Progress in Nonlinear Differential Equations and Their Applications
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Parabolic Problems

The Herbert Amann Festschrift

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Contents

Prof. Herbert Amann

Preface

Herbert Amann studied at the universities of Freiburg, Basel, and München in the early 1960s. In 1965 he received his doctoral degree under the supervision of Joachim Nitsche from the University of Freiburg. At that time, Herbert Amann's research revolved around the use of Monte Carlo simulations in connection with the resolution of elliptic problems [1]. His research interests then shifted toward the area of nonlinear integral equations, with a particular focus on the Hammerstein equation [2, 3]. In 1970 Herbert Amann moved from Freiburg to Bloomington, Indiana, and, the following year, to Lexington, Kentucky, where he held visiting professor positions. During the years spent in the US, his interests evolved toward nonlinear elliptic problems and the use of topological methods for their analysis. He was appointed full professor at the Ruhr-Universität Bochum in 1972 where he continued these investigations. Of this time are some of his most frequently cited and influential research papers about the topological degree [4, 5], the sub- and supersolution method [6, 7, 8], and multiplicity of solutions for nonlinear elliptic problems [9, 10]. Of outstanding importance is his consistently highly cited review article [11] on fixed point theory in ordered Banach spaces.

Herbert Amann moved to the Christian-Albrechts-Universität zu Kiel in 1978, and then to the Universität Zürich in 1979. During his tenure in Zürich, he continued his studies on qualitative features of nonlinear elliptic boundary value problems [12, 13], and then immersed himself in the study of nonlinear parabolic problems. A deep and careful understanding of the fundamental properties of general evolution systems together with the development of the interpolation-extrapolation framework were an important breakthrough in the study of nonlinear parabolic problems [14, 15, 16]. The full strength of this abstract approach is apparent in the dynamic theory for general quasilinear systems of parabolic type [17, 18, 19, 20]. A successful implementation in applications, like, e.g., coagulation-fragmentation processes [21], requires a thorough insight into the theory of function spaces and multiplier results, particularly also in the Banach space valued setting. Among the most important contributions in this context are [20, 22, 23, 24, 25, 35]. In recent years, Herbert Amann also contributed to the development of the theory of maximal regularity. His comprehensive view on complex structures allowed him to derive far-reaching results on Navier-Stokes equations, non-Newtonian fluids, image processing, and evolution equations with memory [26, 27, 28, 29]. Besides more than 100 research papers, Herbert Amann also has written important monographs [30, 31] and successful text books [32, 33, 34].

Herbert Amann has been a steady source of new ideas, and he has influenced many researchers. His unwavering dedication to research and teaching has been an example to all of his colleagues and students, in particular to his 24 doctoral students. In 2001 he became foreign corresponding member of the Real Academia de Ciencias Exactas, Físicas y Naturales, Madrid, and, one year later, received a Doctor Honoris Causa from the Universidad Complutense, Madrid. As of 2004, Herbert Amann is Professor Emeritus of the Universität Zürich.

During his long and ongoing career he has enjoyed the invaluable support of his wife, Gisela Amann.

The present volume contains original research papers and reflects the wide-ranging scientific interests of Herbert Amann. It is inspired by the conference "Nonlinear Parabolic Problems: In honor of Herbert Amann" held May 10–16, 2009, at the Banach Center in Bedlewo, Poland.

We are grateful to all the participants of the conference and all the contributors of this volume.

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Double Obstacle Limit for a Navier-Stokes/Cahn-Hilliard System

Helmut Abels

Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. We consider the double obstacle limit for a Navier-Stokes/Cahn-Hilliard type system. The system describes a so-called diffuse interface model for the two-phase flow of two macroscopically immiscible incompressible viscous fluids in the case of matched densities, also known as Model H. Starting with a suitable class of singular free energies, which keep the concentration strictly inside the physically reasonable interval $[a, b]$, we analyze a certain singular limit, where the equation for the chemical potential converges to a differential inclusion related to the subgradient of the indicator function of $[a, b]$.

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Keywords. Two-phase flow, diffuse interface model, mixtures of viscous fluids, Cahn-Hilliard equation, Navier-Stokes equation, double obstacle problem.

1. Introduction and main result

In the present contribution we study a system describing the flow of viscous incompressible Newtonian fluids of the same density, but different viscosity. Although it is assumed that the fluids are macroscopically immiscible, the model takes a partial mixing on a small length scale measured by a parameter $\varepsilon > 0$ into account. Therefore the classical sharp interface between both fluids is replaced by an interfacial region and an order parameter related to the concentration difference of both fluids is introduced. This makes it possible to describe the flow beyond the occurrence of topological singularities of the separating interface (e.g., coalescence or formation of droplets), cf. Anderson and McFadden [5] for a review on that topic.

This model, also known as "model H", cf. Hohenberg and Halperin [10] and Gurtin et al. [9], leads to a coupled Navier-Stokes/Cahn-Hilliard system:

$$\partial_t v + v \cdot \nabla v - \operatorname{div}(\nu(c)Dv) + \nabla p = -\varepsilon \operatorname{div}(\nabla c \otimes \nabla c) \qquad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$\operatorname{div} v = 0 \qquad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

$$\partial_t c + v \cdot \nabla c = m\Delta\mu \qquad \text{in } \Omega \times (0,\infty), \tag{1.3}$$

$$\mu = \varepsilon^{-1}\phi(c) - \varepsilon\Delta c \qquad \text{in } \Omega \times (0,\infty). \tag{1.4}$$

Here v is the mean velocity, $Dv = \frac{1}{2}(\nabla v + \nabla v^T)$, p is the pressure, c is an order parameter related to the concentration of the fluids (e.g., the concentration difference or the concentration of one component), and Ω is a suitable bounded domain. Moreover, $\nu(c) > 0$ is the viscosity of the mixture, $\varepsilon > 0$ is a (small) parameter, which will be related to the "thickness" of the interfacial region, and $\phi = \Phi'$, where Φ is the homogeneous free energy density specified below.

It is assumed that the densities of both components as well as the density of the mixture are constant and for simplicity equal to one. We note that capillary forces due to surface tension are modeled by an extra contribution $\varepsilon\nabla c \otimes \nabla c$ in the stress tensor leading to the term on the right-hand side of (1.1). Moreover, we note that in the modeling diffusion of the fluid components is taken into account. Therefore $m\Delta\mu$ is appearing in (1.3), where $m > 0$ is the mobility coefficient, which is assumed to be constant.

We close the system by adding the boundary and initial conditions

$$v|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{1.5}$$

$$\partial_n c|_{\partial\Omega} = \partial_n \mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{1.6}$$

$$(v,c)|_{t=0} = (v_0, c_0) \qquad \text{in } \Omega. \tag{1.7}$$

Here (1.5) is the usual no-slip boundary condition for viscous fluids, n is the exterior normal on $\partial\Omega$, $\partial_n \mu|_{\partial\Omega} = 0$ means that there is no flux of the components through the boundary, and $\partial_n c|_{\partial\Omega} = 0$ describes a "contact angle" of $\pi/2$ of the diffused interface and the boundary of the domain.

We note that (1.1) can be replaced by

$$\partial_t v + v \cdot \nabla v - \operatorname{div}(\nu(c)Dv) + \nabla g = \mu\nabla c \tag{1.8}$$

with $g = p + \frac{\varepsilon}{2}|\nabla c|^2 + \varepsilon^{-1}\Phi(c)$ since

$$\mu\nabla c = \nabla\left(\frac{\varepsilon}{2}|\nabla c|^2 + \varepsilon^{-1}\Phi(c)\right) - \varepsilon \operatorname{div}(\nabla c \otimes \nabla c). \tag{1.9}$$

The total energy of the system above is given by $E(c,v) = E_{\text{free}}(c) + E_{\text{kin}}(v)$, where

$$E_{\text{free}}(c) = \frac{1}{2}\int_\Omega \varepsilon|\nabla c(x)|^2\,dx + \int_\Omega \varepsilon^{-1}\Phi(c(x))\,dx, \tag{1.10}$$

$$E_{\text{kin}}(v) = \frac{1}{2}\int_\Omega |v(x)|^2\,dx.$$

Here the free energy $E_{\mathrm{free}}(c)$ describes an interfacial energy associated with the region where c is not close to the minima of $\Phi(c)$ and $E_{\mathrm{kin}}(v)$ is the kinetic energy of the fluid. The system is dissipative. More precisely, for sufficiently smooth solutions

$$\frac{d}{dt}E(c(t),v(t)) = -\int_\Omega \nu(c(t))|Dv(t)|^2\,dx - m\int_\Omega |\nabla\mu(t)|^2\,dx.$$

Since we will consider (1.1)–(1.7) only for fixed $\varepsilon > 0$, we will assume for simplicity that $\varepsilon = 1$ in the following. But all statements remain true for arbitrary $\varepsilon > 0$ (with constants depending on ε).

The assumptions on the homogeneous free energy density Φ are motivated by the so-called regular solution model free energy suggested by Cahn and Hilliard [7]:

$$\Phi(c) = \frac{\theta}{2}\left((1+c)\ln(1+c) + (1-c)\ln(1-c)\right) - \frac{\theta_c}{2}c^2 \qquad (1.11)$$

with $\theta, \theta_c > 0$. We note that $\Phi(c)$ is not convex if and only if $0 < \theta < \theta_c$. But we have the decomposition

$$\Phi(s) = \theta\Phi_0(s) - \frac{\theta_c}{2}s^2, \qquad \phi(s) = \theta\phi_0(s) - \theta_c s$$

where $\Phi_0 \in C([-1,1]) \cap C^\infty((-1,1))$ is convex and $\theta, \theta_c > 0$. Finally, we note that

$$\phi_0(s) \to_{s\to\pm1} \pm\infty.$$

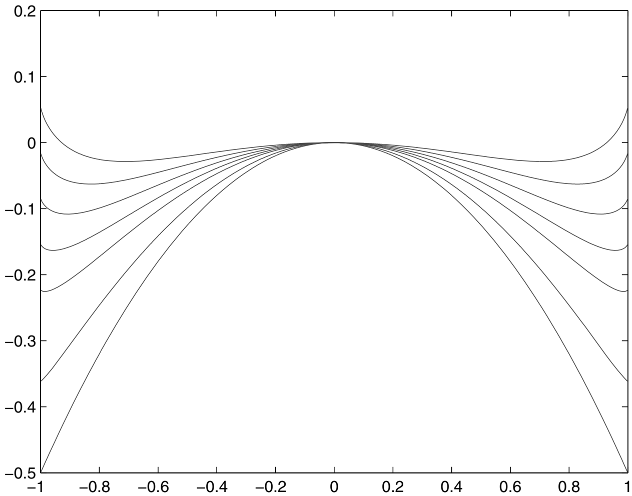

FIGURE 1. Logarithmic free energy density for $\theta = 0.8$, 0.7, 0.6, 0.5, 0.4, 0.2, 0.0 (from top to bottom) and $\theta_c = 1$

In Figure 1 the free energy density is plotted for some choices of θ, θ_c. Note that, if $\theta > 0$, then the minima are never ± 1. (Although Figure 1 for $\theta = 0.2$ suggests that the minima are ± 1.) But the minima are close to ± 1 if θ is small in comparison with θ_c. Since for two macroscopically immiscible fluids mixing costs a lot of energy, we think that a small θ in comparison with θ_c is physically the most meaningful choice. Since qualitatively the free energy density for $\theta = 0.2$ in Figure 1 looks already the same as for $\theta = 0$, it is very reasonable to choose $\theta = 0$ directly, i.e., to choose the free energy density

$$\Phi(c) = \begin{cases} -\frac{\theta_c}{2}c^2 & \text{if } c \in [-1,1] \\ +\infty & \text{else.} \end{cases}$$

The free energy density is called to be of double obstacle type because of the constraint $c \in [-1,1]$ for all c with $\Phi(c) < \infty$. The Cahn-Hilliard equation with the latter free energy was first studied by Blowey and Elliott [6]. Elliott and Luckhaus [8] have shown that as $\theta \to 0$ the solutions of the Cahn-Hilliard equation with the logarithmic free energy density converge to solutions of the latter free energy of double obstacle type. Physically this limit describes the dynamics of phase separating binary mixture, where the absolute temperature θ is far from the critical temperature θ_c below which phase separation occurs.

The main result of this contribution is that weak solutions of the Model H (1.1)–(1.7) converge as $\theta \to 0$ (for a suitable subsequence) to weak solutions of the corresponding Navier-Stokes/Cahn-Hilliard system, where (1.4) is replaced by a differential inclusion related to the subgradient of the double obstacle free energy, cf. Section 4 for details. In the following we will assume a slightly more general form of the free energy than (1.11). More precisely, we assume that

$$\Phi(s) = \theta\Phi_0(s) - \frac{\theta_c}{2}s^2, \qquad \phi(s) = \theta\phi_0(s) - \theta_c s \tag{1.12}$$

where $\theta, \theta_c > 0$, $\Phi_0 \in C([a,b]) \cap C^2((a,b))$ is convex, $\phi(s) = \Phi'(s)$, $a < b$, and

$$\phi_0(s) \to_{s\to a} -\infty \qquad \phi_0(s) \to_{s \to b} \infty. \tag{1.13}$$

We note that this assumption implies the (θ-independent) assumption made in [3] for the free energy density.

The structure of the article is as follows: First we fix some notation and recall some basic lemmas in Section 2. Then we study the double obstacle limit for the convex part of the free energy E_{free} and the convective Cahn-Hilliard equation (1.3)–(1.4) in Section 3. In Section 4 we state and prove our main result on convergence of weak solutions of (1.1)–(1.7) as $\theta \to 0$. We conclude with two results on uniqueness and regularity of weak solutions for the limit system, which are the same in [3] for the case $\theta > 0$. These results are part of the author's Habilitation thesis [1].

2. Notation and preliminaries

Throughout the article $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, will denote a bounded domain with C^3-boundary and $Q_T := \Omega \times (0, T), Q := Q_\infty$.

We use the same notation as in [3] and refer to the latter article for precise definitions and references. Let us just recall some notation. The inner product of a Hilbert space H is denoted by $(.,.)_H$ and we use the abbreviation $(.,.)_M$ for $(.,.)_{L^2(M)}$. The duality product of a Banach space X and its dual X' is denoted by $\langle .,.\rangle_{X',X}$ or just $\langle .,.\rangle$.

Moreover,

$$L^2_\sigma(\Omega) = \left\{ f \in L^2(\Omega)^d : \operatorname{div} f = 0, n \cdot f|_{\partial\Omega} = 0 \right\},$$

$$V^s_2(\Omega) = \begin{cases} H^s(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega) & \text{if } s \geq 1, \\ H^s(\Omega)^d \cap L^2_\sigma(\Omega) & \text{if } 0 \leq s < \frac{1}{2} \end{cases}$$

and $V_2(\Omega) := V^1_2(\Omega)$.

For $m \in \mathbb{R}$ we set

$$L^q_{(m)}(\Omega) := \left\{ f \in L^q(\Omega) : m(f) := \frac{1}{|\Omega|} \int_\Omega f(x)\, dx = m \right\}, \qquad 1 \leq q \leq \infty,$$

and $P_0 f := f - m(f)$ is the orthogonal projection onto $L^2_{(0)}(\Omega)$. Furthermore, we define

$$H^1_{(0)} \equiv H^1_{(0)}(\Omega) = H^1(\Omega) \cap L^2_{(0)}(\Omega), \qquad H^{-1}_{(0)} \equiv H^{-1}_{(0)}(\Omega) = H^1_{(0)}(\Omega)'.$$

We equip $H^1_{(0)}(\Omega)$ with the inner product $(c, d)_{H^1_{(0)}(\Omega)} := (\nabla c, \nabla d)_{L^2(\Omega)}$. Then the Riesz isomorphism $\mathcal{R}\colon H^1_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ is given by

$$\langle \mathcal{R}c, d\rangle_{H^{-1}_{(0)}, H^1_{(0)}} = (c, d)_{H^1_{(0)}} = (\nabla c, \nabla d)_{L^2}, \qquad c, d \in H^1_{(0)}(\Omega),$$

i.e., $\mathcal{R} = -\Delta_N$ is the negative (weak) Laplace operator with Neumann boundary conditions. We note that, if $u \in H^1_{(0)}(\Omega)$ solves $\Delta_N u = f$ for some $f \in L^q_{(0)}(\Omega)$, $1 < q < \infty$, and $\partial\Omega$ is of class C^2, then it follows from standard elliptic theory that $u \in W^2_q(\Omega)$ and $\Delta u = f$ a.e. in Ω and $\partial_n u|_{\partial\Omega} = 0$ in the sense of traces. If additionally $f \in W^1_q(\Omega)$ and $\partial\Omega \in C^3$, then $u \in W^3_q(\Omega)$. Moreover,

$$\|u\|_{W^{k+2}_q(\Omega)} \leq C_q \|f\|_{W^k_q(\Omega)} \qquad \text{for all } f \in W^k_q(\Omega) \cap L^q_{(0)}(\Omega), k = 0, 1, \tag{2.1}$$

with a constant C_q depending only on $1 < q < \infty$, d, k, and Ω. Finally we denote

$$W^2_{p,N}(\Omega) = \left\{ u \in W^2_p(\Omega) : \partial_n u|_{\partial\Omega} = 0 \right\},$$

where $1 < p < \infty$.

Concerning vector-valued spaces, we recall that $BC(0, T; X)$ is the Banach space of all bounded and continuous $f\colon [0, T) \to X$ equipped with the supremum norm and $BUC(0, T; X)$ is the subspace of all bounded and uniformly continuous functions, where X is a Banach space. Moreover, we define $BC_w(0, T; X)$ as the topological vector space of all bounded and weakly continuous functions

$f\colon [0,T) \to X$. Furthermore, $f \in L^q_{\rm loc}([0,\infty);X)$ if and only if $f \in L^q(0,T;X)$ for every $T > 0$. Moreover, $L^q_{\rm uloc}([0,\infty);X)$ denotes the *uniformly local* variant of $L^q(0,\infty;X)$ consisting of all strongly measurable $f\colon [0,\infty) \to X$ such that

$$\|f\|_{L^q_{\rm uloc}([0,\infty);X)} = \sup_{t\geq 0} \|f\|_{L^q(t,t+1;X)} < \infty.$$

In order to derive some regularity estimates we will use vector-valued Besov spaces $B^s_{q\infty}(I;X)$, where $s \in (0,1)$, $1 \leq q \leq \infty$, I is an interval, and X is a Banach space. They are defined as

$$B^s_{q\infty}(I;X) = \left\{ f \in L^q(I;X) : \|f\|_{B^s_{q\infty}(I;X)} < \infty \right\},$$
$$\|f\|_{B^s_{q\infty}(I;X)} = \|f\|_{L^q(I;X)} + \sup_{0<h\leq 1} \|\Delta_h f(t)\|_{L^q(I_h;X)},$$

where $\Delta_h f(t) = f(t+h) - f(t)$ and $I_h = \{t \in I : t+h \in I\}$. Moreover, we set $C^s(I;X) = B^s_{\infty\infty}(I;X)$, $s \in (0,1)$. Finally, $B^s_{q\infty,\rm uloc}([0,\infty);X)$ is defined in the obvious way replacing $L^q(0,\infty;X)$-norms by $L^q_{\rm uloc}([0,\infty);X)$-norms.

Let us conclude with two useful lemmas for the following.

Lemma 2.1. *Let X, Y be two Banach spaces such that $Y \hookrightarrow X$ and $X' \hookrightarrow Y'$ densely and let $0 < T \leq \infty$. Then $L^\infty(0,T;Y) \cap BUC([0,T];X) \hookrightarrow BC_w([0,T];Y)$.*

We refer to [2, Lemma 4.1] for a proof.

Lemma 2.2. *Let $E\colon [0,T) \to [0,\infty)$, $0 < T \leq \infty$, be a lower semi-continuous function and let $D\colon (0,T) \to [0,\infty)$ be an integrable function. Then*

$$E(0)\varphi(0) + \int_0^T E(t)\varphi'(t)\,dt \geq \int_0^T D(t)\varphi(t)\,dt \qquad (2.2)$$

holds for all $\varphi \in W^1_1(0,T)$ with $\varphi(T) = 0$ if and only if

$$E(t) + \int_s^t D(\tau)\,d\tau \leq E(s) \qquad (2.3)$$

holds for all $s \leq t < T$ and almost all $0 \leq s < T$ including $s = 0$.

See [2, Lemma 4.3] for a proof.

3. Double obstacle limit for the Cahn-Hilliard equation

3.1. Limit of the energy

In this section we study the "convex part" of $E_{\rm free}$ as in (1.10), namely

$$E_\theta(c) = \int_\Omega \frac{|\nabla c|^2}{2}\,dx + \int_\Omega \theta\Phi_0(c(x))\,dx, \qquad \theta > 0, \qquad (3.1)$$

as $\theta \to 0$, where Φ_0 is the same as in (1.12), (1.13).

Firstly, E_θ is defined on $L^2_{(m)}(\Omega)$, $m \in (a,b)$, with

$$\operatorname{dom} E_\theta = \left\{ c \in H^1(\Omega) \cap L^2_{(m)}(\Omega) : c(x) \in [a,b] \text{ a.e.} \right\}.$$

But we will assume in the following $m = 0$ without loss of generality. By a simple shift of c and Φ by m we can always reduce to this case.

We denote by $\partial E_\theta(c)\colon L^2_{(0)}(\Omega) \to \mathcal{P}(L^2_{(0)}(\Omega))$ the subgradient of E_θ at $c \in \operatorname{dom} E_\theta$, i.e., $w \in \partial E_\theta(c)$ if and only if

$$(w, c' - c)_{L^2} \leq E_\theta(c') - E_\theta(c) \qquad \text{for all } c' \in L^2_{(0)}(\Omega).$$

From [3, Corollary 1], see also [4, Corollary 4.4], we recall:

Lemma 3.1. *Let E_θ be as above and extend E_θ to a functional $\widetilde{E}_\theta\colon H^{-1}_{(0)}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by setting $\widetilde{E}_\theta(c) = E_\theta(c)$ if $c \in \operatorname{dom} E_\theta$ and $\widetilde{E}_\theta(c) = +\infty$ else. Then $\widetilde{E}_\theta$ is a proper, convex, and lower semi-continuous functional, $\partial\widetilde{E}_\theta$ is a maximal monotone operator with $\partial\widetilde{E}_\theta(c) = -\Delta_N \partial E_\theta(c)$, and*

$$\mathcal{D}(\partial\widetilde{E}_\theta) = \left\{ c \in \mathcal{D}(\partial E_\theta) : \partial E_\theta(c) = -\Delta c + \theta P_0 \phi_0(c) \in H^1_{(0)}(\Omega) \right\}, \tag{3.2}$$

where

$$\mathcal{D}(\partial E_\theta) = \left\{ c \in H^2(\Omega) \cap L^2_{(0)}(\Omega) : \phi_0(c) \in L^2, \phi_0'(c)|\nabla c|^2 \in L^1, \partial_n c|_{\partial\Omega} = 0 \right\}$$

and $\partial E_\theta(c) = -\Delta c + \theta P_0 \phi_0(c)$. Moreover, for every $c \in \mathcal{D}(\partial\widetilde{E}_\theta)$

$$\|c\|_{W^2_r} + \|\phi_0(c)\|_r \leq C_{r,\theta} \left(\|\partial E_\theta(c)\|_{H^1_{(0)}} + \|c\|_2 + 1 \right), \tag{3.3}$$

where $r = 6$ if $d = 3$ and $2 \leq r < \infty$ is arbitrary if $d = 2$.

Remark 3.2. We note that in [3, Corollary 1] the case $\theta = 1$ is considered. This implies the lemma for every $\theta > 0$ where $C_{r,\theta}$ in (3.3) depends on $\theta > 0$. But one crucial observation for the following is that the estimate (3.3) is valid with a constant C_r independent of $0 < \theta \leq 1$.

Proposition 3.3. *Let E_θ, $0 < \theta \leq 1$, be as above, let $\widetilde{E}_\theta$ be the extension to $H^{-1}_{(0)}(\Omega)$, let $R > 0$ and let $r = 6$ if $d = 3$ and $2 \leq r < \infty$ arbitrary if $d = 2$. Then there are constants $C(R), C'(r, R) > 0$ independent of $0 < \theta \leq 1$ such that*

$$\|c\|_{H^2(\Omega)} + \theta \|\phi_0(c)\|_{L^2(\Omega)} \leq C(R) \left(\|\partial E_\theta(c)\|_{L^2(\Omega)} + 1 \right) \tag{3.4}$$

for all $c \in \mathcal{D}(\partial E_\theta)$ with $\|c\|_{L^2(\Omega)} \leq R$ and

$$\|c\|_{W^2_r(\Omega)} + \theta \|\phi_0(c)\|_{L^r(\Omega)} \leq C'(r, R) \left(\|\partial\widetilde{E}_\theta(c)\|_{H^{-1}_{(0)}(\Omega)} + 1 \right) \tag{3.5}$$

for all $c \in \mathcal{D}(\partial\widetilde{E}_\theta)$ with $\|c\|_{L^2(\Omega)} \leq R$.

Proof. Let $c \in \mathcal{D}(\partial E_\theta)$. First we show a suitable estimate for Δc and $\theta\phi_0(c)$ in $L^2(\Omega)$ which is independent of $\theta \in (0, 1]$. Taking the L^2-inner product of $\partial E_\theta(c) = -\Delta c + P_0 \theta \phi_0(c)$ and $-\Delta c$, we conclude that

$$\int_\Omega |\Delta c|^2 \, dx + \theta \int_\Omega \phi_0'(c) |\nabla c|^2 \, dx = -(\partial E_\theta(c), \Delta c)_\Omega.$$

Therefore

$$\|\Delta c\|_{L^2}^2 + \|\theta P_0 \phi_0(c)\|_{L^2}^2 + \theta \int_\Omega \phi_0'(c)|\nabla c|^2\,dx \le C\|\partial E_\theta(c)\|_{L^2}^2 \tag{3.6}$$

uniformly in $0 < \theta \le 1$ because of $\theta P_0\phi_0(c) = \partial E_\theta(c) + \Delta c$. In order to estimate $\theta m(\phi_0(c))$, we follow some arguments which also can be found in [8, §4] and [11, Lemma 5.2]. To this end we multiply $-\Delta c + P_0\theta\phi_0(c)$ with $c \in L^2_{(0)}(\Omega)$ and obtain

$$\int_\Omega |\nabla c|^2\,dx + \theta\int_\Omega \phi_0(c)c\,dx = (\partial E_\theta(c), c)_\Omega. \tag{3.7}$$

Now we choose $\varepsilon > 0$ so small that $\phi_0(c) \le 0, -c \ge \varepsilon$ for all $c \in (a, a+\varepsilon]$, $\phi_0(c) \ge 0, c \ge \varepsilon$ for all $c \in [b-\varepsilon, b)$. This is possible because of $\phi_0(c) \to_{c\to a} -\infty, \phi_0(c) \to_{c\to b} \infty$. (Note that ε depends only on ϕ_0 and $m = 0 \in (a,b)$.) Then

$$\begin{aligned}\int_\Omega \phi_0(c)c\,dx &= \int_{\{c(x)\in(a,a+\varepsilon]\}} \phi_0(c)c\,dx + \int_{\{c(x)\in(a+\varepsilon,b-\varepsilon)\}} \phi_0(c)c\,dx \\ &\quad + \int_{\{c(x)\in[b-\varepsilon,b)\}} \phi_0(c)c\,dx \\ &\ge \varepsilon \int_{\{c(x)\in(a,a+\varepsilon]\cup[b-\varepsilon,b)\}} |\phi_0(c)|\,dx - C(\varepsilon,\Omega)\|c\|_{L^2(\Omega)}\end{aligned}$$

because of $|\phi_0(c)| \le C_\varepsilon$ on $[a+\varepsilon, b-\varepsilon]$. Using (3.7), we conclude

$$\theta\int_\Omega |\phi_0(c)|\,dx \le C(R,\varepsilon)(\|\partial E_\theta(c)\|_{L^2} + 1)$$

provided that $\|c\|_{L^2} \le R$. Combining this with (3.6), we obtain (3.4).

In order to prove (3.5), we multiply $-\Delta c + \theta\phi_0(c)$ with $\theta^{r-1}|\phi_0(c)|^{r-2}\phi_0(c) \in L^{r'}(\Omega)$ and obtain

$$\begin{aligned}(r-1)\theta^{r-1}\int_\Omega |\phi_0(c)|^{r-2}\phi_0'(c)|\nabla c|^2\,dx + \theta^r\int_\Omega |\phi_0(c)|^r\,dx \\ \le C(\Omega, r, R)\left(\|\partial E_\theta(c)\|_{L^r(\Omega)} + \theta m(\phi_0(c))\right)\|\theta\phi_0(c)\|_r^{r-1}.\end{aligned}$$

Hence

$$\theta\|\phi_0(c)\|_{L^r(\Omega)} \le C(R)(\|\partial E_\theta(c)\|_{L^r(\Omega)} + 1) \tag{3.8}$$

uniformly in $0 < \theta \le 1$ provided that $\|c\|_{L^2} \le R$. This implies (3.5) because of (2.1) and $\partial_n c|_{\partial\Omega} = 0$. $\square$

Now we consider the limit $\theta \to 0$ of E_θ. Since

$$\lim_{\theta\to 0} E_\theta(c) = \int_\Omega \frac{|\nabla c|^2}{2}\,dx + \int_\Omega I_{[a,b]}(c(x))\,dx =: E_0(c) \tag{3.9}$$

for all $c \in H^1(\Omega)$, where $I_{[a,b]}$ denotes the indicator function of $[a,b]$, which is defined as

$$I_{[a,b]}(s) = \begin{cases} 0 & \text{if } s \in [a,b], \\ +\infty & \text{else,}\end{cases}$$

we expect that ∂E_θ converges to the subgradient of ∂E_0 in a suitable sense and under suitable conditions. We note that

$$\partial I_{[a,b]}(c) = \begin{cases} 0 & \text{if } c \in (a,b), \\ [0,\infty) & \text{if } c = b, \\ (-\infty, 0] & \text{if } c = a, \\ \emptyset & \text{else.} \end{cases}$$

Moreover, E_0 is lower semi-continuous on $H = L^2_{(0)}(\Omega)$ and $H = H^{-1}_{(0)}(\Omega)$ since $\liminf_{k\to\infty} E_0(c_k) < \infty$ and $c_k \to_{k\to\infty} c$ in H implies

$$c_{k_j} \in \operatorname{dom} E_0, \quad \sup_{j\in\mathbb{N}_0} \|\nabla c_{k_j}\|^2_{L^2} < \infty, \quad c_{k_j} \rightharpoonup_{j\to\infty} c \quad \text{in } H^1(\Omega) \text{ and a.e.}$$

for some subsequence $(c_{k_j})_{j\in\mathbb{N}}$. Therefore

$$E_0(c) = \frac{1}{2}\|\nabla c\|_2^2 \leq \liminf_{k\to\infty} \frac{1}{2}\|\nabla c_k\|^2_{L^2} = \liminf_{k\to\infty} E_0(c_k).$$

The following corollary will be the essential tool for passing to the limit in the convective Cahn-Hilliard equation (1.3)–(1.4).

Corollary 3.4. *Let $c_k \in L^\infty(0,T;L^2_{(0)}(\Omega))$, $0 < T < \infty$, $k \in \mathbb{N}_0$, be a bounded sequence, let $\theta_k > 0$ be such that $\theta_k \to_{k\to\infty} 0$, and assume that $c_k(t) \to_{k\to\infty} c(t)$ in $L^2_{(0)}(\Omega)$ and $c_k(t) \in \mathcal{D}(\partial E_{\theta_k})$ for almost every $t \in (0,T)$. If*

$$\partial E_{\theta_k}(c_k) \rightharpoonup_{k\to\infty} \mu_0 \quad \textit{in } L^q(0,T;L^2(\Omega))$$

for some $\mu_0 \in L^q(0,T;L^2_{(0)}(\Omega))$ and some $1 < q < \infty$, then $c_k \rightharpoonup_{k\to\infty} c$ in $L^q(0,T;H^2(\Omega))$, $\mu_0(t) \in \partial E_0(c(t))$ for almost every $t \in (0,T)$, where E_0 is considered as a functional on $L^2_{(0)}(\Omega)$, and

$$-\Delta c + P_0 f = \mu_0 \quad \textit{with } f(x,t) \in \partial I_{[a,b]}(c(x,t)) \textit{ for almost all } (x,t) \in Q_T,$$

as well as $\partial_n c(t)|_{\partial\Omega} = 0$ for almost every $t \in (0,T)$. Moreover, if additionally $\partial E_{\theta_k}(c_k)$ is bounded in $L^q(0,T;L^r(\Omega))$, for some $2 \leq r < \infty$, then

$$\|c\|_{L^q(0,T;W^2_r(\Omega))} \leq \sup_{k\in\mathbb{N}_0} C(R)\left(1 + \|\partial E_{\theta_k}(c_k)\|_{L^q(0,T;L^r(\Omega))}\right). \tag{3.10}$$

Proof. Because of (3.4), there is a subsequence $k_j \to_{j\to\infty} \infty$

$$\theta_{k_j}\phi_0(c_{k_j}) \rightharpoonup_{j\to\infty} f \qquad \text{in } L^q(0,T;L^2(\Omega)).$$

Moreover, since $c_k(t) \to_{k\to\infty} c(t)$ in $L^2(\Omega)$ for almost every $t \in (0,T)$ and $(c_k)_{k\in\mathbb{N}}$, is bounded in $L^\infty(0,T;L^2(\Omega))$, $c_k \to_{k\to\infty} c$ in $L^r(0,T;L^2(\Omega))$ for all $1 \leq r < \infty$ by Lebesgue's theorem on dominated convergence. Together with the boundedness of c_k in $L^q(0,T;H^2(\Omega))$ due to (3.4) this implies $c_k \to_{k\to\infty} c$ in $L^q(0,T;C^0(\overline{\Omega}))$ because of

$$\|f\|_\infty \leq C\|f\|_{L^2}^{1-\frac{d}{4}}\|f\|_{H^2}^{\frac{d}{4}},$$

cf. [3, Equation (2.15)] for a reference.

Therefore for a suitable subsequence $c_{k_j}(t) \to c(t)$ in $C^0(\overline{\Omega})$ as $j \to \infty$ for almost every $t \in (0,T)$. Therefore

$$\theta_{k_j}\phi_0(c_{k_j}(x,t)) \to_{j\to\infty} 0 = f(x,t) \quad \text{a.e. in} \quad \{(x,t) \in Q_T : c(x,t) \in (a,b)\}\,.$$

On the other hand, if $c(x,t) = a$ for some $x \in \overline{\Omega}$ and some $t \in (0,T)$ such that $c_{k_j}(t)$ converges strongly in $C(\overline{\Omega})$, then $\phi_0(c_{k_j}(x,t)) \leq 0$ for sufficiently large j and therefore $f(x,t) \leq 0$, i.e., $f(x,t) \in \partial I_{[a,b]}(a)$, almost everywhere on $\{c(x,t) = a\}$. By the same argument $f(x,t) \geq 0$, i.e., $f(x,t) \in \partial I_{[a,b]}(b)$ almost everywhere on $\{c(x,t) = b\}$. On the other hand

$$P_0 f = \lim_{j\to\infty} P_0\theta_{k_j}\phi_0(c_{k_j}) = \lim_{k\to\infty} (\partial E_{\theta_k}(c_k) + \Delta c_k) = \mu_0 + \Delta c$$

weakly in $L^q(0,T;L^2(\Omega))$. Hence it only remains to prove $\mu_0(t) \in \partial E_0(c(t))$ for almost every $t \in (0,T)$. But this follows from the fact that

$$\begin{aligned}\int_0^T \eta(t)(\mu_0(t), c' - c(t))_{L^2_{(0)}}\, dt &= \lim_{k\to\infty}\int_0^T \eta(t)(\partial E_{\theta_k}(c_k(t)), c' - c_k(t))_{L^2_{(0)}}\, dt \\ &\leq \lim_{k\to\infty}\int_0^T \eta(t)\left(E_{\theta_k}(c') - E_{\theta_k}(c_k(t))\right) dt \\ &= \int_0^T \eta(t)\left(E_0(c') - E_0(c(t))\right) dt\end{aligned}$$

for all $c' \in \operatorname{dom} E_0 = \operatorname{dom} E_{\theta_k}$ and $\eta \in C_0^\infty(0,T)$ with $\eta \geq 0$ since $\theta_k\Phi_0(s) \to_{k\to\infty} 0$ uniformly in $s \in [a,b]$ and $c_k(t) \to_{k\to\infty} c(t)$ in $C(\overline{\Omega})$ for almost all $t \in (0,T)$. Finally, (3.10) follows from (3.8), $\partial_n c_k|_{\partial\Omega} = 0$, and (2.1). $\square$

For completeness we give a characterization of ∂E_0 and $\mathcal{D}(\partial E_0)$, which is based on the results of Kenmochi et al. [11].

Lemma 3.5. *Let E_0 be as above and let ∂E_0 be its subgradient with respect to $L^2_{(0)}(\Omega)$. Then*

$$\mathcal{D}(\partial E_0) = \left\{c \in H^2(\Omega) \cap L^2_{(0)}(\Omega) : c(x) \in [a,b] \text{ a.e., } \partial_n c|_{\partial\Omega} = 0\right\} \qquad (3.11)$$

and $\mu_0 \in \partial E_0(c)$, $c \in \mathcal{D}(\partial E_0)$, if and only if $c \in H^2(\Omega) \cap L^2_{(0)}(\Omega)$, $\partial_n c|_{\partial\Omega} = 0$, and

$$\beta(x) := \mu_0(x) + \Delta c(x) + \overline{\mu} \in \partial I_{[a,b]}(c(x)) \qquad \text{a.e. in } \Omega \qquad (3.12)$$

for some $\overline{\mu} \in \mathbb{R}$. Moreover, $\|c\|_{H^2} + \|P_0\beta\|_{L^2} \leq C\left(\|\mu_0\|_{L^2} + \|c\|_{L^2}\right)$.

Finally, if $\widetilde{E}_0$ is the extension of E_0 to $H^{-1}_{(0)}(\Omega)$, then $w \in \partial\widetilde{E}_0(c)$ for some $c \in \mathcal{D}(\partial\widetilde{E}_0(c))$ if and only if $w = -\Delta_N \mu_0$, where $\mu_0 \in H^1(\Omega) \cap \partial E_0(c)$.

Proof. First let $c \in \mathcal{D}(\partial E_0)$ and let $\mu_0 \in \partial E_0(c)$. Then by definition

$$(\mu_0, c' - c)_{L^2} \leq E_0(c') - E_0(c) \qquad \text{for all } c' \in \operatorname{dom}(E_0).$$

Hence c is a minimizer of the functional $F(c') = E_0(c') - (\mu_0, c')_{L^2}$ defined on $L^2_{(0)}(\Omega)$. Since $F(c')$ is strictly convex, the minimizer is unique. Therefore [11, Proposition 5.1, Lemma 5.3] imply that $\Delta_N c \in L^2(\Omega)$ and

$$-\Delta_N c + \beta = \mu_0 + \overline{\mu},$$

where $\beta(x) \in \partial I_{[a,b]}(c(x))$ for almost all $x \in \Omega$, $\overline{\mu} \in \mathbb{R}$, and $\|\Delta_N c\|_{L^2} + \|P_0\beta\|_{L^2} \leq C\|\mu_0\|_{L^2}$. Hence $c \in H^2(\Omega)$, $\partial_n c|_{\partial\Omega} = 0$, and (3.12) holds. This proves one implication and one set inclusion in (3.11).

To prove the converse implication let $c \in H^2(\Omega) \cap L^2_{(0)}(\Omega)$, $\partial_n c|_{\partial\Omega} = 0$, and let $\mu_0 \in L^2_{(0)}(\Omega)$ satisfy (3.12). Then

$$(\mu_0 + \Delta c + \overline{\mu}, c' - c)_\Omega \leq \int_\Omega \left(I_{[a,b]}(c') - I_{[a,b]}(c)\right)\, dx = 0$$

for all $c' \in \operatorname{dom} E_0$ since $\mu_0 + \Delta c + \overline{\mu} \in \partial I_{[a,b]}(c)$. Moreover, using

$$(\Delta c, c' - c)_\Omega = \frac{1}{2}\|\nabla c\|^2_{L^2} - \frac{1}{2}\|\nabla c'\|^2_{L^2} + \frac{1}{2}\|\nabla(c - c')\|^2_{L^2},$$

we conclude

$$(\mu_0, c' - c)_\Omega \leq E_0(c') - E_0(c) \qquad \text{for all } c' \in \operatorname{dom} E_0,$$

i.e., $\mu_0 \in \partial E_0(c)$. Furthermore, for every $c \in H^2(\Omega) \cap L^2_{(0)}(\Omega)$ with $\partial_n c|_{\partial\Omega} = 0$ we have that $-\Delta c \in \partial E_0(c)$ since

$$E_0(c') - E_0(c) = \int_\Omega \nabla c \cdot \nabla(c' - c)\, dx + \frac{1}{2}\int_\Omega |\nabla(c' - c)|^2\, dx \geq -\int_\Omega \Delta c(c' - c)\, dx$$

for all $c' \in \operatorname{dom}(E_0)$. Therefore (3.11) holds.

In order to prove the last statement, let $w \in \partial\widetilde{E}_0(c)$. Then $\mu_0 = -\Delta_N^{-1} w \in H^1(\Omega) \cap \partial E_0(c)$ since

$$(\mu_0, c' - c)_{L^2} = (w, c' - c)_{H^{-1}_{(0)}} \leq \widetilde{E}_0(c') - \widetilde{E}_0(c) = E_0(c') - E_0(c)$$

for all $c' \in \operatorname{dom} E_0 = \operatorname{dom} \widetilde{E}_0$. Conversely, if $\mu_0 \in H^1(\Omega) \cap \partial E_0(c)$, then $w := -\Delta_N \mu_0 \in H^{-1}_{(0)}(\Omega) \in \partial\widetilde{E}_0(c)$ by the same calculation as before. □

3.2. Convective Cahn-Hilliard equation

In this section we consider

$$\partial_t c + v \cdot \nabla c = \Delta\mu \qquad \text{in } \Omega \times (0, \infty), \tag{3.13}$$

$$\mu = \phi(c) - \Delta c \qquad \text{in } \Omega \times (0, \infty), \tag{3.14}$$

$$\partial_n c|_{\partial\Omega} = \partial_n \mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0, \infty), \tag{3.15}$$

$$c|_{t=0} = c_0 \qquad \text{in } \Omega \tag{3.16}$$

for given c_0 with $E_{\mathrm{free}}(c_0) < \infty$ and $v \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; H^1(\Omega))$. Here $\phi = \Phi'$ where Φ is as in (1.12)–(1.13) and $E_{\mathrm{free}}(c)$ is as in (1.10). In particular (1.12) yields the decomposition

$$E_{\mathrm{free}}(c) = E_\theta(c) - \frac{\theta_c}{2}\|c\|^2_{L^2(\Omega)},$$

where $E_\theta(c)$ is as in Section 3.1. We can apply Corollary 3.1 to $E_\theta(c)$, respectively to its extension to $H^{-1}_{(0)}(\Omega)$, which is denoted by $\widetilde{E}_\theta(c)$.

We note that we can consider (3.13)–(3.16) as an evolution equation on $H^{-1}_{(0)}(\Omega)$:

$$\partial_t c(t) + \mathcal{A}_\theta(c(t)) + \mathcal{B}(v(t))c(t) = 0, \qquad \text{for } t > 0, \tag{3.17}$$

$$c|_{t=0} = c_0 \tag{3.18}$$

where $\mathcal{A}_\theta(c) = \partial \widetilde{E}_\theta(c)$ and

$$\langle \mathcal{B}(v)c, \varphi\rangle_{H^{-1}_{(0)}, H^1_{(0)}} = (v\cdot\nabla c, \varphi)_{L^2} - \theta_c(\nabla c, \nabla\varphi)_{L^2}$$

for all $c, \varphi \in \mathcal{D}(\mathcal{B}(v)) = H^1_{(0)}(\Omega)$. This means that $\mathcal{A}_\theta(c) = \Delta_N(\Delta c - \theta P_0\phi_0'(c))$ due to Corollary 3.1 and $\mathcal{B}(v)c = v\cdot\nabla c + \theta_c\Delta_N c$, where $\Delta_N\colon H^1_{(0)}(\Omega) \subset H^{-1}_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ is the Laplace operator with Neumann boundary conditions, which is considered as an unbounded operator on $H^{-1}_{(0)}(\Omega)$. Finally, we note that $\mathcal{A}_\theta$ is a strictly monotone operator since

$$\begin{aligned}(\mathcal{A}_\theta(c_1) - \mathcal{A}_\theta(c_2), c_1 - c_2)_{H^{-1}_{(0)}} \\ = (-\Delta(c_1 - c_2) + \theta\phi_0(c_1) - \theta\phi_0(c_2), c_1 - c_2)_{L^2} \geq \|\nabla(c_1 - c_2)\|^2_{L^2}\end{aligned} \tag{3.19}$$

for all $c_1, c_2 \in \mathcal{D}(\mathcal{A}_\theta)$.

From [3] we recall:

Theorem 3.6. *Let $v \in L^2(0,\infty; H^1(\Omega)) \cap L^\infty(0,\infty; L^2_\sigma(\Omega))$. Then for every $c_0 \in H^1_{(0)}(\Omega)$ with $M := E_{\mathrm{free}}(c_0) < \infty$ there is a unique solution*

$$c \in BC([0,\infty); H^1_{(0)}(\Omega))$$

of (3.13)–(3.16) with $\partial_t c \in L^2(0,\infty; H^{-1}_{(0)}(\Omega))$, $\mu \in L^2_{\mathrm{uloc}}([0,\infty); H^1(\Omega))$. This solution satisfies

$$E_{\mathrm{free}}(c(t)) + \int_{Q_t} |\nabla\mu|^2\, d(x,\tau) = E_{\mathrm{free}}(c_0) - \int_{Q_t} v\cdot\mu\nabla c\, d(x,\tau) \tag{3.20}$$

for all $t \in [0,\infty)$ and

$$\|c\|^2_{L^\infty(0,\infty;H^1)} + \|\partial_t c\|^2_{L^2(0,\infty;H^{-1}_{(0)})} + \|\nabla\mu\|^2_{L^2(Q)} \leq C\left(M + \|v\|^2_{L^2(Q)}\right) \tag{3.21}$$

$$\|c\|^2_{L^2_{\mathrm{uloc}}([0,\infty);W^2_r)} + \|\phi(c)\|^2_{L^2_{\mathrm{uloc}}([0,\infty);L^r)} \leq C_r\left(M + \|v\|^2_{L^2(Q)}\right) \tag{3.22}$$

where $r = 6$ if $d = 3$ and $1 < r < \infty$ is arbitrary if $d = 2$. Here C, C_r are independent of v, c_0. Moreover, for every $R, T > 0$ the solution

$$c \in Y := L^2(0,T; W^2_r(\Omega)) \cap H^1(0,T; H^{-1}_{(0)}(\Omega))$$

depends continuously on

$$(c_0, v) \in X := H^1(\Omega) \times L^1(0,T; L^2_\sigma(\Omega)) \quad \text{such that } E_{\text{free}}(c_0) + \|v\|_{L^2(0,\infty;H^1)} \le R$$

with respect to the weak topology on Y and the strong topology on X.

Remark 3.7. For fixed θ ($\theta = 1$) the theorem coincides with [3, Theorem 6], where we note that (3.21) is only based on the energy estimate and (3.22) is based on (3.3). Since the constant in (3.3) can be chosen independent of $0 < \theta \le 1$, cf. (3.5), it is easy to observe from the proof of [3, Theorem 6] that the inequalities (3.21)–(3.22) hold uniformly with respect to $0 < \theta \le 1$ due to Proposition 3.3.

The following improved regularity statement will be important to get higher regularity of solutions to the Navier-Stokes/Cahn-Hilliard system.

Lemma 3.8. *Let the assumption of Theorem 3.6 be satisfied and let (c, μ) be the corresponding solution of (3.13)–(3.16). Moreover, let $0 \le s < \frac{1}{2}$, let $\omega \equiv 1$ if $c_0 \in \mathcal{D}(\partial \widetilde{E})$ and let $\omega(t) = \left(\frac{t}{1+t}\right)^{\frac{1}{2}}$ else.*

1. *If $\partial_t v \in L^{\frac{4}{3}}_{\text{uloc}}([0,\infty); H^{-s}(\Omega))$ and r is as in Theorem 3.6, then (c, μ) satisfies*

$$\omega \partial_t c \in L^\infty(0,\infty; H^{-1}_{(0)}(\Omega)) \cap L^2_{\text{uloc}}([0,\infty); H^1(\Omega)),$$

$$\omega c \in L^\infty(0,\infty; W^2_r(\Omega)), \ \omega \phi(c) \in L^\infty(0,\infty; L^r(\Omega)), \ \omega \mu \in L^\infty(0,\infty; H^1(\Omega)).$$

2. *If $v \in B^\alpha_{\frac{4}{3}\infty,\text{uloc}}([0,\infty); H^{-s}(\Omega))$ for some $\alpha \in (0,1)$, then*

$$\omega c \in C^\alpha([0,\infty); H^{-1}_{(0)}(\Omega)) \cap B^\alpha_{2\infty,\text{uloc}}([0,\infty); H^1(\Omega)). \tag{3.23}$$

Finally, the same statements hold true if $[0,\infty)$ is replaced by $[0,T]$, $0 < T < \infty$,

Remark 3.9. The lemma coincides with [3, Lemma 3], where $\theta > 0$ was fixed. As in Remark 3.7, the estimates in the proof of the latter lemma hold uniformly with respect to $0 < \theta \le 1$ since they are based essentially on (3.21)–(3.22) and (3.19). Hence Proposition 3.3 implies that all estimates obtained in the proof of [3, Lemma 3] are independent of $\theta \in (0,1]$.

3.3. Limit for the convective Cahn-Hilliard equation

In this section we show that solutions of the convective Cahn-Hilliard equation (1.3)–(1.4) with $\phi(c) = \theta\phi_0(c) - \theta_c c$ converge as $\theta \to 0$ to solutions of the system

$$\partial_t c + v \cdot \nabla c = \Delta \mu \qquad \text{in } \Omega \times (0,\infty), \tag{3.24}$$

$$\mu + \Delta c + \theta_c c \in \partial I_{[a,b]}(c) \qquad \text{in } \Omega \times (0,\infty), \tag{3.25}$$

$$\partial_n c|_{\partial\Omega} = \partial_n \mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{3.26}$$

$$c|_{t=0} = c_0 \qquad \text{in } \Omega \tag{3.27}$$

for given c_0 with $E^0_{\mathrm{free}}(c_0) < \infty$ and $v \in L^\infty(0,\infty;L^2_\sigma(\Omega)) \cap L^2(0,\infty;H^1(\Omega))$, where

$$E^0_{\mathrm{free}}(c) = \int_\Omega \frac{|\nabla c|^2}{2}\,dx + \int_\Omega \left(I_{[a,b]}(c) - \theta_c \frac{c^2}{2}\right) dx. \tag{3.28}$$

Again we have a decomposition

$$E^0_{\mathrm{free}}(c) = E_0(c) - \frac{\theta_c}{2}\|c\|^2_{L^2(\Omega)},$$

where $E_0(c)$ is as in (3.9) and we assume w.l.o.g. that $\int_\Omega c_0(x)\,dx = 0$. As before, (3.24)–(3.27) can be written as an abstract evolution equation in $H^{-1}_{(0)}(\Omega)$:

$$\partial_t c(t) + \mathcal{A}_0(c(t)) + \mathcal{B}(v(t))c(t) \ni 0, \qquad t > 0, \tag{3.29}$$

$$c|_{t=0} = c_0 \tag{3.30}$$

where $\mathcal{A}_0(c) = \partial \widetilde{E}_0(c)$, cf. Lemma 3.5, is now a multi-valued maximal monotone operator and $\mathcal{B}$ is as before. Moreover, we have as before

$$\begin{aligned}(w_1 - w_2, c_1 - c_2)_{L^2(0,\infty;H^{-1}_{(0)})} &= -(\Delta(c_1 - c_2) + f_1 - f_2, c_1 - c_2)_{L^2(Q)} \\ &\geq \|c_1 - c_2\|^2_{L^2(0,\infty;H^1_{(0)})} \end{aligned}\tag{3.31}$$

where $w_j(t) \in \partial \widetilde{E}_0(c_j(t))$, $j = 1,2$, for almost every $t \in (0,\infty)$ and $f_j(x,t) \in \partial I_{[a,b]}(c_j(x,t))$, $j = 1,2$, almost everywhere.

More precisely, we show

Theorem 3.10. *Let $0 < \theta_k \leq 1$, $k \in \mathbb{N}$, be such that $\theta_k \to_{k\to\infty} 0$. Moreover, assume $c_{0,k} \in H^1_{(0)}(\Omega)$ with $c_{0,k}(x) \in [a,b]$ almost everywhere for all $k \in \mathbb{N}_0$ and assume that $v_k \in L^2(0,\infty;H^1(\Omega)) \cap L^\infty(0,\infty;L^2_\sigma(\Omega))$ such that*

$$v_k \rightharpoonup_{k\to\infty} v \quad \text{in } L^2(0,T;H^1(\Omega)), \qquad c_{0,k} \to_{k\to\infty} c_0 \quad \text{in } H^1_{(0)}(\Omega)$$

for all $0 < T < \infty$, and let (c_k, μ_k) denote the unique solutions of (3.13)–(3.16) with $(v, c_0, \phi(c))$ replaced by $(v_k, c_{0,k}, \phi_k(c))$, where $\phi_k(c) = \Phi_k'(c)$ and $\Phi_k(c) = \theta_k \Phi_0(c) - \frac{\theta_c}{2}c^2$. Then

$$\begin{aligned} c_k &\rightharpoonup_{k\to\infty} c && \text{in } L^2(0,T;W^2_r(\Omega)), \\ \nabla\mu_k &\rightharpoonup_{k\to\infty} \nabla\mu && \text{in } L^2(Q), \\ \theta_k\phi_0(c_k) &\rightharpoonup_{k\to\infty} f && \text{in } L^2(0,T;L^r(\Omega)) \end{aligned}$$

for all $0 < T < \infty$, where $f = \mu + \Delta c \in \partial I_{[a,b]}(c)$ for almost all $(x,t) \in Q$, $r = 6$ if $d = 3$, $2 \leq r < \infty$ is arbitrary if $d = 2$, and $(c,\mu) \in BC([0,\infty);H^1(\Omega)) \cap L^2_{\mathrm{uloc}}([0,\infty);H^1(\Omega))$ is the unique solution of (3.24)–(3.27). Moreover, for every $t > 0$

$$E^0_{\mathrm{free}}(c(t)) + \int_{Q_t} |\nabla\mu|^2\,d(x,\tau) = E^0_{\mathrm{free}}(c_0) - \int_{Q_t} v \cdot \mu\nabla c\,d(x,\tau). \tag{3.32}$$

Proof. First of all, because of Remark 3.7, (3.21)–(3.22) with $(c,\mu,\phi(c),v,c_0)$ replaced by $(c_k,\mu_k,\phi_k(c),v_k,c_{0,k})$ hold true with constants independent of θ_k. Hence for a suitable subsequence $k_j \to_{j\to\infty} \infty$

$$\begin{aligned} c_{k_j} &\rightharpoonup_{j\to\infty} c && \text{in } L^2(0,T;W^2_r(\Omega)),\\ \nabla\mu_{k_j} &\rightharpoonup_{j\to\infty} \nabla\mu && \text{in } L^2(Q),\\ \theta_{k_j}\phi_0(c_{k_j}) &\rightharpoonup_{j\to\infty} f && \text{in } L^2(0,T;L^r(\Omega)) \end{aligned}$$

for all $0<T<\infty$. Moreover, since $\partial_t c_k$ is bounded in $L^2(0,\infty;H^{-1}_{(0)}(\Omega))$ due to (3.21),

$$c_{k_j} \to_{j\to\infty} c \qquad \text{in } L^2(0,T;W^1_6(\Omega))$$

for all $0<T<\infty$ due to the lemma of Aubin-Lions. Hence

$$\mathcal{B}(v_{k_j})c_{k_j} \rightharpoonup_{j\to\infty} \mathcal{B}(v)c \qquad \text{in } L^2(0,\infty;H^{-1}_{(0)}(\Omega))$$

and (3.24) holds in the sense of distributions.

Moreover, since $c_k\in H^1(0,T;H^{-1}_{(0)}(\Omega))$ is bounded and c_{k_j} converges strongly in $L^2(0,T;H^1(\Omega))$ for all $0<T<\infty$, c_{k_j} converges weakly in $H^1(0,T;H^{-1}_{(0)}(\Omega))\hookrightarrow C^{\frac12}([0,T];H^{-1}_{(0)}(\Omega))$ for every $0<T<\infty$, which implies that $c|_{t=0}=c_0=\lim_{j\to\infty}c_{0,k_j}$ holds in $H^{-1}_{(0)}(\Omega)$. Hence $c\in BC_w([0,\infty);H^1(\Omega))$ due to Lemma 2.1 and since $c_k\in L^\infty(0,\infty;H^1(\Omega))$ is bounded. Furthermore, since c_{k_j} converges strongly in $L^2(0,T;W^1_6(\Omega))$ for every $0<T<\infty$, a suitable subsequence of $c_{k_j}(t)$ (again denoted by $c_{k_j}(t)$) converges in $W^1_6(\Omega)\hookrightarrow C^0(\overline{\Omega})$ for almost all $t\in(0,\infty)$. Because of Corollary 3.4, we conclude that

$$P_0(\mu+\theta_c c) = -\Delta c + P_0 f \in \partial E_0(c)$$

with $f(x,t)\in\partial I_{[a,b]}(c(x,t))$ almost everywhere. Defining $m(\mu)$ by the equation $m(\mu)+\theta_c m(c)=m(f)$, we obtain (3.25). Moreover,

$$\begin{aligned} \partial\widetilde{E}_{\theta_{k_j}}(c_{k_j}) &= -\Delta_N(-\Delta c_{k_j}+P_0\theta_{k_j}\phi_0(c_{k_j}))\\ &= -\Delta_N(\mu_{k_j}+\theta_c c_{k_j}) \rightharpoonup_{j\to\infty} -\Delta_N(\mu+\theta_c c) = -\Delta_N(-\Delta c+P_0 f) \end{aligned}$$

in $L^2(0,T;H^{-1}_{(0)}(\Omega))$ for every $0<T<\infty$. Hence $-\Delta_N(-\Delta c(t)+P_0 f(t))\in\partial\widetilde{E}_0(c(t))$ for almost every $0<t<\infty$ because of Lemma 3.5 and (3.17) holds.

Finally, the uniqueness and (3.32) is proved in the same way in Theorem 3.6 using (3.31), cf. [3, Proof of Theorem 6]. Since every convergent subsequence converges to some (c,μ) solving (3.24)–(3.27) and the limit is unique, the complete sequence converges to the unique solution of (3.24)–(3.27). $\square$

Finally, we state the analogous result to Lemma 3.8 for the limit system (3.24)–(3.27).

Lemma 3.11. *Let $v\in L^2(0,\infty;H^1(\Omega))\cap L^\infty(0,\infty,L^2_\sigma(\Omega))$, $c_0\in H^1(\Omega)$ with $c_0(x)\in[a,b]$ almost everywhere and let (c,μ) be the corresponding solution of (3.24)–(3.27). Moreover, let $\omega\equiv 1$ if $c_0\in\mathcal{D}(\partial\widetilde{E}_0)\cap H^3(\Omega)$ and let $\omega(t)=\left(\frac{t}{1+t}\right)^{\frac12}$ else.*

1. *If* $\partial_t v \in L^{\frac{4}{3}}_{\mathrm{uloc}}([0,\infty); H^{-s}(\Omega))$ *for some* $0 \leq s < \frac{1}{2}$ *and* r *is as in Theorem* 3.10*, then* (c,μ) *satisfy*

$$\omega\partial_t c \in L^\infty(0,\infty; H^{-1}_{(0)}(\Omega)) \cap L^2_{\mathrm{uloc}}([0,\infty); H^1(\Omega))$$
$$\omega c \in L^\infty(0,\infty; W^2_r(\Omega)),\ \omega\phi(c) \in L^\infty(0,\infty; L^r(\Omega)),\ \omega\mu \in L^\infty(0,\infty; H^1(\Omega)).$$

2. *If* $v \in B^\alpha_{\frac{4}{3}\infty,\mathrm{uloc}}([0,\infty); H^{-s}(\Omega))$ *for some* $0 \leq s < \frac{1}{2}$ *and* $\alpha \in (0,1)$*, then*

$$\omega c \in C^\alpha([0,\infty); H^{-1}_{(0)}(\Omega)) \cap B^\alpha_{2\infty,\mathrm{uloc}}([0,\infty); H^1(\Omega)). \tag{3.33}$$

Finally, the same statements hold true if $[0,\infty)$ *is replaced by* $[0,T)$*,* $T < \infty$*,*

Proof. Let $\theta_k = \frac{1}{k}$, $k \in \mathbb{N}$, $v_k = v$, $c_{0,k} = c_0$ if $c_0 \not\in \mathcal{D}(\partial\widetilde{E}_0) \cap H^3(\Omega)$. If $c_0 \in \mathcal{D}(\partial E_0) \cap H^3(\Omega)$, then let $c_{0,k} = c_0(\varepsilon_k x)$, where $\varepsilon_k > 0$ is chosen such that $|\frac{1}{k}\phi_0(s)| + \frac{1}{k}\phi_0'(s) \leq C$ on $[\varepsilon_k a, \varepsilon_k b]$ and $\varepsilon_k \to_{k\to\infty} 0$, where we assume w.l.o.g. $m = 0 \in (a,b)$. Then $c_{0,k} \in \mathcal{D}(\partial\widetilde{E}_{\theta_k})$ and $\|\partial\widetilde{E}_{\theta_k}(c_{0,k})\|_{H^1} \leq C(\|c_0\|_{H^3} + 1)$. Moreover, let c_k, μ_k be as in Theorem 3.10. Then, because of Proposition 3.3, the estimates in the proof of Lemma 3.8 hold uniformly in $0 < \theta_k \leq 1$, cf. Remark 3.9. Hence c_k are bounded in the corresponding spaces and the lemma follows. □

4. Main result: Double obstacle limit for the model H

First of all, we recall the definition of weak solutions to (1.1)–(1.7) and a theorem on existence of weak solutions from [3]:

Definition 4.1. (Weak Solution)
Let $0 < T \leq \infty$. A triple (v,c,μ) such that

$$v \in BC_w(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; V(\Omega)),$$
$$c \in BC_w(0,T; H^1(\Omega)),\ \phi(c) \in L^2_{\mathrm{loc}}([0,T); L^2(\Omega)), \nabla\mu \in L^2(Q_T)$$

is called a weak solution of (1.1)–(1.7) on $(0,T)$ if

$$-(v,\partial_t\psi)_{Q_T} - (v_0, \psi|_{t=0})_\Omega + (v\cdot\nabla v, \psi)_{Q_T} + (\nu(c)Dv, D\psi)_{Q_T} = (\mu\nabla c, \psi)_{Q_T} \tag{4.1}$$

for all $\psi \in C^\infty_{(0)}([0,T)\times\Omega)^d$ with $\operatorname{div}\psi = 0$,

$$-(c,\partial_t\varphi)_{Q_T} - (c_0, \varphi|_{t=0})_\Omega + (v\cdot\nabla c, \varphi)_{Q_T} = -(\nabla\mu, \nabla\varphi)_{Q_T} \tag{4.2}$$
$$(\mu,\varphi)_{Q_T} = (\phi(c),\varphi)_{Q_T} + (\nabla c, \nabla\varphi)_{Q_T} \tag{4.3}$$

for all $\varphi \in C^\infty_{(0)}([0,T)\times\overline{\Omega})$, and if the (strong) energy inequality

$$E(v(t),c(t)) + \int_{Q_{(t_0,t)}} \nu(c)|Dv|^2\,d(x,\tau) + \int_{Q_{(t_0,t)}} |\nabla\mu|^2\,d(x,\tau) \leq E(v(t_0),c(t_0)) \tag{4.4}$$

holds for almost all $0 \leq t_0 < T$ including $t_0 = 0$ and all $t \in [t_0,T)$.

Theorem 4.2. (Global Existence of Weak Solutions, [3, Theorem 1]**)**
For every $v_0 \in L^2_\sigma(\Omega)$, $c_0 \in H^1(\Omega)$ *with* $c_0(x) \in [a,b]$ *almost everywhere there is a weak solution* (v, c, μ) *of* (1.1)–(1.7) *on* $(0,\infty)$. *Moreover, if* $d = 2$, *then* (4.4) *holds with equality for all* $0 \leq t_0 \leq t < \infty$. *Finally, every weak solution on* $(0,\infty)$ *satisfies*

$$\nabla^2 c, \phi(c) \in L^2_{\mathrm{loc}}([0,\infty); L^r(\Omega)), \frac{t^{\frac{1}{2}}}{1+t^{\frac{1}{2}}} c \in BUC(0,\infty; W^1_q(\Omega)) \tag{4.5}$$

where $r = 6$ *if* $d = 3$ *and* $1 < r < \infty$ *is arbitrary if* $d = 2$ *and* $q > 3$ *is independent of the solution and initial data. If additionally* $c_0 \in H^2_N(\Omega) := \{c \in H^2(\Omega) : \partial_n c|_{\partial\Omega} = 0\}$ *and* $-\Delta c_0 + \theta\phi_0(c_0) \in H^1(\Omega)$, *then* $c \in BUC(0,\infty; W^1_q(\Omega))$.

We show that weak solutions of (1.1)–(1.7) converge as $\theta \to 0$ (for a suitable subsequence) to a weak solution of

$$\partial_t v + v \cdot \nabla v - \operatorname{div}(\nu(c) Dv) + \nabla p = \mu_0 \nabla c \qquad \text{in } \Omega \times (0,\infty), \tag{4.6}$$
$$\operatorname{div} v = 0 \qquad \text{in } \Omega \times (0,\infty), \tag{4.7}$$
$$\partial_t c + v \cdot \nabla c = \Delta\mu \qquad \text{in } \Omega \times (0,\infty), \tag{4.8}$$
$$\mu + \Delta c + \theta_c c \in \partial I_{[a,b]}(c) \qquad \text{in } \Omega \times (0,\infty) \tag{4.9}$$

together with (1.5)–(1.7). The definition of weak solutions to (4.6)–(4.9), (1.5)–(1.7) is the same as in Definition 4.1 just replacing $E_{\mathrm{free}}(c)$ by $E^0_{\mathrm{free}}(c)$ and (4.3) by (4.9) together with $\partial_n c|_{\partial\Omega} = 0$, assuming $c \in L^2_{\mathrm{loc}}([0,\infty); H^2(\Omega))$ in the definition of weak solutions. Here $E^0_{\mathrm{free}}(c) = E_0(c) - \frac{\theta_c}{2}\|c\|^2_{L^2(\Omega)}$ is as in (3.28).

Our main result of this section is the following:

Theorem 4.3. *Let* $d = 2, 3$, $\theta_k > 0$, $k \in \mathbb{N}_0$ *such that* $\theta_k \to_{k\to\infty} 0$. *Moreover, let* (v_k, c_k, μ_k) *be weak solutions of* (1.1)–(1.7) *with initial values* $(v_{0,k}, c_{0,k}) \to_{k\to\infty} (v_0, c_0)$ *in* $L^2_\sigma(\Omega) \times H^1(\Omega)$ *with* $c_{0,k}(x) \in [a,b]$ *for almost all* $x \in \Omega$ *and all* $k \in \mathbb{N}$. *Then there is a subsequence* k_j, $j \in \mathbb{N}_0$, $k_j \to_{j\to\infty} \infty$ *such that*

$$(v_{k_j}, \nabla\mu_{k_j}) \rightharpoonup_{j\to\infty} (v, \nabla\mu) \qquad \text{in } L^2(0,\infty; H^1(\Omega)^d \times L^2(\Omega)), \tag{4.10}$$
$$(v_{k_j}, c_{k_j}) \rightharpoonup^*_{j\to\infty} (v, c) \qquad \text{in } L^\infty(0,\infty; L^2(\Omega)^d \times H^1(\Omega)), \tag{4.11}$$
$$(c_{k_j}, \mu_{k_j}) \rightharpoonup_{j\to\infty} (c, \mu) \qquad \text{in } L^2(0,T; W^2_r(\Omega) \times L^2(\Omega)) \tag{4.12}$$

for all $0 < T < \infty$ *with* $r = 6$ *if* $d = 3$ *and* $2 \leq r < \infty$ *arbitrary if* $d = 2$ *and* (v, c, μ) *is a weak solution of* (4.6)–(4.9), (1.5)–(1.7). *Moreover, every weak solution of* (4.6)–(4.9), (1.5)–(1.7) *satisfies*

$$\nabla^2 c, \mu \in L^2_{\mathrm{uloc}}([0,\infty); L^r(\Omega)), \quad \kappa(t) c \in BUC([0,\infty); W^1_q(\Omega))$$

for some $q > d$ *and with* $\kappa \equiv 1$ *if* $c_0 \in \mathcal{D}(\partial\widetilde{E}_0)$ *and* $\kappa(t) = t^{\frac{1}{2}}/(1+t)^{\frac{1}{2}}$ *else.*

Proof. By the energy estimate $v_k \in L^\infty(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; H^1(\Omega))$, $c_k \in L^\infty(0,\infty; H^1(\Omega))$, $\nabla\mu_k \in L^2(Q)$ are uniformly bounded. Hence there is a subsequence such that (4.10)–(4.11) holds. Moreover, since (3.21)–(3.22) hold uniformly in $k \in \mathbb{N}$, cf. Remark 3.7, we can extract a subsequence such that (4.12) holds too. Therefore c is a solution of (3.24)–(3.27) due to Theorem 3.10. Using (4.6) and the bounds on (v_k, c_k, μ_k), one obtains that $\partial_t v_k \in L^{\frac{4}{3}}_{\mathrm{uloc}}([0,\infty); V_2(\Omega)')$ is uniformly bounded. Hence $v_{k_j} \to_{j\to\infty} v$ in $L^2(0,T;H^s(\Omega))$ for all $s<1$. Moreover, since $\partial_t c_k \in L^2(0,\infty;H^{-1}_{(0)}(\Omega))$ is uniformly bounded due to (3.21), $c_{k_j} \to_{k\to\infty} c$ in $L^2(0,T;W^1_6(\Omega))$ for all $0<T<\infty$ because of the Lemma of Aubin-Lions. Hence we can pass to the limit in (4.1) for $(v_{k_j}, c_{k_j}, \mu_{k_j})$ and the initial values (v_{0,k_j}, c_{0,k_j}) and conclude that (v,c,μ) solve (4.1) too. Moreover, since $v_{k_j} \rightharpoonup_{j\to\infty} v \in W^1_{\frac{4}{3}}(0,T;V_2(\Omega)')$ and $c_{k_j} \rightharpoonup_{j\to\infty} c \in H^1(0,T;H^{-1}_{(0)}(\Omega))$ for all $0<T<\infty$, we obtain $(v_{0,k_j}, c_{0,k_j}) = (v_{k_j}, c_{k_j})|_{t=0} \rightharpoonup_{j\to\infty} (v_0,c_0) = (v,c)|_{t=0}$ weakly in $V_2(\Omega)' \times H^{-1}_{(0)}(\Omega)$. Because of Lemma 2.1. $(v,c) \in BC_w(0,\infty;L^2\times H^1)$.

Furthermore, because of Lemma 2.2, the energy inequality (4.4) for (v_k,c_k,μ_k) is equivalent to

$$\begin{aligned}\int_0^\infty & D_k(c_k(t),v_k(t),\mu_k(t))\varphi(t)\,dt\\ &\leq E_k(c_{0,k},v_{0,k})\varphi(0) + \int_0^\infty E_k(c_k(t),v_k(t))\varphi'(t)\,dt\end{aligned}$$

for all $\varphi \in W^1_1(0,\infty)$, $\varphi \geq 0$, where $E_k(c,v)$ denotes the total energy $E(c,v)$ with respect to $\Phi_{\theta_k}(c) = \theta_k \Phi_0(c) - \theta_c \frac{c^2}{2}$ and

$$D_k(c_k(t),v_k(t),\mu_k(t)) = \int_\Omega 2\nu(c_k(t))|Dv_k(t)|^2\,dx + \|\nabla\mu_k(t)\|_2^2.$$

Since

$$\begin{aligned}&D(c(t),v(t),\mu(t))\\ &\quad := \int_\Omega 2\nu(c(t))|Dv(t)|^2\,dx + \|\nabla\mu(t)\|_2^2 \leq \liminf_{k\to\infty} D_k(c_k(t),v_k(t),\mu_k(t))\end{aligned}$$

by the weak lower semi-continuity of the L^2-norm and

$$E_k(c_{k_j}(t),v_{k_j}(t)) \to_{j\to\infty} E(c(t),v(t)) \qquad \text{for almost all } 0<t<\infty,$$

we obtain

$$E(c_0,v_0)\varphi(0) + \int_0^\infty E(c(t),v(t))\varphi'(t)\,dt \geq \int_0^\infty D(c(t),v(t),\mu(t))\varphi(t)\,dt$$

for all $\varphi \in W^1_1(0,\infty)$, $\varphi \geq 0$, where $E(c,v) = E^0_{\mathrm{free}}(c) + E_{\mathrm{kin}}(v)$. Using Lemma 2.2 again, we have proved (4.4).

Finally, the regularity statements for c,μ follow from Theorem 3.10 since for given v (4.8)–(4.9) together with (1.6)–(1.7) has a unique solution (c,μ). $\square$

Since the Cahn-Hilliard equation (4.8)–(4.9) has the same structure as in the case $\theta > 0$ and the same regularity results, cf. Lemma 3.11 are available, it is easy to obtain the same uniqueness and regularity results as for $\theta > 0$, cf. [3, Proposition 1, Theorem 2]. These are as follows:

Proposition 4.4. (Uniqueness)
Let $0 < T \leq \infty$, $q = 3$ if $d = 3$ and let $q > 2$ if $d = 2$. Moreover, assume that $v_0 \in W^1_{q,0}(\Omega) \cap L^2_\sigma(\Omega)$ and let $c_0 \in C^{0,1}(\overline{\Omega})$ with $c_0(x) \in [a,b]$ for all $x \in \overline{\Omega}$. If there is a weak solution (v, c, μ) of (4.6)–(4.9), (1.5)–(1.7) on $(0,T)$ with $v \in L^\infty(0,T; W^1_q(\Omega))$ and $\nabla c \in L^\infty(Q_T)$, then any weak solution (v', c', μ') of (4.6)–(4.9), (1.5)–(1.7) on $(0,T)$ with the same initial values and $\nabla c' \in L^\infty(Q_T)$ coincides with (v, c, μ).

Proof. The proof is literally the same as the proof of [3, Lemma 7]. One just has to replace the first equality in the proof by

$$\partial_t \tilde{c} + a_1 - a_2 = -\theta_c \Delta \tilde{c} - w \cdot \nabla c_1 - v_2 \cdot \nabla \tilde{c},$$

where $a_j(t) \in \mathcal{A}_0(c_j(t)), j = 1, 2$, for almost all $t \in (0,T)$ and $w = v_1 - v_2$ and has to use (3.31). □

Theorem 4.5. (Regularity of Weak Solutions)
Assume that $c_0 \in \mathcal{D}(\partial \widetilde{E}_0) \cap H^3(\Omega)$, where $\widetilde{E}_0$ is as in Lemma 3.5.

1. *Let $d = 2$ and let $v_0 \in V_2^{1+s}(\Omega)$ with $s \in (0,1]$, $s \neq \frac{1}{2}$. Then every weak solution (v, c) of (4.6)–(4.9), (1.5)–(1.7) on $(0,\infty)$ satisfies*

   $$v \in L^2(0,\infty; V_2^{2+s'}(\Omega)) \cap H^1(0,\infty; V_2^{s'}(\Omega)) \cap BUC([0,\infty); H^{1+s-\varepsilon}(\Omega))$$

   for all $s' \in [0, \frac{1}{2}) \cap [0, s]$ and all $\varepsilon > 0$ as well as $\nabla^2 c, \phi(c) \in L^\infty(0,\infty; L^r(\Omega))$ for every $1 < r < \infty$. In particular, the weak solution is unique.
2. *Let $d = 2, 3$. Then for every weak solution (v, c, μ) of (4.6)–(4.9), (1.5)–(1.7) on $(0, \infty)$ there is some $T > 0$ such that*

   $$v \in L^2(T,\infty; V_2^{2+s}(\Omega)) \cap H^1(T,\infty; V_2^s(\Omega)) \cap BUC([T,\infty); H^{2-\varepsilon}(\Omega))$$

   for all $s \in [0, \frac{1}{2})$ and all $\varepsilon > 0$ as well as $\nabla^2 c, \phi(c) \in L^\infty(T,\infty; L^r(\Omega))$ with $r = 6$ if $d = 3$ and $1 < r < \infty$ arbitrary if $d = 2$.
3. *If $d = 3$ and $v_0 \in V_2^{s+1}(\Omega)$, $s \in (\frac{1}{2}, 1]$, then there is some $T_0 > 0$ such that every weak solution (v, c) of (4.6)–(4.9), (1.5)–(1.7) on $(0, T_0)$ satisfies*

   $$v \in L^2(0,T_0; V_2^{2+s'}(\Omega)) \cap H^1(0,T_0; V_2^{s'}(\Omega)) \cap BUC([0,T_0]; H^{1+s-\varepsilon}(\Omega))$$

   for all $s' \in [0, \frac{1}{2})$ and all $\varepsilon > 0$ as well as $\nabla^2 c, \phi(c) \in L^\infty(0,T_0; L^6(\Omega))$. In particular, the weak solution is unique on $(0, T_0)$.

Proof. The proof is the same as the one of [3, Theorem 2]. Its proof only relies on the available regularity results for c solving (1.3)–(1.4), which are the same for (4.8)–(4.9), as well as the uniqueness statement of [3, Proposition 1], which is replaced by Proposition 4.4. Therefore the proof directly carries over. □

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Flows of Generalized Oldroyd-B Fluids in Curved Pipes

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Dedicated to Prof. Herbert Amann on the occasion of his 70*th birthday*

Abstract. The aim of this work is to present a numerical study of generalized Oldroyd-B flows with shear-thinning viscosity in a curved pipe of circular cross section and arbitrary curvature ratio. Flows are driven by a given pressure gradient and behavior of the solutions is discussed with respect to different rheologic and geometric flow parameters.

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1. Introduction

Complex rheological phenomena such as shear dependent viscosity, stress relaxation, nonlinear creeping and normal stress differences can be found in many fluids like inks, polymer melts, suspensions, liquids crystals or biological fluids. These properties, which cannot be captured by the classical Navier-Stokes equations, lead to non-constant viscosity or to viscoelastic behavior described by nonlinear relations between the Cauchy stress and the strain tensor. Fluids of this type are called non-Newtonian [20].

There are many ways to generalize the Newtonian law of viscosity. The simplest case is the generalized Newtonian model where the extra-stress incorporates a shear-rate dependent viscosity. However, the generalized Newtonian fluids cannot account for the effects described above, namely the viscoelasticity, but they are often used to model simple flows and to study the flow rate in a pipe, as a function of the pressure drop. Suitable viscoelastic constitutive equations are then

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required. In general terms, non-Newtonian viscoelastic fluids exhibit both viscous and elastic properties and can be classified as fluids of differential type, rate type and integral type. We refer to the monographs [5], [15], [26], [29] for relevant issues related to non-Newtonian fluids behavior and modeling. Models of rate type such as Oldroyd-B fluids can predict stress relaxation and are used to describe flows in polymer processing. However they cannot capture the complex rheological behavior of many real fluids, such as blood in which the non-Newtonian viscosity effects are of major importance.

Over the past twenty years, a significant progress has been made in the mathematical analysis of the equations of motion of non-Newtonian viscoelastic fluids. Usually, the constitutive equations lead to highly nonlinear systems of partial differential equations of a combined parabolic-hyperbolic type (or elliptic-hyperbolic, for steady flows) closed with appropriate initial and/or boundary conditions. The study of the behavior of their solutions in different geometries requires the use of specific techniques of nonlinear analysis, such as fixed-point arguments associated to auxiliary linear sub-problems. We refer to [21] and [22] for an introduction to existence results for viscoelastic flows.

The hyperbolic nature of the constitutive equations is responsible for many of the difficulties associated with the numerical simulation of viscoelastic flows. Some factors including singularities in the geometry, boundary layers in the flow and the dominance of the nonlinear terms in the equations, result in numerical instabilities for high values of Weissenberg number (non-dimensional viscoelastic parameter). A variety of alternative numerical methods have been developed to overcome this difficulty, but many challenges still remain, in particular for viscoelastic flows in complex geometries (see, e.g., [16], [17] and the references cited therein).

It is known since the pioneering experimental works of Williams *et al.* [30], Grindley and Gibson [14], and Eustice ([11], [12]) that flows in curved pipes are very challenging and considerably more complex than flows in straight pipes. Due to fluid inertia, a secondary motion appears in addition to the primary axial flow. It is induced by an imbalance between the cross-stream pressure gradient and the centrifugal force and consists of a pair of counter-rotating vortices, which appear even for the most mildly curved pipe. This results in asymmetrical wall stresses with higher shear and low pressure regions ([4], [18], [27]).

Steady fully developed viscous flows in curved pipes of circular, elliptical and annular cross-section of both Newtonian and non-Newtonian fluids, have been studied by several authors ([1]–[4],[13], [19], [23], [24], [27]) following the fundamental work of Dean ([9], [10]) for circular cross-section pipes. Using regular perturbation methods around the curvature ratio, Dean obtained analytical solutions in the case of Newtonian fluids. These results have been extended for a larger range of curvature ratio and Reynolds number, showing the existence of additional pairs of vortices and multiple solutions ([8], [31]).

The great interest in the study of curved pipe flows is due to its wide range of applications in engineering (e.g., hydraulic pipe systems related to corrosion failure) and in biofluid dynamics, such as blood flow in vascular regions of low shear (in healthy or disease states), where the shear-thinning viscosity and viscoelastic behavior should not be neglected ([6], [7], [25], [28]).

This paper is concerned with the numerical study of the behavior of fully developed flows of shear-thinning generalized Oldroyd-B fluids in curved pipes with circular cross-section and arbitrary curvature ratio, for a prescribed pressure gradient. Numerical results show interesting viscosity and viscoelastic effects: for sufficiently small curvature ratio and certain range of viscosity parameters, the flow field is quite complex, showing counter-clockwise rotation of the secondary streamlines and loss of symmetry of the flow field. Stronger inertial effects result in a deformation of the pair of vortices and rotation of the flow in an opposite direction. These effects become weaker for higher values of the Weissenberg number. We remark that for generalized Oldroyd-B fluids the second normal stress difference is zero, as in the particular case of Oldroyd-B, and consequently the second normal stress difference has no impact on the secondary flows ([13]).

The paper is organized as follows. After introducing the governing equations and formulating the problem in polar toroidal coordinates (Section 2 and 3), we consider in Section 4 the numerical approximation of the steady Oldroyd-B model with non constant shear-dependent viscosity, in the above-described geometry. The original problem is decomposed into a Navier-Stokes system and a tensorial transport equation. Using the finite element method and a fixed point algorithm to couple the auxiliary problems, numerical results are obtained for a certain range of non-dimensional flow parameters (viscosity exponent, Reynolds and Weissenberg numbers) associated to the model. A continuation method is used to find the initial guess of the iterations and to increase the absolute value of the viscosity parameter.

Existence and uniqueness of approximated solutions, as well as a priori error estimates to the coupled full problem have already been proved, under a natural restriction on the curvature ratio (see [2]). In a future work, the systematic numerical study presented in this paper will be complemented by a theoretical analysis to justify the complex qualitative behavior of the combined effects of viscosity, inertial and viscoelastic parameters.

2. Governing equations

This paper is concerned with flows of incompressible viscoelastic Oldroyd-B fluids with shear dependent viscosity in a curved pipe $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$. For these fluids, the extra-stress tensor is related to the kinematic variables through

$$\mathbf{S} + \lambda_1 \overset{\nabla}{\mathbf{S}} = 2\left(\nu + \nu_0(1+|D\mathbf{u}|^2)^q\right) D\mathbf{u} + 2\lambda_2 \overset{\nabla}{D\mathbf{u}}, \tag{2.1}$$

where $\mathbf{u}$ is the velocity field, $D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)_{i,j=1,2}$ denotes the symmetric part of the velocity gradient, $|D\mathbf{u}|$ is the shear rate, q is a real number, ν and ν_0 are nonnegative real numbers satisfying $\nu + \nu_0 > 0$, $\lambda_1 > 0$ and

$\lambda_2 > 0$ are viscoelastic constants. The symbol $\triangledown$ denotes the objective derivative of Oldroyd type defined by

$$\overset{\triangledown}{\mathbf{S}} = \left(\tfrac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{S} - \mathbf{S}\,\nabla\mathbf{u} - (\nabla\mathbf{u})^t\,\mathbf{S}.$$

The Cauchy stress tensor is given by $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$, where p represents the pressure. The equations of conservation of momentum and mass hold in the domain Ω,

$$\rho\left(\tfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p = \nabla\cdot\mathbf{S} + \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

where $\rho > 0$ is the (constant) density of the fluid and $\mathbf{f}$ is an external force. We first decompose the extra-stress tensor $\mathbf{S}$ into the sum of its Newtonian part $\boldsymbol{\tau}_s = 2\frac{\lambda_2}{\lambda_1}\,D\mathbf{u}$ and its viscoelastic part $\boldsymbol{\tau}$. Introducing the quantities

$$x = \frac{\widetilde{x}}{L}, \qquad t = \frac{U\widetilde{t}}{L}, \qquad u = \frac{\widetilde{u}}{U}, \qquad p = \frac{\widetilde{p}L}{(\nu+\nu_0)U},$$
$$\boldsymbol{\tau} = \frac{\widetilde{\boldsymbol{\tau}}L}{(\nu+\nu_0)U}, \qquad \mathbf{f} = \frac{\widetilde{\mathbf{f}}L^2}{(\nu+\nu_0)U},$$

where the symbol $\widetilde{\ }$ is attached to dimensional parameters (L represents a reference length and U a characteristic velocity of the flow). We also set

$$\varepsilon = 1 - \frac{\lambda_2}{\lambda_1(\nu+\nu_0)}, \qquad \eta = \frac{\nu_0}{\nu+\nu_0},$$

and defining the Reynolds number and the Weissenberg number as

$$\mathcal{R}e = \frac{\rho U L}{\nu+\nu_0}, \qquad \mathcal{W}e = \frac{\lambda_1 U}{L}.$$

we can write (2.1)–(2.2) in dimensionless form

$$\begin{cases} -(1-\varepsilon)\,\Delta\mathbf{u} + \mathcal{R}e\left(\tfrac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p = \mathbf{f} + \nabla\cdot\boldsymbol{\tau}, \\ \nabla\cdot\mathbf{u} = 0, \\ \boldsymbol{\tau} + \mathcal{W}e\left(\tfrac{\partial\boldsymbol{\tau}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\tau} - g(\nabla\mathbf{u},\boldsymbol{\tau})\right) = 2\left(\varepsilon + \eta\boldsymbol{\sigma}\left(|D\mathbf{u}|^2\right)\right)D\mathbf{u}, \end{cases} \tag{2.3}$$

with

$$g(\nabla\mathbf{u},\boldsymbol{\tau}) = \boldsymbol{\tau}\,\nabla\mathbf{u} + (\nabla\mathbf{u})^t\,\boldsymbol{\tau}, \qquad \boldsymbol{\sigma}(x) = (1+x)^q - 1.$$

This system is supplemented with a Dirichlet homogeneous boundary condition

$$\mathbf{u} = \mathbf{0} \text{ on } \partial\Omega.$$

In a simple shear this model predicts shear dependent viscosity (shear-thinning for $q < 0$ and shear-thickening for $q > 0$) and normal stress coefficients Ψ_1 and Ψ_2 given by (see, e.g., [5], [17], [29])

$$\Psi_1(|Du|) = 2\left(\varepsilon + \eta\boldsymbol{\sigma}\left(|D\mathbf{u}|^2\right)\right)|D\mathbf{u}|^2$$
$$\Psi_2(|D\mathbf{u}|) = 0.$$

Note that the model reduces to Oldroyd-B when $q = 0$.

3. Equivalent formulation in polar toroidal coordinates

We consider fully developed flows in a curved pipe with circular cross-section (see Figure 1).

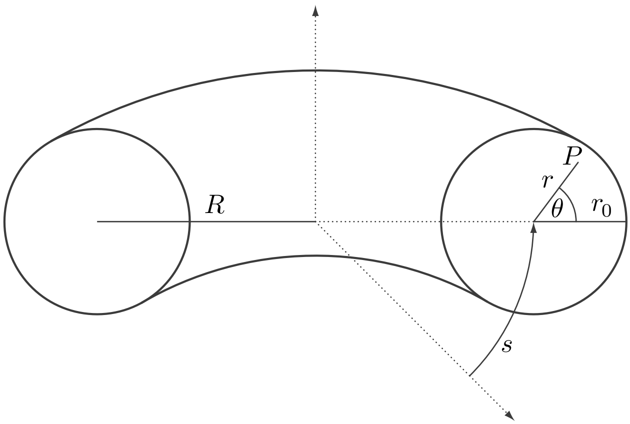


FIGURE 1. Polar toroidal coordinates.

For this pipe geometry, it is more convenient to use the polar toroidal coordinate system, in the variables $(\widetilde{r}, \theta, \widetilde{s})$, defined with respect to the rectangular cartesian coordinates $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ through the relations

$$\widetilde{r} = \sqrt{\widetilde{z}^2 + \left(\sqrt{\widetilde{x}^2 + \widetilde{y}^2} - R\right)^2},$$
$$\theta = \arctan \frac{\widetilde{z}}{\sqrt{\widetilde{x}^2 + \widetilde{y}^2} - R}, \qquad \widetilde{s} = R \arctan \frac{\widetilde{y}}{\widetilde{x}},$$

and the inverse relations

$$\widetilde{x} = (R + \widetilde{r}\cos\theta)\cos\frac{\widetilde{s}}{R}, \qquad \widetilde{y} = (R + \widetilde{r}\cos\theta)\sin\frac{\widetilde{s}}{R}, \qquad \widetilde{z} = \widetilde{r}\sin\theta,$$

with $0 < r_0 < R$, $0 \leqslant \theta < 2\pi$ and $0 \leqslant \widetilde{s} < \pi R$, where $R > \widetilde{r} \geqslant 0$ is the constant centerline radius. Introducing

$$s = \frac{\widetilde{s}}{r_0}, \qquad \delta = \frac{r_0}{R},$$

we see that the corresponding non-dimensional coordinate systems are given by

$$r = \sqrt{z^2 + \left(\sqrt{x^2 + y^2} - \tfrac{1}{\delta}\right)^2},$$
$$\theta = \arctan \frac{z}{\sqrt{x^2 + y^2} - \frac{1}{\delta}}, \qquad s = \frac{1}{\delta} \arctan \frac{y}{x},$$

and the inverse relations

$$x = \left(\frac{1}{\delta} + r\cos\theta\right)\cos(s\delta), \qquad y = \left(\frac{1}{\delta} + r\cos\theta\right)\sin(s\delta), \qquad z = r\sin\theta,$$

with $\delta < 1$, $0 \leqslant \theta < 2\pi$ and $0 \leqslant s < \frac{\pi}{\delta}$.

Let us now formulate system (2.3) in this new coordinate system. For convenience, we keep the notation as for the cartesian system (e.g., $u \equiv u \cdot \mathbf{e_r}$, $v \equiv v \cdot \mathbf{e}_\theta$ and $w \equiv w \cdot \mathbf{e_s}$). To simplify the notation we set

$$\beta_1 \equiv \beta_1(r,\theta) = r\delta \sin\theta, \qquad \beta_2 \equiv \beta_2(r,\theta) = r\delta\cos\theta,$$
$$\beta \equiv \beta(r,\theta) = 1 + r\delta\cos\theta.$$

By using standard arguments, we rewrite the problem (2.3) in the toroidal coordinates (r,θ,s), and we see that the problem reads as

Find $(\mathbf{u} \equiv (u,v,w), p, \boldsymbol{\tau})$ solution of

$$\begin{cases} -\left(\nabla\cdot(2(1-\varepsilon)D\mathbf{u} + \boldsymbol{\tau} - \mathcal{R}e\,\mathbf{u}\otimes\mathbf{u})\right)_r + \mathcal{R}e\frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = 0, \\ -\left(\nabla\cdot(2(1-\varepsilon)D\mathbf{u} + \boldsymbol{\tau} - \mathcal{R}e\,\mathbf{u}\otimes\mathbf{u})\right)_\theta + \mathcal{R}e\frac{\partial v}{\partial t} + \frac{1}{r}\frac{\partial p}{\partial \theta} = 0, \\ -\left(\nabla\cdot(2(1-\varepsilon)D\mathbf{u} + \boldsymbol{\tau} - \mathcal{R}e\,\mathbf{u}\otimes\mathbf{u})\right)_s + \mathcal{R}e\frac{\partial w}{\partial t} + \frac{1}{\beta}\frac{\partial p}{\partial s} = 0, \\ \frac{\partial}{\partial r}(r\beta u) + \frac{\partial}{\partial\theta}(\beta v) + \frac{\partial}{\partial s}(rw) = 0, \\ \boldsymbol{\tau} + \mathcal{W}e\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\theta} + \frac{w}{\beta}\frac{\partial}{\partial s}\right)\boldsymbol{\tau} + \mathcal{W}e\frac{\partial\boldsymbol{\tau}}{\partial t} \\ \qquad = \frac{\mathcal{W}e}{r\beta}\mathbf{F}(\mathbf{u},\boldsymbol{\tau}) + 2\left(\varepsilon - \eta + \frac{\eta}{(r\beta)^{2q}}\left((r\beta)^2 + |r\beta D\mathbf{u}|^2\right)^q\right)D\mathbf{u}, \\ \mathbf{u}_{|\partial\Omega} = 0 \end{cases} \tag{3.1}$$

where for third-order tensor $\boldsymbol{\sigma}$, $\nabla\cdot\boldsymbol{\sigma}$ is given by

$$(\nabla\cdot\boldsymbol{\sigma})_r = \frac{1}{r\beta}\Big(\frac{\partial}{\partial r}(r\beta\boldsymbol{\sigma}_{rr}) + \frac{\partial}{\partial\theta}(\beta\boldsymbol{\sigma}_{r\theta}) + \frac{\partial}{\partial s}(r\boldsymbol{\sigma}_{rs}) - \beta_2\boldsymbol{\sigma}_{ss} - \beta\boldsymbol{\sigma}_{\theta\theta}\Big),$$
$$(\nabla\cdot\boldsymbol{\sigma})_\theta = \frac{1}{r\beta}\Big(\frac{\partial}{\partial r}(r\beta\boldsymbol{\sigma}_{\theta r}) + \frac{\partial}{\partial\theta}(\beta\boldsymbol{\sigma}_{\theta\theta}) + \frac{\partial}{\partial s}(r\boldsymbol{\sigma}_{\theta s}) + \beta_1\boldsymbol{\sigma}_{ss} + \beta\boldsymbol{\sigma}_{r\theta}\Big),$$
$$(\nabla\cdot\boldsymbol{\sigma})_s = \frac{1}{r\beta}\Big(\frac{\partial}{\partial r}(r\beta\boldsymbol{\sigma}_{sr}) + \frac{\partial}{\partial\theta}(\beta\boldsymbol{\sigma}_{s\theta}) + \frac{\partial}{\partial s}(r\boldsymbol{\sigma}_{ss}) - \beta_1\boldsymbol{\sigma}_{\theta s} + \beta_2\boldsymbol{\sigma}_{rs}\Big),$$

and the velocity gradient and the symmetric tensor $\mathbf{F}$ are given by

$$\nabla\mathbf{u} = \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \\ \frac{1}{r}\frac{\partial u}{\partial\theta} - \frac{v}{r} & \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r} & \frac{1}{r}\frac{\partial w}{\partial\theta} \\ \frac{1}{\beta}\frac{\partial u}{\partial s} - \frac{\beta_2}{r\beta}w & \frac{1}{\beta}\frac{\partial v}{\partial s} + \frac{\beta_1}{r\beta}w & \frac{1}{\beta}\frac{\partial w}{\partial s} + \frac{\beta_2}{r\beta}u - \frac{\beta_1}{r\beta}v \end{pmatrix},$$

$$\mathbf{F}_{rr}(\mathbf{u},\boldsymbol{\tau}) = 2\left(r\beta\frac{\partial u}{\partial r}\boldsymbol{\tau}_{rr} + \beta\frac{\partial u}{\partial\theta}\boldsymbol{\tau}_{r\theta} + r\frac{\partial u}{\partial s}\boldsymbol{\tau}_{rs}\right),$$
$$\mathbf{F}_{r\theta}(\mathbf{u},\boldsymbol{\tau}) = \beta\left(r\frac{\partial v}{\partial r} - v\right)\boldsymbol{\tau}_{rr} + \beta\left(\frac{\partial v}{\partial\theta} + u + r\frac{\partial u}{\partial r}\right)\boldsymbol{\tau}_{r\theta}$$
$$\qquad + r\frac{\partial v}{\partial s}\boldsymbol{\tau}_{rs} + \beta\frac{\partial u}{\partial\theta}\boldsymbol{\tau}_{\theta\theta} + r\frac{\partial u}{\partial s}\boldsymbol{\tau}_{\theta s},$$
$$\mathbf{F}_{rs}(\mathbf{u},\boldsymbol{\tau}) = \left(r\beta\frac{\partial w}{\partial r} - \beta_2 w\right)\boldsymbol{\tau}_{rr} + \left(\beta\frac{\partial w}{\partial\theta} + \beta_1 w\right)\boldsymbol{\tau}_{r\theta} + \beta\frac{\partial u}{\partial\theta}\boldsymbol{\tau}_{\theta s}$$
$$\qquad + \left(r\frac{\partial w}{\partial s} + \beta_2 u - \beta_1 v + r\beta\frac{\partial u}{\partial r}\right)\boldsymbol{\tau}_{rs} + r\frac{\partial u}{\partial s}\boldsymbol{\tau}_{ss},$$
$$\mathbf{F}_{\theta\theta}(\mathbf{u},\boldsymbol{\tau}) = 2\left(\beta\left(r\frac{\partial v}{\partial r} - v\right)\boldsymbol{\tau}_{r\theta} + \beta\left(\frac{\partial v}{\partial\theta} + u\right)\boldsymbol{\tau}_{\theta\theta} + r\frac{\partial v}{\partial s}\boldsymbol{\tau}_{\theta s}\right),$$

$$\begin{aligned}\mathbf{F}_{\theta s}(\mathbf{u},\boldsymbol{\tau}) &= \left(r\beta\,\tfrac{\partial w}{\partial r} - \beta_2 w\right)\boldsymbol{\tau}_{r\theta} + \left(\beta\,\tfrac{\partial w}{\partial \theta} + \beta_1 w\right)\boldsymbol{\tau}_{\theta\theta} + \beta\left(r\tfrac{\partial v}{\partial r} - v\right)\boldsymbol{\tau}_{rs}\\ &\quad + \left(r\tfrac{\partial w}{\partial s} + \beta_2 u - \beta_1 v + \beta\tfrac{\partial v}{\partial \theta} + \beta u\right)\boldsymbol{\tau}_{\theta s} + r\tfrac{\partial v}{\partial s}\boldsymbol{\tau}_{ss},\\ \mathbf{F}_{ss}(\mathbf{u},\boldsymbol{\tau}) &= 2\left(\left(r\beta\,\tfrac{\partial w}{\partial r} - \beta_2 w\right)\boldsymbol{\tau}_{rs} + \left(\beta\,\tfrac{\partial w}{\partial \theta} + \beta_1 w\right)\boldsymbol{\tau}_{\theta s}\right)\\ &\quad + 2\left(r\tfrac{\partial w}{\partial s} + \beta_2 u - \beta_1 v\right)\boldsymbol{\tau}_{ss}.\end{aligned}$$

Considering fully developed flows, the velocity, the pressure and the stress tensor $\boldsymbol{\tau}$ are independent of the variable s. Consequently, they satisfy respectively,

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial s} = \frac{\partial w}{\partial s} \equiv 0,\ \frac{\partial p}{\partial s} = -p^* \text{ and } \frac{\partial \widehat{\boldsymbol{\tau}}}{\partial s} \equiv 0. \tag{3.2}$$

Using (3.2), problem (3.1) defined in the set

$$\Sigma = \left\{(r,\theta) \in \mathbb{R}^2 \mid 0 < r < 1, 0 < \theta \leq 2\pi\right\} \tag{3.3}$$

reads as follows

Find $(\mathbf{u} \equiv (u,v,w), p, \widehat{\boldsymbol{\tau}})$ solution of

$$\mathcal{P} = \begin{cases} -\mathcal{A}_{r,\gamma}\Big(\psi_\gamma\left(2(1-\varepsilon)D\mathbf{u} - \mathcal{R}e\,\mathbf{u}\otimes\mathbf{u}\right) + \widehat{\boldsymbol{\tau}}\Big) + \mathcal{R}e\,\frac{\partial u}{\partial t} + \psi_{\gamma+1}\,\frac{\partial p}{\partial r} = 0,\\ -\mathcal{A}_{\theta,\gamma}\Big(\psi_\gamma\left(2\left(1-\varepsilon\right)D\mathbf{u} - \mathcal{R}e\,\mathbf{u}\otimes\mathbf{u}\right) + \widehat{\boldsymbol{\tau}}\Big) + \mathcal{R}e\,\frac{\partial v}{\partial t} + \beta\psi_\gamma\,\frac{\partial p}{\partial \theta} = 0,\\ -\mathcal{A}_{s,\gamma}\Big(\psi_\gamma\left(2\left(1-\varepsilon\right)D\mathbf{u} - \mathcal{R}e\,\mathbf{u}\otimes\mathbf{u}\right) + \widehat{\boldsymbol{\tau}}\Big) + \mathcal{R}e\,\frac{\partial w}{\partial t} = p^* r\psi_\gamma,\\ \frac{\partial}{\partial r}\left(r\beta u\right) + \frac{\partial}{\partial \theta}(\beta v) = 0,\\ r\beta\,\widehat{\boldsymbol{\tau}} + \mathcal{W}e\Big(\frac{\partial}{\partial t} + r\beta u\,\frac{\partial}{\partial r} + \beta v\,\frac{\partial}{\partial \theta}\Big)\widehat{\boldsymbol{\tau}} = \mathbf{G}(\mathbf{u},\widehat{\boldsymbol{\tau}})\\ \mathbf{u}_{|\partial\Sigma} = 0 \end{cases} \tag{3.4}$$

where for a tensor σ,

$$\mathcal{A}_{\xi,\gamma}\left(\boldsymbol{\sigma}\right) = r\beta(\nabla\cdot\boldsymbol{\sigma})_\xi - \gamma\Big((\beta+\beta_2)\boldsymbol{\sigma}_{\xi r} - \beta_1\boldsymbol{\sigma}_{\xi\theta}\Big), \quad \text{for any } \xi = r,\theta,s$$

and

$$\begin{aligned}\mathbf{G}(\mathbf{u},\widehat{\boldsymbol{\tau}}) = \mathcal{W}e\Big(&\mathbf{F}(\mathbf{u},\widehat{\boldsymbol{\tau}}) + \gamma\left(\left(\beta+\beta_2\right)u - \beta_1 v\right)\widehat{\boldsymbol{\tau}}\Big)\\ &+ 2\left(\varepsilon-\eta\right)\psi_\gamma D\mathbf{u}\\ &+ 2\eta\,\psi_{\gamma-2q+1}\left((r\beta)^2 + |r\beta D\mathbf{u}|^2\right)^q D\mathbf{u},\end{aligned} \tag{3.5}$$

where $\gamma = 2\max\left(q,0\right) + 1$, $\psi_\gamma = \left(r\beta\right)^\gamma$ and $\widehat{\boldsymbol{\tau}} \equiv \psi_\gamma\boldsymbol{\tau}$.

4. Numerical approximation and results

We use finite element methods to obtain approximate solutions to problem (3.4). Here and in the remaining sections, only steady solutions will be considered.

4.1. Setting of the approximated problem

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of regular triangulations of the rectangle Σ defined by (3.3), and denote by

$$\mathbf{X}_h = (X_h)^3 = \left\{\mathbf{v}_h \in \mathbf{C}(\overline{\Sigma}) \cap \mathbf{H}_0^1(\Sigma) \mid \mathbf{v}_{h|K} \in \mathbb{P}_2(K),\, \forall K \in \mathcal{T}_h\right\}^3,$$
$$Q_h = \left\{q_h \in C(\overline{\Sigma}) \cap L_0^2(\Sigma) \mid q_{h|K} \in \mathbb{P}_1(K),\, \forall K \in \mathcal{T}_h\right\},$$

and

$$\mathbf{T}_h = (T_h)^{3\times 3} = \{\tau_h \in L^2(\Sigma) \mid \tau_{h|K} \in \mathbb{P}_1,\, \forall K \in \mathcal{T}_h\}^{3\times 3}$$

the finite element spaces. System (3.4) is approximated by the following problem

Find $(\mathbf{u}_h, p_h, \widehat{\boldsymbol{\tau}}_h) \equiv (\mathbf{u}, p, \widehat{\boldsymbol{\tau}}) \in \mathbf{X}_h \times Q_h \times \mathbf{T}_h$ solution of

$$\begin{cases} \Big(-\mathcal{A}_{r,\gamma}\big(\psi_\gamma\,(2(1-\varepsilon)D\mathbf{u} - \mathcal{R}e\,\mathbf{u}\otimes\mathbf{u})\big) + \psi_{\gamma+1}\frac{\partial p}{\partial r}, \boldsymbol{\phi}_1\Big) = \left(\mathcal{A}_{r,\gamma}\left(\widehat{\boldsymbol{\tau}}\right), \boldsymbol{\phi}_1\right), \\ \Big(-\mathcal{A}_{\theta,\gamma}\big(\psi_\gamma\,(2(1-\varepsilon)D\mathbf{u} - \mathcal{R}e\,\mathbf{u}\otimes\mathbf{u})\big) + \beta\psi_\gamma\,\frac{\partial p}{\partial \theta}, \boldsymbol{\phi}_2\Big) = \left(\mathcal{A}_{\theta,\gamma}\left(\widehat{\boldsymbol{\tau}}\right), \boldsymbol{\phi}_2\right), \\ \Big(-\mathcal{A}_{s,\gamma}\big(\psi_\gamma\,(2(1-\varepsilon)D\mathbf{u} - \mathcal{R}e\,\mathbf{u}\otimes\mathbf{u})\big) - p^* r\,\psi_\gamma, \boldsymbol{\phi}_3\Big) = \left(\mathcal{A}_{s,\gamma}\left(\widehat{\boldsymbol{\tau}}\right), \boldsymbol{\phi}_3\right), \\ \big(\frac{\partial}{\partial r}(r\beta u) + \frac{\partial}{\partial\theta}(\beta v), \varphi\big) = 0, \\ \Big(r\beta\,\widehat{\boldsymbol{\tau}}_{ij}, \boldsymbol{\sigma}_\ell\Big) + \mathcal{W}e\,\mathcal{B}_h\big(u, v, \widehat{\boldsymbol{\tau}}_{ij}, \boldsymbol{\sigma}_\ell\big) = \Big(\mathbf{G}_{ij}\left(\mathbf{u}, \widehat{\boldsymbol{\tau}}\right), \boldsymbol{\sigma}_\ell\Big), \\ \mathbf{u}_{|\partial\Sigma} = 0 \end{cases} \tag{4.1}$$

for every $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3, \varphi) \in \mathbf{V}_{\delta,h} \times Q_h$, where

$$\mathbf{V}_{\delta,h} = \{\mathbf{v}_h \in \mathbf{X}_h \mid \nabla' \cdot (\beta\,\mathbf{v}_h) = 0\}, \quad \delta \in [0,1[$$

and every $\boldsymbol{\sigma}_\ell \in T_h$ $(\ell = 1, \dots, 6)$, with $\mathcal{B}_h$ defined by

$$\begin{aligned}\mathcal{B}_h\big(u, v, \tau, \sigma\big) = &\left(r\beta u\frac{\partial\tau}{\partial r} + \beta v\frac{\partial\tau}{\partial\theta} + \frac{1}{2}\left(\frac{\partial}{\partial r}(r\beta u) + \frac{\partial}{\partial\theta}(\beta v)\right)\tau, \sigma\right)_h \\ &- \left\langle \tau^+ - \tau^-, \sigma^+\right\rangle_{h,u,v}\end{aligned}$$

where

$$(\cdot,\cdot)_h = \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_K,$$
$$\langle\sigma,\tau\rangle_{h,u,v} = \sum_{K\in\mathcal{T}_h} \int_{\partial K^-(ru,v)} \tau\sigma\,\beta(run_r + vn_\theta)\,ds,$$
$$\partial K^-(\psi,\zeta) = \{s \in \partial K \mid (\psi,\zeta)\cdot(n_r,n_\theta) < 0\},$$

and where (n_r, n_θ) is the outward unit normal vector to element $K \in \mathcal{T}_h$.

Using standard integration by parts we show that problem (4.1) can be rewritten in the form

$(\mathcal{P}_h)$ Find $(\mathbf{u}_h, p_h, \widehat{\boldsymbol{\tau}}_h) \equiv (\mathbf{u}, p, \widehat{\boldsymbol{\tau}}) \in \mathbf{X}_h \times Q_h \times \mathbf{T}_h$ solution of

$$
\begin{aligned}
2\mathcal{K}_\gamma\Big(\psi_\gamma \tfrac{\partial u}{\partial r}, \boldsymbol{\phi}_1\Big) &+ \mathcal{L}_\gamma\Big(\beta\psi_{\gamma-1}\tfrac{\partial u}{\partial \theta}, \boldsymbol{\phi}_1\Big) + 2\Big(\psi_{\gamma-1}\left(\beta^2+\beta_2^2\right)u, \boldsymbol{\phi}_1\Big)\\
&+ \mathcal{L}_\gamma\Big(\beta\psi_{\gamma-1}\left(r\tfrac{\partial v}{\partial r} - v\right), \boldsymbol{\phi}_1\Big) + 2\Big(\psi_{\gamma-1}\left(\beta^2\tfrac{\partial v}{\partial \theta} - \beta_1\beta_2 v\right), \boldsymbol{\phi}_1\Big)\\
&- \tfrac{1}{1-\varepsilon}\mathcal{K}_{\gamma+1}\Big(\psi_\gamma\, p, \boldsymbol{\phi}_1\Big) + \tfrac{\mathcal{R}e}{1-\varepsilon}\Big(\mathcal{K}_\gamma\Big(\psi_\gamma u^2, \boldsymbol{\phi}_1\Big) + \mathcal{L}_\gamma\Big(\psi_\gamma uv, \boldsymbol{\phi}_1\Big)\Big)\\
&+ \tfrac{\mathcal{R}e}{1-\varepsilon}\Big(\psi_\gamma\left(\beta_2 w^2 + \beta v^2\right), \boldsymbol{\phi}_1\Big)\\
=\; &- \tfrac{1}{1-\varepsilon}\Big(\mathcal{K}_\gamma(\widehat{\boldsymbol{\tau}}_{rr}, \boldsymbol{\phi}_1) + \mathcal{L}_\gamma(\widehat{\boldsymbol{\tau}}_{r\theta}, \boldsymbol{\phi}_1) + \left(\beta_2\widehat{\boldsymbol{\tau}}_{ss} + \beta\widehat{\boldsymbol{\tau}}_{\theta\theta}, \boldsymbol{\phi}_1\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}_\gamma\Big(\beta\psi_{\gamma-1}\big(r\tfrac{\partial v}{\partial r} - v\big), \boldsymbol{\phi}_2\Big) &+ 2\mathcal{L}_\gamma\Big(\beta\psi_{\gamma-1}\tfrac{\partial v}{\partial \theta}, \boldsymbol{\phi}_2\Big)\\
&+ \Big(\psi_{\gamma-1}\left(\left(\beta^2 + 2\beta_1^2\right)v - r\beta^2\tfrac{\partial v}{\partial r}\right), \boldsymbol{\phi}_2\Big) + \mathcal{K}_\gamma\Big(\beta\,\psi_{\gamma-1}\tfrac{\partial u}{\partial \theta}, \boldsymbol{\phi}_2\Big)\\
&+ 2\mathcal{L}_\gamma\Big(\beta\psi_{\gamma-1}u, \boldsymbol{\phi}_2\Big) - \Big(\psi_{\gamma-1}\left(\beta^2\tfrac{\partial u}{\partial \theta} + 2\beta_1\beta_2 u\right), \boldsymbol{\phi}_2\Big)\\
&- \tfrac{1}{1-\varepsilon}\mathcal{L}_{\gamma+1}\Big(\psi_\gamma p, \boldsymbol{\phi}_2\Big) + \tfrac{\mathcal{R}e}{1-\varepsilon}\Big(\mathcal{K}_\gamma\Big(\psi_\gamma uv, \boldsymbol{\phi}_2\Big) + \mathcal{L}_\gamma\Big(\psi_\gamma v^2, \boldsymbol{\phi}_2\Big)\Big)\\
&- \tfrac{\mathcal{R}e}{1-\varepsilon}\Big(\psi_\gamma\left(\beta_1 w^2 + \beta uv\right), \boldsymbol{\phi}_2\Big)\\
=\; &- \tfrac{1}{1-\varepsilon}\Big(\mathcal{K}_\gamma(\widehat{\boldsymbol{\tau}}_{\theta r}, \boldsymbol{\phi}_2) + \mathcal{L}_\gamma(\widehat{\boldsymbol{\tau}}_{\theta\theta}, \boldsymbol{\phi}_2) - \left(\beta_1\widehat{\boldsymbol{\tau}}_{ss} + \beta\widehat{\boldsymbol{\tau}}_{r\theta}, \boldsymbol{\phi}_2\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}_\gamma\Big(\psi_{\gamma-1}\big(r\beta\tfrac{\partial w}{\partial r} - \beta_2 w\big), \boldsymbol{\phi}_3\Big) &+ \mathcal{L}_\gamma\Big(\psi_{\gamma-1}\left(\beta\tfrac{\partial w}{\partial \theta} + \beta_1 w\right), \boldsymbol{\phi}_3\Big)\\
&+ \Big(\psi_{\gamma-1}\Big((r\delta)^2 w + \beta\beta_1\tfrac{\partial w}{\partial \theta} - r\beta\beta_2\tfrac{\partial w}{\partial r}\Big), \boldsymbol{\phi}_3\Big)\\
&- \tfrac{1}{1-\varepsilon}\Big(r\psi_\gamma\, p^*, \boldsymbol{\phi}_3\Big) + \tfrac{\mathcal{R}e}{1-\varepsilon}\Big(\mathcal{K}_\gamma\Big(\psi_\gamma\, uw, \boldsymbol{\phi}_3\Big) + \mathcal{L}_\gamma\Big(\psi_\gamma\, vw, \boldsymbol{\phi}_3\Big)\Big)\\
&- \tfrac{\mathcal{R}e}{1-\varepsilon}\Big(\psi_\gamma\left(\beta_2 u - \beta_1 v\right)w, \boldsymbol{\phi}_3\Big)\\
=\; &- \tfrac{1}{1-\varepsilon}\Big(\mathcal{K}_\gamma(\widehat{\boldsymbol{\tau}}_{sr}, \boldsymbol{\phi}_3) + \mathcal{L}_\gamma(\widehat{\boldsymbol{\tau}}_{s\theta}, \boldsymbol{\phi}_3) + \left(\beta_1\widehat{\boldsymbol{\tau}}_{\theta s} - \beta_2\widehat{\boldsymbol{\tau}}_{rs}, \boldsymbol{\phi}_3\right)\Big),
\end{aligned}
$$

$\Big(\frac{\partial}{\partial r}(r\beta u) + \frac{\partial}{\partial \theta}(\beta v), \varphi\Big) = 0$, for every $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3, \varphi) \in \mathbf{V}_{\delta,h} \times Q_h$, and

$$
\left(\widehat{\boldsymbol{\tau}}_{ij}, r\beta\boldsymbol{\sigma}_\ell\right)_h + \mathcal{W}e\,\mathcal{B}_h\left(u, v, \widehat{\boldsymbol{\tau}}_{ij}, \boldsymbol{\sigma}_\ell\right) = \left(\mathbf{G}_{ij}\left(\mathbf{u}, \widehat{\boldsymbol{\tau}}\right), \boldsymbol{\sigma}_\ell\right)_h
$$

for all $\boldsymbol{\sigma} \in (T_h)^6$ with

$$
\mathcal{B}_h = -\left(r\beta u\frac{\partial \sigma}{\partial r} + \beta v\frac{\partial \sigma}{\partial \theta} - \frac{1}{2}\left(\frac{\partial}{\partial r}(r\beta u) + \frac{\partial}{\partial \theta}(\beta v)\right)\sigma, \tau\right)_h + \langle \tau^-, \sigma^+ - \sigma^-\rangle_{h,u,v},
$$

with $\mathcal{K}_\gamma(\sigma, \varphi) = \Big(\sigma, r\beta\frac{\partial \varphi}{\partial r} + \gamma\left(\beta + \beta_2\right)\varphi\Big)$, $\mathcal{L}_\gamma(\sigma, \varphi) = \Big(\sigma, \beta\frac{\partial \varphi}{\partial \theta} - \gamma\beta_1\varphi\Big)$ and $\mathbf{G}$ given by (3.5).

4.2. Algorithm

Next we define the algorithm to solve the approximated problem $(\mathcal{P}_h)$ (as usual, the index h is dropped to simplify the presentation).

- Given an iterate $\widehat{\boldsymbol{\tau}}^k$, find $\mathbf{u}^k \equiv (u^k, v^k, w^k)$, and p^k solutions of the following Navier-Stokes system $(NS)_k$

$$
\begin{aligned}
2\mathcal{K}_\gamma\Big(\psi_\gamma \tfrac{\partial u^k}{\partial r}, \boldsymbol{\phi}_1\Big) &+ \mathcal{L}_\gamma\left(\beta\psi_{\gamma-1}\tfrac{\partial u^k}{\partial \theta}, \boldsymbol{\phi}_1\right) + 2\left(\psi_{\gamma-1}\left(\beta^2+\beta_2^2\right)u^k, \boldsymbol{\phi}_1\right)\\
&+ \mathcal{L}_\gamma\left(\beta\psi_{\gamma-1}\left(r\tfrac{\partial v^k}{\partial r} - v^k\right), \boldsymbol{\phi}_1\right) + 2\left(\psi_{\gamma-1}\left(\beta^2\tfrac{\partial v^k}{\partial \theta} - \beta_1\beta_2 v^k\right), \boldsymbol{\phi}_1\right)\\
&- \tfrac{1}{1-\varepsilon}\mathcal{K}_{\gamma+1}\Big(\psi_\gamma\, p^k, \boldsymbol{\phi}_1\Big) + \tfrac{\mathcal{R}e}{1-\varepsilon}\mathcal{K}_\gamma\Big(\psi_\gamma\,(u^k)^2, \boldsymbol{\phi}_1\Big)\\
&+ \tfrac{\mathcal{R}e}{1-\varepsilon}\Big(\mathcal{L}_\gamma\Big(\psi_\gamma u^k v^k, \boldsymbol{\phi}_1\Big) + \Big(\psi_\gamma\left(\beta_2(w^k)^2 + \beta(v^k)^2\right), \boldsymbol{\phi}_1\Big)\Big)\\
= -\tfrac{1}{1-\varepsilon}&\Big(\mathcal{K}_\gamma(\widehat{\boldsymbol{\tau}}^k_{rr}, \boldsymbol{\phi}_1) + \mathcal{L}_\gamma(\widehat{\boldsymbol{\tau}}^k_{r\theta}, \boldsymbol{\phi}_1) + \left(\beta_2\widehat{\boldsymbol{\tau}}^k_{ss} + \beta\widehat{\boldsymbol{\tau}}^k_{\theta\theta}, \boldsymbol{\phi}_1\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}_\gamma\Big(\beta\psi_{\gamma-1}(r\tfrac{\partial v^k}{\partial r} - v^k), \boldsymbol{\phi}_2\Big) &+ 2\mathcal{L}_\gamma\left(\beta\psi_{\gamma-1}\tfrac{\partial v^k}{\partial \theta}, \boldsymbol{\phi}_2\right)\\
&+ \left(\psi_{\gamma-1}\left(\left(\beta^2+2\beta_1^2\right)v^k - r\beta^2\tfrac{\partial v^k}{\partial r}\right), \boldsymbol{\phi}_2\right) + \mathcal{K}_\gamma\left(\beta\psi_{\gamma-1}\tfrac{\partial u^k}{\partial \theta}, \boldsymbol{\phi}_2\right)\\
&+ 2\mathcal{L}_\gamma\left(\beta\psi_{\gamma-1}\,u^k, \boldsymbol{\phi}_2\right) - \left(\psi_{\gamma-1}\left(\beta^2\tfrac{\partial u^k}{\partial \theta} + 2\beta_1\beta_2 u^k\right), \boldsymbol{\phi}_2\right)\\
&- \tfrac{1}{1-\varepsilon}\,\mathcal{L}_{\gamma+1}\Big(\psi_\gamma p^k, \boldsymbol{\phi}_2\Big) + \tfrac{\mathcal{R}e}{1-\varepsilon}\,\mathcal{K}_\gamma\Big(\psi_\gamma u^k v^k, \boldsymbol{\phi}_2\Big)\\
&+ \tfrac{\mathcal{R}e}{1-\varepsilon}\Big(\mathcal{L}_\gamma\Big(\psi_\gamma(v^k)^2, \boldsymbol{\phi}_2\Big) - \Big(\psi_\gamma\left(\beta_1(w^k)^2 + \beta u^k v^k\right), \boldsymbol{\phi}_2\Big)\Big)\\
= -\tfrac{1}{1-\varepsilon}&\Big(\mathcal{K}_\gamma(\widehat{\boldsymbol{\tau}}^k_{\theta r}, \boldsymbol{\phi}_2) + \mathcal{L}_\gamma(\widehat{\boldsymbol{\tau}}^k_{\theta\theta}, \boldsymbol{\phi}_2) - \left(\beta_1\widehat{\boldsymbol{\tau}}^k_{ss} + \beta\widehat{\boldsymbol{\tau}}^k_{r\theta}, \boldsymbol{\phi}_2\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}_\gamma\Big(\psi_{\gamma-1}(r\beta\tfrac{\partial w^k}{\partial r} - \beta_2 w^k), \boldsymbol{\phi}_3\Big) &+ \mathcal{L}_\gamma\left(\psi_{\gamma-1}\left(\beta\tfrac{\partial w^k}{\partial \theta} + \beta_1 w^k\right), \boldsymbol{\phi}_3\right)\\
&+ \left(\psi_{\gamma-1}\left((r\delta)^2 w^k + \beta\beta_1\tfrac{\partial w^k}{\partial \theta} - r\beta\beta_2\tfrac{\partial w^k}{\partial r}\right), \boldsymbol{\phi}_3\right)\\
&- \tfrac{1}{1-\varepsilon}\Big(r\psi_\gamma\, p^*, \boldsymbol{\phi}_3\Big) + \tfrac{\mathcal{R}e}{1-\varepsilon}\,\mathcal{K}_\gamma\Big(\psi_\gamma\, u^k w^k, \boldsymbol{\phi}_3\Big)\\
&+ \tfrac{\mathcal{R}e}{1-\varepsilon}\Big(\mathcal{L}_\gamma\Big(\psi_\gamma\, v^k w^k, \boldsymbol{\phi}_3\Big) - \Big(\psi_\gamma\left(\beta_2 u^k - \beta_1 v^k\right)w^k, \boldsymbol{\phi}_3\Big)\Big)\\
= -\tfrac{1}{1-\varepsilon}&\Big(\mathcal{K}_\gamma(\widehat{\boldsymbol{\tau}}^k_{sr}, \boldsymbol{\phi}_3) + \mathcal{L}_\gamma(\widehat{\boldsymbol{\tau}}^k_{s\theta}, \boldsymbol{\phi}_3) + \left(\beta_1\widehat{\boldsymbol{\tau}}^k_{\theta s} - \beta_2\widehat{\boldsymbol{\tau}}^k_{rs}, \boldsymbol{\phi}_3\right)\Big),
\end{aligned}
$$

$\Big(\frac{\partial}{\partial r}\left(r\beta u^k\right) + \frac{\partial}{\partial \theta}(\beta v^k), \varphi\Big) = 0$, for every $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3, \varphi) \in \mathbf{V}_{\delta,\mathbf{h}} \times \mathbf{Q}_\mathbf{h}$.

- Calculate the new iterate $\widehat{\boldsymbol{\tau}}^{k+1}$ as the solution of the transport problem $\Big(\widehat{\boldsymbol{\tau}}^{k+1}_{ij}, r\beta\boldsymbol{\sigma}_\ell\Big) + \mathcal{W}e\mathcal{B}_h\left(u^k, v^k, \widehat{\boldsymbol{\tau}}^{k+1}_{ij}, \boldsymbol{\sigma}_\ell\right) = \Big(\mathbf{G}_{ij}\big(\mathbf{u}^k, \widehat{\boldsymbol{\tau}}^k\big), \boldsymbol{\sigma}_\ell\Big) \quad \forall\,\boldsymbol{\sigma} \in (T_h)^6$.
- Find $\mathbf{u}^{k+1} \equiv (u^{k+1}, v^{k+1}, w^{k+1}, p^{k+1})$ solution of the Navier-Stokes system $(NS)_{k+1}$.

Taking into account this algorithm, our aim is to write the linear systems corresponding to problem $(\mathcal{P}_h)$ at a given iteration k. To simplify the presentation, we will consider the case of creeping non-Newtonian flows, which corresponds to $\mathcal{R}e = 0$.

Given $\widehat{\boldsymbol{\tau}}^k$, expressing the corresponding approximate solutions u^k, v^k, w^k and p^k in the basis of $\mathbf{V}_{\delta,\mathbf{h}}$ and Q_h

$$u^k = \sum_{i=1}^{n_h} u_i^k\, \phi_1^i, \quad v^k = \sum_{i=1}^{n_h} v_i^k\, \phi_2^i, \quad w^k = \sum_{i=1}^{n_h} w_i^k\, \phi_3^i, \quad p^k = \sum_{i=1}^{m_h} p_i^k\, \varphi^i,$$

we obtain the following linear system

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & 0 & \frac{1}{1-\varepsilon}\,\mathbf{A}_3 \\ \mathbf{A}_4 & \mathbf{A}_5 & 0 & \frac{1}{1-\varepsilon}\,\mathbf{A}_6 \\ 0 & 0 & \mathbf{A}_7 & 0 \\ \mathbf{A}_8 & \mathbf{A}_9 & 0 & 0 \end{pmatrix} \begin{pmatrix} u^k \\ v^k \\ w^k \\ p^k \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^k \\ \mathbf{b}_2^k \\ \mathbf{b}_3^k \\ 0 \end{pmatrix}$$

where

$$\begin{aligned}
(\mathbf{A}_1)_{ij} &= 2\mathcal{K}_\gamma\left(\psi_\gamma \tfrac{\partial \phi_1^j}{\partial r}, \phi_1^i\right) + \mathcal{L}_\gamma\left(\beta\psi_{\gamma-1}\,\tfrac{\partial \phi_1^j}{\partial \theta}, \phi_1^i\right) + 2\left(\psi_{\gamma-1}\left(\beta^2+\beta_2^2\right)\phi_1^j, \phi_1^i\right),\\
(\mathbf{A}_2)_{ij} &= \mathcal{L}_\gamma\left(\psi_{\gamma-1}\left(r\beta\tfrac{\partial \phi_2^j}{\partial r} - \beta\phi_2^j\right), \phi_1^i\right) + 2\left(\psi_{\gamma-1}\left(\beta^2\tfrac{\partial \phi_2^j}{\partial \theta} - \beta_1\beta_2\phi_2^j\right), \phi_1^i\right),\\
(\mathbf{A}_3)_{ij} &= -\mathcal{K}_{\gamma+1}\left(\psi_\gamma\varphi^j, \phi_1^i\right),\\
(\mathbf{A}_4)_{ij} &= \mathcal{K}_\gamma\left(\beta\psi_{\gamma-1}\tfrac{\partial \phi_1^j}{\partial \theta}, \phi_2^j\right) + 2\mathcal{L}_\gamma\left(\beta\psi_{\gamma-1}\phi_1^j, \phi_2^j\right)\\
&\qquad - \left(\psi_{\gamma-1}\left(\beta^2\tfrac{\partial \phi_1^j}{\partial \theta} + 2\beta_1\beta_2\phi_1^j\right), \phi_2^i\right),\\
(\mathbf{A}_5)_{ij} &= \mathcal{K}_\gamma\left(\psi_{\gamma-1}\left(r\beta\tfrac{\partial \phi_2^j}{\partial r} - \beta\phi_2^j\right), \phi_2^i\right) + 2\mathcal{L}_\gamma\left(\beta\psi_{\gamma-1}\tfrac{\partial \phi_2^j}{\partial \theta}, \phi_2^i\right)\\
&\qquad + \left(\psi_{\gamma-1}\left(\left(\beta^2+2\beta_1^2\right)\phi_2^j - r\beta^2\,\tfrac{\partial \phi_2^j}{\partial r}\right), \phi_2^i\right),\\
(\mathbf{A}_6)_{ij} &= -\mathcal{L}_{\gamma+1}\left(\psi_\gamma\varphi^j, \phi_2^j\right),\\
(\mathbf{A}_7)_{ij} &= \mathcal{K}_\gamma\left(\psi_{\gamma-1}\left(r\beta\tfrac{\partial \phi_3^j}{\partial r} - \beta_2\phi_3^j\right), \phi_3^i\right) + \mathcal{L}_\gamma\left(\psi_{\gamma-1}\left(\beta\tfrac{\partial \phi_3^j}{\partial \theta} + \beta_1\phi_3^j\right), \phi_3^i\right)\\
&\qquad + \left(\psi_{\gamma-1}\left((r\delta)^2\,\phi_3^j + \beta\beta_1\tfrac{\partial \phi_3^j}{\partial \theta} - r\beta\beta_2\tfrac{\partial \phi_3^j}{\partial r}\right), \phi_3^i\right),\\
(\mathbf{A}_8)_{ij} &= \left(r\beta\tfrac{\partial \phi_1^j}{\partial r} + (\beta+\beta_2)\phi_1^j, \varphi^j\right),\\
(\mathbf{A}_9)_{ij} &= \left(\beta\tfrac{\partial \phi_2^j}{\partial \theta} - \beta_1\,\phi_2^j, \varphi^j\right),
\end{aligned}$$

and where the vectors $\mathbf{b_j}$ $(j = 1, 3)$ are given by

$$\begin{aligned}
\left(\mathbf{b}_1^k\right)_i &= -\tfrac{1}{1-\varepsilon}\Big(\mathcal{K}_\gamma(\widehat{\boldsymbol{\tau}}_{rr}^k, \phi_1^i) + \mathcal{L}_\gamma(\widehat{\boldsymbol{\tau}}_{r\theta}^k, \phi_1^i) + \left(\beta_2\widehat{\boldsymbol{\tau}}_{ss}^k + \beta\widehat{\boldsymbol{\tau}}_{\theta\theta}^k, \phi_1^i\right)\Big),\\
\left(\mathbf{b}_\mathbf{2}^\mathbf{k}\right)_i &= -\tfrac{1}{1-\varepsilon}\Big(\mathcal{K}_\gamma(\widehat{\boldsymbol{\tau}}_{\theta r}^k, \phi_2^i) + \mathcal{L}_\gamma(\widehat{\boldsymbol{\tau}}_{\theta\theta}^k, \phi_2^i) - \left(\beta_1\widehat{\boldsymbol{\tau}}_{ss}^k + \beta\widehat{\boldsymbol{\tau}}_{r\theta}^k, \phi_2^i\right)\Big),
\end{aligned}$$

$$\begin{aligned}\left(\mathbf{b}_3^k\right)_i = & -\tfrac{1}{1-\varepsilon}\Big(\mathcal{K}_\gamma(\widehat{\boldsymbol{\tau}}_{sr}^k,\phi_3^i) + \mathcal{L}_\gamma(\widehat{\boldsymbol{\tau}}_{s\theta}^k,\phi_3^i) + \left(\beta_1\widehat{\boldsymbol{\tau}}_{\theta s}^k - \beta_2\widehat{\boldsymbol{\tau}}_{rs}^k,\phi_3^i\right)\Big)\\ & + \tfrac{1}{1-\varepsilon}\Big(r\psi_\gamma\, p^*,\phi_3^i\Big).\end{aligned}$$

After obtaining (u^k, v^k, w^k, p^k), we consider the transport equation to get $\widehat{\boldsymbol{\tau}}^{k+1} \equiv \left(\widehat{\boldsymbol{\tau}}_{ij}^{k+1}\right)$. Using the local basis functions $\{\zeta_\ell\}_{\ell=1,2,3} \subset T_h$, the local system for the approximate transport problem can be written as

$$\mathbf{A}^k\boldsymbol{\tau}_{ij}^{k+1} + \left(\mathbf{A}^k\right)^-\left(\boldsymbol{\tau}_{ij}^{k+1}\right)^- = \mathbf{G}_{ij}^k, \qquad i,j=1,2,3$$

with

$$\begin{aligned}\mathbf{A}_{\ell m}^k &= \Big(\zeta_m, r\beta\,\zeta_l\Big)_K - \mathcal{W}e\left(\zeta_m, r\beta u^k\tfrac{\partial\zeta_l}{\partial r} + \beta v^k\tfrac{\partial\zeta_l}{\partial\theta}\right)_K\\ &\quad - \tfrac{\mathcal{W}e}{2}\left(\left(r\beta\tfrac{\partial u^k}{\partial r} + \beta\tfrac{\partial v^k}{\partial\theta} + (\beta+\beta_2)u^k - \beta_1 v^k\right)\zeta_m,\zeta_l\right)_K,\\ \left(\mathbf{A^k}\right)_{\ell m}^- &= \mathcal{W}e\int_{\partial K^-(ru_h,v_h)}\beta\zeta_m^-\left(\zeta_l-\zeta_l^-\right)\left(ru^k nr + v^k n_\theta\right)\,ds,\\ \left(\mathbf{G}_{ij}^k\right)_\ell &= \Big(\mathbf{G}_{ij}\big(\mathbf{u}^k,\widehat{\boldsymbol{\tau}}^k\big),\zeta_\ell\Big)_K,\end{aligned}$$

where ζ_ℓ^- denotes the ℓth basis function over the correspondent adjacent element to K by an inflow edge, $\mathbf{G}$ is the function given by (3.5). The local systems lead to a linear system of the form

$$M^k\tau_i^{k+1} = C_i^k,$$

where M^k is a non-symmetric matrix whose dimension is twice the number of nodes in the triangulation.

4.3. Numerical results

The domain Σ defined by (3.3) is discretized using triangles. Referring to the algorithm, we see that a Navier-Stokes system has to be solved for (u,v,p), a Poisson equation for w and a transport equation for $\widehat{\boldsymbol{\tau}}$. The velocity is set to zero on the lateral surface of the pipe. The non-dimensional stream function ψ can be written with respect to the components u and v, as

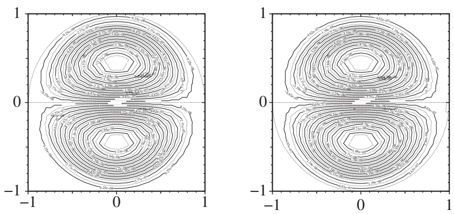
$$u = -\frac{1}{r\beta}\frac{\partial\psi}{\partial\theta}, \qquad v = \frac{1}{\beta}\frac{\partial\psi}{\partial r}.$$

and the wall-shear stress is $\boldsymbol{\tau}_w = -\left(\mathbf{T}\cdot\mathbf{n}\right)\cdot\boldsymbol{\lambda}\,|_{r=1}$. In this particular case, $\boldsymbol{\tau}_w$ is given by

$$\begin{aligned}\boldsymbol{\tau}_w = & -2(1-\varepsilon)\left(\tfrac{\partial u}{\partial r} - \left(\tfrac{\partial v}{\partial\theta} + u\right)\right)|_{r=1}\sin\theta\cos\theta\\ & -(1-\varepsilon)\left(\tfrac{\partial v}{\partial r} + \left(\tfrac{\partial u}{\partial\theta} - v\right)\right)|_{r=1}\left(\sin^2\theta - \cos^2\theta\right)\\ & -\left(\boldsymbol{\tau}_{rr} - \boldsymbol{\tau}_{\theta\theta}\right)|_{r=1}\sin\theta\cos\theta - \boldsymbol{\tau}_{r\theta}|_{r=1}\left(\sin^2\theta - \cos^2\theta\right).\end{aligned}$$

In what follows, we consider the numerical simulation of fully developed steady Oldroyd-B flows with constant and non constant viscosity, in a curved pipe with constant cross section. The behavior of creeping (i.e., Reynolds number set to zero) and inertial flows (non-zero Reynolds number) is analyzed for different values of the parameter involved in the governing equations (the Reynolds number $\mathcal{R}e$, the Weissenberg number $\mathcal{W}e$, the curvature ration δ, the non-dimensional viscosity parameter η and the exponent q appearing in the power-law type viscosity). A continuation method is carried out to implement these different tests.

4.3.1. Creeping generalized Oldroyd-B flows. In this section, we are interested in the qualitative study of creeping flows ($\mathcal{R}e = 0$) for generalized Oldroyd-B fluids, and especially on the behavior of the secondary motions and of the wall shear stress. In order to analyse the combined effect of the viscoelasticity, the non-constant viscosity and the curvature ratio, several calculations were achieved.



(a) $\mathcal{W}e = 1$ (b) $\mathcal{W}e = 5$

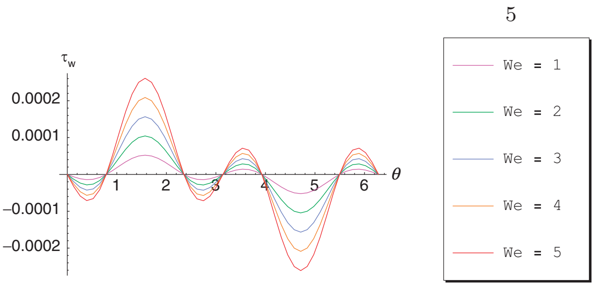


FIGURE 2. Streamlines (top) and wall shear stress (bottom) for creeping Oldroyd-B flows, for the curvature ratio $\delta = 0.001$.

It is well known that in the case of creeping Oldroyd-B fluids, the viscoelasticity promotes secondary flows characterized by two counter-rotating vortices, that the global behavior is stable, and is of Newtonian type. We did not observe any notable changes in the nature of the flow when varying the characteristic parameters (Weissenberg number and curvature ratio). The only difference lies in the values of the stream function and of the wall shear stress, which increase with these parameters, and also in a slight shift to the left of the vortices with increasing curvature ratio. Figure 2 displays the flow behavior for the curvature ratio $\delta = 0.001$.

In a second step, we consider the more general case of Oldroyd-B fluids with non constant viscosity. Fixing the curvature ratio δ and the viscosity parameter η, we implement a continuation method with respect to the exponent q, for different values of the Weissenberg number. The values of the maximum exponent for which convergence is ensured are shown in Table 1. As one can see, and as expected, q_{max} depends on the different parameters. In particular, the values decrease when these parameters increase. We also observe that there is no convergence, for the curvature ratio $\delta = 0.2$, with $\mathcal{W}e = 5$ and the same values of the viscosity parameter η.

$\mathcal{W}e$	1	2	3	4	5
$\delta = 0.001$					
$\eta = 0.4$	21.6	6.80	6.75	7.32	6.93
0.5	4.70	4.57	4.45	4.15	3.95
0.6	3.05	2.85	2.60	2.30	2.00
$\delta = 0.1$					
$\eta = 0.4$	8.92	1.02	0.58	0.45	0.36
0.5	1.53	0.60	0.42	0.32	0.26
0.6	0.80	0.45	0.32	0.26	0.21
$\delta = 0.2$					
$\eta = 0.4$	3.31	0.62	0.42	0.32	–
0.5	0.91	0.45	0.32	0.24	–
0.6	0.60	0.34	0.24	0.19	–

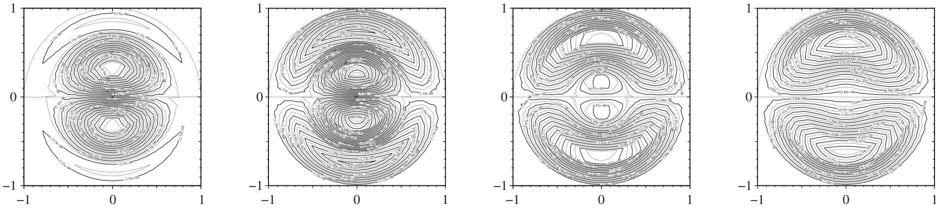
TABLE 1. Maximum values of $|q|$.

Numerical results (using finite element methods) show some changes in flow characteristics and that the viscosity influences its behavior. In summary, we have two phases:

- A phase of variation in the behavior passing from the standard Oldroyd-B type to a new type.
- A phase of stabilisation in the new type.

More precisely, for small values of $|q|$, we observe a surprising phenomenon. Initially, the secondary flows involve non-zero values and are characterized by two counter-rotating vortices. As q increases in absolute value, the streamlines in the core region become less dense, the size of the couple of vortices reduces, and the flow is driven near the wall pipe. We observe the formation of boundary layers flows with a pair of new vortices, initially weak and elongated, strengthening and dominating as the viscosity exponent increases. In contrast, the original secondary flows become more and more weak before vanishing when the level of the exponent viscosity reaches a critical value. The orientation of the new contours is opposite, as well as the sign of the stream function, suggesting a *transition* to a different regime.

It is interesting to observe that this behavior is global, in the sense that it is seems to be independent of the Weissenberg number and of the curvature ratio, and occurs for the same values of the viscosity exponent q. Figure 3 illustrates the behavior of the streamlines in the particular case of a curvature ratio $\delta = 0.001$ and viscosity parameter $\eta = 0.4$. The influence of the viscosity parameter is evident, and as η increases the transition occurs earlier. Moreover, because of the effect of the curvature, the new two vortices are not localized in the center of the cross section but are slightly translated.



(a) $q = -0.17$ (b) $q = -0.18$ (c) $q = -0.19$ (d) $q = -0.2$

FIGURE 3. Qualitative behavior of the streamlines for creeping generalized Oldroyd-B flows, with η=0.4 and for different values of $|q|$ (δ=0.001).

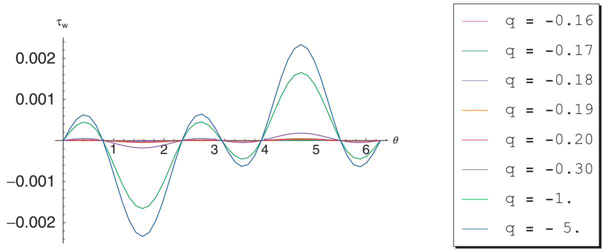


FIGURE 4. Wall shear stress for creeping generalized Oldroyd-B flows, with η=0.4 (δ=0.001).

We also noticed variations in the wall shear stress (see Figure 4). It can be observed that during the *transition*, the amplitude of the wall shear stress decreases and the corresponding curves are inverted in comparison with the Oldroyd-B case.

After the transition phase, and before reaching some "critical" value of the viscosity exponent, the flow is qualitatively more stable. The streamlines are symmetric and the global behavior of the wall shear stress remains unchanged. This critical value of q depends on the viscoelastic parameter, on the viscosity parameter, and particularly on the curvature ratio. Globally, the changes which occur from now on, are similar in some aspects to those already noticed for the generalized Newtonian flows [1, 3].

In particular, for $\eta = 0.4$, for relatively small Weissenberg numbers ($\mathcal{W}e = 2, 3, 4$), and especially in the case of small curvature ratio, we observe a variation

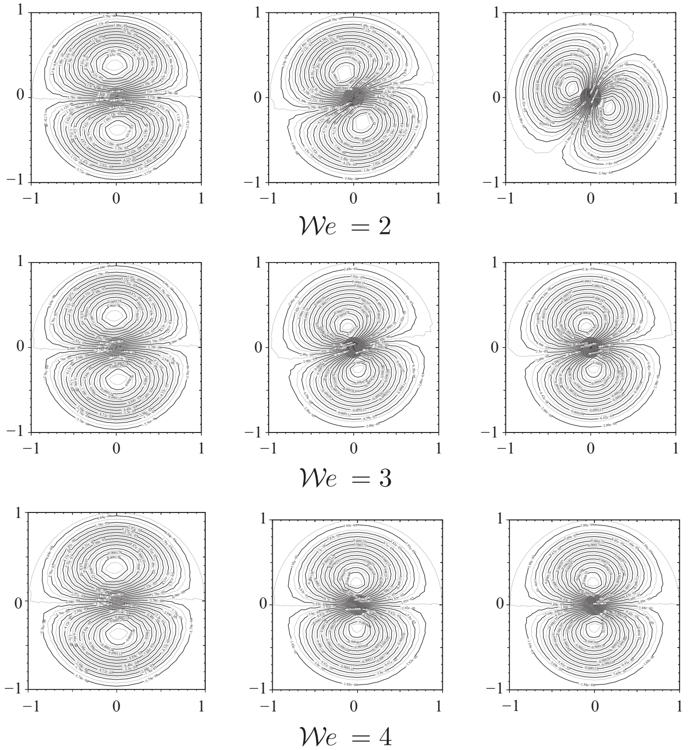


FIGURE 5. Streamlines for creeping generalized Oldroyd-B flows with η=0.4 and $|q| = 6$, 6.4, 6.5 (from left to right), for different Weissenberg numbers $\mathcal{W}e$ (δ=0.001).

in the shape of the vortices, their displacement to the core region, the concentration of the contours in this region and the beginning of a counter-clockwise rotation.

In Figure 5, we plot the streamlines corresponding to this case. The rotation is more pronounced when the Weissenberg number is small suggesting that the viscoelasticity, as well as the inertial forces in the case of generalized Newtonian flows, has an opposite effect [1, 3].

However, contrarily to the generalized Newtonian flows, the viscosity exponent corresponding to the initiation of the rotation is not constant. As can be seen in Table 2, it depends on $\mathcal{W}e$ and as this parameter increases, the rotation initiates earlier. Moreover, the viscoelastic parameter affects the maximum angle and the development of the rotation: for $\mathcal{W}e = 2$, the contours are left-rotating (L), for $\mathcal{W}e = 3$ they are initially left-rotating and then right-rotating (R) before stabilizing symmetrically (S). Finally, when the $\mathcal{W}e$ increases, there is no more rotation and the contours remain symmetric.

