & \ddots & \\ \ell^a_{n1}(n) & \cdots & \ell^a_{n\,n-1}(n) & \ell^a_{nn}(n) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \ell^a_{11}(n)x_1 \\ \sum_{j=1}^{2} \ell^a_{2j}(n)x_j \\ \vdots \\ \sum_{j=1}^{n} \ell^a_{nj}(n)x_j \end{pmatrix},$$

where $x_1, x_2, \ldots, x_n \in X$. For each entries, we have the triangle inequalities

$$\left\| \sum_{j=1}^{i} \ell^a_{ij}(n)x_j \right\| \leqq \sum_{j=1}^{i} \left\| \ell^a_{ij}(n)x_j \right\| \quad (1 \leqq i \leqq n).$$

We revealed the fact that the sum of all differences of the triangle inequalities

$$\left\| \sum_{j=1}^{i} \ell^a_{ij}(n)x_j \right\| \leqq \sum_{j=1}^{i} \left\| \ell^a_{ij}(n)x_j \right\| \quad (1 \leqq i \leqq n)$$

less than the difference of the triangle inequality

$$\left\| \sum_{j=1}^{n} x_j \right\| \leqq \sum_{j=1}^{n} \|x_j\|$$

as follows:

**Theorem 6** [12, Theorem 3.2] *Let* $n \geqq 2$. *With the above notation, take any* $a = (a_{ij})$ *in* $L_n$. *For all elements* $x_1, x_2, \ldots, x_n$ *in a Banach space* $X$, *we have*

$$0 \leqq \sum_{i=1}^{n} \left( \sum_{j=1}^{i} \left\| \ell^a_{ij}(n)x_j \right\| - \left\| \sum_{j=1}^{i} \ell^a_{ij}(n)x_j \right\| \right) \leqq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|.$$

To understand the above inequality, we shall discuss in the cases of $n = 2$ and 3. As the case $n = 2$ we have:

**Theorem 7** (Cf. [4, Theorem 2.2]) *For each* $x$, $y$ *in a Banach space* (*or normed linear space*) $X$, *and* $s$, $t$ *in* $\mathbb{R}$ *with* $0 \leqq s, t \leqq 1$, *we have*

$$0 \leqq \|sx\| + \|ty\| - \|sx + ty\| \leqq \|x\| + \|y\| - \|x + y\|.$$

For $x$ and $y$ in $X$, putting a function $f$ on a product space $[0, 1] \times [0, 1]$ as

$$f(s,t) = \|sx\| + \|ty\| - \|sx + ty\| \quad \big((s,t) \in, [0,1] \times [0,1]\big),$$

then it is clear that $f$ is a continuous function on $[0, 1] \times [0, 1]$ satisfying

$$f(0,0) = 0 \quad \text{and} \quad f(1,1) = \|x\| + \|y\| - \|x + y\|.$$

Since $[0, 1] \times [0, 1]$ is connected, by using the intermediate value theorem, we have

**Corollary 8** [12, Corollary 2.2] *Let* $x, y \in X$. *For each* $\omega$ *with* $0 \leqq \omega \leqq \|x\| + \|y\| - \|x + y\|$, *there is* $(s_0, t_0)$ *in* $[0, 1] \times [0, 1]$ *such that*

$$\omega = \|s_0 x\| + \|t_0 y\| - \|s_0 x + t_0 y\|.$$

Corollary 8 not only gives the solution of Problem 2 but also contains Theorem 3. Indeed, for any $x$ and $y$ in $X$ and $s, s_0, t, t_0 \in \mathbb{R}$ with $0 \leqq s \leqq s_0 \leqq 1$, $0 \leqq t \leqq t_0 \leqq 1$, we see that $s_0 x, t_0 y \in X$, $0 \leqq \frac{s}{s_0} \leqq 1$ and $0 \leqq \frac{t}{t_0} \leqq 1$. Hence, by Theorem 7, we have

$$\begin{aligned} f(s,t) &= \left\| \frac{s}{s_0}(s_0 x) \right\| + \left\| \frac{t}{t_0}(t_0 y) \right\| - \left\| \frac{s}{s_0}(s_0 x) + \frac{t}{t_0}(t_0 y) \right\| \\ &\leqq \|s_0 x\| + \|t_0 y\| - \|s_0 x + t_0 y\| \\ &= f(s_0, t_0). \end{aligned}$$

Thus $f$ is a nondecreasing continuous function on $[0, 1] \times [0, 1]$, and so if $\|y\| \leqq \|x\|$, then we have

$$\begin{aligned} 0 &= f(0,0) \leqq f(s_1, 1) \leqq f\left( \frac{\|y\|}{\|x\|}, 1 \right) \leqq f(s_2, 1) \\ &\leqq f(1,1) = \|x\| + \|y\| - \|x + y\| \quad \left( 0 \leqq \forall s_1 \leqq \frac{\|y\|}{\|x\|} \leqq \forall s_2 \leqq 1 \right). \end{aligned}$$

Since

$$f\left( \frac{\|y\|}{\|x\|}, 1 \right) = \left( 2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \min\big\{\|x\|, \|y\|\big\},$$

by Theorem 7, we have Theorem 3 as

$$0 \leqq \left( 2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right) \min\big\{\|x\|, \|y\|\big\} \leqq \|x\| + \|y\| - \|x + y\|.$$

Next, we consider a geometric meaning of these inequalities as $X = \mathbb{R}^2$. Let $x, y \in X$ with $\|y\| \leqq \|x\|$. Theorem 7 shows the relation of the differences of the inequalities concerning two triangles in Fig. 1.

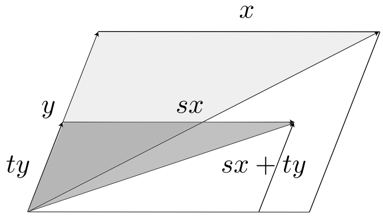


**Fig. 1** Theorem 7

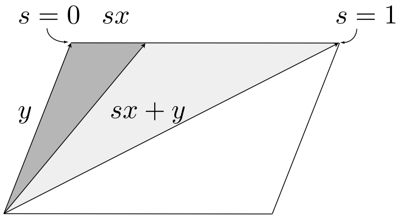


**Fig. 2** Case with $t = 1$

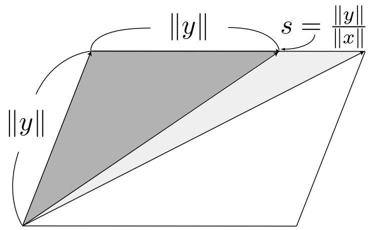


**Fig. 3** Theorem 3

Especially, Theorem 7 is as follows for $t = 1$

$$\begin{aligned} 0 &\leqq \|sx\| + \|y\| - \|sx + y\| \\ &= f(s, 1) \\ &\leqq \|x\| + \|y\| - \|x + y\| \quad (0 \leqq \forall s \leqq 1). \end{aligned}$$

Figure 2 is useful to understand this relation.

In this case, if the value of $s$ is continuously moved from 0 to 1, all the values between $f(0, 1) = 0$ and $f(1, 1) = \|x\| + \|y\| - \|x + y\|$ can be obtained. And, when the value of $s$ is just $\|y\|/\|x\|$, it is Theorem 3 (see Fig. 3).

In the case of $n = 3$, by Theorem 6, we have as follows:

**Theorem 9** [12, Theorem 2.3] *For each $x$, $y$, $z$ in $X$, and $\alpha, \beta, \gamma, \lambda, \mu$ in $\mathbb{R}$ with $0 \leqq \alpha, \beta, \gamma, \lambda, \mu \leqq 1$, we have*

$$\begin{aligned} 0 \leqq &\left(\|\alpha x\| + \|\beta y\| + \|\gamma z\| - \|\alpha x + \beta y + \gamma z\|\right) \\ &+ \left(\left\|\lambda(1-\alpha)x\right\| + \left\|\mu(1-\beta)y\right\| - \left\|\lambda(1-\alpha)x + \mu(1-\beta)y\right\|\right) \\ &\leqq \|x\| + \|y\| + \|z\| - \|x + y + z\|. \end{aligned}$$

As well as in the case of $n = 2$, for $x, y, z$ in $X$, if we put a function $g$ on the product space $\prod_{i=1}^{5}[0,1] = \underbrace{[0,1] \times \cdots \times [0,1]}_{5 \text{ times}}$ as

$$g(\alpha, \beta, \gamma, \lambda, \mu) = \big(\|\alpha x\| + \|\beta y\| + \|\gamma z\| - \|\alpha x + \beta y + \gamma z\|\big) + \big(\big\|\lambda(1-\alpha)x\big\| + \big\|\mu(1-\beta)y\big\| - \big\|\lambda(1-\alpha)x + \mu(1-\beta)y\big\|\big),$$

then it can be easily checked that, for each $(\alpha_1, \beta_1, \gamma_1, \lambda_1, \mu_1)$, $(\alpha_2, \beta_2, \gamma_2, \lambda_2, \mu_1)$ in $\prod_{i=1}^{5}[0,1]$,

$$\begin{aligned} &\big|g(\alpha_1, \beta_1, \gamma_1, \lambda_1, \mu_1) - g(\alpha_2, \beta_2, \gamma_2, \lambda_2, \mu_1)\big| \\ &\quad \leqq 2\big(2|\alpha_1 - \alpha_2| + |\lambda_1 - \lambda_2|\big) \cdot \|x\| + 2\big(2|\beta_1 - \beta_2| + |\mu_1 - \mu_2|\big) \cdot \|y\| \\ &\qquad + 2|\gamma_1 - \gamma_2| \cdot \|z\|, \end{aligned}$$

and so, $g$ is a continuous function on $\prod_{i=1}^{5}[0,1]$. Moreover, we see that

$$g(0,0,0,0,0) = 0 \quad \text{and}$$
$$g(1,1,1,\lambda,\mu) = \|x\| + \|y\| + \|z\| - \|x + y + z\| \quad (0 \leqq \forall\lambda, \mu \leqq 1).$$

Thus we have

**Corollary 10** [12, Corollary 2.4] *Let $x, y, z \in X$. For each $\omega$ with $0 \leqq \omega \leqq \|x\| + \|y\| + \|z\| - \|x + y + z\|$, there exists $(s_1, s_2, s_3, s_4, s_5) \in \prod_{i=1}^{5}[0,1]$ such that*

$$\omega = \big(\|s_1 x\| + \|s_2 y\| + \|s_3 z\| - \|s_1 x + s_2 y + s_3 z\|\big) + \big(\big\|s_4(1-s_1)x\big\| + \big\|s_5(1-s_2)y\big\| - \big\|s_4(1-s_1)x + s_5(1-s_2)y\big\|\big).$$

There are infinitely many paths to take the value from 0 to $\|x\| + \|y\| + \|z\| - \|x + y + z\|$ by how to choose variables. We can obtain Theorems 4 and 5 in the case $n = 3$ by choosing it well. Note that Theorems 4 and 5 are as follows respectively for $n = 3$:

**Theorem 11** (Cf. Theorem 4) *For all nonzero elements $x$, $y$, $z$ in $X$ with $\|x\| \geqq \|y\| \geqq \|z\|$,*

$$0 \leqq \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|z\| \leqq \|x\| + \|y\| + \|z\| - \|x + y + z\|.$$

**Theorem 12** (Cf. Theorem 5) *For all nonzero elements $x$, $y$, $z$ in $X$ with $\|x\| \geqq \|y\| \geqq \|z\|$,*

$$\begin{aligned} 0 &\leqq \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|z\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\big(\|y\| - \|z\|\big) \\ &\leqq \|x\| + \|y\| + \|z\| - \|x + y + z\|. \end{aligned}$$

To obtain these theorems, the following example gives one of the paths.

*Example 13* We may assume that $\|x\| \neq \|z\|$. For each $t$ with $0 \leqq t \leqq \frac{\|z\|}{\|x\|}$, put

$$s_1^t = t, \qquad s_2^t = \frac{\|x\|}{\|y\|}t, \qquad s_3^t = \frac{\|x\|}{\|z\|}t,$$
$$s_4^t = \frac{\|x\|}{\|z\|} \cdot \frac{\|y\| - \|z\|}{\|x\| - \|z\|} \cdot t, \qquad s_5^t = \frac{\|x\|}{\|z\|}t,$$

and for each $t$ with $\frac{\|z\|}{\|x\|} \leqq t \leqq 1$, put

$$s_1^t = t, \qquad s_2^t = \frac{\|z\|}{\|y\|} + \frac{\|x\|}{\|y\|} \cdot \frac{\|y\| - \|z\|}{\|x\| - \|z\|} \cdot \left(t - \frac{\|z\|}{\|x\|}\right), \qquad s_3^t = 1,$$
$$s_4^t = \frac{\|y\| - \|z\|}{\|x\| - \|z\|} + \frac{(\|x\| - \|y\|)(\|x\|t - \|z\|)}{(\|x\| - \|z\|)^2}, \qquad s_5^t = 1,$$

then we see that $(s_1^t, s_2^t, s_3^t, s_4^t, s_5^t) \in \prod_{i=1}^5$ satisfying

$$\left(s_1^0, s_2^0, s_3^0, s_4^0, s_5^0\right) = (0, 0, 0, 0, 0), \qquad \left(s_1^1, s_2^1, s_3^1, s_4^1, s_5^1\right) = (1, 1, 1, 1, 1) \quad \text{and}$$
$$\left(s_1^{\frac{\|z\|}{\|x\|}}, s_2^{\frac{\|z\|}{\|x\|}}, s_3^{\frac{\|z\|}{\|x\|}}, s_4^{\frac{\|z\|}{\|x\|}}, s_5^{\frac{\|z\|}{\|x\|}}\right) = \left(\frac{\|z\|}{\|x\|}, \frac{\|z\|}{\|y\|}, \frac{\|z\|}{\|z\|}, \frac{\|y\| - \|z\|}{\|x\| - \|z\|}, 1\right).$$

Hence if we define a function $f$ on [0, 1] by

$$f(t) = g\left(s_1^t, s_2^t, s_3^t, s_4^t, s_5^t\right),$$

then $f$ is a continuous function on [0, 1] such that

$$f(0) = 0 \quad \text{and} \quad f(1) = \|x\| + \|y\| + \|z\| - \|x + y + z\|.$$

Moreover

$$\begin{aligned}
f\left(\frac{\|z\|}{\|x\|}\right) &= g\left(s_1^{\frac{\|z\|}{\|x\|}}, s_2^{\frac{\|z\|}{\|x\|}}, s_3^{\frac{\|z\|}{\|x\|}}, s_4^{\frac{\|z\|}{\|x\|}}, s_5^{\frac{\|z\|}{\|x\|}}\right) \\
&= g\left(\frac{\|z\|}{\|x\|}, \frac{\|z\|}{\|y\|}, \frac{\|z\|}{\|z\|}, \frac{\|y\| - \|z\|}{\|x\| - \|z\|}, 1\right) \\
&= \left(\left\|\frac{\|z\|}{\|x\|}x\right\| + \left\|\frac{\|z\|}{\|y\|}y\right\| + \left\|\frac{\|z\|}{\|z\|}z\right\| - \left\|\frac{\|z\|}{\|x\|}x + \frac{\|z\|}{\|y\|}y + \frac{\|z\|}{\|z\|}z\right\|\right) \\
&\quad + \left\{\left\|\frac{\|y\| - \|z\|}{\|x\| - \|z\|}\left(1 - \frac{\|z\|}{\|x\|}\right)x\right\| + \left\|\left(1 - \frac{\|z\|}{\|y\|}\right)y\right\|\right. \\
&\quad \left. - \left\|\frac{\|y\| - \|z\|}{\|x\| - \|z\|}\left(1 - \frac{\|z\|}{\|x\|}\right)x + \left(1 - \frac{\|z\|}{\|y\|}\right)y\right\|\right\}
\end{aligned}$$

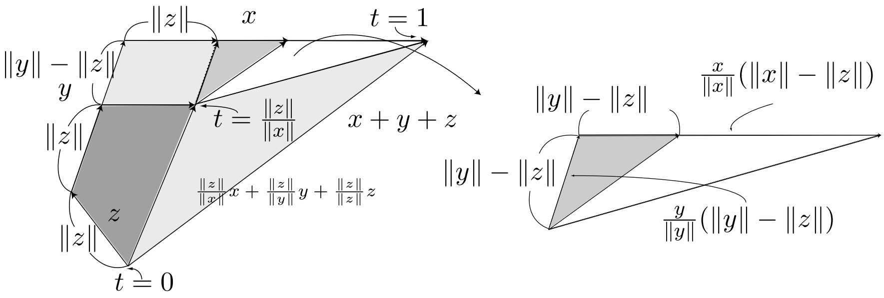


**Fig. 4** Example 13

$$= \left(3 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} \right\|\right)\|z\|$$
$$+ \left( \left\| \frac{\|y\| - \|z\|}{\|x\|} \cdot x \right\| + \left\| \frac{\|y\| - \|z\|}{\|y\|} \cdot y \right\| \right.$$
$$\left. - \left\| \frac{\|y\| - \|z\|}{\|x\|} \cdot x + \frac{\|y\| - \|z\|}{\|y\|} \cdot y \right\| \right)$$
$$= \left(3 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|} \right\|\right)\|z\| + \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|\right)(\|y\| - \|z\|).$$

Since $f(0) \leqq f(\frac{\|z\|}{\|x\|}) \leqq f(1)$, we have Theorem 12.

Figure 4 shows that Theorem 12 is a combination of Theorems 3 and 11. That is, we obtain Theorem 12, applying Theorem 3 for two vectors $\frac{y}{\|y\|}(\|y\| - \|z\|)$ and $\frac{x}{\|x\|}(\|x\| - \|z\|)$ in addition to Theorem 11.

Furthermore, if we put, for each $t, u$ with $0 \leqq t, u \leqq \frac{\|z\|}{\|x\|}$,

$$s_1^t = t, \qquad s_2^t = \frac{\|x\|}{\|y\|}t, \qquad s_3^t = \frac{\|x\|}{\|z\|}t,$$
$$s_4^u = \frac{\|x\|}{\|z\|} \cdot \frac{\|y\| - \|z\|}{\|x\| - \|z\|} \cdot u, \qquad s_5^u = \frac{\|x\|}{\|z\|}u,$$

and for each $t, u$ with $\frac{\|z\|}{\|x\|} \leqq t, u \leqq 1$, put

$$s_1^t = t, \qquad s_2^t = \frac{\|z\|}{\|y\|} + \frac{\|x\|}{\|y\|} \cdot \frac{\|y\| - \|z\|}{\|x\| - \|z\|} \cdot \left(t - \frac{\|z\|}{\|x\|}\right), \qquad s_3^t = 1,$$
$$s_4^u = \frac{\|y\| - \|z\|}{\|x\| - \|z\|} + \frac{(\|x\| - \|y\|)(\|x\|u - \|z\|)}{(\|x\| - \|z\|)^2}, \qquad s_5^t = 1,$$

then we can define a function $h$ on $[0, 1] \times [0, 1]$ by

$$h(t, u) = g\left(s_1^t, s_2^t, s_3^t, s_4^u, s_5^u\right).$$

It contains both Theorems 11 and 12 at the same time, that is, if we take $(t,u) = (\frac{\|z\|}{\|x\|}, 0)$, then we have Theorem 11 and if we take $(t,u) = (\frac{\|z\|}{\|x\|}, \frac{\|z\|}{\|x\|})$, then we also have Theorem 12.

Finally, we consider Problem 2 for a general case.

Let $x_1, x_2, \ldots, x_n$ in $X$. For each $a$ in $L_n$, if we put

$$f(a) = \sum_{i=1}^{n}\left(\sum_{j=1}^{i}\|\ell_{ij}^{a}(n)x_j\| - \left\|\sum_{j=1}^{i}\ell_{ij}^{a}(n)x_j\right\|\right),$$

then we see that

$$f(a_0) = 0 \quad \text{and} \quad f(a_1) = \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\|,$$

where $a_0, a_1 \in L_n$ with

$$a_0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad a_1 = \begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,n-1} & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Moreover, considering $f$ for a function on $\prod_{i=1}^{n(n+1)/2}[0,1]$, we see that $f$ is continuous on $\prod_{i=1}^{n(n+1)/2}[0,1]$. Therefore, as a solution of Problem 2, we obtain the following

**Corollary 14** [12, Corollary 3.3] *Keep the notations as above. Let $x_1, x_2, \ldots, x_n \in X$. For each $\omega$ with $0 \leqq \omega \leqq \sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n}x_j\|$, there exists $a$ in $L_n$ such that*

$$\omega = \sum_{i=1}^{n}\left(\sum_{j=1}^{i}\|\ell_{ij}^{a}(n)x_j\| - \left\|\sum_{j=1}^{i}\ell_{ij}^{a}(n)x_j\right\|\right).$$

Of course Theorem 6 contains Theorems 4 and 5 as well as the cases $n = 2, 3$.

**Corollary 15** [7, Lemma 2] *For all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space $X$, we have*

$$0 \leqq \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\min_{1\leqq j\leqq n}\|x_j\| \leqq \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\|.$$

**Corollary 16** [13, Theorem 1] *For all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space $X$, we have*

$$0 \leqq \sum_{i=2}^{n} \left( i - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| \right) \left( \|x_i^*\| - \|x_{i+1}^*\| \right) \leqq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|,$$

*where $x_1^*, \ldots, x_n^*$ are the rearrangement of $x_1, x_2, \ldots, x_n$ which satisfies $\|x_1^*\| \geqq \|x_2^*\| \geqq \cdots \geqq \|x_n^*\|$ and $x_{n+1}^* = 0$.*

*Note* The other direction of Problem 1.2, namely; characterize all values $D$ which satisfy

$$0 \leqq \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\| \leqq D$$

by using $x_1, x_2, \ldots, x_n$ in $X$, has been studied in [9] and [13]. We extend these results by using a continuous mapping. The results will be appeared in [11].

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# A Dunkl-Williams Inequality and the Generalized Operator Version

**Kichi-Suke Saito and Masaru Tominaga**

**Abstract** C.F. Dunkl and K.S. Williams (Am. Math. Mon. 71, 53–54 (1964)) showed that for any nonzero elements $x$, $y$ in a normed linear space $\mathcal{X}$

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{4\|x-y\|}{\|x\|+\|y\|}.$$

Recently, J. Pečarić and R. Rajić (J. Math. Inequal. 4, 1–10 (2010)) gave a refinement and, moreover, a generalization to operators $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|$, $|B|$ are invertible as follows:

$$\left|A|A|^{-1} - B|B|^{-1}\right|^2 \le |A|^{-1}\left(p|A-B|^2 + q\left(|A|-|B|\right)^2\right)|A|^{-1}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In this note, we review some results concerning the Dunkl-Williams inequality and the generalization of the operator version of J. Pečarić and R. Rajić.

**Keywords** Dunkl-Williams inequality · Bohr inequality · Operator inequality

**Mathematics Subject Classification** 26D15 · 46B20 · 47A63

## 1 Introduction

In 1964, C.F. Dunkl and K.S. Williams [4] showed that for any nonzero elements $x$, $y$ in a normed linear space $\mathcal{X}$

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{1}$$

Recently, J. Pečarić and R. Rajić in [15] gave the following refinement of a Dunkl-Williams inequality (1): For any nonzero elements $x$, $y$ in a normed linear space $\mathcal{X}$

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{\sqrt{2\|x-y\|^2 + 2(\|x\|-\|y\|)^2}}{\max\{\|x\|, \|y\|\}}. \tag{2}$$

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In this note, let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, we denote the absolute value operator of $A$ by $|A|$, that is, $|A| = (A^*A)^{\frac{1}{2}}$, where $A^*$ stands for the adjoint operator of $A$. We denote the closure of $A\mathcal{H}$ by $[A\mathcal{H}]$ and the orthogonal projection onto $[A\mathcal{H}]$ by $P_{[A\mathcal{H}]}$. Let $A = U|A|$ be the polar decomposition of $A \in \mathcal{B}(\mathcal{H})$ with $U^*U = P_{[|A|\mathcal{H}]}$.

Recently, Pečarić and R. Rajić in [15] generalized the inequality (2) to the (invertible) operator case and studied its equality conditions (cf. [15, Theorem 2.1]):

**Theorem 1** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be operators where $|A|$ and $|B|$ are invertible, and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left|A|A|^{-1} - B|B|^{-1}\right|^2 \le |A|^{-1}\left(p|A - B|^2 + q\left(|A| - |B|\right)^2\right)|A|^{-1}. \tag{3}$$

*The equality in* (3) *holds if and only if*

$$p(A - B)|A|^{-1} = qB\left(|A|^{-1} - |B|^{-1}\right). \tag{4}$$

Our aim in this note is to review some results concerning the Dunkl-Williams inequality and the generalization of the operator version of J. Pečarić and R. Rajić. In Sect. 2, we shall introduce a Dunkl-Williams inequality and some related results, that is, its refinement and reverse inequalities. In Sect. 3, we shall cite a classical Bohr inequality, its operator version and moreover generalizations. In Sect. 4, we shall show that the inequality (3) can be generalized by using polar decompositions $A = U|A|$ and $B = V|B|$ of operators $A, B \in \mathcal{B}(\mathcal{H})$ with $U^*U = P_{[|A|\mathcal{H}]}$ and $V^*V = P_{[|B|\mathcal{H}]}$. As a result, Theorem 1 is extended without demanding the invertibility of $|A|$ and $|B|$. Moreover, we investigate the equality condition (4) and give a refinement of the equality conditions without additional assumptions related to inverse conditions in Sect. 5. The obtained results are generalizations of [15]. The content of Sect. 4 and Sect. 5 is a survey of our paper [16].

## 2 Dunkl-Williams Inequalities and Its Related Results

In this section, we introduce the following interested inequality which is given by C.F. Dunkl and K.S. Williams in 1964, and its proof:

**Theorem 2** *Let $\mathcal{X}$ be a normed linear space. Then for every $x, y (\neq 0)$ in $\mathcal{X}$*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{4\|x - y\|}{\|x\| + \|y\|}. \tag{5}$$

*Proof* We calculate the following related to the norm:

$$\begin{aligned}\|x\|\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| &\le \|x\|\left\|\frac{x}{\|x\|}-\frac{y}{\|x\|}\right\|+\|x\|\left\|\frac{y}{\|x\|}-\frac{y}{\|y\|}\right\| \\ &= \|x-y\|+\frac{\|(\|y\|-\|x\|)y\|}{\|y\|} \\ &= \|x-y\|+\big|\|y\|-\|x\|\big| \le 2\|x-y\|.\end{aligned}$$

We obtain the following by a similar method:

$$\|y\|\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \le 2\|x-y\|.$$

So we have the desired inequality (1). □

In particular, if $\mathcal{X}$ is an inner product space, then a more precise inequality is obtained:

**Theorem 3** *Let $\mathcal{X}$ be an inner product space. Then for every $x, y (\neq 0)$ in $\mathcal{X}$*

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \le \frac{2\|x-y\|}{\|x\|+\|y\|}. \tag{6}$$

*Proof* We calculate the following related to the norm and the inner product:

$$\begin{aligned}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2 &= \left\langle \frac{x}{\|x\|}-\frac{y}{\|y\|}, \frac{x}{\|x\|}-\frac{y}{\|y\|}\right\rangle \\ &= \frac{1}{\|x\|\|y\|}\left\{2\|x\|\|y\|-2\,\mathrm{Re}\langle x, y\rangle\right\} \\ &= \frac{1}{\|x\|\|y\|}\left\{2\|x\|\|y\|-\left(\|x\|^2+\|y\|^2-\|x-y\|^2\right)\right\} \\ &= \frac{\|x-y\|^2-(\|x\|-\|y\|)^2}{\|x\|\|y\|}.\end{aligned}$$

So we have

$$\begin{aligned}&\|x-y\|^2-\left(\frac{\|x\|+\|y\|}{2}\right)^2\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2 \\ &= \frac{(\|x\|-\|y\|)^2}{4\|x\|\|y\|}\left\{\left(\|x\|+\|y\|\right)^2-\|x-y\|^2\right\} \ge 0.\end{aligned}$$

The proof of Theorem 3 is completed. □

In a bit later 1964, W. Kirk and M. Smiley [9] characterized the inner product spaces:

**Theorem 4** *Let $\mathcal{X}$ be a normed linear space. Then for every $x, y(\neq 0)$ in $\mathcal{X}$*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\|+\|y\|} \tag{7}$$

*if and only if $\mathcal{X}$ is an inner product space.*

E.R. Lorch [10] showed that $\mathcal{X}$ is an inner product space if and only if the following relation holds: If every vectors $x, y \in \mathcal{X}$ holds $\|x\| = \|y\|$, then the inequality $\|\alpha x + \alpha^{-1} y\| \geq \|x+y\|$ satisfies for any positive real number $\alpha$.

A certain kind of evaluation values "4" and "2" that appears to inequalities (1) and (6), respectively are interested numbers. In 2008, the following constant was introduced in [7]:

$$\mathrm{DW}(\mathcal{X}) := \sup\big\{\mathrm{dw}(x,y) : x, y \in \mathcal{X}, x \neq 0, y \neq 0, x \neq y\big\},$$

where $\mathrm{dw}(x,y) = \frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$. We denote this constant by the Dunkl-Williams constant. This constant has the following interesting properties:

(1) $2 \leq \mathrm{DW}(\mathcal{X}) \leq 4$.
(2) $\mathrm{DW}(\mathcal{X}) = 2 \Leftrightarrow \mathcal{X}$ is a inner product space.
(3) $\mathrm{DW}(\mathbb{R}^2, \|\cdot\|_1) = \mathrm{DW}(\mathbb{R}^2, \|\cdot\|_\infty) = 4$.
(4) $\mathrm{DW}(\mathcal{X}) < 4 \Leftrightarrow \mathcal{X}$ is a uniformly nonsquare, that is, if there exists $\delta > 0$ such that for any pair $x, y \in \mathcal{B}_{\mathcal{X}} (= \{x \in \mathcal{X} : \|x\| \leq 1\})$, then $\min\{\|x+y\|, \|x-y\|\} \leq \delta$.

Next we consider the inequality (1) more precisely. Recently, the first author studied the following related to the triangle inequality [8] (cf. [11, 12]):

**Theorem 5** *Let $\mathcal{X}$ be a normed linear space. Then for every $x, y(\neq 0)$ in $\mathcal{X}$*

$$\begin{aligned} \|x+y\| &+ \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\min\big\{\|x\|, \|y\|\big\} \\ &\leq \|x\| + \|y\| \\ &\leq \|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\max\big\{\|x\|, \|y\|\big\}. \end{aligned}$$

In the second inequality of Theorem 5 we replace $y$ with $-y$. Then the following theorem estimates the inequality (1) more precisely:

**Theorem 6** *Let $\mathcal{X}$ be a normed linear space. Then for every $x, y(\neq 0)$ in $\mathcal{X}$*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x-y\| + \big|\|x\| - \|y\|\big|}{\max\{\|x\|, \|y\|\}}. \tag{8}$$

Moreover, a reverse inequality of (1) is given as follows:

**Theorem 7** *Let $\mathcal{X}$ be a normed linear space. Then for every $x, y (\neq 0)$ in $\mathcal{X}$*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geq \frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}}. \tag{9}$$

Next we introduce Massera-Schaffer's inequality which is obtained by the inequalities (8) and $\|x-y\| + |\|x\| - \|y\|| \leq 2\|x-y\|$ (cf. [13]):

**Theorem 8** *Let $\mathcal{X}$ be a normed linear space. Then for every $x, y (\neq 0)$ in $\mathcal{X}$*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}}. \tag{10}$$

On the other hand, for every $x, y (\neq 0)$ in $\mathcal{X}$ we have

$$\|x-y\| + \big|\|x\| - \|y\|\big| \leq \sqrt{2}\sqrt{\|x-y\|^2 + \big(\|x\| - \|y\|\big)^2} \leq 2\|x-y\|.$$

As a result, the following inequality holds (cf. [15]):

**Theorem 9** *Let $\mathcal{X}$ be a normed linear space. Then for every $x, y (\neq 0)$ in $\mathcal{X}$*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\sqrt{2}\sqrt{\|x-y\|^2 + (\|x\| - \|y\|)^2}}{\max\{\|x\|, \|y\|\}}. \tag{11}$$

## 3 Bohr Inequality and Its Related Results

In this section, we introduce the operator version of the classical Bohr inequality. First, the classical Bohr inequality in [2, 14] says that

$$|a+b|^2 \leq p|a|^2 + q|b|^2$$

for all scalars $a$, $b$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. The equality holds if and only if $(p-1)a = b$. In [6], Hirzallah proposed an operator version of Bohr inequality:

**Theorem 10** *If $A, B \in \mathcal{B}(\mathcal{H})$ and $q \geq p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$|A-B|^2 + \big|(p-1)A + B\big|^2 \leq p|A|^2 + q|B|^2.$$

Moreover Hirzallah gave the following inequality and pointed out an equality condition:

**Corollary 11** [6, Corollary 1] *Let $A, B \in \mathcal{B}(\mathcal{H})$, and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$|A-B|^2 \leq p|A|^2 + q|B|^2 \tag{12}$$

*with equality if and only if $pA = -qB$.*

Afterwards, several authors have presented generalizations of Bohr inequality (cf. [1, 3, 5, 6]). We show that generalized Bohr inequalities are covered by this theorem. On the other hand, a generalized parallelogram law also implies generalized Bohr inequalities. It is essentially same as the discussion in [1]. Here, we cite Bohr type inequalities from [3] and [6].

**Theorem 12** *If* $A, B \in \mathcal{B}(\mathcal{H})$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $1 < p \leq 2$, *then*

(i) $|A - B|^2 + |(p-1)A + B|^2 \leq p|A|^2 + q|B|^2$,
(ii) $|A - B|^2 + |A + (q-1)B|^2 \geq p|A|^2 + q|B|^2$.

*On the other hand, if either* $p < 1$ *or* $p \geq 2$, *then*

(iii) $|A - B|^2 + |(p-1)A + B|^2 \geq p|A|^2 + q|B|^2$.

Moreover, M. Fujii and H. Zuo in [5] mentioned an approach to Bohr inequality related to the parallelogram law, whose idea is essentially same as that of Abramovich, Barić and Pečarić [1].

**Theorem 13** *If* $A$ *and* $B$ *are operators on a Hilbert space and* $t \neq 0$, *then*

$$|A + B|^2 + \frac{1}{t}|tA - B|^2 = (1+t)|A|^2 + \left(1 + \frac{1}{t}\right)|B|^2.$$

Theorem 13 gives a unified proof to Theorem 12.

## 4 A Generalization of Dunkl-Williams Operator Inequality

First, we consider a generalization of the inequality (3):

**Theorem 14** *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be operators with polar decompositions* $A = U|A|$ *and* $B = V|B|$, *and let* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\big|(U - V)|A|\big|^2 \leq p|A - B|^2 + q\big(|A| - |B|\big)^2. \tag{13}$$

*The equality in* (13) *holds if and only if*

$$p(A - B) = qV\big(|B| - |A|\big) \quad \text{and} \quad U^*U = V^*V. \tag{14}$$

For the proof of the above theorem, we use the following lemma:

**Lemma 15** *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be operators with polar decompositions* $A = U|A|$ *and* $B = V|B|$, *and let* $p, q \in \mathbb{R}$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Suppose that* $p(A - B) = qV(|B| - |A|)$. *Then*

$$p|A - B|^2 \leq q\big(|A|^2 - |B|^2\big).$$

*In particular, if $p > 1$, then $|A| \geq |B|$ and $U^*U \geq V^*V$. Further if $U^*U = V^*V$, then $p|A-B|^2 = q(|A|^2 - |B|^2)$.*

*Proof* It follows from the equality $p(A-B) = qV(|B|-|A|)$ that

$$\big|p(A-qB)\big|^2 = \big|-qV|A|\big|^2 = q^2|A|V^*V|A| \leq q^2|A|^2.$$

On the other hand, we have

$$\big|p(A-qB)\big|^2 = p^2\big\{(1-q)|A|^2 + \big(q^2-q\big)|B|^2 + q\big(|A-B|^2\big)\big\}.$$

So we have by $q - 1 = \frac{q}{p}$

$$-|A|^2 + q|B|^2 + p|A-B|^2 \leq \frac{q}{p}|A|^2, \quad \text{that is,} \quad p|A-B|^2 \leq q\big(|A|^2 - |B|^2\big).$$

If $p > 1$, then $|A|^2 \geq |B|^2$. Hence we have $|A| \geq |B|$. If $U^*U = V^*V$, then $V^*V|A| = |A|$. So we have $p|A-B|^2 = q(|A|^2 - |B|^2)$. □

By Corollary 11 and the above lemma, we shall prove Theorem 14:

*Proof of Theorem 14* We have

$$\big|(U-V)|A|\big|^2 = \big|A - B - V\big(|A|-|B|\big)\big|^2.$$

Applying Corollary 11 to operators $A - B$ and $V(|A|-|B|)$, we have

$$\big|(U-V)|A|\big|^2 \leq p|A-B|^2 + q\big|V\big(|A|-|B|\big)\big|^2 \tag{15}$$

$$\leq p|A-B|^2 + q\big(|A|-|B|\big)^2 \quad \big(\text{by } V^*V \leq I\big). \tag{16}$$

By Corollary 11 the equality in (15) holds if and only if $p(A-B) = qV(|B|-|A|)$. The equality in (16) holds if and only if $V^*V|A| = |A|$. This implies $U^*U \leq V^*V$. On the other hand, the condition $p(A-B) = qV(|B|-|A|)$ gives $U^*U \geq V^*V$ by Lemma 15. Hence we have $U^*U = V^*V$.

Conversely, the condition $U^*U = V^*V$ gives $V^*V(|A|-|B|) = |A|-|B|$. So equalities in (15) and (16) hold, which implies equality in (13). □

**Corollary 16** *Theorem 14 gives Theorem 1.*

*Proof* We have $U = A|A|^{-1}$, $V = B|B|^{-1}$, $U^*U = V^*V = I$. This implies

$$\begin{aligned}\big|A|A|^{-1} - B|B|^{-1}\big|^2 &= |U-V|^2 \\ &= |A|^{-1}\big|(U-V)|A|\big|^2|A|^{-1} \\ &\leq |A|^{-1}\big(p|A-B|^2 + q\big(|A|-|B|\big)^2\big)|A|^{-1} \quad \big(\text{by } (13)\big).\end{aligned}$$

The equality in (3) holds if and only if

$$\begin{aligned} p(A-B)|A|^{-1} &= qV\big(|B|-|A|\big)|A|^{-1} \\ &= qB\big(|A|^{-1}-|B|^{-1}\big). \end{aligned}$$

Hence we have Theorem 1. □

Putting $p=q=2$ in (13) and taking the square root of each side of it, we have the following corollary.

**Corollary 17** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be operators with polar decompositions $A=U|A|$ and $B=V|B|$. Then*

$$\big|(U-V)|A|\big| \le \sqrt{2}\big(|A-B|^2+\big(|A|-|B|\big)^2\big)^{\frac{1}{2}}. \tag{17}$$

## 5 Equality Conditions in Theorem 14

In this section, we give a refinement of the equality conditions (14) in Theorem 14. First we have the following proposition which is related to the equality in Theorem 14.

**Proposition 18** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be operators with polar decompositions $A=U|A|$ and $B=V|B|$, and let $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Suppose that $U^*U=V^*V$. Then the following statements are equivalent*:

(i) $p(A-B)=qV(|B|-|A|)$.
(ii) $|A|=|B|+\frac{p}{q}|A-B|$ *and* $A-B=-V|A-B|$.

*Proof* We shall show (i) ⇒ (ii). We have

$$p^2|A-B|^2=\big|qV\big(|B|-|A|\big)\big|^2=q^2\big(|B|-|A|\big)^2$$

from $U^*U=V^*V$. Moreover, since the condition (i) gives $|A|\ge|B|$ by Lemma 15, we have $q(|A|-|B|)=p|A-B|$, and so the first condition of (ii) holds. On the other hand, since $A-B=\frac{q}{p}V(|B|-|A|)=-V|A-B|$, we have the second condition of (ii).

The proof of (ii) ⇒ (i) is trivial. □

By the equality condition of Theorem 14 and Proposition 18, we have the following corollary.

**Corollary 19** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be operators with polar decompositions $A=U|A|$ and $B=V|B|$, and let $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Suppose that*

$$\big|(U-V)|A|\big|^2=p|A-B|^2+q\big(|A|-|B|\big)^2.$$

*Then*

$$|A| = |B| + \frac{p}{q}|A - B| \quad \textit{and} \quad A - B = -V|A - B|.$$

Next we will give characterizations of the equality in (13). For this purpose, we need the following lemmas.

**Lemma 20** [15, Lemma 2.9] *Let $S, T \in \mathcal{B}(\mathcal{H})$ be positive operators such that $ST + TS = tS^2$ for some $t \in \mathbb{R}$. Then the following statements hold*:

(i) *If $t < 0$, then $S = 0$.*
(ii) *If $t \geq 0$, then $ST = TS = \frac{1}{2}tS^2$.*

*Proof* We give a simple proof. The operator $S^2T(= S(tS^2 - TS)$ is self-adjoint. So, since $S^2$ and $T$ are commuting, from $S, T > 0$ we have $S$ and $T$ are commuting. If $t < 0$, then $2ST = tS^2 \leq 0$. So we have $S = 0$. On the other hand, if $t \geq 0$, then $2ST = tS^2$. □

**Lemma 21** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be operators with polar decompositions $A = U|A|$ and $B = V|B|$, and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that*

$$\big|(U - V)|A|\big|^2 = p|A - B|^2 + q\big(|A| - |B|\big)^2. \tag{18}$$

*Then*

$$|B||A - B| + |A - B||B| = (2 - p)|A - B|^2. \tag{19}$$

*Proof* Since the equality (18) implies $C(:= A - B) = -V|C|$ by Corollary 19, we have $B^*C = -|B|V^*V|C| = -|B||C|$. Hence we have

$$|C + B|^2 = |C|^2 - |C||B| - |B||C| + |B|^2.$$

On the other hand, by Corollary 19 we have

$$|C + B|^2 = \left(|B| + \frac{p}{q}|C|\right)^2.$$

The combination of above two equalities gives

$$\left(\frac{p^2}{q^2} - 1\right)|C|^2 + \left(\frac{p}{q} + 1\right)|B||C| + \left(\frac{p}{q} + 1\right)|C||B| = 0.$$

It follows from $\frac{p}{q} - 1 = p - 2$ that

$$|B||C| + |C||B| = (2 - p)|C|^2,$$

and this lemma is proved. □

Now we give a characterization of the equality in (13) by dividing the real numbers $p > 1$ into two cases ($p \geq 2$ in Theorem 22 and $1 < p < 2$ in Theorem 23) according to Lemma 20.

**Theorem 22** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be operators with polar decompositions $A = U|A|$ and $B = V|B|$, and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $p \geq 2$. Then the following statements are equivalent*:

(i) $|(U - V)|A||^2 = p|A - B|^2 + q(|A| - |B|)^2$.
(ii) $A = B$.

*Proof* (i) ⇒ (ii) Let $C = A - B$. The condition (i) gives an inequality $|B||C| + |C||B| = (2 - p)|C|^2$ by Lemma 21. If $p > 2$, then $A - B = 0$ by Lemma 20(i). Next we suppose $p = 2$. Then it holds $|C||B| = 0$ and so $|C|V^*V = 0$ by $V^*V = P_{[|B|\mathcal{H}]}$. Since $C = -V|C|$ from Corollary 19, we have $|C|^2 = |C|V^*V|C| = 0$, and hence $C = 0$.

(ii) ⇒ (i) It is obvious that $A = B$ gives $U = V$ by the uniqueness of the polar decomposition. □

**Theorem 23** *Let $A, B \in \mathcal{B}(\mathcal{H})$ be operators with polar decompositions $A = U|A|$ and $B = V|B|$, and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $1 < p < 2$ Then the following statements are equivalent*:

(i) $|(U - V)|A||^2 = p|A - B|^2 + q(|A| - |B|)^2$.
(ii) $\begin{cases} A = B(I - \frac{2}{2-p}W^*W), \\ |A| = |B|(I + \frac{2p}{(2-p)q}W^*W), \end{cases}$

*where $A - B = W|A - B|$ is the polar decomposition of $A - B$.*

*Proof* (i) ⇒ (ii) We put $C = A - B$. The condition (i) gives equalities

$$C = -V|C| \quad \text{and} \quad |B||C| + |C||B| = (2 - p)|C|^2$$

by Corollary 19 and Lemma 21, respectively. Then we have from $1 < p < 2$

$$|B||C| = |C||B| = \frac{1}{2}(2 - p)|C|^2$$

in Lemma 20. So it holds $B|C| = \frac{1}{2}(2 - p)V|C|^2 = \frac{1}{2}(p - 2)C|C|$, that is, $A|C| = \frac{p}{p-2}B|C|$. It follows from $W^*W\mathcal{H} = [|C|\mathcal{H}]$ that

$$AW^*W = \frac{p}{p - 2}BW^*W.$$

On the other hand, we have $(I - W^*W)\mathcal{H} = \operatorname{Ker} C$. This implies

$$A\big(I - W^*W\big) = B\big(I - W^*W\big).$$

So we have

$$A = AW^*W + A\big(I - W^*W\big) = B\left(I - \frac{2}{2-p}W^*W\right),$$

and so the first equality of (ii) holds.

Next we have by Corollary 19

$$-V|C| = A - B = -\frac{2}{2-p}V|B|W^*W. \tag{20}$$

Here, since the condition (i) implies $U^*U = V^*V$ by Theorem 14, we note that $V^*V \geq W^*W$. It follows from $U^*U = V^*V$ that $[A^*\mathcal{H}] = [B^*\mathcal{H}]$. So we have $[|C|\mathcal{H}] = [(A-B)^*\mathcal{H}] \subset [B^*\mathcal{H}] = [|B|\mathcal{H}]$, and so $V^*V \geq W^*W$. Hence the equality (20) gives $|C| = \frac{2}{2-p}|B|W^*W$. By Corollary 19, we have

$$|A| = |B| + \frac{p}{q}|C| = |B|\left(I + \frac{2p}{(2-p)q}W^*W\right).$$

(ii) ⇒ (i) We obtain that

$$qV\big(|B| - |A|\big) = -\frac{2p}{(2-p)}V|B|W^*W = p(A-B).$$

Moreover, since the operator $I + \frac{2p}{(2-p)q}W^*W (= (I - W^*W) + (1 + \frac{2p}{(2-p)q})W^*W)$ is invertible, we have $[|A|\mathcal{H}] = [|B|(I + \frac{2p}{(2-p)q}W^*W)\mathcal{H}] = [|B|\mathcal{H}]$. Hence we have $U^*U = V^*V$. And so we get the condition (i) from Theorem 14. □

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# Part 4
# Generalized Convexity

# Jordan Type Representation of Functions with Generalized High Order Bounded Variation

**Gabriela Cristescu**

**Abstract** The aim of this work is to identify few classes of functions with generalized type of bounded variation for which a decomposition theorem of Jordan type holds. We refer especially to functions with $n$th order bounded variation with respect to a Tchebycheff system. The particular case of trigonometric Tchebycheff systems bring interesting results.

**Keywords** Function with $n$th order bounded variation · Jordan decomposition · $n$th order convex function · Tchebycheff system

**Mathematics Subject Classification** Primary 26A45 · Secondary 26A51 · 26D05

## 1 Introduction. Functions with Bounded Variation of Superior Order

### *1.1 Divided Differences and Classical Convexity of Superior Order*

Throughout the chapter $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ will be used to denote the sets of all positive integers, integers, rational numbers and real numbers respectively. Let $[a,b] \subseteq \mathbb{R}$ and a function $f : [a,b] \to \mathbb{R}$. The $n$th order divided difference of $f$ on the points $x_k \in [a,b]$, $k \in \{0, 1, \ldots, n\}$ is defined, according to the results from [13], by the following recurrence:

$$[x_0; f] = f(x_0); \qquad [x_0, \ldots, x_n; f] = \frac{[x_0, \ldots, x_{n-1}; f] - [x_1, \ldots, x_n; f]}{x_0 - x_n}. \quad (1)$$

As usual (see [16]), $f$ is said to be $n$th order convex on $[a,b]$ if

$$[x_0, \ldots, x_{n+1}; f] \geq 0 \quad (2)$$

whenever $x_k \in [a,b]$ for $k \in \{0, 1, \ldots, n+1\}$. The set of all $n$th order convex functions on $[a,b]$ is a convex cone denoted, as in [9] by $K_n[a,b]$. Taking $n = -1$ one obtains the class of non-negative functions and taking $n = 0$ one gets the non-decreasing functions. The case $n = 1$ was defined in [5] and the general case was introduced in [6] for functions defined on an interval.

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## 1.2 Representation Theorems for Functions with Bounded Variation of Superior Order

Let $(d)$ be a partition of $[a, b]$,

$$(d): \quad a = x_0 < x_1 < \cdots < x_m = b, \quad m \geq n + 1. \tag{3}$$

By the standard definition (see [16]), the $n$th order variation of $f$ on $(d)$ is the number

$$V_n(d; f) = \sum_{i=1}^{n} \big| [x_i, \ldots, x_{i+n}; f] - [x_{i+1}, \ldots, x_{i+n+1}; f] \big|. \tag{4}$$

If $D_{n+1} = \{(d) | (d) = (a = x_0 < x_1 < \cdots < x_m = b), m \in \mathbb{N}, m \geq n + 1\}$ then the total variation of function $f$ on $[a, b]$ is the number

$$V_n\big([a, b]; f\big) = \sup\big\{V_n(d; f) | (d) \in D_{n+1}\big\}. \tag{5}$$

According to the definition of Camille Jordan [7], generalized to superior order (cf. [16]), $f$ is said to be of $n$th order bounded variation if $V_n([a, b]; f) < +\infty$. The real linear space of functions having $n$th order bounded variation on $[a, b]$ is denoted by $BV_n[a, b]$. The symbol $BV_0[a, b]$ is used to denote the set of functions having the classical type of bounded variation [7]. The decomposition theorem of Jordan [7] states, in fact, that

**Theorem 1** $BV_0[a, b] = K_0[a, b] - K_0[a, b]$.

It means that the real linear space of bounded variation (in classical sense) functions is the minimal linear space containing the cone of all increasing functions on $[a, b]$. In [16] is proven the following decomposition theorem within the space $BV_n[a, b]$:

**Theorem 2** *Every continuous function* $f \in BV_n[a, b]$ *is the difference between two functions, which are convex of* $-1, 0, 1, 2, \ldots, n$*th order on* $[a, b]$.

This result was obtained by A. Winternitz [20] for $n = 1$ and by E. Hopf [6] in case of real functions defined on an interval. Tiberiu Popoviciu [16] proved it considering arbitrary sets of real numbers as the domain of functions.

In this chapter we intend to extend these results of Tiberiu Popoviciu to the functions with bounded variation of superior order with respect to a Tchebycheff system. The next section is devoted to the presentation of the main concepts that are necessary in the sequel and the state of art of the domain. The third section contains the main results. A particular case, referring to a trigonometric complete Tchebycheff system and the related convexity and bounded variation concepts is presented in the last section.

## 2 Tchebycheff Systems and Related Convexity and Bounded Variation Concepts

### 2.1 *Divided Differences and Convexity with Respect to a Tchebycheff System*

As usual (see [8]), a set of functions $\mathcal{F}_n = \{\varphi_1, \varphi_2, \ldots, \varphi_{n+1}\}$ is said to be a Tchebycheff system on $[a, b]$ if for every set of $n + 1$ distinct points $x_k \in [a, b]$, $k \in \{1, 2, \ldots, n + 1\}$,

$$V(\mathcal{F}_n; x_1, \ldots, x_{n+1}) = \det \begin{pmatrix} \varphi_1(x_1) & \cdots & \varphi_{n+1}(x_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(x_{n+1}) & \ldots & \varphi_{n+1}(x_{n+1}) \end{pmatrix} \neq 0. \tag{6}$$

E. Phragmen and E. Lindelöf [14] considered in 1907, for the first time, the trigonometric Tchebycheff system consisting in the two functions $f_0 = \cos x$, $f_1 = \sin x$ on $(0, \pi)$. T. Popoviciu [17] used in 1936 the concept of T-systems referring to the Tchebycheff systems. E. Moldovan (Popoviciu) [10] used in 1955 the concept of interpolation set of functions, in order to elaborate a nonlinear theory of convexity with respect to a family of given functions and mean theorems for this type of functions. She followed the line of research opened by T. Popoviciu in 1936 [17] and E.F. Beckenbach in 1937 [1]. Basically, all the concepts used by these authors come to sets $\mathcal{M}$ of functions having the following property: for every system of $n$ distinct points $x_1, x_2, \ldots, x_n$ from $[a, b]$ and for every real numbers $y_1, y_2, \ldots, y_n$ there is a unique function $f \in \mathcal{M}$ such that $f(x_i) = y_i$, for every $i \in \{1, 2, \ldots, n\}$.

If $\mathcal{F}_n = \{\varphi_1, \varphi_2, \ldots, \varphi_{n+1}\}$ is a Tchebycheff system on $[a, b]$ then the $n$th order divided difference of a function $f : [a, b] \to \mathbb{R}$ on the distinct points $x_k \in [a, b]$, $k \in \{1, 2, \ldots, n + 1\}$, can be defined, due to the results from [11], by the following recurrence:

$$[\mathcal{F}_0; x_1] = \frac{f(x_1)}{\varphi_1(x_1)}, \tag{7}$$

$$[\mathcal{F}_n; x_1, \ldots, x_{n+1}; f] = \frac{[\mathcal{F}_{n-1}; x_2, \ldots, x_{n+1}; f] - [\mathcal{F}_{n-1}; x_1, \ldots, x_n; f]}{[\mathcal{F}_{n-1}; x_2, \ldots, x_{n+1}; \varphi_{n+1}] - [\mathcal{F}_{n-1}; x_1, \ldots, x_n; \varphi_{n+1}]}. \tag{8}$$

G. Cristescu [4] obtained in 1977 another recurrence formula for divided differences in a trigonometric case Tchebycheff system. It will be presented in the last section of this chapter. More recently, M. Bessenyei [2] obtained characterization results of the Tchebycheff systems and the related divided differences by means of an interesting geometry associated to these kinds of sets of functions. Let $w_0, w_1, \ldots, w_n : [a, b] \to \mathbb{R}$ be positive and continuous functions. Define

$$\varphi_1(x) = w_0(x),$$

$$\varphi_2(x) = w_0(x) \int_a^x w_1(t_1)\, dt_1,$$

$$\vdots$$

$$\varphi_{n+1}(x) = w_0(x) \int_a^x w_1(t_1) \int_a^{t_1} w_2(t_2) \cdots \int_a^{t_{n-1}} w_n(t_n)\, dt_n \cdots dt_2\, dt_1.$$

The set $\mathcal{F}_n = \{\varphi_1, \varphi_2, \ldots, \varphi_{n+1}\}$ is said to be a *complete Tchebycheff system* on $[a, b]$, i.e. $\mathcal{F}_k$ is a Tchebycheff system for $k \in \{1, 2, \ldots, n+1\}$ (see [8]).

## *2.2 Convexity with Respect to Tchebycheff Systems*

T. Popoviciu [17] considered in 1936 high order convex functions with respect to a T-system. E.F. Beckenbach [1] independently formulated in 1937 the same definition as T. Popoviciu using a family of functions with interpolation properties. E. Moldovan (Popoviciu) also defined the high order convex functions with respect to an interpolating set of functions in 1955. While the most of the results obtained in generalized convexity by E. Moldovan (Popoviciu) are in the linear framework, she also approached few non-linear topics in her research [15]. For example, she develops mean theorems for $n$th order convex functions with respect to a Tchebycheff system, where the convexity is defined by means of the manner of sign changing of the difference between the value of the function and the interpolation operator of corresponding order with respect to the Tchebycheff system. Unfortunately [15] is published in Romanian language only. So, [2] and [18] contain some of the results of E. Popoviciu, independently derived, since the authors could not read [15].

As usual (see [9]), one can define convexity concepts with respect to a Tchebycheff system:

**Definition 3** A function $f : [a, b] \to \mathbb{R}$ is said to be $n$th order convex (resp. non-concave, polynomial, non-convex, concave) with respect to the Tchebycheff system $\mathcal{F}_n$ if

$$[\mathcal{F}_n; x_1, \ldots, x_{n+1}; f] > (\geq, =, \leq, <)\, 0, \tag{9}$$

whenever $x_k \in [a, b]$, $k \in \{1, 2, \ldots, n+1\}$, are distinct points.

The cone of non-concave functions of order $n$ with respect to $\mathcal{F}_n$ is denoted by $K_n(\mathcal{F}_n; [a, b])$.

The definition of T. Popoviciu and E.F. Beckenbach is the same, but T. Popoviciu followed, basically, the linear direction opened by definition 3, while E.F. Beckenbach developed the nonlinear line. Recently, M. Bessenyei [3] and S. Wąsowicz

[18] obtained both identic and new results on the convexity with respect to Tchebycheff systems and also with respect to interpolation set of functions, in the nonlinear case.

## 2.3 Functions with Bounded Variation with Respect to a Tchebycheff System

The succession of generalizations of the concept of function with bounded variation started in 1924 by N. Wiener's results [19]. He introduced the concept of $p$-bounded variation in order to generalize the classical concept of function with bounded variation. He replaced the absolute value of the differences of values of function on the division points by the $p$-power, discussing the properties related to various types of $p$. L.C. Young (1937) defined functions with $\Phi$-bounded variation, replacing the $p$-power of N. Wiener by a general function $\Phi$, having some smoothness properties (see [21]). This direction of research resulted in Musielak and Orlicz type of bounded variation [12]. T. Popoviciu [16] defined in 1934 the concept of $n$th bounded variation with respect to T-systems. E. Moldovan (Popoviciu) [10] defined in 1957 the $n$th bounded variation with respect to an interpolation set of functions. The concept of function with $n$th bounded variation with respect to a particular trigonometric Tchebycheff system was studied in 1977 by G. Cristescu [4]. In the same year, L. Lupaş [9] approached the notion of $n$th bounded variation with respect to a Tchebycheff system. As one can see, this topic was largely studied during 1930–1980, within the convexity and approximation school opened by T. Popoviciu in Cluj-Napoca. This domain of convexity research moved a little west, to Debrecen, during the last two decades of the XXth Century and the beginning of the third millennium.

**Definition 4** The *nth order variation of $f$ with respect to $\mathcal{F}_n$ on $(d) \in D_{n+1}$* is the number

$$V_n(\mathcal{F}_n; d; f) = \sum_{i=1}^{n} \big| [\mathcal{F}_n; x_i, \dots, x_{i+n}; f] - [\mathcal{F}_n; x_{i+1}, \dots, x_{i+n+1}; f] \big|. \tag{10}$$

The *total variation of function $f$ with respect to $\mathcal{F}_n$ on* $[a, b]$ is the number

$$V_n\big(\mathcal{F}_n; [a, b]; f\big) = \sup\big\{ V_n(\mathcal{F}_n; d; f) \mid (d) \in D_{n+1} \big\}. \tag{11}$$

Function $f$ is said to be *of nth order bounded variation with respect to $\mathcal{F}_n$* if $V_n(\mathcal{F}_n; [a, b]; f)$ is finite.

In the sequel, $BV_n(\mathcal{F}_n; [a, b])$ denotes the real linear space of functions with $n$th order bounded variation with respect to the Tchebycheff system $\mathcal{F}_n$. We remind few properties of this kind of functions. First, if a function has its $n$th bounded variation with respect to $\mathcal{F}_n$ then its $n$th order divided difference with respect to $\mathcal{F}_n$ is bounded on every subset of $(a, b)$. If $n > 0$, if the functions of the system $\mathcal{F}_n$ are continuous on $[a, b]$ and if $f$ is a function with $n$th order bounded variation with respect to the Tchebycheff system $\mathcal{F}_n$ then $f$ is continuous.

## *2.4 Representation Theorems for Functions with Bounded Variation with Respect to a Tchebycheff System in Particular Cases*

L. Lupaş (1977) obtained the following Jordan type representation theorems for functions with bounded variation with respect to a complete Tchebycheff system in cases $n = 0$ and $n = 1$.

**Theorem 5**

$$BV_0(\mathcal{F}_0; [a, b]) = K_0(\mathcal{F}_0; [a, b]) - K_0(\mathcal{F}_0; [a, b]). \tag{12}$$

**Definition 6** If $w$ is an increasing function on $[a, b]$ then *the derivative of $f$ with respect to $w$ on $x_0 \in [a, b]$* is $D_w f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{w(x) - w(x_0)}$.

It is obvious that if both $w$ and $f$ are differentiable in classical sense and $w'(x_0) \neq 0$ then $f$ has a finite derivative with respect to $w$. Also, if $w$ is differentiable in classical sense, with $w'(x_0) \neq 0$, and $f$ has a finite derivative with respect to $w$ then $f$ is differentiable in classical sense.

This kind of derivative makes useful the Stieltjes integration in the framework of behaviors with respect to complete Tchebycheff systems, in order to deduce properties as the following one.

**Theorem 7** *If $\mathcal{F}_1 = \{\varphi_1, \varphi_2\}$ is a complete Tchebycheff system defined as above by means of the increasing continuous functions $w_0, w_1$, then*

$$BV_1(\mathcal{F}_1; [a, b]) = K_1(\mathcal{F}_1; [a, b]) - K_1(\mathcal{F}_1; [a, b]). \tag{13}$$

The proof of this theorem is in [9], using successive Stieltjes integration of $f$ with respect to $w_0$, $w_1$. This theorem is generalized in the sequel. In particular smooth cases the Riemann integration may replace the Stieltjes integration. This happens, for example, in the trigonometric case that will be discussed in the last section of this chapter.

# 3 Representation Theorems for Functions with Bounded Variation of Superior Order with Respect to a Tchebycheff System

## 3.1 Representation Theorem in Case of Complete Tchebycheff Systems

**Theorem 8** *If $\mathcal{F}_n$ is a complete Tchebycheff system defined as above by means of increasing continuous functions, then*

$$BV_n\big(\mathcal{F}_n;[a,b]\big) = K_n\big(\mathcal{F}_n;[a,b]\big) - K_n\big(\mathcal{F}_n;[a,b]\big). \tag{14}$$

*Proof* Starting from the result from [9], we proceed by induction. The proofs of the particular cases $n=0$ and $n=1$, mentioned in the previous section. Now, let us define

$$g_n = \frac{1}{2}\big[V_n\big(\mathcal{F}_n;[a,x];f\big) + D_{w_n w_{n-1}\cdots w_1} f_-(x) - D_{w_n w_{n-1}\cdots w_1} f_+(a)\big], \tag{15}$$

$$h_n = \frac{1}{2}\big[V_n\big(\mathcal{F}_n;[a,x];f\big) - D_{w_n w_{n-1}\cdots w_1} f_-(x) + D_{w_n w_{n-1}\cdots w_1} f_+(a)\big]. \tag{16}$$

We Stieltjes integrate $g_n$ and $h_n$ $n$ times, successively with respect to $w_1, w_2, \ldots, w_n$. The results are two $n$th order non-concave functions with respect to $\mathcal{F}_n$, denoted $G_n$ and $H_n$. Now it is easy to prove that $f = G_n - H_n$. □

## 3.2 Representation Theorem in Case of General Tchebycheff Systems

Like in the classical $n$th order convexity [16], one can prove the following auxiliary results.

**Lemma 9** *Every function $f : \{x_1, x_2, \ldots, x_m\} \to \mathbb{R}$ is the difference, $f = \alpha - \beta$, between two non-concave functions of orders $-1, 0, 1, 2, \ldots, n$ with respect to $\mathcal{F}_n$.*

*Proof* Let us consider $x_1 < x_2 < \cdots < x_m$ and let us remark that it is easy to chose

$$\frac{\alpha(x_1)}{\varphi_1(x_1)} > 0$$

and large enough such that we obtain also

$$\frac{\beta(x_1)}{\varphi_1(x_1)} > 0.$$

Also, one can take

$$\frac{\alpha(x_2)}{\varphi_1(x_2)} > 0$$

large enough to satisfy the following inequalities:

$$[\mathcal{F}; x_1, x_2; \alpha] > 0,$$
$$\beta(x_2) > 0,$$
$$[\mathcal{F}; x_1, x_2; \beta] > 0,$$

etc. Let us consider the system of $m$ linear equations

$$[\mathcal{F}_n; x_i, \dots, x_{k+1}; \alpha] = \lambda_k, \quad \text{if } k \in \{0, 1, \dots, n\},$$
$$[\mathcal{F}_n; x, x_{i+1}, \dots, x_{i+n+1}; \alpha] = \mu_i, \quad \text{if } i \in \{1, 2, \dots, m-n-1\}$$

with the unknowns $\alpha(x_1), \alpha(x_2), \dots, \alpha(x_m)$. Taking into account that the points are increasingly ordered and using the recurrence formula for the divided differences with respect to $\mathcal{F}_n$ one can determine the unknowns $\alpha(x_k)$ as a linear function of $\lambda_i$ and $\mu_j$ numbers with nonnegative coefficients. Let us take the following system of inequalities

$$[\mathcal{F}_n; x_i, \dots, x_{i+k}; \alpha] \geq \frac{|[\mathcal{F}_n; x_i, \dots, x_{i+k}; f]| + [\mathcal{F}_n; x_i, \dots, x_{i+k}; f]}{2}, \tag{17}$$

for $i = 1$ and $k \in \{0, 1, \dots, n\}$, respectively for $k = n+1$ and $i \in \{1, 2, \dots, m-n+1\}$ and $\alpha^*$ a solution of this system. Let $\alpha$ be the function which satisfies this system in the extreme case of equalities. From the above reasoning it follows that

$$\alpha^*(x_i) \geq \alpha(x_i) \tag{18}$$

for $i \in \{1, 2, \dots, m\}$. All the equalities hold in the extreme case of equality of the previous inequalities system. The decomposition of $f$ may be achieved in non-unique manner, depending on the parameters used in the above described construction. Among all these solutions there is one which is minimal, with respect to the punctual order between the functions. It is easy to prove that it is the one obtained in the previously discussed equality case. □

**Lemma 10** *If $f : \{x_1, \dots, x_m\} \to \mathbb{R}$ then there are two non-concave functions $\alpha : \{x_1, \dots, x_m\} \to \mathbb{R}$ and $\beta : \{x_1, \dots, x_m\} \to \mathbb{R}$, of orders $-1, 0, 1, 2, \dots, n$, such that $f = \alpha - \beta$ and*

1. $\max\{|[\mathcal{F}_n; x_k, \dots, x_{k+n+1}; \alpha]| \mid k \in \{1, \dots, m-k-1\}\} \leq \max\{|[\mathcal{F}_n; x_k, \dots, x_{k+n+1}; f]| \mid k \in \{1, \dots, m-k-1\}\}$ *and* $\max\{|[\mathcal{F}_n; x_k, \dots, x_{k+n+1}; \beta]| \mid k \in \{1, \dots, m-k-1\}\} \leq \max\{|[\mathcal{F}_n; x_k, \dots, x_{k+n+1}; f]| \mid k \in \{1, \dots, m-k-1\}\}$.
2. $V_n(\mathcal{F}_n; [a,b]; \alpha) \leq V_n(\mathcal{F}_n; [a,b]; f)$ *and* $V_n(\mathcal{F}_n; [a,b]; \beta) \leq V_n(\mathcal{F}_n; [a,b]; f)$, *where* $[a,b]$ *is the minimum interval containing* $\{x_1, \dots, x_m\}$.

3. $\max|\alpha|$ *and* $\max|\beta|$ *depend on the properties, up to level* $n$, *of* $f$ *and on the variation of the divided differences of the functions of the Tchebycheff system between the first n and last n of the points* $\{x_1, \ldots, x_m\}$.

*Proof* The proof of the inequalities from (1) and (2) are immediate since one can obtain functions $\alpha$ and $\beta$ by taking into account the previous construction of the decomposition on a finite domain. The extremal conditions are a consequence of the fact that functions $\alpha$ and $\beta$ are chosen as the functions making sharp the system of inequalities (17). So, they are determined to satisfy the system of equations:

$$f(x) = \alpha(x) - \beta(x)$$

on $\{x_1, \ldots, x_m\}$ and

$$[\mathcal{F}_n; x_i, \ldots, x_{i+k}; \alpha] = \frac{|[\mathcal{F}_n; x_i, \ldots, x_{i+k}; f]| + [\mathcal{F}_n; x_i, \ldots, x_{i+k}; f]}{2}, \tag{19}$$

for $i = 1$, $k \in \{1, 2, \ldots, n+1\}$ and if $k = n+2$ then $1 = 1, 2, \ldots, m-n-2$.

Now, in order to prove (3), one has:

$$\big|[\mathcal{F}_n; x_i, \ldots, x_{i+k}; \alpha]\big| \leq \max\big\{\big|[\mathcal{F}_n; x_i, \ldots, x_{i+k}; f]\big| \mid k \in \{1, \ldots, m-k\}\big\},$$

for $k \in \{1, 2, \ldots, n\}$, and

$$\begin{aligned}&\big|[\mathcal{F}_n; x_i, \ldots, x_{i+k}; \alpha]\big| \\ &\quad \leq \frac{1}{2}\big[V_n\big(\mathcal{F}_n; [a,b]; f\big) + 2\max\big\{\big|[\mathcal{F}_n; x_i, \ldots, x_{i+n}; f]\big| \mid k \in \{1, \ldots, m-n\}\big\}\big],\end{aligned}$$

for $k \in \{1, 2, \ldots, m-n\}$. This inequality is obtained by adding (19) in a convenient manner. Let us denote by

$$\begin{aligned}\Delta_k(f) &= \max\big\{\big|[\mathcal{F}_n; x_i, \ldots, x_{i+k}; f]\big| \mid k \in \{1, \ldots, m-k\}\big\},\\ \mathrm{Diff}_k &= [\mathcal{F}_k; x_{m-k+1}, \ldots, x_m; \varphi_{k+1}] - [\mathcal{F}_k; x_1, \ldots, x_k; \varphi_{k+1}].\end{aligned}$$

Then, using (8), one gets

$$\big|[\mathcal{F}_n; x_k, \ldots, x_{k+n-1}; \alpha]\big| < \Delta_{n-1}(f) + \frac{n}{2}\mathrm{Diff}_n\big[V_n\big(\mathcal{F}_n; [a,b]; f\big) + \Delta_n(f)\big],$$

for $k \in \{2, \ldots, m-n+1\}$. Iteratively repeating the procedure, one obtains

$$|\alpha| < \sum_{i=1}^{n} i!\Delta_i(f)\prod_{k=1}^{i}\mathrm{Diff}_k + \frac{n!}{2}\big[V_n\big(\mathcal{F}_n; [a,b]; f\big) + \Delta_n(f)\big]\prod_{k=1}^{n}\mathrm{Diff}_k,$$

on $\{x_1, \ldots, x_m\}$. A similar formula can be deduced for function $\beta$. It is obvious that it completely proves the lemma. □

If the Tchebycheff system $\mathcal{F}_n$ reduces to the classical polynomial case, then the inequalities obtained in order to prove Lemma 10 reduce to the inequalities from [16, p. 29].

**Theorem 11** *Let us suppose that $\mathcal{F}_n$ contains continuous functions. If function $f : M \to \mathbb{R}$, $M \subseteq [a, b]$ an interval, has the nth bounded variation with respect to $\mathcal{F}_n$ then there are two functions $\alpha : \{x_1, \ldots, x_m\} \to \mathbb{R}$ and $\beta : \{x_1, \ldots, x_m\} \to \mathbb{R}$, non-concave with respect to $\mathcal{F}_n$ of orders $-1, 0, 1, 2, \ldots, n$, such that*

$$f = \alpha - \beta \tag{20}$$

*and also*

$$V_n(\mathcal{F}_n; M; \alpha) \le V_n(\mathcal{F}_n; M; f), \tag{21}$$

$$V_n(\mathcal{F}_n; M; \beta) \le V_n(\mathcal{F}_n; M; f). \tag{22}$$

*Proof* The proof follows the density reasoning, as that one from the classical polynomial case (see [16]). Since $f$ is continuous on $M$ then it is completely determined by its values on a countable subset $M^*$ having the property that $M \setminus M^*$ is included into the derived set of $M^*$. It is possible to generate set $M^*$ by a sequence of finite sets $M_1^* \subset M_2^* \subset \cdots \subset M_s^* \subset \cdots$. Using Lemma 10 on each member of this sequence of subsets of $M$, one generates two sequences of functions:

$$\alpha_1, \alpha_2, \ldots, \alpha_s, \ldots$$

$$\beta_1, \beta_2, \ldots, \beta_s, \ldots$$

such that

$$f = \alpha_k - \beta_k,$$

for $k = 1, 2, \ldots$. All the members of the two sequences of functions are equally bounded, non-concave with respect to $\mathcal{F}_n$ of orders $-1, 0, 1, 2, \ldots, n$ and their $n$th order total variation with respect to $\mathcal{F}_n$ do not exceed $V_n(\mathcal{F}_n; M; f)$. Then the two sequences are convergent respectively to functions $\alpha : M^* \to \mathbb{R}$ and $\beta : M^* \to \mathbb{R}$, that have the properties:

$$V_n\big(\mathcal{F}_n; M^*; \alpha\big) \le V_n(\mathcal{F}_n; M; f),$$

$$V_n\big(\mathcal{F}_n; M^*; \beta\big) \le V_n(\mathcal{F}_n; M; f),$$

with $\alpha$ and $\beta$ non-concave with respect to $\mathcal{F}_n$ of orders $-1, 0, 1, 2, \ldots, n$ and

$$f = \alpha - \beta,$$

on $M^*$. The continuity of all the functions help in extending the result to all $M$. □

The problem of finding a minimal, according to some meaning of minimality that should be described, can also be approached. We suspect that there is an extremal property in the general case, which is quite similar to that one from the classical polynomial Tchebycheff system.

# 4 Example

## 4.1 Divided Differences with Respect to a Trigonometric Tchebycheff System

Let us consider the trigonometric Tchebycheff system

$$\mathcal{T}_n = \left\{\sin x, \sin 2x, \dots, \sin(n+1)x\right\}, \tag{23}$$

which is a complete Tchebycheff system on every interval $[a, b] \subseteq (0, \pi)$. It is generated by mens of the above described method, using functions $w_k(x) = c_k \sin x$, with specific $c_k \in \mathbb{R}$, $k \in \{0, 1, 2, \dots, n\}$. In particular, $w_0(x) = \sin x$, $w_1(x) = -2\sin x$, $w_2(x) = -4\sin x$, $w_3(x) = -6\sin x$, etc. In this case, the following recurrence formula was deduced in [4]:

**Proposition 12** *The divided difference of a function $f : [a, b] \to \mathbb{R}$, with respect to the Tchebycheff system $\mathcal{T}_n$, on knots $x_1, \dots, x_{n+1}$ satisfies the following recurrence relation*:

$$[\mathcal{T}_n; x_1, \dots, x_{n+1}; f] = \frac{[\mathcal{T}_{n-1}; x_2, \dots, x_{n+1}; f] - [\mathcal{T}_{n-1}; x_1, \dots, x_n; f]}{2(\cos x_{n+1} - \cos x_1)} \tag{24}$$

*for $n \geq 1$.*

*Proof* The proof of this formula consists either in direct calculus, as in [4], or in the remark that the divided difference of order $n$ is the coefficient of the term of the highest order of the interpolation operator $L(\mathcal{T}_n; x_1, \dots, x_{n+1}; f)$ of function $f$ defined by means of the Tchebycheff system $\mathcal{T}_n$ on $x_1, \dots, x_{n+1}$. Denoting by

$$u(x) = \prod_{k=1}^{n+1}(\cos x - \cos x_k); \qquad u_i(x) = \prod_{k=1, k\neq i}^{n+1}(\cos x - \cos x_k), \tag{25}$$

one can write

$$L(\mathcal{T}_n; x_1, \dots, x_{n+1}; f) = \sum_{i=1}^{n+1} f(x_i)\frac{u_i(x)\sin x}{u_i(x_i)\sin x_i} \tag{26}$$

and deduce the following recurrence formulas:

$$L(\mathcal{T}_n; x_1, \ldots, x_{n+1}; f) = L(\mathcal{T}_{n-1}; x_1, \ldots, x_n; f) + [\mathcal{T}_n; x_1, \ldots, x_{n+1}; f] \frac{2^n u(x) \sin x}{\cos x - \cos x_{n+1}}, \tag{27}$$

$$L(\mathcal{T}_n; x_1, \ldots, x_{n+1}; f) = L(\mathcal{T}_{n-1}; x_2, \ldots, x_{n+1}; f) + [\mathcal{T}_n; x_1, \ldots, x_{n+1}; f] 2^n u_1(x) \sin x. \tag{28}$$

Computing the difference of the two formulas one gets

$$L(\mathcal{T}_n; x_1, \ldots, x_{n+1}; f) = \frac{(\cos x - \cos x_1) L(\mathcal{T}_{n-1}; x_2, \ldots, x_{n+1}; f) - (\cos x - \cos x_{n+1}) L(\mathcal{T}_{n-1}; x_1, \ldots, x_n; f)}{\cos x_{n+1} - \cos x_1}. \tag{29}$$

From this last recurrence one can easily deduce the above mentioned recurrence formula for the divided difference with respect to $\mathcal{T}_n$. □

*Remark 13* The $n$th order derivative successively computed with respect to the functions of the particular trigonometric Tchebycheff system $\mathcal{T}_n$ considered above, reduces to the classical $n$th order derivative, according to the following transformation:

$$D_{w_n w_{n-1} \cdots w_1}(f)(x) = \frac{(-1)^{n-1}}{2^{n-1} (\sin x)^n} \frac{f^{(n-1)}(x)}{(n-1)!}. \tag{30}$$

## 4.2 Representation Theorems for Functions of Bounded Variation with Respect to a Trigonometric Tchebycheff System

Theorem 8 becomes, in the framework of the trigonometric Tchebycheff system $\mathcal{T}_n$

**Proposition 14** *If function $f$ is with nth order bounded variation with respect to $\mathcal{F}_n = \mathcal{T}_n$ then,*

$$BV_n(\mathcal{T}_n; [a, b]) = K_n(\mathcal{T}_n; [a, b]) - K_n(\mathcal{T}_n; [a, b]). \tag{31}$$

In this case, on a convenient domain $[a, b]$, the two functions $g_n$ and $h_n$ become

$$g_n = \frac{1}{2} \left[ V_n(\mathcal{T}_n; [a, x]; f) + \frac{(-1)^{n-1}}{2^{n-1} (\sin x)^n} \frac{f_-^{(n-1)}(x)}{(n-1)!} - \frac{(-1)^{n-1}}{2^{n-1} (\sin a)^n} \frac{f_+^{(n-1)}(a)}{(n-1)!} \right], \tag{32}$$

$$h_n = \frac{1}{2} \left[ V_n(\mathcal{T}_n; [a, x]; f) - \frac{(-1)^{n-1}}{2^{n-1} (\sin x)^n} \frac{f_-^{(n-1)}(x)}{(n-1)!} + \frac{(-1)^{n-1}}{2^{n-1} (\sin a)^n} \frac{f_+^{(n-1)}(a)}{(n-1)!} \right] \tag{33}$$

and the $n$ times successive Riemann integration gives the required decomposition.

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# On Vector Hermite-Hadamard Differences Controlled by Their Scalar Counterparts

**Roman Ger and Josip Pečarić**

**Abstract** We present a new, in a sense direct, proof that the system of two functional inequalities

$$\left\| F\left(\frac{x+y}{2}\right) - \frac{1}{y-x}\int_x^y F(t)\,dt \right\| \le \frac{1}{y-x}\int_x^y f(t)\,dt - f\left(\frac{x+y}{2}\right)$$

and

$$\left\| \frac{F(x)+F(y)}{2} - \frac{1}{y-x}\int_x^y F(t)\,dt \right\| \le \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)\,dt$$

is satisfied for functions $F$ and $f$ mapping an open interval $I$ of the real line $\mathbb{R}$ into a Banach space and into $\mathbb{R}$, respectively, if and only if $F$ yields a delta-convex mapping with a control function $f$.