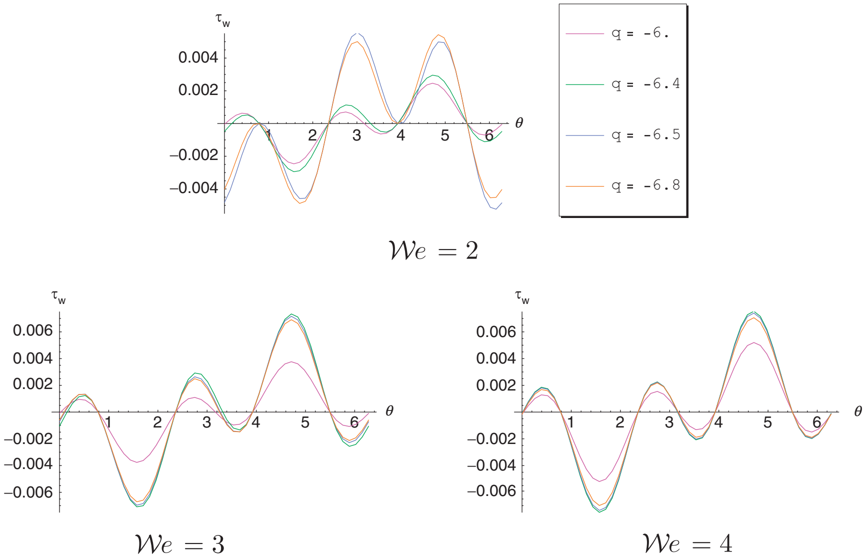


FIGURE 6. Wall shear stress for creeping generalized Oldroyd-B creeping flows with η=0.4 and $|q| = 6$, 6.4, 6.5, 6.8 (δ=0.001).

$\mathcal{W}e$	2	3	4	5
$\delta = 0.001$				
$\eta = 0.4$	6.30	6.25	–	–
	L	L-S	S	S
0.5	4.50	4.30	–	–
	R	R(slight)	S	–

TABLE 2. Values of $|q|$ initiating the rotation.

Parallel modifications can be observed concerning the wall shear stress. Figure 6 shows the corresponding curves for $\eta = 0.4$ and for different values of the Weissenberg number. For $\mathcal{W}e = 5$, the curves corresponding to different viscosity exponents q are identical, suggesting that the wall shear stress is stable in this case. For $\mathcal{W}e = 4$, the global behavior of the curves is similar but with variations in the amplitudes. When the viscoelastic parameter is set to 3, small modifications could be observed in comparison to the previous case. In particular, for q greater than the critical value q_{var} initiating the rotation, we lose the symmetry with respect to the horizontal axis, and the wall shear stress takes negative values at $\theta = 0$ and 2π. This fact is more pronounced for $\mathcal{W}e = 2$, with lost of symmetry with respect to the axis $\theta = \pi$. The same differences are obtained when $\eta = 0.5$. The sign of the wall shear stress for values of q greater than q_{var} is positive. This strongly suggests the existence of a relation between the sign of $\boldsymbol{\tau}_w$ at the points $\theta = 0$ and 2π, and the orientation of the rotation.

A final observation is related to the maximum values of the stream function. Independently of the viscoelastic and viscosity parameters, the maximum values dramatically increase in the neighborhood of the critical value q_{var}. For the cases where the maximum value of the exponent q is big enough, we observe that after this peak, the maximum values decrease before stabilizing.

4.3.2. Inertial generalized Oldroyd-B flows. In the previous subsection, we studied the behavior of the generalized Oldroyd-B flows in the absence of inertia (creeping flows). Our aim here is to consider the more general case of inertial flows, and to analyse the effect of the Reynolds number in combination with the Weissenberg number, the viscosity parameter η, the exponent q, and the curvature ratio δ.

We first consider a pipe with a small curvature ratio ($\delta = 0.001$) in the case of a constant viscosity (inertial Oldroyd-B fluid). The secondary flows exist and the corresponding stream function and wall shear stress have globally the same behavior as the creeping Oldroyd-B flows. At this stage, the nature of the flow is qualitatively identical to that of a Newtonian fluid.

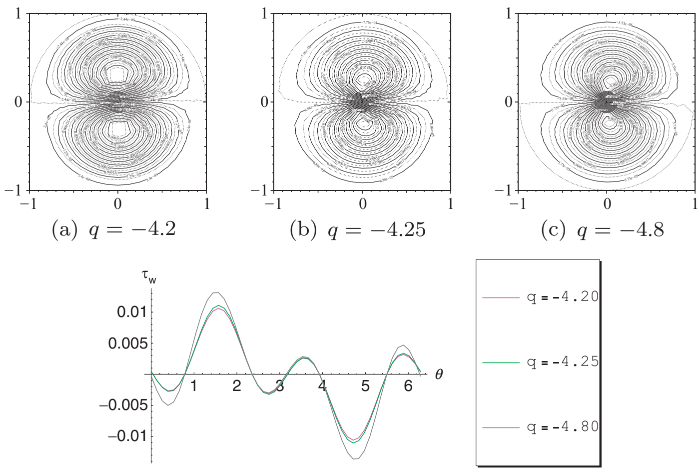
In order to compare with the case of generalized creeping flows, the viscosity parameter η is set to 0.4, 0.5, 0.6 and as previously, several tests were performed for different values of the Weissenberg and the Reynolds numbers, with a continuation in the exponent q.

One of the first remarks is that the *transition* phenomenon observed in the case of generalized creeping flows does not hold, even for relatively small Reynolds number ($\mathcal{R}e = 15$). This fact is evident when η takes the values 0.4 and 0.5 and the behavior in these two cases is close to that of inertial generalized Newtonian flows.

Fixing $\eta = 0.4$, and varying the Reynolds and the Weissenberg numbers, the flow is globally stable for q less than some critical value. From the contours of the stream function and of the wall shear stress for the exponent $|q| \leq 5$, we observe that the qualitative behavior is similar and independent of both inertia and viscoelasticity. The only difference lies in the values and the magnitude of these quantities, which clearly depend on the involved parameters. In the neighborhood of a critical value of q, the behavior presents some changes and is no more uniform. The rotation already observed initiates, and its orientation depend on $\mathcal{R}e$ and $\mathcal{W}e$. Indeed, fixing the Reynolds number to 15, we can see that for $\mathcal{W}e = 2$, a counter-clockwise rotation occurs for $q_{\text{var}} = -6$ and that the contours remain stable till we reach the maximum value for which the convergence is ensured. For $\mathcal{W}e = 3$, the stream function initiates a very slight counter-clockwise rotation at the same value q_{var}, but recovers the symmetry very quickly. Finally, for $\mathcal{W}e = 3$, the same behavior is captured, but with a very slight clockwise rotation. The wall shear stresses behave in an analogous way. In order to emphasize the role of the inertia, we fixed the Weissenberg number and increase the Reynolds number.

In a second step, we consider the case $\eta = 0.5$. For $\mathcal{R}e = 15$, we can observe that the Weissenberg number favorite the clockwise rotation (cf. Figure 7). Indeed, when the Weissenberg number is small (even with larger exponent viscosity), the contours remain symmetric. As $\mathcal{W}e$ increases, the rotation initiates and holds earlier. On the other hand, the Reynolds number does not seem to have a significative influence on the nature of the flow. We implement several tests corresponding to larger values of this characteristic parameter ($\mathcal{R}e = 1$, 15, 30, 50, 70) and did not observe any significative difference.

Table 3 summarizes the results of the maximum values of viscosity parameter $|q|$ with respect to the viscoelasticity ($\mathcal{W}e$) and the inertia ($\mathcal{R}e$) parameters, as a function of the curvature ratio (δ). For fixed δ and $\mathcal{W}e$, the value of $|q|$ decreases when $\mathcal{R}e$ increases. The same occurs if δ and $\mathcal{R}e$ are fixed: the value of $|q|$ decreases when $\mathcal{W}e$ increases. The curvature ratio associated with ($\mathcal{R}e$) and ($\mathcal{W}e$) has a strong influence on the convergence, since it can be shown that when these parameters increase, the values of $|q|$ decrease considerably and in some cases convergence is not achieved.



(a) $q = -4.2$ (b) $q = -4.25$ (c) $q = -4.8$

Figure 7. Streamlines and wall shear stress for inertial generalized Oldroyd-B flows with $\eta = 0.5$, $Re = 15$, $\mathcal{W}e = 1$ and different values of the $|q|$ ($\delta = 0.001$).

The case $\eta = 0.6$ is certainly the more surprising. For large values of the Reynolds number and for the achieved viscosity exponents, the behavior seems to be stable and no notable fact can be observed. The more interesting variations were observed for relatively small Reynolds numbers. Fixing for example $\mathcal{R}e = 15$ and varying the Weissenberg number, we observe that the behavior is qualitatively stable for small values of this parameter. For $\mathcal{W}e = 3$, 4 and 5, some new phenomenon initiates. The characteristics are very similar to those observed in the case

$\mathcal{W}e$	1	2	3	4	5
$\delta = 0.001$					
$\mathcal{R}e = 1$	5.12	4.65	4.45	4.22	4.0
15	4.81	4.36	4.25	4.21	3.95
30	4.78	3.76	3.54	3.37	3.25
50	3.60	3.03	2.81	2.62	2.46
70	3.44	2.57	2.23	1.94	1.72
$\delta = 0.1$					
$\mathcal{R}e = 1$	2.47	0.7	0.46	0.36	0.3
15	0.51	0.15	–	–	–
30	0.05	–	–	–	–
50	–	–	–	–	–
$\delta = 0.2$					
$\mathcal{R}e = 1$	1.26	0.53	0.36	–	–
15	0.23	–	–	–	–
30	–	–	–	–	–

TABLE 3. Maximum values of $|q|$ with $\eta = 0.5$.

of generalized creeping flows during the *phase transition*: formation of boundary layers and a pair of new vortices or strengthening of the new contours and weakening of the original ones. However, in contrast with the creeping flows, the new state is not stable and at some level the inverse phenomenon initiates (Figure 8).

For this particular viscosity parameter, Table 4 shows the results of the maximum values of $|q|$, obtained for different $\mathcal{W}e$ and $\mathcal{R}e$ numbers, in the case of $\delta = 0.001$. Comparing with Table 3 for the same curvature ratio, the same effects of viscoelasticity and inertia on $|q|$ can be observed.

$\mathcal{W}e$	1	2	3	4	5
$\delta = 0.001$					
$\mathcal{R}e = 1$	3.10	2.89	2.68	2.6	2.56
15	2.88	2.67	2.60	2.55	2.51
70	1.69	1.39	1.20	1.05	0.95

TABLE 4. Maximum values of $|q|$ with $\eta = 0.6$.

5. Conclusion

This paper is devoted to finite element simulations of flows of incompressible viscoelastic non-Newtonian fluids of Oldroyd-type through pipes of uniform circular cross-section, and follows the work already published in [1] and [3] for generalized Newtonian fluids. We compare the quantitative and qualitative behavior of the

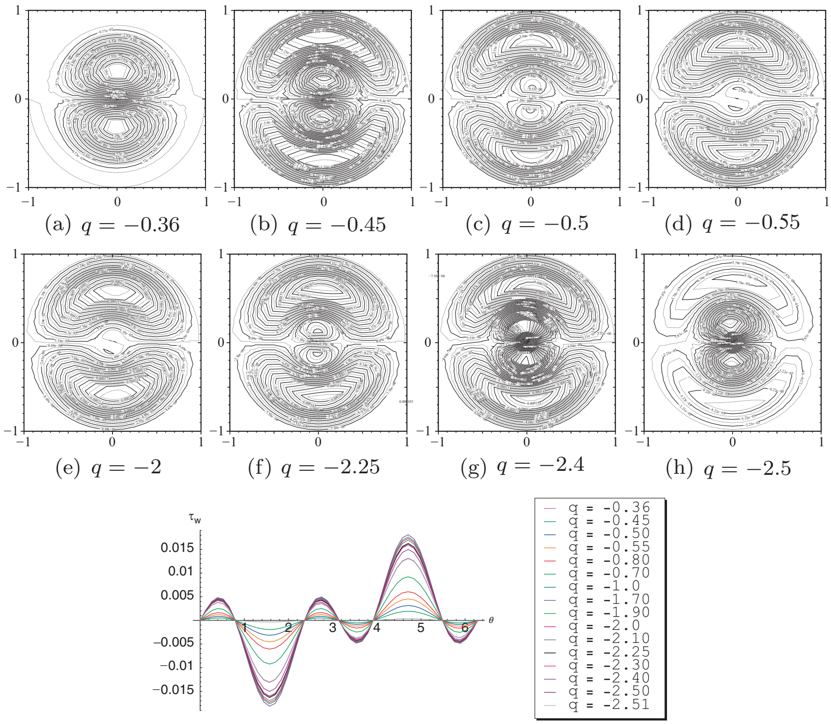


FIGURE 8. Streamlines and wall shear stress for inertial generalized Oldroyd-B flows with η=0.6, $\mathcal{R}e$=15 and $\mathcal{W}e$=5 (δ=0.001).

secondary streamlines and the wall shear stress for creeping and inertial generalized Oldroyd-B flows, performing computations for different values of the Reynolds number, the Weissenberg number, the curvature ratio and the non-dimensional viscosity parameters involved in the governing equations.

In particular, we observe interesting viscosity effects such that, for small curvature ratio and within a certain range of viscosity parameters, the secondary streamlines contours undergo a counter-clockwise rotation and lose symmetry. The complexity of the flow characteristics shown in the numerical tests suggest that further theoretical analysis is needed to study the existence of more than one solution and investigate the corresponding stability, for a range of appropriate non-dimensional parameters.

More detailed discussion and numerical results can be found in [19] where the generalized Newtonian flows are obtained as a particular case of generalized Oldroyd-B flows, in the limit of vanish Weissenberg number (neglected viscoelastic effects). The numerical validation of the present results, using the perturbation method [24] is a work in progress.

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Remarks on Maximal Regularity

Pascal Auscher and Andreas Axelsson

In honour of H. Amann

Abstract. We prove weighted estimates for the maximal regularity operator. Such estimates were motivated by boundary value problems. We take this opportunity to study a class of weak solutions to the abstract Cauchy problem. We also give a new proof of maximal regularity for closed and maximal accretive operators following from Kato's inequality for fractional powers and almost orthogonality arguments.

Mathematics Subject Classification (2000). Primary 47D06; Secondary 35K90, 47A60.

Keywords. Maximal regularity, weighted estimates, abstract Cauchy problem, Kato's inequality, fractional powers, Cotlar's lemma.

1. Weighted estimates for the maximal regularity operator

Assume $-A$ is a densely defined, closed linear operator, generating a bounded analytic semigroup $\{e^{-zA}, |\arg z| < \delta\}$, $0 < \delta < \pi/2$, on a Hilbert space $\mathcal{H}$. Equivalently, A is sectorial of type $\omega(A) = \pi/2 - \delta$. Let $\mathsf{D}(A)$ denote its domain. The maximal regularity operator is defined by the formula

$$\mathcal{M}_+ f(t) = \int_0^t Ae^{-(t-s)A} f(s)\, ds.$$

This operator is associated to the forward abstract evolution equation

$$\dot{u}(t) + Au(t) = f(t), t > 0; \quad u(0) = 0$$

as for appropriate f, $Au(t) = \mathcal{M}_+ f(t)$. An estimate on $\mathcal{M}_+ f$ in the same space as f gives therefore bounds on $\dot{u}$ and Au separately. See Section 2.

The integral defining $\mathcal{M}_+ f$ converges strongly in $\mathcal{H}$ for each $t > 0$ and $f \in L^2(0, \infty; dt, \mathsf{D}(A))$. The estimate $\|Ae^{-(t-s)A}\| \le C(t-s)^{-1}$ following from the analyticity of the semigroup shows that the integral is singular if one only assumes $f(s) \in \mathcal{H}$. The maximal regularity operator is an example of a singular integral operator with operator-valued kernel. The celebrated theorem by de Simon [4] asserts

Theorem 1.1. *Assume $-A$ generates a bounded holomorphic semigroup in $\mathcal{H}$. The operator $\mathcal{M}_+$, initially defined on $L^2(0,\infty; dt, \mathsf{D}(A))$, extends to a bounded operator on $L^2(0,\infty; dt, \mathcal{H})$.*

Motivated by boundary value problems for some second-order elliptic equations, we proved in [3] the following result.

Theorem 1.2. *Assume $-A$ generates a bounded holomorphic semigroup in $\mathcal{H}$ and furthermore that A has bounded holomorphic functional calculus, then $\mathcal{M}_+$, initially defined on $L^2_c(0,\infty; dt, \mathsf{D}(A))$, extends to a bounded operator on $L^2(0,\infty; t^\beta dt, \mathcal{H})$ for all $\beta \in (-\infty, 1)$.*

Here and in what follows the subscript $_c$ means with compact support.

The proof given there uses the operational calculus defined in the thesis of Albrecht [1]. It used as an assumption that A has bounded holomorphic functional calculus as defined by McIntosh [9]. Under this assumption estimates of integral operators more general than the maximal regularity operator, with operator-kernels defined through functional calculus of A, were proved and gave other useful information to understand also the case $\beta = 1$ needed for the boundary value problems. However, not all generators of bounded analytic semigroups have a bounded holomorphic functional calculus (see [10], and Kunstmann and Weis [6, Section 11] for a list of equivalent conditions.) So if we only consider the maximal regularity operator, it is natural to ask whether one can drop the assumption on bounded holomorphic functional calculus in Theorem 1.2. It is indeed the case and as we shall see the proof is extremely simple assuming we know Theorem 1.1.

Theorem 1.3. *Let $-A$ be the generator of a bounded analytic semigroup on $\mathcal{H}$. Then $\mathcal{M}_+$, initially defined on $L^2_c(0,\infty; dt, \mathsf{D}(A))$, extends to a bounded operator on $L^2(0,\infty; t^\beta dt, \mathcal{H})$ for all $\beta \in (-\infty, 1)$.*

The subscript $_c$ means with compact support in $(0,\infty)$. Set $|||f(t)|||^2 = \int_0^\infty \|f(t)\|^2 \frac{dt}{t}$ (we leave in the t-variable in the notation for convenience). As we often use it, we recall the following simplified version of Schur's lemma: if $U(t,s)$, $s,t > 0$, are bounded linear operators on $\mathcal{H}$ with bounds $\|U(t,s)\| \le h(t/s)$ and $C = \int_0^\infty h(u)\frac{du}{u} < \infty$, then

$$\left|\left|\left| \int_0^\infty U(t,s) f(s) \frac{ds}{s} \right|\right|\right| \le C \, |||f(s)||| \, .$$

Proof of Theorem 1.3. Let $\beta < 1$. For $\beta = 0$, this is Theorem 1.1. Assume $\beta \neq 0$ and set $\alpha = \beta/2$. Observe that

$$\|\mathcal{M}_+ f(t)\|_{L^2(t^\beta dt, \mathcal{H})} = \|t^\alpha \mathcal{M}_+ f(t)\|_{L^2(dt,\mathcal{H})}.$$

We have, with $f_\alpha(s) = s^\alpha f(s)$,

$$t^\alpha \mathcal{M}_+ f(t) = \mathcal{M}_+(f_\alpha)(t) + \int_0^t Ae^{-(t-s)A}(t^\alpha - s^\alpha)f(s)\,ds.$$

For the first term apply Theorem 1.1. For the second, write

$$\left\| \int_0^t Ae^{-(t-s)A}(t^\alpha - s^\alpha)f(s)\,ds \right\|_{L^2(dt,\mathcal{H})} = \left|\left|\left| \int_0^\infty U(t,s)g(s)\frac{ds}{s} \right|\right|\right|$$

with $g(s) = s^{1/2+\alpha}f(s)$ and $U(t,s) = Ae^{-(t-s)A}(t^\alpha - s^\alpha)s^{1/2-\alpha}t^{1/2}$ for $s < t$ and 0 otherwise. Since $|||g(t)||| = \|f\|_{L^2(t^\beta dt,\mathcal{H})}$, it remains to estimate the norm of $U(t,s)$ on $\mathcal{H}$. We have

$$\|U(t,s)\| \le C\frac{|t^\alpha - s^\alpha|}{|t-s|}s^{1/2-\alpha}t^{1/2}, \quad s < t.$$

It is easy to see that it is on the order of $(s/t)^{1/2-\max(\alpha,0)}$ as $s < t$. We conclude by applying Schur's lemma. $\square$

Let

$$\mathcal{M}_- f(t) = \int_t^\infty Ae^{-(s-t)A}f(s)\,ds.$$

This operator is associated to the backward abstract evolution equation

$$\dot{v}(t) - Av(t) = f(t), t > 0; \quad v(\infty) = 0$$

as for appropriate f, $Av(t) = -\mathcal{M}_- f(t)$.

Corollary 1.4. *Assume that $-A$ generates a bounded analytic semigroup on $\mathcal{H}$. Then $\mathcal{M}_-$, initially defined on $L^2_c(0,\infty; dt, \mathsf{D}(A))$, extends to a bounded operator on $L^2(0,\infty; t^\beta dt, \mathcal{H})$ for all $\beta \in (-1,\infty)$.*

Proof. Observe that the adjoint of $\mathcal{M}_-$ in $L^2(0,\infty;t^\beta dt,\mathcal{H})$ for the duality defined by $L^2(0,\infty;dt,\mathcal{H})$ is $\mathcal{M}_+$ in $L^2(0,\infty; t^{-\beta}dt, \mathcal{H})$ associated to A^* and apply Theorem 1.3. $\square$

We next show that the range of β is optimal in both results.

Theorem 1.5. *For any non zero $-A$ generating a bounded analytic semigroup on $\mathcal{H}$ and $\beta \ge 1$, $\mathcal{M}_+$ is not bounded on $L^2(0,\infty; t^\beta dt, \mathcal{H})$ and $\mathcal{M}_-$ is not bounded on $L^2(0,\infty; t^{-\beta}dt, \mathcal{H})$.*

Proof. It suffices to consider $\mathcal{M}_-$. Since $A \ne 0$, $\overline{\mathsf{R}(A)}$, the closure of the range of A, contains non zero elements. As $\overline{\mathsf{R}(A)} \cap \mathsf{D}(A)$ is dense in it, pick $u \in \overline{\mathsf{R}(A)} \cap \mathsf{D}(A)$, $u \ne 0$, and set $f(t) = u$ for $1 \le t \le 2$ and 0 elsewhere. Then $f \in L^2_c(0,\infty; dt, \mathsf{D}(A))$ and $f \in L^2(0,\infty; t^{-\beta}dt, \mathcal{H})$ with $\|f(t)\|_{L^2(0,\infty;t^{-\beta}dt,\mathcal{H})} = c_\beta\|u\| < \infty$. For $t < 1$, one has

$$\mathcal{M}_- f(t) = (e^{-(1-t)A} - e^{-(2-t)A})u,$$

which converges to $(e^{-A} - e^{-2A})u$ in $\mathcal{H}$ when $t \to 0$.

We claim that $(e^{-A} - e^{-2A})u \neq 0$ so

$$\|\mathcal{M}_- f(t)\|^2_{L^2(0,\infty;t^{-\beta}dt,\mathcal{H})} \geq \int_0^1 \|(e^{-(1-t)A} - e^{-(2-t)A})u\|^2 \frac{dt}{t^\beta} = \infty.$$

To prove the claim, we argue as follows. Assume it is 0, then $e^{-2A}u = e^{-A}u$ so that an iteration yields $e^{-nA}u = e^{-A}u$ for all integers $n \geq 2$. If $n \to \infty$, $e^{-nA}u$ tends to 0 in $\mathcal{H}$ because $u \in \overline{\mathsf{R}(A)}$. Thus $e^{-A}u = 0$ and it follows that $e^{-tA}u = e^{-(t-1)A}e^{-A}u = 0$ for all $t > 1$. The analytic function $z \to e^{-zA}u$ is thus identically 0 for $|\arg z| < \delta$. On letting $z \to 0$, we get $u = 0$ which is a contradiction. $\square$

We have seen that $\mathcal{M}_-$ cannot map $L^2(0,\infty;t^{-1}dt,\mathcal{H})$ into itself and that it seems due to the behavior of $\mathcal{M}_- f(t)$ at $t = 0$ for some f. We shall make this precise and general: under a further assumption on A which we introduce next, we define $\mathcal{M}_- : L^2(0,\infty;t^{-1}dt,\mathcal{H}) \to L^2_{\text{loc}}(0,\infty;dt,\mathcal{H})$ and show that controlled behavior at 0 of $\mathcal{M}_- f$ guarantees $\mathcal{M}_- f \in L^2(0,\infty;t^{-1}dt,\mathcal{H})$.

We begin by writing whenever $f \in L^2_c(0,\infty;\, dt, \mathsf{D}(A))$ and denoting $f_{-1/2}(s) = s^{-1/2}f(s)$,

$$\begin{aligned}
&\mathcal{M}_- f(t) - e^{-tA}\int_0^\infty Ae^{-sA}f(s)\,ds \\
&\quad = t^{1/2}\mathcal{M}_-(f_{-1/2})(t) + \int_t^{2t} Ae^{-(s-t)A}(s^{1/2} - t^{1/2})s^{1/2}f(s)\,\frac{ds}{s} \\
&\qquad + \int_{2t}^\infty A(e^{-(s-t)A} - e^{-(s+t)A})(s^{1/2} - t^{1/2})s^{1/2}f(s)\,\frac{ds}{s} \\
&\qquad - \int_{2t}^\infty Ae^{-(s+t)A}t^{1/2}s^{1/2}f(s)\,\frac{ds}{s} \\
&\qquad - \int_0^{2t} Ae^{-(s+t)A}sf(s)\,\frac{ds}{s}.
\end{aligned}$$

The right-hand side is seen to belong to $L^2(0,\infty;t^{-1}dt,\mathcal{H})$ with an estimate $C\,|||f(s)|||$ using Theorem 1.1 for the first term and Schur's lemma for the other four terms. Hence, by density, the right-hand side defines a bounded linear operator $\widetilde{\mathcal{M}}_-$ on $L^2(0,\infty;t^{-1}dt,\mathcal{H})$. Also, the integral $\int_0^\infty Ae^{-sA}f(s)\,ds$ is defined as a Bochner integral in $\mathcal{H}$ whenever $f \in L^2_c(0,\infty;dt,\mathcal{H})$. Thus, by density of $\mathsf{D}(A)$ in $\mathcal{H}$, one can set for $f \in L^2_c(0,\infty;dt,\mathcal{H})$,

$$\mathcal{M}_- f(t) := \widetilde{\mathcal{M}}_- f(t) + e^{-tA}\int_0^\infty Ae^{-sA}f(s)\,ds \quad \text{in } L^2_{\text{loc}}(0,\infty;dt,\mathcal{H}). \tag{1.1}$$

Let E be the space of $f \in L^2(0,\infty;t^{-1}dt,\mathcal{H})$ such that the integrals $\int_\delta^R Ae^{-sA}f(s)\,ds$ converge weakly in $\mathcal{H}$ as $\delta \to 0$ and $R \to \infty$. Then the above equality extends to $f \in E$. Assuming, in addition, that A^* satisfies the quadratic estimate

$$\left|\left|\left| sA^*e^{-sA^*}h \right|\right|\right| \leq C\|h\|_{\mathcal{H}} \quad \text{for all } h \in \mathcal{H}, \tag{1.2}$$

we have $E = L^2(0,\infty;t^{-1}dt,\mathcal{H})$. Indeed, for all $f \in L^2(0,\infty;t^{-1}dt,\mathcal{H})$ and $h \in \mathcal{H}$,

$$\int_0^\infty \left|(sAe^{-sA}f(s),h)\right| \frac{ds}{s} \le |||f(s)||| \left|\left|\left|sA^*e^{-sA^*}h\right|\right|\right| \lesssim |||f(s)||| \, \|h\|_{\mathcal{H}} \tag{1.3}$$

and the weak convergence of the truncated integrals follows easily. Thus, the right-hand side of (1.1) makes sense for all $f \in L^2(0,\infty;t^{-1}dt,\mathcal{H})$ under (1.2) and this defines $\mathcal{M}_-f$. Moreover, it follows from (1.3) that

$$\sup_{\tau>0} \frac{1}{\tau} \int_\tau^{2\tau} \|\mathcal{M}_-f(t)\|_{\mathcal{H}}^2 \, dt \le C\, |||f(s)|||^2 . \tag{1.4}$$

Then remark that

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_\tau^{2\tau} \mathcal{M}_-f(t)\, dt = \int_0^\infty Ae^{-sA}f(s)\, ds \quad \text{in } \mathcal{H}, \tag{1.5}$$

as the corresponding limit for $\widetilde{\mathcal{M}}_-f$ is 0 and $e^{-tA} \to I$ strongly when $t \to 0$.

All this yields the following result.

Proposition 1.6. *Let $-A$ be the generator of a bounded analytic semigroup in $\mathcal{H}$ and assume that the quadratic estimate* (1.2) *holds for A^*. Then* (1.1) *defines $\mathcal{M}_-f \in L^2_{\mathrm{loc}}(0,\infty;dt,\mathcal{H})$ with estimates* (1.4) *and limit* (1.5) *for all $f \in L^2(0,\infty;t^{-1}dt,\mathcal{H})$. In particular,*

$$\mathcal{M}_-f \in L^2(0,\infty;t^{-1}dt,\mathcal{H})$$

if and only if

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_\tau^{2\tau} \mathcal{M}_-f(t)\, dt = 0.$$

The last condition defines a closed subspace of $L^2(0,\infty;t^{-1}dt,\mathcal{H})$ and there is a constant C such that for all f in this subspace

$$\|\mathcal{M}_-f(t)\|_{L^2(0,\infty;t^{-1}dt,\mathcal{H})} \le C\|f(t)\|_{L^2(0,\infty;t^{-1}dt,\mathcal{H})}.$$

Note that (1.2) holds if A has bounded holomorphic functional calculus by McIntosh's theorem [9].

Remark 1.7. For $\mathcal{M}_+$, the analysis is not that satisfactory (for $\beta = 1$). One can show similarly that

$$\left\|\mathcal{M}_+f(t) - Ae^{-tA}\int_0^\infty e^{-sA}f(s)\, ds\right\|_{L^2(0,\infty;tdt,\mathcal{H})} \le C\|f(t)\|_{L^2(0,\infty;tdt,\mathcal{H})}$$

provided $f \in L^2_c(0,\infty;dt,\mathsf{D}(A))$. If the quadratic estimate (1.2) holds for A, this allows to extend $\mathcal{M}_+$ to the space $\{f \in L^2_{\mathrm{loc}}(0,\infty;dt,\mathcal{H}); \int_0^\infty e^{-sA}f(s)\, ds$ converges weakly in $\mathcal{H}\}$. However, there is no simple description of this space.

2. Applications to the abstract Cauchy problem

In this section, we assume throughout that $-A$ generates a bounded analytic semigroup in $\mathcal{H}$.

Let $f \in L^2_{\text{loc}}(0,\infty; dt, \mathcal{H})$. We say that u is a weak solution to $\dot{u}(t) + Au(t) = f(t), t > 0$, if $u \in L^2_{\text{loc}}(0,\infty; dt, \mathcal{H})$,

$$\sup_{0<\tau<1} \frac{1}{\tau} \int_{\tau}^{2\tau} \|u(s)\|_{\mathcal{H}} \, ds < \infty \tag{2.1}$$

and for all $\phi \in C^1_c(0,\infty; \mathcal{H}) \cap C^0_c(0,\infty; \mathsf{D}(A^*))$,

$$\int_0^\infty (u(s), -\dot{\phi}(s) + A^*\phi(s)) \, ds = \int_0^\infty (f(s), \phi(s)) \, ds. \tag{2.2}$$

The notion of weak solution here differs from the one in Amann's book [2, Chapter 5] called weak $L_{p,loc}$ solution ($p \in [1,\infty]$) specialized to $p = 2$. We assume a uniform control through (2.1) near $t = 0$ and assume ϕ compactly supported in $(0,\infty)$ in (2.2) instead of specifying the initial value at $t = 0$ and taking ϕ compactly supported in $[0,\infty)$ as in [2].

Lemma 2.1. *Let $\beta \in (-\infty, 1)$ and $f \in L^2(0,\infty; t^\beta dt, \mathcal{H})$. Then*

$$v(t) = \int_0^t e^{-(t-s)A} f(s) \, ds \tag{2.3}$$

satisfies

(1) *$v \in C^0([0,\infty); \mathcal{H})$ and for all $t > 0$, $\|v(t)\|^2_{\mathcal{H}} \le Ct^{1-\beta} \int_0^t s^\beta \|f(s)\|^2_{\mathcal{H}} \, ds$,*
(2) *v is a weak solution to $\dot{u}(t) + Au(t) = f(t), t > 0$,*
(3) *$Av(t) = \mathcal{M}_+ f(t)$ in $L^2_{\text{loc}}(0,\infty; dt, \mathcal{H})$, and*

$$\|\dot{v}(t)\|_{L^2(0,\infty;t^\beta dt,\mathcal{H})} + \|Av(t)\|_{L^2(0,\infty;t^\beta dt,\mathcal{H})} \le C\|f(t)\|_{L^2(0,\infty;t^\beta dt,\mathcal{H})}.$$

Here, by $\mathcal{M}_+$ we mean the bounded extension to $L^2(0,\infty; t^\beta dt, \mathcal{H})$.

Proof. The inequality in (1) follows from the uniform boundedness of the semigroup and Cauchy-Schwarz inequality, and this shows that the integral defining $v(t)$ norm converges in $\mathcal{H}$, thus inferring continuity on $[0,\infty)$, and also (2.1). To check (2.2), it suffices to change order of integration and calculate. The equality $\mathcal{M}_+ f = Av$ is proved by duality against a ϕ as in (2.2) since such ϕ form a dense subspace in $L^2_c(0,\infty; dt, \mathcal{H})$. Finally, the inequalities in (3) are consequences of Theorem 1.3. $\square$

We now state that all weak solutions have an explicit representation and a trace at $t = 0$.

Proposition 2.2. *Let $\beta \in (-\infty, 1)$ and $f \in L^2(0,\infty; t^\beta dt, \mathcal{H})$. Let u be a weak solution to $\dot{u}(t) + Au(t) = f(t), t > 0$. Then, there exists $h \in \mathcal{H}$ such that*

$$u(t) = e^{-tA}h + v(t) \quad \text{in } L^2_{\text{loc}}(0,\infty; dt, \mathcal{H}), \tag{2.4}$$

with v defined by (2.3). In particular, $t \mapsto u(t)$ can be redefined on a null set to be $C^0([0,\infty); \mathcal{H})$ with trace h at $t = 0$.

This immediately implies the following existence and uniqueness results.

Corollary 2.3. *Let $u_0 \in \mathcal{H}$. The initial value problem $\dot{u}(t) + Au(t) = 0, t > 0$, $\lim_{\tau\to 0} \frac{1}{\tau}\int_\tau^{2\tau} u(t)\,dt = u_0$ in $\mathcal{H}$, has a unique weak solution given by $u(t) = e^{-tA}u_0$ for almost every $t > 0$. In particular, up to redefining $t \mapsto u(t)$ on a null set, $u \in C^\infty(0,\infty; D(A))$ and is a strong solution.*

Corollary 2.4. *Let $\beta \in (-\infty, 1)$ and $f \in L^2(0,\infty; t^\beta dt, \mathcal{H})$. The initial value problem $\dot{u}(t) + Au(t) = f(t), t > 0$, with $\lim_{\tau\to 0} \frac{1}{\tau}\int_\tau^{2\tau} u(t)\,dt = 0$ in $\mathcal{H}$, has a unique weak solution given by v defined by (2.3), up to redefining $t \mapsto u(t)$ on a null set.*

Proof of Proposition 2.2. Define $\eta(s)$ to be the piecewise linear continuous function with support $[1,\infty)$, which equals 1 on $(2,\infty)$ and is linear on $(1,2)$. Let $t > 0$. For $0 < \epsilon < t/4$ and $s > 0$, let

$$\eta_\epsilon(t,s) := \eta(s/\epsilon)\eta((t-s)/\epsilon).$$

Let $\phi_0 \in \mathcal{H}$ be any boundary element, and choose $\phi(s) := \eta_\epsilon(t,s)e^{-(t-s)A^*}\phi_0 \in \mathrm{Lip}_c(0,\infty; \mathsf{D}(A^*))$ as test function (by approximating $\eta_\epsilon(t,s)$ by a smooth function, this can be done). A calculation yields (in this proof, $(\ ,\ )$ denotes inner product in $\mathcal{H}$)

$$\begin{aligned} &-\frac{1}{\epsilon}\int_\epsilon^{2\epsilon} \left(e^{-(t-s)A}u(s), \phi_0\right) ds + \frac{1}{\epsilon}\int_\epsilon^{2\epsilon} \left(e^{-sA}u(t-s), \phi_0\right) ds \\ &= \int_0^\infty \left(\eta_\epsilon(t,s)e^{-(t-s)A}f(s), \phi_0\right) ds \end{aligned}$$

and since this is true for arbitrary $\phi_0 \in \mathcal{H}$ and η_ϵ has compact support, we deduce that

$$-\frac{1}{\epsilon}\int_\epsilon^{2\epsilon} e^{-(t-s)A}u(s)\,ds + \frac{1}{\epsilon}\int_\epsilon^{2\epsilon} e^{-sA}u(t-s)\,ds = \int_0^\infty \eta_\epsilon(t,s)e^{-(t-s)A}f(s)\,ds.$$

Now, we let $\epsilon \to 0$ as follows. First, $\eta_\epsilon(t,s)$ tends to the indicator function of $(0,t)$ so that the right-hand side is easily seen to converge to $v(t)$ in $\mathcal{H}$ for any fixed $t > 0$ by dominated convergence. Fix now $0 < a < b < \infty$ and integrate in $t \in (a,b)$ the left-hand side. Remark that $\frac{1}{\epsilon}\int_a^b\int_\epsilon^{2\epsilon} e^{-sA}u(t)\,dsdt$ converges to $\int_a^b u(t)\,dt$ in $\mathcal{H}$. Subtracting this quantity from the second term in the right-hand side and using $u \in L^2_{\mathrm{loc}}(0,\infty;\mathcal{H})$, Lebesgue's theorem yields

$$\int_a^b \left\|\frac{1}{\epsilon}\int_\epsilon^{2\epsilon} e^{-sA}(u(t-s)-u(t))\,ds\right\|_{\mathcal{H}}^2 dt \le \frac{C}{\epsilon}\int_a^b\int_\epsilon^{2\epsilon} \|u(t-s)-u(t)\|_{\mathcal{H}}^2\,dsdt \to 0.$$

For the first term, using $\|e^{-(t-s)A} - e^{-tA}\| \le Cs/t$ from analyticity and (2.1), one sees that

$$\left\|\frac{1}{\epsilon}\int_\epsilon^{2\epsilon} (e^{-(t-s)A} - e^{-tA})u(s)\,ds\right\|_{\mathcal{H}} \to 0 \tag{2.5}$$

for each $t > 0$. Thus

$$h_\epsilon(t) := e^{-tA}h_\epsilon, \quad \text{with} \quad h_\epsilon := \frac{1}{\epsilon}\int_\epsilon^{2\epsilon} u(s)\,ds,$$

has a limit, say $h(t)$, in $L^2(a,b;\mathcal{H})$. The semigroup property yields

$$h_\epsilon(t) = e^{-(t-\tau)A}h_\epsilon(\tau) \quad \text{for all } t \geq \tau.$$

Thus,

$$\begin{aligned}\|h_\epsilon(t) - h_{\epsilon'}(t)\|_{\mathcal{H}} &\leq \frac{1}{b-a}\int_a^b \|e^{-(t-\tau)A}(h_\epsilon(\tau) - h_{\epsilon'}(\tau))\|_{\mathcal{H}} d\tau \\ &\leq C\bigg(\int_a^b \|h_\epsilon(\tau) - h_{\epsilon'}(\tau)\|^2_{\mathcal{H}} d\tau\bigg)^{1/2},\end{aligned}$$

when $t > b$. Hence, since (a,b) is arbitrary, $h_\epsilon(t)$ converges in $\mathcal{H}$ to $h(t)$ for each $t > 0$. Thus, for any $\phi_0 \in \mathcal{H}$ and $t > 0$, we have

$$(h_\epsilon, e^{-tA^*}\phi_0) = (h_\epsilon(t), \phi_0) \to (h(t), \phi_0).$$

Since $(h_\epsilon)_{\epsilon<1}$ is a bounded sequence in $\mathcal{H}$ by (2.1) and the elements $e^{-tA^*}\phi_0$, $t > 0$, $\phi_0 \in \mathcal{H}$, form a dense set of $\mathcal{H}$, we infer that h_ϵ has a weak limit in $\mathcal{H}$. Calling h this weak limit we have $(h, e^{-tA^*}\phi_0) = (h(t), \phi_0)$, hence $h(t) = e^{-tA}h$ as desired. Summarizing, we have obtained $-e^{-tA}h + u(t) = v(t)$ in $L^2(a,b;\mathcal{H})$ for all $0 < a < b < \infty$.

Thus, u agrees almost everywhere with the continuous function $t \mapsto v(t) + e^{-tA}h$ which has limit h at $t = 0$. □

Remark 2.5. The only time analyticity is used in this proof is in (2.5). If we had incorporated the existence of an initial value as in [2] in our definition of a weak solution then analogous proposition and corollaries would hold for all generators of bounded C^0-semigroups.

3. A proof of maximal regularity via Kato's inequality for fractional powers

There are many proofs of the de Simon's theorem, via Fourier transform or operational calculus, and various extensions to Banach spaces. We refer to [6, Section 1].

Here, we wish to provide a proof using "almost orthogonality arguments" (Cotlar's lemma), and Kato's inequality for fractional powers [5, Theorem 1.1] which we recall for the reader's convenience.

Theorem 3.1. *Let A be closed and maximal accretive. For any $0 \leq \alpha < 1/2$, the operators A^α and $A^{*\alpha}$ have same domains and satisfy*

$$\|A^{*\alpha}f\| \leq \tan\frac{\pi(1+2\alpha)}{4}\|A^\alpha f\|. \tag{3.1}$$

If, moreover, A is injective then $A^\alpha A^{*-\alpha}$ extends to a bounded operator on $\mathcal{H}$ for $-1/2 < \alpha < 1/2$.

Maximal accretive means that $\mathrm{Re}(Au,u) \geq 0$ for every $u \in \mathsf{D}(A)$ and $(\lambda - A)^{-1}$ is bounded whenever $\mathrm{Re}\lambda < 0$. Note that (3.1) holds true with different constants for operators which are similar to a closed and maximal accretive operator. Assume A is sectorial of type $\omega(A) < \pi/2$ and injective. Le Merdy showed in [7] that A is similar to a maximal accretive operator if and only if A has bounded imaginary powers (i.e., A^{it} is bounded for all $t \in \mathbf{R}$). (See also [8] for a more general result and [11] for explicit examples.) But, following earlier works of Yagi [13], McIntosh showed in his seminal paper [9] that A has bounded imaginary powers if and only if A has a bounded holomorphic functional calculus. (See [6, Section 11] for extensive discussions with historical notes.) So proving maximal regularity (i.e., Theorem 1.1) assuming maximal accretivity is the same as proving maximal regularity assuming bounded holomorphic functional calculus. Nevertheless, this direct argument below could be of interest.

Proof of Theorem 1.1 *under further assumption of maximal accretivity.* Let $g \in L^2(0,\infty; dt, \mathsf{D}(A))$. We prove that $\mathcal{M}_+ g \in L^2(0,\infty; dt, \mathcal{H})$ with norm controlled by that of g in $L^2(0,\infty; dt, \mathcal{H})$ Since $Ae^{-(t-s)A}$ annihilates $\mathsf{N}(A)$, the null space of A, we may assume $g(s) \in \overline{\mathsf{R}(A)}$ for all $s > 0$. Alternately, we may factor out the null space of A and assume that A is injective, which we do (A is sectorial, so $\mathcal{H}$ splits topologically as $\mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}$).

Then one can write $g(s) = \int_0^\infty uAe^{-uA} g(s) \frac{du}{u}$ and so we have the representation of $\mathcal{M}_+$ as

$$\mathcal{M}_+ g(t) = \int_0^\infty (T_u g)(t) \frac{du}{u}, \quad \text{with } (T_u g)(t) = \mathcal{M}_+(uAe^{-uA} g)(t).$$

By Cotlar's lemma (see [12, Chapter VII]) it is enough to show in operator norm on $L^2(0,\infty;\mathcal{H})$ that $\|T_u T_v^*\| + \|T_u^* T_v\| \le h(u/v)$ with $C = \int_0^\infty h(x) \frac{dx}{x} < \infty$ to conclude that $\mathcal{M}_+$ is bounded on $L^2(0,\infty;\mathcal{H})$ with norm less than or equal to C. We show that for all $\alpha \in (0,1/2)$ one can take $h(x) = C_\alpha \min(x^\alpha, x^{-\alpha})$.

We begin with $T_u T_v^*$ for fixed (u,v). Since $\|T_u T_v^*\| = \|T_v T_u^*\|$, we may assume $u \le v$. A computation yields

$$(T_u T_v^*)(g)(t) = \int_0^\infty K_{(u,v)}(t,\tau) g(\tau)\, d\tau$$

where

$$K_{(u,v)}(t,\tau) = \int_0^{\min(t,\tau)} uA^2 e^{-(t-s+u)A} vA^{*2} e^{-(\tau-s+v)A^*}\, ds.$$

We turn to estimate the operator norm on $\mathcal{H}$ of $K_{(u,v)}(t,\tau)$ for fixed (t,τ). (Recall we fixed (u,v) with $u \le v$.) Since A is maximal accretive and injective, we have $\|A^\alpha A^{*-\alpha}\| \le C(\alpha)$ for $\alpha \in (0,1/2)$. So we write

$$\begin{aligned} &uA^2 e^{-(t-s+u)A} vA^{*2} e^{-(\tau-s+v)A^*} \\ &\quad = uA^{2-\alpha} e^{-(t-s+u)A} (A^\alpha A^{*-\alpha}) vA^{*(2+\alpha)} e^{-(\tau-s+v)A^*}, \end{aligned}$$

and by analyticity the operator norm on $\mathcal{H}$ is bounded by constant times $a(s)b(s)$ with

$$a(s) = \frac{u}{(t-s+u)^{2-\alpha}}, \quad b(s) = \frac{v}{(\tau-s+v)^{2+\alpha}}.$$

Plug this estimate into the integral. If $t \le \tau$, bound $b(s)$ by $b(t)$ and get

$$\|K_{(u,v)}(t,\tau)\| \le Cu^\alpha b(t) = C(u/v)^\alpha \frac{v^{1+\alpha}}{(\tau - t + v)^{2+\alpha}}.$$

If $\tau \le t$, bound $a(s)$ by $a(\tau)$ and get

$$\|K_{(u,v)}(t,\tau)\| \le Ca(\tau)v^{-\alpha} = C(u/v)^\alpha \frac{u^{1-\alpha}}{(t - \tau + u)^{2-\alpha}}.$$

It follows that

$$\sup_{\tau>0} \int_0^\infty (\|K_{(u,v)}(t,\tau)\| + \|K_{(u,v)}(\tau,t)\|)\, dt \le C(u/v)^\alpha.$$

By Schur's lemma we obtain $\|T_u T_v^*\| \le C(u/v)^\alpha$ when $u \le v$.

We now turn to estimate $T_u^* T_v$. By symmetry under taking adjoints again, it is enough to assume $u \le v$. We obtain

$$(T_u^* T_v)(g)(t) = \int_0^\infty \tilde{K}_{(u,v)}(t,\tau) g(\tau)\, d\tau$$

where

$$\tilde{K}_{(u,v)}(t,\tau) = \int_{\max(t,\tau)}^\infty uA^{*2} e^{-(s-t+u)A^*} vA^2 e^{-(s-\tau+v)A}\, ds.$$

This time we use the bound $\|A^{*\alpha} A^{-\alpha}\| \le C(\alpha)$ for $\alpha \in (0, 1/2)$ to obtain, if $\tau \le t$,

$$\|\tilde{K}_{(u,v)}(t,\tau)\| \le C(u/v)^\alpha \frac{v^{1+\alpha}}{(\tau - t + v)^{2+\alpha}}$$

and if $t \le \tau$,

$$\|\tilde{K}_{(u,v)}(t,\tau)\| \le C(u/v)^\alpha \frac{u^{1-\alpha}}{(t - \tau + u)^{2-\alpha}}.$$

So,

$$\sup_{\tau>0} \int_0^\infty (\|\tilde{K}_{(u,v)}(t,\tau)\| + \|\tilde{K}_{(u,v)}(\tau,t)\|)\, dt \le C(u/v)^\alpha$$

and by Schur's lemma, $\|T_u T_v^*\| \le C(u/v)^\alpha$ when $u \le v$. $\square$

As Kato's inequality holds for all $\alpha \in (-1/2, 1/2)$, the argument above can be used to prove that $\mathcal{M}_+$ is bounded on $L^2(0,\infty; t^\beta dt, \mathcal{H})$ but for $\beta \in (-1, 1)$. We leave details to the reader.

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On the Classical Solvability of Boundary Value Problems for Parabolic Equations with Incompatible Initial and Boundary Data

Galina Bizhanova

Dedicated to Professor Herbert Amann on the occasion of his 70th birthday

Abstract. We study the first and second boundary value problems for parabolic equations in a half-space $\mathbb{R}^n_+$, $n \geq 2$, with incompatible initial and boundary data on the boundary $x_n = 0$ of a domain. The existence, uniqueness and estimates of the solutions in the Hölder and weighted spaces are proved. We show that nonfulfillment of the compatibility conditions leads to appearance of the solutions, which are singular in the vicinity of a boundary of a domain as $t \to 0$.

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Keywords. Parabolic equation, boundary value problem, incompatible initial and boundary data, existence, uniqueness, estimate, classical solution.

1. Introduction. Statement of the problems. Main definitions

To study boundary value problems for parabolic equations in the Hölder space $C_x^{2+l,1+l/2}_{\;t}(\overline{\Omega}_T)$ we require fulfillment of the compatibility conditions of the boundary and initial data on the boundary of a domain at $t = 0$. These conditions provide continuity of the solution and its derivatives and boundedness of the Hölder constants of the higher derivatives in $\overline{\Omega}_T$. The compatibility conditions represent the functional identities connecting the given functions on the boundary of the domain at the initial moment.

Assume that the problem we study is a mathematical model of a certain physical process (in particular, heat, diffusive) beginning at $t = T^*$. Let this process go continuously. If we choose an initial moment $T_0 > T^*$ in the problem, then the compatibility conditions will be fulfilled.

If we study the problem since $t = T^*$ or since the moment of a jump of all characteristics of the process (given functions, coefficients, parameters in the problem), then, in general, the compatibility conditions are not fulfilled, but the physical process continues, and the problem can also have a solution.

To study the first and second boundary value problems for the parabolic equations in the classes $C_x^{2,}{}_t^1(\Omega_T) \cap C(\overline{\Omega}_T)$ and $C_x^{2,}{}_t^1(\Omega_T) \cap C_x^{1,}{}_t^0(\overline{\Omega}_T)$ respectively, we assume that the compatibility conditions of zero order are fulfilled in these problems, because we look for, in the closure of a domain $\overline{\Omega}_T$, a continuous solution of the first boundary value problem and a continuous solution together with all its derivatives of first order with respect to the spatial variables of the second boundary value problem.

Solutions of boundary value problems in a weighted Hölder space $C_s^l(\Omega_T)$, $s \leq l$, introduced by V.S. Belonosov, permits us to get rid of one compatibility condition [1, 2, 8]. Considering the first boundary value problem in this class we must require fulfillment of the compatibility condition of zero order, but the first-order compatibility condition can not take place. Y. Martel and Ph. Souplet in [7] proved that the solution of the first boundary value problem for the parabolic equation with incompatible data is not continuous in the closure of a domain. One-dimensional boundary value problems with incompatible data were studied in [3, 4].

We study the first and second boundary value problems for heat equations in the half-space $\mathbb{R}_+^n$, $n \geq 2$, with incompatible initial and boundary data on the boundary $x_n = 0$ of a domain at $t = 0$. The existence, uniqueness and estimates of the solutions are proved in Hölder and weighted spaces. Nonfulfillment of the compatibility conditions of initial and boundary data in the first and second boundary value problems leads to appearance of the functions $z_j(x', t)\,\mathrm{erfc}\frac{x_n}{2\sqrt{at}}$, $W_j(x, t)$, $j = 0, 1$, (see Theorems 2.1, 2.2) and $-2\sqrt{at}\, z_2(x', t)\mathrm{ierfs}\,\frac{x_n}{2\sqrt{at}}$ (see Theorems 2.3, 2.4) in the solutions of these problems respectively, which are singular in the vicinity of a boundary of a domain as $t \to 0$. These functions permit us to reduce the original problems to problems with a fulfilled compatibility conditions of all necessary orders.

In Chapter 1 the Hölder and weighted spaces are determined, the definition of the special functions – repeated integrals of the probability and the compatibility conditions for the considered problems are given. The main results of the paper are formulated in Chapter 2. In Chapter 3 there are constructed and studied the singular solutions of the auxiliary problems. In Chapters 4 and 5 with the help of these singular solutions the original first and second boundary value problems are reduced to problems that have unique solutions in the weighted and classical Hölder spaces. In the Appendix the auxiliary first boundary value problem is studied.

Let

$$D := \mathbb{R}^n_+ = \{x = (x', x_n) \mid x' \in \mathbb{R}^{n-1},\ x_n > 0)\}, \quad n \geq 2,$$
$$R := \{x \mid x' \in \mathbb{R}^{n-1},\ x_n = 0\}, \quad x' = (x_1, \dots, x_{n-1}),$$
$$D_T = D \times (0, T), \quad R_T = R \times [0, T].$$

We consider two problems. We are required to find the solution $u(x,t)$ of the first boundary value problem – Problem 1

$$\partial_t u - a\,\Delta\, u = f(x,t) \text{ in } D_T, \tag{1.1}$$

$$u|_{t=0} = u_0(x) \ \text{ in } \ D, \tag{1.2}$$

$$u|_{x_n=0} = \varphi(x', t) \text{ on } R_T, \tag{1.3}$$

and the solution $u(x,t)$ of the second boundary value problem – Problem 2, which satisfies an equation (1.1), initial condition (1.2) and the boundary condition

$$\partial_{x_n} u|_{x_n=0} = \psi(x', t) \text{ on } R_T. \tag{1.4}$$

Here $a = \text{const} > 0$, $\partial_t = \partial/\partial t$, $\partial_{x_n} = \partial/\partial x_n$, $\Delta = \partial^2_{x_1x_1} + \dots + \partial^2_{x_nx_n}$. By $c_1, c_2, \dots$ we shall denote positive constants.

Determine the weighted and classical Hölder spaces.

Let l be a positive non-integer. By $C^l_s(D_T)$, $s \leq l$, we shall denote a weighted Hölder space defined by V.S. Belonosov with the norm [1, 2, 8],

$$|u|^{(l)}_{s,D_T} = \sup_{t\leq T} t^{\frac{l-s}{2}} [u]^{(l)}_{D'_t} + \sum_{s<2k+|m|<l} \sup_{t\leq T} t^{\frac{2k+|m|-s}{2}} |\partial^k_t \partial^m_x u|_{D'_t} + \begin{cases} |u|^{(s)}_{D_T}, & s \geq 0, \\ 0, & s < 0, \end{cases} \tag{1.5}$$

where $D'_t = D \times [t/2, t]$, $|\,v\,|_{D_T} = \sup_{(x,t)\in \overline{D}_T} |\,v(x,t)\,|$,

$$[u]^{(l)}_{D_T} = \sum_{2k+|m|=[l]} [\partial^k_t \partial^m_x u]^{(l-[l])}_{x,D_T} + \sum_{0<l-2k-|m|<2} [\partial^k_t \partial^m_x u]^{\left(\frac{l-2k-|m|}{2}\right)}_{t,D_T}, \tag{1.6}$$

$$[v]^{(\alpha)}_{x,D_T} = \sup_{(x,t),(z,t)\in\overline{D}_T} \big|v(x,t) - v(z,t)\big|\,|x-z|^{-\alpha}, \tag{1.7}$$

$$[v]^{(\alpha)}_{t,D_T} = \sup_{(x,t),(x,t_1)\in\overline{D}_T} \big|v(x,t) - v(x,t_1)\big|\,|t-t_1|^{-\alpha}, \ \alpha \in (0,1), \tag{1.8}$$

$|u|^{(s)}_{D_T}$ is the norm of the classical Hölder space $C^{s,\,s/2}_{x\ \ t}(\overline{D}_T)$ [6],

$$|u|^{(s)}_{D_T} = \sum_{2k+|m|\leq[s]} |\partial^k_t \partial^m_x u|_{D_T} + \begin{cases} 0, & s \text{ an integer}, \\ [u]^{(s)}_{D_T}, & s \text{ not an integer}, \end{cases} \tag{1.9}$$

where $[u]^{(s)}_{D_T}$ is determined by (1.6)–(1.8).

For $s = l$, $C^l_l(D_T)$ is the space $C^{l,\,l/2}_{x\ \ t}(\overline{D}_T)$.

From the norm (1.5) we can see that for $s < l$ the function $u(x,t)$ has a singularity with respect to t as $t \to 0$ in the whole domain D including its boundary. For instance, if an initial function $u_0(x)$ in the Cauchy problem for the parabolic equation is from the Hölder space $C^s(\overline{D})$, $s < l$, then the solution of this problem belongs to $C^l_s(D_T)$.

We introduce a weighted space $C^2_{s,\,\delta_0}(D_T)$, $\delta_0 > 0$, $2 \le s \le 2+\alpha$, of the functions $u(x,t)$ with the norm

$$
\begin{aligned}
|u|^{(2)}_{s,\delta_0,D_T} := {} & \sup_{t\le T} t^{-1}\big|u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D'_t} + \sum_{\mu=1}^{n-1}\sup_{t\le T} t^{\frac{1-s}{2}}\big|\partial_{x_\mu}u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D'_t} \\
& + \sup_{t\le T} t^{-1/2}\big|\partial_{x_n}u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D'_t} + \sum_{i=1}^{n}\sum_{\mu=1}^{n-1}\sup_{t\le T} t^{\frac{2-s}{2}}\big|\partial^2_{x_ix_\mu}u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D'_t} \\
& + \big|\partial^2_{x_nx_n}u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D_T} + \big|\partial_t u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D_T},
\end{aligned}
\tag{1.10}
$$

and, in particular, for $s = 2+\alpha$,

$$
\begin{aligned}
|u|^{(2)}_{2+\alpha,\delta_0,D_T} := {} & \sup_{t\le T} t^{-1}\big|u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D'_t} + \sum_{\mu=1}^{n-1}\sup_{t\le T} t^{-\frac{1+\alpha}{2}}\big|\partial_{x_\mu}u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D'_t} \\
& + \sup_{t\le T} t^{-1/2}\big|\partial_{x_n}u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D'_t} + \sum_{i=1}^{n}\sum_{\mu=1}^{n-1}\sup_{t\le T} t^{-\frac{\alpha}{2}}\big|\partial^2_{x_ix_\mu}u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D'_t} \\
& + \big|\partial^2_{x_nx_n}u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D_T} + \big|\partial_t u\,e^{\delta_0\frac{x_n^2}{t}}\big|_{D_T}.
\end{aligned}
\tag{1.11}
$$

We write $|u|^{(2)}_{s,\,\delta_0,D_T} := M$. From (1.10), (1.11) we shall have

$$
\begin{gathered}
\big|\partial^2_{x_ix_\mu}u\big| \le M\,e^{-\delta_0\frac{x_n^2}{t}},\ i = 1,\dots,n,\ \mu = 1,\dots,n-1, \quad \text{for } s = 2, \\
\big|\partial_t u(x,t)\big|,\ \big|\partial^2_{x_nx_n}u(x,t)\big| \le M\,e^{-\delta_0\frac{x_n^2}{t}} \ \text{for } 2 \le s \le 2+\alpha.
\end{gathered}
\tag{1.12}
$$

Let the point x be in the interior of the domain: $x_n \ge r_0 = \text{const} > 0$, then from (1.12) we obtain that the derivatives $\partial^2_{x_ix_\mu}u$ for $s = 2$, and $\partial_t u(x,t)$, $\partial^2_{x_nx_n}u(x,t)$ for $2 \le s \le 2+\alpha$ tend to zero exponentially as $t \to 0$ and on the boundary $x_n = 0$ of D they are bounded and can not be equal to zero at $t = 0$, i.e., they can be discontinuous in $\overline{D}_T$.

The negative powers of the t weights in the norms (1.10), (1.11) mean that the function and its derivatives with respect to x with such weights tend to zero as $t \to 0$ on the boundary $x_n = 0$ of a domain and exponentially in the interior of a domain due to the weight $e^{\delta_0\frac{x_n^2}{t}}$.

We point out also that x_n in exponential power is the distance between a point $x = (x_1,\dots,x_n)$ and a boundary of a domain $x_n = 0$.

An example of a function from $C^2_{s,\,\delta_0}(D_T)$ is a solution of problem (A.1) in the Appendix (see Theorems A.1 and A.2).

We do not include the Hölder constants of the derivatives into the norms (1.10), (1.11). But from these norms we see the behavior of the function and its derivatives in the domain and on its boundary.

We define the compatibility conditions of the boundary and initial functions for Problems 1 and 2.

Let

$$
\begin{aligned}
&A_0(x') := \varphi(x',0) - u_0(x)\big|_{x_n=0},\\
&A_1(x') := \partial_t\varphi(x',t)\big|_{t=0} - \big(a\,\Delta u_0(x) + f(x,0)\big)\big|_{x_n=0},\\
&B_0(x') := \psi(x',0) - \partial_{x_n} u_0(x)\big|_{x_n=0}.
\end{aligned}
$$

The compatibility conditions of zero and first orders on the boundary $x_n = 0$ for Problem 1 (1.1), (1.2), (1.3) are $A_0(x') = 0,\ A_1(x') = 0$ on R and of zero order for Problem 2 (1.1), (1.2), (1.4) – $B_0(x') = 0$ on R. Evidently, nonfulfillment of the compatibility conditions of zero and first orders means $A_0(x') \neq 0$, $A_1(x') \neq 0$, $B_0(x') \neq 0$ on R.

Later on we shall apply special functions – iterated integrals of the probability $\mathrm{i}^n\mathrm{erfc}\,\zeta$; they are determined by the formulas [5], Ch. 7.2,

$$\mathrm{i}^n\mathrm{erfc}\,\zeta := \int_\zeta^\infty \mathrm{i}^{n-1}\mathrm{erfc}\,\xi\,d\,\xi,\ \ n = 0,1,2,\ldots,$$

$$\mathrm{i}^{-1}\mathrm{erfc}\,\zeta := \frac{2}{\sqrt{\pi}}\,e^{-\zeta^2},\ \ \mathrm{i}^0\mathrm{erfc}\,\zeta := \mathrm{erfc}\,\zeta = \frac{2}{\sqrt{\pi}}\int_\zeta^\infty e^{-\xi^2}\,d\,\xi,\ \ \mathrm{i}^1\mathrm{erfc}\,\zeta := \mathrm{ierfc}\,\zeta.$$

The following relations for them hold:

$$
\begin{aligned}
&\frac{d}{d\zeta}\,\mathrm{i}^n\mathrm{erfc}\,\zeta = -\mathrm{i}^{n-1}\mathrm{erfc}\,\zeta, \qquad n = 0,1,2,\ldots, &&(1.13)\\
&\mathrm{i}^n\mathrm{erfc}\,0 = \frac{1}{2^n\,\Gamma(n/2+1)}, \quad n = -1,0,1,\ldots,
\end{aligned}
$$

where $\Gamma(\cdot)$ – Euler gamma – function,

$$
\begin{aligned}
&\mathrm{i}^n\mathrm{erfc}\,\zeta = \frac{1}{2n}\,\mathrm{i}^{n-2}\,\mathrm{erfc}\,\zeta - \frac{\zeta}{n}\,\mathrm{i}^{n-1}\mathrm{erfc}\,\zeta, \quad n = 1,2,\ldots. &&(1.14)\\
&\mathrm{i}^n\mathrm{erfc}\,\zeta \le \mathrm{i}^n\mathrm{erfc}\,0, \quad \zeta \ge 0, \qquad n = -1,0,1,\ldots.
\end{aligned}
$$

2. Main results

We formulate the main results for Problems 1, 2.

Theorem 2.1. *Let $\alpha \in (0,1)$, $s \in (\alpha,\, 2+\alpha]$. For all functions $u_0(x) \in C^s(\overline{D})$, $f(x,t) \in C^\alpha_{s-2}(D_T)$, $\varphi(x',t) \in C^{2+\alpha}_s(R_T)$ that do not satisfy, on the boundary $x_n = 0$ of the domain D, compatibility conditions of zero order for $s \in (\alpha,2)$ ($A_0(x') := \varphi(x',0) - u_0(x)\big|_{x_n=0} \neq 0$ on R) and of zero and first orders for $s \in [2,\, 2+\alpha]$ ($A_0(x') \neq 0$, $A_1(x') := \partial_t\varphi(x',t)\big|_{t=0} - \big(a\,\Delta u_0(x) + f(x,0)\big)\big|_{x_n=0} \neq 0$ on R),* *Problem* 1 (1.1), (1.2), (1.3) *has a unique solution* $u(x,t) = V_0(x,t) + v_1(x,t)$ *for* $s \in (\alpha, 2)$ *and* $u(x,t) = V_0(x,t) + W_0(x,t) + V_1(x,t) + W_1(x,t) + v_2(x,t)$ *for* $s \in [2, 2+\alpha]$, *where* $V_j(x,t) = z_j(x',t)\,\mathrm{erfc}\frac{x_n}{2\sqrt{at}}$, $z_j \in C_s^{2+\alpha}(R_T)$, $W_j \in C^2_{s,\delta_0}(D_T)$, $j = 0,1$, $\delta_0 = \frac{1}{8a}$, $v_i \in C_s^{2+\alpha}(D_T)$, $i = 1,2$, *and the following estimates for them hold:*

$$|z_j|^{(2+\alpha)}_{s,R_T} \le c_1 |A_j|_R^{(s-2j)}, \ |W_j|^{(2)}_{s,\delta_0,D_T} \le c_2 |A_j|_R^{(s-2j)}, \quad j = 0,1, \tag{2.1}$$

$$|v_i|^{(2+\alpha)}_{s,D_T} \le c_3\Big(|u_0|_D^{(s)} + |f|^{(\alpha)}_{s-2,D_T} + |\varphi|^{(2+\alpha)}_{s,R_T}\Big), \qquad i = 1,2. \tag{2.2}$$

From this theorem for $s = 2+\alpha$ we obtain the following one.

Theorem 2.2. *For all functions* $u_0(x) \in C^{2+\alpha}(\overline{D})$, $f(x,t) \in C_x^{\alpha,\alpha/2}{}_t(\overline{D}_T)$, $\varphi(x',t) \in C_{x'}^{2+\alpha,1+\alpha/2}{}_t(R_T)$, $\alpha \in (0,1)$ *that do not satisfy, on the boundary* $x_n = 0$ *of the domain* D, *compatibility conditions of zero and first orders* ($A_0(x') \neq 0$, $A_1(x') \neq 0$ *on* R), *Problem* 1 (1.1), (1.2), (1.3) *has a unique solution* $u(x,t) = V_0(x,t) + W_0(x,t) + V_1(x,t) + W_1(x,t) + v_2(x,t)$ *where* $V_j(x,t) = z_j(x',t)\,\mathrm{erfc}\frac{x_n}{2\sqrt{at}}$, $z_j \in C_{x'}^{2+\alpha,1+\alpha/2}{}_t(R_T)$, $W_j \in C^2_{2+\alpha,\delta_0}(D_T)$, $j = 0,1$, $\delta_0 = \frac{1}{8a}$, $v_2 \in C_x^{2+\alpha,1+\alpha/2}{}_t(D_T)$, *and the following estimates for them hold:*

$$|z_j|^{(2+\alpha)}_{R_T} \le c_4 |A_j|_R^{(2+\alpha-2j)}, \ |W_j|^{(2)}_{2+\alpha,\delta_0,D_T} \le c_5 |A_j|_R^{(2+\alpha-2j)}, \quad j = 0,1,$$
$$|v_2|^{(2+\alpha)}_{D_T} \le c_6\Big(|u_0|_D^{(2+\alpha)} + |f|^{(\alpha)}_{D_T} + |\varphi|^{(2+\alpha)}_{R_T}\Big).$$

Theorem 2.3. *Let* $\alpha \in (0,1)$, $s \in (\alpha, 2+\alpha]$. *For all functions* $u_0(x) \in C^s(\overline{D})$, $f(x,t) \in C^\alpha_{s-2}(D_T)$, $\psi(x',t) \in C^{1+\alpha}_{s-1}(R_T)$, $s \in (\alpha, 1)$, *and if* $s \in [1, 2+\alpha]$, *then for all functions* u_0, f, ψ *that do not satisfy, on the boundary* $x_n = 0$ *of the domain* D, *the compatibility condition of zero order* ($B_0(x') := \psi(x',0) - \partial_{x_n} u_0(x)\big|_{x_n=0} \neq 0$ *on* R), *Problem* 2 (1.1), (1.2), (1.4) *has a unique solution* $u(x,t) = v_1(x,t)$ *for* $s \in (\alpha, 1)$ *and* $u(x,t) = -2\sqrt{at}\, z_2(x',t)\mathrm{ierfs}\,\frac{x_n}{2\sqrt{at}} + v_2(x,t)$ *for* $s \in [1, 2+\alpha, 2]$, *where* $z_2 \in C^{2+\alpha}_{s-1}(R_T)$, $v_i \in C^{2+\alpha}_s(D_T)$, $i = 1,2$, *and the following estimates for them hold:*

$$\begin{aligned} |z_2|^{(2+\alpha)}_{s-1,R_T} &\le c_7 |B_0|_R^{(s-1)}, \\ |v_i|^{(2+\alpha)}_{s,D_T} &\le c_8\Big(|u_0|_D^{(s)} + |f|^{(\alpha)}_{s-2,D_T} + |\psi|^{(1+\alpha)}_{s-1,R_T}\Big), \ i = 1,2. \end{aligned} \tag{2.3}$$

From this theorem for $s = 2+\alpha$ we obtain the following one.

Theorem 2.4. *For all functions* $u_0(x) \in C^{2+\alpha}(\overline{D})$, $f(x,t) \in C_x^{\alpha,\alpha/2}{}_t(\overline{D}_T)$, $\psi(x',t) \in C_{x'}^{1+\alpha,\frac{1+\alpha}{2}}{}_t(R_T)$ *that do not satisfy, on the boundary* $x_n = 0$ *of the domain* D, *the compatibility condition of zero order* ($B_0(x') \neq 0$ *on* R), *Problem* 2 (1.1), (1.2), (1.4) *has a unique solution* $u(x,t) = -2\sqrt{at}\, z_2(x',t)\mathit{ierfs}\,\frac{x_n}{2\sqrt{at}} + v_2(x,t)$, *where*

$z_2 \in C^{2+\alpha}_{1+\alpha}(R_T)$, $v_2 \in C^{2+\alpha,\,1+\alpha/2}_{x\quad t}(\overline{D}_T)$ *and the following estimates for them hold:*

$$|z_2|^{(2+\alpha)}_{1+\alpha,R_T} \le c_9 |B_0|^{(1+\alpha)}_R,$$
$$|v_2|^{(2+\alpha)}_{s,D_T} \le c_{10}\Big(|u_0|^{(2+\alpha)}_D + |f|^{(\alpha)}_{D_T} + |\psi|^{(1+\alpha)}_{R_T}\Big).$$

For Theorems 2.1–2.4, the functions $z_j(x',t)$, $z_2(x',t)$ and $W_j(x,t)$, $j=0,1$, are defined in Theorems 3.3, 3.4 and 3.5, 3.6, A.1, A.2 respectively.

Remark 2.5. If in Problem 1 the compatibility condition of zero (first) order is fulfilled, i.e., $A_0(x')=0$ $(A_1(x')=0)$ on R, then in Theorems 2.1, 2.2 $V_0(x,t)=0$, $W_0(x,t)=0$ $(V_1(x,t)=0,\ W_1(x,t)=0)$ in D_T. If $A_0(x')=0$, $A_1(x')=0$ on R, then $V_j(x,t)=0$, $W_j(x,t)=0$ in D_T, $j=0,1$.

If in Problem 2 the compatibility condition of zero order is fulfilled, i.e., $B_0(x')=0$ on R, then in Theorems 2.3, 2.4 $z_2(x',t)=0$ on R_T.

3. Auxiliary problems

We recall that $D := \mathbb{R}^n_+$, $n \ge 2$, $R := \{x = (x', x_n)\,|\, x' \in \mathbb{R}^{n-1}, x_n = 0\}$, $D_T = D \times (0,T)$, $R_T = R \times [0,T]$, $\Delta' := \partial^2_{x_1} + \cdots + \partial^2_{x_{n-1}}$.

First, we construct a function $z(x',t)$ satisfying the conditions

$$z\big|_{t=0} = \mu_0(x'),\quad \partial_t z\big|_{t=0} = \mu_1(x') \ \text{ in } R. \tag{3.1}$$

Lemma 3.1. [6] *Let $\mu_0(x') \in C^s(R)$, $\mu_1(x') \in C^{s-2}(R)$, $s \in [2,\, 2+\alpha]$, $\alpha \in (0,1)$. Then there exists a unique function $z(x',t) \in C^{2+\alpha}_s(R_T)$, for which the following estimate holds:*

$$|z|^{(2+\alpha)}_{s,R_T} \le c_1\big(|\mu_0|^{(s)}_R + |\mu_1|^{(s-2)}_R\big). \tag{3.2}$$

Lemma 3.2. [6] *Let $\mu_0(x') \in C^{2+\alpha}(R)$, $\mu_1(x') \in C^{\alpha}(R)$, $\alpha \in (0,1)$. Then there exists a unique function $z(x',t) \in C^{2+\alpha,1+\alpha/2}_{x'\quad t}(R_T)$, for which the following estimate holds:*

$$|z|^{(2+\alpha)}_{R_T} \le c_2\big(|\mu_0|^{(2+\alpha)}_R + |\mu_1|^{(\alpha)}_R\big).$$

This lemma follows from Lemma 3.1 for $s = 2+\alpha$.

Proof of Lemma 3.1. Consider the Cauchy problem

$$\partial_t z - a\,\Delta' z = z^{(1)}(x',t) \ \text{ in } \ R_T, \tag{3.3}$$
$$z|_{t=0} = \mu_0(x') \quad \text{in} \quad R, \tag{3.4}$$

where $z^{(1)}(x',t)$ is a solution of the Cauchy problem

$$\partial_t z^{(1)} - a\,\Delta' z^{(1)} = 0 \ \text{ in } \ R_T, \tag{3.5}$$
$$z^{(1)}|_{t=0} = \mu_1(x') - a\Delta'\mu_0(x') \in C^{s-2}(R) \quad \text{in} \quad R. \tag{3.6}$$

Problem (3.5), (3.6) has a unique solution [1, 2, 6, 8] $z^{(1)}(x',t) \in C^{2+\alpha}_{s-2}(R_T)$, such that

$$|z^{(1)}|^{(2+\alpha)}_{s-2,R_T} \le c_3\big(|\mu_0|^{(s)}_R + |\mu_1|^{(s-2)}_R\big).$$

Due to embedding $C^{2+\alpha}_{s-2}(R_T) \subset C^{\alpha}_{s-2}(R_T)$ we obtain

$$|z^{(1)}|^{(\alpha)}_{s-2,R_T} \le c_4\big(|\mu_0|^{(s)}_R + |\mu_1|^{(s-2)}_R\big). \tag{3.7}$$

Then problem (3.3), (3.4) has a unique solution $z \in C^{2+\alpha}_s(R_T)$, for which a valid estimate is

$$|z|^{(2+\alpha)}_{s,R_T} \le c_5\big(|z^{(1)}|^{(\alpha)}_{s-2,R_T} + |\mu_0|^{(s)}_R\big).$$

From here by (3.7) we shall have an estimate (3.2). Moreover, $z(x',t)$ satisfies the conditions (3.1). Really, the first condition in (3.1) is fulfilled due to (3.4) and from (3.3) and (3.6) we obtain

$$\partial_t z|_{t=0} = a\Delta'\mu_0(x') + z^{(1)}|_{t=0} = \mu_1(x'),$$

i.e., the second condition in (3.1) also holds. □

We recall that the compatibility conditions of zero and first orders for Problem 1 and of zero order for Problem 2 are not fulfilled, i.e., $A_0(x') := \varphi(x',0) - u_0(x)\big|_{x_n=0} \ne 0$, $A_1(x') := \partial_t\varphi(x',t)\big|_{t=0} - \big(a\,\Delta u_0(x) + f(x,0)\big)\big|_{x_n=0} \ne 0$ on R and $B_0(x') := \psi(x',0) - \partial_{x_n}u_0(x)\big|_{x_n=0} \ne 0$ on R.

To extract from the solutions of Problems 1 and 2 the singular parts, which appear due to a nonfulfillment of the compatibility conditions on R, we extend the functions $A_j(x')$, $j = 0,1$, $B_0(x')$ into R_T and then into D_T. For that, first, we construct the functions $z_j(x',t)$, $j = 0,1$, and $z_2(x',t)$ under the conditions

$$z_0|_{t=0} = A_0(x') \text{ in } R, \qquad s \in (\alpha,\, 2), \tag{3.8}$$

$$z_0|_{t=0} = A_0(x'),\quad \partial_t z_0|_{t=0} = 0 \text{ in } R, \quad s \in [2,\, 2+\alpha], \tag{3.9}$$

$$z_1|_{t=0} = 0,\quad \partial_t z_1|_{t=0} = A_1(x') \ \text{ in } R, \quad s \in [2,\, 2+\alpha] \tag{3.10}$$

for Problem 1 (1.1)–(1.3) and

$$z_2|_{t=0} = B_0(x') \ \text{ in } R, \quad s \in [1,\, 2+\alpha] \tag{3.11}$$

for Problem 2 (1.1), (1.2), (1.4).

Theorem 3.3. *Let* $A_0(x') \in C^s(R)$, $s \in (\alpha,\, 2+\alpha]$, $A_1(x') \in C^{s-2}(R)$, $s \in [2,\, 2+\alpha]$, $B_0(x') \in C^{s-1}(R)$, $s \in [1,\, 2+\alpha]$, $\alpha \in (0,1)$. *Then there exist unique functions* $z_j(x',t) \in C^{2+\alpha}_s(R_T)$, $j = 0,1$, $z_2(x',t) \in C^{2+\alpha}_{s-1}(R_T)$, *which satisfy the conditions* (3.8)–(3.11) *respectively, and the estimates for them hold:*

$$|z_0|^{(2+\alpha)}_{s,R_T} \le c_6|A_0|^{(s)}_R, \qquad s \in (\alpha,\, 2+\alpha], \tag{3.12}$$

$$|z_1|^{(2+\alpha)}_{s,R_T} \le c_7|A_1|^{(s-2)}_R, \qquad s \in [2,\, 2+\alpha], \tag{3.13}$$

$$|z_1| \le c_8|A_1|^{(s-2)}_R\, t \text{ in } R_T, \quad s \in [2,\, 2+\alpha], \tag{3.14}$$

$$|z_2|^{(2+\alpha)}_{s-1,R_T} \le c_9|B_0|^{(s-1)}_R, \qquad s \in [1,\, 2+\alpha]. \tag{3.15}$$

Theorem 3.4. *Let* $A_0(x') \in C^{2+\alpha}(R)$, $A_1(x') \in C^{\alpha}(R)$, $B_0(x') \in C^{1+\alpha}(R)$, $\alpha \in (0,1)$. *Then there exist unique functions* $z_j(x',t) \in C^{2+\alpha,1+\alpha/2}_{x'\ \ t}(R_T)$, $j = 0,1$,

$z_2(x',t) \in C^{2+\alpha}_{1+\alpha}(R_T)$, *which satisfy the conditions* (3.8)–(3.11) *respectively, and the estimates for them hold:*

$$\begin{aligned}
|z_0|^{(2+\alpha)}_{R_T} &\le c_{10}|A_0|^{(2+\alpha)}_{R},\\
|z_1|^{(2+\alpha)}_{R_T} &\le c_{11}|A_1|^{(\alpha)}_{R}, \quad |z_1| \le c_{11}|A_1|^{(\alpha)}_{R}\, t \text{ in } R_T,\\
|z_2|^{(2+\alpha)}_{1+\alpha,R_T} &\le c_{12}|B_0|^{(1+\alpha)}_{R}.
\end{aligned}$$

This theorem follows from Theorem 3.3 for $s = 2+\alpha$.

Proof of Theorem 3.3. 1. For $s \in (\alpha, 2)$ we can take the function $z_0(x',t)$ as a solution of the Cauchy problem in $R_T := \mathbb{R}^{n-1} \times [0,T]$ for the equation

$$\partial_t\, z_0 - a\,\Delta'\, z_0 = 0 \;\; \text{in } R_T \tag{3.16}$$

with an initial data (3.8). This solution belongs to $C^{2+\alpha}_s(R_T)$, and an estimate (3.12) for it holds [1, 2, 8].

2. Let $s \in [2,\, 2+\alpha]$. With the help of Lemma 3.1 we construct the functions $z_0(x',t)$, $z_1(x',t)$ as solutions of the problems for the equations

$$\partial_t\, z_j - a\,\Delta'\, z_j = z_j^{(1)}(x',t) \;\; \text{in } R_T,\ j = 0,1, \tag{3.17}$$

with the initial conditions (3.9) for $j=0$ and (3.10) for $j=1$. Here the functions $z_j^{(1)}(x',t)$ are the solutions of the following Cauchy problems:

$$\partial_t\, z_j^{(1)} - a\,\Delta'\, z_j^{(1)} = 0 \;\; \text{in } R_T,\ j = 0,1, \tag{3.18}$$

$$z_0^{(1)}|_{t=0} = -a\Delta' A_0(x') \text{ in } R; \;\; z_1^{(1)}|_{t=0} = A_1(x') \text{ in } R. \tag{3.19}$$

The solutions $z_j^{(1)}$ of problems (3.18), (3.19) exist, belong to $C^{2+\alpha}_{s-2}(R_T) \subset C^{\alpha}_{s-2}(R_T)$ and the estimates (3.7) for them are fulfilled with $\mu_0 = A_0$, $\mu_1 = 0$ for $z_0^{(1)}$ and with $\mu_0 = 0$, $\mu_1 = A_1$ for $z_1^{(1)}$, i.e.,

$$|z_j^{(1)}|^{(\alpha)}_{s-2,R_T} \le c_{13}|A_j|^{(s-2j)}_{R}, \quad j = 0,1. \tag{3.20}$$

By Lemma 3.1 the functions $z_j(x',t)$, $j=0,1$, belong to $C^{2+\alpha}_s(R_T)$ and the estimates (3.12), (3.13) for them are valid.

Due to (3.13) and the first condition (3.10) we shall have an estimate (3.14) for z_1, really,

$$|z_1(x',t)| = \left| \int_0^t \partial_\tau z_1(x',\tau)\, d\tau \right| \le c_8 |A_1|^{(s-2)}_{R}\, t, \quad s \in [2,\, 2+\alpha].$$

3. Let $s \in [1,\, 2+\alpha]$. We determine a function $z_2(x',t)$ as a solution of the Cauchy problem

$$\partial_t\, z_2 - a\,\Delta'\, z_2 = 0 \;\; \text{in } R_T, \quad z_2|_{t=0} = B_0(x') \;\; \text{in } R. \tag{3.21}$$

By [1, 2, 8] this problem has a unique solution $z_2(x',t) \in C^{2+\alpha}_{s-1}(R_T)$ and it satisfies an estimate (3.15). $\square$

Thus, we have extended the functions $A_j(x')$, $j=0,1$, and $B_0(x')$ into R_T by the functions $z_j(x',t)$, $j=0,1$, and $z_2(x',t)$ respectively. Now we extend $z_j(x',t)$, $j=0,1$, and $z_2(x',t)$ into $D_T := \mathbb{R}^n_+ \times (0,T)$.

Consider two of the first boundary value problems with unknown functions $Z_j(x,t)$, $j=0,1$,

$$\partial_t Z_j - a\,\Delta\, Z_j = 0 \quad \text{in} \quad D_T, \tag{3.22}$$

$$Z_j|_{t=0} = 0 \quad \text{in} \quad D,\ Z_j|_{x_n=0} = z_j(x',t) \quad \text{on} \quad R_T, \tag{3.23}$$

where the function z_0 is the solution of the problems (3.16), (3.8) for $s \in (\alpha,\, 2)$ and (3.17), $j=0$, (3.9) for $s \in [2,\, 2+\alpha]$, and z_1 is the solution of the problem (3.17), $j=1$, (3.10).

In the problem for $Z_0(x,t)$ the compatibility condition of zero order is not fulfilled and the first-order compatibility condition holds by (3.8), (3.9) and for $Z_1(x,t)$ inversely, the compatibility condition of zero order is fulfilled and of the first order is not, by (3.10).

We remind that $\operatorname{erfs}\zeta = \frac{2}{\sqrt{\pi}}\int_\zeta^\infty e^{-\xi^2}d\xi$ and the norms $|u|^{(2)}_{s,\delta_0,D_T}$, $|u|^{(2)}_{2+\alpha,\delta_0,D_T}$ are determined by (1.10), (1.11).

Theorem 3.5. *Let* $A_0(x') \in C^s(R)$, $s \in (\alpha,\, 2+\alpha]$, $A_1(x') \in C^{s-2}(R)$, $s \in [2,\, 2+\alpha]$. *Then each of the problems* (3.22), (3.23) *has the unique solution*

$$Z_0(x,t) = V_0(x,t) + \begin{cases} 0, & s \in (\alpha,\, 2), \\ W_0(x,t), & s \in [2,\, 2+\alpha], \end{cases} \tag{3.24}$$

$$Z_1(x,t) = V_1(x,t) + W_1(x,t),\ s \in [2,\, 2+\alpha], \tag{3.25}$$

$$V_j(x,t) = z_j(x',t)\operatorname{erfc}\frac{x_n}{2\sqrt{at}},\ j=0,1,$$

where the functions $z_j(x',t)$ *are defined in Theorem* 3.3*:* $z_j(x',t) \in C^{2+\alpha}_s(R_T)$ *and*

$$|z_j|^{(2+\alpha)}_{s,D_T} \le c_{14}|A_j|^{(s-2j)}_R, \quad j=0,1,$$

$W_j(x,t) \in C^{2+\alpha}_{s,\delta_0}(R_T)$, $\delta_0 = \frac{1}{8a}$, *and*

$$|W_j|^{(2)}_{s,\delta_0,D_T} \le c_{15}|A_j|^{(s-2j)}_R, \quad j=0,1. \tag{3.26}$$

Theorem 3.6. *Let* $A_0(x') \in C^{2+\alpha}(R)$, $A_1(x') \in C^\alpha(R)$, $\alpha \in (0,1)$. *Then each of problems* (3.22), (3.23) *has the unique solution* $Z_j(x,t) = V_j(x,t) + W_j(x,t)$, $j=0,1$, *where* $V_j = z_j(x',t)\operatorname{erfc}\frac{x_n}{2\sqrt{at}}$, *the functions* $z_j(x',t)$ *are defined in Theorem* 3.4*:* $z_j(x',t) \in C^{2+\alpha,1+\alpha/2}_{x'\ \ t}(R_T)$ *and*

$$|z_j|^{(2+\alpha)}_{D_T} \le c_{16}|A_j|^{(2+\alpha-2j)}_R, \quad j=0,1,$$

$W_j(x,t) \in C^2_{2+\alpha,\delta_0}(R_T)$, $\delta_0 = \frac{1}{8a}$, *and*

$$|W_j|^{(2)}_{2+\alpha,\delta_0,D_T} \le c_{17}|A_j|^{(2+\alpha-2j)}_R, \quad j=0,1.$$

This theorem follows from Theorem 3.5 for $s = 2+\alpha$.

Proof of Theorem 3.5. The solutions of problems (3.22), (3.23) may be represented in the explicit form

$$Z_j(x,t) = -2a \int_0^t d\tau \int_{\mathbb{R}^{n-1}} z_j(y',\tau)\partial_{x_n}\Gamma(x'-y',x_n,\tau)dy', \tag{3.27}$$

where $\Gamma(x,t)$ is a fundamental solution of the heat equation (3.22),

$$\Gamma(x,t) = \frac{1}{(2\sqrt{a\pi})^n} e^{-\frac{x^2}{4at}}.$$

We construct the solutions of problems (3.22), (3.23) in more suitable forms than (3.27) to see the character of their singularity. For this, first, we write the potential (3.27) as follows:

$$Z_j(x,t) = V_j(x,t) - 2a \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \big(z_j(y',\tau) - z_j(x',t)\big)\partial_{x_n}\Gamma(x'-y',x_n,\tau)dy', \tag{3.28}$$

$$\begin{aligned} V_j(x,t) :=& \ -2az_j(x',t) \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \partial_{x_n}\Gamma(x'-y',x_n,\tau)dy' \\ =& \ z_j(x',t) \operatorname{erfc}\frac{x_n}{2\sqrt{at}}, \quad j=0,1. \end{aligned}$$

1. Let $s \in (\alpha,\, 2)$. Consider the function $V_0(x,t) = z_0(x',t) \operatorname{erfc}\frac{x_n}{2\sqrt{at}}$. It satisfies the homogenous heat equation. Really, z_0 was constructed in Theorem 3.3 as a solution of a Cauchy problem for equation (3.16) with an initial condition (3.8): $z_0|_{t=0} = A_0(x')$. By direct computation with the help of formula (1.13) we can see that

$$\partial_t \operatorname{erfc}\frac{x_n}{2\sqrt{at}} - a\partial^2_{x_n} \operatorname{erfc}\frac{x_n}{2\sqrt{at}} = 0, \tag{3.29}$$

but then

$$\partial_t V_0 - a\Delta V_0 = \operatorname{erfc}\frac{x_n}{2\sqrt{at}}\big(\partial_t z_0 - a\Delta' z_0\big) + z_0(x',t)\big(\partial_t - a\partial^2_{x_n x_n}\big)\operatorname{erfc}\frac{x_n}{2\sqrt{at}} = 0.$$

Moreover, due to the relations $\operatorname{erfc}\infty = 0$, $\operatorname{erfc} 0 = 1$ the function $V_0(x,t)$ satisfies the conditions (3.23),

$$V_0|_{t=0} = z_0(x',t) \operatorname{erfc}\frac{x_n}{2\sqrt{at}}\Big|_{t=0} = 0 \ \text{ in } \ D, \quad V_0|_{x_n=0} = z_0(x',t) \ \text{ on } \ R_T.$$

Thus, the function $V_0(x,t)$ is a solution of problem (3.22), (3.23), $j=0$. We point out that the last potential in (3.28) satisfies equation (3.22) and initial and boundary conditions (3.23) with zero in the right-hand sides.

2. Let $s \in [2,\, 2+\alpha]$. The functions $V_j(x,t) = z_j(x',t) \operatorname{erfc}\frac{x_n}{2\sqrt{at}}$, $j=0,1$, are solutions of the nonhomogeneous equations

$$\partial_t V_j - a\,\Delta\, V_j = z_j^{(1)}(x',t)\operatorname{erfc}\frac{x_n}{2\sqrt{at}}$$

by (3.17), (3.29). So we determine the functions $W_j(x,t)$ as solutions of the first boundary value problems

$$\begin{gathered}\partial_t W_j - a\Delta W_j = -z_j^{(1)}(x',t)\,\mathrm{erfc}\frac{x_n}{2\sqrt{at}} \quad \text{in } D_T,\\ W_j\big|_{t=0} = 0 \quad \text{in } D, \quad W_j\big|_{x_n=0} = 0 \text{ on } R_T, \ j=0,1,\end{gathered} \tag{3.30}$$

then the sum $Z_j(x,t) = V_j(x,t) + W_j(x,t)$ will satisfy the homogeneous equation $\partial_t Z_j - a\Delta Z_j = 0$, $j=0,1$, i.e., (3.22).

We can see that in problem (3.30) the compatibility conditions of zero order is fulfilled

(3.30) is the problem (A.1). In Theorem 3.3 it was proved that the function $z_j^{(1)}$, $j=0,1$, belongs to $C^{\alpha}_{s-2}(R_T)$, but then Theorem A.1 is valid for problem (3.30). In accordance to this theorem, problem (3.30) has a unique solution $W_j(x,t) \in C^2_{s,\delta_0}(D_T)$, $\delta_0 = \frac{1}{8a}$, and is subjected to an estimate

$$|W_j|^{(2)}_{s,\delta_0,D_T} \le c_{18}|z_j^{(1)}|^{(\alpha)}_{s-2,R_T} \le c_{19}|A_j|^{(s-2j)}_R, \ j=0,1,$$

due to (A.3) and (3.20).

By direct substitution of the function $Z_j(x,t) = z_j(x',t)\,\mathrm{erfc}\frac{x_n}{2\sqrt{at}} + W_j(x,t)$, $j=0,1$, into initial and boundary conditions (3.23), we are convinced that it satisfies these conditions. Thus $Z_j(x,t)$ are the unique solutions of problem (3.22), (3.23). $\square$

Now we study the second boundary value problem with unknown function $Z_2(x,t)$,

$$\partial_t Z_2 - a\Delta Z_2 = 0 \quad \text{in} \quad D_T, \tag{3.31}$$

$$Z_2|_{t=0} = 0 \quad \text{in } D, \quad \partial_{x_n} Z_2|_{x_n=0} = z_2(x',t) \quad \text{on} \quad R_T, \tag{3.32}$$

where a function $z_2(x',t)$ was constructed in Theorems 3.3, 3.4 as a solution of a Cauchy problem (3.21) with an initial data $z_2|_{t=0} = B_0(x')$ in R, where $B_0(x') = \psi(x',0) - \partial_{x_n} u_0(x)|_{x_n=0} \neq 0$. We can see that in this problem the compatibility condition of zero order is not fulfilled for $s \in [1,\, 2+\alpha]$. We recall also that

$$\mathrm{ierfc}\,\zeta = \int_\zeta^\infty \mathrm{erfc}\,\xi\, d\xi, \quad \mathrm{erfc}\,\zeta = \frac{2}{\sqrt{\pi}}\int_\zeta^\infty e^{-\zeta^2}\, d\zeta.$$

Theorem 3.7. *Let $B_0(x') \in C^{s-1}(R)$, $s \in [1,\, 2+\alpha]$, $\alpha \in (0,1)$. Then problem* (3.31), (3.32) *has the unique solution*

$$V_2(x,t) = -2\sqrt{at}\, z_2(x',t)\,\mathrm{ierfs}\,\frac{x_n}{2\sqrt{at}}, \tag{3.33}$$

where $z_2(x',t)$ is defined in Theorem 3.3*: $z_2 \in C^{2+\alpha}_{s-1}(R_T)$, $|z_2|^{(2+\alpha)}_{s-1,R_T} \le C_9|B_0|^{(s-1)}_R$.*

Theorem 3.8. *Let $B_0(x') \in C^{1+\alpha}(R)$, $\alpha \in (0,1)$. Then problem* (3.31), (3.32) *has the unique solution $V_2(x,t)$ defined by formula* (3.33)*, where $z_2(x',t)$ is determined in Theorem* 3.4*: $z_2 \in C^{2+\alpha}_{1+\alpha}(R_T)$, $|z_2|^{(2+\alpha)}_{1+\alpha,\,R_T} \le C_{12}|B_0|^{(1+\alpha)}_R$.*

This theorem follows from Theorem 3.7 for $s = 2+\alpha$.

Proof of Theorem 3.7. The solution of problem (3.31), (3.32) may be written in the explicit form

$$\begin{aligned} Z_2(x,t) &= -2a\int_0^t d\tau \int_{\mathbb{R}^{n-1}} z_2(y',\tau)\Gamma(x'-y',x_n,\tau)dy' \\ &= V_2(x,t) - 2a\int_0^t d\tau \int_{\mathbb{R}^{n-1}} \big(z_2(y',\tau)-z_2(x',t)\big)\Gamma(x'-y',x_n,\tau)dy', \end{aligned} \tag{3.34}$$

where

$$V_2(x,t) = -2a z_2(x',t)\int_0^t d\tau \int_{\mathbb{R}^{n-1}} \Gamma(x'-y',x_n,\tau)dy' = -2\sqrt{at}\, z_2(x',t)\,\mathrm{ierfs}\frac{x_n}{2\sqrt{at}}.$$

The function $\sqrt{t}\,\mathrm{ierfs}\,\frac{x_n}{2\sqrt{at}}$ satisfies an equation

$$\left(\partial_t - a\partial^2_{x_n x_n}\right)\sqrt{t}\,\mathrm{ierfs}\,\frac{x_n}{2\sqrt{at}} = 0.$$

This is confirmed by direct computations with the help of formulas (1.13), (1.14) for the iterated integrals of the probability. The function z_2 is a solution of a Cauchy problem (3.21) with an initial condition $z_2|_{t=0} = B_0(x')$. But then the function $V_2(x,t)$ is a solution of an equation (3.31),

$$\begin{aligned} \partial_t V_2 - a\Delta V_2 = &-2\sqrt{at}\,\mathrm{ierfc}\frac{x_n}{2\sqrt{at}}\big(\partial_t z_2 - a\Delta' z_2\big) \\ &- 2\sqrt{a}z_2(x',t)\big(\partial_t - a\partial^2_{x_n x_n}\big)\sqrt{t}\,\mathrm{ierfc}\frac{x_n}{2\sqrt{at}} = 0. \end{aligned}$$

Moreover, by the relations $\mathrm{ierfc}\,\infty = 0$, $\mathrm{erfc}\,0 = 1$ we have

$$V_2\big|_{t=0} = 0, \quad \partial_{x_n}V_2\big|_{x_n=0} = z_2(x',t)\,\mathrm{erfc}\frac{x_n}{2\sqrt{at}}\big|_{x_n=0} = z_2(x',t).$$

We remark that the last potential in (3.34) satisfies an equation (3.31) and the homogeneous conditions (3.32).

Thus, we have shown that the function $V_2(x,t)$ is a unique solution of problem (3.31), (3.32). □

4. Problem 1

Consider the first boundary value problem (1.1)–(1.3) – Problem 1.

Proof of Theorem 2.1. It is known [1, 2, 8] that the solution of Problem 1 belongs to $C_s^{2+\alpha}(R_T)$, if there are fulfilled the compatibility conditions of zero order for $s \in (\alpha, 2)$: $A_0(x') = 0$ on R, and of zero and first orders for $s \in [2, 2+\alpha]$: $A_0(x') = 0$, $A_1(x') = 0$ on R, where

$$A_0(x') := \varphi(x',0) - u_0(x)\big|_{x_n=0} \text{ on } R, \tag{4.1}$$

$$A_1(x') := \partial_t\varphi(x',t)\big|_{t=0} - \big(a\,\Delta u_0(x) + f(x,0)\big)\big|_{x_n=0} \text{ on } R. \tag{4.2}$$

We study the problem under the conditions $A_0(x') \neq 0$, $A_1(x') \neq 0$ on R.

In Theorems 3.3, 3.4 we have extended the functions $A_j(x)$ into R_T by the functions $z_j(x',t)$, which satisfy conditions (3.8)–(3.10) and in Theorems 3.5, 3.6 we have continued $z_j(x',t)$ into D_T by the functions $Z_j(x,t)$, $j=0,1$, as the solutions of problems (3.22), (3.23):

$$\partial_t Z_j - a\,\Delta\, Z_j = 0 \text{ in } D_T,\ Z_j|_{t=0} = 0 \ \text{ in } \ D,\ Z_j|_{x_n=0} = z_j(x',t) \text{ on } R_T, \quad (4.3)$$

and have represented them in the form (3.24), (3.25),

$$Z_0(x,t) = V_0(x,t) + \begin{cases} 0, & s\in(\alpha,\,2), \\ W_0(x,t), & s\in[2,\,2+\alpha], \end{cases} \quad (4.4)$$

$$Z_1(x,t) = V_1(x,t) + W_1(x,t),\ s\in[2,\,2+\alpha], \quad (4.5)$$

$$V_j(x,t) = z_j(x',t)\,\mathrm{erfc}\frac{x_n}{2\sqrt{at}},\ j=0,1. \quad (4.6)$$

With the help of the functions $Z_j(x,t)$, $j=0,1$, we reduce the original Problem 1 to a problem with the fulfilled compatibility conditions of the necessary orders.

We make the substitution

$$\begin{aligned} u(x,t) &= Z_0(x,t) + v_1(x,t), && s\in(\alpha,\,2), \\ u(x,t) &= Z_0(x,t) + Z_1(x,t) + v_2(x,t), && s\in[2,\,2+\alpha], \end{aligned} \quad (4.7)$$

in Problem 1 (1.1)–(1.3), where $v_1,\ v_2$ are the new unknowns. For these functions due to (4.3) we shall have the problems

$$\partial_t\, v_i - a\,\Delta\, v_i = f(x,t) \text{ in } D_T,\ i=1,2, \quad (4.8)$$

$$v_1|_{t=0} = u_0(x) \ \text{ in } \ D,\ v_1|_{x_n=0} = \varphi(x',t) - z_0(x',t) \text{ on } R_T, \quad (4.9)$$

and

$$v_2|_{t=0} = u_0(x) \ \text{ in } \ D,\ v_2|_{x_n=0} = \varphi(x',t) - \big(z_0(x',t) + z_1(x',t)\big) \text{ on } R_T. \quad (4.10)$$

In problems (4.8), $i=1$, (4.9) and (4.8), $i=2$, (4.10) the compatibility conditions of zero and of zero and first orders are fulfilled respectively. Really, taking into account the initial conditions (3.8), (3.9), (3.10) for $z_j(x',t)$, $\partial_t z_j(x',t)$, $j=0,1$, and formulas (4.1), (4.2) we derive the identities (compatibility conditions) on the boundary $x_n=0$ at $t=0$ for problem (4.8), $i=1$, (4.9)

$$v_1\big|_{\substack{x_n=0,\\ t=0}} \equiv u_0(x',0) = \varphi(x',0) - z_0(x',0) = \varphi(x',0) - A_0(x') \equiv u_0(x',0)$$

and for problem (4.8), $i=2$, (4.10),

$$\begin{aligned} v_2\big|_{\substack{x_n=0,\\ t=0}} &\equiv u_0(x',0) = \varphi(x',0) - \big(z_0(x,'t) + z_1(x,'t)\big)\big|_{t=0} \\ &= \varphi(x',0) - A_0(x') \equiv u_0(x',0), \end{aligned}$$

$$\begin{aligned} \partial_t v_2\big|_{\substack{x_n=0,\\ t=0}} &\equiv \big(a\,\Delta u_0(x) + f(x,0)\big)\big|_{x_n=0} \\ &= \partial_t\varphi(x',t)\big|_{t=0} - \big(\partial_t z_0(x',t) + \partial_t z_1(x',t)\big)\big|_{t=0} \\ &= \partial_t\varphi(x',t)\big|_{t=0} - A_1(x') \equiv \big(a\,\Delta u_0(x) + f(x,0)\big)\big|_{x_n=0}. \end{aligned}$$

Then each of problems (4.8), $i = 1$, (4.9) and (4.8), $i = 2$, (4.10) has a unique solution $v_i \in C_s^{2+\alpha}(D_T)$, $i = 1, 2$, and for v_i the following estimates are valid [1, 2, 6, 8]:

$$\begin{aligned} |v_1|_{s,D_T}^{(2+\alpha)} &\le c_1\Big(|u_0|_D^{(s)} + |f|_{s-2,D_T}^{(\alpha)} + |\varphi|_{s,R_T}^{(2+\alpha)} + |z_0|_{s,D_T}^{(2+\alpha)}\Big), \\ |v_2|_{s,D_T}^{(2+\alpha)} &\le c_2\Big(|u_0|_D^{(s)} + |f|_{s-2,D_T}^{(\alpha)} + |\varphi|_{s,R_T}^{(2+\alpha)} + |z_0|_{s,D_T}^{(2+\alpha)} + |z_1|_{s,D_T}^{(2+\alpha)}\Big). \end{aligned} \tag{4.11}$$

Here the functions $z_j(x',t)$, $j = 0, 1$, satisfy the estimates (3.12), (3.13): $|z_j|_{s,D_T}^{(2+\alpha)} \le c_3|A_j|_R^{(s-2j)}$, and the norms of A_0 and A_1 (see (4.1), (4.2)) are evaluated by the norms of the given functions u_0, f, φ respectively. Thus, (4.11) leads to the estimate (2.2). The functions Z_0, Z_1 in the solution (4.7) of Problem 1 are expressed via the functions $z_j(x',t)$, $W_j(x,t)$, $j = 0, 1$, satisfying the estimates (3.12), (3.13), (3.26), i.e., (2.1).

The solution (4.7), (4.4)–(4.6) of Problem 1 contains the function

$$h_0(x_n,t) := \operatorname{erfc}\frac{x_n}{2\sqrt{at}} = \frac{2}{\sqrt{\pi}}\int_{\frac{x_n}{2\sqrt{at}}}^{\infty} e^{-\xi^2}d\xi.$$

Consider this function and its derivatives

$$\partial_{x_n}h_0(x_n,t) = -\frac{1}{\sqrt{a\pi t}}\,e^{-\frac{x_n^2}{4at}}, \quad \partial_t h_0 = a\partial^2_{x_nx_n}h_0 = \frac{x_n}{2\sqrt{a\pi}t^{3/2}}\,e^{-\frac{x_n^2}{4at}}.$$

The function h_0 and its derivatives $\partial_{x_n}h_0$, $\partial_t h_0$, $a\partial^2_{x_nx_n}h_0$ have different limits at the point $((x',0),0)$ depending on the approach of the point (x,t) to this point $((x',0),0)$. Really,

$$\lim_{t\to0}\lim_{x_n\to0}h_0(x_n,t) = 1, \qquad \lim_{x_n\to0}\lim_{t\to0}h_0(x_n,t) = 0, \tag{4.12}$$

$$\lim_{t\to0}\lim_{x_n\to0}\partial_{x_n}h_0(x_n,t) = -\infty, \quad \lim_{x_n\to0}\lim_{t\to0}\partial_{x_n}h_0(x_n,t) = 0. \tag{4.13}$$

Let the point $x = (x_1,\dots,x_n)$ tend to the boundary $x_n = 0$ of a domain as (x', l_0t^β), $t \to 0$, $\beta > 0$, $l_0 = \text{const} > 0$, then

$$h_0(l_0t^\beta,t) = \operatorname{erfs}\Big(\frac{l_0}{2\sqrt{a}}t^{\beta-1/2}\Big),$$

$$\partial_{x_n}h_0(x_n,t)\big|_{x_n=l_0t^\beta} = -\frac{1}{\sqrt{a\pi t}}\,e^{-\frac{l_0^2}{4a}t^{2\beta-1}},$$

$$\partial_t h_0(x_n,t)\big|_{x_n=l_0t^\beta} = a\partial^2_{x_nx_n}h_0(x_n,t)\big|_{x_n=l_0t^\beta} = \frac{l_0}{2\sqrt{a\pi}}t^{\beta-3/2}\,e^{-\frac{l_0^2}{4a}t^{2\beta-1}}$$

and

$$\lim_{t\to0}h_0(l_0t^\beta,t) = \begin{cases} 0, & 0<\beta<1/2, \\ \operatorname{erfc}\frac{l_0}{2\sqrt{a}}, & \beta = 1/2, \\ 1, & \beta > 1/2, \end{cases} \tag{4.14}$$

$$\lim_{t\to0}\partial_{x_n}h_0(x_n,t)\big|_{x_n=l_0t^\beta} = \begin{cases} 0, & 0<\beta<1/2, \\ -\infty, & \beta\ge 1/2, \end{cases} \tag{4.15}$$

$$\lim_{t\to 0}\partial_t h_0(x_n,t)\big|_{x_n=l_0 t^\beta} = \lim_{t\to 0} a\partial^2_{x_n x_n} h_0(x_n,t)\big|_{x_n=l_0 t^\beta}$$
$$= \begin{cases} 0, & 0<\beta<1/2, \\ \infty, & 1/2\le\beta<3/2, \\ \frac{l_0}{2\sqrt{a\pi}}, & \beta=3/2, \\ 0, & \beta>3/2. \end{cases} \tag{4.16}$$

In particular, for $x_n = l_0\sqrt{t}$, l_0 =const>0, we have

$$h_0(l_0\sqrt{t},t) = \operatorname{erfc}\frac{l_0}{2\sqrt{a}} = \text{const}, \tag{4.17}$$

$$\partial_{x_n} h_0(x_n,t)\big|_{x_n=l_0\sqrt{t}} = -\frac{1}{\sqrt{a\pi t}}\, e^{-\frac{l_0^2}{4a}}, \tag{4.18}$$

$$\partial_t h_0(x_n,t)\big|_{x_n=l_0\sqrt{t}} = a\partial^2_{x_n x_n} h_0(x_n,t)\big|_{x_n=l_0\sqrt{t}} = \frac{l_0}{2\sqrt{a\pi}}\frac{1}{t}\, e^{-\frac{l_0^2}{4a}}. \tag{4.19}$$

From here it is seen that the derivatives $\partial_{x_n} h_0\big|_{x_n=l_0\sqrt{t}}$ and $\partial_t h_0\big|_{x_n=l_0\sqrt{t}}$, $\partial^2_{x_n x_n} h_0\big|_{x_n=l_0\sqrt{t}}$ tend to $-\infty$ and ∞ as $-1/\sqrt{t}$ and $1/t$ respectively, when $t\to 0$ $\big((x|_{x_n=l_0\sqrt{t}}\,,\,t)\to((x',0),0)\big)$.

Let $x_n \ge r_0$=const>0, then applying the estimates $|\xi|^\beta \le c_\beta\, e^{-\xi^2/2}$, $\beta\ge 0$, $\operatorname{erfc}\zeta \le \sqrt{2}\operatorname{erfs}\frac{\zeta}{\sqrt{2}}\, e^{-\zeta^2/2} \le \sqrt{2}\, e^{-\zeta^2/2}$ we shall have

$$\begin{gathered} h_0(x_n,t) \le c_4\, e^{-\frac{x_n^2}{8at}}, \\ |\partial_{x_n} h_0(x_n,t)| \le c_5\frac{x_n}{\sqrt{t}}\frac{1}{x_n}\, e^{-\frac{x_n^2}{4at}} \le c_6\frac{1}{r_0}\, e^{-\frac{x_n^2}{8at}}, \\ |\partial_t h_0(x_n,t)|,\ |\partial^2_{x_n x_n} h_0(x_n,t)| \le c_7\frac{1}{r_0^2}\, e^{-\frac{x_n^2}{8at}},\ x_n\ge r_0. \end{gathered} \tag{4.20}$$

These inequalities show that the function $h_0 = \operatorname{erfc}\frac{x_n}{2\sqrt{at}}$ and its derivatives $\partial_{x_n} h_0$, $\partial_t h_0$, $\partial^2_{x_n x_n} h_0$ tend to zero exponentially as $t\to 0$ in the interior of the domain $x_n \ge r_0$.

We see that nonfulfillment of the compatibility conditions of zero ($A_0 \ne 0$) and first ($A_1 \ne 0$) orders in Problem 1 leads to appearance of the singular functions $Z_0(x,t)$ and $Z_1(x,t)$ in the solution respectively (see (4.7), (4.4)–(4.6)), the principle parts of which are

$$V_j(x,t) = z_j(x',t)\operatorname{erfc}\frac{x_n}{2\sqrt{at}},\ j=0,1,$$

where $z_j(x',t)\in C_s^{2+\alpha}(R_T)$, $|z_0|^{(2+\alpha)}_{s,R_T} \le c_8|A_0|^{(s)}_R$, $s\in(\alpha,\,2+\alpha]$, $|z_1|^{(2+\alpha)}_{s,R_T} \le c_9|A_1|^{(s-2)}_R$, $s\in[2,\,2+\alpha]$.

Consider the functions $V_j(x,t)$, $j=0,1$, and its derivatives:

$$\begin{aligned}\partial_{x'}V_j(x,t) &= \operatorname{erfc}\frac{x_n}{2\sqrt{at}}\,\partial_{x'}z_j(x',t),\\ \partial^2_{x'x'}V_j(x,t) &= \operatorname{erfc}\frac{x_n}{2\sqrt{at}}\,\partial^2_{x'x'}z_j(x',t),\end{aligned} \tag{4.21}$$

$$\begin{aligned}\partial_{x_n}V_j(x,t) &= -\frac{1}{\sqrt{at}}\,e^{-\frac{x_n^2}{4at}}\,z_j(x',t),\\ \partial^2_{x'x_n}V_j(x,t) &= -\frac{1}{\sqrt{a\pi t}}\,e^{-\frac{x_n^2}{4at}}\,\partial_{x'}z_j(x',t),\end{aligned} \tag{4.22}$$

$$\begin{aligned}\partial^2_{x_nx_n}V_j(x,t) &= \frac{x_n}{2a\sqrt{a\pi}t^{3/2}}\,e^{-\frac{x_n^2}{4at}}\,z_j(x',t),\\ \partial_tV_j(x,t) &= \operatorname{erfc}\frac{x_n}{2\sqrt{at}}\,\partial_tz_j(x',t)+a\,\partial^2_{x_nx_n}V_j(x,t),\quad j=0,1,\end{aligned} \tag{4.23}$$

where $z_0(x',0)=A(x')\neq 0$, $z_1(x',0)=0$ on R.

Due to the function $h_0(x_n,t):=\operatorname{erfc}\frac{x_n}{2\sqrt{at}}$ satisfying the estimates (4.20) for $x_n\geq r_0$ V_0, ∂_xV_0, $\partial^2_{xx}V_0$, ∂_tV_0 go to zero exponentially as $t\to 0$ in the interior of the domain. V_0 is bounded, but discontinuous at the point $((x',0),\,0)$, as it was shown (see (4.12), (4.14), (4.17)). The derivatives $\partial_{x_n}V_0$ and $\partial^2_{x_nx_n}V_0$ containing $\partial_{x_n}h_0$ and $\partial^2_{x_nx_n}h_0$ respectively have finite or infinite limits as $(x,\,t)\to((x',0),\,0)$ in accordance with (4.13), (4.15), (4.18) and (4.16), (4.19). We can see also that the derivatives $\partial^2_{x_nx_n}V_0$, ∂_tV_0 are integrable functions with respect to $t\in(0,T)$, $T>0$. The derivatives $\partial_{x'}V_0$, $\partial^2_{x'x}V_0$, ∂_tV_0 and $\operatorname{erfc}\frac{x_n}{2\sqrt{a\pi t}}\partial_tz_0(x',t)$ in (4.23) are subjected to the estimates

$$|\partial_{x'}V_0|\leq c_{10}\,t^{\frac{s-1}{2}}\,e^{-\frac{x_n^2}{4at}},\quad s\in(\alpha,\,1),$$

$$|\partial_{xx'}V_0|,\ \operatorname{erfc}\frac{x_n}{2\sqrt{a\pi t}}\,|\partial_tz_0(x',t)|\leq c_{11}\,t^{\frac{s-2}{2}}\,e^{-\frac{x_n^2}{8at}},\quad s\in(\alpha,\,2),$$

the derivatives $\partial_{x'}z_0$ for $s\in[1,\,2+\alpha]$ and $\partial^2_{x'x}z_0$, ∂_tz_0 for $s\in[2,\,2+\alpha]$, in formulas (4.21), (4.22), (4.23) are bounded functions.

Thanks to the function $z_1(x',t)$ satisfying an estimate (3.14): $|z_1|\leq c_{12}$ $|A_1|_R^{(s-2)}$ t in R_T, $s\in[2,\,2+\alpha]$, the functions $V_1(x,t)$ and $\partial_xV_1(x,t)$ are continuous, the higher derivatives $\partial_tV_1(x,t)$, $\partial^2_{xx}V_1(x,t)$ are bounded, but discontinuous in the vicinity of a boundary $x_n=0$ of a domain as $t\to 0$.

If $A_0(x')\neq 0$, but $A_1(x')=0$ on R, then $Z_1(x,t)=0$ in D_T and the solution of Problem 1 takes the form

$$u(x,t)=Z_0(x,t)+\begin{cases}v_1(x,t),\ s\in(\alpha,2),\\ v_2(x,t),\ s\in[2,\,2+\alpha].\end{cases}$$

If $A_0(x')=0$, but $A_1(x')\neq 0$ on R, then $Z_0(x,t)=0$ in D_T and

$$u(x,t)=\begin{cases}v_1(x,t),\ s\in(\alpha,2),\\ Z_1(x,t)+v_2(x,t),\ s\in[2,\,2+\alpha].\end{cases}$$

For $A_0(x') = 0,\;\; A_1(x') = 0$ on R we have

$$u(x,t) = \begin{cases} v_1(x,t), \;\; s \in (\alpha, 2), \\ v_2(x,t), \;\; s \in [2,\, 2+\alpha]. \end{cases}$$

The derivatives $\partial_x v_1;\;\; \partial_t v_1,\; \partial^2_{xx} v_1$ and their Hölder constants have singularities of orders $t^{\frac{s-1}{2}},\;\; s \in (\alpha,\, 1);\;\; t^{\frac{s-2}{2}},\;\; s \in (\alpha,\, 2)$, and $t^{\frac{s-2-\alpha}{2}}$, $s \in (\alpha,\, 2+\alpha)$, respectively in $\overline{D}$ for $t \to 0$. The function $v_2(x,t)$ possesses the bounded derivatives $\partial_t\, v_2(x,t)$, $\partial^2_x\, v_2(x,t)$, and their Hölder constants are unbounded of $t^{\frac{s-2-\alpha}{2}}$ order, $s \in [2,\, 2+\alpha)$ and bounded for $s = 2+\alpha$ as $t \to 0$.

We can see that the character of the singularities of the functions $Z_0(x,t)$, $Z_1(x,t)$ and $v_1(x,t),\;\; v_2(x,t)$ is different. Nonfulfillment of the compatibility conditions of the given functions on the boundary $x_n = 0$ at $t = 0$ leads to appearance of the functions $Z_0(x,t),\;\; Z_1(x,t)$, which are singular only in the vicinity of a boundary $x_n = 0$ of a domain as $t \to 0$. The derivatives of the function $v_1(x,t)$ and the Hölder constants of the higher derivatives of $v_1(x,t)$ and $v_2(x,t)$ may be singular as $t \to 0$ in the closure of a domain $\overline{D}$ and their singularities depend on an initial function $u_0(x)$ of Problem 1 belonging to $C^s(\overline{D})$. □

5. Problem 2

Consider the second boundary value problem (1.1), (1.2), (1.4) – Problem 2.

Proof of Theorem 2.3. 1. For $s \in (\alpha,\, 1)$ the derivatives $\partial_x u$ have singularities of order $t^{\frac{s-1}{2}}$ as $t \to 0$, so the compatibility condition can not be fulfilled and it is not required. In this case Problem 2 has a unique solution $u(x,t) := v_1(x,t) \in C^{2+\alpha}_s(D_T)$, and an estimate (2.3), $i = 1$, for it is valid [1, 2, 8]. The higher derivatives $\partial_t\, v_1(x,t),\;\; \partial^2_x\, v_1(x,t)$ and their Hölder constants have in $\overline{D}$ singularities of orders $t^{\frac{s-2}{2}}$ and $t^{\frac{s-2-\alpha}{2}}$ respectively as $t \to 0$.

2. Let $s \in [1,\, 2+\alpha]$. This case requires the fulfillment of the zero-order compatibility condition: $B_0(x') = 0$ on R, where

$$B_0(x') := \psi(x',0) - \partial_{x_n} u_0(x)\big|_{x_n=0} \in C^{s-1}(R_T) \text{ on } R \tag{5.1}$$

for the solution of Problem 2 to belong to $C^{2+\alpha}_s(D_T)$ [1, 2, 6, 8].

We study the problem under the condition $B_0(x') \neq 0$ on R. In Theorem 3.3 we have extended the function $B_0(x')$ into R_T by the function $z_2(x',t)$ as a solution of the Cauchy problem (3.21) satisfying an initial condition

$$z_2|_{t=0} = B_0(x') \text{ on } R, \tag{5.2}$$

then we have extended the function $z_2(x',t)$ into D_T by the function

$$V_2(x,t) = -2\,\sqrt{a\,t}\, z_2(x',t)\, \mathrm{ierfs}\, \frac{x_n}{2\sqrt{at}} \tag{5.3}$$

which is a solution of problem (3.31), (3.32),

$$\partial_t V_2 - a\Delta V_2 = 0 \text{ in } D_T,\ \ V_2|_{t=0} = 0 \text{ in } D,\ \ \partial_{x_n} V_2|_{x_n=0} = z_2(x',t) \text{ on } R_T. \quad (5.4)$$

We recall that

$$\operatorname{ierfs}\zeta = \int_\zeta^\infty \operatorname{erfs}\xi\, d\xi,\ \ \operatorname{erfs}\zeta = \frac{2}{\sqrt{\pi}} \int_\zeta^\infty e^{-\xi^2} d\xi.$$

After the substitution

$$u(x,t) = V_2(x,t) + v_2(x,t)$$

in Problem 2 (1.1), (1.2), (1.4) and taking into account that $V_2(x,t)$ is a solution of a problem (5.4) we obtain the following problem for the new unknown function $v_2(x,t)$,

$$\begin{aligned} &\partial_t v_2 - a\,\Delta\, v_2 = f(x,t) \ \text{ in } \ D_T, \quad v_2\big|_{t=0} = u_0(x) \ \text{ in } \ D, \\ &\quad \partial_{x_n} v_2\big|_{x_n=0} = \psi(x',t) - \partial_{x_n} V_2\big|_{x_n=0} \\ &\qquad\qquad \equiv \psi(x',t) - z_2(x',t)\operatorname{erfs}\frac{x_n}{2\sqrt{at}}\Big|_{x_n=0} \equiv \psi(x',t) - z_2(x',t). \end{aligned} \quad (5.5)$$

Due to (5.2), (5.1) in this problem the compatibility condition on the boundary $x_n = 0$ at $t = 0$ is fulfilled:

$$\partial_{x_n} v_2\big|_{\substack{x_n=0,\\ t=0}} \equiv \partial_{x_n} u_0(x)\big|_{x_n=0} = \psi(x',0) - B_0(x') \equiv \partial_{x_n} u_0(x)\big|_{x_n=0}.$$

As it was shown in Theorem 3.3, $z_2(x',t) \in C^{2+\alpha}_{s-1}(R_T) \subset C^{1+\alpha}_{s-1}(R_T)$ and an estimate (3.15) for it holds: $|z_2|^{(2+\alpha)}_{s-1,R_T} \le c_9 |B_0|^{(s-1)}_R$, here $B_0(x')$ is evaluated by the norms of the functions u_0, ψ. But then problem (5.5) has a unique solution $v_2(x,t) \in C^{2+\alpha}_s(R_T)$ and it satisfies an estimate (2.3) [1, 2, 6, 8].

Consider the function $V_2(x,t)$ (see (5.3)) and its derivatives. V_2 is a continuous function in $\overline{D}_T$ due to the cofactor $\sqrt{t}$. In the vicinity of a boundary $x_n = 0$ the derivative

$$\partial_{x_n} V_2(x,t) = z_2(x't)\operatorname{erfs}\frac{x_n}{2\sqrt{at}}$$

is bounded, but discontinuous as $(x,t) \to ((x',0),\,0)$ (see (4.12), (4.14), (4.17)), and the second derivative

$$\partial^2_{x_n x_n} V_2(x,t) = -\frac{1}{\sqrt{a\pi t}}\, z_2(x't)\, e^{-\frac{x_n^2}{4at}}$$

is singular as $(x,t) \to ((x',0),\,0)$ (see (4.13), (4.15), (4.18)).

In formula (5.3) the function $z_2(x',t)$ belongs to $C^{2+\alpha}_{s-1}(R_T)$, but thanks to a cofactor $\sqrt{t}$ the orders of the singularities with respect to t as $t \to 0$ of the derivatives $\partial_{x'} V_2$, $\partial^2_{x'x'} V_2$, $\sqrt{t}\operatorname{ierfs}\frac{x_n}{2\sqrt{at}}\,\partial_t z_2(x',t)$ decrease.

In the interior of a domain $x_n \ge r_0$=const > 0 the function V_2 and its derivatives $\partial_x V_2$, $\partial^2_{xx} V_2$, $\partial_t V_2$ tend to zero as $t \to 0$ exponentially due to the estimates $\operatorname{erfs}\zeta \le \sqrt{2}\ \operatorname{erfs}\frac{\zeta}{\sqrt{2}}\, e^{-\zeta^2/2} \le \sqrt{2}\, e^{-\zeta^2/2}$, $\ \operatorname{ierfs}\zeta \le 2\ \operatorname{ierfs}\frac{\zeta}{\sqrt{2}}\, e^{-\zeta^2/2} \le \sqrt{\pi}\, e^{-\zeta^2/2}$ and (4.20).

Consider the function $v_2(x,t)$. The higher derivatives $\partial_t v_2$, $\partial^2_{xx} v_2$ and their Hölder constants have in $\overline{D}$ the singularities of orders $t^{\frac{s-2}{2}}$ for $s \in [1,2)$ and $t^{\frac{s-2-\alpha}{2}}$, $s \in [1, 2+\alpha)$ respectively as $t \to 0$. These singularities are caused by the initial function $u_0(x)$ belonging to $C^s(\overline{D})$. Nonfulfillment of the boundary and initial functions leads to appearance of the function $V_2(x,t)$ in the solution of Problem 2, which is singular only in the vicinity of a boundary $x_n = 0$ as $t \to 0$. □

Appendix

We have let $D := \mathbb{R}^n_+$, $n \geq 2$, $R := \{x = (x', x_n) \,|\, x' \in \mathbb{R}^{n-1}, x_n = 0\}$, $x' = (x_1, \ldots, x_{n-1})$, $D_T = D \times (0,T)$, $R_T = R \times [0,T]$.

In Theorems 3.5, 3.6 for $s \in [2,\, 2+\alpha]$ we have constructed the singular functions $Z_j(x,t) = V_j(x,t) + W_j(x,t)$, $V_j(x,t) = z_j(x',t)\mathrm{erfc}\frac{x_n}{2\sqrt{at}}$, $j = 0,1$, (see (3.24), (3.25)), which appear due to the incompatible boundary and initial data in Problem 1. The functions $V_j(x,t)$, $j = 0,1$, are the principle parts of the singular solutions $Z_j(x,t)$, $j = 0,1$. The functions $W_j(x,t)$, $j = 0,1$, are those that remain after extracting from the singular solutions their principle parts $V_j(x,t)$, $j = 0,1$, and they are the solutions of problems (3.30).

Consider this problem with unknown function $W(x,t)$:

$$\begin{gathered} \partial_t W - a\,\Delta\, W = g(x',t)\,\mathrm{erfs}\frac{x_n}{2\sqrt{at}} \quad \text{in } D_T, \\ W\big|_{t=0} = 0 \ \text{ in } \ D, \ W\big|_{x_n=0} = 0 \ \text{ on } \ R_T, \end{gathered} \tag{A.1}$$

where

$$\mathrm{erfs}\,\zeta = \frac{2}{\sqrt{\pi}} \int_\zeta^\infty e^{-\xi^2} d\xi \leq \sqrt{2}\,\mathrm{erfs}\frac{\zeta}{\sqrt{2}} e^{-\zeta^2/2} \leq \sqrt{2}\, e^{-\zeta^2/2}, \ \zeta \geq 0. \tag{A.2}$$

We can see that in problem (A.1) the compatibility condition of zero order is fulfilled.

Theorem A.1. *Let $s \in [2,\, 2+\alpha]$, $\alpha \in (0,1)$. For every function $g(x',t) \in C^{\alpha}_{s-2}(R_T)$ the problem* (A.1) *has a unique solution $W(x,t) \in C^2_{s,\delta_0}(D_T)$, $\delta_0 = \frac{1}{8a}$, which satisfies an estimate*

$$|W|^{(2)}_{s,\delta_0,D_T} \leq c_1 |g|^{(\alpha)}_{s-2,R_T}. \tag{A.3}$$

From this theorem for $s = 2+\alpha$ we have the following one.

Theorem A.2. *For every function $g(x',t) \in C^{\alpha,\,\alpha/2}_{x'\ \ t}(R_T)$, $\alpha \in (0,1)$, the problem* (A.1) *has a unique solution $W(x,t) \in C^2_{2+\alpha,\delta_0}(D_T)$, $\delta_0 = \frac{1}{8a}$, which satisfies an estimate*

$$|W|^{(2)}_{2+\alpha,\delta_0,D_T} \leq c_2 |g|^{(\alpha)}_{R_T}.$$

Here the norms $|g|^{(\alpha)}_{s-2,R_T}$, $|g|^{(\alpha)}_{R_T}$ and $|W|^{(2)}_{s,\delta_0,D_T}$, $|W|^{(2)}_{2+\alpha,\delta_0,D_T}$ are defined by formulas (1.5), (1.9) and (1.10), (1.11) respectively.

Proof of Theorem A.1. The solution of problem (A.1) may be written in the explicit form

$$W(x,t) = \int_0^t d\tau \int_{\mathbb{R}^n_+} g(y',\tau)\,\mathrm{erfc}\frac{y_n}{2\sqrt{a\tau}}\big(\Gamma_n(x-y,t-\tau) - \Gamma_n(x-y^*,t-\tau)\big)\,dy \tag{A.4}$$

$$\equiv \int_0^t d\tau \int_{\mathbb{R}^{n-1}} g(y',\tau)\,\Gamma_{n-1}(x'-y',t-\tau)\,dy'$$
$$\times \int_0^\infty \mathrm{erfc}\frac{y_n}{2\sqrt{a\tau}}\big(\Gamma_1(x_n-y_n,t-\tau) - \Gamma_1(x_n+y_n,t-\tau)\big)\,dy_n;$$

here $y^* = (y_1,\dots,y_{n-1},-y_n)$,

$$\Gamma_n(x,t) = \frac{1}{(2\sqrt{a\pi t})^n} e^{-\frac{x^2}{4at}} \tag{A.5}$$

is a fundamental solution of a heat equation satisfying an estimate

$$|\partial_t^k \partial_x^m \Gamma_n(x,t)| \le c_3 \frac{1}{t^{\frac{n+2k+|m|}{2}}} e^{-\frac{x^2}{8at}}. \tag{A.6}$$

We evaluate the norm (1.10) of the potential (A.4). First, with the help of a tabular formula

$$\int_{-\infty}^\infty e^{-A(x-y)^2 - B(y-z)^2}\,dy = \frac{\sqrt{\pi}}{\sqrt{A+B}}\, e^{-\frac{AB(x-z)^2}{A+B}}, \quad A,B>0,$$

we compute an integral

$$\begin{aligned} J_1(x_n,\,t-\tau,\,\tau;\,k) &= \int_0^\infty \big(e^{-\frac{(x_n-y_n)^2}{ka(t-\tau)}} + e^{-\frac{(x_n+y_n)^2}{ka(t-\tau)}}\big) e^{-\frac{y_n^2}{ka\tau}}\,dy_n \\ = \int_{-\infty}^\infty e^{-\frac{(x_n-y_n)^2}{ka(t-\tau)}}\, e^{-\frac{y_n^2}{ka\tau}}\,dy_n &= \sqrt{\pi a k}\frac{\sqrt{\tau(t-\tau)}}{\sqrt{t}}\, e^{-\frac{x_n^2}{kat}},\ \tau\in(0,t),\ k>0, \end{aligned} \tag{A.7}$$

and estimate it as

$$J_1(x_n,\,t-\tau,\,\tau;\,k) \le \sqrt{\pi a k}\sqrt{t-\tau}\, e^{-\frac{x_n^2}{kat}}. \tag{A.8}$$

Let $R'_t = R\times[t/2,t]$,

$$M_1 := \sup_{t\le T} |\,g(x',t)|_R, \tag{A.9}$$

$$M_2 := \sup_{t\le T} t^{\frac{2+\alpha-s}{2}} \sup_{(x',t),(z',t)\in R'_t} \big|g(x',t)-g(z',t)\big|\,|x'-z'|^{-\alpha}, \tag{A.10}$$

$$M_3 := \sup_{t\le T} t^{\frac{2+\alpha-s}{2}} \sup_{(x',t),(x',t_1)\in R'_t} \big|g(x',t)-g(x',t_1)\big|\,|t-t_1|^{-\alpha/2}. \tag{A.11}$$

We can represent the derivatives $\partial^j_{x_i}\partial_{x_\mu}W(x,t)$, $i=1,\dots,n$, $\mu=1,\dots,n-1$, $j=0,1$, in the form

$$\partial^j_{x_i}\partial_{x_\mu}W(x,t)=\int_0^t d\tau\int_{\mathbb{R}^n_+}\big(g(y',\tau)-g(x',\tau)\big)\mathrm{erfc}\frac{y_n}{2\sqrt{a\tau}}$$
$$\times\,\partial^j_{x_i}\partial_{x_\mu}\big(\Gamma_n(x-y,t-\tau)-\Gamma_n(x-y^*,t-\tau)\big)\,dy$$

by an identity

$$\int_{\mathbb{R}^{n-1}}\partial_{x_\mu}\big(\Gamma_n(x-y,t-\tau)-\Gamma_n(x-y^*,t-\tau)\big)\,dy'=0.$$

Applying the notations (A.9), (A.10) and the estimates (A.6) for $\Gamma_n(x,t)$, (A.2) for erfc $\frac{x_n}{2\sqrt{at}}$, (A.8) for J_1,

$$|\xi|^\beta e^{-\xi^2}\le c_\beta\,e^{-\xi^2/2},\quad \beta\ge 0, \tag{A.12}$$

and integrating with respect to y' we obtain

$$|W(x,t)|\le c_4\,M_1\int_0^t\frac{1}{\sqrt{t-\tau}}\,J_1(x_n,\,t-\tau,\,\tau;\,8)\,d\tau\le c_5\,M_1\,t\,e^{-\frac{x_n^2}{8at}}; \tag{A.13}$$

$$|\partial^j_{x_i}\partial_{x_\mu}W(x,t)|\le c_6\,M_2\int_0^t\tau^{\frac{s-2-\alpha}{2}}\frac{1}{(t-\tau)^{1+j/2-\alpha/2}}\,J_1(x_n,\,t-\tau,\,\tau;\,8)\,d\tau$$
$$\le c_7\,M_2\,t^{\frac{s-1-j}{2}}\,e^{-\frac{x_n^2}{8at}},\ j=0,1,$$

and from here we shall have the estimates

$$|\partial_{x_\mu}W(x,t)|\le c_7\,M_2\,t^{\frac{s-1}{2}}\,e^{-\frac{x_n^2}{8at}},\quad |\partial_{x_i}\partial_{x_\mu}W(x,t)|\le c_7\,M_2\,t^{\frac{s-2}{2}}\,e^{-\frac{x_n^2}{8at}}, \tag{A.14}$$

$i=1,\dots,n$, $\mu=1,\dots,n-1$, $s\in[2,\,2+\alpha]$.

We evaluate $\partial_{x_n}W(x,t)$, $\partial^2_{x_nx_n}W(x,t)$. For that we write down the derivative $\partial_{x_n}W(x,t)$ as follows:

$$\partial_{x_n}W(x,t)=\int_0^t d\tau\int_{\mathbb{R}^{n-1}}g(y',\tau)\Gamma_{n-1}(x'-y',t-\tau)J_2(x_n,t-\tau,\tau)\,dy', \tag{A.15}$$
$$J_2(\cdot)=-\int_0^\infty\mathrm{erfc}\frac{y_n}{2\sqrt{a\tau}}\partial_{y_n}\big(\Gamma_1(x_n-y_n,t-\tau)+\Gamma_1(x_n+y_n,t-\tau)\big)\,dy_n,$$

where Γ_{n-1}, Γ_1 are defined by (A.5). Integrating J_2 by parts and applying formula (A.7) we obtain

$$J_2(x_n,t-\tau,\tau)=2\,\Gamma_1(x_n,t-\tau)-\frac{1}{2a\pi\sqrt{(t-\tau)\tau}}J_1(x_n,t-\tau,\tau;4)$$
$$=2\big(\Gamma_1(x_n,t-\tau)-\Gamma_1(x_n,t)\big)$$

and the derivative (A.15) takes the form

$$\partial_{x_n} W(x,t) = 2\int_0^t \big(\Gamma_1(x_n, t-\tau) - \Gamma_1(x_n,t)\big)d\tau \times \int_{\mathbb{R}^{n-1}} g(y',\tau)\Gamma_{n-1}(x'-y',t-\tau)\,dy'.$$

From here with the help of the formula

$$\int_{\mathbb{R}^{n-1}} \Gamma_{n-1}(x'-y',t-\tau)\,dy' = 1 \tag{A.16}$$

we derive

$$\begin{aligned}\partial^2_{x_n x_n} W(x,t) = 2&\int_0^t \partial_{x_n}\big(\Gamma_1(x_n,t-\tau) - \Gamma_1(x_n,t)\big)d\tau \\ &\times \int_{\mathbb{R}^{n-1}} \big(g(y',\tau) - g(x',t)\big)\Gamma_{n-1}(x'-y',t-\tau)\,dy' \\ &+ 2g(x',t)\int_0^t \partial_{x_n}\big(\Gamma_1(x_n,t-\tau) - \Gamma_1(x_n,t)\big)d\tau \\ := J_3&(x,t) + J_4(x,t).\end{aligned} \tag{A.17}$$

Taking into account the notations (A.9)–(A.11), formula (A.16), the inequalities (A.12) and

$$e^{-\frac{x_n^2}{4a(t-\tau)}} \le e^{-\frac{x_n^2}{4at}},\ \ \tau \in (0,t),$$

then integrating with respect to τ we find

$$|\partial_{x_n} W(x,t)| \le c_8\, M_1\, t^{1/2}\, e^{-\frac{x_n^2}{4at}} \tag{A.18}$$

and

$$\begin{aligned}|J_3(x,t)| &\le c_9\,(M_2+M_3)\int_0^t \tau^{\frac{s-2-\alpha}{2}}\left(\frac{x_n}{(t-\tau)^{\frac{3-\alpha}{2}}}\, e^{-\frac{x_n^2}{4a(t-\tau)}} + \frac{x_n(t-\tau)^{\alpha/2}}{t^{3/2}}\, e^{-\frac{x_n^2}{4at}}\right)d\tau \\ &\le c_{10}\,(M_2+M_3)\, t^{\frac{s-2}{2}}\, e^{-\frac{x_n^2}{4at}},\end{aligned} \tag{A.19}$$

$$J_4(x,t) = \frac{1}{a}\, g(x',t)\left(\frac{x_n}{2\sqrt{a\pi t}}\, e^{-\frac{x_n^2}{4at}} - \operatorname{erfc}\frac{x_n}{2\sqrt{at}}\right), \tag{A.20}$$

$$J_4(x,t) \le c_{11}\, M_1\, e^{-\frac{x_n^2}{8at}}. \tag{A.21}$$

We make use of estimates (A.19), (A.21) in (A.17) and take into account that $M_1 + M_2 + M_3 = |g|^{\alpha}_{s-2,R_T},\ \ e^{-\xi^2} \le e^{-\xi^2/2}$, then we obtain

$$|\partial^2_{x_n x_n} W(x,t)| \le c_{12}\left(M_1 + t^{\frac{s-2}{2}}\,(M_2+M_3)\right) e^{-\frac{x_n^2}{8at}} \le c_{13}|g|^{(\alpha)}_{s-2,R_T}\ e^{-\frac{x_n^2}{8at}} \tag{A.22}$$

(here $s \in [2,\, 2+\alpha]$).

From the equation in (A.1) with the help of formulas (A.17), (A.20) we find the time derivative $\partial_t W(x,t)$,

$$\partial_t W(x,t) = a\Delta' W(x,t) + aJ_3(x,t) + g(x',t)\,\frac{x_n}{2\sqrt{a\pi t}}\,e^{-\frac{x_n^2}{4at}}$$

and applying the estimates (A.14), (A.19), (A.12) to it we derive

$$|\partial_t W(x,t)| \le c_{14}\left(M_1 + t^{\frac{s-2}{2}}\,(M_2+M_3)\right)e^{-\frac{x_n^2}{8at}} \le c_{15}|g|^{(\alpha)}_{s-2,R_T}\;e^{-\frac{x_n^2}{8at}}. \qquad \text{(A.23)}$$

Gathering obtained estimates (A.13), (A.14), (A.18), (A.22), (A.23) we shall have an estimate (A.3) of the norm $|W|^{(2)}_{s,\delta_0,D_T}$, $\delta_0 = \frac{1}{8a}$, defined by (1.10). $\square$

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On the Maxwell-Stefan Approach to Multicomponent Diffusion

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Dedicated to Herbert Amann on the occasion of his 70^{th} *anniversary*

Abstract. We consider the system of Maxwell-Stefan equations which describe multicomponent diffusive fluxes in non-dilute solutions or gas mixtures. We apply the Perron-Frobenius theorem to the irreducible and quasi-positive matrix which governs the flux-force relations and are able to show normal ellipticity of the associated multicomponent diffusion operator. This provides local-in-time wellposedness of the Maxwell-Stefan multicomponent diffusion system in the isobaric, isothermal case.

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1. Introduction

On the macroscopic level of continuum mechanical modeling, fluxes of chemical components (species) are due to convection and molecular fluxes, where the latter essentially refers to diffusive transport. The almost exclusively employed constitutive "law" to model diffusive fluxes within continuum mechanical models is Fick's law, stating that the flux of a chemical component is proportional to the gradient of the concentration of this species, directed against the gradient. There is no influence of the other components, i.e., cross-effects are ignored although well known to appear in reality. Actually, such cross-effects can completely divert the diffusive fluxes, leading to so-called reverse diffusion (up-hill diffusion in direction of the gradient) or osmotic diffusion (diffusion without a gradient). This has been proven in several experiments, e.g., in a classical setting by Duncan and Toor; see [7].

To account for such important phenomena, a multicomponent diffusion approach is required for realistic models. The standard approach within the theory of Irreversible Thermodynamics replaces Fickian fluxes by linear combinations of the gradients of all involved concentrations, respectively chemical potentials. This requires the knowledge of a full matrix of binary diffusion coefficients and this diffusivity matrix has to fulfill certain requirements like positive semi-definiteness in order to be consistent with the fundamental laws from thermodynamics. The Maxwell-Stefan approach to multicomponent diffusion leads to a concrete form of the diffusivity matrix and is based on molecular force balances to relate all individual species velocities. While the Maxwell-Stefan equations are successfully used in engineering applications, they seem much less known in the mathematical literature. In fact we are not aware of a rigorous mathematical analysis of the Maxwell-Stefan approach to multicomponent diffusion, except for [8] which mainly addresses questions of modeling and numerical computations, but also contains some analytical results which are closely related to the present considerations.

2. Continuum mechanical modeling of multicomponent fluids

We consider a multicomponent fluid composed of n chemical components A_i. Starting point of the Maxwell-Stefan equations are the individual mass balances, i.e.,

$$\partial_t \rho_i + \operatorname{div}(\rho_i \mathbf{u}_i) = R_i^{\text{tot}}, \tag{1}$$

where $\rho_i = \rho_i(t, \mathbf{y})$ denotes the mass density and $\mathbf{u}_i = \mathbf{u}_i(t, \mathbf{y})$ the individual velocity of species A_i. Note that the spatial variable is denoted as $\mathbf{y}$, while the usual symbol $\mathbf{x}$ will refer to the composition of the mixture. The right-hand side is the total rate of change of species mass due to all chemical transformations. We assume conservation of the total mass, i.e., the production terms satisfy $\sum_{i=1}^n R_i^{\text{tot}} = 0$. Let ρ denote the total mass density and $\mathbf{u}$ be the barycentric (i.e., mass averaged) velocity, determined by

$$\rho := \sum_{i=1}^n \rho_i, \qquad \rho\, \mathbf{u} := \sum_{i=1}^n \rho_i\, \mathbf{u}_i.$$

Summation of the individual mass balances (1) then yields

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{2}$$

i.e., the usual continuity equation.

In principle, a full set of n individual momentum balances should now be added to the model; cf. [11]. But in almost all engineering models, a single set of Navier-Stokes equations is used to describe the evolution of the velocity field, usually without accounting for individual contributions to the stress tensor. One main reason is a lack of information about appropriate constitutive equations for the stress in multicomponent mixtures; but cf. [16]. For the multicomponent, single momentum model the barycentric velocity $\mathbf{u}$ is assumed to be determined by the Navier-Stokes equations. Introducing the mass diffusion fluxes

$$\mathbf{j}_i := \rho_i(\mathbf{u}_i - \mathbf{u}) \tag{3}$$

and the mass fractions $Y_i := \rho_i/\rho$, the mass balances (1) can be rewritten as

$$\rho\partial_t Y_i + \rho\mathbf{u}\cdot\nabla Y_i + \operatorname{div}\mathbf{j}_i = R_i^{\text{tot}}. \tag{4}$$

In the present paper, main emphasis is on the aspect of multicomponent diffusion, including the cross-diffusion effects. Therefore, we focus on the special case of *isobaric, isothermal* diffusion. The (thermodynamic) pressure p is the sum of partial pressures p_i and the latter correspond to $c_i RT$ in the general case with c_i denoting the molar concentration, R the universal gas constant and T the absolute temperature; here $c_i = \rho_i/M_i$ with M_i the molar mass of species A_i. Hence isobaric conditions correspond to the case of constant total molar concentration c_{tot}, where $c_{\text{tot}} := \sum_{i=1}^n c_i$. Still, species diffusion can lead to transport of momentum because the M_i are different. Instead of $\mathbf{u}$ we therefore employ the molar averaged velocity defined by

$$c_{\text{tot}}\mathbf{v} := \sum_{i=1}^n c_i\mathbf{u}_i. \tag{5}$$

Note that other velocities are used as well; only the diffusive fluxes have to be adapted; see, e.g., [20]. With the molar averaged velocity, the species equations (1) become

$$\partial_t c_i + \operatorname{div}(c_i\mathbf{v} + \mathbf{J}_i) = r_i^{\text{tot}} \tag{6}$$

with $r_i^{\text{tot}} := R_i^{\text{tot}}/M_i$ and the diffusive molar fluxes

$$\mathbf{J}_i := c_i(\mathbf{u}_i - \mathbf{v}). \tag{7}$$

Below we exploit the important fact that

$$\sum_{i=1}^n \mathbf{J}_i = 0. \tag{8}$$

As explained above we may now assume $\mathbf{v} = 0$ in the isobaric case. In this case the species equations (6) simplify to a system of reaction-diffusion equations given by

$$\partial_t c_i + \operatorname{div}\mathbf{J}_i = r_i^{\text{tot}}, \tag{9}$$

where the individual fluxes $\mathbf{J}_i$ need to be modeled by appropriate constitutive equations. The most common constitutive equation is Fick's law which states that

$$\mathbf{J}_i = -D_i \operatorname{grad} c_i \tag{10}$$

with diffusivities $D_i > 0$. The diffusivities are usually assumed to be constant, while they indeed depend in particular on the composition of the system, i.e., $D_i = D_i(\mathbf{c})$ with $\mathbf{c} := (c_1, \ldots, c_n)$. Even if the dependence of the D_i is taken into account, the above definition of the fluxes misses the cross-effects between the diffusing species. In case of concentrated systems more realistic constitutive equations are hence required which especially account for such mutual influences. Here a common approach is the general constitutive law

$$\mathbf{J}_i = -\sum_{j=1}^n D_{ij} \operatorname{grad} c_j \tag{11}$$

with binary diffusivities $D_{ij} = D_{ij}(\mathbf{c})$. Due to the structure of the driving forces, as discussed below, the matrix $\mathbf{D} = [D_{ij}]$ is of the form $\mathbf{D}(\mathbf{c}) = \mathbf{L}(\mathbf{c})\, G''(\mathbf{c})$ with a positive definite matrix $G''(\mathbf{c})$, the Hessian of the Gibbs free energy. Then, from general principles of the theory of Irreversible Thermodynamics, it is assumed that the matrix of transport coefficients $\mathbf{L} = [L_{ij}]$ satisfies

- $\mathbf{L}$ is *symmetric* (the Onsager reciprocal relations)
- $\mathbf{L}$ is *positive semidefinite* (the second law of thermodynamics).