A similar result is obtained for delta-convexity of higher orders with detailed proofs given in the case of delta-convexity of the second order, i.e. when the functional inequality

$$\left\| 3F\left(\frac{x+2y}{3}\right) + F(x) - 3F\left(\frac{2x+y}{3}\right) - F(y) \right\|$$
$$\le 3f\left(\frac{2x+y}{3}\right) + f(y) - 3f\left(\frac{x+2y}{3}\right) - f(x)$$

holds true provided that $x, y \in I$, $x \le y$.

**Keywords** Hermite-Hadamard type inequalities · Delta-convex map · Control function · Delta-convex map of higher order

**Mathematics Subject Classification** 26B25 · 39B72 · 39B62

## 1 Background

The notion of delta-convexity in Banach spaces has been introduced by L. Veselý and L. Zajiček in [8] as a generalization of functions which are representable as a

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difference of two convex functions. The latter ones yield a natural analogue of functions of bounded variation which are representable as a difference of two monotonic functions. The definition of delta-convexity proposed by these two authors reads as follows:

Given two real normed linear spaces $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ and a nonempty open and convex subset $D \subset X$, we say that a map $F : D \to Y$ is *delta-convex* provided that there exists a continuous convex functional $f : D \to \mathbb{R}$ such that $f + y^* \circ F$ is continuous and convex for any member $y^*$ of the space $Y^*$ dual to $Y$ with $\|y^*\| = 1$. If that is the case then $F$ is called to be *controlled* by $f$ or that $F$ is a delta-convex mapping with a *control function* $f$.

It turns out that a *continuous* function $F : D \to Y$ is a delta-convex mapping controlled by a *continuous* function $f : D \to \mathbb{R}$ if and only if the functional inequality

$$\left\| F\left(\frac{x+y}{2}\right) - \frac{F(x)+F(y)}{2} \right\| \le \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)$$

is satisfied for all $x, y \in D$ (see Corollary 1.18 in [8]).

L. Veselý and L. Zajiček have shown also that any delta-convex mapping is locally Lipschitzian (cf. Proposition 1.10 in [8]). Both continuity assumptions may considerably be weakened (see R. Ger's paper [3]).

In the present paper we offer an alternative (direct) proof of a result from recent paper [5] of R. Ger concerning vector counterparts of the celebrated Hermite-Hadamard inequalities

$$\varphi\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y \varphi(t)\,dt \le \frac{\varphi(x)+\varphi(y)}{2}$$

satisfied for convex real functions on an interval. The focus of our attention lies in exhibiting strict connections between the inequalities discussed with the notion of delta-convexity, also of higher orders.

## 2 An Alternative Proof

We begin with a somewhat technical lemma.

**Lemma** *Let $I$ stand for an open interval in $\mathbb{R}$ and let $(Y, \|\cdot\|)$ be a real Banach space. If that a locally Bochner integrable function $F : I \to Y$ and a locally Lebesgue integrable function $f : I \to \mathbb{R}$ satisfy the functional inequality*

$$\left\| F\left(\frac{x+y}{2}\right) - \frac{F(x)+F(y)}{2} \right\| \le \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right), \quad x, y \in I, \tag{1}$$

*then for every $\lambda$ from the unit interval $[0, 1]$ and for all $x$, $y$ from $I$ one has*

$$\begin{aligned} &\big\| F\big(\lambda x + (1-\lambda)y\big) - \lambda F(x) - (1-\lambda)F(y) \big\| \\ &\quad \le \lambda f(x) + (1-\lambda) f(y) - f\big(\lambda x + (1-\lambda)y\big). \end{aligned} \tag{2}$$

*Proof* With the aid of Theorem 1 from R. Ger's paper [3] we obtain easily the delta-convexity of $F$. Inequality (2) results now from Proposition 1.13 (assertion (iii)) in L. Veselý & L. Zajiček [8]. □

Among others, the following theorem has been proved in [5].

**Theorem 1** *Under the assumptions of Lemma the following two inequalities hold true*:

$$\left\| F\left(\frac{x+y}{2}\right) - \frac{1}{y-x}\int_x^y F(t)\,dt \right\| \leq \frac{1}{y-x}\int_x^y f(t)\,dt - f\left(\frac{x+y}{2}\right) \tag{3}$$

*and*

$$\left\| \frac{F(x)+F(y)}{2} - \frac{1}{y-x}\int_x^y F(t)\,dt \right\| \leq \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)\,dt \tag{4}$$

*for all* $x$, $y$ *from* $I$, $x \neq y$. *In particular*, $F$ *yields is a delta-convex mapping with a control function* $f$.

*Conversely*, *assuming that a locally Bochner integrable function* $F : I \to Y$ *and a locally Lebesgue integrable function* $f : I \to \mathbb{R}$ *satisfy the system of functional inequalities* (3) *and* (4) *we have* (1) *and hence also* (2).

The basic idea of the proof presented in [5] was to reduce the problem to the scalar case and to apply the classical Hermite-Hadamard inequalities. In what follows we present a completely different (in a sense "direct") proof of the result in question.

*An alternative proof of Theorem 1* To show that inequality (3) holds true, fix arbitrarily $x$, $y$ from $I$, $x \neq y$, and put, for brevity, $u := (x+y)/2$, $v := (y-x)/2$. Then, by a standard change of variables, one has

$$\int_{-1}^{1} F(u+tv)\,dt = \frac{2}{y-x}\int_x^y F(t)\,dt = \int_{-1}^{1} F(u-tv)\,dt$$

and *a fortiori*,

$$\begin{aligned}
&\left\| F\left(\frac{x+y}{2}\right) - \frac{1}{y-x}\int_x^y F(t)\,dt \right\| \\
&= \left\| F(u) - \frac{1}{4}\int_{-1}^{1} \left\{ F(u+tv) + F(u-tv) \right\} dt \right\| \\
&= \frac{1}{2}\left\| \int_{-1}^{1} \left( F(u) - \frac{F(u+tv)+F(u-tv)}{2} \right) dt \right\| \\
&\leq \frac{1}{2}\int_{-1}^{1} \left\| F(u) - \frac{F(u+tv)+F(u-tv)}{2} \right\| dt
\end{aligned}$$

$$\leq \frac{1}{2}\int_{-1}^{1}\left(\frac{f(u+tv)+f(u-tv)}{2}-f(u)\right)dt$$
$$=\frac{1}{y-x}\int_{x}^{y} f(t)\,dt - f\left(\frac{x+y}{2}\right),$$

by means of (1).

Similarly,

$$\left\|\frac{F(x)+F(y)}{2}-\frac{1}{y-x}\int_{x}^{y} F(t)\,dt\right\|$$
$$=\left\|\frac{F(x)+F(y)}{2}-\frac{1}{2}\int_{-1}^{1} F\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right)dt\right\|$$
$$=\left\|\frac{1}{2}\int_{-1}^{1}\left\{\frac{1-t}{2}F(x)+\frac{1+t}{2}F(y)-F\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right)\right\}dt\right\|$$
$$\leq \frac{1}{2}\int_{-1}^{1}\left\|\frac{1-t}{2}F(x)+\frac{1+t}{2}F(y)-F\left(\frac{1-t}{2}x+\frac{1+t}{2}y\right)\right\|dt$$
$$\leq \frac{1}{2}\int_{-1}^{1}\left\{\frac{1-t}{2}f(x)+\frac{1+t}{2}f(y)-f(\frac{1-t}{2}x+\frac{1+t}{2}y)\right\}dt$$
$$=\frac{f(x)+f(y)}{2}-\frac{1}{y-x}\int_{x}^{y} f(t)\,dt,$$

by means of the Lemma and the inequality (4) is proved. □

*Remark 1* To prove inequality (3) we were not using the Lemma. Thus our present proof of (3) is even "more direct".

*Remark 2* Inequality (2) is equivalent to delta-convexity for mappings defined on an open interval or, more generally, on a nonempty open convex subset of $\mathbb{R}^n$. That equivalence fails to hold for infinite dimensional domains. Indeed, it suffices to take an arbitrary discontinuous linear functional $F$ and an arbitrary convex (not necessarily continuous) functional $f$ to get (2) and to miss delta-convexity.

## 3 Hermite-Hadamard Type Inequalities for 2-Convex Functions

The method used in [5] may, however, be applied in numerous different contexts concerned with inequalities. To visualize it, in what follows, we will examine delta-convexity of higher orders.

It is well-known that the functional equation

$$\Delta_h^{n+1}\varphi(x)=0,$$

where $\Delta_h^p$ stands for the $p$th iterate of the difference operator $\Delta_h\varphi(x) := \varphi(x+h) - \varphi(x)$, of *polynomial functions* characterizes the usual polynomials of at most $n$th degree in the class of continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$. Continuous solutions $\varphi : \mathbb{R} \to \mathbb{R}$ of the functional inequality

$$\Delta_h^{n+1}\varphi(x) \geq 0,$$

where $x \in \mathbb{R}$, $h \in (0,\infty)$, are just $C^{n-1}$-functions whose derivatives $\varphi^{(n-1)}$ are convex (see e.g. M. Kuczma [7, Chapter XV]). Therefore, such functions are used to be called *n-convex functions.* For $n = 1$ the latter inequality states that

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{\varphi(x)+\varphi(y)}{2}, \quad x, y \in \mathbb{R},$$

which is nothing else but the functional inequality defining Jensen convexity. Motivated by this fact, in what follows, we shall be using the operator

$$\delta_y^n\varphi(x) := \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\varphi\left(\left(1-\frac{j}{n+1}\right)x + \frac{j}{n+1}y\right),$$

instead of $\Delta_h^{n+1}$. We have

$$\delta_y^n\varphi(x) = \Delta_{\frac{y-x}{n+1}}^{n+1}\varphi(x);$$

thus $\varphi$ is $n$-convex if and only if

$$x \leq y \quad \Longrightarrow \quad \delta_y^n\varphi(x) \geq 0. \tag{5}$$

In a natural way, this leads also to the notion of delta-convexity of higher orders (see [4]):

A map $F : I \to Y$ is termed *delta-convex of nth order* if and only if there exists a (control) functional $f : I \to \mathbb{R}$ such that for all $x, y \in I$ one has

$$x \leq y \quad \Longrightarrow \quad \left\|\delta_y^n F(x)\right\| \leq \delta_y^n f(x). \tag{6}$$

For $n = 2$ relation (5) may equivalently be written in the form

$$x \leq y \quad \Longrightarrow \quad \frac{1}{4}\varphi(x) + \frac{3}{4}\varphi\left(\frac{1}{3}x + \frac{2}{3}y\right) \leq \frac{3}{4}\varphi\left(\frac{2}{3}x + \frac{1}{3}y\right) + \frac{1}{4}\varphi(y).$$

It turns out (see M. Bessenyei [1, Corollary 1.13]) that for continuous 2-convex function $\varphi : I \to \mathbb{R}$ the following Hermite-Hadamard type inequality holds true:

$$x, y \in I, \quad x < y \quad \Longrightarrow \quad \frac{1}{4}\varphi(x) + \frac{3}{4}\varphi\left(\frac{1}{3}x + \frac{2}{3}y\right) \le \frac{1}{y-x}\int_x^y \varphi(t)\,dt \le \frac{3}{4}\varphi\left(\frac{2}{3}x + \frac{1}{3}y\right) + \frac{1}{4}\varphi(y).$$

This gives rise to look for analogues of Theorem 1 for higher order delta-convexity. For the sake of simplicity, we shall confine ourselves to order 2.

**Theorem 2** *Let $I$ stand for an open interval in $\mathbb{R}$ and let $(Y, \|\cdot\|)$ be a real Banach space. Assume that a locally Bochner integrable function $F : I \to Y$ and a locally Lebesgue integrable function $f : I \to \mathbb{R}$ satisfy the functional inequality*

$$\left\| 3F\left(\frac{x+2y}{3}\right) + F(x) - 3F\left(\frac{2x+y}{3}\right) - F(y) \right\| \le 3f\left(\frac{2x+y}{3}\right) + f(y) - 3f\left(\frac{x+2y}{3}\right) - f(x) \tag{7}$$

*whenever $x, y \in I$, $x \le y$. Then the following two inequalities hold true:*

$$\left\| \frac{3}{4}F\left(\frac{x+2y}{3}\right) + \frac{1}{4}F(x) - \frac{1}{y-x}\int_x^y F(t)dt \right\| \le \frac{1}{y-x}\int_x^y f(t)\,dt - \frac{3}{4}f\left(\frac{x+2y}{3}\right) - \frac{1}{4}f(x) \tag{8}$$

*and*

$$\left\| \frac{3}{4}F\left(\frac{2x+y}{3}\right) + \frac{1}{4}F(y) - \frac{1}{y-x}\int_x^y F(t)\,dt \right\| \le \frac{3}{4}f\left(\frac{2x+y}{3}\right) + \frac{1}{4}f(y) - \frac{1}{y-x}\int_x^y f(t)\,dt \tag{9}$$

*for all $x, y$ from $I$, $x < y$.*

*Conversely, assuming that a locally Bochner integrable function $F : I \to Y$ and a locally Lebesgue integrable function $f : I \to \mathbb{R}$ satisfy the system of functional inequalities (8) and (9) we have (7), i.e. $F$ yields a delta-convex mapping of the 2nd order with a control function $f$.*

*Proof* Assume (7) and fix arbitrarily a continuous linear functional $y^*$ from the unit ball of the space $Y^*$ dual to $Y$. Since, for all points $x, y \in I$ such that $x \le y$ one has

$$\left\| 3F\left(\frac{x+2y}{3}\right) + F(x) - 3F\left(\frac{2x+y}{3}\right) - F(y) \right\| = \sup_{z^* \in Y^*, \|z^*\| \le 1} \left| z^*\left( 3F\left(\frac{x+2y}{3}\right) + F(x) - 3F\left(\frac{2x+y}{3}\right) - F(y) \right) \right|$$

$$= \sup_{z^* \in Y^*, \|z^*\| \le 1} \left| 3z^* \circ F\left(\frac{x+2y}{3}\right) + z^* \circ F(x) - 3z^* \circ F\left(\frac{2x+y}{3}\right) - z^* \circ F(y) \right|,$$

by means of (7), we infer that

$$3y^* \circ F\left(\frac{x+2y}{3}\right) + y^* \circ F(x) - 3y^* \circ F\left(\frac{2x+y}{3}\right) - y^* \circ F(y)$$
$$\le 3f\left(\frac{2x+y}{3}\right) + f(y) - 3f\left(\frac{x+2y}{3}\right) - f(x), \quad x, y \in I,\ x \le y,$$

or, equivalently,

$$\big(y^* \circ F + f\big)(x) + 3\big(y^* \circ F + f\big)\left(\frac{x+2y}{3}\right)$$
$$\le 3\big(y^* \circ F + f\big)\left(\frac{2x+y}{3}\right) + \big(y^* \circ F + f\big)(y), \quad x, y \in I,\ x \le y,$$

which states nothing else but the 2-convexity of the functional $\varphi := y^* \circ F + f$. Since, on account of the local Bochner integrability of $F$ and the local Lebesgue integrability of $f$, the functional $\varphi$ is locally Lebesgue integrable as well (in particular, Lebesgue measurable), we infer that $\varphi$ is continuous (see e.g. M. Kuczma [7, Theorem 15.5.4] or R. Ger [2]). With the aid of M. Bessenyei's result presented in [1, Corollary 1.13] we deduce the inequalities

$$\frac{3}{4}\varphi\left(\frac{x+2y}{3}\right) + \frac{1}{4}\varphi(x) \le \frac{1}{y-x}\int_x^y \varphi(t)\,dt \le \frac{3}{4}\varphi\left(\frac{2x+y}{3}\right) + \frac{1}{4}\varphi(y)$$

valid for all $x, y$ from $I$, $x < y$. This due to the well known theorem of E. Hille (see e.g. Hille-Philips [6, Theorem 3.7.12]) and the definition of $\varphi$ implies that

$$y^*\left(\frac{3}{4}F\left(\frac{x+2y}{3}\right) + \frac{1}{4}F(x) - \frac{1}{y-x}\int_x^y F(t)\,dt\right)$$
$$\le \frac{1}{y-x}\int_x^y f(t)\,dt - \frac{3}{4}f\left(\frac{x+2y}{3}\right) - \frac{1}{4}f(x),$$

valid for all $x, y \in I$, $x < y$. Replacing here $y^*$ by $-y^*$ gives

$$\left| y^*\left(\frac{3}{4}F\left(\frac{x+2y}{3}\right) + \frac{1}{4}F(x) - \frac{1}{y-x}\int_x^y F(t)\,dt\right) \right|$$
$$\le \frac{1}{y-x}\int_x^y f(t)\,dt - \frac{3}{4}f\left(\frac{x+2y}{3}\right) - \frac{1}{4}f(x),$$

for all $x, y \in I$, $x < y$, as well. Now, since the choice of the functional $y^*$ was unrestricted, we conclude that the inequality (8) remains valid for all $x, y \in I$, as claimed. Inequality (9) may be shown quite analogously. Since the latter part of the assertions results immediately from summing up inequalities (8) an (9) side by side and applying the triangle inequality, the proof has been completed. □

## 4 Concluding Remark

Right before the formulation of Theorem 2 we have mentioned that (for the sake of simplicity) we shall confine ourselves to delta-convexity of order 2. Actually, it is easily seen that our proof method carries over to the delta-convexity of higher order because the corresponding higher order estimates of the integral mean (higher order Hermite-Hadamard inequalities) established in M. Bessenyei's dissertation [1] are at our disposal. In particular, for delta-convexity of order 3 an analogue of Theorem 2 reads as follows.

**Theorem 3** *Let $I$ stand for an open interval in $\mathbb{R}$ and let $(Y, \|\cdot\|)$ be a real Banach space. Assume that a locally Bochner integrable function $F : I \to Y$ and a locally Lebesgue integrable function $f : I \to \mathbb{R}$ satisfy the functional inequality*

$$\left\| F(x) - 4F\left(\frac{3x+y}{4}\right) + 6F\left(\frac{x+y}{2}\right) - 4F\left(\frac{x+3y}{4}\right) + F(y) \right\|$$
$$\leq f(x) + 6f\left(\frac{x+y}{2}\right) + f(y) - 4f\left(\frac{3x+y}{4}\right) - 4f\left(\frac{x+3y}{4}\right)$$

*whenever $x, y \in I$, $x \leq y$. Then the following two inequalities hold true*:

$$\left\| \frac{1}{2}F\left(\frac{3+\sqrt{3}}{6}x + \frac{3-\sqrt{3}}{6}y\right) + \frac{1}{2}F\left(\frac{3-\sqrt{3}}{6}x + \frac{3+\sqrt{3}}{6}y\right) - \frac{1}{y-x}\int_x^y F(t)\,dt \right\|$$
$$\leq \frac{1}{y-x}\int_x^y f(t)\,dt - \frac{1}{2}f\left(\frac{3+\sqrt{3}}{6}x + \frac{3-\sqrt{3}}{6}\right) - \frac{1}{2}f\left(\frac{3-\sqrt{3}}{6}x + \frac{3+\sqrt{3}}{6}\right)$$

*and*

$$\left\| \frac{1}{6}F(x) + \frac{2}{3}F\left(\frac{x+y}{2}\right) + \frac{1}{6}F(y) - \frac{1}{y-x}\int_x^y F(t)\,dt \right\|$$
$$\leq \frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y) - \frac{1}{y-x}\int_x^y f(t)\,dt$$

*for all $x, y$ from $I$, $x < y$.*

*Conversely, if a locally Bochner integrable function $F : I \to Y$ and a locally Lebesgue integrable function $f : I \to \mathbb{R}$ satisfy the system of the latter two functional inequalities, then $F$ yields a delta-convex mapping of the 3rd order with a control functional $f$.*

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# Functions Generating Strongly Schur-Convex Sums

**Kazimierz Nikodem, Teresa Rajba, and Szymon Wąsowicz**

*Dedicated to the Memory of Professor Wolfgang Walter*

**Abstract** The notion of strongly Schur-convex functions is introduced and functions generating strongly Schur-convex sums are investigated. The results presented are counterparts of the classical Hardy–Littlewood–Pólya majorization theorem and the theorem of Ng characterizing functions generating Schur-convex sums. It is proved, among others, that for some (for every) $n \geq 2$, the function $F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)$ is strongly Schur-convex with modulus $c$ if and only if $f$ is of the form $f(x) = g(x) + a(x) + c\|x\|^2$, where $g$ is convex and $a$ is additive.

**Keywords** Schur-convex functions · Strongly Schur-convex functions · Strongly convex functions · Strongly Jensen-convex functions · Strongly Wright-convex functions · Doubly stochastic matrices · Majorization

**Mathematics Subject Classification** Primary 26B25 · Secondary 39B62

## 1 Introduction

Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathcal{I}^n$, where $n \geq 2$ (throughout the paper $n$ will be always assumed to be a natural number). Following Schur [14] (cf. also [6]) we say that $x$ *is majorized by* $y$, and write $x \preccurlyeq y$, if there exists a doubly stochastic $n \times n$ matrix $P$ (i.e. matrix containing nonnegative elements with all rows and columns summing up to 1) such that $x = y \cdot P$. A function $F : \mathcal{I}^n \to \mathbb{R}$ is said to be *Schur-convex* if $F(x) \leq F(y)$ whenever $x \preccurlyeq y$, $x, y \in \mathcal{I}^n$.

It is known, by the classical works of Schur [14], Hardy–Littlewood–Pólya [2] and Karamata [4] that if a function $f : \mathcal{I} \to \mathbb{R}$ is convex then it *generates Schur-convex sums*, that is the function $F : \mathcal{I}^n \to \mathbb{R}$ defined by

$$F(x) = F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)$$

is Schur-convex. It is also known that the convexity of $f$ is a sufficient but not necessary condition under which $F$ is Schur-convex. In 1987 C.T. Ng [9] gave a full

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characterization of functions (defined on a convex and open subset of $\mathbb{R}^n$) generating Schur-convex sums.

In this note we introduce the notion of strong Schur-convexity (in $X^n$, where $X$ is an inner product space) and we present a counterpart of the Ng representation theorem for functions generating strongly Schur-convex sums.

Let $(X, \|\cdot\|)$ be a (real) inner product space. We consider the space $X^n$ $(n \geq 2)$ with the product norm

$$|||x||| = \sqrt{\|x_1\|^2 + \cdots + \|x_n\|^2}, \qquad x = (x_1, \ldots, x_n) \in X^n.$$

Similarly as in the classical case we define the majorization in $X^n$. Namely, given two $n$-tuples $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X^n$ we say that $x$ *is majorized by* $y$, written $x \preccurlyeq y$, if

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \cdot P$$

for some doubly stochastic $n \times n$ matrix $P$.

Note that if $x \preccurlyeq y$ then $|||x|||^2 \leq |||y|||^2$. It follows, for instance, from the fact that the function $\|\cdot\|^2 : X \to \mathbb{R}$ is convex and so it generates Schur-convex sums (the proof is exactly the same as in the classical case of $X = \mathbb{R}$; cf. also the proof of Theorem 1 below, where we repeat the argument for the sake of completeness).

Let $D$ be a convex subset of $X$ and let $c > 0$. Recall that a function $f : D \to \mathbb{R}$ is called *strongly convex with modulus c* (cf. e.g. [3, 11, 13]) if

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2 \tag{1}$$

for all $x, y \in D$ and $t \in [0, 1]$; $f$ is called *strongly Jensen-convex with modulus c* if condition (1) is assumed only for $t = \frac{1}{2}$, that is

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2, \quad x, y \in D.$$

Recall also that $f$ is said to be *strongly Wright-convex with modulus c* (see [8]) if

$$f\big(tx + (1-t)y\big) + f\big((1-t)x + ty\big) \leq f(x) + f(y) - 2ct(1-t)\|x - y\|^2$$

for all $x, y \in D$ and $t \in [0, 1]$.

The concept of strong convexity was inspired by the optimization theory and many properties and applications of it can be found in the literature. For some recent results concerning strongly (Jensen-, Wright-) convex functions the reader is referred to [1, 7, 8, 10, 12].

Motivated by the above definitions we propose a strengthening of the notion of Schur-convexity. Let $D$ be a convex subset of $X$, $c > 0$ and $n \geq 2$. We say that a function $F : D^n \to \mathbb{R}$ is *strongly Schur-convex with modulus c* if

$$x \preccurlyeq y \quad \Longrightarrow \quad F(x) \leq F(y) - c\big(|||y|||^2 - |||x|||^2\big)$$

for all $x, y \in D$. Note that the usual Schur-convexity corresponds to the case $c = 0$.

## 2 Schur-Convex Sums

In this section we prove that strongly convex functions generate strongly Schur-convex sums and functions generating strongly Schur-convex sums are strongly Jensen-convex (and hence, under some regularity assumptions they are strongly convex).

**Theorem 1** *Let $D$ be a convex subset of an inner product space $(X, \|\cdot\|)$ and $c > 0$. If a function $f : D \to \mathbb{R}$ is strongly convex with modulus $c$, then for every $n \geq 2$ the function $F : D^n \to \mathbb{R}$ given by*

$$F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n), \quad (x_1, \ldots, x_n) \in D^n,$$

*is strongly Schur-convex with modulus $c$.*

*Proof* Assume that $f : D \to \mathbb{R}$ is strongly convex with modulus $c$. Since $X$ is an inner product space, the function $h : D \to \mathbb{R}$ given by $h(x) = f(x) - c\|x\|^2$, $x \in D$, is convex (see [10, Lemma 2.1]). Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in D^n$ and $x \preccurlyeq y$. There exists a doubly stochastic $n \times n$ matrix $P = [t_{ij}]$ such that $x = y \cdot P$. Then

$$x_j = \sum_{i=1}^{n} t_{ij} y_i, \quad j = 1, \ldots, n,$$

and, by the convexity of $h$, we obtain

$$\begin{aligned} h(x_1) + \cdots + h(x_n) &= \sum_{j=1}^{n} h\left(\sum_{i=1}^{n} t_{ij} y_i\right) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} t_{ij} h(y_i) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij} h(y_i) = \sum_{i=1}^{n} h(y_i) \sum_{j=1}^{n} t_{ij} = h(y_1) + \cdots + h(y_n). \end{aligned}$$

Consequently,

$$\begin{aligned} F(x) &= f(x_1) + \cdots + f(x_n) \\ &= h(x_1) + \cdots + h(x_n) + c\big(\|x_1\|^2 + \cdots + \|x_n\|^2\big) \\ &\leq h(y_1) + \cdots + h(y_n) + c\big(\|x_1\|^2 + \cdots + \|x_n\|^2\big) \\ &= f(y_1) + \cdots + f(y_n) - c\big(\|y_1\|^2 + \cdots + \|y_n\|^2\big) + c\big(\|x_1\|^2 + \cdots + \|x_n\|^2\big) \\ &= F(y) - c\big(|||y|||^2 - |||x|||^2\big). \end{aligned}$$

This shows that $F$ is strongly Schur-convex with modulus $c$, which was to be proved. □

*Remark 2* The converse theorem is not true. For instance, if $a : \mathbb{R} \to \mathbb{R}$ is an additive discontinuous function, then $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a(x) + x^2$, $x \in \mathbb{R}$, is not strongly convex with any $c > 0$ (because it is not continuous) but it generates strongly Schur-convex sums. To see this take $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ $(n \geq 2)$ such that $x \preccurlyeq y$. Then $x = y \cdot P$ for some doubly stochastic $n \times n$ matrix $P = [t_{ij}]$. By the additivity of $a$ we have

$$a(x_1) + \cdots + a(x_n) = a(x_1 + \cdots + x_n) = a\left(\sum_{j=1}^{n}\sum_{i=1}^{n} t_{ij} y_i\right)$$
$$= a\left(\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij} y_i\right) = a\left(\sum_{i=1}^{n} y_i \sum_{j=1}^{n} t_{ij}\right) = a(y_1) + \cdots + a(y_n).$$

Hence,

$$\begin{aligned} &f(x_1) + \cdots + f(x_n) \\ &\quad = a(x_1) + \cdots + a(x_n) + x_1^2 + \cdots + x_n^2 \\ &\quad = a(y_1) + \cdots + a(y_n) + y_1^2 + \cdots + y_n^2 - \left(y_1^2 + \cdots + y_n^2 - x_1^2 - \cdots - x_n^2\right) \\ &\quad = f(y_1) + \cdots + f(y_n) - \left(\|y\|^2 - \|x\|^2\right). \end{aligned}$$

This proves that $F : \mathbb{R}^n \to \mathbb{R}$ defined by $F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)$ is strongly Schur-convex with modulus 1.

The next result shows that the strong Jensen-convexity is a necessary condition under which $f$ generates strongly Schur-convex sums.

**Theorem 3** *Let $D$ be a convex subset of an inner product space $(X, \|\cdot\|)$, $c > 0$ and $f : D \to \mathbb{R}$. If for some $n \geq 2$ the function $F : D^n \to \mathbb{R}$ given by*

$$F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n), \quad (x_1, \ldots, x_n) \in D^n$$

*is strongly Schur-convex with modulus $c$, then $f$ is strongly Jensen-convex with modulus $c$.*

*Proof* Take $y_1, y_2 \in D$ and put $x_1 = x_2 = \frac{1}{2}(y_1 + y_2)$. Consider the points

$$y = (y_1, y_2, y_2, \ldots, y_2), \qquad x = (x_1, x_2, y_2, \ldots, y_2)$$

(if $n = 2$, then we take $y = (y_1, y_2)$, $x = (x_1, x_2)$). Now, if

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

then $x = y \cdot P$ and $x \preccurlyeq y$. Therefore, by the strong Schur-convexity of $F$,

$$F(x) \le F(y) - c\big(|||y|||^2 - |||x|||^2\big),$$

whence

$$2f\left(\frac{y_1 + y_2}{2}\right) \le f(y_1) + f(y_2) - c\left(\|y_1\|^2 + \|y_2\|^2 - 2\left\|\frac{y_1 + y_2}{2}\right\|^2\right). \quad (2)$$

By the parallelogram law we have

$$\|y_1\|^2 + \|y_2\|^2 = \frac{1}{2}\|y_1 + y_2\|^2 + \frac{1}{2}\|y_1 - y_2\|^2.$$

Consequently, by (2),

$$f\left(\frac{y_1 + y_2}{2}\right) \le \frac{f(y_1) + f(y_2)}{2} - \frac{c}{4}\|y_1 - y_2\|^2,$$

which means that $f$ is strongly Jensen-convex with modulus $c$. □

*Remark 4* The converse theorem is not true. For instance, let $a : \mathbb{R} \to \mathbb{R}$ be an additive discontinuous function such that $a(1) = 0$ and let $t \in (0, 1)$ with $a(t) \neq 0$. Then the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |a(x)| + x^2$, $x \in \mathbb{R}$, is strongly Jensen-convex with modulus 1 (because $x \mapsto f(x) - x^2 = |a(x)|$ is a Jensen-convex function, cf. [10, Lemma 2.1]), but it does not generate strongly Schur-convex sums with modulus 1. Indeed, if $n = 2$, $x = (t, 1 - t)$ and $y = (1, 0)$, then $x \preccurlyeq y$, but

$$F(x) = \big|a(t)\big| + \big|a(1-t)\big| + t^2 + (1-t)^2 > t^2 + (1-t)^2 = F(y) - \big(\|y\|^2 - \|x\|^2\big).$$

Since continuous strongly Jensen-convex functions are strongly convex (see [1, Corollary 2.2]), as an immediate consequence of Theorem 3 we obtain the following result.

**Corollary 5** *Let $D$ be a convex subset of an inner product space $(X, \|\cdot\|)$ and $c > 0$. If $f : D \to \mathbb{R}$ is continuous and for some $n \ge 2$ the function $F : D^n \to \mathbb{R}$ given by $F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)$, $(x_1, \ldots, x_n) \in D^n$, is strongly Schur-convex with modulus $c$, then $f$ is strongly convex with modulus $c$.*

*Remark 6* In the above corollary the assumption that $f$ is continuous can be replaced by other, formally weaker, regularity conditions. For instance, we may assume that $f$ is bounded from above on a set with nonempty interior or $f$ is Lebesgue measurable (if $X = \mathbb{R}^n$). It is a consequence of Bernstein–Doetsch or Sierpiński type results stating that under such assumptions strongly Jensen-convex functions are strongly convex (cf. [1]; for comprehensive review of theorems guaranteeing continuity of Jensen-convex functions see also [5]).

## 3 A Characterization

In this section we characterize the functions generating Schur-convex sums without any regularity assumptions. Our result is a counterpart of the celebrated Ng's Theorem [9] giving a representation of functions generating Schur-convex sums.

**Theorem 7** *Let $D$ be a convex subset of an inner product space $(X, \|\cdot\|)$, $f : D \to \mathbb{R}$ and $c > 0$. The following conditions are equivalent.*

(i) *For every $n \geq 2$ the function $F : D^n \to \mathbb{R}$ defined by*

$$F(x_1, \dots, x_n) = f(x_1) + \dots + f(x_n), \quad (x_1, \dots, x_n) \in D^n, \tag{3}$$

*is strongly Schur-convex with modulus $c$.*

(ii) *For some $n \geq 2$ the function $F$ given by (3) is strongly Schur-convex with modulus $c$.*

(iii) *The function $f$ is strongly Wright-convex with modulus $c$.*

(iv) *There exist a convex function $g : D \to \mathbb{R}$ and an additive function $a : X \to \mathbb{R}$ such that*

$$f(x) = g(x) + a(x) + c\|x\|^2, \quad x \in D. \tag{4}$$

*Proof* The implication (i) $\Rightarrow$ (ii) is obvious.

To prove (ii) $\Rightarrow$ (iii) fix $y_1, y_2 \in D$ and $t \in (0, 1)$. Put

$$x_1 = ty_1 + (1-t)y_2, \qquad x_2 = (1-t)y_1 + ty_2$$

and, if $n > 2$, take additionally $x_i = y_i = z \in D$ for $i = 3, \dots, n$. Then, by the similar argumentation as in the proof of Theorem 3, we have

$$x = (x_1, \dots, x_n) \preccurlyeq y = (y_1, \dots, y_n).$$

Therefore, using the strong convexity of $F$, we obtain

$$F(x) \leq F(y) - c\big(|||y|||^2 - |||x|||^2\big),$$

and hence

$$\begin{aligned}
&f\big(ty_1+(1-t)y_2\big)+f\big((1-t)y_1+ty_2\big)\\
&\quad\le f(y_1)+f(y_2)-c\Big(\|y_1\|^2+\|y_2\|^2-\big\|ty_1+(1-t)y_2\big\|^2\\
&\qquad-\big\|(1-t)y_1+ty_2\big\|^2\Big).
\end{aligned}\tag{5}$$

Using elementary properties of the inner product we get

$$\begin{aligned}
&\|y_1\|^2+\|y_2\|^2-\big\|ty_1+(1-t)y_2\big\|^2-\big\|(1-t)y_1+ty_2\big\|^2\\
&\quad=\|y_1\|^2+\|y_2\|^2\\
&\qquad-\big(t^2\|y_1\|^2+(1-t)^2\|y_2\|^2+(1-t)^2\|y_1\|^2+t^2\|y_2\|^2+4t(1-t)\langle y_1|y_2\rangle\big)\\
&\quad=2t(1-t)\big(\|y_1\|^2-2\langle y_1|y_2\rangle+\|y_2\|^2\big)=2t(1-t)\|y_1-y_2\|^2.
\end{aligned}$$

Consequently, from (5) we get

$$f\big(ty_1+(1-t)y_2\big)+f\big((1-t)y_1+ty_2\big)\le f(y_1)+f(y_2)-2ct(1-t)\|y_1-y_2\|^2,$$

which means that $f$ is strongly Wright-convex with modulus $c$.

The implication (iii) $\Rightarrow$ (iv) follows from the characterization of strongly Wright-convex functions given in [8, Corollary 5].

To see that (iv) $\Rightarrow$ (i) assume that $f$ has the representation (4). Then the function $h=g+c\|\cdot\|^2$ is strongly convex with modulus $c$ (cf. [10]) and hence, by Theorem 1, it generates strongly Schur-convex sums. Therefore, for any $x=(x_1,\dots,x_n)\preccurlyeq y=(y_1,\dots,y_n)$ we have

$$h(x_1)+\cdots+h(x_n)\le h(y_1)+\cdots+h(y_n)-c\big(|||y|||^2-|||x|||^2\big).$$

Consequently, using the additivity of $a$ (similarly as in Remark 2), we arrive at

$$\begin{aligned}
F(x)&=f(x_1)+\cdots+f(x_n)=h(x_1)+\cdots+h(x_n)+a(x_1)+\cdots+a(x_n)\\
&\le h(y_1)+\cdots+h(y_n)-c\big(|||y|||^2-|||x|||^2\big)+a(y_1)+\cdots+a(y_n)\\
&=f(y_1)+\cdots+f(y_n)-c\big(|||y|||^2-|||x|||^2\big)=F(y)-c\big(|||y|||^2-|||x|||^2\big),
\end{aligned}$$

which shows that $F$ is strongly Schur-convex with modulus $c$. This finishes the proof. □

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# Strongly Convex Sequences

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**Abstract** Let $\omega \geq 0$ be a given number and $I$ a subinterval of $\mathbb{Z}$. We say that a sequence $(f_k)_{k\in I}$ is *$\omega$-midconvex* if

$$f_k \leq \frac{f_{k-1} + f_{k+1}}{2} - \omega \quad \text{for } k-1, k, k+1 \in I.$$

We give various characterizations of $\omega$-midconvex sequences.

We also show that in a natural way one can derive from the above definition classical notions of convexity and strong convexity for functions defined on subintervals of $\mathbb{R}$.

**Keywords** Convex function · Strongly convex function

**Mathematics Subject Classification** 26B25 · 39B62

## 1 Introduction

The notion of convexity, its generalizations and modifications are usually defined for functions with convex domains [2]. But there are natural needs (motivated in particular by applications) to consider also convexity in a more general settings [1, 4]. Using a computer we generally deal with a discrete case. In natural way there appears the idea of convexity and strong convexity for sequences. Our main aim is to show that classical convexity, strong convexity of functions can be derived from very simple and natural conditions of real sequences. In this chapter we define their sequential analogue. Our definition is inspired by [4] (where the notion of convex functions were defined for abelian groups), [3] (where so called $\delta$-convex were defined) and some considerations in [7]. However contrary to [3] we do not weaken the notion of convexity but strengthen it. One more important remark should be mentioned here. The definition of concave (affine) function we obtain by replacing in the definition of convex function "$\leq$" by "$\geq$" (respectively by "$=$"). In this chapter we will use analogous procedure. It occurred that it leads to interesting consequences.

In the whole chapter $I$ denotes an interval in $\mathbb{Z}$, i.e. intersection of an interval in $\mathbb{R}$ and $\mathbb{Z}$. To avoid trivial cases we assume that $I$ contains at least three elements. We assume also that $\omega \geq 0$ is a given number.

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**Definition 1** We say that a sequence $(f_k)_{k\in I} \subset \mathbb{R}$ is

(i) *ω-midconvex* if

$$f_k \le \frac{f_{k-1} + f_{k+1}}{2} - \omega \quad \text{for } k-1, k, k+1 \in I;$$

(ii) *ω-midconcave* if

$$f_k \ge \frac{f_{k-1} + f_{k+1}}{2} - \omega \quad \text{for } k-1, k, k+1 \in I;$$

(iii) *ω-midaffine* if

$$f_k = \frac{f_{k-1} + f_{k+1}}{2} - \omega \quad \text{for } k-1, k, k+1 \in I.$$

We present various characterizations of sequences satisfying Definition 1. At the end of the chapter we show that our results naturally yield some convexity notions for functions defined on subintervals of $\mathbb{R}$.