Under this assumption the quasilinear reaction-diffusion system

$$\partial_t \mathbf{c} + \operatorname{div}\,(-\mathbf{D}(\mathbf{c})\,\nabla \mathbf{c}) = \mathbf{r}(\mathbf{c}), \tag{12}$$

satisfies – probably after a reduction to $n-1$ species – parabolicity conditions sufficient for local-in-time wellposedness. Here $\mathbf{r}(\mathbf{c})$ is short for $(r_1^{\text{tot}}(\mathbf{c}), \ldots, r_n^{\text{tot}}(\mathbf{c}))$.

A main problem now is how realistic diffusivity matrices together with their dependence on the composition vector $\mathbf{c}$ can be obtained.

Let us note in passing that Herbert Amann has often been advocating that general flux vectors should be considered, accounting both for concentration dependent diffusivities and for cross-diffusion effects. For a sample of his contributions to the theory of reaction-diffusion systems with general flux vectors see [1], [2] and the references given there.

3. The Maxwell-Stefan equations

The Maxwell-Stefan equations rely on inter-species force balances. More precisely, it is assumed that the thermodynamical driving force $\mathbf{d}_i$ of species A_i is in local equilibrium with the total friction force. Here and below it is often convenient to work with the molar fractions $x_i := c_i/c_{\text{tot}}$ instead of the chemical concentrations. From chemical thermodynamics it follows that for multicomponent systems which are locally close to thermodynamical equilibrium (see, e.g., [20]) the driving forces under isothermal conditions are given as

$$\mathbf{d}_i = \frac{x_i}{RT} \operatorname{grad} \mu_i \tag{13}$$

with μ_i the chemical potential of species A_i. Equation (13) requires some more explanation. Recall first that the chemical potential μ_i for species A_i is defined as

$$\mu_i = \frac{\partial G}{\partial c_i}, \tag{14}$$

where G denotes the (volume-specific) density of the Gibbs free energy. The chemical potential depends on c_i, but also on the other c_j as well as on pressure and temperature. In the engineering literature, from the chemical potential a part μ_i^0 depending on pressure and temperature is often separated and, depending on the context, a gradient may be applied only to the remainder. To avoid confusion, the common notation in use therefore is

$$\nabla \mu_i = \nabla_{T,p} \mu_i + \frac{\partial \mu_i}{\partial p} \nabla p + \frac{\partial \mu_i}{\partial T} \nabla T.$$

Here $\nabla_{T,p}\mu_i$ means the gradient taken under constant pressure and temperature. In the isobaric, isothermal case this evidently makes no difference. Let us also note that G is assumed to be a convex function of the c_i for single phase systems, since this guarantees thermodynamic stability, i.e., no spontaneous phase separations. For concrete mixtures, the chemical potential is often assumed to be given by

$$\mu_i = \mu_i^0 + RT \ln a_i \tag{15}$$

with a_i the so-called activity of the ith species; equation (15) actually implicitly defines a_i. In (15), the term μ_i^0 depends on pressure and temperature. For a mixture of ideal gases, the activity a_i equals the molar fraction x_i. The same holds for solutions in the limit of an ideally dilute component, i.e., for $x_i \to 0+$. This is no longer true for non-ideal systems in which case the activity is written as

$$a_i = \gamma_i \, x_i \tag{16}$$

with an activity coefficient γ_i which itself depends in particular on the full composition vector $\mathbf{x}$.

The mutual friction force between species i and j is assumed to be proportional to the relative velocity as well as to the amount of molar mass. Together with the assumption of balance of forces this leads to the relation

$$\mathbf{d}_i = -\sum_{j \neq i} f_{ij} \, x_i \, x_j (\mathbf{u}_i - \mathbf{u}_j) \tag{17}$$

with certain drag coefficients $f_{ij} > 0$; here $f_{ij} = f_{ji}$ is a natural mechanical assumption. Insertion of (13) and introduction of the so-called Maxwell-Stefan (MS) diffusivities $Đ_{ij} = 1/f_{ij}$ yields the system

$$\frac{x_i}{RT} \operatorname{grad} \mu_i = -\sum_{j \neq i} \frac{x_j \mathbf{J}_i - x_i \mathbf{J}_j}{c_{\text{tot}} \, Đ_{ij}} \qquad \text{for } i = 1, \ldots, n. \tag{18}$$

The set of equations (18) together with (8) forms the Maxwell-Stefan equations of multicomponent diffusion. The matrix $[Đ_{ij}]$ of MS-diffusivities is assumed to be symmetric in accordance with the symmetry of $[f_{ij}]$. Let us note that for ideal gases the symmetry can be obtained from the kinetic theory of gases; cf. [9] and [14]. The MS-diffusivities $Đ_{ij}$ will in general depend on the composition of the system.

Due to the symmetry of $[Đ_{ij}]$, the model is in fact consistent with the Onsager reciprocal relations (cf. [18] as well as below), but notice that the $Đ_{ij}$ are not to be inserted into (11), i.e., they do not directly correspond to the D_{ij} there. Instead, the MS equations have to be inverted in order to provide the fluxes $\mathbf{J}_i$.

Note also that the Ansatz (17) implies $\sum_i \mathbf{d}_i = 0$ because of the symmetry of $[f_{ij}]$, resp. of $[Đ_{ij}]$. Hence $\sum_i \mathbf{d}_i = 0$ is necessary in order for (17) to be consistent. It in fact holds because of (and is nothing but) the Gibbs-Duhem relation, see, e.g., [12]. The relation $\sum_i \mathbf{d}_i = 0$ will be important below.

Example (Binary systems). For a system with two components we have

$$\mathbf{d}_1(=-\mathbf{d}_2) = -\frac{1}{c_{\text{tot}}\,Đ_{12}}\big(x_2\mathbf{J}_1 - x_1\mathbf{J}_2\big). \tag{19}$$

Using $x_1 + x_2 = 1$ and $\mathbf{J}_1 + \mathbf{J}_2 = 0$ one obtains

$$\mathbf{J}_1(=-\mathbf{J}_2) = -\frac{Đ_{12}}{RT}c_1\,\text{grad}\,\mu_1. \tag{20}$$

Writing c and $\mathbf{J}$ instead of c_1 and $\mathbf{J}_1$, respectively, and assuming that the chemical potential is of the form $\mu = \mu^0 + RT\ln(\gamma c)$ with the activity coefficient $\gamma = \gamma(c)$ this finally yields

$$\mathbf{J} = -Đ_{12}\Big(1 + \frac{c\,\gamma'(c)}{\gamma(c)}\Big)\text{grad}\,c. \tag{21}$$

Inserting this into the species equation leads to a nonlinear diffusion equation, namely

$$\partial_t c - \Delta\phi(c) = r(c), \tag{22}$$

where the function $\phi : \mathbb{R} \to \mathbb{R}$ satisfies $\phi'(s) = Đ_{12}(1 + s\gamma'(s)/\gamma(s))$ and, say, $\phi(0) = 0$. Equation (22) is also known as the filtration equation (or, the generalized porous medium equation) in other applications. Note that well-known pde-theory applies to (22) and especially provides well-posedness as soon as ϕ is continuous and nondecreasing; cf., e.g., [21]. The latter holds if $s \to s\gamma(s)$ is increasing which is nothing but the fact that the chemical potential μ of a component should be an increasing function of its concentration. This is physically reasonable in systems without phase separation.

4. Inversion of the flux-force relations

In order to get constitutive equations for the fluxes $\mathbf{J}_i$ from the Maxwell-Stefan equations, which need to be inserted into (9), we have to invert (18). Now (18) alone is not invertible for the fluxes, since these are linearly dependent. Elimination of $\mathbf{J}_n$ by means of (8) leads to the reduced system

$$c_{\text{tot}}\begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_{n-1} \end{bmatrix} = -\,\mathbf{B}\begin{bmatrix} \mathbf{J}_1 \\ \vdots \\ \mathbf{J}_{n-1} \end{bmatrix}, \tag{23}$$

where the $(n-1) \times (n-1)$-matrix $\mathbf{B}$ is given by

$$B_{ij} = \begin{cases} x_i\Big(\dfrac{1}{Đ_{1n}} - \dfrac{1}{Đ_{ij}}\Big) & \text{for } i \neq j, \\[2ex] \dfrac{x_i}{Đ_{in}} + \displaystyle\sum_{k\neq i}^{n} \dfrac{x_k}{Đ_{ik}} & \text{for } i = j \ \ (\text{with } x_n = 1 - \textstyle\sum_{m<n} x_m). \end{cases} \tag{24}$$

Assuming for the moment the invertibility of $\mathbf{B}$ and letting μ_i be functions of the composition expressed by the molar fractions $\mathbf{x} = (x_1, \ldots, x_n)$, the fluxes are given by

$$\begin{bmatrix} \mathbf{J}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{J}_{n-1} \end{bmatrix} = -c_{\text{tot}} \mathbf{B}^{-1} \, \mathbf{\Gamma} \begin{bmatrix} \nabla x_1 \\ \cdot \\ \cdot \\ \cdot \\ \nabla x_{n-1} \end{bmatrix}, \tag{25}$$

where

$$\mathbf{\Gamma} = [\Gamma_{ij}] \quad \text{with} \;\; \Gamma_{ij} = \delta_{ij} + x_i \, \frac{\partial \ln \gamma_i}{\partial x_j} \tag{26}$$

captures the thermodynamical deviations from the ideally diluted situation; here δ_{ij} denotes the Kronecker symbol.

Example (Ternary systems). We have

$$\mathbf{B} = \begin{bmatrix} \frac{1}{Đ_{13}} + x_2\left(\frac{1}{Đ_{12}} - \frac{1}{Đ_{13}}\right) & -x_1\left(\frac{1}{Đ_{12}} - \frac{1}{Đ_{13}}\right) \\ -x_2\left(\frac{1}{Đ_{12}} - \frac{1}{Đ_{23}}\right) & \frac{1}{Đ_{23}} + x_1\left(\frac{1}{Đ_{12}} - \frac{1}{Đ_{23}}\right) \end{bmatrix} \tag{27}$$

and $\det(\mathbf{B} - t\mathbf{I}) = t^2 - \operatorname{tr} \mathbf{B}\, t + \det \mathbf{B}$ with

$$\det \mathbf{B} = \frac{x_1}{Đ_{12}\, Đ_{13}} + \frac{x_2}{Đ_{12}\, Đ_{23}} + \frac{x_3}{Đ_{13}\, Đ_{23}} \geq \min \left\{ \frac{1}{Đ_{12}\, Đ_{13}}, \frac{1}{Đ_{12}\, Đ_{23}}, \frac{1}{Đ_{13}\, Đ_{23}} \right\} \tag{28}$$

and

$$\operatorname{tr} \mathbf{B} = \frac{x_1 + x_2}{Đ_{12}} + \frac{x_1 + x_3}{Đ_{13}} + \frac{x_2 + x_3}{Đ_{23}} \geq 2 \min \left\{ \frac{1}{Đ_{12}}, \frac{1}{Đ_{13}}, \frac{1}{Đ_{23}} \right\}. \tag{29}$$

It is easy to check that $(\operatorname{tr} \mathbf{B})^2 \geq 3 \det \mathbf{B}$ for this particular matrix and therefore the spectrum of $\mathbf{B}^{-1}$ is in the right complex half-plane within a sector of angle less than $\pi/6$. This implies normal ellipticity of the differential operator $\mathbf{B}^{-1}(\mathbf{x})(-\Delta \mathbf{x})$. Recall that a second-order differential operator with matrix-valued coefficients is said to be *normally elliptic* if the symbol of the principal part has it's spectrum inside the open right half-plane of the complex plane; see section 4 in [2] for more details. This notion has been introduced by Herbert Amann in [1] as the appropriate concept for generalizations to more general situations with operator-valued coefficients.

Consequently, the Maxwell-Stefan equations for a ternary system are locally-in-time wellposed if $\mathbf{\Gamma} = \mathbf{I}$, i.e., in the special case of *ideal solutions*. The latter refers to the case when the chemical potentials are of the form (15) with $\gamma_i \equiv 1$ for all i. Of course this extends to any $\mathbf{\Gamma}$ which is a small perturbations of $\mathbf{I}$, i.e., to slightly non-ideal solutions.

Let us note that Theorem 1 below yields the local-in-time wellposedness also for general non-ideal solutions provided the Gibbs energy is strongly convex. Note also that the reduction to $n-1$ species is the common approach in the engineering literature, but invertibility of $\mathbf{B}$ is not rigorously checked. For $n=4$, the 3×3-matrix $\mathbf{B}$ can still be shown to be invertible for any composition due to $x_i \geq 0$ and $\sum_i x_i = 1$. Normal ellipticity can no longer be seen so easily. For general n this approach is not feasible and the invariant approach below is preferable.

Valuable references for the Maxwell-Stefan equations and there applications in the Engineering Sciences are in particular the books [4], [9], [20] and the review article [12].

5. Wellposedness of the Maxwell-Stefan equations

We first invert the Maxwell-Stefan equations using an invariant formulation. For this purpose, recall that $\sum_i u_i = 0$ holds for both $u_i = \mathbf{J}_i$ and $u_i = \mathbf{d}_i$. We therefore have to solve

$$A\,\mathbf{J} = c_{\mathrm{tot}}\,\mathbf{d} \qquad \text{in } E = \{u \in \mathbb{R}^n : \sum_i u_i = 0\}, \tag{30}$$

where $A = A(\mathbf{x})$ is given by

$$A = \begin{bmatrix} -s_1 & & d_{ij} \\ & \ddots & \\ d_{ij} & & -s_n \end{bmatrix} \qquad \text{with } s_i = \sum_{k\neq i} \frac{x_k}{Đ_{ik}}, \quad d_{ij} = \frac{x_i}{Đ_{ij}}.$$

The matrix A has the following properties, where $\mathbf{x} \gg 0$ means $x_i > 0$ for all i:

(i) $N(A) = \mathrm{span}\{\mathbf{x}\}$ for $\mathbf{x} = (x_1, \ldots, x_n)$.

(ii) $R(A) = \{\mathbf{e}\}^\perp$ for $\mathbf{e} = (1, \ldots, 1)$.

(ii) $A = [a_{ij}]$ is quasi-positive, i.e., $a_{ij} \geq 0$ for $i \neq j$.

(iv) If $\mathbf{x} \gg 0$ then A is irreducible, i.e., for every disjoint partition $I \cup J$ of $\{1, \ldots, n\}$ there is some $(i,j) \in I \times J$ such that $a_{ij} \neq 0$.

Due to (i) and (ii) above, the Perron-Frobenius theorem in the version for quasi-positive matrices applies; cf. [10] or [17]. This yields the following properties of the spectrum $\sigma(A)$: The spectral bound $s(A) := \max\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}$ is an eigenvalue of A, it is in fact a simple eigenvalue with a strictly positive eigenvector. All other eigenvalues do not have positive eigenvectors or positive generalized eigenvectors. Moreover,

$$\mathrm{Re}\,\lambda < s(A) \qquad \text{for all } \lambda \in \sigma(A),\ \lambda \neq s(A).$$

From now on we assume that in the present case $\mathbf{x}$ is strictly positive. Then, since $\mathbf{x}$ is an eigenvector to the eigenvalue 0, it follows that

$$\sigma(A) \subset \{0\} \cup \{z \in \mathbb{C} : \mathrm{Re}\,z < 0\}.$$

Unique solvability of (30) already follows at this point. In addition, the same arguments applied to $A_\mu := A - \mu(\mathbf{x} \otimes \mathbf{e})$ for $\mu \in \mathbb{R}$ yield

$$\sigma(A_\mu) \subset \{-\mu\} \cup \{z \in \mathbb{C} : \mathrm{Re}\,z < -\mu\} \qquad \text{for all small } \mu > 0.$$

In particular, A_μ is invertible for sufficiently small $\mu > 0$ and

$$\mathbf{J} = -c_{\mathrm{tot}} \big(A - \mu(\mathbf{x} \otimes \mathbf{e})\big)^{-1} \mathbf{d} \tag{31}$$

is the unique solution of (30). Note that $A_\mu \mathbf{y} = \mathbf{d}$ with $\mathbf{d} \perp \mathbf{e}$ implies $\mathbf{y} \perp \mathbf{e}$ and $A\mathbf{y} = \mathbf{d}$. A similar representation of the inverted Maxwell-Stefan equations can be found in [8].

The information on the spectrum of A can be significantly improved by symmetrization. For this purpose let $X = \operatorname{diag}(x_1, \dots x_n)$ which is regular due to $\mathbf{x} \gg 0$. Then $A_S := X^{-\frac{1}{2}} A X^{\frac{1}{2}}$ satisfies

$$A_S = \begin{bmatrix} -s_1 & & \hat{d}_{ij} \\ & \ddots & \\ \hat{d}_{ij} & & -s_n \end{bmatrix}, \quad s_i = \sum_{k \neq i} \frac{x_k}{Đ_{ik}}, \quad \hat{d}_{ij} = \frac{\sqrt{x_i x_j}}{Đ_{ij}},$$

i.e., A_S is symmetric with $N(A_S) = \operatorname{span}\{\sqrt{\mathbf{x}}\}$, where $\sqrt{\mathbf{x}}_i := \sqrt{x_i}$. Hence the spectrum of A_S and, hence, that of A is real. Moreover,

$$A_S(\alpha) = A_S - \alpha\sqrt{\mathbf{x}} \otimes \sqrt{\mathbf{x}}$$

has the same properties as A_S for sufficiently small $\alpha > 0$. In particular, A_S is quasi-positive, irreducible and $\sqrt{\mathbf{x}} \gg 0$ is an eigenvector for the eigenvalue $-\alpha$. This holds for all $\alpha < \delta := \min\{1/Đ_{ij} : i \neq j\}$. Hence we obtain the improved inclusion

$$\sigma(A) \setminus \{0\} = \sigma(A_S(\alpha)) \setminus \{-\alpha\} \text{ for all } \alpha \in [0, \delta).$$

Therefore

$$\sigma(A) \subset (-\infty, -\delta] \cup \{0\}, \tag{32}$$

which provides a uniform spectral gap for A sufficient to obtain normal ellipticity of the associated differential operator.

In order to work in a subspace of the composition space $\mathbb{R}^n$ instead of a hyperplane, let $u_i = c_i - c^0_{\text{tot}}/n$ such that $\sum_i c_i \equiv$ const is the same as $u \in E = \{u \in \mathbb{R}^n : \sum_i u_i = 0\}$. Above we have shown in particular that $A_{|E} : E \to E$ is invertible and

$$[\mathbf{J}_i] = X^{\frac{1}{2}} (A_{S|\hat{E}})^{-1} X^{-\frac{1}{2}} [\mathbf{d}_i] = \frac{1}{RT} X^{\frac{1}{2}} (A_{S|\hat{E}})^{-1} X^{\frac{1}{2}} [\nabla \mu_i] \tag{33}$$

with the symmetrized form A_S of A and $\hat{E} := X^{\frac{1}{2}} E = \{\sqrt{\mathbf{x}}\}^\perp$. Note that this also shows the consistency with the Onsager relations. To proceed, we employ (14) to obtain the representation

$$[\mathbf{J}_i] = X^{\frac{1}{2}} (A_{S|\hat{E}})^{-1} X^{\frac{1}{2}} G''(\mathbf{x}) \nabla \mathbf{x}. \tag{34}$$

Inserting (34) into (9) and using $c_{\text{tot}} x_i = u_i + c^0_{\text{tot}}/n$, we obtain the system of species equations with multicomponent diffusion modeled by the Maxwell-Stefan equations. Without chemical reactions and in an isolated domain $\Omega \subset \mathbb{R}^n$ (with ν the outer normal) we obtain the initial boundary value problem

$$\partial_t u + \operatorname{div}(-\mathbf{D}(u) \nabla u) = 0, \quad \partial_\nu u_{|\partial\Omega} = 0, \; u_{|t=0} = u_0, \tag{35}$$

which we will consider in $L^p(\Omega; E)$. Note that $X^{\frac{1}{2}} (A_{S|\hat{E}})^{-1} X^{\frac{1}{2}} G''(\mathbf{x})$ from (34) corresponds to $-\mathbf{D}(u)$ here.

Applying well-known results for quasilinear parabolic systems based on L_p-maximal regularity, e.g., from [3] or [15], we obtain the following result on local-in-time wellposedness of the Maxwell-Stefan equations in the isobaric, isothermal case. Below we call $G \in C^2(V)$ strongly convex if $G''(\mathbf{x})$ is positive definite for all $\mathbf{x} \in V$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ be open bounded with smooth $\partial\Omega$. Let $p > \frac{N+2}{2}$ and $u_0 \in W_p^{2-\frac{2}{p}}(\Omega; E)$ such that $c_i^0 > 0$ in $\bar{\Omega}$ and $c_{\rm tot}^0$ is constant in Ω. Let the diffusion matrix $\mathbf{D}(u)$ be given according to (34), i.e., by*

$$\mathbf{D}(u) = X^{\frac{1}{2}}(A_{S|\hat{E}})^{-1} X^{\frac{1}{2}} G''(\mathbf{x}) \;\; \textit{with } c_{\rm tot} x_i = u_i + c_{\rm tot}^0/n,$$

where $G : (0,\infty)^n \to \mathbb{R}$ is smooth and strongly convex. Then there exists – locally in time – a unique strong solution (in the L^p-sense) of (35). This solution is in fact classical.

Concerning the proof let us just mention that

$$\operatorname{div}(-\mathbf{D}(u)\nabla u) = \mathbf{D}(u)\,(-\Delta u) \; + \text{ lower order terms},$$

hence the system of Maxwell-Stefan equations is locally-in-time wellposed if the principal part $\mathbf{D}(u)\,(-\Delta u)$ is normally elliptic for all $u \in E$ such that $\mathbf{c}(u) := u + c_{\rm tot}^0 \mathbf{e}$ is close to $\mathbf{c}^0$. The latter holds if, for some angle $\theta \in (0, \frac{\pi}{2})$, the spectrum of $\mathbf{D}(u) \in \mathcal{L}(E)$ satisfies

$$\sigma(\mathbf{D}(u)) \subset \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\} \tag{36}$$

for all $u \in E$ such that $|\mathbf{c}(u) - \mathbf{c}^0|_\infty < \epsilon$ for $\epsilon := \min_i c_i^0/2$, say. For such an $u \in E$, let $\lambda \in \mathbb{C}$ and $v \in E$ be such that $\mathbf{D}(u)\,v = \lambda v$. Let $\mathbf{x} := \mathbf{c}(u)/c_{\rm tot}(u) \in (0,\infty)^n$ and $X = \operatorname{diag}(x_1, \ldots, x_n)$. Then

$$X^{\frac{1}{2}}(A_{S|\hat{E}})^{-1} X^{\frac{1}{2}} G''(\mathbf{x})\, v = \lambda v.$$

Taking the inner product with $G''(\mathbf{x})\,v$ yields

$$\langle (A_{S|\hat{E}})^{-1} X^{\frac{1}{2}} G''(\mathbf{x})\, v, X^{\frac{1}{2}} G''(\mathbf{x})\, v\rangle = \lambda \langle v, G''(\mathbf{x})\, v\rangle.$$

Note that $X^{\frac{1}{2}} G''(\mathbf{x})\, v \in \{\sqrt{\mathbf{x}}\}^\perp$, hence the left-hand side is strictly positive due to the analysis given above. Moreover $\langle v, G''(\mathbf{x})\, v\rangle > 0$ since G is strongly convex, hence $\lambda > 0$. This implies (36) for any $\theta \in (0, \frac{\pi}{2})$ and, hence, local-in-time existence follows.

6. Final remarks

A straightforward extension of Theorem 1 to the inhomogeneous case with locally Lipschitz continuous right-hand side $f : \mathbb{R}^n \to \mathbb{R}^n$, say, is possible if $f(u) \in E$ holds for all u. Translated back to the original variables (keeping the symbol f) this yields a local-in-time solution of

$$\partial_t \mathbf{c} + \operatorname{div}(-\mathbf{D}(\mathbf{c})\nabla \mathbf{c}) = f(\mathbf{c}), \qquad \partial_\nu \mathbf{c}_{|\partial\Omega} = 0, \quad \mathbf{c}_{|t=0} = \mathbf{c}_0$$

for appropriate initial values $\mathbf{c}_0$. Then a natural question is whether the solution stays componentwise nonnegative. This can only hold if f satisfies

$$f_i(\mathbf{c}) \geq 0 \quad \text{whenever} \quad \mathbf{c} \geq 0 \text{ with } c_i = 0,$$

which is called *quasi-positivity* as in the linear case. In fact, under the considered assumption, quasi-positivity of f forces any classical solution to stay nonnegative as long as it exists. The key point here is the structure of the Maxwell-Stefan equations (18) which yields

$$\mathbf{J}_i = -D_i(\mathbf{c}) \operatorname{grad} c_i + c_i \mathbf{F}_i(\mathbf{c}, \operatorname{grad} \mathbf{c})$$

with

$$D_i(\mathbf{c}) = 1 / \sum_{j \neq i} \frac{x_j}{Đ_{ij}} \quad \text{and} \quad \mathbf{F}_i(\mathbf{c}, \operatorname{grad} \mathbf{c}) = D_i(\mathbf{c}) \sum_{j \neq i} \frac{1}{Đ_{ij}} \mathbf{J}_j.$$

Note that $D_i(\mathbf{c}) > 0$ and $\mathbf{J}_i$ becomes proportional to $\operatorname{grad} c_i$ at points where c_i vanishes, i.e., the diffusive cross-effects disappear. Moreover, it is easy to check that

$$\operatorname{div} \mathbf{J}_i = D_i(\mathbf{c}) \Delta c_i \geq 0 \quad \text{if } c_i = 0 \text{ and } \operatorname{grad} c_i = 0.$$

To indicate a rigorous proof for the nonnegativity of solutions, consider the modified system

$$\partial_t c_i + \operatorname{div} \mathbf{J}_i(\mathbf{c}) = f_i(t, \mathbf{c}^+) + \epsilon, \qquad \partial_\nu \mathbf{c}_{|\partial\Omega} = 0, \quad \mathbf{c}_{|t=0} = \mathbf{c}_0 + \epsilon \mathbf{e}, \tag{37}$$

where $r^+ := \max\{r, 0\}$ denotes the positive part. Assume that the right-hand side f is quasi-positive and that (37) has a classical solution $\mathbf{c}^\epsilon$ for all small $\epsilon > 0$ on a common time interval $[0, T)$. Now suppose that, for some i, the function $m_i(t) = \min_{\mathbf{y} \in \bar{\Omega}} c_i^\epsilon(t, \mathbf{y})$ has a first zero at $t_0 \in (0, T)$. Let the minimum of $c_i^\epsilon(t_0, \cdot)$ be attained at $\mathbf{y}_0$ and assume first that $\mathbf{y}_0$ is an interior point. Then $c_i^\epsilon(t_0, \mathbf{y}_0) = 0$, $\partial_t c_i^\epsilon(t_0, \mathbf{y}_0) \geq 0$, $\operatorname{grad} c_i^\epsilon(t_0, \mathbf{y}_0) = 0$ and $\Delta c_i^\epsilon(t_0, \mathbf{y}_0) \geq 0$ yields a contradiction since $f_i(t_0, c_i^\epsilon(t_0, \mathbf{y}_0)) \geq 0$. Here, because of the specific boundary condition and the fact that Ω has a smooth boundary, the same argument works also if $\mathbf{y}_0$ is a boundary point. In the limit $\epsilon \to 0+$ we obtain a nonnegative solution for $\epsilon = 0$, hence a nonnegative solution of the original problem. This finishes the proof since strong solutions are unique.

Note that non-negativity of the concentrations directly implies L^∞-bounds in the considered isobaric case due to $0 \leq c_i \leq c_{\text{tot}} \equiv c_{\text{tot}}^0$, which is an important first step for global existence.

The considerations in Section 5 are helpful to verify that the Maxwell-Stefan multicomponent diffusion is consistent with the second law from thermodynamics. Indeed, (33) directly yields

$$-[\mathbf{J}_i] : [\nabla \mu_i] = \frac{1}{RT} \Big((-A_{S|\hat{E}})^{-1} X^{\frac{1}{2}} [\nabla \mu_i] \Big) : \Big(X^{\frac{1}{2}} [\nabla \mu_i] \Big) \geq 0,$$

i.e., the entropy inequality is satisfied. The latter is already well known in the engineering literature, but with a different representation of the dissipative term using the individual velocities; cf. [18].

For sufficiently regular solutions and under appropriate boundary conditions the entropy inequality can be used as follows. Let $V(\mathbf{x}) = \int_\Omega G(\mathbf{x})\,d\mathbf{x}$ with G the Gibbs free energy density. Let

$$W(\mathbf{x}, \nabla\mathbf{x}) = -\int_\Omega [\mathbf{J}_i] : [\nabla\mu_i]\,d\mathbf{x} \geq 0.$$

Then (V, W) is a Lyapunov couple, i.e.,

$$V(\mathbf{x}(t)) + \int_0^t W(\mathbf{x}(s), \nabla\mathbf{x}(s))\,ds \leq V(\mathbf{x}(0)) \text{ for } t > 0$$

and all sufficiently regular solutions. For ideal systems this yields a priori bounds on the quantities $|\nabla c_i|^2/c_i$, hence, equivalently, L_2-bounds on $\nabla\sqrt{c_i}$. This type of a priori estimates is well known in the theory of reaction-diffusion systems without cross-diffusion; see [5], [6] and the references given there for more details.

In the present paper we considered the isobaric and isothermal case because it allows to neglect convective transport and, hence, provides a good starting point. The general case of a multicomponent flow is much more complicated, even in the isothermal case. This case leads to a Navier-Stokes-Maxwell-Stefan system which will be studied in future work.

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Global Existence vs. Blowup in a One-dimensional Smoluchowski-Poisson System

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Dedicated to Herbert Amann

Abstract. We prove that, unlike in several space dimensions, there is no critical (nonlinear) diffusion coefficient for which solutions to the one-dimensional quasilinear Smoluchowski-Poisson equation with small mass exist globally while finite time blowup could occur for solutions with large mass.

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1. Introduction

In a previous paper [4] we investigate the influence of the diffusion coefficient a on the life span of solutions to the one-dimensional Smoluchowski-Poisson system

$$\partial_t u = \partial_x \left(a(u)\partial_x u - u\partial_x v\right) \quad \text{in} \quad (0,\infty)\times(0,1), \tag{1.1}$$

$$0 = \partial_x^2 v + u - M \quad \text{in} \quad (0,\infty)\times(0,1), \tag{1.2}$$

$$a(u)\partial_x u = \partial_x v = 0 \quad \text{on} \quad (0,\infty)\times\{0,1\}, \tag{1.3}$$

$$u(0) = u_0 \geq 0 \quad \text{in} \quad (0,1), \quad \int_0^1 v(t,x)dx = 0 \quad \text{for any} \quad t\in(0,\infty), \tag{1.4}$$

where

$$M := \langle u_0 \rangle = \int_0^1 u_0(x)dx$$

denotes the mean value of u_0, and uncover a fundamental difference with the quasilinear Smoluchowski-Poisson system in higher space dimensions. More precisely, when the space dimension n is greater or equal to two, there is a critical diffusion

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$a_*(r) := (1+r)^{(n-2)/n}$ which separates different behaviours for the quasilinear Smoluchowski-Poisson system. Roughly speaking,

(a) if the diffusion coefficient a is stronger than a_* (in the sense that $a(r) \geq C(1+r)^\alpha$ for some $\alpha > (n-2)/n$ and $C > 0$), then all solutions exist globally whatever the value of the mass of the initial condition u_0 [5],
(b) if the diffusion coefficient a is weaker than a_* (in the sense that $a(r) \leq C(1+r)^\alpha$ for some $\alpha < (n-2)/n$ and $C > 0$), then there exists for all $M > 0$ an initial condition u_0 with $\langle u_0 \rangle = M$ for which the corresponding solution to the quasilinear Smoluchowski-Poisson system blows up in finite time (in the sense that $\|u(t)\|_{L^\infty(0,M)} \to \infty$ as $t \to T$ for some $T \in (0,\infty)$) [3, 5, 7],
(c) if the diffusion coefficient a behaves as a_* for large values of r, solutions starting from initial data u_0 with small mass $\langle u_0 \rangle$ exist globally while there are initial data with large mass for which the corresponding solution to the quasilinear Smoluchowski-Poisson system blows up in finite time [3, 7].

Observe that, in space dimension $n = 2$, the critical diffusion is constant and a more precise description of the situation **(c)** is actually available. Namely, when $a \equiv 1$, there is a threshold mass M_* such that, if $\langle u_0 \rangle < M_*$, the corresponding solution is global while, for any $M > M_*$, there are initial data with $\langle u_0 \rangle = M$ for which the corresponding solution blows up in finite time [6, 7, 8]. The threshold mass M_* is known explicitly ($M_* = 4\pi$) but it is worth mentioning that for radially symmetric solutions in a ball, the threshold mass is 8π. Similar results are also available for the quasilinear Smoluchowski-Poisson system in R^n, $n \geq 2$ [1, 2, 9, 10].

Most surprisingly, the above description fails to be valid in one space dimension and we prove in particular in [4] that all solutions are global for the diffusion $a(r) = (1+r)^{-1}$ though it is a natural candidate to be critical. We actually identify two classes of diffusion coefficients a in [4], one for which all solutions exist globally as in **(a)** and the other for which there are solutions blowing up in finite time starting from initial data with an arbitrary positive mass as in **(b)**, but the situation **(c)** does not seem to occur in one space dimension. The purpose of this note is to show that the dichotomy **(a)** or **(b)** can be extended to larger classes of diffusion, thereby extending the analysis performed in [4].

Theorem 1.1. *Let the diffusion coefficient $a \in C^1((0,\infty))$ be a positive function.*

(i) *Assume first that $a \in L^1(1,\infty)$ and one of the following assumptions is satisfied, either*

$$\gamma := \sup_{r\in(0,1)} r \int_r^\infty a(s)ds < \infty, \tag{1.5}$$

or there exist $\vartheta > 0$ and $\alpha \in (\vartheta/(1+\vartheta), 2]$ such that

$$\gamma_\vartheta := \sup_{r\in(0,1)} r^{2+\vartheta} a(r) < \infty \quad \text{and} \quad C_\infty := \sup_{r\geq 1} r^\alpha a(r) < \infty. \tag{1.6}$$

For any $M > 0$, there exists a positive initial condition $u_0 \in C([0,1])$ such that $\langle u_0 \rangle = M$ and the corresponding classical solution to (1.1)–(1.4) *blows up in finite time.*

(ii) *Assume next that* $a \notin L^1(1,\infty)$ *and consider an initial condition* $u_0 \in C([0,1])$ *such that* $u_0 \geq m_0 > 0$ *and* $\langle u_0 \rangle = M$ *for some* $M > 0$ *and* $m_0 \in (0,M)$. *Then the corresponding classical solution to* (1.1)–(1.4) *exists globally.*

As already mentioned, Theorem 1.1 extends the results obtained in [4]. More precisely, in [4, Theorem 5], the assertion (ii) of Theorem 1.1 is proved under the additional assumption that, for each $\varepsilon \in (0,\infty)$, there is $\kappa_\varepsilon > 0$ for which

$$a(r) \leq \varepsilon\, ra(r) + \frac{\kappa_\varepsilon}{r} \quad \text{for} \quad r \in (0,1)\,,$$

which roughly means that a cannot have a singularity stronger than $1/r$ near $r = 0$. This assumption turns out to be unnecessary for global existence but nevertheless ensures the global boundedness of the solution in L^∞. Under the sole assumption of Theorem 1.1 (ii), our proof does not exclude that the solution to (1.1)–(1.4) becomes unbounded as $t \to \infty$. Concerning Theorem 1.1 (i), it is established in [4, Theorem 10] for $a \in L^1(1,\infty)$ such that there is a concave function B for which

$$0 \leq -rA(r) \leq B(r) \quad \text{with} \quad A(r) = -\int_r^\infty a(s)\, ds\,, \qquad r \in (0,\infty)\,, \tag{1.7}$$

$$\lim_{r\to\infty} \frac{B(r)}{r} = 0\,. \tag{1.8}$$

We make this criterion more explicit here by showing that the integrability of a on $(1,\infty)$ and (1.5) guarantee the existence of a concave function B satisfying (1.7) and (1.8), see Lemma 3.1 below. Let us point out here that the assumption (1.5) somehow means that a cannot have a singularity stronger that $1/r^2$ near $r = 0$. However, the result remains true if a has an algebraic singularity of higher order near $r = 0$ which is allowed by (1.6) provided a decays suitably at infinity. Observe that the second condition in (1.6) is compatible with the integrability of a at infinity as $\vartheta/(1+\vartheta) < 1$.

Summarizing the outcome of Theorem 1.1, we realize that, for a given diffusion coefficient a with a singularity weaker than $1/r^2$ near $r = 0$, the integrability or non-integrability of a at infinity completely determines whether we are in the situation **(a)** or **(b)** described above and excludes the situation **(c)**. There is thus no critical diffusion in this class. The same comment applies to the class of diffusion coefficients satisfying (1.6) with an algebraic singularity stronger than $1/r^2$ near $r = 0$. In particular there is no critical nonlinearity in the class of functions $C([0,\infty)) \cap C^1((0,\infty))$.

The paper is organized as follows: in Section 2 we recall some statements from [4]. Section 3 is devoted to proving the finite time blowup of solutions to (1.1)–(1.4) when $a \in L^1(1,\infty)$. Global existence of solutions for all initial data when a is not integrable at infinity is proved in the last section.

2. Preliminaries

In this section we summarize some results and methods introduced in [4]. Let $a \in C^1((0,\infty))$ be a positive function and consider an initial condition $u_0 \in C([0,1])$ such that $u_0 \geq m_0 > 0$ and $\langle u_0 \rangle = M$ for some $M > 0$ and $m_0 \in (0,M)$. By [4, Propositions 2 and 3] there is a unique maximal classical solution (u,v) to (1.1)–(1.4) defined on $[0,T_{\max})$ which satisfies

$$\min_{x\in[0,1]} u(t,x) > 0\,, \quad \langle u(t)\rangle := \int_0^1 u(t,x)\ dx = M\,, \quad \text{and} \tag{2.1}$$
$$\langle v(t)\rangle := \int_0^1 v(t,x)\ dx = 0$$

for $t \in (0,T_{\max})$. In addition, $T_{\max} = \infty$ or $T_{\max} < \infty$ with $\|u(t)\|_{L^\infty(0,M)} \to \infty$ as $t \to T_{\max}$.

We next recall the approach introduced in [4] which will be used herein as well. Owing to the positivity (2.1) and the regularity of u, the indefinite integral

$$U(t,x) := \int_0^x u(t,z)dz\,, \quad x \in [0,1]\,,$$

is a smooth increasing function from $[0,1]$ onto $[0,M]$ for each $t \in [0,T_{\max})$ and has a smooth inverse F defined by

$$U(t,F(t,y)) = y\,, \qquad (t,y) \in [0,T_{\max}) \times [0,M]\,. \tag{2.2}$$

Introducing $f(t,y) := \partial_y F(t,y)$, we have

$$f(t,y)\ u(t,F(t,y)) = 1\,, \qquad (t,y) \in [0,T_{\max}) \times [0,M]\,, \tag{2.3}$$

and it follows from (1.1)–(1.4) that f solves

$$\partial_t f = \partial_y^2 \Psi(f) - 1 + Mf\,, \qquad (t,y) \in (0,T_{\max}) \times (0,M)\,, \tag{2.4}$$
$$\partial_y f(t,0) = \partial_y f(t,M) = 0\,, \qquad t \in (0,T_{\max})\,, \tag{2.5}$$
$$f(0,y) = f_0(y) := \frac{1}{u_0(F(0,y))}\,, \qquad y \in (0,M)\,, \tag{2.6}$$

where

$$\Psi'(r) := \frac{1}{r^2}\ a\left(\frac{1}{r}\right) \ \text{ for any } \ r > 0\,, \qquad \Psi(1) := 0\,. \tag{2.7}$$

Moreover the conservation of mass (2.1) yields

$$\int_0^M f(t,y)dy = F(t,M) - F(t,0) = 1\,, \qquad t \in [0,T_{\max})\,. \tag{2.8}$$

At this point, the crucial observation is that, thanks to (2.3), finite time blowup of u is equivalent to the vanishing (or touch-down) of f in finite time. In other words, u exists globally if the minimum of $f(t)$ is positive for each $t > 0$. We refer to [4, Proposition 1] for a more detailed description.

A salient property of (1.1)–(1.4) is the existence of a Liapunov function [4, Lemma 8] which we recall now:

Lemma 2.1. *The function*

$$L_1(t) := \frac{1}{2} \int_0^M |\partial_y \Psi(f(t,y))|^2 \ dy + \int_0^M (\Psi(f(t,y)) - M \ \Psi_1(f(t,y))) \ dy$$

is a non-increasing function of time on $[0, T_{\max})$, *the function* Ψ_1 *being defined by*

$$\Psi_1(1) := 0 \quad \textit{and} \quad \Psi_1'(r) := r\Psi'(r) = \frac{1}{r} \ a\left(\frac{1}{r}\right), \qquad r \in (0,\infty). \tag{2.9}$$

3. Finite time blowup

In this section we prove the blowup assertion of Theorem 1.1. To this end we first prove that the condition (1.5) allows us to construct a concave function B satisfying (1.7) and (1.8) so that [4, Theorem 10] can be applied.

Lemma 3.1. *Let* $a \in C^1((0,\infty))$ *be a positive function such that* $a \in L^1(1,\infty)$ *and* (1.5) *holds. Then there exists a concave function* $B \in C([0,\infty))$ *such that for all* $r \geq 0$

$$B(r) \geq r \int_r^\infty a(s) ds \tag{3.1}$$

and

$$\lim_{r\to\infty} \frac{B(r)}{r} = 0. \tag{3.2}$$

Proof of Lemma 3.1. We construct $B : [0,\infty) \to [0,\infty)$ in the following way: we put

$$b_i := \int_{2^i}^\infty a(s) ds, \qquad i \geq 0,$$

and notice that $\{b_i\}_{i\geq 0}$ is a decreasing sequence converging to zero as $i \to \infty$. We next define

$$B(r) = \begin{cases} b_0 r + \gamma & \text{if} \quad r \in [0,2], \\ b_i r + \displaystyle\sum_{j=0}^{i-1} (b_j - b_{j+1}) 2^{j+1} + \gamma & \text{if} \quad r \in (2^i, 2^{i+1}] \ \text{ and } \ i \geq 1. \end{cases} \tag{3.3}$$

Clearly, $B \in C([0,\infty))$ and

$$B'(r) = \begin{cases} b_0 & \text{if} \quad r \in (0,2), \\ b_i & \text{if} \quad r \in (2^i, 2^{i+1}) \ \text{ and } \ i \geq 1. \end{cases} \tag{3.4}$$

Hence B is concave as a consequence of the fact that the sequence $\{b_i\}_{i\geq 0}$ is decreasing. Furthermore, for $r \in [0,1]$, we have

$$B(r) \geq \gamma \geq r \int_r^\infty a(s) ds,$$

and for $r \in [2^i, 2^{i+1}]$, $i \geq 0$,

$$B(r) \geq b_i r = r \int_{2^i}^\infty a(s) ds \geq r \int_r^\infty a(s) ds.$$

Therefore, B satisfies (3.1).

Finally, let $k \geq 1$. If $i \geq k+1$ and $r \in (2^i, 2^{i+1}]$, then

$$\begin{aligned}
\frac{B(r)}{r} &= b_i + \frac{\gamma}{r} + \sum_{j=0}^{i-1} (b_j - b_{j+1}) \frac{2^{j+1}}{r} \\
&\leq b_i + \frac{\gamma}{r} + \sum_{j=k}^{i-1} (b_j - b_{j+1}) + \sum_{j=0}^{k-1} (b_j - b_{j+1}) \frac{2^{j+1}}{r} \\
&\leq b_i + \frac{1}{r} \left(\gamma + 2^k \sum_{j=0}^{k-1} (b_j - b_{j+1}) \right) + (b_k - b_i) \\
&\leq b_k + \frac{1}{r} \left(\gamma + 2^k b_0 \right).
\end{aligned}$$

Consequently,

$$\limsup_{r \to \infty} \frac{B(r)}{r} \leq b_k \text{ for all } k \geq 1\,.$$

Letting $k \to \infty$, we obtain (3.2) since $b_k \to 0$ as $k \to \infty$ and Lemma 3.1 is proved. □

Proof of Theorem 1.1 (i)*, Part* 1. When a belongs to $L^1(1,\infty)$ and satisfies (1.5), it follows from Lemma 3.1 that the conditions (1.7) and (1.8) are satisfied so that Theorem 1.1 (i) follows from [4, Theorem 10]. □

To handle the other case, we proceed in a different way by showing an upper bound for the function f defined in Section 2. We first observe that the function Ψ defined in (2.7) satisfies

$$\Psi(r) = \int_1^r \frac{1}{s^2} a\left(\frac{1}{s}\right)\, ds = \int_{1/r}^1 a(s)\, ds\,, \quad r \in (0,\infty)\,,$$

so that, if $a \in L^1(1,\infty)$, $\Psi(r)$ has a finite limit $\Psi(0) := -\|a\|_{L^1(1,\infty)}$ as $r \to 0$. We then define

$$\tilde{\Psi}(r) := \Psi(r) - \Psi(0) = \int_0^r \frac{1}{s^2} a\left(\frac{1}{s}\right)\, ds = \int_{1/r}^\infty a(s)\, ds\,, \quad r \in (0,\infty)\,. \tag{3.5}$$

Lemma 3.2. *Let $a \in C^1((0,\infty))$ be a positive function such that $a \in L^1(1,\infty)$. There exists a positive constant $\mu_M > 0$ depending only on M and a such that, for any non-negative function $g \in H^1(0,M)$ satisfying $\|g\|_{L^1(0,M)} = 1$, we have*

$$\|\tilde{\Psi}(g)\|^2_{L^\infty(0,M)} \leq 32 M \mathcal{L}_1(g) + \mu_M\,, \tag{3.6}$$

with

$$\mathcal{L}_1(g) := \frac{1}{2}\|\partial_y \Psi(g)\|^2_{L^2(0,M)} + \int_0^M \left(\Psi(g) - M\Psi_1(g)\right)(y)\ dy\,, \tag{3.7}$$

the functions Ψ *and* Ψ_1 *being defined in* (2.7) *and* (2.9)*, respectively.*

Proof of Lemma 3.2. We set $G := \|g\|_{L^\infty(0,M)}$ which is finite owing to the continuous embedding of $H^1(0,M)$ in $L^\infty(0,M)$. Assume first that $G > 1$. Then, for $y \in (0,M)$ and $z \in (0,M)$, we have

$$\tilde\Psi(g(y)) = \tilde\Psi(g(z)) + \int_z^y \partial_x \tilde\Psi(g(x))\ dx \le \tilde\Psi(g(z)) + M^{1/2}\|\partial_y \Psi(g)\|_{L^2(0,M)}.$$

Integrating the above inequality over $(0,M)$ with respect to z gives

$$\begin{aligned}
M\tilde\Psi(g(y)) &\le \int_0^M \tilde\Psi(g(z))\ dz + M^{3/2}\|\partial_y\Psi(g)\|_{L^2(0,M)} \\
&\le \int_0^M \mathbf{1}_{[0,2/M]}(g(z))\tilde\Psi(g(z))dz \\
&\quad + \int_0^M \mathbf{1}_{(2/M,\infty)}(g(z))\tilde\Psi(g(z))dz + M^{3/2}\|\partial_y\Psi(g)\|_{L^2(0,M)} \\
&\le M\tilde\Psi\left(\frac{2}{M}\right) + \frac{M\tilde\Psi(G)}{2} \\
&\quad \times \int_0^M \mathbf{1}_{(2/M,\infty)}(g(z))g(z)\ dz + M^{3/2}\|\partial_y\Psi(g)\|_{L^2(0,M)} \\
&\le M\tilde\Psi\left(\frac{2}{M}\right) + \frac{M\tilde\Psi(G)}{2} + M^{3/2}\|\partial_y\Psi(g)\|_{L^2(0,M)}\,,
\end{aligned}$$

where we have used the property $\|g\|_{L^1(0,M)} = 1$ to obtain the last inequality. Taking the supremum over $y \in (0,M)$ and using the monotonicity and non-negativity of $\tilde\Psi$, we deduce that

$$\tilde\Psi(G) \le 2\tilde\Psi\left(\frac{2}{M}\right) + 2M^{1/2}\|\partial_y\Psi(g)\|_{L^2(0,M)}. \tag{3.8}$$

We next observe that the integrability of a at infinity also ensures that $\Psi_1(0) > -\infty$, so that $\tilde\Psi_1 := \Psi_1 - \Psi_1(0)$ is well defined and satisfies

$$\tilde\Psi_1(r) = \int_0^r s\Psi'(s)\ ds \le r\tilde\Psi(r)\,, \quad r \in (0,\infty)\,. \tag{3.9}$$

Since $\|g\|_{L^1(0,M)} = 1$, it follows from (3.8) and (3.9) that

$$\begin{aligned}
\int_0^M \tilde\Psi_1(g)\ dy &\le \int_0^M g\tilde\Psi(g)\ dy \le \tilde\Psi(G)\ \int_0^M g dy \\
&\le 2\tilde\Psi\left(\frac{2}{M}\right) + 2M^{1/2}\|\partial_y\Psi(g)\|_{L^2(0,M)}.
\end{aligned} \tag{3.10}$$

We next infer from (3.10) and the non-negativity of $\tilde{\Psi}$ that

$$
\begin{aligned}
\mathcal{L}_1(g) &\geq \frac{1}{2}\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 + \int_0^M \tilde{\Psi}(g)\ dy + M\Psi(0) - M\int_0^M \tilde{\Psi}_1(g)\ dy \\
&\geq \frac{1}{2}\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 + M\Psi(0) - 2M\tilde{\Psi}\left(\frac{2}{M}\right) - 2M^{3/2}\|\partial_y \Psi(g)\|_{L^2(0,M)} \\
&\geq \frac{1}{4}\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 + \left(\frac{1}{2}\|\partial_y \Psi(g)\|_{L^2(0,M)} - 2M^{3/2}\right)^2 \\
&\quad - 4M^3 + M\Psi(0) - 2M\tilde{\Psi}\left(\frac{2}{M}\right) \\
&\geq \frac{1}{4}\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 - 4M^3 + M\Psi(0) - 2M\tilde{\Psi}\left(\frac{2}{M}\right),
\end{aligned}
$$

whence

$$
\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 \leq 4\mathcal{L}_1(g) + 16M^3 - 4M\Psi(0) + 8M\tilde{\Psi}\left(\frac{2}{M}\right).
$$

It then follows from (3.8) and the above inequality that

$$
\begin{aligned}
\tilde{\Psi}(G)^2 &\leq 8\tilde{\Psi}\left(\frac{2}{M}\right)^2 + 8M\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 \\
&\leq 8\tilde{\Psi}\left(\frac{2}{M}\right)^2 + 32M\mathcal{L}_1(g) + 128M^4 - 32M^2\Psi(0) + 64M^2\tilde{\Psi}\left(\frac{2}{M}\right) \\
&\leq 32M\mathcal{L}_1(g) + \mu_M,
\end{aligned}
$$

with

$$
\mu_M := 1 + 128M^4 - 32M^2\Psi(0) + 64M^2\tilde{\Psi}\left(\frac{2}{M}\right) + 8\tilde{\Psi}\left(\frac{2}{M}\right)^2 + \Psi(0)^2 - 32M\Psi(0).
$$

We have thus shown Lemma 3.2 when $G = \|g\|_{L^\infty(0,M)} > 1$. To complete the proof, we finally consider the case $G \in [0,1]$ and notice that, in that case,

$$
0 \leq \tilde{\Psi}(G) \leq -\Psi(0) \quad \text{and} \quad \mathcal{L}_1(g) \geq \int_0^M \tilde{\Psi}(g)\ dy + M\Psi(0) \geq M\Psi(0),
$$

since $\Psi_1 \leq 0$ in $(0,1)$ and $\tilde{\Psi} \geq 0$. Consequently,

$$
\tilde{\Psi}(G)^2 \leq \Psi(0)^2 = 32M\Psi(0) + \Psi(0)^2 - 32M\Psi(0) \leq 32M\mathcal{L}_1(g) + \mu_M,
$$

and the proof of Lemma 3.2 is complete. □

As an obvious consequence of Lemmas 2.1 and 3.2 we have the following result:

Corollary 3.3. *Let $a \in C^1((0,\infty))$ be a positive function such that $a \in L^1(1,\infty)$. For $t \in [0, T_{\max})$ and $y \in [0,M]$, we have*

$$
0 \leq \tilde{\Psi}(f(t,y)) \leq \left(32M\max\{\mathcal{L}_1(f_0), 0\} + \mu_M\right)^{1/2}.
$$

Proof of Corollary 3.3. Clearly

$$\mathcal{L}_1(f(t)) = L_1(t) \le L_1(0) = \mathcal{L}_1(f_0) \le \max\{\mathcal{L}_1(f_0), 0\}$$

for $t \in [0, T_{\max})$ by Lemma 2.1 and Corollary 3.3 readily follows from Lemma 3.2. □

Remark 3.4. Corollary 3.3 provides an L^∞-bound on f only if $\Psi(r) \to \infty$ as $r \to \infty$, that is, if $a \notin L^1(0,1)$. In that case, it gives a positive lower bound for u by (2.3).

We next turn to the proof of the second part of Theorem 1.1 for which we develop further the arguments from [4, Theorem 10].

Proof of Theorem 1.1 (i)*, Part* 2. Assume now that $a \in L^1(1,\infty)$ and satisfies (1.6). We fix $M > 0$, $q > 2$, and $\varepsilon_M \in (0,1)$ such that

$$q > \max\left\{3+\vartheta, \frac{5+3\vartheta}{\alpha(\vartheta+1)-\vartheta}\right\} \quad \text{and} \quad \frac{q(q+1)}{M^2}\int_{1/\varepsilon_M}^\infty a(s)\ ds \le \frac{1}{2}\,, \tag{3.11}$$

the existence of ε_M being guaranteed by the integrability of a at infinity.

For

$$\delta \in \left(0, \min\left\{1, 2M, (2M)^{-1/q}\right\}\right)\,, \tag{3.12}$$

we put

$$f_0(y) := \frac{2(1-M\delta^q)}{\delta^2}\ (\delta-y)_+ + \delta^q \ge \delta^q > 0\,, \quad y \in [0,M]\,. \tag{3.13}$$

Then

$$\int_0^M f_0(y)\ dy = 1, \quad \|f_0\|_{L^\infty(0,M)} = \frac{2(1-M\delta^q)}{\delta} + \delta^q \le \frac{2}{\delta}\,. \tag{3.14}$$

Denoting the corresponding solution to (2.4)–(2.6) by f we next introduce

$$m_q(t) := \int_0^M y^q f(t,y)\ dy\,, \quad t \in [0, T_{\max})\,,$$

and we have

$$m_q(0) = \left(\frac{2(1-M\delta^q)}{(q+1)(q+2)} + \frac{M^{q+1}}{q+1}\right)\ \delta^q \le C_1\delta^q \tag{3.15}$$

with

$$C_1 := \left(\frac{2+(q+2)M^{q+1}}{(q+1)(q+2)}\right)\,.$$

It follows from (2.4), (2.5), and the non-negativity of $\tilde\Psi$ that

$$\begin{aligned}\frac{dm_q}{dt} &= -q\ \int_0^M y^{q-1}\partial_y\tilde\Psi(f)\ dy + Mm_q - \frac{M^{q+1}}{q+1},\\ \frac{dm_q}{dt} &\le q(q-1)\ \int_0^M y^{q-2}\tilde\Psi(f)\ dy + Mm_q - \frac{M^{q+1}}{q+1}\\ &.\end{aligned} \tag{3.16}$$

We shall now estimate the integral on the right-hand side of (3.16): to this end, we split the domain of integration into three parts which we handle differently. As a preliminary step, we notice that, by (1.6),

$$\Psi'(r) \le \gamma_\vartheta r^\vartheta \quad \text{and} \quad \Psi(r) \le \frac{\gamma_\vartheta}{\vartheta+1}\, r^{\vartheta+1} \le \gamma_\vartheta\, r^{\vartheta+1}\,, \quad r \ge 1\,. \tag{3.17}$$

We next define

$$K_0 := \left(32M \max\{\mathcal{L}_1(f_0), 0\} + \mu_M\right)^{1/2(2+\vartheta)} > 1\,,$$

and consider $(t,y) \in [0, T_{\max}) \times [0, M]$.

- If $f(t,y) \in (0, \varepsilon_M]$, it follows from (3.11) and the monotonicity of $\tilde{\Psi}$ that

$$\tilde{\Psi}(f(t,y)) \le \tilde{\Psi}(\varepsilon_M) = \int_{1/\varepsilon_M}^\infty a(s)\, ds \le \frac{M^2}{2q(q+1)}\,. \tag{3.18}$$

- If $f(t,y) \in (\varepsilon_M, K_0)$, then (3.17) and the monotonicity of $\tilde{\Psi}$ yield

$$\begin{aligned} \tilde{\Psi}(f(t,y)) = \frac{\tilde{\Psi}(f(t,y))}{f(t,y)}\, f(t,y) &\le \frac{\Psi(K_0) - \Psi(0)}{\varepsilon_M}\, f(t,y) \\ &\le \frac{\gamma_\vartheta K_0^{\vartheta+1} - \Psi(0)}{\varepsilon_M}\, f(t,y)\,. \end{aligned} \tag{3.19}$$

- If $f(t,y) \ge K_0$, Corollary 3.3 ensures that

$$\tilde{\Psi}(f(t,y)) = \frac{\tilde{\Psi}(f(t,y))}{f(t,y)}\, f(t,y) \le \frac{K_0^{\vartheta+2}}{K_0}\, f(t,y) \le K_0^{\vartheta+1}\, f(t,y)\,. \tag{3.20}$$

Consequently, recalling that $K_0 > 1$ and $\Psi(0) < 0$, we deduce from (3.16) and (3.18)–(3.20) that

$$\begin{aligned} \frac{dm_q}{dt} &\le q(q-1) \int_0^M y^{q-2} \tilde{\Psi}(f)\, \mathbf{1}_{(0,\varepsilon_M]}(f)\, dy \\ &\quad + q(q-1) \int_0^M y^{q-2} \tilde{\Psi}(f)\, \mathbf{1}_{(\varepsilon_M, K_0)}(f)\, dy \\ &\quad + q(q-1) \int_0^M y^{q-2} \tilde{\Psi}(f)\, \mathbf{1}_{[K_0,\infty)}(f)\, dy + M m_q - \frac{M^{q+1}}{q+1} \\ &\le \frac{(q-1)M^2}{2(q+1)} \int_0^M y^{q-2}\, dy + q(q-1)\, \frac{\gamma_\vartheta K_0^{\vartheta+1} - \Psi(0)}{\varepsilon_M} \int_0^M y^{q-2} f\, dy \\ &\quad + q(q-1) K_0^{\vartheta+1} \int_0^M y^{q-2} f\, dy + M m_q - \frac{M^{q+1}}{q+1} \\ &\le C_2\, K_0^{\vartheta+1} \int_0^M y^{q-2} f\, dy + M m_q - \frac{M^{q+1}}{2(q+1)}\,, \end{aligned}$$

with $C_2 := q(q-1)(\gamma_\vartheta - \Psi(0) + \varepsilon_M)/\varepsilon_M$. We next use Hölder's inequality and (2.8) to conclude that

$$\frac{dm_q}{dt} \le C_2\ K_0^{\vartheta+1}\ m_q^{(q-2)/q} + Mm_q - \frac{M^{q+1}}{2(q+1)}\,. \tag{3.21}$$

It remains to estimate K_0 and in fact $\mathcal{L}_1(f_0)$. Since Ψ is negative on $(0,1)$ and Ψ_1 is bounded from below by $\Psi_1(0)$, it follows from (3.12) and (3.13) that

$$\begin{aligned}\mathcal{L}_1(f_0) &\le \frac{2}{\delta^4}\ (1-M\delta^q)^2\ \int_0^\delta |\Psi'(f_0)|^2\ dy + \int_0^\delta \Psi(f_0)\ dy - M^2\Psi_1(0)\\ &\le \frac{2}{\delta^4}\ \int_0^\delta |\Psi'(f_0)|^2\ dy + \int_0^\delta \Psi(f_0)\ dy - M^2\Psi_1(0).\end{aligned}$$

On the one hand, we infer from (3.14), (3.17), and the monotonicity of Ψ that

$$\int_0^\delta \Psi(f_0)\ dy \le \delta\ \Psi\left(\frac{2}{\delta}\right) \le \gamma_\vartheta 2^{\vartheta+1}\ \delta^{-\vartheta}\,.$$

On the other hand, we have

$$\begin{aligned}&f_0(y) \ge 1 \quad \text{for} \quad y\in[0,y_\delta] \quad \text{with} \quad y_\delta := \delta - \frac{1-\delta^q}{2(1-M\delta^q)}\delta^2 > 0\,,\\ &f_0(y)\in[\delta^q,1] \quad \text{for} \quad y\in[y_\delta,\delta]\,,\end{aligned}$$

so that, if $y\in[0,y_\delta]$,

$$\Psi'(f_0(y))^2 \le \gamma_\vartheta^2 f_0(y)^{2\vartheta} \le \gamma_\vartheta 4^\vartheta\ \delta^{-2\vartheta}$$

by (3.14) and (3.17), while, if $y\in(y_\delta,\delta]$,

$$\Psi'(f_0(y))^2 \le \frac{1}{f_0(y)^4} a\left(\frac{1}{f_0(y)}\right)^2 \le C_\infty^2\ f_0(y)^{2(\alpha-2)} \le C_\infty^2\ \delta^{-2q(2-\alpha)}$$

by (1.6) since $\alpha \le 2$. Therefore,

$$\begin{aligned}\mathcal{L}_1(f_0) \le{}& \frac{2}{\delta^4}\left[\int_0^{y_\delta} \gamma_\vartheta 4^\vartheta\ \delta^{-2\vartheta}\ dy + \int_{y_\delta}^\delta C_\infty^2\ \delta^{-2q(2-\alpha)}\ dy\right]\\ &+ \gamma_\vartheta 2^{\vartheta+1}\ \delta^{-\vartheta} - M^2\Psi_1(0)\\ \le{}& \gamma_\vartheta 4^{\vartheta+1}\ \delta^{-3-2\vartheta} + C_\infty^2\ \frac{1-\delta^q}{2(1-M\delta^q)}\ \delta^{-2-2q(2-\alpha)}\\ &+ \gamma_\vartheta 2^{\vartheta+1}\ \delta^{-\vartheta} - M^2\Psi_1(0)\\ \le{}& \gamma_\vartheta 4^{\vartheta+1}\ \delta^{-2(2+\vartheta)} + C_\infty^2\ \delta^{-2-2q(2-\alpha)} + \gamma_\vartheta 2^{\vartheta+1}\ \delta^{-\vartheta} - M^2\Psi_1(0)\\ \le{}& C_3\ \left(\delta^{-2(2+\vartheta)} + \delta^{-2-2q(2-\alpha)}\right)\end{aligned}$$

with $C_3 := \gamma_\vartheta 4^{\vartheta+2} + C_\infty^2 - M^2\Psi_1(0)$. Therefore,

$$K_0^{\vartheta+1} \le C_4\ \left(\delta^{-(\vartheta+1)} + \delta^{-(\vartheta+1)(1+q(2-\alpha))/(\vartheta+2)}\right) \tag{3.22}$$

for some constant $C_4 > 0$ depending only on M and a.

For $C_5 := C_2 C_4$ we define

$$\Lambda_\delta(m_q) := C_5 \ \left(\delta^{-(\vartheta+1)} + \delta^{-(\vartheta+1)(1+q(2-\alpha))/(\vartheta+2)}\right) \ m_q^{(q-2)/q} \ + M m_q - \frac{M^{q+1}}{2(q+1)} .$$

Combining (3.21) and (3.22) yields

$$\frac{dm_q}{dt} \le \Lambda_\delta(m_q) \tag{3.23}$$

for $t \in [0, T_{\max})$. At this point, we note that the monotonicity of Λ_δ and (3.23) imply that $\Lambda_\delta(m_q(t)) \le \Lambda_\delta(m_q(0))$ for $t \in [0, T_{\max})$ if $\Lambda_\delta(m_q(0)) < 0$, the latter condition being satisfied for δ small enough since

$$\Lambda_\delta(m_q(0)) \le C_1^{(q-2)/q} C_5 \ \left(\delta^{q-3-\vartheta} + \delta^{(q(\alpha(\vartheta+1)-\vartheta)-3\vartheta-5)/(\vartheta+2)}\right) + M C_1 \ \delta^q - \frac{M^{q+1}}{2(q+1)}$$

by (3.11) and (3.15).

Summarizing, we have shown that, if δ satisfies (3.12) and

$$C_1^{(q-2)/q} C_5 \ \left(\delta^{q-3-\vartheta} + \delta^{(q(\alpha(\vartheta+1)-\vartheta)-3\vartheta-5)/(\vartheta+2)}\right) + M C_1 \ \delta^q < \frac{M^{q+1}}{2(q+1)} , \tag{3.24}$$

we have

$$\frac{dm_q}{dt}(t) \le \Lambda_\delta(m_q(t)) \le \Lambda_\delta(m_q(0)) < 0 \,, \quad t \in [0, T_{\max}) \,,$$

an inequality which can only be true on a finite time interval owing to the non-negativity of m_q. Therefore, $T_{\max} < \infty$ in that case and, for any $M > 0$, we have found an initial condition u_0 given by (2.2), (2.3), and (3.13) (for δ small enough according to the above analysis) such that $\langle u_0 \rangle = M$ and the first component u of the corresponding solution to (1.1)–(1.4) blows up in finite time. $\square$

4. Global existence

The proof of Theorem 1.1 (ii) also relies on the study of the function L_1 defined in Lemma 2.1. For that purpose, we first recall another property from [4]. We define the function E_1 by

$$E_1(h) := \frac{1}{2} \|\partial_y h\|_{L2(0,M)} + \int_0^M \mathbf{1}_{(-\infty,0)}(h(y)) \ h(y) \ dy \,, \quad h \in H^1(0,M) \,, \tag{4.1}$$

for which we have the following lower bound.

Lemma 4.1. [4, Lemma 9] *For $M > 0$, we have*

$$E_1(h) \ge \frac{1}{4} \ \|\partial_y h\|_{L2(0,M)} - M^3 - M \left| \Psi\left(\frac{1}{M}\right) \right| , \tag{4.2}$$

and

$$\|h\|_{L1(0,M)} \le M^{3/2} \|\partial_y h\|_{L2(0,M)} + M \left| \Psi\left(\frac{1}{M}\right) \right| \tag{4.3}$$

for every $h \in H^1(0, M)$ *satisfying*

$$\int_0^M \Psi^{-1}(h)(y)\ dy = 1\,. \tag{4.4}$$

We now show that the non-integrability of a at infinity allows us to show that $T_{\max} = \infty$. To this end, we use the alternative formulation (2.4)–(2.6) as in [4] and prove that f cannot vanish in finite time.

Proof of Theorem 1.1 (ii). Owing to (2.6) and the assumptions made on u_0, we have

$$0 < f_0(y) \le \frac{1}{m_0}\,, \quad y \in [0, M].$$

Introducing $\Sigma(t) := M^{-1} + e^{Mt}\left(m_0^{-1} - M^{-1}\right)$ for $t \ge 0$ we have

$$\partial_t \Sigma - \partial_y^2 \Psi(\Sigma) - M\Sigma + 1 = M\left(\Sigma - \frac{1}{M}\right) - M\Sigma + 1 = 0,$$
$$\Sigma(0) = \frac{1}{m_0} \ge f_0(y), \quad y \in (0, M),$$

and the comparison principle warrants that

$$f(t, y) \le \Sigma(t), \quad (t, y) \in [0, T_{\max}) \times [0, M]. \tag{4.5}$$

We now follow the strategy of the proof of [4, Theorem 5] and first use the properties of Ψ, Ψ_1, and (4.5) to estimate the function L_1 (defined in Lemma 2.1) from below. Indeed, since $\Psi \ge 0$ on $(1, \infty)$ and $\Psi_1 \le 0$ on $(0, 1)$ we arrive at

$$\begin{aligned}
L_1(0) \ge L_1(t) &= \frac{1}{2}\, \|\partial_y \Psi(f(t))\|_{L^2(0,M)}^2 \\
&\quad + \int_0^M \mathbf{1}_{(0,1)}(f(t,y))(\Psi - M\Psi_1)(f(t,y))\ dy \\
&\quad + \int_0^M \mathbf{1}_{(1,\infty)}(f(t,y))(\Psi - M\Psi_1)(f(t,y))\ dy \\
&\ge \frac{1}{2}\, \|\partial_y \Psi(f(t))\|_{L^2(0,M)}^2 + \int_0^M \mathbf{1}_{(-\infty,0)}(\Psi(f(t,y)))\Psi(f(t,y))\ dy \\
&\quad - M \int_0^M \mathbf{1}_{(1,\infty)}(f(t,y))\Psi_1(f(t,y))\ dy \\
&\ge E_1(\Psi(f(t))) - M^2 \Psi_1(\Sigma(t)),
\end{aligned}$$

where E_1 is defined in (4.1) and we have used (4.5) to obtain the last inequality. Next, by Lemma 4.1 and (2.8), we have

$$L_1(0) \ge \frac{1}{4}\, \|\partial_y \Psi(f(t))\|_{L^2(0,M)}^2 - M^3 - M\left|\Psi\left(\frac{1}{M}\right)\right| - M^2 \Psi_1(\Sigma(t)),$$

whence

$$\frac{1}{4}\,\|\partial_y\Psi(f(t))\|_{L^2(0,M)}^2 \le L_1(0) + M^3 + M\left|\Psi\left(\frac{1}{M}\right)\right| + M^2\Psi_1(\Sigma(t)). \tag{4.6}$$

Using again Lemma 4.1, we have

$$\begin{aligned}\|\Psi(f(t))\|_{L^1(0,M)} &\le M^{3/2}\,\|\partial_y\Psi(f(t))\|_{L^2(0,M)} + M\left|\Psi\left(\frac{1}{M}\right)\right| \\ &\le 2M^{3/2}\left(L_1(0) + M^3 + M\left|\Psi\left(\frac{1}{M}\right)\right| + M^2\Psi_1(\Sigma(t))\right)^{1/2} + M\left|\Psi\left(\frac{1}{M}\right)\right|.\end{aligned}$$

Combining the previous inequality with (4.6) and the Poincaré inequality leads us to the bound

$$\|\Psi(f(t))\|_{H^1(0,M)} \le C_6(T)\,, \quad t\in[0,T]\cap[0,T_{\max})\,, \tag{4.7}$$

for all $T>0$. Together with the continuous embedding of $H^1(0,M)$ in $L^\infty(0,M)$, (4.7) gives

$$-C_7(T) \le \Psi(f(t,y)) \le C_7(T)\,, \quad (t,y)\in([0,T]\cap[0,T_{\max}))\times[0,M]\,.$$

Since

$$\lim_{r\to 0}\Psi(r) = -\infty$$

due to $a\notin L^1(1,\infty)$, the above lower bound on $\Psi(f)$ ensures that $f(t)$ cannot vanish in finite time, from which Theorem 1.1 (ii) follows as already discussed in Section 2. □

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Perturbation Results for Multivalued Linear Operators

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Abstract. We give some perturbation theorems for multivalued linear operators in a Banach space. Two different approaches are suggested: the resolvent approach and the modified resolvent approach. The results allow us to handle degenerate abstract Cauchy problems (inclusions). A very wide application of obtained abstract results to initial boundary value problems for degenerate parabolic (elliptic-parabolic) equations with lower-order terms is studied. In particular, integro-differential equations have been considered too.

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1. Introduction

Degenerate evolution equations in Banach spaces and their applications to partial differential equations constitute a very wide field of mathematical research. Many different methods exist to handle this subject (see, e.g., [11]). Two different approaches were introduced in [6]. The first one relates to multivalued linear operators, while the second uses a modified resolvent and operational method, extending G. Da Prato and P. Grisvard's approach (see also [7] for applications to nonlinear equations). In whichever case, a basic role is played by the resolvent estimates of the operators involved. A number of applications to singular linear parabolic differential equations has been given in [4], [5], which improved the previous results in [8] and [6]. In particular, [5] deals directly with second-order differential operators with lower-order terms.

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In this paper we obtain the relevant perturbation theorems for such equations. To the best of our knowledge, such perturbation results have not appeared in the existing literature.

More precisely, Section 2 describes the resolvent approach of perturbing a multivalued linear operator by another (possibly) multivalued linear operator and satisfying some estimate. This goal is reached by some arguments from [1]. Section 3 deals with the modified resolvent approach which is, on the other hand, strictly related to the first type. Section 4 furnishes a large number of concrete examples from partial differential equations to which the developed abstract theory applies. We note that in Examples 5–7, an alternative approach to certain equations of [5], related to gradient estimates, is indicated.

2. Resolvent approach

Denote by X, Y Banach spaces, by $\mathcal{L}(X,Y)$ a space of all bounded linear operators from X into Y, by $M\mathcal{L}(X,Y)$ a space of all multivalued linear operators from X into Y, by A, B multivalued linear operators from $M\mathcal{L}(X,Y)$. If $X = Y$ then denote $\mathcal{L}(X) := \mathcal{L}(X,Y)$ and $M\mathcal{L}(X) := M\mathcal{L}(X,Y)$. The norm of Au is defined as follows

$$\|Au\| := \inf\{\|y\|_Y \ : \ y \in Au\}, \quad \forall u \in D(A),$$

and

$$\|A\|_{M\mathcal{L}(X,Y)} := \sup_{\|u\|_X \le 1} \|Au\|_Y.$$

By A_0 we denote a selection (or a single-valued linear part) of A, i.e.,

$$A = A_0 + A - A, \qquad D(A_0) = D(A).$$

Obviously,

$$Au = A_0 u + A(0), \qquad \forall u \in D(A).$$

Here and everywhere, $A(0)$ stands for $A0$.

Definition 2.1. Given an operator $A \in M\mathcal{L}(X)$. By $\rho(A) \subset \mathbb{C}$ we denote the **resolvent set** of A, i.e., $\lambda \in \rho(A)$ if and only if the inverse operator $(\lambda I - A)^{-1}$ is a linear bounded single-valued operator defined on the whole space X. In this case we denote $R(\lambda, A) := (\lambda I - A)^{-1}$ and call it the **resolvent** of A.

Let E be a linear subspace of X. Denote by P_E the multivalued linear projection given by $P_E x := E$ for any $x \in X$. Thus the graph of P_E, $G(P_E) = X \times E$. Obviously, the kernel of P_E, $N(P_E) = X$ and $P_E(0) = E$. Next, define $I_E \in M\mathcal{L}(X)$ by $I_E := I + P_E$. Then, we have a series of simple facts: $N(I_E) = E$, the image $R(I_E) = X$, $I_E(0) = E$, $G(I_E) = \{(x, x+e) : x \in X, e \in E\}$, $I_E x = x + E$, for all $x \in X$, $\|I_E\| = \sup\limits_{x \neq 0} \frac{\|I_E x\|}{\|x\|} = \sup\limits_{x \neq 0} \frac{\|x+E\|}{\|x\|} \le 1$ (since $\|I_E x\| = \|x+E\| = \inf\{\|x+y\| :$

$y \in E\}$), and the minimum modulus (cf. [1, II.2.2])

$$\gamma(I_E) = \begin{cases} \infty, & \text{if } E \text{ is dense in } X, \\ \inf\limits_{x \notin \overline{E}} \frac{\|x+E\|}{\|x+E\|}, & \text{otherwise} \end{cases} = \begin{cases} \infty, & \text{if } \overline{E} = X, \\ 1, & \text{if } \overline{E} \neq X. \end{cases}$$

Proposition 2.2. *Let $S \in M\mathcal{L}(X)$ satisfy $D(S) = X$ and $\|S\| < 1$. Then, the operator $I - S$ has a dense range.*

Proof. Write $E := S(0)$ (it is known, [1, I.2.4], that $S(0)$ is a linear subspace of X). Since $\gamma(I_E) > 0$, I_E is open (see [1, II.3.2(b)]). Also, $R(I_E) = X$, $S(0) = E = I_E(0)$, and $\|S\| < 1 \leq \gamma(I_E)$. Hence, by [1, III.7.5], $I_E - S$ has a dense range. Since $I - S = I_E - S$, the result follows. □

Obviously, if X is finite dimensional, then $D((I - S)^{-1}) = X$.

Proposition 2.3. *Let $S \in M\mathcal{L}(X)$ satisfy $D(S) = X$ and $\|S\| < 1$, and let $S(0)$ be closed. Then, $(I - S)^{-1} \in M\mathcal{L}(X)$ is continuous if and only if $R(I - S)$ is closed.*

Proof. By Proposition 2.2, if $R(I - S)$ is closed then $I - S$ is surjective, therefore, $D((I - S)^{-1}) = X$ which implies, by the closed graph theorem (see [1, III.4.2]), that $(I - S)^{-1}$ is continuous. Conversely, if $(I - S)^{-1}$ is continuous then $I - S$ is open (see [1, II.3.1]) and, by [1, III.4.2(b)], $R(I - S)$ is closed. □

Let us now prove the main proposition.

Proposition 2.4. *Let $S \in M\mathcal{L}(X)$ satisfy $D(S) = X$ and $\|S\| < 1$, and let $S(0)$ be closed. Then, $(I - S)^{-1} \in M\mathcal{L}(X)$ is everywhere defined and continuous.*

Proof. In order to prove that the operator $(I - S)^{-1}$ is continuous, it is enough, by Proposition 2.3, to show that $R(I - S)$ is closed. By [1, III.4.4], $R(I - S)$ is closed if and only if the range of the adjoint $R((I - S)')$ is closed. By [1, III.1.5(b)], $(I - S)' = I - S'$. Thus, we are going to show that $R(I - S')$ is closed.

By [1, II.3.2(a)] and [1, III.1.13], the adjoint operator S' is continuous. Then, by [1, III.4.2(a)], $D(S')$ is a closed subspace of X'. Since $D(S) = X$ then, by [1, III.1.4(b)], S' is single-valued. By [1, III.1.13], $\|S'\| \leq \|S\| < 1$. Denote $\|S'\| = 1 - d$, where $0 < d < 1$. Some trivial calculations show that the operator $I - S'$ is injective. Indeed, let $(I - S')x' = 0$. Then, $0 = \|(I - S')x'\| \geq \|x'\| - \|S'x'\| \geq \|x'\| - \|S'\|\|x'\| = d\|x'\| \geq 0$, i.e., $x' = 0$. Let us now show that $I - S'$ is open. By [1, II.2.1], since $I - S'$ is injective, $\gamma(I - S') = \inf\limits_{x' \neq 0, x' \in D(S')} \frac{\|(I-S')x'\|}{\|x'\|} \geq d > 0$. This proves, by [1, II.3.2(b)], that $I - S'$ is open as a map from $D(I - S') = D(S')$ onto $R(I - S')$, i.e., the inverse $(I - S')^{-1}$ is continuous, by [1, II.3.1] and, therefore, $D(S')$ and $R(I - S')$ are isomorphic. This implies, in turn, that $R(I - S')$ is closed since $D(S')$ is closed. Therefore, as mentioned above, $R(I - S)$ is closed and, by Proposition 2.3, the operator $(I - S)^{-1}$ is continuous.

Combining Proposition 2.2 with the fact that $R(I - S)$ is closed, we conclude that $R(I - S) = X$. □

Note that for $I - S$ to be only surjective one should not claim that $S(0)$ is closed or $\|S\| < 1$. Let us give some other sufficient conditions for that.

Proposition 2.5. *If $S \in M\mathcal{L}(X)$ (with $D(S) = X$) has a continuous selection S_0 such that $\|S_0\| < 1$ then $I - S$ is surjective.*

Proof. If $\|S_0\| < 1$ then $I - S_0$ is surjective (from the single-valued theory). On the other hand, $I - S = I - S_0 + S - S$, i.e., $(I - S)x = (I - S_0)x + S(0)$ for all $x \in X$. Hence, $R(I - S) \supset R(I - S_0) = X$. □

Proposition 2.6. *If $S \in M\mathcal{L}(X)$ (with $D(S) = X$), $\|S\| < 1$, and there exists a linear projection, with the norm equal to 1, defined on $R(S)$ with kernel $S(0)$, then $I - S$ is surjective.*

Proof. Let P be the norm-1 projection and let $S_0 = PS$. Then, by [1, II.3.13], $\|S_0\| = \|PS\| \le \|P\|\|S\| = \|S\| < 1$. Now apply Proposition 2.5. □

Note that if X is a Hilbert space then there exists a norm-1 linear projection defined on $R(S)$ with kernel $S(0)$.

We now pass to the main perturbation theorems.

Theorem 2.7. *Let the following conditions be satisfied:*

1. *$A \in M\mathcal{L}(X)$ and, for some $\eta \in (0, 1]$,*

$$\|R(\lambda, A)\| \le C|\lambda|^{-\eta}, \quad \text{for sufficiently large } \lambda \in \Gamma,$$

where Γ is an unbounded set of the complex plane;

2. *$B \in M\mathcal{L}(X)$, $D(B) \supset D(A)$, $B(0)$ is closed, and, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that*

$$\|Bu\| \le \varepsilon\|Au\|^{\eta}\|u\|^{1-\eta} + C(\varepsilon)\|u\|, \quad \forall u \in D(A).$$

Then, for every sufficiently large $\lambda \in \Gamma$, the multivalued linear operator $(\lambda I - A - B)^{-1}$ is defined on the whole space X and

$$\|(\lambda I - A - B)^{-1}\| \le C|\lambda|^{-\eta}, \quad \text{for sufficiently large } \lambda \in \Gamma.$$

Proof. By, e.g., [6, Theorem 1.7, p. 24], $-I + \lambda R(\lambda, A) \subset AR(\lambda, A)$, $\forall \lambda \in \rho(A)$. Then, for sufficiently large $\lambda \in \Gamma$, by condition (1),

$$\|AR(\lambda, A)\| \le 1 + |\lambda|\|R(\lambda, A)\| \le C|\lambda|^{1-\eta}.$$

Hence, by conditions 1 and 2, for sufficiently large $\lambda \in \Gamma$ and for any $v \in X$,

$$\begin{aligned}\|BR(\lambda, A)v\| &\le \varepsilon\|AR(\lambda, A)v\|^{\eta}\|R(\lambda, A)v\|^{1-\eta} + C(\varepsilon)\|R(\lambda, A)v\| \\ &\le (\varepsilon C + C(\varepsilon)|\lambda|^{-\eta})\|v\|.\end{aligned}$$

Therefore,

$$\|BR(\lambda, A)\| \le q < 1, \quad \text{for sufficiently large } \lambda \in \Gamma.$$

Then, by Proposition 2.4, the multivalued linear operator $(I - BR(\lambda, A))^{-1}$ is everywhere defined and continuous for the same λ, i.e., by [1, II.3.2(a)], the operator has a bounded norm

$$\|(I - BR(\lambda, A))^{-1}\| \leq C, \quad \text{for sufficiently large } \lambda \in \Gamma. \tag{2.1}$$

Further, since $\lambda I - A$ is injective (see, e.g., [1, Exercise VI.1.2, p. 220]) then, by, e.g., [1, formula I.1.3/(9), p. 3] and [9, Proposition A.1.1/(e), p. 281], on $D(A)$,

$$(I - BR(\lambda, A))(\lambda I - A) \subset \lambda I - A - BR(\lambda, A)(\lambda I - A) = \lambda I - A - B. \tag{2.2}$$

Then, by the definition of the inverse operator of multivalued linear operators (see, e.g., [1, p. 1])

$$[(I - BR(\lambda, A))(\lambda I - A)]^{-1} \subset (\lambda I - A - B)^{-1}.$$

From this, taking into account, e.g., [9, Proposition A.1.1/(a), p. 280], we get, for sufficiently large $\lambda \in \Gamma$,

$$R(\lambda, A)(I - BR(\lambda, A))^{-1} \subset (\lambda I - A - B)^{-1}. \tag{2.3}$$

In fact, the left-hand side operator is defined on the whole space X, which means that the image of the left-hand side operator in (2.2) is X, i.e., also the image $R(\lambda I - A - B) = X$. This says that the operator $(\lambda I - A - B)^{-1}$ is defined on the whole space X for sufficiently large $\lambda \in \Gamma$. On the other hand, by (2.1), (2.3), [1, Corollary II.3.13, p. 38], and condition 1, we get

$$\begin{aligned} \|(\lambda I - A - B)^{-1}\| &\leq \|R(\lambda, A)\| \; \|(I - BR(\lambda, A))^{-1}\| \\ &\leq C|\lambda|^{-\eta}, \quad \text{for sufficiently large } \lambda \in \Gamma. \end{aligned}$$ □

Conditions of Theorem 2.7 do not guarantee that the operator $(\lambda I - A - B)^{-1}$ is single valued, that is why we could not say that $(\lambda I - A - B)^{-1} = R(\lambda, A + B)$. Let us now formulate the results which state that $(\lambda I - A - B)^{-1} = R(\lambda, A + B)$.

Theorem 2.8. *Let the following conditions be satisfied:*

1. $A \in M\mathcal{L}(X)$ *and, for some* $\eta \in (0, 1]$,

$$\|R(\lambda, A)\| \leq C|\lambda|^{-\eta}, \quad \textit{for sufficiently large } \lambda \in \Gamma,$$

where Γ *is an unbounded set of the complex plane;*

2. $B \in M\mathcal{L}(X)$, $D(B) \supset D(A)$, $B(0)$ *is closed, and, for any* $\varepsilon > 0$, *there exists* $C(\varepsilon) > 0$ *such that*

$$\|Bu\| \leq \varepsilon \|Au\|^{\eta} \|u\|^{1-\eta} + C(\varepsilon)\|u\|, \quad \forall u \in D(A);$$

3. *For sufficiently large* $\lambda \in \Gamma$, *the operator* $(I - R(\lambda, A)B)^{-1}$ *is single valued.*

Then, every sufficiently large $\lambda \in \Gamma$ *belongs to* $\rho(A + B)$ *and*

$$\|R(\lambda, A + B)\| \leq C|\lambda|^{-\eta}, \quad \textit{for sufficiently large } \lambda \in \Gamma.$$

Proof. From $I \subset (\lambda I - A)R(\lambda, A)$ and [9, Proposition A.1.1/(d) and (e), pp.280-281], for sufficiently large $\lambda \in \Gamma$,

$$\lambda I - A - B \subset \lambda I - A - (\lambda I - A)R(\lambda, A)B \subset (\lambda I - A)(I - R(\lambda, A)B).$$

Then, by the definition of the inverse operator of multivalued linear operators (see, e.g., [1, p.1]) and, e.g., [9, Proposition A.1.1/(a), p.280],

$$(\lambda I - A - B)^{-1} \subset (I - R(\lambda, A)B)^{-1} R(\lambda, A), \tag{2.4}$$

for the same λ. Since the right-hand side operator is single valued, for sufficiently large $\lambda \in \Gamma$, then $(\lambda I - A - B)^{-1}$ is also single valued for the same λ. Combining this with the result of Theorem 2.7, we get that these λ belong to $\rho(A + B)$ and $(\lambda I - A - B)^{-1} = R(\lambda, A + B)$. □

Theorem 2.9. *Let the following conditions be satisfied:*

1. $A \in M\mathcal{L}(X)$ *and, for some* $\eta \in (0, 1]$,
$$\|R(\lambda, A)\| \le C|\lambda|^{-\eta}, \quad \textit{for sufficiently large } \lambda \in \Gamma,$$
where Γ *is an unbounded set of the complex plane;*
2. B *is a single valued linear operator in* X, $D(B) \supset D(A)$, *and, for any* $\varepsilon > 0$, *there exists* $C(\varepsilon) > 0$ *such that*
$$\|Bu\| \le \varepsilon \|Au\|^{\eta} \|u\|^{1-\eta} + C(\varepsilon)\|u\|, \quad \forall u \in D(A);$$
3. *There exists a Banach space* Z, $D(A) \subset Z \subset D(B)$, *such that, for some* $\theta \in (0, 1]$,
$$\|R(\lambda, A)\|_{\mathcal{L}(X,Z)} \le C|\lambda|^{-\theta}, \quad \textit{for sufficiently large } \lambda \in \Gamma.$$

Then, every sufficiently large $\lambda \in \Gamma$ *belongs to* $\rho(A + B)$ *and*

$$\|R(\lambda, A + B)\| \le C|\lambda|^{-\eta}, \quad \textit{for sufficiently large } \lambda \in \Gamma.$$

Proof. From conditions 1 and 2 we get conditions 1 and 2 of Theorem 2.7. Therefore, the result of Theorem 2.7 is true. Thus, in order to get the assertion of Theorem 2.9, it is now enough to show that the operator $(\lambda I - A - B)^{-1}$ is single valued.

Since B is single valued then, for sufficiently large $\lambda \in \Gamma$, $R(\lambda, A)B$ is also single valued. On the other hand, by condition 3,

$$\|R(\lambda, A)B\|_{\mathcal{L}(Z)} \le \|R(\lambda, A)\|_{\mathcal{L}(X,Z)} \|B\|_{\mathcal{L}(Z,X)} \le C|\lambda|^{-\theta} < 1,$$

for sufficiently large $\lambda \in \Gamma$. Therefore, $I - R(\lambda, A)B$ has a bounded single valued inverse operator $(I - R(\lambda, A)B)^{-1}$ in $\mathcal{L}(Z)$. Then, since the image $R((\lambda I - A - B)^{-1}) = D(A)$ and the image $R(R(\lambda, A)) = D(A)$, we obtain, by (2.4), that the operator $(\lambda I - A - B)^{-1}$ is single valued (in fact, $(\lambda I - A - B)^{-1} = (I - R(\lambda, A)B)^{-1} R(\lambda, A)$, for sufficiently large $\lambda \in \Gamma$). □

Remark 2.10. The same conclusion, as in Theorem 2.9, holds if instead of the inequality in condition 2 it is assumed that, for some $\sigma > 0$,

$$\|BR(\lambda, A)\| \le C|\lambda|^{-\sigma}, \quad \text{for sufficiently large } \lambda \in \Gamma. \tag{2.5}$$

The proof is the same as that of Theorem 2.9, where the inequality in condition 2 is needed in order to apply Theorem 2.7. In turn, the inequality in Theorem 2.7 has been only used for the proving that $\|BR(\lambda, A)\| \le q < 1$, for sufficiently large $\lambda \in \Gamma$. The latter inequality is now obvious due to (2.5).

3. Modified resolvent approach

Definition 3.1. Let M and L be two single-valued, closed linear operators in a Banach space X, $D(L) \subset D(M)$, $0 \in \rho(L)$. The set $\{\lambda \in \mathbb{C} \,:\, \lambda M - L$ has a single-valued and bounded inverse defined on $X\}$ is called the M **modified resolvent set of** L (or simply the M **resolvent set of** L) and is denoted by $\rho_M(L)$. The bounded operator $(\lambda M - L)^{-1}$ is called the M **modified resolvent of** L (or simply the M **resolvent of** L).

Theorem 3.2. *Let the following conditions be satisfied:*

1. *Operators $M(t)$ and $L(t)$ are single valued, closed linear operators in a Banach space X which depend on a parameter t, $D(L(t)) \subset D(M(t))$, every $\lambda \in \Gamma$, $|\lambda| \to \infty$, belongs to $\rho_{M(t)}(L(t))$, and, for some $\eta \in (0, 1]$,*

   $$\|M(t)(\lambda M(t) - L(t))^{-1}\| \le C|\lambda|^{-\eta}, \quad \textit{for sufficiently large } \lambda \in \Gamma,$$

   uniformly on t, where Γ is an unbounded set of the complex plane;
2. *An operator $L_1(t)$ is a single-valued, closed linear operator in X, $D(L(t)) \subset D(L_1(t))$ and, for any $\varepsilon > 0$,*

   $$\|L_1(t)u\| \le \varepsilon \|L(t)u\|^{\eta} \|M(t)u\|^{1-\eta} + C(\varepsilon)\|M(t)u\|, \quad \forall u \in D(L(t)),$$

   uniformly on t.

Then, every sufficiently large $\lambda \in \Gamma$ belongs to $\rho_{M(t)}(L(t) + L_1(t))$ and

$$\|M(t)[\lambda M(t) - (L(t) + L_1(t))]^{-1}\| \le C|\lambda|^{-\eta}, \quad \textit{for sufficiently large } \lambda \in \Gamma,$$

uniformly on t.

Proof. Since, for $\lambda \in \rho_{M(t)}(L(t))$,

$$\begin{aligned} L(t)(\lambda M(t) - L(t))^{-1} &= (-(\lambda M(t) - L(t)) + \lambda M(t))(\lambda M(t) - L(t))^{-1} \\ &= -I + \lambda M(t)(\lambda M(t) - L(t))^{-1} \end{aligned}$$

then, by condition 1, for sufficiently large $\lambda \in \Gamma$, we have

$$\|L(t)(\lambda M(t) - L(t))^{-1}\| \le C|\lambda|^{1-\eta},$$

uniformly on t. Hence, by conditions 1 and 2, for sufficiently large $\lambda \in \Gamma$ and for any $v \in X$,

$$\begin{aligned}\|L_1(t)(\lambda M(t) - L(t))^{-1}v\| &\le \varepsilon \|L(t)(\lambda M(t) - L(t))^{-1}v\|^{\eta} \|M(t)(\lambda M(t) \\ &\qquad - L(t))^{-1}v\|^{1-\eta} + C(\varepsilon)\|M(t)(\lambda M(t) - L(t))^{-1}v\| \\ &\le (\varepsilon C + C(\varepsilon)|\lambda|^{-\eta})\|v\|,\end{aligned}$$

uniformly on t. Therefore,

$$\|L_1(t)(\lambda M(t) - L(t))^{-1}\| \le q < 1, \quad \text{for sufficiently large } \lambda \in \Gamma,$$

uniformly on t, which implies, for the same λ,

$$\|[I - L_1(t)(\lambda M(t) - L(t))^{-1}]^{-1}\| \le C, \tag{3.1}$$

uniformly on t.

On the other hand, for $\lambda \in \rho_{M(t)}(L(t))$,

$$\lambda M(t) - (L(t) + L_1(t)) = [I - L_1(t)(\lambda M(t) - L(t))^{-1}](\lambda M(t) - L(t)),$$

i.e., for sufficiently large $\lambda \in \Gamma$,

$$[\lambda M(t) - (L(t) + L_1(t))]^{-1} = (\lambda M(t) - L(t))^{-1}[I - L_1(t)(\lambda M(t) - L(t))^{-1}]^{-1}$$

or

$$\begin{aligned}&M(t)[\lambda M(t) - (L(t) + L_1(t))]^{-1} \\ &\quad = M(t)(\lambda M(t) - L(t))^{-1}[I - L_1(t)(\lambda M(t) - L(t))^{-1}]^{-1}\end{aligned}$$

which, by condition 1 and inequality (3.1), completes the proof. □

Remark 3.3. The same conclusion, as in Theorem 3, holds if instead of condition (2) it is assumed that $L_1(t)$ is a single-valued, closed linear operator in X, $D(L(t)) \subset D(L_1(t))$, and, for some $\sigma > 0$,

$$\|L_1(t)(\lambda M(t) - L(t))^{-1}\| \le C|\lambda|^{-\sigma}, \quad \text{for sufficiently large } \lambda \in \Gamma,$$

uniformly on t. In this case, the proof of (3.1) is obvious since

$$\|L_1(t)(\lambda M(t) - L(t))^{-1}\| \le q < 1, \quad \text{for sufficiently large } \lambda \in \Gamma,$$

uniformly on t.

Corollary 3.4. *Let the following conditions be satisfied:*

1. *Operators $M(t)$ and $L(t)$ are single valued, closed linear operators in a Banach space X which depend on a parameter t, $D(L(t)) \subset D(M(t))$, every sufficiently large $\lambda \in \Gamma$ belongs to $\rho_{M(t)}(L(t))$, and, for some $\eta \in (0, 1]$,*

   $$\|M(t)(\lambda M(t) - L(t))^{-1}\| \le C|\lambda|^{-\eta}, \quad \textit{for sufficiently large } \lambda \in \Gamma,$$

   uniformly on t, where Γ is an unbounded set of the complex plane;

2. *An operator* $L_1(t)$ *is a single-valued, closed linear operator in* X, $D(L(t)) \subset D(L_1(t))$ *and, for some* $\mu \in [0, \eta)$,
$$\|L_1(t)u\| \leq C(\|L(t)u\|^{\mu}\|M(t)u\|^{1-\mu} + \|M(t)u\|), \quad \forall u \in D(L(t)),$$
uniformly on t.

Then, every sufficiently large $\lambda \in \Gamma$ *belongs to* $\rho_{M(t)}(L(t) + L_1(t))$ *and*
$$\|M(t)(\lambda M(t) - (L(t) + L_1(t)))^{-1}\| \leq C|\lambda|^{-\eta}, \quad \textit{for sufficiently large } \lambda \in \Gamma,$$
uniformly on t.

Proof. By the Young inequality, we have
$$\begin{aligned}\|L(t)u\|^{\mu}\|M(t)u\|^{1-\mu} &= \|M(t)u\|^{1-\eta}\|L(t)u\|^{\mu}\|M(t)u\|^{\eta-\mu}\\ &\leq \|M(t)u\|^{1-\eta}(\varepsilon\|L(t)u\|^{\eta} + C(\varepsilon)\|M(t)u\|^{\eta})\\ &\leq \varepsilon\|L(t)u\|^{\eta}\|M(t)u\|^{1-\eta} + C(\varepsilon)\|M(t)u\|,\end{aligned}$$
which implies that the assertion follows from Theorem 3.2. □

Theorem 3.5. *Let* $M(t)$ *and* $L(t)$ *be single valued, closed linear operators in a Banach space* X *which depend on a parameter* t, $D(L(t)) \subset D(M(t))$, *and*
$$\|M(t)(\lambda M(t) - L(t))^{-1}\| \leq K|\lambda|^{-1}, \quad \arg\lambda = \varphi,\ \lambda \textit{ sufficiently large},$$
uniformly on t.

Then, for some $\alpha > 0$,
$$\|M(t)(\lambda M(t) - L(t))^{-1}\| \leq C|\lambda|^{-1}, \quad |\arg\lambda - \varphi| < \alpha,\ \lambda \textit{ sufficiently large},$$
uniformly on t.

Proof. If $\mu \in \rho_{M(t)}(L(t))$, then
$$\begin{aligned}\lambda M(t) - L(t) &= \mu M(t) - L(t) + (\lambda - \mu)M(t)\\ &= [I - (\mu - \lambda)M(t)(\mu M(t) - L(t))^{-1}](\mu M(t) - L(t)).\end{aligned}$$
Then, for $|\lambda - \mu| < \|M(t)(\mu M(t) - L(t))^{-1}\|^{-1}$,
$$\begin{aligned}&M(t)(\lambda M(t) - L(t))^{-1}\\ &\quad = M(t)(\mu M(t) - L(t))^{-1}[I - (\mu - \lambda)M(t)(\mu M(t) - L(t))^{-1}]^{-1},\end{aligned}$$
or
$$\begin{aligned}&M(t)(\lambda M(t) - L(t))^{-1}\\ &\quad = M(t)(\mu M(t) - L(t))^{-1}\sum_{k=0}^{\infty}[M(t)(\mu M(t) - L(t))^{-1}]^k(\mu - \lambda)^k.\end{aligned}$$
Fix $q < 1$. Therefore, in the circle $|\lambda - s| < K^{-1}|s|q$, with a center point $s \in \rho_{M(t)}(L(t))$ on $\arg s = \varphi$ and s is large enough, by the condition of the theorem, we have
$$\|M(t)(\lambda M(t) - L(t))^{-1}\| \leq K|s|^{-1}\sum_{k=0}^{\infty}(K|s|^{-1})^k(K^{-1}|s|q)^k = K|s|^{-1}(1 - q)^{-1},$$

uniformly on t. Since $K^{-1}|s|q > |\lambda - s| \geq |\lambda| - |s|$, then $|s|^{-1} < (K^{-1}q+1)|\lambda|^{-1}$. So, in the circle $|\lambda - s| < K^{-1}|s|q$, for sufficiently large s on $\arg s = \varphi$, we have

$$\|M(t)(\lambda M(t) - L(t))^{-1}\| \leq K(1-q)^{-1}(K^{-1}q+1)|\lambda|^{-1} \leq C|\lambda|^{-1},$$

uniformly on t. Since the circles $|\lambda - s| < K^{-1}|s|q$ cover the angle $|\arg \lambda - \varphi| < \alpha$, where $\alpha = \arctan(K^{-1}q)$, then, for $\alpha = \arctan(K^{-1}q)$, the necessary estimate is fulfilled. $\square$

4. Application to PDEs

We give various examples of possible applications of the obtained abstract perturbation results.

Example 1. Consider an initial boundary value problem for an equation of parabolic type in the domain $[0,1] \times [0,T]$

$$\begin{cases} \frac{\partial}{\partial t}\left\{\left(1 - \frac{\partial^2}{\partial x^2}\right) v(x,t)\right\} = -\frac{\partial^4 v(x,t)}{\partial x^4} + \sum_{j=0}^{3}\sum_{i=1}^{N_j} b_{ji}(x)\frac{\partial^j v(\varphi_{ji}(x),t)}{\partial x^j} \\ \qquad + \sum_{j=0}^{4}\int_0^1 B_j(x,y)\frac{\partial^j v(y,t)}{\partial y^j}dy + f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ v(0,t) = v(1,t) = \frac{\partial^2 v}{\partial x^2}(0,t) = \frac{\partial^2 v}{\partial x^2}(1,t) = 0, \quad 0 < t < T, \\ \left(1 - \frac{\partial^2}{\partial x^2}\right) v(x,0) = u_0(x), \quad 0 < x < 1, \end{cases} \tag{4.1}$$

where $b_{ji}(x) \in L^2(0,1)$, $\varphi_{ji}(x)$ are functions mapping the segment [0,1] into itself and belong to $C[0,1]$, $B_j(x,y)$ are kernels such that, for some $\sigma > 1$, $\int_0^1 |B_j(x,y)|^\sigma dy + \int_0^1 |B_j(x,y)|^\sigma dx \leq C$.

Denoting $X := L^2(0,1)$, $M := I - \frac{d^2}{dx^2}$ with $D(M) := H^2(0,1) \cap H_0^1(0,1)$, $L := -\frac{d^4}{dx^4}$ with $D(L) := H^4(0,1; u(0) = u(1) = u''(0) = u''(1) = 0)$, and

$$L_1 u := \sum_{j=0}^{3}\sum_{i=1}^{N_j} b_{ji}(x)u^{(j)}(\varphi_{ji}(x)) + \sum_{j=0}^{4}\int_0^1 B_j(x,y)u^{(j)}(y)dy$$

with $D(L_1) = D(L)$, and, using [6, Example 3.1, p. 73], [12, Examples 3 and 4, p. 201], [12, Lemma 1.2.8/3], [6, Theorem 3.8], and Theorem 3.2 (with $\eta = 1$), one can get that for any $f \in C^s([0,T]; L^2(0,1))$, $0 < s$ (≤ 1), and any $u_0 \in L^2(0,1)$, there exists a unique strict solution $v(x,t)$ of problem (4.1) such that

$$Mv \in C^1((0,T]; L^2(0,1)), \quad Lv, L_1 v \in C((0,T]; L^2(0,1))$$

provided that $Mv(x,0) = u_0(x)$ is understood in the seminorm sense, i.e.,

$$\|M(\gamma M - L)^{-1}(Mv(\cdot,t) - u_0(\cdot))\|_{L^2(0,1)} \to 0 \quad as \quad t \to 0,$$

where $\gamma > 0$ is sufficiently large.

Example 2. Modify now the boundary conditions in (4.1) and consider, e.g., the following problem

$$\begin{cases} \frac{\partial}{\partial t}\left\{\left(1-\frac{\partial^2}{\partial x^2}\right)v(x,t)\right\} = -\frac{\partial^4 v(x,t)}{\partial x^4} + \sum_{j=0}^2 b_j(x)\frac{\partial^j v(x,t)}{\partial x^j} \\ \qquad\qquad + \sum_{j=0}^2 \int_0^1 B_j(x,y)\frac{\partial^j v(y,t)}{\partial y^j}dy + f(x,t), \quad 0<x<1,\ 0<t<T, \\ v(0,t) = v(1,t) = \frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(1,t) = 0, \quad 0<t<T, \\ \left(1-\frac{\partial^2}{\partial x^2}\right)v(x,0) = u_0(x), \quad 0<x<1, \end{cases} \tag{4.2}$$

where $b_j(x) \in L^\infty(0,1)$, $B_j(x,y) \in L^2((0,1)\times(0,1))$.

Denoting $X := L^2(0,1)$, $M := I - \frac{d^2}{dx^2}$ with $D(M) := H^2(0,1) \cap H_0^1(0,1)$, $L := -\frac{d^4}{dx^4}$ with $D(L) := H^4(0,1; u(0) = u(1) = u'(0) = u'(1) = 0)$, and

$$L_1 u := \sum_{j=0}^{2} b_j(x)u^{(j)}(x) + \sum_{j=0}^{2}\int_0^1 B_j(x,y)u^{(j)}(y)dy$$

with $D(L_1) = D(M)$, and, using [6, Example 3.2, p. 73], [6, Theorem 3.8], and Corollary 3.4 (with $\eta = \frac{1}{2}$ and any $\mu \in [0, \frac{1}{2})$), one can get a similar result as in the previous example also for problem (4.2). Note only that, for any $u \in D(L)$, obviously

$$\begin{aligned} \|L_1 u\|_{L^2(0,1)} &\le C\|u\|_{H^2(0,1)} \le C\|u\|^{\mu}_{H^4(0,1)}\|u\|^{1-\mu}_{H^2(0,1)} \\ &\le C(\|Lu\|^{\mu}_{L^2(0,1)} + \|u\|^{\mu}_{L^2(0,1)})\|Mu\|^{1-\mu}_{L^2(0,1)} \\ &\le C(\|Lu\|^{\mu}_{L^2(0,1)} + \|Mu\|^{\mu}_{L^2(0,1)})\|Mu\|^{1-\mu}_{L^2(0,1)} \\ &\le C(\|Lu\|^{\mu}_{L^2(0,1)}\|Mu\|^{1-\mu}_{L^2(0,1)} + \|Mu\|_{L^2(0,1)}). \end{aligned}$$

Example 3. Consider now an initial boundary value problem for an equation of parabolic type in the domain $[0,\ell\pi]\times[0,T]$

$$\begin{cases} \frac{\partial}{\partial t}\left\{\left(1+\frac{\partial^2}{\partial x^2}\right)v(x,t)\right\} = \frac{\partial^2 v(x,t)}{\partial x^2} + a(x)\int_0^x \left(v(s,t) + \frac{\partial^2 v(s,t)}{\partial s^2}\right)ds \\ \qquad\qquad\qquad\qquad + f(x,t), \quad 0<x<\ell\pi,\ 0<t<T, \\ v(0,t) = v(\ell\pi,t) = 0, \quad 0<t<T, \\ \left(1+\frac{\partial^2}{\partial x^2}\right)v(x,0) = \left(1+\frac{\partial^2}{\partial x^2}\right)v_0(x), \quad 0<x<\ell\pi, \end{cases} \tag{4.3}$$

where ℓ is a positive integer, $a(x)$ is a continuous function on $[0,\ell\pi]$.

Denoting $X := C([0,\ell\pi]; u(0) = u(\ell\pi) = 0)$ and $M := I + \frac{d^2}{dx^2}$ with $D(M) := C^2([0,\ell\pi]; u(0) = u(\ell\pi) = u''(0) = u''(\ell\pi) = 0)$, and, using [6, Example 3.10, p. 86], the corresponding calculations in [6, p. 85], [6, Theorem 3.8], and Corollary 3.4 (with $\eta = 1$ and $\mu = 0$), one can get that for any $f \in C^s([0,T]; X)$, $0 < s \le 1$, and any $v_0 \in D(M)$, there exists a unique strict solution $v(x,t)$ of problem (4.3) such that

$$Mv \in C^1((0,T]; X)$$

provided that $Mv(x,0) = Mv_0(x)$ is understood in the seminorm sense, i.e.,

$$\|M(\gamma M + I)^{-1}M(v(\cdot,t) - v_0(\cdot))\|_X \to 0 \;\; as \;\; t \to 0,$$

where $\gamma > 0$ is sufficiently large.

Example 4. Consider an initial value problem

$$\begin{cases} \frac{d}{dt}(t^\ell v(t)) = Lv(t) + L_1(t)v(t) - av(t) + f(t), \quad 0 < t < T, \\ \lim_{t\to 0^+}(t^\ell v(t)) = 0. \end{cases} \tag{4.4}$$

Theorem 4.1. *Let the following conditions be satisfied:*

1. *The operator L is a single-valued, closed linear operator in a Banach space X, $D(L)$ is dense in X, the resolvent set $\rho(L)$ contains the region $\{\lambda \in \mathbb{C} : \Re e\lambda \geq -c(|\Im m\lambda| + 1)\}$, and, for these λ,*
$$\|R(\lambda, L)\| \leq \frac{C}{|\lambda| + 1},$$
for some positive constants c and C;
2. *The operator $L_1(t)$, for any $t \in [0,T]$, is a single-valued, closed linear operator in X, $D(L) \subset D(L_1(t))$, $L_1(\cdot) \in C^{1+\mu}([0,T]; B(D(L), X))$, where $\mu = \min\{\ell - 1, 1\}$, and, for any $\varepsilon > 0$,*
$$\|L_1(t)u\| \leq \varepsilon\|Lu\| + C(\varepsilon)\|u\|, \quad \forall u \in D(L),$$
uniformly on $t \in [0,T]$;
3. *$\ell > 1$; $a = 0$ if $L_1(t) \equiv 0$, otherwise $a > 0$ is sufficiently large;*
4. *$f \in C^\sigma([0,T]; X)$ with $0 < \sigma < \mu(1 - \frac{1}{\ell})$.*

Then, there exists a unique strict solution of (4.4) *such that $t^\ell v \in C^{1+\sigma}([0,T]; X)$ and $v \in C^\sigma([0,T]; D(L))$. Moreover, for the solution, $\lim_{t\to 0^+} \frac{d}{dt}(t^\ell v(t)) = 0$ and $Lv(0) + L_1(0)v(0) - av(0) + f(0) = 0$.*

Proof. If $L_1(t) \equiv 0$ (i.e., by condition 3, also $a = 0$) then the theorem has been proved in [6, pp. 111–112]. So, a new case is a perturbed equation in (4.4), i.e., $L_1(t) \not\equiv 0$. In this case, by conditions 1 and 2, from Theorem 3.2 (with $M(t) \equiv I$, $\eta = 1$), we have

$$\|R(\lambda, L + L_1(t))\| \leq \frac{C}{|\lambda| + 1},$$

uniformly on $t \in [0,T]$, in the region $\{\lambda \in \mathbb{C} : \Re e(\lambda - a) \geq -c(|\Im m\lambda| + 1)\}$, for $a > 0$ sufficiently large, or, equivalently,

$$\|R(\lambda, L + L_1(t) - aI)\| \leq \frac{C}{|\lambda| + 1},$$

uniformly on $t \in [0,T]$, in the region $\{\lambda \in \mathbb{C} : \Re e\lambda \geq -c(|\Im m\lambda| + 1)\}$. Therefore, $M(t) := t^\ell I$ and $L(t) := L + L_1(t) - aI$ satisfy [6, formula (4.14), p. 106] with $\alpha = \beta = 1$ and $\gamma = 0$. Moreover, from

$$t^{\ell-1}\|(\lambda t^\ell - (L + L_1(t) - aI))^{-1}\| \leq \frac{Ct^{\ell-1}}{|\lambda t^\ell| + 1} \leq \frac{C}{|\lambda|^{1-\frac{1}{\ell}}}, \quad \Re e\lambda \geq -c_0(|\Im m\lambda| + 1),$$

uniformly on $t \in [0,T]$, where $c_0 > 0$ is some suitable constant, [6, formula (4.19), p. 108] is verified with $\tilde{\nu} = 1 - \frac{1}{\ell}$. Formula (4.20) in [6, p. 108], by condition 2 and that $M(t) = t^\ell I$, is obvious with $\mu = \min\{\ell - 1, 1\}$. Then, the proof is completed by using [6, Proposition 4.15]. $\square$

Remark 4.2. Using Example 1, for application of (4.4), we can take, in $X := L^2(0,1)$, an operator $L := -\frac{d^4}{dx^4}$ with $D(L) := H^4(0,1; u(0) = u(1) = u''(0) = u''(1) = 0)$ and, e.g., an operator

$$L_1(t)u := b(t)\left(\sum_{j=0}^{3}\sum_{i=1}^{N_j} b_{ji}(x)u^{(j)}(\varphi_{ji}(x)) + \sum_{j=0}^{4}\int_0^1 B_j(x,y)u^{(j)}(y)dy\right)$$

with $D(L_1) = D(L)$, where $b(t) \in C^{1+\mu}[0,T]$, and, by Theorem 4.1, get the corresponding result.

The following examples are variations of initial boundary value problems of the type

$$\begin{cases} \dfrac{\partial}{\partial t}(m(x)u(x,t)) + \mathcal{L}u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,\tau], \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\tau], \\ m(x)u(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4.5}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$,

$$\mathcal{L} = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} + a_0(x),$$

a_{ij}, a_i, a_0 are real-valued functions on $\overline{\Omega}$ such that $a_{ji} = a_{ij} \in C(\overline{\Omega})$; $\frac{\partial a_{ij}}{\partial x_j}$, a_i, $\frac{\partial a_i}{\partial x_i}$, $a_0 \in L^\infty(\Omega)$, $i,j = 1,\ldots,n$;

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c_0 \sum_{j=1}^{n} \xi_j^2, \quad \forall x \in \overline{\Omega},\ \exists c_0 > 0;$$

$a_0(x) \geq c_1 > 0$, $\forall x \in \overline{\Omega}$, $\exists c_1 > 0$; $m(x) \geq 0$, $m \in L^\infty(\Omega)$; and the initial condition in (4.5) is understood in the seminorm sense $\|m\mathcal{L}^{-1}(mu - v_0)\|_{L^p(\Omega)} \to 0$, as $t \to 0^+$.

Generalizing to the degenerate case the well-known classical result in [2, Theorem 4.43], it is shown in [4] that if $D(L) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $Lu := \mathcal{L}u$, $a_i(x) \equiv 0$, for $i = 1,\ldots,n$, $D(M) := L^p(\Omega)$, $Mu := m(\cdot)u$, and m is ρ-regular in the sense that $m \in C^1(\overline{\Omega})$ and, for some $\rho \in (0,1)$,

$$|\nabla m(x)| \leq Cm(x)^\rho, \quad \forall x \in \overline{\Omega}, \tag{4.6}$$

then, for $p \geq 2$, the estimate

$$\|M(\lambda M - L)^{-1}\|_{\mathcal{L}(L^p(\Omega))} \leq C(1 + |\lambda|)^{-\frac{2}{p(2-\rho)}} \tag{4.7}$$

holds in the sector $\Sigma_1 := \{\lambda \in \mathbb{C} \,:\, \Re e\lambda \geq -c(1 + |\Im m\lambda|)\}$, for some $c > 0$. Note that, in [5], it was considered the general problem (4.5) also with Robin boundary condition $\sum_{i,j=1}^{n} a_{ij}(x)\nu_j(x)\frac{\partial u(x,t)}{\partial x_i} + b(x)u(x,t) = 0$ on $\partial\Omega \times (0,\tau]$ (instead of the Dirichlet boundary condition), where $\nu(x) = (\nu_1(x),\ldots,\nu_n(x))$ is the unit outer normal vector at x on $\partial\Omega$, $b \in L^\infty(\partial\Omega)$. Under some additional restrictions on the coefficients of the problem, like $a_0(x) - \frac{1}{p}\sum_{i=1}^{n}\frac{\partial a_i(x)}{\partial x_i} \geq c_1 > 0$, $b(x) + \frac{1}{p}\sum_{i=1}^{n} a_i(x)\nu_i(x) \geq 0$, $\forall x \in \partial\Omega$, estimate (4.7) was again proved.

In what follows, we indicate different types of conditions on the lower-order terms which allow us to apply the perturbation results in Sections 2 and 3.

Example 5. We introduce now a problem, to which the resolvent approach (see Section 2) can be applied. Consider an initial boundary value problem for an elliptic-parabolic equation of the type

$$\begin{aligned}\frac{\partial}{\partial t}(m(x)u(x,t)) = \nabla\bullet(a(x)\nabla u(x,t)) + \sum_{i=1}^{n} a_i(x)\frac{\partial(m(x)u(x,t))}{\partial x_i}&\\ + c_0(x)u(x,t) + f(x,t), \quad (x,t) \in \Omega \times (0,\tau],& \qquad (4.8)\end{aligned}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\tau], \qquad (4.9)$$

$$m(x)u(x,0) = v_0(x), \quad x \in \Omega. \qquad (4.10)$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ of class C^2, $n = 2,3$; ∇ denotes the gradient vector with respect to x variable; $m(x) \geq 0$ on $\overline{\Omega}$ and $m \in C^2(\overline{\Omega})$; $a(x)$, $a_i(x)$, $c_0(x)$ are real-valued smooth functions on $\overline{\Omega}$; $a(x) \geq \delta > 0$ and $c_0(x) \leq 0$ for all $x \in \overline{\Omega}$, there exists $\delta > 0$. Moreover, (4.6) is satisfied.

Take the space $X := L^2(\Omega)$, the operators $Mu := m(\cdot)u$, $D(M) := X$, $Lu := \nabla\bullet(a(\cdot)\nabla u) + c_0(\cdot)$, $D(L) := H^2(\Omega) \cap H_0^1(\Omega)$. Introduce the new unknown function $v := Mu$ and, in X, consider the operators

$$A := LM^{-1}, \quad D(A) := M(D(L)),$$

$$Bv := \sum_{i=1}^{n} a_i(x)\frac{\partial v}{\partial x_i}, \quad D(B) := H^1(\Omega).$$

Note that A is a multivalued linear operator. Then, problem (4.8)–(4.10) is reduced to the multivalued Cauchy problem (inclusion)

$$\begin{cases} \dfrac{\partial v}{\partial t} \in Av + Bv + f(t), \quad t \in (0,\tau], \\ v(0) = v_0, \end{cases} \qquad (4.11)$$

where $f(t) := f(t,\cdot)$ and $v_0 := v_0(\cdot)$.

We are going to use Theorem 2.9 together with Remark 2.10 for the multivalued linear operator $A + B$. We get, from (4.7), that

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} = \|M(\lambda M - L)^{-1}\|_{\mathcal{L}(X)} \leq C(1 + |\lambda|)^{-\frac{1}{2-\rho}}$$

holds for any λ in the sector Σ_1, i.e., condition 1 of Theorem 2.9 is fulfilled with $\eta = \frac{1}{2-\rho}$. Further, using arguments in [7, pp. 443–444], introduce the Hilbert space

$Z := H^{\frac{n}{2}+\varepsilon}(\Omega)$, $0 < \varepsilon < \frac{1}{2}$. Obviously, $Z = [L^2(\Omega), H^2(\Omega)]_{\frac{n}{4}+\frac{\varepsilon}{2}}$. Then, by the interpolation theory (see [7, p. 444] for the details), it can be shown that

$$\|R(\lambda, A)\|_{\mathcal{L}(X,Z)} = \|M(\lambda M - L)^{-1}\|_{\mathcal{L}(X,Z)} \leq C(1+|\lambda|)^{\frac{n}{4}+\frac{\varepsilon}{2}-\frac{1}{2-\rho}}$$

in the sector Σ_1. If $n = 2$ then for any $\rho \in (0,1)$ there exists $\varepsilon > 0$ such that the last exponent will be negative, while if $n = 3$ then for any $\rho \in (\frac{2}{3}, 1)$ there exists $\varepsilon > 0$ such that the last exponent will be negative. Therefore, condition 3 of Theorem 2.9 is fulfilled with $\theta = \frac{1}{2-\rho} - \frac{n}{4} - \frac{\varepsilon}{2} > 0$. Observe now that $Z \subset H^1(\Omega)$ and B is bounded from $H^1(\Omega)$ into $L^2(\Omega)$. Therefore, the operator $BR(\lambda, A)$ satisfies the estimate in Remark 2.10 (with $\sigma = \theta = \frac{1}{2-\rho} - \frac{n}{4} - \frac{\varepsilon}{2}$) and thus, by Theorem 2.9,

$$\|R(\lambda, A+B)\|_{\mathcal{L}(X)} \leq C|\lambda|^{-\frac{1}{2-\rho}}$$

for any sufficiently large λ in the sector Σ_1. Without loss of generality, we can assume this estimate to be hold for all λ in the sector Σ_1 and not only for $|\lambda|$ big enough. To this end, it is enough to make a change of the unknown function $u = e^{kt}w$, for $k > 0$ big enough, in problem (4.8)–(4.10). In this way, $A - kI$ substitutes A.

We now apply [6, Theorem 3.7] (with $\alpha = 1, \beta = \frac{1}{2-\rho}$) and conclude that, for any $f \in C^\sigma([0,\tau]; L^2(\Omega))$, $\frac{1-\rho}{2-\rho} < \sigma \leq 1$, and $v_0 \in L^2(\Omega)$, there exists a unique strict solution $u(x,t)$ of problem (4.11) and, therefore, of problem (4.8)–(4.10), i.e., $m(x)u \in C^1((0,\tau]; L^2(\Omega))$, $\nabla \bullet (a(x)\nabla u) + \sum_{i=1}^n a_i(x)\frac{\partial(m(x)u)}{\partial x_i} + c_0(x)u \in C((0,\tau]; L^2(\Omega))$, u satisfies (4.8)–(4.9), and (4.10) holds in the seminorm sense $\|m(L+L_1)^{-1}(mu - v_0)\|_{L^2(\Omega)} \to 0$, as $t \to 0^+$, where $L_1 u := \sum_{i=1}^n a_i(\cdot)\frac{\partial(m(\cdot)u)}{\partial x_i}$, $D(L_1) := \{u \in L^2(\Omega) \ : \ mu \in H^1(\Omega)\}$. Note that we assume (4.6) to be fulfilled for $0 < \rho < 1$, if $n = 2$, while for $\frac{2}{3} < \rho < 1$, if $n = 3$.

Example 6. In this example, we show how using new gradient estimates we can perturb the main operator of the equation. To this end, for the sake of simplicity, we detail the case $n = 1$. Consider the following resolvent equation with the Dirichlet boundary conditions

$$\lambda m(x)u(x) - u''(x) + c(x)u(x) = f(x), \quad x \in (0,1), \tag{4.12}$$

$$u(0) = u(1) = 0. \tag{4.13}$$

In the space $X = L^2(0,1)$, denote by $Lu := u'' - c(\cdot)u$, $D(L) := H^2(0,1) \cap H_0^1(0,1)$ and $Mu := m(\cdot)u$, $D(M) := X$. Assume that $m \in C^1[0,1]$, $m(x) \geq 0$, $|m'(x)| \leq Cm(x)^\rho$, $0 < \rho < 1$, and c is a measurable, bounded function on $[0,1]$, $c(x) \geq \delta > 0$. Multiplying (4.12) by $\overline{u(x)}$ and integrating the obtained equality on $(0,1)$, we easily get (integrating by parts the second summand in the left-hand side of the equality and using (4.13))

$$\lambda\|\sqrt{m}u\|_X^2 + \|u'\|_X^2 + \int_0^1 c(x)|u(x)|^2 dx = \int_0^1 f(x)\overline{u(x)}dx,$$

implying

$$\Re e\lambda\|\sqrt{m}u\|_X^2 + \|u'\|_X^2 + \int_0^1 c(x)|u(x)|^2 dx = \Re e \int_0^1 f(x)\overline{u(x)}dx,$$
$$\Im m\lambda\|\sqrt{m}u\|_X^2 = \Im m \int_0^1 f(x)\overline{u(x)}dx.$$

Therefore,

$$(\Re e\lambda + |\Im m\lambda|)\|\sqrt{m}u\|_X^2 + \|u'\|_X^2 + \int_0^1 c(x)|u(x)|^2 dx$$
$$= \Re e \int_0^1 f(x)\overline{u(x)}dx + \left|\Im m \int_0^1 f(x)\overline{u(x)}dx\right|.$$

Then, by using Poincaré's lemma and Cauchy-Schwartz inequality, there is a positive constant $c_0 > 0$ such that

$$\begin{aligned}
&(\Re e\lambda + |\Im m\lambda| + c_0)\|\sqrt{m}u\|_X^2 + \frac{1}{2}\|u'\|_X^2 + \int_0^1 c(x)|u(x)|^2 dx \\
&\le \Re e \int_0^1 f(x)\overline{u(x)}dx + \left|\Im m \int_0^1 f(x)\overline{u(x)}dx\right| \\
&\le 2\left|\int_0^1 f(x)\overline{u(x)}dx\right| \\
&\le 2\|f\|_X\|u\|_X.
\end{aligned}$$

Take $\lambda \in \mathbb{C}$, $\Re e\lambda + |\Im m\lambda| + c_0 \ge c_1 > 0$. Hence, for all such λ,

$$\|u'\|_X^2 \le 4\|f\|_X\|u\|_X, \qquad \|u\|_X^2 \le \frac{2}{\delta}\|f\|_X\|u\|_X$$

and thus, there exists $C > 0$ such that, for $\lambda \in \mathbb{C}$, $\Re e\lambda + |\Im m\lambda| + c_0 \ge c_1 > 0$,

$$\|u\|_X \le C\|f\|_X, \qquad \|u'\|_X \le C\|f\|_X. \tag{4.14}$$

Multiply now (4.12) by $m(x)\overline{u(x)}$ and integrate the obtained equality on $(0,1)$. Then,

$$\lambda\|Mu\|_X^2 - \int_0^1 m(x)u''(x)\overline{u(x)}dx + \int_0^1 m(x)c(x)|u(x)|^2 dx = \int_0^1 m(x)f(x)\overline{u(x)}dx$$

gives, after integrating by parts in the first integral and using (4.13),

$$\begin{aligned}
&\lambda\|Mu\|_X^2 + \int_0^1 m(x)|u'(x)|^2 dx + \int_0^1 m(x)c(x)|u(x)|^2 dx \\
&= \int_0^1 m(x)f(x)\overline{u(x)}dx - \int_0^1 m'(x)u'(x)\overline{u(x)}dx.
\end{aligned} \tag{4.15}$$

Further, from (4.15), we have

$$\begin{aligned}\int_0^1 m(x)|u'(x)|^2dx = & -\lambda\|Mu\|_X^2 - \int_0^1 m(x)c(x)|u(x)|^2dx \\ & + \int_0^1 m(x)f(x)\overline{u(x)}dx - \int_0^1 m'(x)u'(x)\overline{u(x)}dx.\end{aligned} \tag{4.16}$$

On the other hand, using (4.6) (for $n = 1$) and the Hölder inequality, we get

$$\begin{aligned}\left|\int_0^1 m'(x)u'(x)\overline{u(x)}dx\right| &\le \left(\int_0^1 |m'(x)|^2|u(x)|^2dx\right)^{\frac{1}{2}} \|u'\|_X \\ &\le C\left(\int_0^1 m(x)^{2\rho}|u(x)|^2dx\right)^{\frac{1}{2}} \|u'\|_X \\ &= C\left(\int_0^1 m(x)^{2\rho}|u(x)|^{2\rho}|u(x)|^{2-2\rho}dx\right)^{\frac{1}{2}} \|u'\|_X \\ &\le C\left(\int_0^1 m(x)^2|u(x)|^2dx\right)^{\frac{\rho}{2}} \left(\int_0^1 |u(x)|^2dx\right)^{\frac{1-\rho}{2}} \|u'\|_X \\ &= C\|Mu\|_X^{\rho}\|u\|_X^{1-\rho}\|u'\|_X.\end{aligned} \tag{4.17}$$

Note that there exist $\tilde{c} > 0$ and $\tilde{c}_1 > 0$ such small that

$$\tilde{\Sigma}_1 := \{\lambda \in \mathbb{C} \ : \ \Re e\lambda \ge -\tilde{c}(1 + |\Im m\lambda|)\} \subset \Sigma_1,$$
$$\tilde{\Sigma}_1 \subset \{\lambda \in \mathbb{C} \ : \ \Re e\lambda + |\Im m\lambda| + c_0 \ge \tilde{c}_1 > 0\}.$$

Therefore, (4.7) and (4.14) hold in $\tilde{\Sigma}_1$. Then, from (4.16), using (4.7), (4.14), and (4.17), for $\lambda \in \tilde{\Sigma}_1$, we get

$$\begin{aligned}\int_0^1 m(x)|u'(x)|^2dx \le & -\lambda\|Mu\|_X^2 + \int_0^1 m(x)f(x)\overline{u(x)}dx - \int_0^1 m'(x)u'(x)\overline{u(x)}dx \\ \le & \, |\lambda|\|Mu\|_X^2 + \left|\int_0^1 m(x)f(x)\overline{u(x)}dx\right| + \left|\int_0^1 m'(x)u'(x)\overline{u(x)}dx\right| \\ \le & \, C\frac{|\lambda|}{(1+|\lambda|)^{\frac{2}{2-\rho}}}\|f\|_X^2 + \|Mu\|_X\|f\|_X \\ & + C\|Mu\|_X^{\rho}\|u\|_X^{1-\rho}\|u'\|_X \\ \le & \, C|\lambda|^{1-\frac{2}{2-\rho}}\|f\|_X^2 + C(1+|\lambda|)^{-\frac{1}{2-\rho}}\|f\|_X^2 \\ & + C(1+|\lambda|)^{-\frac{\rho}{2-\rho}}\|f\|_X^{\rho}\|f\|_X^{1-\rho}\|f\|_X \\ \le & \, C(|\lambda|^{-\frac{\rho}{2-\rho}} + |\lambda|^{-\frac{1}{2-\rho}})\|f\|_X^2,\end{aligned}$$

i.e.,

$$\int_0^1 m(x)|u'(x)|^2dx \le C|\lambda|^{-\frac{\rho}{2-\rho}}\|f\|_X^2, \quad \lambda \in \tilde{\Sigma}_1, \ |\lambda| \ge 1.$$

Therefore, the gradient estimate reads

$$\left(\int_0^1 m(x)|u'(x)|^2 dx\right)^{\frac{1}{2}} \leq C|\lambda|^{-\frac{\rho}{2(2-\rho)}}\|f\|_X, \quad \lambda \in \tilde{\Sigma}_1,\ |\lambda| \geq 1, \tag{4.18}$$

extending the well-known result for the regular case (see [10, Theorem 3.1.3]).

This argument extends to an arbitrary bounded domain $\Omega \in \mathbb{R}^n$ with the smooth boundary $\partial\Omega$, in view of Green's formula (and $u = 0$ on $\partial\Omega$)

$$\int_\Omega m\Delta u\overline{u}dx = -\int_\Omega \nabla(m\overline{u}) \bullet \nabla u dx = -\int_\Omega m|\nabla u|^2 dx - \sum_{j=1}^n \int_\Omega \frac{\partial m}{\partial x_j}\overline{u}\frac{\partial u}{\partial x_j}dx,$$

where m satisfies (4.6). So, if $(\lambda m(x) - \Delta + c(x))u = f \in L^2(\Omega)$, $u = 0$ on $\partial\Omega$ (as (4.12)–(4.13)) then, similarly to (4.18), for sufficiently large $\lambda \in \tilde{\Sigma}_1$,

$$\left(\int_\Omega m(x)|\nabla u(x)|^2 dx\right)^{\frac{1}{2}} \leq C|\lambda|^{-\frac{\rho}{2(2-\rho)}}\|f\|_{L^2(\Omega)}. \tag{4.19}$$

As a consequence, consider the initial boundary value problem

$$\frac{\partial}{\partial t}(m(x)u(x,t)) = \Delta u(x,t) + \sqrt{m(x)}\sum_{i=1}^n a_i(x)\frac{\partial u(x,t)}{\partial x_i} - c(x)u(x,t)$$
$$+ f(x,t),\ (x,t) \in \Omega \times (0,\tau], \tag{4.20}$$
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\tau], \tag{4.21}$$
$$m(x)u(x,0) = v_0(x), \quad x \in \Omega. \tag{4.22}$$

Suppose that $m, a_i \in C^1(\overline{\Omega})$, (4.6) is satisfied, c is a measurable bounded function on $\overline{\Omega}$, $c(x) \geq \delta > 0$, $f \in C^\sigma([0,\tau]; L^2(\Omega))$, $\frac{1-\rho}{2-\rho} < \sigma \leq 1$, and $v_0 \in L^2(\Omega)$. In $X := L^2(\Omega)$, consider the operator $Mu := m(\cdot)u$, $D(M) := X$, $Lu := \Delta u - c(\cdot)u$, $D(L) := H^2(\Omega) \cap H_0^1(\Omega)$, $L_1 u := \sqrt{m(\cdot)}\sum_{i=1}^n a_i(\cdot)\frac{\partial u}{\partial x_i}$, $D(L_1) := H^1(\Omega)$. Then, from (4.7) (which is true in $\tilde{\Sigma}_1$), we get condition 1 of Theorem 3.2 (with $\eta = \frac{1}{2-\rho}$) and, from (4.19), we get the estimate in Remark 3.3 (with $\sigma = \frac{\rho}{2(2-\rho)}$). So, the conclusion of Theorem 3.2 for the operator $L + L_1$ is true. Hence, by [6, Theorem 3.8] (with $\alpha = 1$, $\beta = \frac{1}{2-\rho}$, $\gamma > 0$ big enough), problem (4.20)–(4.22) admits a unique strict solution $u(x,t)$, i.e., $m(x)u \in C^1((0,\tau]; L^2(\Omega))$, $(L+L_1)u \in C((0,\tau]; L^2(\Omega))$, u satisfies (4.20)–(4.21), and condition (4.22) is understood in the seminorm sense $\|m(L+L_1)^{-1}(mu - v_0)\|_{L^2(\Omega)} \to 0$, as $t \to 0^+$.

Example 7. In the last example, we use the result in [3, Theorem 4.4]. Given an unbounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, with a boundary $\partial\Omega$ of class C^2 and an admissible domain according to [3, Definition 4.1]. We consider the linear second-order differential expression in divergence form

$$A(x, D_x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) - a_0(x)$$

with $a_{ij} = a_{ji} \in C_b^1(\overline{\Omega})$, $i, j = 1, \dots, n$, $a_0 \in C_b(\overline{\Omega})$, $a_0 \geq \nu_1 > 0$, $\exists \nu_1 > 0$,

$$\nu_2|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \nu_3|\xi|^2, \quad \forall (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n, \ \exists \nu_2, \nu_3 > 0,$$

$m \in C_b^2(\overline{\Omega})$ is non-negative and $|\nabla m(x)| \leq Km(x)$, $(A(x, D_x), m)$ is an admissible pair according to [3, Definition 4.2]. Note that $m = e^v$, where $v \in C^{0,1}(\overline{\Omega})$ and $\sup_{x \in \Omega} v(x) < +\infty$ satisfies the assumptions above.

According to the quoted [3, Theorem 4.4], for all $p \in (1, +\infty)$ there exists $\omega_p \geq 0$ such that if $\lambda \in \omega_p + \Sigma_1$, where again $\Sigma_1 := \{\lambda \in \mathbb{C} \ : \ \Re e\lambda \geq -c(1 + |\Im m\lambda|)\}$ for some $c > 0$, the spectral equation $\lambda m(x)u - A(x, D_x)u = f$, $f \in L^p(\Omega)$, admits a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying the estimates

$$\|mu\|_{L^p(\Omega)} \leq C|\lambda|^{-\beta}\|f\|_{L^p(\Omega)}, \tag{4.23}$$

$$\|m|\nabla u|\|_{L^p(\Omega)} \leq C|\lambda|^{-\beta+\frac{1}{2}}\|f\|_{L^p(\Omega)}, \tag{4.24}$$

where

$$\beta = \begin{cases} 1, & \text{if } \ p \in (1,2], \\ \frac{2}{p}, & \text{if } \ p \in [2, +\infty). \end{cases}$$

Consider now the following initial boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t}(m(x)u(x,t)) = & \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u(x,t)}{\partial x_j}) + m(x)\sum_{i=1}^{n} a_i(x)\frac{\partial u(x,t)}{\partial x_i} \\ & - a_0(x)u(x,t) + f(x,t), \ (x,t) \in \Omega \times (0, \tau], \end{aligned} \tag{4.25}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0, \tau], \tag{4.26}$$

$$m(x)u(x,t) = v_0(x), \quad x \in \Omega, \tag{4.27}$$

where $a_i \in C_b(\overline{\Omega})$. Moreover, assume $1 < p < 4$ (so that $1 \geq \beta > \frac{1}{2}$). Take $X = L^p(\Omega)$ and denote $D(M) := X$, $Mu := m(\cdot)u$, $D(L) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $Lu := A(x, D_x)u$, $D(L_1) := W^{1,p}(\Omega)$, $L_1 u := m(\cdot)\sum_{i=1}^{n} a_i(\cdot)\frac{\partial u}{\partial x_i}$. Then, the inequalities (4.23) and (4.24) provide the conclusion of Theorem 3.2 for the operator $L + L_1$. More precisely, the inequality (4.23) implies condition 1 of Theorem 3.2 (with $\eta = \beta$) and the inequality (4.24) implies the estimate in Remark 3.3 (with $\sigma = \beta - \frac{1}{2}$). Therefore, using [6, Theorem 3.8] (with $\alpha = 1$, $\beta = \beta$, $\gamma = \omega_p$), we get the following result. For all $f \in C^\sigma([0, \tau]; L^p(\Omega))$, $1 - \beta < \sigma \leq 1$, $v_0 \in L^p(\Omega)$, problem (4.25)–(4.27) has a unique strict solution $u(x,t)$, i.e., $m(x)u \in C^1((0, \tau]; L^p(\Omega))$, $(L + L_1)u \in C((0, \tau]; L^p(\Omega))$, u satisfies (4.25)–(4.26), and (4.27) is satisfied in the seminorm sense $\|m(L + L_1)^{-1}(mu - v_0)\|_{L^p(\Omega)} \to 0$, as $t \to 0^+$.

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Semilinear Stochastic Integral Equations in L_p

Wolfgang Desch and Stig-Olof Londen

Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. We consider a semilinear parabolic stochastic integral equation

$$u(t,\omega,x) = Aa_\alpha * u(t,\omega,x) + \sum_{k=1}^{\infty} a_\beta \star G^k(t,\omega,u(t,\omega,\cdot))(x) + a_\gamma * F(t,\omega,u(t,\omega,\cdot))(x) + u_0(\omega,x) + tu_1(\omega,x).$$

Here $t \in [0,T]$, ω in a probability space Ω, x in a σ-finite measure space B with (positive) measure Λ. The kernels $a_\mu(t)$ are multiples of $t^{\mu-1}$. The operator $A : \mathcal{D}(A) \subset L_p(B) \to L_p(B)$ is such that $(-A)$ is a nonnegative operator. The convolution integrals $a_\beta \star G^k$ are stochastic convolutions with respect to independent scalar Wiener processes w^k. $F : [0,T] \times \Omega \times \mathcal{D}((-A)^\theta) \to L_p(B)$ and $G : [0,T] \times \Omega \times \mathcal{D}((-A)^\theta) \to L_p(B,l_2)$ are nonlinear with suitable Lipschitz conditions.

We establish an L_p-theory for this equation, including existence and uniqueness of solutions, and regularity results in terms of fractional powers of $(-A)$ and fractional derivatives in time.

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Keywords. Semilinear stochastic integral equations, stochastic fractional differential equation, regularity, nonnegative operator, Volterra equation, singular kernel.

1. Introduction

Since our equation (i.e., (1.1) below) reads somewhat complicated and involved, let us start with a casual motivation. The setting of spaces, domains and operators will be given more precisely as soon as we state (1.1). The prototype of an equation for modelling diffusion processes is the heat equation

$$\frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + f(t,x).$$

The operator Δ, as usual, denotes the Laplacian in space. Instead of the source term $f(t,x)$ one might consider white noise. Then the equation turns into the stochastic heat equation

$$\frac{\partial}{\partial t}u(t,x,\omega) = \Delta u(t,\omega,x) + g(t,\omega,x)\,\dot{W}_t.$$

Here, typically, W_t is standard Brownian motion, although fractional Brownian motion has also been considered ([2]). There is an abundance of literature on the stochastic heat equation and its generalizations. Frequently Δ is replaced by a general elliptic partial differential operator or even an abstract operator A (where $(-A)$ is a positive operator), in Hilbert space as well as (more recently) in L_p or general Banach spaces. The key issues in handling such equations are stochastic integration in Banach spaces, and exploiting the regularity properties of parabolic partial differential equations.

Diffusion according to the heat equation, i.e., according to Fick's law, has the property of exponentially decaying memory in time. However, many processes in physics, chemistry and finance exhibit time memory decaying only at a power law rather than exponentially. For a survey of such phenomena see [19, Chapter 8]. Such anomalous diffusion is frequently referred to as subdiffusion or superdiffusion, depending on whether the net motion of particles happens more slowly or quickly than random diffusion according to Fick's law. Anomalous diffusion can be modelled by a fractional differential equation

$$D_t^\alpha u(t,x) = \Delta u(t,x) + f(t,x).$$

Here, D_t^α means the fractional derivative of order $\alpha \in (0,2)$ with respect to time. The case $\alpha < 1$ describes subdiffusion, while $\alpha > 1$ corresponds to superdiffusion. The limiting case $\alpha = 2$, of course, yields the wave equation. Deterministic fractional differential – partial differential equations with parabolic differential operators in space and fractional derivatives in time have been subject to thorough investigations, both from theory and applications ([19], [24], [25]). Notice that the equation may be integrated to give

$$u(t,x) = u_0(x) + \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\Delta u(s,x)\,ds + \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} f(s,x)\,ds.$$

(If $\alpha > 1$, we need an additional term $tu_1(x)$ to take care of the initial condition $\frac{d}{dt}u(0) = u_1$.) This equation can be considered as a parabolic abstract evolutionary equation. An extensive theory is available to treat such equations ([29]).

Again, the source term can be replaced by a stochastic additive perturbation of the system. The equation now turns into

$$D_t^\alpha u(t,\omega,x) = \Delta u(t,\omega,x) + D_t^{\beta-\alpha} g(t,\omega,x)\dot{W}_t.$$

Here W_t is Brownian motion, but the introduction of the fractional derivative or fractional integral $D_t^{\beta-\alpha}$ allows us to model smoother ($\beta < \alpha$) or rougher ($\beta > \alpha$) stochastic perturbation. To our knowledge, the first attempt to tackle this equation in L^p with $p \neq 2$ was made in [11] and extended in [13].

In this paper we go a step further and investigate state dependent forcing, i.e., semilinear equations. The prototype of the equation to be treated here is

$$D_t^\alpha u(t,\omega,x) = \Delta u(t,\omega,x) + G(t,u(t,\omega,x))\dot{W}_t(\omega) + F(t,u(t,\omega,x)).$$

We treat semilinear feedback on the stochastic (volatility) term as well as on the deterministic forcing (drift). Semilinear stochastic heat equations (with $\frac{d}{dt}$ instead of the fractional derivative D_t^α) in spaces more general than than Hilbert spaces have been recently in the center of interest of several research groups, just to mention a few, we refer to [21], as well as work based on abstract stochastic integration in Banach spaces like [4], [27], [38] and others. To our knowledge, the present paper is the first attempt to deal with the fractional derivative case in L_p with $p \neq 2$.

One of our central tasks is to balance space and time regularity. To have as much freedom as possible, we put again fractional integrals in front of all forcing terms:

$$D_t^\alpha u(t,\omega,x) = \Delta u(t,\omega,x) + D_t^{\beta-\alpha} G(t,\omega,u(t,\omega,x))\dot{W}_t + D_t^{\gamma-\alpha} F(t,\omega,u(t,\omega,x)).$$

Regularity in space will be expressed by fractional powers of $(-\Delta)$. In integrated form and full generality, our equation finally reads:

$$\begin{aligned} u(t,\omega,x) = A &\int_0^t a_\alpha(t-s)u(s,\omega,x)\,ds \\ &+ \sum_{k=1}^\infty \int_0^t a_\beta(t-s)G^k(s,\omega,u(s,\omega,\cdot))(x)\,dw_s^k \\ &+ \int_0^t a_\gamma(t-s)F(s,\omega,u(s,\omega,\cdot))(x)\,ds + u_0(\omega,x) + tu_1(\omega,x). \end{aligned} \tag{1.1}$$

The real scalar-valued solution $u(t,\omega,x)$ depends on $t \in [0,T]$, ω in a probability space Ω, and x in a measure space B. The convolution kernels a_μ are defined by

$$a_\mu(t) := \frac{1}{\Gamma(\mu)}\, t^{\mu-1}. \tag{1.2}$$

We assume $\alpha \in (0,2)$, $\beta > \frac{1}{2}$, and $\gamma > 0$. While the parameter α is the order of the fractional time derivative, the parameter β regulates the time regularity of the stochastic semilinear feedback, and γ regulates the time regularity of the deterministic feedback. The operator $A : \mathcal{D}(A) \subset L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ (with $2 \leq p < \infty$) is such that $(-A)$ is a nonnegative linear operator (see Section 2 below). In particular we have in mind elliptic partial differential operators on a sufficiently smooth (bounded or unbounded) domain $B \subset \mathbb{R}^n$, but formally we require only that $(-A)$ is sectorial and the state space is an L_p-space on some measure space B. The processes w_s^k are scalar-valued, independent Wiener processes. F and G^k are nonlinear and satisfy suitable Lipschitz estimates with respect to u. The functions u_0 and u_1 are given initial data. For the precise conditions, see Section 3.

Our goal is to establish existence and uniqueness of solutions for the semilinear equation (1.1) in an L_p-framework with $p \in [2, \infty)$. Regularity results will be stated in terms of fractional powers of $-A$ (for spatial regularity) and fractional time integrals and derivatives as well as Hölder continuity (for time regularity).

Technically we rely primarily on results concerning a linear integral equation where the forcing terms F and G are replaced by functions independent of u, i.e., (5.1). In recent work [13] we have developed an L_p-theory for (5.1), albeit without the deterministic part and without the u_1-term. These results need, however, – for the purpose of analyzing (1.1) – to be extended and to be made more precise.

Our linear results build on an approach due to Krylov, developed for parabolic stochastic partial differential equations. This approach uses the Burkholder-Davis-Gundy inequality and estimates on the solution and on its spatial gradient. To analyze the integral equation (5.1) we combine Krylov's approach with transformation techniques and estimates involving both fractional powers of $-A$, and fractional time-derivatives (integrals) of the solution. Krylov's approach is very efficient in obtaining maximal regularity, however, it relies on a highly nontrivial Paley-Littlewood inequality [20]. A counterpart of this estimate can be given for general sectorial A by straightforward estimates on the Dunford integral, when we allow for an infinitesimal loss of regularity.

We also include results for the deterministic convolution and for the u_1-term. Obviously, no originality is claimed for these results.

To obtain result on the semilinear equation (1.1) we combine our linear theory with a standard contraction approach.

The paper is organized as follows: Before we can state our main results, we need to collect some facts about sectorial operators and fractional differentiation and integration in Section 2. Section 3 states the hypotheses and results for the semilinear equation. In Section 4 we provide the tools to define a stochastic integral and a stochastic convolution in L_p-spaces. The central part of this section is an application of the Burkholder-Davis-Gundy inequality to lift scalar-valued Ito-integrals to stochastic integrals in L_p. This approach is adapted from [21]. Section 5 deals with the linear fractional differential equation. In the beginning we give the results on existence and regularity which are basic to obtain similar results on the semilinear equation. We construct the solution via the resolvent operator and a variation of parameters formula. The contribution of the initial data and of the forcing F, which enters as a Lebesgue integral, are well known ([29], [39]). The contribution of the stochastic integral containing G is handled by a recent result [13]. We collect these results in a unified way to allow a comparison of the various requirements on regularity. In Section 6 we arrive at the proof of our main results on the semilinear equation by a standard contraction procedure. In Section 7 we make some comments on available maximal regularity results for the linear equation and their implications for the semilinear equation. Finally, in Section 8 we compare our results to some recent results on parabolic stochastic differential equations obtained recently using an abstract theory of stochastic integration in Banach spaces.

2. Fractional powers and fractional derivatives

In this paper $A : \mathcal{D}(A) \subset L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ will be a linear operator such that $(-A)$ is nonnegative. Here $p \in [2,\infty)$, but fixed. Regularity in space will be expressed in terms of the fractional powers $(-A)^\theta$ of A. Regularity in time will be expressed in terms of fractional time derivatives $D_t^\eta f$. In corollaries we will also give regularity results in terms of the function spaces $h^\gamma_{0\to 0}([0,T];X)$, i.e., the little Hölder-continuous functions with $f(0) = 0$.

In this section we summarize briefly the definitions and some known results about nonnegative operators, their fractional powers, and about fractional integration and differentiation.

Let X be a complex Banach space and let $\mathcal{L}(X)$ be the space of bounded linear operators on X. Let B be a closed, linear map of $\mathcal{D}(B) \subset X$ into X. The operator $-B$ is said to be nonnegative if $\rho(B)$, the resolvent set of B, contains $(0,\infty)$, and

$$\sup_{\lambda>0} \|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(X)} < \infty.$$

An operator is positive if it is nonnegative and, in addition, $0 \in \rho(B)$. For $\omega \in [0,\pi)$, we define

$$\Sigma_\omega := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\ \lambda| < \omega\}.$$

Recall that if $(-B)$ is nonnegative, then there exists a number $\eta \in (0,\pi)$ such that $\rho(B) \supset \Sigma_\eta$, and

$$\sup_{\lambda\in\Sigma_\eta} \|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(X)} < \infty. \tag{2.1}$$

The spectral angle of $(-B)$ is defined by

$$\phi_{(-B)} := \inf\Big\{\omega \in (0,\pi] \mid \rho(B) \supset \Sigma_{\pi-\omega},\ \sup_{\lambda\in\Sigma_{\pi-\omega}} \|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(X)} < \infty\Big\}.$$

We will rely on the concept of fractional powers of $(-B)$: Let $(-B)$ be a densely defined nonnegative linear operator on X, and $\theta > 0$. If $(-B)$ is positive, then $(-B)^{-1}$ is a bounded operator, and $(-B)^{-\theta}$ can be defined by integral formulas [5, Ch. 3] or [22, Section 2.2.2]. As usual,

$$(-B)^\theta := ((-B)^{-\theta})^{-1}, \quad \theta > 0. \tag{2.2}$$

If $(-B)$ is nonnegative with $0 \in \sigma(-B)$, we proceed as in [5, Ch. 5]: Since $(-B+\epsilon I)$ is a positive operator if $\epsilon > 0$, its fractional power $(-B+\epsilon I)^\theta$ is well defined according to (2.2). We define

$$\mathcal{D}(((-B)^\theta)) := \Big\{y \in \bigcap_{0<\epsilon\le\epsilon_0} \mathcal{D}(((-B+\epsilon I)^\theta)) \mid \lim_{\epsilon\to 0+} (-B+\epsilon I)^\theta y \text{ exists}\Big\}, \tag{2.3}$$

$$(-B)^\theta y := \lim_{\epsilon\to 0+} (-B+\epsilon I)^\theta y \quad \text{for } y \in \mathcal{D}(((-B)^\theta)). \tag{2.4}$$

Lemma 2.1. *Let $-B$ be a nonnegative linear operator on a Banach space X with spectral angle $\phi_{(-B)}$, and let $\theta > 0$.*

1) *$(-B)^\theta$ is closed and $\overline{\mathcal{D}(((-B)^\theta))} = \overline{\mathcal{D}((-B))}$.*
2) *Assume that $\theta\phi_{(-B)} < \pi$. Then $(-B)^\theta$ is nonnegative and has spectral angle $\theta\phi_{(-B)}$.*

Proof. For (1) see [5, p. 109, 142], also [8, Theorem 10]. For (2) see [5, p. 123]. □

Lemma 2.2. *Let $-B$ be a nonnegative linear operator on a Banach space X with spectral angle $\phi_{(-B)}$. Then for $\eta \in [0, \pi - \phi_{(-B)})$*

$$\sup_{|\arg\ \mu| \leq \eta,\, \mu \neq 0} \|(-B)^\theta \mu^{1-\theta}(\mu I - B)^{-1}\|_{\mathcal{L}(X)} < \infty. \tag{2.5}$$

Proof. In case $\eta = 0$, see [5, Th. 6.1.1, p. 141]. The general case can be reduced to the case $\mu > 0$, [15, p.314]. See also [13, Lemma 3.3]. □

We turn now to fractional differentiation and integration in time:

Definition 2.3. Let X be a Banach space and $\alpha \in (0,1)$, let $u \in L_1((0,T);X)$ for some $T > 0$.

1) Fractional integration in time of order α is defined by $D_t^{-\alpha}u := \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * u$.
2) We say that u has a fractional derivative of order $\alpha > 0$ provided $u = D_t^{-\alpha}f$, for some $f \in L_1((0,T);X)$. If this is the case, we write $D_t^\alpha u = f$.

Remark 2.4. Suppose that u has a fractional derivative of order $\alpha \in (0,1)$. Then $\frac{1}{\Gamma(1-\alpha)}t^{-\alpha} * u$ is differentiable a.e. and absolutely continuous with $D_t^\alpha u = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}t^{-\alpha} * u\right)$.

For the equivalence of fractional derivatives in L_p and fractional powers of the realization of the derivative in L_p, we have the following lemma.

Lemma 2.5. [9, Prop.2] *Let $p \in [1,\infty)$, X a Banach space and define*

$$\mathcal{D}(L) := \{u \in W^{1,p}((0,T);X) \mid u(0) = 0\},\ Lu = u' \textit{ for } u \in \mathcal{D}(L).$$

Then, with $\beta \in (0,1)$,

$$L^\beta u \;=\; D_t^\beta u, \quad u \in \mathcal{D}(L^\beta), \tag{2.6}$$

where $\mathcal{D}(L^\beta)$ coincides with the set of functions u having a fractional derivative in L_p, i.e.,

$$\mathcal{D}(L^\beta) = \{u \in L_p((0,T);X) \mid \frac{1}{\Gamma(1-\beta)}t^{-\beta} * u \in W_0^{1,p}((0,T);X)\}.$$

In particular, D_t^β is closed.

We refer to [9] for further properties of the operator D_t^β.

3. The main result

Hypothesis 3.1. Let $(B, \mathcal{A}, \Lambda)$ be a σ-finite measure space and fix $2 \leq p < \infty$. Let $(-A) : \mathcal{D}(A) \subset L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ be a nonnegative linear operator with spectral angle $\phi_{(-A)}$, and such that $\mathcal{D}(A) \cap L_1(B;\mathbb{R}) \cap L_\infty(B;\mathbb{R})$ is dense in $L_p(B;\mathbb{R})$.

Hypothesis 3.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with an increasing, right continuous filtration $\{\mathcal{F}_t \mid t \geq 0\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$ for all $t \geq 0$. Let $\mathcal{P}$ denote the predictable σ-algebra on $[0,\infty) \times \Omega$ generated by $\{\mathcal{F}_t\}$, and assume that $\{w_s^k \mid k = 1, 2, 3, \dots\}$ is an independent family of (scalar-valued) $\mathcal{F}_t$-adapted Wiener processes on $(\Omega, \mathcal{F}, \mathbb{P})$.

Remark 3.3. On $[0,T] \times \Omega$, measurability will always be understood with respect to the predictable σ-algebra $\mathcal{P}$, and the product measure of the Lebesgue measure on $[0,T]$ and $\mathbb{P}$.

Hypothesis 3.4. For suitable $\theta \in [0,1)$ and $\epsilon \in [0,1)$, the function

$$F : [0,T] \times \Omega \times \mathcal{D}((-A)^\theta) \to \mathcal{D}((-A)^\epsilon)$$

satisfies the following assumptions:

(a) For fixed $u \in \mathcal{D}((-A)^\theta)$, the function $F(\cdot,\cdot,u)$ is measurable from $[0,T] \times \Omega$ into $\mathcal{D}((-A)^\epsilon)$.

(b) There exists a constant $M_F > 0$, such that for all $t \in [0,T]$, and all $u_1, u_2 \in \mathcal{D}((-A)^\theta)$ the following Lipschitz estimate holds

$$\|F(t,\omega,u_1) - F(t,\omega,u_2)\|_{\mathcal{D}((-A)^\epsilon)} \leq M_F \|u_1 - u_2\|_{\mathcal{D}((-A)^\theta)} \quad \text{for a.e. } \omega \in \Omega. \tag{3.1}$$

(c) For $u = 0$ we have

$$\left[\int_\Omega \int_0^T \|F(t,\omega,0)\|^p_{\mathcal{D}((-A)^\epsilon)}\, dt\, d\mathbb{P}\right]^{1/p} = M_{F,0} < \infty. \tag{3.2}$$

Hypothesis 3.5. For the same $\theta \in [0,1)$ as in Hypothesis 3.4, the function

$$G : [0,T] \times \Omega \times \mathcal{D}((-A)^\theta) \to L_p(B; l_2)$$
$$[G(t,\omega,u)](x) := \big(G^k(t,\omega,u)(x)\big)_{k=1}^\infty$$

satisfies the following assumptions:

(a) For fixed $u \in \mathcal{D}((-A)^\theta)$, the function $G(\cdot,\cdot,u)$ is measurable from $[0,T] \times \Omega$ into $L_p(B; l_2)$.

(b) There exists a constant $M_G > 0$, such that for all $t \in [0,T]$, and all $u_1, u_2 \in \mathcal{D}((-A)^\theta)$ the following Lipschitz estimate holds:

$$\|G(t,\omega,u_1) - G(t,\omega,u_2)\|_{L_p(B;l_2)} \leq M_G \|u_1 - u_2\|_{\mathcal{D}((-A)^\theta)} \quad \text{for a.e. } \omega \in \Omega. \tag{3.3}$$

(c) For $u = 0$ we have

$$\left[\int_\Omega \int_0^T \|G(t,\omega,0)\|^p_{L_p(B;l_2)}\, dt\, d\mathbb{P}\right]^{1/p} = M_{G,0} < \infty. \tag{3.4}$$

Theorem 3.6. *Let the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and the Wiener processes* $\left(w_s^k\right)_{k=1}^{\infty}$ *be as in Hypothesis* 3.2. *Let* $p \in [2, \infty)$, *let the measure space* $(B, \mathcal{A}, \Lambda)$ *and the operator* $A : \mathcal{D}(A) \subset L_p(B; \mathbb{R}) \to L_p(B; \mathbb{R})$ *satisfy Hypothesis* 3.1. *Let* $\alpha \in (0, 2)$, $\beta > \frac{1}{2}$ *and* $\gamma > 0$. *Let* $T > 0$ *and assume that* $F : [0, T] \times \Omega \times \mathcal{D}((-A)^{\theta}) \to \mathcal{D}((-A)^{\epsilon})$ *and* $G : [0, T] \times \Omega \times \mathcal{D}((-A)^{\theta}) \to L_p(B; l_2)$ *satisfy Hypotheses* 3.4 *and* 3.5 *with suitable* $\theta, \epsilon \in [0, 1]$. *Let* $u_0 \in L_p(\Omega; \mathcal{D}((-A)^{\delta_0}))$, $u_1 \in L_p(\Omega; \mathcal{D}((-A)^{\delta_1}))$, *with suitable* $\delta_i \in [0, 1]$, *both* u_i *measurable with respect to* $\mathcal{F}_0$. *Suppose that the following inequalities hold:*

$$\alpha\theta < \gamma + \alpha\epsilon, \tag{3.5}$$

$$\frac{1}{2} + \alpha\theta < \beta, \tag{3.6}$$

$$\alpha\theta < \frac{1}{p} + \alpha\delta_0, \tag{3.7}$$

$$\alpha\theta < 1 + \frac{1}{p} + \alpha\delta_1. \tag{3.8}$$

Then there exists a unique function $u \in L_p([0, T] \times \Omega; \mathcal{D}((-A)^{\theta}))$ *such that for almost all* $t \in [0, T]$

$$\int_0^t a_\alpha(t - s) u(s, \omega, \cdot)\, ds \in \mathcal{D}(A) \quad \text{for a.e. } \omega \in \Omega,$$

and (1.1) *is satisfied for almost all* $t \in [0, T]$ *and almost all* $\omega \in \Omega$.

The following theorem provides additional regularity results in terms of fractional power domains $\mathcal{D}((-A)^{\zeta})$ and of fractional time derivatives. However, depending on the parameters, $u(t) - u_0$ may sometimes exhibit more regularity than $u(t)$ itself. Similar considerations hold for $u(t) - tu_1$ and $u(t) - a_\gamma * F(\cdot, \omega, u)$. To handle all cases in one term, we will introduce the function v in (3.13).

Theorem 3.7. *Let the assumptions of Theorem* 3.6 *hold. Moreover, assume that* $\eta \in (-1, 1)$, $\zeta \in [0, 1]$ *are such that*

$$\eta + \alpha\zeta < \gamma + \alpha\epsilon, \tag{3.9}$$

$$\frac{1}{2} + \eta + \alpha\zeta < \beta, \tag{3.10}$$

$$\eta + \alpha\zeta < \frac{1}{p} + \alpha\delta_0, \tag{3.11}$$

$$\eta + \alpha\zeta < 1 + \frac{1}{p} + \alpha\delta_1. \tag{3.12}$$

With the notation $1_{\{a>b\}} = 1$ *if* $a > b$ *and* $1_{\{a>b\}} = 0$ *if* $a \leq b$, *we put*

$$\begin{aligned} v(t) = u(t) &- 1_{\{\delta_0 > \zeta\}} u_0 - 1_{\{\delta_1 > \zeta\}} t u_1 \\ &- 1_{\{\epsilon > \zeta\}} \int_0^t a_\gamma(t - s) F(s, \omega, u(s))\, ds. \end{aligned} \tag{3.13}$$

(a) *Then, if* $\eta > 0$*, the function* v*, considered as a Banach space valued function* $v : [0,T] \to L_p(\Omega; \mathcal{D}((-A)^\zeta))$*, has a fractional derivative of order* η*.*
(b) *If* $\eta < 0$*, the function* $v : [0,T] \to L_p(\Omega; L_p(B; \mathbb{R}))$ *has a fractional integral of order* $-\eta$*. Moreover,* $D_t^\eta v$ *takes values in* $L_p(\Omega; \mathcal{D}((-A)^\zeta))$*.*
(c) *If* $\eta = 0$*, of course, we denote* $D_t^0 v = v$*.*

In any case, there exists a constant M_u*, depending on* A*,* p*,* T*,* α*,* β*,* γ*,* δ_i*,* ϵ*,* ζ*,* η*,* θ*,* M_F*,* M_G *such that*

$$\begin{aligned} &\|D_t^\eta v\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\zeta))} \\ &\quad \le M_u \left[\|u_0\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_0}))} + \|u_1\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_1}))} + M_{F,0} + M_{G,0}\right]. \end{aligned} \tag{3.14}$$

Corollary 3.8. *Let the Assumptions of Theorem* 3.6 *hold. Let* $\zeta \in [0,1]$*. Let* u *be the solution of* (1.1) *and* v *be defined by* (3.13)*.*

(1) *Let* $p < q < \infty$ *be such that*

$$\frac{1}{p} - \frac{1}{q} + \alpha\zeta < \gamma + \alpha\epsilon, \qquad \frac{1}{2} + \frac{1}{p} - \frac{1}{q} + \alpha\zeta < \beta,$$
$$\alpha\zeta - \frac{1}{q} < \alpha\delta_0, \qquad \alpha\zeta - \frac{1}{q} < 1 + \alpha\delta_1.$$

Then $v \in L_q([0,T]; L_p(\Omega; \mathcal{D}((-A)^\zeta)))$*.*

(2) *Let* $\mu \in (0, 1 - \frac{1}{p})$ *be such that*

$$\frac{1}{p} + \mu + \alpha\zeta < \gamma + \alpha\epsilon, \qquad \frac{1}{2} + \frac{1}{p} + \mu + \alpha\zeta < \beta,$$
$$\mu + \alpha\zeta < \alpha\delta_0, \qquad \mu + \alpha\zeta < 1 + \alpha\delta_1.$$

Then $v \in h^\mu_{0\to 0}([0,T]; L_p(\Omega; \mathcal{D}((-A)^\zeta)))$*.*

Hypothesis 3.9. Let $F_1, F_2 : [0,T] \times \Omega \times \mathcal{D}((-A)^\theta) \to \mathcal{D}((-A)^\epsilon)$ satisfy Hypothesis 3.4, $G_1, G_2 : [0,T] \times \Omega \times \mathcal{D}((-A)^\theta) \to L_p(B; l_2)$ satisfy Hypothesis 3.5, and suppose that there are nonnegative functions $\mu_{\Delta F}$, $\mu_{\Delta G} \in L_p([0,T] \times \Omega; \mathbb{R})$ such that for all $t \in [0,T]$ and $u \in \mathcal{D}((-A)^\theta)$, and almost all $\omega \in \Omega$

$$\|F_1(t,\omega,u) - F_2(t,\omega,u)\|_{\mathcal{D}((-A)^\epsilon)} \le \mu_{\Delta F}(t,\omega), \tag{3.15}$$
$$\|G_1(t,\omega,u) - G_2(t,\omega,u)\|_{L_p(B;l_2)} \le \mu_{\Delta G}(t,\omega). \tag{3.16}$$

Remark 3.10. The standard example of F_i, G_i satisfying Hypothesis 3.9 is (for $i = 1,2$):

$$F_i(t,\omega,u) = F(t,\omega,u) + f_i(t,\omega),$$
$$G_i(t,\omega,u) = G(t,\omega,u) + g_i(t,\omega),$$

where F and G satisfy Hypotheses 3.4 and 3.5, respectively, and $f_i \in L_p([0,T] \times \Omega; \mathcal{D}((-A)^\epsilon))$, $g_i \in L_p([0,T] \times \Omega; L_p(B; l_2))$. Here we take

$$\mu_{\Delta F}(t,\omega) = \|f_1(t,\omega) - f_2(t,\omega)\|_{\mathcal{D}((-A)^\epsilon)},$$
$$\mu_{\Delta G}(t,\omega) = \|g_1(t,\omega) - g_2(t,\omega)\|_{L_p(B;l_2)}.$$

Theorem 3.11. *Let the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and the Wiener processes* w_s^k *be as in Hypothesis* 3.2. *Let* $p \in [2, \infty)$, *let the measure space* $(B, \mathcal{A}, \Lambda)$ *and the operator* $A : \mathcal{D}(A) \subset L_p(B; \mathbb{R}) \to L_p(B; \mathbb{R})$ *satisfy Hypothesis* 3.1.

Let $T > 0$, $\alpha \in (0, 2)$, $\beta > \frac{1}{2}$, $\gamma > 0$, *and* δ_0, δ_1, $\epsilon \in [0, 1]$ *be such that* (3.5), (3.6), (3.7), *and* (3.8) *hold. Let* $\eta \in (-1, 1)$ *and* $\zeta \in [0, 1]$ *be such that* (3.9), (3.10), (3.11), (3.12) *hold. Then there exists a constant* $M_{\Delta u} > 0$, *dependent on* $p, T, \alpha, \beta, \gamma, \delta_0, \delta_1, \epsilon, \zeta, M_F, M_G$, *such that the following Lipschitz estimate holds:*

Let F_1, F_2, G_1, G_2 *satisfy Hypotheses* 3.4, 3.5 *and* 3.9 *with* ϵ, θ *as above. For* $i = 1, 2$ *let the initial data* $u_{0,i} \in L_p(\Omega; \mathcal{D}((-A)^{\delta_0}))$ *and* $u_{1,i} \in L_p(\Omega; \mathcal{D}((-A)^{\delta_1}))$ *be* $\mathcal{F}_0$*-measurable, and let* $u_1(t, \omega, x)$, $u_2(t, \omega, x)$ *be the solutions of* (1.1) *with* F, G, u_0, u_1 *replaced by* $F_i, G_i, u_{0,i}, u_{1,i}$. *Let* v_i *be defined according to* (3.13) *with* u *replaced by* u_i. *Then*

$$\begin{aligned} &\|D_t^\eta v_1 - D_t^\eta v_2\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\zeta))} \qquad (3.17)\\ &\le\ M_{\Delta u} \Big[\|u_{0,1} - u_{0,2}\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_0}))} + \|u_{1,1} - u_{1,2}\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_1}))} \\ &\qquad + \|\mu_{\Delta F}(t, \omega) + \mu_{\Delta G}(t, \omega)\|_{L_p([0,T]\times\Omega;\mathbb{R})}\Big]. \end{aligned}$$

4. Stochastic lemmas

Lemma 4.1 ([21], Theorem 3.10). *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfy Hypothesis* 3.2. *Let* Y *be a dense subspace of* $L_p(B; \mathbb{R})$, $0 < T \le \infty$, *and* $g \in L_p([0, T] \times \Omega; L_p(B; l_2))$. *Then there exists a sequence of functions* $g_j \in L_p([0, T] \times \Omega; L_p(B; l_2))$ *converging to* g *in* $L_p([0, T] \times \Omega, L_p(B; l_2))$ *such that each* $g_j = (g_j^k)_{k=1}^\infty$ *is of the form*

$$g_j^k(t, \omega, x) = \begin{cases} \sum_{i=1}^j I_{\tau_{i-1}^j(\omega) < t \le \tau_i^j(\omega)}(t) g_{j,i}^k(x) & \text{if } k \le j, \\ 0 & \text{else,} \end{cases} \qquad (4.1)$$

where $\tau_0^j \le \tau_1^j \le \cdots \tau_j^j$ *are bounded stopping times with respect to the filtration* $\mathcal{F}_t$, *and* $g_{j,i}^k \in Y$. (*Here, for any set* A, I_A *denotes its indicator function.*)

Remark 4.2. We will apply Lemma 4.1 with $Y = \mathcal{D}(A) \cap L_1(B; \mathbb{R}) \cap L_\infty(B; \mathbb{R})$.

Lemma 4.3. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *and the Wiener processes* w_t^k *be as in Hypothesis* 3.2. *Let* $p \in [2, \infty)$. *Let* Y *be a dense subspace of* $L_p(B; \mathbb{R})$, *let* $T > 0$, *and let* $g_j \in L_p([0, T] \times \Omega; L_p(B; l_2))$ *be of the simple structure given in* (4.1). *For* $t \in [0, T]$, *let* $V(t) : Y \to L_p(B; \mathbb{R})$ *be a linear operator such that the function* $t \mapsto V(t)y$ *is in* $L_2([0, T]; L_p(B; \mathbb{R}))$ *for each* $y \in Y$. *Then there exists a constant* M, *depending only on* p *and* T, *such that for all* $t \in (0, T]$

$$\begin{aligned} &\int_B \int_\Omega \left| \sum_{k=1}^j \int_0^t [V(t - s) g_j^k(s, \omega)](x)\, dw_s^k \right|^p d\mathbb{P}(\omega)\, d\Lambda(x) \\ &\quad \le M \int_B \int_\Omega \left(\int_0^t |[V(t - s) g_j(s, \omega)](x)|_{l_2}^2\, ds \right)^{\frac{p}{2}} d\mathbb{P}(\omega)\, d\Lambda(x). \end{aligned} \qquad (4.2)$$

Proof. First fix some $t \in (0,T]$. For $x \in B$, $r > 0$ we define

$$Y_j(r,\omega,x) = \sum_{k=1}^{j} \int_0^r [V(t-s)g_j^k(s,\omega)](x)\,dw_s^k.$$

By the elementary structure of g_j,

$$\int_0^r \left|[V(t-s)g_j^k(s,\omega)](x)\right|^2 ds < \infty$$

for almost all $x \in B$, so that $Y_j(r,\omega,x)$ is well defined as an Ito integral for such x, and it is a martingale. Since the Wiener processes w_s^k are independent, the quadratic variation of $Y_j(\cdot,\cdot,x)$ is

$$\sum_{k=1}^{j} \int_0^r \left|[V(t-s)g_j^k(s,\omega)](x)\right|^2 ds.$$

Now the Burkholder-Davis-Gundy inequality (see [18, p. 163]) yields for $r \in [0,t]$ and each $x \in B$,

$$\begin{aligned}
&\int_\Omega \Big|\sum_{k=1}^{j} \int_0^r [V(t-s)g_j^k(s,\omega)](x)\ dw_s^k\Big|^p d\mathbb{P}(\omega)\\
&\le M \int_\Omega \left(\int_0^r \sum_{k=1}^{j} |[V(t-s)g_j^k(s,\omega)](x)|^2\ ds\right)^{\frac{p}{2}} d\mathbb{P}(\omega) \qquad (4.3)\\
&= M \int_\Omega \left(\int_0^r |V(t-s)g_j(s,\omega)](x)|_{l_2}^2\ ds\right)^{\frac{p}{2}} d\mathbb{P}(\omega).
\end{aligned}$$

In (4.3), take $r = t$ and integrate over B:

$$\begin{aligned}
&\int_B \int_\Omega \Big|\sum_{k=1}^{j} \int_0^t [V(t-s)g_j^k(s,\omega)](x)\ dw_s^k\Big|^p d\mathbb{P}(\omega)\,d\Lambda(x)\\
&\le M \int_B \int_\Omega \left(\int_0^t |[V(t-s)g_j(s,\omega)](x)|_{l_2}^2\ ds\right)^{\frac{p}{2}} d\mathbb{P}(\omega)\,d\Lambda(x). \qquad \square
\end{aligned}$$

Lemma 4.4. *Let $(\Omega,\mathcal{F},\mathbb{P})$ and the Wiener processes w_t^k satisfy Hypothesis* 3.2. *Let $T > 0$, $2 \le p < \infty$, and $g \in L_p([0,T]\times\Omega; L_p(B;l_2))$, moreover, let $\{g_j\}$ be a sequence approximating g in the sense of Lemma* 4.1. *Let $\beta > \frac{1}{2}$, $\eta \in [0,1)$ such that $\beta - \eta > \frac{1}{2}$. Then the functions*

$$D_t^\eta \sum_{k=1}^{\infty} \int_0^t a_\beta(t-s)g_j^k(s,\omega,x)\,dw_s^k(\omega)$$

converge in $L_p([0,T]\times\Omega; L_p(B;\mathbb{R}))$, as $j \to \infty$.

Proof. Put $h^k_{i,j} := g^k_j - g^k_i$. The stochastic Fubini theorem implies that

$$D_t^{-\eta}\int_0^t a_{\beta-\eta}(t-s)h^k_{i,j}(s,\omega,x)\,dw^k_s = \int_0^t a_\beta(t-s)h^k_{i,j}(s,\omega,x)\,dw^k_s,$$

i.e.,

$$\int_0^t a_{\beta-\eta}(t-s)h^k_{i,j}(s,\omega,x)\,dw^k_s = D_t^{\eta}\int_0^t a_\beta(t-s)h^k_{i,j}(s,\omega,x)\,dw^k_s.$$

We use Lemma 4.3 and the fact that $a^2_{\beta-\eta} \in L_1([0,T];\mathbb{R})$:

$$\begin{aligned}
&\int_0^T\int_B\int_\Omega \left|D_t^\eta \sum_{k=1}^\infty \int_0^t a_\beta(t-s)h^k_{i,j}(s,\omega,x)\,dw^k_s\right|^p d\mathbb{P}(\omega)\,d\Lambda(x)\,dt\\
&= \int_0^T\int_B\int_\Omega \left|\sum_{k=1}^\infty \int_0^t a_{\beta-\eta}(t-s)h^k_{i,j}(s,\omega,x)\,dw^k_s\right|^p d\mathbb{P}(\omega)\,d\Lambda(x)\,dt\\
&\le M\int_0^T\int_B\int_\Omega \left(\int_0^t |a_{\beta-\eta}(t-s)h_{i,j}(s,\omega,x)|^2_{l_2}\,ds\right)^{\frac{p}{2}} d\mathbb{P}(\omega)\,d\Lambda(x)\,dt\\
&\le M\int_B\int_\Omega \left[\int_0^T a^2_{\beta-\eta}(s)\,ds\right]^{\frac{p}{2}}\left[\int_0^T |h_{i,j}(s,\omega,x)|^p_{l_2}\,ds\right] d\mathbb{P}(\omega)\,d\Lambda(x)\\
&\le M\|h_{i,j}\|^p_{L_p([0,T]\times\Omega;L_p(B;l_2))}.
\end{aligned}$$

As $i,j\to\infty$, we have $h_{i,j}\to 0$ in $L_p([0,T]\times\Omega;L_p(B;l_2))$, thus $D_t^\eta\sum_{k=1}^\infty\int_0^t a_\beta(t-s)g^k_j(s,\omega,x)\,dw^k_s(\omega)$ is a Cauchy sequence in $L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))$. □

Definition 4.5. Let $(\Omega,\mathcal{F},\mathbb{P})$ and the Wiener processes w^k_t satisfy Hypothesis 3.2. Let $T>0$, $2\le p<\infty$, and $g\in L_p([0,T]\times\Omega;L_p(B;l_2))$, moreover, let $\{g_j\}$ be a sequence approximating g in the sense of Lemma 4.1. Let $\beta>\frac{1}{2}$. Then we define

$$\begin{aligned}
(a_\beta\star g)(t,\omega) &:= \sum_{k=1}^\infty\int_0^t a_\beta(t-s)g^k(s,\omega,x)\,dw^k_s(\omega)\\
&:= \lim_{j\to\infty}\sum_{k=1}^\infty\int_0^t a_\beta(t-s)g^k_j(s,\omega,x)\,dw^k_s(\omega).
\end{aligned}$$

5. Linear theory

In this section we replace the semilinear inhomogeneity in (1.1) by inhomogeneities independent of u, so that we obtain a linear integral equation:

$$\begin{aligned}
u(t,\omega,x) = A\int_0^t a_\alpha(t-s)u(s,\omega,x)\,ds + \sum_{k=1}^\infty\int_0^t a_\beta(t-s)g^k(s,\omega,x)\,dw^k_s\\
+ \int_0^t a_\gamma(t-s)f(s,\omega,x)\,ds \;+\; u_0(\omega,x)+tu_1(\omega,x). \qquad (5.1)
\end{aligned}$$

We will prove the following propositions by a chain of lemmas:

Proposition 5.1. *Let the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and the Wiener processes* w_s^k *be as in Hypothesis* 3.2. *Let* $p \in [2, \infty)$, *let the measure space* $(B, \mathcal{A}, \Lambda)$ *and the operator* $A : \mathcal{D}(A) \subset L_p(B; \mathbb{R}) \to L_p(B; \mathbb{R})$ *satisfy Hypothesis* 3.1. *Assume that* $T > 0$ *and let* $f \in L_p([0,T] \times \Omega; L_p(B; \mathbb{R}))$, *and* $g \in L_p([0,T] \times \Omega; L_p(B, l_2))$. *Let* $u_0 \in L_p(\Omega; L_p(B; \mathbb{R}))$ *and* $u_1 \in L_p(\Omega; L_p(B; \mathbb{R}))$ *be* $\mathcal{F}_0$*-measurable.*

Let $\alpha \in (0, 2)$, $\beta > \frac{1}{2}$, $\gamma > 0$. *Then there exists a unique function* $u \in L_p([0,T] \times \Omega; L_p(B, \mathbb{R}))$ *such that for almost all* $t \in [0, T]$

$$\int_0^t a_\alpha(t-s) u(s, \omega, \cdot)\, ds \in \mathcal{D}(A) \quad \textit{for a.e. } \omega \in \Omega,$$

and (5.1) *holds for almost all* $\omega \in \Omega$ *and almost all* $t \in [0, T]$.

Proposition 5.2. *Let the assumptions of Proposition* 5.1 *hold. Suppose that* $f \in L_p([0,T] \times \Omega; \mathcal{D}((-A)^\epsilon))$, $u_0 \in L_p(\Omega; \mathcal{D}((-A)^{\delta_0}))$ *and* $u_1 \in L_p(\Omega; \mathcal{D}((-A)^{\delta_1}))$ *with suitable* $\epsilon, \delta_0, \delta_1 \in [0, 1)$. *Let* u *be as in Proposition* 5.1. *Let* $\eta \in (-1, 1)$, $\zeta \in [0, 1]$ *satisfy*

$$\eta + \alpha\zeta < \gamma + \alpha\epsilon, \tag{5.2}$$

$$\frac{1}{2} + \eta + \alpha\zeta < \beta, \tag{5.3}$$

$$\eta + \alpha\zeta < \frac{1}{p} + \alpha\delta_0, \tag{5.4}$$

$$\eta + \alpha\zeta < 1 + \frac{1}{p} + \alpha\delta_1. \tag{5.5}$$

With the notation $1_{\{a>b\}} = 1$ *if* $a > b$ *and* $1_{\{a>b\}} = 0$ *else, we put*

$$v(t) = u(t) - 1_{\{\delta_0 > \zeta\}} u_0 - 1_{\{\delta_1 > \zeta\}} t u_1 - 1_{\{\epsilon > \zeta\}} \int_0^t a_\gamma(t-s) f(s)\, ds.$$

(a) *Then, if* $\eta > 0$, *the function* v, *considered as a Banach space valued function* $v : [0, T] \to L_p(\Omega; \mathcal{D}((-A)^\zeta))$, *has a fractional derivative of order* η.

(b) *If* $\eta < 0$, *the function* $v : [0,T] \to L_p(\Omega; L_p(B; \mathbb{R}))$ *has a fractional integral of order* $-\eta$. *Moreover,* $D_t^\eta v$ *takes values in* $L_p(\Omega; \mathcal{D}((-A)^\zeta))$.

(c) *If* $\eta = 0$, *clearly* $D_t^0 v = v$.

In either case, there exist constants M_{init}, $M_{T,\text{Leb}}$, *and* $M_{T,\text{Ito}}$ *depending on* p, T, α, β, γ, δ_0, δ_1, ϵ, ζ, η *such that*

$$\begin{aligned} &\|D_t^\eta v(t)\|_{L_p([0,T] \times \Omega; \mathcal{D}((-A)^\zeta))} \\ &\quad \le M_{\text{init}} \left[\|u_0\|_{L_p(\Omega; \mathcal{D}((-A)^{\delta_0}))} + \|u_1\|_{L_p(\Omega; \mathcal{D}((-A)^{\delta_1}))} \right] \\ &\qquad + M_{T,\text{Leb}} \|f\|_{L_p([0,T] \times \Omega; \mathcal{D}((-A)^\epsilon))} + M_{T,\text{Ito}} \|g\|_{L_p([0,T] \times \Omega; L_p(B, l_2))}. \end{aligned} \tag{5.6}$$

Moreover, the constants $M_{T,\text{Leb}}$ *and* $M_{T,\text{Ito}}$ *can be made arbitrarily small by choosing the time interval* $[0, T]$ *sufficiently short.*

The proof of the propositions above relies on the concept of a resolvent operator (see [29]), introduced by the following definition:

Definition 5.3. Let A satisfy Hypothesis 3.1, let $\alpha \in (0,2)$ and $\beta > 0$. For $t > 0$ we define the resolvent operator $S_{\alpha,\beta}(t) : L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ by

$$S_{\alpha,\beta}(t)x := \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x\,d\lambda \tag{5.7}$$

along the contour

$$\Gamma_{\rho,\phi}(t) = \begin{cases} (t-\phi+\rho)e^{i\phi} & \text{for } t > \phi, \\ \rho e^{it} & \text{for } t \in (-\phi,\phi), \\ (-t-\phi+\rho)e^{-i\phi} & \text{for } t < -\phi, \end{cases}$$

with $\rho > 0$, $\phi > \frac{\pi}{2}$, $\alpha\phi + \phi_A < \pi$.

For $\beta = 1$, this definition coincides with the known notion of a resolvent operator, c.f. [29]. For $\beta > 1$, $S_{\alpha,\beta}$ could be obtained by fractional integration of $S_{\alpha,1}$.

Equation (5.1) is formally solved by the variation of parameters formula

$$\begin{aligned} u(t) &= S_{\alpha,1}(t)u_0 + S_{\alpha,2}(t)u_1 \\ &\quad + \int_0^t S_{\alpha,\gamma}(t-s)f(s)\,ds + \int_0^t \sum_{k=1}^\infty S_{\alpha,\beta}(t-s)g(s)\,dw_s^k. \end{aligned} \tag{5.8}$$

The task of the proof is to make sense of this formal expression in suitable function spaces, and to show that it gives a solution of (5.1). Moreover, the estimates claimed in Proposition 5.2 need to be verified. Since the equation is linear, all terms u_0, u_1, f, g can be treated separately. This is done in the following Lemmas 5.6, 5.7, and 5.9. Uniqueness can be proved by the standard reduction to a deterministic homogeneous equation with zero initial data, which has only the zero solution by the well-known theory of deterministic evolutionary integral equations (see [29]).

First we collect some basic facts about the resolvent operator:

Lemma 5.4. *Let A satisfy Hypothesis 3.1, let $\alpha \in (0,2)$ and $\beta > 0$. The resolvent operator defined above has the following properties:*

1) *For all $t > 0$ and all $\zeta \in [0,1]$, the operator $S_{\alpha,\beta}(t)$ is a bounded linear operator $L_p(B,\mathbb{R}) \to \mathcal{D}((-A)^\zeta)$.*
2) *For all $x \in L_p(B;\mathbb{R})$, the function $t \mapsto S_{\alpha,\beta}(t)x$ can be extended analytically to some sector in the right half-plane.*

3) *For all* $x \in L_p(B;\mathbb{R})$ *and all* $t > 0$*, we have*

$$\int_0^t a_\alpha(t-s)\|S_{\alpha,\beta}(s)x\|_{L_p(B;\mathbb{R})}\,ds < \infty,$$

$$\int_0^t a_\alpha(t-s)S_{\alpha,\beta}(s)x\,ds \in \mathcal{D}(A),$$

$$S_{\alpha,\beta}(t)x = A\int_0^t a_\alpha(t-s)S_{\alpha,\beta}(s)x\,ds + a_\beta(t)x. \tag{5.9}$$

4) *Let* $T > 0$*,* $\delta,\zeta \in [0,1]$*, and* $\eta \in (-1,1)$ *such that*

$$\eta + \alpha\zeta < \beta + \alpha\delta. \tag{5.10}$$

Let $x \in \mathcal{D}((-A)^\delta)$ *and put*

$$v(t) = \begin{cases} S_{\alpha,\beta}(t)x & \text{if } \delta \le \zeta, \\ S_{\alpha,\beta}(t)x - a_\beta(t)x & \text{if } \delta > \zeta. \end{cases}$$

(a) *Then, if* $\eta > 0$*, the function* v*, considered as a Banach-space valued function* $v : [0,T] \to \mathcal{D}((-A)^\zeta)$*, admits a fractional derivative of order* η*.*

(b) *If* $\eta < 0$*, the function* $v : [0,T] \mapsto L_p(B;\mathbb{R})$*, has a fractional integral of order* $-\eta$*. Moreover,* $D_t^\eta v$ *takes values in* $\mathcal{D}((-A)^\zeta)$*.*

(c) *If* $\eta = 0$*, we write* $D_t^0 v = v$*.*

In either case, there exists some $M > 0$ *(dependent on* $A, \alpha, \beta, \zeta, \delta, \eta$*) such that for all* $t \in (0,T]$ *and all* $x \in \mathcal{D}((-A)^\delta)$*,*

$$\|D_t^\eta v(t)\|_{\mathcal{D}((-A)^\zeta)} \le M t^{(\beta+\alpha\delta)-(\eta+\alpha\zeta)-1}\|x\|_{\mathcal{D}((-A)^\delta)}. \tag{5.11}$$

Remark 5.5. In fact, if $x \in \mathcal{D}((-A)^\delta)$ with $\delta \ge \zeta$ and $\beta > \eta$, the function $t \mapsto a_\beta(t)x$ admits a fractional derivative $D_t^\eta a_\beta x = a_{\beta-\eta}x$ in $\mathcal{D}((-A)^\zeta)$. In this case, (5.10) holds, and both functions, $S_{\alpha,\beta}(t)x$ and $S_{\alpha,\beta}(t)x - a_\beta(t)x$ admit fractional derivatives of order η in $\mathcal{D}((-A)^\zeta)$. On the other hand, evidently, if $\beta \le \eta$ or $x \notin \mathcal{D}((-A)^\zeta)$, at most one of the two functions above can have a fractional derivative of order η in $\mathcal{D}((-A)^\zeta)$.

Proof. All these results come out of standard estimates of the contour integral, along with the usual analyticity arguments. Since the estimate (5.11) is crucial in the sequel, we give a more detailed proof.

First we consider the case $\delta \le \zeta$ where we can utilize Lemma 2.2 with $\theta = 0$ for ρ in a suitable sector:

$$\|(\rho - A)^{-1}x\|_{\mathcal{D}((-A)^\zeta)} \le M|\rho|^{\zeta-\delta-1}\|x\|_{\mathcal{D}((-A)^\delta)}.$$

Formally, the Laplace transform of $D_t^\eta S_{\alpha,\beta}x$ is $\lambda^{\eta+\alpha-\beta}(\lambda^\alpha - A)^{-1}x$. We show that the contour integral

$$w(t) := \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{\eta+\alpha-\beta}(\lambda^\alpha - A)^{-1}x\,d\lambda$$

exists in $\mathcal{D}((-A)^\zeta)$, if (5.10) holds.

$$\begin{aligned}
&\left\|\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\lambda^{\eta+\alpha-\beta}(\lambda^\alpha - A)^{-1}x\,d\lambda\right\|_{\mathcal{D}((-A)^\zeta)}\\
&\quad= \left\|\int_{\Gamma_{t\rho,\phi}} e^{\mu}\,\left(\frac{\mu}{t}\right)^{\eta+\alpha-\beta}\,(\left(\frac{\mu}{t}\right)^\alpha - A)^{-1}x\,\frac{1}{t}d\mu\right\|_{\mathcal{D}((-A)^\zeta)}\\
&\quad= t^{\beta-\alpha-\eta-1}\left\|\int_{\Gamma_{1,\phi}} e^{\mu}\,\mu^{\eta+\alpha-\beta}\,(\left(\frac{\mu}{t}\right)^\alpha - A)^{-1}x\,d\mu\right\|_{\mathcal{D}((-A)^\zeta)}\\
&\quad\le t^{\beta-\alpha-\eta-1}\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}\,|\mu|^{\alpha+\eta-\beta}\,\left\|\left(\left(\frac{\mu}{t}\right)^\alpha - A\right)^{-1}x\right\|_{\mathcal{D}((-A)^\zeta)}\,|d\mu|\\
&\quad\le t^{\beta-\alpha-\eta-1}\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}|\mu|^{\alpha+\eta-\beta}\,M\left|\frac{\mu}{t}\right|^{\alpha(\zeta-\delta-1)}\,\|x\|_{\mathcal{D}((-A)^\delta)}\,|d\mu|\\
&\quad= t^{\beta-\eta-\alpha\zeta+\alpha\delta-1}M\|x\|_{\mathcal{D}((-A)^\delta)}\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}|\mu|^{\eta-\beta+\alpha(\zeta-\delta)}\,|d\mu|.
\end{aligned}$$

Because of (5.10), w is locally integrable and admits a Laplace transform. It requires some standard complex analysis, to show that $\hat{w}(\lambda) = \lambda^{\eta+\alpha-\beta}(\lambda^\alpha - A)^{-1}x$. Now we have to show that in fact $w = D_t^\eta S_{\alpha,\beta}x$.

First consider the case $\eta > 0$: By the convolution theorem for Laplace transforms we have

$$\widehat{[D_t^{-\eta}w]}(\lambda) = \lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x,$$

whence $w = D_t^\eta S_{\alpha,\beta}x$. In case $\eta < 0$, the convolution theorem yields

$$\widehat{D_t^\eta S_{\alpha,\beta}}x(\lambda) = \lambda^\eta\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x = \hat{w}(\lambda).$$

To handle the case $\delta > \zeta$, we will use Lemma 2.2 with $\theta = 1$:

$$\|A(\rho - A)^{-1}x\|_{\mathcal{D}((-A)^\zeta)} \le M\rho^{\zeta-\delta}\|x\|_{\mathcal{D}((-A)^\delta)}.$$

Notice first that $\hat{a}_\beta(\lambda) = \lambda^{-\beta}$, and

$$a_\beta(t) = \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{-\beta}\,d\lambda.$$

Therefore,

$$\begin{aligned}
S_{\alpha,\beta}(t)x - a_\beta(t)x &= \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,[\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x - \lambda^{-\beta}x]\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{-\beta}A(\lambda^\alpha - A)^{-1}x\,d\lambda.
\end{aligned}$$

Now we estimate similarly as above

$$\begin{aligned}&\left\|\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{\eta-\beta}A(\lambda^\alpha-A)^{-1}x\,d\lambda\right\|_{\mathcal{D}((-A)^\zeta)}\\&\quad\le\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}\left|\frac{\mu}{t}\right|^{\eta-\beta}M\left|\frac{\mu}{t}\right|^{\alpha(\zeta-\delta)}\|x\|_{\mathcal{D}((-A)^\delta)}\frac{1}{t}|d\mu|\\&\quad=M\|x\|_{\mathcal{D}((-A)^\delta)}\,t^{-\eta+\beta-\alpha\zeta+\alpha\delta-1}\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}|\mu|^{\eta-\beta+\alpha\zeta-\alpha\delta}\,|d\mu|.\end{aligned}$$

Thus, for $t>0$, the following integral exists in $\mathcal{D}((-A)^\zeta)$:

$$w_1(t):=\frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{\eta-\beta}A(\lambda^\alpha-A)^{-1}x\,d\lambda,$$

$$\|w_1(t)\|_{\mathcal{D}((-A)^\zeta)}\le Mt^{(\beta+\alpha\delta)-(\eta+\alpha\zeta)-1}.$$

In the end one verifies again that in fact $w_1(t)=D_t^\eta v(t)$. □

Lemma 5.6 (Contribution of the initial conditions u_0, u_1). *Let A satisfy Hypothesis* 3.1, *let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, $p\in[2,\infty)$. Let $\alpha\in(0,2)$, $0<T<\infty$, and u_0, $u_1\in L_p(\Omega;L_p(B;\mathbb{R}))$. We define $u(t):=S_{\alpha,1}(t)u_0+S_{\alpha,2}(t)u_1$.*

1) *The function u exists in $L_\infty([0,T];L_p(\Omega\times B;\mathbb{R}))$. For all $t>0$ we have*

$$\int_0^t a_\alpha(t-s)u(s)\,ds\in\mathcal{D}(A),$$

$$u(t)=A\int_0^t a_\alpha(t-s)u(s)\,ds+u_0+tu_1.$$

2) *Moreover, suppose that $u_i\in L_p(\Omega;\mathcal{D}((-A)^{\delta_i}))$ with some $\delta_i\in[0,1]$. Let $\zeta\in[0,1]$, $\eta\in(-1,1)$ be such that*

$$\eta+\alpha\zeta<\frac{1}{p}+\alpha\delta_0, \tag{5.12}$$

$$\eta+\alpha\zeta<1+\frac{1}{p}+\alpha\delta_1, \tag{5.13}$$

Put

$$v(t)=u(t)-1_{\zeta<\delta_0}u_0-1_{\zeta<\delta_1}tu_1.$$

Then v has a fractional derivative of order η (if $\eta<0$: a fractional integral of order $-\eta$) which is in $L_p([0,T]\times\Omega;\mathcal{D}((-A)^\zeta))$ and satisfies

$$\begin{aligned}&\|D_t^\eta v(t,\omega)\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\zeta))}\\&\quad\le M\big[\|u_0\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_0}))}+\|u_1\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_1}))}\big]\end{aligned}$$

with a constant M depending on $p,A,T,\alpha,\delta_0,\delta_1,\zeta,\eta$.

Proof. First let $u_1 = 0$ so that $u(t) = S_{\alpha,1}u_0$. We will apply Lemma 5.4 with $\beta = 1$. Notice that $a_1(t) = 1$, so that $v(t) = S_{\alpha,1}u_0 - 1_{\zeta<\delta_0}u_0$. To prove Part (1), put $\zeta = \delta_0 = 0$ and take any $\eta < 1$. By Lemma 5.4 (3) we have, pointwise for all $\omega \in \Omega$,

$$u(t,\omega) = A\int_0^t a_\alpha(t-s)u(s,\omega)\,ds + u_0.$$

Moreover, with these parameters, $v(t) = u(t)$, and (5.10) holds. By Lemma 5.4, Part (4) and (5.11), $u(\cdot,\omega)$ admits a fractional time derivative of order η with values in $L_p(B;\mathbb{R})$, such that the following estimate holds:

$$\|D_t^\eta u(t,\omega)\|_{L_p(B;\mathbb{R})} \le Mt^{-\eta}\|u_0(\omega)\|_{L_p(B;\mathbb{R})}.$$

Taking convolution with $t^{\eta-1}/\Gamma(\eta)$, we obtain with a suitable constant M_1 independent of t,

$$\|u(t,\omega)\|_{L_p(B;\mathbb{R})} \le M_1\|u_0(\omega)\|_{L_p(B;\mathbb{R})}.$$

Integrating with respect to ω we obtain

$$\|u(t)\|_{L_p(\Omega\times B;\mathbb{R})} \le M_1\|u_0\|_{L_p(\Omega\times B;\mathbb{R})}.$$

Thus $u \in L_\infty([0,T];L_p(\Omega\times B;\mathbb{R}))$.

To prove Part (2), take again $\beta = 1$ and let (5.12) hold. Then (5.10) (with δ_0 instead of δ) holds a fortiori. Let $v(t) = S_{\alpha,1}u_0 - 1_{\zeta<\delta_0}u_0$. Fix $\omega \in \Omega$. By Lemma 5.4 (4), $v(\cdot,\omega)$ has a fractional time derivative of order η which satisfies an estimate

$$\|D_t^\eta v(t,\omega)\|_{\mathcal{D}((-A)^\zeta)} \le Mt^{\alpha\delta_0-\eta-\alpha\zeta}\|u_0(\omega)\|_{\mathcal{D}((-A)^{\delta_0})}.$$

Since, by (5.12), $\alpha\delta_0 - \eta - \alpha\zeta > -\frac{1}{p}$, the estimate above implies

$$\|D_t^\eta v(\cdot,\omega)\|_{L_p([0,T];\mathcal{D}((-A)^\zeta))} \le M_1\|u_0(\omega)\|_{\mathcal{D}((-A)^{\delta_0})}.$$

Integrating again with respect to Ω, we obtain Part (2).

The case $u_0 = 0$, $u_1 \ne 0$, $u(t) = S_{\alpha,2}u_1$ is treated similarly with $\beta = 2$. Notice that $a_2(t) = t$. In the end we can combine both cases. □

Lemma 5.7 (Contribution of *f*). *Let A satisfy Hypothesis* 3.1*, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, $p \in [2,\infty)$. Let $\alpha \in (0,2)$, $\gamma > 0$, $0 < T < \infty$, and $f \in L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))$.*

1) *For almost $t \in [0,T]$, the following integral exists in $L_p(B;\mathbb{R})$, pointwise for almost all $\omega \in \Omega$, as well as in $L_p(\Omega;L_p(B;\mathbb{R}))$:*

$$u(t,\omega) = \int_0^t S_{\alpha,\gamma}(t-s)f(s,\omega)\,ds. \tag{5.14}$$

Moreover, $u \in L_p([0,T] \times \Omega; L_p(B;\mathbb{R}))$, *and for almost all* $\omega \in \Omega$ *and almost all* $t \in [0,T]$,

$$\int_0^t a_\alpha(t-s)u(s,\omega)\,ds \in \mathcal{D}(A),$$

$$u(t,\omega) = A\int_0^t a_\alpha(t-s)u(s,\omega)\,ds + \int_0^t a_\gamma(t-s)f(s,\omega)\,ds.$$

2) *Suppose, in addition, that* $f \in L_p([0,T] \times \Omega; \mathcal{D}((-A)^\epsilon))$ *with some* $\epsilon \in [0,1]$, *let* $\eta \in (-1,1)$, $\zeta \in [0,1]$ *be such that*

$$\eta + \alpha\zeta < \gamma + \alpha\epsilon. \tag{5.15}$$

Put

$$v(t) = \begin{cases} u(t) & \text{if } \zeta \geq \epsilon, \\ u(t) - \int_0^t a_\gamma(t-s)f(s)\,ds & \text{if } \zeta < \epsilon. \end{cases}$$

Then, if $\eta > 0$, *the function* $t \mapsto v(t) \in L_p(\Omega; \mathcal{D}((-A)^\zeta))$ *has a fractional derivative of order* η *in* $L_p([0,T] \times \Omega; \mathcal{D}((-A)^\zeta))$. *If* $\eta < 0$, *the function* $t \mapsto v(t) \in L_p(\Omega; L_p(B;\mathbb{R}))$ *has a fractional integral of order* $-\eta$ *with values in* $L_p(\Omega; \mathcal{D}((-A)^\zeta))$. *If* $\eta = 0$, *we define* $D_t^\eta v = v$. *In either case there exists a constant* $M_{T,\mathrm{Leb}}$ *dependent on* $A, T, p, \alpha, \gamma, \epsilon, \zeta, \eta$ *such that*

$$\|D_t^\eta v\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\zeta))} \leq M_{T,\mathrm{Leb}}\|f\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\epsilon))}.$$

Moreover, the constant $M_{T,\mathrm{Leb}}$ *can be made arbitrarily small by taking the time interval* $[0,T]$ *sufficiently short.*

Proof. The function $t \mapsto \int_0^t (t-s)^{\gamma-1}\|f(s)\|_{L_p(\Omega\times B,\mathbb{R})}\,ds$ is the convolution of an L_1-function and an L_p-function, therefore it is in $L_p([0,T],\mathbb{R})$. From (5.11) with $\delta = \zeta = \eta = 0$ we obtain $\|S_{\alpha,\gamma}(t)\|_{L_p(B;\mathbb{R})\to L_p(B;\mathbb{R})} \leq Mt^{\gamma-1}$. Consequently, the integral

$$u(t) = \int_0^t S_{\alpha,\gamma}(t-s)f(s)\,ds$$

exists as an integral in $L_p(\Omega \times B;\mathbb{R})$ for a.e. t, and $u \in L_p([0,T] \times \Omega, L_p(B,\mathbb{R}))$. By standard arguments the integral (5.14) exists also in $L_p(B;\mathbb{R})$ for a.e. $\omega \in \Omega$ and a.e. $t \in [0,T]$. Now (5.9) implies (almost everywhere in Ω and $[0,T]$)

$$\begin{aligned} u(t) - \int_0^t a_\gamma(t-s)f(s,\omega)\,ds &= \int_0^t [S_{\alpha,\gamma}(t-s)f(s,\omega) - a_\gamma(t-s)f(s,\omega)]\,ds \\ &= \int_0^t A[\int_0^{t-s} a_\alpha(\sigma)S_{\alpha,\gamma}(t-s-\sigma)f(s,\omega)\,d\sigma]\,ds. \end{aligned}$$

We use the closedness of A and interchange the order of integrals to obtain

$$u(t) - \int_0^t a_\gamma(t-s)f(s,\omega)\,ds = A\int_0^t a_\alpha(\sigma)u(t-\sigma,\omega)\,d\sigma.$$

This proves Part (1) of the lemma.

To prove Part (2), let η, ζ, ϵ be such that (5.15) holds. For shorthand put

$$V(t)x = \begin{cases} D_t^\eta S_{\alpha,\gamma} x & \text{if } \epsilon \le \zeta, \\ D_t^\eta [S_{\alpha,\gamma}(t)x - a_\gamma x] & \text{else.} \end{cases}$$

From (5.11) with β replaced by γ, and δ replaced by ϵ, we have

$$\|V(t)x\|_{\mathcal{D}((-A)^\zeta)} \le M t^{(\gamma+\alpha\epsilon)-(\eta+\alpha\zeta)-1} \|x\|_{\mathcal{D}((-A)^\epsilon)}.$$

We obtain by a straightforward convolution argument that

$$\| \int_0^t V(t-s) f(s)\, ds \|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\zeta))}\, ds \le M_{t,\mathrm{Leb}} \|f\|_{L_p(\Omega,\mathcal{D}((-A)^\epsilon))}.$$

with

$$M_{T,\mathrm{Leb}} = M \int_0^T t^{(\gamma+\alpha\epsilon)-(\eta+\alpha\zeta)-1}\, dt.$$

Clearly, $M_{T,\mathrm{Leb}}$ converges to 0 as $T \to 0$. All we have to show is that in fact

$$D_t^\eta v(t) = \int_0^t V(t-s) f(s)\, ds.$$

We treat the case $\eta > 0$, $\epsilon > \zeta$, the other cases are done similarly. The definition of $V(t)x$ yields

$$\int_0^t a_\eta(s) V(t-s)x\, dx = S_{\alpha,\gamma}(t)x - a_\gamma(t)x.$$

Fubini's Theorem implies

$$\int_0^t a_\eta(s) \int_0^{t-s} V(t-s-\sigma) f(\sigma)\, d\sigma\, ds = \int_0^t \int_0^{t-\sigma} a_\eta(s) V(t-\sigma-s) f(\sigma)\, ds\, d\sigma$$
$$= \int_0^t [S_{\alpha,\gamma}(t-\sigma) - a_\gamma(t-\sigma)]\, f(\sigma)\, d\sigma = v(t).$$

Thus $v(t)$, considered as a function with values in $\mathcal{D}((-A)^\zeta)$, admits a fractional derivative of order η which is $V * f$. □

The following lemma is the key to estimate the contribution of the stochastic integral:

Lemma 5.8. *Let A satisfy Hypothesis 3.1, $p \in [2, \infty)$. Let $\alpha \in (0, 2)$, $\beta > \frac{1}{2}$, $\zeta \in [0, 1]$ and $\eta \in (-1, 1)$, such that (5.3) holds, i.e., $\frac{1}{2} + \eta + \alpha\zeta < \beta$. Let $T > 0$. Then there exists a constant $\tilde{M}_{T,\mathrm{Ito}} > 0$ depending on $A, p, T, \alpha, \beta, \eta, \zeta$ such that for all $h \in L_p([0, T]; L_p(B; l_2))$,*

$$\int_0^T \int_B \left(\int_0^t |(-A)^\zeta D_t^\eta S_{\alpha,\beta}(t-s) h(s,x)|_{l_2}^2\, ds \right)^{\frac{p}{2}} d\Lambda(x)\, dt$$
$$\le \tilde{M}_{T,\mathrm{Ito}} \int_0^T \int_B |h(s,x)|_{l_2}^p\, d\Lambda(x)\, ds.$$

Moreover, the constant $\tilde{M}_{T,\mathrm{Ito}}$ can be made arbitrarily small by taking the time interval $[0,T]$ sufficiently short.

Proof. Write $V(t) := (-A)^{\zeta} D_t^{\eta} S_{\alpha,\beta}(t)$ and notice that by (5.11) (with $\delta = 0$),

$$\|V(t)\|_{L_p(B,l_2)\to L_p(B,l_2)} \le M t^{\beta-(\eta+\alpha\zeta)-1}.$$

First assume that $p > 2$. Notice that $\frac{p}{2}$ and $\frac{p}{p-2}$ are conjugate exponents. Take $f : [0,T]\times B \to \mathbb{R}^+$ such that $\int_0^T \int_B f^{\frac{p}{p-2}}(t,x)\,d\Lambda(x)\,dt = 1$. We estimate

$$\begin{aligned}
&\int_0^T \int_B f(t,x) \int_0^t |V(t-s)h(s,x)|_{l_2}^2\,ds\,d\Lambda(x)\,dt \\
&= \int_0^T \int_0^t \int_B f(t,x)\,|V(t-s)h(s,x)|_{l_2}^2\,d\Lambda(x)\,ds\,dt \\
&\le \int_0^T \int_0^t \left[\int_B f(t,x)^{\frac{p}{p-2}}\,d\Lambda(x)\right]^{\frac{p-2}{p}} \left[\int_B |V(t-s)h(s,x)|_{l_2}^p\,d\Lambda(x)\right]^{\frac{2}{p}} ds\,dt \\
&\le \int_0^T \|f(t,\cdot)\|_{L_{\frac{p}{p-2}}(B;\mathbb{R})} \int_0^t \|V(t-s)\|^2_{L_p(B;l_2)\to L_p(B;l_2)} \|h(s,\cdot)\|^2_{L_p(B;l_2)}\,ds\,dt \\
&\le \left[\int_0^T \|f(t,\cdot)\|^{\frac{p}{p-2}}_{L_{\frac{p}{p-2}}(B;\mathbb{R})}\,dt\right]^{\frac{p-2}{p}} \\
&\quad\times \left[\int_0^T |\int_0^t \|V(t-s)\|^2_{L_p(B;l_2)\to L_p(B;l_2)} \|h(s,\cdot)\|^2_{L_p(B;l_2)}\,ds|^{\frac{p}{2}}\,dt\right]^{\frac{2}{p}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left[\int_0^T \int_B \left(\int_0^t |V(t-s)h(s,x)|_{l_2}^2\,ds\right)^{\frac{p}{2}} d\Lambda(x)\,dt\right]^{\frac{2}{p}} \\
&\quad\le \left[\int_0^T |\int_0^t \|V(t-s)\|^2_{L_p(B;l_2)\to L_p(B;l_2)} \|h(s,\cdot)\|^2_{L_p(B;l_2)}\,ds|^{\frac{p}{2}}\,dt\right]^{\frac{2}{p}}.
\end{aligned} \tag{5.16}$$

For $p = 2$, the estimate (5.16) is obvious. In either case, we obtain (by estimating the convolution with respect to s)

$$\begin{aligned}
&\left[\int_0^T \int_B \left(\int_0^t |V(t-s)h(s,x)|_{l_2}^2\,ds\right)^{\frac{p}{2}} d\Lambda(x)\,dt\right]^{\frac{2}{p}} \\
&\quad\le \left(\int_0^T \|V(s)\|^2_{L_p(B;l_2)\to L_p(B;l_2)}\,ds\right) \left(\int_0^T \|h(s,.)\|^p_{L_p(B;l_2)}\,ds\right)^{\frac{2}{p}} \\
&\quad\le M^2 \int_0^T s^{2(\beta-(\eta+\alpha\zeta)-1)}\,ds\,\|h\|^2_{L_p([0,T];L_p(B;l_2))}.
\end{aligned}$$

By (5.3) we infer that $s^{2(\beta-(\eta+\alpha\zeta)-1)}$ is integrable on $[0,T]$ so that

$$\tilde{M}_{T,\mathrm{Ito}} := \left[M^2 \int_0^T s^{2(\beta-(\eta+\alpha\zeta)-1)}\,ds \right]^{\frac{p}{2}}$$

is finite and converges to 0 as $T \to 0+$. $\square$

Lemma 5.9 (Contribution of g). *Let A satisfy Hypothesis* 3.1*, and let the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and the Wiener processes w_t^k be as in Hypothesis* 3.2*. Let $T>0$, $2\le p<\infty$, $\alpha\in(0,2)$, and $\beta>\frac{1}{2}$. Let $g\in L_p([0,T]\times\Omega;L_p(B;l_2))$ and $\{g_j\}$ be a sequence approximating g in the sense of Lemma* 4.1*, where the values of g_j^k are in $\mathcal{D}(A)\cap L_1(B;\mathbb{R})\cap L_\infty(B;\mathbb{R})$. Let $a_\beta\star g$ be given by Definition* 4.5*. For $j\in\mathbb{N}$ put*

$$u_j(t) = \sum_{k=1}^{j} \int_0^t S_{\alpha,\beta}(t-s) g_j^k(s)\,dw_s^k.$$

1) *The limit $u(t)=\lim_{j\to\infty} u_j(t)$ exists in $L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))$. Moreover, for almost all $t\in[0,T]$ and almost all $\omega\in\Omega$,*

$$u(t,\omega) = A\int_0^t a_\alpha(t-s)u(s,\omega)\,ds + (a_\beta\star g)(t,\omega).$$

2) *Suppose $0\le\zeta\le 1$ and $\eta\in(-1,1)$ are such that*

$$\eta+\alpha\zeta+\frac{1}{2}<\beta. \tag{5.17}$$

Then, if $\eta>0$, the function $u:[0,T]\to L_p(\Omega,\mathcal{D}((-A)^\zeta))$ has a fractional derivative of order η. If $\eta<0$, then $u:[0,T]\to L_p(\Omega;L_p(\Omega,\mathbb{R}))$ has a fractional integral of order $-\eta$ with values in $L_p(\Omega;\mathcal{D}((-A)^\zeta))$. If $\eta=0$, we denote $D_t^0 u=u$. In either case there exists a constant $M_{T,\mathrm{Ito}}$ dependent on $A,p,\alpha,\beta,\eta,\zeta$ such that

$$\|D_t^\eta u\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\zeta))} \le M_{T,\mathrm{Ito}}\|g\|_{L_p([0,T]\times\Omega;L_p(B;l_2))}.$$

Moreover, the constant $M_{T,\mathrm{Ito}}$ can be made arbitrarily small by choosing the time interval $[0,T]$ sufficiently short.

Proof. First, let $h\in L_p([0,T]\times\Omega;L_p(B;l_2))$ be of the elementary structure like the g_j in Lemma 4.1. Evidently, the following integral exists

$$\sum_{k=1}^\infty \int_0^t S_{\alpha,\beta}(t-s)h^k(s)\,dw_s^k = \sum_{k=1}^\infty\sum_{i=1}^\infty \int_{\tau_{i-1}^k}^{\tau_i^k} S_{\alpha,\beta}(t-s)h_i^k\,dw_s^k$$

where τ_i^k are suitable stopping times, $h_i^k\in\mathcal{D}(A)\cap L_1(B;\mathbb{R})\cap L_\infty(B;\mathbb{R})$, and both sums are in fact only finite sums. For $\eta\in(-1,1)$, $\zeta\in[0,1]$, satisfying (5.17), put $V(t)x = (-A)^\zeta D_t^\eta S_{\alpha,\beta}(t)x$. We apply Lemma 4.3 and integrate for $t \in [0,T]$. Subsequently we apply Lemma 5.8:

$$\begin{aligned}
&\int_0^T \int_\Omega \left\| \sum_{k=1}^\infty \int_0^t V(t-s)h^k(s)\,dw_s^k \right\|_{L_p(B;\mathbb{R})}^p d\mathbb{P}(\omega)\,dt \qquad (5.18)\\
&\le M \int_0^T \int_B \int_\Omega \left(\int_0^t |[V(t-s)h(s,\omega)](x)|_{l_2}^2\,ds \right)^{\frac{p}{2}} d\mathbb{P}(\omega)\,d\Lambda(x)\,dt\\
&\le M\tilde{M}_{T,\mathrm{Ito}} \|h\|_{L_p([0,T]\times\Omega;L_p(B;l_2))}^p .
\end{aligned}$$

In particular, with $\zeta = \eta = 0$, and $h = g_j - g_m$, we have

$$\|u_j - u_m\|_{L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))} \le M\|g_j - g_m\|_{L_p([0,T]\times\Omega;L_p(B;l_2))}$$

so that $\{u_j\}$ is a Cauchy sequence in $L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))$ and has a limit u. Without loss of generality, taking a subsequence, if necessary, we may assume that u_j converges also pointwise for almost all $t \in [0,T]$. Again we use the simple structure of g_j, in particular that $g_j^k(t,\omega) \in \mathcal{D}(A)$. From (5.9) and the Stochastic Fubini Theorem we obtain

$$\begin{aligned}
&\int_0^t [S_{\alpha,\beta}(t-\sigma)g_j^k(\sigma,\omega) - a_\beta(t-\sigma)g_j^a(\sigma,\omega)]\,dw_\sigma^k\\
&\quad= \int_0^t A[\int_0^{t-\sigma} a_\alpha(s)S_{\alpha,\beta}(t-\sigma-s)g_j^k(\sigma,\omega)\,ds]\,dw_\sigma^k\\
&\quad= A\int_0^t \int_0^{t-\sigma} a_\alpha(s)S_{\alpha,\beta}(t-\sigma-s)g_j^k(\sigma,\omega)\,ds\,dw_\sigma^k\\
&\quad= A\int_0^t a_\alpha(s) \int_0^{t-s} S_{\alpha,\beta}(t-\sigma-s)g_j^k(\sigma,\omega)\,dw_\sigma^k\,ds.
\end{aligned}$$

Taking the sum over $k = 1,\dots,j$ we obtain

$$u_j(t,\omega) - a_\beta \star g_j(t,\omega) = A\int_0^t a_\alpha(s)u_j(t-s,\omega)\,ds.$$

Taking limits for $j \to \infty$ (pointwise a.e. in $[0,T]$), and using the closedness of A we have for almost all $t \in [0,T]$

$$u(t,\omega) - a_\beta \star g(t,\omega) = A\int_0^t a_\alpha(s)u(t-s,\omega)\,ds.$$

Thus Part (1) of the lemma is proved.

To prove Part (2), let $\eta \in (-1,1)$ and $\zeta \in [0,1]$ satisfy (5.17). With $V(t)x = (-A)^\zeta D_t^\eta S_{\alpha,\beta}x$ and $h = g_j$, we obtain from (5.18),

$$\left\| \int_0^t \sum_{k=1}^j V(t-s)g_j^k(s)\,dw_s^k \right\|_{L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))} \le M_{T,\mathrm{Ito}}\|g_j\|_{L_p([0,T]\times\Omega;L_p(B;l_2))},$$

with a suitable constant $M_{T,\mathrm{Ito}}$ which converges to 0 as $T \to 0$. We have to show that in fact

$$\sum_{k=1}^{j} \int_0^t V(t-s) g_j^k(s)\, dw_s^k = (-A)^\zeta D_t^\eta u_j(t)$$

First let $\eta > 0$. By definition we know that

$$\int_0^t a_\eta(s) V(t-s) x\, ds = (-A)^\zeta S_{\alpha,\beta}(t) x.$$

Taking integrals and using the Stochastic Fubini Theorem, we obtain

$$\begin{aligned}
(-A)^\zeta u_j(t) &= \sum_{k=1}^{j} \int_0^t (-A)^\zeta S_{\alpha,\beta}(t-\sigma) g_j^k(\sigma,\omega)\, dw_\sigma^k \\
&= \sum_{k=1}^{j} \int_0^t \int_0^{t-\sigma} a_\eta(s) V(t-\sigma-s) g_j^k(\sigma,\omega)\, ds\, dw_\sigma^k \\
&= \int_0^t a_\eta(s) \sum_{k=1}^{j} \int_0^{t-s} V(t-s-\sigma) g_j^k(\sigma,\omega)\, dw_\sigma^k\, ds
\end{aligned}$$

Thus $(-A)^\zeta u_j$ has a fractional derivative of order η which is $V \star g_j$. Taking the limit for $j \to \infty$ we infer that $D_t^\eta (-A)^\zeta u = V \star g$. Now let $\eta < 0$. Similarly as above, the Stochastic Fubini Theorem yields

$$\sum_{k=1}^{j} \int_0^t V(t-\sigma) g_j^k(\sigma)\, dw_\sigma^k = (-A)^\zeta \int_0^t a_{-\eta}(s) u_j(t-s)\, ds.$$

Again we take the limit for $j \to \infty$ and use the closedness of A, to see that the fractional integral $D_t^\eta u$ takes values in $\mathcal{D}((-A)^\zeta)$ with $(-A)^\zeta D_t^\eta u = V \star g$. $\square$

6. The semilinear equation

This section is devoted to the proof of Theorems 3.6, 3.7, 3.11, and Corollary 3.8.

Lemma 6.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let A satisfy Hypothesis* 3.1*, and let F and G satisfy Hypotheses* 3.4 *and* 3.5*. We define the operators*

$$\begin{aligned}
\mathcal{N}_F &: L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta)) \to L_p([0,T] \times \Omega; \mathcal{D}((-A)^\epsilon)), \\
\mathcal{N}_G &: L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta)) \to L_p([0,T] \times \Omega; L_p(B; l_2))
\end{aligned}$$

by

$$[\mathcal{N}_F v](t,\omega) := F(t,\omega,v(t)), \quad [\mathcal{N}_G v](t,\omega) := G(t,\omega,v(t)).$$

(1) *Then $\mathcal{N}_F$ and $\mathcal{N}_G$ are well defined and Lipschitz continuous with Lipschitz constants M_F, M_G, respectively.*

(2) *Let* F_1, F_2, G_1, G_2 *satisfy Hypotheses* 3.4, 3.5, *and* 3.9. *Let* $v \in L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$. *Then*

$$\|\mathcal{N}_{F_1} v - \mathcal{N}_{F_2} v\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\epsilon))} \le \|\mu_{\Delta F}\|_{L_p([0,T]\times\Omega;\mathbb{R})},$$
$$\|\mathcal{N}_{G_1} v - \mathcal{N}_{G_2} v\|_{L_p([0,T]\times\Omega;L_p(B;l_2))} \le \|\mu_{\Delta G}\|_{L_p([0,T]\times\Omega;\mathbb{R})}.$$

Here the constants M_F, M_G *and the functions* $\mu_{\Delta F}$ *and* $\mu_{\Delta G}$ *are as in Hypotheses* 3.4, 3.5, *and* 3.9.

Proof. These are straightforward estimates. □

Lemma 6.2. *Let the assumptions of Theorem* 3.6 *hold, in addition assume that* $\delta_0 \le \theta$, $\delta_1 \le \theta$. *For* $v \in L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$ *let* $\mathcal{T}_{[F,G,u_0,u_1]} v : [0,T] \times \Omega \to L_p(B;\mathbb{R})$ *be the unique solution* u *of*

$$\begin{aligned} u(t,\omega) = A \int_0^t a_\alpha(t-s) u(s,\omega)\, ds + u_0 + t u_1 \\ + \sum_{k=1}^\infty \int_0^t a_\beta(t-s) G_k(s,\omega,v(s))\, dw_s^k \\ + \int_0^t a_\gamma(t-s) F(s,\omega,v(s))\, ds. \end{aligned}$$

in the sense of Proposition 5.1.

(1) *Then* $\mathcal{T}_{[F,G,u_0,u_1]}$ *is well defined as a nonlinear operator from* $L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$ *into* $L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$. *Moreover,* $\mathcal{T}_{[F,G,u_0,u_1]}$ *is globally Lipschitz continuous with a Lipschitz constant dependent on* A, p, T, α, β, γ, ϵ, θ, M_F, M_G.

(2) *There exists an equivalent norm on* $L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$, *such that the Lipschitz constant of* $\mathcal{T}_{[F,G,u_0,u_1]}$ *is smaller than* 1. *This norm depends on* $T, p, A, \alpha, \beta, \gamma, \theta, \epsilon, M_F, M_G$.

(3) *There exists a constant* M, *depending on* $A, T, p, M_F, M_G, \alpha, \beta, \epsilon, \theta, \delta_0, \delta_1$, *such that the following Lipschitz estimate holds:*
If F_1, F_2, G_1, G_2 *satisfy Hypotheses* 3.4, 3.5, *and* 3.9, *if* $u_{0,1}$, $u_{0,2}$ *are in* $L_p(\Omega; \mathcal{D}((-A)^{\delta_0}))$ *and* $u_{1,1}, u_{1,2} \in L_p(\Omega; \mathcal{D}((-A)^{\delta_1}))$, *measurable with respect to* $\mathcal{F}_0$, *then for any* $v \in L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$ *we have*

$$\begin{aligned} &\|\mathcal{T}_{[F_1,G_1,u_{0,1},u_{1,1}]} v - \mathcal{T}_{[F_2,G_2,u_{0,2},u_{1,2}]} v\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))} \\ &\quad \le M\Big[\|u_{0,1} - u_{0,2}\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_0}))} + \|u_{1,1} - u_{1,2}\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_1}))} \\ &\qquad + \|\mu_{\Delta F}\|_{L_p([0,T]\times\Omega;\mathbb{R})} + \|\mu_{\Delta G}\|_{L_p([0,T]\times\Omega;\mathbb{R})}\Big]. \end{aligned}$$

(4) $\mathcal{T}_{[F,G,u_0,u_1]}$ *has a unique fixed point* $u_{[F,G,u_0,u_1]} \in L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$. *Moreover, there exists a constant* M *dependent on* A, p, T, M_F, M_G, α, β, ϵ, θ, δ_0, δ_1 *such that the following Lipschitz estimate holds:*

If $u_{i,j}$, F_i, G_i *are as in* (3), *then*

$$\begin{aligned}
&\|u_{[F_1,G_1,u_{0,1},u_{1,1}]} - u_{[F_2,G_2,u_{0,2},u_{1,2}]}\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))} \\
&\quad\le M\Big[\|u_{0,1}-u_{0,2}\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_0}))} + \|u_{1,1}-u_{1,2}\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_1}))} \\
&\qquad\qquad + \ \|\mu_{\Delta F}\|_{L_p([0,T]\times\Omega;\mathbb{R})} + \|\mu_{\Delta G}\|_{L_p([0,T]\times\Omega;\mathbb{R})}\Big].
\end{aligned}$$

Proof. We recall Proposition 5.2 with $\eta = 0$ and ζ replaced by θ. Notice that the conditions (5.2), and (5.3), (5.4), (5.5) are satisfied. Let u solve

$$u = Aa_\alpha * u + a_\gamma * f + a_\beta \star g + u_0 + tu_1,$$

Notice that with the present choice of coefficients the function v in (5.6) is simply u. Thus $u \in L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))$ with

$$\begin{aligned}
&\|u(t)\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))} \qquad\qquad (6.1)\\
&\quad\le M_{\text{init}}\left[\|u_0\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_0}))} + \|u_1\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_1}))}\right] \\
&\qquad\quad + \ M_{T,\text{Leb}}\|f\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\epsilon))} + \ M_{T,\text{Ito}}\|g\|_{L_p([0,T]\times\Omega;L_p(B,l_2))}.
\end{aligned}$$

Given $v \in L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))$, we put $f = \mathcal{N}_F v$ and $g = \mathcal{N}_G v$ as in Lemma 6.1. Then $f \in L_p([0,T]\times\Omega;\mathcal{D}((-A)^\epsilon))$ and $g \in L_p([0,T]\times\Omega;L_p(B;l_2))$. Thus, by (6.1), $u = \mathcal{T}_{[F,G,u_0,u_1]}v \in L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))$. In particular for $v = 0$ we have

$$\begin{aligned}
&\|\mathcal{T}_{[F,G,u_0,u_1]}(0)\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))} \\
&\le M_{\text{init}}\big[\|u_0\|_{L_p(\Omega,\mathcal{D}((-A)^{\delta_0}))} + \|u_1\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_1}))}\big] + M_{T,\text{Leb}}M_{F,0} + M_{T,\text{Ito}}M_{G,0}.
\end{aligned}$$

We could immediately get a Lipschitz estimate for $\mathcal{T}_{[F,G,u_0,u_1]}$ by (6.1), but we will get a better (contraction) estimate in an equivalent norm below.

To prove (2), we recall from Proposition 5.2 that $M_{T,\text{Leb}}$ and $M_{T,\text{Ito}}$ can be taken arbitrarily small, if the time intervals are sufficiently short. In particular, there exists $m \in \mathbb{N}$ such that

$$M_{T/m,\text{Leb}}M_F + M_{T/m,\text{Ito}}M_G < \frac{1}{4}.$$

With some $\kappa > 0$ to be specified below, we define for $v \in L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))$,

$$|||v||| := \sum_{q=1}^m \kappa^q \left[\int_{T(q-1)/m}^{Tq/m}\int_\Omega \|v(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\,d\mathbb{P}(\omega)\,dt\right]^{1/p}.$$

For $q = 1,\ldots,m$ we put

$$\begin{aligned}
F_q(t,\omega,v) &:= I_{(q-1)T/m\le t<qT/m}(t)\,F(t,\omega,v(t)),\\
G_q(t,\omega,v) &:= I_{(q-1)T/m\le t<qT/m}(t)\,G(t,\omega,v(t)).
\end{aligned}$$

If $v,\tilde{v} \in L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))$, then

$$\mathcal{T}_{[F,G,u_0,u_1]}v - \mathcal{T}_{[F,G,u_0,u_1]}\tilde{v} = \sum_{q=1}^m w_q,$$

where w_q solves

$$w_q = Aa_\alpha * w_q + a_\gamma * [F_q(v) - F_q(\tilde v)] + a_\beta \star [G_q(v) - G_q(\tilde v)].$$

Now $w_q = 0$ on $[0, \frac{T(q-1)}{m}]$. Lemma 6.1(1) and (6.1) imply

$$\begin{aligned}
&\left(\int_{T(q-1)/m}^{Tq/m} \int_\Omega \|w_q(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right)^{1/p} \\
&\quad \le M_{T/m,\mathrm{Leb}} M_F \left(\int_{T(q-1)/m}^{Tq/m} \int_\Omega \|v(t,\omega) - \tilde v(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right)^{1/p} \\
&\qquad + \; M_{T/m,\mathrm{Ito}} M_G \left(\int_{T(q-1)/m}^{Tq/m} \int_\Omega \|v(t,\omega) - \tilde v(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right)^{1/p} \\
&\quad \le \frac{1}{4}\left(\int_{T(q-1)/m}^{Tq/m} \int_\Omega \|v(t,\omega) - \tilde v(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right)^{1/p}.
\end{aligned}$$

On the intervals $\left[\frac{(r-1)T}{m}, \frac{rT}{m}\right]$ with $r > q$ we have the estimate

$$\begin{aligned}
&\left(\int_{T(r-1)/m}^{Tr/m} \int_\Omega \|w_q(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right)^{1/p} \\
&\quad \le \left(\int_0^T \int_\Omega \|w_q(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right)^{1/p} \\
&\quad \le M\left(\int_{T(q-1)/m}^{Tq/m} \int_\Omega \|v(t,\omega) - \tilde v(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right)^{1/p}.
\end{aligned}$$

with $M = M_F M_{T,\mathrm{Leb}} + M_G M_{T,\mathrm{Ito}}$. We choose $\kappa \in (0,1)$ sufficiently small, such that $M\sum_{r=1}^\infty \kappa^r < \frac{1}{4}$. We have therefore

$$\begin{aligned}
|||w_q||| &= \sum_{r=q}^m \kappa^r \left[\int_{T(r-1)/m}^{Tr/m} \int_\Omega \|w_q(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right]^{1/p} \\
&\le \left[\frac{1}{4} + M\sum_{r=q+1}^m \kappa^{r-q}\right] \kappa^q \left[\int_{T(q-1)/m}^{Tq/m} \|v(t,\omega) - \tilde v(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right]^{1/p} \\
&\le \frac{1}{2}\kappa^q \left[\int_{T(q-1)/m}^{Tq/m} \|v(t,\omega) - \tilde v(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right]^{1/p}.
\end{aligned}$$

Summing for $q = 1, \dots, m$ we obtain

$$\begin{aligned}
&|||\mathcal{T}_{[F,G,u_0,u_1]} v - \mathcal{T}_{[F,G,u_0,u_1]} \tilde v||| \le \sum_{q=1}^m |||w_q||| \\
&\le \frac{1}{2}\sum_{q=1}^m \kappa^q \left[\int_{T(q-1)/m}^{Tq/m} \|v(t,\omega) - \tilde v(t,\omega)\|^p_{\mathcal{D}((-A)^\theta)}\, d\mathbb{P}(\omega)\, dt\right]^{1/p} = \frac{1}{2}|||v - \tilde v|||.
\end{aligned}$$

Part (3) is a straightforward application of (6.1) and Lemma 6.1 (2).

Finally, since for all F, G, u_0, u_1 the operator $\mathcal{T}_{[F,G,u_0,u_1]}$ is a strict contraction with Lipschitz constant $\frac{1}{2} < 1$ on $L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$ (with the norm $|||\cdot|||$), and since $\mathcal{T}_{[F,G,u_0,u_1]}v$ depends Lipschitz on F, G, u_0, u_1 by Part (3), the standard contraction arguments yield Part (4). □

We are now ready to finish the proofs of the main results:

Proof of Theorem 3.6. We may assume without loss of generality that δ_0, $\delta_1 \leq \theta$. (If any δ_i is greater that θ, it may be replaced by θ.) Obviously, the unique solution of (1.1) in $L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$ is exactly the unique fixed point of $\mathcal{T}_{[F,G,u_0,u_1]}$ constructed in Lemma 6.2. □

Proof of Theorem 3.7. Let u be the solution of (1.1), thus, with $f = \mathcal{N}_F u \in L_p([0,T] \times \Omega; \mathcal{D}((-A)^\epsilon))$ and $g = \mathcal{N}_G u \in L_p([0,T] \times \Omega; L_p(B; l_2))$ we have that u solves (5.1). Let v be defined by (3.13). Now, $\zeta, \eta, \delta_0, \delta_1, \epsilon$ satisfy the conditions of Proposition 5.2, which yields immediately the required additional regularity results. □

Proof of Corollary 3.8. To prove Part (1), choose η such that

$$\frac{1}{p} - \frac{1}{q} < \eta < 1,$$

and such that the conditions (3.9), (3.10), (3.11), and (3.12) from Theorem 3.7 are satisfied. Then $D_t^\eta v \in L_p([0,T]; L_p(\Omega; \mathcal{D}((-A)^\zeta)))$. Notice that $q < \frac{p}{1-p\eta}$, so that we infer from [9, p. 421] that $v \in L_q([0,T]; L_p(\Omega; \mathcal{D}((-A)^\zeta)))$.

To prove Part (2), put $\mu + \frac{1}{p} = \eta$. Consequently conditions (3.9), (3.10), (3.11), and (3.12) from Theorem 3.7 hold. Then $D_t^\eta v \in L_p([0,T]; L_p(\Omega; \mathcal{D}((-A)^\zeta)))$. Then by [9, p. 421] we infer that $v \in h_{0\to 0}^{\eta - p^{-1}}([0,T]; L_p(\Omega; \mathcal{D}((-A)^\zeta)))$. □

Proof of Theorem 3.11. For $i = 1, 2$, let $u_{[F_i,G_i,u_{0,i},u_{1,i}]}$ be the solution of (1.1) with u_0 replaced by $u_{0,i}$, etc. Let $v_{[F_i,G_i,u_{0,i},u_{1,i}]}$ be defined by (3.13) with the obvious modifications. From Lemma 6.2, Part (4) we have a Lipschitz estimate

$$\begin{aligned}
&\|u_{[F_1,G_1,u_{0,1},u_{1,1}]} - u_{[F_2,G_2,u_{0,2},u_{1,2}]}\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\theta))} \leq Md \text{ with} \\
&d = \big[\|u_{0,1} - u_{0,2}\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_0}))} + \|u_{1,1} - u_{1,2}\|_{L_p(\Omega;\mathcal{D}((-A)^{\delta_1}))} \\
&\quad + \ \|\mu_{\Delta F}\|_{L_p([0,T]\times\Omega;\mathbb{R})} + \|\mu_{\Delta G}\|_{L_p([0,T]\times\Omega;\mathbb{R})}\big].
\end{aligned}$$

Now let $f_i = \mathcal{N}_F u_{[F_i,G_i,u_{0,i},u_{1,i}]}$ and $g_i = \mathcal{N}_G u_{[F_i,G_i,u_{0,i},u_{1,i}]}$. By Lemma 6.1(1) we have

$$\|f_1 - f_2\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\epsilon))} \leq M_F M d, \quad \|g_1 - g_2\|_{L_p([0,T]\times\Omega;L_p(B,l_2))} \leq M_G M d.$$

The difference $v = v_{[F_1,G_1,u_{0,1},u_{1,1}]} - v_{[F_2,G_2,u_{0,2},u_{1,2}]}$ solves (5.1) with u_0 replaced by $u_{0,1} - u_{0,2}$, etc. Proposition 5.2 yields now

$$\|v_{[F_1,G_1,u_{0,1},u_{1,1}]} - v_{[F_2,G_2,u_{0,2},u_{1,2}]}\|_{L_p([0,T]\times\Omega;\mathcal{D}((-A)^\zeta))} \leq Md$$

with a suitable constant M. □

7. Maximal regularity considerations

In this section, we consider the case that $B = \mathbb{R}^n$ and $A = \Delta : W^{2,p}(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$, the Laplacian in $L_p(\mathbb{R}^n)$. In this case, a maximum regularity result can be proved. To keep the paper at a reasonable size we concentrate on the stochastic part and confine ourselves to the equation

$$\begin{aligned} u(t,\omega,x) = \Delta \int_0^t a_\alpha(t-s)u(s,\omega,x)\,ds \\ + \sum_{k=1}^\infty \int_0^t a_\beta(t-s)G^k(s,\omega,u(s,\omega,x))\,dw_s^k \end{aligned} \tag{7.1}$$

and the linear equation

$$u(t,\omega,x) = \Delta \int_0^t a_\alpha(t-s)u(s,\omega,x)\,ds + \sum_{k=1}^\infty \int_0^t a_\beta(t-s)g^k(s,\omega,x)\,dw_s^k. \tag{7.2}$$

Notice that various results on maximal regularity with respect to deterministic forcing functions (see, e.g., [39]) and to initial data (e.g., [9]) are available. These could be combined with the results given here and adapted to the semilinear case.

For (7.2) we obtain

Proposition 7.1 ([13], Theorem 4.14). *For a positive integer n, let $\Delta : W^{2,p}(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ be the Laplacian with $1 < p < \infty$. Suppose that the probability space Ω and the Wiener processes w^k satisfy Hypothesis 3.2. Let $T > 0$, $\beta > \frac{1}{2}$, $\alpha \in (0,2)$, and $g \in L_p([0,T] \times \Omega; L_p(\mathbb{R}^n, l_2))$.*

(a) *Then there exists a unique function $u \in L_p([0,T] \times \Omega, L_p(\mathbb{R}^n))$ such that for almost all $t \in [0,T]$,*

$$\int_0^t a_\alpha(t-s)u(s)\,ds \in W^{2,p}(\mathbb{R}^n)$$

and (7.2) holds.

(b) *Moreover, if $\zeta \in [0,1]$ is such that*

$$\alpha\zeta + \frac{1}{2} \le \beta, \tag{7.3}$$

then $u \in L_p([0,T] \times \Omega, \mathcal{D}((-\Delta)^\zeta))$, and

$$\|u\|_{L_p([0,T]\times\Omega,\mathcal{D}((-\Delta)^\zeta))} \le M\|g\|_{L_p([0,T]\times\Omega,l_2)} \tag{7.4}$$

with a constant M dependent on $n, T, p, \alpha, \beta, \zeta$.

(c) *If strict inequality holds in (7.3), then M in (7.4) can be obtained arbitrarily small by taking sufficiently small T.*

Proof. Of course, if strict inequality holds in (7.3), then the assertions above are just a special case of Proposition 5.2 with $A = \Delta$, $u_0 = u_1 = 0$, and $\eta = 0$. But for such A and η, the assertion of Lemma 5.8 holds also if equality holds in (7.3), with the only exception that $\tilde{M}_{T,\mathrm{Ito}}$ cannot be made small by taking small T. See [12, Theorem 1.2]. (To prove this, the general estimates from Lemma 5.4 are replaced by a more sophisticated analysis of the resolvent kernel for the Laplacian, using the heat kernel and its self-similarity properties. This has been done for the heat equation by Krylov in [20], and generalized to the case of integral equations in [12].) Once Lemma 5.8 is established, the proof continues exactly as in Section 5. More details can be found in [13]. □

Since M in (7.4) cannot be controlled simply by taking short time intervals, we need a more sophisticated Lipschitz condition. (For the heat equation, compare [21, Assumption 5.6].)

Hypothesis 7.2. There exists some $\theta \in (0,1)$ such that

$$G : [0,T] \times \Omega \times \mathcal{D}((-\Delta)^\theta) \to L_p(\mathbb{R}^n; l_2)$$
$$[G(t,\omega,u)](x) := \left(G^k(t,\omega,u)(x)\right)_{k=1}^\infty$$

satisfies the following assumptions:

(a) For fixed $u \in \mathcal{D}((-\Delta)^\theta)$, the function $G(\cdot,\cdot,u)$ is measurable from $[0,T] \times \Omega$ into $L_p(\mathbb{R}^n; l_2)$.
(b) For each $\epsilon > 0$, there exists a constant $M_G(\epsilon) > 0$, such that for all $t \in [0,T]$, and all $u_1, u_2 \in \mathcal{D}((-\Delta)^\theta)$ the following Lipschitz estimate holds:

$$\begin{aligned} &\|G(t,\omega,u_1) - G(t,\omega,u_2)\|_{L_p(\mathbb{R}^n;l_2)} \\ &\quad \le \left[\epsilon^p \|u_1 - u_2\|^p_{\mathcal{D}((-\Delta)^\theta)} + M_G(\epsilon)^p \|u_1 - u_2\|^p_{L_p(\mathbb{R}^n)}\right]^{1/p} \quad \text{for } \omega \in \Omega \text{ a.e..} \end{aligned} \tag{7.5}$$

(c) For $u = 0$ we have

$$\left[\int_\Omega \int_0^T \|G(t,\omega,0)\|^p_{L_p(\mathbb{R}^n;l_2)}\, dt\, d\mathbb{P}\right]^{1/p} = M_{G,0} < \infty. \tag{7.6}$$

Theorem 7.3. *Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Wiener processes $\left(w_s^k\right)_{k=1}^\infty$ be as in Hypothesis 3.2. Let $p \in [2,\infty)$, and Δ be the Laplacian on $L_p(\mathbb{R}^n)$. Let $\alpha \in (0,2)$, $\beta > \frac{1}{2}$, and $T > 0$. Assume that $G : [0,T] \times \Omega \times \mathcal{D}((-\Delta)^\theta) \to L_p(\mathbb{R}^n; l_2)$ satisfies Hypothesis 7.2 with suitable $\theta \in (0,1)$, such that*

$$\alpha\theta + \frac{1}{2} = \beta. \tag{7.7}$$

Then there exists a unique function $u \in L_p([0,T] \times \Omega; \mathcal{D}((-\Delta)^\theta))$ such that for almost all $t \in [0,T]$

$$\int_0^t a_\alpha(t-s) u(s,\omega,\cdot)\, ds \in \mathcal{D}(\Delta) \quad \textit{for a.e. } \omega \in \Omega,$$

and (7.1) is satisfied for almost all $\omega \in \Omega$.

Proof. We refine the contraction argument from Section 6. As in Lemma 6.1 we define for $v \in L_p([0,T] \times \Omega; \mathcal{D}((-\Delta)^\theta))$

$$\mathcal{N}_G(v) : \begin{cases} [0,T] \times \Omega & \to L_p(\mathbb{R}^n; l_2), \\ t \times \omega & \mapsto G(t, \omega, v(t,\omega)). \end{cases}$$

For $g \in L_p([0,T] \times \Omega; L_p(\mathbb{R}^n, l_2))$ we define $\mathcal{S}g := u \in L_p([0,T] \times \Omega; \mathcal{D}((-\Delta)^\theta))$, where u is the solution of (7.2) according to Proposition 7.1 with forcing function g. As in Section 6 the desired solution u is a fixed point of the operator $\mathcal{T} := \mathcal{S} \circ \mathcal{N}_G$ which maps $L_p([0,T] \times \Omega; \mathcal{D}((-\Delta)^\theta))$ into itself.

By (7.4) for $\zeta = 0$ and for $\zeta = \theta$ we infer

$$\|\mathcal{S}g\|_{L_p([0,T]\times\Omega;L_p(\mathbb{R}^n))} \leq M_0(T)\|g\|_{L_p([0,T]\times\Omega;L_p(\mathbb{R}^n;l_2))},$$
$$\|\mathcal{S}g\|_{L_p([0,T]\times\Omega;\mathcal{D}((-\Delta)^\theta))} \leq M_\theta\|g\|_{L_p([0,T]\times\Omega;L_p(\mathbb{R}^n;l_2))},$$

with fixed M_θ, while $M_0(T)$ can be made arbitrarily small by taking T sufficiently small. We fix $\epsilon > 0$ such that $M_\theta \epsilon < \frac{1}{8}$ and choose the corresponding $M_G(\epsilon)$ according to Hypothesis 7.2. On $L_p([0,T]\times\Omega;\mathcal{D}((-\Delta)^\theta))$ we introduce the following equivalent norm

$$\|v\|^p_{L_p([0,T]\times\Omega;\mathcal{D}((-\Delta)^\theta)),\text{equiv}} := \int_0^T \int_\Omega \left[\epsilon^p \|v(t,\omega)\|^p_{\mathcal{D}((-\Delta)^\theta)} + M_G^p(\epsilon)\|v(t,\omega)\|^p_{L_p(\mathbb{R}^n)}\right] d\mathbb{P}(\omega)\, dt.$$

With respect to this norm, the nonlinear operator

$$\mathcal{N}_G : L_p([0,T] \times \Omega; \mathcal{D}((-\Delta)^\theta)) \to L_p([0,T] \times \Omega; L_p(\mathbb{R}^n; l_2))$$

has Lipschitz constant 1 by Hypothesis 7.2. On the other hand

$$\|\mathcal{S}g\|_{L_p([0,T]\times\Omega;\mathcal{D}((-\Delta)^\theta)),\text{equiv}} \leq (\epsilon^p M_\theta^p + M_G(\epsilon)^p M_0(T)^p)^{1/p} \|g\|_{L_p([0,T]\times\Omega;L_p(\mathbb{R}^n;l_2))}.$$

We infer that $\mathcal{T}$ is Lipschitz on $L_p([0,T] \times \Omega; \mathcal{D}((-\Delta)^\theta))$ with respect to the equivalent norm $\|\cdot\|_{L_p([0,T]\times\Omega;\mathcal{D}((-\Delta)^\theta)),\text{equiv}}$, and if T is sufficiently small, so that $(\epsilon^p M_\theta^p + M_G(\epsilon)^p M_0(T)^p)^{1/p} < \frac{1}{4}$, then the Lipschitz constant of $\mathcal{T}$ is less than $\frac{1}{4}$. We can now proceed as in Lemma 6.2 (2) to construct an equivalent norm on $L_p([0,T] \times \Omega; \mathcal{D}((-A)^\theta))$ which makes $\mathcal{T}$ a strict contraction also for large T. □

8. Krylov's approach versus B-space valued stochastic integration

At the center of the study of stochastic integral equations in Banach spaces is the problem of defining and estimating stochastic integrals, in particular stochastic convolutions, in Banach spaces. Krylov's approach, which is used in this paper, is elementary in the sense that stochastic integrals are taken pointwise, so they are classical Ito-integrals of scalar-valued processes. The Burkholder-Davis-Gundy inequality provides the step from L_2-estimates to L_p. Of course, this can only be done for sufficiently "nice" integrands. The final step is to extend the results obtained for smooth initial data and elementary forcing terms to more general L_p-data by a completion argument.

On the other hand, the recent progress on stochastic integration in Banach spaces provides a convenient tool to handle stochastic convolutions directly in the Banach space. In [4] stochastic convolution is developed to the point where semilinear stochastic differential equations can be treated in M-type 2 UMD spaces. (This covers our case $X = L_p$.) In fact, the key is a generalized version of the Burkholder-Davis-Gundy inequality. Further developments of stochastic integration can be found in [26].

In [13] we compared our linear results with those obtained in [14], [37]. In the context of the present paper it appears interesting to make a similar brief comparison concerning semilinear equations. Note that the aim of the present paper is to treat fractional differential equations and not only the differential equation case $\alpha = \beta = \gamma = 1$. To our knowledge, no results are available for stochastic fractional integral equations using abstract integration methods in spaces other than Hilbert spaces. However, there are several papers dealing with differential equations, in settings that are quite more general than ours. For instance, nonautonomous problems ([38]) and local Lipschitz conditions ([4], [27]) can be treated. And, as stated above, the abstract methods are not confined to the space L_p.

With $\alpha = \beta = \gamma = 1$ our equation (1.1) reduces to the stochastic nonlinear differential equation

$$du(t) = Au(t)\,dt + G(t,\omega,u(t))\,dW_t + F(t,\omega,u(t))\,dt. \tag{8.1}$$

It is this case, where we can compare our results to the results obtained by the abstract integration theory. Note that in abstract notation, W_t is a cylindrical Wiener process in a separable Hilbert space H and that, for fixed u, $G \in L_p([0,T]\times\Omega;\gamma(H,L_p(B)))$ where $\gamma(H,L_p(B))$ denotes the space of γ-radonifying operators $H \to L_p(B)$. This is equivalent to writing the stochastic forcing in Krylov's notation

$$G = \sum_{k=1}^{\infty} G^k w_s^k.$$

with (for fixed u) $\{G^k\}_{k=1}^{\infty} \in L_p([0,T]\times\Omega;L_p(B,l_2))$ (use, e.g., [37, Proposition 3.2.3]).

Possibly the results which can be most easily compared with ours are those of [4], which are formulated for any type 2 UMD space. Rewritten to our notation (and reduced to the globally Lipschitzian case, and L_p instead of general X), the essential conditions in [4] read:

- There are constants $\epsilon \in (-1,0)$, $\nu \in (0,1]$, $\vartheta \geq 0$, and

$$0 \leq \theta_2 < \theta_1 < \theta_2 + p^{-1} < \min\left(\epsilon + 1, \frac{1}{2} - \vartheta\right), \tag{8.2}$$

  such that the following conditions hold:

- $G : [0,T] \times \Omega \times \mathcal{D}((-A)^{\theta_1}) \to L_p(B; l_2)$ satisfies a Lipschitz condition

$$\begin{aligned}&\|(-A)^{-\vartheta} G(t,\omega,u_1) - (-A)^{-\vartheta} G(t,\omega,u_2)\|_{L_p(B;l_2)}\\ &\quad \le K \|u_1 - u_2\|^{\nu}_{\mathcal{D}((-A)^{\theta_1})} \|u_1 - u_2\|^{1-\nu}_{\mathcal{D}((-A)^{\theta_2})}\end{aligned}$$

  for almost all $\omega \in \Omega$, and all $t \in [0,T]$, $u_1, u_2 \in \mathcal{D}((-A)^{\theta_1})$.
- $F : [0,T] \times \Omega \times \mathcal{D}((-A)^{\theta_1}) \to \mathcal{D}((-A)^{\epsilon})$ satisfies a Lipschitz condition

$$\|F(t,\omega,u_1) - F(t,\omega,u_2)\|_{\mathcal{D}((-A)^{\epsilon})} \le K \|u_1 - u_2\|^{\nu}_{\mathcal{D}((-A)^{\theta_1})} \|u_1 - u_2\|^{1-\nu}_{\mathcal{D}((-A)^{\theta_2})}$$

  for almost all $\omega \in \Omega$, and all $t \in [0,T]$, $u_1, u_2 \in \mathcal{D}((-A)^{\theta_1})$.
- $u_0 \in L_p(\Omega, \mathcal{F}_0, \mathcal{D}((-A)^{\theta_1})$.
- In addition, suitable conditions for measurability and linear growth of F and G in u are given.

With these conditions, (8.1) admits a unique mild solution

$$u \in L_p([0,T] \times \Omega; \mathcal{D}((-A)^{\theta_1})) \cap L_p(\Omega; \mathcal{C}([0,T]; \mathcal{D}((-A)^{\theta_2}))).$$

This assertion is stronger than our Theorem 3.6 since it ensures that trajectories are continuous a.s. in the some space $\mathcal{D}((-A)^{\theta_2})$, while our theorem only provides a solution in $L_p([0,T] \times \Omega; \mathcal{D}((-A)^{\theta_1}))$. Accordingly, the conditions on θ_2 will not have a counterpart in Theorem 3.6. The Lipschitz condition on G can be compared to Hypothesis 3.5 if we identify θ_1 above with our θ and put $\vartheta = 0$, and (with some abuse) forget about the role of θ_2. The Lipschitz condition on F is more general than Hypothesis 3.4, since it allows for negative ϵ. We expect that it might be a minor technical task to sharpen our arguments to match this situation. With $\alpha = \beta = \gamma = 1$, our conditions (3.5), (3.6) and (3.7) can be rewritten in the form

$$\theta_1 < 1 + \epsilon, \quad \theta_1 < \frac{1}{2}, \quad \theta_1 < \frac{1}{p} + \delta_0.$$

The conditions on θ_1 and ϵ are exactly what is left from (8.2) if we forget about θ_2. Our Theorem 3.6 allows for $u_0 \in \mathcal{D}((-A)^{\delta_0})$ where $\delta_0 > \theta_1 - \frac{1}{p}$. In [4] slightly more regular $u_0 \in \mathcal{D}((-A)^{\theta_1})$ is required, with the payoff that solutions are continuous at least as functions with values in $\mathcal{D}((-A)^{\theta_2})$. The continuity assertion may be (with some caution) compared to our Corollary 3.8 (2), if we identify θ_2 from [4] with our ζ. Then the conditions of our Corollary 3.8 require $\mu \in (0, 1 - \frac{1}{p})$ such that

$$\frac{1}{p} + \mu + \theta_2 < \min\left(1 + \epsilon, \frac{1}{2}\right), \quad \mu + \theta_2 < \delta_0.$$

Such μ can be found if (8.2) holds. Our corollary states that in this case $u \in h^{\mu}_{0\to 0}([0,T]; L_p(\Omega; \mathcal{D}((-A)^{\theta_2})))$.

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On the Motion of Several Rigid Bodies in an Incompressible Viscous Fluid under the Influence of Selfgravitating Forces

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Abstract. The global existence of weak solutions is proved for the problem of the motion of several rigid bodies in non-newtonian fluid of power-law with selfgraviting forces.

Mathematics Subject Classification (2000). 35Q35.

Keywords. Existence of weak solutions, motion of several rigid bodies, non-Newtonian fluid, selfgravitating forces.

1. Introduction

In the last decade, a large activity has been devoted to the study of motions of rigid or elastic bodies in a fluid. The motion of particles in a viscous liquid has become an important goal of applied research. The presence of particles affects the flow of the liquid, and the fluid, in turn, affects the motion of particles, so that the problem of determining the flow characteristics is a strongly coupled one.

There exists now a lot of numerical studies and simulations concerning this system and its theoretical aspects. We deal here with the problem of several rigid bodies embedded into a viscous fluid.The fluid and rigid bodies are contained in a fixed open bounded set of R^3. We will suppose that this fluid is non-Newtonian with a sufficiently high viscosity and also we consider selfgraviting forces. We are interested in the existence of weak solutions and also in no existence of collisions in finite time. We will apply two different techniques of penalization. The first one was introduced by Conca, San Martin, Tucsnak and Starovoitov [34, 5] and the second one by Bost, Cottet and Maitre [2]. We want to compare efficiency of both methods.

Historically, the weak formulation of the problem has been introduced and studied by Judakov see [39] and after that by many authors: Desjardins and Esteban [6, 7], Hoffmann and Starovoitov [29, 30], San Martin, Starovoitov, Tucsnak [34], Serre [35], Galdi [23], among others.

Concerning the problem of the existence of collisions, let us mention that in the case of compressible fluids there is a result obtained by E. Feireisl.

In [16], E. Feireisl considered a rigid sphere surrounded by a compressible viscous fluid inside a cavity. He constructed a solution to the subsequent system in which the sphere sticks to the ceiling of the cavity without falling down. On the other hand, in the incompressible case, Hesla [26] and Hillairet [27] proved a no-collision result when there is only one body in a bounded two-dimensional cavity. Later the result was extended to the three-dimensional situation by Hillairet and Takahashi [28].

Starovoitov in [38] showed that collisions, if any, must occur with zero relative *translational* velocity if the boundaries of the rigid objects are smooth and the gradient of the underlying velocity field is square integrable – a hypothesis satisfied by any Newtonian fluid flow of finite energy. The possibility or impossibility of collisions in a viscous fluids is related to the properties of the velocity gradient. A simple argument reveals that the velocity gradient must become singular (unbounded) at the contact point, since otherwise the streamlines would be well defined, in particular, they could never meet each other.

Indeed Starovoitov [38] showed that collisions of two or more rigid objects are impossible if:

- the physical domain $\Omega \subset R^3$ as well as the rigid objects in its interior have boundaries of class $C^{1,1}$;
- the pth power of the velocity gradient is integrable, with $p \geq 4$.

Inspired by work of Starovoitov, Feireisl et al. [17] have considered the motion of several rigid bodies in a non-Newtonian fluid of power-law type (see Chapter 1 in Málek et al. [33] for details), where the viscous stress tensor $\mathbb{S}$ depends on the symmetric part $\mathbb{D}[\mathbf{u}]$,

$$\mathbb{D}[\mathbf{u}] = \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}$$

of the gradient of the velocity field $\mathbf{u}$ in the following way:

$$\mathbb{S} = \mathbb{S}[\ \mathbb{D}[\mathbf{u}]\ ],\ \mathbb{S} : R^{3\times 3}_{\text{sym}} \to R^{3\times 3}_{\text{sym}} \text{ is continuous,} \tag{1.1}$$

$$\Big(\mathbb{S}[\mathbb{M}] - \mathbb{S}[\mathbb{N}]\Big) : \Big(\mathbb{M} - \mathbb{N}\Big) > 0 \text{ for all } \mathbb{M} \neq \mathbb{N}, \tag{1.2}$$

and

$$c_1|\mathbb{M}|^p \leq \mathbb{S}[\mathbb{M}] : \mathbb{M} \leq c_2(1 + |\mathbb{M}|^p) \text{ for a certain } p \geq 4 \tag{1.3}$$

and they showed not only the existence of a weak solution but also that collisions cannot occur in such viscous fluids.

The question how the smoothness of boundary has influence on the existence of collisions was investigated in the work of Gerard-Varet and Hillairet [24].

The present work is an extension of that of Feireisl et al. (see [17]) to the case where selfgravitating forces are present and two different possibilities of penalization technique are used.We have to use a slightly different approximation scheme by means of artificial viscosity terms to handle the selfgravitating force. Moreover, we have to assume the regularity of boundary $C^{2,\nu}$, $\nu \in (0,1)$ to get the strong solution of the regularized continuity equation.

2. Formulation of the problem

2.1. Bodies and motions

A *rigid body* can be identified with a connected compact subset $\bar{\mathbf{B}}$ of the Euclidean space R^3. The motion is represented as a mapping $\eta : (0,T) \times R^3 \to R^3$, which defines an isometry

$$\eta(t,\cdot) : R^3 \to R^3 \tag{2.1}$$

for any time $t \in (0,T)$.

We adopt the Eulerian (spatial) description of motion, where the coordinate system is attached to a fixed region of the physical space currently occupied by the fluid. The *position* $\mathbf{x}$ and the *time* $t \in (0,T)$ play the role of independent variables. The mappings $\eta(t,\cdot)$ are isometries satisfying the following identity

$$\eta(t,\mathbf{x}) = \mathbf{X}_g(t) + \mathcal{O}(t)(\mathbf{x} - \mathbf{X}_g(0)),$$

where $\mathbf{X}_g$; the position of the center of mass at a time t and $\mathcal{O}(t)$ is a matrix satisfying $\mathcal{O}^T\mathcal{O} = \mathcal{I}$. Moreover, the translation and angular velocities can be expressed respectively by

$$\frac{d}{dt}\mathbf{X}_g = \mathbf{U}_g \tag{2.2}$$

and

$$\Big(\frac{d}{dt}\mathcal{O}(t)\Big)\mathcal{O}^T(t) = \mathcal{Q}(t). \tag{2.3}$$

The solid velocity in the Eulerian coordinate system can be written as

$$\mathbf{u}^S(t,\mathbf{x}) = \frac{\partial \eta}{\partial t}(t,\eta^{-1}(t,\mathbf{x})) = \mathbf{U}_g(t) + \mathcal{Q}(t)(\mathbf{x} - \mathbf{X}_g(t)).$$

The total force $\mathbf{F}^{B_i}$ acting on the body $\mathbf{B_i}$ is a sum of the *gravitation force* and the *contact force*

$$\mathbf{F}^{B_i}(t) = \int_{\mathbf{B_i}(t)} \rho^{B_i} g^{B_i}\ \mathrm{d}x + \int_{\partial\mathbf{B_i}(t)} \mathbb{S}\mathbf{n}\ d\sigma,$$

where $\mathbb{S}$ is the Cauchy stress, $\mathbf{g}^{B_i} = G\nabla \int_{R^3} \sum_{j\neq i} \frac{\rho^{S_j}}{|x-y|}$ is the gravitation force and $\bar{\mathbf{B}}_\mathbf{i}(t) = \eta(t,\bar{\mathbf{B}}_\mathbf{i})$. Under the assumption of the continuity of stress, the balance of linear momentum for body $\mathbf{B_i}$ is expressed by **Newton's second law**

$$m\frac{d}{dt}\mathbf{U}_g(t) = \frac{d}{dt}\int_{\mathbf{B_i}(t)} \rho^{B_i}\mathbf{u}^{B_i}\ d\mathbf{x} = \int_{\mathbf{B_i}(t)} \rho^{B_i}\mathbf{g}^{B_i}\ \mathrm{d}x + \int_{\partial\mathbf{B_i}(t)} \mathbb{S}\mathbf{n}\ d\sigma, \tag{2.4}$$

where m is the total mass of the body.

The matrix $\mathcal{Q}$ is skew-symmetric, therefore it can be represented by a vector ω such that

$$\mathcal{Q}(t)(\mathbf{x}-\mathbf{X}_g)=\omega(t)\times(\mathbf{x}-\mathbf{X}_g).$$

Moreover the balance of angular momentum reads

$$\frac{d}{dt}(\mathcal{J}\omega)=\frac{d}{dt}\int_{\mathbf{B_i}(t)}\rho^{B_i}(\mathbf{x}-\mathbf{X}_g)\times\mathbf{u}^{B_i}\ \mathrm{d}x \tag{2.5}$$

$$=\int_{\partial\mathbf{B_i}(t)}\rho^{B_i}(\mathbf{x}-\mathbf{X}_g)\times\mathbb{S}\mathbf{n}\ \mathrm{d}\sigma+\int_{\mathbf{B_i}(t)}\rho^{B_i}(\mathbf{x}-\mathbf{X}_g)\times g^{B_i}\ \mathrm{d}x,$$

where $\mathcal{J}$ is the *inertia tensor*

$$\mathcal{J}\mathbf{a}\cdot\mathbf{b}=\int_{\mathbf{B_i}(t)}\rho^{B_i}(\mathbf{a}\times(\mathbf{x}-\mathbf{X}_g))\cdot(\mathbf{b}\times(\mathbf{x}-\mathbf{X}_g))\ \mathrm{d}x.$$

2.2. The fluid motion

The fluid is completely determined by the density ρ^f, and the velocity $\mathbf{u}^f$. The standard mass and momentum balance equations are the following:

$$\left.\begin{array}{l}\partial_t\rho^f+\mathrm{div}(\rho^f\mathbf{u}^f)=0,\\ \mathrm{div}\mathbf{u^f}=0,\\ \partial_t(\rho^f\mathbf{u}^f)+\mathrm{div}(\rho^f\mathbf{u}^f\otimes\mathbf{u}^f)+\nabla P=\mathrm{div}\ \mathbb{S}+\rho^f G\nabla\int_{R^3}\frac{\rho^f}{|x-y|}dy,\end{array}\right\}\ \text{in } \Omega_f, \tag{2.6}$$

where $\Omega_f=\Omega\setminus\bigcup_{i=1}^N\overline{\mathbf{B}}_\mathbf{i}$, P is the pressure and $\mathbb{S}$ is the viscous stress tensor. We also consider a no-slip boundary condition for velocity

$$\mathbf{u}^f=0 \text{ on } \partial\Omega. \tag{2.7}$$

We will introduce the notation

$$\begin{aligned}Q&:=I\times\Omega,\\ Q^i&=\{(t,x)|t\in I,x\in\bar{\mathbf{B}}^\mathbf{i}(t)\},\\ Q^s&:=\bigcup_{i=1}^N Q^i, Q^f:=Q\setminus Q^s,\end{aligned}$$

and define the quantities

$$\rho(t,\mathbf{x})=\begin{cases}\rho^f(t,\mathbf{x}) \text{ on } Q^f,\\ \rho^{B_i}(t,\mathbf{x}) \text{ on } Q^i,\\ 0 \text{ on } R^3\setminus\Omega,\end{cases}$$

and

$$\mathbf{u}(t,\mathbf{x})=\begin{cases}\mathbf{u}^f(t,\mathbf{x}) \text{ on } Q^f,\\ \mathbf{u}^{B_i}(t,\mathbf{x}) \text{ on } Q^i,\\ 0 \text{ on } R^3\setminus\Omega.\end{cases}$$

We will restrict ourselves to the particular case where the density is nonconstant only on the solid part and constant on the fluid part to get local estimates on the pressure.

3. Weak formulation

We begin with a description of the initial position of a set of rigid bodies. We assume that the initial position of the rigid bodies is determined through a family of domains

$$\mathbf{B}_i \subset R^3,\ i = 1, \ldots, n,$$

each of them being diffeomorphic to the unit ball in R^3. In addition, in order to facilitate the analysis, the boundaries of all the rigid bodies are assumed to be regular, more specifically, there exists $\delta_0 > 0$ such that for any $\mathbf{x} \in \partial\mathbf{B}_i$, there are two closed balls $\mathbf{B}^{\text{int}}$, $\mathbf{B}^{\text{ext}}$ of radius δ_0 such that

$$\mathbf{x} \in \mathbf{B}^{\text{int}} \cap \mathbf{B}^{\text{ext}},\ \mathbf{B}^{\text{int}} \subset \overline{\mathbf{B}}_i,\ \mathbf{B}^{\text{ext}} \subset R^3 \setminus \mathbf{B}_i. \tag{3.1}$$

Similarly, the underlying physical space $\Omega \subset R^3$, occupied by the fluid containing the rigid bodies, is supposed to be a domain such that for any $\mathbf{x} \in \partial\Omega$, there are two closed balls $\mathbf{B}^{\text{int}}$, $\mathbf{B}^{\text{ext}}$ of radius δ_0 such that

$$\mathbf{x} \in \mathbf{B}^{\text{int}} \cap \mathbf{B}^{\text{ext}},\ \mathbf{B}^{\text{int}} \subset \overline{\Omega},\ \mathbf{B}^{\text{ext}} \subset R^3 \setminus \Omega. \tag{3.2}$$

The motion η_i associated to the body $\mathbf{B}_i$ is a mapping

$$\eta_i = \eta_i(t, \mathbf{x}),\ t \in [0, T),\ \mathbf{x} \in R^3,\ \eta_i(t, \cdot) : R^3 \to R^3,$$

together with the initial condition

$$\eta_i(0, \mathbf{x}) = \mathbf{x} \text{ for all } \mathbf{x} \in R^3, \quad i = 1, \ldots, n,$$

that is an isomorphism $R^3 \to R^3$.

Accordingly, the position of the body $\mathbf{B}_i$ at a time t is given by the formula

$$\mathbf{B}_i(t) = \eta_i(t, \mathbf{B}_i),\ i = 1, \ldots, n.$$

We proceed with the definition of a *weak solution for fluid structure interaction*, which was introduced by Judakov [39] based on the Eulerian reference system and a class of test functions depending on the position of the specified rigid bodies (see Desjardins and Esteban [6, 7], Galdi [22], [23], Hoffmann and Starovoitov [29], San Martin et al. [34], Serre [35], among others) however we will use a slightly different definition in the last paragraph.

The mass density $\varrho = \varrho(t, \mathbf{x})$ and the velocity field $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ at a time $t \in (0, T)$ and the spatial position $\mathbf{x} \in \Omega$ satisfy the integral identity

$$\int_0^T \int_\Omega \Big(\rho \partial_t \phi + \rho \mathbf{u} \cdot \nabla_x \phi\Big) \mathrm{d}x\ \mathrm{d}t = -\int_\Omega \rho_0 \phi\ \mathrm{d}x,\ \phi \in C^1([0, T) \times \bar{\Omega}), \tag{3.3}$$

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\rho \mathbf{u} \cdot \partial_t \varphi + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x[\varphi] - \mathbb{S} : \mathbf{D}[\varphi]\Big)\ \mathrm{d}x\ \mathrm{d}t \\ &\quad = -\int_0^T \int_\Omega \rho G \nabla_x \int_{R^3} \frac{\rho}{|x - y|} dy \cdot \varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \rho_0 \mathbf{u}_0 \cdot \varphi\ \mathrm{d}x\ \mathrm{d}t, \end{aligned} \tag{3.4}$$

$$\varphi \in C^1([0, T) \times \bar{\Omega}),\ \varphi(t, \cdot) \in \mathcal{R}(t), \tag{3.5}$$

$$\mathcal{R}(t) = \{\phi \in C^1(\bar{\Omega}) \mid \text{div } \Phi = 0 \text{ in } \Omega,\ \phi = 0 \text{ on a neighborhood of } \partial\Omega, \qquad (3.6)$$
$$\mathbf{D}[\Phi] = 0 \text{ on a neighborhood of} \cup_{i=1}^{n} \bar{\mathbf{B}}_i(t)\},$$

where

$$\int_0^T \int_\Omega \rho G \nabla_x \Big(\int_{R^3} \frac{\rho}{|x-y|} dy \Big) \varphi \ dx \ dt = \int_0^T \int_\Omega \rho G \nabla_x F \ dx \ \mathrm{d}t,$$

with

$$F = \Big(\sum_{i \neq j} \int_{R^3} \frac{\rho_j^{B_j}}{|x-y|} \ dy + \int_{R^3} \frac{\rho^f}{|x-y|} \ dy \Big).$$

Finally, we require the velocity field u to be compatible with the motion of the bodies. As the mappings $\eta_i(t,i)$ are isometries on R^3, they can be written in the form

$$\eta_i(t, \mathbf{x}) = x_i(t) + \mathcal{O}_i(t)\mathbf{x}.$$

Accordingly, we require the velocity field u to be compatible with the family of motions $\{\eta_1, \dots, \eta_n\}$ if

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{u}^{B_i}(t, \mathbf{x}) = \mathbf{U}_i(t) + \mathcal{Q}_i(t)(\mathbf{x} - x_i(t)) \text{ for a.a. } x \in \bar{\mathbf{B}}_i(t),\ i = 1, \dots, n \quad (3.7)$$

for a.a. $t \in [0, T)$, where

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i = \mathbf{U}_i,\ \Big(\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{O}_i \Big) \mathcal{O}_i^T = \mathcal{Q}_i \text{ a.a. on } (0, T). \qquad (3.8)$$

We introduce now some notation:

- *Given a collection* $\widetilde{\mathbf{B}}_{i=1,\dots,m}$ *of open relatively compact connected subsets in* Ω, we denote by
$$\mathrm{d}[\cup \widetilde{\mathbf{B}}_i] := \inf\{ \inf_{i,j=1,\dots,m} \mathrm{dist}(\mathbf{B}_i, \mathbf{B}_j), \inf_{i=1,\dots,m} \mathrm{dist}(\mathbf{B}_i, \partial\Omega)\}$$
the minimal interdistance "solids-boundary".
- *The signed distance to the boundary* is
$$\mathbf{db}_S(x) = \mathrm{dist}\,[\mathbf{x}; R^3 \setminus S] - \mathrm{dist}\,[\mathbf{x}, S].$$

Given a subset $S \subset R^3$, and $\alpha > 0$, we denote

$$[S]_\alpha = \mathbf{db}_S^{-1}((-\infty, -\alpha)), \quad]S[_\alpha = \mathbf{db}_S^{-1}((-\infty, \alpha)). \qquad (3.9)$$

In the sense of J.A. San Martin, M. Tucsnak and V. Starovoitov, $[S]_\alpha$ is the α-neighborhood of S and $]S[_\alpha$ is the α-kernel of $\mathcal{O}$.

$$\mathcal{V} := \{\phi \in \mathcal{C}_c^\infty(\Omega),\ \text{such that } \mathrm{div}(\phi) = 0\}.$$

1. V_p stands for the closure of $\mathcal{V}$ in $\mathcal{W}_0^{1,p}(\Omega)$ and V^s for the closure of $\mathcal{V}$ in $\mathcal{W}^{s,2}(\Omega)$. For simplicity $V = V^1 = V_2$.
2. Given $\mathbf{B} \subset \Omega$, we write
$$\mathcal{K}_p(\mathbf{B}) = \{\mathbf{v} \in V_p \text{ with } D(\mathbf{v}) = \mathbf{0} \text{ in } \mathbf{B}\},$$
$$\mathcal{K}^s(\mathbf{B}) = \{\mathbf{v} \in V^s \text{ with } D(\mathbf{v}) = \mathbf{0} \text{ in } \mathbf{B}\}.$$

3. Given a subset $\mathbf{B}$ of Ω, $\mathcal{P}^k\{\mathbf{B}\}$ is the orthogonal projection of V^s onto $\mathcal{K}^s([\mathbf{B}]_\sigma)$.

   We will introduce the energy inequality (EI) of the problem as follows:

$$\frac{1}{2}\Big\{ \int_{\Omega_{t_2}} \rho|\mathbf{u}|^2(t_2)\, dx - \int_{\Omega_{t_2}} G\rho(t_2,\mathbf{x})F dx\Big\} + \int_{t_1}^{t_2}\int_\Omega \mathbb{S} : \mathbf{D}[u]\, dx\, dt$$
$$\leq \frac{1}{2}\Big\{ \int_{\Omega_{t_1}} \rho|\mathbf{u}|^2(t_1)\, dx - \int_{\Omega_{t_1}} G\rho(t_1,\mathbf{x})F\, dx\Big\}.$$

Problem P. *Let the initial distribution of the density and the velocity field be determined through given functions ρ_0, $\mathbf{u}_0$, respectively. The initial position of the rigid bodies is $\mathbf{B^i} \subset \Omega$, $i = 1, \ldots, m$. We say that a family ρ, $\mathbf{u}$, η^i, $i = 1, \ldots, m$, represent a variational solution of* **problem (P)** *on a time interval $(0,T)$ if the following conditions are satisfied:*

- The density ρ is a non-negative bounded function, the velocity field $\mathbf{u}$ belongs to the space $L^\infty(0,T;L^2(\Omega;R^3)) \cap L^p(0,T;W_0^{1,p}(\Omega;R^3))$, and they satisfy energy inequality (EI) for $t_1 = 0$ and a.a. $t_2 \in (0,T)$.
- The continuity equation holds on $(0,T) \times R^3$ provided ρ and $\mathbf{u}$ are extended to be zero outside Ω.
- The momentum equation (the integral identity) holds for any admissible test function $\mathbf{w} \in \mathcal{R}(t)$.
- The mappings η^i, $i = 1, \ldots, m$ are affine isometries of R^3 compatible with the velocity field $\mathbf{u}$ in the sense of compatibility conditions.

4. Main result I

Let us formulate one of our main existence results.

Theorem 4.1. *Let the initial position of the rigid bodies be given through a family of open sets*

$$\mathbf{B}_i \subset \Omega \subset R^3,\ \mathbf{B}_i \text{ diffeomorphic to the unit ball for } i = 1, \ldots, n,$$

where both $\partial\mathbf{B}_i$, $i = 1, \ldots, n$, and $\partial\Omega$ belong to the regularity class specified in (3.1), (3.2)*. In addition, suppose that*

$$\mathrm{dist}[\overline{\mathbf{B}}_i, \overline{\mathbf{B}}_j] > 0 \text{ for } i \neq j,\ \mathrm{dist}[\overline{\mathbf{B}}_i, R^3 \setminus \Omega] > 0 \text{ for any } i = 1, \ldots, n$$

and we assume that the boundaries of Ω and $\mathbf{B}_i$ belong to $C^{2,\nu}$, $\nu \in (0,1)$. Furthermore, let the viscous stress tensor $\mathbb{S}$ satisfy hypotheses (1.1)–(1.3)*, with $p \geq 4$.*

Finally, let the initial distribution of the density be given as

$$\varrho_0 = \begin{cases} \varrho_f = \mathrm{const} > 0 \text{ in } \Omega \setminus \cup_{i=1}^n \overline{\mathbf{B}}_i, \\ \varrho_{\mathbf{B}_i} \text{ on } S_i, \text{ where } \varrho_{\mathbf{B}_i} \in L^\infty(\Omega),\ \mathrm{ess\,inf}_{\mathbf{B}_i}\, \varrho_{\mathbf{B}_i} > 0,\ i = 1, \ldots, n, \end{cases}$$

while

$$\mathbf{u}_0 \in L^2(\Omega;R^3),\ \mathrm{div}_x \mathbf{u}_0 = 0 \text{ in } \mathcal{D}'(\Omega),\ \mathbb{D}[\mathbf{u}_0] = 0 \text{ in } \mathcal{D}'(\mathbf{B}_i;R^{3\times 3}) \text{ for } i = 1, \ldots, n.$$

Then there exist a density function ϱ,

$$\varrho \in C([0,T];L^1(\Omega)),\ 0 < \operatorname{ess}\inf_{\Omega} \varrho(t,\cdot) \le \operatorname{ess}\sup_{\Omega} \varrho(t,\cdot) < \infty \textit{ for all } t \in [0,T],$$

a family of isometries $\{\eta_i(t,\cdot)\}_{i=1}^n$, $\eta_i(0,\cdot) = \mathrm{I}$, *and a velocity field* $\mathbf{u}$,

$$\mathbf{u} \in C_{\mathrm{weak}}([0,T];L^2(\Omega;R^3)) \cap L^p(0,T;W_0^{1,p}(\Omega;R^3)),$$

compatible with $\{\eta_i\}_{i=1}^n$ *in the sense specified in* (3.7), (3.8), *such that* ϱ, $\mathbf{u}$ *satisfy the integral identity* (3.3) *for any test function* $\phi \in C^1([0,T) \times R^3)$, *and the integral identity* (3.4) *for any* φ *satisfying* (3.5), (3.6).

5. Approximate problem

We will construct weak solutions based on a three-level approximation scheme which consists in solving the system of equations

$$\partial_t \varrho + \operatorname{div}_x(\varrho[\mathbf{u}]_\delta) = d\Delta\rho, \tag{5.1}$$

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}_x(\varrho\mathbf{u} \otimes [\mathbf{u}]_\delta) + \nabla_x P + d\nabla\rho\nabla\mathbf{u} = \operatorname{div}_x([\mu_\varepsilon]_\delta \mathbb{S}) + \varrho\nabla_x F - \chi_\varepsilon \mathbf{u}, \tag{5.2}$$

$$\partial_t \mu_\varepsilon + \operatorname{div}_x(\mu_\varepsilon[\mathbf{u}]_\delta) = 0, \tag{5.3}$$

$$\operatorname{div}_x \mathbf{u} = 0, \tag{5.4}$$

where

$$[\mathbf{u}]_\delta = \sigma_\delta * \mathbf{u} \text{ (spatial convolution) for } 0 < \delta < \delta_0,$$

with

$$\sigma_\delta(x) = \frac{1}{\delta^3}\sigma\Big(\frac{|x|}{\delta}\Big), \tag{5.5}$$

$$\sigma \in \mathcal{D}(-1,1),\ \sigma(z) > 0 \text{ for } -1 < z < 1,\ \sigma(z) = \sigma(-z),\ \int_{-1}^{1} \sigma(z)\ \mathrm{d}z = 1.$$

As Ω is bounded, we can assume that $\Omega \subset [-L,L]^3$ for a certain $L > 0$ and consider system (5.1–5.4) on the spatial torus

$$\mathcal{T} = [(-L,L)|_{\{-L,L\}}]^3,$$

meaning that all quantities are assumed to be spatially periodic with period $2L$.

System (5.1)–(5.4) is supplemented with the initial conditions

$$\varrho(0,\cdot) = \varrho_{0,\delta} = \varrho_f + \sum_{i=1}^n \varrho_{\mathbf{B}_i,\delta}, \tag{5.6}$$

where

$$\varrho_{\mathbf{B}_i} \in \mathcal{D}(\mathbf{B}_i),\ \varrho_{\mathbf{B}_i,\delta}(x) = 0 \text{ whenever } \operatorname{dist}[x,\partial\mathbf{B}_i] < \delta, i = 1,\dots,n. \tag{5.7}$$

Similarly, we prescribe

$$\mu(0,\cdot) = \mu_{0,\varepsilon} = 1 + \frac{1}{\varepsilon}\sum_{i=1}^n \mu_{B_i}, \tag{5.8}$$

where

$$\left.\begin{cases} \mu_{B_i} \in \mathcal{D}(\mathbf{B}_i),\ \mu_{B_i}(x) = 0 \text{ whenever } \operatorname{dist}[x, \partial\mathbf{B}_i] < \delta, \\ \mu_{B_i}(x) > 0 \text{ for } x \in \mathbf{B}_i,\ \operatorname{dist}[x, \partial\mathbf{B}_i] > \delta \end{cases}\right\} i = 1, \ldots, n. \tag{5.9}$$

We take

$$\chi_\varepsilon = \frac{1}{\varepsilon}\chi,\ \chi \in \mathcal{D}(\mathcal{T}),\ \chi > 0 \text{ on } \mathcal{T} \setminus \Omega,\ \chi = 0 \text{ in } \overline{\Omega}. \tag{5.10}$$

Finally we add the homogeneous Neumann boundary condition

$$\nabla\rho \cdot n = 0 \qquad \text{on } \partial\Omega \cup \bigcup_{i=1}^{n} \partial\mathbf{B}_i, \tag{5.11}$$

and we assume that

$$\rho_{0,\delta} \in C^{2,\nu}(\overline{\Omega}). \tag{5.12}$$

The parameters ε, d and δ are small positive numbers. The ε-dependent initial distribution of the "viscosity" μ can be easily identified as the penalization introduced by Hoffmann and Starovoitov [29] and San Martin et al. [34], where the rigid bodies are replaced by a fluid of high viscosity becoming singular for $\varepsilon \to 0$. Here, the extra parameter $\delta > 0$ has been introduced to keep the density constant in the approximate fluid region in order to construct the local pressure. Moreover, we regularized the continuity equation by an artificial viscosity term to get the estimation of the selfgravitating forces. We will also introduce in the last section another possibility of penalization which was introduced by Bost et al. [2] and gives the main idea of the proof. As for $\varepsilon > 0$, $\delta > 0$ and $d > 0$ fixed, we report the following existence result that can be proved in a standard way through the solution of the continuity equation through L^p-L^q regularity (see for more details [18], Lemma 3.1) and by means of a standard monotonicity argument (see Málek et al. [33]) which was extended to nonhomogeneous fluids by Frehse et al. [20], [21].

Proposition 5.1. *Suppose that $p \geq 4$. Let the initial distribution of ϱ, μ be given through* (5.6)–(5.9)*, with fixed $\varepsilon > 0$, $\delta > 0$, $d > 0$. Moreover, assume that*

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \in \mathcal{T},\ \mathbf{u}_0 \in L^2(\mathcal{T}; R^3),\ \operatorname{div}_x \mathbf{u}_0 = 0 \text{ in } \mathcal{D}'(\mathcal{T}; R^3), \tag{5.13}$$

and $\chi_\varepsilon \in C^\infty(\mathcal{T})$, where χ_ε is determined by (5.10) *and let us assume* 5.12.

Then problem (5.1)–(5.4)*, supplemented with the initial data* (5.6)–(5.9), (5.12) *and additional boundary conditions* (5.11)*, possesses a* (*weak*) *solution ϱ, μ, $\mathbf{u}$ belonging to the class*

$$\mu \in C^{1,2}([0,T] \times \mathcal{T}),$$
$$\varrho \in C([0,T]; C^{2,\nu}(\mathcal{T})), \partial_t\varrho \in C([0,T]; C^{0,\nu}(\mathcal{T})),$$
$$\mathbf{u} \in C_{\text{weak}}([0,T]; L^2(\mathcal{T}; R^3)) \cap L^p(0,T; W^{1,p}(\mathcal{T}; R^3)).$$

In addition, the solution satisfies the energy inequality

$$\int_{\mathcal{T}} \frac{1}{2}\varrho|\mathbf{u}|^2(\tau)\ \mathrm{d}x + \int_s^\tau \int_{\mathcal{T}} [\mu_\varepsilon]_\delta \mathbb{S} : \nabla_x \mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \int_s^\tau \int_{\mathcal{T}} \chi_\varepsilon |\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_{\mathcal{T}} \frac{1}{2}\varrho|\mathbf{u}|^2(s)\,\mathrm{d}x \tag{5.14}$$

for a.a. $0 \le s < \tau \le T$ *including* $s = 0$.

Let us remark that, in the weak formulation, equation (5.2) is represented by the integral identity

$$\begin{aligned} &\int_0^T \int_{\mathcal{T}} \varrho\mathbf{u}\cdot\partial_t\varphi + \varrho(\mathbf{u}\otimes[\mathbf{u}]_\delta) : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t + \int_{\mathcal{T}} \delta\nabla\rho\nabla\mathbf{u}\ \varphi\ dx\ dt \\ &\quad = \int_0^T \int_{\mathcal{T}} [\mu_\varepsilon]_\delta\mathbb{S} : \mathbb{D}[\varphi]\ \mathrm{d}x\ \mathrm{d}t - \int_0^T \int_{\mathcal{T}} \rho G\cdot\nabla_x\left\{\int_{R^3} \frac{\rho(t,y)}{|x-y|}dy\right\}\varphi\ \mathrm{d}x\ \mathrm{d}t \\ &\qquad - \int_0^T \int_{\mathcal{T}} \chi_\varepsilon\mathbf{u}\cdot\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_{\mathcal{T}} \varrho_{0,\delta}\mathbf{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x \end{aligned} \tag{5.15}$$

to be satisfied for any test function $\varphi \in \mathcal{D}([0,T)\times\mathcal{T};R^3)$, such that $\mathrm{div}_x\varphi = 0$.

Remark. From [18] [Chapter 3, Lemma 3.1] it follows that in the level of approximation δ, ϵ the density satisfies the following bounds:

$$\rho \in L^2(0,T;W^{2,2}\mathcal{T}) \cap C(0,T;W^{1,2}(\mathcal{T})), \partial_t\rho \in L^2((0,T)\times\mathcal{T}) \tag{5.16}$$

and also through a bootstrap argument we have

$$\begin{aligned} \|\rho\|_{C([0,T];W^{2-2/p,p}(\mathcal{T}))} + \|\rho\|_{L^p(0,T;W^{2,p}(\mathcal{T}))} \qquad\qquad & \qquad (5.17)\\ +\|\partial_t\rho\|_{L^p((0,T)\times\mathcal{T})} \le c\|\rho_0\|_{W^{2-2/p,p}(\mathcal{T})}, & \\ \|\rho\|_{C([0,T];C^{2,\nu}(\bar{\mathcal{T}}))} \le c, & \qquad (5.18)\\ \|\partial_t\rho\|_{C([0,T];C^{0,\nu}(\bar{\mathcal{T}}))} \le c. & \qquad (5.19) \end{aligned}$$

The term with selfgraviting force is estimated by the following way:

$$\int_0^T \int_{\mathcal{T}} \rho G\nabla_x\left\{\int_{R^3} \frac{\rho(t,y)}{|x-y|}dy\right\}\phi dxdt \le c\|\rho\|_{L^3(\mathcal{T}}\|\rho\|_{L^2\mathcal{T}}\|u\|_{L^6(\mathcal{T})}$$

for more details see [8, 9].

6. Artificial viscosity limit

Our first task is to identify the limit problem resulting from (5.1)–(5.9) for fixed $\delta > 0$ and $\varepsilon \to 0$. To this end, let us denote by $\{\varrho_\varepsilon, \mu_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ the associated family of approximate solutions, the existence of which is guaranteed by Proposition 5.1. As already pointed out, there are two major issues to be addressed, namely the strong (pointwise) convergence of the velocity fields, and strong convergence of the velocity gradients in order to pass to the limit in the non-linearity of the stress tensor. The first goal is accomplished basically in the same way as in [34] and repeated in [17], so we will give only the main ideas of the proof. Note that in the present setting, the analysis is considerably simplified thanks to the no-collision result by Starovoitov [38].

Let us start with the following stability result for solutions to the transport equation established in [16, Proposition 5.1]:

Proposition 6.1. *Let* $\mathbf{v}_n = \mathbf{v}_n(t,x)$ *be a family of vector fields such that*

$$\{\mathbf{v}_n\}_{n=1}^{\infty} \text{ is bounded in } L^2(0,T;W^{1,\infty}(R^3;R^3)).$$

Let $\eta_n(t,\cdot) : R^3 \to R^3$ *be the solution operator associated to the family of characteristic curves generated by* $\mathbf{v}_n$*, specifically,*

$$\frac{\partial}{\partial t}\eta_n(t,x) = \mathbf{v}_n(t,\eta_n(t,x)),\ \ \eta_n(0,x) = x \text{ for all } x \in R^3.$$

Then, at least for a suitable subsequence,

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly-(*) in } L^2(0,T;W^{1,\infty}(\Omega;R^3)),$$

$$\eta_n(t,\cdot) \to \eta(t,\cdot) \text{ in } C_{\mathrm{loc}}(R^3) \text{ uniformly for } t \in [0,T],$$

where η *is the unique solution of*

$$\frac{\partial}{\partial t}\eta(t,x) = \mathbf{v}(t,\eta(t,x)),\ \ \eta(0,x) = x,\ \ x \in R^3.$$

If, in addition,

$$S_n \xrightarrow{b} S,$$

then

$$\eta_n(t,S_n) \equiv S_n(t) \xrightarrow{b} S(t) \equiv \eta(t,S)$$

which means that

$$\mathbf{db}_{S_n(t)} \to \mathbf{db}_{S(t)} \text{ in } C_{\mathrm{loc}}(R^3) \text{ uniformly with respect to } t \in [0,T].$$

The energy inequality (5.14), together with the coercivity hypothesis (1.3), that $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ give us that $\{u_\epsilon\}$ is a bounded sequence in $L^p(0,T;W^{1,p}(\mathcal{T};R^3))$. Consequently, passing to a suitable subsequence as the case may be, we can assume

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^p(0,T;W^{1,p}(\mathcal{T};R^3))$$

where the limit velocity field satisfies $\mathrm{div}_x\mathbf{u} = 0$ a.a. on $(0,T)\times\mathcal{T}$. Accordingly, the regularized sequence $\{[\mathbf{u}_\varepsilon]_\delta\}_{\varepsilon>0}$ satisfies

$$[\mathbf{u}_\varepsilon]_\delta \to [\mathbf{u}]_\delta \text{ weakly-(*) in } L^p(0,T;W^{1,\infty}(\mathcal{T};R^3)),\ \mathrm{div}_x[\mathbf{u}]_\delta = 0. \tag{6.1}$$

Moreover, using hypothesis (5.10) combined with (5.14), we get

$$\mathbf{u} = 0 \text{ a.a. in the set } (0,T)\times(\mathcal{T}\setminus\Omega).$$

As Ω is regular, this yields

$$\mathbf{u}|_{\partial\Omega} = 0;\ \text{whence } \mathbf{u} \in L^p(0,T;V^{1,p}).$$

Seeing that $\{\varrho_\varepsilon\}_{\varepsilon>0}$ solves the modified transport equation (5.1), we can use Proposition 6.1 and (6.1) to deduce that

$$\varrho_\varepsilon \to \varrho \text{ in } C([0,T]\times\mathcal{T}), \tag{6.2}$$

where

$$\inf_{x\in\mathcal{T}} \varrho_{0,\delta} \le \inf_{x\in\mathcal{T}} \varrho_\varepsilon(t,x) \le \sup_{x\in\mathcal{T}} \varrho_\varepsilon(t,x) \le \sup_{x\in\mathcal{T}} \varrho_{0,\delta},$$

in particular,

$$\inf_{x\in\mathcal{T}} \varrho_{0,\delta} \le \inf_{x\in\mathcal{T}} \varrho(t,x) \le \sup_{x\in\mathcal{T}} \varrho(t,x) \le \sup_{x\in\mathcal{T}} \varrho_{0,\delta}. \tag{6.3}$$

Consequently, employing once more the energy inequality (5.14), we conclude that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^\infty(0,T;L^2(\mathcal{T};R^3)).$$

Clearly, the limit density ϱ satisfies the equation of continuity

$$\partial_t\varrho + \mathrm{div}_x(\varrho[\mathbf{u}]_\delta) = d\Delta\varrho \text{ in } (0,T)\times R^3 \tag{6.4}$$

provided that ϱ has been extended to be ϱ_f outside Ω. Moreover, in accordance with Proposition 6.1 and hypothesis (5.7),

$$\varrho = \varrho_f \text{ on the set } \Big((0,T)\times\Omega\Big) \setminus \cup_{t\in[0,T]} \cup_{i=1}^n \eta(t,[\mathbf{B}_i]_\delta), \tag{6.5}$$

where η solves

$$\partial_t\eta(t,x) = [\mathbf{u}]_\delta(t,\eta(t,x)),\ \ \eta(0,x) = x \tag{6.6}$$

and $[\mathbf{B}_i]_\delta$ denotes the δ-kernel introduced in Section 3.