## 2 *ω*-Midconvex Sequences

It occurs that $\omega$-midconvexity ($\omega$-midconcavity, $\omega$-midaffinity) is closely related to convexity (concavity, affinity).

**Theorem 1** *Let $(f_k)_{k\in I} \subset \mathbb{R}$ be a given sequence. Then the following statements are equivalent*:

(i) *$(f_k)_{k\in I}$ is ω-midconvex* (*ω-midconcave*, *ω-midaffine*);
(ii) *there exist $b, c \in \mathbb{R}$ such that the sequence $(f_k - (\omega k^2 + bk + c))_{k\in I}$ is convex* (*concave*, *affine*);
(iii) *for every $b, c \in \mathbb{R}$ the sequence $(f_k - (\omega k^2 + bk + c))_{k\in I}$ is convex* (*concave*, *affine*).

*Proof* We consider only the case when $(f_k)_{k\in I}$ is $\omega$-midconvex (the proof in the midconcave and midaffinity case is analogous as it consists on replacing in the following considerations "≤" by "≥" or by "=", respectively).

Implication (iii) ⇒ (ii) is trivial.

(ii) ⇒ (i). Assume that the sequence $(f_k - (\omega k^2 + bk + c))_{k\in I}$ is convex. Then we have for $k-1, k, k+1 \in I$

$$\begin{aligned} f_k &= f_k - \big(\omega k^2 + bk + c\big) + \big(\omega k^2 + bk + c\big) \\ &\le \frac{f_{k-1} - [\omega(k-1)^2 + b(k-1) + c] + f_{k+1} - [\omega(k+1)^2 + b(k+1) + c]}{2} \\ &\quad + \omega k^2 + bk + c \end{aligned}$$

$$= \frac{f_{k-1} + f_{k+1}}{2} - \omega.$$

(i) $\Rightarrow$ (iii). Suppose that $(f_k)_{k\in I}$ is $\omega$-midconvex. Fix arbitrarily $b, c \in \mathbb{R}$ and define

$$g_k := f_k - \big(\omega k^2 + bk + c\big) \quad \text{for } k \in I.$$

Then we have for $k-1, k, k+1 \in I$

$$g_k = f_k - \big(\omega k^2 + bk + c\big) \le \frac{f_{k-1} + f_{k+1}}{2} - \omega - \big(\omega k^2 + bk + c\big)$$
$$= \frac{f_{k-1} - (\omega(k-1)^2 + b(k-1) + c) + f_{k+1} - (\omega(k+1)^2 + b(k+1) + c)}{2}$$
$$= \frac{g_{k-1} + g_{k+1}}{2}. \qquad \square$$

Now we are going to estimate $f_k$ by the convex combination of elements $f_{k-l}$, $f_{k+m}$ and some antidilatation depending on $\omega$, $k$, $l$, $m$.

**Theorem 2** *Let $(f_k)_{k\in I}$ be a given sequence. Then*

(i) $(f_k)_{k\in I}$ *is $\omega$-convex if and only if*

$$f_k \le \frac{m-k}{m-l} f_l + \frac{k-l}{m-l} f_m - \omega(m-k)(k-l) \tag{1}$$

*for $l, k, m \in I$, $l < m$, $l \le k \le m$;*

(ii) $(f_k)_{k\in I}$ *is $\omega$-concave if and only*

$$f_k \ge \frac{m-k}{m-l} f_l + \frac{k-l}{m-l} f_m - \omega(m-k)(k-l) \tag{2}$$

*for $l, k, m \in I$, $l < m$, $l \le k \le m$;*

(iii) $(f_k)_{k\in I}$ *is $\omega$-affine if and only*

$$f_k = \frac{m-k}{m-l} f_l + \frac{k-l}{m-l} f_m - \omega(m-k)(k-l) \tag{3}$$

*for $l, k, m \in I$, $l < m$, $l \le k \le m$.*

*Proof* Again we present the proof only for the convexity case.

Clearly, if (1) holds for all $l, k, m \in I$: $l \le k \le m$, then taking in (1) $l = k-1$, $m = k+1$ we obtain condition (i) of Definition 1.

Assume that $(f_k)_{k\in I}$ is $\omega$-midconvex. We are going to show that (1) holds. We fix arbitrarily $l, m \in I$, $l < m$ and define a sequence

$$g_k = f_k + \left(\omega(m-k)(k-l) - \frac{m-k}{m-l} f_l - \frac{k-l}{m-l} f_m\right) \quad \text{for } k \in I,\ l \le k \le m.$$

By Theorem 1 the sequence $(g_k)_{k\in[l,m]\cap\mathbb{Z}}$ is convex. Furthermore $g_l = 0 = g_m$. One can notice easily that for our assertion it is sufficient to prove that

$$B := \max\{g_k : k \in [l,m] \cap \mathbb{Z}\} \le 0.$$

Suppose for the proof by contradiction that $B > 0$. Let $k_0 \in [l,m] \cap \mathbb{Z}$ be the smallest element such that $g_{k_0} = B$. Then $l < k_0 < m$ and $g_{k_0-1} < B$. Consequently by convexity of $(g_k)$ we have

$$B = g_{k_0} \le \frac{g_{k_0-1} + g_{k_0+1}}{2} < \frac{B+B}{2} = B,$$

a contradiction. □

The formulas (1), (2), (3) can be rewritten in the following form (we write $f(n)$ instead of $f_n$):

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) - \omega t(1-t)(x-y)^2,$$
$$f\big(tx + (1-t)y\big) \ge tf(x) + (1-t)f(y) - \omega t(1-t)(x-y)^2,$$
$$f\big(tx + (1-t)y\big) = tf(x) + (1-t)f(y) - \omega t(1-t)(x-y)^2,$$

for $x, y \in I$, $t \in (0,1)$: $tx + (1-t)y \in I$.

In a natural way this leads to the notions of $\omega$-midconvexity ($\omega$-midconcavity, $\omega$-midaffinity) for functions defined on subintervals of $\mathbb{R}$.

**Definition 2** Let $P$ be an interval in $\mathbb{R}$. We say that a function $F : P \to \mathbb{R}$ is

(i) *$\omega$-midconvex* if

$$F\big(tx + (1-t)y\big) \le tF(x) + (1-t)F(y) - \omega t(1-t)(x-y)^2$$

for $x, y \in P$, $t \in (0,1)$;

(ii) *$\omega$-midconcave* if

$$F\big(tx + (1-t)y\big) \ge tF(x) + (1-t)F(y) - \omega t(1-t)(x-y)^2$$

for $x, y \in P$, $t \in (0,1)$;

(iii) *$\omega$-midaffine* if

$$F\big(tx + (1-t)y\big) = tF(x) + (1-t)F(y) - \omega t(1-t)(x-y)^2$$

for $x, y \in P$, $t \in (0,1)$.

*Remark 1* It is easy to verify that a function $F : P \to \mathbb{R}$ is $\omega$-midconvex ($\omega$-midconcave, $\omega$-midaffine) if and only if $F(x) - \omega x^2$ is convex (concave, affine, respectively).

Now we make some comments on Definition 2. It leads to known notions, which is an argument that it is very natural.

If $\omega = 0$ we obtain the standard definitions of convex, concave and affine functions.

When $\omega > 0$ in the midconvexity case we obtain the definition of $\omega$-strongly convex function. It was introduced by B. Polyak [6]. For some recent results and references concerning the subject we refer the reader to [5].

A function $F$ is $\omega$-midconcave if and only if $-F$ is $\omega$-semiconvex [1]. Clearly $\omega$-midaffine functions are simply quadratic functions with leading parameter $\omega$.

Using Definitions 2 we obtain another characterization of $\omega$-midconvex ($\omega$-midconcave, $\omega$-midaffine) sequences. We show that they are restrictions of $\omega$-midconvex ($\omega$-midconcave) functions.

**Theorem 3** *Let $(f_k)_{k\in I}$ be a given sequence. Then the following statements are equivalent*:

(i) *$(f_k)_{k\in I}$ is $\omega$-midconvex* (*$\omega$-midconcave*, *$\omega$-midaffine*);
(ii) *there exists an $\omega$-midconvex* (*$\omega$-midconcave*, *$\omega$-midaffine*) *function $F : \operatorname{conv} I \to \mathbb{R}$ such that $f = F|_I$*;
(iii) *for every $b, c \in \mathbb{R}$ there exists a convex* (*concave*, *affine*) *function $G : \operatorname{conv} I \to \mathbb{R}$ such that*

$$f_k = G(k) + \omega k^2 + bk + c \quad \textit{for } k \in I.$$

*Proof* We give the proof for the convex case.

(i) $\Rightarrow$ (iii). Assume that $(f_k)_{k\in I}$ is $\omega$-midconvex and consider arbitrary $b, c \in \mathbb{R}$. By Theorem 1 the sequence

$$g_k = f_k - \big(\omega k^2 + bk + c\big) \quad \text{for } k \in I$$

is convex. We define $G : \operatorname{conv} I \to \mathbb{R}$ as the piecewise linear function such that

$$G(k) = g(k) \quad \text{for } k \in I.$$

Since $(g_k)_{k\in I}$ is convex function $G$ is locally convex. But locally convex function is convex. So $G$ is convex and

$$f_k = g_k + \omega k^2 + bk + c = G(k) + \omega k^2 + bk + c \quad \text{for } k \in I.$$

(iii) $\Rightarrow$ (ii). We put $F(x) = G(x) + \omega x^2 + bx + c$. Now the implication follows directly from Remark 1.

(ii) $\Rightarrow$ (i). Obvious. □

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# Part 5
# Convexity and Related Inequalities

# Refinement of Inequalities Related to Convexity via Superquadracity, Weaksuperquadracity and Superterzacity

**Shoshana Abramovich, Slavica Ivelić, and Josip Pečarić**

**Abstract** This paper is about inequalities satisfied by functions called superterzatic and their relations to convex and to superquadratic functions. In analogy to inequalities satisfied by convex and by superquadratic functions that are reduced to equalities when $f(x)=x$, $f(x)=x^2$, $x\ge 0$ respectively, the inequalities satisfied by superterzatic functions reduce to equalities when $f(x)=x^3$, $x\ge 0$.

In particular, we deal here with the generalization of the inequality

$$\frac{x^q+y^q}{2}-\left(\frac{x+y}{2}\right)^q-\left(\left|\frac{x-y}{2}\right|\right)^{q-1}\left(\frac{x+y}{2}\right)$$
$$\ge (q-1)\left(\frac{x-y}{2}\right)^2\left(\frac{x+y}{2}\right)^{q-2},$$

$x, y\ge 0$, $q\ge 3$, that reduces to equality for $q=3$.

**Keywords** Superquadracity · Weaksuperquadracity · Superterzacity · Weaksuperterzacity · Convexity · Jensen inequality · Jensen-Steffensen inequality

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## 1 Introduction

In this chapter we present a new set of functions that includes all the power functions

$$x^p,\quad x\ge 0,\quad p\ge 3,$$

which reduce to equality for $p=3$.

This is in analogy to the convex functions that include all the power functions

$$x^p,\quad x\ge 0,\quad p\ge 1,$$

which reduce to equality for $p=1$, and the superquadratic functions (defined in (4)) that include all the power functions

$$x^p,\quad x\ge 0,\quad p\ge 2,$$

and which reduce to equality for $p=2$.

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Inspired by [11], which refines Hölder and Minkowski inequalities, extension and refinement of inequalities satisfied by convex functions are proved in [6]. There, similarly to the three well known inequalities:

$$f(y) \geq f(x) + C_f(x)(y - x), \tag{1}$$

$$f\left(\int h(s)\, d\mu(s)\right) \leq \int f\big(h(s)\big)\, d\mu(s), \tag{2}$$

($\mu$ is a probability measure and $h$ is and $\mu$-integrable function), and

$$\sum_{i=1}^{n} \alpha_i f(x_i) - f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \geq 0, \quad x_i \in (a, b), \quad \alpha_i \geq 0, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} \alpha_i = 1, \tag{3}$$

that are satisfied by convex functions $f : (a, b) \to \mathbb{R}$ and reduce to equalities for $f(x) = x$, the three inequalities (4), (5) and (6) are stated and shown to be satisfied by what is called superquadratic functions $f : [0, b) \to \mathbb{R}$:

A function $f : [0, b) \to \mathbb{R}$ is superquadratic provided that for all $x \in [0, b)$ there exists a constant $C_f(x) \in \mathbb{R}$ such that the inequality

$$f(y) \geq f(x) + C_f(x)(y - x) + f\big(|y - x|\big), \tag{4}$$

holds for all $y \in [0, b)$ (see [6, Definition 2.1]).

According to [6, Theorem 2.2] the inequality

$$f\left(\int h(s)\, d\mu(s)\right) \leq \int f\big(h(s)\big) - f\left(\left|h(s) - \int h(s)\, d\mu(s)\right|\right) d\mu(s) \tag{5}$$

holds for all probability measures $\mu$ and all nonnegative $\mu$-integrable $h$, if and only if $f$ is superquadratic.

The discrete version of (5) is

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \left(f(x_i) - f\left(\left|x_i - \sum_{j=1}^{n} \alpha_j x_j\right|\right)\right), \tag{6}$$

$$x_i \in [0, b), \quad \alpha_i \geq 0, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} \alpha_i = 1.$$

These inequalities reduce to equalities for the superquadratic function $f(x) = x^2$ (see [6]).

Since the appearance of [6], numerous results related to superquadracity were published, part of which are in our reference list.

Now we present the new set of functions which we call superterzatic functions.

A function $g : [0, b) \to \mathbb{R}$ is called superterzatic provided that for all $\bar{x} \in [0, b)$ there exists a constant $C(\bar{x}) \in \mathbb{R}$ such that the inequality

$$\begin{aligned}
&\sum_{i=1}^{n} \alpha_i g(x_i) - g(\bar{x}) \\
&\quad \geq \sum_{i=1}^{n} \alpha_i x_i \left[ (x_i - \bar{x}) C(\bar{x}) + \left( |x_i - \bar{x}| \right)^{-1} g\left( |x_i - \bar{x}| \right) \right] \\
&\quad = \sum_{i=1}^{n} \alpha_i (x_i - \bar{x})^2 C(\bar{x}) + \sum_{i=1}^{n} \alpha_i x_i \left( |x_i - \bar{x}| \right)^{-1} g\left( |x_i - \bar{x}| \right), \qquad (7)
\end{aligned}$$

holds for all $x_i \in [0, b)$ and $\alpha_i \geq 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \alpha_i = 1$ where $\bar{x} = \sum_{i=1}^{n} \alpha_i x_i$. (The equality $\sum_{i=1}^{n} \alpha_i x_i (x_i - \bar{x}) = \sum_{i=1}^{n} \alpha_i (x_i - \bar{x})^2$ follows from direct calculation and also from the equality the superquadratic function $f(x) = x^2$ satisfies.)

This name is given to $g(x)$ because (7) holds for $g(x) = x^p$, $p \geq 3$, with equality for $p = 3$.

In Sect. 2, Theorem 1, we present sufficient conditions for a function to be superterzatic. Using Theorem 1 we get in Corollary 1 refinements of some of the inequalities that convex and/or superquadratic functions satisfy. In the same section we show a set of functions that are both superquadratic and superterzatic, or superquadratic and subterzatic.

At the end of Sect. 2 we deal with obvious relations between superterzatic functions and strongly convex functions that are dealt with in [14]. There the authors present a function $g : I \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval) which is called strongly-convex with modulus $C > 0$ if

$$\alpha g(x) + (1 - \alpha) g(y) - g\big(\alpha x + (1 - \alpha) y\big) \geq C \alpha (1 - \alpha)(y - x)^2 \qquad (8)$$

holds for any $x, y \in I$, $\alpha \in [0, 1]$ (see [14]).

In Sect. 3 we compare the inequalities satisfied by weaksuperquadratic functions and weaksuperterzatic functions as defined here (in [10, 12], and [13] and in other publications the set of weaksuperquadratic functions are called superquadratic functions):

**Definition 1** A function $f$ that satisfies

$$\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right) \geq f\left(\left|\frac{y - x}{2}\right|\right), \quad 0 \leq x, y \leq A, \qquad (9)$$

is called weaksuperquadratic on $[0, A]$ (see [1, 9, 10] and [13], where it is dealt with in other domains).

It is obvious that a function which is superquadratic on [0, $A$], as proved in [6], satisfies

$$\begin{aligned}&\alpha f(x)+(1-\alpha)f(y)-f\big(\alpha x+(1-\alpha)y\big)\\&\quad\geq \alpha f\big((1-\alpha)|y-x|\big)+(1-\alpha)f\big(\alpha|y-x|\big),\\&\qquad 0\leq x\leq A,\quad 0\leq\alpha\leq 1,\end{aligned}\tag{10}$$

is also weaksuperquadratic on [0, $A$].

**Definition 2** A function $g(x)=xf(x)$ is weaksuperterzatic on [0, $A$] if for $0\leq x, y\leq A$, there exists a $C(x)\in\mathbb{R}$ such that

$$\begin{aligned}&\frac{g(x)+g(y)}{2}-g\left(\frac{x+y}{2}\right)\\&\quad\geq\left(\frac{y-x}{2}\right)^2C\left(\frac{x+y}{2}\right)+\frac{x+y}{2}f\left(\left|\frac{y-x}{2}\right|\right)\end{aligned}\tag{11}$$

is satisfied.

It is obvious that if $g(x)$ is superterzatic, that is, $g(x)=xf(x)$ satisfies

$$\begin{aligned}&\alpha g(x)+(1-\alpha)g(y)-g\big(\alpha x+(1-\alpha)y\big)\\&\quad=\alpha xf(x)+(1-\alpha)yf(y)-\big(\alpha x+(1-\alpha)y\big)f\big(\alpha x+(1-\alpha)y\big)\\&\quad\geq\alpha(1-\alpha)(y-x)^2C\big(\alpha x+(1-\alpha)y\big)+\alpha xf\big((1-\alpha)|y-x|\big)\\&\qquad+(1-\alpha)yf\big(\alpha|y-x|\big),\quad 0\leq\alpha\leq 1,\end{aligned}\tag{12}$$

it is also weaksuperterzatic.

In Sect. 3 we show also an example of a function $g$ that is superterzatic but not part of the set dealt with in Theorem 1. In this example $f$ is not superquadratic but $g(x)=xf(x)$ is superterzatic.

In [9], A. Gilányi suggested to deal with a different set of functions called here $W3$. As in the case of superterzatic functions, the inequalities that the functions in this set satisfy reduce to equality for $p=3$, where $x^p$, $x\geq 0$.

The hierarchal chain dealt in [9] starts with superadditive functions, then weak-superquadratic function followed by $W3$ function and can continue to $Wn$, where $n$ is an integer $n\geq 3$. Whereas, our chain deals with convex function, then superquadratic functions followed by superterzatic functions.

## 2 Main Results

In this section we use part of [5, Lemma 1] and [5, Theorem 1] to prove our Theorem 1:

**Lemma A** [5, Lemma 2.1] *Let $f$ be continuously differentiable on $[0,b)$ and $f'$ be superadditive on $[0,b)$. Then the function $D:[0,b)\to\mathbb{R}$ defined by*

$$D(y)=f(y)-f(z)-f'(z)(y-z)-f\big(|y-z|\big)+f(0) \tag{13}$$

*is nonnegative on $[0,b)$, nonincreasing on $[0,z)$, and nondecreasing on $[z,b)$ for $0\le z<b$.*

Using this lemma we get the same inequality (6) that defines superquadracity but this time related to Jensen-Steffensen conditions (14):

**Theorem A** [5, Theorem 2.2] *Let a function $f:[0,b)\to\mathbb{R}$ be continuously differentiable on $[0,b)$. Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ and $\mathbf{x}=(x_1,x_2,\dots,x_n)$ be real n-tuples satisfying*

$$\begin{gathered} 0\le A_j\le A_n,\quad j=1,\dots,n,\quad A_n>0,\\ A_j=\sum_{i=1}^{j}\alpha_i,\quad \bar{A}_j=\sum_{i=j}^{n}\alpha_i,\quad j=1,\dots,n.\\ x_i\in[0,b),\quad i=1,\dots,n,\quad x_1\le x_2\le\dots\le x_n. \end{gathered} \tag{14}$$

*Then*

i) *If $f(0)\le 0$ and $f'$ is superadditive on $[0,b)$, $f$ is superquadratic and*

$$\sum_{i=1}^{n}\alpha_i f(x_i)-A_n f(\bar{x})-\sum_{i=1}^{n}\alpha_i f\big(|x_i-\bar{x}|\big)\ge 0 \tag{15}$$

*holds, where $\bar{x}=\frac{1}{A_n}\sum_{i=1}^{n}\alpha_i x_i$.*

ii) *If $f(0)\ge 0$ and $f'$ is subadditive on $[0,b)$, $f$ is subquadratic and the reverse inequality in (15) holds.*

In Theorem 1 we show sufficient conditions for a function $g:[0,b)\to\mathbb{R}$ to be superterzatic.

**Theorem 1** *Let $f:[0,b)\to\mathbb{R}$ and let $g:[0,b)\to\mathbb{R}$ be the function defined by $g(x)=xf(x)$. Let $\mathbf{x}=(x_1,\dots,x_n)\in[0,b)^n$, $\alpha=(\alpha_1,\dots,\alpha_n)$ be real n-tuples and $C_f(x)$ be as in (4).*

**Case A** *Suppose that $\alpha_i\ge 0$, $i=1,\dots,n$, with $\sum_{i=1}^{n}\alpha_i=1$ and $\bar{x}=\sum_{i=1}^{n}\alpha_i x_i$. Then*

i) *If $f$ is superquadratic on $[0,b)$, we have*

$$\sum_{i=1}^{n}\alpha_i g(x_i)-g(\bar{x})=\sum_{i=1}^{n}\alpha_i x_i f(x_i)-\bar{x}f(\bar{x})$$

$$\geq \sum_{i=1}^{n} \alpha_i x_i (x_i - \bar{x}) \cdot C_f(\bar{x}) + \sum_{i=1}^{n} \alpha_i x_i f\big(|x_i - \bar{x}|\big)$$

$$= \sum_{i=1}^{n} \alpha_i (x_i - \bar{x})^2 C_f(\bar{x}) + \sum_{i=1}^{n} \alpha_i x_i \big(|x_i - \bar{x}|\big)^{-1} g\big(|x_i - \bar{x}|\big). \quad (16)$$

*Hence $g$ is superterzatic.*

ii) *If $f$ is subquadratic on $[0,b)$, the reverse inequality in (16) holds, that is $g$ is subterzatic.*

*Inequality in (16) becomes equality for $g(x) = xf(x) = x^3$.*

**Case B** *Suppose that $f$ is continuously differentiable on $[0,b)$, $\alpha$ and $\mathbf{x}$ satisfy (14) with $\sum_{i=1}^{n} \alpha_i = 1$ and $\bar{x} = \sum_{i=1}^{n} \alpha_i x_i$. Then*

i) *If $f(0) \leq 0$ and $f'$ is superadditive on $[0,b)$, inequality in (16) holds and $g$ is superterzatic.*

ii) *If $f(0) \geq 0$ and $f'$ is subadditive on $[0,b)$, the reverse inequality in (16) holds and $g$ is subterzatic.*

*Inequality in (16) becomes equality for $g(x) = xf(x) = x^3$.*

*Proof* **Case A**. Let $f$ be superquadratic. Then as $\alpha_i \geq 0$ and $0 \leq x_i < b$, $i = 1, \ldots, n$, replacing in (4) $y$ by $x_i$ and $x$ by $\bar{x}$, multiplying by $\alpha_i x_i$, $i = 1, \ldots, n$ and summing we get that (16) holds.

**Case B**. As $\alpha$ and $\mathbf{x}$ satisfy (14) we get

$$B_j = \sum_{i=1}^{j} b_i = \sum_{i=1}^{j} \alpha_i x_i = \bar{A}_1 x_1 + \sum_{i=2}^{j} \bar{A}_i (x_i - x_{i-1}) \geq 0$$

for $j = 1, \ldots, n$, and

$$\bar{B}_{j+1} = B_n - B_j = \sum_{i=j+1}^{n} a_i x_i = \sum_{i=1}^{n} \alpha_i y_i = \bar{A}_1 y_1 + \sum_{i=2}^{j} \bar{A}_i (y_i - y_{i-1})$$

where $y_1 = y_2 = \cdots = y_j = 0$, $y_i = x_i$, $j+1 \leq i \leq n$. As $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is nondecreasing, we get that $B_n - B_j \geq 0$, $j = 1, \ldots, n$, and therefore, $b_i$, $i = 1, \ldots, n$ satisfies (14) too.

Now we replace in (13) $y$ by $x_i$ and $z$ by $\bar{x}$, and instead of multiplying (13) by $\alpha_i$, we multiply it by $b_i = a_i x_i$, $i = 1, \ldots, n$. As it was proved in [2] and [3] that $0 \leq x_1 \leq \bar{x} \leq x_n < b$, we can choose $z = \bar{x} = \sum_i^n a_i x_i$. Then from Lemma A we get

$$\sum_{i=1}^{n} b_i \big(f(x_i) - f(\bar{x}) - f'(\bar{x})(x_i - \bar{x}) - f\big(|x_i - \bar{x}|\big)\big) \geq 0.$$

Hence, (16) holds for $g(x) = xf(x)$. □

*Remark 1* In a similar way to the proof of Theorem 1, under the same conditions as in Case A and Case B on $\alpha$ and $\mathbf{x}$, we get for a convex function $f$ that

$$\sum_{i=1}^{n} \alpha_i x_i f(x_i) - \bar{x} f(\bar{x}) \geq \sum_{i=1}^{n} \alpha_i x_i C(\bar{x})(x_i - \bar{x}) \tag{17}$$

holds, where $C(x) = f'(x)$ is any value *from the interval* $[f'_-(x), f'_+(x)]$. The inequality (17) can also be derived from [8, Lemma 1a].

For $n = 2$ we have

$$\alpha g(x) + (1-\alpha)g(y) - g(\bar{x}) \geq \alpha(1-\alpha)C(\bar{x})(y-x)^2, \tag{18}$$

where $\bar{x} = \alpha x + (1-\alpha)y$, $0 \leq \alpha \leq 1$.

In Corollary 1 we summarize some results which follow immediately from Inequality (6), Theorem 1 and Remark 1.

**Corollary 1** *Let* $f : [0, b) \to \mathbb{R}$ *and let* $g : [0, b) \to \mathbb{R}$ *be the function defined by* $g(x) = xf(x)$. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in [0, b)^n$ *and* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *be real n-tuples. Let* $C_f(x)$*be as in* (4) *and* $f'(x)$ *be any value from the interval* $[f'_-(x), f'_+(x)]$.

**Case A** *Suppose that* $\alpha_i \geq 0$, $i = 1, \ldots, n$, *with* $\sum_{i=1}^{n} \alpha_i = 1$ *and* $\bar{x} = \sum_{i=1}^{n} \alpha_i x_i$. *Then*

i) *If* $f$ *and* $g$ *are superquadratic*, *we have*

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_i g(x_i) - g(\bar{x}) \\ &\quad = \sum_{i=1}^{n} \alpha_i x_i f(x_i) - \bar{x} f(\bar{x}) \\ &\quad \geq Max\Bigg\{\sum_{i=1}^{n} \alpha_i x_i (x_i - \bar{x}) C_f(\bar{x}) + \sum_{i=1}^{n} \alpha_i x_i f\big(|x_i - \bar{x}|\big), \\ &\qquad \sum_{i=1}^{n} \alpha_i |x_i - \bar{x}| f\big(|x_i - \bar{x}|\big)\Bigg\}. \end{aligned} \tag{19}$$

ii) *If* $f$ *is subquadratic and* $g$ *is superquadratic*, *we have*

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i |x_i - \bar{x}| f\big(|x_i - \bar{x}|\big) &= \sum_{i=1}^{n} \alpha_i g\big(|x_i - \bar{x}|\big) \\ &\leq \sum_{i=1}^{n} \alpha_i g(x_i) - g(\bar{x}) \end{aligned}$$

$$= \sum_{i=1}^{n} \alpha_i x_i f(x_i) - \bar{x} f(\bar{x})$$

$$\le \sum_{i=1}^{n} \alpha_i x_i \big[(x_i - \bar{x}) C_f(\bar{x}) + f\big(|x_i - \bar{x}|\big)\big]. \quad (20)$$

iii) *If $f$ is convex and subquadratic and $g$ is superquadratic, we have*

$$Max\left\{\sum_{i=1}^{n} \alpha_i x_i (x_i - \bar{x}) f'(\bar{x}), \sum_{i=1}^{n} \alpha_i |x_i - \bar{x}| f\big(|x_i - \bar{x}|\big)\right\}$$

$$\le \sum_{i=1}^{n} \alpha_i x_i f(x_i) - \bar{x} f(\bar{x})$$

$$\le \sum_{i=1}^{n} \alpha_i x_i \big[(x_i - \bar{x}) C_f(\bar{x}) + f\big(|x_i - \bar{x}|\big)\big]. \quad (21)$$

iv) *If $f$ is concave and $g$ is convex, we have*

$$0 \le \sum_{i=1}^{n} \alpha_i g(x_i) - g(\bar{x}) = \sum_{i=1}^{n} \alpha_i x_i f(x_i) - \bar{x} f(\bar{x})$$

$$\le \sum_{i=1}^{n} \alpha_i x_i (x_i - \bar{x}) f'(\bar{x}). \quad (22)$$

**Case B** *Suppose that $f$ is continuously differentiable on $[0, b)$, $f(0) = 0$, and $\alpha$ and $\mathbf{x}$ satisfy (14) with $\sum_{i=1}^{n} \alpha_i = 1$ and $\bar{x} = \sum_{i=1}^{n} \alpha_i x_i$. Then*

i) *If $f'$ and $g'$ are superadditive on $[0, b)$, inequality (19) holds.*
ii) *If $f'$ is subadditive and $g'$ is superadditive on $[0, b)$, inequalities (20) hold.*
iii) *If $f$ is convex, $f'$ is subadditive and $g'$ is superadditive on $[0, b)$, inequalities (21) hold.*
iv) *If $f$ is concave and $g$ is convex, inequalities (22) hold.*

The inequalities obtained in Corollary 1 are derived for functions which are at least in two of the three sets: the set of convex functions, the set of superquadratic functions and the set of superterzatic functions. To show that such possibilities are not trivial we present the following examples.

In the first example we present a set of functions that are both superquadratic and superterzatic.

*Example 1* Suppose that a function $f : [0, \infty) \to [0, \infty)$ is continuously differentiable and superquadratic. Then according to [6, Lemma 2.1], $f$ is also increasing and convex and $f(0) = f'(0) = 0$. Therefore, according to [6, Lemma 3.2] the function $\frac{f(x)}{x}$ is increasing. Let the function $g : [0, \infty) \to \mathbb{R}$ be defined by $g(x) = x f(x)$.

Then $\frac{g'(x)}{x} = \frac{(xf(x))'}{x} = \frac{f(x)}{x} + f'(x)$ is increasing and since $g(0) = 0$, according to [6, Lemma 3.1], we conclude that $g$ is superquadratic beside being superterzatic, and Theorem 1 Case A i) and Corollary 1 Case A i) hold.

*Example 2* Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = x^p$, $p \geq 0$. Then for $p \geq 2$ the function $f$ is superquadratic on $[0, \infty)$ and for $0 \leq p \leq 2$ $f$ is subquadratic on $[0, \infty)$. Let the function $g : [0, \infty) \to \mathbb{R}$ be defined by $g(x) = xf(x) = x^{p+1}$. Then

a) If $p \geq 2$, $g(x) = x^{p+1}$ is both superquadratic and superterzatic, and Theorem 1, Corollary 1 Case A i) and Case B i) hold.
b) If $1 \leq p \leq 2$, $g(x) = x^{p+1}$ is superquadratic and subterzatic, and Corollary 1 Case A ii) and Case B ii) hold.
c) If $0 \leq p \leq 1$, $g(x) = x^{p+1}$ is convex and subquadratic, and Corollary 1 Case A iv) and Case B iv) hold.

In these cases $C_f(\bar{x}) = p\bar{x}^{p-1}$.

*Example 3* Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = x^2 \ln x^{-1}$, $f(0) = 0$. Let $g : [0, \infty) \to \mathbb{R}$ be the function defined by $g(x) = xf(x)$. Then

a) $f$ is subquadratic on $[0, \infty)$ and $g$ is subterzatic on $[0, \infty)$.
b) $g$ is superquadratic on $[0, e^{-4/3})$, hence Corollary 1 Case A ii) and Case B ii) hold on $[0, e^{-4/3})$.
c) Since $f$ is convex on $[0, e^{-3/2})$ and $g$ is superquadratic on $[0, e^{-4/3})$, Corollary 1 Case A iii) and Case B iii) hold on $[0, e^{-3/2})$.
d) Since $f$ is concave on $(e^{-3/2}, \infty)$ and $g$ is convex on $[0, e^{-5/6})$, Corollary 1 Case A iv) and Case B iv) hold on $(e^{-3/2}, e^{-5/6})$.

To prove Theorem 2, that deals like Theorem 1 with sufficient conditions for $g : [0, b) \to \mathbb{R}$ to be superterzatic, we need Lemma 1 which uses similar techniques as in [8, Lemma 1b].

**Lemma 1** *Let* $f : [0, b) \to \mathbb{R}$ *be a function satisfying* $f(0) = f'(0) = 0$ *and* $f''$ *be increasing on* $[0, b)$. *Let* $z \in [0, b)$ *be fixed. Then the function* $\bar{D} : [0, b) \to \mathbb{R}$ *defined by*

$$\bar{D}(y) = f(z) - f(y) - f'(y)(z - y) - f\big(|y - z|\big) \tag{23}$$

*is nonnegative on* $[0, b)$ *nonincreasing on* $[\frac{z}{2}, z]$ *and nondecreasing on* $[z, b)$.

*Proof* Since $f''$ is increasing on $[0, b)$ and $f(0) = f'(0) = 0$, then according to [7, Lemma 2.2], $f$ is superquadratic, which means that for every $y \in [0, b)$ we have $\bar{D}(y) \geq 0$.

Further, we have

$$\bar{D}'(y) = f''(y)(y - z) - f'\big(|y - z|\big)\operatorname{sgn}(y - z).$$

Therefore, since $f''$ is increasing on $[0, b)$, then for $0 \le z \le y < b$ we have

$$\begin{aligned}\bar{D}'(y) &= f''(y)(y-z) - f'(y-z) \\ &\ge f''(y-z)(y-z) - f\big(|y-z|\big) \ge 0,\end{aligned}$$

which means that $\bar{D}(y)$ increases when $0 \le z \le y < b$.

Similarly, for $0 \le \frac{z}{2} \le y \le z < b$ we get that $\bar{D}'(y) \le 0$, which means that $\bar{D}(y)$ decreases when $0 \le \frac{z}{2} \le y \le z < b$. □

**Theorem 2** *Let $f : [0, b) \to \mathbb{R}$ and let $g : [0, b) \to \mathbb{R}$ be the function defined by $g(x) = xf(x)$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in [0, b)^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ be real n-tuples and $C_f(x)$ be as in (4).*

**Case A** *If $f$ is superquadratic on $[0, b)$, $\alpha_i \ge 0$, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} \alpha_i = 1$ and $\bar{x} = \sum_{i=1}^{n} \alpha_i x_i$, then $g$ is superterzatic and*

$$\begin{aligned}&\sum_{i=1}^{n} \alpha_i g(x_i) - g(\bar{x}) \\ &\quad\le \sum_{i=1}^{n} \alpha_i x_i (x_i - \bar{x}) C_f(x_i) - \sum_{i=1}^{n} \alpha_i x_i \big(|x_i - \bar{x}|\big)^{-1} g\big(|x_i - \bar{x}|\big). \qquad (24)\end{aligned}$$

**Case B** *If $f$ satisfies $f(0) = f'(0) = 0$ and $f''$ is nondecreasing, $\alpha$ and $\mathbf{x}$ satisfy (14) with $\sum_{i=1}^{n} \alpha_i = 1$, $x_n \le 2x_1$ and $\bar{x} = \sum_{i=1}^{n} \alpha_i x_i$, then $g$ is superterzatic and (24) holds.*

*In both cases equality holds in (24) when $g(x) = x^3$, $x \ge 0$.*

*Proof* **Case A**. From Theorem 1 we know that $g(x) = xf(x)$ is superterzatic. Since $f$ is superquadratic, $\alpha_i \ge 0$ and $0 \le x_i < b$, $i = 1, \ldots, n$, then by replacing in (4) $y$ by $\bar{x}$ and $x$ by $x_i$, multiplying by $\alpha_i x_i$, $i = 1, \ldots, n$, and summing, we get that (24) holds.

**Case B**. Since $f''$ increases on $[0, b)$ and $f(0) = f'(0) = 0$, then according to [7, Lemma 2.2] $f$ is superquadratic and therefore $g(x) = xf(x)$ is superterzatic.

In the proof of Theorem 1 Case B we showed that since $\alpha$ and $\mathbf{x}$ satisfy (14) then

$$B_j = \sum_{i=1}^{j} b_i = \sum_{i=1}^{j} \alpha_i x_i \ge 0 \quad \text{for } j = 1, \ldots, n$$

and

$$\bar{B}_{j+1} = B_n - B_j \ge 0 \quad \text{for } j = 1, \ldots, n,$$

and therefore, $b_i$, $i = 1, \ldots, n$ satisfies (14) too.

As mentioned before, see [2] and [3], we have

$$x_1 \le \cdots \le x_k \le \bar{x} \le x_{k+1} \le \cdots \le x_n,$$

and since $x_n \leq 2x_1$, we get

$$0 \leq \bar{x} - x_1 \leq x_n - x_1 \leq x_1,$$

i.e.

$$\frac{\bar{x}}{2} \leq x_1 \leq \cdots \leq x_k \leq \bar{x}.$$

Let $\bar{D}(x_i) = f(\bar{x}) - f(x_i) - f'(x_i)(\bar{x} - x_i) - f(|x_i - \bar{x}|)$, $i = 1, \ldots, n$.

From Lemma 1 we know that $\bar{D}(x_i) \geq 0$ for all $i = 1, \ldots, n$. Also we have

$$\bar{D}(x_1) \geq \bar{D}(x_2) \geq \cdots \geq \bar{D}(x_k) \geq 0$$

and

$$0 \leq \bar{D}(x_{k+1}) \leq \bar{D}(x_{k+2}) \leq \cdots \leq \bar{D}(x_n).$$

Denoting $B_0 = 0$ and $\bar{B}_{n+1} = 0$, it follows

$$\begin{aligned}
b_i &= B_i - B_{i-1}, \quad i = 1, \ldots, n, \\
\bar{B}_k &= \sum_{i=k}^{n} b_i = B_n - B_{k-1}, \quad k = 1, \ldots, n, \\
b_i &= \bar{B}_i - \bar{B}_{i+1}, \quad i = 1, \ldots, n,
\end{aligned}$$

and therefore, we have

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i x_i \bar{D}(x_i) &= \sum_{i=1}^{n} b_i \bar{D}(x_i) \\
&= \sum_{i=1}^{k-1} B_i \big( \bar{D}(x_i) - \bar{D}(x_{i+1}) \big) + B_k \bar{D}(x_k) + \bar{B}_{k+1} \bar{D}(x_{k+1}) \\
&\quad + \sum_{i=k+2}^{n} \bar{B}_i \big( \bar{D}(x_i) - \bar{D}(x_{i-1}) \big) \\
&\geq 0.
\end{aligned}$$

Hence, we get

$$\sum_{i=1}^{n} \alpha_i x_i \bar{D}(x_i) = \sum_{i=1}^{n} \alpha_i x_i \big( f(\bar{x}) - f(x_i) - (\bar{x} - x_i) f'(x_i) - f\big(|x_i - \bar{x}|\big) \big) \geq 0,$$

i.e. the inequality (24) holds, where $C_f(x_i) = (\frac{g(x_i)}{x_i})' = f'(x_i)$. □

In Corollary 1 we got inequalities resulting from superterzatic functions, proved in Theorem 1, together with inequalities satisfied by superquadratic functions. Similarly, inequalities obtained in Theorem 2 for superterzatic functions with inequalities satisfied by superquadratic functions can bring about new inequalities, but not presented here.