6.1. Identifying the position of the rigid bodies

In order to identify the position of the rigid bodies, we proceed through several steps. We will give only the main points since we apply a technique similar to that used in the work of Feireisl et al., see [17].

- **Step 1:** We have to prove that

  $$\mathbb{D}[\mathbf{u}] = 0 \text{ a.a. on the set } \cup_{t\in[0,T]} \cup_{i=1}^n]\eta(t,[\mathbf{B}_i]_\omega)[_\delta \text{ for any } \omega > \delta, \tag{6.7}$$

  where η is determined by (6.6), and the symbols $]\cdot[$, $[\cdot]$ are specified in (3.9). Note that the kernels $[\mathbf{B}_i]_\omega$ as well as their images $\eta(t,[\mathbf{B}_i]_\omega)$ are non-empty connected open sets when $0<\delta<\omega<\delta_0/2$, where δ_0 has been introduced in hypothesis (3.1).
- **Step 2:** In accordance with (6.7), the limit velocity $\mathbf{u}$ coincides with a rigid velocity field $\mathbf{u}^{B_i}$ on the δ-neighborhood of each of the sets $\eta(t,[\mathbf{B}_i]_\omega)$, $\omega>\delta$, $i=1,\dots,n$; in particular, we deduce that

  $$\begin{gathered}\mathbf{u}(t,x) = \mathbf{u}^{B_i}(t,x) = [\mathbf{u}]_\delta(t,x) \text{ for } t\in[0,T],\\ x\in\eta(t,[\mathbf{B}_i]_\delta),\ \ i=1,\dots,n.\end{gathered} \tag{6.8}$$

  Note that rigid velocity fields coincide with their regularization, here $[\mathbf{u}^{B_i}]_\delta = \mathbf{u}^{B_i}$.

- **Step 3:** Letting $\varepsilon \to 0$ in the momentum equation (5.15) we deduce that

$$\int_0^T \int_\Omega \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \overline{(\mathbf{u} \otimes [\mathbf{u}]_\delta)} : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t + d \int_0^T \int_\Omega \nabla \rho \overline{\nabla \mathbf{u}} \, \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{6.9}$$
$$= \int_0^T \int_\Omega \overline{\mathbb{S}} : \mathbb{D}[\varphi] \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{\mathcal{T}} \varrho \overline{\nabla_x} F \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_{\mathcal{T}} \varrho_{0,\delta} \mathbf{u}_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x$$

for any test function $\varphi \in C^1([0,T) \times \overline{\Omega})$,

$$\varphi(t, \cdot) \in [\mathcal{R}M](t),$$

where

$$[\mathcal{R}M](t) = \{\phi \in C^1(\overline{\Omega}) \mid \mathrm{div}_x \phi = 0 \text{ in } \Omega, \ \phi = 0 \text{ on a neighborhood of } \partial\Omega,$$

$$\mathbb{D}[\phi] = 0 \text{ on a neighborhood of} \cup_{i=1}^n \overline{\mathbf{B}}_i(t)\},$$

with

$$[\mathbf{B}_i(t) = \eta_i(t, \mathbf{B}_i), \ i = 1, \ldots, n.$$

Indeed, because the η_i are isometries, it implies that

$$]\eta_i(t, [\mathbf{B}_i]_\delta)[_\delta = \eta_i(t, \mathbf{B}_i), \ i = 1, \ldots, n.$$

Consequently, $[\mu_\varepsilon]_\delta$ converges uniformly locally to 1 in the complement of $\cup_{i=1}^n \overline{\mathbf{B}_i(t)}$ for any $t \in [0,T]$. It yields (6.9), in which the bar denotes weak limits:

$$\begin{aligned} \mathbf{u}_\varepsilon \otimes [\mathbf{u}_\varepsilon]_\delta &\to \overline{\mathbf{u} \otimes [\mathbf{u}]_\delta} \text{ weakly in } L^2(0,T; L^2(\Omega; R^3)), \\ \nabla \rho_\epsilon &\to \nabla \rho \text{ strongly in } L^2(Q), \\ \nabla \rho_\epsilon \nabla \mathbf{u}_\epsilon &\to \rho \nabla \mathbf{u} \text{ in } \mathcal{D}'((0,T) \times \Omega) \end{aligned} \tag{6.10}$$

and from a monotonicity argument together with results of Frehse et al. [20, 21] it follows that

$$\mathbb{S}_\varepsilon \to \overline{\mathbb{S}} \text{ weakly in } L^{p'}((0,T) \times \Omega; R^{3\times 3}), \ \frac{1}{p} + \frac{1}{p'} = 1. \tag{6.11}$$

6.2. Convergence of the selfgravitating force

From the strong convergence of density in $C(0,T,C^{2,\nu})$ and the $\nabla\rho$ in $L^2((0,T) \times \Omega)$ there follows the weak * convergence in $L^2(0,T,L^\infty(\Omega))$ of the term

$$-\int_0^T \int_\Omega G \rho \nabla \Big(\int_{R^3} \frac{\rho(t,y)}{|x-y|} dy \Big) \varphi.$$

6.3. Pointwise convergence of the velocities

Our aim is to identify the weak limit in (6.10); more specifically, we show that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2(0,T;L^2(\Omega;R^3)). \tag{6.12}$$

Note that the main difficulty here is the possible existence of oscillations of the velocity fields in time.

We know from the result of Starovoitov [38, Theorem 3.1], that collisions between two rigid objects are eliminated, because the fluid is incompressible, and that the velocity gradients are bounded in the Lebesgue space L^p, with $p \geq 4$. Although originally stated for only one body in a bounded domain, it is easy to see that this result extends directly to the case of several bodies. Here we will use the terminology introduced in Section 3,

$$\mathrm{d}(\cup_{i=1}^n \mathbf{B}_i(t)) = d(t) > 0 \text{ uniformly for } t \in [0,T], \tag{6.13}$$

and, in agreement with Proposition 8.1,

$$\mathrm{d}(\cup_{i=1}^n \mathbf{B}_i^\varepsilon(t)) = d_\varepsilon \to d \text{ in } C[0,T], \tag{6.14}$$

where we have set $\mathbf{B}_i^\varepsilon(t) = \eta_\varepsilon(t,\mathbf{B}_i)$.

The absence of contacts facilitates considerably the proof of compactness of the velocity fields that can be carried over by means of the same method as in [34].

To begin with, as

$$\mathbf{B}_i^\varepsilon(t) \xrightarrow{b} \mathbf{B}_i(t) \text{ uniformly with respect to } t \in [0,T],\ i = 1,\dots,n,$$

we have, for any fixed $\sigma > 0$,

$$\mathbf{B}_i(t) \subset]\mathbf{B}_i^\varepsilon(t)[_\sigma,\ \mathbf{B}_i^\varepsilon(t) \subset]\mathbf{B}_i(t)[_\sigma, \text{ for all } t \in [0,T],\ i = 1,\dots,n,$$

and all $\varepsilon < \varepsilon_0(\sigma)$ small enough.

Lemma 6.1. *Given a family of smooth open sets $\{\mathbf{B}_i\}_{i=1}^n \subset \Omega$, $0 < k < 1/2$, there exists a function $h : (0,\sigma_0) \to \mathbb{R}^+$, with $h(\sigma) \to 0$ when $\sigma \to 0$, such that, for arbitrary $\mathbf{v} \in V^{1,p}$:*

$$\left\|\mathbf{v} - \mathcal{P}^k\left(\cup_{i=1}^n]\mathbf{B}_i[_\sigma\right)\mathbf{v}\right\|_{W^{1,k}(\Omega;R^3)} \leq c\Big(\|\mathbb{D}(\mathbf{v})\|_{L^2(\cup_{i=1}^n \mathbf{B}_i;R^{3\times 3})} + h(\sigma)\|\mathbf{v}\|_{W^{1,p}(\Omega;R^3)}\Big) \tag{6.15}$$

with an absolute constant $c < \infty$. Moreover, h and c are independent of the position of $\mathbf{B}_i$ inside Ω as long as $\mathrm{d}[\cup_{i=1}^n \mathbf{B}_i] > 2\sigma_0$.

Proof. See [17]. □

At this stage, we use a local-in-time Lions-Aubin argument in order to show the following:

Lemma 6.2. *For all $\sigma > 0$ sufficiently small, and $0 < k < 1/2$, we have*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{P}^k\left(\cup_{i=1}^n]\mathbf{B}_i(t)[_\sigma\right)[\mathbf{u}_\varepsilon]\mathrm{d}x\ \mathrm{d}t = \int_0^T\int_\Omega \varrho\mathbf{u}\cdot\mathcal{P}^k\left(\cup_{i=1}^n]\mathbf{B}_i(t)[_\sigma\right)[\mathbf{u}]\mathrm{d}x\ \mathrm{d}t.$$

Proof. See [17]. □

Combining Lemmas 6.1, 6.2 we deduce

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t = \int_0^T\int_\Omega \rho|\mathbf{u}|^2 \ \mathrm{d}x \ \mathrm{d}t \tag{6.16}$$

yielding the desired conclusion (6.12).

6.4. Compactness of the velocity gradients

Our ultimate goal is to establish strong convergence of the velocity gradients in the "fluid" part of the cylinder $(0,T)\times\Omega$. To this end, we consider equation (5.15) on the set $I\times A$, where $I\subset(0,T)$ is an interval and $A\subset\Omega$ is a ball. In accordance with (6.5), we may assume that $\varrho=\varrho_f$ in $I\times A$. In particular, we have

$$\begin{aligned}&\int_0^T\int_\Omega \varrho_f\mathbf{u}_\varepsilon\cdot\partial_t\varphi + (\varrho_f\mathbf{u}_\varepsilon\otimes[\mathbf{u}_\varepsilon]_\delta - \mathbb{S}[\mathbf{u}_\varepsilon]) : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t\\ &\quad = -\int_0^T\int_\Omega \rho_f\nabla\int_{R^3}\frac{\rho_f}{|x-y|}\varphi \mathrm{d}x\mathrm{d}t\end{aligned} \tag{6.17}$$

for any test function

$$\varphi\in\mathcal{D}(I\times A;R^3),\ \mathrm{div}_x\varphi=0.$$

At this stage, the problem must be localized by separating the fluid part from the rigid bodies. To this end, we introduce a "local" pressure

$$p = p_{\mathrm{reg}} + \partial_t p_{\mathrm{harm}}, \tag{6.18}$$

where p_{reg} enjoys the same regularity properties as the sum of the convective and viscous terms, while p_{harm} is a harmonic function. The basic idea of the concept of local pressure was developed by Wolf [44, Theorem 2.6]. A similar global result was proved by H. Koch and V. Solonnikov [32]. Note, however, that our construction gives a result different from that of Wolf [44]. In particular, the regular part is given in the form of Riesz transforms suitable for application to problems with non-standard growth conditions.

Lemma 6.3. *Let $I=(T_1,T_2)$ be a time interval and $A\subset R^3$ a domain with regular $C^{2+\mu}$ boundary. Assume that $\mathbf{U}\in L^\infty(I;L^2(A;R^3))$, $\mathrm{div}_x\mathbf{U}=0$, and $\mathbb{T}\in L^q(I\times A,R^{3\times 3})$, $1<q<2$, satisfy the integral identity*

$$\int_I\int_A\Big(\mathbf{U}\cdot\partial_t\varphi + \mathbb{T}:\nabla_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{6.19}$$

for all $\varphi\in\mathcal{D}(I\times A;R^3)$, $\mathrm{div}_x\varphi=0$.

Then there exist two functions

$$p_{\mathrm{reg}}\in L^q(I\times A),$$

$$p_{\mathrm{harm}}\in L^\infty(I;L^q(\Omega)),\ \Delta_x p_{\mathrm{harm}}=0 \ \textit{in}\ \mathcal{D}'(I\times A),\ \int_B p_{\mathrm{harm}}(t,\cdot)\ \mathrm{d}x=0$$

satisfying

$$\int_I\int_A\Big(\mathbf{U}\cdot\partial_t\varphi+\mathbb{T}:\nabla_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t = \int_I\int_A\Big(p_{\mathrm{reg}}\mathrm{div}_x\varphi + p_{\mathrm{harm}}\partial_t\mathrm{div}_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t$$

for any $\varphi \in \mathcal{D}(I \times A; R^3)$. *In addition,*

$$\|p_{\rm reg}\|_{L^q((0,T)\times A)} \le c(q)\|\mathbb{T}\|_{L^q(I\times A;R^3)}, \tag{6.20}$$

$$\|p_{\rm harm}\|_{L^\infty(I;L^q(\Omega))} \le c(q,I,A)\Big(\|\mathbb{T}\|_{L^q(I\times A;R^3)} + \|\mathbf{U}\|_{L^\infty(0,T;L^2(\Omega;R^3))}\Big). \tag{6.21}$$

Proof. See [17]. □

Accordingly, for any $\varepsilon > 0$, there exist two scalar functions $p^\varepsilon_{\rm reg}$, $p^\varepsilon_{\rm harm}$ such that

$$p^\varepsilon_{\rm reg} \in L^{p'}(I; L^{p'}(A)),\ p^\varepsilon_{\rm harm} \in L^\infty(I; L^{p'}(A)), \quad \text{are uniformly bounded} \tag{6.22}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega \Big[(\varrho_f \mathbf{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm})\cdot \partial_t\varphi\Big]\, {\rm d}x\ dx\ {\rm d}t \\ &\quad + \int_0^T \int_\Omega \Big[\Big(\varrho_f \mathbf{u}_\varepsilon \otimes [\mathbf{u}_\varepsilon]_\delta - \mathbb{S}[\mathbf{u}_\varepsilon] + p^\varepsilon_{\rm reg}\mathbb{I}\Big)\nabla_x\varphi\Big]\, {\rm d}x\ dx\ {\rm d}t \\ &\quad + \int_0^T \int_\Omega \Big[\rho_f\Big(\int_{R^3} \frac{\rho_f}{|x-y|}\Big) : \nabla_x\varphi\Big]\, dx\ {\rm d}t = 0 \end{aligned} \tag{6.23}$$

for any test function $\varphi \in \mathcal{D}(I \times A; R^3)$.

Moreover,

$$\Delta p^\epsilon_{\rm harm} = 0, \quad \int_A p^\varepsilon_{\rm harm}(t,\cdot)\, dx = 0, \quad \forall\, t \in I.$$

Consequently, the standard elliptic theory implies that $p^\varepsilon_{\rm harm}$ is uniformly bounded in $L^\infty(I; W^{1,2}_{\rm loc}(A))$. The standard Lions-Aubin argument yields

$$\varrho_f \mathbf{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm} \to \varrho_f \mathbf{u} + \nabla_x p_{\rm harm} \text{ in } L^2(I; L^2(A'; R^3)), \tag{6.24}$$

for arbitrary $A' \subset\subset A$, where $p_{\rm harm}$ is a harmonic function in x.

On the other hand, by virtue of (6.12), the velocity field $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ is precompact in $L^2(0,T; L^2(\Omega; R^3))$; whence we are allowed to conclude that

$$\nabla_x p^\varepsilon_{\rm harm} \to \nabla_x p_{\rm harm} \text{ in } L^2(I; W^{1,2}(A'; R^3)). \tag{6.25}$$

As the argument is valid for any A', letting $\varepsilon \to 0$ in (6.23) we get

$$\int_0^T \int_\Omega \Big[(\varrho_f \mathbf{u} + \nabla_x p_{\rm harm})\cdot \partial_t\varphi + (\varrho_f \mathbf{u} \otimes [\mathbf{u}]_\delta - \overline{\mathbb{S}[\mathbf{u}]} + p_{\rm reg}\mathbb{I}) : \nabla_x\varphi\Big]\, {\rm d}x\ {\rm d}t = 0 \tag{6.26}$$

for any test function $\varphi \in \mathcal{D}(I \times A; R^3)$, where

$$\mathbb{S}[\mathbf{u}_\varepsilon] \to \overline{\mathbb{S}[\mathbf{u}]} \text{ weakly in } L^{p'}(0,T; L^{p'}(\Omega; R^{3\times 3}_{\rm sym})),$$

and

$$p^\varepsilon_{\rm reg} \to p_{\rm reg} \text{ weakly in } L^{p'}(I \times A).$$

Finally, taking

$$\phi_\varepsilon = \psi(t) r(x)(\varrho_f \mathbf{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm}),\ \psi \in \mathcal{D}(I),\ r \in \mathcal{D}(A),$$

and

$$\phi = \psi(t) r(x)(\varrho_f \mathbf{u} + \nabla_x p_{\rm harm})$$

as a test function in (6.23), and (6.26), respectively, and letting $\varepsilon \to 0$ we deduce the desired conclusion

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \psi r \; \mathbb{S}[\mathbf{u}_\varepsilon] : \nabla_x[\mathbf{u}_\varepsilon] \; {\rm d}x \; {\rm d}t = \int_0^T \int_\Omega \psi r \; \overline{\mathbb{S}[\mathbf{u}]} : \nabla_x[\mathbf{u}] \; {\rm d}x \; {\rm d}t \, .$$

This yields, by means of the standard monotonicity argument,

$$\mathbb{S}[\mathbf{u}_\varepsilon] \to \mathbb{S}[\mathbf{u}] \text{ a.e. in } I \times A. \tag{6.27}$$

Indeed, we have

$$\int_0^T \int_\Omega \Big(\varrho_f \mathbf{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm} \Big) \cdot \partial_t \phi_\varepsilon \; {\rm d}x \; {\rm d}t = \int_0^T \int_\Omega \frac{1}{2} \Big| \varrho_f \mathbf{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm} \Big|^2 r \partial_t \psi \; {\rm d}x \; {\rm d}t$$
$$\to \int_0^T \int_\Omega \frac{1}{2} \Big| \varrho_f \mathbf{u} + \nabla_x p_{\rm harm} \Big|^2 r \partial_t \psi \; {\rm d}x \; {\rm d}t = \int_0^T \int_\Omega \Big(\varrho_f \mathbf{u} + \nabla_x p_{\rm harm} \Big) \cdot \partial_t \phi \; {\rm d}x \; {\rm d}t,$$

while

$$\int_0^T \int_\Omega p^\varepsilon_{\rm reg} {\rm div}_x \phi_\varepsilon \; {\rm d}x \; {\rm d}t = \int_0^T \int_\Omega \psi p^\varepsilon_{\rm reg} \nabla_x r \cdot (\varrho_f \mathbf{u}_\varepsilon + \nabla_x p^\varepsilon_{\rm harm}) \; {\rm d}x \; {\rm d}t$$
$$\to \int_0^T \int_\Omega \psi p_{\rm reg} \nabla_x r \cdot (\varrho_f \mathbf{u} + \nabla_x p_{\rm harm}) \; {\rm d}x \; {\rm d}t = \int_0^T \int_\Omega p_{\rm reg} {\rm div}_x \phi \; {\rm d}x \; {\rm d}t$$

and

$$\int_0^T \int_\Omega \rho_f \int_{R^3} \left(\frac{\rho_f}{|x-y|} \right) : \nabla_x \phi_\epsilon \; {\rm d}t \to \int_0^T \int_\Omega \rho_f \int_{R^3} \left(\frac{\rho_f}{|x-y|} \right) : \nabla_x \phi \; {\rm d}t$$

as $\varepsilon \to 0$.

6.5. Conclusion

In accordance with the results obtained in the preceding two sections, relation (6.9) reduces to

$$\int_0^T \int_\Omega \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho (\mathbf{u} \otimes [\mathbf{u}]_\delta) : \nabla_x \varphi \; {\rm d}x \; {\rm d}t + \int_0^T \int_\Omega d \nabla \rho \nabla u \; {\rm d}x \; {\rm d}t \tag{6.28}$$
$$= \int_0^T \int_\Omega \mathbb{S}[\mathbf{u}] : \mathbb{D}[\varphi] \; {\rm d}x \; {\rm d}t - \int_0^T \int_{\mathcal{T}} \varrho \nabla_x F \cdot \varphi \; {\rm d}x \; {\rm d}t - \int_{\mathcal{T}} \varrho_{0,\delta} \mathbf{u}_0 \cdot \varphi(0, \cdot) \; {\rm d}x$$

for any test function $\varphi \in C^1([0,T) \times \overline{\Omega})$,

$$\varphi(t, \cdot) \in [\mathcal{R}M](t),$$

where

$$[\mathcal{R}M](t) = \{ \phi \in C^1(\overline{\Omega}) \mid {\rm div}_x \phi = 0 \text{ in } \Omega, \; \phi = 0 \text{ on a neighborhood of } \partial\Omega,$$
$$\mathbb{D}[\phi] = 0 \text{ on a neighborhood of} \cup_{i=1}^n \overline{\mathbf{B}}_i(t) \},$$

with

$$\mathbf{B}_i(t) = \eta_i(t, \mathbf{B}_i), \; i = 1, \ldots, n.$$

Moreover, the limit solution satisfies the energy inequality

$$\begin{aligned}&\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2(\tau)\ \mathrm{d}x + \int_s^\tau \int_\Omega \mathbb{S} : D(\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t\\ &\quad\leq \int_{\mathcal{T}} \frac{1}{2}\varrho|\mathbf{u}|^2(s)\ \mathrm{d}x + \int_s^\tau \int_\Omega \varrho\nabla_x F\cdot \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t\end{aligned} \tag{6.29}$$

for any τ and a.a. $s \in (0,T)$ including $s = 0$.

7. The passage to the limit for $d \longrightarrow 0$ and $\delta \longrightarrow 0$

Our final goal is to let $d \to 0$, $\delta \to 0$ in the system of equations

$$\partial_t\varrho + \mathrm{div}_x(\varrho[\mathbf{u}]_\delta) = d\Delta\rho \text{ in } (0,T)\times R^3, \tag{7.1}$$

(6.4), (6.22) as well as in the associated family of isometries $\{\eta_i\}_{i=1}^n$.

First we are passing with $d \to 0$. Let us denote by $\{\varrho_d, \mathbf{u}_d, \{\eta_i^d\}_{i=1}^n\}_{d>0}$ the corresponding solutions, the applying the results from [18] we get

$$\begin{aligned}d\nabla\rho_d\nabla\mathbf{u}_d &\to 0 \text{ in } L^1((0,T)\times\Omega),\\ d\Delta\rho_d &\to 0 \text{ in } L^2(0,T;W^{-1,2}(\Omega)).\end{aligned}$$

Let us denote by $\{\varrho_\delta, \mathbf{u}_\delta, \{\eta_i^\delta\}_{i=1}^n\}_{\delta>0}$ the corresponding solutions constructed in the previous section.

To begin with, the theory of transport equations developed by DiPerna and P.-L. Lions [10] can be used in order to show that

$$\varrho_\delta \to \varrho \text{ in } C([0,T];L^1(\Omega)). \tag{7.2}$$

In order to see this, observe first that the initial data $\varrho_{B_i,\delta}$ in (5.6) can be taken in such a way that

$$\|\varrho_{B_i,\delta}\|_{L^\infty(\Omega)} \leq c,\ \ \varrho_f + \varrho_{B_i,\delta} \to \varrho_{B_i}\ \text{ as } \delta \to 0 \text{ in } L^1(\Omega),\ \ i = 1,\dots,n,$$

where $\{\varrho_{B_i}\}_{i=1}^n$ are the initial distributions of the mass of the rigid bodies in Theorem 4.1.

In addition, by virtue of the energy inequality (6.29), we have

$$\mathbf{u}_\delta \to \mathbf{u} \text{ weakly in } L^p(0,T;W^{1,p}(\Omega;R^3))$$

where both $\mathbf{u}_\delta$ as well as the limit velocity $\mathbf{u}$ are solenoidal. In particular, the continuity equation (7.1) reduces to a transport equation

$$\partial_t\varrho + \mathbf{u}\cdot\nabla_x\varrho = 0,$$

for which the abstract theory developed by DiPerna and P.-L. Lions [10] yields (7.2).

The rest of the convergence proof can be done repeating step by step the arguments of the preceding section.

8. An alternative proof

8.1. Definition of weak solution II

We give hereafter another formulation of a weak solution.

Given initial conditions $H^{0i} = \chi_{B^i(0)}$, $\rho^0 = \rho_{B_i} H^{0i} + \rho_f(1 - H^{0i})$ and $u = u^0 \in \mathcal{R}M(0)$, find $(x,t) \to (\rho(x,t), u(x,t), H^i(x,t))$ such that

$$\left.\begin{aligned} &u \in L^\infty(0,T,L^2(\Omega)) \cap L^p(0,T,W^{1,p}(\Omega)),\\ &H^i, \rho \in C(0,T,L^q(\Omega)), \text{for all } 1 < q < \infty, \end{aligned}\right\} \tag{8.1}$$

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\rho\mathbf{u} \cdot \partial_t \varphi + \rho\mathbf{u} \otimes \mathbf{u} : \nabla_x[\varphi] - \mathbb{S} : \mathbf{D}[\varphi] \Big) \mathrm{d}x\mathrm{d}t \\ &\quad = -\int_0^T \int_\Omega \left[\rho G \nabla_x \sum_i \int_{R^3} \frac{\rho}{|x-y|} dy \cdot \varphi \right] \mathrm{d}x\mathrm{d}t - \int_\Omega \rho_0 \mathbf{u}_0 \cdot \varphi \ \mathrm{d}x \end{aligned} \tag{8.2}$$

$$\varphi \in C^1([0,T) \times \bar{\Omega}),\ \varphi(t,\cdot) \in \mathcal{R}(t), \tag{8.3}$$

$$\int_0^T \int_\Omega \Big(\rho \partial_t \phi + \rho \mathbf{u} \cdot \nabla_x \phi \Big) \mathrm{d}x\mathrm{d}t = -\int_\Omega \rho_0 \phi dx,\ \ \phi \in C^1([0,T) \times \bar{\Omega}), \tag{8.4}$$

$$\int_0^T \int_\Omega \left(H^i \frac{\partial \phi}{\partial t} + H^i u \nabla \phi \right) dxdt + \int_\Omega H^{0i} \phi(0) dx = 0. \tag{8.5}$$

Here we have again used that $\rho = \rho_{B^i} H^i + \rho_f(1 - H^i)$.

Theorem 8.1. *Let the initial position of the rigid bodies be given through a family of open sets*

$$\mathbf{B}_i \subset \Omega \subset R^3,\ \mathbf{B}_i \textit{ diffeomorphic to the unit ball for } i = 1, \dots, n,$$

where both $\partial[\mathbf{B}_i]$, $i = 1, \dots, n$, *and* $\partial\Omega$ *belong to the regularity class specified in* (3.1), (3.2). *In addition, suppose that*

$$\operatorname{dist}[\overline{\mathbf{B}}_i, \overline{\mathbf{B}}_j] > 0 \textit{ for } i \neq j,\ \operatorname{dist}[\overline{\mathbf{B}}_i, R^3 \setminus \Omega] > 0 \textit{ for any } i = 1, \dots, n$$

and we assume that $\partial\Omega$ *and* $\partial\mathbf{B}_i$, $i = 1, \dots, n$ *belong to* $C^{2,\nu}$. *Furthermore, let the viscous stress tensor* $\mathbb{S}$ *satisfy hypotheses* (1.1)–(1.3), *with* $p \geq 4$.

Finally, let the initial distribution of the density be given as

$$\varrho_0 = \begin{cases} \varrho_f = \text{const} > 0 \textit{ in } \Omega \setminus \cup_{i=1}^n \overline{\mathbf{B}}_i, \\ \varrho_{B_i} \textit{ on } \mathbf{B}_i,\ \textit{where } \varrho_{B_i} \in L^\infty(\Omega),\ \operatorname{ess\,inf}_{\mathbf{B}_i} \varrho_{B_i} > 0,\ i = 1, \dots, n, \end{cases}$$

while

$$\mathbf{u}_0 \in L^2(\Omega; R^3),\ \operatorname{div}_x \mathbf{u}_0 = 0 \textit{ in } \mathcal{D}'(\Omega),\ \mathbb{D}[\mathbf{u}_0] = 0 \textit{ in } \mathcal{D}'(B_i; R^{3\times3}) \textit{ for } i = 1, \dots, n.$$

Then there exist a density function ϱ,

$$\varrho \in C([0,T]; L^1(\Omega)),\ 0 < \operatorname*{ess\,inf}_\Omega \varrho(t,\cdot) \leq \operatorname*{ess\,sup}_\Omega \varrho(t,\cdot) < \infty \textit{ for all } t \in [0,T],$$

a family of isometries $\{\eta_i(t,\cdot)\}_{i=1}^n$, $\eta_i(0,\cdot) = \mathrm{I}$, *and a velocity field* $\mathbf{u}$,

$$\mathbf{u} \in C_{\text{weak}}([0,T]; L^2(\Omega; R^3)) \cap L^p(0,T; W_0^{1,p}(\Omega; R^3)),$$

compatible with $\{\eta_i\}_{i=1}^n$ *in the sense specified in* (3.7), (3.8), *such that* ϱ, $\mathbf{u}$ *satisfy the integral identity* (8.3) *for any test function* $\varphi \in C^1([0,T)\times R^3)$, *and the integral identity* (8.4) *for any* φ *satisfying* (8.5).

We introduce an $\epsilon - \delta$ scheme; for the penalization part see [2] or [19].

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon [\mathbf{u}]_\delta) = d\Delta\rho_\epsilon, \tag{8.6}$$

$$\begin{aligned}\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes [\mathbf{u}]_\delta) + \nabla_x P_\epsilon + d\nabla\rho_\epsilon \nabla u_\epsilon \\ = \operatorname{div}_x([\mu]\mathbb{S}_\varepsilon) + \varrho_\varepsilon \nabla_x F_\varepsilon - \chi_\varepsilon \mathbf{u}_\varepsilon + \frac{1}{\epsilon}\rho_\varepsilon H_\varepsilon^i(\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon),\end{aligned} \tag{8.7}$$

$$\partial_t H_\varepsilon^i + \operatorname{div}_x(\tilde{\mathbf{u}}_\varepsilon H_\varepsilon^i) = 0, \tag{8.8}$$

$$\operatorname{div}_x \mathbf{u}_\varepsilon = 0, \tag{8.9}$$

$$\tilde{\mathbf{u}}_\varepsilon = \frac{1}{M_\varepsilon}\int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon H_\varepsilon^i dx + \left(J_\varepsilon^{-1}\int_\Omega \rho_\varepsilon (r_\varepsilon \times \mathbf{u}_\varepsilon) H_\varepsilon^i dx\right) \times R \tag{8.10}$$

where

$$[\mathbf{u}]_\delta = \sigma_\delta * \mathbf{u} \text{ (spatial convolution)},\ 0 < \delta < \delta_0,$$

with

$$\sigma_\delta(x) = \frac{1}{\delta^3}\sigma\Big(\frac{|x|}{\delta}\Big), \tag{8.11}$$

$$\sigma \in \mathcal{D}(-1,1),\ \sigma(z) > 0 \text{ for } -1 < z < 1,\ \sigma(z) = \sigma(-z),\ \int_{-1}^1 \sigma(z)\ \mathrm{d}z = 1,$$

$$M_\varepsilon = \int_\Omega \rho_\varepsilon H_\varepsilon^i dx, \qquad |\mathbf{B}_i(t)| = |\mathbf{B}_i(0)|,$$

since $\tilde{u}$ is divergence free and H_i vanishes on $\partial\Omega$.

The inertia tensor is defined by

$$J_\varepsilon = \int_\Omega \rho_\varepsilon H_\varepsilon^i (r_\varepsilon^2 I - r_\varepsilon \times r_\varepsilon) dx$$

with $r = x - x_G = x - \int_\Omega \rho_\varepsilon H_\varepsilon^i$. For $a \in R^3 - \{0\}$,

$$a^T J_\varepsilon a = \int_\Omega \rho_\varepsilon |r_\varepsilon \times a|^2 dx \geq \min(\rho_{B_i}, \rho_f)\int_{\mathbf{B}_i(t)} |r_\varepsilon \times a|^2 dx.$$

Moreover, we supplemented the system with the initial conditions

$$\rho(0,.) = \rho_{0,\delta} = \rho_f + \sum_{i=1}^n \rho_{B_i,\delta},$$

where

$$\rho_{B_i} \in \mathcal{D}(\mathbf{B}_i), \rho_{B_i,\delta} = 0 \text{ whenever } \operatorname{dist}[x, \partial\mathbf{B}_i] < \delta,\ i = 1,\dots,n.$$

Finally, we add the homogeneous Neumann boundary condition

$$\nabla\rho \cdot n = 0 \text{ on } \partial\Omega \cup \bigcup_{i=1}^n \partial\mathbf{B}_i,$$

and we assume the following regularity of data

$$\rho_{0,\delta} \in C^{2,\nu}(\overline{\Omega}).$$

8.2. ϵ limit

For fixed δ we identify the limit problem for $\epsilon \to 0$. We will show, in this limit, the strong convergence of u_ε in L^2 of u and the strong convergence of H^i_ϵ in $C(0,T;L^q(\Omega))$ for all $q \geq 1$. The other part (strong convergence of the gradient of the velocity field) is the same as in the previous proof.

Proposition 8.1. *Let ξ be a rigid velocity field, i.e., such that $\xi(x) = V + \omega \times r(x)$ for some constant vectors $V \in R^3$ and $\omega \in R^3$.Then if $(\tilde{u}_\varepsilon)$ is defined by (8.10) the identity*

$$\int_\Omega \rho_\epsilon H^\epsilon_i(u_\epsilon - \tilde{u}_\varepsilon) \cdot \xi dx = 0$$

is satisfied.

Proof. See [2]. □

8.2.1. Estimates for transport equations and momentum equations. Using standard estimates for transport equations, we get

$$\rho_\varepsilon, H^i_\varepsilon \in L^\infty(0,T,L^\infty(\Omega)).$$

Moreover

$$\rho_{\min} := \min(\rho_s, \rho_f) \leq \rho_\varepsilon(x,t) \leq \max(\rho_s, \rho_f),$$
$$H^i_\varepsilon \in \{0,1\} \text{ a.e. } x \in \Omega.$$

Multiplying the momentum equation by u_ε we get

$$\begin{aligned}
&\int_{\mathcal{T}} \Big\{\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes [\mathbf{u}^\varepsilon]_\delta)\Big\}\mathbf{u}_\varepsilon \; dx \\
&\quad = \int_{\mathcal{T}} \Big\{\mathrm{div}_x([\mu]\mathbb{S}_\varepsilon) + \Big(\varrho_\varepsilon \nabla \int_{R^3} \frac{\rho^\varepsilon}{|x-y|} dy\Big)\Big\}\mathbf{u}_\varepsilon \mathrm{d}x \\
&\quad\quad + \int_{\mathcal{T}} \Big\{ -\chi_\varepsilon \mathbf{u}_\varepsilon + \frac{1}{\epsilon}\rho_\varepsilon H^\varepsilon_i(\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon)\Big\}\mathbf{u}_\varepsilon \mathrm{d}x.
\end{aligned} \tag{8.12}$$

Since

$$\int_{\mathcal{T}} \rho_\varepsilon H^i_\varepsilon(\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon) \cdot \mathbf{u}_\varepsilon \; dx = 0, \qquad \sqrt{H^i_\varepsilon} = H^i_\varepsilon,$$

applying again the results from [18] Chapter 3, Lemma 3.1 we have that the density satisfies the bounds

$$\rho_\varepsilon \in L^2(0,T;W^{2,2}(\mathcal{T})) \cap C(0,T;W^{1,2}(\mathcal{T})), \partial_t \rho_\varepsilon \in L^2((0,T) \times \mathcal{T}),$$

and also through a bootstrap argument we have

$$\begin{aligned}
\|\rho_\varepsilon\|_{C([0,T];W^{2-2/p,p}(\mathcal{T}))} &+ \|\rho_\varepsilon\|_{L^p(0,T;W^{2,p}(\mathcal{T}))} \\
&+ \||\partial_t \rho_\varepsilon\|_{L^p((0,T)\times\mathcal{T})} \leq c\|\rho_{\varepsilon,0}\|_{W^{2-2/p,p}(\mathcal{T})}, \\
&\|\rho_\varepsilon\|_{C([0,T];C^{2,\nu}(\bar{\mathcal{T}}))} \leq c \\
&\|\partial_t \rho_\varepsilon\|_{C([0,T];C^{0,\nu}(\bar{\mathcal{T}}))} \leq c.
\end{aligned}$$

Then we will get from 8.12,

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|^2_{L^2(\mathcal{T})} + \mu\|\mathbf{D}(u_\varepsilon)|^p_{L^p(Q)} + \frac{2}{\varepsilon}\|\sqrt{\rho_\varepsilon}u_\varepsilon H^i_\varepsilon(u_\varepsilon - \tilde{u}_\varepsilon)|^2_{L^2(Q)} \le 0. \quad (8.13)$$

This gives us the following bounds:

- $\mathbf{u}_\varepsilon$ is bounded in $L^p(0,T,W^{1,p}(\mathcal{T}))$,
- $\sqrt{\rho_\varepsilon}u_\varepsilon$ and $(u)_\varepsilon$ are bounded in $L^\infty(0,T,L^2(\mathcal{T}))$,
- $\frac{1}{\sqrt{\varepsilon}}\sqrt{\rho_\varepsilon}H^i_\varepsilon(u_\varepsilon - \tilde{u}_\varepsilon)$ and $\frac{1}{\varepsilon}H^i_\varepsilon(\mathbf{u}_\varepsilon - \tilde{u})$ are bounded in $L^2(0,T;L^2(\mathcal{T}))$.

Then we can extract subsequences such that

- $\mathbf{u}_\varepsilon \to \mathbf{u}$ weakly in $L^p(0,T,W^{1,p}(\Omega))$,
- $[\mathbf{u}_\varepsilon]_\delta \to [\mathbf{u}]_\delta$ weakly * in $L^p(0,T,W^{1,\infty}(\Omega)), \operatorname{div}[\mathbf{u}]_\delta = 0$.

Since Ω is regular then $u = 0$ on $\partial\Omega$.

Observing that $\{\varrho_\varepsilon\}_{\varepsilon>0}$ solves the transport equation (5.1), we can use Proposition 8.1 and (6.1) to deduce that

$$\varrho_\varepsilon \to \varrho \text{ in } C([0,T]\times\mathcal{T}), \quad (8.14)$$

where

$$\inf_{x\in\mathcal{T}} \varrho_{0,\delta} \le \inf_{x\in\mathcal{T}} \varrho_\varepsilon(t,x) \le \sup_{x\in\mathcal{T}} \varrho_\varepsilon(t,x) \le \sup_{x\in\mathcal{T}} \varrho_{0,\delta}.$$

In particular,

$$\inf_{x\in\mathcal{T}} \varrho_{0,\delta} \le \inf_{x\in\mathcal{T}} \varrho(t,x) \le \sup_{x\in\mathcal{T}} \varrho(t,x) \le \sup_{x\in\mathcal{T}} \varrho_{0,\delta}. \quad (8.15)$$

Consequently, employing once more the energy inequality (5.14), we conclude that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^\infty(0,T;L^2(\mathcal{T};R^3)).$$

Clearly, the limit density ϱ satisfies the approximate equation of continuity

$$\partial_t\varrho + \operatorname{div}_x(\varrho[\mathbf{u}]_\delta) = d\Delta\rho \text{ in } (0,T)\times R^3, \quad (8.16)$$

provided ϱ has been extended to be ϱ_f outside Ω.
Moreover,

- $(\sqrt{\rho_\varepsilon}H^i_\varepsilon\mathbf{u}_\varepsilon - \sqrt{\rho_\varepsilon}H^i_\varepsilon\tilde{\mathbf{u}}) \to 0$ in $L^2(0,T,L^2(\Omega))$ strongly, and
- $H^i_\varepsilon\mathbf{u}_\varepsilon - H^i_\varepsilon\tilde{\mathbf{u}} \to 0$ in $L^2(0,T,L^2(\Omega))$ strongly.

8.2.2. Passing to the limit in the rigid velocity. The rigid velocity is defined as

$$\tilde{\mathbf{u}}_\varepsilon(\mathbf{x},t) = \mathbf{u}_{\varepsilon,G}(t) + \omega_\varepsilon(t)\times r_\varepsilon(x,t),$$

with

$$\mathbf{u}_{\varepsilon,G}(t) = \frac{1}{M_\varepsilon}\int_\Omega \rho_\varepsilon\mathbf{u}_\varepsilon H^i_\varepsilon dx, \qquad \omega_\varepsilon(t) = J^{-1}\int_\Omega \rho_\varepsilon(r_\varepsilon\times\mathbf{u}_\varepsilon)H^i_\varepsilon dx\,.$$

Then it follows that

- $\mathbf{u}_{\varepsilon,G}(t)$ is bounded in $L^\infty(0,T)$,
- $\omega_\varepsilon(t)$ is bounded in $L^\infty(0,T)$.

It implies that

$$\tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{u}} \text{ in the weak } ^* \text{ sense in } L^\infty(0,T,L^\infty(\Omega)).$$

Taking gradients of the rigid velocity $\tilde{\mathbf{u}}_\varepsilon$ implies that

$$\tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{u}} \text{ in } L^2(0,T,W^{1,\infty}(\Omega)) \text{ weak }^*.$$

Applying the compactness results to the transport equation on H^i_ε gives us

$$H^i_\varepsilon \to H^i \text{ a.e. in } C(0,T,L^p(\Omega)) \text{ strongly for } p \in [1,\infty[,$$

satisfying the transport equation

$$H^i_t + \tilde{\mathbf{u}} \cdot \nabla H^i = 0, \qquad H^i(0,x) = H^i(0).$$

This implies the strong convergence of r_ε in $C(0,T,L^p(\Omega))$, for all $p \geq 1$ and we can also pass to the limit in the expression $\tilde{u}_\varepsilon$ and ω_ε.

8.2.3. Strong convergence of $\mathbf{u}_\varepsilon$. To prove the strong convergence of a subsequence of $\mathbf{u}_\varepsilon$ in $L^2(Q)$ we write

$$\int_0^T \int_\Omega |\mathbf{u}_\varepsilon - \mathbf{u}|^2 \mathrm{d}x\mathrm{d}t \leq \frac{1}{\rho_{\min}} \Big(\int_0^T \int_\Omega |\rho(\mathbf{u}_\varepsilon^2 - \mathbf{u}^2)| \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |2\rho \mathbf{u}(\mathbf{u} - \mathbf{u}_\varepsilon| \mathrm{d}x\mathrm{d}t \Big). \tag{8.17}$$

Since the second term in (8.17) converges to 0, then

$$\int_0^T |\mathbf{u}_\varepsilon - \mathbf{u}|^2 \mathrm{d}x\mathrm{d}t \leq \frac{1}{\rho_{\min}} \Big(\int_0^T \int_\Omega |\rho_\varepsilon \mathbf{u}_\varepsilon^2 - \rho \mathbf{u}_\varepsilon^2| \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |(-\rho_\varepsilon + \rho)\mathbf{u}_\varepsilon^2 \mathrm{d}x\mathrm{d}t \Big). \tag{8.18}$$

From (8.14) and boundedness of $\mathbf{u}_\varepsilon$ in $L^\infty(0,T,L^2)$ we get that convergence of the second term of (8.18) converges to 0, and

$$\begin{aligned}
&\int_0^T \int_\Omega |\mathbf{u}_\varepsilon - \mathbf{u}|^2 \mathrm{d}x\mathrm{d}t \\
&\quad \leq \frac{1}{\rho_{\min}} \bigg(\int_0^T \int_\Omega |\rho_\varepsilon \mathbf{u}_\varepsilon \mathcal{P}^k\Big(\cup_{i=1}^n \mathbf{B}_i(t)[_\sigma \Big)[\mathbf{u}_\varepsilon] - \rho \mathbf{u} \cdot \mathcal{P}^k\Big(\cup_{i=1}^n] \mathbf{B}_i(t)[_\sigma \Big)[\mathbf{u}_\varepsilon])| \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega |\rho_\varepsilon \mathbf{u}_\varepsilon \Big(\mathbf{u}_\varepsilon - \mathcal{P}^k\Big(\cup_{i=1}^n] \mathbf{B}_i(t)[_\sigma \Big)[\mathbf{u}_\varepsilon]) \Big)| \mathrm{d}x dt \\
&\qquad + \int_0^T \int_\Omega |\rho \mathbf{u} \cdot (\mathcal{P}^k\Big(\cup_{i=1}^n] \mathbf{B}_i(t)[_\sigma \Big)[\mathbf{u}] - \mathbf{u})| \mathrm{d}x dt + l_\varepsilon \bigg) \\
&\quad \leq \frac{1}{\rho_{\min}} \Big\{ |\rho_\varepsilon \mathbf{u}_\varepsilon \mathcal{P}^k\Big(\cup_{i=1}^n] \mathbf{B}_i(t)[_\sigma \Big)[\mathbf{u}_\varepsilon] - \rho \mathbf{u} \cdot \mathcal{P}^k\Big(\cup_{i=1}^n] \mathbf{B}_i(t)[_\sigma \Big)[\mathbf{u}](\mathbf{u})|_{L^1(Q)} \\
&\qquad + C\|(\mathbf{u}_\varepsilon - \mathcal{P}^k\Big(\cup_{i=1}^n] \mathbf{B}_i(t)[_\sigma \Big)[\mathbf{u}_\varepsilon]\|_{L^2(Q)} \\
&\qquad + c\|(\mathbf{u} - \mathcal{P}^k\Big(\cup_{i=1}^n] \mathbf{B}_i(t)[_\sigma \Big)[\mathbf{u}])\|_{L^2(Q)} + l_\varepsilon \Big\},
\end{aligned} \tag{8.19}$$

where $l_\varepsilon \to 0$ when $\varepsilon \to 0$.

It implies the strong convergence of $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $L^2(Q)$. Now from previous estimates and Sobolev imbedding we get

- $H^i_\varepsilon \mathbf{u}_\varepsilon \to H^i u$ weakly in $L^p(0,T,L^{\frac{3p}{3-p}}(\Omega))$,
- $H^i_\varepsilon \tilde{\mathbf{u}}_\varepsilon \to H^i \tilde{u}$ weakly in $L^p(Q)$,

- $H^i_\varepsilon \tilde{\mathbf{u}}_\varepsilon - H^i \tilde{\mathbf{u}}_\varepsilon \to H^i \mathbf{u} - H^i \tilde{\mathbf{u}}$ weakly in $L^p(0,T,L^{\frac{3p}{3-p}}(\Omega))$,
- $H^i_\varepsilon \mathbf{u}_\varepsilon - H_\varepsilon \tilde{\mathbf{u}} \to 0$ strongly in $L^2(Q)$.

Thus
$$H\mathbf{u} = H\tilde{\mathbf{u}}, \qquad \mathrm{div}\ (\mathbf{u}H) = \mathrm{div}(\tilde{\mathbf{u}}H),$$
and finally we obtain
$$H^i\mathbf{u} = H^i\tilde{\mathbf{u}}.$$
The proof of the inequality (8.19) goes exactly in the same way as in Section 6.3. Finally it remains to show the strong convergence of the gradient of velocity and to pass to the limit, see Section 6.4. Then we pass to the limit with d similarly as in Feireisl and Novotný [18]. The last step is passing to the limit with δ which must proceed as with the ε limit together with using the transport theory developed by DiPerna-Lions [10].

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Geometric Aspects of the Periodic μ-Degasperis-Procesi Equation

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Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. We consider the periodic μDP equation (a modified version of the Degasperis-Procesi equation) as the geodesic flow of a right-invariant affine connection ∇ on the Fréchet Lie group $\mathrm{Diff}^\infty(\mathbb{S}^1)$ of all smooth and orientation-preserving diffeomorphisms of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. On the Lie algebra $\mathrm{C}^\infty(\mathbb{S}^1)$ of $\mathrm{Diff}^\infty(\mathbb{S}^1)$, this connection is canonically given by the sum of the Lie bracket and a bilinear operator. For smooth initial data, we show the short time existence of a smooth solution of μDP which depends smoothly on time and on the initial data. Furthermore, we prove that the exponential map defined by ∇ is a smooth local diffeomorphism of a neighbourhood of zero in $\mathrm{C}^\infty(\mathbb{S}^1)$ onto a neighbourhood of the unit element in $\mathrm{Diff}^\infty(\mathbb{S}^1)$. Our results follow from a general approach on non-metric Euler equations on Lie groups, a Banach space approximation of the Fréchet space $\mathrm{C}^\infty(\mathbb{S}^1)$, and a sharp spatial regularity result for the geodesic flow.

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1. Introduction

In recent years, several nonlinear equations arising as approximations to the governing model equations for water waves attracted a considerable amount of attention in the fluid dynamics research community (cf. [22]). The Korteweg-de Vries (KdV) equation is a well-known model for wave-motion on shallow water with small amplitudes over a flat bottom. This equation is completely integrable, allows for a Lax pair formulation and the corresponding Cauchy problem was the subject of many studies. However, it was observed in [3] that solutions of the KdV equation do not break as physical water waves do: the flow is globally well posed for square integrable initial data (see also [23, 24] for further results).

The Camassa-Holm (CH) equation

$$u_t + 3uu_x = 2u_x u_{xx} + uu_{xxx} + u_{txx}$$

was introduced to model the shallow-water medium-amplitude regime (see [4]). Closely related to the CH equation is the Degasperis-Procesi (DP) equation

$$u_t + 4uu_x = 3u_x u_{xx} + uu_{xxx} + u_{txx},$$

which was discovered in a search for integrable equations similar to the CH equation (see [12]). Both equations are higher-order approximations in a small amplitude expansion of the incompressible Euler equations for the unidirectional motion of waves at a free surface under the influence of gravity (cf. [10]). They have a bi-Hamiltonian structure, are completely integrable and allow for wave breaking and peaked solitons [6, 13, 16, 21]. The Cauchy problem for the periodic CH equation in spaces of classical solutions has been studied extensively (see, e.g., [5, 33]); in [7] and [11] the authors explain that this equation is also well posed in spaces which include peakons, showing in this way that peakons are indeed meaningful solutions of CH. Well-posedness for the periodic DP equation and various features of solutions of the DP on the circle are discussed in [20]. Both, the CH equation and the DP equation, are embedded into the family of b-equations

$$m_t = -(m_x u + bmu_x), \quad m := u - u_{xx}, \tag{1.1}$$

where $u(t,x)$ is a function of a spatial variable $x \in \mathbb{S}^1$ and a temporal variable $t \in \mathbb{R}$. Note that the family (1.1) can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic-order accuracy for any $b \neq -1$ by an appropriate Kodama transformation [14]. For $b = 2$, we recover the CH equation and for $b = 3$, we get the DP equation. Note that the b-equation is integrable only if $b = 2$ or $b = 3$. For further results and references we refer to [19].

Since the pioneering works [1, 15], geometric interpretations of evolution equations led to several interesting results in the applied analysis literature. A detailed discussion of the CH equation in this framework was given by [26]. The geometrical aspects of some metric Euler equations are explained in [8, 9, 25, 32]. Studying the b-equations as a geodesic flow on the diffeomorphism group $\mathrm{Diff}^\infty(\mathbb{S}^1)$, it was shown recently in [17] that for smooth initial data u_0, there is a unique short-time solution $u(t,x)$ of (1.1), depending smoothly on (t, u_0). The crucial idea is to define an affine (not necessarily Riemannian) connection ∇ on $\mathrm{Diff}^\infty(\mathbb{S}^1)$, given at the identity by the sum of the Lie bracket and a bilinear symmetric operator B, so that $B(u,u) = -u_t$. Most importantly, this approach also works for b-equations of non-metric type and it motivates the study of geometric quantities like curvature or an exponential map for the family (1.1). In particular, the authors of [17] proved that the exponential map for ∇ is a smooth local diffeomorphism near zero in $\mathrm{C}^\infty(\mathbb{S}^1)$. Recently it has been shown in [18] that the b-equation can be realized as a metric Euler equation *only* if $b = 2$. In all other cases $b \neq 2$ there is no Riemannian metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ such that the corresponding geodesic flow is re-expressed by the b-equation. Geometric aspects of some novel nonlinear PDEs related to CH and DP are discussed in [31].

In this paper, we study the μDP equation

$$\mu(u_t) - u_{txx} + 3\mu(u)u_x - 3u_x u_{xx} - uu_{xxx} = 0, \tag{1.2}$$

where μ denotes the projection $\mu(u) = \int_0^1 u \, \mathrm{d}x$ and $u(t,x)$ is a spatially periodic real-valued function of a temporal variable $t \in \mathbb{R}$ and a space variable $x \in \mathbb{S}^1$. The μDP equation belongs to the family of μ-b-equations which follows from (1.1) by replacing $m = \mu(u) - u_{xx}$. The study of μ-variants of (1.1) is motivated by the following key observation: Letting $m = -\partial_x^2 u$, equation (1.1) for $b = 2$ becomes the Hunter-Saxton (HS) equation

$$2u_x u_{xx} + uu_{xxx} + u_{txx} = 0,$$

which possesses various interesting geometric properties, see, e.g., [29, 30], whereas the choice $m = (1-\partial_x^2)u$ leads to the CH equation as explained above. In the search for integrable equations that are given by a perturbation of $-\partial_x^2$, the μ-b-equation has been introduced and it could be shown that it behaves quite similarly to the b-equation; see [31] where the authors discuss local and global well-posedness as well as finite time blow-up and peakons. Our study of the μDP equation is inspired by the results in [17]. In fact using the approach of [17] we shall conceptualise a geometric picture of the μDP equation.

Our study is mostly performed in the C^∞-category. Elements of $\mathrm{C}^\infty(\mathbb{S}^1)$ are sometimes also called smooth for brevity.

We will reformulate the μDP equation in terms of a geodesic flow on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ to obtain the following main result: Given a smooth initial data $u_0(x)$, for which $\|u_0\|_{\mathrm{C}^3(\mathbb{S}^1)}$ is small, there is a unique smooth solution $u(t,x)$ of (1.2) which depends smoothly on (t, u_0). More precisely, we have

Theorem 1.1. *There exists an open interval J centered at zero and $\delta > 0$ such that for each $u_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$ with $\|u_0\|_{\mathrm{C}^3(\mathbb{S}^1)} < \delta$, there exists a unique solution $u \in \mathrm{C}^\infty(J, \mathrm{C}^\infty(\mathbb{S}^1))$ of the μDP equation such that $u(0) = u_0$. Moreover, the solution u depends smoothly on $(t, u_0) \in J \times \mathrm{C}^\infty(\mathbb{S}^1)$.*

It is known that the Riemannian exponential mapping on general Fréchet manifolds fails to be a smooth local diffeomorphism from the tangent space back to the manifold, cf. [8]. Therefore the following result is quite remarkable.

Theorem 1.2. *The exponential map* exp *at the unity element for the μDP equation on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is a smooth local diffeomorphism from a neighbourhood of zero in $\mathrm{C}^\infty(\mathbb{S}^1)$ onto a neighbourhood of* id *in $\mathrm{Diff}^\infty(\mathbb{S}^1)$.*

Our paper is organized as follows: In Section 2, we rewrite (1.2) in terms of a local flow $\varphi \in \mathrm{Diff}^n(\mathbb{S}^1)$, $n \geq 3$, and explain the geometric setting. The resulting equation is an ordinary differential equation and in Section 3, we apply the Theorem of Picard-Lindelöf to obtain a solution of class $\mathrm{C}^n(\mathbb{S}^1)$ with smooth dependence on t and $u_0(x)$. In addition, we show that this solution in $\mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$ does neither lose nor gain spatial regularity as t varies through the associated interval of existence. We then approximate the Fréchet Lie group $\mathrm{Diff}^\infty(\mathbb{S}^1)$ by the topological groups $\mathrm{Diff}^n(\mathbb{S}^1)$ and the Fréchet space $\mathrm{C}^\infty(\mathbb{S}^1)$ by the Banach spaces $\mathrm{C}^n(\mathbb{S}^1)$ to obtain an analogous existence result for the geodesic equation on $\mathrm{Diff}^\infty(\mathbb{S}^1)$. Finally, in Section 4, we make again use of a Banach space approximation to prove that the exponential map for the μDP is a smooth local diffeomorphism nero zero as a map $\mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{Diff}^\infty(\mathbb{S}^1)$.

2. Geometric reformulation of the μDP equation

We write $\mathrm{Diff}^\infty(\mathbb{S}^1)$ for the smooth orientation-preserving diffeomorphisms of the unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $\mathrm{Vect}^\infty(\mathbb{S}^1)$ for the space of smooth vector fields on $\mathbb{S}^1$. Clearly, $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is a Lie group and it is easy to see that its Lie algebra is $\mathrm{Vect}^\infty(\mathbb{S}^1)$: If $t \mapsto \varphi(t)$ is a smooth path in $\mathrm{Diff}^\infty(\mathbb{S}^1)$ with $\varphi(0) = \mathrm{id}$, then $\varphi_t(0, x) \in T_x\mathbb{S}^1$ for all $x \in \mathbb{S}^1$ and thus the Lie algebra element $\varphi_t(0, \cdot)$ is a smooth vector field on $\mathbb{S}^1$. Furthermore, since $T\mathbb{S}^1 \simeq \mathbb{S}^1 \times \mathbb{R}$ is trivial, we can identify the Lie algebra $\mathrm{Vect}^\infty(\mathbb{S}^1)$ with $\mathrm{C}^\infty(\mathbb{S}^1)$. Note that $[u, v] = u_x v - v_x u$ is the corresponding Lie bracket. In the following, we will also use that $\mathrm{Diff}^\infty(\mathbb{S}^1)$ has a smooth manifold structure modelled over the Fréchet space $\mathrm{C}^\infty(\mathbb{S}^1)$. In particular, $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is a Fréchet Lie group and thus it is parallelizable, i.e., $T\mathrm{Diff}^\infty(\mathbb{S}^1) \simeq \mathrm{Diff}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1)$. Let $\mathrm{Diff}^n(\mathbb{S}^1)$ denote the group of orientation-preserving diffeomorphisms of $\mathbb{S}^1$ which are of class $\mathrm{C}^n(\mathbb{S}^1)$. Similarly, $\mathrm{Diff}^n(\mathbb{S}^1)$ has a smooth manifold structure modelled over the Banach space $\mathrm{C}^n(\mathbb{S}^1)$. Note that $\mathrm{Diff}^n(\mathbb{S}^1)$ is only a topological group but not a Banach Lie group, since the composition and inversion maps are continuous but not smooth. Furthermore, the trivialization $T\mathrm{Diff}^n(\mathbb{S}^1) \simeq \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$ is only topological and not smooth.

In this section, we write (1.2) as an ordinary differential equation on the tangent bundle $\mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$, where $n \geq 3$. In a first step, we rewrite (1.2) using the operator $A := \mu - \partial_x^2$. Here μ denotes the linear map given by $f \mapsto \int_0^1 f(t, x)\,\mathrm{d}x$ for any function $f(t, x)$ depending on time t and space $x \in \mathbb{S}^1$. Observe that $\mu(\partial_x^k f) = 0$ for $k \geq 1$ if f and its derivatives are continuous functions on $\mathbb{S}^1$. Furthermore, $\mu(f)$ is still depending on the time variable t. The following lemma establishes the invertibility of A as an operator acting on $\mathrm{C}^n(\mathbb{S}^1)$ for $n \geq 2$.

Lemma 2.1. *Given $n \geq 2$, the operator $A = \mu - \partial_x^2$ maps $\mathrm{C}^n(\mathbb{S}^1)$ isomorphically onto $\mathrm{C}^{n-2}(\mathbb{S}^1)$. The inverse is given by*

$$(A^{-1}f)(x) = \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{13}{12}\right)\int_0^1 f(a)\,\mathrm{d}a + \left(x - \frac{1}{2}\right)\int_0^1\int_0^a f(b)\,\mathrm{d}b\,\mathrm{d}a$$
$$- \int_0^x\int_0^a f(b)\,\mathrm{d}b\,\mathrm{d}a + \int_0^1\int_0^a\int_0^b f(c)\,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a.$$

Proof. Clearly, $\mu(A^{-1}f) = \mu(f)$ and $(A^{-1}f)_{xx} = \mu(f) - f$ so that $A(A^{-1}f) = f$. To verify that A is surjective, we observe that $\partial_x^k(A^{-1}f)(0) = \partial_x^k(A^{-1}f)(1)$ for all $k \in \{0, \dots, n\}$. To see that A is injective, assume that $Au = 0$ for $u \in \mathrm{C}^n(\mathbb{S}^1)$ and $n \geq 2$. Then there are constants $c, d \in \mathbb{R}$ such that $u = \frac{1}{2}\mu(u)x^2 + cx + d$. By periodicity we first conclude that $c = 0$ and $\mu(u) = 0$. Hence d has to vanish as well. □

Lemma 2.2. *Assume that* $u \in \mathrm{C}((-T, T), \mathrm{C}^n(\mathbb{S}^1)) \cap \mathrm{C}^1((-T, T), \mathrm{C}^{n-1}(\mathbb{S}^1))$ *is a solution of* (1.2) *for some* $n \geq 3$ *with* $T > 0$. *Then the* μDP *equation can be written as*

$$u_t = -A^{-1}(u(Au)_x + 3(Au)u_x). \tag{2.1}$$

Proof. Writing (1.2) in the form

$$\mu(u_t) - u_{txx} = uu_{xxx} - 3u_x(\mu(u) - u_{xx}),$$

we see that it is equivalent to

$$Au_t = -u(Au)_x - 3(Au)u_x.$$

Thus u is a solution of (1.2) if and only if (2.1) holds true. □

As explained in [27, 28], the vector field $u(t, x)$ admits a unique local flow φ of class $\mathrm{C}^n(\mathbb{S}^1)$, i.e.,

$$\varphi_t(t, x) = u(t, \varphi(t, x)), \quad \varphi(0, x) = x$$

for all $x \in \mathbb{S}^1$ and all t in some open interval $J \subset \mathbb{R}$. We will use the short-hand notation $\varphi_t = u \circ \varphi$ for $\varphi_t(t, x) = u(t, \varphi(t, x))$; i.e., $\circ$ denotes the composition with respect to the spatial variable. Particularly, we have that $u = \varphi_t \circ \varphi^{-1}$. Moreover, given $(\varphi, \xi) \in \mathrm{C}^1(J, \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1))$, then $\varphi^{-1}(t)$ is a $\mathrm{C}^n(\mathbb{S}^1)$-diffeomorphism for all $t \in J$ and $\xi \circ \varphi^{-1} \in \mathrm{C}^1(J, \mathrm{C}^n(\mathbb{S}^1))$.

In this paper, we are mainly interested in smooth diffeomorphisms on $\mathbb{S}^1$. For the reader's convenience we briefly recall the basic geometric setting. Let us consider the Fréchet manifold $\mathrm{Diff}^\infty(\mathbb{S}^1)$ and a continuous non-degenerate inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{C}^\infty(\mathbb{S}^1)$, i.e., $u \mapsto \langle u, u \rangle$ is continuous (and hence smooth) and $\langle u, v \rangle = 0$ for all $v \in \mathrm{C}^\infty(\mathbb{S}^1)$ forces $u = 0$. To define a weak right-invariant Riemannian metric on $\mathrm{Diff}^\infty(\mathbb{S})$, we extend the inner product $\langle \cdot, \cdot \rangle$ to any tangent space by right-translations, i.e., for all $g \in \mathrm{Diff}^\infty(\mathbb{S}^1)$ and all $u, v \in T_g\mathrm{Diff}^\infty(\mathbb{S}^1)$, we set

$$\langle u, v \rangle_g = \left\langle (R_{g^{-1}})_* u, (R_{g^{-1}})_* v \right\rangle_e,$$

where e denotes the identity. Observe that any open set in the topology induced by this inner product is open in the Fréchet space topology of $\mathrm{C}^\infty(\mathbb{S}^1)$ but the converse is not true. We therefore call $\langle \cdot, \cdot \rangle$ a *weak* Riemannian metric on $\mathrm{Diff}^\infty(\mathbb{S})$, cf. [8]. We next define a bilinear operator $B : \mathrm{Vect}^\infty(\mathbb{S}^1) \times \mathrm{Vect}^\infty(\mathbb{S}^1) \to \mathrm{Vect}^\infty(\mathbb{S}^1)$ by

$$B(u, v) = \frac{1}{2}((\mathrm{ad}_u)^*(v) + (\mathrm{ad}_v)^*(u)),$$

where $(\mathrm{ad}_u)^*$ is the adjoint (with respect to $\langle \cdot, \cdot \rangle$) of the natural action of the Lie algebra on itself given by $\mathrm{ad}_u : v \mapsto [u, v]$. Observe that B defines a right-invariant affine connection ∇ on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ by

$$\nabla_{\xi_u}\xi_v = \frac{1}{2}[\xi_u, \xi_v] + B(\xi_u, \xi_v), \tag{2.2}$$

where ξ_u and ξ_v are the right-invariant vector fields on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ with values u, v at the identity. It can be shown that a smooth curve $t \mapsto g(t)$ in $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is a geodesic if and only if $u = (R_{g^{-1}})_*\dot{g}$ solves the *Euler equation*

$$u_t = -B(u, u); \tag{2.3}$$

here, u is the *Eulerian velocity* (cf. [2]). Hence the Euler equation (2.3) corresponds to the geodesic flow of the affine connection ∇ on the diffeomorphism group $\mathrm{Diff}^\infty(\mathbb{S}^1)$. Paradigmatic examples are the following: In [8], the authors show that the Euler equation for the right-invariant L^2-metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is given by the inviscid Burgers equation. Equipping on the other hand $\mathrm{C}^\infty(\mathbb{S}^1)$ with the H^1-metric, one obtains the Camassa-Holm equation. Similar correspondences for the general H^k-metrics are explained in [9].

Conversely, starting with an equation of type $u_t = -B(u, u)$ with a bilinear operator B, one associates an affine connection ∇ on $\mathrm{Diff}^\infty(\mathbb{S})$ by formula (2.2). It is however by no means clear that this connection corresponds to a Riemannian structure on $\mathrm{Diff}^\infty(\mathbb{S})$. It is worthwhile to mention that the connection ∇ corresponding to the family of b-equations is compatible with some metric only for $b = 2$: In [18] the authors explain that for any $b \neq 2$, the b-equation (1.1) cannot be realized as an Euler equation on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ for any regular inertia operator. This motivates the notion of *non-metric Euler equations*. An analogous result holds true for the μ-b-equations from which we conclude that the μDP equation belongs to the class of non-metric Euler equations. Although we have no metric for the μDP equation, we will obtain some geometric information by using the connection ∇, defined in the following way.

Let $X(t) = (\varphi(t), \xi(t))$ be a vector field along the curve $\varphi(t) \in \mathrm{Diff}^\infty(\mathbb{S}^1)$. Furthermore let

$$B(v, w) := \frac{1}{2}A^{-1}(v(Aw)_x + w(Av)_x + 3(Av)w_x + 3(Aw)v_x).$$

Lemma 2.2 shows that

$$B(u, u) = A^{-1}(u(Au)_x + 3(Au)u_x) = -u_t,$$

if u is a solution to the μDP equation. Next, the covariant derivative of $X(t)$ in the present case is given by

$$\frac{\mathrm{D}X}{\mathrm{D}t}(t) = \left(\varphi(t), \xi_t + \frac{1}{2}[u(t), \xi(t)] + B(u(t), \xi(t))\right),$$

where $u = \varphi_t \circ \varphi^{-1}$. We see that u is a solution of the μDP if and only if its local flow φ is a geodesic for the connection ∇ defined by B via (2.2).

Although we are mainly interested in the smooth category, we will first discuss flows $\varphi(t)$ on $\mathrm{Diff}^n(\mathbb{S}^1)$ for technical purposes. Regarding $\mathrm{Diff}^n(\mathbb{S}^1)$ as a smooth

Banach manifold modelled over $\mathrm{C}^n(\mathbb{S}^1)$, the following result has to be understood locally, i.e., in any local chart of $\mathrm{Diff}^n(\mathbb{S}^1)$.

Proposition 2.3. *Given $n \geq 3$, the function $u \in \mathrm{C}(J, \mathrm{C}^n(\mathbb{S}^1)) \cap \mathrm{C}^1(J, \mathrm{C}^{n-1}(\mathbb{S}^1))$ is a solution of* (1.2) *if and only if $(\varphi, \xi) \in \mathrm{C}^1(J, \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1))$ is a solution of*

$$\begin{cases} \varphi_t &= \quad \xi, \\ \xi_t &= \; -P_\varphi(\xi), \end{cases} \tag{2.4}$$

where $P_\varphi := R_\varphi \circ P \circ R_{\varphi^{-1}}$ and $P(f) := 3A^{-1}(f_x f_{xx} + (Af)f_x)$.

Proof. The function u and the corresponding flow $\varphi \in \mathrm{Diff}^n(\mathbb{S}^1)$ satisfy the relation $\varphi_t = u \circ \varphi$. Setting $\varphi_t = \xi$, the chain rule implies that

$$\xi_t = (u_t + uu_x) \circ \varphi.$$

Applying Lemma 2.2, we see that u is a solution of the μDP equation (1.2) if and only if

$$\begin{aligned} u_t + uu_x &= -A^{-1}(u(Au)_x - A(uu_x) + 3(Au)u_x) \\ &= -A^{-1}(-uu_{xxx} + u_{xx}u_x + uu_{xxx} + 2u_x u_{xx} + 3(Au)u_x) \\ &= -3A^{-1}(u_x u_{xx} + (Au)u_x) \\ &= -P(u). \end{aligned}$$

Recall that

$$\mu(uu_x) = \int_0^1 uu_x \,\mathrm{d}x = \frac{1}{2}\int_0^1 \partial_x(u^2)\,\mathrm{d}x = \frac{1}{2}(u^2(1) - u^2(0)) = 0,$$

since u is continuous on $\mathbb{S}^1$. With $u = \xi \circ \varphi^{-1}$ the desired result follows. □

3. Short time existence of geodesics

We now define the vector field

$$F(\varphi, \xi) := (\xi, -P_\varphi(\xi))$$

such that $(\varphi_t, \xi_t) = F(\varphi, \xi)$. We know that

$$F : \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1) \to \mathrm{C}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1),$$

since P is of order zero. We aim to prove smoothness of the map F. It is worth to mention that this will not follow from the smoothness of P since neither the composition nor the inversion are smooth maps on $\mathrm{Diff}^n(\mathbb{S}^1)$. The following lemma will be crucial for our purposes.

Lemma 3.1. *Assume that p is a polynomial differential operator of order r with coefficients depending only on μ, i.e.,*

$$p(u) = \sum_{\substack{I=(\alpha_0,\dots,\alpha_r), \\ \alpha_i \in \mathbb{N}\cup\{0\},\, |I|<K}} a_I(\mu(u))\, u^{\alpha_0}(u')^{\alpha_1} \cdots (u^{(r)})^{\alpha_r}.$$

Then the action of $p_\varphi := R_\varphi \circ p \circ R_{\varphi^{-1}}$ *is*

$$p_\varphi(u) = \sum_I a_I \left(\int_0^1 u(y)\varphi_x(y)\,\mathrm{d}y \right) q_I(u; \varphi_x, \dots, \varphi^{(r)}),$$

where q_I *are polynomial differential operators of order* r *with coefficients being rational functions of the derivatives of* φ *up to the order* r*. Moreover, the denominator terms only depend on* φ_x*.*

Proof. It is sufficient to consider a monomial

$$m(u) = a(\mu(u))u^{\alpha_0}(u')^{\alpha_1} \cdots (u^{(r)})^{\alpha_r}.$$

We have

$$m_\varphi(u) = a(\mu(u \circ \varphi^{-1}))u^{\alpha_0}[(u \circ \varphi^{-1})' \circ \varphi]^{\alpha_1} \cdots [(u \circ \varphi^{-1})^{(r)} \circ \varphi]^{\alpha_r},$$

where $\circ$ denotes again the composition with respect to the spatial variable. First, we observe that

$$\mu(u \circ \varphi^{-1}) = \int_{\mathbb{S}^1} u(\varphi^{-1}(x))\,\mathrm{d}x = \int_0^1 u(y)\varphi_x(y)\,\mathrm{d}y,$$

where we have omitted the time dependence of u and φ. Recall that $\varphi(\mathbb{S}^1) = \mathbb{S}^1$, $\varphi_x > 0$ and that $\mu(u \circ \varphi^{-1})$ is a constant with respect to the spatial variable $x \in \mathbb{S}^1$. Let us introduce the notation

$$a_k = (u \circ \varphi^{-1})^{(k)} \circ \varphi, \quad k = 1, 2, \dots, r.$$

Then, by the chain rule,

$$a_1 = (\partial_x(u \circ \varphi^{-1})) \circ \varphi = \frac{u_x \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \circ \varphi = \frac{u_x}{\varphi_x}$$

and

$$a_{k+1} = (\partial_x(u \circ \varphi^{-1})^{(k)}) \circ \varphi = (\partial_x(a_k \circ \varphi^{-1})) \circ \varphi = \frac{\partial_x a_k}{\varphi_x},$$

so that our theorem follows by induction. $\square$

Recall that in the Banach algebras $\mathrm{C}^n(\mathbb{S}^1)$, $n \geq 1$, addition and multiplication as well as the mean value operation μ and the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ are smooth maps. We therefore conclude that if the coefficients a_I are smooth functions for any multi-index I and u and φ are at least r times continuously differentiable, then $p_\varphi(u)$ depends smoothly on (φ, u).

Proposition 3.2. *The vector field*

$$F : \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1) \to \mathrm{C}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$$

is smooth for any $n \geq 3$*.*

Proof. We write $F = (F_1, F_2)$. Since $F_1 : (\varphi, \xi) \mapsto \xi$ is smooth, it remains to check that $F_2 : (\varphi, \xi) \mapsto -P_\varphi(\xi)$ is smooth. For this purpose, we consider the map

$$\tilde{P} : \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1) \to \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$$

defined by

$$\tilde{P}(\varphi,\xi) = (\varphi, (R_\varphi \circ P \circ R_{\varphi^{-1}})(\xi)).$$

Observe that we have the decomposition $\tilde{P} = \tilde{A}^{-1} \circ \tilde{Q}$ with

$$\tilde{A}(\varphi,\xi) = (\varphi, (R_\varphi \circ A \circ R_{\varphi^{-1}})(\xi))$$

and

$$\tilde{Q}(\varphi,\xi) = (\varphi, (R_\varphi \circ Q \circ R_{\varphi^{-1}})(\xi)),$$

where $Q(f) := 3(f_x f_{xx} + (Af)f_x)$. We now apply Lemma 3.1 to deduce that

$$\tilde{A}, \tilde{Q} : \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1) \to \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^{n-2}(\mathbb{S}^1)$$

are smooth. To show that $\tilde{A}^{-1} : \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^{n-2}(\mathbb{S}^1) \to \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$ is smooth, we compute the derivative $D\tilde{A}$ at an arbitrary point (φ,ξ). We have the following directional derivatives of the components $\tilde{A}_1$ and $\tilde{A}_2$:

$$D_\varphi \tilde{A}_1 = \mathrm{id}, \quad D_\xi \tilde{A}_1 = 0, \quad D_\xi \tilde{A}_2 = R_\varphi \circ A \circ R_{\varphi^{-1}}.$$

It remains to compute $(D_\varphi \tilde{A}_2(\varphi,\xi))(\psi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{A}_2(\varphi+\varepsilon\psi,\xi)\big|_{\varepsilon=0}$. In a first step, we calculate

$$\begin{aligned}
\partial_x^2(\xi \circ (\varphi+\varepsilon\psi)^{-1}) &= \partial_x\left[\left(\frac{\xi_x}{\varphi_x+\varepsilon\psi_x}\right)\circ(\varphi+\varepsilon\psi)^{-1}\right] \\
&= \left(\frac{\xi_{xx}}{(\varphi_x+\varepsilon\psi_x)^2} - \xi_x\frac{\varphi_{xx}+\varepsilon\psi_{xx}}{(\varphi_x+\varepsilon\psi_x)^3}\right)\circ(\varphi+\varepsilon\psi)^{-1},
\end{aligned}$$

from which we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\partial_x^2(\xi\circ(\varphi+\varepsilon\psi)^{-1})\circ(\varphi+\varepsilon\psi)\right) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\frac{\xi_{xx}}{(\varphi_x+\varepsilon\psi_x)^2} - \xi_x\frac{\varphi_{xx}+\varepsilon\psi_{xx}}{(\varphi_x+\varepsilon\psi_x)^3}\right) \\
&= -2\frac{\xi_{xx}\psi_x}{(\varphi_x+\varepsilon\psi_x)^3} - \frac{\xi_x\psi_{xx}}{(\varphi_x+\varepsilon\psi_x)^3} \\
&\quad + 3\frac{\xi_x\psi_x}{(\varphi_x+\varepsilon\psi_x)^4}(\varphi_{xx}+\varepsilon\psi_{xx})
\end{aligned}$$

and finally

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\partial_x^2(\xi\circ(\varphi+\varepsilon\psi)^{-1})\circ(\varphi+\varepsilon\psi)\right)\bigg|_{\varepsilon=0} = -2\frac{\xi_{xx}\psi_x}{\varphi_x^3} - \frac{\xi_x\psi_{xx}}{\varphi_x^3} + 3\frac{\varphi_{xx}\xi_x\psi_x}{\varphi_x^4}.$$

Secondly, we observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mu(\xi\circ(\varphi+\varepsilon\psi)^{-1})\bigg|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\mathbb{S}^1}\xi(y)(\varphi_x+\varepsilon\psi_x)(y)\,\mathrm{d}y\bigg|_{\varepsilon=0} \\
&= \int_{\mathbb{S}^1}\xi(y)\psi_x(y)\,\mathrm{d}y,
\end{aligned}$$

since $\varphi+\varepsilon\psi \in \mathrm{Diff}^n(\mathbb{S}^1)$ for small $\varepsilon > 0$. Hence

$$(D_\varphi\tilde{A}_2(\varphi,\xi))(\psi) = \int_{\mathbb{S}^1}\xi(y)\psi_x(y)\,\mathrm{d}y + 2\frac{\xi_{xx}\psi_x}{\varphi_x^3} + \frac{\xi_x\psi_{xx}}{\varphi_x^3} - 3\frac{\varphi_{xx}\xi_x\psi_x}{\varphi_x^4}$$

and

$$D\tilde{A}(\varphi,\xi)=\begin{pmatrix} \mathrm{id} & 0 \\ D_\varphi \tilde{A}_2(\varphi,\xi) & R_\varphi\circ A\circ R_{\varphi^{-1}} \end{pmatrix}.$$

It is easy to check that $D\tilde{A}(\varphi,\xi)$ is an invertible bounded linear operator $\mathrm{C}^n(\mathbb{S}^1)\times \mathrm{C}^n(\mathbb{S}^1)\to \mathrm{C}^n(\mathbb{S}^1)\times \mathrm{C}^{n-2}(\mathbb{S}^1)$. By the open mapping theorem, $D\tilde{A}$ is a topological isomorphism and, by the inverse mapping theorem, $\tilde{A}^{-1}$ is smooth. □

Since F is smooth, we can apply the Banach space version of the Picard-Lindelöf Theorem (also known as *Cauchy-Lipschitz Theorem*) as explained in [27], Chapter XIV-3. This yields the following theorem about the existence and uniqueness of integral curves for the vector field F.

Theorem 3.3. *Given $n\geq 3$, there is an open interval J_n centered at zero and an open ball $B(0,\delta_n)\subset \mathrm{C}^n(\mathbb{S}^1)$ such that for any $u_0\in B(0,\delta_n)$ there exists a unique solution $(\varphi,\xi)\in \mathrm{C}^\infty(J_n,\mathrm{Diff}^n(\mathbb{S}^1)\times \mathrm{C}^n(\mathbb{S}^1))$ of (2.4) with initial conditions $\varphi(0)=\mathrm{id}$ and $\xi(0)=u_0$. Moreover, the flow (φ,ξ) depends smoothly on (t,u_0).*

From the above theorem, we get a unique short-time solution $u=\xi\circ\varphi^{-1}$ in $\mathrm{C}^n(\mathbb{S}^1)$ of the μDP equation with continuous dependence on (t,u_0). We now aim to obtain a similar result for smooth initial data u_0. But since $\mathrm{C}^\infty(\mathbb{S}^1)$ is a Fréchet space, classical results like the Picard-Lindelöf Theorem or the local inverse theorem for Banach spaces are no longer valid in $\mathrm{C}^\infty(\mathbb{S}^1)$. In the proof of our main theorem, we will make use of a Banach space approximation of the Fréchet space $\mathrm{C}^\infty(\mathbb{S}^1)$. First, we shall establish that any solution u of the μDP equation does neither lose nor gain spatial regularity as t increases or decreases from zero. For this purpose, the following conservation law is quite useful. In its formulation we use the notation $m_0(x):=(Au)(0,x)=\mu(u_0)-(u_0)_{xx}$.

Lemma 3.4. *Let u be a $\mathrm{C}^3(\mathbb{S}^1)$-solution of the μDP equation on $(-T,T)$ and let φ be the corresponding flow. Then*

$$(Au)(t,\varphi(t,x))\varphi_x^3(t,x)=m_0,$$

for all $t\in(-T,T)$.

Proof. We compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(\mu(u)-u_{xx}\circ\varphi)\varphi_x^3]\\
&\quad=[\mu(u_t)-u_{xxt}\circ\varphi-(u_{xxx}\circ\varphi)\varphi_t]\,\varphi_x^3+3\varphi_x^2\varphi_{tx}(\mu(u)-u_{xx}\circ\varphi)\\
&\quad=[\mu(u_t)-u_{xxt}\circ\varphi-(u_{xxx}\circ\varphi)(u\circ\varphi)]\,\varphi_x^3+3\varphi_x^2(u\circ\varphi)_x(\mu(u)-u_{xx}\circ\varphi)\\
&\quad=[(\mu(u_t)-u_{xxt}-u_{xxx}u)\circ\varphi]\,\varphi_x^3+3\varphi_x^2(u_x\circ\varphi)\varphi_x(\mu(u)-u_{xx}\circ\varphi)\\
&\quad=[(\mu(u_t)-u_{xxt}-u_{xxx}u)\circ\varphi]\,\varphi_x^3+3\varphi_x^3[u_x(\mu(u)-u_{xx})]\circ\varphi\\
&\quad=[(3u_xu_{xx}-3\mu(u)u_x)\circ\varphi]\,\varphi_x^3-3\varphi_x^3(u_xu_{xx}-\mu(u)u_x)\circ\varphi\\
&\quad=0.
\end{aligned}$$

Since $\varphi(0)=\mathrm{id}$ and $\varphi_x(0)=1$, the proof is completed. □

Lemma 3.5. *Let* $(\varphi, \xi) \in \mathrm{C}^\infty(J_3, \mathrm{Diff}^3(\mathbb{S}^1) \times \mathrm{C}^3(\mathbb{S}^1))$ *be a solution of* (2.4) *with initial data* (id, u_0)*, according to Theorem* 3.3. *Then, for all* $t \in J_3$*,*

$$\varphi_{xx}(t) = \varphi_x(t)\left(\int_0^t \mu(u)\varphi_x(s)\,\mathrm{d}s - m_0\int_0^t \varphi_x(s)^{-2}\,\mathrm{d}s\right) \tag{3.1}$$

and

$$\xi_{xx}(t) = \xi_x(t)\frac{\varphi_{xx}(t)}{\varphi_x(t)} + \varphi_x(t)[\mu(u)\varphi_x(t) - m_0\varphi_x(t)^{-2}]. \tag{3.2}$$

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varphi_{xx}}{\varphi_x}\right) = \frac{\varphi_{xxt}\varphi_x - \varphi_{xt}\varphi_{xx}}{\varphi_x^2}.$$

Since $\varphi_t = u \circ \varphi$,

$$\varphi_{xt} = \varphi_{tx} = \partial_x(u \circ \varphi) = (u_x \circ \varphi)\varphi_x$$

and

$$\begin{aligned}\varphi_{xxt} &= \varphi_{txx}\\ &= \partial_x^2(u \circ \varphi)\\ &= \partial_x[(u_x \circ \varphi)\varphi_x]\\ &= (u_{xx} \circ \varphi)\varphi_x^2 + (u_x \circ \varphi)\varphi_{xx}.\end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varphi_{xx}}{\varphi_x}\right) = (u_{xx} \circ \varphi)\varphi_x.$$

According to the previous lemma, we know that

$$u_{xx} \circ \varphi = \mu(u) - m_0\varphi_x^{-3}.$$

Integrating

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varphi_{xx}}{\varphi_x}\right) = \mu(u)\varphi_x - m_0\varphi_x^{-2}$$

over $[0, t]$ leads to equation (3.1) and taking the time derivative of (3.1) yields (3.2). □

Remark 3.6. Since the μDP equation is equivalent to the quasi-linear evolution equation

$$u_t + uu_x + 3\mu(u)\partial_x A^{-1}u = 0,$$

we see that $\mu(u_t) = 0$ and hence $\mu(u) = \mu(u_0)$ so that $\mu(u)$ can in fact be written in front of the first integral sign in equation (3.1).

Corollary 3.7. *Let* (φ, ξ) *be as in Lemma* 3.5. *If* $u_0 \in \mathrm{C}^n(\mathbb{S}^1)$ *then we have* $(\varphi(t), \xi(t)) \in \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$ *for all* $t \in J_3$*.*

Proof. We proceed by induction on n. For $n = 3$ the result is immediate from our assumption on $(\varphi(t), \xi(t))$. Let us assume that $(\varphi(t), \xi(t)) \in \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$ for some $n \geq 3$. Then Lemma 3.5 shows that, if $u_0 \in \mathrm{C}^{n+1}(\mathbb{S}^1)$, then $(\varphi(t), \xi(t)) \in \mathrm{Diff}^{n+1}(\mathbb{S}^1) \times \mathrm{C}^{n+1}(\mathbb{S}^1)$, finishing the proof. □

Corollary 3.8. *Let* (φ, ξ) *be as in Lemma* 3.5. *If there exists a nonzero* $t \in J_3$ *such that* $\varphi(t) \in \mathrm{Diff}^n(\mathbb{S}^1)$ *or* $\xi(t) \in \mathrm{C}^n(\mathbb{S}^1)$ *then* $\xi(0) = u_0 \in \mathrm{C}^n(\mathbb{S}^1)$.

Proof. Again, we use a recursive argument. For $n = 3$, there is nothing to do. For some $n \geq 3$, suppose that $u_0 \in \mathrm{C}^n(\mathbb{S}^1)$. By the previous corollary, $(\varphi(t), \xi(t)) \in \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$ for all $t \in J_3$. Assume that there is $0 \neq t_0 \in J_3$ such that $\varphi(t_0) \in \mathrm{Diff}^{n+1}(\mathbb{S}^1)$ or $\xi(t_0) \in \mathrm{C}^{n+1}(\mathbb{S}^1)$. Since $\varphi_x > 0$, Lemma 3.5 immediately implies that also $u_0 \in \mathrm{C}^{n+1}(\mathbb{S}^1)$. $\square$

Now we discuss Banach space approximations of Fréchet spaces.

Definition 3.9. Let X be a Fréchet space. A *Banach space approximation* of X is a sequence $\{(X_n, \|\cdot\|_n);\ n \in \mathbb{N}_0\}$ of Banach spaces such that

$$X_0 \supset X_1 \supset X_2 \supset \cdots \supset X, \quad X = \bigcap_{n=0}^{\infty} X_n$$

and $\{\|\cdot\|_n;\ n \in \mathbb{N}_0\}$ is a sequence of norms inducing the topology on X with

$$\|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq \cdots$$

for any $x \in X$.

We have the following result. For a proof, we refer to [17].

Lemma 3.10. *Let* X *and* Y *be Fréchet spaces with Banach space approximations* $\{(X_n, \|\cdot\|_n);\ n \in \mathbb{N}_0\}$ *and* $\{(Y_n, \|\cdot\|_n);\ n \in \mathbb{N}_0\}$. *Let* $\Phi_0 : U_0 \to V_0$ *be a smooth map between the open subsets* $U_0 \subset X_0$ *and* $V_0 \subset Y_0$. *Let*

$$U := U_0 \cap X \quad \text{and} \quad V := V_0 \cap Y,$$

as well as

$$U_n := U_0 \cap X_n \quad \text{and} \quad V_n := V_0 \cap Y_n,$$

for any $n \geq 0$. *Furthermore, we assume that, for each* $n \geq 0$, *the following properties are satisfied:*

(1) $\Phi_0(U_n) \subset V_n$,
(2) *the restriction* $\Phi_n := \Phi_0|_{U_n} : U_n \to V_n$ *is a smooth map.*

Then $\Phi_0(U) \subset V$ *and the map* $\Phi := \Phi_0|_U : U \to V$ *is smooth.*

Now we come to our main theorem which we first formulate in the geometric picture.

Theorem 3.11. *There exists an open interval* J *centered at zero and* $\delta > 0$ *such that for all* $u_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$ *with* $\|u_0\|_{\mathrm{C}^3(\mathbb{S}^1)} < \delta$, *there exists a unique solution* $(\varphi, \xi) \in \mathrm{C}^\infty(J, \mathrm{Diff}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1))$ *of* (2.4) *such that* $\varphi(0) = \mathrm{id}$ *and* $\xi(0) = u_0$. *Moreover, the flow* (φ, ξ) *depends smoothly on* $(t, u_0) \in J \times \mathrm{C}^\infty(\mathbb{S}^1)$.

Proof. Theorem 3.3 for $n = 3$ shows that there is an open interval J centered at zero and an open ball $U_3 = B(0, \delta) \subset \mathrm{C}^3(\mathbb{S}^1)$ so that for any $u_0 \in U_3$ there exists a unique solution $(\varphi, \xi) \in \mathrm{C}^\infty(J, \mathrm{Diff}^3(\mathbb{S}^1) \times \mathrm{C}^3(\mathbb{S}^1))$ of (2.4) with initial data (id, u_0) and a smooth flow

$$\Phi_3 : J \times U_3 \to \mathrm{Diff}^3(\mathbb{S}^1) \times \mathrm{C}^3(\mathbb{S}^1).$$

Let

$$U_n := U_3 \cap \mathrm{C}^n(\mathbb{S}^1) \quad \text{and} \quad U_\infty := U_3 \cap \mathrm{C}^\infty(\mathbb{S}^1).$$

By Corollary 3.7, we have

$$\Phi_3(J \times U_n) \subset \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$$

for any $n \geq 3$ and the map

$$\Phi_n := \Phi_3|_{J \times U_n} : J \times U_n \to \mathrm{Diff}^n(\mathbb{S}^1) \times \mathrm{C}^n(\mathbb{S}^1)$$

is smooth. Lemma 3.10 implies that

$$\Phi_3(J \times U_\infty) \subset \mathrm{Diff}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1),$$

completing the proof of the short-time existence for smooth initial data u_0. Moreover, the mapping

$$\Phi_\infty := \Phi_3|_{J \times U_\infty} : J \times U_\infty \to \mathrm{Diff}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1)$$

is smooth, proving the smooth dependence on time and on the initial condition. □

Under the assumptions of Theorem 3.11, the map

$$\mathrm{Diff}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1), \quad (\varphi, \xi) \mapsto \xi \circ \varphi^{-1} = u$$

is smooth. Thus we obtain the result stated in Theorem 1.1.

4. The exponential map

For a Banach manifold M equipped with a symmetric linear connection, the exponential map is defined as the time one of the geodesic flow, i.e., if $t \mapsto \gamma(t)$ is the (unique) geodesic in M starting at $p = \gamma(0)$ with velocity $\gamma_t(0) = u \in T_pM$ then $\exp_p(u) = \gamma(1)$. Roughly speaking, the map $\exp_p(\cdot)$ is a projection from T_pM to the manifold M. Since the derivative of $\exp_p$ at zero is the identity, the exponential map is a smooth diffeomorphism from a neighbourhood of zero of T_pM to a neighbourhood of $p \in M$. However, this fails for Fréchet manifolds like $\mathrm{Diff}^\infty(\mathbb{S}^1)$ in general. We know that the Riemannian exponential map for the L^2-metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is not a local C^1-diffeomorphism near the origin, cf. [9]. For the Camassa-Holm equation and more general for the H^k-metrics, $k \geq 1$, the Riemannian exponential map in fact is a smooth local diffeomorphism. This result was generalized to the family of b-equations, see [17], and in this section we obtain a similar result for the μDP equation.

The basic idea of the proof of Theorem 1.2 is to consider a perturbed problem: Let $(\varphi^\varepsilon, \xi^\varepsilon)$ denote the local expression of an integral curve of (2.4) in $T\mathrm{Diff}^n(\mathbb{S}^1)$

with initial data $(\mathrm{id}, u+\varepsilon w)$, where $u, w \in \mathrm{C}^n(\mathbb{S}^1)$. Let

$$\psi(t) := \left.\frac{\partial \varphi^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

By the homogeneity of the geodesics,

$$\varphi^{\varepsilon}(t) = \exp(t(u+\varepsilon w)),$$

so that

$$\psi(t) = D\left(\exp(tu)\right) tw =: L_n(t,u)w,$$

where $L_n(t,u)$ is a bounded linear operator on $\mathrm{C}^n(\mathbb{S}^1)$.

Lemma 4.1. *Suppose that $u \in \mathrm{C}^{n+1}(\mathbb{S}^1)$. Then, for $t \neq 0$,*

$$L_n(t,u)(\mathrm{C}^n(\mathbb{S}^1)\backslash \mathrm{C}^{n+1}(\mathbb{S}^1)) \subset \mathrm{C}^n(\mathbb{S}^1)\backslash \mathrm{C}^{n+1}(\mathbb{S}^1).$$

Proof. First, we write down equation (3.1) for $\varphi^{\varepsilon}(t)$,

$$\varphi^{\varepsilon}_{xx}(t) = \varphi^{\varepsilon}_x(t)\left[\mu(u+\varepsilon w)\int_0^t \varphi^{\varepsilon}_x(s)\,\mathrm{d}s - m_0^{\varepsilon}\int_0^t \varphi^{\varepsilon}_x(s)^{-2}\,\mathrm{d}s\right],$$

and take the derivative with respect to ε,

$$\begin{aligned}
\frac{\partial \varphi^{\varepsilon}_{xx}}{\partial \varepsilon}(t) = {} & \frac{\partial \varphi^{\varepsilon}_{x}}{\partial \varepsilon}(t)\left[\mu(u+\varepsilon w)\int_0^t \varphi^{\varepsilon}_x(s)\,\mathrm{d}s - m_0^{\varepsilon}\int_0^t \varphi^{\varepsilon}_x(s)^{-2}\,\mathrm{d}s\right] \\
& + \varphi^{\varepsilon}_x(t)\left[\mu(w)\int_0^t \varphi^{\varepsilon}_x(s)\,\mathrm{d}s + \mu(u+\varepsilon w)\int_0^t \frac{\partial \varphi^{\varepsilon}_x}{\partial \varepsilon}(s)\,\mathrm{d}s\right] \\
& - \varphi^{\varepsilon}_x(t)\left[\frac{\partial m_0^{\varepsilon}}{\partial \varepsilon}\int_0^t \varphi^{\varepsilon}_x(s)^{-2}\,\mathrm{d}s + m_0^{\varepsilon}\int_0^t \frac{\partial}{\partial \varepsilon}\varphi^{\varepsilon}_x(s)^{-2}\,\mathrm{d}s\right].
\end{aligned}$$

Notice that

$$\frac{\partial m_0^{\varepsilon}}{\partial \varepsilon} = \mu(w) - w_{xx} = Aw$$

and that $m_0^{\varepsilon} \to m_0 = Au$ as $\varepsilon \to 0$. Hence

$$\begin{aligned}
\psi_{xx}(t) = {} & \psi_x(t)\left[\mu(u)\int_0^t \varphi_x(s)\,\mathrm{d}s - m_0\int_0^t \varphi_x(s)^{-2}\,\mathrm{d}s\right] \\
& + \varphi_x(t)\left[\mu(w)\int_0^t \varphi_x(s)\,\mathrm{d}s + \mu(u)\int_0^t \psi_x(s)\,\mathrm{d}s\right] \\
& - \varphi_x(t)\left[(\mu(w) - w_{xx})\int_0^t \varphi_x(x)^{-2}\,\mathrm{d}s - 2m_0\int_0^t \varphi_x(s)^{-3}\psi_x(s)\,\mathrm{d}s\right] \\
= {} & a(t)\psi_x(t) + b(t)\int_0^t c(s)\psi_x(s)\,\mathrm{d}s + d(t) + e(t)w_{xx}
\end{aligned}$$

with $a(t), b(t), c(t), d(t), e(t) \in \mathrm{C}^{n-1}(\mathbb{S}^1)$ and $e(t) \neq 0$ for $t \neq 0$. Finally, if

$$w \in \mathrm{C}^n(\mathbb{S}^1)\backslash \mathrm{C}^{n+1}(\mathbb{S}^1),$$

then

$$\psi(t) = L_n(t,u)w \in \mathrm{C}^n(\mathbb{S}^1)\backslash \mathrm{C}^{n+1}(\mathbb{S}^1). \qquad \square$$

Let us now turn to the proof of Theorem 1.2. Since $\mathrm{C}^3(\mathbb{S}^1)$ is a Banach space and $\mathrm{Diff}^3(\mathbb{S}^1)$ is a Banach manifold modelled over $\mathrm{C}^3(\mathbb{S}^1)$, we know that the exponential map is a smooth diffeomorphism near zero, i.e., there are neighbourhoods U_3 of zero in $\mathrm{C}^3(\mathbb{S}^1)$ and V_3 of id in $\mathrm{Diff}^3(\mathbb{S}^1)$ such that

$$\exp_3 := \exp|_{U_3} : U_3 \to V_3$$

is a smooth diffeomorphism. For $n \geq 3$, we now define

$$U_n := U_3 \cap \mathrm{C}^n(\mathbb{S}^1) \quad \text{and} \quad V_n = V_3 \cap \mathrm{Diff}^n(\mathbb{S}^1).$$

Let $\exp_n := \exp_3|_{U_n}$. Since $\exp_n$ is a restriction of $\exp_3$, it is clearly injective. We now use Corollary 3.7 and Corollary 3.8 to deduce that $\exp_n$ is also surjective, more precisely, $\exp_n(U_n) = V_n$. If the geodesic φ with $\varphi(1) = \exp(u)$ starts at id $\in \mathrm{Diff}^n(\mathbb{S}^1)$ with velocity vector u belonging to $\mathrm{C}^n(\mathbb{S}^1)$, then $\varphi(t) \in \mathrm{Diff}^n(\mathbb{S}^1)$ for any t and hence $\exp_n(U_n) \subset V_n$. Conversely, if $v \in V_n$ is given, then there is $u \in U_3$ with $\exp_3(u) = v$. Corollary 3.8 immediately implies that $u \in \mathrm{C}^n(\mathbb{S}^1)$; hence $u \in U_n$ and $\exp_n(u) = v$. Note that $\exp_n$ is a bijection from U_n to V_n. Furthermore, $\exp_n$ is a smooth map and diffeomorphic $U_n \to V_n$. We now show that $\exp_n$ is a smooth diffeomorphism; precisely, we show that $\exp_n^{-1} : V_n \to U_n$ is smooth by virtue of the inverse mapping theorem. For each $u \in \mathrm{C}^n(\mathbb{S}^1)$, $D\exp_n(u)$ is a bounded linear operator $\mathrm{C}^n(\mathbb{S}^1) \to \mathrm{C}^n(\mathbb{S}^1)$. Notice that

$$D\exp_n(u) = D\exp_3(u)|_{\mathrm{C}^n(\mathbb{S}^1)},$$

from which we conclude that $D\exp_n(u)$ is injective. Let us prove the surjectivity of $D\exp_n(u)$, $n \geq 3$, by induction. For $n = 3$, this follows from the fact that $\exp_3 : U_3 \to V_3$ is diffeomorphic and hence a submersion. Assume that $D\exp_n(u)$ is surjective for some $n \geq 3$ and that $u \in \mathrm{C}^{n+1}(\mathbb{S}^1)$. We have to show that this implies the surjectivity of $D\exp_{n+1}(u)$. But this is a direct consequence of $D\exp_n(u) = L_n(1, u)$ and the previous lemma: Let $f \in \mathrm{C}^{n+1}(\mathbb{S}^1)$. We have to find $g \in \mathrm{C}^{n+1}(\mathbb{S}^1)$ with the property $D\exp_{n+1}(u)g = f$. By our assumption, there is $g \in \mathrm{C}^n(\mathbb{S}^1)$ such that $D\exp_n(u)g = f$. Suppose that $g \notin \mathrm{C}^{n+1}(\mathbb{S}^1)$. But then $f = L_n(1, u)g \notin \mathrm{C}^{n+1}(\mathbb{S}^1)$ in contradiction to the choice of f. Thus $g \in \mathrm{C}^{n+1}(\mathbb{S}^1)$ and $D\exp_{n+1}(u)g = f$. Now we can apply the open mapping theorem to deduce that for any $n \geq 3$ and any $u \in \mathrm{C}^n(\mathbb{S}^1)$ the map

$$D\exp_n(u) : \mathrm{C}^n(\mathbb{S}^1) \to \mathrm{C}^n(\mathbb{S}^1)$$

is a topological isomorphism. By the inverse function theorem, $\exp_n : U_n \to V_n$ is a smooth diffeomorphism. If we define

$$U_\infty := U_3 \cap \mathrm{C}^\infty(\mathbb{S}^1) \quad \text{and} \quad V_\infty := V_3 \cap \mathrm{Diff}^\infty(\mathbb{S}^1),$$

Lemma 3.10 yields that

$$\exp_\infty := \exp_3|_{U_\infty} : U_\infty \to V_\infty$$

as well as

$$\exp_\infty^{-1} : V_\infty \to U_\infty$$

are smooth maps. Thus $\exp_\infty$ is a smooth diffeomorphism between U_∞ and V_∞.

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Global Leray-Hopf Weak Solutions of the Navier-Stokes Equations with Nonzero Time-dependent Boundary Values

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Abstract. In a bounded smooth domain $\Omega \subset \mathbb{R}^3$ and a time interval $[0, T)$, $0 < T \leq \infty$, consider the instationary Navier-Stokes equations with initial value $u_0 \in L^2_\sigma(\Omega)$ and external force $f = \operatorname{div} F$, $F \in L^2(0, T; L^2(\Omega))$. As is well known there exists at least one weak solution in the sense of J. Leray and E. Hopf with vanishing boundary values satisfying the strong energy inequality. In this paper, we extend the class of global in time Leray-Hopf weak solutions to the case when $u|_{\partial\Omega} = g$ with non-zero time-dependent boundary values g. Although there is no uniqueness result for these solutions, they satisfy a strong energy inequality and an energy estimate. In particular, the long-time behavior of energies will be analyzed.

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Keywords. Instationary Navier-Stokes equations; weak solutions; energy inequality; non-zero boundary values; time-dependent data; long-time behavior.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary of class $C^{1,1}$, and let $[0, T)$, $0 < T \leq \infty$, be a time interval. In $\Omega \times [0, T)$ we consider the instationary Navier-Stokes system in the form

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \quad \operatorname{div} u = 0 \\ u|_{\partial\Omega} = g, \quad u(0) = u_0 \end{aligned} \tag{1.1}$$

with viscosity $\nu = 1$ and data f, g, u_0.

For the data we assume the following:

$$\begin{aligned} &f = \operatorname{div} F,\ F \in L^2\big(0,T;L^2(\Omega)\big),\ u_0 \in L^2_\sigma(\Omega),\\ &g \in L^4\big(0,T;W^{-\frac14,4}(\partial\Omega)\big) \cap L^s\big(0,T;W^{-\frac1q,q}(\partial\Omega)\big),\\ &\frac2s + \frac3q = 1,\ 2<s<\infty,\ 3<q<\infty; \end{aligned} \tag{1.2}$$

the initial data u_0 has to satisfy further assumptions to be introduced later, see Section 5.

First we recall the well-known definition of a weak solution u in the sense of Leray-Hopf with $u|_{\partial\Omega} = 0$ and initial value $u_0 \in L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}$, and describe their main properties.

Definition 1.1. Let $f = \operatorname{div} F$, $F \in L^2(0,T;L^2(\Omega))$, $0 < T \le \infty$, and $u_0 \in L^2_\sigma(\Omega)$ be given. Then the vector field u is called a *Leray-Hopf weak solution* of the system (1.1) with data f, u_0 and $g = 0$ if the following conditions are satisfied:

(i) $u \in L^\infty\big(0,T;L^2_\sigma(\Omega)\big) \cap L^2\big(0,T;W^{1,2}_0(\Omega)\big)$,

(ii) for each test function $w \in C^\infty_0\big([0,T);C^\infty_{0,\sigma}(\Omega)\big)$

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T},$$

(iii) for $0 \le t < T$

$$\frac12\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\,d\tau \le \frac12\|u_0\|_2^2 - \int_0^t \langle F, \nabla u\rangle_\Omega\,d\tau.$$

As is well known [20] there exists at least one weak solution u of (1.1) in the sense of Definition 1.1; moreover, u may be chosen to be weakly continuous from $[0,T)$ to $L^2_\sigma(\Omega)$ and to satisfy beyond the *energy inequality* (iii) in Definition 1.1 the so-called *strong energy inequality*

$$\frac12\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u\|_2^2\,d\tau \le \frac12\|u(t_0)\|_2^2 - \int_{t_0}^t (F, \nabla u)_\Omega\,d\tau \tag{1.3}$$

for almost all $t_0 \in [0,T)$ including $t_0 = 0$ and for all $t \in [t_0,T)$. We note that the strong energy inequality (1.3) yields the *energy estimate*

$$\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u\|_2^2\,d\tau \le \|u(t_0)\|_2^2 + \int_{t_0}^t \|F\|_2^2\,d\tau$$

for a.a. $t_0 \in (0,T)$, for $t_0 = 0$ and all $t \in [t_0,T)$. An application of Hölder's inequality implies that

$$u \in L^s\big(0,T;L^q(\Omega)\big), \quad \frac2s + \frac3q = \frac32,\ 2 \le s \le \infty,\ 2 \le q \le 6.$$

In order to extend Definition 1.1 to the more general case of time-dependent boundary data $u|_{\partial\Omega} = g$ we will reduce the system (1.1) to a perturbed Navier-Stokes system with $g = 0$. For this purpose we have to find first of all a (so-called) very weak solution of the inhomogeneous Stokes system

$$\begin{aligned} E_t - \Delta E + \nabla h = f_0, \quad \operatorname{div} E = 0 \\ E\big|_{\partial\Omega} = g, \qquad E(0) = E_0, \end{aligned} \tag{1.4}$$

see [2]–[5], in $\Omega \times [0,T)$ with suitable data $f_0 = \operatorname{div} F_0$ and E_0; here ∇h means the associated pressure. At first sight, it seems to suffice to choose $f_0 = 0$, $F_0 = 0$, but for later application it will be helpful to consider general data f_0, F_0, see Corollary 1.5 and (1.15) below. Setting

$$v = u - E, \ \tilde{p} = p - h, \ f_1 = f - f_0, \ v_0 = u_0 - E_0 \tag{1.5}$$

we write (1.1) in the form

$$\begin{aligned} v_t - \Delta v + (v+E)\cdot\nabla(v+E) + \nabla\tilde{p} = f_1, \quad \operatorname{div} v = 0, \\ v\big|_{\partial\Omega} = 0, \qquad v(0) = v_0 \end{aligned} \tag{1.6}$$

which is a perturbed Navier-Stokes system with homogeneous data $v\big|_{\partial\Omega} = 0$, but with the new perturbation terms

$$(v+E)\cdot\nabla(v+E) = \operatorname{div}\big(vv + (Ev + vE) + EE\big); \tag{1.7}$$

here $Ev = E \otimes v = (E_i v_j)_{i,j=1,2,3}$ denotes the dyadic product of the vector fields E and v, and the divergence is taken columnwise.

To deal with Leray-Hopf type weak solutions v of (1.6), see Definition 1.2 below, we need that E in (1.4) has the following properties:

$$\begin{aligned} & E \in L^4\big(0,T;L^4(\Omega)\big) \cap L^s\big(0,T;L^q(\Omega)\big) \\ & \frac{2}{s} + \frac{3}{q} = 1, \ 2 < s \le \infty, \ 3 \le q < \infty. \end{aligned} \tag{1.8}$$

Actually, the condition $E \in L^4(0,T;L^4(\Omega))$ in (1.8) is needed for estimates of the term EE in (1.7) in the space $L^2(0,T;L^2(\Omega))$, whereas the second condition $E \in L^s(0,T;L^q(\Omega))$ will help to estimate the terms vE and Ev. Note that $E \in L^4(0,T;L^4(\Omega))$ is the classical condition on weak solutions of the Navier-Stokes system to satisfy the energy identity, and it automatically holds true when $4 \le s \le 8$, $4 \le q \le 6$. To guarantee (1.8) for the solution E of (1.4) the data f_0, g, E_0 have to satisfy certain assumptions known from the theory of the very weak Stokes system. However, looking at (1.6), it suffices to assume (1.8) and $v_0 \in L^2_\sigma(\Omega)$, $f_1 = \operatorname{div} F_1$, $F_1 \in L^2(0,T;L^2(\Omega))$, in order to define Leray-Hopf type weak solutions of (1.6); later concrete conditions on g, u_0, E_0 will be described to satisfy these assumptions, see Section 5.

In this respect, this paper mainly deals with the perturbed Navier-Stokes system (1.6) rather than with (1.1).

Definition 1.2. Let E satisfy (1.8) and assume $v_0 \in L^2_\sigma(\Omega)$, $f_1 = \operatorname{div} F_1$, $F_1 \in L^2(0,T;L^2(\Omega))$. Then a vector field v on $\Omega \times [0,T)$ is a *Leray-Hopf type weak solution* of the perturbed Navier-Stokes system (1.6) if the following conditions are satisfied:

(i) $v \in L^\infty_{\rm loc}\big([0,T); L^2_\sigma(\Omega)\big) \cap L^2_{\rm loc}\big([0,T); W^{1,2}_0(\Omega)\big)$,

(ii) for each test function $w \in C^\infty_0\big([0,T); C^\infty_{0,\sigma}(\Omega)\big)$

$$\begin{aligned}\langle v, w_t\rangle_{\Omega,T} - \langle \nabla v, \nabla w\rangle_{\Omega,T} - \langle (v+E)(v+E), \nabla w\rangle_{\Omega,T}\\ = \langle v_0, w(0)\rangle_\Omega - \langle F_1, \nabla w\rangle_{\Omega,T}\,,\end{aligned} \tag{1.9}$$

(iii) the energy inequality

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\,d\tau \le \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F_1 - (v+E)E, \nabla v\rangle_\Omega\,d\tau \tag{1.10}$$

holds for $0 < t < T$.

Now our main theorem reads as follows:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{1,1}$, and let $f_1 = \operatorname{div} F_1$, $F_1 \in L^2(0,T; L^2(\Omega))$, $v_0 \in L^2_\sigma(\Omega)$. Assume that E satisfies* (1.8) *where in the case $s = \infty$, $q = 3$ this condition is replaced by*

$$E \in C^0\big([0,T); L^3(\Omega)\big), \quad \|E\|_{C^0([0,T);L^3(\Omega))} < \alpha_0 \tag{1.11}$$

for a sufficiently small constant $\alpha_0 = \alpha_0(\Omega) > 0$. Then the perturbed Navier-Stokes system

$$\begin{aligned}v_t - \Delta v + (v+E)\cdot\nabla(v+E) + \nabla\tilde p = f_1, \quad \operatorname{div} v = 0,\\ v|_{\partial\Omega} = 0, \qquad v(0) = v_0\end{aligned} \tag{1.12}$$

has at least one Leray-Hopf type weak solution v in the sense of Definition 1.2*. In addition to the energy inequality* (1.10) *v satisfies the strong energy inequality*

$$\frac{1}{2}\|v(t)\|_2^2 + \int_{t_0}^t \|\nabla v\|_2^2\,d\tau \le \frac{1}{2}\|v(t_0)\|_2^2 - \int_{t_0}^t \langle F_1 - (v+E)E, \nabla v\rangle\,d\tau \tag{1.13}$$

for a.a. $t_0 \in [0,T)$ including $t_0 = 0$ and for all $t_0 < t < T$. Moreover, for these t_0, t the energy estimate

$$\begin{aligned}&\|v(t)\|_2^2 + \int_{t_0}^t \|\nabla v\|_2^2\,d\tau\\ &\quad \le c\Big(\|v(t_0)\|_2^2 + 4\int_{t_0}^t (\|F_1\|_2^2 + \|E\|_4^4)\,d\tau\Big)\exp\Big(c\int_{t_0}^t \|E\|_q^s\,d\tau\Big)\end{aligned} \tag{1.14}$$

holds, where $c = c(\Omega, q) > 0$ means a constant.

In view of (1.1), (1.4)–(1.6) and Definition 1.2 $u = v + E$ is called a Leray-Hopf type weak solution of (1.1) with boundary data $u|_{\partial\Omega} = g = E|_{\partial\Omega}$ and initial value $u(0) = v_0 + E_0$. In the most general setting of very weak solutions, cf. [19, 18], these terms are not well defined separately from each other and from f, but have to be interpreted in the generalized sense that $v = u - E$ satisfies $v|_{\partial\Omega} = 0$ and $v(0) = v_0$. For a more concrete situation and precise assumptions on g, u_0 we refer to Section 5.

Remark 1.4. (i) The weak solution v in Theorem 1.3 may be modified on a null set of $(0,T)$ such that $v:[0,T)\to L^2_\sigma(\Omega)$ is weakly continuous. Hence $v(0)=v_0$ is well defined, $v|_{\partial\Omega}=0$ is well defined for a.a. $t\in[0,T)$ in the sense of traces, and there exists a distribution $\tilde p$ on $\Omega\times(0,T)$ such that

$$v_t-\Delta v+(v+E)\cdot\nabla(v+E)+\nabla\tilde p=f_1.$$

(ii) Assume $T=\infty$ and

$$E\in L^4\big(0,\infty;L^4(\Omega)\big)\cap L^s_{\mathrm{loc}}\big([0,\infty);L^q(\Omega)\big),\ 2<s<\infty,$$

such that $\|E\|^s_{q,s;t}=\int_0^t\|E\|_q^s\,d\tau$ is increasing at least linearly as $t\to\infty$. We note that in this case the proof in Section 4 will easily show the existence of a weak solution v in $(0,\infty)$. Then the energy estimate (1.14) (with $t_0=0$) yields for the kinetic energy $\frac12\|v(t)\|_2^2$ only an exponentially increasing bound as $t\to\infty$. This worst case estimate reflects the fact that nonzero boundary values could imply a permanent flux of energy through the boundary into the domain.

(iii) If $s=\infty$ and $\|E\|_{C^0([0,\infty);L^3(\Omega))}$ is sufficiently small, cf. (1.11), then due to energy dissipation the scenario of (ii) will not occur, and (1.14) is replaced by the better estimate

$$\|v(t)\|_2^2+\int_{t_0}^t\|\nabla v\|_2^2\,d\tau\le\|v(t_0)\|_2^2+4\int_{t_0}^t\big(\|F_1\|_2^2+\|E\|_4^4\big)\,d\tau$$

for a.a. $t_0\in[0,\infty)$ including $t_0=0$ and for all $t\in(t_0,\infty)$. The proof of this estimate is based on the energy estimate (3.12) below and arguments in Sections 3–4.

The smallness assumptions in Remark 1.4 (iii), see also (1.11), is too restrictive for further applications. Next we consider an assumption on E known as Leray's inequality in the context of stationary Navier-Stokes equations in multiply connected domains: Let E satisfy the conditions

$$\begin{aligned}&E\in L^\infty\big(0,\infty;L^3(\Omega)\big)\quad\text{and}\\&\Big|\int_\Omega w_1E(t)\cdot\nabla w_2\,dt\Big|\le\frac14\|\nabla w_1\|_2\,\|\nabla w_2\|_2\\&\text{for all }w_1,w_2\in W_0^{1,2}(\Omega)\cap L^2_\sigma(\Omega)\text{ and a.a. }t\in(0,\infty).\end{aligned}\tag{1.15}$$

We recall that in a (multiply connected) bounded domain $\Omega\subset\mathbb{R}^3$ with $\partial\Omega=\cup_{j=0}^L\Gamma_j\in C^{1,1}$ $(L\in\mathbb{N})$ to any boundary data $g\in W^{1/2,2}(\partial\Omega)$ satisfying the *restricted flux condition*

$$\int_{\Gamma_j}g\cdot N\,do=0\text{ for each boundary component }\Gamma_j\subset\partial\Omega,\quad j=1,\dots,L,$$

and any $\varepsilon>0$ there exists an extension $E_\varepsilon\in W^{1,2}(\Omega)$ with following properties:

$$\begin{aligned}&E_\varepsilon\in W^{1,2}(\Omega),\ \operatorname{div}E_\varepsilon=0,\ E_\varepsilon|_{\partial\Omega}=g\\&\Big|\int_\Omega wE_\varepsilon\cdot\nabla w\,dx\Big|\le\varepsilon\,\|\nabla w\|_2^2\quad\text{for all }w\in W_0^{1,2}(\Omega)\cap L^2_\sigma(\Omega).\end{aligned}\tag{1.16}$$

In particular this result holds for a simply connected domain (having only one boundary component $\Gamma = \partial\Omega$). In our case, it suffices to consider $\varepsilon = \frac{1}{4}$ in $(1.15)_2$ since the viscosity $\nu = 1$.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{1,1}$, let $v_0 \in L^2_\sigma(\Omega)$ and let*

$$f_1 = \operatorname{div} F_1,\ F_1 \in L^\infty\big(0,\infty; L^2(\Omega)\big). \tag{1.17}$$

Furthermore, assume that E satisfies (1.15). *Then the perturbed Navier-Stokes system* (1.12) *has a global in time Leray-Hopf type weak solution v satisfying the energy estimate*

$$\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\, d\tau \le \|v_0\|_2^2 + c\int_0^t \big(\|F_1\|_2^2 + \|E\|_4^4\big)\, d\tau\,, \tag{1.18}$$

where $c = c(\Omega) > 0$ is a constant. To be more precise, there exists a bound κ^* (depending on the norms $\|F_1\|_{2,\infty;\infty}$ and $\|E\|_{4,\infty;\infty}$, but not an $\|v_0\|_2$) and a time $T_0 = T_0(\|v_0\|_2)$ such that

$$\|v(t)\|_2^2 \le \kappa^* \text{ for all } t \ge T_0.$$

Corollary 1.5, see also Corollary 5.6, strictly exploits the assumption (1.15) on E and the dissipativity of the Navier-Stokes system (1.1), (1.12). It is closely related to the work [14] of E. Hopf where a similar result is proved in the context of moving bodies for given smooth solutions. The impact of the final result of Corollary 1.5 is the fact that all solutions are bounded after a finite time by a bound independent of the initial value.

There are many applications of the two-dimensional Navier-Stokes system with nonhomogeneous boundary values in optimal control theory since the $2D$-system admits global smooth solutions and uniqueness. For the three-dimensional case there are only few results on the existence of global or weak solutions. We mention the existence of local in time strong solutions by A.V. Fursikov, M.D. Gunzburger and L.S. Hou [7], and results in a scale of Besov spaces by G. Grubb [12]. The existence of global in time weak solutions is proved by J.-P. Raymond [17] for boundary data in a fractional Sobolev space on $\partial\Omega \times (0,T)$ with derivatives in space and time of fractional order 3/4 for domains with boundary $\partial\Omega \in C^3$.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{1,1}$, let $0 < T \le \infty$ and $1 \le q$, $s \le \infty$ with conjugate exponents $1 \le q'$, $s' \le \infty$. We will use standard notation for Lebesgue spaces $(L^q(\Omega), \|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q)$ and for Bochner spaces $L^s(0,T;L^q(\Omega)), \|\cdot\|_{L^s(0,T;L^q(\Omega))} = \|\cdot\|_{q,s;T})$. Here $v \in L^s_{\mathrm{loc}}([0,T);L^q(\Omega))$ means that $v \in L^s(0,T';L^q(\Omega))$ for each finite $0 < T' \le T$. The pairing of functions (or vector fields) in Ω and $\Omega \times (0,T)$ is denoted by $\langle\cdot,\cdot\rangle_\Omega$ and $\langle\cdot,\cdot\rangle_{\Omega,T}$, respectively. Sobolev spaces are denoted by $(W^{m,q}(\Omega), \|\cdot\|_{W^{m,q}}), m \in \mathbb{N}$, the corresponding trace space by $(W^{m-1/q,q}(\partial\Omega), \|\cdot\|_{W^{m-1/q,q}})$ when $1 < q < \infty$. The dual space to

$W^{1-1/q',q'}(\partial\Omega)$ is denoted by $W^{-1/q,q}(\partial\Omega)$, the corresponding pairing is $\langle\cdot,\cdot\rangle_{\partial\Omega}$. Concerning smooth functions we need the spaces $C_0^\infty(\Omega), C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$, and in the context of very weak solutions

$$C_{0,\sigma}^2(\overline{\Omega}) = \{w \in C^2(\overline{\Omega}) : w = 0 \text{ on } \partial\Omega, \operatorname{div} w = 0\},$$

see Section 5. Note that $W_0^{1,q}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,q}}}$ and $L_\sigma^q(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$. By $w \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$, the space of test functions in Definitions 1.1 and 1.2, we mean that $w \in C_0^\infty([0,T) \times \Omega)$ satisfies $\operatorname{div}_x w = 0$ for all $t \in [0,T)$ (taking the divergence with respect to $x \in \Omega$).

For $1 < q < \infty$ let $P_q : L^q(\Omega) \to L_\sigma^q(\Omega)$ be the Helmholtz projection and let

$$A_q = -P_q\Delta : \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega) \subset L_\sigma^q(\Omega) \to L_\sigma^q(\Omega)$$

denote the Stokes operator. For its fractional powers $A_q^\alpha : \mathcal{D}(A_q^\alpha) \to L_\sigma^q(\Omega), -1 \le \alpha \le 1$, we know that $\mathcal{D}(A_q) \subseteq \mathcal{D}(A_q^\alpha) \subseteq L_\sigma^q(\Omega)$ and $\mathcal{R}(A_q^\alpha) = L_\sigma^q(\Omega)$ for $0 \le \alpha \le 1$; finally, $(A_q^\alpha)^{-1} = A_q^{-\alpha}$ for $-1 \le \alpha \le 1$. In particular, for $A = A_2$, one has $\|A^{\frac12}v\|_2 = \|\nabla v\|_2$ for $v \in \mathcal{D}(A^{\frac12})$. Moreover, we note the following embedding estimates (with constants $c = c(q,\Omega) > 0$):

$$\|v\|_q \le c\|A^\alpha v\|_2, \quad 0 \le \alpha \le \frac12,\ 2 \le q < \infty,\ 2\alpha + \frac3q = \frac32,\ v \in \mathcal{D}(A^\alpha) \tag{2.1}$$

$$\|A^\alpha v\|_2 \le \|Av\|_2^\alpha\, \|v\|_2^{1-\alpha}, \quad 0 \le \alpha \le 1,\ v \in \mathcal{D}(A) \tag{2.2}$$

$$\|v\|_q \le c\|v\|_{W^{1,2}}^\beta \|v\|_2^{1-\beta}, \quad 2 \le q \le 6, \beta = 3(\frac12 - \frac1q), v \in W^{1,2}(\Omega). \tag{2.3}$$

Finally we note that $-A_q$ generates a bounded, exponentially decaying analytic semigroup $\{e^{-tA_q} : t \ge 0\}$ on $L_\sigma^q(\Omega)$ satisfying the estimate

$$\|A_q^\alpha e^{-tA_q} v\|_q \le c\, t^{-\alpha}\|v\|_q, \quad 0 \le \alpha \le 1, t > 0, v \in L_\sigma^q(\Omega) \tag{2.4}$$

with a constant $c = c\,(q,\Omega) > 0$ and $c = 1$ if $q = 2$.

To find approximate solutions of the Navier-Stokes system in Section 3 we need Yosida's approximation operators

$$J_m = (I + \frac1m A^{\frac12})^{-1},\ m \in \mathbb{N},$$

where I denotes the identity on $L_\sigma^2(\Omega)$. The following properties are well known:

$$\begin{aligned} &\|J_m v\|_2 \le \|v\|_2,\ \|\frac1m A^{\frac12} J_m v\|_2 \le \|v\|_2, \\ &\lim_{m\to\infty} J_m v = v \text{ for all } v \in L_\sigma^2(\Omega), \\ &\|\nabla J_m v\|_2 \le \|\nabla v\|_2 \text{ for all } v \in W_0^{1,2}(\Omega) \cap L_\sigma^2(\Omega) = \mathcal{D}(A^{\frac12}); \end{aligned} \tag{2.5}$$

for the proof of the last inequality we use that $\|A^{\frac12} v\|_2 = \|\nabla v\|_2$ and the commutativity of J_m with $A^{\frac12}$.

For these and further properties of the Stokes operator and Yosida' s approximation we refer, e.g., to [9], [10] and [20].

3. The approximate system

There are several proofs of Theorem 1.3 in the special case when $g = 0$, see [13], [15], [16], [20], [22]. The proofs rest on three steps: (1) an approximation procedure yielding a sequence of solutions, (u_m), (2) an energy estimate for u_m with bounds independent of $m \in \mathbb{N}$ to show that each u_m exists on the maximal interval of existence, $[0, T)$, and (3) weak and strong convergence properties of a suitable subsequence of $\{u_m\}$ to construct a weak solution of the Navier-Stokes equation. One possibility in (1) is the use of the Yosida approximation yielding the approximate system

$$u_t - \Delta u + (J_m u) \cdot \nabla u + \nabla p = f,\ \operatorname{div} u = 0$$

together with $u_{|_{\partial\Omega}} = 0$, $u(0) = u_0$ on $\Omega \times [0, T)$.

In our case (1.12) let $v = v_m$, $m \in \mathbb{N}$, be a weak solution of the *approximate perturbed Navier-Stokes system*

$$\begin{aligned} v_t - \Delta v + (J_m v + E) \cdot \nabla(v + E) + \nabla\tilde{p} = f_1, \quad \operatorname{div} v = 0 \\ v_{|_{\partial\Omega}} = 0, \qquad v(0) = v_0 \end{aligned} \tag{3.1}$$

where $v_0 \in L^2_\sigma(\Omega)$, $f_1 = \operatorname{div} F_1$, $F_1 \in L^2\big(0, T; L^2(\Omega)\big)$, and E results from (1.4) as a (very) weak solution satisfying (1.8). However, as already explained, we will only need that the vector field E satisfies (1.8) without referring to (1.4). Then

$$v = v_m \in L^\infty_{\text{loc}}\big([0, T); L^2_\sigma(\Omega)\big) \cap L^2_{\text{loc}}\big([0, T); W_0^{1,2}(\Omega)\big) \tag{3.2}$$

is called a (Leray-Hopf type) weak solution of (3.1) in $\Omega \times [0, T)$ if the relation

$$\begin{aligned} -\langle v, w_t\rangle_{\Omega,T} + \langle \nabla v, \nabla w\rangle_{\Omega,T} - \langle (J_m v + E)(v + E), \nabla w\rangle_{\Omega,T} \\ = \langle v_0, w(0)\rangle_\Omega - \langle F_1, \nabla w\rangle_{\Omega,T} \end{aligned} \tag{3.3}$$

is satisfied for every $w \in C_0^\infty([0, T; C^\infty_{0,\sigma}(\Omega))$ and the energy inequality

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2\, d\tau \le \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F_1 - (J_m v + E)E, \nabla v\rangle_\Omega\, d\tau, \tag{3.4}$$

$0 \le t < T$, holds.

Lemma 3.1. *Let $v_0 \in L^2_\sigma(\Omega)$, $f_1 = \operatorname{div} F_1$, $F_1 \in L^2(0, T; L^2(\Omega))$, and E satisfying* (1.8) *be given. If $s = \infty$, suppose the smallness assumption* (1.11) *on E as well. Then there exists some $T' = T'(v_0, F_1, E, m) \in (0, \min(1, T)]$ such that* (3.1) *has a unique weak solution $v = v_m$ on $\Omega \times (0, T')$, i.e., v satisfies* (3.2)–(3.4) *with T replaced by T'.*

Proof. Assume that $v = v_m$ is a solution of (3.1) on $\Omega \times (0, T')$, $0 < T' \le 1$. Hence v is contained in the space

$$X_{T'} := L^\infty\big(0, T'; L^2_\sigma(\Omega)\big) \cap L^2\big(0, T'; W_0^{1,2}(\Omega)\big)$$

with

$$\|v\|_{X_{T'}} := \|v\|_{2,\infty;T'} + \|A^{\frac{1}{2}}v\|_{2,2;T'} < \infty.$$

Then we obtain for any $0 < T' \leq \min(1,T)$, using Hölder's inequality, the properties (2.1)–(2.3) and (2.5), the following estimates for $(J_m v + E)(v + E)$ with some constant $c = c(\Omega) > 0$:

$$\begin{aligned}
\|(J_m v)v\|_{2,2;T'} &\leq \|J_m v\|_{6,4;T'}\,\|v\|_{3,4;T'} \leq c\,\|A^{\frac{1}{2}}J_m v\|_{2,4;T'}\,\|v\|_{X_{T'}}\\
&\leq c\,m\,\|v\|_{2,4;T'}\,\|v\|_{X_{T'}} \leq c\,m\,(T')^{1/4}\,\|v\|^2_{X_{T'}},\\
\|(J_m v)E\|_{2,2;T'} &\leq \|J_m v\|_{4,4;T'}\,\|E\|_{4,4;T'} \leq c\,\|J_m v\|_{6,4;T'}\,\|E\|_{4,4;T'}\\
&\leq c\,m\,(T')^{1/4}\,\|v\|_{X_{T'}}\,\|E\|_{4,4;T'},\\
\|E\,v\|_{2,2;T'} &\leq \|E\|_{q,s;T'}\,\|v\|_{(\frac{1}{2}-\frac{1}{q})^{-1},(\frac{1}{2}-\frac{1}{s})^{-1},T'}\\
&\leq c\,\|E\|_{q,s;T'}\,\|v\|_{X_{T'}}.
\end{aligned}$$

Since $\|EE\|_{2,2;T'} \leq \|E\|^2_{4,4;T'}$, we proved the estimate

$$\begin{aligned}
\|(J_m v + E)(v + E)\|_{2,2;T'} \leq{}& c\,m(T')^{1/4}\,\|v\|_{X_{T'}}\big(\|E\|_{4,4;T'} + \|v\|_{X_{T'}}\big)\\
&+ \|E\|^2_{4,4;T'} + c\,\|E\|_{q,s;T'}\,\|v\|_{X_{T'}}.
\end{aligned} \tag{3.5}$$

Obviously, (3.5) also holds in the limit case $s = \infty$, $q = 3$.

With the definition

$$\hat{F}_1(v) = F_1 - (J_m v + E)(v + E)$$

we write the system (3.1) in the form

$$\begin{aligned}
v_t - \Delta v + \nabla p = \operatorname{div}\hat{F}_1(v), \quad \operatorname{div} v = 0\\
v = 0 \text{ on } \partial\Omega, \qquad v(0) = v_0.
\end{aligned}$$

Since $v_0 \in L^2_\sigma(\Omega)$ and $\hat{F}_1(v) \in L^2(0,T';L^2(\Omega))$, we apply classical L^2-results [20, Ch. IV] on weak solutions of the instationary Stokes system to get that $v \in C^0([0,T');L^2_\sigma(\Omega))$ and satisfies the fixed point relation

$$v = \mathcal{F}_{T'}(v) \text{ in } X_{T'}, \tag{3.6}$$

where

$$\big(\mathcal{F}_{T'}(v)\big)(t) = e^{-tA}v_0 - \int_0^t A^{\frac{1}{2}}\,e^{-(t-\tau)A}\,A^{-\frac{1}{2}}P_2\mathrm{div}\;\hat{F}_1(v)(\tau)\,d\tau\,;$$

see [20, III.2.6] concerning the operator $A^{-\frac{1}{2}}P_2\mathrm{div}$. Moreover, v satisfies even an energy equality for $t \in [0,T')$ instead of the energy inequality (3.4), and, by (3.5), the energy estimate

$$\|\mathcal{F}_{T'}(v)\|_{X_{T'}} \leq a\,\|v\|^2_{X_{T'}} + b\,\|v\|_{X_{T'}} + d \tag{3.7}$$

where

$$a = c\,m(T')^{1/4},\; b = c\,\|E\|_{q,s;T'} + c\,m(T')^{1/4}\,\|E\|_{4,4;T'}\,, \qquad d = c\big(\|v_0\|_2 + \|E\|^2_{4,4;T'} + \|F_1\|_{2,2;T'}\big) \tag{3.8}$$

with constants $c > 0$ independent of v, m and T'.

By analogy, we get for two elements $v_1, v_2 \in X_{T'}$ the estimate

$$\begin{aligned}\|\mathcal{F}_{T'}(v_1) - \mathcal{F}_{T'}(v_2)\|_{X_{T'}} & \\ &\le c\,m(T')^{1/4}\,\|v_1 - v_2\|_{X_{T'}}\big(\|v_1\|_{X_{T'}} + \|v_2\|_{X_{T'}} + \|E\|_{4,4;T'}\big) \\ &\quad + c\,\|v_1 - v_2\|_{X_{T'}}\,\|E\|_{q,s;T'} \\ &\le \|v_1 - v_2\|_{X_{T'}}\big(a\,(\|v_1\|_{X_{T'}} + \|v_2\|_{X_{T'}}) + b\big).\end{aligned} \tag{3.9}$$

Up to now, to derive (3.7), (3.9), we considered a given solution $v = v_m \in X_{T'}$ of (3.1).

In the next step, we treat (3.6) as a fixed point problem in $X_{T'}$. Assuming the smallness condition

$$4\,a\,d + 2\,b < 1 \tag{3.10}$$

we easily see that the quadratic equation $y = ay^2 + by + d$ has a minimal positive root y_1 which also satisfies $2ay_1 + b < 1$. Hence, under the assumption (3.10), $\mathcal{F}_{T'}$ maps the closed ball $B_{T'} = \{v \in X_{T'} : \|v\|_{X_{T'}} \le y_1\}$ into itself. Moreover, (3.9), (3.10) imply that $\mathcal{F}_{T'}$ is a strict contradiction on $B_{T'}$. Now Banach's Fixed Point Theorem yields the existence of a unique solution $v = v_m \in B_{T'}$ of the fixed point problem (3.6). This solution is a weak solution of the approximate perturbed Navier-Stokes system (3.1). Moreover, v satisfies an energy identity and $v \in C^0([0,T'); L^2_\sigma(\Omega))$.

To satisfy the smallness assumption (3.10) (for fixed $m \in \mathbb{N}$), it suffices in view of (3.8) to choose $T' \in (0, \min(1,T))$ sufficiently small in the case that $s < \infty$. However, if $s = \infty$, looking at the term $\|E\|_{q,s;T'}$ in (3.8) we have to assume that $\|E\|_{C^0([0,T);L^3(\Omega))}$ is sufficiently small.

Finally, we show that the solution just found, $v = v_m$, which is unique in $B_{T'}$, is even unique in $X_{T'}$. Indeed, consider any solution $\tilde{v} \in X_{T'}$ of (3.1). Then there exists $0 < T^* \le \min(1,T')$ such that $\|\tilde{v}\|_{X_{T^*}} \le y_1$, and the estimate (3.9) with T' replaced by $T^* \in (0, \min(1,T))$ implies that $\|v - \tilde{v}\|_{X_{T^*}} = \|\mathcal{F}_{T^*}(v) - \mathcal{F}_{T^*}(\tilde{v})\|_{X_{T^*}} \le \|v - \tilde{v}\|_{X_{T^*}}(2ay_1 + b)$. Since $2ay_1 + b < 4ad + b < 1$, we conclude that $v = \tilde{v}$ on $[0,T^*]$. When $T^* < T'$, we repeat this step finitely many times to see that $v = \tilde{v}$ on $[0,T']$. $\square$

To prove that the approximate solution $v = v_m$ does not only exist on an interval $[0,T')$ where $T' = T'(m)$, but on $[0,T)$, and to pass to the limit $m \to \infty$, we need a global (in time) and uniform (in $m \in \mathbb{N}$) energy estimate of v_m.

Lemma 3.2. *Let $v = v_m$, $m \in \mathbb{N}$, be a weak solution of the approximate perturbed Navier-Stokes system* (3.1) *on some interval $[0,T') \subseteq [0,T)$ where $v_0 \in L^2_\sigma(\Omega)$,*

$f_1 = \operatorname{div} F_1$, $F_1 \in L^2(0,T;L^2(\Omega))$, and let E satisfy (1.8)*. If $2 < s < \infty$, $3 < q < \infty$, then v satisfies the energy estimate*

$$\begin{aligned} &\|v\|^2_{2,\infty;t} + \|\nabla v\|^2_{2,2;t} \\ &\qquad \le \left(\|v_0\|_2^2 + 4\,\|F_1\|^2_{2,2;t} + 4\,\|E\|^4_{4,4;t}\right) \exp\left(c\,\|E\|^s_{q,s;t}\right) \end{aligned} \tag{3.11}$$

for all $t \in [0,T')$ where $c = c(\Omega,q) > 0$ is a constant. If $s = \infty$, $q = 3$, under the smallness assumption (1.11)*, v satisfies the energy estimate*

$$\|v(t)\|_2^2 + \|\nabla v\|^2_{2,2;t} \le \|v_0\|_2^2 + 4\,\|E\|^4_{4,4;t} + 4\,\|F_1\|_{2,2;t}. \tag{3.12}$$

Proof. In view of the energy inequality (3.4) we have to estimate the crucial term $\int_0^t \langle (J_m v + E)E, \nabla v\rangle_\Omega \, d\tau$. By Hölder's inequality, (2.1)–(2.3) and (2.5), we get

$$\begin{aligned} \left|\int_0^t \langle (J_m v)E, \nabla v\rangle_\Omega \, d\tau\right| &\le \int_0^t \|J_m v\|_{(\frac{1}{2}-\frac{1}{q})^{-1}} \|E\|_q \,\|\nabla v\|_2 \, d\tau \\ &\le c \int_0^t \|J_m v\|_2^\alpha \,\|\nabla J_m v\|_2^{1-\alpha}\,\|E\|_q \,\|\nabla v\|_2 \, d\tau \\ &\le c \int_0^t \|v\|_2^\alpha \,\|E\|_q \,\|\nabla v\|_2^{2-\alpha} \, d\tau \end{aligned} \tag{3.13}$$

where $\alpha = 1 - \frac{3}{q} = \frac{2}{s}$, cf. (2.3). Hence, if $2 < s < \infty$, by Young's inequality

$$\left|\int_0^t \langle (J_m v)E, \nabla v\rangle_\Omega \, d\tau\right| \le \frac{1}{8}\,\|\nabla v\|^2_{2,2;t} + c \int_0^t \|v\|_2^2 \,\|E\|_q^s \, d\tau \tag{3.14}$$

with a constant $c = c(q,\Omega) > 0$. In the case $s = \infty$, $q = 3$, we have $\alpha = 0$ and get

$$\left|\int_0^t \langle (J_m v)E, \nabla v\rangle_\Omega \, d\tau\right| \le c\,\|E\|_{3,\infty;t}\,\|\nabla v\|^2_{2,2;t}. \tag{3.15}$$

Moreover,

$$\left|\int_0^t \langle EE, \nabla v\rangle_\Omega \, d\tau\right| \le \int_0^t \|E\|_4^2\,\|\nabla v\|_2 \, d\tau \le \frac{1}{8}\,\|\nabla v\|^2_{2,2;t} + 2\|E\|^4_{4,4;t}\,; \tag{3.16}$$

the term $\int_0^t \langle F, \nabla v\rangle_\Omega \, d\tau$ is treated similarly. Inserting these estimates into (3.4) we are led to the estimate ($2 < s < \infty$)

$$\|v(t)\|_2^2 + \|\nabla v\|^2_{2,2;t} \le \|v_0\|_2^2 + 4\,\|E\|^4_{4,4;t} + 4\,\|F_1\|^2_{2,2;t} + c\int_0^t \|v\|_2^2\,\|E\|_q^s \, d\tau. \tag{3.17}$$

Then Gronwall's Lemma proves (3.11).

To get a similar result when $s = \infty$ we have to assume in view of (3.15) with $\alpha = 0$ that $c\,\|E\|_{3,\infty;t} \le \frac{1}{4}$; then we immediately get (3.12). □

Lemma 3.3. *Under the assumptions of Lemma* 3.1 *for every* $m \in \mathbb{N}$ *there exists a unique weak solution* $v = v_m$ *of* (3.1) *on* $[0,T)$. *This solution* $v \in C^0([0,T); L^2_\sigma(\Omega))$ *satisfies in addition to the energy inequality* (3.4) *the strong energy identity*

$$\frac{1}{2}\|v(t)\|_2^2 + \int_{t_0}^t \|\nabla v\|_2^2\,d\tau = \frac{1}{2}\|v(t_0)\|_2^2 - \int_{t_0}^t \langle F_1 - (J_m v + E)E, \nabla v\rangle_\Omega\,d\tau \quad (3.18)$$

for all $t_0 \in [0,T)$ *and* $t_0 < t < T$, *and it holds*

$$\frac{d}{dt}\left(\frac{1}{2}\|v(t)\|_2^2\right) + \|\nabla v(t)\|_2^2 = -\langle F_1 - (J_m v + E)E, \nabla v\rangle_\Omega\,(t) \quad (3.19)$$

in the sense of distributions on $[0,T)$.

Proof. Let $[0,T^*) \subseteq [0,T)$ be the largest interval of existence of $v = v_m$, and assume that $T^* < T$. Since $v \in C^0([0,T^*); L^2_\sigma(\Omega))$, we find $0 < T_0 < T^*$ arbitrarily close to T^* with $v(T_0) \in L^2_\sigma(\Omega)$ which will be taken as initial value at T_0 in (3.1) in order to extend v beyond T_0. Since the length δ of the interval of existence $[T_0, T_0+\delta)$ of this unique extension depends only on $\|v(T_0)\|_2$ and $\|F_1\|_{2,2;T}$, $\|E\|_{4,4;T}$, $\|E\|_{q,s;T}$ by Lemma 3.1, we see that v can be extended beyond T^* in contradiction with the assumption.

Since $v = v_m$ in Lemma 3.1 satisfies an energy identity instead of only an energy inequality, $v = v_m$ will satisfy the strong energy equality (3.18) on $[0,T)$. Since both integrands in (3.18) are L^1-functions, the corresponding integrals are absolutely continuous in t; hence we get the differential identity (3.19) in the sense of distributions. □

4. Proof of Theorem 1.3

Let (v_m) denote the sequence of approximate solutions on $[0,T)$ constructed in Section 3 and let $0 < T' \le T$ be finite. By (3.11) we find a constant $c > 0$ such that

$$\|v_m\|_{2,\infty;T'} + \|\nabla v_m\|_{2,2;T'} \le c \quad \text{for all } m \in \mathbb{N}. \quad (4.1)$$

Hence there exists a subsequence of (v_m) which for simplicity will again be denoted by (v_m) with the following properties:

There exists a vector field $v \in L^\infty(0,T'; L^2_\sigma(\Omega)) \cap L^2(0,T'; H_0'(\Omega))$:

$$\begin{array}{rcll} v_m \overset{*}{\rightharpoonup} v & \text{in} & L^\infty(0,T'; L^2_\sigma(\Omega)) & \text{(weakly-}*) \\ v_m \rightharpoonup v & \text{in} & L^2(0,T'; H_0^1(\Omega)) & \text{(weakly)} \\ v_m \to v & \text{in} & L^2(0,T'; L^2(\Omega) & \text{(strongly)} \\ v_m(t) \to v(t) & \text{in} & L^2(\Omega) \text{ for a.a. } t \in [0,T) & \text{(strongly).} \end{array} \quad (4.2)$$

We note that the third property is based on compactness arguments just as for the classical Navier-Stokes system and that the fourth property is a well-known consequence of the strong convergence in $L^2(0,T'; L^2(\Omega))$. Moreover, for all $t \in [0,T)$

$$\begin{aligned} \|\nabla v\|_{2,2;t} &\leq \liminf_{m\to\infty} \|\nabla v_m\|_{2,2;t}, \\ \|v(t)\|_2 &\leq \liminf_{m\to\infty} \|v_m(t)\|_2. \end{aligned} \tag{4.3}$$

By Hölder's inequality, (4.1) and (4.2) we also conclude (after extracting a further subsequence again denoted by (v_m)) that

$$\begin{aligned} v_m &\rightharpoonup v \quad \text{in} \quad L^{s_1}(0,T';L^{q_1}(\Omega)),\ \tfrac{2}{s_1}+\tfrac{3}{q_1}=\tfrac{3}{2},\ 2\leq s_1,q_1<\infty \\ v_m v_m &\rightharpoonup vv \quad \text{in} \quad L^{s_2}(0,T';L^{q_2}(\Omega)),\ \tfrac{2}{s_2}+\tfrac{3}{q_2}=3,\ 1\leq s_1,q_1<\infty \\ v_m\cdot\nabla v_m &\rightharpoonup v\cdot\nabla v \quad \text{in} \quad L^{s_3}(0,T';L^{q_3}(\Omega)),\ \tfrac{2}{s_3}+\tfrac{3}{q_3}=4,\ 1\leq s_1,q_1<\infty. \end{aligned} \tag{4.4}$$

When passing to the limit in the weak formulation (3.3) only some terms in $\langle (J_m v_m+E)(v_m+E), \nabla w\rangle_{\Omega,T'}$ need a special consideration. Concerning $(J_m v_m)v_m$ we first note that

$$J_m v_m = J_m(v_m - v) + J_m v \to v \text{ in } L^2(0,T';L^2(\Omega));$$

hence, as $m\to\infty$, by $(4.4)_1$ and Hölder's inequality

$$\begin{aligned} &(J_m v_m)v_m - vv = (J_m v_m - v_m)v_m + v_m\cdot(v_m - v) + (v_m - v)v \rightharpoonup 0 \\ &\text{in } L^{s_4}(0,T';L^{q_4}(\Omega)), \quad \tfrac{2}{s_4}+\tfrac{3}{q_4}=4, \quad 1\leq s_4,\ q_4\leq 2. \end{aligned} \tag{4.5}$$

Proceeding similarly with all other terms we prove that v is a weak solution of the perturbed Navier-Stokes system satisfying Definition 1.2 (i),(ii).

It remains to show that v satisfies the energy inequality (1.13). To this aim we consider the energy equality (3.18) for v_m and those $t_0\in[0,T)$ where the strong convergence $v_m(t_0)\to v(t_0)$ in $L^2(\Omega)$ holds, see $(4.2)_4$. By (4.3) and $(4.2)_4$ the first three terms in (3.18) pose no problems for $m\to\infty$. The same holds true for the terms $\langle F,\nabla v_m\rangle_\Omega$ and $\langle EE,\nabla v_m\rangle$. To treat the remaining term we have to prove that

$$\int_{t_0}^t \langle (J_m v_m)E, \nabla v_m\rangle_\Omega \, d\tau \to \int_{t_0}^t \langle vE,\nabla v\rangle_\Omega \, d\tau \tag{4.6}$$

as $m\to\infty$.

Since $E\in L^s(t_0,t;L^q(\Omega))$ and $C_0^\infty((t_0,t)\times\Omega)$ is dense in $L^s(t_0,t;L^q(\Omega))$ for $1\leq s,\ q<\infty$, it suffices to show (4.6) for any smooth $\tilde{E}$ and that the sequence $((J_m v_m)\nabla v_m)$ is bounded in $L^{s'}(t_0,t;L^{q'}(\Omega))$. Indeed, for $\tilde{E}\in C_0^\infty((t_0,t)\times\Omega)$

$$\begin{aligned} &\int_{t_0}^t \langle (J_m v_m)\tilde{E},\nabla v_m\rangle_\Omega - \langle v\tilde{E},\nabla v\rangle_\Omega)d\tau \\ &\quad = -\int_{t_0}^t \langle (J_m v_m)v_m - vv, \nabla\tilde{E}\rangle_\Omega \, d\tau \ \to \ 0 \quad \text{as} \quad m\to\infty \end{aligned}$$

due to (4.5). Moreover,

$$\|(J_m v_m)\nabla v_m\|_{q',s';t} \leq \|\nabla v_m\|_{2,2;t}\, \|J_m v_m\|_{(\frac{1}{q'}-\frac{1}{2})^{-1},(\frac{1}{s'}-\frac{1}{2})^{-1};t}$$

is uniformly bounded in $m \in \mathbb{N}$ by (4.1) and since $2(\frac{1}{s'} - \frac{1}{2}) + 3(\frac{1}{q'} - \frac{1}{2}) = \frac{3}{2}$ where $\frac{2}{s} + \frac{3}{q} = 1, 2 \leq s \leq \infty, 3 \leq q \leq \infty$. In the case $E \in C^0((0,T); L^3(\Omega))$ the same argument applies; note that only for this argument we needed $E \in C^0$ instead of $E \in L^\infty$. Summarizing the previous ideas we prove (4.6). □

Proof of Corollary 1.5. Since the proof is based on a differential inequality and not on Gronwall's Lemma applied to an integral inequality we have to consider the sequence of approximate solutions (v_m) first of all. By the differential equation (3.19) for $v = v_m$ we get the estimate

$$\frac{1}{2}\frac{d}{dt}\|v_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2 \leq (\|F_1\|_2 + \|E\|_4^2)\|\nabla v_m\|_2 + \langle (J_m v_m)E, \nabla v_m\rangle_\Omega$$

where the last term due to the assumption (1.15) can be estimated as follows:

$$|((J_m v_m)E, \nabla v_m)| \leq \frac{1}{4}\|\nabla J_m v_m\|_2 \|\nabla v_m\|_2 \leq \frac{1}{4}\|\nabla v_m\|_2^2.$$

Then Young's inequality and an absorption argument lead to the estimate

$$\frac{d}{dt}\|v_m(t)\|_2^2 + \|\nabla v_m(t)\|_2^2 \leq 4(\|F_1\|_2^2 + \|E\|_4^4)(t) \tag{4.7}$$

for a.a. $t \geq 0$. Let $\mu_0 > 0$ denote the smallest eigenvalue of the Dirichlet Laplacian on Ω. Then $\|\nabla v_m(t)\|_2^2 \geq \mu_0 \|v_m(t)\|_2^2$ for a.a. $t > 0$, and we get that

$$\frac{d}{dt}\|v_m(t)\|_2^2 + \mu_0\|v_m(t)\|_2^2 \leq 4(\|F_1\|_2^2 + \|E\|_4^4)(t).$$

This estimate together with the assumption (1.17) immediately implies that

$$\begin{aligned} \|v_m(t)\|_2^2 &\leq e^{-\mu_0 t}\|v_0\|_2^2 + e^{-\mu_0 t}\int_0^t e^{\mu_0\tau} 4(\|F_1\|_2^2 + \|E\|_4^4)d\tau \\ &\leq e^{-\mu_0 t}\|v_0\|_2^2 + \tfrac{4}{\mu_0}(\|F_1\|^2_{L^\infty(0,\infty;L^2(\Omega))} + \|E\|^4_{L^\infty(0,\infty;L^4(\Omega))}). \end{aligned} \tag{4.8}$$

Hence v_m is uniformly bounded on $(0,\infty)$ with a bound independent of $m \in \mathbb{N}$. By the pointwise convergence property $(4.2)_4$ $v(t)$ satisfies the same bound, first of all for a.a. $t \geq 0$, but due to its weak continuity property in $L^2(\Omega)$ even for all $t \geq 0$.

Moreover, (4.8) applied to v yields a time $T_0 = T_0(\|v_0\|_2)$ such that $\|v(t)\|_2^2$ is bounded by κ^* for all $t \geq T_0$ where κ^* is chosen larger than the last two terms in (4.8).

Finally, integrating (4.7) with respect to time and passing to the limit with $m \to \infty$, we get the estimate (1.18). □

5. Construction of the vector field *E*

To apply Theorem 1.3 and Corollary 1.5 and to find solutions u of the Navier-Stokes system (1.1) in the form $u = v + E$ we have to construct a suitable vector field E solving (1.4); the solution should satisfy the assumptions (1.8) to apply Theorem 1.3 and (1.15) to apply Corollary 1.5, respectively. First we consider very weak solutions E of (1.8), see [2]–[5], for given suitable data g, E_0 and f_0. For their definition we introduce the space of initial values, $\mathcal{J}_\sigma^{q,s}(\Omega)$, by

$$\mathcal{J}_\sigma^{q,s}(\Omega) = \Big\{u_0 \in \mathcal{D}(A_{q'})' : \|u_0\|_{\mathcal{J}_\sigma^{q,s}} = \Big(\int_0^\infty \|A_q e^{-\tau A_q}(A_q^{-1}P_q u_0)\|_q^s \, d\tau\Big)^{\frac{1}{s}} < \infty\Big\}. \tag{5.1}$$

Here, $\mathcal{D}(A_{q'})$ is equipped with the graph norm or equivalently with the norm $\|A_{q'} \cdot \|_{q'}$, and the term $A_q^{-1}P_q u_0$ for $u_0 \in \mathcal{D}(A_{q'})'$ denotes the unique element $u^* \in L^q_\sigma(\Omega)$ such that $\langle A_q^{-1}P_q u, \varphi\rangle = \langle u^*, \varphi\rangle = \langle u, P_{q'}A_{q'}^{-1}\varphi\rangle$ for all $\varphi \in L_\sigma^{q'}$.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{1,1}$, let $0 < T \le \infty$ and let $1 < s, q, r < \infty$ satisfy $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$. Assume that $f_0 = \operatorname{div} F_0$,*

$$F_0 \in L^s(0,T; L^r(\Omega)),\ g \in L^s(0,T; W^{-\frac{1}{q},q}(\partial\Omega)) \tag{5.2}$$

such that $\langle g(t), N\rangle_{\partial\Omega} = 0$ for a.a. $t \in (0,T)$ and let $E_0 \in \mathcal{J}_\sigma^{q,s}(\Omega)$. Then the Stokes system (1.4) *has a unique very weak solution*

$$E \in L^s(0,T; L^q(\Omega)) \tag{5.3}$$

in the sense that for all test functions $w \in C_0^1([0,T); C^2_{0,\sigma}(\overline{\Omega}))$

$$\begin{aligned} &-\langle E, w_t\rangle_{\Omega,T} - \langle E, \Delta w\rangle_{\Omega,T} = \langle E_0, w(0)\rangle_\Omega - \langle F_0, \nabla w\rangle_{\Omega,T} - \langle g, N\cdot\nabla w\rangle_{\partial\Omega,T} \\ &\operatorname{div} E = 0 \ \ \textit{in} \ \ \Omega \times (0,T),\ E\cdot N = g\cdot N \ \ \textit{on} \ \ \partial\Omega \times (0,T). \end{aligned} \tag{5.4}$$

This solution satisfies the a priori estimate

$$\|E\|_{q,s;T} \le c\,(\|F_0\|_{r,s;T} + \|g\|_{L^s(0,T;W^{-\frac{1}{q},q}(\partial\Omega))} + \|u_0\|_{\mathcal{J}_\sigma^{q,s}}) \tag{5.5}$$

with a constant $c = c(q,r,s,\Omega) > 0$ independent of T and of the data.

For more details on very weak solutions we refer to [1]–[5]. See, e.g., [5, Chapters 2.1, 2.3], for the well-definedness of all terms in (5.1) and (5.4); Proposition 5.1 is a special case of [5, Theorem 2.14] and a remark in [5, §1.3] on the extension of results in [3], [4] where $\partial\Omega \in C^{2,1}$ to the case $\partial\Omega \in C^{1,1}$. Note that Serrin's condition $\frac{2}{s} + \frac{3}{q} = 1$ is not needed in the linear theory. Moreover, $s = \infty$ is not included in Proposition 5.1; hence the case $s = \infty$ will not be dealt with in the next result.

Corollary 5.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{1,1}$, let $0 < T \le \infty$ and let $1 < s, q, r < \infty$ satisfy $\frac{2}{s} + \frac{3}{q} = 1$, $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$. Assume that $f_0 = \operatorname{div} F_0$,*

$$\begin{aligned} &F_0 \in L^s(0,T; L^r(\Omega)) \cap L^4(0,T; L^{\frac{12}{7}}(\Omega)), \\ &g \in L^s(0,T; W^{-\frac{1}{q},q}(\partial\Omega)) \cap L^4(0,T; W^{-\frac{1}{4},4}(\partial\Omega)), \\ &E_0 \in \mathcal{J}_\sigma^{q,s}(\Omega) \cap \mathcal{J}_\sigma^{4,4}(\Omega) \end{aligned} \tag{5.6}$$

such that $\langle g(t), N\rangle_{\partial\Omega} = 0$ *for a.a.* $t \in (0,T)$. *Then the inhomogeneous Stokes system* (1.4) *has a unique very weak solution* E *satisfying* (1.8)*, i.e.,*

$$E \in L^s(0,T;L^q(\Omega)) \cap L^4(0,T;L^4(\Omega)), \tag{5.7}$$

and the a priori estimate

$$\begin{aligned}\|E\|_{q,s;T} + \|E\|_{4,4;T} \le c\,(&\|F_0\|_{q,s;T} + \|F_0\|_{\frac{12}{7},4;T}\\ &+\|g\|_{L^s(0,T;W^{-\frac{1}{q},q}(\partial\Omega))} + \|g\|_{L^4(0,T;W^{-\frac{1}{4},4}(\partial\Omega))}\\ &+\|u_0\|_{\mathcal{J}_\sigma^{q,s}} + \|u_0\|_{\mathcal{J}_\sigma^{4,4}})\end{aligned} \tag{5.8}$$

holds with a constant $c = c(q,r,s,\Omega) > 0$ *independent of* T.

Proof. We apply Proposition 5.1 with the exponents s,q,r and $4,4,\frac{12}{7}$. Since the very weak solution E of (1.4) in [3, 4, 5] is constructed in a finite number of steps where each of them yields the same result for s,q,r and for $4,4,\frac{12}{7}$, it is easily seen that the unique solution E satisfies (5.7), (5.8). □

Remark 5.3. (i) In the case $4 \le s \le 8$, $4 \le q \le 6$ and T finite the $L^s(L^q)$-conditions in (5.6) imply the $L^4(L^4)$-conditions; then (5.6)–(5.8) simplify considerably.

(ii) For the system (1.1) consider data $f = \operatorname{div} F$, $F \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and boundary data g as in $(5.6)_3$ satisfying $\langle g(t), N\rangle_{\partial\Omega} = 0$ for a.a. $t \in (0,T)$. Then solve (1.4) with data $f_0 = 0$, $E_0 = 0$ and g to get a (unique) very weak solution E satisfying (5.7) and the a priori estimate

$$\|E\|_{q,s;T} + \|E\|_{4,4;T} \le c\left(\|g\|_{L^s(0,T;W^{-\frac{1}{q},q}(\partial\Omega))} + \|g\|_{L^4(0,T;W^{-\frac{1}{4},4}(\partial\Omega))}\right).$$

Next, by Theorem 1.3, we find a weak solution v of the perturbed Navier-Stokes system (1.12) with data $f_1 = f = \operatorname{div} F$, $F_1 = F$, and $v_0 = u_0$ satisfying (1.13), (1.14). Then $u = v + E$ is a weak solution of (1.1) split into a weak and a very weak part, v and E. It is an easy exercise to write down a corresponding energy estimate for u in terms of u_0, f and g only.

(iii) Assuming more regularity on the boundary data g better properties of $u = v + E$ can be achieved; we refer to [3, 4] and to the forthcoming paper [6] for such results.

In the second part of this Section we consider the assumption (1.15) and Corollary 1.5. Assume that the bounded domain $\Omega \subset \mathbb{R}^3$ with $\partial\Omega = \bigcup_{j=0}^{L} \Gamma_j \in C^{1,1}$ has boundary components $\Gamma_0,\dots,\Gamma_L$ with Γ_0 being the "outer" boundary of Ω and $\Gamma_j, 1 \le j \le L$, being the boundary of "holes" Ω_j'. Further, let the boundary data g with $g(t) \in W^{\frac{1}{2},2}(\partial\Omega)$ for a.a. $t \in (0,T)$ satisfy the restricted flux condition

$$\int_{\Gamma_j} g(t)\cdot N\,do = 0,\; 0 \le j \le L. \tag{5.9}$$

Then, due to a construction in [14], there exists a solenoidal extension $E = E_\varepsilon \in W^{1,2}(\Omega)$ of g for a.a. $t \in (0,T)$ satisfying (1.16) (for arbitrary but fixed $\varepsilon > 0$ and for a.a. t). However, we do need also an estimate of E and E_t in terms of g and g_t, respectively.

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain as above and let the boundary function*

$$g \in L^\infty(0,\infty; W^{\frac{1}{2},2}(\partial\Omega)),\ g_t \in L^\infty(0,\infty; W^{-\frac{1}{2},2}(\partial\Omega)) \tag{5.10}$$

satisfy the restricted flux condition (5.9)*. Then there exists an extension*

$$E \in L^\infty(0,\infty; W^{1,2}(\Omega)),\ E_t \in L^\infty(0,\infty; W^{-1,2}(\Omega)) \tag{5.11}$$

of g satisfying inequality $(1.15)_2$ *and the a priori estimate*

$$\begin{aligned} \|E\|_{L^\infty(0,\infty;W^{1,2}(\Omega))} &\le c\,\|g\|_{L^\infty(0,\infty;W^{\frac{1}{2},2}(\partial\Omega))} \\ \|E_t\|_{L^\infty(0,\infty;W^{-1,2}(\Omega))} &\le c\,\|g_t\|_{L^\infty(0,\infty;W^{-\frac{1}{2},2}(\partial\Omega))} \end{aligned} \tag{5.12}$$

with a constant $c = c(\Omega) > 0$.

Proof. We follow the ideas of E. Hopf [14] as described in [8, 11] to find an extension E of g written as the curl of a suitable vector potential and defined by a bounded linear operator $g \mapsto E$.

Ignoring $t \in (0,T)$ for a moment we consider $g \in W^{1/2,2}(\partial\Omega)$ satisfying the restricted flux condition as in (5.9). Then we use the theory of very weak solutions, see [3]–[5], to find a solution $u_j \in L^2(\Omega_j)$, $1 \le j \le L$, of the stationary Stokes system

$$-\Delta u_j + \nabla p_j = 0,\ \operatorname{div} u_j = 0 \text{ in } \Omega_j,\ u = g \text{ on } \partial\Omega_j \tag{5.13}$$

for each hole Ω_j, $1 \le j \le L$. By definition

$$\begin{aligned} -(u_j, \Delta w)_{\Omega_j} + \langle g, N\cdot\nabla w\rangle_{\partial\Omega_j} = 0 \quad &\text{for all } w \in C^2_{0,\sigma}(\overline{\Omega}_j) \\ \operatorname{div} u_j = 0 \text{ in } \Omega_j, \quad &u_j \cdot N = g\cdot N \text{ on } \partial\Omega_j, \end{aligned}$$

and [4, Theorem 3], yields the existence of a unique very weak solution u_j satisfying the a priori estimate

$$\|u_j\|_{2,\Omega_j} \le c\,\|g\|_{W^{-1/2,2}(\partial\Omega_j)}\,;$$

here the necessary compatibility condition $\langle g, N\rangle_{\partial\Omega_j} = 0$ is fulfilled due to (5.9) for each j. By analogy, we find a very weak solution $u_0 \in W^{1,2}(\Omega)$ such that

$$-\Delta u_0 + \nabla p_0 = 0,\ \operatorname{div} u_0 = 0 \text{ in } \Omega,\ u = g \text{ on } \partial\Omega,$$

again taking into account (5.9), and get that

$$\|u_0\|_{2,\Omega} \le c\,\|g\|_{W^{-1/2,2}(\partial\Omega)}.$$

Finally, we consider $A = B_R \backslash (\overline{\Omega} \cup \bigcup_{j=1}^L \overline{\Omega}_j)$ for a ball of radius R and center 0 such that $\overline{\Omega} \subset B_R$, and find a unique very weak solution $u_A \in W^{1,2}(A)$ of the Stokes system

$$-\Delta u_A + \nabla p_A = 0,\ \operatorname{div} u_A = 0 \text{ in } A,\ u|_{\Gamma_0} = g,\ u|_{\partial B_R} = 0$$

since $\langle g, N\rangle_{\Gamma_0} = 0$ by (5.9). Moreover, $\|u_A\|_{2,A} \le c\,\|g\|_{W^{-1/2,2}(\partial\Omega)}$.

Since $g \in W^{1/2,2}(\partial\Omega) \subset W^{-1/2,2}(\partial\Omega)$, the very weak solutions u_j, u_0, u_A constructed so far are also weak solutions, and, in particular, $u_j \in W^{1,2}(\Omega_j)$ and $\|u_j\|_{W^{1,2}(\Omega_j)} \le c\,\|g\|_{W^{1/2,2}(\partial\Omega_j)}$; for this regularity argument see [4, Remarks 2(1)].

Next we define u on $\mathbb{R}^3$ by $u = u_j$ in Ω_j, $1 \le j \le L$, $u = u_0$ in Ω, $u = u_A$ in A and $u = 0$ in $\mathbb{R}^3 \backslash \overline{A}$. Obviously, $u \in W^{1,2}(\mathbb{R}^3)$, $\operatorname{div} u = 0$ in $\mathbb{R}^3$, and u satisfies the estimates

$$\begin{aligned} \|\nabla u\|_2 &\le c\,\|g\|_{W^{1/2,2}(\partial\Omega)}, \\ \|u\|_2 &\le c\,\|g\|_{W^{-1/2,2}(\partial\Omega)}. \end{aligned} \tag{5.14}$$

Using L^2-Fourier analysis we find a vector potential $\psi \in W^{2,2}(\mathbb{R}^3)$ such that

$$u = \operatorname{rot}\psi, \ \ \|\nabla\psi, \nabla^2\psi\|_2 \le c\,\|u\|_{W^{1,2}(\mathbb{R}^3)}. \tag{5.15}$$

Indeed, since $\operatorname{div} u = 0$, the equation $\operatorname{rot}\psi = u$ has a unique solenoidal solution ψ defined in Fourier space via $|\xi|^2\hat{\psi} = i\,\xi \times \hat{u}$. Obviously, ψ satisfies (5.15). To prove that $\psi \in L^2(\mathbb{R}^3)$ we introduce the space $\hat{W}^{1,q}(\mathbb{R}^n)$, $1 < q < \infty$, as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\nabla(\cdot)\|_q$, and its dual space $\hat{W}^{-1,q}(\mathbb{R}^n) \equiv \hat{W}^{1,q'}(\mathbb{R}^n)^*$. Evidently, $\psi \in L^2(\mathbb{R}^n)$ and $\|\psi\|_2 \le c\,\|u\|_{\hat{W}^{-1,2}}$ provided that $u \in \hat{W}^{-1,2}(\mathbb{R}^3)$. To see the latter assertion we exploit the fact that u has compact support in $\overline{B}_R$ and refer to the following Lemma 5.5 the proof of which will be postponed to the end of the paper.

Lemma 5.5. *Let D be a bounded domain in $\mathbb{R}^n$ and let $1 < q < \infty$. If $u \in L^q(\mathbb{R}^n)$ with* supp $u \subset D$*, then $u \in \hat{W}^{-1,q}(\mathbb{R}^n)$ and satisfies the estimate*

$$\|u\|_{\hat{W}^{-1,q}(\mathbb{R}^n)} \le c\|u\|_{L^q(D)}, \tag{5.16}$$

where $c = c(D, n, q)$ is a constant independent of u.

Summarizing (5.15), the estimate $\|\psi\|_2 \le c\,\|u\|_{\hat{W}^{-1,2}}$ and (5.16) with $q = 2$ we get that $u = \operatorname{rot}\psi$,

$$\begin{aligned} \|\nabla\psi\|_2 + \|\nabla^2\psi\|_2 &\le c\,\|g\|_{W^{1/2,2}(\partial\Omega)} \\ \|\psi\|_2 &\le c\,\|g\|_{W^{-1/2,2}(\partial\Omega)} \end{aligned} \tag{5.17}$$

with a constant $c = c(\Omega) > 0$. Moreover, the map $g \mapsto \psi$ is linear.

In the next step we define the vector field $E = E_\varepsilon$ by $E = \operatorname{rot}(\theta_\varepsilon\psi)$ where $\theta_\varepsilon \in W^{1,\infty}$ is a cut-off function with support in an ε-neighborhood of $\partial\Omega$. Following [11, pp. 288–290] or [22, Ch. II, §1, Lemma 1.9, Lemma 1.10] for pointwise estimates of θ_ε and E we get for all $w_1, w_2 \in W^{1,2}_{0,\sigma}(\Omega)$ the estimates

$$|\langle w_1 E, \nabla w_2\rangle_\Omega| \le \|w_1 E\|_2 \|\nabla w_2\|_2$$

and with $\chi_\varepsilon = \chi_{\operatorname{supp}\theta_\varepsilon}$ and $d(x) = \operatorname{dist}(x, \partial\Omega)$

$$\begin{aligned}\|w_1 E\|_2^2 &\le c \int_\Omega |w_1|^2 \Big(\frac{\varepsilon}{d(x)}|\psi(x)| + |\nabla\psi(x)|\chi_\varepsilon\Big)^2 dx\\ &\le c\,\varepsilon^2 \left(\int_\Omega \Big|\frac{w_1}{d(\cdot)}\Big|^2 dx\right) \|\psi\|_\infty^2 + c\,\|w_1\|_6^2 \|\nabla\psi\|_6^2 \|\chi_\varepsilon\|_6^2\\ &\le c\,\|\nabla w_1\|_2^2\ \|\psi\|_{H^2(\mathbb{R}^3)}^2\ (\varepsilon^2 + \|\chi_\varepsilon\|_6^2);\end{aligned}$$

here $\varepsilon > 0$ may be chosen arbitrarily small and is related to the size of $\operatorname{supp}\theta_\varepsilon$ which shrinks when $\varepsilon \to 0$. Hence $(1.15)_2$ can be fulfilled for a.a. fixed $t > 0$ in the sense that

$$|\langle w_1 E, \nabla w_2\rangle_\Omega| \le \frac{1}{4}\|\nabla w_1\|_2\,\|\nabla w_2\|_2,\ w_1, w_2 \in W^{1,2}_{0,\sigma}(\Omega)\,. \tag{5.18}$$

Furthermore, since $\langle E, \varphi\rangle_\Omega = \langle \theta\psi, \operatorname{rot}\varphi)\rangle_\Omega$ for all $\varphi \in W^{1,2}_0(\Omega)$ we get by $(5.17)_2$ the estimate

$$\|E\|_{W^{-1,2}(\Omega)} \le c\,\|g\|_{W^{-\frac{1}{2},2}(\partial\Omega)}. \tag{5.19}$$

In the final step we define $E(\cdot)$ as in (5.11) satisfying $(1.15)_2$. Given $g \in L^\infty(0,\infty; W^{\frac{1}{2},2}(\partial\Omega))$ fulfilling (5.9) for a.a. $t > 0$ we find by the previous arguments for a.a. $t > 0$ a vector field $E(t) = \operatorname{rot}(\theta\psi(t))$ satisfying (5.18) and, due to (5.17),

$$\|E(t)\|_{W^{1,2}(\Omega)} \le c\,\|g(t)\|_{W^{\frac{1}{2},2}(\partial\Omega)} \quad \text{for a.a. } t \in (0,T).$$

Hence $E \in L^\infty(0,\infty; W^{1,2}(\Omega))$, and $(5.12)_1$, $(1.15)_2$ are easy consequences. Since the map $g \mapsto E$ is linear and $g_t \in L^\infty(0,\infty; W^{-1/2,2}(\partial\Omega))$, the previous arguments, the method of difference quotients and (5.19) also imply that $E_t \in L^\infty(0,\infty; W^{-1/2,2}(\partial\Omega))$ and that $(5.12)_2$ holds.

Now Proposition 5.4 is completely proved. □

To apply Proposition 5.4 to the Navier-Stokes system (1.1) via Theorem 1.3 we have to consider the Stokes system (1.4) for E more closely. In this setting where E has already been defined by the boundary data g we have to determine f_0 and E_0 in (1.4). Let $h \equiv 0$ so that by the construction in the proof of Proposition 5.4

$$f_0 = E_t - \Delta E, \quad E = \operatorname{rot}(\theta\psi),$$

which may be written also in the form $f_0 = \operatorname{div} F_0$. By (5.12) we easily get that $F_0 \in L^\infty(0,\infty; L^2(\Omega))$ and that

$$\begin{aligned}\|F_0\|_{2,\infty;\infty} &\le c\left(\|E_t\|_{L^\infty(0,\infty;W^{-1,2}(\Omega))} + \|E\|_{L^\infty(0,\infty;W^{1,2}(\Omega))}\right)\\ &\le c\left(\|g_t\|_{L^\infty(0,\infty;W^{-\frac{1}{2},2}(\partial\Omega))} + \|g\|_{L^\infty(0,\infty;W^{\frac{1}{2},2}(\partial\Omega))}\right).\end{aligned} \tag{5.20}$$

Moreover, since $E \in L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega))$ and $E_t \in L^2_{\mathrm{loc}}([0,\infty); W^{-1,2}(\Omega))$, a classical interpolation result states that $E \in C^0([0,\infty); L^2(\Omega))$, the initial value

$E_0 = E(0) \in L^2(\Omega)$ is well defined and there exists a constant $c > 0$ such that

$$\begin{aligned}\|E_0\|_2 &\le c\left(\|E\|_{L^\infty(0,\infty;W^{1,2}(\Omega))} + \|E_t\|_{L^\infty(0,\infty;W^{-1,2}(\Omega))}\right)\\ &\le c\left(\|g\|_{L^\infty(0,\infty;W^{\frac12,2}(\partial\Omega))} + \|g_t\|_{L^\infty(0,\infty;W^{-\frac12,2}(\partial\Omega))}\right).\end{aligned} \tag{5.21}$$

Furthermore, $\operatorname{div} E_0 = 0$ and $E_0|_{\partial\Omega} = g(0)$ where $g(0)$ is well defined in $L^2(\partial\Omega)$.

Now we are ready to state our final result.

Corollary 5.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^{1,1}$ and boundary components Γ_j, $0 \le j \le L$. Assume that $f = \operatorname{div} F$, $F \in L^\infty(0,\infty; L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and that g satisfies* (5.10) *and the restricted flux condition* (5.9).

Then the Navier-Stokes system (1.1) *has a global weak solution $u = v + E$ where E satisfies* (5.11), (5.12) *and*

$$\|v(t)\|_2^2 \le e^{-\mu_0 t}\left(\|v_0\|_2^2 + 2\int_0^t e^{\mu_0\tau}\left(\|F_1\|_2^2 + \|E\|_4^4\right)d\tau\right);$$

here $F_1 = F + F_0$ satisfies (5.20), *$v_0 = u_0 - E_0$ where E_0 is subject to* (5.21), *and with μ_0 from* (4.8).

In particular, v and also u are bounded globally in time in $L^2(\Omega)$. There exists a bound κ^* independent of u_0 such that for all $t > T_0 = T_0(u_0, g, g_t)$ we have $\|v(t)\|_2 \le \kappa^*$.

Proof of Lemma 5.5. Let us first consider the case $n' = \frac{n}{n-1} < q < \infty$. For such q, it holds $1 < q' = \frac{q}{q-1} < n$, and the Sobolev inequality states that $\|\varphi\|_{L^{\frac{nq'}{n-q'}}(\mathbb{R}^n)} \le C\|\nabla\varphi\|_{L^{q'}(\mathbb{R}^n)}$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Hence taking $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\eta(x) \equiv 1$ for $x \in D$, we have

$$\begin{aligned}|\langle u, \varphi\rangle| = |\langle u, \eta\varphi\rangle| &\le \|u\|_{L^q(D)}\|\eta\varphi\|_{L^{q'}(D)} \le c\|u\|_{L^q(D)}\|\varphi\|_{L^{\frac{nq'}{n-q'}}(\mathbb{R}^n)}\\ &\le c\|u\|_{L^q(D)}\|\nabla\varphi\|_{L^{q'}(\mathbb{R}^n)}\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Hence $u \in \hat{W}^{-1,q}(\mathbb{R}^n)$, and u satisfies the estimate (5.16).

We next consider the case $1 < q \le n'$, i.e., $n \le q' < \infty$. For such q, we see that the subspace $S_D \equiv \{\varphi \in C_0^\infty(\mathbb{R}^n); \int_D \varphi(x)dx = 0\}$ is dense in $\hat{W}^{1,q'}(\mathbb{R}^n)$. For a moment, let us assume this density. Then it follows from the Poincaré-Friedrichs inequality $\|\varphi\|_{L^{q'}(D)} \le c\|\nabla\varphi\|_{L^{q'}(D)}$ for $\varphi \in S_D$ that

$$|\langle u, \varphi\rangle| \le \|u\|_{L^q(D)}\|\eta\varphi\|_{L^{q'}(D)} \le c\|u\|_{L^q(D)}\|\varphi\|_{L^{q'}(D)} \le c\|u\|_{L^q(D)}\|\nabla\varphi\|_{L^{q'}(\mathbb{R}^n)}$$

for all $\varphi \in S_D$. Since S_D is dense in $\hat{W}^{1,q'}(\mathbb{R}^n)$, the above estimate implies that $u \in \hat{W}^{-1,q}(\mathbb{R}^n)$ with (5.16).

It remains to prove the density of the space S_D in $\hat{W}^{1,q'}(\mathbb{R}^n)$. Take a function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $\zeta(x) = 1$ for $|x| \le 1$ and $\zeta(x) = 0$ for $|x| > 2$, and define $\zeta_k(x) \equiv \zeta(x/k)$ for $k \in \mathbb{N}$. For every $\varphi \in C_0^\infty(\mathbb{R}^n)$, we choose a sequence $\{\varphi_k\}_{k=1}^\infty$ by

$$\varphi_k(x) = \varphi(x) - \left(\frac{1}{|D|}\int_D \varphi(y)dy\right)\zeta_k(x), \quad x \in \mathbb{R}^n,\ k \in \mathbb{N}.$$

For sufficiently large k we have $\zeta_k(x) \equiv 1$ for all $x \in D$, and hence we may assume that $\{\varphi_k\}_{k=1}^{\infty} \subset S_D$. For $1 < q < n'$, i.e., $n < q' < \infty$, it holds that

$$\|\nabla\varphi_k - \nabla\varphi\|_{L^{q'}(\mathbb{R}^n)} \leq c\|\nabla\zeta_k\|_{L^{q'}(\mathbb{R}^n)} \leq ck^{-1+\frac{n}{q'}} \to 0 \quad \text{as } k \to \infty,$$

from which we conclude that S_D is dense in $\hat{W}^{1,q'}(\mathbb{R}^n)$ provided $1 < q < n'$.

For $q = n'$, i.e., $q' = n$, we have $\sup_{k\in\mathbb{N}} \|\nabla\varphi - \nabla\varphi_k\|_{L^n(\mathbb{R}^n)} < \infty$, and we easily conclude that

$$\nabla\varphi_k \rightharpoonup \nabla\varphi \quad \text{weakly in } L^n(\mathbb{R}^n) \text{ as } k \to \infty.$$

Applying Mazur's lemma to the sequence $\{\varphi_k\}_{k=1}^{\infty}$, we may select a sequence $\{\bar{\varphi}_k\}_{k=1}^{\infty}$ of convex combinations of $\{\varphi_k\}_{k=1}^{\infty}$ so that

$$\nabla\bar{\varphi}_k \to \nabla\varphi \quad \text{strongly in } L^n(\mathbb{R}^n) \text{ as } k \to \infty,$$

from which we also deduce the density of S_D in $\hat{W}^{1,n}(\mathbb{R}^n)$. This proves the lemma. □

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Time and Norm Optimality of Weakly Singular Controls

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Abstract. Let $\bar{u}(t)$ be a control that satisfies the infinite-dimensional version of Pontryagin's maximum principle for a linear control system, and let $z(t)$ be the costate associated with $\bar{u}(t)$. It is known that integrability of $\|z(t)\|$ in the control interval $[0,T]$ guarantees that $\bar{u}(t)$ is time and norm optimal. However, there are examples where optimality holds (or does not hold) when $\|z(t)\|$ is not integrable. This paper presents examples of both cases for a particular semigroup (the right translation semigroup in $L^2(0,\infty)$).

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1. Introduction

We consider two optimal control problems for the system

$$y'(t) = Ay(t) + u(t)\,, \quad y(0) = \zeta \tag{1.1}$$

with controls $u(\cdot) \in L^\infty(0,T;E)$, where A generates a strongly continuous semigroup $S(t)$ in a Banach space E. The first is the *norm optimal* problem, where we drive the initial point ζ to a point target,

$$y(T) = \bar{y} \tag{1.2}$$

in a fixed time interval $0 \le t \le T$ minimizing $\|u(\cdot)\|_{L^\infty(0,T;E)}$. The second is the *time optimal* problem, where we drive to the target with a bound on the norm of the control (say $\|u(\cdot)\|_{L^\infty(0,T;E)} \le 1$) in optimal time T. The *solution* or *trajectory* of (1.1) is the continuous function

$$y(t) = y(t,\zeta,u) = S(t)\zeta + \int_0^t S(t-\sigma)u(\sigma)d\sigma\,. \tag{1.3}$$

For the time optimal problem, controls in $L^\infty(0,T;E)$ with $\|u(\cdot)\|_{L^\infty(0,T;E)} \le 1$ are named *admissible.*

Necessary and sufficient conditions for norm and time optimality can be given in terms of the maximum principle (1.5) below which requires the construction of spaces of multipliers (final values of costates). We summarize this construction from [4] or [5], Section 2.3. When A has a bounded inverse, we define the space E^*_{-1} as the completion of E^* in the norm

$$\|y^*\|_{E^*_{-1}} = \|(A^{-1})^* y^*\|_{E^*} \,.$$

Each $S(t)^*$ can be extended to an operator $S(t)^* : E^*_{-1} \to E^*_{-1}$, and $Z_w(T) \subseteq E^*_{-1}$ consists of all $z \in E^*_{-1}$ such that $S(t)^* z \in E^*$ and[1]

$$\|z\|_{Z_w(T)} = \int_0^T \|S(t)^* z\|_{E^*} dt < \infty \,. \tag{1.4}$$

The space $Z_w(T)$ equipped with $\|\cdot\|_{Z_w(T)}$ is a Banach space. All spaces $Z_w(T)$ coincide (that is, $Z_\omega(T) = Z_\omega(T')$ for any $T, T' > 0$ and the norms $\|\cdot\|_{Z_w(T)}$, $\|\cdot\|_{Z_w(T')}$ are equivalent). $Z_w(T)$ is an example of a *multiplier space,* an arbitrary linear space $\mathcal{Z} \supseteq E^*$ to which $S(t)^*$ can be extended in such a way that $S(t)^*\mathcal{Z} \subseteq E^*$ for $t > 0$. When A does not have a bounded inverse, the construction of the spaces is modified as follows. Since A is a semigroup generator, $(\lambda I - A)^{-1}$ exists for $\lambda > \omega$ and E^*_{-1} is the completion of E^* in any of the equivalent norms

$$\|y^*\|_{E^*_{-1},\lambda} = \|((\lambda I - A)^{-1})^* y^*\|_{E^*} \,, \quad (\lambda > \omega) \,.$$

The definition of $Z_w(T)$ (and of multiplier spaces) is the same. See [5], Section 2.3 for proofs of these results and additional details.

A control $\bar u(\cdot) \in L^\infty(0,T;E)$ satisfies *Pontryagin's maximum principle* if

$$\langle S(T-t)^* z, \bar u(t)\rangle = \max_{\|u\| \le \rho} \langle S(T-t)^* z, u\rangle \quad \text{a.e. in } 0 \le t < T \,, \tag{1.5}$$

where $\langle \cdot\,, \cdot\rangle$ is the duality of the space E and the dual E^*, with $\rho = \|\bar u(\cdot)\|_{L^\infty(0,T;E)}$ and z in some multiplier space $\mathcal{Z}$. We call z the *multiplier* and $z(t) = S(T-t)^* z$ the *costate corresponding to* (or *associated with*) the control $\bar u(t)$. We assume that (1.5) is nontrivial; this means $S(T-t)^* z$ is not identically zero in the interval $0 \le t < T$, although it may be zero in part of the interval (in which part (1.5) says nothing about $\bar u(t)$). That (1.5) is nontrivial implies that $z \ne 0$. The maximum principle is especially simple when E is a Hilbert space; it reduces to

$$\bar u(t) = \rho \, \frac{S(T-t)^* z}{\|S(T-t)^* z\|} \qquad (T - \delta < \sigma \le T) \,, \tag{1.6}$$

where $0 \le t < \delta$ is the maximal interval where $S(t)^* z \ne 0$; if $\delta \ge T$ the interval is $0 < \sigma \le T$.

[1]Without further assumptions, the semigroup $S(t)^*$ may not be strongly continuous, or even strongly measurable (consider, for instance, the translation semigroup $S(t)y(x) = y(x-t)$ in $E = L^1(\infty, \infty)$). However, $S(t)^*$ is always E-weakly continuous, which guarantees that $\|S(t)^*\|$ is lower semicontinuous, hence measurable. This justifies the integral (1.4).

A large part of the theory of optimal controls for the system (1.1) deals with the relation between optimality and the maximum principle (1.5). All one has (at present) are separate necessary and sufficient conditions for optimality based on the maximum principle (Theorem 1.1 below). We call an optimal control $\bar{u}(t)$ *regular* if it satisfies (1.5) with $z \in Z_w(T)$.

Theorem 1.1. *Assume $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T, \zeta, \bar{u})$ time or norm optimally in the interval $0 \le t \le T$ and that*

$$\bar{y} - S(T)\zeta \in D(A)\,. \tag{1.7}$$

Then $\bar{u}(t)$ is regular. Conversely, let $\bar{u}(t)$ be a regular control. Then $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T, \zeta, \bar{u})$ norm optimally in the interval $0 \le t \le T$; if $\rho = 1$ the drive is time optimal.

For the proof see [4], Theorem 5.1, [5], Theorem 2.5.1; we note that in the sufficiency half of Theorem 1.1 no conditions of the type of (1.7) are put on the initial value ζ or the target $\bar{y}$.[2]

Following the terminology in [5] we call a control *weakly singular* if it satisfies the maximum principle (1.5) but the costate does not satisfy the integrability condition (1.4) (that is, $z \notin Z_w(T)$). The following question arises: *is a weakly singular control* (*norm, time*) *optimal?* The answer to this question is "not necessarily" and examples of weakly singular controls that are (or are not) optimal are known. It is proved in [2] (see [5], Section 3.4) that for the (self-adjoint) multiplication operator

$$\mathcal{A}u(\lambda) = -\lambda u(\lambda)$$

in $L^2(0, \infty)$, which generates the analytic semigroup

$$\mathcal{S}(t)u(\lambda) = e^{-\lambda t}u(t) \tag{1.8}$$

there exist optimal controls for (1.1) satisfying the maximum principle (1.5) where the growth of the costate $z(t)$ as t approaches the final time T is $\approx C/(T-t)$ in the sense that

$$(T-t)\|z(t)\|_{E^*} = (T-t)\|S(T-t)^*z\|_{E^*} \to C \quad \text{as } t \to T \tag{1.9}$$

with $0 < C < \infty$. These controls cannot satisfy (1.4), thus they are weakly singular. On the other hand there exist controls satisfying (1.5) and

$$(T-t)^\alpha\|z(t)\|_{E^*} = (T-t)^\alpha\|S(T-t)^*z\|_{E^*} \to C \quad \text{as } t \to T \tag{1.10}$$

with $\alpha > 1$ and $0 < C < \infty$ (thus weakly singular) that are not time or norm optimal. We provide in this paper similar examples for the *right translation* semigroup

[2]The statement on time optimality, however, needs additional assumptions on the initial condition ζ and the target $\bar{y}$. These conditions are satisfied if either $\zeta = 0$ or $\bar{y} = 0$ [5], Theorem 2.5.7. We point out that the conditions are on the "size" of ζ $\bar{y}$, not on their smoothness like (1.7); for instance, for $\zeta = 0$, $\bar{y}$ may be an arbitrary element of E. We also need to assume that $S(t)^*z \neq 0$ in the entire interval $0 \le t \le T$.

$S(t)$ in $L^2(0,\infty)$ defined by

$$S(t)y(x) = \begin{cases} y(x-t) & (x \geq t) \\ 0 & (x < t)\,. \end{cases} \tag{1.11}$$

Although the technical means are totally different, the examples are of the same sort as those in [2]; there are controls that satisfy (1.9) and are optimal, whereas there are controls with faster increase of $\|z(t)\|$ which are not optimal. What is remarkable about the examples in this paper is that they resemble similar examples for semigroups as different as (1.8), analytic with (1.1) an abstract parabolic equation. The semigroup (1.11) under study here is isometric, with associated equation (1.1) having a finite velocity of propagation, thus qualifying as "abstract hyperbolic".

On the basis of this similar behavior of controls for very different semigroups it is tempting to guess that there must exist some sort of classification of weakly singular controls (as optimal or nonoptimal) which is based on the growth of the norm of the costate $z(t) = S(T-t)^*z$ as $t \to T$ and holds for arbitrary semigroups. There seems to be no such general result except [3] Lemma 8.3, [5] Lemma 3.5.9 where the generator is self-adjoint in Hilbert space and $\|S(t)^*z\|$ has "hyperpower growth" as $t \to 0$; this means

$$\int_0^1 \frac{\|S(r\sigma)^*z\|}{\sigma\|S(\sigma)^*z\|}d\sigma < \infty \tag{1.12}$$

for some r in the range $1 < r < 2$ (the adjoint may be omitted since the semigroup $S(t)$ is self-adjoint). Under (1.12), the control corresponding to the multiplier z is not optimal. Condition (1.12) cannot hold if (1.10) is satisfied for any $\alpha > 0$; in fact, in this case

$$\frac{\|S(r\sigma)^*z\|}{\sigma\|S(\sigma)^*z\|} \approx \frac{C\sigma^\alpha}{\sigma C(r\sigma)^\alpha} = \frac{1}{r^\alpha\sigma}$$

making the integral infinite. However, there is a wide gap between hyperpower growth and power growth like (1.9), and nothing is known for intermediate growths.

We mention in passing the results on multipliers in [6]. When the semigroup satisfies

$$S(t)E = E \quad (t > 0) \tag{1.13}$$

then every multiplier space satisfies $\mathcal{Z} = Z_w(T) = E^*$, that is, all multipliers in (1.5) automatically belong to E^*; this makes moot the question of the growth of $\|z(t)\|$ as $t \to T$. It is also shown in [6] that (under the assumption that E is reflexive and separable) (1.13) is a necessary condition for all multipliers to belong to E^*. Moreover, condition (1.7) can be dropped from Theorem 1.1 in case (1.12) holds: all time or norm optimal controls satisfy (1.5) with a multiplier $z \in E^*$.

2. The right translation semigroup

The space is $E = L^2(0,\infty)$. Its elements $y(x)$ (defined in $x \geq 0$) are extended as $y(x) = 0$ for $x < 0$. The *right translation* semigroup $S(t)$ defined by (1.11) is strongly continuous and isometric in $L^2(0,\infty)$. The adjoint semigroup is the *left translation* (and chop-off) semigroup

$$S(t)^*y(x) = \begin{cases} y(x+t) & (x \ge 0) \\ 0 & (x < 0)\,. \end{cases} \tag{2.1}$$

We have

$$S(t)^*S(t) = I\,, \quad S(t)S(t)^*y(x) = \chi_t(x)y(x) \tag{2.2}$$

where $\chi_t(x)$ is the characteristic function of $[t,\infty)$. The infinitesimal generator A of $S(t)$ is

$$Ay(x) = -y'(x)\,, \tag{2.3}$$

with domain $D(A) = \{\text{all } y(\cdot) \in L^2(0,\infty) \text{ with } y'(\cdot) \text{ in } L^2(0,\infty) \text{ and } y(0)=0\}$, the derivative understood in the sense of distributions. The semigroup $S(t)$ is associated with the control system

$$\begin{gathered} \frac{\partial y(t,x)}{\partial t} = -\frac{\partial y(t,x)}{\partial x} + u(t,x) \\ y(0,x) = \zeta(x)\,, \qquad y(t,0) = 0\,, \end{gathered} \tag{2.4}$$

in the sense that $S(t)$ is the propagation semigroup of the homogeneous equation $(u(t,x)=0)$. Formula (1.3) for the control $u(t)(x) = u(t,x)$ is

$$\begin{aligned} y(t,x,\zeta,u) = y(t,\zeta,u)(x) &= \left(S(t)\zeta + \int_0^t S(t-\sigma)u(\sigma)d\sigma\right)(x) \\ &= \zeta(x-t) + \int_0^t u(\sigma, x-(t-\sigma))d\sigma\,, \end{aligned} \tag{2.5}$$

thus the contribution of $u(\sigma,x)$ to $y(t,x,\zeta,u)$ is the integral of $u(\sigma,x)$ over the intersection with the positive quadrant of the characteristic line $(\sigma, x-(t-\sigma))$ joining $(0,x-t)$ with (t,x), as shown in Figure 1.

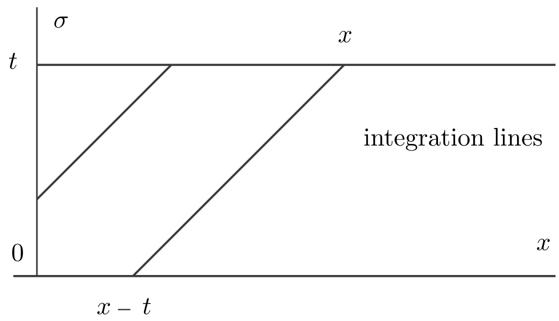

FIGURE 1

We name $\mathcal{Z}$ the space of all measurable $z(x)$ defined in $x > 0$ and such that

$$\kappa(\sigma, z) = \|S(\sigma)^* z(\cdot)\| = \sqrt{\int_0^\infty z(x+\sigma)^2 dx} = \sqrt{\int_\sigma^\infty z(x)^2 dx} < \infty$$

in $\sigma > 0$. The space[3] $Z(T)$ consists of all $z(\cdot) \in \mathcal{Z}$ with

$$\int_0^T \|S(\sigma)^* z(\cdot)\| d\sigma = \int_0^T \kappa(\sigma, z) d\sigma < \infty\,. \tag{2.6}$$

Since we are in a Hilbert space (1.6) applies and any control that satisfies the maximum principle (1.5) is given a.e. by

$$\bar{u}(\sigma, x) = \rho \frac{S(T-\sigma)^* z(x)}{\|S(T-\sigma)^* z(\cdot)\|} = \rho \chi_0(x) \frac{z(x + (T-\sigma))}{\kappa(T-\sigma, z)} \quad (T - \delta < \sigma \le T)\,, \tag{2.7}$$

where $0 \le t < \delta$ is the maximal interval where $S(t)^* z \ne 0$; if $\delta \ge T$, the interval in (2.7) is $0 < \sigma \le T$.

Using the second equality (2.2) and assuming for simplicity that $\rho = 1$ we obtain

$$S(T-\sigma)\bar{u}(\sigma)(x) = \frac{S(T-\sigma)S(T-\sigma)^* z(x)}{\|S(T-\sigma)^* z(\cdot)\|} = \frac{\chi_{T-\sigma}(x) z(x)}{\kappa(T-\sigma, z)} \quad (0 \le \sigma < T) \tag{2.8}$$

whenever $\kappa(T-\sigma, z) \ne 0$. Formula (2.5) for $t = T$ becomes

$$\begin{aligned} y(T, x, \zeta, \bar{u})(x) &= S(T)\zeta(x) + \int_0^T S(T-\sigma)\bar{u}(\sigma, x) d\sigma \\ &= \zeta(x-T) + \int_0^T \frac{\chi_{T-\sigma}(x) z(x)}{\kappa(T-\sigma, z)} d\sigma = \zeta(x-T) + z(x) \int_0^T \frac{\chi_{T-\sigma}(x)}{\kappa(T-\sigma, z)} d\sigma \\ &= \zeta(x-T) + z(x) \int_0^T \frac{\chi_\sigma(x)}{\kappa(\sigma, z)} d\sigma = \zeta(x-T) + z(x)\omega(T, x, z)\,, \end{aligned} \tag{2.9}$$

where

$$\omega(T, x, z) = \int_0^T \frac{\chi_\sigma(x)}{\kappa(\sigma, z)} d\sigma = \int_0^{\min(x,T)} \frac{d\sigma}{\kappa(\sigma, z)}\,. \tag{2.10}$$

If we drive from 0 to $\bar{y}(x)$ in time T, the target $\bar{y}(x)$ and the costate $z(x)$ are related by

$$\bar{y}(x) = y(T, x, 0, \bar{u}) = z(x)\omega(T, x, z)\,, \tag{2.11}$$

so that the target $\bar{y}(x)$ is a multiple of the multiplier $z(x)$ in $x \ge T$.

[3] We drop the subindex w since $S(t)^*$ is strongly continuous.

3. Weakly singular controls, I

We use the family of multipliers

$$z(x) = \frac{1}{x^\alpha} \quad \Big(\alpha > \frac{1}{2}\Big) \tag{3.1}$$

associated with the controls

$$\bar{u}_\alpha(\sigma, x) = \frac{\chi_0(x)}{\kappa(T-\sigma, z)(x + (T-\sigma))^\alpha} \quad (0 \le \sigma \le T)\,. \tag{3.2}$$

We have

$$\kappa(\sigma, z)^2 = \int_\sigma^\infty \frac{dx}{x^{2\alpha}} = \frac{\sigma^{1-2\alpha}}{2\alpha - 1} = \frac{1}{(2\alpha-1)\sigma^{2\alpha-1}}\,,$$

thus

$$\kappa(\sigma, z) = \frac{1}{\sqrt{2\alpha-1}\,\sigma^{\alpha-1/2}} \tag{3.3}$$

and, in view of (2.6) the control $\bar{u}_\alpha(\sigma, x)$ is regular ($z(\cdot) \in Z(T)$) if and only if

$$\alpha - \frac{1}{2} < 1 \quad \Longleftrightarrow \quad \alpha < \frac{3}{2}\,;$$

on the other hand $z(\cdot) \in \mathcal{Z}$ (thus $\bar{u}_\alpha(\sigma, x)$ is weakly singular) for arbitrary $\alpha \ge 3/2$. Combining (3.2) and (3.3)

$$\bar{u}_\alpha(\sigma, x) = \frac{\sqrt{2\alpha-1}\chi_0(x)(T-\sigma)^{\alpha-1/2}}{(x+(T-\sigma))^\alpha} \quad (0 \le \sigma \le T)\,. \tag{3.4}$$

We have

$$\begin{aligned}\int_\sigma^\infty \bar{u}_\alpha(\sigma, x)^2 dx &= (2\alpha-1)(T-\sigma)^{2\alpha-1} \int_\sigma^\infty (x+(T-\sigma))^{-2\alpha} dx \\ &= (2\alpha-1)(T-\sigma)^{2\alpha-1} \frac{(x+(T-\sigma))^{1-2\alpha}}{1-2\alpha}\Big|_{x=\sigma}^{x=\infty} = \Big(\frac{T-\sigma}{T}\Big)^{2\alpha-1}, \end{aligned} \tag{3.5}$$

$$\begin{aligned}\int_0^\sigma \bar{u}_\alpha(\sigma, x)^2 dx &= (2\alpha-1)(T-\sigma)^{2\alpha-1} \int_0^\sigma (x+(T-\sigma))^{-2\alpha} dx \\ &= (2\alpha-1)(T-\sigma)^{2\alpha-1} \frac{(x+(T-\sigma))^{1-2\alpha}}{1-2\alpha}\Big|_{x=0}^{x=\sigma} = 1 - \Big(\frac{T-\sigma}{T}\Big)^{2\alpha-1}. \end{aligned} \tag{3.6}$$

Since $2\alpha - 1 > 0$ we have the proof of

Lemma 3.1. *Let* $0 \le \sigma \le T$. *Let*

$$I(\alpha) = \int_0^\sigma \bar{u}_\alpha(\sigma, x)^2 dx\,, \quad J(\alpha) = \int_\sigma^\infty \bar{u}_\alpha(\sigma, x)^2 dx \quad \Big(\alpha > \frac{1}{2}\Big)\,.$$

Then $I(\alpha)$ (*resp.* $J(\alpha)$) *is a decreasing* (*resp. increasing*) *function of* α.

From (2.10) and (3.3) we have

$$\omega(T, x, z) = \sqrt{2\alpha-1}\int_0^x \sigma^{\alpha-1/2} d\sigma = \frac{\sqrt{2\alpha-1}}{\alpha+1/2} x^{\alpha+1/2}$$

for $x \le T$; for $x \ge T$

$$\omega(T, x, z) = \sqrt{2\alpha - 1} \int_0^T \sigma^{\alpha - 1/2} d\sigma = \frac{\sqrt{2\alpha - 1}}{\alpha + 1/2} T^{\alpha + 1/2}.$$

Formula (2.11) implies that the control $\bar{u}_\alpha(\sigma, x)$ in (3.4) drives 0 to the target

$$\begin{aligned} \bar{y}(x) &= \frac{\sqrt{2\alpha - 1}}{\alpha + 1/2} x^{1/2} \qquad (x \le T), \\ \bar{y}(x) &= \frac{\sqrt{2\alpha - 1}}{\alpha + 1/2} \frac{T^{\alpha + 1/2}}{x^\alpha} \quad (x > T). \end{aligned} \tag{3.7}$$

Figure 2 shows $\bar{y}_{3/2}(x)$ for $T = 1$.

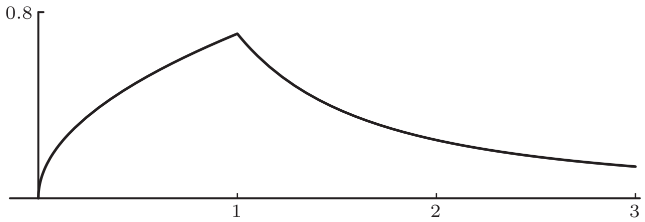

FIGURE 2

As a consequence of Theorem 1.1 we obtain

Theorem 3.2. *If $\alpha < 3/2$ the control $\bar{u}_\alpha(\sigma, x)$ is regular, thus it drives 0 to $\bar{y}_\alpha(x)$ time and norm optimally.*

We prove below that this is no longer true if $\alpha > 3/2$. The proof is based on the fact that the function

$$K(\alpha) = \frac{\sqrt{2\alpha - 1}}{\alpha + 1/2}$$

(the factor in (3.7)) has the unique maximum 3/2 in $\alpha \ge 1/2$ (with $K(3/2) = 1/\sqrt{2}$). The graph of $K(\alpha)$ is shown in Figure 3.

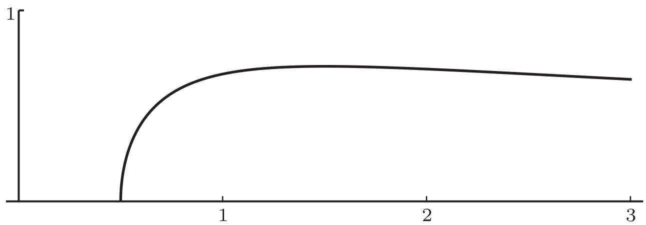

FIGURE 3

Let $\alpha > 3/2$, so that $K(3/2) > K(\alpha)$. We improve the norm-performance provided by the control $\bar{u}_\alpha(\sigma, x)$ constructing a control $\tilde{u}_\alpha(\sigma, x)$ "by pieces" as follows (see Figure 4).

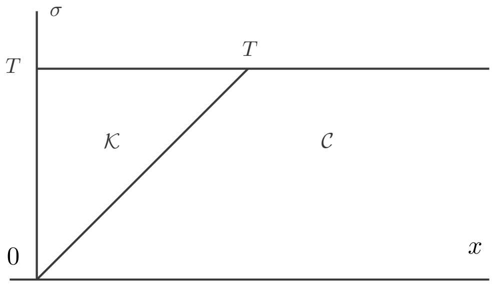

FIGURE 4

In the triangle $\mathcal{K}$ we define

$$\tilde{u}_\alpha(\sigma, x) = \frac{K(\alpha)}{K(3/2)} \bar{u}_{3/2}(\sigma, x) \,. \tag{3.8}$$

The integration formula (2.5) (see also Figure 1) shows that this new definition affects the target $\bar{y}(x)$ only in the interval $0 \le t \le T$. There is actually no change in the target, since in this interval the target hit by $\tilde{u}_\alpha(\sigma, x)$ is

$$\bar{y}(x) = \frac{K(\alpha)}{K(3/2)} \bar{y}_{3/2}(x) = \bar{y}_\alpha(x) \,.$$

Using the first part of Lemma 3.1 we obtain

$$\int_0^\sigma \tilde{u}_\alpha(\sigma, x)^2 dx = \frac{K(\alpha)}{K(3/2)} \int_0^\sigma \bar{u}_{3/2}(\sigma, x)^2 dx \le \frac{K(\alpha)}{K(3/2)} \int_0^\sigma \bar{u}_\alpha(\sigma, x)^2 dx \,. \tag{3.9}$$

As a first approximation we don't modify the control $\bar{u}_\alpha(\sigma, x)$ in the complement $\mathcal{C}$ of $\mathcal{K}$, so that we drive to the target $\bar{y}_\alpha(x)$ with

$$v(\sigma, x) = \begin{cases} \tilde{u}_\alpha(\sigma, x) & (\sigma, x) \in \mathcal{K} \\ \bar{u}_\alpha(\sigma, x) & (\sigma, x) \in \mathcal{C} \,. \end{cases} \tag{3.10}$$

Using (3.5), (3.6) and (3.9),

$$\begin{aligned} \|v(\sigma, \cdot)\|_{L^2(0,\infty)} &= \int_0^\infty v(\sigma, x)^2 dx \\ &= \int_0^\sigma v(\sigma, x)^2 dx + \int_\sigma^\infty v(\sigma, x)^2 dx \\ &\le \frac{K(\alpha)}{K(3/2)} \int_0^\sigma \bar{u}_\alpha(\sigma, x)^2 dx + \int_\sigma^\infty \bar{u}_\alpha(\sigma, x)^2 dx \\ &= \frac{K(\alpha)}{K(3/2)} \Big(1 - \Big(\frac{T - \sigma}{T} \Big)^{2\alpha - 1} \Big) + \Big(\frac{T - \sigma}{T} \Big)^{2\alpha - 1} = \eta(\sigma) \end{aligned}$$

The fact that $\eta(\alpha) < 1$ in $0 < \sigma \le T$ implies right away that $v(\cdot,\cdot)$ (thus $\bar{u}_\alpha(\cdot,\cdot)$) is not time optimal, since the bang-bang theorem [1] Theorem 2.2, [5] Theorem 2.1.3 says that time optimal controls $\bar{u}(\sigma)$ for (1.1) must satisfy $\|\bar{u}(\sigma)\| = 1$ almost everywhere. However, $\eta(\sigma) < 1$ does not imply that $v_\alpha(\cdot,\cdot)$ is not norm optimal since $\eta(0) = 1$ (time optimal $\to$ norm optimal but the converse is not true). Thus, we must prove directly that $\bar{u}_\alpha(\cdot,\cdot)$ is not norm optimal, which requires modification of $v(\cdot,\cdot)$ in $\mathcal{C}$. This will be done by further subdivision of $\mathcal{C}$ into two regions $\mathcal{C}_1$ and $\mathcal{C}_2$ indicated in Figure 5; the parameter $N > 0$ will be determined later.

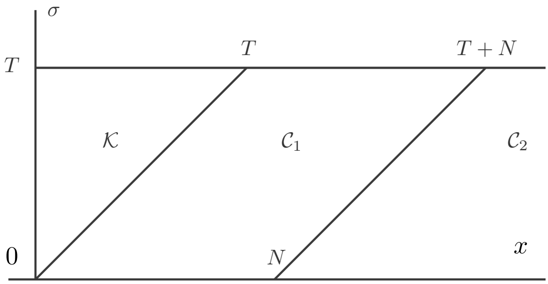

Figure 5

A computation entirely similar to (3.6) shows

$$\int_\sigma^{\sigma+N} \bar{u}_\alpha(\sigma,x)^2 dx = \Big(\frac{T-\sigma}{T}\Big)^{2\alpha-1} - \Big(\frac{T-\sigma}{T+N}\Big)^{2\alpha-1}. \tag{3.11}$$

We introduce the control

$$\bar{v}_\alpha(\sigma,x) = \frac{\sqrt{2\alpha-1}\chi_0(x)\sigma^{\alpha-1/2}}{(x+(T-\sigma))^\alpha} \quad (0 \le \sigma \le T)\,. \tag{3.12}$$

After another computation like (3.6),

$$\int_{\sigma+N}^\infty \bar{v}_\alpha(\sigma,x)^2 dx = \Big(\frac{\sigma}{T+N}\Big)^{2\alpha-1}.$$

Moreover,

$$\bar{v}_\alpha(\sigma, x-(T-\sigma)) = \frac{\sqrt{2\alpha-1}\chi_0(x)\sigma^{\alpha-1/2}}{x^\alpha}.$$

Since

$$\bar{u}_\alpha(\sigma, x-(T-\sigma)) = \frac{\sqrt{2\alpha-1}\chi_0(x)(T-\sigma)^{\alpha-1/2}}{x^\alpha}$$

formula (2.5) and the change of variables $\sigma \to T-\sigma$ show that both controls drive 0 to $\bar{y}_\alpha(x)$ in $x \ge T$. This can also be directly verified for $\bar{v}_\alpha(\sigma,x)$:

$$\int_0^T \bar{v}_\alpha(\sigma, x-(T-\sigma))d\sigma = \frac{\sqrt{2\alpha-1}}{x^\alpha}\int_0^T \sigma^{\alpha-1/2}d\sigma = \frac{\sqrt{2\alpha-1}}{\alpha+1/2}\frac{T^{\alpha+1/2}}{x^\alpha}.$$

The control $v(\sigma, x)$ is now defined by

$$v(\sigma, x) = \begin{cases} \tilde{u}_\alpha(\sigma, x) & (\sigma, x) \in \mathcal{K} \\ \bar{u}_\alpha(\sigma, x) & (\sigma, x) \in \mathcal{C}_1 \\ \bar{v}_\alpha(\sigma, x) & (\sigma, x) \in \mathcal{C}_2 \end{cases} \tag{3.13}$$

and we have

$$\begin{aligned} \|v(\sigma, \cdot)\|_{L^2(0,\infty)} &= \int_0^\infty v(\sigma, x)^2 dx \\ &= \int_0^\sigma v(\sigma, x)^2 dx + \int_\sigma^{\sigma+N} v(\sigma, x)^2 dx + \int_{\sigma+N}^\infty v(\sigma, x)^2 dx \\ &\leq \frac{K(\alpha)}{K(3/2)} \int_0^\sigma \bar{u}_\alpha(\sigma, x)^2 dx + \int_\sigma^{\sigma+N} \bar{u}_\alpha(\sigma, x)^2 dx + \int_{\sigma+N}^\infty \bar{v}_\alpha(\sigma, x)^2 dx \\ &= \frac{K(\alpha)}{K(3/2)} \Big(1 - \Big(\frac{T-\sigma}{T}\Big)^{2\alpha-1}\Big) + \Big(\Big(\frac{T-\sigma}{T}\Big)^{2\alpha-1} - \Big(\frac{T-\sigma}{T+N}\Big)^{2\alpha-1}\Big) \\ &\quad + \Big(\frac{\sigma}{T+N}\Big)^{2\alpha-1} = \eta(\sigma)\,. \end{aligned} \tag{3.14}$$

We have

$$\begin{aligned} \eta(0) &= 1 - \Big(\frac{T}{T+N}\Big)^{2\alpha-1} < 1 \\ \eta(T) &= \frac{K(\alpha)}{K(3/2)} + \Big(\frac{T}{T+N}\Big)^{2\alpha-1} < 1\,, \end{aligned} \tag{3.15}$$

the second inequality taking place for N large enough. Since $\alpha \geq 3/2$ we have $2\alpha - 1 \geq 2$ and each of the three terms in (3.14) that make up $\eta(\sigma)$ have positive second derivative. It follows that $\eta(\sigma)$ is convex, thus (3.15) implies

$$\eta(\sigma) \leq \max(\eta(0), \eta(T)) < 1$$

and the control $v(\sigma, x)$ reaches the target $\bar{y}_\alpha(x)$ with smaller norm than $\bar{u}_\alpha(\sigma, x)$ thus proving that the latter is not norm optimal.

4. Weakly singular controls, II

We show in this section that the control $u_{3/2}(\sigma, x)$, although weakly singular, is time and norm optimal. To this end, we assume it not time optimal: then there exists an admissible control $u(\sigma, x)$ driving 0 to $\bar{y}_{3/2}(x)$ in time $T - \delta < T$. We show below that this implies that, for some $\alpha < 3/2$ sufficiently close to 3/2 the control $\bar{u}_\alpha(\sigma, x)$ is not norm optimal, which contradicts Theorem 3.2.

We construct a control $v(\sigma, x)$ that drives 0 to $\bar{y}_\alpha(x)$ in time T. This control is also constructed by pieces; the different domains are in Figure 6.
Since $u(\sigma, x)$ drives 0 to $\bar{y}_{3/2}(x)$ in time $T - \delta$, the control

$$\begin{cases} 0 & (0 \leq \sigma < \delta) \\ u(\sigma - \delta, x) & (\delta \leq \sigma \leq T) \end{cases}$$

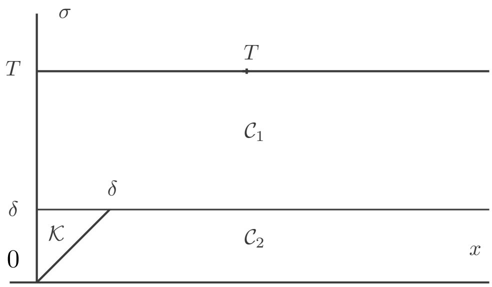

FIGURE 6

drives 0 to $\bar{y}_{3/2}(x)$ in time T. In view of (3.7), the control

$$v_1(\sigma, x) = \begin{cases} 0 & (0 \le \sigma < \delta) \\ \dfrac{K(\alpha)}{K(3/2)} u(\sigma - \delta, x) & (\delta \le \sigma \le T) \end{cases} \tag{4.1}$$

drives 0 to

$$v_1(\sigma, x) = \begin{cases} \dfrac{K(\alpha)}{K(3/2)} \bar{y}_{3/2}(x) = \bar{y}_\alpha(x) & (0 \le x \le T) \\ \dfrac{K(\alpha)T^2}{y^{3/2}} & (T < x < \infty) \end{cases} \tag{4.2}$$

in time T. We define next

$$v_2(\sigma, x) = \begin{cases} \dfrac{K(\alpha)}{\delta} \Big(\dfrac{T^{\alpha+1/2}}{(x + (T - \sigma))^\alpha} - \dfrac{T^2}{(x + (T - \sigma))^{3/2}} \Big) & (\sigma, x) \in \mathcal{C}_2 \\ 0 & (\sigma, x) \in \mathcal{K} \cup \mathcal{C}_1 \,. \end{cases}$$

This control drives to a target which is $= 0$ in $0 \le x \le T$. Over the paths of integration $(\sigma, x - (T - \sigma))$ in formula (2.5) for $x \ge T$ we have

$$v_2(\sigma, x - (T - \sigma)) = \begin{cases} \dfrac{K(\alpha)}{\delta} \Big(\dfrac{T^{\alpha+1/2}}{x^\alpha} - \dfrac{T^2}{x^{3/2}} \Big) & (0 \le \sigma \le \delta) \\ 0 & (\delta < \sigma < T) \end{cases}$$

thus

$$\int_0^T v_2(\sigma, x - (T - \sigma)) d\sigma = K(\alpha) \Big(\frac{T^{\alpha+1/2}}{x^\alpha} - \frac{T^2}{x^{3/2}} \Big) \quad (x \ge T)$$

and it follows that $v_2(\sigma, x)$ drives to a "corrector" target $\bar{y}^2(x)$ such that $\bar{y}^2(x) = 0$ in $0 \le x \le T$ and

$$\bar{y}^1(x) + \bar{y}^2(x) = \bar{y}_\alpha(x) \quad (0 \le x < \infty) \,.$$

Accordingly, the control

$$v(\sigma, x) = v_1(\sigma, x) + v_2(\sigma, x) \tag{4.3}$$

drives 0 to the "right" target $\bar{y}_\alpha(x)$. It remains to select α so that $v(\sigma, x)$ does the drive with norm < 1. On the one hand, we have

$$\begin{aligned} \|v(\sigma,\cdot)\|_{L^2(0,\infty)} &= \frac{K(\alpha)}{K(3/2)}\|u(\sigma-\delta,x)\|_{L^2(0,\infty)} \\ &\le \frac{K(\alpha)}{K(3/2)} < 1 \quad (\delta \le \sigma \le T)\,. \end{aligned} \tag{4.4}$$

On the other hand, $\|v(\sigma,\cdot)\|^2_{L^2(0,\infty)} = \|v_2(\sigma,\cdot)\|^2_{L^2(0,\infty)}$ is constant in $0 \le \sigma \le \delta$ thus

$$\begin{aligned} &\|v(\sigma,\cdot)\|^2_{L^2(0,\infty)} = \|v_2(0,\cdot)\|^2_{L^2(0,\infty)} \\ &\quad = \frac{K(\alpha)^2}{\delta^2}\int_0^\infty \Big(\frac{T^{\alpha+1/2}}{(x+T)^\alpha} - \frac{T^2}{(x+T)^{3/2}}\Big)^2 dx = \frac{K(\alpha)^2}{\delta^2}\int_T^\infty \Big(\frac{T^{\alpha+1/2}}{x^\alpha} - \frac{T^2}{x^{3/2}}\Big)^2 \\ &\quad = \frac{K(\alpha)^2}{\delta^2}\Big(T^{2\alpha+1}\int_T^\infty x^{-2\alpha}dx - 2T^{\alpha+5/2}\int_T^\infty x^{-\alpha-3/2}dx + T^4\int_T^\infty x^{-3}dx\Big) \\ &\quad = \frac{K(\alpha)^2T^2}{\delta^2}\Big(\frac{1}{2\alpha-1} - \frac{2}{\alpha+1/2} + \frac{1}{2}\Big) \\ &\quad = \frac{K(\alpha)^2T^2(2\alpha-3)^2}{\delta^2(8\alpha^2-2)} = L(\alpha) \to 0 \quad \text{as } \alpha \to \frac{3}{2} \quad (0 \le \sigma \le \delta) \end{aligned} \tag{4.5}$$

thus using (4.4) and (4.5) and taking α sufficiently close to $3/2$ in (4.5) we insure that

$$\|v(\sigma,\cdot)\|_{L^2(0,\infty)} \le L(\alpha) < 1 \quad 0 \le \sigma \le \delta\,. \tag{4.6}$$

In view of (4.4) and (4.6) we have constructed a control $v(\sigma, x)$ that drives 0 to $\bar{y}_\alpha(x)$ improving the norm of $\bar{u}_\alpha(\sigma, x)$. But $\bar{u}_\alpha(\sigma, x)$ is norm optimal by virtue of Theorem 3.2, thus a contradiction ensues and we are all done.

5. Weakly singular controls, III

The second counterexample involves the multiplier

$$z(x) = \frac{e^{1/2x}}{x} \quad (x > 0)\,. \tag{5.1}$$

We have

$$\kappa(\sigma,z)^2 = \int_\sigma^\infty \frac{e^{1/x}}{x^2}dx = -\int_\sigma^\infty \big(e^{1/x}\big)'dx = e^{1/\sigma} - 1\,,$$

so that

$$\kappa(\sigma,z) = \sqrt{e^{1/\sigma}-1} = e^{1/2\sigma}\sqrt{1-e^{-1/\sigma}}\,.$$

To estimate $\kappa(\sigma, z)$ near zero, we note that the positive function $f(\sigma) = e^{-1/\sigma}/\sigma$ tends to zero for $\sigma \to 0$, $\sigma \to \infty$ and (since $f'(\sigma) = e^{-1/\sigma}(1-\sigma)/\sigma^3$) has a maximum at $\sigma = 1$ where $f(1) = 1/e$. It follows that $e^{-1/\sigma} \le \sigma/e$ everywhere so that (giving up a lot)

$$\kappa(\sigma,z) = e^{1/2\sigma}(1 + O(e^{-1/\sigma})) = e^{1/2\sigma}(1 + O(\sigma))\,. \tag{5.2}$$

This estimation shows that $\kappa(\sigma, z)$ is far from integrable in $[0, \infty)$, thus the function (5.1) is a multiplier in $\mathcal{Z}$ but it does not belong to $Z(T)$. We have

$$\frac{1}{\kappa(\sigma, z)} = \frac{e^{-1/2\sigma}}{1 + O(\sigma)} = e^{-1/2\sigma}(1 + O(\sigma)) \tag{5.3}$$

for σ near zero, thus

$$\begin{aligned}\omega(T, x, z) &= \int_0^x \frac{d\sigma}{\kappa(\sigma, z)} = \int_0^x e^{-1/2\sigma} d\sigma + \int_0^x O(\sigma) e^{-1/2\sigma} d\sigma \\ &= (1 + O(x)) \int_0^x e^{-1/2\sigma} d\sigma \end{aligned} \tag{5.4}$$

since

$$\int_0^x |O(\sigma)| e^{-1/2\sigma} d\sigma \le C \int_0^x \sigma e^{-1/2\sigma} d\sigma < Cx \int_0^x e^{-1/2\sigma} d\sigma \,.$$

Integrating by parts twice the last integral in (5.4) we obtain

$$\begin{aligned}\int_0^x e^{-1/2\sigma} d\sigma &= 2 \int_0^x \sigma^2 \big(e^{-1/2\sigma}\big)' d\sigma = 2x^2 e^{-1/2x} - 4 \int_0^x \sigma e^{-1/2\sigma} d\sigma \\ &= 2x^2 e^{-1/2x} - 8 \int_0^x \sigma^3 \big(e^{-1/2\sigma}\big)' d\sigma \\ &= 2x^2 e^{-1/2x} - 8x^3 e^{-1/2x} + 24 \int_0^x \sigma^2 e^{-1/2\sigma} dx \,. \end{aligned} \tag{5.5}$$

The function $g(\sigma) = \sigma^2 e^{-1/2\sigma}$ has derivative $g'(\sigma) = e^{-1/2\sigma}(2\sigma + 1/2)$ thus $g(\sigma)$ is increasing and we can estimate the last integral in (5.5) as follows:

$$\int_0^x \sigma^2 e^{-1/2\sigma} d\sigma \le x^2 e^{-1/2x} \int_0^x d\sigma = x^3 e^{-1/2x} \,. \tag{5.6}$$

Putting together (5.4), (5.5) and (5.6) we deduce that the behavior of $\omega(T, x, z)$ near zero is described by

$$\omega(T, x, z) = 2x^2 e^{-1/2x}(1 + O(x)) \,. \tag{5.7}$$

Lemma 5.1. *The control*

$$\bar{u}(\sigma, x) = \frac{S(T-\sigma)^* z(x)}{\|S(T-\sigma)^* z(\cdot)\|} = \chi_0(x) \frac{z(x + (T - \sigma))}{\kappa(T - \sigma, z)} \quad (0 \le t \le T)$$

associated with the multiplier (5.1) *drives* 0 *to a target* $\bar{y}(\cdot) \in D(A)$ *in time* T.

The proof of Lemma 5.1 requires checking that both $\bar{y}(\cdot)$ and $\bar{y}'(\cdot)$ belong to $L^2(0, \infty)$. Observe first that $\kappa(x, z)$ is infinitely differentiable in $x > 0$; since $\kappa(x, z) \ne 0$ the same is true of $1/\kappa(x, z)$, and it follows from formula (2.10) that $\omega(T, x, z)$ is infinitely differentiable in $0 < x \le T$ and constant in $x \ge T$, the right- and left-sided derivatives different at T,

$$\omega_l'(T, T, z) = \frac{1}{\kappa(T, z)} = \frac{1}{\sqrt{e^{1/T} - 1}} \,, \quad \omega_r'(T, T, z) = 0 \,.$$

From (5.7) and (5.1) we have

$$\bar{y}(x) = \frac{e^{1/2x}}{x} 2x^2 e^{-1/2x}(1 + O(x)) = 2x(1 + O(x))$$

near $x = 0$, thus $\bar{y}(x)$ is continuous in $x \geq 0$ and the boundary condition $\bar{y}(0) = 0$ is satisfied. On the other hand

$$\begin{aligned}\bar{y}'(x) &= z'(x)\omega(T, x, z) + z(x)\omega'(T, x, z) = z'(x)\omega(T, x, z) + \frac{z(x)}{\kappa(x, z)} \\ &= \Big(- \frac{e^{1/2x}}{2x^3} - \frac{e^{1/2x}}{x^2}\Big) 2x^2 e^{-1/2x}(1 + O(x)) + \frac{e^{1/2x}}{x} e^{-1/2x}(1 + O(x))\end{aligned}$$

where we have used the equality

$$\omega'(T, x, z) = \frac{1}{\kappa(x, z)} \quad (0 < x < T)$$

(consequence of (2.10)) and (5.3) to estimate $\omega'(T, x, z)$. The bad terms cancel out and $\bar{y}(x)$ is continuously differentiable in $0 \leq x \leq T$, thus the proof of Lemma 5.1 is over.

We note that the fact that the target $\bar{y}(x)$ has a "corner" at $x = T$ is typical of targets for equation (2.4) (see for instance the graph of $\bar{y}_\alpha(x)$ in Figure 2).

Theorem 5.2. *Let $T > 0$ be arbitrary. Then the control $\bar{u}(\sigma, x)$ does not drive 0 to $\bar{y}(x)$ time or norm optimally.*

Proof. Assume $\bar{u}(\sigma, x)$ drives 0 to $\bar{y}(x)$ time or norm optimally. Then, by Theorem 1.1 $\bar{u}(\sigma, x)$ is regular, thus there exists a multiplier $\zeta(\cdot) \in Z(T)$ such that

$$\bar{u}(\sigma, x) = \frac{S(T - \sigma)^* \zeta(x)}{\|S(T - \sigma)^* \zeta(\cdot)\|} = \chi_0(x) \frac{\zeta(x + (T - \sigma))}{\lambda(T - \sigma, z)} \quad (0 \leq t \leq T)\,, \tag{5.8}$$

where

$$\lambda(x, \zeta) = \sqrt{\int_x^\infty \zeta(\sigma)^2 d\sigma}$$

and

$$\int_0^T \lambda(x, z) dx < \infty\,. \tag{5.9}$$

Since $\bar{u}(\sigma, x)$ can be represented both by (5.5) and (5.8) we have

$$\frac{\zeta(x + (T - \sigma))}{\lambda(T - \sigma, z)} = \frac{z(x + (T - \sigma))}{\kappa(T - \sigma, z)} \quad (0 \leq \sigma \leq T)\,.$$

Over a characteristic $(\sigma, x - (T - \sigma))$ we have

$$\frac{\zeta(x)}{\lambda(T - \sigma, z)} = \frac{z(x)}{\kappa(T - \sigma, z)} \quad (0 \leq \sigma \leq T)\,,$$

which implies

$$\lambda(\sigma, z) = \frac{z(x)}{\zeta(x)} \kappa(\sigma, z)\,.$$

However, this is absurd in view of (5.9) and of the fact that

$$\int_0^T \kappa(\sigma, z)d\sigma = \infty\,.$$

This ends the proof. $\square$

The multiplier (5.1) used in this example roughly corresponds to the multiplier used in [2], Section 5 for the semigroup (1.8) which satisfies (a) $\|S(T-t)^*z\|$ increases very fast as $t \to 0$, (b) $\bar{u}(t)$ drives 0 to a target $\bar{y} \in D(A)$. This example was elevated into a theorem in [3], Lemma 8.3 but the result is restricted to self-adjoint analytic semigroups, thus it cannot be applied to the right translation semigroup. It is remarkable that the present example, similar to the one in [2] works for the right translation semigroup.

The examples in this paper and [2] prompt the conjecture that growth (1.9) of the costate is the most that can be allowed for optimality irrespective of the semigroup $S(t)$; in other words, that a control associated with a costate that satisfies

$$\lim_{t\to T}(T-t)\|z(t)\|_{E^*} = \lim_{t\to T}(T-t)\,\|S(T-t)^*z\|_{E^*} = \infty \tag{5.10}$$

cannot be optimal. The evidence, of course, is insufficient to support this and it is not clear that the manipulations in [2] (much less those in this paper) could be twisted into proving a general result. It is also unknown whether there exist nonoptimal controls with associated costate satisfying (1.9).

6. Adjoints

It is worth noting that optimal problems for the system

$$y'(t) = A^*y(t) + u(t) \tag{6.1}$$

with A^* the adjoint of the operator A in (2.3) behave in a totally different way than those for the system (1.1) The operator A^* is given by

$$A^*y(x) = y'(x) \tag{6.2}$$

with domain $D(A) = \{\text{all } y(\cdot) \in L^2(0,\infty) \text{ with } y'(\cdot) \in D(A)\}$ (no boundary conditions) The semigroup $S(t)^*$ generated by A^* is the left translation semigroup (2.1) and satisfies (1.12) so that all multipliers $z^*(\cdot)$ belong to $L^2(0,\infty)$ and all norm or time optimal controls satisfy (1.5) with no conditions whatsoever on the target $\bar{y}^*(x)$. The semigroup $S(t)^*$ is associated with the control system

$$\frac{\partial y(t,x)}{\partial t} = \frac{\partial y(t,x)}{\partial x} + u(t,x)\,, \quad y(0,x) = \zeta(x)\,, \quad (0 \le t,\, x < \infty)\,, \tag{6.3}$$

in the sense that $S(t)$ is the propagation semigroup of the homogeneous equation $(u(t,x) = 0)$. Since $(S(t)^*)^* = S(t)$ all time and norm optimal controls $\bar{u}(t)$ for this system satisfy

$$\bar{u}(t) = \frac{S(T-t)z}{\|S(T-t)z\|} = \frac{S(T-t)z}{\|z\|}\,,$$

the second equality coming from the fact that $S(t)$ is isometric. This is also a sufficient condition for optimality. The first equality (2.2) implies that optimal trajectories starting at ζ are of the form

$$\begin{aligned} y^*(t) &= S(t)^*\zeta + \int_0^t S(t-\sigma)^*\bar{u}(\sigma)d\sigma \\ &= S(t)^*\zeta + \frac{1}{\|z\|}\int_0^t S(t-\sigma)^*S(T-\sigma)zd\sigma \\ &= S(t)^*\zeta + \frac{1}{\|z\|}\int_0^t S(t-\sigma)^*S(t-\sigma)S(T-t)zd\sigma \\ &= S(t)^*\zeta + \frac{S(T-t)}{\|z\|}\int_0^t zd\sigma = S(t)^*\zeta + \frac{tS(T-t)z}{\|z\|} \end{aligned}$$

and hit the target

$$\bar{y}(T) = S(T)\zeta + \frac{Tz}{\|z\|}\,.$$

The control system (6.1) is essentially the only truly infinite-dimensional example where "everything can be easily calculated". This is far from true for the control system (2.4) treated in this paper, whose only difference with (6.3) consists of the presence of a boundary condition.

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Asymptotic Behavior of a Leray Solution around a Rotating Obstacle

Giovanni P. Galdi and Mads Kyed

To Professor Herbert Amann on the occasion of his 70th birthday

Abstract. We consider a body, $\mathfrak{B}$, that rotates, without translating, in a Navier-Stokes liquid that fills the whole space exterior to $\mathfrak{B}$. We analyze asymptotic properties of steady-state motions, that is, time-independent solutions to the equation of motion written in a frame attached to the body. We prove that "weak" steady-state solutions in the sense of J. Leray that satisfy the energy inequality are Physically Reasonable in the sense of R. Finn, provided the "size" of the data is suitably restricted

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1. Introduction

Consider a rigid body, $\mathfrak{B}$, whose particles move with prescribed (Eulerian) velocity $\omega\times x$ in a Navier-Stokes liquid. Here, $\omega \in \mathbb{R}^3$, $\omega \neq 0$, and x is the spatial variable. It is well known that a prescribed velocity field of this form corresponds to a uniform rotation of $\mathfrak{B}$ with angular velocity ω.

We assume the liquid fills the whole exterior of $\mathfrak{B}$. More precisely, we assume that, at each time t, $\mathfrak{B}$ occupies a compact set of $\mathbb{R}^3$ with a connected boundary, so that, at each time t, the liquid fills an exterior domain, $\mathfrak{D} = \mathfrak{D}(t)$, of $\mathbb{R}^3$. As customary in this problem, it is convenient to refer the motion of the liquid to a frame, $\mathfrak{S}$, attached to $\mathfrak{B}$. In this way, the region occupied by the liquid becomes a

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time-independent domain, Ω, of $\mathbb{R}^3$. We shall suppose that, with respect to $\mathfrak{S}$, the motion of the liquid is steady and that it reduces to rest at large spatial distances. Thus, the equations governing the motion of the liquid in $\mathfrak{S}$ can be written in the following non-dimensional form (see, *e.g.*, [8])

$$\begin{cases} \Delta v - \nabla p - \operatorname{Re} v \cdot \nabla v + \operatorname{Ta}\big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) = f & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = v_* & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with

$$\lim_{|x|\to\infty} v = 0. \tag{1.2}$$

Here, v and p are velocity and pressure fields of the liquid in $\mathfrak{S}$, while f and v_* are prescribed functions of x. The Reynolds number Re and Taylor number Ta are dimensionless constants with $\operatorname{Re}, \operatorname{Ta} > 0$.

Mostly over the past decade, the study of the properties of solutions to (1.1), (1.2) has attracted the attention of many mathematicians, who have investigated basic issues like existence, uniqueness and asymptotic (in space) behavior; see, *e.g.*, [2, 4, 3, 9, 10, 11, 12, 13, 14] and the literature cited therein.

We wish to recall and to emphasize that the characteristic difficulty related to the investigation of (1.1), (1.2) is the presence of the term $\omega \times x \cdot \nabla v$, whose coefficient becomes unbounded as $|x| \to \infty$. For this reason, the above problem can *not* be treated as a "perturbation" to the analogous one with $\omega = 0$, even for "small" $|\omega|$.

Concerning the *existence* of solutions, there are, basically, two types of results.

On one hand, one can show that, for any f and v_* in a suitable (and quite large) class with $\int_{\partial\Omega} v_* \cdot n = 0$, there corresponds a pair (v, p), such that

$$v \in L^6(\Omega), \quad \nabla v \in L^2(\Omega)\,, \tag{1.3}$$

and $p \in L^2_{\mathrm{loc}}(\Omega)$ satisfying (1.1) in the sense of distribution, and (1.2) in an appropriate generalized sense; see [1]. In addition, v and p obey the energy *inequality*:

$$\begin{aligned} 2\int_\Omega |\mathbf{D}(v)|^2 \,\mathrm{d}x \le -\int_\Omega f \cdot v \,\mathrm{d}x + \int_{\partial\Omega} \big(\mathrm{T}(v,p)\cdot n\big)\cdot v_* \,\mathrm{d}S \\ - \frac{\operatorname{Re}}{2}\int_{\partial\Omega} |v_*|^2\, v_* \cdot n \,\mathrm{d}S + \frac{\operatorname{Ta}}{2}\int_{\partial\Omega} |v_*|^2\, \mathrm{e}_1 \times x \cdot n \,\mathrm{d}S\,, \end{aligned} \tag{1.4}$$

where $\mathrm{T}(v,p)$ and $\mathbf{D}(v)$ are the Cauchy stress and stretching tensor, respectively; see (2.1). Finally, if Ω and the data are sufficiently smooth, then v and p are likewise smooth and satisfy both (1.1) and (1.2) in the ordinary sense; see [8]. This type of solution is usually called *Leray solution*, in that they were first found by J. Leray in the case $\omega = 0$; see [15]. It must be emphasized that a Leray solution carries very little information about the behavior of v as $|x| \to \infty$, namely, (1.3), while no information at all is available for the pressure field p. It is just for this reason that in (1.4) there appears an inequality sign (instead of an equality sign) that may cast shadows about the physical meaning of Leray solution.

On the other hand, if f is sufficiently smooth and decays sufficiently fast as $|x| \to \infty$, and provided the size of the data is suitably restricted, one can show the existence of a solution (v, p) with a suitable asymptotic behavior that, in fact, verifies the energy *equality*

$$\begin{aligned} 2\int_\Omega |\mathbf{D}(v)|^2\,\mathrm{d}x = & -\int_\Omega f\cdot v\,\mathrm{d}x + \int_{\partial\Omega} \big(\mathrm{T}(v,p)\cdot n\big)\cdot v_*\,\mathrm{d}S \\ & -\frac{\mathrm{Re}}{2}\int_{\partial\Omega} |v_*|^2\, v_*\cdot n\,\mathrm{d}S + \frac{\mathrm{Ta}}{2}\int_{\partial\Omega} |v_*|^2\ \mathrm{e}_1\times x\cdot n\,\mathrm{d}S\,, \end{aligned} \tag{1.5}$$

see [10, 3]. In particular, in [10] it is shown the existence of a solution that (besides satisfying (1.5)) decays like the Stokes fundamental solution as $|x| \to \infty$, namely,

$$\begin{aligned} v(x) = O\big(|x|^{-1}\big)\,, \quad \nabla v(x) = O\big(|x|^{-2}\big)\,, \\ p(x) = O\big(|x|^{-2}\big)\,, \quad \nabla p(x) = O\big(|x|^{-3}\big)\,. \end{aligned} \tag{1.6}$$

Keeping the nomenclature introduced by R. Finn [5] for the case $\omega = 0$, solutions possessing this type of properties are called *Physically Reasonable.*

Now, while it is quite obvious that a Physically Reasonable solution is also a Leray solution, the converse is by no means obvious, even in the case of small data.

Objective of this paper is to prove that every Leray solution corresponding to data of restricted size, with f decaying sufficiently fast at large distances, is Physically Reasonable; see Theorem 4.1. The proof of this theorem exploits the method introduced in [6] for the case $\omega = 0$, and it is based on a uniqueness argument. Precisely, we shall show that a Physically Reasonable solution is *unique* (for small data) in the class of Leray solutions (see Lemma 3.3), so that the desired result follows from the existence result proved in [10]. However, for this argument to work, it is crucial to show that the pressure, p, associated to a Leray solution possesses the summability property $p \in L^3(\Omega)$. Now, while in the case $\omega = 0$ the proof of this property is quite straightforward [6], in the case at hand the proof is far from being obvious, due to the presence of the term $\omega \times x \cdot \nabla v$. Actually, it requires a detailed analysis that we develop through Lemmas 3.1 and 3.2.

The plan of the paper is the following. After recalling some standard notation in Section 2, in Section 3 we begin to establish appropriate global summability property for the pressure of a Leray solution. Successively, using also this property, we show the uniqueness of a Physically Reasonable solution corresponding to "small" data in the class of Leray solutions. Finally, in Section 4, as a corollary to this latter result and with the help of the existence theorem established in [10], we prove that every Leray solution corresponding to "small" data is, in fact, Physically Reasonable.

2. Notation

We let $L^q(\Omega)$ and $W^{m,q}(\Omega)$ denote Lebesgue and Sobolev spaces, respectively, and $\|\cdot\|_q$, $\|\cdot\|_{m,q}$ the associated norms. We write $D^{m,q}(\Omega)$ and $|\cdot|_{m,q}$ to denote homogeneous Sobolev spaces and their (semi-)norms, respectively. We will initially explicitly indicate when a function space consists of vector- or tensor-valued functions, for example $L^q(\Omega)^3$, but may omit the indication when no confusion can arise.

We will make use of the weighted norms

$$[\![f]\!]_{\alpha,A} := \operatorname*{ess\,sup}_{x\in A}\,[(1+|x|^{\alpha})|f(x)|]$$

for A a domain of $\mathbb{R}^3$, and $f : A \to \mathbb{R}^3$ measurable and $\alpha \in \mathbb{N}$. If no confusion arises, we will omit the subscript "A".

We denote by

$$\mathrm{T}(v,p) := 2\mathbf{D}(v) - pI, \quad \mathbf{D}(v) := \frac{1}{2}\big(\nabla v + \nabla v^T\big) \tag{2.1}$$

the usual Cauchy stress and stretching tensors, respectively, of a Navier-Stokes liquid corresponding to the non-dimensional form of the equations (1.1).

In what follows, $\Omega \subset \mathbb{R}^3$ will denote an exterior domain of class C^2. Without loss of generality, we assume $0 \in \mathbb{R}^3\setminus\overline{\Omega}$. For $\rho > 0$, we put $\mathrm{B}_\rho := \{x \in \mathbb{R}^3 \mid |x| < \rho\}$, $\mathrm{B}^\rho := \{x \in \mathbb{R}^3 \mid |x| \geq \rho\}$, and set $\Omega_\rho := \Omega \cap \mathrm{B}_\rho$ and $\Omega^\rho := \Omega \cap \mathrm{B}^\rho$. Moreover, we put $\mathrm{B}_{\rho_2,\rho_1} := \mathrm{B}_{\rho_2} \setminus \mathrm{B}_{\rho_1}$.

As noted in the introduction, Re and Ta are positive real constants.

We use small letters for constants $(c_1, c_2, \ldots)$ that appear only in a single proof, and capital letters $(C_1, C_2, \ldots)$ for global constants.

3. Preliminaries

In this section, we will establish, in a series of preliminary lemmas, some properties of weak solutions to (1.1).

We start by recalling the well-known inequality

$$\|v\|_6 \leq C_1 |v|_{1,2} \tag{3.1}$$

which holds for all $v \in D^{1,2}(\Omega) \cap L^6(\Omega)$ (see [7, Theorem II.5.1]). We shall frequently use (3.1) without reference.

In the first lemma, we establish (global) higher-order regularity of a weak solution.

Lemma 3.1. *Let $f \in L^2(\Omega)^3$, $v_* \in W^{\frac{3}{2},2}(\partial\Omega)^3$, and $(v,p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2_{\mathrm{loc}}(\Omega)$ be a solution to* (1.1). *Then $v \in D^{2,2}(\Omega)$.*

Proof. By standard regularity theory for elliptic systems, $v \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})$ and $p \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. We therefore only need to show $v \in D^{2,2}(\Omega^\rho)$ for some $\rho > 0$.

Choose $r > 0$ so that $\mathbb{R}^3 \backslash \Omega \subset \mathrm{B}_r$. Moreover, choose for any $R > 2r$ a function $\psi_R \in C^\infty(\mathbb{R}^3;\mathbb{R})$ with $0 \le \psi_R \le 1$, $\psi_R = 0$ in B_r, $\psi_R = 1$ in $\mathrm{B}_{R,2r}$, $\psi_R = 0$ in B^{2R}, and $|\mathrm{D}^\alpha \psi_R| \le \frac{c_1}{|x|^{|\alpha|}}$ with c_1 independent of R.

We shall test $(1.1)_1$ with $-\nabla\times(\psi_R^2\nabla\times v)$. Note that $-\nabla\times(\psi_R^2\nabla\times v) \in L^2(\mathbb{R}^3)$, has bounded support,

$$\operatorname{div}\big[-\nabla\times(\psi_R^2\nabla\times v)\big] = 0, \tag{3.2}$$

and

$$-\nabla\times(\psi_R^2\nabla\times v) = \psi_R^2\Delta v + (\nabla\times v)\times\nabla[\psi_R^2]. \tag{3.3}$$

Thus, we compute

$$\begin{aligned}
&\left|\int_\Omega (\mathrm{e}_1 \times v)\cdot\big(-\nabla\times(\psi_R^2\nabla\times v)\big)\,\mathrm{d}x\right| \\
&\quad = \left|\int_\Omega \psi_R^2(\mathrm{e}_1 \times v)\cdot(\nabla\times(\nabla\times v)) + (\mathrm{e}_1 \times v)\cdot\big((\nabla\times v)\times\nabla[\psi_R^2]\big)\,\mathrm{d}x\right| \\
&\quad = \left|\int_\Omega -(\nabla\times\psi_R^2(\mathrm{e}_1 \times v))\cdot(\nabla\times v) + \nabla[\psi_R^2]\cdot\big((\mathrm{e}_1 \times v)\times(\nabla\times v)\big)\,\mathrm{d}x\right| \qquad (3.4)\\
&\quad \le c_2\bigg(\int_\Omega |\nabla v|^2\,\mathrm{d}x + \int_{\mathrm{B}_{2R,R}} \frac{1}{R}|v||\nabla v|\,\mathrm{d}x + \int_{\mathrm{B}_{2r,r}} \frac{1}{r}|v||\nabla v|\,\mathrm{d}x\bigg) \\
&\quad \le c_3\bigg(\int_\Omega |\nabla v|^2\,\mathrm{d}x + \|v\|_6\|\nabla v\|_2\bigg) \le c_4\,|v|_{1,2}^2.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\int_\Omega (\mathrm{e}_1 \times x\cdot\nabla v)\cdot\big(-\nabla\times(\psi_R^2\nabla\times v)\big)\,\mathrm{d}x \\
&\quad = \int_\Omega \psi_R^2\,(\mathrm{e}_1 \times x\cdot\nabla v)\cdot\Delta v\,\mathrm{d}x + \int_\Omega (\mathrm{e}_1 \times x\cdot\nabla v)\cdot\big((\nabla\times v)\times\nabla[\psi_R^2]\big)\,\mathrm{d}x \\
&\quad = -\int_\Omega \nabla[\psi_R^2]\otimes(\mathrm{e}_1 \times x\cdot\nabla v):\nabla v\,\mathrm{d}x - \int_\Omega \psi_R^2\,\nabla(\mathrm{e}_1 \times x\cdot\nabla v):\nabla v\,\mathrm{d}x \\
&\qquad + \int_\Omega (\mathrm{e}_1 \times x\cdot\nabla v)\cdot\big((\nabla\times v)\times\nabla[\psi_R^2]\big)\,\mathrm{d}x.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_\Omega \psi_R^2\,\nabla(\mathrm{e}_1 \times x\cdot\nabla v):\nabla v\,\mathrm{d}x \\
&\quad = \int_\Omega \psi_R^2\,\partial_j\partial_k v_i\,(\mathrm{e}_1 \times x)_k\,\partial_j v_i\,\mathrm{d}x + \int_\Omega \psi_R^2\,\partial_k v_i\,\partial_j[\mathrm{e}_1 \times x]\,\partial_j v_i\,\mathrm{d}x \\
&\quad = -\frac{1}{2}\int_\Omega \partial_k\big[\psi_R^2\,(\mathrm{e}_1 \times x)_k\big]\,(\partial_j v_i)^2\,\mathrm{d}x + \int_\Omega \psi_R^2\,\partial_k v_i\,\partial_j[\mathrm{e}_1 \times x]\,\partial_j v_i\,\mathrm{d}x,
\end{aligned}$$

and observing that $|\partial_i\psi_R\,(\mathrm{e}_1\times x)_j| \leq c_5$ for any $i,j=1,2,3$, we may conclude

$$\left|\int_\Omega (\mathrm{e}_1\times x\cdot\nabla v)\cdot\big(-\nabla\times(\psi_R^2\nabla\times v)\big)\,\mathrm{d}x\right| \leq c_6\,|v|_{1,2}^2. \tag{3.5}$$

Next, we estimate

$$\begin{aligned}\left|\int_\Omega (v\cdot\nabla v)\cdot(\psi_R^2\Delta v)\,\mathrm{d}x\right| &\leq \int_\Omega |\psi_R\nabla v|\,|v|\,|\psi_R\Delta v|\,\mathrm{d}x\\ &\leq \|\psi_R\nabla v\|_3\,\|v\|_6\,\|\psi_R\Delta v\|_2\\ &= \|\nabla[\psi_R v]-v\otimes\nabla\psi_R\|_3\,\|v\|_6\,\|\psi_R\Delta v\|_2\\ &\leq \big(\|\nabla[\psi_R v]\|_3+\|v\otimes\nabla\psi_R\|_3\big)\,\|v\|_6\,\|\psi_R\Delta v\|_2.\end{aligned} \tag{3.6}$$

By the Nirenberg inequality, we have

$$\begin{aligned}\|\nabla[\psi_R v]\|_{3,\mathbb{R}^3} &\leq c_7\,\|\nabla[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12}\,\|\nabla^2[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12}\\ &\leq c_8\,\|\nabla[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12}\,\|\Delta[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12}\\ &\leq c_8\,\|\nabla[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12}\,\big(\|\psi_R\Delta v\|_2+2\|\nabla v\cdot\nabla\psi_R\|_2+\|\Delta\psi_R v\|_2\big)^{\frac12}.\end{aligned}$$

Since

$$\begin{aligned}\|\nabla[\psi_R v]\|_2 &\leq \|v\otimes\nabla\psi_R\|_2+\|\psi_R\nabla v\|_2\\ &\leq c_9\bigg(\int_{\mathrm{B}_{2R,R}}\frac{|v|^2}{R^2}\,\mathrm{d}x+\int_{\mathrm{B}_{2r,r}}\frac{|v|^2}{r^2}\,\mathrm{d}x\bigg)^{\frac12}+\|\nabla v\|_2\\ &\leq c_{10}\|v\|_6+\|\nabla v\|_2\leq c_{11}\,|v|_{1,2},\end{aligned}$$

and similarly

$$\|\Delta\psi_R v\|_2\leq c_{12}\,|v|_{1,2},$$

we see that

$$\|\nabla[\psi_R v]\|_{3,\mathbb{R}^3}\leq c_{13}\big(|v|_{1,2}^{\frac12}\,\|\psi_R\Delta v\|_2^{\frac12}+|v|_{1,2}\big).$$

Also,

$$\|v\otimes\nabla\psi_R\|_3\leq c_{14}\left(\int_{\mathrm{B}_{2R,R}}\frac{|v|^3}{R^3}\,\mathrm{d}x+\int_{\mathrm{B}_{2r,r}}\frac{|v|^3}{r^3}\,\mathrm{d}x\right)^{\frac13}\leq c_{15}\,\|v\|_6\leq c_{16}\,|v|_{1,2}.$$

Thus, from (3.6) we conclude that

$$\begin{aligned}\left|\int_\Omega (v\cdot\nabla v)\cdot(\psi_R^2\Delta v)\,\mathrm{d}x\right| &\leq c_{17}\big(|v|_{1,2}^{\frac12}\,\|\psi_R\Delta v\|_2^{\frac12}+|v|_{1,2}\big)\,|v|_{1,2}\,\|\psi_R\Delta v\|_2\\ &\leq c_{18}\big(|v|_{1,2}^{\frac32}\,\|\psi_R\Delta v\|_2^{\frac32}+|v|_{1,2}^2\,\|\psi_R\Delta v\|_2\big)\\ &\leq c_{19}(\varepsilon)\,\big(|v|_{1,2}^6+|v|_{1,2}^4\big)+\varepsilon\,\|\psi_R\Delta v\|_2^2\end{aligned} \tag{3.7}$$

for any $\varepsilon > 0$. In a similar manner, we estimate

$$\begin{aligned}\left|\int_\Omega (v\cdot\nabla v)\cdot\big((\nabla\times v)\times\nabla[\psi_R^2]\big)\,\mathrm{d}x\right| \\ &\le c_{20}\int_\Omega |v|\,|\nabla v|\,|\nabla\psi_R||\psi_R\nabla v|\,\mathrm{d}x \\ &\le c_{21}\,\|v\|_6\,\|\nabla v\|_2\,\|\psi_R\nabla v\|_3 \\ &\le c_{22}\,|v|_{1,2}^2\,\|\psi_R\nabla v\|_2^{\frac12}\,\|\nabla[\psi_R\nabla v]\|_2^{\frac12} \\ &\le c_{23}\,|v|_{1,2}^{\frac52}\big(\|\psi_R\Delta v]\|_2+|v|_{1,2}\big)^{\frac12} \\ &\le c_{23}\big(|v|_{1,2}^{\frac52}\,\|\psi_R\Delta v\|_2^{\frac12}+|v|_{1,2}^3\big) \\ &\le c_{24}(\varepsilon)|v|_{1,2}^{\frac{10}{3}}+\varepsilon\,\|\psi_R\Delta v\|_2^2+c_{23}|v|_{1,2}^3\end{aligned}\tag{3.8}$$

for any $\varepsilon > 0$. We also have

$$\begin{aligned}\left|\int_\Omega \Delta v\cdot\big((\nabla\times v)\times\nabla[\psi_R^2]\big)\,\mathrm{d}x\right| &\le \int_\Omega |\psi_R\Delta v|\,|\nabla v|\,|\nabla\psi_R|)\,\mathrm{d}x \\ &\le \varepsilon\,\|\psi_R\Delta v\|_2^2+c_{25}(\varepsilon)|v|_{1,2}^2\end{aligned}\tag{3.9}$$

for any $\varepsilon > 0$. Finally, we can estimate

$$\begin{aligned}\left|\int_\Omega f\cdot\big(-\nabla\times(\psi_R^2\nabla\times v)\big)\,\mathrm{d}x\right| \\ &\le \int_\Omega |f\cdot\psi_R^2\Delta v|\,\mathrm{d}x+\int_\Omega |f\cdot\big((\nabla\times v)\times\nabla[\psi_R^2]\big)|\,\mathrm{d}x \\ &\le c_{26}(\varepsilon)\,\|f\|_2^2+\varepsilon\,\|\psi_R\Delta v\|_2^2+c_{27}|v|_{1,2}\,\|f\|_2\end{aligned}\tag{3.10}$$

for any $\varepsilon > 0$. Combining now (3.4), (3.5), (3.7), (3.8), (3.9), (3.10) and recalling (3.2) and (3.3), we conclude that multiplication of $(1.1)_1$ by $-\nabla\times(\psi_R^2\nabla\times v)$ and subsequent integration over Ω yields

$$\int_\Omega \psi_R^2\,|\Delta v|^2\,\mathrm{d}x \le c_{28}(\varepsilon)\big(|v|_{1,2}^2+|v|_{1,2}^6+\|f\|_2^2\big)+\varepsilon\,\|\psi_R\Delta v\|_2^2 \tag{3.11}$$

for any $\varepsilon > 0$. Hence, by choosing $0<\varepsilon<1$ and letting $R\to\infty$ in (3.11), we infer that $\Delta v\in L^2(\Omega^r)$. It follows that $v\in D^{2,2}(\Omega^\rho)$ for $\rho > r$. In fact, by an easy calculation that takes into account the properties of the "cut-off" ψ_R, we obtain

$$\sum_{|\alpha|=2}\|\psi_R\,\mathrm{D}^\alpha v\|_{2,\Omega^r}^2 \le c_{29}\left(\|\frac{v}{|x|^2}\|_{2,\Omega^r}^2+\|\frac{\nabla v}{|x|}\|_{2,\Omega^r}^2+\sum_{|\alpha|=2}\|\mathrm{D}^\alpha(\psi_R\,v)\|_{2,\Omega^r}^2\right).$$

However, since $\psi_R v$ is of compact support, we have

$$\sum_{|\alpha|=2}\|\mathrm{D}^\alpha(\psi_R\,v)\|_{2,\mathbb{R}^3}^2 \le c_{30}\|\Delta(\psi_R\,v)\|_{2,\mathbb{R}^3}^2,$$

with c_{30} independent of R, and so, the previous inequality implies

$$\sum_{|\alpha|=2} \|\psi_R \, \mathrm{D}^\alpha v\|^2_{2,\Omega^r} \leq c_{31} \left(\|\frac{v}{|x|^2}\|^2_{2,\Omega^r} + \|\frac{\nabla v}{|x|}\|^2_{2,\Omega^r} + \|\psi_R(\Delta v)\|^2_{2,\Omega^r} \right).$$

where c_{31} is independent of R. If we use the assumption $v \in D^{1,2}(\Omega) \cap L^6(\Omega)$ in this relation, along with a Hardy-type inequality (see for example [7, Theorem II.5.1]) and the fact that $\Delta v \in L^2(\Omega^r)$, we deduce

$$\sum_{|\alpha|=2} \|\psi_R \, \mathrm{D}^\alpha v\|^2_{2,\Omega^r} \leq c_{32}, \tag{3.12}$$

where c_{32} is independent of R. The desired property for $\mathrm{D}^2 v$ then follows by letting $R \to \infty$ in (3.12). □

In the next lemma, we establish $L^3(\Omega)$-summability of the pressure. More precisely, we have:

Lemma 3.2. *Let $f \in L^2(\Omega)^3 \cap L^{\frac{3}{2}}(\Omega)^3$, $v_* \in W^{\frac{3}{2},2}(\partial\Omega)^3$, and let $(v,p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2_{\mathrm{loc}}(\Omega)$ be a corresponding solution to* (1.1)*. Then $p+c \in L^3(\Omega)$ for some constant $c \in \mathbb{R}$.*

Proof. Standard regularity theory for elliptic systems again yields $p \in W^{1,2}_{loc}(\overline{\Omega})$. Consequently, by Sobolev embedding, we have $p \in L^3_{\mathrm{loc}}(\overline{\Omega})$. We therefore only need to show $p + c \in L^3(\Omega^\rho)$ for some $\rho > 0$ and $c \in \mathbb{R}$.

Let $\rho > \mathrm{diam}(\mathbb{R}^3 \setminus \Omega)$ and $\psi \in C^\infty(\mathbb{R}^3; \mathbb{R})$ be a "cut-off" function with $\psi = 0$ on B_ρ and $\psi = 1$ on $\mathbb{R}^3 \setminus \mathrm{B}_{2\rho}$. Moreover, let

$$\sigma(x) := \left(\int_{\partial \mathrm{B}_{2\rho}} v \cdot n \, \mathrm{d}x \right) \nabla \mathfrak{E}, \quad \mathfrak{E}(x) := \frac{1}{4\pi|x|}. \tag{3.13}$$

Since

$$\begin{aligned} \int_{\mathrm{B}_{2\rho}} \nabla\psi \cdot (v + \sigma) \, dx &= \int_{\mathrm{B}_{2\rho}} \mathrm{div} \left[\psi(v + \sigma) \right] \mathrm{d}x \\ &= \int_{\partial \mathrm{B}_{2\rho}} v \cdot n \, \mathrm{d}x + \int_{\partial \mathrm{B}_{2\rho}} \sigma \cdot n \, \mathrm{d}x = 0, \end{aligned}$$

there exists (see [7, Theorem III.3.2]) a field

$$H \in W^{3,2}(\mathbb{R}^3), \quad \mathrm{supp}\, H \subset \mathrm{B}_{2\rho}, \quad \mathrm{div}\, H = \nabla\psi \cdot (v + \sigma). \tag{3.14}$$

Put

$$w = \psi v + \psi\sigma - H, \quad \pi = \psi p.$$

Using the fact that $\mathrm{e}_1 \times x \cdot \nabla\sigma - \mathrm{e}_1 \times \sigma = 0$, we find that

$$\begin{cases} \Delta w - \nabla\pi + \mathrm{Ta} \left(\mathrm{e}_1 \times x \cdot \nabla w - \mathrm{e}_1 \times w \right) = \psi f + G + \mathrm{Re}\, \psi \, v \cdot \nabla v & \text{in } \mathbb{R}^3, \\ \mathrm{div}\, w = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.15}$$

where $G \in L^2(\mathbb{R}^3)$ with $\operatorname{supp}(G) \subset \mathrm{B}_{2\rho}$. Taking divergence on both sides in (3.15) yields

$$-\Delta\pi = \operatorname{div}\big[\psi f\big] + \operatorname{div} G + \operatorname{Re} \operatorname{div}\big[\psi\, v \cdot \nabla v\big] \quad \text{in } \mathbb{R}^3 \tag{3.16}$$

in the sense of distributions. We now observe that we can write f as follows (again in the sense of distributions)

$$f = \operatorname{div} F, \quad F \in L^3(\Omega). \tag{3.17}$$

In fact, it is enough to choose $F_k = \nabla\mathfrak{E} * f_k$, where $\{f_k\}_{k=1}^{\infty} \subset C_0^{\infty}(\Omega)$ converges to f in $L^3(\Omega)$, and then pass to the limit $k \to \infty$, in the sense of distributions. We can express, again in the sense of distributions,

$$\psi\, f = \psi \operatorname{div} F = \operatorname{div}[\psi F] - F \cdot \nabla\psi.$$

Thus, introducing

$$\tilde{G} := G - F \cdot \nabla\psi, \quad \tilde{F} := \psi F, \quad \text{and} \quad \tilde{f} := \operatorname{div} \tilde{F},$$

from (3.16) we have

$$-\Delta\pi = \operatorname{div} \tilde{f} + \operatorname{div} \tilde{G} + \operatorname{Re} \operatorname{div}\big[\psi\, v \cdot \nabla v\big] \quad \text{in } \mathbb{R}^3, \tag{3.18}$$

where $\tilde{f} = \operatorname{div} \tilde{F} \in L^2(\mathbb{R}^3)$, $\tilde{F} \in L^3(\mathbb{R}^3)$, and $\tilde{G} \in L^2(\mathbb{R}^3)$ with $\operatorname{supp}(\tilde{G}) \subset \mathrm{B}_{2\rho}$. Consider now the three separate equations

$$-\Delta\pi_1 = \operatorname{div} \tilde{f} \quad \text{in } \mathbb{R}^3, \tag{3.19}$$

$$-\Delta\pi_2 = \operatorname{div} \tilde{G} \quad \text{in } \mathbb{R}^3, \tag{3.20}$$

$$-\Delta\pi_3 = \operatorname{Re} \operatorname{div}\big[\psi\, v \cdot \nabla v\big] \quad \text{in } \mathbb{R}^3, \tag{3.21}$$

with respect to unknowns π_1,π_2,π_3. Using the Riesz transformations,

$$\mathfrak{R}_j : L^q(\mathbb{R}^3) \to L^q(\mathbb{R}^3),\ \forall q > 1, \quad \mathfrak{R}_j(u) := \mathfrak{F}^{-1}\left(\frac{\xi_j}{|\xi|}\mathfrak{F}(u)\right),$$

where $\mathfrak{F}$ denotes the Fourier transformation, we find that

$$\pi_1 := \mathfrak{F}^{-1}\left(\frac{i\xi_j}{|\xi|^2}\mathfrak{F}(\tilde{f}_j)\right) = \mathfrak{F}^{-1}\left(\frac{-\xi_j\xi_k}{|\xi|^2}\mathfrak{F}(\tilde{F}_{jk})\right) = -\mathfrak{R}_j \circ \mathfrak{R}_k(\tilde{F}_{jk}) \tag{3.22}$$

is a solution to (3.19) with $\pi_1 \in L^3(\mathbb{R}^3)$. Moreover, since clearly $\tilde{G} \in L^{\frac{3}{2}}(\mathbb{R}^3)$, we can use the Riesz potential

$$\mathfrak{I} : L^{\frac{3}{2}}(\mathbb{R}^3) \to L^3(\mathbb{R}^3), \quad \mathfrak{I}(u) := \mathfrak{F}^{-1}\left(\frac{1}{|\xi|}\mathfrak{F}(u)\right)$$

to obtain a solution

$$\pi_2 := \mathfrak{F}^{-1}\left(\frac{i\xi_j}{|\xi|^2}\mathfrak{F}\big(\tilde{G}_j\big)\right) = i\,\mathfrak{R}_j \circ \mathfrak{I}\big(\tilde{G}_j\big) \tag{3.23}$$

to (3.20) with $\pi_2 \in L^3(\mathbb{R}^3)$. Similarly, putting $h := \operatorname{Re}\psi\, v\cdot\nabla v$, we have $h \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and obtain by

$$\pi_3 := \mathfrak{F}^{-1}\Big(\frac{i\xi_j}{|\xi|^2}\mathfrak{F}(h_j)\Big) = i\,\mathfrak{R}_j \circ \mathfrak{I}(h_j) \tag{3.24}$$

a solution to (3.21) with $\pi_3 \in L^3(\mathbb{R}^3)$. We furthermore conclude that

$$\partial_k \pi_1 = \mathfrak{F}^{-1}\Big(\frac{-\xi_k\xi_j}{|\xi|^2}\mathfrak{F}(\tilde{f}_j)\Big) \;= -\mathfrak{R}_j \circ \mathfrak{R}_k(\tilde{f}_j) \in L^{\frac{3}{2}}(\mathbb{R}^3), \tag{3.25}$$

$$\partial_k \pi_2 = \mathfrak{F}^{-1}\Big(\frac{-\xi_k\xi_j}{|\xi|^2}\mathfrak{F}\big(\tilde{G}_j\big)\Big) = -\mathfrak{R}_j \circ \mathfrak{R}_k\big(\tilde{G}_j\big) \in L^{\frac{3}{2}}(\mathbb{R}^3), \tag{3.26}$$

$$\partial_k \pi_3 = \mathfrak{F}^{-1}\Big(\frac{-\xi_k\xi_j}{|\xi|^2}\mathfrak{F}(h_j)\Big) \;= -\mathfrak{R}_j \circ \mathfrak{R}_k(h_j) \in L^{\frac{3}{2}}(\mathbb{R}^3), \tag{3.27}$$

for $k = 1, 2, 3$. We therefore deduce that

$$\bar{\pi}(x) := \pi_1(x) + \pi_2(x) + \pi_3(x) \tag{3.28}$$

is a solution to (3.18) with $\bar{\pi} \in L^3(\mathbb{R}^3)$ and $\nabla\bar{\pi} \in L^{\frac{3}{2}}(\mathbb{R}^3)$. Since also π satisfies the same equation, it follows that $Z := \nabla(\bar{\pi} - \pi)$ is harmonic in $\mathbb{R}^3$, so that, by the mean-value theorem, we have for each fixed $x \in \mathbb{R}^3$,

$$Z(x) = \frac{c_1}{R^3}\int_{\mathrm{B}_R(x)} \nabla(\bar{\pi} - \pi)\,\mathrm{d}y =: \frac{c_1}{R^3}\big(I_1(R) + I_2(R)\big). \tag{3.29}$$

By the Hölder inequality we find

$$|I_1(R)| \le \|\nabla\bar{\pi}\|_{\frac{3}{2}}\,|\mathrm{B}_R|^{\frac{1}{3}} \le c_2\,R. \tag{3.30}$$

Moreover, from Lemma 3.1, we have $v \in D^{2,2}(\Omega)$. Thus, $\Delta w \in L^2(\mathbb{R}^3)$, and from $(3.15)_1$ we infer

$$\frac{\nabla\pi}{(1+|x|)} \in L^2(\mathbb{R}^3).$$

Therefore, by Schwarz inequality,

$$|I_2(R)| \le c_3\,R\,\|\nabla\pi/(1+|y|)\|_2\,|\mathrm{B}_R|^{\frac{1}{2}} \le c_4\,R^{\frac{5}{2}}. \tag{3.31}$$

Combining (3.29)–(3.31) and letting $R \to \infty$, we find $Z(x) = 0$ for all $x \in \mathbb{R}^3$. Hence, $\bar{\pi} = \pi + c$, for some constant c, which concludes the proof of the lemma. $\square$

In the next lemma, we show that a weak solution satisfying the energy inequality and a solution decaying like $\frac{1}{|x|}$ must coincide under a suitable smallness condition. The proof follows essentially that of the main theorem in [6].

Lemma 3.3. *Let $f \in L^2(\Omega)^3 \cap L^{\frac{6}{5}}(\Omega)^3$, and $v_* \in W^{\frac{3}{2},2}(\partial\Omega)^3$. Moreover, let $(v, p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2_{\mathrm{loc}}(\Omega)$ be a solution to* (1.1) *that satisfies the energy inequality* (1.4). *If $(w, \pi) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2(\Omega)$ is another solution to* (1.1) *and $[\![w]\!]_1 < \frac{1}{8\mathrm{Re}}$, then $(w, \pi) = (v, p)$. In this case, (v, p) satisfies the energy equality* (1.5).

Proof. By standard regularity theory for elliptic systems, we have $(v,p),(w,\pi) \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega}) \times W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. We can thus multiply $(1.1)_1$ with w and integrate over Ω_R ($R > \mathrm{diam}(\mathbb{R}^3 \setminus \Omega)$). By partial integration, we then obtain

$$\begin{aligned} &-\int_{\Omega_R} \nabla v : \nabla w \,\mathrm{d}x + \int_{\partial \mathrm{B}_R} (\nabla v \cdot n) \cdot w \,\mathrm{d}S - \int_{\partial \mathrm{B}_R} p\,(w \cdot n)\,\mathrm{d}S \\ &\quad - \mathrm{Re} \int_{\Omega_R} (v \cdot \nabla v) \cdot w \,\mathrm{d}x + \mathrm{Ta} \int_{\Omega_R} \big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) \cdot w \,\mathrm{d}x \\ &\quad = -\int_{\partial\Omega} \big((\nabla v - pI) \cdot n\big) \cdot w \,\mathrm{d}S + \int_{\Omega_R} f \cdot w \,\mathrm{d}x. \end{aligned} \tag{3.32}$$

Analogously, by switching the roles of v and w, we get

$$\begin{aligned} &-\int_{\Omega_R} \nabla v : \nabla w \,\mathrm{d}x + \int_{\partial \mathrm{B}_R} (\nabla w \cdot n) \cdot v \,\mathrm{d}S - \int_{\partial \mathrm{B}_R} \pi\,(v \cdot n)\,\mathrm{d}S \\ &\quad - \mathrm{Re} \int_{\Omega_R} (w \cdot \nabla w) \cdot v \,\mathrm{d}x + \mathrm{Ta} \int_{\Omega_R} \big(\mathrm{e}_1 \times x \cdot \nabla w - \mathrm{e}_1 \times w\big) \cdot v \,\mathrm{d}x \\ &\quad = -\int_{\partial\Omega} \big((\nabla w - pI) \cdot n\big) \cdot v \,\mathrm{d}S + \int_{\Omega_R} f \cdot v \,\mathrm{d}x. \end{aligned} \tag{3.33}$$

We shall now examine the integrals over $\partial\,\mathrm{B}_R$ in (3.32) and (3.33) in the limit as $R \to \infty$. For this purpose, we utilize Lemma 3.2 and obtain $p \in L^3(\Omega^\rho)$ for some $\rho > 0$. Consequently, we can find a sequence $\{R_n\}_{n=1}^\infty \subset [\rho,\infty]$ so that $\lim_{n\to\infty} R_n = \infty$ and

$$\lim_{n\to\infty} \left[R_n \int_{\partial \mathrm{B}_{R_n}} |p|^3 + |\nabla v|^2 + |v|^6 + |\pi|^2 + |\nabla w|^2 + |w|^6 \,\mathrm{d}x \right] = 0. \tag{3.34}$$

We conclude that

$$\begin{aligned} \left| \int_{\partial \mathrm{B}_{R_n}} (\nabla v \cdot n) \cdot w \,\mathrm{d}S \right| &\le c_1 [\![w]\!]_1 \int_{\partial \mathrm{B}_{R_n}} \frac{|\nabla v|}{R_n} \,\mathrm{d}S \\ &\le c_2 [\![w]\!]_1 \bigg(\int_{\partial \mathrm{B}_{R_n}} |\nabla v|^2 \,\mathrm{d}S \bigg)^{\frac{1}{2}} \to 0 \quad \text{as } n \to \infty \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} \left| \int_{\partial \mathrm{B}_{R_n}} p\,(w \cdot n) \,\mathrm{d}S \right| &\le c_3 [\![w]\!]_1 \int_{\partial \mathrm{B}_{R_n}} \frac{|p|}{R_n} \,\mathrm{d}S \\ &\le c_4 [\![w]\!]_1 \bigg(R_n \int_{\partial \mathrm{B}_{R_n}} |p|^3 \,\mathrm{d}S \bigg)^{\frac{1}{3}} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.36}$$

Furthermore,

$$\left|\int_{\partial\,\mathrm{B}_{R_n}} (\nabla w\cdot n)\cdot v\,\mathrm{d}S\right| \leq c_5\left(\int_{\partial\,\mathrm{B}_{R_n}} |\nabla w|^2\,\mathrm{d}S\right)^{\frac{1}{2}}\left(\int_{\partial\,\mathrm{B}_{R_n}} |v|^6\,\mathrm{d}S\right)^{\frac{1}{6}} R_n^{\frac{2}{3}}$$
$$= c_5\left(R_n\int_{\partial\,\mathrm{B}_{R_n}} |\nabla w|^2\,\mathrm{d}S\right)^{\frac{1}{2}}\left(R_n\int_{\partial\,\mathrm{B}_{R_n}} |v|^6\,\mathrm{d}S\right)^{\frac{1}{6}}$$
$$\to 0 \quad \text{as } n\to\infty \tag{3.37}$$

and

$$\left|\int_{\partial\,\mathrm{B}_{R_n}} \pi\,(v\cdot n)\,\mathrm{d}S\right| \leq c_6\left(\int_{\partial\,\mathrm{B}_{R_n}} |\pi|^2\,\mathrm{d}S\right)^{\frac{1}{2}}\left(\int_{\partial\,\mathrm{B}_{R_n}} |v|^6\,\mathrm{d}S\right)^{\frac{1}{6}} R_n^{\frac{2}{3}} \to 0 \quad \text{as } n\to\infty. \tag{3.38}$$

We now turn our attention to the limits as $R_n\to\infty$ of the integrals over Ω_{R_n} in (3.32) and (3.33). We begin to observe that, by using a Hardy-type inequality (see for example [7, Theorem II.5.1]), we find

$$\int_\Omega |(v\cdot\nabla v)\cdot w|\,\mathrm{d}x \leq [\![w]\!]_1\left(\int_\Omega |\nabla v|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega \frac{|v|^2}{(1+|x|)^2}\,\mathrm{d}x\right)^{\frac{1}{2}} < \infty\,. \tag{3.39}$$

Consequently,

$$\lim_{n\to\infty}\int_{\Omega_{R_n}} (v\cdot\nabla v)\cdot w\,\mathrm{d}x = \int_\Omega (v\cdot\nabla v)\cdot w\,\mathrm{d}x. \tag{3.40}$$

Similarly, we have

$$\int_\Omega |(w\cdot\nabla w)\cdot v|\,\mathrm{d}x \leq [\![w]\!]_1\left(\int_\Omega |\nabla w|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega \frac{|v|^2}{(1+|x|)^2}\,\mathrm{d}x\right)^{\frac{1}{2}} < \infty$$

and thus

$$\lim_{n\to\infty}\int_{\Omega_{R_n}} (w\cdot\nabla w)\cdot v\,\mathrm{d}x = \int_\Omega (w\cdot\nabla w)\cdot v\,\mathrm{d}x. \tag{3.41}$$

Concerning the integrals involving the data f, we observe that they are both well defined, in the sense of Lebesgue, because $f\in L^{\frac{6}{5}}(\Omega)$ and $w,v\in L^6(\Omega)$. We thus find

$$\lim_{n\to\infty}\int_{\Omega_{R_n}} f\cdot v\,\mathrm{d}x = \int_\Omega f\cdot v\,\mathrm{d}x. \tag{3.42}$$

and

$$\lim_{n\to\infty}\int_{\Omega_{R_n}} f\cdot w\,\mathrm{d}x = \int_\Omega f\cdot w\,\mathrm{d}x. \tag{3.43}$$

Now put $u := v - w$. Then

$$\begin{aligned}\int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla u - \mathrm{e}_1 \times u\big) \cdot u \,\mathrm{d}x &= \int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla u\big) \cdot u \,\mathrm{d}x \\ &= \frac{1}{2} \int_{\partial \mathrm{B}_{R_n}} |u|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S = 0,\end{aligned} \tag{3.44}$$

where the last equality holds since $n = \frac{x}{|x|}$ on $\partial \mathrm{B}_{R_n}$. By the same argument, we also have

$$\int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) \cdot v \,\mathrm{d}x = \frac{1}{2} \int_{\partial\Omega} |v_*|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S \tag{3.45}$$

and

$$\int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla w - \mathrm{e}_1 \times w\big) \cdot w \,\mathrm{d}x = \frac{1}{2} \int_{\partial\Omega} |v_*|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S. \tag{3.46}$$

It follows from (3.44), (3.45), and (3.46) that

$$\begin{aligned}&\int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) \cdot w \,\mathrm{d}x + \int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla w - \mathrm{e}_1 \times w\big) \cdot v \,\mathrm{d}x \\ &\quad = \int_{\partial\Omega} |v_*|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S.\end{aligned} \tag{3.47}$$

Adding together (3.32) and (3.33), utilizing (3.47), and finally letting $n \to \infty$, we find that

$$\begin{aligned}&-2 \int_\Omega \nabla v : \nabla w \,\mathrm{d}x \\ &\qquad = \mathrm{Re}\left(\int_\Omega (v \cdot \nabla v) \cdot w \,\mathrm{d}x + \int_\Omega (w \cdot \nabla w) \cdot v \,\mathrm{d}x \right) \\ &\qquad\quad + \int_\Omega f \cdot v \,\mathrm{d}x - \int_{\partial\Omega} \big((\nabla v - pI) \cdot n\big) \cdot v_* \,\mathrm{d}S \\ &\qquad\quad + \int_\Omega f \cdot w \,\mathrm{d}x - \int_{\partial\Omega} \big((\nabla w - \pi I) \cdot n\big) \cdot v_* \,\mathrm{d}S \\ &\qquad\quad - \mathrm{Ta} \int_{\partial\Omega} |v_*|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S.\end{aligned} \tag{3.48}$$

We can now write

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x = \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega |\nabla w|^2 \,\mathrm{d}x - 2 \int_\Omega \nabla v : \nabla w \,\mathrm{d}x. \tag{3.49}$$

By assumption, (v, p) satisfies the energy inequality

$$\begin{aligned}\int_\Omega |\nabla v|^2 \,\mathrm{d}x \leq &- \int_\Omega f \cdot v \,\mathrm{d}x + \int_{\partial\Omega} \big((\nabla v - pI) \cdot n\big) \cdot v_* \,\mathrm{d}S \\ &- \frac{\mathrm{Re}}{2} \int_{\partial\Omega} |v_*|^2 \, v_* \cdot n \,\mathrm{d}S + \frac{\mathrm{Ta}}{2} \int_{\partial\Omega} |v_*|^2 \; \mathrm{e}_1 \times x \cdot n \,\mathrm{d}S.\end{aligned} \tag{3.50}$$

From the decay properties of (w,π), it is easy to verify that (w,π) satisfies the energy equality

$$\begin{aligned}\int_\Omega |\nabla w|^2 \,\mathrm{d}x = -\int_\Omega f\cdot w\,\mathrm{d}x + \int_{\partial\Omega} \big((\nabla w - \pi I)\cdot n\big)\cdot v_*\,\mathrm{d}S \\ - \frac{\mathrm{Re}}{2}\int_{\partial\Omega} |v_*|^2\, v_*\cdot n\,\mathrm{d}S + \frac{\mathrm{Ta}}{2}\int_{\partial\Omega} |v_*|^2\ \mathrm{e}_1\times x\cdot n\,\mathrm{d}S.\end{aligned} \tag{3.51}$$

Combining now (3.48), (3.49), (3.50), and (3.51), we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2\,\mathrm{d}x \le \mathrm{Re}\Big(\int_\Omega (v\cdot\nabla v)\cdot w\,\mathrm{d}x + \int_\Omega (w\cdot\nabla w)\cdot v\,\mathrm{d}x\Big) \\ - \mathrm{Re}\int_{\partial\Omega} |v_*|^2\, v_*\cdot n\,\mathrm{d}S.\end{aligned}$$

Next, we observe that

$$\begin{aligned}&\int_\Omega (u\cdot\nabla u)\cdot w\,\mathrm{d}x - \int_\Omega (w\cdot\nabla u)\cdot u\,\mathrm{d}x \\ &\quad = \int_\Omega (v\cdot\nabla v)\cdot w\,\mathrm{d}x + \int_\Omega (w\cdot\nabla w)\cdot v\,\mathrm{d}x - \int_{\partial\Omega} |v_*|^2\, v_*\cdot n\,\mathrm{d}S.\end{aligned}$$

By an argument similar to (3.39) and (3.40), all integrals above are well defined and finite. We can now conclude that

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x \le \mathrm{Re}\Big(\int_\Omega (u\cdot\nabla u)\cdot w\,\mathrm{d}x - \int_\Omega (w\cdot\nabla u)\cdot u\,\mathrm{d}x\Big)$$

and thus estimate, using again the Hardy-type inequality, this time in form

$$\int_\Omega \frac{|u|^2}{|x|^2}\,\mathrm{d}x \le 4\int_\Omega |\nabla u|^2\,\mathrm{d}x,$$

valid for all fields vanishing at the boundary $\partial\Omega$,

$$\begin{aligned}\int_\Omega |\nabla u|^2\,\mathrm{d}x &\le 2\,\mathrm{Re}\,[\![w]\!]_1 \Big(\int_\Omega \frac{|u|}{1+|x|}\,|\nabla u|\,\mathrm{d}x\Big) \\ &\le 2\,\mathrm{Re}\,[\![w]\!]_1 \Big(\int_\Omega \frac{|u|^2}{(1+|x|)^2}\,\mathrm{d}x\Big)^{\frac12}\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^{\frac12} \\ &\le 8\,\mathrm{Re}\,[\![w]\!]_1 \int_\Omega |\nabla u|^2\,\mathrm{d}x.\end{aligned} \tag{3.52}$$

We finally conclude that $u=0$ when $8\,\mathrm{Re}\,[\![w]\!]_1 < 1$. □

4. Main theorem

Our main theorem follows as a consequence of Lemma 3.3 and the fact that a solution (w,π) with the in Lemma 3.3 required properties exists, provided the data are suitably restricted [10].

Theorem 4.1. *Let* $\Omega \subset \mathbb{R}^3$ *be an exterior domain of class* C^2 *and* $\mathrm{Re}\,, \mathrm{Ta} \in (0, B]$, *for some* $B > 0$. *Suppose* $v_* \in W^{\frac{3}{2},2}(\partial\Omega)^3$ *and* $f = \operatorname{div} F$, *with*

$$\mathbf{F} := \big([\![F]\!]_2 + [\![f]\!]_3 + [\![\operatorname{div}\operatorname{div} F]\!]_4 \big) < \infty. \tag{4.1}$$

Then, there is a constant $M_1 = M_1(\Omega, B) > 0$ *such that if*

$$\mathrm{Re}\,\big(\mathbf{F} + \|v_*\|_{W^{\frac{3}{2},2}(\partial\Omega)}\big) < M_1, \tag{4.2}$$

then a weak solution $(v, p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2_{\mathrm{loc}}(\Omega)$ *to* (1.1) *that satisfies the energy inequality* (1.4)*, that is, a* Leray solution*, also satisfies, for some constant* $c \in \mathbb{R}$*,*

$$|v|_{2,2} + [\![v]\!]_1 + [\![\nabla v]\!]_2 + [\![p + c]\!]_2 + [\![\nabla p]\!]_{3,\Omega^R} \le C_2\,\big(\mathbf{F} + \|v_*\|_{W^{\frac{3}{2},2}(\partial\Omega)}\big), \tag{4.3}$$

where $C_2 = C_2(\Omega, B, R)$. *Moreover,* (v, p) *satisfies the energy equality* (1.5)*. Finally,* (v, p) *is unique* (*up to addition of a constant to* p) *in the class of weak solutions satisfying* (1.4)*.*

Proof. The existence of a solution (w, π) satisfying the properties stated for (v, p) has been established in [10, Theorem 2.1 and Remark 2.1] in the case $v_* \equiv 0$. Moreover, in [9] the methods from [10] have been further developed to also consider this more general case. Now, from (4.3) – written with w and π in place of v and p – and from (4.2), it follows that, if M_1 is taken "sufficiently small", we find, in particular, $[\![w]\!]_1 < \frac{1}{8\mathrm{Re}}$. Therefore, the stated properties for (v, p) at once follow from the uniqueness Lemma 3.3. □

Remark 4.2. The properties satisfied by the Leray solution (v, p) in Theorem 4.1 imply that (v, p) is, in fact, physically reasonable in the sense of Finn [5].

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A Remark on Maximal Regularity of the Stokes Equations

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Dedicated to Prof. Herbert Amann on the occasion of his 70th birthday

Abstract. Assuming that the Helmholtz decomposition exists in $L^q(\Omega)^n$ it is proved that the Stokes equation has maximal L^p-regularity in $L^s_\sigma(\Omega)$ for $s \in [\min\{q, q'\}, \max\{q, q'\}]$. Here $\Omega \subset \mathbb{R}^n$ is an (ε, ∞) domain with uniform C^3-boundary.

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1. Introduction and main result

For the understanding of nonlinear parabolic equations the property of maximal L^p-regularity has been proved to be very useful. For equations considered in a Hilbert space the question whether we have maximal L^p-regularity estimates reduces to the question whether the associated operator generates an analytic C_0-semigroup. But the question is more difficult to answer in a general Banach space setting. See, e.g., [Ama95], [Ama97], [DHP03], [KW01] and the references therein.

In this paper we study the property of maximal L^p-regularity for the Stokes equations that are given by

$$\begin{aligned} \partial_t u - \Delta u + \nabla p = f &\qquad \text{in } \Omega \times (0,T) \\ \operatorname{div} u = 0 &\qquad \text{in } \Omega \times (0,T) \\ u = 0 &\qquad \text{on } \partial\Omega \times (0,T) \\ u(\cdot, 0) = 0 &\qquad \text{in } \Omega \end{aligned} \tag{1.1}$$

in a possibly unbounded domain $\Omega \subset \mathbb{R}^n$. Here u denotes the velocity of the fluid and p the pressure. We are going to study this set of equations in the L^q-setting, where $q \neq 2$.

As noted above the case $q = 2$ is easier to handle. It is well known that the Stokes operator is a semibounded self-adjoint operator in $L^2_\sigma(\Omega)$. Hence, it is the generator of an analytic semigroup $(e^{tA})_{t\geq 0}$ on $L^2_\sigma(\Omega)$. Note that $L2(\Omega)^n = L2_\sigma(\Omega) \oplus G_2(\Omega)$ for all open sets Ω. For a proper definition of these spaces see (1.2).

Assuming that the Helmholtz decomposition exists in $L^q(\Omega)^n$ there is associated a projection P_q onto $L^q_\sigma(\Omega)$ called the Helmholtz projection. In this case and if Ω has a sufficiently smooth boundary we define the Stokes operator by $A_q = P_q\Delta$ with domain $D(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ in $L^q_\sigma(\Omega)$.

We say that the problem (1.1) has maximal L^p-regularity in $L^q_\sigma(\Omega)$, $1 < p, q < \infty$, if the operator

$$(\partial_t - A_q) : L^p((0,T); D(A_q)) \cap W^{1,p}((0,T); L^q_\sigma(\Omega)) \to L^p((0,T); L^q(\Omega)^n)$$

is an isomorphism.

An affirmative answer to the above question for bounded or exterior domains with smooth boundaries was first given by Solonnikov ([Sol77]). His proof makes use of potential theoretic arguments. Lateron, further proofs were obtained, e.g., by combining Giga's result on bounded imaginary powers of the Stokes operator ([Gig85]) with the Dore-Venni theorem, by Giga and Sohr [GS91a], by Grubb and Solonnikov [GS91b] using pseudo-differential techniques and by Fröhlich [Frö07] making use of the concept of weighted estimates with respect to Muckenhoupt weights. For related results see also [Gig81] and [FS94]. The half-space case was studied, e.g., in [Uka87] and [DHP01]. For results concerning infinite layers we refer to the work of Abe and Shibata [AS03b] and [AS03a], Abels [Abe05] and Abels and Wiegner [AW05]. In [Fra00], [His04] the case of an aperture domain is discussed and in [FR08] it was shown that the Stokes Operator has maximal L^p-regularity in $L^q_\sigma(\Omega)$ on tube-like domains Ω. Moreover, by applying pseudo-differential techniques, a rather large class of domains could be treated in [Abe10]. For applications of these results to the equations of Navier-Stokes, see, e.g., [Kat84], [Ama00] and [Soh01].

One problem in treating domains with unbounded boundary is that the Helmholtz decomposition does not exist in $L^q(\Omega)^n$ in general. For example Bogovskiĭ gave in [Bog86] examples of unbounded domains Ω with smooth boundaries for which the Helmholtz decomposition of $L^q(\Omega)^n$ exists only for certain values of q. For details, see also [Gal94]. In [FKS07] the authors sail around this problem by changing the basic Banach space to $L^q(\Omega) + L^2(\Omega)$ or to $L^q(\Omega) \cap L^2(\Omega)$ for $p \geq 2$ or $p \leq 2$, respectively. The majority of the works cited above treat classes of domains that yield maximal regularity in $L^q_\sigma(\Omega)$ for any $1 < q < \infty$.

In [GHHS09] the authors prove maximal L^p-regularity in $L^s_\sigma(\Omega)$ for (1.1) assuming that the Helmholtz decomposition exists in $L^s(\Omega)^n$. Indeed, their main result is the following

Proposition 1.1 ([GHHS09]). *Let $1 < p, q < \infty$, $J = (0,T)$ for some $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a domain with uniform C^3-boundary. Assume that the Helmholtz* *projection P exists for $L^q(\Omega)^n$. Then problem* (1.1) *has maximal L^p-regularity in $L^q_\sigma(\Omega)$.*

In this paper we are concerned with the question to which spaces $L^r_\sigma(\Omega)$ we can extend the maximal L^p-regularity property in this case.

Let $q \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$. We say that the Helmholtz decomposition on $L^q(\Omega)^n$ exists if

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus G^q(\Omega), \tag{1.2}$$

where

$$G^q(\Omega) := \{f \in L^q(\Omega)^n : \exists g \in L^q_{\mathrm{loc}}(\Omega) : f = \nabla g\} \text{ and } L^q_\sigma(\Omega) := G^{q'}(\Omega)^\perp.$$

Here $M^\perp \subset X'$ denotes the annihilator of $M \subset X$ for some Banach space X and $q' = \frac{q}{q-1}$ denotes the dual exponent of q. In this case, there exists the Helmholtz projection P_q from $L^q(\Omega)^n$ onto $L^q_\sigma(\Omega)$. Since $G^q(\Omega)$ and $L^q_\sigma(\Omega)$ are both reflexive we obtain

$$(L^q_\sigma(\Omega))^\perp = \left(G^{q'}(\Omega)^\perp\right)^\perp = G^{q'}(\Omega).$$

Hence, $P_{q'}$ is a projection from $L^{q'}(\Omega)^n$ onto $L^{q'}_\sigma(\Omega)$. However, it is not clear whether P_q and $P_{q'}$ are consistent, i.e.,

$$P_q f = P_{q'} f, \quad f \in L^q(\Omega) \cap L^{q'}(\Omega).$$

The main assumption on the underlying domain Ω will be the (ε,∞) property. Let $\varepsilon > 0$ and $\delta \in (0,\infty]$. Then we say that a domain $\Omega \subset \mathbb{R}^n$ is an (ε,δ) domain if for any $x, y \in \Omega$ satisfying $|x-y| < \delta$ there exists a rectifiable curve $\gamma \subset \Omega$ connecting x with y such that for any point z on γ

$$\ell(\gamma) \le \frac{1}{\varepsilon}|x-y| \tag{1.3}$$

$$\mathrm{dist}(z,\Omega^c) \ge \varepsilon \frac{|x-z||y-z|}{|x-y|} \tag{1.4}$$

holds. Here $\ell(\gamma)$ denotes the length of the path γ and $\mathrm{dist}(z,M)$ is the distance of the point z to the set M.

Now we can formulate the main result of this paper.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an (ε,∞) domain for some $\varepsilon > 0$ with uniform C^3-boundary. Moreover, assume that the Helmholtz decomposition exists in $L^q(\Omega)^n$ for some $1 < q < \infty$. Then the problem* (1.1) *has maximal L^p-regularity in $L^s_\sigma(\Omega)$ for $s \in [\min\{q,q'\}, \max\{q,q'\}]$.*

Remark 1.3. Consider the cone $K \subset \mathbb{R}^2$ which is the set of all points between the rays $M_\pm := \{x \in \mathbb{R}^2 : x_1 > 0, x_2 = \pm\kappa x_1\}$ with $(1,0) \in K$. Denote by θ the angle between the rays $M_\pm$ measured across the domain K. Using piecewise circular paths connecting the points $x, y \in K$ it is easy to see that K is an (ε,∞) domain. By a result of Bogovskiĭ for $n = 2$ the Helmholtz decomposition in $L^q(K)$, where K is a cone with angle $\theta > \pi$, exists if and only if $2/(1+\pi/\theta) < q <$

$2/(1-\pi/\theta)$. The same statement is true for cones with smooth apex. This means that the results in this paper yield maximal L^p-regularity in $L^q_\sigma(K)$ for (1.1) and $q \in (2/(1+\pi/\theta), 2/(1-\pi/\theta))$.