We deal now with the following inequalities of the type dealt in [4]. There in [4, Theorem 2.1] it was proved that if $f$ is nonnegative superquadratic function then:

$$\sum_{i=1}^{n} a_i f\left(\frac{x_i}{a_i}\right) - \left(\sum_{i=1}^{n} a_i\right) f\left(\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} a_i}\right)$$
$$\geq \underset{1\leq i<j\leq n}{Max} \left\{ a_i f\left(\left|\frac{x_i}{a_i} - \frac{x_i + x_j}{a_i + a_j}\right|\right) + a_j f\left(\left|\frac{x_j}{a_j} - \frac{x_i + x_j}{a_i + a_j}\right|\right)\right\}$$

From there we know that if $g$ is convex then

$$\sum_{i=1}^{n} a_i g(x_i) - \left(\sum_{i=1}^{n} a_i\right) g\left(\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i}\right)$$
$$\geq \underset{1\leq i<j\leq n}{Max} \left\{ a_i g(x_i) + a_j g(x_j) - (a_i + a_j) g\left(\frac{a_i x_j + a_j x_i}{a_i + a_j}\right)\right\}. \quad (25)$$

Hence, let $f(x)$ be superquadratic and nonnegative on $[0, b)$. According to [6, Lemma 2.1] $f(x)$ is also convex increasing. Therefore the function $g(x) = x f(x)$, $x \geq 0$, is convex on $[0, b)$ and (16) holds too. By the substitution

$$\alpha_i = \frac{a_i}{\sum_{j=1}^{n} a_j}, \quad y_i = \frac{x_i}{a_i}, \quad i = 1, \ldots, n \quad \text{for } a_i > 0,\ x_i \geq 0,\ i = 1, \ldots, n,$$

we get

$$\sum_{i=1}^{n} a_i g\left(\frac{x_i}{a_i}\right) - \left(\sum_{i=1}^{n} a_i\right) g\left(\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} a_i}\right)$$
$$= \sum_{i=1}^{n} x_i f\left(\frac{x_i}{a_i}\right) - \left(\sum_{i=1}^{n} x_i\right) f\left(\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} a_i}\right)$$
$$\geq \underset{1\leq i<j\leq n}{Max} \left\{ x_i f\left(\frac{x_i}{a_i}\right) + x_j f\left(\frac{x_j}{a_j}\right) - (x_i + x_j) f\left(\frac{x_i + x_j}{a_i + a_j}\right)\right\}$$
$$= x_k f\left(\frac{x_k}{a_k}\right) + x_m f\left(\frac{x_m}{a_m}\right) - (x_k + x_m) f\left(\frac{x_k + x_m}{a_k + a_m}\right)$$
$$\geq \frac{(a_k x_m - a_m x_k)^2}{a_k a_m (a_k + a_m)} C_f\left(\frac{x_k + x_m}{a_k + a_m}\right)$$
$$+ x_k f\left(\left|\frac{a_k x_m - a_m x_k}{a_k (a_k + a_m)}\right|\right) + x_m f\left(\left|\frac{a_k x_m - a_m x_k}{a_k (a_k + a_m)}\right|\right).$$

If $g(x) = xf(x) = x^p$, $p \geq 3$, then

$$\sum_{i=1}^{n} \frac{x_i^p}{a_i^{p-1}} - \frac{(\sum_{i=1}^{n} x_i)^p}{(\sum_{i=1}^{n} a_i)^{p-1}}$$

$$\geq \underset{1 \leq i < j \leq n}{Max} \left\{ \frac{x_i^p}{a_i^{p-1}} + \frac{x_j^p}{a_j^{p-1}} - \frac{(x_i + x_j)^p}{(a_i + a_j)^{p-1}} \right\}$$

$$= \frac{x_k^p}{a_k^{p-1}} + \frac{x_m^p}{a_m^{p-1}} - \frac{(x_k + x_m)^p}{(a_k + a_m)^{p-1}}$$

$$\geq \frac{(a_k x_m - a_k x_m)^2 (p-1)(x_k + x_m)^{p-2}}{a_k a_m (a_k + a_m)^{p-1}} + \frac{x_k (|a_m x_k - a_k x_m|)^{p-1}}{(a_k (a_k + a_m))^{p-1}}$$

$$+ \frac{x_m (|a_m x_k - a_k x_m|)^{p-1}}{(a_m (a_k + a_m))^{p-1}}.$$

We conclude this section by looking at the inequalities (12) satisfied by superterzatic functions and inequality (18) satisfied by $g(x) = xf(x)$ where $f$ is convex, and observing similarities between these inequalities and (8) that defines strongly convex functions as in [14]. These similarities lead to the results below and are obtained easily from the discussion in the previous sections, therefore the proofs are omitted.

In particular, we will deal here with sufficient conditions for functions to be strongly convex.

I. Let $f : [A, B] \to \mathbb{R}$ be convex increasing. Then the function $g(x) = xf(x)$ is strongly convex on $[a, B]$, $A < a \leq B$, with modulus $C = f'_-(a)$ (the left derivative at $a$) and $g(x) - f'_-(a)x^2$ is convex on $[a, B]$ (see in [14] the characterization of strongly convex function).
II. Let $f : [0, b) \to \mathbb{R}$ be superquadratic. Then a function $f(x) - Cx^2 - dx - e$, where $C \in \mathbb{R}$, $d, e \in \mathbb{R}^+$, is also superquadratic. If $f$ is also positive on $[0, b)$, then the function $g(x) = xf(x)$ is convex, increasing, superquadratic and superterzatic on $[0, b)$. Therefore, $g$ is strongly convex on $[a, b)$, $0 < a < b$, with modulus $C = f'_-(a)$.

## 3 Superterzacity and Weak Superterzacity, Superquadracity and Weak Superquadracity

We state here two main problems that are still open.

**Problem I** Finding a nontrivial example such that $f(x)$ is NOT superquadratic but $g(x) = xf(x)$ is superterzatic.

**Problem II**

**Case a** Under what conditions, if $g(x)$ is superterzatic then $g(x)$ is also superquadratic, and under what conditions the lower bound of

$$\alpha g(x) + (1-\alpha)g(y) - g\big(\alpha x + (1-\alpha)y\big),$$

reached by the superterzacity of $g(x)$, is better than the lower bound reached by the superquadracity of $g(x)$.

**Case b** under what conditions, if $g(x)$ is superquadratic then $g(x)$ is also superterzatic, and under what conditions the lower bound of

$$\alpha g(x) + (1-\alpha)g(y) - g\big(\alpha x + (1-\alpha)y\big)$$

reached by the superquadracity of $g(x)$ is better than the lower bound reached by the superterzacity of $g(x)$.

If $f(x)$ is a superquadratic function that is not always positive, it is not necessary that $C_f(x) = f'(x)$, ($C_f(x)$ as appears in (4)), even if $f(x)$ has a continuous derivative (see [6, Example 4.2], where it is shown that the function $f(x) = -(1+x^{\frac{1}{p}})^p$, $p > 0$, is superquadratic with $C_f(x) = 0$).

Here we use Definition 1 of weaksuperquadracity, and Definition 2 of weaksuperterzacity, to deal with Problem I and Problem II:

We want to remind here that in [1] there are two examples that show that weaksuperquadracity does not necessarily lead to superquadracity. In [1, Example 1], it is shown that

$$f(x) = \begin{cases} 2x-3, & 0 \le x \le 1, \\ -1, & x > 1, \end{cases}$$

is weaksuperquadratic but not superquadratic. The same holds for the function

$$f(x) = \begin{cases} -3, & x = 0, \\ -1, & x > 0, \end{cases} \tag{26}$$

(see [1, Example 2] and [10]).

In order to solve Problem I, we want to find conditions that ensure that although $f$ is not superquadratic, that means (10) does not hold, $g(x) = xf(x)$ is superterzatic which means that (12) is satisfied.

The following is a trivial example:

*Example 4* Let a function $f$ be defined as in (26). As proved in [1] and [10], this function is not superquadratic but $g(x) = xf(x) = -x$, $x \ge 0$, is superquadratic and also superterzatic (choose $C(\bar{x}) \equiv 0$ in (12)).

Here we deal with Problem II for weaksuperquadracity and weaksuperterzacity cases:

From (10) it follows that if $g(x) = xf(x)$ is superquadratic on $[0, b)$, then

$$\begin{aligned} &\alpha g(x) + (1-\alpha)g(y) - g\big(\alpha x + (1-\alpha)y\big) \\ &\quad \geq \alpha(1-\alpha)|y-x|\big(f\big((1-\alpha)|y-x|\big) + f\big(\alpha|y-x|\big)\big) \end{aligned} \tag{27}$$

holds for $0 \leq \alpha \leq 1$, and for $\alpha = \frac{1}{2}$ we get

$$\frac{g(x)+g(y)}{2} - g\left(\frac{x+y}{2}\right) \geq \left|\frac{y-x}{2}\right| f\left(\left|\frac{y-x}{2}\right|\right). \tag{28}$$

Therefore, $g(x) = xf(x)$ is weaksuperquadratic.

*Problem II. Case a*: From (27) and (12) it follows that in order to solve this case it has to be shown under what conditions

$$\begin{aligned} &\alpha(1-\alpha)(y-x)^2 C\big(\alpha x + (1-\alpha)y\big) + \alpha x f\big((1-\alpha)|y-x|\big) \\ &\quad + (1-\alpha)y f\big(\alpha|y-x|\big) \\ &\geq \alpha(1-\alpha)|y-x|\big(f\big((1-\alpha)|y-x|\big) + f\big(\alpha|y-x|\big)\big) \end{aligned} \tag{29}$$

holds.

In Example 1, it is shown that if $f$ is nonnegative superquadratic function, so is $g(x) = xf(x)$.

Using this result, in the following we show an example that (29) is satisfied for $\alpha = \frac{1}{2}$.

*Example 5* Let $\alpha = \frac{1}{2}$ and $f$ be nonnegative superquadratic function. Hence, $C(x) \geq 0$ and $C(x) = f'(x)$ almost everywhere (see [6, Lemma 2.1]). Then as $x, y \geq 0$ inequality

$$\left(\frac{y-x}{2}\right)^2 C\left(\frac{x+y}{2}\right) + \frac{x+y}{2} f\left(\left|\frac{y-x}{2}\right|\right) \geq \left|\frac{y-x}{2}\right| f\left(\left|\frac{y-x}{2}\right|\right) \tag{30}$$

holds and by (28) and (11) we see that (11) for the weaksuperterzatic $g(x) = xf(x)$ is tighter than (28) for the weaksuperquadratic $g(x) = xf(x)$. But from here it does not necessarily follow that for every $0 \leq \alpha \leq 1$, (12) gives a better lower bound than (27) under the same conditions on $f$.

*Problem II. Case b*: For negative superquadratic function $g(x) = xf(x)$ and $C(x) \leq 0$, it is clear that for $x, y \geq 0$ inequality

$$\begin{aligned} &\left|\frac{y-x}{2}\right| f\left(\left|\frac{y-x}{2}\right|\right) \\ &\quad \geq \left(\frac{y-x}{2}\right)^2 C\left(\frac{x+y}{2}\right) + \frac{x+y}{2} f\left(\left|\frac{y-x}{2}\right|\right) \end{aligned}$$

holds and therefore, from the superquadracity of $g(x) = xf(x)$ it follows that $g(x)$ is also weaksuperterzatic. Again, it does not necessarily follow that for every $0 \leq \alpha \leq 1$ inequality

$$\begin{aligned} &\alpha(1-\alpha)|y-x|\big(f\big((1-\alpha)|y-x|\big) + f\big(\alpha|y-x|\big)\big) \\ &\quad \geq \alpha(1-\alpha)(y-x)^2 C\big(\alpha x + (1-\alpha)y\big) + \alpha x f\big((1-\alpha)|y-x|\big) \\ &\qquad + (1-\alpha)yf\big(\alpha|y-x|\big) \end{aligned} \tag{31}$$

holds.

*Example 6* Let $g(x) = -(1 + x^{\frac{1}{p}})^p$, $p > 0$. This function is superquadratic with $C(x) \equiv 0$ (see [6, Example 4.2]). Therefore, $g$ is also weaksuperterzatic, but it is not proved here that $g$ is superterzatic.

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# On Two Different Concepts of Subquadraticity

**Attila Gilányi, Csaba Gábor Kézi, and Katarzyna Troczka-Pawelec**

*Dedicated to the memory of Wolfgang Walter*

**Abstract** In the recent years, subquadratic functions have been investigated by several authors. However, two different concepts of subquadraticity have been considered. Based on a simple modification of the geometric notion of concave functions a function $f\colon [0,\infty[\to\mathbb{R}$ is called subquadratic if, for each $x \ge 0$, there exists a constant $c_x \in \mathbb{R}$ such that the inequality

$$f(y) - f(x) \le c_x(y-x) + f\big(|y-x|\big)$$

is valid for all nonnegative $y$.

Related to the concept of quadratic functions, a function $f\colon \mathbb{R}\to\mathbb{R}$ is said to be subquadratic if it fulfils the inequality

$$f(x+y) + f(x-y) \le 2f(x) + 2f(y)$$

for all $x, y \in \mathbb{R}$. In the present paper, the connections between these two concepts are described and a third inequality related to these concepts is studied.

**Keywords** Subquadratic function · Quadratic function · Subadditive function · Concave function

**Mathematics Subject Classification** 26A51 · 26D07 · 39B62

## 1 Introduction

In the recent years, subquadratic functions have been investigated by several authors. However, two different concepts of subquadraticity have been considered.

Using a simple modification of the geometric notion of concave functions, in their papers [8] and [9] S. Abramovich, G. Jameson and G. Sinnamon introduced

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the concept calling a function $f: [0, \infty[\to \mathbb{R}$ subquadratic if, for each $x \geq 0$, there exists a constant $c_x \in \mathbb{R}$ such that the inequality

$$f(y) - f(x) \leq c_x(y - x) + f\big(|y - x|\big) \tag{1}$$

is valid for all nonnegative $y$. Functions satisfying this inequality have been studied by several authors. Among others, S. Abramovich, S. Banić, J. Barić, S.S. Dragomir, S. Ivelić, M. Klaričić Bakula, M. Matić, J.A. Oguntuase, L.-E. Persson, J. Pečarić, S. Varošanec published results in the papers [1, 4–7, 10–13, 19]. (More precisely, Abramovich, Jameson and Sinnamon studied superquadratic functions in [8] and [9] taking the inequality in the opposite direction in (1). However, it is easy to see that a function $f$ is superadditive if and only if $-f$ is subadditive, therefore, we may investigate either of these concepts.)

The other concept of subquadraticity is related to quadratic functions: a function $f: \mathbb{R} \to \mathbb{R}$ is called quadratic if it satisfies the square-norm (in other terminologies also called parallelogram or Jordan–von Neumann) equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathbb{R}$. Based on this notion and analogously to the concepts of additive and subadditive functions (cf., e.g., [17, 18, 20]), $f: \mathbb{R} \to \mathbb{R}$ is called subquadratic if it fulfils the inequality

$$f(x + y) + f(x - y) \leq 2f(x) + 2f(y)$$

for all $x, y \in \mathbb{R}$. Subquadratic functions, in this sense, have been studied, among others, by Z. Kominek, K. Troczka–Pawelec, W. Smajdor and A. Gilányi [15, 16, 21, 22].

The problem of investigating the connections between the concepts above arose during the Conference on Inequalities and Applications '07 held in Noszvaj, Hungary in 2007. Results on the problem were presented in several talks and papers (cf., e.g., [2, 3, 6, 14] and [22]). In this chapter, summarizing and extending these results, we completely describe the relation between the two concepts of subquadraticity. In the first section of the chapter, we give this description using a terminology which distinguishes between the two concepts based on their connection. In the second part, we introduce and investigate a third inequality related to both concepts of subquadraticity. We note that the main results of the chapter (2.1, 2.6 and 2.7), in a slightly different form, but essentially with the same content and independently from our research, were obtained and published by S. Abramovich in [3].

## 2 Subquadraticity Concepts and Their Relation

**Definition 1** A function $f: [0, \infty[\to \mathbb{R}$ is called strongly subquadratic (subquadratic for short) if, for each $x \geq 0$, there exists a constant $c_x \in \mathbb{R}$ such that

$$f(y) - f(x) \leq c_x(y - x) + f\big(|y - x|\big) \quad (y \geq 0). \tag{2}$$

**Definition 2** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be weakly subquadratic, if

$$f(x+y) + f(x-y) \leq 2f(x) + 2f(y) \quad (x, y \in \mathbb{R}). \tag{3}$$

Analogously to the concepts of subadditive and superadditive (or convex and concave) functions, a function $f$ is called strongly superquadratic if $-f$ is strongly subquadratic, while $f$ is said to be weakly superquadratic if $-f$ is weakly subquadratic. Therefore, properties of superquadratic functions can easily be obtained from those of subquadratic functions. By this reason, we only consider subquadratic functions in the following part of the chapter.

**Lemma 3** *Let $f\colon [0,\infty[\to \mathbb{R}$ be a function. If $f$ is strongly subquadratic then it satisfies the inequality*

$$f(x+y) + f(x-y) \leq 2f(x) + 2f(y) \quad (x \geq y \geq 0). \tag{4}$$

*Proof* Assume that $f\colon [0,\infty[\to \mathbb{R}$ is a strongly subquadratic function and let $\bar{x}$ and $\bar{y}$ be fixed real numbers with the property $\bar{x} \geq \bar{y} \geq 0$. Writing $\bar{x} - \bar{y}$ and $\bar{x} + \bar{y}$ instead of $y$ in inequality (2), we get

$$f(\bar{x} - \bar{y}) - f(\bar{x}) \leq c_{\bar{x}}(-\bar{y}) + f(\bar{y})$$

and

$$f(\bar{x} + \bar{y}) - f(\bar{x}) \leq c_{\bar{x}}\bar{y} + f(\bar{y}),$$

respectively. Adding these inequalities side by side, we obtain

$$f(\bar{x} + \bar{y}) + f(\bar{x} - \bar{y}) \leq 2f(\bar{x}) + 2f(\bar{y}),$$

which proves our statement. □

**Lemma 4** *If a function $f\colon [0,\infty[\to \mathbb{R}$ satisfies inequality (4) then its even extension $\bar{f} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \geq 0, \\ f(-x), & \text{if } x < 0, \end{cases} \tag{5}$$

*is weakly subquadratic.*

*Proof* Suppose that a function $f\colon [0,\infty[\to \mathbb{R}$ fulfils (4) and let $\bar{x} \in \mathbb{R}$ and $\bar{y} \in \mathbb{R}$ be given.

If $\bar{x} \geq \bar{y} \geq 0$ then (4) immediately gives our statement.

In the case when $\bar{y} \geq \bar{x} \geq 0$, the substitutions $x = \bar{y}$ and $y = \bar{x}$ in (4) also give the validity of (3). This means, that inequality (4) implies

$$\bar{f}(x+y) + \bar{f}(x-y) \leq 2\bar{f}(x) + 2\bar{f}(y) \quad (x, y \geq 0). \tag{6}$$

If $\bar{x} < 0$ and $\bar{y} < 0$ then writing $x = -\bar{x}$ and $y = -\bar{y}$ in inequality (6), we obtain

$$\bar{f}(-\bar{x} - \bar{y}) + \bar{f}(-\bar{x} + \bar{y}) \leq 2\bar{f}(-\bar{x}) + 2\bar{f}(-\bar{y}),$$

which, by the evenness of $\bar{f}$, yields

$$\bar{f}(\bar{x} + \bar{y}) + \bar{f}(\bar{x} - \bar{y}) \leq 2\bar{f}(\bar{x}) + 2\bar{f}(\bar{y}), \tag{7}$$

that is, (3) for $\bar{x} < 0$ and $\bar{y} < 0$.

In the case when $\bar{x} < 0$ and $\bar{y} \geq 0$, the substitutions $x = -\bar{x}$ and $y = \bar{y}$ in (7) give

$$\bar{f}(-\bar{x} + \bar{y}) + \bar{f}(-\bar{x} - \bar{y}) \leq 2\bar{f}(-\bar{x}) + 2\bar{f}(\bar{y}),$$

which, using the evenness of $\bar{f}$ again, gives (3).

Finally, in the remaining case when $\bar{x} \geq 0$ and $\bar{y} < 0$, writing $x = \bar{x}$ and $y = -\bar{y}$ in (6), similarly to the case above, we obtain the validity of (3). □

**Theorem 5** *Let $f : [0, \infty[\to \mathbb{R}$ be a function. If $f$ is strongly subquadratic then its even extension $\bar{f} : \mathbb{R} \to \mathbb{R}$ defined in (5) is weakly subquadratic.*

*Proof* The statement can be obtained as a combination of Lemmas 3 and 4. □

*Remark 6* The converse of the theorem above is not true. More precisely, there exists a weakly subquadratic function $f : \mathbb{R} \to \mathbb{R}$ such that its restriction $f : [0, \infty[\to \mathbb{R}$ is not strongly subquadratic. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3, & \text{if } x = 0, \\ 1, & \text{if } x \neq 0. \end{cases} \tag{8}$$

Obviously, $f$ is weakly subquadratic. However, the substitutions $x = 1$, $y = 3$ and $x = 1$, $y = 0$ in (2) yield that that $f$ is not strongly subquadratic.

## 3 Another Inequality Related to Subquadraticity

**Lemma 7** *Let $f : [0, \infty[\to \mathbb{R}$. The function $f$ fulfils inequality (4) if and only if it satisfies*

$$f(x + y) + f\big(|x - y|\big) \leq 2f(x) + 2f(y) \quad (x, y \geq 0). \tag{9}$$

*Proof* Obviously, if a function $f : [0, \infty[\to \mathbb{R}$ satisfies inequality (9) then it fulfils (4). On the other hand, if $x \geq y$ then inequality (4) gives (9). Finally, in the remaining case when $x < y$, interchanging the role of $x$ and $y$ in (4), we obtain (9). □

**Theorem 8** *If $f : [0, \infty[\to \mathbb{R}$ is a strongly subquadratic function then it satisfies inequality (9).*

*Proof* The statement is a consequence of Lemmas 3 and 7. □

*Remark 9* The converse of Theorem 8 is not valid. Namely, it is easy to see that the function given in (8) fulfils inequality (9) but it is not strongly subquadratic.

**Theorem 10** *If* $f\colon [0,\infty[\to \mathbb{R}$ *is a strongly subquadratic function then its even extension* $\bar{f}:\mathbb{R}\to\mathbb{R}$ *defined in* (5) *satisfies the inequality*

$$\bar{f}(x+y)+\bar{f}\big(|x-y|\big)\le 2\bar{f}(x)+2\bar{f}(y)\quad (x,y\in\mathbb{R}). \tag{10}$$

*Proof* Let $f\colon [0,\infty[\to\mathbb{R}$ be strongly subquadratic. According to Theorem 8, it satisfies (9). Let us consider its extension $\bar{f}$ as above and let $\bar{x}\in\mathbb{R}$ and $\bar{y}\in\mathbb{R}$ be fixed. We prove the validity of (10) for $\bar{f}$ in four steps.

Evidently, if $\bar{x}\ge 0$ and $\bar{y}\ge 0$ then (10) is valid.

If $\bar{x}<0$ and $\bar{y}<0$ then the substitutions $x=-\bar{x}$ and $y=-\bar{y}$ in (9) give

$$f(-\bar{x}-\bar{y})+f\big(|-\bar{x}+\bar{y}|\big)\le 2f(-\bar{x})+2f(-\bar{y}),$$

thus, by the evenness of $\bar{f}$, we have (10).

If $\bar{x}<0$ and $\bar{y}\ge 0$ then writing $x=-\bar{x}$ and $y=\bar{y}$ in (9) we get

$$f(-\bar{x}+\bar{y})+f\big(|-\bar{x}-\bar{y}|\big)\le 2f(-\bar{x})+2f(\bar{y}). \tag{11}$$

Here, if $|\bar{x}|\le|\bar{y}|$ then we have $-\bar{x}+\bar{y}=|\bar{x}-\bar{y}|$ and $|-\bar{x}-\bar{y}|=\bar{x}+\bar{y}$, furthermore, by the evenness of $\bar{f}$, $f(-\bar{x})=f(\bar{x})$, therefore, inequality (11) can be written as

$$f\big(|\bar{x}-\bar{y}|\big)+f(\bar{x}+\bar{y})\le 2f(\bar{x})+2f(\bar{y}), \tag{12}$$

that is, (10) holds. In the case when $|\bar{x}|>|\bar{y}|$, we have $-\bar{x}+\bar{y}=|\bar{x}-\bar{y}|$ and $|-\bar{x}-\bar{y}|=-(\bar{x}+\bar{y})$, thus, using the evenness of $\bar{f}$, we obtain again (12), that is (10).

Finally, if $\bar{x}\ge 0$ and $\bar{y}<0$ then substituting $x=\bar{x}$ and $y=-\bar{y}$ in (9) and using a similar argumentation as above, we obtain our statement. □

**Theorem 11** *If* $f:\mathbb{R}\to\mathbb{R}$ *is a weakly subquadratic function then it satisfies the inequality*

$$f(x+y)+f\big(|x-y|\big)\le 2f(x)+2f(y)\quad (x,y\in\mathbb{R}). \tag{13}$$

*Proof* Assume that $f:\mathbb{R}\to\mathbb{R}$ is a weakly subquadratic function and let $\bar{x}$ and $\bar{y}$ be fixed. If $\bar{x}\ge\bar{y}$ then inequality (3) immediately gives (13). In the case when $\bar{x}<\bar{y}$, substituting $\bar{x}=y$ and $\bar{y}=x$ in (3), we also obtain (13). □

*Remark 12* The converse of Theorem 11 is not true. It is easy to verify that the function $f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases}1, & \text{if } x\ge 0,\\ 4, & \text{if } x<0,\end{cases}$$

satisfies (13) but, writing $x = 1$ and $y = 2$ in (3), we get $5 \leq 4$, thus, $f$ does not fulfil (3).

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# Connections Between the Jensen and the Chebychev Functionals

**Flavia Corina Mitroi**

**Abstract** This work is devoted to the study of connections between *the Jensen functional* and *the Chebychev functional* for convex, superquadratic and strongly convex functions. We give a more general definition of these functionals and establish some inequalities involving them. The entire discussion incorporates both the discrete and the continuous approach.

**Keywords** Jensen functional · Chebychev functional · Superquadratic · Convex · Strong convex function

**Mathematics Subject Classification** Primary 26B25 · Secondary 26E60 · 26D15

## 1 Introduction

The aim of this chapter is to establish a connection between the Jensen and the Chebychev functionals, to study and compare several notions of convex analysis. We will consider respectively the cases of convex, superquadratic and strong convex functions.

For the convenience of the reader we briefly recall some basic facts.

We consider a real valued function $f$ defined on an interval $I$, $x_1, x_2, \ldots, x_n \in I$ and $p_1, p_2, \ldots, p_n \in (0,1)$ with $\sum_{i=1}^n p_i = 1$. We denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. The *Jensen functional* is defined by

$$\mathcal{J}(f, \mathbf{p}, \mathbf{x}) = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right),$$

and the *Chebychev functional* is defined by

$$\mathcal{T}(f, \mathbf{p}, \mathbf{x}) = \sum_{i=1}^n p_i \left(x_i - \sum_{j=1}^n p_j x_j\right) f(x_i).$$

(See [3] and [7] for details.)

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To avoid trivialities we consider that the domain is not a singleton. The interior of $I$ is denoted $int(I)$. We denote by $\partial f$ the multivalued mapping called the *subdifferential*

$$\partial f = \{\varphi : I \to R : f(y) - f(x) \geq (y-x)\varphi(x) \text{ for all } y, x \in I\}.$$

The set $dom\partial f$ consists of all points $x$ in $I$ where $f$ has a support line. For each convex function we have $int(I) \subset dom\partial f$.

Relating the Jensen functional to the Chebychev functional the following inequality has been established using the notion of subdifferential (C.P. Niculescu [7]).

**Proposition 1** *Suppose* $x_1, x_2, \ldots, x_n \in dom\partial f$ *and* $p_1, p_2, \ldots, p_n \in (0,1)$ *with* $\sum_{i=1}^n p_i = 1$. *If* $f$ *is a convex function then*

$$0 \leq \mathcal{J}(f, \mathbf{p}, \mathbf{x}) \leq \mathcal{T}(\varphi, \mathbf{p}, \mathbf{x}) \tag{1}$$

*holds for all* $\varphi \in \partial f$.

(See also S.S. Dragomir and N.M. Ionescu [2].)

**Definition 1** A function $f$ defined on an interval $I$ is *strongly convex with modulus* $c$ if

$$f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)f(x) + \lambda f(y) - c(1-\lambda)\lambda(y-x)^2$$

for all $x, y \in I$ and all $\lambda \in [0,1]$.

Strongly convex functions were introduced by B.T. Polyak [8].

In the paper by N. Merentes and K. Nikodem [5, Theorem 4] we find the proof of the following result:

**Proposition 2** *Let* $p_i \geq 0$, $i = 1, \ldots, n$, *with* $\sum_{i=1}^n p_i = 1$, $\bar{x} = \sum_{i=1}^n p_i x_i$ *and the function* $f$ *strongly convex with modulus* $c$. *Then*

$$\mathcal{T}(c \cdot Id, \mathbf{p}, \mathbf{x}) = c\sum_{i=1}^{n} p_i(x_i - \bar{x})^2 \leq \mathcal{J}(f, \mathbf{p}, \mathbf{x})$$

*where Id denotes the identity function.*

This is re-proved using the probabilistic approach in the paper by T. Rajba and Sz. Wąsowicz [9, Corollary 2.3].

A result due to Rockafellar [10] states the following.

**Proposition 3** *The function* $f$ *defined on interval* $I$ *is strongly convex with modulus* $c$ *if and only if*

$$f(y) \geq f(x) + \varphi(x)(y - x) + c(y - x)^2$$

*for all* $x \in dom\partial f$, $\varphi \in \partial f$, $y \in I$.

Since every strongly convex function is convex, we are now able to refine the inequality (1) via Proposition 3, for strongly convex functions with modulus $c$.

**Proposition 4** *Suppose* $x_1, x_2, \ldots, x_n \in dom\partial f$ *and* $p_1, p_2, \ldots, p_n \in (0, 1)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $f$ *is a strongly convex functions with modulus* $c$ *then*

$$\mathcal{J}(f, \mathbf{p}, \mathbf{x}) \leq \mathcal{T}(\varphi, \mathbf{p}, \mathbf{x}) - \mathcal{T}(c \cdot Id, \mathbf{p}, \mathbf{x}),$$

*for all* $\varphi \in \partial f$.

*Proof* Let us substitute in Proposition 3 for $i = 1, \ldots, n$

$$x = x_i,$$
$$y = \bar{x}.$$

Multiplying the obtained inequalities by $p_i$ and summing them we get the required result. □

A result related to strong convexity due to Hiriart-Urruty and Lemaréchal [4] says that:

**Proposition 5** *The function* $f$ *is strongly convex with modulus* $c$ *if and only if the function* $g(x) = f(x) - cx^2$ *is convex.*

**Definition 2** A function $f$ defined on an interval $I = [0, a]$ or $[0, \infty)$ is *superquadratic* if for each $x$ in $I$ there exists a real number $C(x)$ such that

$$f(y) - f(x) \geq f\big(|y - x|\big) + C(x)(y - x)$$

for all $y \in I$.

According to S. Abramovich and S.S. Dragomir [1, Theorem 9] we have the following result.

**Proposition 6** *Suppose* $f : [0, \infty) \to \mathbb{R}$ *is a superquadratic function. Then*:

$$\sum_{j=1}^{n} p_j f\big(|x_j - \bar{x}|\big) \leq \mathcal{J}(f, \mathbf{p}, \mathbf{x}) \leq \mathcal{T}(C, \mathbf{p}, \mathbf{x}) - \sum_{j=1}^{n} p_j f\big(|x_j - \bar{x}|\big). \tag{2}$$

The next section contains a more general definition of *Jensen* and *Chebychev* functionals and analogues results.

# 2 Main Results

## 2.1 Discrete Case

Assume that we have a real valued function $f$ defined on an interval $I$, real numbers $p_{ij}$, $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ such that $p_{ij} > 0$, $\sum_{j=1}^{n_i} p_{ij} = 1$ for all $i = 1, \ldots, k$ (we denote $\mathbf{p}_i = (p_{i1}, p_{i2}, \ldots, p_{in_i})$), $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in_i}) \in I^{n_i}$ for all $i = 1, \ldots, k$ and $\mathbf{q} = (q_1, q_2, \ldots, q_k)$, $q_i > 0$ such that $\sum_{i=1}^{k} q_i = 1$.

We generalize now the Jensen and the Chebychev functionals defined above.

**Definition 3** We define the *generalized Jensen functional* by

$$\begin{aligned}&\mathcal{J}_k(f, \mathbf{p}_1, \ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \ldots, \mathbf{x}_k)\\ &\quad:= \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1} \cdots p_{kj_k} f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right) - f\left(\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij}\right)\end{aligned}$$

and the *generalized Chebychev functional* by:

$$\begin{aligned}&\mathcal{T}_k(f, \mathbf{p}_1, \ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \ldots, \mathbf{x}_k)\\ &\quad= \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1} \cdots p_{kj_k} \sum_{i=1}^{k} q_i \left(x_{j_i} - \sum_{j=1}^{n_i} p_{ij} x_{ij}\right) f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right).\end{aligned}$$

For more results related to the generalized Jensen functional the reader is referred to [6]. We denote throughout this section

$$\bar{x} = \sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij}.$$

We may now state and prove our main results that give us connections between this two important functionals. We extend the results listed in Introduction.

**Theorem 1** *If $f$ is a convex function and $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in_i}) \in (dom\partial f)^{n_i}$, $i = 1, \ldots, k$, then the following inequalities hold*

$$0 \le \mathcal{J}_k(f, \mathbf{p}_1, \ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \ldots, \mathbf{x}_k) \le \mathcal{T}_k(\varphi, \mathbf{p}_1, \ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \ldots, \mathbf{x}_k)$$

*for all $\varphi \in \partial f$.*

*Proof* We know that

$$\prod_{i=1}^{k} (p_{i1} + \cdots + p_{in_i}) = \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1} \cdots p_{kj_k} = 1.$$

Since $f$ is convex, we have

$$f\left(\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k}\sum_{i=1}^{k} q_i x_{ij_i}\right) \le \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k} f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right).$$

Combining this inequality with the following result

$$\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k}\sum_{i=1}^{k} q_i x_{ij_i} = \sum_{i=1}^{k} q_i\left(\sum_{j=1}^{n_i} p_{ij}\varphi(x_{ij})\right), \tag{3}$$

we obtain the claimed result.

In order to prove the second inequality, due to convexity and utilizing the definition of the subdifferential we have

$$f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right) - f(\bar{x}) \le \varphi\left(\sum_{i=1}^{k} q_i x_{ij_i}\right)\left(\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right).$$

We get, by multiplying with $p_{1j_1}\cdots p_{kj_k}$ and summing over $1 \le j_i \le n_i$, $i = 1,\ldots,k$:

$$\begin{aligned}
&\mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)\\
&= \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k} f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right) - f(\bar{x})\\
&\le \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k}\varphi\left(\sum_{i=1}^{k} q_i x_{ij_i}\right)\left(\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right)\\
&= \mathcal{T}_k(\varphi,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k).
\end{aligned}$$

This completes the proof. □

In the case of strongly convex functions we improve these bounds by the following theorem.

**Theorem 2** *If a function $f$ is strongly convex with modulus $c$ and $\mathbf{x}_i = (x_{i1}, x_{i2},\ldots, x_{in_i}) \in (dom\partial f)^{n_i}$, $i = 1,\ldots,k$ then the following inequalities*

$$\begin{aligned}
0 &\le \mathcal{T}_k(c\cdot Id,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k) \le \mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)\\
&\le \mathcal{T}_k(\varphi,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k) - \mathcal{T}_k(c\cdot Id,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k),
\end{aligned}$$

*where Id denotes the identity function, hold for all $\varphi \in \partial f$.*

*Proof* The first inequality is easily deduced via the Chebychev's inequality, taking into account the convexity of the function $c\cdot Id$.

The second inequality is proved by a short computation:

$$\begin{aligned}
&\mathcal{J}_k(f, \mathbf{p}_1, \ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \ldots, \mathbf{x}_k) \\
&\quad \geq c \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1} \cdots p_{kj_k} \left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right)^2 \\
&\quad = c \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1} \cdots p_{kj_k} \sum_{i=1}^{k} q_i x_{ij_i} \left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right) \\
&\quad = c\mathcal{T}_k(id, \mathbf{p}_1, \ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \ldots, \mathbf{x}_k).
\end{aligned}$$

In order to prove the third inequality we use Proposition 3 substituting

$$x = \sum_{i=1}^{k} q_i x_{j_i},$$
$$y = \bar{x}.$$

We obtain

$$f\left( \sum_{i=1}^{k} q_i x_{j_i} \right) - f(\bar{x}) \leq \varphi\left( \sum_{i=1}^{k} q_i x_{j_i} \right)\left( \sum_{i=1}^{k} q_i x_{j_i} - \bar{x} \right) - c\left( \bar{x} - \sum_{i=1}^{k} q_i x_{j_i} \right)^2.$$

We multiply these inequalities by $p_{1j_1} \cdots p_{kj_k}$ and sum them over $1 \leq j_i \leq n_i$, $i = 1, \ldots, k$. □

Notice that this theorem is a refinement of Theorem 1 for strongly convex functions.

**Theorem 3** *If a function* $f : [0, \infty) \to \mathbb{R}$ *is superquadratic then*:

$$\begin{aligned}
&\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1} \cdots p_{kj_k} f\left( \left| \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right| \right) \\
&\quad \leq \mathcal{J}_k(f, \mathbf{p}_1, \ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \ldots, \mathbf{x}_k) \\
&\quad \leq \mathcal{T}_k(C, \mathbf{p}_1, \ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \ldots, \mathbf{x}_k) - \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1} \cdots p_{kj_k} f\left( \left| \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right| \right).
\end{aligned}$$

*Proof* In order to prove the first inequality, by the definition of superquadratic functions, we have

$$f\left( \sum_{i=1}^{k} q_i x_{ij_i} \right) - f(\bar{x}) \geq C(\bar{x})\left( \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right) + f\left( \left| \sum_{i=1}^{k} q_i x_{ij_i} - \bar{x} \right| \right).$$

Since

$$\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k} = \prod_{i=1}^{k}\left(\sum_{j=1}^{n_i} p_{ij}\right) = 1,$$

this yields us to

$$\begin{aligned}
&\mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)\\
&\quad\geq C(\bar{x})\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k}\left(\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right)\\
&\qquad+\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k} f\left(\left|\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right|\right).
\end{aligned}$$

Obviously we have

$$\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k}\left(\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right) = 0,$$

and the first inequality is proved.