2. Proof of the main result

In this section we give the proof of our main result. We start with some comments about the Helmholtz projection. It is well known that existence of the Helmholtz decomposition on $L^q(\Omega)^n$ is equivalent to the solvability of the following weak Neumann problem, see [SS92] and the references therein:

There exists $C > 0$ such that for any $f \in L^q(\Omega)^n$ there exists a unique $u \in \widehat{W}^{1,q}(\Omega) := \{f \in L^q_{\mathrm{loc}}(\Omega) : \nabla f \in L^q(\Omega)^n\}$ satisfying $\|u\|_{\widehat{W}^{1,q}(\Omega)} \leq C\|f\|_{L^q(\Omega)^n}$ and

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega \langle f, \nabla \varphi \rangle, \quad \varphi \in \widehat{W}^{1,q'}(\Omega). \tag{2.1}$$

In this case, $P_q f = f - \nabla u$.

If (2.1) is uniquely solvable the mapping $J : \widehat{W}^{1,q}(\Omega) \to \left(\widehat{W}^{1,q'}(\Omega)\right)'$ defined by

$$\langle J\varphi, \Psi \rangle := \int_\Omega \langle \nabla \varphi, \nabla \Psi \rangle, \quad \Psi \in \widehat{W}^{1,q'}(\Omega), \tag{2.2}$$

is an isomorphism (even the converse is true, see [SS92, Chapter 6] for a similar result). Indeed, let $\varphi \in \widehat{W}^{1,q}(\Omega)$. Then,

$$|\langle J\varphi, \Psi \rangle| = |\int_\Omega \langle \nabla \varphi, \nabla \Psi \rangle| \leq \|\varphi\|_{\widehat{W}^{1,q}(\Omega)} \|\Psi\|_{\widehat{W}^{1,q'}(\Omega)}, \quad \Psi \in \widehat{W}^{1,q'}(\Omega).$$

Note that by assumption J is one-to-one. On the other hand, let $\varphi \in \left(\widehat{W}^{1,q'}(\Omega)\right)'$. Since $\nabla \Psi \in G^{q'}(\Omega) \subset L^{q'}(\Omega)$ there exists $f_\varphi \in L^q(\Omega)^n$ such that

$$\langle \varphi, \Psi \rangle = \int_\Omega \langle f_\varphi, \nabla \Psi \rangle = \int_\Omega \langle \nabla u, \nabla \Psi \rangle, \quad \Psi \in \widehat{W}^{1,q'}(\Omega),$$

where $u \in \widehat{W}^{1,q}(\Omega)$ is the unique solution of (2.1). Therefore J is onto as well.

The next proposition allows us to apply [GHHS09, Theorem 1.2] to our situation.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an (ε, ∞) domain with a uniform C^1 boundary. Assume that the Helmholtz projection P_p exists for some $p \in (1, \infty)$. Then the Helmholtz projection P_q exists for $q \in [\min\{p, p'\}, \max\{p, p'\}]$ where $1/p' + 1/p = 1$. Moreover, $P_p = P_q$ in $L^q(\Omega)^n \cap L^p(\Omega)^n$.*

In order to prove Proposition 2.1 we need some preparation. By adapting the arguments given in [SS92, Lemma 3.8(b)] to our situation, we obtain the following lemma.

Lemma 2.2. *Let $1 < r < q < \infty$, $\Omega \subset \mathbb{R}^n$ an (ε, ∞) domain and $\nabla p \in G_q(\Omega)$ with*

$$\sup_{0 \neq \nabla v \in C_c^\infty(\overline{\Omega})} \left| \frac{\langle \nabla p, \nabla v \rangle}{\|v\|_{\widehat{W}^{1,r'}(\Omega)}} \right| < \infty,$$

where r' is the dual exponent of r. Moreover, assume that the Helmholtz projection exists in $L^r(\Omega)^n$. Then $\nabla p \in G_r(\Omega)$.

The next proposition states some basic properties of homogeneous Sobolev spaces.

Proposition 2.3. *Let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an (ε, ∞) domain. Then $C_c^\infty(\overline{\Omega})$ is dense in $\widehat{W}^{1,p}(\Omega)$. Moreover, for $q \in (1, \infty)$ and $u \in \widehat{W}^{1,p}(\Omega) \cap \widehat{W}^{1,q}(\Omega)$ there exists $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\overline{\Omega})$ such that $\lim_{n\to\infty} u_n = u$ in $\widehat{W}^{1,p}(\Omega) \cap \widehat{W}^{1,q}(\Omega)$.*

Proof. Let $f \in \widehat{W}^{1,p}(\Omega)$. By [Jon81, Theorem 2], there exists an extension operator $E \in \mathcal{L}(\widehat{W}^{1,p}(\Omega), \widehat{W}^{1,p}(\mathbb{R}^n))$, *independent* of $p \in (1, \infty)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $\widehat{W}^{1,p}(\mathbb{R}^n)$, see [Sob63] for instance, there exists $(\varphi_n) \subset C_c^\infty(\mathbb{R}^n)$ such that $\lim_{n\to\infty} \|\varphi_n - Ef\|_{\widehat{W}^{1,p}(\mathbb{R}^n)} = 0$. Hence,

$$\lim_{n\to\infty} \|\varphi_n|_\Omega - f\|_{\widehat{W}^{1,p}(\Omega)} \leq \lim_{n\to\infty} \|\varphi_n - Ef\|_{\widehat{W}^{1,p}(\Omega)} = 0. \qquad \square$$

Remark 2.4.

1. The proof of Proposition 2.3 is based on the existence of an extension operator $E \in \mathcal{L}(\widehat{W}^{1,p}(\Omega), \widehat{W}^{1,p}(\mathbb{R}^n))$ only. In particular, whenever there exists an extension operator for some $p \in (1, \infty)$ then $C_c^\infty(\overline{\Omega})$ is dense in $\widehat{W}^{1,p}(\Omega)$.
2. O.V. Besov constructed in [Bes67] an extension operator
$$E \in \mathcal{L}(\widehat{W}^{1,p}(\Omega), \widehat{W}^{1,p}(\mathbb{R}^n)), \quad p \in (1, \infty),$$
for domains $\Omega \subset \mathbb{R}^n$ satisfying an interior horn condition.
3. Note that Proposition 2.3 does not hold for arbitrary domains as, for instance, $\widehat{W}^{1,p}(\Omega) \neq \overline{C_c^\infty(\overline{\Omega})}^{\widehat{W}^{1,p}(\Omega)}$ for aperture domains, see [Fra00].
4. Let $p \in (1, \infty)$ and $\theta \in (0, 1)$. Since the extension operator E given in Proposition 2.3 is a coretraction it follows from real interpolation theory, see, e.g., [Tri78], that
$$\left(\widehat{W}^{1,p}(\Omega), \widehat{W}^{1,p'}(\Omega)\right)_{\theta,r} = \widehat{W}^{1,r}(\Omega),$$
where $1/r = \theta/p + (1-\theta)/p'$.

We are now in the position to prove Proposition 2.1.

Proof of Proposition 2.1. Let P_p denote the Helmholtz projection on $L^p(\Omega)^n$. Then $(P_p)'$ is a projection from $L^{p'}(\Omega)^n$ onto $L^{p'}_\sigma(\Omega)$. In order to show that P_p and $(P_p)'$

are consistent note that it is enough to show that the solution u of (2.1) is consistent for $q = p, p'$. We may assume without loss of generality that $p > 2$.

Let $f \in L^p(\Omega)^n \cap L^{p'}(\Omega)^n$ and denote the corresponding solutions of (2.1) in $\widehat{W}^{1,p}(\Omega)$ and $\widehat{W}^{1,p'}(\Omega)$ by u_p and $u_{p'}$, respectively. Obviously, we have

$$\int_\Omega \langle \nabla u_p, \nabla\varphi\rangle = \int_\Omega \langle f, \nabla\varphi\rangle \le \|f\|_{L^{p'}(\Omega)}\|\varphi\|_{\widehat{W}^{1,p}(\Omega)}, \quad \varphi \in C_c^\infty(\overline{\Omega}).$$

Hence, it follows from Lemma 2.2 that $u_p \in \widehat{W}^{1,p'}(\Omega)$. Since $C_c^\infty(\overline{\Omega})$ is dense in $\widehat{W}^{1,p}(\Omega)$ we obtain

$$\int_\Omega \langle \nabla u_p, \nabla\varphi\rangle = \int_\Omega \langle f, \nabla\varphi\rangle, \quad \varphi \in \widehat{W}^{1,p}(\Omega).$$

Finally, by uniqueness of solutions to (2.1) in $\widehat{W}^{1,p'}(\Omega)$, we obtain $u_p = u_{p'}$.

Therefore P_p and $(P_p)'$ are consistent and we may write $P_{p'} = (P_p)'$. Now the proposition follows from interpolation theory.

□

Remark 2.5. 1. In order to show that $P_{p'}$ exists whenever P_p exists, we might use $J : \widehat{W}^{1,p}(\Omega) \to (\widehat{W}^{1,p'}(\Omega))'$ and duality theory.

2. The result that $P_{p'}$ exists whenever P_p exists is due to [Bog86].

Finally, we are prepared to give the

Proof of Theorem 1.2. We assume w.l.o.g. that $p > 2$. Once we establish the consistency of the resolvent problem in $L^q_\sigma(\Omega)$ and in $L^{q'}_\sigma(\Omega)$ we can use interpolation theory and Proposition 1.1 (see [GHHS09]) in order to get the claim. The consistency of the Helmholtz projection was proved in Proposition 2.1.

That means that the solution operator of the weak Neumann problem exists in $L^s(\Omega)$ for $s \in [p', p]$ and is consistent. The consistency of the Stokes resolvent problem now follows from the solution formula for the Stokes resolvent problem which was developed in [GHHS09]. In this representation the authors use an explicit solution formula for the half-space. Noting that in the half-space we have consistent resolvents it is clear that the consistency is preserved in the general case. □

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On Linear Elliptic and Parabolic Problems in Nikol'skij Spaces

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Dedicated to professor Herbert Amann, in occasion of his seventieth birthday

Abstract. We consider estimates depending on a parameter for general linear elliptic boundary value problems, with nonhomogeneous boundary conditions, in Nikol'skij spaces. These estimates are then employed to study general linear nonautonomous parabolic systems, again with nonhomogeneous boundary conditions. Maximal regularity results are proved.

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0. Introduction and notations

The aim of this paper is to illustrate some improvements concerning estimates depending on a parameter for elliptic boundary value problems with nonhomogeneous boundary conditions, and to apply them to mixed parabolic systems. The main spaces we shall work with are Nikol'skij spaces, which are a category of the class of Besov spaces: if $\beta \in \mathbb{R}$, $p \in [1, \infty]$ and Ω is a domain in $\mathbb{R}^n$, the Nikol'skij space $N_p^\beta(\Omega)$ coincides with the Besov space $B_{p,\infty}^\beta(\Omega)$. The main reason why we are interested in $N_p^\beta(\Omega)$ is that results of maximal regularity in closed subspaces of $B([0,T]; N_p^\beta(\Omega))$, the space of bounded functions with values in $N_p^\beta(\Omega)$, are available. Maximal regularity estimates in $L^p(0,T; L^p(\Omega))$ $(1 < p < \infty)$ (even for a very general class of systems) were obtained by V.A. Solonnikov (see [11]). In the case of homogeneous boundary conditions, the extension to $L^p(0,T; L^q(\Omega))$ $(1 < p, q < \infty)$ is essentially due to G. Dore and A. Venni as an application of their celebrated theorem (see [4]). An extension of this result to the case of operator-valued coefficients is proposed by R. Denk, M. Hieber, J. Prüss in [2]. The case of nonhomogeneous boundary conditions (again in the L^p-L^q setting) is treated by the same authors in [3]. They prove a remarkable intrinsic characterization of the boundary data in such a way that a solution in the class $W^{1,p}(0,T;L^q(\Omega)) \cap L^p(0,T;W^{2m,q}(\Omega))$ exists. Such characterization is in terms of vector-valued Triebel-Lizorkin spaces. However, for problems of convergence to a stationary state and asymptotic expansion of the solutions, it seems preferable to work in spaces of continuous, or, at least, bounded functions in the time variable. If A is the infinitesimal generator of an analytic semigroup in X, maximal regularity results of this type for the corresponding Cauchy problem are available in closed subspaces of the real interpolation space $(X, D(A))_{\theta,\infty}$ (see, for example, [9]). If $X = L^p(\Omega)$ and $D(A)$ is a closed subspace of $W^{2m,p}(\Omega)$, $(X, D(A))_{\theta,\infty}$ is typically a closed subspace of the Nikol'skiy space $N_p^{2m\theta}(\Omega)$ (see Section 1 for precise definitions). This makes our choice quite natural.

We pass to describe the content of the paper.

In Section 1, we recall some definitions and results concerning Nikol'skij spaces, which we shall use throughout the article. The Nikol'skij space $N_p^s(\Omega)$ ($s \in \mathbb{R}$, $p \in [1, \infty]$) coincides with the Besov space $B^s_{p,\infty}(\Omega)$. We refer to [6] for a summary of results concerning Besov spaces. Much more complete expositions can be found (for example) in [14], [10], and (in the case $s > 0$) in [1], Chapter 7. For the sake of simplicity, we have chosen to consider only the case of domains with "smooth" (that is, C^∞) boundaries, following the lines of [14], [10], [6].

In Section 2 we deal with estimates depending on a parameter for general nonhomogeneous elliptic boundary value problems in Nikol'skij spaces. Here also we shall limit ourselves to the case of smooth coefficients. It is quite well known that estimates of this type are basic to treat parabolic systems. The main result of this Section is Theorem 2.3, in which we prove an estimate depending on a parameter in the case that the solution annihilates the elliptic operator with the parameter and the boundary conditions are nonhomogeneous, and extending Proposition 2.16 in [6]. Theorem 2.3 has Theorem 2.7, containing a general estimate depending on a parameter, as a corollary.

In Section 3 we consider general parabolic systems. The estimates obtained in Section 2 allow to prove maximal regularity results for problems with nonhomogeneous boundary conditions at first in the case of autonomous systems. In this direction, the main result is Theorem 3.6. The method of the proof follows an old idea by B. Terreni (see [13]). Such idea was already employed in [5], where I treated similar problems in little-Nikol'skij spaces (the closure of smooth functions in Nikol'skij spaces). Apart the slightly different setting, here I have been able to extend considerably the range of indexes to which the results are applicable. Roughly speaking, if $N_p^\beta(\Omega)$ is the basic space, here β is allowed to vary in the interval $(\mu + p^{-1} - 2m, \nu + p^{-1})$, with ν and μ, respectively, the minimum and maximum order of the boundary conditions, while in [5] I treated only the case $\beta \in (0, p^{-1})$ (compare Theorem 3.6 with Theorem 2.1 in [5]). So we are able to treat even cases with $\beta < 0$ and the elements of the basic space are distributions. Finally we have put an extension, which does not seem completely trivial, to nonautonomous parabolic systems.

We conclude this introductory section with the indication of some notations that we shall use in the paper.

If X and Y are Banach spaces, we shall indicate with $\mathcal{L}(X,Y)$ the Banach space of linear bounded operators from X to Y. In case $X=Y$, we shall write $\mathcal{L}(X)$. If Ω is an open subset of $\mathbb{R}^n$ with smooth boundary and x belongs to the boundary $\partial\Omega$, we shall indicate with $\nu(x)$ the unit vector, which is normal to $\partial\Omega$ in x and points outside Ω, with $\frac{\partial f}{\partial \nu}(x)$ the normal derivative, and with $T_x(\partial\Omega)$ the tangent space to $\partial\Omega$ in x. Given $\theta \in (0,1)$ and $q \in [1,\infty]$, we shall indicate with $(.,.)_{\theta,q}$ the real interpolation functor, $W^{m,p}(\Omega)$ ($m \in \mathbb{N}_0$, $p \in [1,\infty]$) will be the standard Sobolev spaces. If A is a set and X is a normed space, we shall indicate with $B(A;X)$ the space of bounded functions from A to X, with its natural norm. If $\beta \in \mathbb{R}$, we set

$$[\beta] := \max\{j \in \mathbb{Z} : j < \beta\}, \{\beta\} := \beta - [\beta]. \tag{0.1}$$

If $\rho \in (0,1)$, $T \in \mathbb{R}^+$, X is a Banach space with norm $\|.\|$, and $f:[0,T] \to X$, we set

$$\|f\|_{C^\rho([0,T];X)} := \max\left\{ \|f\|_{B([0,T];X)}, \sup_{0 \le s < t \le T} \frac{\|f(t)-f(s)\|}{(t-s)^\rho} \right\}.$$

Of course, $C^\rho([0,T];X)$ is the set of functions f such that $\||f\|_{C^\rho([0,T];X)} < +\infty$. If $m \in \mathbb{N}$, $C^{m+\rho}([0,T];\, X)$ is the set of functions f in $C^m([0,T];X)$ such that $f^{(m)} \in C^\rho([0,T];X)$. We shall equip it with the norm

$$\|f\|_{C^{m+\rho}([0,T];X)} := \max\{\|f\|_{C^m([0,T];X)}, \|f^{(m)}\|_{C^\rho([0,T];X)}\}.$$

Finally, C will be used to indicate a positive constant which we are not interested to precise, and may be different from time to time. We shall often use the notation C_0, $C_1, \dots$ if we have a sequence of estimates. Other notations, concerning Nikol'skij spaces, will be introduced in Section 1.

1. Nikol'skij spaces

We start by introducing Nikol'skij spaces.

Definition 1.1. Let $n \in \mathbb{N}$, $p \in [1,+\infty[$, $\beta \in (0,1]$. We define

$$\begin{aligned} N_p^\beta(\mathbb{R}^n) &:= \{f \in L^p(\mathbb{R}^n) : [f]_{\beta,p,\mathbb{R}^n} \\ &:= \sup\nolimits_{h \in \mathbb{R}^n \setminus \{0\}} |h|^{-\beta} \|f(.+h) - 2f + f(.-h)\|_{L^p(\mathbb{R}^n)} < +\infty\}. \end{aligned} \tag{1.1}$$

If $\beta \in (1,+\infty)$, we set

$$N_p^\beta(\mathbb{R}^n) := \{f \in W^{[\beta],p}(\mathbb{R}^n) : \forall \alpha \in \mathbb{N}_0^n, |\alpha| \le [\beta], \partial^\alpha f \in N_p^{\{\beta\}}(\mathbb{R}^n)\} \tag{1.2}$$

(we recall that $\{\beta\} \in (0,1]$).

Remark 1.2. $N_p^\beta(\mathbb{R}^n)$, with the norm

$$\|f\|_{\beta,p,\mathbb{R}^n} := \|f\|_{W^{[\beta],p}(\mathbb{R}^n)} + \sum_{|\alpha|=[\beta]} [f]_{\{\beta\},p,\mathbb{R}^n}$$

is a Banach space. If $\beta \in (0,1)$ an equivalent norm can be obtained replacing in (1.1) $\|f(.+h)-2f+f(.-h)\|_{L^p(\mathbb{R}^n)}$ with $\|f(.+h)-f\|_{L^p(\mathbb{R}^n)}$ (see [12], Theorem 4).

Now we define the space $N_p^\beta(\mathbb{R}^n)$ in case $\beta \le 0$.

Definition 1.3. Let $\beta \in \mathbb{R}$, $\beta \le 0$, $p \in [1,+\infty]$. We set

$$N_p^\beta(\mathbb{R}^n) := \Big\{ f = \sum_{|s|\le|[\beta]|} \partial^s f_s : f_s \in N_p^{\{\beta\}}(\mathbb{R}^n) \Big\},$$

equipped with the norm

$$\|f\|_{\beta,p,\mathbb{R}^n} := \inf\Big\{ \sum_{|s|\le|[\beta]|} \|f_s\|_{\{\beta\},p,\mathbb{R}^n} : f = \sum_{|s|\le|[\beta]|} \partial^s f_s, f_s \in N_p^{\{\beta\}}(\mathbb{R}^n) \Big\}. \quad (1.3)$$

Definition 1.4. Let $n \in \mathbb{N}$, $p \in [1,\infty]$, $\beta \in \mathbb{R}$ and let Ω be an open subset of $\mathbb{R}^n$. We set

$$N_p^\beta(\Omega) := \{ f_{|\Omega} : f \in N_p^\beta(\mathbb{R}^n)\}. \quad (1.4)$$

If $g \in N_p^\beta(\Omega)$, we set

$$\|g\|_{\beta,p,\Omega} := \inf\{\|f\|_{\beta,p,\mathbb{R}^n} : f_{|\Omega} = g\}. \quad (1.5)$$

We shall write $\mathcal{C}^\beta(\Omega)$ in alternative to $N_\infty^\beta(\Omega)$.

Remark 1.5. If $\beta \in \mathbb{R}^+ \setminus \mathbb{Z}$, $\mathcal{C}^\beta(\Omega)$ coincides with the space of elements in $C^{[\beta]}(\overline{\Omega})$ whose derivatives of order $[\beta]$ are Hölder continuous of exponent $\{\beta\}$.

For the sake of simplicity, here we shall consider only the case that the boundary $\partial\Omega$ of Ω is "smooth", that is, $\partial\Omega$ is a C^∞ submanifold of $\mathbb{R}^n$, and Ω lies on one side of $\partial\Omega$, although this restrictive condition could be considerably relaxed.

The spaces $N_p^\beta(\Omega)$ are locally invariant with respect to compositions with smooth (that is, C^∞) diffeomorphisms (see [14], 2.10). This implies that we can define by local charts the spaces $N_p^\beta(S)$ if S is a smooth submanifold of $\mathbb{R}^n$. The following fact holds (see [6], Theorem 1.19):

Theorem 1.6. *Assume* $1 \le p \le +\infty$, $\beta > m + p^{-1}$, *with* $m \in \mathbb{N}_0$. *Then there exists* $T \in \mathcal{L}(N_p^\beta(\Omega), \prod_{j=0}^m N_p^{\beta-j-1/p}(\partial\Omega))$, *such that, if* $u \in N_p^\beta(\Omega) \cap C^m(\overline{\Omega})$, $Tu = (u_{|\partial\Omega}, \dots, \frac{\partial^m u}{\partial\nu^m})$.

Nikol'skij spaces arise in real interpolation theory. If $\theta \in (0,1)$ and $m_0, m_1 \in \mathbb{N}_0$, with $m_0 < m_1$, we have

$$(W^{m_0,p}(\Omega), W^{m_1,p}(\Omega))_{\theta,\infty} = N_p^{(1-\theta)m_0+\theta m_1}(\Omega), \quad (1.6)$$

with equivalent norms (see [1], 7.32). In general, if $\beta_0, \beta_1 \in \mathbb{R}$ and $\theta \in (0,1)$,

$$(N_p^{\beta_0}(\Omega), N_p^{\beta_1}(\Omega))_{\theta,\infty} = N_p^{(1-\theta)\beta_0+\theta\beta_1}(\Omega), \quad (1.7)$$

again with equivalent norms (see [14], 3.3.6). Finally, if β_0, $\beta_1 \in \mathbb{R}$, $m \in \mathbb{N}_0$, $\beta_0 < m < \beta_1$, and $m = (1-\theta)\beta_0 + \theta\beta_1$, with $\theta \in (0,1)$, $\forall f \in N_p^{\beta_1}(\Omega)$,

$$\|f\|_{W^{m,p}(\Omega)} \le C\|f\|_{\beta_0,p,\Omega}^{1-\theta}\|f\|_{\beta_1,p,\Omega}^{\theta}, \quad (1.8)$$

with $C > 0$, independent of f. This is a consequence of the fact that

$$(N_p^{\beta_0}(\Omega), N_p^{\beta_1}(\Omega))_{\theta,1} = B^m_{p,1}(\Omega) \hookrightarrow W^{m,p}(\Omega),$$

(see [14, 3.3.6]), where $B^m_{p,1}(\Omega)$ is a Besov space.

2. Elliptic problems depending on a parameter with nonhomogeneous boundary conditions

Let now Ω be a bounded open subset of $\mathbb{R}^n$, with smooth boundary $\partial\Omega$. We consider the partial linear operator of order $2m$ ($m \in \mathbb{N}$)

$$A(x, \partial_x) = \sum_{|s|\le 2m} a_s(x)\partial_x^s, \tag{2.1}$$

with smooth coefficients in $\overline{\Omega}$. We indicate with $A_0(x, \partial_x)$ its principal part $\sum_{|s|=2m} a_s(x)\partial_x^s$. For $k = 1, \dots, m$, we introduce partial linear operators

$$B_k(x, \partial_x) = \sum_{|s|\le \mu_k} b_{ks}(x)\partial_x^s, \tag{2.2}$$

again with smooth coefficients in $\overline{\Omega}$. We assume the following:

(H1) *$A(x, \partial_x)$ is strongly elliptic, in the sense that there exists $c > 0$, such that, $\forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n$, $(-1)^m \operatorname{Re}(A_0(x, \xi)) \le -c|\xi|^{2m}$.*

(H2) *$0 \le \mu_1 < \mu_2 < \dots < \mu_m \le 2m - 1$; moreover, if we indicate with $B_{k0}(x, \partial_x)$ the principal part of $B_k(x, \partial_x)$, $B_{k0}(x, \nu(x)) \ne 0$ for each $k = 1, \dots, m$.*

(H3) *$\forall \theta_0 \in [-\pi/2, \pi/2]$, $\forall x \in \partial\Omega$, $\forall (\xi', r) \in (T_x(\partial\Omega) \times [0, +\infty)) \setminus \{0, 0)\}$, the o.d.e. problem*

$$\begin{cases} (A_0(x, i\xi' + \nu(x)\partial_t) - r^{2m} e^{i\theta_0})v(t) = 0, & t \ge 0, \\ B_{k0}(x, i\xi' + \nu(x)\partial_t)v(0) = 0, & k = 1, \dots, m, \\ v \text{ bounded in } \mathbb{R}^+ \end{cases}$$

has only the trivial solution.

The following theorem summarizes in its statement Lemmas 2.12 and 2.14 (using also Theorem 2.17) in [6]:

Theorem 2.1. *Let Ω be a bounded open subset in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$, and let $A(x, \partial_x)$ and, for $k = 1, \dots, m$, $B_1(x, \partial_x), \dots, B_m(x, \partial_x)$ operators satisfying the conditions* (H1)–(H3). *We assume that $\beta \in \mathbb{R}$, $1 \le p \le \infty$, $\mu + p^{-1} - 2m < \beta$, with $\mu := \max_{1\le k\le m} \mu_k$. We consider the problem*

$$\begin{cases} \lambda u(x) - A(x, \partial_x)u(x) = 0, & x \in \Omega, \\ B_k(x', \partial_x)u(x') - g_k(x') = 0 & x' \in \partial\Omega, \quad k = 1, \dots, m, \end{cases} \tag{2.3}$$

with $\lambda \in \mathbb{C}$, $\operatorname{Re}(\lambda) \ge 0$, for $k = 1, \dots, m$, $g_k \in N_p^{2m+\beta-\mu_k}(\Omega)$. Then:

(I) *there exist* r_0 *and* C *positive, such that, if* $|\lambda| \geq r_0$, (2.3) *has a unique solution* u *belonging to* $N_p^{2m+\beta}(\Omega)$.

(II) *The following estimate holds:*

$$\begin{aligned} &|\lambda|^{1+\frac{\beta}{2m}} \|u\|_{L^p(\Omega)} + |\lambda| \|u\|_{\beta,p,\Omega} + \|u\|_{2m+\beta,p,\Omega} \\ &\quad \leq C \sum_{k=1}^{m} (\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k}{2m}} \|g_k\|_{L^p(\Omega)}). \end{aligned} \tag{2.4}$$

(III) *If, moreover,* $0 < \{\beta\} < p^{-1}$, *the following estimate holds:*

$$\begin{aligned} &|\lambda|^{1+\frac{\beta}{2m}} \|u\|_{L^p(\Omega)} + |\lambda| \|u\|_{\beta,p,\Omega} + \|u\|_{2m+\beta,p,\Omega} \\ &\quad \leq C \sum_{k=1}^{m} (\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+[\beta]-\mu_k}{2m}} \|g_k\|_{\{\beta\},p,\Omega}). \end{aligned} \tag{2.5}$$

Remark 2.2. In the statements of Lemmas 2.12 and 2.14 in [6], concerning the formula corresponding to (2.4), there appear summands of the form $|\lambda|^\rho \|g_k\|_{W^{l,p}(\Omega)}$ and $|\lambda|^\rho \|g_k\|_{l,p,\Omega}$, with l intermediate between, respectively, $2m+\beta-\mu_k$ and 0, and $2m+\beta-\mu_k$ and $\{\beta\}$. They can be all skipped, using (1.8) and (1.7).

We want to generalize estimate (2.5). More precisely, we want to prove the following

Theorem 2.3. *Assume that the assumptions of Theorem* 2.1 *are satisfied. Let* $\lambda \in \mathbb{C}$, *with* $\mathrm{Re}(\lambda) \geq 0$ *and* $|\lambda| \geq r_0$, *with* r_0 *as in the statement of Theorem* 2.1. *Further, let* $\theta \in (0, 1/p)$. *Then, there exists* $C > 0$, *independent of* λ *and* g_k $(1 \leq k \leq m)$, *such that*

$$\begin{aligned} &|\lambda|^{1+\frac{\beta}{2m}} \|u\|_{L^p(\Omega)} + |\lambda| \|u\|_{\beta,p,\Omega} + \|u\|_{2m+\beta,p,\Omega} \\ &\quad \leq C \Big(\sum_{k=1}^{m} \|g_k\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}} \|g_k\|_{\theta,p,\Omega} \Big). \end{aligned} \tag{2.6}$$

To this aim, we need the following lemmas:

Lemma 2.4. *Assume* $1 \leq p < +\infty$, $0 < \beta \leq \theta < \frac{1}{p}$, $g \in N_p^\theta(\mathbb{R}^n_+)$ *and vanishes if* $x_n > r$. *Then:*

(I) *there exists* $C > 0$, *independent of* g, *such that,* $\forall r \in \mathbb{R}^+$,

$$\|g\|_{L^p(\mathbb{R}^n_+)} \leq C r^\theta \|g\|_{\theta,p,\mathbb{R}^n_+};$$

(II) *there exists* $C > 0$, *independent of* g, *such that,* $\forall r \in \mathbb{R}^+$,

$$\|g\|_{\beta,p,\mathbb{R}^n_+} \leq C r^{\theta-\beta} \|g\|_{\theta,p,\mathbb{R}^n_+};$$

Proof. Concerning (I), see Lemma 2.13 in [6].

Concerning (II), from (I) and the fact that

$$N_p^\beta(\mathbb{R}^n_+) = (L^p(\mathbb{R}^n_+), N_p^\theta(\mathbb{R}^n_+))_{\beta/\theta,\infty},$$

we obtain

$$\|g\|_{\beta,p,\mathbb{R}^n_+} \leq C_1 \|g\|_{L^p(\mathbb{R}^n)}^{1-\beta/\theta} \|g\|_{\theta,p,\mathbb{R}^n_+}^{\beta/\theta} \leq C_2 (r^\theta \|g\|_{\theta,p,\mathbb{R}^n_+})^{1-\beta/\theta} \|g\|_{\theta,p,\mathbb{R}^n_+}^{\beta/\theta}. \qquad \square$$

Lemma 2.5. *Assume* $1 \le p < +\infty$, $\beta \in (0,1]$, $\theta \in (0, \frac{1}{p} \wedge \beta)$, *and let* $\{\chi_r\}_{0<r\le 1}$ *be a family of* C^1 *functions in* $[0,+\infty)$, *such that*

(a) $\chi_r(t) = 0$ *if* $t \ge r$ $\forall r \in (0,1]$;

(b) $|\chi_r(t)| + r|\chi_r'(t)| \le A$, $\forall r \in (0,1]$, $\forall t \ge 0$, *for some* $A \in \mathbb{R}^+$.

Then:

(I) *there exists* $C \in \mathbb{R}^+$, *depending only on* θ, p, A, *such that,* $\forall g \in N_p^\theta(\mathbb{R}^n_+)$, $\forall r \in (0,1]$, *if* $g_r(x', x_n) = g(x', x_n)\chi_r(x_n)$,

$$\|g_r\|_{\theta,p,\mathbb{R}^n_+} \le C\|g\|_{\theta,p,\mathbb{R}^n_+};$$

(II) *assume, moreover, that the functions* χ_r *are of class* C^2, *and* $r^2|\chi_r''(t)| \le A, \forall r \in (0,1]$, $\forall t \ge 0$. *Then,* $\forall g \in N_p^\beta(\mathbb{R}^n_+)$, $\forall r \in (0,1]$,

$$\|g_r\|_{\beta,p,\mathbb{R}^n_+} \le C(\|g\|_{\beta,p,\mathbb{R}^n_+} + r^{\theta-\beta}\|g\|_{\theta,p,\mathbb{R}^n_+}).$$

Proof. (I) is essentially proved in [5], Lemma 1.6 (see also [6], Lemma 2.13).

Concerning (II), we shall use the following equivalent norm in $N_p^\beta(\mathbb{R}^n_+)$ (see for this [8], Ch. VII, Theorem 2.1):

$$\|g\|'_{\beta,p,\mathbb{R}^n_+}. := \|g\|_{L^p(\mathbb{R}^n_+)} + \sum_{j=1}^n \sup_{t>0} t^{-\beta}\|g(. + 2te^j) - 2g(. + te^j) + g\|_{L^p(\mathbb{R}^n_+)}.$$

Now, let $g \in N_p^\beta(\mathbb{R}^n_+)$. Obviously,

$$\|g_r\|_{L^p(\mathbb{R}^n_+)} + \sum_{j=1}^{n-1} \sup_{t>0} t^{-\beta}\|g_r(. + 2te^j) - 2g_r(. + te^j) + g_r\|_{L^p(\mathbb{R}^n_+)} \le C\|g\|'_{\beta,p,\mathbb{R}^n_+},$$

with C depending only on A. Moreover,

$$\begin{aligned}
&\|g_r(. + 2te^n) - 2g_r(. + te^n) + g_r\|_{L^p(\mathbb{R}^n_+)} \\
&\quad \le \|(\chi_r(x_n + 2t) - \chi_r(x_n + t))(g(. + 2te^n) - g(. + te^n))\|_{L^p(\mathbb{R}^n_+)} \\
&\qquad + \|(\chi_r(x_n + 2t) - 2\chi_r(x_n + t) + \chi_r(x_n))g(. + te^n))\|_{L^p(\mathbb{R}^n_+)} \\
&\qquad + \|(\chi_r(x_n + t) - \chi_r(x_n))(g(. + 2te^n) - g(. + te^n))\|_{L^p(\mathbb{R}^n_+)} \\
&\qquad + \|\chi_r(x_n)(g(. + 2te^n) - 2g(. + te^n) + g)\|_{L^p(\mathbb{R}^n_+)} := I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.7}$$

First of all,

$$t^{-\beta}I_4 \le C\|g\|'_{\beta,p,\mathbb{R}^n_+}. \tag{2.8}$$

Next,

$$t^{-\beta}I_1 \le C\|\chi_r\|_{C^{\beta-\theta}([0,+\infty[)}\|g\|_{\theta,p,\mathbb{R}^n_+} \le Cr^{\theta-\beta}\|g\|_{\theta,p,\mathbb{R}^n_+}, \tag{2.9}$$

owing to well-known interpolatory inequalities, which are consequences of (a)–(b). $t^{-\beta}I_3$ can be estimated similarly. Concerning $t^{-\beta}I_2$, we start by observing that

$$(\chi_r(x_n + 2t) - 2\chi_r(x_n + t) + \chi_r(x_n))g(x + te^n) = 0$$

if $x_n > r$. So, from Lemma 2.4, we have

$$t^{-\beta}I_2 \le Cr^\theta t^{-\beta}\|(\chi_r(x_n + 2t) - 2\chi_r(x_n + t) + \chi_r(x_n))g(. + te^n)\|_{\theta,p,\mathbb{R}^n_+}.$$

Assume first that $t \geq r$. Then, by (I)

$$\begin{aligned} & r^{\theta} t^{-\beta} \|(\chi_r(x_n+2t) - 2\chi_r(x_n+t) + \chi_r(x_n))g(.+te^n)\|_{\theta,p,\mathbb{R}^n_+} \\ & \quad = r^{\theta} t^{-\beta} \|\chi_r(x_n) g(.+te^n)\|_{\theta,p,\mathbb{R}^n_+} \leq C r^{\theta} t^{-\beta} \|g\|_{\theta,p,\mathbb{R}^n_+} \leq C r^{\theta-\beta} \|g\|_{\theta,p,\mathbb{R}^n_+}. \end{aligned}$$

Finally, assume $0 < t < r$. We define

$$c_{r,t}(s) := r t^{-1}(\chi_r(s+2t) - 2\chi_r(s+t) + \chi_r(s)).$$

Then, $\forall r, t$,

$$|c_{r,t}(s)| + r|c'_{r,t}(s)| \leq 4A.$$

From (I) we deduce

$$\begin{aligned} t^{-\beta} I_2 &= r^{-1} t^{1-\beta} \|c_{r,t}(x_n) g(.+te^n)\|_{L^p(\mathbb{R}^n_+)} \\ &\leq C r^{\theta-1} t^{1-\beta} \|c_{r,t}(x_n) g(.+te^n)\|_{\theta,p,\mathbb{R}^n_+} \\ &\leq C r^{\theta-1} t^{1-\beta} \|g\|_{\theta,p,\mathbb{R}^n_+}, \leq C r^{\theta-\beta} \|g\|_{\theta,p,\mathbb{R}^n_+}. \end{aligned}$$

Collecting together the previous estimates, we obtain the conclusion. □

Remark 2.6. An example of family of functions satisfying conditions (a) and (b) in the statement of Lemma 2.5 can be obtained setting $\chi_r(t) := \chi(\frac{t}{r})$, with $\chi : [0,+\infty) \to \mathbb{C}$ suitably smooth.

Now, we are able to prove Theorem 2.3:

Proof of Theorem 2.3. For every $x^0 \in \partial\Omega$ there exist an open neighbourhood U of x^0, and a smooth diffeomorphism $\Phi : U \to \Phi(U) \subseteq \mathbb{R}^n$, such that $\Phi(x^0) = 0$, $\Phi(U \cap \partial\Omega) = \Phi(U) \cap \{(y', 0) : y' \in \mathbb{R}^{n-1}\}$ and $\Phi(U \cap \Omega) = \Phi(U) \cap \mathbb{R}^n_+$. As Ω is bounded, there exist $x^1, \dots, x^N$, with corresponding diffeomorphisms $\Phi^1, \dots, \Phi^N$, such that $\partial\Omega \subseteq \cup_{j=1}^N U_j$. We add an open subset U_0 of Ω, with $\overline{U_0} \subseteq \Omega$, such that $\overline{\Omega} \subseteq \cup_{j=0}^N U_j$. Let $\{\psi_0, \dots, \psi_N\}$ be a smooth partition of unity in a neighbourhood of $\overline{\Omega}$, such that the support of ψ_j is contained in U_j $(j = 0, \dots, N)$. Let $(\chi_r)_{0<r\leq 1}$ be a family of smooth functions of domain $[0,+\infty)$, such that $\chi_r(t) = 1$ if $0 \leq t \leq r/2$, $\chi_r(t) = 0$ if $t \geq r$, $|\chi_r^{(l)}(t)| \leq C_l r^{-l}$, $\forall r \in (0,1]$, $\forall l \in \mathbb{N}_0$, $\forall t \geq 0$. We put, for $k = 1, \dots, m$, $r > 0$ sufficiently small

$$g_{kr}(x) := \sum_{j=1}^{N} \psi_j(x) \chi_r(\Phi_j(x)_n) g_k(x).$$

Then, applying Theorem 2.1 (II), we have $\forall r \in (0, r_0]$,

$$\begin{aligned} & |\lambda|^{1+\frac{\beta}{2m}} \|u\|_{L^p(\Omega)} + |\lambda| \|u\|_{\beta,p,\Omega} + \|u\|_{2m+\beta,p,\Omega} \\ & \quad \leq C \sum_{k=1}^{m} (\|g_{kr}\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k}{2m}} \|g_{kr}\|_{L^p(\Omega)}), \end{aligned} \tag{2.10}$$

with $C > 0$, independent of λ, g_k $(1 \le k \le m)$ and $r \in (0, r_0]$. We have

$$\|g_{kr}\|_{L^p(\Omega)} \le \sum_{j=1}^{N} \|\psi_j \chi_r(\Phi_j(x)_n) g_k\|_{L^p(\Omega)} \le C_1 \sum_{j=1}^{N} \|\chi_r(y_n)(\psi_j g_k \circ \Phi_j^{-1})\|_{L^p(\mathbb{R}^n_+)}.$$

We set $g^* := \psi_j g_k \circ \Phi_j^{-1}$. Then, by Lemmas 2.4 and 2.5,

$$\|\chi_r(y_n) g^*\|_{L^p(\mathbb{R}^n_+)} \le C_1 r^\theta \|\chi_r(y_n) g^*\|_{\theta,p,\mathbb{R}^n_+}, \le C_2 r^\theta \|g^*\|_{\theta,p,\mathbb{R}^n_+}, \le C_3 r^\theta \|g_k\|_{\theta,p,\Omega}. \tag{2.11}$$

Moreover,

$$\begin{aligned}\|g_{kr}\|_{2m+\beta-\mu_k,p,\Omega} &\le \sum_{j=1}^{N} \|\psi_j \chi_r(\Phi_j(x)_n) g_k\|_{2m+\beta-\mu_k,p,\Omega} \\ &\le C_4 \sum_{j=1}^{N} \|\chi_r(y_n)(\psi_j g_k \circ \Phi_j^{-1})\|_{2m+\beta-\mu_k,p,\mathbb{R}^n_+}.\end{aligned}$$

We have

$$\|\chi_r(y_n) g^*\|_{2m+\beta-\mu_k,p,\mathbb{R}^n_+} \le C_5 \sum_{s_1+|s_2|\le 2m+[\beta]-\mu_k} \|\chi_r^{(s_1)}(y_n) \partial^{s_2} g^*\|_{\{\beta\},p,\mathbb{R}^n_+}. \tag{2.12}$$

Assume first that $\{\beta\} \le \theta$. Then, if $|s_2| \ge 1$, we have, employing Lemma 2.5,

$$\begin{aligned}\|\chi_r^{(s_1)}(y_n)\partial^{s_2} g^*\|_{\{\beta\},p,\mathbb{R}^n_+} &\le r^{-s_1} \|r^{s_1}\chi_r^{(s_1)}(y_n)\partial^{s_2} g^*\|_{\{\beta\},p,\mathbb{R}^n_+} \\ &\le C r^{-s_1} \|\partial^{s_2} g^*\|_{\{\beta\},p,\mathbb{R}^n_+} \le C r^{-s_1} \|g^*\|_{|s_2|+\{\beta\},p,R^n_+} \\ &\le C r^{|s_2|-(2m+[\beta]-\mu_k)} \|g_k\|_{|s_2|+\{\beta\},p,\Omega}.\end{aligned} \tag{2.13}$$

If $s_2 = 0$,

$$\begin{aligned}\|\chi_r^{(s_1)}(y_n) g^*\|_{\{\beta\},p,\mathbb{R}^n_+} &\le C r^{\theta-\{\beta\}} \|\chi_r^{(s_1)}(y_n) g^*\|_{\theta,p,\mathbb{R}^n_+} \le C r^{\theta-s_1-\{\beta\}} \|g^*\|_{\theta,p,\mathbb{R}^n_+} \\ &\le C r^{\theta-(2m+\beta-\mu_k)} \|g_k\|_{\theta,p,\Omega}.\end{aligned} \tag{2.14}$$

Instead, we assume that $\theta < \{\beta\}$. Again applying Lemma 2.5, we have

$$\begin{aligned}&\|\chi_r^{(s_1)}(y_n)\partial^{s_2} g^*\|_{\{\beta\},p,\mathbb{R}^n_+} = r^{-s_1} \|r^{s_1}\chi_r^{(s_1)}(y_n)\partial^{s_2} g^*\|_{\{\beta\},p,\mathbb{R}^n_+} \\ &\le C r^{-s_1} (\|\partial^{s_2} g^*\|_{\{\beta\},p,\mathbb{R}^n_+} + r^{\theta-\{\beta\}} \|\partial^{s_2} g^*\|_{\theta,p,\mathbb{R}^n_+}) \qquad (2.15) \\ &\le C(r^{|s_2|+\{\beta\}-(2m+\beta-\mu_k)} \|g_k\|_{|s_2|+\{\beta\},p,\Omega} + r^{|s_2|+\theta-(2m+\beta-\mu_k)} \|g_k\|_{|s_2|+\theta,p,\Omega}).\end{aligned}$$

Observe now that the last expressions in (2.13), (2.14), (2.15) can be all majorized with

$$C(\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + r^{\theta-(2m+\beta-\mu_k)} \|g_k\|_{\theta,p,\Omega}). \tag{2.16}$$

In fact, if $\theta < \rho < 2m+\beta-\mu_k$, by (1.7),

$$\begin{aligned}r^{\rho-(2m+\beta-\mu_k)} \|g_k\|_{\rho,p,\Omega} &\le C(r^{\theta-(2m+\beta-\mu_k)} \|g_k\|_{\theta,p,\Omega})^{\frac{2m+\beta-\mu_k-\rho}{2m+\beta-\mu_k-\theta}} \|g_k\|_{2m+\beta-\mu_k,p,\Omega}^{\frac{\rho-\theta}{2m+\beta-\mu_k-\theta},\infty} \\ &\le C(\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + r^{\theta-(2m+\beta-\mu_k)} \|g_k\|_{\theta,p,\Omega},).\end{aligned}$$

So, from (2.10), we obtain, $\forall r \in (0, r_0]$,

$$
\begin{aligned}
&|\lambda|^{1+\frac{\beta}{2m}}\|u\|_{L^p(\Omega)} + |\lambda|\|u\|_{\beta,p,\Omega} + \|u\|_{2m+\beta,p,\Omega} \\
&\quad \leq C\sum_{k=1}^{m}(\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + r^{\theta-(2m+\beta-\mu_k)}\|g_k\|_{\theta,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k}{2m}}r^{\theta}\|g_k\|_{\theta,p,\Omega}).
\end{aligned} \tag{2.17}
$$

Choosing $r = |\lambda|^{-1/(2m)}$, we obtain the conclusion. □

As a consequence, we obtain the following refinement of Propositions 2.15–2.16 in [6]:

Theorem 2.7. *Let Ω be a bounded open subset in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$, and let $A(x,\partial_x)$ and, for $k = 1,\dots,m$, $B_1(x,\partial_x),\dots,B_m(x,\partial_x)$ operators satisfying the conditions* (H1)–(H3)*. We assume that $\beta \in \mathbb{R}$, $\mu + p^{-1} - 2m < \beta < \nu + p^{-1}$, with $\mu := \max_{1\leq k\leq m}\mu_k$, $\nu := \min_{1\leq k\leq m}\mu_k$. Let $\theta \in (0,p^{-1})$ be such that $\nu+\theta \geq \beta$. We consider the problem*

$$
\begin{cases} \lambda u(x) - A(x,\partial_x)u(x) = f(x), & x \in \Omega, \\ B_k(x',\partial_x)u(x') - g_k(x') = 0 & x' \in \partial\Omega, \quad k = 1,\dots,m, \end{cases} \tag{2.18}
$$

with $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq 0$, $f \in N_p^{\beta}(\Omega)$, for $k = 1,\dots,m$ $g_k \in N_p^{2m+\beta-\mu_k}(\Omega)$.

Then, there exist r_0 and C positive, such that, if $|\lambda| \geq r_0$, (2.18) *has a unique solution u belonging to $N_p^{2m+\beta}(\Omega)$. Moreover, the following estimate holds:*

$$
\begin{aligned}
&|\lambda|\|u\|_{\beta,p,\Omega} + \|u\|_{2m+\beta,p,\Omega} \\
&\quad \leq C(\|f\|_{\beta,p,\Omega} + \sum_{k=1}^{m}\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|g_k\|_{\theta,p,\Omega}).
\end{aligned} \tag{2.19}
$$

Proof. Consider a strongly elliptic operator $\overline{A}(x,\partial)$ with coefficients which are smooth and bounded with all their derivatives in $\mathbb{R}^n$, such that the restriction of $\overline{A}(x,\partial)$ to Ω coincides with $A(x,\partial)$ (see the proof of Proposition 2.15 in [6] for the construction of $\overline{A}(x,\partial)$). We fix $f' \in N_p^{\beta}(\mathbb{R}^n)$, such that $f'_{|\Omega} = f$ and

$$
\|f'\|_{\beta,p,\mathbb{R}^n} \leq C\|f\|_{\beta,p,\Omega}. \tag{2.20}
$$

Then, if $|\lambda|$ is large enough and $\mathrm{Re}(\lambda) \geq 0$, owing to Proposition 2.5 in [6], there exists a unique $u' \in N_p^{2m+\beta}(\mathbb{R}^n)$, such that

$$
\lambda u' - \overline{A}(x,\partial)u' = f' \quad \text{in} \quad \mathbb{R}^n \tag{2.21}
$$

and

$$
|\lambda|\|u'\|_{\beta,p,\mathbb{R}^n} + \|u'\|_{2m+\beta,p,\mathbb{R}^n} \leq C\|f'\|_{\beta,p,\mathbb{R}^n}. \tag{2.22}
$$

This implies that, $\forall \rho \in [\beta, 2m+\beta]$,

$$
\|u'\|_{\rho,p,\mathbb{R}^n} \leq C(\rho)|\lambda|^{\frac{\rho-(2m+\beta)}{2m}}\|f'\|_{\beta,p,\mathbb{R}^n}. \tag{2.23}
$$

We indicate with v the restriction of u' to Ω. Now we consider the problem

$$\begin{cases} \lambda z(x) - A(x,\partial_x)z(x) = 0, \quad x \in \Omega, \\ B_k(x',\partial_x)z(x') - (g_k(x') - B_k(x',\partial_x)v(x')) = 0, \quad x' \in \partial\Omega, \quad k = 1,\dots,m, \end{cases} \tag{2.24}$$

By Theorem 2.3, if $|\lambda|$ is sufficiently large, (2.24) has a unique solution z in $N_p^{2m+\beta}(\Omega)$. Clearly, $u := v + z$ is the unique solution of (2.18) in $N_p^{2m+\beta}(\Omega)$. Moreover, employing (2.23),

$$\begin{aligned}
&|\lambda|\|u\|_{\beta,p,\Omega} + \|u\|_{2m+\beta,p,\Omega} \\
&\quad \leq |\lambda|\|u'\|_{\beta,p,\mathbb{R}^n} + \|u'\|_{2m+\beta,p,\mathbb{R}^n} + |\lambda|\|z\|_{\beta,p,\Omega} + \|z\|_{2m+\beta,p,\Omega} \\
&\quad \leq C[\|f'\|_{\beta,p,\mathbb{R}^n} + \sum_{k=1}^{m}(\|g_k - B_k(.,\partial)v\|_{2m+\beta-\mu_k,p,\Omega} \\
&\qquad + |\lambda|^{\frac{(2m+\beta-\mu_k)-\theta}{2m}}\|g_k - B_k(.,\partial)v\|_{\theta,p,\Omega})] \\
&\quad \leq C[\|f\|_{\beta,p,\Omega} + \sum_{k=1}^{m}(\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|g_k\|_{\theta,p,\Omega}) \\
&\qquad + \sum_{k=1}^{m}(\|B_k(.,\partial)v\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|B_k(.,\partial)v\|_{\theta,p,\Omega})] \\
&\quad \leq C[\|f\|_{\beta,p,\Omega} + \sum_{k=1}^{m}(\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|g_k\|_{\theta,p,\Omega}) \\
&\qquad + \|v\|_{2m+\beta,p,\Omega} + \sum_{k=1}^{m}|\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|v\|_{\mu_k+\theta,p,\Omega}] \\
&\quad \leq C[\|f\|_{\beta,p,\Omega} + \sum_{k=1}^{m}(\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|g_k\|_{\theta,p,\Omega})]. \qquad \square
\end{aligned}$$

We conclude the section with the following simple result of perturbation:

Theorem 2.8. *Assume that the assumptions of Theorem* 2.7 *are satisfied. Let* $\beta' \in \mathbb{R}$, $\beta' < \beta$, $P \in \mathcal{L}(N_p^{2m+\beta'}(\Omega), N_p^{\beta}(\Omega))$ *and let* $\lambda \in \mathbb{C}$. *We consider the problem*

$$\begin{cases} \lambda u(x) - A(x,\partial_x)u(x) - Pu(x) = f(x), \quad x \in \Omega, \\ B_k(x',\partial_x)u(x') - g_k(x') = 0, \quad x' \in \partial\Omega, \quad k = 1,\dots,m. \end{cases} \tag{2.25}$$

Then there exist $\theta_0 \in (\pi/2,\pi)$, $r_0, C \in \mathbb{R}^+$, *such that, if* $|\lambda| \geq r_0$, $|Arg(\lambda)| \leq \theta_0$, $f \in N_p^{\beta}(\Omega)$, *for* $k = 1,\dots,m$ $g_k \in N_p^{2m+\beta-\mu_k}(\Omega)$, (2.18) *has a unique solution* u *belonging to* $N_p^{2m+\beta}(\Omega)$. *Moreover, an estimate like* (2.19) *holds.*

Proof. Obviously, we may assume that $2m + \beta' \geq \beta$. By the well-known continuation method, it suffices to prove an a priori estimate. So, let $u \in N_p^{2m+\beta}(\Omega)$ be a solution of (2.25). We fix $\mu \in \mathbb{C}$, such that $|\mu| = |\lambda|$, and $\mathrm{Re}(\mu) \geq 0$. Then (2.25) can be written in the equivalent form

$$\begin{cases} \mu u(x) - A(x,\partial_x)u(x) = f(x) + (\mu-\lambda)u(x) + Pu(x), \quad x \in \Omega, \\ B_k(x',\partial_x)u(x') - g_k(x') = 0 \quad x' \in \partial\Omega, \quad k = 1,\dots,m, \end{cases} \tag{2.26}$$

so that, if $|\lambda|$ is sufficiently large, we obtain from Theorem 2.7 the estimate

$$\begin{aligned} &|\lambda|\|u\|_{\beta,p,\Omega} + |\lambda|^{\frac{\beta-\beta'}{2m}}\|u\|_{2m+\beta',p,\Omega} + \|u\|_{2m+\beta,p,\Omega} \\ &\quad \leq C_0(\|f\|_{\beta,p,\Omega} + \sum_{k=1}^m \|g_k\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|g_k\|_{\theta,p,\Omega} \\ &\qquad + \|u\|_{2m+\beta',p,\Omega} + |\lambda-\mu|\|u\|_{\beta,p,\Omega}). \end{aligned} \tag{2.27}$$

If $\theta_0 - \frac{\pi}{2}$ is sufficiently small, it is possible to choose μ in such a way that $C_0\frac{|\lambda-\mu|}{|\lambda|} \leq \frac{1}{2}$. If $|\lambda|$ is so large that $C_0|\lambda|^{\frac{\beta'-\beta}{2m}} \leq \frac{1}{2}$, from (2.27) we immediately deduce (3.10). □

Remark 2.9. In case $f = 0$, the following more general estimate holds: if $\beta' \leq \alpha \leq 2m+\beta$,

$$\|u\|_{\alpha,p,\Omega} \leq C|\lambda|^{\frac{\alpha-\beta}{2m}-1}\sum_{k=1}^m(\|g_k\|_{2m+\beta-\mu_k,p,\Omega} + |\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|g_k\|_{\theta,p,\Omega}). \tag{2.28}$$

This follows from Theorem 2.8 if $\beta \leq \alpha \leq 2m+\beta$. The case $\beta' \leq \alpha < \beta$ follows from

$$\|u\|_{\alpha,p,\Omega} = |\lambda|^{-1}\|[A(.,\partial_x)+P]u\|_{\alpha,p,\Omega} \leq C|\lambda|^{-1}\|u\|_{2m+\alpha,p,\Omega}.$$

3. Parabolic problems

We start with the autonomous parabolic system

$$\begin{cases} D_t u(t,x) = A(x,\partial_x)u(t,x) + Pu(t,x) + f(t,x), \quad (t,x) \in [0,T]\times\Omega, \\ B_k(x',\partial_x)u(t,x') - g_k(t,x') = 0, \quad k = 1,\dots,m, \quad (t,x') \in [0,T]\times\partial\Omega \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{3.1}$$

under the following assumptions:

(I1) *Ω is an open bounded subset of $\mathbb{R}^n$, with smooth boundary $\partial\Omega$,*

$$A(x,\partial_x) = \sum_{|s|\leq 2m} a_s(x)\partial_x^s$$

is a linear partial differential operator of order $2m$ ($m \in \mathbb{N}$), and, for $k = 1,\dots,m$

$$B_k(x,\partial_x) = \sum_{|s|\leq \mu_k} b_{ks}(x)\partial_x^s$$

is a linear partial differential operator, all with smooth coefficients in $\overline{\Omega}$.

(I2) *The conditions* (H1)–(H3) *are fulfilled.*

(I3) $\beta \in (\mu + p^{-1} - 2m, \nu + p^{-1})$, *with* $\mu := \max_{1 \le k \le m} \mu_k$, $\nu := \min_{1 \le k \le m} \mu_k$.

(I4) $P \in \mathcal{L}(N_p^{2m+\beta'}(\Omega), N_p^{\beta}(\Omega))$, *for some* $\beta' < \beta$.

We introduce the following operator in $N_p^{\beta}(\Omega)$:

$$\begin{cases} D(A) := \{u \in N_p^{2m+\beta}(\Omega) : B_k(., \partial_x)u_{|\partial\Omega} = 0, k = 1, \dots, m\}, \\ \quad Au := A(., \partial_x)u + Pu. \end{cases} \tag{3.2}$$

By Theorem 2.8, A is a sectorial operator (with not dense domain) in $N_p^{\beta}(\Omega)$. We use θ_0 with the same meaning as in the statement of Theorem 2.8. Then we can construct an analytic semigroup $\{T(t) : t > 0\}$ (not strongly continuous in 0) in a standard way (see [9]): we fix a piecewise C^1 path γ, describing $\{\lambda \in \mathbb{C} \setminus \{0\} : |Arg(\lambda)| = \theta_1, |\lambda| \ge r_0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : |Arg(\lambda)| \le \theta_1, |\lambda| = r_0\}$, with r_0 as in the statement of Theorem 2.8, $\theta_1 \in (\pi/2, \theta_0)$, γ oriented from $\infty e^{-i\theta_1}$ to $\infty e^{i\theta_1}$, and set, for $t > 0$,

$$T(t) := \frac{1}{2\pi i} \int_\gamma e^{\lambda t} (\lambda - A)^{-1} d\lambda. \tag{3.3}$$

If $T \in \mathbb{R}^+$, for $t \in (0, T]$ $T(t)$ satisfies the estimate

$$\|T(t)f\|_{\beta,p,\Omega} + t\|T(t)f\|_{2m+\beta,p,\Omega} \le C(T)\|f\|_{\beta,p,\Omega}. \tag{3.4}$$

We consider also the operator

$$T^{(-1)}(t) := \int_0^t T(s)ds = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} \lambda^{-1} (\lambda - A)^{-1} d\lambda. \tag{3.5}$$

This operator satisfies, for $T \in \mathbb{R}^+$, an estimate of the form

$$\|T^{(-1)}(t)f\|_{\beta,p,\Omega} + t\|T^{(-1)}(t)f\|_{2m+\beta,p,\Omega} \le C(T)t\|f\|_{\beta,p,\Omega}. \tag{3.6}$$

It is convenient to fix $\beta'' \in (\beta', \beta) \cap (\mu + p^{-1} - 2m, \nu + p^{-1})$, and consider the analogues of A, $T(t)$ and $T^{(-1)}(t)$ in $N_p^{\beta''}(\Omega)$. We shall indicate these operators with $A_{\beta''}$, $T_{\beta''}(t)$ and $T_{\beta''}^{(-1)}(t)$ respectively. Clearly they satisfy estimates of the form (3.4) and (3.6), with β'' replacing β. Of course, A, $T(t)$ and $T^{(-1)}(t)$ are the parts of $A_{\beta''}$, $T_{\beta''}(t)$ and $T_{\beta''}^{(-1)}(t)$ in $N_p^{\beta}(\Omega)$.

We pass to consider nonhomogeneous boundary conditions. Let $i \in \{1, \dots, m\}$ and take $g \in N_p^{2m+\beta-\mu_i}(\Omega)$. We indicate with $N_i(\lambda)g$ the solution of system (2.25), in case $f = 0$, $g_k = 0$ if $k \ne i$, $g_i = g$. If $t > 0$, we set

$$K_i(t)g := (2\pi i)^{-1} \int_\gamma e^{\lambda t} N_i(\lambda) g d\lambda. \tag{3.7}$$

We have:

Lemma 3.1. *Let* $T \in \mathbb{R}^+$, $k \in \{1, \dots, m\}$, $g \in N_p^{2m+\beta-\mu_k}(\Omega)$, $\beta' \le \alpha \le 2m + \beta$, $\theta \in (0, p^{-1})$, *with* $\nu + \theta \ge \beta$. *Then:*

(I) *there exists $C > 0$, depending only on T and α, such that*

$$\|K_k(t)g\|_{\alpha,p,\Omega} \leq Ct^{-\frac{\alpha-\beta}{2m}}(\|g\|_{2m+\beta-\mu_k,p,\Omega} + t^{-\frac{2m+\beta-\mu_k-\theta}{2m}}\|g\|_{\theta,p,\Omega});$$

(II) $K_k \in C^1(\mathbb{R}^+; \mathcal{L}(N_p^{2m+\beta-\mu_k}(\Omega); N_p^\alpha(\Omega))$ *and*

$$K_k'(t) = (2\pi i)^{-1}\int_\gamma e^{\lambda t}\lambda N_k(\lambda)gd\lambda; \tag{3.8}$$

(III) $\|K_k'(t)g\|_{\alpha,p,\Omega} \leq Ct^{-1-\frac{\alpha-\beta}{2m}}(\|g\|_{2m+\beta-\mu_k,p,\Omega} + t^{-\frac{2m+\beta-\mu_k-\theta}{2m}}\|g\|_{\theta,p,\Omega});$

(IV) $\forall t \in \mathbb{R}^+$ $[A(.,\partial_x) + P]K_k(t)g = K_k'(t)g;$

(V) $\forall t \in \mathbb{R}^+$, $\forall i \in \{1,\dots,m\}$, $B_i(.,\partial_x)K_k(t)g_{|\partial\Omega} = 0;$

(VI) *if ξ is positive and sufficiently large, $\xi \in \rho(A)$ and*

$$(\xi - A)^{-1}K_k(t)g = (2\pi i)^{-1}\int_\gamma e^{\lambda t}(\xi-\lambda)^{-1}N_k(\lambda)gd\lambda.$$

Proof. Modifying γ, if necessary, we may assume, owing to Cauchy's theorem, that $|\lambda| \geq Tr_0$ $\forall\lambda \in \gamma$. So, by Remark 2.9,

$$\begin{aligned}
\|K_k(t)g\|_{\alpha,p,\Omega} &= \|(2\pi i)^{-1}\int_{t^{-1}\gamma} e^{\lambda t}N_i(\lambda)gd\lambda\|_{\alpha,p,\Omega} \\
&= \|(2\pi it)^{-1}\int_\gamma e^{\lambda}N_i(t^{-1}\lambda)gd\lambda\|_{\alpha,p,\Omega} \\
&\leq Ct^{-1}\int_\gamma e^{\mathrm{Re}(\lambda)}|t^{-1}\lambda|^{\frac{\alpha-\beta}{2m}-1}(\|g\|_{2m+\beta-\mu_k,p,\Omega} + |t^{-1}\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|g\|_{\theta,p,\Omega})|d\lambda|,
\end{aligned}$$

which implies the conclusion.

(II) is obvious.

(III) can be obtained from (II), with the same method of (I).

(IV) follows from

$$\begin{aligned}
K_k(t)g &= (2\pi i)^{-1}\int_\gamma e^{\lambda t}[A(.,\partial_x) + P]N_k(\lambda)gd\lambda \\
&= (2\pi i)^{-1}\int_\gamma e^{\lambda t}\lambda N_k(\lambda)gd\lambda.
\end{aligned}$$

(V) follows from

$$\begin{aligned}
B_i(.,\partial_x)K_k(t)g_{|\partial\Omega} &= (2\pi i)^{-1}\int_\gamma e^{\lambda t}[B_i(.,\partial_x)N_k(\lambda)g]_{|\partial\Omega}d\lambda \\
&= (2\pi i)^{-1}\int_\gamma e^{\lambda t}\delta_{ik}g_{|\partial\Omega}d\lambda = 0.
\end{aligned}$$

(VI) The first statement is a consequence of the fact that A is a sectorial operator. Concerning the formula, observe first that, if $\xi \neq \lambda$,

$$(\xi - A)^{-1}N_k(\lambda)g = (\xi-\lambda)^{-1}[N_k(\lambda) - N_k(\xi)]g.$$

It follows

$$(\xi - A)^{-1}K_k(t)g = (2\pi i)^{-1}\int_\gamma e^{\lambda t}(\xi-\lambda)^{-1}N_k(\lambda)gd\lambda - (2\pi i)^{-1} \times \int_\gamma e^{\lambda t}(\xi-\lambda)^{-1}N_k(\xi)gd\lambda,$$

and the second integral is 0. □

We deduce from Lemma 3.1, by choosing $\theta > \beta - \nu$, that, at least as an element of $\mathcal{L}(N_p^{2m+\beta-\mu_k}(\Omega), N_p^{\beta}(\Omega))$, we can define the following operator

$$K_k^{(-1)}(t) := \int_0^t K_k(s)ds = (2\pi i)^{-1}\int_\gamma \frac{e^{\lambda t}}{\lambda}N_k(\lambda)d\lambda. \tag{3.9}$$

We have:

Lemma 3.2. *Let $T \in \mathbb{R}^+$, $k \in \{1,\dots,m\}$, $g \in N_p^{2m+\beta-\mu_k}(\Omega)$, $\beta' \le \alpha \le 2m+\beta$, $\theta \in (0, p^{-1})$, with $\nu + \theta \ge \beta$. Then:*

(I) *there exists $C > 0$, depending only on T and α, such that*

$$\|K_k^{(-1)}(t)g\|_{\alpha,p,\Omega} \le Ct^{1-\frac{\alpha-\beta}{2m}}(\|g\|_{2m+\beta-\mu_k,p,\Omega} + t^{-\frac{2m+\beta-\mu_k-\theta}{2m}}\|g\|_{\theta,p,\Omega});$$

(II) $\forall t \in \mathbb{R}^+$ $[A(.,\partial_x)+P]K_k^{(-1)}(t)g = K_k(t)g;$

(III) $\forall t \in \mathbb{R}^+$, $\forall i \in \{1,\dots,m\}$, $B_i(.,\partial_x)K_k^{(-1)}(t)g_{|\partial\Omega} = \delta_{ik}g_{|\partial\Omega}$.

Proof. (I) can be obtained with the same arguments in the proof of Lemma 3.1.

(II) follows from

$$\begin{aligned} K_k^{(-1)}(t)g &= (2\pi i)^{-1}\int_\gamma \frac{e^{\lambda t}}{\lambda}[A(.,\partial_x)+P]N_k(\lambda)gd\lambda \\ &= (2\pi i)^{-1}\int_\gamma e^{\lambda t}N_k(\lambda)gd\lambda. \end{aligned}$$

Concerning (III),

$$\begin{aligned} B_i(.,\partial_x)K_k^{(-1)}(t)g_{|\partial\Omega} &= (2\pi i)^{-1}\int_\gamma \frac{e^{\lambda t}}{\lambda}[B_i(.,\partial_x)N_k(\lambda)g]_{|\partial\Omega}d\lambda \\ &= (2\pi i)^{-1}\int_\gamma \frac{e^{\lambda t}}{\lambda}\delta_{ik}g_{|\partial\Omega}d\lambda = \delta_{ik}g_{|\partial\Omega}. \end{aligned}$$ □

After these preliminaries, we pass to consider system (3.1). We begin with the case $u_0 = 0$ and $g_k \equiv 0$ for each $k = 1,\dots,m$. Here we have ([7], Theorem 2.7) that

$$N_p^{\beta}(\Omega) = (N_p^{\beta''}(\Omega), D(A_{\beta''}))_{\frac{\beta-\beta''}{2m}}. \tag{3.10}$$

As $A_{\beta''}$ is a sectorial operator, the following characterization of $N_p^\beta(\Omega)$ holds (see [9], Proposition 2.2.2): is $[\xi_0, \infty) \subseteq \rho(A_{\beta''})$,

$$N_p^\beta(\Omega) = \Big\{ f \in N_p^{\beta''}(\Omega) : \sup_{\xi \geq \xi_0} \|\xi^{\frac{\beta-\beta''}{2m}} A_{\beta''}(\xi - A_{\beta''})^{-1} f\|_{\beta'',p,\Omega} < \infty \Big\}. \tag{3.11}$$

Moreover, an equivalent norm in $N_p^\beta(\Omega)$ is

$$f \to \|f\|_{\beta'',p,\Omega} + \sup_{\xi \geq \xi_0} \|\xi^{\frac{\beta-\beta''}{2m}} A_{\beta''}(\xi - A_{\beta''})^{-1} f\|_{\beta'',p,\Omega}.$$

Employing Theorem 4.3.8 in [9], we deduce the following

Lemma 3.3. *Assume that the assumptions* (I1)–(I4) *are satisfied. Consider the system* (3.1) *in case* $f \in C([0,T]; N_p^{\beta''}(\Omega)) \cap B([0,T]; N_p^\beta(\Omega))$, $g_k \equiv 0$ *for each* $k = 1, \dots, m$, *and* $u_0 = 0$. *Then there is a unique solution* u *belonging to* $C^1([0,T]; N_p^{\beta''}(\Omega)) \cap B([0,T]; N_p^{2m+\beta}(\Omega))$, *with* $D_t u \in B([0,T]; N_p^\beta(\Omega))$. u *can be represented by the variation of parameter formula*

$$u(t) = \int_0^t T(t-s) f(s) ds, \quad t \in [0,T]. \tag{3.12}$$

Next, we consider the case $f \equiv 0$, $u_0 = 0$, $g_i \equiv 0$ if $i \neq k$ ($k \in \{1, \dots, m\}$). We shall often use the following

Lemma 3.4. *Let* $T \in \mathbb{R}^+$, *let* β_0, β_1 *be real numbers, such that* $\beta_0 < \beta_1$, $\rho \in \mathbb{R}^+$, $p \in [1, +\infty]$, *and let* $g \in B([0,T]; N_p^{\beta_1}(\Omega)) \cap C^\rho([0,T]; N_p^{\beta_0}(\Omega))$. *Then, if* $\xi \in (0,1)$ *and* $(1-\xi)\rho \notin \mathbb{Z}$, $g \in C^{(1-\xi)\rho}([0,T]; N_p^{(1-\xi)\beta_0 + \xi\beta_1}(\Omega))$. *Moreover, there exists* $C > 0$, *independent of* g, *such that*

$$\|g\|_{C^{(1-\xi)\rho}([0,T]; N_p^{(1-\xi)\beta_0+\xi\beta_1}(\Omega))} \leq C(\|g\|_{B([0,T]; N_p^{\beta_1}(\Omega))} + \|g\|_{C^\rho([0,T]; N_p^{\beta_0}(\Omega))}).$$

Proof. Employing the characterization of $C^{(1-\xi)\rho}([0,T]; N_p^{(1-\xi)\beta_0+\xi\beta_1}(\Omega))$ by higher-order differences (see [14], 3.4.2), we have, taking $l \in \mathbb{N}$, $l \geq \beta_1$, $h \in (0, T/l]$ and $t \in [0, T - lh]$,

$$\begin{aligned} & h^{-(1-\xi)\rho} \Big\| \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} g(t+jh) \Big\|_{(1-\xi)\beta_0+\xi\beta_1,p,\Omega} \\ & \leq C_0 \Big(h^{-\rho} \Big\| \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} g(t+jh) \Big\|_{\beta_0,p,\Omega} \Big)^{1-\xi} \Big\| \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} g(t+jh) \Big\|_{\beta_1,p,\Omega}^{\xi} \\ & \leq C_1 \|g\|_{C^\rho([0,T]; N_p^{\beta_0}(\Omega))}^{1-\xi} \|g\|_{B([0,T]; N_p^{\beta_1}(\Omega))}^{\xi}. \end{aligned}$$

□

Now we consider the first case with nonhomogeneous boundary conditions.

Lemma 3.5. *Assume that the assumptions* (I1)–(I4) *are satisfied. Consider the system* (3.1) *in case* $f \equiv 0$, $u_0 = 0$, $g_i \equiv 0$ *for each* $i = 1, \dots, m$ *and* $i \neq k$,

$g_k = g$. *We assume that* $g \in B([0,T]; N_p^{2m+\beta-\mu_k}(\Omega)) \cap C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T]; N_p^{\theta}(\Omega))$ *for some* $\theta \in (0, p^{-1})$, *and* $g(0) = 0$. *Consider the function*

$$u(t) := \int_0^t K_k(t-s)g(s)ds. \tag{3.13}$$

Then:

(I) u *is well defined in* $[0,T]$*;*

(II) u *belongs to*

$$C^1([0,T]; N_p^{\beta''}(\Omega)) \cap B([0,T]; N_p^{2m+\beta}(\Omega))), \ D_t u \in B([0,T];\ N_p^{\beta}(\Omega)));$$

(III) u *satisfies* (3.1).

Proof. By Lemma 3.4, it is not restrictive to assume that

$$\theta > \beta - \nu \geq \beta - \mu_k, \tag{3.14}$$

in such a way that $\frac{2m+\beta-\mu_k-\theta}{2m} < 1$. Now we observe that

$$u(t) = K_k^{(-1)}(t)g(t) + \int_0^t K_k(t-s)[g(s) - g(t)]ds := v_0(t) + v_1(t). \tag{3.15}$$

From Lemma 3.2 and the assumptions on g, we deduce that $v_0 \in B([0,T]; N_p^{2m+\beta}$ $(\Omega))$ and

$$[A(.,\partial_x) + P]v_0(t) = K_k(t)g(t), \tag{3.16}$$

$\forall i \in \{1, \ldots, m\}$,

$$B_i(.,\partial_x)v_0(t)_{|\partial\Omega} = \delta_{ik}g(t)_{|\partial\Omega}. \tag{3.17}$$

Now we consider v_1. By Lemma 3.1, we have that, at least, $v_1 \in B([0,T]; N_p^{2m+\beta''}$ $(\Omega))$. Moreover,

$$[A(.,\partial_x) + P]v_1(t) = \int_0^t K_k'(t-s)[g(s) - g(t)]ds, \tag{3.18}$$

and, for each $i \in \{1, \ldots, m\}$,

$$B_i(.,\partial_x)v_1(t)_{|\partial\Omega} = 0. \tag{3.19}$$

In order to prove that v_1 is bounded with values in $B([0,T]; N_p^{2m+\beta}(\Omega))$, we recall again (3.11). Owing to (3.19), recalling that the domain of A is a subspace of $N_p^{2m+\beta}(\Omega)$, we can try to show that $A_{\beta''}v_1$ is bounded with values in $(N_p^{\beta''}(\Omega), D(A_{\beta''}))_{\frac{\beta-\beta''}{2m}}$. This is equivalent to prove that, for some $\xi_0 > 0$,

$$\|\xi^{\frac{\beta-\beta''}{2m}} A_{\beta''}(\xi - A_{\beta''})^{-1} A_{\beta''} v_1(t)\|_{\beta'',p,\Omega} \leq C, \tag{3.20}$$

with C independent of $\xi \geq \xi_0$ and $t \in [0,T]$. We have from Lemma 3.1

$$\begin{aligned}
&A_{\beta''}(\xi - A_{\beta''})^{-1}A_{\beta''}v_1(t) = \xi^2(\xi - A_{\beta''})^{-1}v_1(t) - A_{\beta''}v_1(t) - \xi v_1(t)\\
&\quad = (2\pi i)^{-1}\int_0^t \left(\int_\gamma e^{\lambda(t-s)}\frac{\lambda^2}{\xi-\lambda}N_k(\lambda)[g(s)-g(t)]d\lambda\right)ds\\
&\quad = \sum_{j=0}^1 (2\pi i)^{-1}\int_0^t \left(\int_{\gamma_j} e^{\lambda(t-s)}\frac{\lambda^2}{\xi-\lambda}N_k(\lambda)[g(s)-g(t)]d\lambda\right)ds := \sum_{j=0}^1 I_j(\xi,t).
\end{aligned}$$

with γ_0 describing $\{\lambda \in \mathbb{C}\setminus\{0\} : |Arg(\lambda)| = \theta_1, |\lambda| \geq r_0\}$, γ_1 describing $\{\lambda \in \mathbb{C}\setminus\{0\} : |Arg(\lambda)| \leq \theta_1, |\lambda| = r_0\}$.

If $\lambda \in \gamma$, we have from Remark 2.9, recalling our assumptions on g,

$$\begin{aligned}
\|N_k(\lambda)[g(s)-g(t)]\|_{\beta'',p,\Omega} &\leq C|\lambda|^{\frac{\beta''-\beta}{2m}-1}\Big(\|g\|_{B([0,T];N_p^{2m+\beta-\mu_k}(\Omega))}\\
&\quad + |\lambda|^{\frac{2m+\beta-\mu_k-\theta}{2m}}(t-s)^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|g\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))}\Big).
\end{aligned}$$

So we can easily deduce that

$$\|I_1(\xi,t)\|_{\beta'',p,\Omega} \leq C\xi^{-1}\|g\|_{B([0,T];N_p^{2m+\beta-\mu_k}(\Omega))},$$

if $\xi \geq \xi_0$, with ξ_0 suitably large. Moreover,

$$\begin{aligned}
\|I_0(\xi,t)\|_{\beta'',p,\Omega} &\leq C_0\int_0^t\left(\int_{\mathbb{R}^+} e^{r(t-s)\cos(\theta_1)}\frac{r^{1+\frac{\beta''-\beta}{2m}}}{r+\xi}(\|g\|_{B([0,T];N_p^{2m+\beta-\mu_k}(\Omega))}\right.\\
&\quad \left. + r^{\frac{2m+\beta-\mu_k-\theta}{2m}}(t-s)^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|g\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))})dr\right)ds\\
&\leq C_1\int_{\mathbb{R}^+}\frac{r^{\frac{\beta''-\beta}{2m}}}{r+\xi}dr \quad \left(\|g\|_{B([0,T];N_p^{2m+\beta-\mu_k}(\Omega))} + \|g\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))}\right)\\
&\leq C_2\xi^{\frac{\beta''-\beta}{2m}}\left(\|g\|_{B([0,T];N_p^{2m+\beta-\mu_k}(\Omega))} + \|g\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))}\right).
\end{aligned}$$

Next, we show that $u \in C^1([0,T];N_p^{\beta''}(\Omega))$ and $D_t u = [A(.,\partial_x)+P]u$. To this aim, we follow a standard argument: let $\epsilon \in (0,T]$ and define, for $t \in [\epsilon, T]$,

$$u_\epsilon(t) := \int_0^{t-\epsilon} K_k(t-s)g(s)ds. \tag{3.21}$$

Then we have

$$\begin{aligned}
D_t u_\epsilon(t) &= K_k(\epsilon)g(t-\epsilon) + \int_0^{t-\epsilon} K_k'(t-s)g(s)ds \qquad (3.22)\\
&= K_k(t)g(t) + \int_0^{t-\epsilon} K_k'(t-s)[g(s)-g(t)]ds + K_k(\epsilon)[g(t-\epsilon)-g(t)],
\end{aligned}$$

which, by Lemma 3.1, converges in $B([\delta,T];N_p^{\beta''})$, for every $\delta\in(0,T]$, to

$$K_k(t)g(t)+\int_0^t K_k'(t-s)[g(s)-g(t)]ds=[A(.,\partial_x)+P]u(t), \tag{3.23}$$

by (3.16) and (3.18). This can be extended to $t=0$, observing that the final expression in (3.23) converges to 0 in $N_p^{\beta''}(\Omega)$ as $t\to 0$.

In conclusion, we have proved that $u\in C^1([0,T];N_p^{\beta''}(\Omega))\cap B([0,T];N_p^{2m+\beta}(\Omega))$. By interpolation, $u\in C([0,T];N_p^{\rho}(\Omega))$ for every $\rho<2m+\beta$. Finally, from $D_tu=[A(.,\partial_x)+P]u$ we deduce that $D_tu\in B([0,T];N_p^{\beta}(\Omega))$.

The proof is complete. □

Now we are able to prove the main result of existence and uniqueness of a solution in the autonomous case.

Theorem 3.6. *Assume that the assumptions* (I1)–(I4) *hold. We consider system* (2.1) *and assume that:*

(I) *for some* $\beta''<\beta$, $f\in B([0,T];N_p^{\beta}(\Omega))\cap C([0,T];N_p^{\beta''}(\Omega))$;

(II) *for each* $k\in\{1,\dots,m\}$, *and for some* $\theta\in(0,p^{-1})$, $g_k\in B([0,T];N_p^{2m+\beta-\mu_k}(\Omega))\cap C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^{\theta}(\Omega))$;

(III) $u_0\in N_p^{2m+\beta}(\Omega)$;

(IV) *for each* $k\in\{1,\dots,m\}$, $(B_k(.,\partial_x)u_0)_{|\partial\Omega}=g_k(0)_{|\partial\Omega}$.

Then (3.1) *has a unique solution u belonging to* $C^1([0,T];N_p^{\beta''}(\Omega))\cap B([0,T];N_p^{2m+\beta}(\Omega)))$, *with* $D_tu\in B([0,T];N_p^{\beta}(\Omega)))$. *$u$ can be represented by the variation of parameter formula*

$$u(t)=T(t)u_0+\int_0^t T(t-s)f(s)ds+\sum_{k=1}^m\int_0^t K_k(t-s)g_k(s)ds. \tag{3.24}$$

Proof. The uniqueness follows from well-known properties of sectorial operators. Concerning the existence, we already know, from Lemmas 3.3 and 3.5, that it holds if $u_0=0$ and $g_k(0)=0$ for each $k=1,\dots,m$. Now we consider the general case. Subtracting to the unknown u the initial value u_0, and setting $v(t):=u(t)-u_0$, we are reduced to the system

$$\begin{cases} D_tv(t,x)=A(x,\partial_x)v(t,x)+Pv(t,x)+f(t,x)+A(x,\partial_x)u_0(x)+Pu_0(x),\\ (t,x)\in[0,T]\times\Omega,\\ B_k(x',\partial_x)v(t,x')-(g_k(t,x')-B_k(x',\partial_x)u_0(x'))=0,\quad k=1,\dots,m,\\ (t,x')\in[0,T]\times\partial\Omega\\ v(0,x)=0,\quad x\in\Omega. \end{cases} \tag{3.25}$$

From assumption (IV), we have that, if we set $\tilde g_k:=g_k-B_k(.,\partial_x)u_0$, $\tilde g_k(0)$ vanishes in $\partial\Omega$. So, if we replace $\tilde g_k$ with $\tilde g_k-\tilde g_k(0)$, (3.25) does not change and we are reduced to a case already treated. We deduce that (3.1) has a unique solution with the desired properties.

It remains to show that the representation (3.24) holds. From (3.25) observing that $K_k(t)g = 0$ if $g_{|\partial\Omega} \equiv 0$, we have

$$u(t) = u_0 + \int_0^t T(t-s)f(s)ds + \sum_{k=1}^m \int_0^t K_k(t-s)g_k(s)ds \\ + T^{(-1)}(t)[A(.,\partial_x)u_0 + Pu_0] - \sum_{k=1}^m K_k^{(-1)}(t)B_k(.,\partial_x)u_0. \tag{3.26}$$

We observe that for every $\lambda \in \rho(A)$

$$u_0 = (\lambda - A)^{-1}(\lambda - A(.,\partial_x) - P)u_0 + \sum_{k=1}^m N_k(\lambda)B_k(.,\partial_x)u_0. \tag{3.27}$$

So,

$$\begin{aligned} & u_0 + T^{(-1)}(t)[A(.,\partial_x)u_0 + Pu_0] - \sum_{k=1}^m K_k^{(-1)}(t)B_k(.,\partial_x)u_0 \\ &= u_0 + (2\pi i)^{-1}\int_\gamma \frac{e^{\lambda t}}{\lambda}\{(\lambda - A)^{-1}[A(.,\partial_x)u_0 + Pu_0] - \sum_{k=1}^m N_k(\lambda)B_k(.,\partial_x)u_0\}d\lambda \\ &= u_0 + T(t)u_0 - (2\pi i)^{-1}\int_\gamma \frac{e^{\lambda t}}{\lambda}u_0 d\lambda = T(t)u_0. \end{aligned} \tag{3.28}$$

So the proof is complete. □

We precise Theorem 3.6:

Proposition 3.7. *Consider the system* (3.1), *with the assumptions* (I1)–(I4). *Let* $T_0 \in \mathbb{R}^+$. *Then:*

(I) *if* $0 < T \le T_0$ *and* $\theta > \beta - \nu$, *the solution* u *satisfies the estimate*

$$\begin{aligned} & \|u\|_{B([0,T];N_p^{2m+\beta}(\Omega)} + \|D_t u\|_{B([0,T];N_p^\beta(\Omega)} \\ & \le C(T_0)\Big(\|f\|_{B([0,T];N_p^\beta(\Omega))} + \sum_{k=1}^m \|g_k\|_{B([0,T];N_p^{2m-\mu_k+\beta}(\Omega))} \\ & + \sum_{k=1}^m \|g_k\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))} + \|u_0\|_{2m+\beta,p,\Omega}\Big). \end{aligned} \tag{3.29}$$

(II) *Assume that* $u_0 = 0$, $\theta > \beta - \nu$, $\beta \le \alpha \le 2m+\beta$, $0 \le \gamma \le 1 - \frac{\alpha-\beta}{2m}$. *Then,*

$$\begin{aligned} \|u\|_{B([0,T];N_p^\alpha(\Omega))} & \le C(\alpha, T_0)T^{1-\frac{\alpha-\beta}{2m}}\Big(\|f\|_{B([0,T];N_p^\beta(\Omega))} \\ & + \sum_{k=1}^m \|g_k\|_{B([0,T];N_p^{2m-\mu_k+\beta}(\Omega))} + \sum_{k=1}^m \|g_k\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))}\Big), \end{aligned} \tag{3.30}$$

$$\|u\|_{C^\gamma([0,T];N_p^\alpha(\Omega))}$$
$$\leq C(\alpha,T_0)T^{1-\frac{\alpha-\beta}{2m}-\gamma}\bigg(\|f\|_{B([0,T];N_p^\beta(\Omega))}+\sum_{k=1}^m\|g_k\|_{B([0,T];N_p^{2m-\mu_k+\beta}(\Omega))}$$
$$+\sum_{k=1}^m\|g_k\|_{B([0,T];N_p^{2m-\mu_k+\beta}(\Omega))}+\sum_{k=1}^m\|g_k\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))}\bigg). \tag{3.31}$$

Proof. We extend f and g_k to $[0,T_0]$, setting

$$\tilde{f}(t)=\begin{cases} f(t) & \text{if} \quad 0\leq t\leq T,\\ f(T) & \text{if} \quad T\leq t\leq T_0,\end{cases} \qquad \tilde{g}_k(t)=\begin{cases} g_k(t) & \text{if} \quad 0\leq t\leq T,\\ g_k(T) & \text{if} \quad T\leq t\leq T_0.\end{cases}$$

As $\theta>\beta-\nu$, $\frac{2m+\beta-\mu_k-\theta}{2m}<1$ for each $k=1,\dots,m$, so that

$$\|\tilde{g}_k\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T_0];N_p^\theta(\Omega))}=\|g_k\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))}.$$

We call $\tilde{u}$ the corresponding solution (again with initial datum u_0) in $[0,T_0]$. Of course, u is the restriction of $\tilde{u}$ to $[0,T]$. We deduce that

$$\begin{aligned}
&\|u\|_{B([0,T];N_p^{2m+\beta}(\Omega))}+\|\|D_tu\|_{B([0,T];N_p^\beta(\Omega))}\\
&\quad\leq\|\tilde{u}\|_{B([0,T];N_p^{2m+\beta}(\Omega))}+\|\|D_t\tilde{u}\|_{B([0,T];N_p^\beta(\Omega))}\\
&\quad\leq C(T_0)\bigg(\|\|\tilde{f}\|_{B([0,T];N_p^\beta(\Omega))}+\sum_{k=1}^m\|\tilde{g}_k\|_{B([0,T];N_p^{2m-\mu_k+\beta}(\Omega))}\\
&\qquad+\sum_{k=1}^m\|\tilde{g}_k\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))}+\|u_0\|_{2m+\beta,p,\Omega}\bigg)\\
&\quad=C(T_0)\bigg(\|f\|_{B([0,T];N_p^\beta(\Omega))}+\sum_{k=1}^m\|g_k\|_{B([0,T];N_p^{2m-\mu_k+\beta}(\Omega))}\\
&\qquad+\sum_{k=1}^m\|g_k\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))}+\|u_0\|_{2m+\beta,p,\Omega}\bigg).
\end{aligned} \tag{3.32}$$

Concerning (II), by (I) we have

$$\begin{aligned}
\|u\|_{B([0,T];N_p^\beta(\Omega))}&\leq T\|D_tu\|_{B([0,T];N_p^\beta(\Omega))}\\
&\leq C(T_0)T\bigg(\|f\|_{B([0,T];N_p^\beta(\Omega))}\\
&\quad+\sum_{k=1}^m\|g_k\|_{B([0,T];N_p^{2m-\mu_k+\beta}(\Omega))}\\
&\quad+\sum_{k=1}^m\|g_k\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))}\bigg).
\end{aligned} \tag{3.33}$$

The intermediate cases follow from the interpolation inequalities. □

We conclude extending Theorem 3.6 to the nonautonomous case. We shall use the following

Theorem 3.8. *Let Ω be a bounded open subset of $\mathbb{R}^n$, with smooth boundary. Then:*

(I) *if $\beta, \rho \in \mathbb{R}$ and $\rho > |\beta|$, $\mathcal{C}^\rho(\Omega)$ is a space of pointwise multipliers for $N_p^\beta(\Omega)$.*

(II) *In case $\beta > 0$, $\mathcal{C}^\beta(\Omega)$ is a space of pointwise multipliers for $N_p^\beta(\Omega)$; moreover, there exists $C \in \mathbb{R}^+$ such that, $\forall a \in \mathcal{C}^\beta(\Omega)$, $\forall f \in N_p^\beta(\Omega)$, $\forall \alpha \in (0, \beta)$,*

$$\begin{aligned}\|af\|_{\beta,p,\Omega} &\leq C_0(\|a\|_{L^\infty(\Omega)}\|f\|_{\beta,p,\Omega} + \|a\|_{\beta,\infty,\Omega}\|f\|_{L^p(\Omega)}) \\ &\leq C(\alpha)(\|a\|_{L^\infty(\Omega)}\|f\|_{\beta,p,\Omega} + \|a\|_{\beta,\infty,\Omega}\|f\|_{\alpha,p,\Omega}).\end{aligned}$$

Proof. Concerning (I), see [14], 3.3.2.

(II) is a particular case of Lemma 1(II) in [10, Chapter 5.3.7]. The constant σ_p in the reference is defined in Chapter 2.1.3. □

We pass to consider the following system in the unknown u:

$$\begin{cases} D_t u(t,x) = A(t,x,\partial_x)u(t,x) + P(t)u(t,.)(x) + f(t,x), \quad (t,x) \in [0,T]\times\Omega, \\ B_k(t,x',\partial_x)u(t,x') - g_k(t,x') = 0, \quad k = 1,\dots,m, \quad (t,x') \in [0,T]\times\partial\Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{3.34}$$

with the following assumptions:

(J1) *Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, for every $t \in [0,T]$ the operators $A(t,x,\partial_x)$ and $B_k(t,x',\partial_x)$ $(1 \leq k \leq m)$ satisfy the conditions* (H1)–(H3), *with m, $\mu_1, \dots, \mu_m$ independent of t;*

(J2) *$p \in [1,+\infty)$, $\beta \in (\mu + p^{-1} - 2m, \nu + p^{-1})$, with $\nu := \min_{1\leq k\leq m}\mu_k$, $\mu := \max_{1\leq k\leq m}\mu_k$;*

(J3) *the coefficients $a_s(t,x)$ of $A(t,x,\partial_x)$ $(|s| \leq 2m)$ belong to $C([0,T];\mathcal{C}^{|\beta|+\epsilon}(\overline{\Omega}))$ for some $\epsilon \in \mathbb{R}^+$ if $\beta \leq 0$, to $B([0,T];C^\beta(\overline{\Omega})) \cap C([0,T];C(\overline{\Omega}))$ if $\beta > 0$;*

(J4) *for each $k = 1,\dots,m$, the coefficients b_{ks} of $B_k(t,x,\partial_x)$ $(|s| \leq 2m - \mu_k)$ belong to $B([0,T];\, \mathcal{C}^{2m+\beta-\mu_k}(\overline{\Omega})) \cap C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];\mathcal{C}^\theta(\overline{\Omega}))$ for some $\theta \in (0,p^{-1})$;*

(J5) *$f \in B([0,T];N_p^\beta(\Omega)) \cap C([0,T];N_p^{\beta''}(\Omega))$, for some $\beta'' < \beta$;*

(J6) *for each $k \in \{1,\dots,m\}$, $g_k \in B([0,T];N_p^{2m+\beta-\mu_k}(\Omega)) \cap C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,T];N_p^\theta(\Omega))$;*

(J7) *$u_0 \in N_p^{2m+\beta}(\Omega)$ and, for each $k = 1,\dots,m$,*

$$(B_k(0,.,\partial_x)u_0)_{|\partial\Omega} = g_k(0)_{|\partial\Omega};$$

(J8) *For some $\beta' < \beta$,*

$$P \in B([0,T];\mathcal{L}(N_p^{2m+\beta'}(\Omega), N_p^\beta(\Omega))) \cap C([0,T];\mathcal{L}(N_p^{2m+\beta'}(\Omega), N_p^{\beta''}(\Omega))).$$

Remark 3.9. Owing to the interpolation inequalities, it is not restrictive to assume that

$$\theta > \beta - \nu, \quad \beta' \vee (\mu + p^{-1} - 2m) < \beta''. \tag{3.35}$$

We want to prove the following

Theorem 3.10. *Assume that the assumptions* (J1)–(J8) *are satisfied. Then the system* (3.34) *has a unique solution* u *belonging to* $C^1([0,T];N_p^{\beta''}(\Omega)) \cap B([0,T];N_p^{2m+\beta}(\Omega))$, *with* $D_t u \in B([0,T];N_p^{\beta}(\Omega))$.

Proof. It is convenient to define $A(t,x,\partial_x) := A(T,x,\partial_x)$, $P(t) := P(T)$, $B_k(t,x,\partial_x) := B_k(T,x,\partial_x)$ if $t > T$.

As $[0,T]$ is compact, the conclusion will be proved if we show the following:

(P) *let* $s_0 \in [0,T]$ *and consider the problem*

$$\begin{cases} D_t v(t,x) = A(s+t,x,\partial_x)v(t,x) + P(s+t)v(t,.)(x) + r(t,x), \\ (t,x) \in [0,\delta] \times \Omega, \\ B_k(s+t,x',\partial_x)v(t,x') - r_k(t,x') = 0, \quad k = 1,\dots,m, \quad (t,x') \in [0,\delta] \times \partial\Omega, \\ v(0,x) = v_0(x), \quad x \in \Omega, \end{cases} \tag{3.36}$$

with $\delta \in \mathbb{R}^+$, $s \in (s_0-\delta, s_0+\delta) \cap [0,+\infty)$, $r \in B([0,\delta];N_p^{\beta}(\Omega)) \cap C([0,\delta];N_p^{\beta''}((\Omega))$ *for some* $\beta'' < \beta$, *for each* $k \in \{1,\dots,m\}$, *and for a fixed* $\theta \in (0 \vee (\beta-\nu), p^{-1})$, $r_k \in B([0,\delta];N_p^{2m+\beta-\mu_k}(\Omega)) \cap C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];\, N_p^{\theta}(\Omega))$, $v_0 \in N_p^{2m+\beta}(\Omega)$ *and, for each* $k \in \{1,\dots,m\}$, $(B_k(s,.,\partial_x)v_0)_{|\partial\Omega} = r_k(0)_{|\partial\Omega}$. *Then there exists* $\delta > 0$, *depending only on* s_0, *such that* (3.36) *has a unique solution* v *in* $C^1([0,\delta];N_p^{\beta''}((\Omega)) \cap B([0,\delta];N_p^{2m+\beta}(\Omega))$, *with* $D_t v \in B([0,\delta];N_p^{\beta}(\Omega))$.

By subtracting the constant function v_0, we may limit ourselves to study (3.36) in the case $v_0 = 0$. We consider the family of problems

$$\begin{cases} D_t v(t,x) = A(s_0,x,\partial_x)v(t,x) + \tau\{[A(s+t,x,\partial_x) - A(s_0,x,\partial_x)]v(t,x) \\ \;+ P(s+t)v(t,.)(x)\} + r(t,x), \quad (t,x) \in [0,\delta] \times \Omega, \\ B_k(s_0,x',\partial_x)v(t,x') - \tau\{B_k(s_0,x',\partial_x)v(t,x') - B_k(s+t,x',\partial_x)v(t,x')\} \\ \;- r_k(t,x') = 0, \quad k = 1,\dots,m, \quad (t,x') \in [0,\delta] \times \partial\Omega, \\ v(0,x) = 0, \quad x \in \Omega, \end{cases} \tag{3.37}$$

depending on the parameter $\tau \in [0,1]$. As the problem is uniquely solvable in case $\tau = 0$, by the continuation method, it suffices to obtain an a priori estimate of a solution, which is independent of τ We are going to show that this is possible if δ is sufficiently small.

We recall that we are assuming $\theta > \beta - \nu$ and $\beta' \vee (\mu + p^{-1} - 2m) < \beta''$.

Let

$$v \in C^1([0,\delta];\ N_p^{\beta''}((\Omega)) \cap B([0,\delta]; N_p^{2m+\beta}(\Omega)),$$

with

$$D_t v \in B([0,\delta]; N_p^{\beta}\ (\Omega)) \quad \text{and} \quad v(0) = 0,$$

and let $\eta \in \mathbb{R}^+$.

Then, by Theorem 3.8 and (J3), we have, $\forall \tau \in [0,1]$, if δ is sufficiently small,

$$\begin{aligned} &\|\tau\{[A(s+.,.,\partial_x) - A(s_0,.,\partial_x)]v + P(s+.)v\}\|_{B([0,\delta];N_p^{\beta}(\Omega))} \\ &\quad \leq \eta\|v\|_{B([0,\delta];N_p^{2m+\beta}(\Omega))} + C_0\|v\|_{B([0,\delta];N_p^{\alpha}(\Omega))}, \end{aligned} \tag{3.38}$$

for some $\alpha \in [\beta, 2m+\beta)$. Analogously, employing (J4), we have

$$\begin{aligned} &\|\tau\{[B_k(s+.,.,\partial_x) - B_k(s_0,.,\partial_x)]v\}\|_{B([0,\delta];N_p^{2m+\beta-\mu_k}(\Omega))} \\ &\quad \leq \eta\|v\|_{B([0,\delta];N_p^{2m+\beta}(\Omega))} + C_0\|v\|_{B([0,\delta];N_p^{\alpha}(\Omega))}. \end{aligned} \tag{3.39}$$

Now we estimate

$$\begin{aligned} &\|\tau\{[B_k(s+.,.,\partial_x) - B_k(s_0,.,\partial_x)]v\}\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];N_p^{\theta}(\Omega))} \\ &\quad \leq \|\{[B_k(s+.,.,\partial_x) - B_k(s_0,.,\partial_x)]v\}\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];N_p^{\theta}(\Omega))}. \end{aligned}$$

If $t, t' \in [0,\delta]$,

$$\begin{aligned} &|[B_k(s+t,.,\partial_x) - B_k(s_0,.,\partial_x)]v(t,.) \\ &\qquad - B_k(s+t',.,\partial_x) - B_k(s_0,.,\partial_x)]v(t',.)\|_{\theta,p,\Omega} \\ &\quad \leq \|[B_k(s+t,.,\partial_x) - B_k(s+t',.,\partial_x)]v(t,.)\|_{\theta,p,\Omega} \\ &\qquad + \|[B_k(s+t',.,\partial_x) - B_k(s_0,.,\partial_x)][v(t,.) - v(t',.)]\|_{\theta,p,\Omega} := I_1 + I_2. \end{aligned}$$

By Theorem 3.8, we have, if $|r| \leq \mu_k$, using (J4),

$$\begin{aligned} &\|[b_{kr}(s+t,.) - b_{kr}(s+t',.)]\partial_x^r v(t,.))\|_{\theta,p,\Omega} \\ &\quad \leq C\|b_{kr}(s+t,.) - b_{kr}(s+t',.)\|_{\theta,\infty,\Omega}\|v(t,.))\|_{\mu_k+\theta,p,\Omega} \\ &\quad \leq C|t-t'|^{\frac{2m+\beta-\mu_k-\theta}{2m}}\|v\|_{C([0,\delta];N_p^{\mu_k+\theta}(\Omega))}, \end{aligned}$$

$$\begin{aligned} &\|[b_{kr}(s+t',.) - b_{kr}(s_0,.)]\partial_x^r[v(t,.) - v(t',.)]\|_{\theta,p,\Omega} \\ &\quad \leq C[\|b_{kr}(s+t',.) - b_{kr}(s_0,.)\|_{C(\overline{\Omega})}\|v(t,.) - v(t',.)\|_{\mu_k+\theta,p,\Omega} \\ &\qquad + \|b_{kr}(s+t',.) - b_{kr}(s_0,.)\|_{\theta,\infty,\Omega}\|v(t,.) - v(t',.)\|_{\alpha_k,p,\Omega}] \\ &\quad \leq C_0\eta|t-t'|^{\frac{2m+\beta-\mu_k-\theta}{2m}}[\|v\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];N_p^{\mu_k+\theta}(\Omega))} \\ &\qquad + \|v\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];N_p^{\alpha_k}(\Omega))}], \end{aligned}$$

for fixed $\alpha_k \in (0, \mu_k + \theta)$. So, applying Proposition 3.7, we obtain the following a priori estimate, in case δ is sufficiently small:

$$
\begin{aligned}
&\|v\|_{B([0,\delta];N_p^{2m+\beta}(\Omega))} + \|D_t v\|_{B([0,\delta];N_p^{\beta}(\Omega))} + \delta^{\frac{\alpha-\beta}{2m}-1}\|v\|_{B([0,\delta];N_p^{\alpha}(\Omega))} \\
&\quad + \delta^{\frac{\beta'-\beta}{2m}}\|v\|_{B([0,\delta];N_p^{2m+\beta'}(\Omega))} + \sum_{k=1}^{m} \delta^{\frac{\alpha_k-\mu_k-\theta}{2m}}\|v\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];N_p^{\alpha_k}(\Omega)(\Omega))} \\
&\le C\Bigg(\|r\|_{B([0,\delta];N_p^{\beta}(\Omega))} + \sum_{k=1}^{m}\|r_k\|_{B([0,\delta];N_p^{2m+\beta-\mu_k}(\Omega))} \\
&\quad + \sum_{k=1}^{m}\|r_k\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];N_p^{\theta}(\Omega))} + \eta\|v\|_{B([0,\delta];N_p^{2m+\beta}(\Omega))} \\
&\quad + \|v\|_{B([0,\delta];N_p^{2m+\beta'}(\Omega))} + \|v\|_{B([0,\delta];N_p^{\alpha}(\Omega))} \\
&\quad + \eta\sum_{k=1}^{m}\|v\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];N_p^{\mu_k+\theta}(\Omega))} + \sum_{k=1}^{m}\|v\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];N_p^{\alpha_k}(\Omega)(\Omega))}\Bigg),
\end{aligned}
\tag{3.40}
$$

and the conclusion follows from the inequalities

$$
\|v\|_{C^{\frac{2m+\beta-\mu_k-\theta}{2m}}([0,\delta];N_p^{\mu_k+\theta}(\Omega))} \le C(\|v\|_{B([0,\delta];N_p^{2m+\beta}(\Omega)} + \|D_t v\|_{B([0,\delta];N_p^{\beta}(\Omega)}),
\tag{3.41}
$$

with C independent of δ, choosing η and δ suitably small. □

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Parabolic Equations in Anisotropic Orlicz Spaces with General $\mathcal{N}$-functions

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Abstract. In the present paper we study the existence of weak solutions to an abstract parabolic initial-boundary value problem. On the operator appearing in the equation we assume the coercivity conditions given by an $\mathcal{N}$-function (i.e., convex function satisfying conditions specified in the paper). The main novelty of the paper consists in the lack of any growth restrictions on the $\mathcal{N}$-function combined with an anisotropic character of the $\mathcal{N}$-function, namely we allow the dependence on all the directions of the gradient, not only on its absolute value.

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1. Introduction

We concentrate on an abstract parabolic equation

$$u_t = \operatorname{div} A(t, x, \nabla u) \quad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$u(0,x) = u_0 \quad \text{in } \Omega, \tag{1.2}$$

$$u(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded set with a Lipschitz boundary $\partial\Omega$, $(0,T)$ is the time interval with $T<\infty$, $u : (0,T)\times\Omega \to \mathbb{R}$ and the operator A satisfies the following conditions:

(A1) A is a Carathéodory function (i.e., measurable w.r.t. t and x and continuous w.r.t. the last variable).

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(A2) There exists a function $M : \mathbb{R}^d \to \mathbb{R}_+$ and a constant $c > 0$ such that

$$A(t, x, \xi) \cdot \xi \geq c(M(\xi) + M^*(A(t, x, \xi)))$$

where M is an $\mathcal{N}$-function, i.e., it is convex, has superlinear growth, $M(\xi) = 0$ iff $\xi = 0$ and $M(\xi) = M(-\xi)$, $M^*(\eta) = \sup_{\xi \in \mathbb{R}^d}(\eta \cdot \xi - M(\xi))$.

(A3) For all $\xi, \eta \in \mathbb{R}^d$,

$$(A(t, x, \xi) - A(t, x, \eta)) \cdot (\xi - \eta) \geq 0.$$

We consider the problem of existence of weak solutions to the initial boundary value problem (1.1)–(1.3). In the present paper we assume a full anisotropy of the $\mathcal{N}$-function, namely it depends on a vector-valued argument and we do not put any assumptions on the growth of an $\mathcal{N}$-function. Indeed we do not assume that neither M nor M^* satisfies the so-called Δ_2 condition[1]. The following pair of functions can serve as an example in $d = 2$:

$$M(\xi) = M(\xi_1, \xi_2) = e^{|\xi_1|} - |\xi_1| - 1 + (1 + |\xi_2|)\ln(1 + |\xi_2|) - |\xi_2|,$$

$$M^*(\eta) = M^*(\eta_1, \eta_2) = (1 + |\eta_1|)\ln(1 + |\eta_1|) - |\eta_1| + e^{|\xi_2|} - |\eta_2| - 1.$$

The result in the anisotropic case is not a straightforward extension of the isotropic one. The new difficulty arising here concerns the density of compactly supported smooth functions w.r.t. the modular topology of the gradients. The detailed analysis of this issue appears in Section 3.

We can cite some nontrivial examples of the operator A:

- $A(t, x, \xi) = a(t, x)\xi \exp(|\xi|^2)$ where $0 < c_1 \leq a(t, x) \leq c_2 < \infty$,
- $A(t, x, \xi) = a(t, x)\xi \ln(1 + |\xi|)$ where $0 < c_1 \leq a(t, x) \leq c_2 < \infty$,
- $A(t, x, \xi_1, \xi_2) = a_1(t, x)\xi_1 \exp(|\xi_1|^2) + a_2(t, x)\xi_2 \ln(1 + |\xi_2|)$ where $0 < c_1 \leq a_i(t, x) \leq c_2 < \infty$ for $i = 1, 2$.

In [2] the operator A was assumed to be an elliptic second-order operator in divergence form and monotone. The growth and coercivity conditions were more general than the standard growth conditions in L^p, namely the $\mathcal{N}$-function formulation was stated[2]. Under the assumptions on the $\mathcal{N}$-function M: $\xi^2 << M(|\xi|)$ (i.e., ξ^2 grows essentially less rapidly than $M(|\xi|)$) and M^* satisfies a Δ_2-condition, the existence results to (1.1)–(1.3) was established. The restrictions on the growth of M were abandoned in [3], but still M had an isotropic character.

The review paper [7] summarizes the monotone-like mappings techniques in Orlicz and Orlicz–Sobolev spaces[3].

---

[1] We say that an $\mathcal{N}$-function M satisfies the Δ_2-condition if for some constant $C > 0$ it holds that $M(2\xi) \leq CM(\xi)$ for all $\xi \in \mathbb{R}^d$.

[2] Assumption **(A2)** is in fact a generalization of the growth and coercivity conditions assumed in [2] for the case of M dependent on a vector-valued argument.

[3] $W^m L_M$ is the Orlicz–Sobolev space of functions in L_M with all distributional derivatives up to order m in L_M.

Before stating the main result let us recall the standard notation. For brevity we write $Q := \Omega \times (0,T)$. By the generalized Orlicz class $\mathcal{L}_M(Q)$ we mean the set of all measurable functions $\xi : Q \to \mathbb{R}^d$ for which the modular

$$\rho_M(\xi) = \int_Q M(\xi(t,x))dxdt$$

is finite. By $L_M(Q)$ we denote the generalized Orlicz space which is the set of all measurable functions $\xi : Q \to \mathbb{R}^d$ for which $\rho_M(\alpha\xi) \to 0$ as $\alpha \to 0$. This is a Banach space with respect to the norm

$$\|\xi\|_M = \sup\left\{\int_Q \eta \cdot \xi dxdt : \eta \in L_{M^*}(Q), \int_Q M^*(\eta)dxdt \le 1\right\}.$$

By $E_M(Q)$ we denote the closure of all bounded functions in $L_M(Q)$. The space $L_{M^*}(Q)$ is the dual space of $E_M(Q)$. We will say that a sequence z^j converges modularly to z in $L_M(Q)$ if there exists $\lambda > 0$ such that

$$\rho_M\left(\frac{z^j - z}{\lambda}\right) \to 0.$$

We will use the notation $z^j \xrightarrow{M} z$ for the modular convergence in $L_M(Q)$. Contrary to [6] we consider the $\mathcal{N}$-function M not dependent only on $|\xi|$, but on the whole vector ξ. Below, we formulate the main result of the present paper. The space Z_0^M appearing in the theorem is defined in Section 3.

Theorem 1.1. *Let M be an $\mathcal{N}$-function and let A satisfy conditions* (A1)–(A3)*. Given $u_0 \in L^2(\Omega)$ there exists $u \in Z_0^M$ such that*

$$\int_Q (-u\varphi_t + A(t,x,\nabla u) \cdot \nabla\varphi)\,dxdt = \int_\Omega u_0(x)\varphi(0,x)dx \tag{1.4}$$

holds for all $\varphi \in \mathcal{D}((-\infty,T) \times \Omega)$.

2. Useful facts about Orlicz spaces

In this short section we collect some facts about $\mathcal{N}$-functions and Orlicz spaces, which are used in the proof of the main theorem. All the proofs to the collected lemmas and propositions can be found, e.g., in [5].

Lemma 2.1. *Let $z^j : Q \to \mathbb{R}^d$ be a measurable sequence. Then $z^j \xrightarrow{M} z$ in $L_M(Q)$ modularly if and only if $z^j \to z$ in measure and there exist some $\lambda > 0$ such that the sequence $\{M(\lambda z^j)\}$ is uniformly integrable, i.e.,*

$$\lim_{R\to\infty}\left(\sup_{j\in\mathbb{N}} \int_{\{(t,x):|M(\lambda z^j)|\ge R\}} M(\lambda z^j)dxdt\right) = 0.$$

Lemma 2.2. *Let M be an $\mathcal{N}$-function and for all $j \in \mathbb{N}$ let $\int_Q M(z^j) \le c$. Then the sequence $\{z^j\}$ is uniformly integrable.*

Proposition 2.3. *Let M be an $\mathcal{N}$-function and M^* its complementary function. Suppose that the sequences $\psi^j : Q \to \mathbb{R}^d$ and $\phi^j : Q \to \mathbb{R}^d$ are uniformly bounded in $L_M(Q)$ and $L_{M^*}(Q)$ respectively. Moreover $\psi^j \xrightarrow{M} \psi$ modularly in $L_M(Q)$ and $\phi^j \xrightarrow{M^*} \phi$ modularly in $L_{M^*}(Q)$. Then $\psi^j \cdot \phi^j \to \psi \cdot \phi$ strongly in $L^1(Q)$.*

Proposition 2.4. *Let ϱ^j be a standard mollifier, i.e., $\varrho \in C^\infty(\mathbb{R})$, ϱ has a compact support and $\int_{\mathbb{R}} \varrho(\tau)d\tau = 1, \varrho(t) = \varrho(-t)$. We define $\varrho^j(t) = j\varrho(jt)$. Moreover let $*$ denote a convolution in the variable t. Then for any function $\psi : Q \to \mathbb{R}^d$ such that $\psi \in L^1(Q)$ it holds that*

$$(\varrho^j * \psi)(t,x) \to \psi(t,x) \quad \text{in measure.}$$

Proposition 2.5. *Let ϱ^j be defined as in Proposition 2.4. Given an $\mathcal{N}$-function M and a function $\psi : Q \to \mathbb{R}^d$ such that $\psi \in \mathcal{L}(Q)$ the sequence $\{M(\varrho^j * \psi)\}$ is uniformly integrable.*

3. Closures of compactly supported smooth functions

In the present section we concentrate on the issue of closures. We consider the closure of $C_c^\infty((-\infty,T) \times \Omega)$ in the three different topologies:

1. strong (norm) topology of L_M(Q) and we denote this space by X_0^M, namely
$$\begin{aligned} X_0^M =&\{\varphi \in L^\infty(0,T;L^2(\Omega)), \nabla\varphi \in L_M(Q) \mid \exists\, \{\varphi^j\}_{j=1}^\infty \subset C_c^\infty((-\infty,T)\times\Omega) : \\ &\varphi^j \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ and } \nabla\varphi^j \to \nabla\varphi \text{ strongly in } L_M(Q)\},\end{aligned}$$
2. modular topology of $L_M(Q)$, which we denote by Y_0^M, namely
$$\begin{aligned} Y_0^M =&\{\varphi \in L^\infty(0,T;L^2(\Omega)), \nabla\varphi \in L_M(Q) \mid \exists\, \{\varphi^j\}_{j=1}^\infty \subset C_c^\infty((-\infty,T)\times\Omega) : \\ &\varphi^j \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ and } \nabla\varphi^j \xrightarrow{M} \nabla\varphi \text{ modularly in } L_M(Q)\},\end{aligned}$$
3. weak-star topology of $L_M(Q)$, which we denote by Z_0^M, namely
$$\begin{aligned} Z_0^M =&\{\varphi \in L^\infty(0,T;L^2(\Omega)), \nabla\varphi \in L_M(Q) \mid \exists\, \{\varphi^j\}_{j=1}^\infty \subset C_c^\infty((-\infty,T)\times\Omega) : \\ &\varphi^j \overset{*}{\rightharpoonup} \varphi \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ and } \nabla\varphi^j \overset{*}{\rightharpoonup} \nabla\varphi \text{ weakly star in } L_M(Q)\}.\end{aligned}$$

The closures in the strong and weak topology of the ∇u in $L_M(Q)$ are equal if and only if M satisfies the Δ_2-condition, cf. [4]. Hence as we do not require Δ_2-condition, this will not hold and we only concentrate on the relation between modular and weak-star closures.

Lemma 3.1. *Let M be an $\mathcal{N}$-function and Y_0^M, Z_0^M be the function spaces defined in 2. and 3. above. Then $Y_0^M = Z_0^M$.*

Proof. Clearly $Y_0^M \subset Z_0^M$ (modular topology is stronger than weak-star). The proof of an opposite inclusion we will split into two steps. In the first step we will assume that Ω is a star-shaped domain, whereas in the second step we extend the idea for arbitrary Lipschitz domains.

Step 1. *Star-shaped domains.* We are aiming to show that

$$Z_0^M \subset Y_0^M. \tag{3.1}$$

For readability of the proof we introduce the notation

$$V_M := \{u \in L^1(0,T;W_0^{1,1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \mid \nabla u \in L_M(Q)\}.$$

The starting point is extending the function u by zero outside of Ω to the whole $\mathbb{R}^d$ to mollify it. To assume the extension is properly defined we split (3.1) as follows:

$$Z_0^M \subset V_M \tag{3.2}$$

and

$$V_M \subset Y_0^M. \tag{3.3}$$

Inclusion (3.2) is obvious and to prove (3.3) we define

$$u^\lambda(t,x) := u(t, \lambda(x-x_0)+x_0)$$

where x_0 is a vantage point of Ω. Let $\varepsilon_\lambda = \frac{1}{2}\mathrm{dist}\,(\partial\Omega, \lambda\Omega)$ where $\lambda\Omega := \{y = \lambda\cdot(x-x_0)+x_0 \mid x \in \Omega\}$. Define then

$$u^{\lambda,\varepsilon}(t,x) := \varrho_\varepsilon * u^\lambda(t,x)$$

where ϱ_ε is a standard mollifier, the convolution is done w.r.t. x and t and $\varepsilon < \varepsilon_\lambda$.

This approximation has the property that if $u \in V_M$, then $u^{\lambda,\varepsilon} \in V_M$ for $\varepsilon < \varepsilon_\lambda$. First we pass to the limit with $\varepsilon \to 0$ and hence $u^{\lambda,\varepsilon} \xrightarrow{\varepsilon\to 0} u^\lambda$ in $L^1(0,T;W_0^{1,1}(\Omega))$ and $\nabla u^{\lambda,\varepsilon} \xrightarrow{\varepsilon\to 0} \nabla u^\lambda$ modularly in $L_M(Q)$. Then we pass with $\lambda \to 1$ and obtain that $u^\lambda \xrightarrow{\lambda\to 1} u$ in $L^1(0,T;W_0^{1,1}(\Omega))$ and $\nabla u^\lambda \xrightarrow{\lambda\to 1} \nabla u$ modularly in $L_M(Q)$.

Step 2. *Arbitrary Lipschitz domains.* Since we consider the domain with Lipschitz boundary, then there exists a countable family of star-shaped Lipschitz domains $\{\Omega_i\}$ such that (cf. [8])

$$\Omega = \bigcup_{i\in J} \Omega_i.$$

We introduce the partition of unity θ_i with $0 \le \theta_i \le 1$, $\theta_i \in C_0^\infty(\Omega_i)$, $\mathrm{supp}\,\theta_i = \Omega_i$, $\sum_{i\in J}\theta_i(x) = 1$ for $x \in \Omega$. The proof of (3.1) we split also into two parts. First we show that

$$\overline{V_M \cap L^\infty(Q)}^M = V_M, \tag{3.4}$$

where the closure above is meant w.r.t. the modular topology. Indeed, define $T_n(u)$ – the truncation of the function u, namely

$$T_n(u) = \begin{cases} u & \text{if} \quad |u| \le n, \\ n & \text{if} \quad u > n, \\ -n & \text{if} \quad u < -n. \end{cases} \tag{3.5}$$

Note that if $u \in V_M$ then $T_n(u) \in V_M$. Moreover, as $n \to \infty$ we observe the convergence

$$T_n(u) \to u \quad \text{strongly in} \quad L^1(0,T;W_0^{1,1}(\Omega)).$$

Additionally it holds that $M(\nabla T_n(u(t,x))) \leq M(\nabla u(t,x))$ a.e. in Q. Indeed, this inequality can be easily concluded if we consider separately three subsets: the set where $T_n(u)$ and u coincide and the remaining two sets, where $T_n(u)$ is equal to n or $-n$. Since $T_n(u) \in L^1(0,T;W_0^{1,1}(\Omega))$, then $\nabla T_n(u)$ is equal to zero a.e. in these two sets. Consequently $M(\nabla T_n(u(t,x)))$ is uniformly integrable, which combined with pointwise convergence provides

$$\nabla T_n(u) \to \nabla u \quad \text{modularly in} \quad L_M(Q).$$

In the next step we will show that

$$(V_M \cap L^\infty(Q)) \subset Y_0^M,$$

which together with (3.4) and (3.2) will prove (3.1). If $u \in V_M \cap L^\infty(Q)$ then

$$u \cdot \theta_i \in L^1(0,T;W_0^{1,1}(\Omega_i)) \cap L^\infty(0,T;L^2(\Omega_i)) \cap L^\infty((0,T)\times\Omega_i))$$

and

$$\nabla u \cdot \theta_i + u \cdot \nabla\theta_i = \nabla(u \cdot \theta_i) \in L_M((0,T)\times\Omega_i),$$

where $\Omega_i = \operatorname{supp}\,\theta_i$. Now we follow the case of star-shaped domains to complete the proof. $\square$

Remark 3.2. Note that in the classical case $M = M(|\cdot|)$ (i.e., the modular does not depend on the direction) the proof is simpler. The problem which arises here is the lack of proper Poincaré inequality, cf. [1], which in the isotropic case has the form

$$\int_Q M(|\nabla u(t,x)|)dxdt \geq c \int_Q M(|u|)dxdt.$$

4. Existence result

In the current section we will prove Theorem 1.1. The finite-dimensional approximate problem is constructed by means of the Galerkin method. The basis consisting of eigenvectors of the Laplace operator is chosen and by u^n we mean the solution to the problem projected to n vectors of the chosen basis. Let $Q^s := (0,s)\times\Omega$ with $0 < s < T$. In the standard manner we conclude that, for $0 < s < T$,

$$\int_{Q^s} A(t,x,\nabla u^n)\cdot\nabla u^n dxdt = \frac{1}{2}\|u^n(0)\|_2^2 - \frac{1}{2}\|u^n(s)\|_2^2 \tag{4.1}$$

holds, the energy estimates are derived and convergence of appropriate sequences is concluded, namely

$$\begin{aligned}
\nabla u^n &\overset{*}{\rightharpoonup} \nabla u && \text{weakly-star in} \quad L_M(Q),\\
u^n &\rightharpoonup u && \text{weakly in} \quad L^1(0,T;W^{1,1}(\Omega)),\\
A(\cdot,\nabla u^n) &\overset{*}{\rightharpoonup} \chi && \text{weakly-star in} \quad L_{M^*}(Q),\\
u^n &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;L^2(\Omega)),\\
u_t^n &\overset{*}{\rightharpoonup} u_t && \text{weakly-star in} \quad W^{-1,\infty}(0,T;L^2(\Omega)).
\end{aligned} \tag{4.2}$$

After passing to the limit we conclude the limit identity

$$-\int_Q u\varphi_t dxdt + \int_Q \chi \cdot \nabla\varphi dxdt = \int_\Omega u_0(x)\varphi(0,x)dx \tag{4.3}$$

for each compactly supported and smooth function φ. In the remaining part of the proof we will concentrate on characterizing the limit χ. We are aiming to integrate by parts in (4.3) although the solution is not an admissible test function. According to the results of Section 3 a function of the form

$$\varphi^j = \varrho^j * \varrho^j * u \tag{4.4}$$

is already a proper test function with $\varrho \in C^\infty(\mathbb{R})$, ϱ having a compact support, $\varrho(\tau) = \varrho(-\tau)$, $\int_{\mathbb{R}} \varrho(\tau)d\tau = 1$ and defining $\varrho^j(t) = j\varrho(jt)$. Indeed, if we approximate the function of the form (4.4) by a sequence of smooth functions, then in the case $\chi \in E_{M^*}(Q)$ we pass to the limit with weak-star convergence. But if $\chi \notin E_{M^*}(Q)$, which indeed is the case when $E_{M^*}(Q) \neq L_{M^*}(Q)$, we will pass to the limit by means of modular convergence. This is the reason why in Section 3 we concentrated on showing that weak-star and modular limits coincide.

Then, we can observe that for $0 < s_0 < s < T$ it follows that

$$\begin{aligned}\int_{s_0}^s \langle u_t, \varphi^j\rangle \, dt &= \int_{s_0}^s \langle u_t, (\varrho^j * \varrho^j * u)\rangle \, dt = \int_{s_0}^s \langle(\varrho^j * u_t), (\varrho^j * u)\rangle \, dt \\ &= \int_{s_0}^s \frac{1}{2}\frac{d}{dt}\|\varrho^j * u\|_2^2 \, dt = \frac{1}{2}\|\varrho^j * u(s)\|_2^2 - \frac{1}{2}\|\varrho^j * u(s_0)\|_2^2.\end{aligned}$$

Next, we pass to the limit with $j \to \infty$ and obtain for almost all s_0, s, namely for all Lebesgue points of the function $u(t)$, that

$$\lim_{j\to\infty} \int_{s_0}^s \langle u_t, u^j\rangle \, dt = \frac{1}{2}\|u(s)\|_2^2 - \frac{1}{2}\|u(s_0)\|_2^2. \tag{4.5}$$

Let us pass now with $j \to \infty$ in other terms. First we concentrate for $0 < s_0 < s_1 < T$ on the term

$$\int_{s_0}^{s_1} \int_\Omega \chi \cdot (\varrho^j * \varrho^j * \nabla u)dxdt = \int_{s_0}^{s_1} \int_\Omega (\varrho^j * \chi) \cdot (\varrho^j * \nabla u)dxdt.$$

Obviously, for any $\psi \in L^1(Q)$ it holds that $(\varrho^j * \psi) \to \psi$ in measure. Hence

$$\varrho^j * \chi \to \chi \quad \text{in measure}$$

and

$$\varrho^j * \nabla u \to \nabla u \quad \text{in measure.}$$

Since M and M^* are convex and nonnegative, then weak lower semicontinuity and a priori estimates provide that the integrals are finite,

$$\int_Q M(\nabla u)dxdt \quad \text{and} \quad \int_Q M^*(\chi)dxdt$$

and consequently the sequences $\{M(\varrho^j * \nabla u)\}$ and $\{M^*(\varrho^j * \chi)\}$ are uniformly integrable. Thus by Lemma 2.1,

$$\varrho^j * \nabla u \xrightarrow{M} \nabla u \quad \text{modularly in } L_M(Q),$$
$$\varrho^j * \chi \xrightarrow{M^*} \chi \qquad \text{modularly in } L_{M^*}(Q).$$

By Proposition 2.3 we conclude that

$$\lim_{j\to\infty} \int_{s_0}^{s_1} \int_\Omega (\varrho^j * \chi) \cdot (\varrho^j * \nabla u) dxdt = \int_{s_0}^{s_1} \int_\Omega \chi \cdot \nabla u dxdt. \tag{4.6}$$

We are aiming to show that

$$\frac{1}{2}\|u(s)\|_2^2 - \frac{1}{2}\|u_0\|_2^2 + \int_{Q^s} \chi \cdot \nabla u dxdt = 0, \tag{4.7}$$

which according to (4.3)–(4.6) holds for some $0 < s_0 < T$, not necessarily equal to zero. To pass to the limit with $s_0 \to 0$ we need to establish the weak continuity of u in $L^2(\Omega)$ w.r.t. time. For this purpose we consider the sequence $\{\frac{du^n}{dt}\}$ and provide uniform estimates. By P^n we mean the orthogonal projection of $L^2(\Omega)$ on the first n eigenvectors of the Laplace operator. Let $\varphi \in L^\infty(0,T; W_0^{r,2}(\Omega))$, $\|\varphi\|_{L^\infty(0,T;W_0^{r,2})} \le 1$, where $r > \frac{d}{2} + 1$ and observe that

$$\left\langle \frac{du^n}{dt}, \varphi \right\rangle = \left\langle \frac{du^n}{dt}, P^n\varphi \right\rangle = -\int_\Omega A(t,x,\nabla u^n) \cdot \nabla(P^n\varphi) dx.$$

Since $\|P^n\varphi\|_{W_0^{r,2}} \le \|\varphi\|_{W_0^{r,2}}$ and $W^{r-1,2}(\Omega) \subset L^\infty(\Omega)$ we estimate as follows:

$$\begin{aligned} \Big| \int_0^T \int_\Omega A(t,x,\nabla u^n) \cdot \nabla(P^n\varphi) dxdt \Big| &\le \int_0^T \|A(t,\cdot,\nabla u^n)\|_{L^1(\Omega)} \|\nabla(P^n\varphi)\|_{L^\infty(\Omega)} dt \\ \le c\int_0^T \|A(t,\cdot,\nabla u^n)\|_{L^1(\Omega)} \|P^n\varphi\|_{W_0^{r,2}} dt &\le c\|A(\cdot,\cdot,\nabla u^n)\|_{L^1(Q)} \|\varphi\|_{L^\infty(0,T;W_0^{r,2})}. \end{aligned} \tag{4.8}$$

Hence we conclude that $\frac{du^n}{dt}$ is bounded in $L^1(0,T; W^{-r,2}(\Omega))$. From the energy estimates and Lemma 2.2 we conclude existence of a monotone, continuous function $L : \mathbb{R}_+ \to \mathbb{R}_+$, with $L(0) = 0$ which is independent of n and

$$\int_{s_1}^{s_2} \|A(t,\cdot,\nabla u^n)\|_{L^1(\Omega)} \le L(|s_1 - s_2|)$$

for any $s_1, s_2 \in [0,T]$. Consequently, estimate (4.8) provides that

$$\left| \int_{s_1}^{s_2} \left\langle \frac{du^n}{dt}, \varphi \right\rangle dt \right| \le L(|s_1 - s_2|)$$

for all φ with supp $\varphi \subset (s_1, s_2) \subset [0,T]$ and $\|\varphi\|_{L^\infty(0,T;W_0^{r,2})} \le 1$. Since

$$\|u^n(s_1) - u^n(s_2)\|_{W^{-r,2}} = \sup_{\|\psi\|_{W_0^{r,2}} \le 1} \left| \left\langle \int_{s_1}^{s_2} \frac{du^n(t)}{dt}, \psi \right\rangle \right| \tag{4.9}$$

then

$$\sup_{n\in\mathbb{N}} \|u^n(s_1) - u^n(s_2)\|_{W^{-r,2}} \le L(|s_1 - s_2|) \tag{4.10}$$

which provides that the family of functions $u^n : [0,T] \to W^{-r,2}(\Omega)$ is equicontinuous. Together with a uniform bound in $L^\infty(0,T;L^2(\Omega))$ it yields that the sequence $\{u^n\}$ is relatively compact in $C([0,T];W^{-r,2}(\Omega))$ and $u \in C([0,T];W^{-r,2}(\Omega))$. Consequently we can choose a sequence $\{s_0^i\}_i$, $s_0^i \to 0^+$ as $i \to \infty$ such that

$$u(s_0^i) \stackrel{i\to\infty}{\longrightarrow} u(0) \quad \text{in } W^{-r,2}(\Omega). \tag{4.11}$$

The limit coincides with the weak limit of $\{u(s_0^i)\}$ in $L^2(\Omega)$ and hence we conclude that

$$\liminf_{i\to\infty} \|u(s_0)\|_{L^2(\Omega)} \ge \|u_0\|_{L^2(\Omega)}. \tag{4.12}$$

Consequently we obtain from (4.1) for any Lebesgue point s of u that

$$\begin{aligned}
\limsup_{n\to\infty} \int_{Q_s} A(t,x,\nabla u^n)\cdot\nabla u^n &= \frac{1}{2}\|u_0\|_2^2 - \liminf_{k\to\infty} \frac{1}{2}\|u^n(s)\|_2^2 \\
&\le \frac{1}{2}\|u_0\|_2^2 - \frac{1}{2}\|u(s)\|_2^2 \\
&\stackrel{(4.12)}{\le} \liminf_{i\to\infty} \left(\frac{1}{2}\|u(s_0^i)\|_2^2 - \frac{1}{2}\|u(s)\|_2^2 \right) \\
&= \lim_{i\to\infty} \int_{s_0^i}^s \int_\Omega \chi\cdot\nabla u dxdt = \int_0^s \int_\Omega \chi\cdot\nabla u dxdt.
\end{aligned} \tag{4.13}$$

Having the above estimate we can complete the proof using the monotonicity of A, namely

$$\int_{Q^s} (A(t,x,\bar v) - A(t,x,\nabla u^n))\cdot(\bar v - \nabla u^n) dxdt \ge 0 \tag{4.14}$$

for all $\bar v \in L^\infty(Q)$. Observe that for $\bar v \in L^\infty(Q)$ it also holds that $A(t,x,\bar v) \in L^\infty(Q)$. Indeed, assume the opposite, i.e., $A(t,x,\bar v)$ is unbounded. Then, since M is nonnegative, by **(A2)**, the following estimate holds:

$$|\bar v| \ge \frac{M^*(x, A(t,x,\bar v))}{|A(t,x,\bar v)|}.$$

Taking into account the superlinear growth of M we observe that the right-hand side tends to infinity, which contradicts that $\bar v \in L^\infty(Q)$. Before passing to the limit with $n \to \infty$, we rewrite (4.14):

$$\int_{Q^s} A(t,x,\nabla u^n)\cdot\nabla u^n dxdt \ge \int_{Q^s} A(t,x,\nabla u^n)\cdot\bar v dxdt + \int_{Q^s} A(t,x,\bar v)\cdot(\nabla u^n - \bar v) dxdt \tag{4.15}$$

hence

$$\int_{Q^s} \chi\cdot\nabla u dxdt \ge \int_{Q^s} \chi\cdot\bar v dxdt + \int_{Q^s} A(t,x,\bar v)\cdot(\nabla u - \bar v) dxdt \tag{4.16}$$

and consequently

$$\int_{Q^s} (A(t,x,\bar{v}) - \chi) \cdot (\bar{v} - \nabla u) dxdt \geq 0. \tag{4.17}$$

We fix $k > 0$ and define Q_k as

$$Q_k = \{(t,x) \in Q^s : |\nabla u(t,x)| \leq k \quad \text{a.e. in } Q^s\}.$$

Let now $0 < j < i$ be arbitrary and $h > 0$. We make the following choice of $\bar{v}$,

$$\bar{v} = (\nabla u)\mathbb{1}_{Q_i} + h\bar{z}\mathbb{1}_{Q_j},$$

with an arbitrary $\bar{z} \in L^\infty(Q)$. Using (4.17) we obtain

$$-\int_{Q^s \setminus Q_i} (A(t,x,0) - \chi) \cdot \nabla u dxdt + h \int_{Q_j} (A(t,x,\nabla u + h\bar{z}) - \chi) \cdot \bar{z} dxdt \geq 0. \tag{4.18}$$

Since **(A2)** implies that $A(t,x,0) = 0$, then obviously

$$-\int_{Q^s \setminus Q_i} (A(t,x,0) - \chi) \cdot \nabla u dxdt = \int_Q \chi \cdot \nabla u \; \mathbb{1}_{Q^s \setminus Q_i} dxdt$$

and since

$$\int_Q |\chi \cdot \nabla u| dxdt < \infty$$

we obtain, while passing to the limit with $i \to \infty$,

$$\chi \cdot \nabla u \; \mathbb{1}_{Q^s \setminus Q_i} \to 0 \qquad \text{a.e. in } Q.$$

Hence by the Lebesgue dominated convergence theorem

$$\lim_{i \to \infty} \int_{Q^s \setminus Q_i} \chi \cdot \nabla u dxdt = 0.$$

Letting $i \to \infty$ in (4.18) and dividing by h yields

$$\int_{Q_j} (A(t,x,\nabla u + h\bar{z}) - \chi) \cdot \bar{z} dxdt \geq 0.$$

Observe that $\nabla u + h\bar{z} \to \nabla u$ a.e. in Q_j as $h \to 0^+$ and $A(t,x,\nabla u + h\bar{z})$ is uniformly bounded in $L^\infty(Q_j)$, $|Q_j| < \infty$, hence by Vitali's theorem we conclude that

$$A(t,x,\nabla u + h\bar{z}) \to A(t,x,\nabla u) \text{ in } L^1(Q_j)$$

and

$$\int_{Q_j} (A(t,x,\nabla u + h\bar{z}) - \chi) \cdot \bar{z} dxdt \to \int_{Q_j} (A(t,x,\nabla u) - \chi) \cdot \bar{z} dxdt$$

as $h \to 0^+$. Consequently,

$$\int_{Q_j} (A(t,x,\nabla u) - \chi) \cdot \bar{z} dxdt \geq 0$$

for all $\bar{z} \in L^\infty(Q)$. The choice $\bar{z} = -\mathrm{sgn}\,(A(t,x,\nabla u) - \chi)$ yields

$$\int_{Q_j} |A(t,x,\nabla u) - \chi| dxdt \leq 0.$$

Hence

$$A(t, x, \nabla u) = \chi \quad \text{a.e. in } Q_j. \tag{4.19}$$

The above identity holds for arbitrary j, hence (4.19) holds a.e. in Q^s. Since it holds for almost all s such that $0 < s < T$ then

$$\chi = A(t, x, \nabla u) \quad \text{a.e. in } Q,$$

which completes the proof of Theorem 1.1 □

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Maximal Parabolic Regularity for Divergence Operators on Distribution Spaces

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Abstract. We show that elliptic second-order operators A of divergence type fulfill maximal parabolic regularity on distribution spaces, even if the underlying domain is highly non-smooth, the coefficients of A are discontinuous and A is complemented with mixed boundary conditions. Applications to quasilinear parabolic equations with non-smooth data are presented.

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Keywords. Maximal parabolic regularity, quasilinear parabolic equations, mixed Dirichlet-Neumann conditions.

1. Introduction

It is the aim of this paper to provide an abridged version of our work [40]. The goal is to provide a text with only few proofs and with a considerable reduction of the sophisticated technicalities. In particular, we present a direct way to carry over maximal parabolic regularity from L^p spaces to the distribution spaces, avoiding the Dore-Venni argument. So our hope is to produce a more readable text for colleagues who are only interested in the principal ideas and results of [40]. On the other hand, due to discussions with K. Gröger, we succeeded in eliminating the crucial supposition in [40] that the local bi-Lipschitz charts for the boundary of the domain have to be volume preserving – at least what concerns maximal parabolic regularity. Thus the conditions get a lot easier to control in examples from beyond the class of strong Lipschitz domains. Naturally, the proof of this is pointed out below, see Section 4.

Our motivation was to find a concept which allows us to treat nonlinear parabolic equations of the formal type

$$\begin{cases} u' - \nabla \cdot \mathcal{G}(u)\mu\nabla u = \mathcal{R}(t, u), \\ \qquad u(T_0) = u_0, \end{cases} \tag{1.1}$$

combined with mixed, nonlinear boundary conditions:

$$\nu \cdot \mathcal{G}(u)\mu\nabla u + b(u) = g \text{ on } \Gamma \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega \setminus \Gamma, \tag{1.2}$$

where Γ is a suitable open subset of $\partial\Omega$.

The main feature is here – in contrast to [43] – that inhomogeneous Neumann conditions and the appearance of distributional right-hand sides (e.g., surface densities) should be admissible. Thus, one has to consider the equations in suitably chosen distribution spaces. The concept to solve (1.1) is to apply a theorem of Prüss (see [51], see also [15]) which bases on maximal parabolic regularity. This has the advantage that right-hand sides are admissible which depend discontinuously on time, which is desirable in many applications. Pursuing this idea, one has, of course, to prove that the occurring elliptic operators satisfy maximal parabolic regularity on the chosen distribution spaces.

In fact, we show that, under very mild conditions on the domain Ω, the Dirichlet boundary part $\partial\Omega \setminus \Gamma$ and the coefficient function, elliptic divergence operators with real, symmetric L^∞-coefficients satisfy maximal parabolic regularity on a huge variety of spaces, among which are Sobolev, Besov and Lizorkin-Triebel spaces, provided that the differentiability index is between 0 and -1 (cf. Theorem 5.18). We consider this as the first main result of this work, also interesting in itself. Up to now, the only existing results for mixed boundary conditions in distribution spaces (apart from the Hilbert space situation) are, to our knowledge, that of Gröger [36] and the recent one of Griepentrog [31]. Concerning the Dirichlet case, compare [10] and references therein.

Let us point out some ideas, which will give a certain guideline for the paper:

In principle, our strategy for proving maximal parabolic regularity for divergence operators on $H^{-1,q}_\Gamma$ was to show an analog of the central result of [9], this time in case of mixed boundary conditions, namely that

$$\bigl(-\nabla \cdot \mu\nabla + 1\bigr)^{-1/2} : L^q \to H^{1,q}_\Gamma \tag{1.3}$$

provides a topological isomorphism for suitable q. This would give the possibility of carrying over the maximal parabolic regularity, known for L^q, to the dual of $H^{1,q'}_\Gamma$, because, roughly spoken, $(-\nabla \cdot \mu\nabla + 1)^{-1/2}$ commutes with the corresponding parabolic solution operator. Unfortunately, we were only able to prove the continuity of (1.3) within the range $q \in \,]1, 2]$, due to a result of Duong and M$^{\text{c}}$Intosh [22], see also [49], but did not succeed in proving the continuity of the inverse in general.

It turns out, however, that (1.3) provides a topological isomorphism, if $\Omega \cup \Gamma$ is the image under a bi-Lipschitz mapping of one of Gröger's model sets [35], describing the geometric configuration in neighborhoods of boundary points of Ω. Thus, in these cases one may carry over the maximal parabolic regularity from L^q to $H^{-1,q}_\Gamma$. Knowing this, we localize the linear parabolic problem, use the 'local' maximal parabolic information and interpret this again in the global context at the end. Interpolation with the L^p result then yields maximal parabolic regularity on the corresponding interpolation spaces.

Let us explicitly mention that the concept of Gröger's regular sets, where the domain itself is a Lipschitz domain, seems adequate to us, because it covers many realistic geometries that fail to be domains with Lipschitz boundary. One striking example are the two crossing beams, cf. [40, Subsection 7.3].

The strategy for proving that (1.1), (1.2) admit a unique local solution is as follows. We reformulate (1.1) by adding the distributional terms, corresponding to the boundary condition (1.2) to the right-hand side of (1.1). Assuming additionally that the elliptic operator $-\nabla \cdot \mu \nabla + 1 : H^{1,q}_\Gamma \to H^{-1,q}_\Gamma$ provides a topological isomorphism for a q larger than the space dimension d, the above-mentioned result of Prüss for abstract quasilinear equations applies to the resulting quasilinear parabolic equation. The detailed discussion how to assure all requirements of [51], including the adequate choice of the Banach space, is presented in Section 6. Let us further emphasize that the presented setting allows for coefficient functions that really jump at hetero interfaces of the material and permits mixed boundary conditions, as well as domains which do not possess a Lipschitz boundary. It is well known that this is highly desirable when modelling real world problems. One further advantage is that nonlinear, nonlocal boundary conditions are admissible in our concept, despite the fact that the data is highly non-smooth, compare [2]. It is remarkable that, irrespective of the discontinuous right-hand sides, the solution is Hölder continuous simultaneously in space and time, see Corollary 6.17 below.

In Section 7 we give examples for geometries, Dirichlet boundary parts and coefficients in three dimensions for which our additional supposition, the isomorphy $-\nabla \cdot \mu \nabla + 1 : H^{1,q}_\Gamma \to H^{-1,q}_\Gamma$ really holds for a $q > d$.

Finally, some concluding remarks are given in Section 8.

2. Notation and general assumptions

Throughout this article the following assumptions are valid.

- $\Omega \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain (cf. Assumption 3.1) and Γ is an open subset of $\partial\Omega$.
- The coefficient function μ is a Lebesgue measurable, bounded function on Ω taking its values in the set of real, symmetric, positive definite $d \times d$ matrices, satisfying the usual ellipticity condition.

Remark 2.1. Concerning the notions 'Lipschitz domain' and 'domain with Lipschitz boundary' (synonymous: strongly Lipschitz domain) we follow the terminology of Grisvard [34].

Remark 2.2. Since the requirement 'Lipschitz domain' does not become apparent explicitly in the subsequent considerations, let us briefly comment on this: it assures the existence of a continuous extension operator $\mathfrak{E} : L^1(\Omega) \to L^1(\mathbb{R}^d)$ whose restriction to $H^{1,2}(\Omega)$ maps this space continuously into $H^{1,2}(\mathbb{R}^d)$. This property is fundamental for nearly all harmonic analysis techniques applied below, see [49, Ch. 6.3] and [7].

For $\varsigma \in]0,1]$ and $1 < q < \infty$ we define $H^{\varsigma,q}_\Gamma(\Omega)$ as the closure of

$$C^\infty_\Gamma(\Omega) := \{\psi|_\Omega : \psi \in C^\infty(\mathbb{R}^d),\ \mathrm{supp}(\psi) \cap (\partial\Omega \setminus \Gamma) = \emptyset\}$$

in the Sobolev space $H^{\varsigma,q}(\Omega)$. Concerning the dual of $H^{\varsigma,q}_\Gamma(\Omega)$, we have to distinguish between the space of linear and the space of anti-linear forms on this space. We define $H^{-\varsigma,q}_\Gamma(\Omega)$ as the space of continuous, linear forms on $H^{\varsigma,q'}_\Gamma(\Omega)$ and $\breve{H}^{-\varsigma,q}_\Gamma(\Omega)$ as the space of anti-linear forms on $H^{\varsigma,q'}_\Gamma(\Omega)$ if $1/q + 1/q' = 1$. Note that L^p spaces may be viewed as part of $\breve{H}^{-\varsigma,q}_\Gamma$ for suitable ς, q via the identification of an element $f \in L^p$ with the anti-linear form $H^{\varsigma,q'}_\Gamma \ni \psi \mapsto \int_\Omega f\overline{\psi}\,\mathrm{dx}$.

If misunderstandings are not to be expected, we drop the Ω in the notation of spaces, i.e., function spaces without an explicitly given domain are to be understood as function spaces on Ω.

By K we denote the open unit cube $]-1,1[^d$ in $\mathbb{R}^d$, by K_- the lower half-cube $K \cap \{\mathrm{x} : x_d < 0\}$, by $\Sigma = K \cap \{\mathrm{x} : x_d = 0\}$ the upper plate of K_- and by Σ_0 the left half of Σ, i.e., $\Sigma_0 = \Sigma \cap \{\mathrm{x} : x_{d-1} < 0\}$.