In order to prove the second inequality, we use one more time the definition of the superquadratic functions that leads to the fact that

$$f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right) - f(\bar{x}) \leq C\left(\sum_{i=1}^{k} q_i x_{ij_i}\right)\left(\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right) - f\left(\left|\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right|\right).$$

We get, by multiplying by $p_{1j_1}\cdots p_{kj_k}$ and summing over $1 \leq j_i \leq n_i$, $i = 1,\ldots,k$:

$$\begin{aligned}
&\mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)\\
&\quad\leq \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k} C\left(\sum_{i=1}^{k} q_i x_{ij_i}\right)\left(\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right)\\
&\qquad-\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k} f\left(\left|\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right|\right)\\
&\quad= \mathcal{T}_k(C,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k) - \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\cdots p_{kj_k} f\left(\left|\sum_{i=1}^{k} q_i x_{ij_i} - \bar{x}\right|\right).
\end{aligned}$$

This completes the proof. □

## 2.2 Continuous Case

In what follows we shall concentrate on the integral analogue of the results from the previous section. We consider $p_i(x)\,dx$ and $r_i(x)\,dx$, $i = 1, \dots, k$ to be absolutely continuous measures, where $p_i, r_i : [a, b] \subset (0, \infty) \to (0, \infty)$ are increasing such that $\int_a^b p_i(x)\,dx = 1$, $\int_a^b r_i(x)\,dx = 1$. We also consider $\mathbf{q} = (q_1, q_2, \dots, q_k)$, $q_i > 0$ with $\sum_{i=1}^k q_i = 1$. We set

$$\mathcal{J}_k(f, p_1, \dots, p_k, \mathbf{q}) := \int_{[a,b]^k} f\left(\sum_{i=1}^k q_i x_i\right) \prod_{i=1}^k \big(p_i(x_i)\,dx_i\big) - f\left(\sum_{i=1}^k q_i \int_a^b x p_i(x)\,dx\right)$$

and

$$\mathcal{T}_k(f, p_1, \dots, p_k, \mathbf{q}) = \int_{[a,b]^k} \sum_{i=1}^k q_i \left(x_i - \int_a^b x p_i(x)\,dx\right) f\left(\sum_{i=1}^k q_i x_i\right) \prod_{i=1}^k \big(p_i(x_i)\,dx_i\big)$$

for all positive integers $k$.

We denote throughout this section

$$\bar{x} = \sum_{i=1}^k q_i \int_a^b x p_i(x)\,dx.$$

Applying the same reasoning as before we obtain:

**Theorem 4** *If $f$ is a convex function then the following inequalities hold*

$$0 \le \mathcal{J}_k(f, p_1, \dots, p_k, \mathbf{q}) \le \mathcal{T}_k(\varphi, p_1, \dots, p_k, \mathbf{q})$$

*for all $\varphi \in \partial f$.*

*Proof* The first inequality can be proved taking into account that

$$\int_{[a,b]} p_i(x)\,dx = 1, \quad \text{for all } i = 1, \dots, k.$$

Since $f$ is convex we have

$$f\left(\int_{[a,b]^k} \sum_{i=1}^k q_i x_i \prod_{i=1}^k \big(p_i(x_i)\,dx_i\big)\right) \le \int_{[a,b]^k} f\left(\sum_{i=1}^k q_i x_i\right) \prod_{i=1}^k \big(p_i(x_i)\,dx_i\big).$$

Combining this inequality with the following result

$$\int_{[a,b]^k} \sum_{i=1}^{k} q_i x_i \prod_{i=1}^{k} \big(p_i(x_i)\, dx_i\big) = \sum_{i=1}^{k} q_i \int_a^b x p_i(x)\, dx$$

we obtain the conclusion.

We prove the second inequality utilizing the definition of the subdifferential. We have

$$f\left(\sum_{i=1}^{k} q_i x_i\right) - f(\bar{x}) \le \varphi\left(\sum_{i=1}^{k} q_i x_i\right)\left(\sum_{i=1}^{k} q_i x_i - \bar{x}\right).$$

By integrating with respect to $\prod_{i=1}^{k} p_i(x_i)\, dx_i$ we get:

$$\begin{aligned}
&\mathcal{J}_k(f, \mathbf{p}_1, \ldots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \ldots, \mathbf{x}_k) \\
&\quad = \int_{[a,b]^k} f\left(\sum_{i=1}^{k} q_i x_i\right) \prod_{i=1}^{k} \big(p_i(x_i)\, dx_i\big) - f(\bar{x}) \\
&\quad \le \int_{[a,b]^k} \varphi\left(\sum_{i=1}^{k} q_i x_i\right)\left(\sum_{i=1}^{k} q_i x_i - \bar{x}\right) \prod_{i=1}^{k} \big(p_i(x_i)\, dx_i\big) \\
&\quad = \mathcal{T}_k(\varphi, p_1, \ldots, p_k, \mathbf{q}).
\end{aligned}$$

This completes the proof. □

For strongly convex functions we refine the bounds form the previous theorem.

**Theorem 5** *If $f$ is a strongly convex function with modulus $c$ then*

$$\begin{aligned}
0 &\le \mathcal{T}_k(c \cdot Id, p_1, \ldots, p_k, \mathbf{q}) \le \mathcal{J}_k(f, p_1, \ldots, p_k, \mathbf{q}) \\
&\le \mathcal{T}_k(\varphi, p_1, \ldots, p_k, \mathbf{q}) - \mathcal{T}_k(c \cdot Id, p_1, \ldots, p_k, \mathbf{q}),
\end{aligned}$$

*where Id is denoting the identity function, for all $\varphi \in \partial f$.*

*Proof* In order to prove the first inequality, we use the fact that the function $f(x) - cx^2$ is convex. We apply the Jensen inequality to it.

$$\begin{aligned}
&f\left(\int_{[a,b]^k} \sum_{i=1}^{k} q_i x_i \prod_{i=1}^{k} \big(p_i(x_i)\, dx_i\big)\right) - c\left(\int_{[a,b]^k} \sum_{i=1}^{k} q_i x_i \prod_{i=1}^{k} \big(p_i(x_i)\, dx_i\big)\right)^2 \\
&\quad \le \int_{[a,b]^k} f\left(\sum_{i=1}^{k} q_i x_i\right) \prod_{i=1}^{k} \big(p_i(x_i)\, dx_i\big) - c \int_{[a,b]^k} \left(\sum_{i=1}^{k} q_i x_i\right)^2 \prod_{i=1}^{k} \big(p_i(x_i)\, dx_i\big).
\end{aligned}$$

Observe that

$$\int_{[a,b]^k}\left(\sum_{i=1}^{k} q_i x_i\right)^2 \prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big) - \left(\int_{[a,b]^k}\sum_{i=1}^{k} q_i x_i \prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big)\right)^2$$
$$= \int_{[a,b]^k}\left(\sum_{i=1}^{k} q_i x_i - \int_{[a,b]^k}\sum_{i=1}^{k} q_i x_i \prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big)\right)^2 \prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big).$$

This yields us to the claimed result:

$$\begin{aligned}\mathcal{J}_k(f, p_1, \ldots, p_k, \mathbf{q}) &= \int_{[a,b]^k} f\left(\sum_{i=1}^{k} q_i x_i\right)\prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big) - f(\bar{x})\\ &\geq c\int_{[a,b]^k}\left(\sum_{i=1}^{k} q_i x_i - \bar{x}\right)^2 \prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big)\\ &= \mathcal{T}_k(c\cdot Id, p_1, \ldots, p_k, \mathbf{q}).\end{aligned}$$

The second inequality has an immediate proof via Proposition 3. We have

$$f\left(\sum_{i=1}^{k} q_i x_i\right) - f(\bar{x}) \leq \varphi\left(\sum_{i=1}^{k} q_i x_i\right)\left(\sum_{i=1}^{k} q_i x_i - \bar{x}\right) - c\left(\sum_{i=1}^{k} q_i x_i - \bar{x}\right)^2.$$

By integrating with respect to $\prod_{i=1}^{k}(p_i(x_i)\,dx_i)$ we get the required inequality. □

The first and second inequalities for the particular case $k = 1$ of this theorem are exactly the integral analogue of Proposition 2 and were already proved in [5] and [9].

On the other hand, for superquadratic functions we have the following bounds.

**Theorem 6** *If a function* $f : [0, \infty) \to \mathbb{R}$ *is superquadratic then*:

$$\begin{aligned}&\int_{[a,b]^k} f\left(\left|\sum_{i=1}^{k} q_i x_i - \bar{x}\right|\right)\prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big)\\ &\quad\leq \mathcal{J}_k(f, p_1, \ldots, p_k, \mathbf{q})\\ &\quad\leq \mathcal{T}_k(f, p_1, \ldots, p_k, \mathbf{q}) - \int_{[a,b]^k} f\left(\left|\sum_{i=1}^{k} q_i x_i - \bar{x}\right|\right)\prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big).\end{aligned}$$

*Proof* We prove only the first inequality, the second one has a similar approach.

By the definition of superquadratic functions, we have

$$f\left(\sum_{i=1}^{k} q_i x_i\right) - f(\bar{x}) \geq C(\bar{x})\left(\sum_{i=1}^{k} q_i x_i - \bar{x}\right) + f\left(\left|\sum_{i=1}^{k} q_i x_i - \bar{x}\right|\right).$$

This yields us to:

$$\begin{aligned}\mathcal{J}_k(f, p_1, \ldots, p_k, \mathbf{q}) \geq C(\bar{x}) &\int_{[a,b]^k}\left(\sum_{i=1}^{k} q_i x_i - \bar{x}\right)\prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big)\\ &+ \int_{[a,b]^k} f\left(\left|\sum_{i=1}^{k} q_i x_i - \bar{x}\right|\right)\prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big).\end{aligned}$$

Since

$$\int_{[a,b]^k}\left(\sum_{i=1}^{k} q_i x_i - \bar{x}\right)\prod_{i=1}^{k}\big(p_i(x_i)\,dx_i\big) = 0,$$

we get the conclusion. □

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# Part 6
# Other Equations and Inequalities

# Functional Inequalities and Equivalences of Some Estimates

**Włodzimierz Fechner**

**Abstract** The purpose of the chapter is to deal with some old and a few new functional inequalities which are motivated by well-known estimates on the real line or on an interval which involve the exponential function. We are concerned with the following six functional inequalities:

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(x)+f(y)}{2},$$

$$f\left(\frac{x+y}{2}\right) \le \frac{f(y)-f(x)}{y-x},$$

$$g(x+h) \le g(x)\varphi(h),$$

$$f(x+h) \le f(x)\psi(h),$$

$$xf(y) \le f(x)+(y-1)f(y)$$

and

$$(1+y)f(x) \le f(x+y).$$

**Keywords** Exponential function · Functional inequality

**Mathematics Subject Classification** 39B12 · 39B62 · 39B72

## 1 Some Estimates Involving the Exponential Function

Throughout the chapter the symbol $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^- = \{x \in \mathbb{R} : x \le 0\}$, $\mathbb{R}^+ = \{x \in \mathbb{R} : x \ge 0\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Further, let the letters $A$, $G$ and $L$ denote the arithmetic, geometric and logarithmic mean, i.e.:

$$A(s,t) = \frac{s+t}{2},$$

$$G(s,t) = \sqrt{s \cdot t},$$

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$$L(s,t) = \frac{t-s}{\log t - \log s} \quad \text{for } s \neq t \quad \text{and} \quad L(s,s) = s$$

for $s, t \in \mathbb{R}$ or $s, t > 0$, respectively.

It is well known that

$$G(s,t) \leq L(s,t) \leq A(s,t), \tag{1}$$

(see F. Burk [3]) and also that:

$$G^{\frac{2}{3}}(s,t) \cdot A^{\frac{1}{3}}(s,t) \leq L(s,t) \leq \frac{2}{3}G(s,t) + \frac{1}{3}A(s,t) \tag{2}$$

for all $s, t > 0$. The first inequality of (2) was proved in 1983 by E.B. Leach and M.C. Sholander [13], whereas the second inequality of (2) was obtained in 1972 by B.C. Carlson [4] (see also F. Burk [3]) and earlier also by G. Pólya and G. Szegő [15]. Let us also note that J. Sándor [17] provided some further refinements of both estimates.

Let us fix arbitrary $x, y \in \mathbb{R}$ such that $x \neq y$ and substitute $s := e^x$ and $t := e^y$ in (1) and (2). Therefore, we see that the exponential function satisfies the following inequalities:

$$e^{\frac{x+y}{2}} \leq \frac{e^y - e^x}{y - x} \leq \frac{e^x + e^y}{2} \tag{3}$$

and

$$6e^{\frac{2}{3}\cdot\frac{x+y}{2}}\left[\frac{e^x + e^y}{2}\right]^{\frac{1}{3}} \leq 6\frac{e^y - e^x}{y - x} \leq 4e^{\frac{x+y}{2}} + e^x + e^y \tag{4}$$

for each $x, y \in \mathbb{R}$ such that $x \neq y$.

Finally, the following estimate:

$$xe^y \leq e^x + (y-1)e^y, \tag{5}$$

for all $x, y \in \mathbb{R}$ was observed by P.L. Duren and H.D. Lipsich [5], see also D.S. Mitrinović [14].

## 2 Functional Inequalities—Known Facts

It is possible to characterize the exponential function using a system of functional equations and inequalities of a single variable (the term *single variable* is used interchangeably with *iterative* and means that only one independent variable appears in functional equation or inequality discussed). M. Kuczma [10] (see also M. Kuczma, B. Choczewski and R. Ger [12, Chapter 10.2B]) proved that without any additional regularity assumptions the map $\varphi = \exp$ is the only real-to-real solution of the following system:

$$\varphi(x) > 0,$$

$$\varphi(x) \geq 1 + x,$$
$$\varphi(2x) = \left[\varphi(x)\right]^2,$$
$$\varphi(-x) = \left[\varphi(x)\right]^{-1},$$

postulated for all $x \in \mathbb{R}$. An earlier result of M. Kuczma [8] (see also M. Kuczma [9, Chapter VI, §12]) states that all solutions of a related equation of a single variable which satisfy some additional smoothness are of the form $\varphi = c \cdot \exp$ with some real $c$.

In 1988 B. Poonen [16], answering a problem proposed by D.J. Shelupsky [18], proved that the general solution $f : \mathbb{R} \to \mathbb{R}$ of the system:

$$\min\left\{f(x), f(y)\right\} \leq \frac{f(y) - f(x)}{y - x} \leq \max\left\{f(x), f(y)\right\} \quad (x \neq y) \tag{6}$$

is of the form $f = c \cdot \exp$, where $c \geq 0$ is an arbitrary constant.

The results of B. Poonen were developed by C. Alsina and J.L. Garcia Roig [1]. To be precise, they studied the following two functional inequalities:

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(x) + f(y)}{2} \quad (x \neq y), \tag{7}$$

and

$$0 \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(x) + f(y)}{2} \quad (x \neq y). \tag{8}$$

They have proved that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies (7) if and only if there exists a nonincreasing function $d : \mathbb{R} \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for all $x \in \mathbb{R}$ [1, Theorem 1]. Further, $f : \mathbb{R} \to \mathbb{R}$ is a solution of (8) if and only if there exists a continuous nonincreasing function $d : \mathbb{R} \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for $x \in \mathbb{R}$ and $d(x + t) \geq e^{-t} d(x)$ for all $x \in \mathbb{R}$ and $t > 0$ [1, Theorem 2].

Inequality

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(y) - f(x)}{y - x} \quad (x \neq y) \tag{9}$$

was considered by C. Alsina and R. Ger [2] and later by the author [6]. In particular, it was shown that all solutions of (9) on an open interval $I$ which enjoy some regularity properties are of the form $f(x) = i(x)e^x$ for all $x \in I$ with a nondecreasing map $i$. The following result is proved in [6].

**Theorem 1** [6, Theorem 1] *Assume that $I$ is an open nonvoid interval, $f : I \to \mathbb{R}$ satisfies* (9) *and*

$$\limsup_{h \to 0+} f(x + h) \geq f(x) \quad (\textit{for all } x \in I). \tag{10}$$

*Then*

$$f(y) \geq e^{y-x} f(x) \quad (x \leq y), \tag{11}$$

*i.e. the mapping* $I \ni x \mapsto i(x) := f(x)e^{-x} \in \mathbb{R}$ *is nondecreasing.*

During the conference we have posted the following two open problems connected with this result.

*Problem 1* The converse of Theorem 1 is not true (see [6, Remark 1]). Find and prove an additional condition upon the map $i$ from Theorem 1 to obtain the "if and only if" result, i.e. to get that each map of the form $f(x) = i(x)e^x$ solves (9).

*Problem 2* Is it possible to drop or weaken the assumption (10) in Theorem 1?

Finally, let us mention that the following two functional inequalities:

$$f\left(\frac{x+y}{2}\right)^2 \cdot \frac{f(x)+f(y)}{2} \leq \left[\frac{f(y)-f(x)}{y-x}\right]^3, \quad (x \neq y) \tag{12}$$

$$6\frac{f(y)-f(x)}{y-x} \leq 4f\left(\frac{x+y}{2}\right) + f(x) + f(y), \quad (x \neq y) \tag{13}$$

which are motivated by (4), are discussed in [7] in the spirit of the above-mentioned studies.

## 3 Results

First, we will observe a certain property of solutions of (7) and (9).

**Proposition 1** *If* $D \subset \mathbb{R}$ *is a set which consists of at least two points,* $f_0 \colon D \to \mathbb{R}$ *solves* (7) *and* $d \colon D \to \mathbb{R}$ *is nonincreasing and both mappings are nonnegative, then* $d \cdot f_0$ *solves* (7), *too.*

*Proof* Assume that $f_0$ and $d$ are as required and denote $f = d \cdot f_0$. Fix arbitrary $x, y \in D$ such that $x < y$; we have

$$\frac{f(y)-f(x)}{y-x} = \frac{d(y)f_0(y)-d(x)f_0(x)}{y-x} \leq d(y)\frac{f_0(y)-f_0(x)}{y-x}$$
$$\leq d(y)\frac{f_0(x)+f_0(y)}{2} \leq \frac{d(x)f_0(x)+d(y)f_0(y)}{2} = \frac{f(x)+f(y)}{2}.$$

□

**Proposition 2** *If* $D \subset \mathbb{R}$ *is a set which consists of at least two points,* $f_0 \colon D \to \mathbb{R}$ *solves* (9) *and* $i \colon D \to \mathbb{R}$ *is nondecreasing and both mappings are nonnegative, then* $i \cdot f_0$ *solves* (9), *too.*

*Proof* If $f_0$ and $i$ are as required, $x, y \in D$ and $x < y$, then

$$f\left(\frac{x+y}{2}\right) = i\left(\frac{x+y}{2}\right) f_0\left(\frac{x+y}{2}\right) \le i(y) f_0\left(\frac{x+y}{2}\right)$$
$$\le i(y)\frac{f_0(y) - f_0(x)}{y-x} \le \frac{i(y) f_0(y) - i(x) f_0(x)}{y-x} = \frac{f(y) - f(x)}{y-x}.$$

□

From Propositions 1 and 2 and from the respective theorems describing solutions of (7) and (9) we conclude that the exponential mapping is in a sense "the best" or "extreme" solution of each inequality. In fact, each nonnegative solution of (7) or (9) can be derived from the particular one $f_0 = \exp$ with the aid of Proposition 1 or 2, respectively.

In [1] and [2], while dealing with (7), (8), (9) and with some other related inequalities, the authors applied a technique which relies on reducing the inequality in question to a special case of the following general functional inequality:

$$g(x+h) \le g(x)\varphi(h) \tag{14}$$

with some $\varphi\colon U \to \mathbb{R}$, where $U$ is a set which satisfies certain additional conditions. Then, an iterative procedure was performed on this inequality to obtain a desired representation of the unknown mapping.

In view of this observation it is of interest to state and prove a more general result which covers the respective parts of reasonings from the above-mentioned papers. We begin with the following lemma.

**Lemma 1** *Assume that $I$ is an open, nonvoid interval, $U \subset \mathbb{R}$ is an interval containing a right neighborhood of zero, $g\colon I \to \mathbb{R}$ is arbitrary and $\varphi\colon U \to \mathbb{R}$ satisfies*

$$\lim_{n\to+\infty} \varphi\left(\frac{h}{n}\right)^n = 1, \tag{15}$$

*for all $h \in U$. If $g$ and $\varphi$ fulfill (14) for each $x \in I$ and each $h \in U$ such that $x + h \in I$, then $g$ is nonincreasing.*

*Moreover, if additionally $U$ contains a left neighborhood of zero, then $g$ is constant.*

*Proof* Fix $x \in I$ and $n \in \mathbb{N}$, and then take $h \in U$ such that $x + nh \in I$ and $\varphi(h) > 0$ (assuming that such $h$ exists). We may verify inductively the following inequality

$$g(x+nh) \le g(x)\varphi(h)^n$$

provided that $h, 2h, \dots, nh \in U$. Indeed, we have

$$g(x+nh) = g\big(x+h+(n-1)h\big) \le g(x+h)\varphi(h)^{n-1} \le g(x)\varphi(h)^n.$$

Next, replace $h$ by $h/n$ to obtain

$$g(x+h) \le g(x)\varphi\left(\frac{h}{n}\right)^n$$

for all $x \in I$ and all $h \in U$ such that $x+h \in I$ provided that $n$ is large enough (by (15) we have $\varphi(h/n) > 0$ for sufficiently large $n \in \mathbb{N}$). Now, let $n$ tend to infinity and use assumption (15) to obtain $g(x+h) \le g(x)$. Therefore, thanks to our assumption upon $U$ we deduce that $g$ is nondecreasing. If additionally $U$ contains a left neighborhood of zero, then $g$ is also nonincreasing and thus constant. □

**Corollary 1** *Assume that $I$ is an open, nonvoid interval, $U \subset \mathbb{R}$ is an interval containing a right neighborhood of zero, $f: I \to \mathbb{R}$ is arbitrary, $\chi: \mathbb{R} \to \mathbb{R}$ is a nonzero solution of the exponential Cauchy equation*:

$$\chi(u+v) = \chi(u)\chi(v),$$

*for each $u, v \in \mathbb{R}$ and $\psi: U \to \mathbb{R}$ satisfies*

$$\lim_{n\to+\infty} \psi\left(\frac{h}{n}\right)^n = \chi(h) \tag{16}$$

*for all $h \in U$. If $f$ and $\psi$ fulfill*

$$f(x+h) \le f(x)\psi(h) \tag{17}$$

*for each $x \in I$ and each $h \in U$ such that $x+h \in I$, then there exists a nonincreasing function $d: I \to \mathbb{R}$ such that $f(x) = d(x)\chi(x)$ for $x \in I$.*

*Moreover, if additionally $U$ contains a left neighborhood of zero, then there exists a constant $c \in \mathbb{R}$ such that $f(x) = c\chi(x)$ for $x \in I$.*

*Proof* First, let us note that since $\chi \ne 0$, then $\chi$ is strictly positive (see M. Kuczma [11, Chapter 13.1]). Therefore, to prove the assertion it suffices to apply Lemma 1 for $g = f\chi^{-1}$ and $\varphi = \psi\chi^{-1}$ and to take $d = g$. □

*Remark 1* An inspection of the proof of Lemma 1 allows us to note that if additionally $g \ge 0$ in Lemma 1 or $f \ge 0$ in Corollary 1, then assumptions (15) and (16) can be replaced by the respective inequalities:

$$0 < \lim_{n\to+\infty} \varphi\left(\frac{h}{n}\right)^n \le 1,$$

and

$$0 < \lim_{n\to+\infty} \psi\left(\frac{h}{n}\right)^n \le \chi(h).$$

*Remark 2* Inequality (7) is a special case of (17). To see this assume that $x < y$ in (7), put $h = y - x$ and rewrite this inequality as

$$f(x+h) \leq f(x)\frac{2+h}{2-h}.$$

It is easy to check that $\psi(h) = \frac{2+h}{2-h}$ defined for $h \in U = (0, 2)$ fulfills condition (16) with $\chi = \exp$ and therefore, all assumptions of Corollary 1 are fulfilled.

*Remark 3* Let us denote the limit functions appearing on the left-hand sides of conditions (15) and (16) by $\Phi$ and $\Psi$, respectively, i.e.

$$\Phi(h) = \lim_{n\to+\infty} \varphi\left(\frac{h}{n}\right)^n, \qquad \Psi(h) = \lim_{n\to+\infty} \psi\left(\frac{h}{n}\right)^n,$$

for $h > 0$ if the set $U$ contains a right neighborhood of zero, or for all $h \in \mathbb{R}$ if zero is an internal point of $U$. One can observe that $\Phi$ and $\Psi$ satisfy a particular case of the iterative functional equation of Böttcher. In fact, for each $k \in \mathbb{N}$ and $h \in I$ we have

$$\Phi(kh) = \lim_{n\to+\infty} \varphi\left(\frac{kh}{n}\right)^n = \lim_{kn\to+\infty} \varphi\left(\frac{kh}{kn}\right)^{kn} = \left(\lim_{n\to+\infty} \varphi\left(\frac{h}{n}\right)^n\right)^k = \Phi(h)^k$$

and analogously $\Psi(kh) = \Psi(h)^k$. Some additional regularity assumptions imposed upon these mappings will force that both $\Phi$ and $\Psi$ are of the form $x \mapsto \exp(\lambda x)$ with some real $\lambda$ (see M. Kuczma [8] or M. Kuczma [9, Chapter VI, §12]).

*Remark 4* One may ask about conditions imposed upon $\varphi$ and $\psi$ which are sufficient for (15) and (16) to hold. Results in this spirit for sequences of iterates can be found in the monograph M. Kuczma, B. Choczewski and R. Ger [12, Chapter 1.3]. However, let us point out that in fact the sequences appearing in (15) and (16) are not really iterates of a function and therefore a direct application of this results may be limited.

In what follows we will study the following functional inequality stemming from the estimate (5):

$$xf(y) \leq f(x) + (y-1)f(y). \tag{18}$$

**Proposition 3** *Assume that the set $D \subset \mathbb{R}$ consists of at least two points and $f: D \to \mathbb{R}$ satisfies (18) for all $x, y \in D$. Then $f$ is nondecreasing on $D$ and nonnegative on $D \setminus \{\inf D\}$.*

*Proof* Arbitrarily fix $x, y \in D$ such that $x \neq y$ and change the roles of variables $x, y$ in (18) to get

$$yf(x) \leq f(y) + (x-1)f(x).$$

Add this side-by-side to (18) to arrive at

$$(x-y)f(y) \le (x-y)f(x).$$

Therefore, if we assume for a moment that $x < y$, then we see that $f$ is nondecreasing. Next, rewrite (18) as follows:

$$(x-y+1)f(y) \le f(x).$$

If we denote $h = y - x$, then we have

$$(1-h)f(x+h) \le f(x). \tag{19}$$

Let us assume that $x < y = x + h$. Thus $h > 0$ and since $f$ is nondecreasing then

$$(1-h)f(x+h) \le f(x) \le f(x+h),$$

and consequently $f(x+h) \ge 0$. □

*Example 1* Solutions of (18) need not to be nonnegative on its whole domain. Indeed, if we take $D = \{-1, 1\}$ and $f: D \to \mathbb{R}$ is given by $f(-1) = -1$ and $f(1) = 1$, then $f$ solves (18).

**Theorem 2** *Assume that $I \subset \mathbb{R}$ is a nonvoid open interval. Then $f: I \to \mathbb{R}$ solves (18) if and only if there exists a constant $c \ge 0$ such that $f(x) = c \cdot \exp(x)$ for each $x \in I$.*

*Proof* The "if" part follows from estimate (5). To prove the "only if" part pick arbitrary $\varepsilon > 0$ and let $I_0 = (\inf I + \varepsilon, \sup I - \varepsilon)$. We may assume that $\varepsilon$ is small enough to ensure us that $I_0 \neq \emptyset$. Then fix arbitrary $x \in I_0$ and $h < 1$ such that $x + h \in I$. By inequality (19) we have

$$f(x+h) \le \frac{1}{1-h} f(x).$$

One can check that all assumptions of Corollary 1 are satisfied with $I = I_0$, $U = (-\varepsilon, \varepsilon)$, $\psi(h) = 1/(1-h)$ for $h \in U$ and $\chi = \exp$. Therefore $f = c \cdot \exp$ on $I_0$ with some real $c$. By Proposition 3 we deduce that $c \ge 0$. Finally, by letting $\varepsilon \to 0$ we obtain $f = c \cdot \exp$ on $I$, as claimed. □

We will terminate the chapter with a one more functional inequality, which is related to the results of M. Kuczma mentioned at the beginning of Sect. 2. It is straightforward to note that

$$(1+y)\exp x \le \exp x \cdot \exp y = \exp(x+y) \tag{20}$$

for all $x, y \in \mathbb{R}^+$. Therefore, we are asking about the general solution of the corresponding functional inequality:

$$(1+y)f(x) \le f(x+y). \tag{21}$$

**Theorem 3** *Assume that* $I \subset \mathbb{R}^+$ *is a nonvoid open interval. Then* $f: I \to \mathbb{R}^+$ *satisfies* (21) *for all* $x, y \in I$ *if and only if there exists a constant* $c \geq 0$ *such that* $f(x) = c\exp(x)$ *for each* $x \in I$.

*Proof* The "if" part follows from (20); we will prove the "only if" part. Note that (21) is a special case of (17) (for the map $f$ replaced by $-f$). Moreover, it is clear that (16) is satisfied by maps $\psi(h) = 1 + h$ and $\chi(h) = \exp(h)$ for $h$ belonging to some interval $U$ such that 0 is an internal point of $U$. Therefore we obtain the desired representation $f(x) = ce^x$ for all $x \in I$ with some $c \geq 0$. □

*Example 2* The assumption $I \subset \mathbb{R}^+$ in Theorem 3 is essential. Indeed, if one take $I = (-2, -1)$, then it is clear that each function $f: I \to \mathbb{R}^+$ solves (21). Moreover, if $I \subset \mathbb{R}^+$, then each function $f: I \to \mathbb{R}$ which is constant and equal to a negative number solves (20) as well.

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# On Measurable Functions Satisfying Multiplicative Type Functional Equations Almost Everywhere

**Antal Járai, Károly Lajkó, and Fruzsina Mészáros**

**Abstract** Using the so-called "almost" variant of a well-known generalization of Steinhaus' theorem, first we prove a general result on the multiplicative type functional equation (3), then we solve functional equations (1) and (2) originated from statistics under such conditions.

**Keywords** Multiplicative type functional equations satisfied a. e. · Density function solutions

**Mathematics Subject Classification** 39B22 · 62E10

## 1 Introduction

Functional equations

$$h_1\big((\alpha+y)x\big)(\alpha+y)f_Y(y)=h_2\big((\beta+x)y\big)(\beta+x)f_X(x), \tag{1}$$

and

$$h_1\left(\frac{x}{\lambda_1(\alpha+y)}\right)\frac{f_Y(y)}{\lambda_1(\alpha+y)}=h_2\left(\frac{y}{\lambda_2(\beta+x)}\right)\frac{f_X(x)}{\lambda_2(\beta+x)}, \tag{2}$$

satisfying almost everywhere on $\mathbb{R}_+^2$ (where $\mathbb{R}_+$ is the set of positive reals), have role in characterizing joint distributions from conditional distributions (see [1, 2, 6, 7]).

Suppose only that the unknown functions in (1) and (2) are density functions of some random variables (i.e. nonnegative and Lebegue integrable with integral 1). Does it follow that they are positive almost everywhere on $\mathbb{R}_+$?

Using a generalization of Steinhaus' theorem (see Corollary 1), we can give an affirmative answer to this question.

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## 2 Some General Results

Let $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. If $r > 0$, consider the closed ball centered at $x$ and having radius $r$. Let us consider the ratio of the Lebesgue outer measure of the intersection of this ball with $A$ divided by the Lebesgue measure of the ball. The density of $A$ at $x$ is the limit (if it exists) of this ratio as $r \to 0$; the upper and the lower densities are the limit superior and the limit inferior, respectively.

**Definition** A point $x \in \mathbb{R}$ is called an *almost inner point* of a set $A \subset \mathbb{R}$, if there is an open interval $(a, b)$ containing $x$ such that $(a, b) \setminus A$ has measure zero.

We need the following corollary of the Main Theorem from the paper [5]

**Corollary 1** *Let $D$ be an open subset of $\mathbb{R} \times \mathbb{R}$, let $N$ be a zero set of $\mathbb{R} \times \mathbb{R}$ and $F : (x, y) \to F(x, y)$ be a continuously differentiable mapping of $D$ into $\mathbb{R}$. Let $A, B \subset \mathbb{R}$ and suppose that $B$ is Lebesgue measurable. If $(a, b) \in D$,*

$$\frac{\partial F}{\partial x}(a, b) \neq 0, \qquad \frac{\partial F}{\partial y}(a, b) \neq 0,$$

*$A$ has density 1 in the point $a$ and $B$ has density 1 in the point $b$, then $F(a, b)$ is an almost inner point of $F((A \times B) \setminus N)$.*

Note that this is an "almost" variant of a well-known generalization of Steinhaus' theorem which states that $F(a, b)$ is an inner point of $F((A \times B))$: see [4, Corollary 3.12].

*Remark 1* The previous corollary may be stated in the following global form: If

$$\frac{\partial F}{\partial x}(x, y) \neq 0, \qquad \frac{\partial F}{\partial y}(x, y) \neq 0$$

for all $(x, y) \in D$ and $A \times B \subset D$, where $A$ and $B$ has positive Lebesgue outer measure and $B$ is Lebesgue measurable, then for any 2-dimensional zero set $N$ we have that $F((A \times B) \setminus N)$ has an almost inner point.

Note again that this is an "almost" variant of a well-known generalization of Steinhaus' theorem which states that $F((A \times B))$ contains an inner point: see [4, Remark 3.13].

**Theorem 1** *For any set $A \subset \mathbb{R}$ the set $V$ of almost inner points of $A$ is an open set. This is the maximal open set for which $V \setminus A$ has measure zero.*

*Proof* For an arbitrary almost inner point of $x$ let us choose rational points $a_x < x$ and $b_x > x$ such that $(a_x, b_x) \setminus A$ is a zero set. The union of the intervals $(a_x, b_x)$ for all almost inner points of $A$ is an open set containing $V$. But each point of any interval $(a_x, b_x)$ is an almost inner point of $A$, hence this union is $V$. To prove that $V \setminus A$ is a null set let us observe that the union is countable, hence $V \setminus A$ is a countable union of zero sets $(a_x, b_x) \setminus A$. Finally, if $U \supset V$, and there exists an $x \in U$ for which $x \notin V$, then $x$ is not an almost inner point of $A$, hence $U \setminus A$ cannot be a zero set. □

**Definition** A set $A \subset \mathbb{R}$ is called *almost open*, if any point of $A$ is an almost inner point of $A$.

Clearly the union of almost open sets is almost open. By the previous theorem taking $V$ as the set of all almost inner points of the almost open set $A$ we have that $V$ is an open set, $A \subset V$ and $N := V \setminus A$ is a zero set, hence $A$ can be represented as $A = V \setminus N$. Trivially, if a set $A$ can be represented as $V \setminus N$, where $V$ is open and $N$ is a zero set, then $A$ is almost open. Such a representation is not unique: for example, if $A$ is the complement of the Cantor set $C$, then it can be represented as $A = A \setminus \emptyset$ and as $A = \mathbb{R} \setminus C$.

Let us consider the functional equation

$$f_1(x) f_2(y) = g_1\big(G_1(x, y)\big) g_2\big(G_2(x, y)\big) h(x, y) \tag{3}$$

with unknown functions $f_1 : X \to \mathbb{C}$, $f_2 : Y \to \mathbb{C}$, $g_1 : U \to \mathbb{C}$, $g_2 : V \to \mathbb{C}$ and given functions $G_1$, $G_2$ and $h$ satisfied for almost all pairs $(x, y) \in X \times Y$ (with respect to the plane Lebesgue measure), where $X, Y, U, V \subset \mathbb{R}$ are nonvoid open intervals, $h$ is nowhere zero on $X \times Y$, the mapping $(x, y) \mapsto G(x, y) := (G_1(x, y), G_2(x, y))$ is a $\mathcal{C}^1$-diffeomorphism of $X \times Y$ onto $U \times V$ with inverse $(u, v) \mapsto F(u, v) := (F_1(u, v), F_2(u, v))$, and all the partial derivatives

$$\frac{\partial G_1}{\partial x}(x, y), \quad \frac{\partial G_1}{\partial y}(x, y), \quad \frac{\partial G_2}{\partial x}(x, y), \quad \frac{\partial G_2}{\partial y}(x, y)$$

and

$$\frac{\partial F_1}{\partial u}(u, v), \quad \frac{\partial F_1}{\partial v}(u, v), \quad \frac{\partial F_2}{\partial u}(u, v), \quad \frac{\partial F_2}{\partial v}(u, v)$$

vanish nowhere on their domain. Let us observe that substituting $u = G_1(x, y)$ and $v = G_2(x, y)$, we obtain the functional equation

$$f_1\big(F_1(u, v)\big) f_2\big(F_2(u, v)\big) = g_1(u) g_2(v) h\big(F_1(u, v), F_2(u, v)\big) \tag{4}$$

satisfied for almost all $(u, v) \in U \times V$; indeed, if (3) is satisfied for all $(x, y) \in X \times Y \setminus N$, where $N \subset X \times Y$ has plane measure zero, then (4) is satisfied for all $(u, v) \in U \times V \setminus M$, where $M = G(N)$ and $M$ has plane measure zero because $G$ is a diffeomorphism.

We shall prove the following theorem:

**Theorem 2** *Suppose that the measurable functions $f_1$, $f_2$, $g_1$, $g_2$ satisfy the functional equation* (3) *almost everywhere. Then either one of the functions $f_1$ and $f_2$*

*and one of the functions $g_1$ and $g_2$ are zero almost everywhere or all of them are almost everywhere nonzero.*

*Proof* We have to prove that if on one side none of the functions are almost everywhere zero then all the functions are almost everywhere nonzero. This will be done in five steps. We will use the notation

$$\{f_1 = 0\} = \{x \in X : f_1(x) = 0\}$$

and the analogous notation $\{f_1 \neq 0\}$, $\{f_2 = 0\}$, etc.

I. First we will prove that the sets $\{f_1 \neq 0\}$, $\{f_2 \neq 0\}$ and $\{g_1 \neq 0\}$, $\{g_2 \neq 0\}$ contain almost inner points. If, for example, $f_1$ and $f_2$ are not almost everywhere zero, then they are nonzero on a Lebesgue measurable set with positive measure. From Remark 1 follows that $\{g_1 \neq 0\}$ and $\{g_2 \neq 0\}$ contain an almost inner point. Now applying the same theorem again for (4) we get that the sets $\{f_1 \neq 0\}$ and $\{f_2 \neq 0\}$ contain almost inner points, too.

II. Let us observe that if $x$ is fixed, then the mappings

$$y \mapsto G_1(x, y) \quad \text{and} \quad y \mapsto G_2(x, y)$$

map almost open set onto almost open set. Indeed, their derivatives are continuous and nowhere zero, hence both mappings are strictly monotonic and map zero sets onto zero sets. The same is true for the mappings

$$\begin{aligned} x &\mapsto G_1(x, y), & x &\mapsto G_2(x, y), \\ v &\mapsto F_1(u, v), & v &\mapsto F_2(u, v), \\ u &\mapsto F_1(u, v), & u &\mapsto F_2(u, v). \end{aligned}$$

III. We prove that if the set $\{f_1 \neq 0\}$ has positive Lebesgue upper density at $x_0$, then $x_0$ is an almost inner point of $\{f_1 \neq 0\}$. Let us choose a sequence $x_n \to x_0$ such that for all $x_n$ we have $x_n \in \{f_1 \neq 0\}$ and the functional equation (3) is satisfied for almost all $y \in Y$, and let $y_0$ be an almost inner point of $\{f_2 \neq 0\}$, say, suppose that $(y_0 - \varepsilon, y_0 + \varepsilon)$ is almost contained in $\{f_2 \neq 0\}$. Let $u_0 = G_1(x_0, y_0)$ and $v_0 = G_2(x_0, y_0)$. The interval $(y_0 - \varepsilon, y_0 + \varepsilon)$ is mapped by $y \mapsto G_1(x_n, y)$ onto an open interval. If $n$ is large enough, then there exists a positive $\delta$ such that $(u_0 - \delta, u_0 + \delta)$ is contained in this interval, and hence $g_1$ is almost everywhere nonzero on $(u_0 - \delta, u_0 + \delta)$. Similarly, if $n$ is large enough, then there exists a positive $\eta$ such that $(v_0 - \eta, v_0 + \eta)$ is contained in the image of $(y_0 - \varepsilon, y_0 + \varepsilon)$ by the mapping $y \mapsto G_2(x_n, y)$, and hence $g_2$ is almost everywhere nonzero on $(v_0 - \eta, v_0 + \eta)$. Now choosing smaller $\delta > 0$ and $\eta > 0$ if necessary, we may suppose that $|u - u_0| < \delta$ and $|v - v_0| < \eta$ imply that $F_2(u, v) \in (y_0 - \varepsilon, y_0 + \varepsilon)$. Let us choose $u_n \to u_0$ such that $u_n \in \{g_1 \neq 0\}$ and for all $u_n$ (4) is satisfied for almost all $v \in V$. The mapping $v \mapsto F_1(u_n, v)$ maps the interval $(v_0 - \eta, v_0 + \eta)$ onto a subinterval of $X$. If $n$ is large enough, then there exists a $\vartheta > 0$ such that this interval covers the interval $(x_0 - \vartheta, x_0 + \vartheta)$. This means, that $(x_0 - \vartheta, x_0 + \vartheta)$ is an almost subset of $\{f_1 \neq 0\}$. We obtain similarly, that if the set $\{f_2 \neq 0\}$, $\{g_1 \neq 0\}$ or $\{g_2 \neq 0\}$ has positive upper Lebesgue density at some point, then this point is an almost inner point of the set $\{f_2 \neq 0\}$, $\{g_1 \neq 0\}$ or $\{g_2 \neq 0\}$, respectively.

IV. Suppose that the upper Lebesgue density of the set $\{f_1 \neq 0\}$ is zero at some point $x_0$, i.e., that $x_0$ is a density point of $\{f_1 = 0\}$. We prove that then $x_0$ is an almost inner point of the set $\{f_1 = 0\}$. Let us choose an almost inner point $y_0$ of the set $\{f_2 \neq 0\}$, say, suppose that $(y_0 - \varepsilon, y_0 + \varepsilon)$ is almost contained in $\{f_2 \neq 0\}$. Then $(x_0, y_0)$ is a density point of the plane set $\{f_1 = 0\} \times \{f_2 \neq 0\}$. Let $u_0 = G_1(x_0, y_0)$ and $v_0 = G_2(x_0, y_0)$. The point $(u_0, v_0)$ is a density point of the set $G(\{f_1 = 0\} \times \{f_2 \neq 0\})$, hence of the set $G(\{f_1 = 0\} \times \{f_2 \neq 0\} \setminus N)$, too, $N$ is the plane zero set where (3) is not satisfied.

We shall prove that the upper density of $\{g_1 = 0\}$ is positive at $u_0$ or the upper density of the set $\{g_2 = 0\}$ is positive at $v_0$. If this was not true, then the density of the set $\{g_1 = 0\} \times V$ and the density of the set $U \times \{g_2 = 0\}$ would be zero at $(u_0, v_0)$ and hence the density of the set $(\{g_1 = 0\} \times V) \cup (U \times \{g_2 = 0\}) \cup M$ would be zero at $(u_0, v_0)$, where $M = G(N)$. But this is impossible: choosing a point $(u, v) \in G(\{f_1 = 0\} \times \{f_2 \neq 0\} \setminus N)$ for which $(u, v) \notin (\{g_1 = 0\} \times V) \cup (U \times \{g_2 = 0\}) \cup M$ the left hand side of (4) is zero and the right hand side is nonzero.

Suppose that the upper density of $\{g_1 = 0\}$ is positive at $u_0$; the other case can be similarly treated. Let us choose a sequence $u_n \in \{g_1 = 0\}$ for which $u_n \to u_0$ and for each $u_n$ (4) is satisfied for almost all $v \in V$. Let us choose $\delta > 0$ and $\eta > 0$ such that $|u - u_0| < \delta$ and $|v - v_0| < \eta$ imply $F_2(u, v) \in (y_0 - \varepsilon, y_0 + \varepsilon)$. The mapping $v \mapsto F_1(u_n, v)$ maps the interval $(v_0 - \eta, v_0 + \eta)$ onto a subinterval of $X$. If $n$ is large enough, then there exists a $\vartheta > 0$ such that this interval covers the interval $(x_0 - \vartheta, x_0 + \vartheta)$. But for almost all $v \in (v_0 - \eta, v_0 + \eta)$, (4) is satisfied, the right hand side is zero, and hence the left hand side has to be zero, too. But for almost all $v \in (v_0 - \eta, v_0 + \eta)$ we have $F_2(u_n, v) \in \{f_2 \neq 0\}$, hence $F_1(u_n, v) \in \{f_1 = 0\}$. This means that $(x_0 - \vartheta, x_0 + \vartheta)$ is almost contained in $\{f_1 = 0\}$, i.e., $x_0$ is an almost inner point of $\{f_1 = 0\}$.

V. Now the statement of the theorem follows: Each point of $X$ is an almost inner point of $\{f_1 \neq 0\}$ or an almost inner point of $\{f_1 = 0\}$. The sets of almost inner points of these sets are disjoint and both are open, hence one of them has to be empty, because $X$ is connected; this can be only the set of almost inner points of $\{f_1 = 0\}$. Similarly we obtain that $f_2$, $g_1$ and $g_2$ are also almost everywhere nonzero. □

*Remark 2* If the measurable functions $f_1$, $f_2$, $g_1$, $g_2$ satisfy functional equation (3) everywhere and on one side none of the functions are almost everywhere zero, then all the functions are everywhere nonzero.

The proof of this remark is similar but much simpler than the proof of Theorem 2 (compare with the proof of Theorem 23.6 in [4]): As step I we get from the Steinhaus-type theorem that the sets $\{f_1 \neq 0\}$, $\{f_2 \neq 0\}$, $\{g_1 \neq 0\}$ and $\{g_2 \neq 0\}$ have inner points. As step III we obtain that $\{f_1 \neq 0\}$ is an open set, because if $f_1(x_0) \neq 0$, then choosing an inner point $y_0$ of $\{f_2 \neq 0\}$ and varying $y$ around $y_0$ by step II we obtain that $u_0 = F_1(x_0, y_0)$ and $v_0 = F_2(x_0, y_0)$ are inner points of $\{g_1 \neq 0\}$ and

$\{g_2 \neq 0\}$, respectively. Now fixing, say, $u_0$ and varying $v$ around $v_0$ we obtain that $x_0 = G_1(u_0, v_0)$ is an inner point of $\{f_1 \neq 0\}$. Similarly we obtain that $\{f_2 \neq 0\}$, $\{g_1 \neq 0\}$ and $\{g_2 \neq 0\}$ are also open. As step IV we prove that the sets $\{f_1 = 0\}$, $\{f_2 = 0\}$, $\{g_1 = 0\}$ and $\{g_2 = 0\}$ are also open. Let, for example, $f_1(x_0) = 0$. Let us choose an $y_0 \in \{f_2 \neq 0\}$. By the functional equation for $u_0 = F_1(x_0, y_0)$ and $v_0 = F_2(x_0, y_0)$ we have $g_1(u_0)g_2(v_0) = 0$. Suppose that $g_1(u_0) = 0$. Varying $v$ around $v_0$ the values $y = G_2(u_0, v)$ remains close to $y_0$ and hence $f_1(x) = 0$ where $x = G_1(u_0, v)$ runs in a neighborhood of $x_0$. Hence $x_0$ is an inner point of $\{f_1 = 0\}$.

## 3 Results for (1)

Now let us consider functional equation

$$h_1\big((\alpha + y)x\big)(\alpha + y) f_Y(y) = h_2\big((\beta + x)y\big)(\beta + x) f_X(x) \tag{1}$$

satisfying for almost all $(x, y) \in \mathbb{R}^2_+$, where the unknown functions $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ are density functions of some random variables and $\alpha, \beta$ are nonnegative constants with $\alpha^2 + \beta^2 \neq 0$. Then $h_1, h_2, f_X, f_Y$ are nonnegative, such that they are positive on some Lebesgue measurable sets of positive Lebesgue measure. Using Theorem 2 we prove the following result.

**Theorem 3** *Let $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ be nonnegative measurable functions satisfying (1) for almost all $(x, y) \in \mathbb{R}^2_+$, such that they are positive on some Lebesgue measurable subsets of $\mathbb{R}_+$ of positive Lebesgue measure. Then $h_1$, $h_2$, $f_X$, $f_Y$ are positive almost everywhere on $\mathbb{R}_+$.*

*Proof* Let us write $\frac{x}{\alpha+y}$ instead of $x$ in (1), hence we get equation

$$h_1(x) f_Y(y) = h_2\left(\left(\beta + \frac{x}{\alpha + y}\right)y\right) f_X\left(\frac{x}{\alpha + y}\right)\frac{\beta(\alpha + y) + x}{(\alpha + y)^2} \tag{5}$$

for almost all $(x, y) \in \mathbb{R}^2_+$, i.e. functional equation (3) for the unknown functions $f_1 = h_1$, $f_2 = f_Y$, $g_1 = h_2$, $g_2 = f_X$ and for the given functions $G_1(x, y) = (\beta + \frac{x}{\alpha+y})y$, $G_2(x, y) = \frac{x}{\alpha+y}$, $h(x, y) = \frac{\beta(\alpha+y)+x}{(\alpha+y)^2}$ $((x, y) \in \mathbb{R}^2_+)$. Observe that $h$ is nowhere zero on $\mathbb{R}^2_+$ and the mapping

$$(x, y) \to G(x, y) = \big(G_1(x, y), G_2(x, y)\big) = \left(\left(\beta + \frac{x}{\alpha + y}\right)y, \frac{x}{\alpha + y}\right)$$

is $\mathcal{C}^1$-diffeomorphism of $\mathbb{R}^2_+$ onto $\mathbb{R}^2_+$ with inverse

$$(u, v) \to F(u, v) = \big(F_1(u, v), F_2(u, v)\big) = \left(\left(\alpha + \frac{u}{\beta + v}\right)v, \frac{u}{\beta + v}\right).$$

All the partial derivatives

$$\frac{\partial G_1}{\partial x} = \frac{y}{\alpha + y}, \quad \frac{\partial G_1}{\partial y} = \beta + \frac{\alpha x}{(\alpha + y)^2}, \quad \frac{\partial G_2}{\partial x} = \frac{1}{\alpha + y}, \quad \frac{\partial G_2}{\partial y} = -\frac{x}{(\alpha + y)^2},$$
$$\frac{\partial F_1}{\partial u} = \frac{v}{\beta + v}, \quad \frac{\partial F_1}{\partial v} = \alpha + \frac{\beta u}{(\beta + v)^2}, \quad \frac{\partial F_2}{\partial u} = \frac{1}{\beta + v}, \quad \frac{\partial F_2}{\partial v} = -\frac{u}{(\beta + v)^2}$$

vanish nowhere on $\mathbb{R}_+^2$.

Substituting $u = (\beta + \frac{x}{\alpha+y})y$, $v = \frac{x}{\alpha+y}$ in (5), we obtain the functional equation

$$h_2(u) f_X(v) = h_1\left(\left(\alpha + \frac{u}{\beta + v}\right)v\right) f_Y\left(\frac{u}{\beta + v}\right) \frac{\alpha(\beta + v) + u}{(\beta + v)^2} \tag{6}$$

for almost all $(u, v) \in \mathbb{R}_+^2$; indeed if (5) is satisfied for all $(x, y) \in \mathbb{R}_+^2 \setminus N$, when $N \subset \mathbb{R}_+^2$ has plane measure zero, then (6) is satisfied for all $(u, v) \in \mathbb{R}_+^2 \setminus G(N)$ and $G(N)$ has plane measure zero, because $G$ is a diffeomorphism. So functional equation (4) is satisfied with functions $h_1 = f_1$, $f_Y = f_2$, $h_2 = g_1$, $f_X = g_2$, $F_1(u, v) = (\alpha + \frac{u}{\beta+v})v$, $F_2(u, v) = \frac{u}{\beta+v}$, $h(u, v) = \frac{\alpha(\beta+v)+u}{(\beta+v)^2}$. All assumptions of Theorem 2 are satisfied and further none of the functions are almost everywhere zero, then all the functions are almost everywhere nonzero. Thus the nonnegativity of functions implies that $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ are almost everywhere positive. □

*Remark 3* Theorem 3 was proved in [8] and [9] by the help of a generalized Steinhaus-type theorem and by interval expansions.

*Remark 4* If the measurable functions $f_X, f_Y, h_1, h_2$ satisfy functional equation (1) everywhere and on one side none of the functions are almost everywhere zero, then all the functions are everywhere nonzero.

Using Theorem 3 and a general result of A. Járai [3], similarly to the proof of Theorem 2 in [7], one can prove the following result.

**Theorem 4** *Let $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ be nonnegative measurable functions, satisfying (1) for almost all $(x, y) \in \mathbb{R}_+^2$ such that they are positive on some Lebesgue measurable subsets of $\mathbb{R}_+$ of positive measure. Then there exist unique continuous functions $\tilde{h}_1, \tilde{h}_2, \tilde{f}_X, \tilde{f}_Y : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\tilde{h}_1 = h_1$, $\tilde{h}_2 = h_2$, $\tilde{f}_X = f_X$ and $\tilde{f}_Y = f_Y$ almost everywhere on $\mathbb{R}_+$, and if $h_1, h_2, f_X, f_Y$ are replaced by $\tilde{h}_1, \tilde{h}_2, \tilde{f}_X, \tilde{f}_Y$, respectively, then (1) is satisfied everywhere on $\mathbb{R}_+^2$.*

*Proof* Theorem 3 shows that functions $h_1, h_2, f_X, f_Y$ are positive almost everywhere on $\mathbb{R}_+$.

First we prove that there exists a unique continuous function $\tilde{h}_1$ which is equal to $h_1$ almost everywhere on $\mathbb{R}_+$ and replacing $h_1$ by $\tilde{h}_1$, (1) is satisfied almost everywhere.

With the substitution $t = (\alpha + y)x$ we get from (1) the equation

$$h_1(t) = \frac{h_2((\beta + \frac{t}{\alpha+y})y)(\beta + \frac{t}{\alpha+y}) f_X(\frac{t}{\alpha+y})}{(\alpha + y) f_Y(y)} \tag{7}$$

which is satisfied for almost all $(t, y) \in \mathbb{R}_+^2$. By Fubini's Theorem it follows that there exists $T' \subseteq \mathbb{R}_+$ of full measure such that for all $t \in T'$ (7) is satisfied for almost every $y \in \{y \in \mathbb{R}_+ | (t, y) \in \mathbb{R}_+^2\} = \mathbb{R}_+$.

Let us define the functions $g_1, g_2, g_3, h$ in the following way:

$$g_1(t, y) = \left(\beta + \frac{t}{\alpha + y}\right) y, \qquad g_2(t, y) = \frac{t}{\alpha + y},$$
$$g_3(t, y) = y, \qquad h(t, y, z_1, z_2, z_3) = \frac{z_1 z_2}{z_3},$$

and let us now apply a theorem of Járai (see [3, Theorem 3]) to (7) with the following casting:

$$h_1(t) = f(t), \qquad h_2(t) = f_1(t), \qquad (\beta + t) f_X(t) = f_2(t), \qquad (\alpha + t) f_Y(t) = f_3(t),$$
$$Z = Z_i = \mathbb{R}_+, \quad T = Y = X_i = \mathbb{R}_+ \quad (i = 1, 2, 3).$$

One can easily verify that all assumptions of Járai's Theorem are satisfied, thus we get that there exists a unique continuous function $\tilde{h}_1 : \mathbb{R}_+ \to \mathbb{R}$ which is almost everywhere equal to $h_1$ on $\mathbb{R}_+$ and $\tilde{h}_1, h_2, f_X, f_Y$ satisfy (1) almost everywhere, which is equivalent to the equation

$$\tilde{h}_1\big((\alpha + y)x\big)(\alpha + y) f_Y(y) = h_2\big((\beta + x)y\big)(\beta + x) f_X(x) \tag{8}$$

for almost all $(x, y) \in \mathbb{R}_+^2$. Furthermore, $\tilde{h}_1$ is positive for almost all $x \in \mathbb{R}_+$.

By a similar argument we can prove the same for the function $h_2$, $f_X$ and $f_Y$, i.e. there exist continuous functions $\tilde{h}_2 : \mathbb{R}_+ \to \mathbb{R}$, $\tilde{f}_X : \mathbb{R}_+ \to \mathbb{R}$ and $\tilde{f}_Y : \mathbb{R}_+ \to \mathbb{R}$ which are almost everywhere equal to $h_2$, $f_X$ and $f_Y$ on $\mathbb{R}_+$, respectively, and the functional equation

$$\tilde{h}_1\big((\alpha + y)x\big)(\alpha + y) \tilde{f}_Y(y) = \tilde{h}_2\big((\beta + x)y\big)(\beta + x) \tilde{f}_X(x) \tag{9}$$

is satisfied almost everywhere on $\mathbb{R}_+^2$.

Both side of (9) define continuous functions on $\mathbb{R}_+^2$, which are equal to each other on a dense subset of $\mathbb{R}_+^2$, therefore we obtain that (9) is satisfied everywhere on $\mathbb{R}_+^2$.

Applying Remark 4 for (9), one can show that if the nonnegative continuous functions $\tilde{h}_1, \tilde{h}_2, \tilde{f}_X, \tilde{f}_Y : \mathbb{R}_+ \to \mathbb{R}$ satisfy functional equation (9) for all $(x, y) \in \mathbb{R}_+^2$, such that they are positive almost everywhere on $\mathbb{R}_+$, then they are positive everywhere on $\mathbb{R}_+$. □

Therefore it suffices to determine the general positive and continuous solutions of (9) for all $(x, y) \in \mathbb{R}_+^2$, apply Theorem 3 in [7], and thus, as an immediate consequence of Theorems 3 and 4 we get the following result for the density function solutions of (1).

**Corollary 2** *Let* $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ *be nonnegative measurable functions, satisfying* (1) *for almost all* $(x, y) \in \mathbb{R}^2_+$ *such that they are positive on some Lebesgue measurable sets of positive Lebesgue measure. Then there exist constants* $c_1, c_2, \gamma, \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}$, *with* $\delta_1 + \delta_3 = \delta_2 + \delta_4$ *such that*

$$\begin{aligned} h_1(x) &= x^{c_1} \exp(\gamma x + \delta_1) \quad a.a.\ x \in \mathbb{R}_+, \\ h_2(x) &= x^{c_2} \exp(\gamma x + \delta_2) \quad a.a.\ x \in \mathbb{R}_+, \\ f_Y(y) &= \frac{y^{c_2}}{(y+\alpha)^{c_1+1}} \exp(\gamma\beta y + \delta_3) \quad a.a.\ y \in \mathbb{R}_+, \\ f_X(x) &= \frac{x^{c_1}}{(x+\beta)^{c_2+1}} \exp(\gamma\alpha x + \delta_4) \quad a.a.\ x \in \mathbb{R}_+. \end{aligned}$$

*Remark 5* Corollary 2 implies the so-called density function solutions if $-\gamma$, $-(c_1 + 1)$, $-(c_2 + 1) \in \mathbb{R}_+$ holds. Further Corollary 2 shows that $h_1$ and $h_2$ are gamma densities with parameters $-\gamma$, $c_1 + 1$ and $-\gamma$, $c_2 + 1$, respectively.

## 4 Results for (2)

Now we consider the functional equation

$$h_1\left(\frac{x}{\lambda_1(\alpha+y)}\right)\frac{1}{\lambda_1(\alpha+y)} f_Y(y) = h_2\left(\frac{y}{\lambda_2(\beta+x)}\right)\frac{1}{\lambda_2(\beta+x)} f_X(x) \qquad (2)$$

for almost all $(x, y) \in \mathbb{R}^2_+$, where the unknown functions $h_1, h_2\ f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ are also density functions of some random variables, $\lambda_1, \lambda_2$ are positive constants and $\alpha, \beta$ are nonnegative constants with $\alpha^2 + \beta^2 \neq 0$. Then $h_1, h_2, f_X, f_Y$ are nonnegative, such that they are positive on some Lebesgue measurable sets of positive Lebesgue measure.

Using again Theorem 2 we prove

**Theorem 5** *Let* $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ *be nonnegative measurable functions satisfying* (2) *for almost all* $(x, y) \in \mathbb{R}^2_+$, *such that they are positive on some Lebesgue measurable subsets of* $\mathbb{R}_+$ *of positive Lebesgue measure. Then* $h_1, h_2, f_X, f_Y$ *are positive almost everywhere on* $\mathbb{R}_+$.

*Proof* Replace $x$ by $\lambda_1(\alpha + y)x$ in (2). Then an easy calculation shows that $h_1, h_2, f_X, f_Y$ satisfy functional equation

$$h_1(x) f_Y(y) = h_2\left(\frac{y}{\lambda_2(\beta+\lambda_1(\alpha+y)x)}\right) f_X\big(\lambda_1(\alpha+y)x\big)\frac{\lambda_1}{\lambda_2}\frac{\alpha+y}{\beta+\lambda_1(\alpha+y)x} \qquad (10)$$

for almost all $(x, y) \in \mathbb{R}_+^2$, i.e. functional equation (3) for the unknown functions $f_1 = h_1$, $f_2 = f_Y$, $g_1 = h_2$, $g_2 = f_X$ and for the given functions $G_1(x, y) = \frac{1}{\lambda_2}\frac{y}{\beta+\lambda_1(\alpha+y)x}$, $G_2(x, y) = \lambda_1(\alpha + y)x$, $h(x, y) = \frac{\lambda_1}{\lambda_2}\frac{\alpha+y}{\beta+\lambda_1(\alpha+y)x}$. Clearly $h$ is nowhere zero on $\mathbb{R}_+^2$ and the mapping

$$(x, y) \to G(x, y) = \big(G_1(x, y), G_2(x, y)\big) = \left(\frac{1}{\lambda_2}\frac{y}{\beta + \lambda_1(\alpha + y)x}, \lambda_1(\alpha + y)x\right)$$

is $\mathcal{C}^1$-diffeomorphism of $\mathbb{R}_+^2$ onto $\mathbb{R}_+^2$ with inverse

$$(u, v) \to F(u, v) = \big(F_1(u, v), F_2(u, v)\big) = \left(\frac{1}{\lambda_1}\frac{v}{\alpha + \lambda_2(\beta + v)u}, \lambda_2(\beta + v)u\right).$$

All the partial derivatives

$$\frac{\partial G_1}{\partial x} = -\frac{\lambda_1}{\lambda_2}\frac{(\alpha + y)y}{(\beta + \lambda_1(\alpha + y)x)^2}, \qquad \frac{\partial G_1}{\partial y} = \frac{1}{\lambda_2}\frac{\beta + \lambda_1\alpha x}{(\beta + \lambda_1(\alpha + y)x)^2},$$
$$\frac{\partial G_2}{\partial x} = \lambda_1(\alpha + y), \qquad \frac{\partial G_2}{\partial y} = \lambda_1 x,$$
$$\frac{\partial F_1}{\partial u} = -\frac{\lambda_2}{\lambda_1}\frac{(\beta + v)v}{(\alpha + \lambda_2(\beta + v)u)^2}, \qquad \frac{\partial F_1}{\partial v} = \frac{1}{\lambda_1}\frac{\alpha + \lambda_2\beta u}{(\alpha + \lambda_2(\beta + v)u)^2},$$
$$\frac{\partial F_2}{\partial u} = \lambda_2(\beta + v), \qquad \frac{\partial F_2}{\partial v} = \lambda_2 u$$

vanish nowhere on $\mathbb{R}_+^2$.

By the transformation

$$u = \frac{1}{\lambda_2}\frac{y}{\beta + \lambda_1(\alpha + y)x} = G_1(x, y), \qquad v = \lambda_1(\alpha + y)x = G_2(x, y),$$

we get from (10) the functional equation

$$h_2(u) f_X(v) = h_1\left(\frac{1}{\lambda_1}\frac{v}{\alpha + \lambda_2(\beta + v)u}\right) f_Y\big(\lambda_2(\beta + v)u\big)\frac{\lambda_2}{\lambda_1}\frac{\beta + v}{\alpha + \lambda_2(\beta + v)u} \tag{11}$$

for almost all $(u, v) \in \mathbb{R}_+^2$; indeed if (10) is satisfied for all $(x, y) \in \mathbb{R}_+^2 \setminus N$, when $N \subset \mathbb{R}_+^2$ has plane measure zero, then (11) is satisfied for all $(u, v) \in \mathbb{R}_+^2 \setminus G(N)$ and $G(N)$ has plane measure zero, because $G$ is a diffeomorphism. So functional equation (4) is satisfied with functions $h_1 = f_1$, $f_Y = f_2$, $h_2 = g_1$, $f_X = g_2$, $F_1(u, v) = \frac{1}{\lambda_1}\frac{v}{\alpha+\lambda_2(\beta+v)u}$, $F_2(u, v) = \lambda_2(\beta + v)u$, $h(u, v) = \frac{\lambda_2}{\lambda_1}\frac{\beta+v}{\alpha+\lambda_2(\beta+v)u}$. All conditions of Theorem 2 are satisfied and further none of the functions are almost everywhere zero, then all the functions are almost everywhere nonzero. Thus the nonnegativity of functions implies that $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ are almost everywhere positive. □

*Remark 6* Theorem 5 was proved in [8] and [9] by the help of a generalized Steinhaus-type theorem and by interval expansions.

*Remark 7* If the measurable functions $f_X, f_Y, h_1, h_2$ satisfy functional equation (2) everywhere and on one side none of the functions are almost everywhere zero, then all the functions are everywhere nonzero.

Using previous theorems and Theorem 3 from [3] one can prove, similarly to the proof of Theorem 4, the following result.

**Theorem 6** *Let $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ be nonnegative measurable functions, satisfying (2) for almost all $(x, y) \in \mathbb{R}^2_+$ such that they are positive on some Lebesgue measurable subsets of $\mathbb{R}_+$ of positive measure. Then there exist unique continuous functions $\tilde{h}_1, \tilde{h}_2, \tilde{f}_X, \tilde{f}_Y : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\tilde{h}_1 = h_1$, $\tilde{h}_2 = h_2$, $\tilde{f}_X = f_X$ and $\tilde{f}_Y = f_Y$ almost everywhere on $\mathbb{R}_+$, and if $h_1, h_2, f_X, f_Y$ are replaced by $\tilde{h}_1, \tilde{h}_2, \tilde{f}_X, \tilde{f}_Y$, respectively, then (2) is satisfied everywhere on $\mathbb{R}^2_+$.*

Therefore it suffices to determine the general positive and continuous solutions of (2) for all $(x, y) \in \mathbb{R}^2_+$, and thus, as an immediate consequence of Theorems 5, 4 and of Theorems 7, 8, 9 in [7], we get the following result for the density function solutions of (2).

**Corollary 3** *If the nonnegative measurable functions $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ satisfy (2) for almost all $(x, y) \in \mathbb{R}^2_+$ and they are positive on some Lebesgue measurable sets of positive Lebesgue measure, then in case $\alpha > 0$, $\beta > 0$*

$$h_1(x) = e^{-d_2}(\lambda_1 x)^{c_2}(\beta + \lambda_1 \alpha x)^{-\frac{\gamma}{\alpha\beta}} \quad a.a.\ x \in \mathbb{R}_+,$$

$$h_2(x) = e^{-d_1}(\lambda_2 x)^{c_1}(\alpha + \lambda_2 \beta x)^{-\frac{\gamma}{\alpha\beta}} \quad a.a.\ x \in \mathbb{R}_+,$$

$$f_X(x) = e^{d_4}\lambda_2 x^{c_2}(x + \beta)^{c_1 - \frac{\gamma}{\alpha\beta} + 1} \quad a.a.\ x \in \mathbb{R}_+,$$

$$f_Y(x) = e^{d_3}\lambda_1 x^{c_1}(x + \alpha)^{c_2 - \frac{\gamma}{\alpha\beta} + 1} \quad a.a.\ x \in \mathbb{R}_+,$$

*where $c_1, c_2, \gamma, d_1, d_2, d_3, d_4 \in \mathbb{R}$ are arbitrary constants with $d_1 + d_3 = d_2 + d_4$.*

**Corollary 4** *If the nonnegative measurable functions $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ satisfy (2) for almost all $(x, y) \in \mathbb{R}^2_+$ and they are positive on some Lebesgue measurable sets of positive Lebesgue measure, then in case $\alpha = 0$, $\beta > 0$*

$$h_1(x) = e^{-d_2}(\lambda_1 x)^{c_2} e^{-\frac{\lambda_1 \gamma}{\beta^2} x} \quad a.a.\ x \in \mathbb{R}_+,$$

$$h_2(x) = e^{-d_1}(\lambda_2 x)^{c_1} e^{-\frac{\gamma}{\beta^2 \lambda_2 x}} \quad a.a.\ x \in \mathbb{R}_+,$$

$$f_X(x) = e^{d_4}\lambda_2(x+\beta)^{c_1+1}x^{c_2} \quad a.a.\ x \in \mathbb{R}_+,$$

$$f_Y(x) = e^{d_3}\lambda_1 x^{c_1+c_2+1}e^{-\frac{\gamma}{\beta x}} \quad a.a.\ x \in \mathbb{R}_+,$$

*where* $c_1, c_2, \gamma, d_1, d_2, d_3, d_4 \in \mathbb{R}$ *are arbitrary constants with* $d_1 + d_3 = d_2 + d_4$.

**Corollary 5** *If the nonnegative measurable functions* $h_1, h_2, f_X, f_Y : \mathbb{R}_+ \to \mathbb{R}$ *satisfy* (2) *for almost all* $(x, y) \in \mathbb{R}_+^2$ *and they are positive on some Lebesgue measurable sets of positive Lebesgue measure, then in case* $\alpha > 0$, $\beta = 0$

$$h_1(x) = e^{-d_2}(\lambda_1 x)^{c_2}e^{-\frac{\gamma}{\alpha^2\lambda_1 x}} \quad a.a.\ x \in \mathbb{R}_+,$$

$$h_2(x) = e^{-d_1}(\lambda_2 x)^{c_1}e^{-\frac{\gamma\lambda_2}{\alpha^2}x} \quad a.a.\ x \in \mathbb{R}_+,$$

$$f_X(x) = e^{d_4}\lambda_2 x^{c_1+c_2+1}e^{-\frac{\gamma}{\alpha x}} \quad a.a.\ x \in \mathbb{R}_+,$$

$$f_Y(x) = e^{d_3}\lambda_1(x+\alpha)^{c_2+1}x^{c_1} \quad a.a.\ x \in \mathbb{R}_+,$$

*where* $c_1, c_2, \gamma, d_1, d_2, d_3, d_4 \in \mathbb{R}$ *are arbitrary constants with* $d_1 + d_3 = d_2 + d_4$.

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# On the $L^1$ Norm of the Weighted Maximal Function of Walsh-Marcinkiewicz Kernels

**Károly Nagy**

**Abstract** The $L^1$ norm of the maximal function of Walsh-Marcinkiewicz kernel is infinite. Thus, we have to use some weight function to "pull it back" to the finite.

The main aim of this chapter is to investigate the integral of the weighted maximal function of the Walsh-Marcinkiewicz kernels. We give a necessary and sufficient conditions for that the weighted maximal function of the Walsh-Marcinkiewicz kernels is in $L^1$. For our motivation we refer the readers to the papers (Gát in Acta Acad. Paedagog. Agriensis Sect. Mat. 30, 55–66 (2003); Mező and Simon in Publ. Math. (Debr.) 71(1–2), 57–65 (2007); Nagy in JIPAM. J. Inequal. Pure Appl. Math. 9(1), 1–9 (2008)).

**Keywords** Walsh system · Marcinkiewicz kernels · Marcinkiewicz means · Maximal operator

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It is easy to see that the $L^1$ norm of $\sup_n |D_n|$ with respect to Walsh-Paley or Walsh-Kaczmarz system is infinite. Gát in [2] raised the following problem: "What happens if we apply some weight function $\alpha$? That is, on what conditions find we the inequality

$$\left\| \sup_n \left| \frac{D_n}{\alpha(n)} \right| \right\|_1 < \infty$$

valid?" He gave necessary and sufficient conditions for both rearrangement of the Walsh system. More details on Walsh-Kaczmarz system can be found in [2, 3, 11, 13]. The main aim of this chapter is to give necessary and sufficient conditions for the maximal function of Walsh-Marcinkiewicz kernels with weight function $\alpha$.

First we give a brief introduction to the theory of dyadic analysis [1, 12].

Denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, that is $\mathbf{Z}_2 = \{0, 1\}$, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbf{Z}_2$ is given in the way that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Let

$$G := \mathop{\times}_{k=0}^{\infty} \mathbf{Z}_2,$$

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$G$ is called the Walsh group. The elements of $G$ can be represented by a sequence $x = (x_0, x_1, \ldots, x_k, \ldots)$, where $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$ $(\mathbf{N} := \{0, 1, \ldots\}, \mathbf{P} := \mathbf{N}\backslash\{0\})$.

The group operation on $G$ is the coordinate-wise addition (denoted by $+$), the measure is the product measure (denoted by $\mu$) and the topology is the product topology. Consequently, $G$ is a compact Abelian group. Dyadic intervals are defined by

$$I_0(x) := G, \qquad I_n(x) := \left\{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\right\}$$

for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of $G$. Let $0 = (0 : i \in \mathbf{N}) \in G$ and $I_n := I_n(0)$ for $n \in \mathbf{N}$.

Furthermore, let $L^p(G)$ denote the usual Lebesgue spaces on $G$ (with the corresponding norm $\|.\|_p$). The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Each natural number $n$ can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i$, $n_i \in \{0, 1\}$ $(i \in \mathbf{N})$, where only a finite number of $n_i$'s different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. That is, $|n|$ is the integral part of the binary logarithm of $n$ and $2^{|n|} \leq n < 2^{|n|+1}$.

Define the Walsh-Paley functions by

$$\omega_n(x) := \prod_{k=0}^{\infty} \left(r_k(x)\right)^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

The Walsh-Paley system is $\omega := (\omega_n : n \in \mathbf{N})$ and the Walsh-Kaczmarz system is denoted by $\kappa$ (for more details see [2, 3, 9, 13]).

Define the Dirichlet and Fejér kernels by

$$D_n^{\omega} := \sum_{k=0}^{n-1} \omega_k, \qquad K_n^{\omega} := \frac{1}{n} \sum_{k=1}^{n} D_k^{\omega},$$

where $D_0^{\omega}, K_0^{\omega} := 0$.

It is known [12] that

$$D_{2^n}^{\omega}(x) = \begin{cases} 2^n, & x \in I_n \\ 0, & \text{otherwise } (n \in \mathbf{N}). \end{cases}$$

Next, we introduce some notation with respect to the theory of two-dimensional system. Let the two-dimensional Walsh group be $G \times G$ and the Dirichlet kernels, the Marcinkiewicz kernels be defined as

$$D_{n_1,n_2}^{\omega}\left(x^1, x^2\right) := D_{n_1}^{\omega}(x^1) D_{n_2}^{\omega}\left(x^2\right), \qquad \mathcal{K}_n^{\omega}\left(x^1, x^2\right) := \frac{1}{n} \sum_{k=0}^{n-1} D_{k,k}^{\omega}\left(x^1, x^2\right).$$

For more details see the work of Weisz [14] and Goginava [7] on Walsh-Marcinkiewicz means and the work of Gát [4] on Vilenkin-Marcinkiewicz means. Moreover, see the paper of Gát and Goginava [5].

Let $\alpha : [0,\infty) \to [1,\infty)$ be a monotone increasing function and define the weighted maximal function of Dirichlet kernels $D_\alpha^{\omega,*}$ and of Fejér kernels $K_\alpha^{\omega,*}$ by

$$D_\alpha^{\omega,*}(x) := \sup_{n\in\mathbf{P}} \frac{|D_n^\omega(x)|}{\alpha([\log n])}, \quad K_\alpha^{\omega,*}(x) := \sup_{n\in\mathbf{P}} \frac{|K_n^\omega(x)|}{\alpha([\log n])} \quad (x \in G),$$

where $[n]$ is the integral part of $n$. For the weighted maximal function of the Dirichlet kernels with respect to Walsh-Paley system $D_\alpha^{\omega,*}$ Gát [2] proved that $D_\alpha^{\omega,*} \in L^1$ if and only if $\sum_{A=0}^\infty \frac{1}{\alpha(A)} < \infty$. Moreover, he proved $\frac{1}{2}\sum_{A=0}^\infty \frac{1}{\alpha(A)} \le \|D_\alpha^{\omega,*}\|_1 \le 2\sum_{A=0}^\infty \frac{1}{\alpha(A)}$. For the Walsh-Kaczmarz system he showed that the situation is changed. The two conditions given by Gát are quiet different for the two rearrangements of the Walsh system.

For $\|K_\alpha^{\omega,*}\|_1$ the author [9] showed that $K_\alpha^{\omega,*} \in L^1$ if and only if $\sum_{A=0}^\infty \frac{1}{\alpha(A)} < \infty$. The author proved the inequality $\frac{1}{4}\sum_{A=0}^\infty \frac{1}{\alpha(A)} \le \|K_\alpha^{\omega,*}\|_1 \le 2\sum_{A=0}^\infty \frac{1}{\alpha(A)}$. Moreover, it was showed that, we could get so good estimation for $\|K_\alpha^{\kappa,*}\|_1$, as we have got for $\|K_\alpha^{\omega,*}\|_1$. That is, the behavior of weighted maximal function of Walsh-Kaczmarz-Fejér kernels is so good as the behavior of weighted maximal function of the Walsh-Fejér kernels.

In 2007 for (bounded and unbounded) Vilenkin systems the weighted maximal Dirichlet and Fejér kernels were investigated by Mező and Simon [8].

Now, we define the weighted maximal function of the Marcinkiewicz kernels $\mathcal{K}_\alpha^{\omega,*}$ by

$$\mathcal{K}_\alpha^{\omega,*}\left(x^1,x^2\right) := \sup_{n\in\mathbf{P}} \frac{|\mathcal{K}_n^\omega(x^1,x^2)|}{\alpha([\log n])} \quad \left(\left(x^1,x^2\right) \in G^2\right).$$

In the present chapter we discuss the behavior of the weighted maximal function of Walsh-Marcinkiewicz means. That is, we prove the following theorem:

**Theorem 1** *There is positive absolute constant $C$ such that*

$$\frac{1}{16}\sum_{A=0}^\infty \frac{1}{\alpha(A)} \le \left\|\mathcal{K}_\alpha^{\omega,*}\right\|_1 \le C\sum_{A=0}^\infty \frac{1}{\alpha(A)}.$$

**Corollary 1** $\mathcal{K}_\alpha^{\omega,*} \in L^1$ *if and only if* $\sum_{A=0}^\infty \frac{1}{\alpha(A)} < \infty$.

Let

$$\mathcal{K}_{a,b}^\omega := \sum_{j=a}^{a+b-1} D_{j,j}^\omega,$$

and $n^{(s)} := \sum_{i=s}^{\infty} n_i 2^i$ $(n, s \in \mathbf{N})$. By simple calculations we get

$$n\mathcal{K}_n^{\omega} = \sum_{s=0}^{|n|} n_s \mathcal{K}^{\omega}_{n^{(s+1)},2^s}.$$

To prove our main theorem we need the following results [10], which estimate the values of the Marcinkiewicz kernels.

Suppose that $s, t, n \in \mathbf{N}$, $(x^1, x^2) \in I_{|n|} \times (I_t \backslash I_{t+1})$. If $s \le t \le |n|$, then

$$\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\left(x^1, x^2\right)\right| \le c2^{s+t}\left(n^{(s+1)} + 2^s\right). \tag{1}$$

If $t < s \le |n|$ then we have

$$\begin{aligned} &\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\left(x^1, x^2\right)\right| \\ &\quad \le \begin{cases} 0 & \text{if } \exists l, t < t+l < s, x^2 - x_t^2 e_t - e_{t+l} \notin I_s, x^2_{t+l} \neq 0, \\ 2^{2t+s+l} & \text{if } \exists l, t < t+l < s, x^2 - x_t^2 e_t - e_{t+l} \in I_s, x^2_{t+l} \neq 0, \\ 2^t n(s,t) & \text{if } x^2 - x_t^2 e_t \in I_s, \end{cases} \end{aligned} \tag{2}$$

where $n(s,t) = [n^{(s+1)}2^{s+1} - 2^t(2^s - 2^{t-1} + \frac{1}{2}) - 2^s(2^s - 2)]$.

Let $s, t^1, t^2, n \in \mathbf{N}$. Suppose that $t^1 \le t^2 < |n|$ and $(x^1, x^2) \in (I_{t^1} \backslash I_{t^1+1}) \times (I_{t^2} \backslash I_{t^2+1})$.

If $s \le t^1 \le t^2 \le |n|$, or $t^1 < s \le t^2 \le |n|$ then

$$\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\left(x^1, x^2\right)\right| \le c2^{t^1+t^2+s}. \tag{3}$$

If $t^1 \le t^2 < s \le |n|$ then

$$\begin{aligned} &\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\left(x^1, x^2\right)\right| \\ &\quad \le \begin{cases} 0 & \text{if } \exists i \in B_1, x_i^1 \neq x_i^2, \\ 0 & \text{if } \forall i \in B_1, x_i^1 = x_i^2, \exists l \in B_2, x^1 - e_{t^1} - e_l \notin I_{t^2+1}, x_l^1 = 1, \\ 2^{s+t^1+l} & \text{if } \forall i \in B_1, x_i^1 = x_i^2, \exists l \in B_2, x^1 - e_{t^1} - e_l \in I_{t^2+1}, x_l^1 = 1, \\ 2^{s+t^1+t^2} & \text{if } x^1 - e_{t^1} \in I_{t^2+1}, (\forall i \in B_1, x_i^1 = x_i^2), \end{cases} \end{aligned} \tag{4}$$

where $B_1 = \{t^2+1, \ldots, s-1\}$, $B_2 = \{t^1+1, \ldots, t^2\}$.

Moreover, we need the following Lemma of Gát, Goginava and the author [6].

**Lemma 1** (Gát, Goginava and Nagy [6]) *Let* $(x^1, x^2) \in (I_{t^1} \backslash I_{t^1+1}) \times (I_{t^2} \backslash I_{t^2+1})$, *where* $t^1 < s \le t^2$. *Then*

$$\left|\mathcal{K}^{w}_{n^{(s+1)},2^s}\left(x^1, x^2\right)\right| \le c2^{t^1+t^2+s} \sum_{m=t^1+1}^{s} \mathbf{1}_{I_s(e_{t^1}+e_m)}\left(x^1\right).$$

*Proof of Theorem 1* The lower estimation comes from the following. On the set $(I_A \setminus I_{A+1})^2$ we have

$$\mathcal{K}_{2^A}^{\phi}\left(x^1, x^2\right) = \frac{1}{2^A} \sum_{k=1}^{2^A-1} k^2 = \frac{(2^A-1)(2^{A+1}-1)}{6},$$

where $\phi = \omega$ or $\kappa$. Thus, we have

$$\begin{aligned}
\left\|\mathcal{K}_{\alpha}^{\phi,*}\right\|_1 &= \sum_{A=0}^{\infty}\sum_{B=0}^{\infty}\int_{(I_A\setminus I_{A+1})\times(I_B\setminus I_{B+1})} \mathcal{K}_{\alpha}^{\phi,*}\left(x^1,x^2\right) d\mu\left(x^1,x^2\right) \\
&\geq \sum_{A=0}^{\infty}\int_{(I_A\setminus I_{A+1})^2} \frac{\mathcal{K}_{2^A}^{\phi}(x^1,x^2)}{\alpha(A)}\, d\mu\left(x^1,x^2\right) \\
&= \sum_{A=0}^{\infty}\frac{1}{\alpha(A)}\int_{(I_A\setminus I_{A+1})^2} \frac{(2^A-1)(2^{A+1}-1)}{6}\, d\mu\left(x^1,x^2\right) \\
&\geq \frac{1}{16}\sum_{A=0}^{\infty}\frac{1}{\alpha(A)}.
\end{aligned}$$

Now, we prove the upper estimation.

$$\begin{aligned}
\left\|\mathcal{K}_{\alpha}^{\omega,*}\right\|_1 &\leq \int_{G^2}\sum_{A=0}^{\infty}\sup_{|n|=A}\frac{|\mathcal{K}_n^{\omega}|}{\alpha(A)}\, d\mu \\
&\leq \sum_{A=0}^{\infty}\frac{1}{\alpha(A)}\int_{G^2}\sup_{|n|=A}\left|\mathcal{K}_n^{\omega}\right| d\mu.
\end{aligned}$$

We show that there exists a $c > 0$ constant such that

$$\int_{G^2}\sup_{|n|=A}\left|\mathcal{K}_n^{\omega}\right| d\mu \leq c$$

holds for all $A \in \mathbf{N}$. This will completes the proof of Theorem 1.

First, we decompose the set $G$ as the following disjoint union:

$$G = I_A \cup \bigcup_{t=0}^{A-1}(I_t \setminus I_{t+1})$$

and introduce the notation $J_t := I_t \setminus I_{t+1}$.

$$\int_{G^2}\sup_{|n|=A}\left|\mathcal{K}_n^{\omega}\right| d\mu = \int_{I_A^2}\sup_{|n|=A}\left|\mathcal{K}_n^{\omega}\right| d\mu + \int_{I_A\times\overline{I_A}}\sup_{|n|=A}\left|\mathcal{K}_n^{\omega}\right| d\mu$$

$$+\int_{\overline{I_A}\times I_A}\sup_{|n|=A}\left|\mathcal{K}_n^{\omega}\right|d\mu+\int_{\overline{I_A}^2}\sup_{|n|=A}\left|\mathcal{K}_n^{\omega}\right|d\mu$$
$$=: S^1+S^2+S^3+S^4.$$

On the set $I_A^2$ the kernel $|\mathcal{K}_n^{\omega}|=\frac{(n-1)(2n-1)}{6}$ and $\sup_{|n|=A}|\mathcal{K}_n^{\omega}|\le\frac{(2^{A+1}-1)(2^{A+2}-1)}{6}$ which immediately gives that $S^1\le c$.

Now, we discuss $S^2$ ($S^3$ goes analogously).

$$S^2\le\sum_{t^2=0}^{A-1}\int_{I_A\times J_{t^2}}\sup_{|n|=A}\frac{1}{n}\sum_{s=0}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right|d\mu.$$

We decompose the set $J_{t^2}$ as a disjoint union

$$J_{t^2}=\bigcup_{q=t^2+1}^{A}J_A^{t^2,q},$$

where $J_A^{t^2,q}:=\{x\in G: x=(0,\ldots,0,x_{t^2}=1,0,\ldots,0,x_q=1,x_{q+1},\ldots,x_{A-1},\ldots)\}=I_{q+1}(e_{t^2}+e_q)$ for $t^2<q<A$ and $J_A^{t^2,A}=I_A(e_{t^2})$ for $q=A$. Thus,

$$S^2\le\sum_{t^2=0}^{A-1}\sum_{q=t^2+1}^{A}\int_{I_A\times J_A^{t^2,q}}\sup_{|n|=A}\frac{1}{n}\sum_{s=0}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right|d\mu.$$

On the set $I_A\times J_A^{t^2,q}$ the kernel $|K_{n^{(s+1)},2^s}|\le cn2^{s+t^2}$ for $0\le s\le q$ (see inequality (1)). That is,

$$S^2\le\sum_{t^2=0}^{A-1}\sum_{q=t^2+1}^{A}\int_{I_A\times J_A^{t^2,q}}\sup_{|n|=A}\frac{1}{n}\left(\sum_{s=0}^{q}cn2^{s+t^2}+\sum_{s=q+1}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right|\right)d\mu.$$

To discuss the last sum, we have to decompose the set $J_A^{t^2,q}$ as the following disjoint union

$$J_A^{t^2,q}=\bigcup_{r=q+1}^{A}J_A^{t^2,q,r},$$

where $J_A^{t^2,q,r}=\{x\in G: x=(0,\ldots,0,x_{t^2}=1,0,\ldots,0,x_q=1,0,\ldots,x_r=1,x_{r+1},\ldots,x_{A-1},\ldots)\}=I_{r+1}(e_{t^2}+e_q+e_r)$ for $r<A$, $J_A^{t^2,q,A}=I_A(e_{t^2}+e_q)$ for $r=A>q$, and $J_A^{t^2,A,A}=I_A(e_{t^2})$. On the set $J_A^{t^2,q,r}$ by inequality (2), $|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}|\le$

$2^{t^2+q+s}$ for $q < s \le r$ and $|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}| = 0$ for $r < s$. Thus, we have

$$\begin{aligned} S^2 &\le \sum_{t^2=0}^{A-1} \sum_{q=t^2+1}^{A} \sum_{r=q+1}^{A} \int_{I_A \times J_A^{t^2,q,r}} \sup_{|n|=A} \frac{1}{n} \left( \sum_{s=0}^{q} cn 2^{s+t^2} + \sum_{s=q+1}^{r} 2^{t^2+q+s} \right) d\mu \\ &\le c \sum_{t^2=0}^{A-1} \sum_{q=t^2+1}^{A} \sum_{r=q+1}^{A} \int_{I_A \times J_A^{t^2,q,r}} \left( \sum_{s=0}^{q} 2^{s+t^2} + \sum_{s=q+1}^{r} 2^{t^2+q+s-A} \right) d\mu \\ &\le c \sum_{t^2=0}^{A-1} \sum_{q=t^2+1}^{A} \sum_{r=q+1}^{A} \int_{I_A \times J_A^{t^2,q,r}} 2^{q+t^2} \, d\mu \\ &\le c \sum_{t^2=0}^{A-1} \sum_{q=t^2+1}^{A} \sum_{r=q+1}^{A} 2^{q+t^2} 2^{-A-r} \\ &\le c \sum_{t^2=0}^{A-1} (A - t^2) 2^{t^2-A} \le c. \end{aligned}$$

Now, we turn our attention to the discussion of $S^4$.

$$\begin{aligned} S^4 &= \sum_{t^1=0}^{A-1} \sum_{t^2=0}^{A-1} \int_{J_{t^1} \times J_{t^2}} \sup_{|n|=A} \left| \mathcal{K}^{\omega}_n \right| d\mu \\ &= \sum_{t^1=0}^{A-1} \sum_{t^2=0}^{t^1-1} S_{t^1,t^2} + \sum_{t^1=0}^{A-1} \sum_{t^2=t^1}^{A-1} S_{t^1,t^2} =: \sum_1 + \sum_2, \end{aligned}$$

where $S_{t^1,t^2} := \int_{J_{t^1} \times J_{t^2}} \sup_{|n|=A} |\mathcal{K}^{\omega}_n| \, d\mu$.

We investigate $\sum_2$ (the investigation of $\sum_1$ goes analogously).

Set $t^1 \le t^2 < A$.

$$\begin{aligned} S_{t^1,t^2} &\le \int_{J_{t^1} \times J_{t^2}} \sup_{|n|=A} \frac{1}{n} \sum_{s=0}^{A} \left| \mathcal{K}^{\omega}_{n^{(s+1)},2^s} \right| d\mu \\ &= \int_{J_{t^1} \times J_{t^2}} \sup_{|n|=A} \frac{1}{n} \sum_{s=0}^{t^1} \left| \mathcal{K}^{\omega}_{n^{(s+1)},2^s} \right| d\mu + \int_{J_{t^1} \times J_{t^2}} \sup_{|n|=A} \frac{1}{n} \sum_{s=t^1+1}^{t^2} \left| \mathcal{K}^{\omega}_{n^{(s+1)},2^s} \right| d\mu \\ &\quad + \int_{J_{t^1} \times J_{t^2}} \sup_{|n|=A} \frac{1}{n} \sum_{s=t^2+1}^{A} \left| \mathcal{K}^{\omega}_{n^{(s+1)},2^s} \right| d\mu =: S^1_{t^1,t^2} + S^2_{t^1,t^2} + S^3_{t^1,t^2}. \end{aligned}$$

By inequality (3) we write

$$S^1_{t^1,t^2} \le c\int_{J_{t^1}\times J_{t^2}} \frac{1}{2^A}\sum_{s=0}^{t^1} 2^{t^1+t^2+s}\,d\mu$$
$$\le c2^{2t^1+t^2-A}2^{-t^1-t^2} \le c2^{t^1-A}.$$

Moreover, this yields

$$\sum_{t^1=0}^{A-1}\sum_{t^2=t^1}^{A-1} S^1_{t^1,t^2} \le c\sum_{t^1=0}^{A-1}\sum_{t^2=t^1}^{A-1} 2^{t^1-A} \le c\sum_{t^1=0}^{A-1}(A-t^1)2^{t^1-A} \le c. \tag{5}$$

By the help of Lemma 1 we discuss $S^2_{t^1,t^2}$.

$$S^2_{t^1,t^2} \le c\sum_{q=t^1+1}^{A}\int_{J_A^{t^1,q}\times J_{t^2}} \frac{1}{2^A}\sum_{s=t^1+1}^{t^2} 2^{t^1+t^2+s}\sum_{m=t^1+1}^{s}\mathbf{1}_{I_s(e_{t^1}+e_m)}(x^1)\,d\mu(x^1,x^2)$$
$$\le c2^{-A}\sum_{q=t^1+1}^{t^2-1}\int_{J_A^{t^1,q}}\sum_{s=t^1+1}^{t^2} 2^{t^1+s}\sum_{m=t^1+1}^{s}\mathbf{1}_{I_s(e_{t^1}+e_m)}(x^1)\,d\mu(x^1)$$
$$+c2^{-A}\sum_{q=t^2}^{A}\int_{J_A^{t^1,q}}\sum_{s=t^1+1}^{t^2} 2^{t^1+s}\sum_{m=t^1+1}^{s}\mathbf{1}_{I_s(e_{t^1}+e_m)}(x^1)\,d\mu(x^1)$$
$$\le c2^{-A}\sum_{q=t^1+1}^{t^2-1}\int_{J_A^{t^1,q}}\sum_{s=t^1+1}^{q} 2^{t^1+s}\mathbf{1}_{I_s(e_{t^1})}(x^1)$$
$$+\sum_{s=q+1}^{t^2} 2^{t^1+s}\mathbf{1}_{I_s(e_{t^1}+e_q)}(x^1)\,d\mu(x^1)$$
$$+c2^{-A}\sum_{q=t^2}^{A}\int_{J_A^{t^1,q}}\sum_{s=t^1+1}^{t^2} 2^{t^1+s}\mathbf{1}_{I_s(e_{t^1})}(x^1)\,d\mu(x^1)$$
$$\le c2^{-A}\sum_{q=t^1+1}^{t^2-1}\left(\sum_{s=t^1+1}^{q} 2^{t^1+s}2^{-q}+\sum_{s=q+1}^{t^2} 2^{t^1}\right)$$
$$+c2^{-A}\sum_{q=t^2}^{A}\sum_{s=t^1+1}^{t^2} 2^{t^1+s-q}$$
$$\le c2^{-A}(A-t^1)^2 2^{t^1}.$$

This gives that

$$\sum_{t^1=0}^{A-1}\sum_{t^2=t^1}^{A-1} S^2_{t^1,t^2} \le c\sum_{t^1=0}^{A-1}\left(A-t^1\right)^3 2^{t^1-A} \le c. \tag{6}$$

At last, we discuss $S^3_{t^1,t^2}$ by the help of inequalities (3) and (4). We use the following disjoint decomposition of $J_{t^1}$:

$$J_{t^1} = \bigcup_{q=t^1+1}^{A}\bigcup_{r=q+1}^{A} J_A^{t^1,q,r}.$$

Thus, we immediately write

$$\begin{aligned}
S^3_{t^1,t^2} &\le \sum_{q=t^1+1}^{A}\sum_{r=q+1}^{A}\int_{J_A^{t^1,q,r}\times J_{t^2}}\frac{1}{2^A}\sum_{s=t^2+1}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu \\
&= \sum_{q=t^1+1}^{A}\int_{J_A^{t^1,q,A}\times J_{t^2}}\frac{1}{2^A}\sum_{s=t^2+1}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu \\
&\quad + \sum_{q=t^1+1}^{A}\int_{J_A^{t^1,q,A-1}\times J_{t^2}}\frac{1}{2^A}\sum_{s=t^2+1}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu \\
&\quad + \sum_{q=t^1+1}^{A}\sum_{r=q+1}^{A-2}\int_{J_A^{t^1,q,r}\times J_{t^2}}\frac{1}{2^A}\sum_{s=t^2+1}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu \\
&=: S^{3,1}_{t^1,t^2} + S^{3,2}_{t^1,t^2} + S^{3,3}_{t^1,t^2}.
\end{aligned}$$

We investigate $S^{3,1}_{t^1,t^2}$ (and $S^{3,2}_{t^1,t^2}$ goes analogously).

$$\begin{aligned}
S^{3,1}_{t^1,t^2} &= \sum_{q=t^1+1}^{t^2}\sum_{r=t^2+1}^{A}\int_{J_A^{t^1,q,A}\times J_A^{t^2,r}} 2^{-A}\sum_{s=t^2+1}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu \\
&\quad + \sum_{q=t^2+1}^{A}\sum_{r=t^2+1}^{A}\int_{J_A^{t^1,q,A}\times J_A^{t^2,r}} 2^{-A}\sum_{s=t^2+1}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu \\
&=: S^{3,1,1}_{t^1,t^2} + S^{3,1,2}_{t^1,t^2}.
\end{aligned}$$

By inequality (4) and $\mathcal{K}^{\omega}_{n^{(s+1)},2^s} \neq 0$ we write that

$$S^{3,1,1}_{t^1,t^2} \leq \sum_{q=t^1+1}^{t^2} \sum_{r=t^2+1}^{A} \int_{J_A^{t^1,q,A} \times J_A^{t^2,r}} 2^{-A} \sum_{s=t^2+1}^{r} 2^{s+t^1+q}\, d\mu$$
$$\leq c \sum_{q=t^1+1}^{t^2} \sum_{r=t^2+1}^{A} 2^{t^1+q-2A} \leq c\big(A-t^2\big)2^{t^1+t^2-2A}$$

and

$$S^{3,1,2}_{t^1,t^2} = \sum_{q=t^2+1}^{A} \sum_{r=t^2+1}^{q-1} \int_{J_A^{t^1,q,A} \times J_A^{t^2,r}} 2^{-A} \sum_{s=t^2+1}^{A} \big|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\big|\, d\mu$$
$$+ \sum_{q=t^2+1}^{A} \int_{J_A^{t^1,q,A} \times J_A^{t^2,q}} 2^{-A} \sum_{s=t^2+1}^{A} \big|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\big|\, d\mu$$
$$+ \sum_{q=t^2+1}^{A} \sum_{r=q+1}^{A} \int_{J_A^{t^1,q,A} \times J_A^{t^2,r}} 2^{-A} \sum_{s=t^2+1}^{A} \big|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\big|\, d\mu.$$

Inequality (4) and $\mathcal{K}^{\omega}_{n^{(s+1)},2^s} \neq 0$ give that

$$S^{3,1,2}_{t^1,t^2} \leq \sum_{q=t^2+1}^{A} \sum_{r=t^2+1}^{q-1} \int_{J_A^{t^1,q,A} \times J_A^{t^2,r}} 2^{-A} \sum_{s=t^2+1}^{r} 2^{s+t^1+t^2}\, d\mu$$
$$+ \sum_{q=t^2+1}^{A} \sum_{p=q+1}^{A} \int_{J_A^{t^1,q,A} \times J_A^{t^2,q,p}} 2^{-A} \sum_{s=t^2+1}^{p} 2^{s+t^1+t^2}\, d\mu$$
$$+ \sum_{q=t^2+1}^{A} \sum_{r=q+1}^{A} \int_{J_A^{t^1,q,A} \times J_A^{t^2,r}} 2^{-A} \sum_{s=t^2+1}^{q} 2^{s+t^1+t^2} + 2^{-A} \sum_{s=q+1}^{r} 2^{s+t^1+q}\, d\mu$$
$$\leq \sum_{q=t^2+1}^{A} \sum_{r=t^2+1}^{q-1} 2^{t^1+t^2-2A} + \sum_{q=t^2+1}^{A} \sum_{p=q+1}^{A} 2^{t^1+t^2-2A}$$
$$+ \sum_{q=t^2+1}^{A} \sum_{r=q+1}^{A} \big(2^{q+t^1+t^2-2A-r} + 2^{t^1+q-2A}\big)$$
$$\leq c2^{t^1+t^2-2A}\big(A-t^2\big)^2 + c2^{t^1-A}\big(A-t^2\big).$$

That is, these immediately yield that

$$\sum_{t^1=0}^{A-1}\sum_{t^2=t^1}^{A-1} S^{3,1}_{t^1,t^2} \le c\sum_{t^1=0}^{A-1}\sum_{t^2=t^1}^{A-1} 2^{t^1+t^2-2A}\left(A-t^2\right)^2 + c\sum_{t^1=0}^{A-1}\sum_{t^2=t^1}^{A-1} 2^{t^1-A}\left(A-t^2\right) \le c.$$

At last, we discuss $S^{3,3}_{t^1,t^2}$. Set $x_{i,j} := \sum_{l=i}^{j} x_l e_l$ and $x'_{i,j} := \sum_{l=i}^{j-1} x_l e_l + (1-x_j)e_j$ (with $x'_{i,A} := \sum_{l=i}^{A-1} x_l e_l$). Thus,

$$\begin{aligned} S^{3,3}_{t^1,t^2} &\le \sum_{q=t^1+1}^{A-3}\sum_{r=q+1}^{A-2}\sum_{\substack{x_i=0\\ i\in\{r+1,\ldots,A-1\}}}^{1}\int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_{t^2}} 2^{-A} \\ &\quad\times \sum_{s=t^2+1}^{A}\left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu \\ &= \sum_{q=t^1+1}^{t^2}(\cdots) + \sum_{q=t^2+1}^{A-3}(\cdots) =: S^{3,3,1}_{t^1,t^2} + S^{3,3,2}_{t^1,t^2}. \end{aligned}$$

Using inequality (4) and $\mathcal{K}^{\omega}_{n^{(s+1)},2^s} \ne 0$, we get that $t^2 < r$ and

$$\begin{aligned} S^{3,3,1}_{t^1,t^2} &\le \sum_{q=t^1+1}^{t^2}\sum_{r=t^2+1}^{A-2}\sum_{\substack{x_i=0\\ i\in\{r+1,\ldots,A-1\}}}^{1}\sum_{p=r+1}^{A}\int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times I_{p+1}(e_{t^2}+e_r+x'_{r+1,p})} \\ &\quad\times 2^{-A}\sum_{s=t^2+1}^{p} 2^{s+t^1+q}\, d\mu \\ &\quad+ \sum_{q=t^1+1}^{t^2}\sum_{r=t^2+1}^{A-2}\sum_{\substack{x_i=0\\ i\in\{r+1,\ldots,A-1\}}}^{1}\sum_{p=r+1}^{A}\int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times I_{p+1}(e_{t^2}+x'_{r+1,p})} \\ &\quad\times 2^{-A}\sum_{s=t^2+1}^{r} 2^{s+t^1+q}\, d\mu \\ &\le 2^{-2A}\sum_{q=t^1+1}^{t^2}\sum_{r=t^2+1}^{A-2} 2^{A-r}(A-r)2^{t^1+q} + 2^{-2A}\sum_{q=t^1+1}^{t^2}\sum_{r=t^2+1}^{A-2} 2^{A-r}2^{t^1+q} \\ &\le 2^{t^1-A}\left(A-t^1\right)^2 \end{aligned}$$

(with the notation $I_{A+1}(e_{t^2}+e_r+x'_{r+1,A}) := I_A(e_{t^2}+e_r+x'_{r+1,A})$). Moreover, we write for $S^{3,3,2}_{t^1,t^2}$ that

$$S^{3,3,2}_{t^1,t^2} = \sum_{q=t^2+1}^{A-3} \sum_{r=q+1}^{A-2} \sum_{\substack{x_i=0\\ i\in\{r+1,\dots,A-1\}}}^{1} \sum_{m=t^2+1}^{A} \sum_{n=m+1}^{A} \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_A^{t^2,m,n}} \times 2^{-A} \sum_{s=t^2+1}^{A} \left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu$$

$\mathcal{K}^{\omega}_{n^{(s+1)},2^s} \neq 0$ and $q > t^2$ yield that

$$\begin{aligned}
&\sum_{m=t^2+1}^{A} \sum_{n=m+1}^{A} \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_A^{t^2,m,n}} 2^{-A} \sum_{s=t^2+1}^{A} \left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu \\
&\le \sum_{m=t^2+1}^{q-1} \sum_{n=m+1}^{A} \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_A^{t^2,m,n}} 2^{-A} \sum_{s=t^2+1}^{m} 2^{s+t^1+t^2}\, d\mu \\
&\quad + \sum_{m=q+1}^{A} \sum_{n=m+1}^{A} \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_A^{t^2,m,n}} 2^{-A} \sum_{s=t^2+1}^{q} 2^{s+t^1+t^2}\, d\mu \\
&\quad + \sum_{n=q+1}^{A} \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_A^{t^2,q,n}} 2^{-A} \sum_{s=t^2+1}^{A} \left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu \\
&\le \sum_{m=t^2+1}^{q-1} \sum_{n=m+1}^{A} 2^{-2A-n} 2^{m+t^1+t^2} + \sum_{m=q+1}^{A} \sum_{n=m+1}^{A} 2^{-2A-n} 2^{q+t^1+t^2} + \sum_{m=q} \\
&\le 2^{-2A+t^1+t^2}\left(q-t^2\right) + \sum_{m=q}.
\end{aligned}$$

$\mathcal{K}^{\omega}_{n^{(s+1)},2^s} \neq 0$ gives again

$$\begin{aligned}
\sum_{m=q}^{A} &\le \sum_{n=q+1}^{r-1} \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_A^{t^2,q,n}} 2^{-A} \sum_{s=t^2+1}^{n} 2^{s+t^1+t^2}\, d\mu \\
&\quad + \sum_{n=r+1}^{A} \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_A^{t^2,q,n}} 2^{-A} \sum_{s=t^2+1}^{r} 2^{s+t^1+t^2}\, d\mu \\
&\quad + \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_A^{t^2,q,r}} 2^{-A} \sum_{s=t^2+1}^{A} \left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu
\end{aligned}$$

$$\leq \sum_{n=q+1}^{r-1} 2^{-2A-n} 2^{n+t^1+t^2} + \sum_{n=r+1}^{A} 2^{-2A-n} 2^{r+t^1+t^2} + \sum_{\substack{m=q\\n=r}}$$

$$\leq (r-q)2^{-2A+t^1+t^2} + \sum_{\substack{m=q\\n=r}}.$$

For fixed $q$, $r$ and $x_{r+1,A-1} \in G$, by $\mathcal{K}^{\omega}_{n^{(s+1)},2^s} \neq 0$ we write, again

$$\sum_{\substack{m=q\\n=r}} = \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times J_A^{t^2,q,r}} 2^{-A} \sum_{s=t^2+1}^{A} \left|\mathcal{K}^{\omega}_{n^{(s+1)},2^s}\right| d\mu$$

$$\leq \sum_{p=r+1}^{A} \int_{I_A(e_{t^1}+e_q+e_r+x_{r+1,A-1})\times I_{p+1}(e_{t^2}+e_q+e_r+x'_{r+1,p})} 2^{-A} \sum_{s=t^2+1}^{p} 2^{s+t^1+t^2}\, d\mu$$

$$\leq \sum_{p=r+1}^{A} 2^{-2A-p} 2^{p+t^1+t^2} \leq (A-r)2^{-2A+t^1+t^2}.$$

Thus, we have

$$S^{3,3,2}_{t^1,t^2} = \sum_{q=t^2+1}^{A-3} \sum_{r=q+1}^{A-2} \sum_{\substack{x_i=0\\ i\in\{r+1,\dots,A-1\}}}^{1} 2^{-2A+t^1+t^2}\left[\left(q-t^2\right)+(r-q)+(A-r)\right]$$

$$\leq c\left(A-t^2\right)2^{t^1-A}$$

and

$$\sum_{t^1=0}^{A-1} \sum_{t^2=t^1}^{A-1} S^{3,3}_{t^1,t^2} \leq c \sum_{t^1=0}^{A-1} \sum_{t^2=t^1}^{A-1} \left(\left(A-t^1\right)^2 2^{t^1-A} + \left(A-t^2\right)2^{t^1-A}\right) \leq c.$$

This completes the proof of our theorem. □

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# Quasimonotonicity as a Tool for Differential and Functional Inequalities

**Peter Volkmann**

*Dedicated to the Memory of Wolfgang Walter,*
*May 2, 1927 - June 26, 2010*

## 1 Introduction

In the context of differential inequalities, the name "quasimonotonicity" had been introduced by Wolfgang Walter [8]. In this monograph also the basic comparison theorems involving ordinary and parabolic differential inequalities, respectively, are treated, the latter being a generalization by Mlak [3] of a theorem of Nagumo [4] to functions having values in $\mathbb{R}^n$.

For both comparison theorems versions are known, where the functions have values in ordered topological vector spaces; cf. [6] for ordinary differential inequalities and the joint paper with Simon [5] for parabolic inequalities. When restricting [5] to the semilinear case, then functions $f(x,t,\xi)$ are involved, whereas in [6] functions $f(t,\xi)$ occur. Here $x$ is a variable in $\mathbb{R}^N$, $t$ is a real variable, and $\xi$ is a variable in an ordered topological vector space $E$; the values of $f$ are in $E$.

Now it turns out that the comparison theorem from [6] can be considered as a special case from [5], when allowing $N = 0$. This will be presented in the next paragraph; for simplicity we only consider the semilinear case. Similarly as in [7], a functional dependence with retarded argument will be admitted, which for $N = 0$ in the case of absence of all derivatives leads to a theorem on functional inequalities; cf. also the joint paper with Baron [1].

Finally the survey article by Herzog [2] on quasimonotonicity has to be mentioned.

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## 2 Comparison Theorem

We consider $N \in \{0, 1, 2, \dots\}$ and a non-empty, open, and bounded set $\Omega \subseteq \mathbb{R}^N$ (hence $\Omega = \{0\}$ and $\partial\Omega = \emptyset$ for $N = 0$). Furthermore, $T > 0$ being fixed, we consider $R = \Omega \times ]0, T]$; the set $\partial_p R = (\Omega \times \{0\}) \cup (\partial\Omega \times [0, T])$ is called the *parabolic boundary* of $R$. We obviously have $R \cup \partial_p R = \bar{R}$, $R \cap \partial_p R = \emptyset$.

Let $E$ be a real Hausdorff topological vector space, and let $K$ be a wedge in $E$, i.e. a non-empty subset satisfying

$$\lambda \geq 0, \quad \xi \in K, \quad \eta \in K \quad \Rightarrow \quad \lambda(\xi + \eta) \in K.$$

We suppose $K$ to be closed and such that

$$\operatorname{Int} K \neq \emptyset.$$

For $\xi, \eta \in E$ we write

$$\xi \leq \eta \quad \Leftrightarrow \quad \eta - \xi \in K,$$
$$\xi \ll \eta \quad \Leftrightarrow \quad \eta - \xi \in \operatorname{Int} K.$$

$K^*$ denotes the dual wedge of $K$, i.e. the set of all linear, continuous $\varphi : E \to \mathbb{R}$ satisfying $\varphi(\xi) \geq 0$ for $\xi \in K$.

According to [6], a function

$$f(x, t, \xi) : D \to E$$

(where $D \subseteq \mathbb{R}^N \times \mathbb{R} \times E$) is *quasimonotone increasing* with respect to $\xi$, if

$$(x, t, \xi) \in D, (x, t, \eta) \in D, \quad \xi \leq \eta, \quad \varphi \in K^*, \quad \varphi(\xi) = \varphi(\eta)$$
$$\Rightarrow \quad \varphi\big(f(x, t, \xi)\big) \leq \varphi\big(f(x, t, \eta)\big).$$

For functions $u : \bar{R} \to E$ and $(x, t) \in R$ the derivatives

$$\frac{\partial u}{\partial x_j}(x, t) \in E, \quad \frac{\partial^2 u}{\partial x_j \partial x_k}(x, t) \in E \quad (j, k = 1, \dots, N) \tag{1}$$

and the left-hand derivatives

$$\frac{\partial^- u}{\partial t}(x, t) \in E \tag{2}$$

are taken in a weak sense: It is sufficient that for any linear, continuous $\varphi : E \to \mathbb{R}$ the function $\omega = \varphi \circ u|_R : R \to \mathbb{R}$ has derivatives $\omega_x$, $\omega_{xx}$ and $\frac{\partial^- \omega}{\partial t} = \lim_{h \uparrow 0} \frac{\omega(x,t+h) - \omega(x,t)}{h}$, which are linked to (1), (2) by

$$\frac{\partial \omega}{\partial x_j}(x, t) = \varphi\left(\frac{\partial u}{\partial x_j}(x, t)\right), \qquad \frac{\partial^2 \omega}{\partial x_j \partial x_k}(x, t) = \varphi\left(\frac{\partial^2 u}{\partial x_j \partial x_k}(x, t)\right),$$

$$\frac{\partial^- \omega}{\partial t}(x,t) = \varphi\left(\frac{\partial^- u}{\partial t}(x,t)\right).$$

**Theorem 1** *For* $(x,t) \in R$ *suppose* $a(x,t) \geq 0$, $b(x,t) \geq 0$, $0 \leq F(x,t) \leq t$, $d_j(x,t) \in \mathbb{R}$ $(j = 1, \ldots, N)$, *and let* $(c_{jk}(x,t))_{j,k=1}^N$ *be positive semidefinite matrices. For two continuous functions* $v, w : \bar{R} \to E$ *suppose*

I) $\quad v(x,t) \ll w(x,t) \quad \big((x,t) \in \partial_{\mathrm{p}} R\big)$,

II)
$$a(x,t)\frac{\partial^- v}{\partial t}(x,t) - b(x,t)v\big(x, F(x,t)\big) - \sum_{j,k=1}^N c_{jk}(x,t)\frac{\partial^2 v}{\partial x_j \partial x_k}(x,t)$$
$$- \sum_{j=1}^N d_j(x,t)\frac{\partial v}{\partial x_j}(x,t) - f\big(x,t,v(x,t)\big)$$
$$\ll a(x,t)\frac{\partial^- w}{\partial t}(x,t) - b(x,t)w\big(x, F(x,t)\big) - \sum_{j,k=1}^N c_{jk}(x,t)\frac{\partial^2 w}{\partial x_j \partial x_k}(x,t)$$
$$- \sum_{j=1}^N d_j(x,t)\frac{\partial w}{\partial x_j}(x,t) - f\big(x,t,w(x,t)\big) \quad \big((x,t) \in R\big),$$

*where* $f(x,t,\xi)$ *has values in* $E$ *and is quasimonotone increasing with respect to* $\xi$ *(at least in the set* $D = \{(x,t,\xi) \mid (x,t) \in R, \xi \in \{v(x,t), w(x,t)\}\}$*). Then the inequality*

$$v(x,t) \ll w(x,t) \quad \big((x,t) \in \bar{R}\big)$$

*follows.*

*Remark 1* For $N = 0$ the sums in II) are considered to be empty (having value zero, and existence of derivatives with respect to $x$ is not required). Furthermore, we have $R = ]0,T] \times \{0\}$ and $\partial_{\mathrm{p}} R = \{0\} \times \{0\}$. When considering the special case $a(0,t) \equiv 1$, $b(0,t) \equiv 0$, and when simply writing

$$v(t) \quad \text{instead of } v(0,t), \qquad w(t) \quad \text{instead of } w(0,t) \quad (0 \leq t \leq T),$$

then we get the comparison theorem from [6]:

**Corollary** *For continuous* $v, w : [0,T] \to E$ *suppose* $v(0) \ll w(0)$ *and*

$$\frac{\partial^- v}{\partial t}(t) - f\big(t, v(t)\big) \ll \frac{\partial^- w}{\partial t}(t) - f\big(t, w(t)\big) \quad (0 < t \leq T),$$

*where* $f(t,\xi)$ *is quasimonotone increasing with respect to* $\xi$. *Then* $v(t) \ll w(t)$ $(0 \leq t \leq T)$ *follows.*

*Remark 2* For $N = 0$ again, let us look at the special case $a(0, t) \equiv 0$, $b(0, t) \equiv 1$. Then Theorem 1 leads to a theorem on functional inequalities which had been examined in the joint paper with Baron [1]; the derivatives $\partial^- v/\partial t$, $\partial^- w/\partial t$ need not exist in this case.

*Remark 3* For $N \geq 1$ it is also possible to consider Theorem 1 in the case $c_{jk}(x, t) \equiv 0$ for $j, k = 1, \dots, N$; then the double sums in II) are zero, and Theorem 1 remains true, without requiring the existence of second derivatives. Further simplification by also taking $d_j(x, t) \equiv 0$ for $j = 1, \dots, N$ makes no sense, because then all the sums in II) disappear, and we can treat the theorem for any $x \in \Omega$ separately with respect to the only variable $t \in [0, T]$.

## 3 Proof of Theorem 1

We suppose the theorem to be false. Then

$$w(x_0, t_0) - v(x_0, t_0) \in E \setminus \operatorname{Int} K$$

would be possible for some $(x_0, t_0) \in R$. We can suppose $t_0 > 0$ to be minimal, then

$$w(x, t) - v(x, t) \in K \quad (x \in \bar{\Omega}, 0 \leq t \leq t_0), \tag{3}$$

$$w(x_0, t_0) - v(x_0, t_0) \in \partial K.$$

According to the Hahn/Banach separation theorem there exists

$$\varphi \in K^* \setminus \{0\} \tag{4}$$

such that

$$\varphi\big(w(x_0, t_0) - v(x_0, t_0)\big) = 0. \tag{5}$$

Then (3), (4) imply

$$\omega(x, t) := \varphi\big(w(x, t) - v(x, t)\big) \geq 0 \quad (x \in \bar{\Omega}, 0 \leq t \leq t_0),$$

hence, according to (5), the function $\omega$ has at $(x_0, t_0)$ the minimum value $\omega(x_0, t_0) = 0$. This implies

$$\frac{\partial^- \omega}{\partial t}(x_0, t_0) \leq 0, \tag{6}$$

$$\omega\big(x_0, F(x_0, t_0)\big) \geq 0 \tag{7}$$

(since $0 \leq F(x_0, t_0) \leq t_0$), and

$$\frac{\partial \omega}{\partial x_j}(x_0, t_0) = 0 \quad (j = 1, \dots, N), \tag{8}$$

$$\left(\frac{\partial^2 \omega}{\partial x_j \partial x_k}(x_0, t_0)\right)_{j,k=1}^{N} \quad \text{is a symmetric, positive semidefinite matrix.} \tag{9}$$

Now we consider II) for $(x, t) = (x_0, t_0)$, and we apply the functional $\varphi$ to both sides. Because of (4), the sign $\ll$ then becomes a $<$-sign, and we get, in abbreviated form,

$$a(x_0, t_0)\varphi\left(\frac{\partial^- v}{\partial t}(x_0, t_0)\right) - \cdots < \cdots - \varphi\big(f\big(x_0, t_0, w(x_0, t_0)\big)\big). \tag{10}$$

Because of (6) we have $\varphi(\frac{\partial^- w}{\partial t}(x_0, t_0) - \frac{\partial^- v}{\partial t}(x_0, t_0)) \leq 0$, therefore we get from (10) a new inequality, when removing from both sides the terms containing left-hand derivatives. Using (7), (8), (9), we can continue this simplification of (10), and we end up with

$$-\varphi\big(f\big(x_0, t_0, v(x_0, t_0)\big)\big) < -\varphi\big(f\big(x_0, t_0, w(x_0, t_0)\big)\big). \tag{11}$$

On the other hand, according to (3), (4), (5) we have

$$v(x_0, t_0) \leq w(x_0, t_0), \quad \varphi \in K^*, \qquad \varphi\big(v(x_0, t_0)\big) = \varphi\big(w(x_0, t_0)\big),$$

and therefore the quasimonotonicity of $f(x_0, t_0, \xi)$ with respect to $\xi$ yields $\varphi(f(x_0, t_0, v(x_0, t_0))) \leq \varphi(f(x_0, t_0, w(x_0, t_0)))$, which is a contradiction to (11).

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