Throughout the paper we will use $\mathrm{x}, \mathrm{y}, \ldots$ for vectors in $\mathbb{R}^d$.

If B is a closed operator on a Banach space X, then we denote by $\mathrm{dom}_X(B)$ the domain of this operator. $\mathcal{L}(X,Y)$ denotes the space of linear, continuous operators from X into Y; if $X = Y$, then we abbreviate $\mathcal{L}(X)$. Furthermore, we will write $\langle \cdot, \cdot \rangle_{X'}$ for the dual pairing of elements of X and the space X' of anti-linear forms on X.

Finally, the letter c denotes a generic constant, not always of the same value.

3. Preliminaries

In this section we will properly define the elliptic divergence operator and afterwards collect properties of the L^p realizations of this operator which will be needed in the subsequent sections.

Let us first recall the concept of regular sets $\Omega \cup \Gamma$, introduced by Gröger in his pioneering paper [35], which will provide us with an adequate geometric framework for all that follows.

Assumption 3.1. For any point $\mathrm{x} \in \partial\Omega$ there is an open neighborhood Υ_x of x and a bi-Lipschitz mapping ϕ_x from Υ_x into $\mathbb{R}^d$, such that $\phi_\mathrm{x}(\mathrm{x}) = 0$ and $\phi_\mathrm{x}\big((\Omega \cup \Gamma) \cap \Upsilon_\mathrm{x}\big) = K_-$ or $K_- \cup \Sigma$ or $K_- \cup \Sigma_0$.

Remark 3.2. It is not hard to see that every Lipschitz domain and also its closure is regular in the sense of Gröger, the corresponding model sets are then K_- or $K_- \cup \Sigma$, respectively, see [34, Ch. 1.2]. In two and three space dimensions one can give the following simplifying characterization for a set $\Omega \cup \Gamma$ to be regular in the sense of Gröger, i.e., to satisfy Assumption 3.1, see [39]:

If $\Omega \subseteq \mathbb{R}^2$ is a bounded Lipschitz domain and $\Gamma \subseteq \partial\Omega$ is relatively open, then $\Omega \cup \Gamma$ is regular in the sense of Gröger, iff $\partial\Omega \setminus \Gamma$ is the finite union of (non-degenerate) closed arc pieces.

In $\mathbb{R}^3$ the following characterization can be proved:

If $\Omega \subseteq \mathbb{R}^3$ is a bounded Lipschitz domain and $\Gamma \subseteq \partial\Omega$ is relatively open, then $\Omega \cup \Gamma$ is regular in the sense of Gröger, iff the following two conditions are satisfied:

i) $\partial\Omega \setminus \Gamma$ is the closure of its interior (within $\partial\Omega$).

ii) For any $\mathrm{x} \in \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ there is an open neighborhood $\mathcal{U}$ of x and a bi-Lipschitz mapping $\kappa : \mathcal{U} \cap \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma) \to]-1, 1[$.

Following [27, Ch. 3.3.4 C], for every Lipschitz hypersurface $\mathcal{H} \subseteq \overline{\Omega}$ one can introduce a surface measure σ on $\mathcal{H}$. This is in particular true for $\mathcal{H} = \partial\Omega$, see also [41]. Having this at hand, one can prove the following trace theorem.

Proposition 3.3. *Assume $q \in]1, \infty[$ and $\theta \in]\frac{1}{q}, 1]$. Let Π be a Lipschitz hypersurface in $\overline{\Omega}$ and let ϖ be any measure on Π which is absolutely continuous with respect to the surface measure σ. If the corresponding Radon-Nikodým derivative is essentially bounded (with respect to σ), then the trace operator* Tr *is continuous from $H^{\theta,q}(\Omega)$ to $L^q(\Pi, \varpi)$.*

Later we will repeatedly need the following interpolation result.

Proposition 3.4. *Let Ω and Γ satisfy Assumption* 3.1 *and let $\theta \in]0, 1[$. Then for $q_0, q_1 \in]1, \infty[$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ one has*

$$H^{1,q}_\Gamma = \big[H^{1,q_0}_\Gamma, H^{1,q_1}_\Gamma\big]_\theta \quad \textit{and} \quad \breve{H}^{-1,q}_\Gamma = \big[\breve{H}^{-1,q_0}_\Gamma, \breve{H}^{-1,q_1}_\Gamma\big]_\theta. \tag{3.1}$$

We define the operator $A : H^{1,2}_\Gamma \to \breve{H}^{-1,2}_\Gamma$ by

$$\langle A\psi, \varphi\rangle_{\breve{H}^{-1,2}_\Gamma} := \int_\Omega \mu\nabla\psi \cdot \nabla\overline{\varphi}\,\mathrm{dx} + \int_\Gamma \varkappa\,\psi\,\overline{\varphi}\,\mathrm{d}\sigma, \quad \psi, \varphi \in H^{1,2}_\Gamma, \tag{3.2}$$

where $\varkappa \in L^\infty(\Gamma, \mathrm{d}\sigma)$. Note that in view of Proposition 3.3 the form in (3.2) is well defined.

In the special case $\varkappa = 0$, we write more suggestively $-\nabla \cdot \mu\nabla$ instead of A.

The L^2 realization of A, i.e., the maximal restriction of A to the space L^2, we denote by the same symbol A; clearly this is identical with the operator which is induced by the form on the right-hand side of (3.2). If B is a self-adjoint operator on L^2, then by the L^p realization of B we mean its restriction to L^p if $p > 2$ and the L^p closure of B if $p \in [1, 2[$.

First, we collect some basic facts on the operator $-\nabla \cdot \mu\nabla$.

Proposition 3.5.

i) *The operator $\nabla \cdot \mu\nabla$ generates an analytic semigroup on $\breve{H}^{-1,2}_\Gamma$.*

ii) *The operator $-\nabla \cdot \mu\nabla$ is self-adjoint on L^2 and bounded by 0 from below. The restriction of $-A$ to L^2 is densely defined and generates an analytic semigroup there.*

iii) *If $\lambda > 0$ then the operator $(-\nabla \cdot \mu\nabla + \lambda)^{1/2} : H^{1,2}_\Gamma \to L^2$ provides a topological isomorphism; in other words: the domain of $(-\nabla \cdot \mu\nabla + \lambda)^{1/2}$ on L^2 is the form domain $H^{1,2}_\Gamma$.*

iv) *Let $\zeta \in L^\infty$ be a real function with a positive lower bound. Then $-\zeta\nabla\cdot\mu\nabla+1$ has its spectrum in $[1,\infty[$ and admits a bounded functional calculus on L^2.*

v) *The operator $\nabla\cdot\mu\nabla$, considered on L^p, $p \in]1,\infty[$, is densely defined and generates a strongly continuous semigroup of contractions there.*

vi) *Let $\zeta \in L^\infty$ be a real function with a positive lower bound. Then, under Assumption* 3.1, *the operator $\zeta\nabla\cdot\mu\nabla$ satisfies the estimate*

$$\|(\zeta\nabla\cdot\mu\nabla - 1 - \lambda)^{-1}\|_{\mathcal{L}(L^p)} \le \frac{M_p}{1+|\lambda|}, \qquad \textit{if} \quad \operatorname{Re}\lambda \ge -\frac{1}{2} \tag{3.3}$$

and, hence, generates a bounded, analytic semigroup on every space L^p, $p \in]1,\infty[$.

Proof. Assertions i)–iii) are standard, while iv) follows from ii) and the subsequent Lemma 3.6. Part v) is proved in [49, Thm. 4.28/Prop. 4.11]. Finally, a proof of (3.3) is contained in [33, Thm. 5.2/Remark 5.1] and the second part of vi) then follows from [50, Thm. 7.7] or [45, Ch. IX.1.6]. □

Lemma 3.6. *Let $w \in L^\infty$ be a real function with a strictly positive lower bound and let B be a selfadjoint, positive operator on L^2. Then the operator $wB+1$ has its spectrum in $[1,\infty[$ and admits a bounded functional calculus on L^2.*

Proof. We equip L^2 with the equivalent scalar product $(f,g)\mapsto \int_\Omega f\overline{g}w^{-1}\mathrm{dx}$ and name the resulting Hilbert space $\widehat{L^2}$. Observe that $C \in \mathcal{L}(L^2)$, iff $C \in \mathcal{L}(\widehat{L^2})$. The equation

$$\int_\Omega (wB+1)f\,\overline{g}w^{-1}\,\mathrm{dx} = \int_\Omega Bf\,\overline{g}\,\mathrm{dx} + \int_\Omega f\,\overline{g}w^{-1}\,\mathrm{dx} = \int_\Omega f\,\overline{Bg}\,\mathrm{dx} + \int_\Omega f\overline{g}w^{-1}\,\mathrm{dx}$$
$$= \int_\Omega f\overline{(wB+1)g}w^{-1}\,\mathrm{dx} \quad \text{for} \quad f,g \in \mathrm{dom}_{L^2}(B)$$

shows that $wB+1$ is symmetric on $\widehat{L^2}$.

Let $\lambda < 1$. Then, by the hypotheses on B and w, the operator

$$w^{-1}(wB+1-\lambda) = B + w^{-1}(1-\lambda)$$

on L^2 is self-adjoint and has a strictly positive lower form bound. Thus, it is continuously invertible on L^2. This now implies invertibility of $wB+1-\lambda$ on L^2, as well as on $\widehat{L^2}$. So, every $\lambda \in]-\infty,1[$ is in the resolvent set of $wB+1$ on $\widehat{L^2}$ and this, together with the symmetry shown above, implies that the operator $wB+1$ is selfadjoint in $\widehat{L^2}$ and has its spectrum in $[1,\infty[$.

Exploiting now a spectral integral representation $wB+1 = \int_1^\infty \lambda dP(\lambda)$, one obtains a bounded functional calculus of $wB+1$ on $\widehat{L^2}$. This implies a bounded functional calculus also on L^2, since both norms are equivalent. □

One essential instrument for our subsequent considerations are (upper) Gaussian estimates.

Theorem 3.7. *Let ζ, V be real, positive L^∞ functions and let ζ admit a positive lower bound. Then the semigroup, generated by $\zeta\nabla\cdot\mu\nabla - V$ satisfies upper Gaussian estimates, precisely:*

$$(\mathrm{e}^{t(\zeta\nabla\cdot\mu\nabla-V)} f)(\mathrm{x}) = \int_\Omega K_t(\mathrm{x},\mathrm{y}) f(\mathrm{y})\,\mathrm{dy}, \quad \mathrm{x}\in\Omega,\ f\in L^2,$$

for some measurable function $K_t : \Omega\times\Omega \to \mathbb{R}_+$ and for every $\varepsilon > 0$ there exist constants $c, b > 0$, such that

$$0 \le K_t(\mathrm{x},\mathrm{y}) \le \frac{c}{t^{d/2}}\ \mathrm{e}^{-b\frac{|\mathrm{x}-\mathrm{y}|^2}{t}}\,\mathrm{e}^{\varepsilon t}, \quad t>0,\ a.a.\ \mathrm{x},\mathrm{y}\in\Omega. \tag{3.4}$$

If V admits a lower bound $V_* > 0$, then ε may be taken as 0.

Proof. Let us first consider the case $\zeta \equiv 1$. If V is only nonnegative, then the estimate in (3.4) follows from [49, Theorem 6.10] (see also [7]). If V admits a strictly positive lower bound V_*, then one may write $V = V - V_* + V_*$ and thus obtains (3.4) with $\varepsilon = 0$.

The case of general ζ is implied by the multiplicative perturbation result in [23]. □

Lemma 3.8. *Let $\zeta \in L^\infty$ be a real function with a strictly positive lower bound and $p \in\,]1,\infty[$. Then, under Assumption 3.1, $\mathrm{dom}_{L^p}\big(-\zeta\nabla\cdot\mu\nabla+1\big)^{1/2} = \mathrm{dom}_{L^p}\big(-\nabla\cdot\mu\nabla+1\big)^{1/2}$, and the norms $\|\big(-\zeta\nabla\cdot\mu\nabla+1\big)^{1/2}\cdot\|_{L^p}$ and $\|\big(-\nabla\cdot\mu\nabla+1\big)^{1/2}\cdot\|_{L^p}$ are equivalent.*

Proof. Since ζ has positive lower and upper bounds, $D_\zeta := \mathrm{dom}_{L^p}(-\zeta\nabla\cdot\mu\nabla+1)$ equals $D := \mathrm{dom}_{L^p}(-\nabla\cdot\mu\nabla+1)$ and the corresponding graph norms are equivalent. Thus, necessarily also $[L^p, D_\zeta]_{1/2} = [L^p, D]_{1/2}$ including the equivalence of the corresponding norms. In order to conclude, we will show that both, $-\nabla\cdot\mu\nabla+1$ and $-\zeta\nabla\cdot\mu\nabla+1$ admit bounded imaginary powers on L^p. Then by [56, Ch. 1.15.3], one obtains the identity

$$\mathrm{dom}_{L^p}\big(-\nabla\cdot\mu\nabla+1\big)^{1/2} = [L^p, D]_{1/2} = [L^p, D_\zeta]_{1/2} = \mathrm{dom}_{L^p}\big(-\zeta\nabla\cdot\mu\nabla+1\big)^{1/2},$$

including the equivalence of the graph norms.

The operator $-\nabla\cdot\mu\nabla+1$ has bounded imaginary powers thanks to Proposition 3.5 v) and [17, Corollary 1]. Concerning $-\zeta\nabla\cdot\mu\nabla+1$, one can argue as follows: this operator has a bounded H^∞ calculus on L^2 by Proposition 3.5 iv) and the associated semigroup admits Gaussian estimates with $\varepsilon = 0$ by Theorem 3.7. Thus the bounded H^∞ functional calculus extrapolates to all spaces L^p, $p \in\,]1,\infty[$, by [24, Theorem 3.4] and this in particular implies bounded imaginary powers. □

4. Mapping properties for $(-\nabla \cdot \mu\nabla + 1)^{1/2}$

In this chapter we prove that, under certain topological conditions on Ω and Γ, the mapping

$$(-\nabla \cdot \mu\nabla + 1)^{1/2} : H^{1,q}_\Gamma \to L^q$$

is a topological isomorphism for $q \in]1, 2[$. We abbreviate $-\nabla \cdot \mu\nabla$ by A_0 throughout this chapter. Let us introduce the following

Assumption 4.1. There is a bi-Lipschitz mapping ϕ from a neighborhood of $\overline{\Omega}$ into $\mathbb{R}^d$ such that $\phi(\Omega \cup \Gamma) = K_-$ or $K_- \cup \Sigma$ or $K_- \cup \Sigma_0$.

The main results of this section are the following three theorems.

Theorem 4.2. *Under the general assumptions made in Section* 2 *the following holds true: For every $q \in]1, 2]$, the operator $(A_0 + 1)^{-1/2}$ is a continuous operator from L^q into $H^{1,q}_\Gamma$. Hence, it continuously maps $\breve{H}^{-1,q}_\Gamma$ into L^q for any $q \in [2, \infty[$.*

Theorem 4.3. *Let in addition Assumption* 4.1 *be fulfilled. Then, for every $q \in]1, 2]$, the operator $(A_0 + 1)^{1/2}$ maps $H^{1,q}_\Gamma$ continuously into L^q. Hence, it continuously maps L^q into $\breve{H}^{-1,q}_\Gamma$ for any $q \in [2, \infty[$.*

Putting these two results together, one immediately gets the following isomorphism property of the square root of $A_0 + 1$.

Theorem 4.4. *Under Assumption* 4.1*, $(A_0 + 1)^{1/2}$ provides a topological isomorphism between $H^{1,q}_\Gamma$ and L^q for $q \in]1, 2]$ and a topological isomorphism between L^q and $\breve{H}^{-1,q}_\Gamma$ for any $q \in [2, \infty[$.*

Remark 4.5. In all three theorems the second assertion follows from the first by the self-adjointness of A_0 on L^2 and duality; thus one may focus on the proof of the first assertions.

Let us first prove the continuity of the operator $(A_0 + 1)^{-1/2} : L^q \to H^{1,q}_\Gamma$. In order to do so, we observe that this follows, whenever

1. The Riesz transform $\nabla(A_0 + 1)^{-1/2}$ is a bounded operator on L^q, and, additionally,
2. $(A_0 + 1)^{-1/2}$ maps L^q into $H^{1,q}_\Gamma$.

The first item is proved in [49, Thm. 7.26]. It remains to show 2. The first point makes clear that $(A_0 + 1)^{-1/2}$ maps L^q continuously into $H^{1,q}$, thus one has only to verify the correct boundary behavior of the images. If $f \in L^2 \hookrightarrow L^q$, then one has $(A_0 + 1)^{-1/2} f \in H^{1,2}_\Gamma \hookrightarrow H^{1,q}_\Gamma$. Thus, the assertion follows from 1. and the density of L^2 in L^q.

Remark 4.6. Theorem 4.2 is not true for other values of q in general, see [8, Ch. 4] for a further discussion.

We now prove Theorem 4.3. It will be deduced from the subsequent deep result on divergence operators with Dirichlet boundary conditions and some permanence principles.

Proposition 4.7 (Auscher/Tchamitchian, [9]). *Let $q \in]1,\infty[$ and Ω be a strongly Lipschitz domain. Then the root of the operator A_0, combined with a homogeneous Dirichlet boundary condition, maps $H_0^{1,q}(\Omega)$ continuously into $L^q(\Omega)$.*

For further reference we mention the following consequence of Theorem 4.2 and Proposition 4.7.

Corollary 4.8. *Under the hypotheses of Proposition 4.7 the operator $(A_0+1)^{-1/2}$ provides a topological isomorphism between L^q and $H_0^{1,q}$, if $q \in]1,2]$.*

Proof. The only thing to show is that the continuity of $A_0^{1/2}$ from Proposition 4.7 carries over to $(A_0+1)^{1/2}$. For this it suffices to show that the mapping $(A_0+1)^{1/2}A_0^{-1/2} = (1+A_0^{-1})^{1/2} : L^2 \to L^2$ extends to a continuous mapping from L^q into itself. Since the operator includes a homogeneous Dirichlet condition, the L^2 spectrum of A_0 is contained in an interval $[\varepsilon,\infty[$ for some $\varepsilon > 0$. But the spectrum of A_0, considered on L^q, is independent from q, see [49, Thm. 7.10]. Hence, A_0^{-1} is well defined and continuous on every L^q. Moreover, the spectrum of $1+A_0^{-1}$, considered as an operator on L^q, is thus contained in a bounded interval $[1,\delta]$ by the spectral mapping theorem, see [45, Ch. III.6.3]. Consequently, $(1+A_0^{-1})^{1/2} : L^q \to L^q$ is also a continuous operator by classical functional calculus, see [21, Ch. VII.3]. $\square$

In view of Assumption 4.1 it is a natural idea to reduce our considerations to the three model constellations mentioned there. In order to do so, we have to show that the assertion of Theorem 4.3 is invariant under bi-Lipschitz transformations of the domain. The proof will stem from the following lemma.

Lemma 4.9. *Assume that ϕ is a bi-Lipschitzian mapping from a neighborhood of $\overline{\Omega}$ into $\mathbb{R}^d$. Let $\phi(\Omega) = \Omega_\triangle$ and $\phi(\Gamma) = \Gamma_\triangle$. Define for any function $f \in L^1(\Omega_\triangle)$,*

$$(\Phi f)(\mathrm{x}) = f(\phi(\mathrm{x})) = (f \circ \phi)(\mathrm{x}), \quad \mathrm{x} \in \Omega.$$

Then

i) *The restriction of Φ to any $L^p(\Omega_\triangle)$, $1 \le p < \infty$, provides a linear, topological isomorphism between this space and $L^p(\Omega)$.*

ii) *For any $p \in]1,\infty[$, the mapping Φ induces a linear, topological isomorphism*

$$\Phi_p : H^{1,p}_{\Gamma_\triangle}(\Omega_\triangle) \to H^{1,p}_\Gamma(\Omega).$$

iii) *$\Phi^*_{p'}$ is a linear, topological isomorphism between $\breve{H}^{-1,p}_\Gamma(\Omega)$ and $\breve{H}^{-1,p}_{\Gamma_\triangle}(\Omega_\triangle)$ for any $p \in]1,\infty[$.*

iv) *One has*

$$\Phi^*_{p'} A_0 \Phi_p = -\nabla \cdot \mu_\triangle \nabla \tag{4.1}$$

with

$$\mu_\triangle(\mathrm{y}) = \frac{1}{\left|\det(D\phi)(\phi^{-1}(\mathrm{y}))\right|}(D\phi)(\phi^{-1}(\mathrm{y}))\,\mu(\phi^{-1}(\mathrm{y}))\,\left(D\phi\right)^T(\phi^{-1}(\mathrm{y}))$$

for almost all $\mathrm{y} \in \Omega_\triangle$. Here, $D\phi$ denotes the derivative of ϕ and $\det(D\phi)$ the corresponding determinant.

v) $\mu_\triangle$ *also is bounded, Lebesgue measurable, elliptic and takes real, symmetric matrices as values.*

vi) *The restriction of* $\Phi_2^*\Phi$ *to* $L^2(\Omega_\triangle)$ *equals the multiplication operator which is given by the function* $\big|\det(D\phi)(\phi^{-1}(\cdot))\big|^{-1}$.

Remark 4.10. It is well known that $\big|\det(D\phi)(\phi^{-1}(\cdot))\big|$ is a function from L^∞ which, additionally, has a positive lower bound, due to the bi-Lipschitz property of ϕ, see [27, Ch. 3]. In the sequel we denote this function by ζ.

Lemma 4.11. *Let* $p \in \,]1,\infty[$. *Then, in the notation of the preceding lemma, the operator* $\big(-\nabla\cdot\mu_\triangle\nabla+1\big)^{1/2}$ *maps* $H^{1,p}_{\Gamma_\triangle}(\Omega_\triangle)$ *continuously into* $L^p(\Omega_\triangle)$, *if* $(A_0+1)^{1/2}$ *maps* $H^{1,p}_\Gamma(\Omega)$ *continuously into* $L^p(\Omega)$.

Proof. We will employ the formula

$$B^{-1/2} = \frac{1}{\pi}\int_0^\infty t^{-1/2}(B+t)^{-1}\,\mathrm{d}t, \tag{4.2}$$

B being a positive operator on a Banach space X, see [56, Ch. 1.14/1.15] or [50, Ch. 2.6].

The operators A_0+1, $-\nabla\cdot\mu_\triangle\nabla+1$ and $-\zeta\nabla\cdot\mu_\triangle\nabla+1$ are positive operators in the sense of [56, Ch. 1.14] on any L^p, see Proposition 3.5. From (4.1) and vi) of the preceding lemma one deduces

$$\Phi_2^*\big(A_0+1+t\big)\Phi_2 = -\nabla\cdot\mu_\triangle\nabla+\zeta^{-1}(1+t)$$

for every $t>0$. This leads to

$$\Phi_2^{-1}\big(A_0+1+t\big)^{-1}\big(\Phi_2^*\big)^{-1} = \big(-\nabla\cdot\mu_\triangle\nabla+\zeta^{-1}(1+t)\big)^{-1} = \big(-\zeta\nabla\cdot\mu_\triangle\nabla+1+t\big)^{-1}\zeta. \tag{4.3}$$

The $H^{1,2}_{\Gamma_\triangle}(\Omega_\triangle) \leftrightarrow \breve{H}^{-1,2}_{\Gamma_\triangle}(\Omega_\triangle)$ duality is the extended $L^2(\Omega_\triangle)$ duality. Thus, when restricting (4.3) to L^2, Φ^* may then be viewed as the adjoint with respect to the L^2 duality. Integrating this equation with weight $\frac{t^{-1/2}}{\pi}$, one obtains, according to (4.2),

$$\Phi^{-1}(A_0+1)^{-1/2}(\Phi_2^*)^{-1} = \big(-\zeta\nabla\cdot\mu_\triangle\nabla+1\big)^{-1/2}\zeta. \tag{4.4}$$

Observe that the corresponding integrals converge in $\mathcal{L}(L^p)$, according to Proposition 3.5. Inverting (4.4), we get the operator equation

$$\Phi^*(A_0+1)^{1/2}\Phi_2 = \zeta^{-1}\big(-\zeta\nabla\cdot\mu_\triangle\nabla+1\big)^{1/2}.$$

From this, the continuity of

$$\big(-\zeta\nabla\cdot\mu_\triangle\nabla+1\big)^{1/2} : H^{1,p}_{\Gamma_\triangle}(\Omega_\triangle) \to L^p(\Omega_\triangle)$$

follows by our supposition on A_0 and straightforward continuity arguments on Φ, see [40] for a detailed discussion. An application of Lemma 3.8 concludes the proof. $\square$

Lemma 4.11 allows us to reduce the proof of Theorem 4.3 to $\Omega = K_-$ and the three cases $\Gamma = \emptyset$, $\Gamma = \Sigma$ or $\Gamma = \Sigma_0$. The first case, $\Gamma = \emptyset$, is already contained in Proposition 4.7. In order to treat the second one, we use a reflection argument. Let us point out the main ideas for this: First, one defines the operator $\mathfrak{E} : L^1(K_-) \to L^1(K)$ which assigns to every function from $L^1(K_-)$ its symmetric extension. Let us further denote by $\mathfrak{R} : L^1(K) \to L^1(K_-)$ the restriction operator. Finally, one defines $-\nabla \cdot \hat{\mu}\nabla : H_0^{1,2}(K) \to \breve{H}^{-1,2}(K)$ as the symmetric extension of $-\nabla \cdot \mu\nabla$ to K. Note that this latter operator is then combined with homogeneous Dirichlet conditions. The definition of $-\nabla \cdot \hat{\mu}\nabla$ in particular implies for $t \geq 0$

$$\big(A_0 + 1 + t\big)^{-1} f = \mathfrak{R}\big(-\nabla \cdot \hat{\mu}\nabla + 1 + t\big)^{-1}\mathfrak{E}f \quad \text{for all} \quad f \in L^2(K_-).$$

Multiplying this equation by $\frac{t^{-1/2}}{\pi}$ and integrating over t, one obtains, in accordance with (4.2),

$$(A_0 + 1)^{-1/2} f = \mathfrak{R}\big(-\nabla \cdot \hat{\mu}\nabla + 1\big)^{-1/2}\mathfrak{E}f, \quad f \in L^2(K_-).$$

This equation extends to all $f \in L^p(K_-)$ with $p \in]1, 2[$. Now one exploits the fact that $\big(-\nabla \cdot \hat{\mu}\nabla + 1\big)^{-1/2}$ is a surjection onto the whole $H_0^{1,p}(K)$ by Corollary 4.8. Then some straightforward arguments show that $(A_0+1)^{-1/2} : L^p(K_-) \to H_\Sigma^{1,p}(K_-)$ also is a surjection. Since, by Theorem 4.2 $(A_0 + 1)^{-1/2} : L^p(K_-) \to H_\Sigma^{1,p}(K_-)$ is continuous, the continuity of the inverse finally is implied by the open mapping theorem.

In order to prove the same for the third model constellation, i.e., $\Gamma = \Sigma_0$, one shows

Lemma 4.12. *There is a bi-Lipschitz mapping $\phi : \mathbb{R}^d \to \mathbb{R}^d$ that maps $K_- \cup \Sigma_0$ onto $K_- \cup \Sigma$.*

Thus, the proof of Theorem 4.3 in the case $\Gamma = \Sigma_0$ results from the case $\Gamma = \Sigma$ and Lemmas 4.11 and 4.12.

Remark 4.13. Let us mention that Lemma 4.11, only applied to $\Omega = K$ and $\Gamma = \emptyset$ (the pure Dirichlet case) already provides a zoo of geometries which is not covered by [9]. Notice in this context that the image of a strongly Lipschitz domain under a bi-Lipschitz transformation need not be a strongly Lipschitz domain at all, cf. [34, Ch. 1.2].

5. Maximal parabolic regularity for A

In this section we intend to prove the first main result of this work announced in the introduction, i.e., maximal parabolic regularity of A in spaces with negative differentiability index. Let us first recall the notion of maximal parabolic L^s regularity.

Definition 5.1. Let $1 < s < \infty$, let X be a Banach space and let $J :=]T_0, T[\subseteq \mathbb{R}$ be a bounded interval. Assume that B is a closed operator in X with dense domain D (in the sequel always equipped with the graph norm). We say that B satisfies *maximal parabolic $L^s(J;X)$ regularity*, if for any $f \in L^s(J;X)$ there exists a unique function $u \in W^{1,s}(J;X) \cap L^s(J;D)$ satisfying

$$u' + Bu = f, \qquad u(T_0) = 0,$$

where the time derivative is taken in the sense of X-valued distributions on J, see [4, Ch. III.1].

Remark 5.2.

i) It is well known that the property of maximal parabolic regularity of an operator B is independent of $s \in]1, \infty[$ and the specific choice of the interval J (cf. [20]). Thus, in the following we will say for short that B admits maximal parabolic regularity on X.
ii) If an operator satisfies maximal parabolic regularity on a Banach space X, then its negative generates an analytic semigroup on X (cf. [20]). In particular, a suitable left half-plane belongs to its resolvent set.
iii) If X is a Hilbert space, the converse is also true: The negative of every generator of an analytic semigroup on X satisfies maximal parabolic regularity, cf. [19] or [20].
iv) If $-B$ is a generator of an analytic semigroup on a Banach space X, and S_X indicates the space of X-valued step functions on J, then we define

$$B\big(\frac{\partial}{\partial t} + B\big)^{-1} : S_X \to C(\overline{J}; X) \hookrightarrow L^s(J;X)$$

by

$$\Big(B\big(\frac{\partial}{\partial t} + B\big)^{-1} f\Big)(t) := B \int_{T_0}^{t} \mathrm{e}^{-(t-s)B} f(s) \, \mathrm{d}s,$$

compare (5.4) below. It is known that S_X is a dense subspace of $L^s(J;X)$, if $s \in [1, \infty[$, see [29, Lemma IV.1.3]. Using this, it is easy to see that B has maximal parabolic regularity on X, if and only if the operator $B\big(\frac{\partial}{\partial t} + B\big)^{-1}$ continuously extends to an operator from $L^s(J;X)$ into itself.
v) Observe that

$$W^{1,s}(J;X) \cap L^s(J;D) \hookrightarrow C(\overline{J}; (X,D)_{1-\frac{1}{s},s}). \tag{5.1}$$

The next lemma, needed below, shows that maximal parabolic regularity is maintained by interpolation:

Lemma 5.3. *Suppose that X, Y are Banach spaces, which are contained in a third Banach space Z with continuous injections. Let B be a linear operator on Z whose restrictions to each of the spaces X, Y induce closed, densely defined operators there. Assume that the induced operators fulfill maximal parabolic regularity on X and Y, respectively. Then B satisfies maximal parabolic regularity on each of the interpolation spaces $[X,Y]_\theta$ and $(X,Y)_{\theta,s}$ with $\theta \in]0,1[$ and $s \in]1,\infty[$.*

The next theorem will be the cornerstone on maximal parabolic regularity of this work and details of the proof will be pointed out below in Subsections 5.1 and 5.2.

Theorem 5.4. *Let* Ω, Γ *fulfill Assumption* 3.1 *and set* $q_{\mathrm{iso}} := \sup M_{\mathrm{iso}}$, *where*

$M_{\mathrm{iso}} := \{q \in [2, \infty[\ :\ -\nabla \cdot \mu\nabla + 1 : H^{1,q}_\Gamma \to \breve{H}^{-1,q}_\Gamma$ *is a topological isomorphism*$\}$.

Then $-\nabla \cdot \mu\nabla$ *satisfies maximal parabolic regularity on* $\breve{H}^{-1,q}_\Gamma$ *for all* $q \in [2, q^*_{\mathrm{iso}}[$, *where by* r^* *we denote the Sobolev conjugated index of* r, *i.e.,*

$$r^* = \begin{cases} \infty, & \text{if } r \geq d, \\ \left(\frac{1}{r} - \frac{1}{d}\right)^{-1}, & \text{if } r \in [1, d[\,. \end{cases}$$

Remark 5.5.

i) If Ω, Γ fulfill Assumption 3.1, then $q_{\mathrm{iso}} > 2$, see [37] and also [35].
ii) It is clear by Lax-Milgram and interpolation (see Proposition 3.4) that M_{iso} is the interval $[2, q_{\mathrm{iso}}[$ or $[2, q_{\mathrm{iso}}]$. Moreover, it can be concluded from a deep theorem of Sneiberg [54] (see also [8, Lemma 4.16]) that the second case cannot occur.

Proposition 5.6. *If* Ω *is a bounded Lipschitz domain and* Γ *is any closed subset of* $\partial\Omega$, *then* $-\nabla \cdot \mu\nabla$ *satisfies maximal parabolic regularity on* L^p *for all* $p \in]1, \infty[$. *In particular,* $\nabla \cdot \mu\nabla$ *generates an analytic semigroup on each* L^p, *cf. Remark* 5.2.

Proof. The operator $-\nabla \cdot \mu\nabla$ possesses upper Gaussian estimates by Theorem 3.7 and this implies maximal parabolic regularity on L^p, if $p \in]1, \infty[$, see [42] or [16].

Alternatively, the assertion of Proposition 5.6 may be deduced as follows: First, the induced semigroup on any L^p is contractive, see Proposition, 3.5 v). Then one applies [47, Cor. 1.1]. □

Lemma 5.7. *Let* Ω, Γ *fulfill Assumption* 4.1. *Then* $\nabla \cdot \mu\nabla$ *generates an analytic semigroup on* $\breve{H}^{-1,q}_\Gamma$ *for all* $q \in [2, \infty[$.

Proof. One has the operator identity

$$\left(-\nabla\cdot\mu\nabla+\lambda\right)^{-1} = \left(-\nabla\cdot\mu\nabla+1\right)^{1/2}\left(-\nabla\cdot\mu\nabla+\lambda\right)^{-1}\left(-\nabla\cdot\mu\nabla+1\right)^{-1/2}, \quad \operatorname{Re}\lambda \geq 0, \tag{5.2}$$

on L^q. Under Assumption 4.1 $(-\nabla\cdot\mu\nabla+1)^{1/2}$ is a topological isomorphism between L^q and $\breve{H}^{-1,q}_\Gamma$ for every $q \in [2, \infty[$, thanks to Theorem 4.4. Thus, via (5.2), the corresponding resolvent estimate carries over from L^q to $\breve{H}^{-1,q}_\Gamma$ by the density of L^q in $\breve{H}^{-1,q}_\Gamma$. □

In the next step we show

Theorem 5.8. *Let* Ω, Γ *fulfill Assumption* 4.1. *Then* $-\nabla \cdot \mu\nabla$ *satisfies maximal parabolic regularity on* $\breve{H}^{-1,q}_\Gamma$ *for all* $q \in [2, \infty[$.

This will be a consequence of Theorem 4.4 and the following two lemmata.

Lemma 5.9. *Assume that the operator B fulfills maximal parabolic regularity on a Banach space X and has no spectrum in $]-\infty, 0]$. If S_X again denotes the space of X-valued step functions on J, then one has, for every $\alpha \in]0,1[$,*

$$B\Big(\frac{\partial}{\partial t} + B\Big)^{-1} \psi = (B+1)^{\alpha} B\Big(\frac{\partial}{\partial t} + B\Big)^{-1} (B+1)^{-\alpha}\psi \quad \textit{for all} \quad \psi \in S_X. \tag{5.3}$$

Proof. First, B satisfies a resolvent estimate $\|(B+\lambda)^{-1}\|_{\mathcal{L}(X)} \le \frac{c}{|\lambda|}$ for all λ from a suitable right half-space. Since, additionally, B has no spectrum in $]-\infty, 0]$, the operators $(B+1)^{-\alpha}$ and $(B+1)^{\alpha}$ are well defined on X.

If $x \in X$ and χ_I denotes the indicator function of an interval $I =]a, b[\subseteq J$, then one calculates, by the definition of $B(\frac{\partial}{\partial t} + B)^{-1}$,

$$\begin{aligned}\Big[B\big(\frac{\partial}{\partial t} + B\big)^{-1}\chi_I x\Big](t) &= B\int_{T_0}^{t} e^{-(t-s)B} x\chi_I(s)\, ds \\ &= \begin{cases} 0, & \text{if } t < a, \\ \big(1 - e^{(a-t)B}\big)x, & \text{if } t \in [a,b], \\ \big(e^{(b-t)B} - e^{(a-t)B}\big)x, & \text{if } t > b, \end{cases}\end{aligned} \tag{5.4}$$

compare [50, Ch. 1.2]. This gives, for every step function $\sum_{l=1}^{N} \chi_{I_l} x_l \in S_X$,

$$\begin{aligned} B\Big(\frac{\partial}{\partial t} + B\Big)^{-1} (B+1)^{-\alpha} \sum_{l=1}^{N} \chi_{I_l} x_l &= \sum_{l=1}^{N} B\Big(\frac{\partial}{\partial t} + B\Big)^{-1} \chi_{I_l} (B+1)^{-\alpha} x_l \\ &= (B+1)^{-\alpha} B\Big(\frac{\partial}{\partial t} + B\Big)^{-1} \sum_{l=1}^{N} \chi_{I_l} x_l, \end{aligned}$$

since $(B+1)^{-\alpha}$ commutes with the semigroup operators e^{-tB}. □

Remark 5.10. By the density of S_X in $L^s(J;X)$ for $s \in [1,\infty[$, equation (5.3) extends to the whole of $L^s(J;X)$, since the left-hand side is a continuous operator on $L^s(J;X)$ by maximal regularity of B.

Lemma 5.11. *Assume that X, Y are Banach spaces, where X continuously and densely injects into Y. Suppose B to be an operator on Y, whose maximal restriction $B|_X$ to X satisfies maximal parabolic regularity there. If $B|_X$ has no spectrum in $]-\infty, 0]$ and $(B+1)^{\alpha}$ provides a topological isomorphism from X onto Y, then B also satisfies maximal parabolic regularity on Y.*

Proof. Let S_X be the set of step functions on J, taking their values in X. By the density of X in Y, S_X is also dense in $L^s(J;Y)$. Due to Lemma 5.9, we may estimate for any $\psi \in S_X$:

$$\begin{aligned}
\left\| B\Big(\frac{\partial}{\partial t}+B\Big)^{-1}\psi\right\|_{\mathcal{L}(L^s(J;Y))} &= \left\|(B+1)^{\alpha}B\Big(\frac{\partial}{\partial t}+B\Big)^{-1}(B+1)^{-\alpha}\psi\right\|_{\mathcal{L}(L^s(J;Y))}\\
\le c\|(B+1)^{\alpha}\|_{\mathcal{L}(L^s(J;X);L^s(J;Y))}&\left\|B\Big(\frac{\partial}{\partial t}+B\Big)^{-1}\right\|_{\mathcal{L}(L^s(J;X))}\|(B+1)^{-\alpha}\psi\|_{L^s(J;X)}\\
\le c\|(B+1)^{\alpha}\|_{\mathcal{L}(X;Y)}\left\|B\Big(\frac{\partial}{\partial t}+B\Big)^{-1}\right\|_{\mathcal{L}(L^s(J;X))}&\|(B+1)^{-\alpha}\|_{\mathcal{L}(Y;X)}\|\psi\|_{L^s(J;Y)}.
\end{aligned}$$

By density, $B\big(\frac{\partial}{\partial t}+B\big)^{-1}$ extends to a continuous operator on the whole of $L^s(J;Y)$ and the assertion follows by Remark 5.2 iv). □

The proof of Theorem 5.8 is now obtained by the isomorphism property $\big(-\nabla\cdot\mu\nabla+1\big)^{1/2} : L^q \to \breve{H}_\Gamma^{-1,q}$, assured by Theorem 4.4, and afterwards applying Proposition 5.6 and Lemma 5.11, putting there $X := L^q$, $Y := \breve{H}_\Gamma^{-1,q}$, $B := -\nabla\cdot\mu\nabla$ and $\alpha := 1/2$.

Now we intend to 'globalize' Theorem 5.8, in other words: We prove that $-\nabla\cdot\mu\nabla$ satisfies maximal parabolic regularity on $\breve{H}_\Gamma^{-1,q}$ for suitable q if Ω, Γ satisfy only Assumption 3.1, i.e., if K_-, $K_-\cup\Sigma$ and $K_-\cup\Sigma_0$ need only to be model sets for the constellation around boundary points. Obviously, then the variety of admissible Ω's and Γ's increases considerably; in particular, Γ may have more than one connected component.

5.1. Auxiliaries

We continue with a result that allows us to 'localize' the elliptic operator.

Lemma 5.12. *Let Ω, Γ satisfy Assumption 3.1 and let $\Upsilon \subseteq \mathbb{R}^d$ be open, such that $\Omega_\bullet := \Omega\cap\Upsilon$ is also a Lipschitz domain. Furthermore, we put $\Gamma_\bullet := \Gamma\cap\Upsilon$ and fix an arbitrary, real-valued function $\eta \in C_0^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\eta)\subseteq\Upsilon$. Denote by $\mu_\bullet$ the restriction of the coefficient function μ to $\Omega_\bullet$ and assume $v \in H_\Gamma^{1,2}(\Omega)$ to be the solution of*

$$-\nabla\cdot\mu\nabla v = f \in \breve{H}_\Gamma^{-1,2}(\Omega).$$

Then the following holds true:

i) *For all $q \in \,]1,\infty[$ the anti-linear form*

$$f_\bullet : w \mapsto \langle f, \widetilde{\eta w}\rangle_{\breve{H}_\Gamma^{-1,2}}$$

(where $\widetilde{\eta w}$ again means the extension of ηw by zero to the whole Ω) is well defined and continuous on $H_{\Gamma_\bullet}^{1,q'}(\Omega_\bullet)$, whenever f is an anti-linear form from $\breve{H}_\Gamma^{-1,q}(\Omega)$. The mapping $\breve{H}_\Gamma^{-1,q}(\Omega) \ni f \mapsto f_\bullet \in \breve{H}_{\Gamma_\bullet}^{-1,q}(\Omega_\bullet)$ is continuous.

ii) *If we denote the anti-linear form*

$$H^{1,2}_{\Gamma_\bullet}(\Omega_\bullet) \ni w \mapsto \int_{\Omega_\bullet} v\mu_\bullet \nabla\eta \cdot \nabla\overline{w}\,\mathrm{dx}$$

by I_v*, then* $u := \eta v|_{\Omega_\bullet}$ *satisfies*

$$-\nabla\cdot\mu_\bullet\nabla u = -\mu_\bullet\nabla v|_{\Omega_\bullet}\cdot\nabla\eta|_{\Omega_\bullet} + I_v + f_\bullet.$$

iii) *For every* $q \geq 2$ *and all* $r \in [2, q^*[$ *(*q^* *denoting again the Sobolev conjugated index of* q*) the mapping*

$$H^{1,q}_\Gamma(\Omega) \ni v \mapsto -\mu_\bullet\nabla v|_{\Omega_\bullet}\cdot\nabla\eta|_{\Omega_\bullet} + I_v \in \breve{H}^{-1,r}_{\Gamma_\bullet}(\Omega_\bullet)$$

is well defined and continuous.

Remark 5.13. It is the lack of integrability for the gradient of v (see the counterexample in [26, Ch. 4]) together with the quality of the needed Sobolev embeddings which limits the quality of the correction terms. In the end it is this effect which prevents the applicability of the localization procedure in Subsection 5.2 in higher dimensions – at least when one aims at a $q > d$.

Remark 5.14. If $v \in L^2(\Omega)$ is a regular distribution, then $v_\bullet$ is the regular distribution $(\eta v)|_{\Omega_\bullet}$.

5.2. Core of the proof of Theorem 5.4

We are now in a position to start the proof of Theorem 5.4. We first note that in any case the operator $-\nabla\cdot\mu\nabla$ admits maximal parabolic regularity on the Hilbert space $\breve{H}^{-1,2}_\Gamma$, since its negative generates an analytic semigroup on this space by Proposition 3.5, cf. Remark 5.2 iii). Thus, defining

$$M_{\mathrm{MR}} := \{q \geq 2 \,:\, -\nabla\cdot\mu\nabla \text{ admits maximal regularity on } \breve{H}^{-1,q}_\Gamma\}$$

and exploiting (3.1) and Lemma 5.3 we see by interpolation that M_{MR} is $\{2\}$ or an interval with left endpoint 2.

The main step of the proof for Theorem 5.4 is contained in the following lemma.

Lemma 5.15. *Let* Ω*,* Γ*,* Υ*,* η*,* $\Omega_\bullet$*,* $\Gamma_\bullet$*,* $\mu_\bullet$ *be as before. Assume that* $-\nabla\cdot\mu_\bullet\nabla$ *satisfies maximal parabolic regularity on* $\breve{H}^{-1,q}_{\Gamma_\bullet}(\Omega_\bullet)$ *for all* $q \in [2,\infty[$ *and that* $-\nabla\cdot\mu\nabla$ *satisfies maximal parabolic regularity on* $\breve{H}^{-1,q_0}_\Gamma(\Omega)$ *for some* $q_0 \in [2, q_{\mathrm{iso}}[$*. If* $r \in [q_0, q_0^*[$ *and* $G \in L^s(J;\breve{H}^{-1,r}_\Gamma(\Omega)) \hookrightarrow L^s(J;\breve{H}^{-1,q_0}_\Gamma(\Omega))$*, then the unique solution* $V \in W^{1,s}(J;\breve{H}^{-1,q_0}_\Gamma(\Omega)) \cap L^s(J;\mathrm{dom}_{\breve{H}^{-1,q_0}_\Gamma(\Omega)}(-\nabla\cdot\mu\nabla))$ *of*

$$V' - \nabla\cdot\mu\nabla V = G, \qquad V(T_0) = 0,$$

even satisfies

$$\eta V \in W^{1,s}(J;\breve{H}^{-1,r}_\Gamma(\Omega)) \cap L^s(J;\mathrm{dom}_{\breve{H}^{-1,r}_\Gamma(\Omega)}(-\nabla\cdot\mu\nabla)).$$

Proof of Theorem 5.4. For every $\mathrm{x} \in \Omega$ let $\Xi_\mathrm{x} \subseteq \Omega$ be an open cube, containing x. Furthermore, let for any point $\mathrm{x} \in \partial\Omega$ an open neighborhood be given according to the supposition of the theorem (see Assumption 3.1). Possibly shrinking this neighborhood to a smaller one, one obtains a new neighborhood Υ_x, and a bi-Lipschitz mapping ϕ_x from a neighborhood of $\overline{\Upsilon_\mathrm{x}}$ into $\mathbb{R}^d$ such that $\phi_\mathrm{x}(\Upsilon_\mathrm{x} \cap (\Omega \cup \Gamma)) = K_-$, $(K_- \cup \Sigma)$ or $(K_- \cup \Sigma_0)$.

Obviously, the Ξ_x and Υ_x together form an open covering of $\overline{\Omega}$. Let the sets $\Xi_{\mathrm{x}_1}, \dots, \Xi_{\mathrm{x}_k}$, $\Upsilon_{\mathrm{x}_{k+1}}, \dots, \Upsilon_{\mathrm{x}_l}$ be a finite subcovering and $\eta_1, \dots, \eta_l$ a C^∞ partition of unity, subordinate to this subcovering. Set $\Omega_j := \Xi_{\mathrm{x}_j} = \Xi_{\mathrm{x}_j} \cap \Omega$ for $j \in \{1, \dots, k\}$ and $\Omega_j := \Upsilon_{\mathrm{x}_j} \cap \Omega$ for $j \in \{k+1, \dots, l\}$. Moreover, set $\Gamma_j := \emptyset$ for $j \in \{1, \dots, k\}$ and $\Gamma_j := \Upsilon_{\mathrm{x}_j} \cap \Gamma$ for $j \in \{k+1, \dots, l\}$.

Denoting the restriction of μ to Ω_j by μ_j, each operator $-\nabla \cdot \mu_j \nabla$ satisfies maximal parabolic regularity in $\breve{H}^{-1,q}_{\Gamma_j}(\Omega_j)$ for all $q \in [2, \infty[$ and all j, according to Theorem 5.8. Thus, we may apply Lemma 5.15, the first time taking $q_0 = 2$. Then $-\nabla \cdot \mu \nabla$ satisfies maximal parabolic regularity on $\breve{H}^{-1,r}_\Gamma(\Omega)$ for all $r \in [2, 2^*[$. Next taking q_0 as any number from the interval $[2, \min(2^*, q_{\mathrm{iso}})[$ and continuing this way, one improves the information on r step by step. Since the augmentation in r increases in every step, any number below q^*_{iso} is indeed achieved. □

Remark 5.16. Note that Theorem 5.4 already yields maximal regularity of $-\nabla \cdot \mu \nabla$ on $\breve{H}^{-1,q}_\Gamma$ for all $q \in [2, 2^*[$ without any additional information on $\mathrm{dom}_{\breve{H}^{-1,q}_\Gamma}(-\nabla \cdot \mu \nabla)$ nor on $\mathrm{dom}_{\breve{H}^{-1,q}_{\Gamma_j}(\Omega_j)}(-\nabla \cdot \mu_j \nabla)$.

In the 2-d case this already implies maximal regularity for every $q \in [2, \infty[$. Taking into account Remark 5.5 i), without further knowledge on the domains we get in the 3-d case every $q \in [2, 6 + \varepsilon[$ and in the 4-d case every $q \in [2, 4 + \varepsilon[$, where ε depends on Ω, Γ, μ.

5.3. The operator A

Next we carry over the maximal parabolic regularity result, up to now proved for $-\nabla \cdot \mu \nabla$ on the spaces $\breve{H}^{-1,q}_\Gamma$, to the operator A and to a much broader class of distribution spaces. For this we first need the following perturbation result on relative boundedness of the boundary part of the operator A.

Lemma 5.17. *Suppose $q \geq 2$, $\varsigma \in]1 - \frac{1}{q}, 1]$ and $\varkappa \in L^\infty(\Gamma, \mathrm{d}\sigma)$ and let Ω, Γ satisfy Assumption* 3.1. *If we define the mapping $Q : \mathrm{dom}_{\breve{H}^{-\varsigma,q}_\Gamma}(-\nabla \cdot \mu \nabla) \to \breve{H}^{-\varsigma,q}_\Gamma$ by*

$$\langle Q\psi, \varphi \rangle_{H^{-\varsigma,q}_\Gamma} := \int_\Gamma \varkappa \, \psi \, \overline{\varphi} \, \mathrm{d}\sigma, \quad \varphi \in H^{\varsigma,q'}_\Gamma,$$

then Q is well defined and continuous. Moreover, it is relatively bounded with respect to $-\nabla \cdot \mu \nabla$, when considered on the space $\breve{H}^{-\varsigma,q}_\Gamma$, and the relative bound may be taken arbitrarily small.

Referring to Lemma 5.3, we can now carry over maximal parabolic regularity from L^q and $\breve{H}^{-1,q}_\Gamma$ to various distribution spaces and thus prove our main result for the operator A.

Theorem 5.18. *Suppose* $q \geq 2$, $\varkappa \in L^\infty(\Gamma, \mathrm{d}\sigma)$ *and let* Ω, Γ *satisfy Assumption* 3.1.

i) *If* $\varsigma \in \,]1 - \frac{1}{q}, 1]$, *then* $\mathrm{dom}_{\breve{H}_\Gamma^{-\varsigma,q}}(-\nabla \cdot \mu \nabla) = \mathrm{dom}_{\breve{H}_\Gamma^{-\varsigma,q}}(A)$.

ii) *If* $\varsigma \in \,]1 - \frac{1}{q}, 1]$ *and* $-\nabla \cdot \mu \nabla$ *satisfies maximal parabolic regularity on* $\breve{H}_\Gamma^{-\varsigma,q}$, *then* A *also does.*

iii) *The operator* A *satisfies maximal parabolic regularity on* L^2. *If* $\varkappa \geq 0$, *then* A *satisfies maximal parabolic regularity on* L^p *for all* $p \in \,]1, \infty[$.

iv) *Suppose that* $-\nabla \cdot \mu \nabla$ *satisfies maximal parabolic regularity on* $\breve{H}_\Gamma^{-1,q}$. *Then* A *satisfies maximal parabolic regularity on any of the interpolation spaces*

$$[L^2, \breve{H}_\Gamma^{-1,q}]_\theta, \quad \theta \in [0,1],$$

and

$$(L^2, \breve{H}_\Gamma^{-1,q})_{\theta,s}, \quad \theta \in [0,1],\ s \in \,]1, \infty[\,.$$

Let $\varkappa \geq 0$ *and* $p \in \,]1, \infty[$ *in case of* $d = 2$ *or* $p \in [\left(\frac{1}{2} + \frac{1}{d}\right)^{-1}, \infty[$ *if* $d \geq 3$. *Then* A *also satisfies maximal parabolic regularity on any of the interpolation spaces*

$$[L^p, \breve{H}_\Gamma^{-1,q}]_\theta, \quad \theta \in [0,1], \tag{5.5}$$

and

$$(L^p, \breve{H}_\Gamma^{-1,q})_{\theta,s}, \quad \theta \in [0,1],\ s \in \,]1, \infty[\,. \tag{5.6}$$

Proof. i) follows from Lemma 5.17 and classical perturbation theory.

ii) The assertion is implied by Lemma 5.17 and a perturbation theorem for maximal parabolic regularity, see [6, Prop. 1.3].

iii) The first assertion follows from Proposition 3.5 ii) and Remark 5.2 iii). The second is shown in [33, Thm. 7.4].

iv) Under the given conditions on p, we have the embedding $L^p \hookrightarrow \breve{H}_\Gamma^{-1,2}$. Thus, the assertion follows from the preceding points and Lemma 5.3. □

Remark 5.19. The interpolation spaces $[L^p, H_\Gamma^{-1,q}]_\theta$, $\theta \in [0,1]$, and $(L^p, H_\Gamma^{-1,q})_{\theta,s}$, $\theta \in [0,1], s \in \,]1,\infty[$, are characterized in [32], see in particular Remark 3.6 therein. Identifying each $f \in L^q$ with the anti-linear form $L^{q'} \ni \psi \to \int_\Omega f \overline{\psi}\,\mathrm{d}x$ and using the retraction/coretraction theorem with the coretraction which assigns to $f \in \breve{H}_\Gamma^{-1,r}$ the linear form $H_\Gamma^{1,r'} \ni \psi \to \langle f, \overline{\psi} \rangle_{\breve{H}_\Gamma^{-1,r}}$, one easily identifies the interpolation spaces in (5.5) and (5.6). In particular, this yields $\left[L^{q_0}, \breve{H}_\Gamma^{-1,q_1}\right]_\theta = \breve{H}_\Gamma^{-\theta,q}$ if $\theta \neq 1 - \frac{1}{q}$.

Corollary 5.20. *Let* Ω *and* Γ *satisfy Assumption* 3.1. *The operator* $-A$ *generates analytic semigroups on all spaces* $\breve{H}_\Gamma^{-1,q}$ *if* $q \in [2, q^*_{\mathrm{iso}}[$ *and on all the interpolation spaces occurring in Theorem* 5.18, *there* q *also taken from* $[2, q^*_{\mathrm{iso}}[$. *Moreover, if* $\varkappa \geq 0$, *the following resolvent estimates are valid:*

$$\|(A + 1 + \lambda)^{-1}\|_{\mathcal{L}(\breve{H}_\Gamma^{-1,q})} \leq \frac{c_q}{1 + |\lambda|}, \quad \operatorname{Re}\lambda \geq 0.$$

6. Nonlinear parabolic equations

In this section we will apply maximal parabolic regularity for the treatment of quasilinear parabolic equations which are of the (formal) type (1.1). Concerning all the occurring operators we will formulate precise requirements in Assumption 6.12 below. In contrast to the previous chapters we now need the (possibly) stronger assumption on the geometry of $\Omega \cup \Gamma$ that the local bi-Lipschitz charts in Assumption 3.1 can be chosen to be volume-preserving. This comes to bear in the proof of Lemma 6.7 (see [40]), what is crucial for the treatment of the nonlinear equations.

Assumption 6.1. Let $\Omega \cup \Gamma$ satisfy Assumption 3.1. Using the notation of this assumption, we assume that for every $\mathrm{x} \in \partial\Omega$ there exists a constant $\alpha > 0$, such that $\alpha\phi_{\mathrm{x}}$ is a volume-preserving map.

The outline of the section is as follows: First we give a motivation for the choice of the Banach space we will regard (1.1)/(1.2) in. Afterwards we show that maximal parabolic regularity, combined with regularity results for the elliptic operator, allows us to solve this problem. Below we will consider (1.1)/(1.2) as a quasilinear problem

$$\begin{cases} u'(t) + \mathcal{B}\big(u(t)\big)u(t) = \mathcal{S}(t, u(t)), \quad t \in J, \\ \qquad\qquad u(T_0) = u_0. \end{cases} \tag{6.1}$$

To give the reader already here an idea what properties of the operators $-\nabla \cdot \mathcal{G}(u)\mu\nabla$ and of the corresponding Banach space are required, we first quote the result on existence and uniqueness for abstract quasilinear parabolic equations (due to Clément/Li [15] and Prüss [51]) on which our subsequent considerations will be based.

Proposition 6.2. *Suppose that B is a closed operator on some Banach space X with dense domain D, which satisfies maximal parabolic regularity on X. For some $s > 1$ suppose further $u_0 \in (X, D)_{1-\frac{1}{s},s}$ and $\mathcal{B} : J \times (X, D)_{1-\frac{1}{s},s} \to \mathcal{L}(D, X)$ to be continuous with $B = \mathcal{B}(T_0, u_0)$. Let, in addition, $\mathcal{S} : J \times (X, D)_{1-\frac{1}{s},s} \to X$ be a Carathéodory map and assume the following Lipschitz conditions on $\mathcal{B}$ and $\mathcal{S}$:*

(**B**) *For every $M > 0$ there exists a constant $C_M > 0$, such that for all $t \in J$*

$$\|\mathcal{B}(t, u) - \mathcal{B}(t, \tilde{u})\|_{\mathcal{L}(D,X)} \le C_M \, \|u - \tilde{u}\|_{(X,D)_{1-\frac{1}{s},s}},$$

if $\|u\|_{(X,D)_{1-\frac{1}{s},s}}, \ \|\tilde{u}\|_{(X,D)_{1-\frac{1}{s},s}} \le M$.

(**S**) *$\mathcal{S}(\cdot, 0) \in L^s(J; X)$ and for each $M > 0$ there is a function $h_M \in L^s(J)$, such that*

$$\|\mathcal{S}(t, u) - \mathcal{S}(t, \tilde{u})\|_X \le h_M(t) \, \|u - \tilde{u}\|_{(X,D)_{1-\frac{1}{s},s}}$$

holds for a.a. $t \in J$, if $\|u\|_{(X,D)_{1-\frac{1}{s},s}}, \|\tilde{u}\|_{(X,D)_{1-\frac{1}{s},s}} \le M$.

Then there exists $T^* \in J$*, such that* (6.1) *admits a unique solution* u *on* $]T_0, T^*[$ *satisfying*

$$u \in W^{1,s}(]T_0, T^*[; X) \cap L^s(]T_0, T^*[; D).$$

Remark 6.3. Up to now we were free to consider complex Banach spaces. But the context of equations like (1.1) requires real spaces, in particular in view of the quality of the operator $\mathcal{G}$ which often is a superposition operator. Therefore, from this moment on we use the real versions of the spaces. In particular, $H_\Gamma^{-\varsigma,q}$ is now understood as the dual of the real space $H_\Gamma^{\varsigma,q'}$ and clearly can be identified with the set of anti-linear forms on the complex space $H_\Gamma^{\varsigma,q'}$ that take real values when applied to real functions.

Fortunately, the property of maximal parabolic regularity is maintained for the restriction of the operator A to the real spaces in case of a real function $\varkappa$, as A then commutes with complex conjugation.

We will now give a motivation for the choice of the Banach space X we will use later. In view of the applicability of Proposition 6.2 and the non-smooth characteristic of (1.1)/(1.2) it is natural to require the following properties.

a) The operators A, or at least the operators $-\nabla \cdot \mu\nabla$, defined in (3.2), must satisfy maximal parabolic regularity on X.
b) As in the classical theory (see [46], [30], [55] and references therein) quadratic gradient terms of the solution should be admissible for the right-hand side.
c) The operators $-\nabla \cdot \mathcal{G}(u)\mu\nabla$ should behave well concerning their dependence on u, see condition (**B**) above.
d) X has to contain certain measures, supported on Lipschitz hypersurfaces in Ω or on $\partial\Omega$ in order to allow for surface densities on the right-hand side or/and for inhomogeneous Neumann conditions.

The condition in a) is assured by Theorems 5.4 and 5.18 for a great variety of Banach spaces, among them candidates for X. Requirement b) suggests that one should have $\mathrm{dom}_X(-\nabla \cdot \mu\nabla) \hookrightarrow H_\Gamma^{1,q}$ and $L^{\frac{q}{2}} \hookrightarrow X$. Since $-\nabla \cdot \mu\nabla$ maps $H_\Gamma^{1,q}$ into $H_\Gamma^{-1,q}$, this altogether leads to the necessary condition

$$L^{\frac{q}{2}} \hookrightarrow X \hookrightarrow H_\Gamma^{-1,q}. \tag{6.2}$$

The Sobolev embedding shows that q cannot be smaller than the space dimension d. Taking into account d), it is clear that X must be a space of distributions which (at least) contains surface densities. In order to recover the desired property $\mathrm{dom}_X(-\nabla \cdot \mu\nabla) \hookrightarrow H_\Gamma^{1,q}$ from the necessary condition in (6.2), we make for all that follows this general

Assumption 6.4. There is a $q > d$, such that $-\nabla \cdot \mu\nabla + 1 : H_\Gamma^{1,q} \to H_\Gamma^{-1,q}$ is a topological isomorphism.

Remark 6.5. By Remark 5.5 i) Assumption 6.4 is always fulfilled for $d = 2$. On the other hand for $d \geq 4$ it is generically false in case of mixed boundary conditions, see [53] for the famous counterexample. Moreover, even in the Dirichlet case, when the domain Ω has only a Lipschitz boundary or the coefficient function μ is constant within layers, one cannot expect $q \geq 4$, see [44] and [25].

In Section 7 we will present examples for domains Ω, coefficient functions μ and Dirichlet boundary parts $\Omega \setminus \Gamma$, for which Assumption 6.4 is fulfilled.

From now on we fix some $q > d$, for which Assumption 6.4 holds.

As a first step, one shows that Assumption 6.4 carries over to a broad class of modified operators:

Lemma 6.6. *Assume that ξ is a real-valued, uniformly continuous function on Ω that admits a lower bound $\underline{\xi} > 0$. Then the operator $-\nabla \cdot \xi\mu\nabla + 1$ also is a topological isomorphism between $H^{1,q}_\Gamma$ and $H^{-1,q}_\Gamma$.*

In this spirit, one could now suggest $X := H^{-1,q}_\Gamma$ to be a good choice for the Banach space, but in view of condition (**S**) the right-hand side of (6.1) has to be a continuous mapping from an interpolation space $(\mathrm{dom}_X(A), X)_{1-\frac{1}{s},s}$ into X. Chosen $X := H^{-1,q}_\Gamma$, for elements $\psi \in (\mathrm{dom}_X(A), X)_{1-\frac{1}{s},s} = (H^{1,q}_\Gamma, H^{-1,q}_\Gamma)_{1-\frac{1}{s},s}$ the expression $|\nabla\psi|^2$ cannot be properly defined and, if so, will not lie in $H^{-1,q}_\Gamma$ in general. This shows that $X := H^{-1,q}_\Gamma$ is not an appropriate choice, but we will see that $X := H^{-\varsigma,q}_\Gamma$, with ς properly chosen, is.

Lemma 6.7. *Put $X := H^{-\varsigma,q}_\Gamma$ with $\varsigma \in [0,1[\setminus \{\frac{1}{q}, 1-\frac{1}{q}\}$. Then*

i) *For every $\tau \in]\frac{1+\varsigma}{2}, 1[$ there is a continuous embedding*
$$(X, \mathrm{dom}_X(-\nabla \cdot \mu\nabla))_{\tau,1} \hookrightarrow H^{1,q}_\Gamma.$$

ii) *If $\varsigma \in [\frac{d}{q}, 1]$, then X has a predual $X_* = H^{\varsigma,q'}_\Gamma$ which admits the continuous, dense injections $H^{1,q'}_\Gamma \hookrightarrow X_* \hookrightarrow L^{(\frac{q}{2})'}$ that by duality clearly imply* (6.2). *Furthermore, $H^{1,q}_\Gamma$ is a multiplier space for X_*.*

Next we will consider requirement c), see condition (**B**) in Proposition 6.2.

Lemma 6.8. *Let q be a number from Assumption* 6.4 *and let X be a Banach space with predual X_* that admits the continuous and dense injections*
$$H^{1,q'}_\Gamma \hookrightarrow X_* \hookrightarrow L^{(\frac{q}{2})'}.$$

i) *If $\xi \in H^{1,q}$ is a multiplier on X_*, then* $\mathrm{dom}_X(-\nabla \cdot \mu\nabla) \hookrightarrow \mathrm{dom}_X(-\nabla \cdot \xi\mu\nabla)$.

ii) *If $H^{1,q}$ is a multiplier space for X_*, then the (linear) mapping $H^{1,q} \ni \xi \mapsto -\nabla \cdot \xi\mu\nabla \in \mathcal{L}(\mathrm{dom}_X(-\nabla \cdot \mu\nabla), X)$ is well defined and continuous.*

Corollary 6.9. *If ξ additionally to the hypotheses of Lemma* 6.8 i) *has a positive lower bound, then*
$$\mathrm{dom}_X(-\nabla \cdot \xi\mu\nabla) = \mathrm{dom}_X(-\nabla \cdot \mu\nabla).$$

Next we will show that functions on $\partial\Omega$ or on a Lipschitz hypersurface, which belong to a suitable summability class, can be understood as elements of the distribution space $H^{-\varsigma,q}_\Gamma$.

Theorem 6.10. *Assume* $q \in]1,\infty[$, $\varsigma \in]1-\frac{1}{q},1[\setminus \{\frac{1}{q}\}$ *and let* Π, ϖ *be as in Proposition* 3.3. *Then the adjoint trace operator* $(\mathrm{Tr})^*$ *maps* $L^q(\Pi)$ *continuously into* $\big(H^{\varsigma,q'}(\Omega)\big)' \hookrightarrow H^{-\varsigma,q}_\Gamma$.

Proof. The result is obtained from Proposition 3.3 by duality. □

Remark 6.11. Here we restricted the considerations to the case of Lipschitz hypersurfaces, since this is the most essential insofar as it gives the possibility of prescribing jumps in the normal component of the current $j := \mathcal{G}(u)\mu\nabla u$ along hypersurfaces where the coefficient function jumps. This case is of high relevance in view of applied problems and has attracted much attention also from the numerical point of view, see, e.g., [1], [11] and references therein.

From now on we fix once and for all a number $\varsigma \in]\max\{1-\frac{1}{q},\frac{d}{q}\},1[$ and set for all that follows $X := H^{-\varsigma,q}_\Gamma$.

Next we introduce the requirements on the data of problem (1.1)/(1.2).

Assumption 6.12. Op) For all that follows we fix a number $s > \frac{2}{1-\varsigma}$.

Ga) The mapping $\mathcal{G} : H^{1,q} \to H^{1,q}$ is locally Lipschitz continuous.

Gb) For any ball in $H^{1,q}$ there exists $\delta > 0$, such that $\mathcal{G}(u) \geq \delta$ for all u from this ball.

Ra) The function $\mathcal{R} : J \times H^{1,q} \to X$ is of Carathéodory type, i.e., $\mathcal{R}(\cdot,u)$ is measurable for all $u \in H^{1,q}$ and $\mathcal{R}(t,\cdot)$ is continuous for a.a. $t \in J$.

Rb) $\mathcal{R}(\cdot,0) \in L^s(J;X)$ and for $M > 0$ there exists $h_M \in L^s(J)$, such that

$$\|\mathcal{R}(t,u) - \mathcal{R}(t,\tilde{u})\|_X \leq h_M(t)\|u-\tilde{u}\|_{H^{1,q}}, \quad t \in J,$$

provided $\max(\|u\|_{H^{1,q}}, \|\tilde{u}\|_{H^{1,q}}) \leq M$.

BC) b is an operator of the form $b(u) = Q(b_\circ(u))$, where $b_\circ$ is a (possibly nonlinear), locally Lipschitzian operator from $C(\overline{\Omega})$ into itself (see Lemma 5.17).

Gg) $g \in L^q(\Gamma)$.

IC) $u_0 \in (X, \mathrm{dom}_X(-\nabla\cdot\mu\nabla))_{1-\frac{1}{s},s}$.

Remark 6.13. i) At first glance the choice of s seems indiscriminate. The point is, however, that generically in applications the explicit time dependence of the reaction term $\mathcal{R}$ is essentially bounded. Thus, in view of condition **Rb)** it is justified to take s as any arbitrarily large number, whose magnitude need not be controlled explicitly.

ii) Note that the requirement on $\mathcal{G}$ allows for nonlocal operators. This is essential if the current depends on an additional potential governed by an auxiliary equation, which is usually the case in drift-diffusion models, see [3], [28] or [52].

iii) The conditions **Ra)** and **Rb)** are always satisfied if $\mathcal{R}$ is a mapping into $L^{q/2}$ with the analog boundedness and continuity properties, see Lemma 6.7 ii).

iv) It is not hard to see that Q in fact is well defined on $C(\overline{\Omega})$, therefore condition **BC)** makes sense. In particular, $b_\circ$ may be a superposition operator, induced by a $C^1(\mathbb{R})$ function. Let us emphasize that in this case the inducing function need not be positive. Thus, non-dissipative boundary conditions are included.

v) Finally, the condition **IC)** is an 'abstract' one and hardly to verify, because one has no explicit characterization of $(X, \mathrm{dom}_X(-\nabla \cdot \mu\nabla))_{1-\frac{1}{s},s}$ at hand. Nevertheless, the condition is reproduced along the trajectory of the solution by means of the embedding (5.1).

In order to solve (1.1)/(1.2), we will consider (6.1) with

$$\mathcal{B}(u) := -\nabla \cdot \mathcal{G}(u)\mu\nabla \tag{6.3}$$

and the right-hand side $\mathcal{S}$

$$\mathcal{S}(t,u) := \mathcal{R}(t,u) - Q(b_\circ(u)) + (\mathrm{Tr})^* g, \tag{6.4}$$

seeking the solution in the space $W^{1,s}(J;X) \cap L^s(J; \mathrm{dom}_X(-\nabla \cdot \mu\nabla))$.

Remark 6.14. Let us explain this reformulation: as it is well known in the theory of boundary value problems, the boundary condition (1.2) is incorporated by introducing the boundary terms $-\varkappa b_\circ(u)$ and g on the right-hand side. In order to understand both as elements from X, we write $Q(b_\circ(u))$ and $(\mathrm{Tr})^* g$, see Lemma 5.17 and Theorem 6.10.

Theorem 6.15. *Let Assumption* 6.4 *be satisfied and assume that the data of the problem satisfy Assumption* 6.12. *Then* (6.1) *has a local-in-time, unique solution in* $W^{1,s}(J;X) \cap L^s(J; \mathrm{dom}_X(-\nabla \cdot \mu\nabla))$*, provided that* $\mathcal{B}$ *and* $\mathcal{S}$ *are given by* (6.3) *and* (6.4)*, respectively.*

Proof. First of all we note that, due to **Op)**, $1 - \frac{1}{s} > \frac{1+\varsigma}{2}$. Thus, if $\tau \in \left]\frac{1+\varsigma}{2}, 1 - \frac{1}{s}\right[$ by a well-known interpolation result (see [56, Ch. 1.3.3]) and Lemma 6.7 i) we have

$$(X, \mathrm{dom}_X(-\nabla \cdot \mu\nabla))_{1-\frac{1}{s},s} \hookrightarrow (X, \mathrm{dom}_X(-\nabla \cdot \mu\nabla))_{\tau,1} \hookrightarrow H^{1,q}. \tag{6.5}$$

Hence, by **IC)**, $u_0 \in H^{1,q}$. Consequently, due to the suppositions on $\mathcal{G}$, both the functions $\mathcal{G}(u_0)$ and $\frac{1}{\mathcal{G}(u_0)}$ belong to $H^{1,q}$ and are bounded from below by a positive constant. Denoting $-\nabla \cdot \mathcal{G}(u_0)\mu\nabla$ by B, Corollary 6.9 gives $\mathrm{dom}_X(-\nabla \cdot \mu\nabla) = \mathrm{dom}_X(B)$. This implies $u_0 \in (X, \mathrm{dom}_X(B))_{1-\frac{1}{s},s}$. Furthermore, the so-defined B has maximal parabolic regularity on X, thanks to (5.5) in Theorem 5.18 with $p = q$.

Condition (**B**) from Proposition 6.2 is implied by Lemma 6.8 ii) in cooperation with Lemma 6.7, the fact that the mapping $H^{1,q} \ni \phi \mapsto \mathcal{G}(\phi) \in H^{1,q}$ is boundedly Lipschitz and (6.5).

It remains to show that the 'new' right-hand side $\mathcal{S}$ satisfies condition (**S**) from Proposition 6.2. We do this for every term in (6.4) separately, beginning from the left: concerning the first, one again uses (6.5) together with the asserted conditions **Ra)** and **Rb)** on $\mathcal{R}$. The assertion for the last two terms results from (6.5), the assumptions **BC)/Gg)**, Lemma 5.17 and Theorem 6.10. $\square$

Remark 6.16. Note that, if $\mathcal{R}$ takes its values only in the space $L^{q/2} \hookrightarrow X$, then – in the light of Lemma 5.17 – the elliptic operators incorporate the boundary conditions (1.2) in a generalized sense, see [29, Ch. II.2] or [14, Ch. 1.2].

Finally, it can be shown that the solution u is Hölder continuous simultaneously in space and time, even more:

Corollary 6.17. *There exist $\alpha, \beta > 0$ such that the solution u of* (1.1)/(1.2) *belongs to the space $C^\beta(J; H^{1,q}_\Gamma(\Omega)) \hookrightarrow C^\beta(J; C^\alpha(\Omega))$.*

7. Examples

In this section we describe geometric configurations for which our Assumption 6.4 holds true and we present concrete examples of mappings $\mathcal{G}$ and reaction terms $\mathcal{R}$ fitting into our framework.

7.1. Geometric constellations

While our results in Sections 4 and 5 on the square root of $-\nabla \cdot \mu \nabla$ and maximal parabolic regularity are valid in the general geometric framework of Assumption 3.1, we additionally had to impose Assumption 6.4 for the treatment of quasilinear equations in Section 6. Here we shortly describe geometric constellations, in which this additional condition is satisfied.

Let us start with the observation that the 2-d case is covered by Remark 5.5 i). Admissible three-dimensional settings may be described as follows.

Proposition 7.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists a $q > 3$ such that $-\nabla \cdot \mu \nabla + 1$ is a topological isomorphism from $H^{1,q}_\Gamma$ onto $H^{-1,q}_\Gamma$, if one of the following conditions is satisfied:*

i) *Ω has a Lipschitz boundary. $\Gamma = \emptyset$ or $\Gamma = \partial\Omega$. $\Omega_\circ \subseteq \Omega$ is another domain which is C^1 and which does not touch the boundary of Ω. $\mu|_{\Omega_\circ} \in BUC(\Omega_\circ)$ and $\mu|_{\Omega \setminus \overline{\Omega_\circ}} \in BUC(\Omega \setminus \overline{\Omega_\circ})$.*
ii) *Ω has a Lipschitz boundary. $\Gamma = \emptyset$. $\Omega_\circ \subseteq \Omega$ is a Lipschitz domain, such that $\partial\Omega_\circ \cap \Omega$ is a C^1 surface and $\partial\Omega$ and $\partial\Omega_\circ$ meet suitably* (*see* [26] *for details*)*. $\mu|_{\Omega_\circ} \in BUC(\Omega_\circ)$ and $\mu|_{\Omega \setminus \overline{\Omega_\circ}} \in BUC(\Omega \setminus \overline{\Omega_\circ})$.*
iii) *Ω is a three-dimensional Lipschitzian polyhedron. $\Gamma = \emptyset$. There are hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_n$ in $\mathbb{R}^3$ which meet at most in a vertex of the polyhedron such that the coefficient function μ is constantly a real, symmetric, positive definite 3×3 matrix on each of the connected components of $\Omega \setminus \cup_{l=1}^n \mathcal{H}_l$. Moreover, for every edge on the boundary, induced by a hetero interface $\mathcal{H}_l$, the angles between the outer boundary plane and the hetero interface do not exceed π and at most one of them may equal π.*
iv) *Ω is a convex polyhedron. $\overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ is a finite union of line segments. $\mu \equiv 1$.*

v) $\Omega \subseteq \mathbb{R}^3$ *is a prismatic domain with a triangle as basis.* Γ *equals either one half of one of the rectangular sides or one rectangular side or two of the three rectangular sides. There is a plane which intersects* Ω *such that the coefficient function* μ *is constant above and below the plane.*

vi) Ω *is a bounded domain with Lipschitz boundary. Additionally, for each* $\mathrm{x} \in \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ *the mapping* ϕ_{x} *defined in Assumption* 3.1 *is a* C^1*-diffeomorphism from* Υ_{x} *onto its image.* $\mu \in BUC(\Omega)$.

The assertions i) and ii) are shown in [26], while iii) is proved in [25] and iv) is a result of Dauge [18]. Recently, v) was obtained in [38] and vi) will be published in a forthcoming paper. □

Remark 7.2. The assertion remains true, if there is a finite open covering $\Upsilon_1, \dots, \Upsilon_l$ of $\overline{\Omega}$, such that each of the pairs $\Omega_j := \Upsilon_j \cap \Omega$, $\Gamma_j := \Gamma \cap \Upsilon_j$ fulfills one of the points i)–vi) after a bi-Lipschitz transformation. This provides a huge zoo of geometries and boundary constellations, for which $-\nabla \cdot \mu\nabla$ provides the required isomorphism. We intend to complete this in the future.

7.2. Nonlinearities and reaction terms

The most common case is that where $\mathcal{G}$ is the exponential or the Fermi-Dirac distribution function $\mathcal{F}_{1/2}$ given by

$$\mathcal{F}_{1/2}(t) := \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{s}}{1 + \mathrm{e}^{s-t}} \,\mathrm{d}s.$$

As a second example we present a nonlocal operator arising in the diffusion of bacteria; see [12], [13] and references therein.

Example 7.3. Let ζ be a continuously differentiable function on $\mathbb{R}$ which is bounded from above and below by positive constants. Assume $\varphi \in L^2(\Omega)$ and define

$$\mathcal{G}(u) := \zeta\left(\int_\Omega u\varphi \,\mathrm{dx}\right), \quad u \in H^{1,q}.$$

Now we give an example for a mapping $\mathcal{R}$.

Example 7.4. Assume $\iota : \mathbb{R} \to]0, \infty[$ to be a continuously differentiable function. Furthermore, let $\mathcal{T} : H^{1,q} \to H^{1,q}$ be the mapping which assigns to $v \in H^{1,q}$ the solution φ of the elliptic problem (including boundary conditions)

$$-\nabla \cdot \iota(v)\nabla\varphi = 0. \tag{7.1}$$

If one defines

$$\mathcal{R}(v) = \iota(v)|\nabla(\mathcal{T}(v))|^2,$$

then, under reasonable suppositions on the data of (7.1), the mapping $\mathcal{R}$ satisfies Assumption **Ra)**.

The example comes from a model which describes electrical heat conduction; see [5] and the references therein.

8. Concluding remarks

Remark 8.1. Under the additional Assumption 6.4, Theorem 5.4 implies maximal parabolic regularity for $-\nabla \cdot \mu \nabla$ on $H_\Gamma^{-1,q}$ for every $q \in [2, \infty[$, as in the 2-d case.

Besides, the question arises whether the limitation for the exponents, caused by the localization procedure, is principal in nature or may be overcome when applying alternative ideas and techniques (cf. Theorem 4.3). We suggest that the latter is the case.

Remark 8.2. Equations of type (1.1)/(1.2) may be treated in an analogous way, if under the time derivative a suitable superposition operator is present, see [40] for details.

Remark 8.3. In the semilinear case, it turns out that one can achieve satisfactory results without Assumption 6.4, at least when the nonlinear term on the right-hand side depends only on the function itself and not on its gradient.

Remark 8.4. Let us explicitly mention that Assumption 6.4 is not always fulfilled in the 3-d case. First, there is the classical counterexample of Meyers, see [48], a simpler (and somewhat more striking) one is constructed in [25], see also [26]. The point, however, is that not the mixed boundary conditions are the obstruction but a somewhat 'irregular' behavior of the coefficient function μ in the inner of the domain. If one is confronted with this, spaces with weight may be the way out.

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On the Relation Between the Large Time Behavior of the Stokes Semigroup and the Decay of Steady Stokes Flow at Infinity

Toshiaki Hishida

Dedicated to Professor Herbert Amann on his 70th birthday

Abstract. Let e^{-tA} be the Stokes semigroup over an unbounded domain Ω. For construction of the Navier-Stokes flow globally in time, it is crucial to derive L^q-L^r decay estimate (1.4) for ∇e^{-tA}; thus, given Ω, we need to ask which (q, r) admits (1.4). The present paper provides a principle which interprets how this question is related to spatial decay properties of steady Stokes flow in the domain Ω under consideration.

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Keywords. Stokes semigroup, L^q-L^r estimate, steady Stokes flow.

1. Introduction

Let Ω be an unbounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Consider the Stokes system for a viscous incompressible fluid that occupies the domain Ω subject to the non-slip boundary condition on $\partial\Omega$. This paper makes it clear that the temporal decay property for $t \to \infty$ of the solution to the initial value problem

$$\begin{cases} \partial_t u - \Delta u + \nabla p = 0 & (x \in \Omega,\ t > 0), \\ \operatorname{div} u = 0 & (x \in \Omega,\ t \geq 0), \\ u = 0 & (x \in \partial\Omega,\ t > 0), \\ u(\cdot, 0) = u_0 & (x \in \Omega), \end{cases} \tag{1.1}$$

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is closely related to the spatial decay property for $|x| \to \infty$ of the solution to the steady problem

$$-\Delta v + \nabla \pi = \operatorname{div} F, \quad \operatorname{div} v = 0 \quad (x \in \Omega); \qquad v|_{\partial\Omega} = 0. \tag{1.2}$$

Here, $u(x,t) = (u_1, \dots, u_n)^T$ and $v(x) = (v_1, \dots, v_n)^T$ are the velocity fields; $p(x,t)$ and $\pi(x)$ are the pressures.

Since the celebrated paper by Kato [28] for the whole space problem, it is well known that L^q-L^r estimates ($q \le r$)

$$\|u(t)\|_r \le C t^{-(n/q-n/r)/2} \|u_0\|_q \qquad (t > 0), \tag{1.3}$$

$$\|\nabla u(t)\|_r \le C t^{-(n/q-n/r)/2-1/2} \|u_0\|_q \qquad (t > 0), \tag{1.4}$$

of the solution (the Stokes semigroup) to (1.1) are important tools for construction of global strong solutions to the Navier-Stokes system with small initial data in $L^n_\sigma(\Omega)$ (the smallness is redundant for the case $n = 2$). Estimates (1.3) and (1.4) play the same role as fractional powers of the Stokes operator, which were employed by Fujita and Kato [19] and Giga and Miyakawa [23] when Ω is a bounded domain. So far, besides those cases – bounded domain, the whole space –, the above-mentioned Navier-Stokes theory ($n \ge 3$) has been established (or essentially available) for the following physically relevant unbounded domains:

(1) half-space ([44], [30]);
(2) exterior domain ([27]);
(3) aperture domain with flux condition ([25], [33]);
(4) perturbed half-space ([34], [35]);
(5) infinite layer ([1]);
(6) asymptotically flat layer ([2], [3]);
(7) infinite cylinder ([40]).

The last three cases (5)–(7) are not kept in mind in this paper because the solution to (1.1) decays exponentially and algebraic decay properties (1.3)–(1.4) do not make sense as optimal decay.

In view of the Navier-Stokes nonlinearity $u \cdot \nabla u$, whether or not one can prove the global existence theorem as above depends crucially on the answer to the following question: Which (q,r) admits the gradient estimate (1.4)? The case $q = r = n$ is particularly important because the initial data are taken from $L^n_\sigma(\Omega)$. On the other hand, it is also interesting to ask optimal decay at space infinity (expected in general) of the solution to (1.2) with $F \in C_0^\infty(\overline{\Omega})^{n\times n}$. The purpose of this paper is to show a principle which interprets the relation between both questions when we do not restrict ourselves to the specified domains (1)–(4) above but consider general unbounded domains which satisfy the reasonable hypothesis (H1) (see Section 2). Roughly speaking (see Theorem 2.3 to be precise), it is proved that if (1.4) holds for $q \le r < q_0$, then the problem (1.2) possesses a solution of class $v \in L^s(\Omega)^n$, $s > p_0$, for each $F \in C_0^\infty(\overline{\Omega})^{n\times n}$, where $1/p_0 = 1 - 1/n - 1/q_0$; thus, (1.4) for larger r implies better summability of the steady flow at infinity. Indeed the opposite implication is hopeless, but the best possible range of (q,r)

which admits (1.4) is suggested as long as we know the optimal rate of decay of generic steady flows in the domain Ω under consideration (as for an example related to this idea, see (iii) in the final section).

The idea of the proof is very simple. We consider the unsteady problem

$$\begin{cases} \partial_t v - \Delta v + \nabla \pi = \operatorname{div} F(t) & (x \in \Omega,\, t \in \mathbb{R}), \\ \operatorname{div} v = 0 & (x \in \Omega,\, t \in \mathbb{R}), \\ v = 0 & (x \in \partial\Omega,\, t \in \mathbb{R}), \end{cases} \tag{1.5}$$

and look for a solution which remains bounded for $t \to -\infty$ in a suitable sense so that the uniqueness of solutions is ensured. We have time-periodic solutions in mind as typical examples, and steady solution can be regarded as a special case of them. So the problem is to derive L^s-summability of the solution $v(t)$ at space infinity by use of (1.4) and we wish to find the summabilty exponent s as small as possible.

For the unbounded domains (1), (3) and (4) mentioned above, we know (1.4) for $1 < q \le r < \infty$ ([44], [33], [34], [35]), so that our result implies the summability $v \in L^s(\Omega)^n$, $s > n/(n-1)$, for (1.2). As for the exterior problem, estimate (1.4) for $1 < q \le r \le n$ was proved by Iwashita [27], Dan and Shibata [12] ($n = 2$) and Maremonti and Solonnikov [36]. Indeed this is fortunately enough to solve the Navier-Stokes system globally in time, but it is interesting to ask whether the restriction $r \le n$ is optimal or not. As a corollary of our result (see Corollary 2.4), it turns out that (1.4) for $1 < q \le r < q_0$ with some $q_0 > n$ is impossible since we know the optimal rate of decay of generic steady flows in exterior domains (Lemma 4.1). This fact was first pointed out by Maremonti and Solonnikov [36]. Although the spatial decay property of exterior flows is the point in [36] as well, their proof is different from ours explained above. In fact, they multiply the equation $(1.1)_1$ by the decaying solution of

$$-\Delta v + \nabla \pi = 0, \quad \operatorname{div} v = 0 \quad (x \in \Omega); \qquad v|_{\partial\Omega} = b \in C^2(\partial\Omega)^n$$

and estimate $\langle v, u(t)\rangle$ by using (1.4) with $r > n$ to accomplish $|\langle v, u_0\rangle| \le C\|u_0\|_{n/2}$ for all $u_0 \in C^\infty_{0,\sigma}(\Omega)$, yielding $v \in L^{n/(n-2)}(\Omega)^n$, a contradiction. Our strategy in the present paper is merely to estimate the solution $v(t)$ of (1.5) and thus the proof is considerably easier.

This paper consists of five sections. In Section 2 we provide our results. Section 3 is devoted to the proof of the main theorem. Section 4 studies the exterior problem to show the corollary. We conclude the paper with some remarks in the final section.

2. Results

To begin with, we fix notation. Let Ω be a domain in $\mathbb{R}^n$. The class $C_0^\infty(\Omega)$ consists of all C^∞-functions whose supports are compact in Ω. Set $C_0^\infty(\overline{\Omega}) = \{f = g|_{\overline{\Omega}};\, g \in C_0^\infty(\mathbb{R}^n)\}$. For $1 \le q \le \infty$ we denote by $L^q(\Omega)$ the usual Lebesgue space with norm $\|\cdot\|_{q,\Omega}$; we simply write $\|\cdot\|_q = \|\cdot\|_{q,\Omega}$ when Ω is a given unbounded domain in

(1.1), (1.2). Let $C^\infty_{0,\sigma}(\Omega)$ be the class of all vector fields $u = (u_1, \dots, u_n)^T$ which satisfy $u_j \in C_0^\infty(\Omega)\,(1 \le j \le n)$ and div $u = 0$. For $1 < q < \infty$ we define $L^q_\sigma(\Omega)$ by the completion of $C^\infty_{0,\sigma}(\Omega)$ in the space $L^q(\Omega)^n$. Note that the space $L^q_\sigma(\Omega)$ may involve an additional side condition for some Ω; for instance, vanishing flux condition through an aperture is hidden in the case of aperture domains, see [18], [39].

Given an unbounded domain Ω in (1.1) with smooth boundary $\partial\Omega$, as usual, the first step is to establish the Helmholtz decomposition

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus G^q(\Omega), \quad G^q(\Omega) = \{\nabla p \in L^q(\Omega)^n; p \in L^q_{\mathrm{loc}}(\overline{\Omega})\}. \tag{2.1}$$

For example, we refer to [38], [42] for exterior domains, and to [18], [39] for aperture domains; also, see [20, Section III.1] and the references therein. When $q = 2$, we have (2.1) for any domain even if assuming no regularity on $\partial\Omega$, where $L^2_{\mathrm{loc}}(\overline{\Omega})$ should be replaced by $L^2_{\mathrm{loc}}(\Omega)$ in $G^2(\Omega)$. However, that is not always the case for $q \ne 2$. In fact, which q admits the decomposition (2.1) depends on the geometry of the domain, see [6], [37]. Assuming (2.1) for some q, we denote by $P = P_q$ the projection operator from $L^q(\Omega)^n$ onto $L^q_\sigma(\Omega)$ along $G^q(\Omega)$. Then the Stokes operator is defined by

$$\begin{cases} D_q(A) = \{u \in W^{2,q}(\Omega)^n \cap L^q_\sigma(\Omega);\ u|_{\partial\Omega} = 0\}, \\ Au = -P\Delta u, \end{cases} \tag{2.2}$$

where $W^{2,q}(\Omega)$ is the usual L^q-Sobolev space of second order.

We make the following assumption on the Stokes operator.

(H1) There is $q_0 \in (2, \infty]$ such that the following properties hold: Set

$$r_0 = \begin{cases} \frac{nq_0}{n-q_0} & (2 < q_0 < n), \\ \infty & (n \le q_0 \le \infty), \end{cases} \qquad r_0' = \begin{cases} 1 & (r_0 = \infty), \\ \frac{r_0}{r_0-1} & (r_0 < \infty). \end{cases}$$

Then, for every $q \in (r_0', r_0)$, the decomposition (2.1) holds and the operator $-A$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t\ge 0}$ on $L^q_\sigma(\Omega)$. Furthermore, $u(t) = e^{-tA}u_0$ with $u_0 \in C^\infty_{0,\sigma}(\Omega)$ satisfies

$$(1.3) \text{ for } r_0' < q \le r < r_0; \qquad (1.4) \text{ for } r_0' < q \le r < q_0.$$

Remark 2.1. Note that $r_0' < q_0$ always holds. It is also natural to assume (1.3) for $r < r_0$ when assuming (1.4) for $r < q_0$ from the viewpoint of embedding relation.

Remark 2.2. When we think of the duality relation between existence and uniqueness for the Neumann problem, it seems to be reasonable to assume (2.1) for $q \in (r_0', r_0)$ with some r_0; and then, we have the relation $P_q^* = P_{q/(q-1)}$. When the boundary $\partial\Omega$ is unbounded, this exponent r_0 depends on the geometry of $\partial\Omega$ near infinity, no matter how smooth it is ([6], [37]).

Our main theorem reads as follows.

Theorem 2.3. *Suppose* (H1) *and define the exponent* p_0 *by*

$$\frac{1}{p_0} = 1 - \frac{1}{n} - \frac{1}{q_0}. \tag{2.3}$$

Then, for each $F \in C_0^\infty(\overline{\Omega})^{n\times n}$*, the problem* (1.2) *necessarily admits a solution* v*, which is of class*

$$v \in L^s_\sigma(\Omega), \qquad \forall s \in (p_0, r_0). \tag{2.4}$$

Here, we note the relation $r_0' < p_0 < r_0$ on account of $q_0 > 2$.

Let us next consider the exterior problem. We know that (1.4) holds for $1 < q \le r \le n$ ([27], [12], [36]). In order to make the linear theory complete, we wish to ask whether or not the restriction $r \le n$ is optimal. The following corollary tells us that the answer is positive.

Corollary 2.4. *Let* Ω *be an exterior domain in* $\mathbb{R}^n$*,* $n \ge 2$*, with smooth boundary* $\partial\Omega$*. Then it is impossible to have* (1.4) *for* $1 < q \le r < q_0$ *with some* $q_0 > n$*.*

Remark 2.5. When $r > n$ for the exterior problem, we have slower decay estimate

$$\|\nabla e^{-tA}u_0\|_r \le Ct^{-\frac{n}{2q}}\|u_0\|_q \qquad (t \ge 1),$$

for $1 < q \le r$, see [27], [12], [36].

3. Proof of Theorem 2.3

This section studies the unsteady problem (1.5) in order to prove Theorem 2.3. Concerning the external force, we may consider very smooth one; say, we make the following assumption.

(H2) $F(t) \in C_0^\infty(\overline{\Omega})^{n\times n}$ for each $t \in \mathbb{R}$ and $\sup_{t\in\mathbb{R}}\|F(t)\|_\infty < \infty$. There is a compact set $K \subset \mathbb{R}^n$ such that Supp $F(t) \subset K$ for all $t \in \mathbb{R}$. Furthermore, there is $\alpha \in (0,1)$ such that div $F \in C^\alpha_{\mathrm{loc}}(\mathbb{R}; L^\infty(\Omega)^n)$.

Under the condition (H1) we rewrite (1.5) as

$$\frac{dv}{dt} + Av = P\,(\mathrm{div}\ F(t)) \qquad (t \in \mathbb{R}) \tag{3.1}$$

in $L^r_\sigma(\Omega)$, $r \in (r_0', r_0)$. We say that $v(t)$ is a solution to (3.1) if it is of class

$$v \in C^1(\mathbb{R}; L^r_\sigma(\Omega)) \cap C(\mathbb{R}; D_r(A))$$

and satisfies (3.1) in $L^r_\sigma(\Omega)$ for some $r \in (r_0', r_0)$. If this solution enjoys the additional condition

$$v(t) \in L^s_\sigma(\Omega), \quad \forall t \in \mathbb{R}; \qquad \sup_{t\in\mathbb{R}}\|v(t)\|_s < \infty \tag{3.2}$$

for some $s \in (r_0', r)$, then it must be of the form

$$v(t) = \int_{-\infty}^t e^{-(t-\tau)A}P\,(\mathrm{div}\ F(\tau))\,d\tau. \tag{3.3}$$

In fact, from

$$\frac{\partial}{\partial \tau}\{e^{-(t-\tau)A}v(\tau)\} = e^{-(t-\tau)A}P\,(\mathrm{div}\ F(\tau))$$

in $L^r_\sigma(\Omega)$ we find

$$v(t) = e^{-(t-\sigma)A}v(\sigma) + \int_\sigma^t e^{-(t-\tau)A}P\,(\mathrm{div}\ F(\tau))\,d\tau \tag{3.4}$$

for $-\infty < \sigma < t < \infty$. Since we have (1.3) for $r_0' < s < r < r_0$ by (H1), the condition (3.2) implies

$$\|e^{-(t-\sigma)A}v(\sigma)\|_r \le C(t-\sigma)^{-(n/s-n/r)/2}\|v(\sigma)\|_s \to 0$$

as $\sigma \to -\infty$. Thus, (3.3) is convergent in $L^r_\sigma(\Omega)$ and it is the unique solution within the class (3.2).

Theorem 3.1. *Suppose* (H1), (H2) *and define the exponent* p_0 *by* (2.3)*. Then the problem* (3.1) *admits a solution* (3.3)*, which is of class* (3.2) *for every* $s \in (p_0, r_0)$*.*

Proof. We first show the summability (3.2) of $v(t)$ given by (3.3). Let $\varphi \in C^\infty_{0,\sigma}(\Omega)$. An integration by parts yields

$$\begin{aligned}\langle v(t), \varphi\rangle &= \int_0^\infty \langle e^{-\tau A}P\,(\mathrm{div}\ F(t-\tau)), \varphi\rangle\,d\tau \\ &= -\int_0^\infty \langle F(t-\tau), \nabla e^{-\tau A}\varphi\rangle\,d\tau = \int_0^1 + \int_1^\infty .\end{aligned} \tag{3.5}$$

Put $M = \sup_{t\in\mathbb{R}}\|F(t)\|_\infty$, then we have $\sup_{t\in\mathbb{R}}\|F(t)\|_q \le M|K|^{1/q}$ for $q \in [1,\infty)$ by (H2), where $|\cdot|$ denotes the Lebesgue measure. Let $s \in (p_0, r_0)$, then $s' \in (r_0', q_0)$, where $1/s' + 1/s = 1$. By (H1) we employ (1.4) to get

$$\left|\int_0^1\right| \le \int_0^1 \|F(t-\tau)\|_s \|\nabla e^{-\tau A}\varphi\|_{s'}\,d\tau \le CM|K|^{1/s}\|\varphi\|_{s'}.$$

Since (2.3) together with $s > p_0$ leads us to

$$\frac{1}{s'} - \frac{1}{q_0} = 1 - \frac{1}{s} - \frac{1}{q_0} > 1 - \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{n} \tag{3.6}$$

we can take $r \in (s', q_0)$ such that $1/s' - 1/r > 1/n$, namely,

$$\frac{n}{2}\left(\frac{1}{s'} - \frac{1}{r}\right) + \frac{1}{2} > 1. \tag{3.7}$$

One can use (1.4) to see from (3.7) that

$$\left|\int_1^\infty\right| \le \int_1^\infty \|F(t-\tau)\|_{r/(r-1)}\|\nabla e^{-\tau A}\varphi\|_r\,d\tau \le CM|K|^{(r-1)/r}\|\varphi\|_{s'}.$$

Hence we find

$$|\langle v(t), \varphi\rangle| \le C\|\varphi\|_{s'} \qquad \forall\varphi \in C^\infty_{0,\sigma}(\Omega)$$

which implies (3.2) for every $s \in (p_0, r_0)$. Next, it is easy to see that $v(t)$ given by (3.3) is actually a solution to (3.1). In fact, for any fixed $\sigma \in \mathbb{R}$, we have the representation (3.4) in $L^s_\sigma(\Omega)$, $s \in (p_0, r_0)$, for $-\infty < \sigma < t < \infty$. It follows from (H2) that $P\,(\mathrm{div}\ F)$ is locally Hölder continuous with values in $L^s_\sigma(\Omega)$; as a consequence ([4, Chapter II]), $v(t)$ is of class

$$v \in C^1((\sigma,\infty); L^s_\sigma(\Omega)) \cap C((\sigma,\infty); D_s(A)) \cap C([\sigma,\infty); L^s_\sigma(\Omega))$$

and satisfies (3.1) for $t \in (\sigma,\infty)$ in $L^s_\sigma(\Omega)$. Since $\mathbb{R} \ni \sigma$ is arbitrary, $v(t)$ is a solution to (3.1) for $t \in \mathbb{R}$. The proof is complete. □

It is also possible to give another shorter proof by employing some Lorentz spaces without dividing integral in (3.5). We will mention this although the elementary proof above by use of less function spaces might be preferred by most of readers. We define the solenoidal Lorentz spaces by

$$L^{q,\beta}_\sigma(\Omega) = (L^p_\sigma(\Omega), L^r_\sigma(\Omega))_{\theta,\beta}$$

where

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{r}, \qquad r_0' < p < q < r < r_0, \qquad 1 \le \beta \le \infty$$

and $(\cdot,\cdot)_{\theta,\beta}$ denotes the real interpolation functor. For the interpolation theory, see for instance Amann [4, Chapter I]. We have the duality relation $L^{q,1}_\sigma(\Omega)^* = L^{q/(q-1),\infty}_\sigma(\Omega)$, and $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^{q,1}_\sigma(\Omega)$. By interpolation we immediately see from (1.4) that

$$\|\nabla e^{-tA}u_0\|_{r,1} \le Ct^{-(n/q-n/r)/2-1/2}\|u_0\|_{q,1} \qquad (t > 0), \tag{3.8}$$

for $r_0' < q \le r < q_0$, where $\|\cdot\|_{q,\beta}$ stands for the norm of $L^{q,\beta}_\sigma(\Omega)$. Following the idea of Yamazaki [45], we apply interpolation to the sublinear operator $u_0 \mapsto \left[t \mapsto \|\nabla e^{-tA}u_0\|_{r,1}\right]$ for fixed r; then, we can deduce from (3.8) that

$$\int_0^\infty \|\nabla e^{-tA}u_0\|_{r,1}\,dt \le C\|u_0\|_{q,1}, \qquad r_0' < q < r < q_0, \quad \frac{1}{q} - \frac{1}{r} = \frac{1}{n} \tag{3.9}$$

in spite of pointwise estimate like $1/t$ of the integrand. Note that the set of pairs (q,r) satisfying the relation (3.9) is not vacuous because $q_0 > 2$ implies $1/r_0' - 1/q_0 > 1/n$.

Another proof of Theorem 3.1. We will show (3.2) for every $s \in (p_0, r_0)$. Given $s \in (p_0, r_0)$, we take $r \in (s', q_0)$ such that $1/s' - 1/r = 1/n$, which is possible because of (3.6) and slightly different from (3.7), where $1/s' + 1/s = 1$. In view of (3.5) it follows from (3.9) that

$$|\langle v(t),\varphi\rangle| \le \int_0^\infty \|F(t-\tau)\|_{r/(r-1),\infty}\|\nabla e^{-\tau A}\varphi\|_{r,1}\,d\tau \le CM|K|^{(r-1)/r}\|\varphi\|_{s',1}$$

for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$. By duality we obtain

$$v(t) \in L^{s,\infty}_\sigma(\Omega), \quad \forall t \in \mathbb{R}; \qquad \sup_{t\in\mathbb{R}} \|v(t)\|_{s,\infty} < \infty$$

for every $s \in (p_0, r_0)$, from which we conclude (3.2) for the same s by interpolation. The proof is complete. □

We are now in a position to show Theorem 2.3.

Proof of Theorem 2.3. When $F(t)$ is periodic in Theorem 3.1, so is the solution $v(t)$ given by (3.3) with the same period. If, in particular, F is independent of t, then v can be regarded as periodic solution with arbitrary period; hence, we conclude that the solution $v \in D_s(A)$ obtained in Theorem 3.1 is a steady flow: $Av = P\,(\mathrm{div}\ F)$ in $L^s_\sigma(\Omega)$. This completes the proof. □

4. Proof of Corollary 2.4

Let Ω be an exterior domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. We will show that we cannot avoid the condition $r \leq n$ for (1.4).

Proof of Corollary 2.4. If (1.4) were correct for $1 < q \leq r < q_0$ with some $q_0 > n \geq 2$, we would have (H1) for such q_0; because we already know the other conditions with $r_0 = \infty$ ([38], [42], [22], [43], [8], [10], [17], [27], [12], [36]). The exponent p_0 given by (2.3) fulfills $p_0 < n/(n-2)$ if $n \geq 3$, and $p_0 < \infty$ if $n = 2$. Hence Theorem 2.3 implies that the steady problem (1.2) has always a smooth solution v of class

$$v \in \begin{cases} L^{n/(n-2)}(\Omega)^n & (n \geq 3), \\ L^s(\Omega)^n \quad \text{for } \exists\, s < \infty & (n = 2), \end{cases} \tag{4.1}$$

for every $F \in C_0^\infty(\overline{\Omega})^{n\times n}$; here, the smoothness of v follows from the regularity theory for the Stokes system, see for instance [20, Section IV.4]. The summability (4.1) is, however, possible only in a special situation due to Lemma 4.1 below and one cannot always have it; see also Remark 4.3. The proof is complete. □

It is well known that the net force exerted on the obstacle by the fluid must vanish whenever the steady flow decays faster than the Stokes fundamental solution. This is usually proved by use of the asymptotic representation for $|x| \to \infty$ of the steady flow, see Chang and Finn [11] and also Galdi [20, Section V.3]. We here give another independent proof of the following lemma by using the summability (4.1) directly.

Lemma 4.1. *Let $F \in C_0^\infty(\overline{\Omega})^{n\times n}$. Suppose that v is a smooth solution to* (1.2) *of class* (4.1). *Then*

$$\int_{\partial\Omega} \nu \cdot \{T(v,\pi) + F\}\, d\sigma = 0, \tag{4.2}$$

where ν is the outward unit normal to $\partial\Omega$,

$$T(v,\pi) = \nabla v + (\nabla v)^T - \pi\mathbb{I} \tag{4.3}$$

is the Cauchy stress tensor and $\mathbb{I}$ is the $n \times n$-identity matrix.

Proof. We fix $\eta \in C^\infty([0,\infty);[0,1])$ such that $\eta(t) = 1\,(0 \leq t \leq 2/3)$ and $\eta(t) = 0\,(t \geq 5/6)$. For $R > 0$ and $x \in \mathbb{R}^n$, we set $\zeta_R(x) = \eta(|x|/R)$; then

$$\|\nabla^2\zeta_R\|_{q,\mathbb{R}^n} \leq CR^{-2+n/q} \qquad (1 \leq q \leq \infty). \tag{4.4}$$

For each $k \in \{1, \ldots, n\}$, we set $\phi_R^{(k)}(x) = \zeta_R(x)e_k$, where e_k stands for the unit vector along x_k-axis. Let $\mathbb{B}_R$ be the Bogovskii operator ([5], [9], [20]) in the annulus $A_R = \{x \in \mathbb{R}^n;\ R/2 < |x| < R\}$ and set

$$\psi_R^{(k)}(x) = \phi_R^{(k)}(x) - \mathbb{B}_R[\partial_k \zeta_R](x).$$

Note that $\mathbb{B}_R[\partial_k \zeta_R] \in C_0^\infty(A_R)^n$ and that

$$\|\nabla^2 \mathbb{B}_R[\partial_k \zeta_R]\|_{q,A_R} \le C\|\nabla \partial_k \zeta_R\|_{q,A_R} \qquad (1 < q < \infty) \tag{4.5}$$

with some constant $C = C(q) > 0$ independent of $R > 0$, see Borchers and Sohr [9, Theorem 2.10]. Since the compatibility condition

$$\int_{A_R} \partial_k \zeta_R \, dx = \int_{A_R} \mathrm{div}\ \phi_R^{(k)} \, dx = \int_{|x|=R/2} \frac{-x_k}{R/2} \, d\sigma = 0$$

is satisfied, we have div $\psi_R^{(k)} = 0$. Combining (4.4) with (4.5), we find

$$\|\nabla^2 \psi_R^{(k)}\|_{q,A_R} \le CR^{-2+n/q} \qquad (1 < q < \infty). \tag{4.6}$$

Set

$$N = (N_1, \ldots, N_n)^T = \int_{\partial\Omega} \nu \cdot \{T(v, \pi) + F\} \, d\sigma$$

and rewrite the equation $(1.2)_1$ as

$$\mathrm{div}\ \{T(v, \pi) + F\} = 0 \qquad (x \in \Omega).$$

Since $\psi_R^{(k)}(x) = 0$ for $|x| \ge R$ and $\psi_R^{(k)}(x) = e_k$ for $|x| \le R/2$, we test this equation with $\psi_R^{(k)}$ to obtain

$$N_k = \int_{A_R} T(v, \pi) : \nabla \psi_R^{(k)} \, dx$$

for $1 \le k \le n$, where $R > 0$ is taken so large that $F(x) = 0$ for $|x| \ge R/2$, and $T : S = \sum_{i,j} T_{ij} S_{ij}$ for matrices T, S. By integration by parts once more and by div $\psi_R^{(k)} = 0$ we find

$$N_k = -\int_{A_R} v \cdot \Delta \psi_R^{(k)} \, dx,$$

from which together with (4.6) it follows that

$$|N_k| \le C\|v\|_{s,A_R} R^{-2+n(1-1/s)}$$

for arbitrary $R > 0$. When $n \ge 3$, we can take $s = n/(n-2)$ by (4.1) to obtain

$$|N_k| \le C\|v\|_{n/(n-2),A_R} \to 0 \qquad (R \to \infty).$$

When $n = 2$, the solution v belongs to $L^s(\Omega)^n$ for some $s < \infty$; therefore, for such s, we find

$$|N_k| \le C\|v\|_{s,A_R} R^{-2/s} \to 0 \qquad (R \to \infty).$$

In any case we conclude $N = 0$, which completes the proof. $\square$

Remark 4.2. Even for the Navier-Stokes flow ($n \geq 3$) subject to $v = 0$ on $\partial\Omega$, the summablity (4.1) or $\nabla v \in L^{n/(n-1)}(\Omega)^{n\times n}$ necessarily implies the net force condition (4.2); for further details, see Kozono and Sohr [31], Borchers and Miyakawa [7], Kozono, Sohr and Yamazaki [32].

Remark 4.3. It is easy to find many solutions of (1.2) which do not satisfy (4.2). Consider, for instance, three-dimensional problem; set $u(x) = \frac{1}{8\pi}(\frac{e_1}{|x|} + \frac{x_1 x}{|x|^3})$, the first column vector of the Stokes fundamental solution, and $p(x) = \frac{x_1}{4\pi|x|^3}$, where $e_1 = (1,0,0)^T$. Assuming $0 \in \operatorname{int}\Omega^c$, we have $-\Delta u + \nabla p = 0, \operatorname{div} u = 0$ in Ω. By $\int_{\partial\Omega} \nu \cdot u \, d\sigma = 0$ there is a vector field $w \in C_0^\infty(\overline{\Omega})^3$ such that $\operatorname{div} w = 0$ in Ω and $w = u$ on $\partial\Omega$. Setting $v = u - w$, we see that (v,p) obeys (1.2) with $F = \nabla w \in C_0^\infty(\overline{\Omega})^{3\times 3}$ and that

$$\int_{\partial\Omega} \nu \cdot \{T(v,p) + F\} \, d\sigma = e_1.$$

The principle is that the solution of (1.2) whose asymptotic rate at infinity is the same as that of the Stokes fundamental solution must possess non-zero net force on $\partial\Omega$. For the Navier-Stokes flow as well, it has been proved by Galdi [21] that flows for which (4.2) happens are rare.

5. Concluding remarks

We conclude this paper with remarks on applications of our idea to some other interesting problems.

(i) The boundary condition $(e^{-\tau A}\varphi)|_{\partial\Omega} = 0$ is needed in (3.5) since $F|_{\partial\Omega} \neq 0$ in general. When $F \in C_0^\infty(\overline{\Omega})^{n\times n}$ is replaced by $F \in C_0^\infty(\Omega)^{n\times n}$, Theorem 2.3 remains valid even for some other boundary conditions provided that the associated Stokes operator can be defined in an appropriate way. Shibata and Shimizu [41] proved (1.4) with $1 < q \leq r < \infty$ (even $r = \infty$ unless $q = \infty$) for the Stokes initial value problem subject to Neumann boundary condition $\nu \cdot T(u,p)|_{\partial\Omega} = 0$ in exterior domains ($n \geq 3$), where $T(u,p)$ is given by (4.3). It is worth while noting that there is no restriction on (q,r) due to the null force on $\partial\Omega$ and that this is consistent with better summability of the steady Stokes flow with Neumann boundary condition when $F \in C_0^\infty(\Omega)^{n\times n}$.

(ii) We have assumed the generation of analytic semigroup in (H1). But the analyticity itself of the semigroup is not essential in our argument. We think of the Stokes operator with rotating effect of an obstacle in 3D exterior domains; actually, the semigroup generated by this operator is no longer analytic ([16], [24]), nevertheless Shibata and the present author [26] proved the L^q-L^r estimates, in particular, (1.4) for $1 < q \leq r \leq 3 \,(= n)$. Recently in [15] Farwig and the present author have derived the asymptotic representation for $|x| \to \infty$ of the steady Stokes flow around a rotating obstacle, from which it follows that the summability (4.1) (with $n = 3$, namely $v \in L^3(\Omega)^3$) does not always hold. This suggests that the restriction $r \leq 3$ above is unavoidable.

(iii) Consider the Oseen operator, where translating effect of an obstacle is taken into account in 3D exterior domains. This operator generates an analytic semigroup ([38]) and it satisfies (1.4) for $1 \le q \le r \le 3\,(= n)$, which was proved by Kobayashi and Shibata [29] (see also [13], [14] for $n \ge 3$). It is obvious that our argument in Section 3 works for the Oseen semigroup as well, so that Theorem 2.3 still remains true. On the other hand, the 3D steady Oseen flow possesses better summability $v \in L^s(\Omega)^3$, $s > 2$, because it has good decay structure outside wake region, see [20, Chapter VII]. This suggests, in view of (2.3) with $p_0 = 2$ and $n = 3$, that the 3D Oseen semigroup may possibly satisfy the gradient estimate (1.4) for $1 < q \le r < 6$ (or even $r \le 6$).

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Well-posedness and Exponential Decay for the Westervelt Equation with Inhomogeneous Dirichlet Boundary Data

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Dedicated to Professor Herbert Amann

Abstract. This paper deals with global solvability of Westervelt equation, which model arises in the context of high intensity ultrasound. The PDE equations derived are evolutionary quasilinear, potentially degenerate damped wave equations defined on a bounded domain in R^n, $n = 1, 2, 3$.

We prove local and global well-posedness as well as exponential decay rates for the Westervelt equation with *inhomogeneous Dirichlet boundary conditions* and small Cauchy data. The local existence proofs are based on an application of Banach's fixed point theorem to an appropriate formulation of these PDEs.

Global well-posedness is shown by exploiting functional analytic properties enjoyed by the semigroups generated by strongly damped wave equations. These include analyticity, dissipativity and suitable characterization of fractional powers of the generator – properties that enable the applicability of the "barrier" method.

The obtained result holds for all times, provided that the Cauchy data are taken from a suitably small (independent on time) ball characterized by the parameters of the equation, and the boundary data satisfy some decay condition.

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1. Introduction

Nonlinear acoustics plays a role in several physical contexts. Our work is especially motivated by high-intensity focused ultrasound (HIFU) being used in technical and medical applications ranging from lithotripsy or thermotherapy to ultrasound cleaning or welding and sonochemistry, see [1], [12], [18], [19], and the references therein.

The Westervelt equation is given by

$$-\frac{1}{c^2}p_{\sim tt} + \Delta p_\sim + \frac{b}{c^2}\Delta(p_{\sim t}) = -\frac{\beta_a}{\varrho_0 c^4}p^2_{\sim tt} \tag{1.1}$$

with $\beta_a = 1 + B/(2A)$, where $p_\sim$ denotes the acoustic pressure fluctuations, c is the speed of sound, b the diffusivity of the sound, ϱ_0 the mass density, and B/A the parameter of nonlinearity. For a detailed derivation of the PDE we refer to [14], [18], [21], [33].

Throughout this paper we will assume that the domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, on which we consider the PDEs is open and bounded with C^2 smooth boundary Γ.

The Westervelt equation can be equivalently rewritten as:

$$(1 - 2ku)u_{tt} - c^2\Delta u - b\Delta(u_t) = 2k(u_t)^2, \tag{1.2}$$

where $k = \beta_a/(\varrho c^2)$ This is a quasilinear strongly damped wave equation with potential degeneracy.

Quasilinear PDE's have attracted considerable attention in the literature with a large arsenal of mathematical tools developed for their treatment. Particularly well studied, with optimal results available, are parabolic equations – see [3, 4, 29] and references therein. In the case of hyperbolic like models, the intrinsic low regularity and oscillatory dynamics puts additional demands on regularity of the data as well as necessitates introduction of some sort of dissipation that may include interior, boundary or partial interior damping – see [32, 34, 23, 5] and references therein.

The distinctive feature of our work is that the model considered corresponds to quasilinear internally damped wave equation with potential degeneracy and *nonhomogeneous boundary forcing*. This calls for a careful setup of the state space where the latter should provide certain topological invariance for the dynamics. The above is achieved through a long chain of estimates that rely critically on recent developments in regularity theory of structurally damped wave equations. The lack of compactness of the resolvent operator is one of the sources of difficulties to be contended with.

In fact, global well-posedness and decay rates for equation (1.2) which is homogeneous on the boundary were obtained in [16].

Our main interest in this present work is global (in time) well-posedness theory of Westervelt equation equipped with *nonhomogeneous boundary data*. It is known, that the presence of nonhomogeneous boundary conditions leads to rather subtle analysis, due to the fact that such inputs are modeled by "unbounded operators" which are not defined on the basic state space. (see [28]). This is clearly seen when inspecting "boundary variation of parameters formula" which has been used recently in the context of studying boundary control problems for parabolic (analytic) semigroups (see [6, 22, 28], [2], [8]). Our methods include derivation of such formula for damped wave equation and their use in the context of studying regularity and long time stability of solutions driven by nonhomogeneity on the boundary. It is our belief that the obtained linear results should also be of independent interest in the context of boundary value problems associated with "overdamping". Analyticity and exponential decay rates valid for the semigroup corresponding to strongly damped wave equation play a crucial role in the analysis.

1.1. Main results

The main goal of this paper is to provide results on (**1**) local existence, (**2**) global existence and (**3**) exponential decay rates for the energy of solutions for the Westervelt equation with Dirichlet

$$u = g \quad \text{on } \partial\Omega \tag{1.3}$$

boundary conditions and given initial data $(u(0), u_t(0)) = (u_0, u_1)$.

In order to formulate our results we introduce the following energy functions:

$$E_{u,0}(t) = \frac{1}{2}\left\{|u_t(t)|^2 + |\nabla u(t)|^2\right\}, t \geq 0$$
$$E_{u,1}(t) = \frac{1}{2}\left\{|u_{tt}(t)|^2 + |\nabla u_t(t)|^2 + |\Delta u(t)|^2\right\}, t > 0$$

where $|u| \equiv |u|_{L_2(\Omega)}$. For $t = 0$,

$$E_{u,1}(0) \equiv \frac{1}{2}\left\{|(1-2ku_0)^{-1}[c^2\Delta u_0 + b\Delta u_1 + 2ku_1^2]|^2 + |\nabla u_1(t)|^2 + |\Delta u_0|^2\right\}.$$

In [16] Westervelt equation with *homogeneous* Dirichlet boundary conditions was considered. The main goal of this paper is the treatment of *nonhomogeneous boundary conditions*, with particular emphasis paid to regularity required to be satisfied by the boundary data which obey the following compatibility conditions:

$$g(t=0) \;=\; u_0|_{\partial\Omega}\,, \quad g_t(t=0) \;=\; u_1|_{\partial\Omega}\,. \tag{1.4}$$

We define:

$$X \equiv C(0,T;H^{3/2}(\partial\Omega)) \cap C^1(0,T;H^{1/2}(\partial\Omega)) \cap C^2(0,T;H^{-1/2}(\partial\Omega)) \\ \cap H^2(0,T;H^{1/2}(\partial\Omega)) \cap H^3(0,T;H^{-3/2}(\partial\Omega)). \tag{1.5}$$

Our first result pertains to local existence and uniqueness of solution:

Theorem 1.1. *Let $T > 0$ be arbitrary. There exist $\rho_T, \tilde{\rho}_T > 0$ such that if*

$$E_{u,0}(0) + E_{u,1}(0) \leq \rho_T, \text{ and } g \in X, \|g\|_X^2 \leq \tilde{\rho}_T$$

with the compatibility conditions (1.4)*, then there exists a unique solution* (u, u_t) *solving the Westervelt equation* (1.2) (*in a weak* $H^{-1}(\Omega)$ *sense*) *and such that*

$$u \in C(0,T; H^2(\Omega)) \cap C^1(0,T; H^1(\Omega)) \cap C^2(0,T; L_2(\Omega)) \cap H_2(0,T; H^1(\Omega)).$$

Our next theorem deals with global well-posedness.

Theorem 1.2. *Let* (u_0, u_1, g) *be such that* $E_{u,0}(0) + E_{u,1}(0) + |g|_X < \infty$*. For any* $M > 0$ *there exist* $\rho > 0$*,* $\tilde{\rho} > 0$*, such that solutions corresponding to initial and boundary data with*

$$E_{u,1}(0) \leq \rho \tag{1.6}$$

$$\sup_{t\geq 0} \left[\sum_{l=0}^{2} \left\| \frac{d^l}{dt^l} g(t) \right\|^2_{H^{3/2-l}(\partial\Omega))} + \sum_{l=0}^{1} \int_0^t \left\| \frac{d^{3-l}}{dt^{3-l}} g \right\|^2_{H^{-3/2+2l}(\partial\Omega))} dt \right] \leq \tilde{\rho}$$

exist for all $t > 0$ *and satisfy* $E_{u,1}(t) + E_{u,0}(t) \leq M$ *for all* $t > 0$*.*

Finally, energy decays for the total energy are presented below:

Theorem 1.3. *With Cauchy data* (u_0, u_1, g) *given in Theorem* 1.2 *and such that for all* $t > 0$ *there exists* $\omega_g > 0$

$$\sum_{l=0}^{3} \left\| \frac{d^l}{dt^l} g(t) \right\|^2_{H^{3/2-l}(\partial\Omega)} + \left\| \frac{d^2}{dt^2} g(t) \right\|^2_{H^{1/2}(\partial\Omega)} \leq C_g e^{-\omega_g t}, \tag{1.7}$$

there exists a constant $\omega > 0$ *such that*

$$E_{u,1}(t) + E_{u,0}(t) \leq C_\rho e^{-\omega t}.$$

Remark 1.4. In one space dimension, one can also consider the case without damping, $b = 0$, which is relevant in certain applications, see, e.g., [15].

In this situation we cannot expect global existence but we still get the a local result for small initial and boundary data, see Theorem 2 in [17].

Like in [16], a first step of the proof is to prove energy estimates for a linear, nonautonomous and nonhomogeneous abstract wave equation related to the nonlinear equation (1.2). The obtained results depend on the analyticity and maximal regularity of abstract damped wave equation with nonhomogeneous boundary data. We shall take advantage of abstract version of variation of parameter formula modeling nonhomogeneous boundary data associated with the strong damping. In the case of purely parabolic problems, such variation of parameters formula was critically used in [22, 2]. One of our technical goals is to extend this semigroup approach to models involving strong damping, which result should be of independent interest.

2. Strongly damped abstract wave equation

In what follows we consider a positive selfadjoint operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a suitable Hilbert space. We shall introduce the following notation

$$|u| \equiv |u|_{\mathcal{H}}\,, \quad (u,v) \equiv (u,v)_{\mathcal{H}};\ L_p(Z) \equiv L_p(0,T;Z)\,, \quad C(Z) \equiv C(0,T;Z)\,.$$

We shall impose

Assumption 2.1. The following continuous embeddings hold

$$\begin{aligned} \mathcal{D}(\mathcal{A}^{1/2}) \subset L_2(\Omega)\,, &\quad \text{with } |w| \leq C_0|\mathcal{A}^{1/2}w|\,,\\ \mathcal{D}(\mathcal{A}^{1/2}) \subset L_6(\Omega)\,, &\quad \text{with } |w|_{L_6(\Omega)} \leq C_1|\mathcal{A}^{1/2}w|\,,\\ \mathcal{D}(\mathcal{A}) \subset C(\Omega)\,, &\quad \text{with } |w|_{L_\infty(\Omega)} \leq C_2|\mathcal{A}w|\,. \end{aligned}$$

In our considerations we take $\mathcal{H} = L_2(\Omega)$ where Ω is a bounded smooth domain in $\mathbb{R}^n$, $n = 1,2,3$. In view of the above, the Assumption 2.1 is automatically satisfied with $\mathcal{A} = -\Delta$ defined on the domain $D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega)$. We will be considering first the following non-homogeneous and nonautonomous abstract strongly damped wave equation:

$$\alpha(t)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = f(t) \tag{2.1}$$

with initial conditions $u(0) = u_0 \in \mathcal{D}(\mathcal{A}^{1/2}),\ u_t(0) = u_1 \in \mathcal{H}$. Here $\alpha(t)$ is a positive multiplier on $\mathcal{H} = L_2(\Omega)$, i.e.,

Assumption 2.2. There exist positive constants $0 < \underline{\alpha}, \overline{\alpha} < \infty$ such that $\forall t \in [0,T]$

$$(\alpha(t)u,u) \geq \underline{\alpha}(t)|u|^2 \quad \geq \underline{\alpha}_0|u|^2 \tag{2.2}$$

$$(\alpha(t)u,v)_{L_2(\Omega)} \leq \overline{\alpha}(t)|u||v| \leq \overline{\alpha}_0|u||v|. \tag{2.3}$$

The above conditions amount to say that $\alpha(t) \in M(\mathcal{H})$ and $|\alpha(t)|_{M(\mathcal{H})} \leq \overline{\alpha}(t)$, where the multipliers space $M(\mathcal{H})$ [30] is equipped with the norm

$$|\alpha|_{M(\mathcal{H})} = \sup_{|u|_{\mathcal{H}}=1} |\alpha u|_{\mathcal{H}}.$$

We are interested in studying regularity properties of solutions u, u_t due to the forcing f and initial conditions u_0, u_1.

The analysis will be conducted for $t \in [0,T]$ where T is either finite or infinite time horizon.

Remark 2.1. In reference to the original equation (1.2), the operator $\mathcal{A} = -\Delta$ where Δ is equipped with zero Dirichlet data. Moreover, $\alpha(t) = 1 - 2ku(t)$ and $f(t) = 2ku_t^2$.

To take into account inhomogeneous boundary conditions, we will decompose $u = u^0 + \bar{g}$ with u^0 having to satisfy homogeneous Cauchy (boundary and initial) conditions and $\bar{g}$ will denote an appropriate extension of the Cauchy data, i.e.,

$$\bar{g} = g \qquad \text{on } \partial\Omega\,, \tag{2.4}$$

$$\bar{g}(t=0) = u_0\,, \quad \bar{g}_t(t=0) \ = \ u_1. \tag{2.5}$$

2.1. Regularity properties of damped wave equation

As in [16] we shall begin with a study of the semigroup generated by nondegenerate operator

$$A = \begin{pmatrix} 0 & I \\ -\frac{c^2}{\alpha}\mathcal{A} & -\frac{b}{\alpha}\mathcal{A} \end{pmatrix}$$

with the domain $D(A) = \{(w,v) \in \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2}), c^2 w + bv \in \mathcal{D}(\mathcal{A})\}$ where $\alpha(t) = \alpha > 0$.

In what follows we shall recall regularity properties of the generator $e^{At} \in \mathcal{L}(H)$ where

$$H \equiv \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{H}.$$

It was shown in [9] that e^{At} is an analytic, strongly continuous semigroup defined on H. Moreover, the following characterization of the domain A is known [9]

$$\begin{aligned} D(A^\theta) &= D(\mathcal{A}^{1/2}) \times D(\mathcal{A}^\theta), \qquad \theta \le 1/2 \\ \text{and } D(A^\theta)]' &= D(\mathcal{A}^{1/2}) \times [D(\mathcal{A}^\theta)]', \quad \theta \le 1/2 \end{aligned} \tag{2.6}$$

and the following analytic estimate is available:

$$|A^\theta e^{At}|_{\mathcal{L}(H)} \le C e^{-\omega t} t^{-\theta}, t > 0 \tag{2.7}$$

where the constant ω is positive and depends on c, b, α [9].

In addition, the following regularity holds:

$$\int_0^\infty |A^{1/2} e^{At} x|_H^2 dt \le C|x|_H^2$$

$$LF(\cdot) \equiv \int_0^\cdot A e^{A(\cdot - s)} F(s) ds \in \mathcal{L}(L_2(H))$$

$$\int_0^\infty |LF(t)|_H^2 dt \le C \int_0^\infty |F(t)|_H^2 dt$$

$$\left| \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds \right|_H^2 \le C \int_0^t e^{-2\omega(t-s)} |\mathcal{A}^{-1/2} f(s)|^2 ds, t > 0. \tag{2.8}$$

Remark 2.2. The regularity listed in (2.8) is well known and has been often used [9] on finite time horizon. It is important for the results of this paper the fact that these estimates are uniform in time. Not surprisingly, this results from the fact that the semigroup e^{At} is exponentially decaying. Indeed, the first estimate, on the strength of (2.6) amounts to showing that for $x \in H$ with $Z(t) \equiv e^{At} x = (z(t), z_t(t))$ one has

$$|\mathcal{A}^{1/2} z| \in L_2(0,\infty), \quad |\mathcal{A}^{1/2} z_t| \in L_2(0,\infty).$$

But the first relation follows just from exponential stability of the semigroup, while the second relation follows from standard energy estimates available for the damped wave equation.

Similarly, the second regularity statement (on $(0,\infty)$) follows from a standard Fourier's transform argument after accounting for the fact that the semigroup is analytic and the spectrum of the generator A is in the left complex plane.

The last estimate in (2.8) follows from standard by now Lyapunov function argument [16].

Throughout the rest of this section we will consider time and space dependent coefficient α, and impose Assumptions 2.1, 2.2.

Theorem 2.3 (Theorem 2.1 of [16]). *Consider* (2.1) *with*

1. $\alpha \in L_\infty(M(\mathcal{H})) \cap C^1(\mathcal{H})$.
2. $f \in L_2(\mathcal{H}) \cap H^1([\mathcal{D}(\mathcal{A}^{1/2})]')$
3. $u_0 \in \mathcal{D}(\mathcal{A})$, $u_1 \in \mathcal{D}(\mathcal{A}^{1/2})$, *and* $\frac{1}{\alpha(0)}(f(0) - c^2\mathcal{A}u_0 - b\mathcal{A}u_1) \in \mathcal{H}$.

Then the following energy estimates hold

$$\begin{aligned} E_u(t) + (b - \hat{\epsilon} - \epsilon) \int_0^t |\mathcal{A}^{1/2}u_t(s)|^2 ds \\ \leq E_u(0) + \frac{1}{4\hat{\epsilon}} \int_0^t |\mathcal{A}^{-1/2}f|^2 ds + C_\epsilon |\alpha_t|^4_{C(\mathcal{H})} \int_0^t |u_t(s)|^2 \, ds \,. \end{aligned} \tag{2.9}$$

$$\begin{aligned} E_{u_t}(t) + (b - \hat{\epsilon} - \epsilon) \int_0^t |\mathcal{A}^{1/2}u_{tt}(s)|^2 ds \\ \leq E_{u_t}(0) + \frac{1}{4\hat{\epsilon}} \int_0^t |\mathcal{A}^{-1/2}f_t|^2 ds + C_\epsilon |\alpha_t|^4_{C(\mathcal{H})} \int_0^t |u_{tt}(s)|^2 ds \,, \end{aligned} \tag{2.10}$$

$$\frac{b}{2}|\mathcal{A}u(t)|^2 \leq \frac{b}{2}|\mathcal{A}u_0|^2 + \frac{1}{2c^2}\int_0^t |f|^2 + \frac{\overline{\alpha}_0^2}{2c^2}\int_0^t |u_{tt}|^2 \,, \tag{2.11}$$

with $C_\epsilon = \frac{27}{32}\frac{C_1^6}{\epsilon^3}$ *and*

$$E_u(t) \equiv \frac{1}{2}\left\{(\alpha(t)u_t(t), u_t(t)) + c^2|\mathcal{A}^{1/2}u(t)|^2\right\} .$$

These, by Gronwall's inequality and $|w_t(s)|^2 \leq \frac{1}{\underline{\alpha}_0}E_w(s)$, *for* $w = u$ *and* $w = u_t$ *imply*

$$u \in C(\mathcal{D}(\mathcal{A})), \;\; u_t \in C(\mathcal{D}(\mathcal{A}^{1/2})), u_{tt} \in C(\mathcal{H}) \cap L_2(\mathcal{D}(\mathcal{A}^{1/2})). \tag{2.12}$$

Note that on the strength of Assumption 2.2 we have

$$E_u(t) \sim E_{u,0}(t) \equiv \frac{1}{2}\left\{|u_t(t)|^2 + |\mathcal{A}^{1/2}u(t)|^2\right\} .$$

Moreover, we will repeatedly make use of the simple estimate

$$\begin{aligned} |w|^2_{L_4(\Omega)} &\leq |w|^{3/2}_{L_6(\Omega)}|w|^{1/2} \\ &\leq C_1^{3/2}|\mathcal{A}^{1/2}w|^{3/2}|w|^{1/2} \leq C_1^{3/2}C_0^{1/2}|\mathcal{A}^{1/2}w|^2 \,, \end{aligned} \tag{2.13}$$

following from Assumption 2.1, using Hölder's inequality and Young's inequality.

2.2. Decay rates for the homogeneous equation

We shall address next the issue of decay of the energy for the abstract wave equation (2.1). We will consider decay rates in both lower level energy E_u and higher level energy

$$\mathcal{E}(t) \equiv E_{u_t}(t) + |\mathcal{A}u(t)|^2 .$$

By Assumption 2.2 we have $E_u(t) \sim E_{u,0}(t) \qquad \mathcal{E}(t) \sim E_{u,1}(t)$.

For comparison, we quote the following result Theorem 2.3 from [16] for the autonomous case.

Theorem 2.4. *Let $f = 0$, and $\alpha_t \equiv 0$. Then there exist $\omega, \omega_1 > 0$ such that*

- $E_u(t) \le e^{1-\omega t} E_u(0)$ *with* $\omega = \omega(\alpha, b, c^2, C_0)$
- $\mathcal{E}(t) \le C e^{-\omega_1 t}\mathcal{E}(0)$ *with* $\omega_1 = \min\{\frac{c^2}{b}, \omega\}$ *if* $\frac{c^2}{b} \ne \omega$ *and* $\omega_1 < \omega$ *if* $\frac{c^2}{b} = \omega$.

2.2.1. Energy estimates for the variable coefficient model. Back in the model with time and space dependent α, we are dealing with

$$\alpha(t,\cdot)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = f$$

where $\underline{\alpha}_0 \le \alpha(t,x) \le \overline{\alpha}_0, \quad (t,x) \in \mathbb{R}\times\Omega$ and $\alpha_t \in C(\mathbb{R};\mathcal{H})$. Our goal is to estimate the lower and the higher level energy in terms of f and α.

Lower level energy

Here we can make use of the following result Proposition 2 in [16].

Proposition 2.5.

$$\begin{aligned} E_u(T) + \gamma_0 \int_0^T E_u(s)\,ds + \frac{b}{2}\int_0^T |\mathcal{A}^{1/2}u_t|^2 ds \le (C^1 + C^2)E_u(0) \\ + \int_0^T \Big(C^3|\mathcal{A}^{-1/2}f|^2 + (C^4|\alpha_t|^8_{C(\mathcal{H})} + C^5|\alpha_t|^4_{C(\mathcal{H})})\,|u_t|^2\Big)\,ds \end{aligned} \tag{2.14}$$

for some constants γ_0, $C^1, \dots, C^5 > 0$ depending only on the coefficients c, b and the constants $\overline{\alpha}_0$, C_0, C_1.

Higher level energy

Here we quote Proposition 3 in [16].

Proposition 2.6.

$$\begin{aligned} \mathcal{E}(T) + \hat{b}\int_0^T \Big[|\mathcal{A}u(t)|^2 + |\mathcal{A}^{1/2}u_t|^2 + |\mathcal{A}^{1/2}u_{tt}|^2\Big]\,dt \\ \le C^1\mathcal{E}(0) + \int_0^T \Big[C^2|\mathcal{A}^{-1/2}f_t|^2 + C^3|f|^2 + C^4|\alpha_t|^4_{C(\mathcal{H})}|u_{tt}(t)|^2\Big]\,dt \end{aligned} \tag{2.15}$$

for some constants γ_0, $C^1, \dots, C^4 > 0$ depending only on the coefficients c, b and the constants $\overline{\alpha}_0$, C_0, C_1.

Remark 2.7. The estimate in Proposition 2.6 will allow us to use the so-called "barrier method" ([5, 32, 23, 34] for establishing global existence.

2.3. Extension of nonhomogeneous boundary data to the interior

The purpose of this section is to provide appropriate extensions into the interior of boundary-initial (Cauchy) data. This procedure allows to homogenize nonlinear equation with a given boundary data. More specifically we will be considering classical harmonic (Dirichlet) extensions that would lead to appropriately defined "parabolic" extensions. We begin by defining the harmonic extension operator – the so-called Dirichlet map defined by

$$v = D^{\Delta} g, \text{ iff } \begin{cases} \Delta v = 0, & \text{in } \Omega \\ \quad v = g & \text{on } \partial\Omega \end{cases} \tag{2.16}$$

which for all $s \in R$ is a bounded mapping

$$D^{\Delta} : H^{s}(\partial\Omega) \to H^{s+1/2}(\Omega)\, \forall s \in R. \tag{2.17}$$

Remark 2.8. It is possible to use different extensions of boundary data, for instance via solutions of wave equation with nonhomogeneous boundary data. In fact, this method, along with "hidden" regularity of wave equations [27], was successfully used in [17] for the study of local existence of solutions. However, since we are interested in global behavior, static harmonic extensions are more appropriate.

2.3.1. Weak solutions with nonhomogeneous boundary data. In this section we introduce a "variation of parameter formula" representing weak solution to the nonhomogeneous boundary value problem driven by non-degenerate ($\alpha = 1$) damped wave equation.

This is done with the help of harmonic extension D^{Δ}. The above construct leads naturally to a definition of "parabolic extension" – which is solution to damped wave equation with non-homogeneous boundary data homogenized via harmonic extension. The corresponding results, while used for the study of nonlinear problem, should be also of independent interest in linear theory pertinent to boundary control, or more generally, theory of damped wave equation with nonhomogeneous boundary data. They provide a counterpart of boundary variation of parameter formula used in the past in the context of heat equation with nonhomogeneous boundary data [22, 2, 8, 28].

We consider

$$\begin{aligned} w_{tt} - c^2 \Delta w - b\Delta w_t &= f(t) \quad && \text{in } (0,T) \times \Omega \\ w &= g && \text{on } \partial\Omega \\ w(0) = w_0\,, \quad w_t(0) &= w_1 && \text{in } \Omega. \end{aligned} \tag{2.18}$$

Recalling the definition of D^{Δ} we rewrite (2.18) as follows:

$$w_{tt} - c^2 \Delta(I - D^{\Delta}\gamma) w - b\Delta(I - D^{\Delta}\gamma) w_t = f(t) \quad \text{in } (0,T) \times \Omega \tag{2.19}$$

$$w = g \quad \text{on } \partial\Omega \tag{2.20}$$

where γ denotes the trace operator, i.e., $\gamma u = u|_{\Gamma}$.

Noting that for smooth and compatible Cauchy data g, w_0, w_1, $w - D^{\Delta}\gamma w \in D(\mathcal{A})$, so that $-\Delta(w - D^{\Delta}g) = \mathcal{A}(w - D^{\Delta}g)$, we rewrite (2.19) as the abstract second-order ODE equation:

$$w_{tt} + c^2\mathcal{A}w + b\mathcal{A}w_t = c^2\mathcal{A}D^{\Delta}g + b\mathcal{A}D^{\Delta}g_t + f(t) \tag{2.21}$$

where in splitting the brackets, we admit representation of the equation in the dual space to $\mathcal{D}(\mathcal{A})$. This procedure is standard by now [28, 8] and references therein. What is less standard, however, is rewriting of this second-order equation as a first-order system. While such procedure can be typically accomplished on $[D(A^*)]'$, (see the references cited above) this is not the case for the damped wave equation considered on H with Dirichlet boundary data. The reason for difficulty is the structure of the domain $D(A)$ along with the lack of global smoothing (hence lack of compactness). We also recall that this difficulty can be circumvented when considering the dynamics on $L_2(\Omega) \times L_2(\Omega)$, where A still generates an analytic semigroup, though not a contraction. In fact, [7] provides a variation of constant formula for boundary control acting on damped wave equation with the state space $L_2(\Omega) \times L_2(\Omega)$. However, the construction in [7], based on the analysis in [6], does not apply to our choice of the state space given by H. In fact this remains true even if we were to penalize the state space by negatives powers of A. The problem is purely algebraic (and the same phenomenon was noticed later in [24, 25, 26] where structurally damped problems with Dirichlet boundary actions were studied in the context of optimization and Riccati theory.

To cope with this issue we need to modify the argument given in [7] and we proceed as follows: Denote $Y \equiv \begin{pmatrix} w - D^{\Delta}g \\ w_t \end{pmatrix}$.

With the above notation and after noting that

$$\begin{pmatrix} -D^{\Delta}g_t \\ b\mathcal{A}D^{\Delta}g_t \end{pmatrix} = A \begin{pmatrix} 0 \\ -D^{\Delta}g_t \end{pmatrix} \in [D(A^*)]'$$

we can rewrite (2.21), in the variable $Y = (w - D^{\Delta}g, w_t)$, as a first-order abstract ODE well-defined on $[D(A^*)]'$

$$Y_t(t) = AY + \begin{pmatrix} -D^{\Delta}g_t \\ b\mathcal{A}D^{\Delta}g_t + f \end{pmatrix}.$$

or equivalently

$$Y_t(t) = AY(t) + A \begin{pmatrix} 0 \\ -D^{\Delta}g_t(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

the equation holding on $[D(A^*)]'$. Consequently, variation of parameters (applied in dual spaces to the domain of A^*) yields

$$Y(t) = e^{At}Y(0) + \int_0^t e^{A(t-s)}A \begin{pmatrix} 0 \\ 0 - D^{\Delta}g_t(s) \end{pmatrix} ds + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds\,. \tag{2.22}$$

The above can be rewritten, with the explicit use of the structure of A and denoting $W(t) \equiv (w(t), w_t(t))$ as

$$W(t) = \begin{pmatrix} D^{\Delta} g_t(t) \\ 0 \end{pmatrix} + e^{At} Y(0) - \int_0^t A e^{A(t-s)} \begin{pmatrix} 0 \\ D^{\Delta} g_t(s) \end{pmatrix} ds + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds.$$

The above leads to the following *variation of parameter formula*, which describes weak ("finite energy") solutions to the boundary model:

Lemma 2.9. *Let $g \in C(H^{1/2}(\Gamma))$, $g_t \in C(H^{-1/2+\epsilon}(\Gamma))$, $\gamma w_0 = g(0)$, $w_0 \in H^1(\Omega)$, $w_1 \in L_2(\Omega)$, $f \in L_2(H^{-1}(\Omega))$.*

The following representation holds pointwisely (in t) with values in $H^1(\Omega) \times L_2(\Omega)$:

$$W(t) = \begin{pmatrix} D^{\Delta} g(t) \\ 0 \end{pmatrix} + e^{At} \begin{pmatrix} w_0 - D^{\Delta} g(0) \\ w_1 \end{pmatrix} - \int_0^t A e^{A(t-s)} \begin{pmatrix} 0 \\ D^{\Delta} g_t(s) \end{pmatrix} ds + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds. \tag{2.23}$$

The solution $W \in C(H^1(\Omega) \times L_2(\Omega))$.

We denote by $C(Z)$ the space of continuous functions on $(0, \infty)$ with values in Z. Similarly, $L_2(Z)$ mean norms over R^+ in time.

Remark 2.10. Variation of parameter formula given in (2.23) is a counterpart of a well-known by know "parabolic boundary control formula" which has been used extensively in both linear and non-linear boundary control theory [6, 28, 2, 8, 22] and more specifically in the context of damped wave equation [7, 24]. This formula provides a direct representation of the solution as depending on the boundary input g. Specific feature of this formula is the presence of unbounded operator A in the integrand involving boundary input g. Analyticity of the semigroup with controlled singularity at zero allows to show that the H norm of this integrand is locally integrable, as long as the range of the operator D^{Δ} belongs to some positive Sobolev space. But this is the case with harmonic extension where $R(D^{\Delta}) \subset H^{1/2}(\Omega)$. The proof, written below, provides the details for this argument.

Proof. The formula (2.23) follows directly from (2.22). The original meaning of all quantities is on the dual space $[D(A^*)]'$, see [28]. We only need to justify the validity of the formula taking values in the state space $H^1(\Omega) \times L_2(\Omega)$. For this, we refer to regularity of the damped wave equation that is inherited from the analyticity of the semigroup.

The most critical term is the third term. Here we notice that by the (2.17) $Dg_t \in C(H^{\epsilon}(\Omega)) = C(\mathcal{D}(\mathcal{A}^{\frac{\epsilon}{2}})$, and by (2.6) $\begin{pmatrix} 0 \\ D^{\Delta} g_t \end{pmatrix} \in C(D(A^{\frac{\epsilon}{2}}))$. Hence,

$$\left| A e^{A(t-s)} \begin{pmatrix} 0 \\ D^{\Delta} g_t(s) \end{pmatrix} \right|_H \leq C_g |t-s|^{1-\frac{\epsilon}{2}} e^{-\omega(t-s)} |g_t(s)|_{H^{-1/2+\epsilon}(\partial\Omega)}$$

which implies $H^1(\Omega) \times L_2(\Omega)$ membership of the third term. Regarding the remaining terms, the analysis is similar. It suffices to notice that $w_0 - Dg(0) \in H_0^1(\Omega) = \mathcal{D}(\mathcal{A}^{1/2})$, $Dg \in C(H^1(\Omega))$ and $\begin{pmatrix} 0 \\ f(s) \end{pmatrix} \in L_2([D(A^{*1/2})]')$.

Thus, the conclusion follows from (2.6), (2.8), after noting that long time behavior is controlled by the exponential decay of the semigroup. □

By using the formula above, along with the aforementioned properties of analytic semigroups given in (2.6), (2.8) the following regularity for less regular Cauchy data is easily deduced

Lemma 2.11. *For*

$$g \in L_2(H^{1/2}(\partial\Omega)),\ g_t \in L_2(H^{-1/2}(\partial\Omega)),\ f \in L_2([\mathcal{D}(\mathcal{A})]') \\ w_0 \in H^1(\Omega), \gamma w_0 = g(0), w_1 \in H^{-1}(\Omega) \tag{2.24}$$

we have $W \in L_2(H^1(\Omega) \times L_2(\Omega))$

After these preliminary considerations, we shall proceed with the study of regularity of solutions, assuming that the boundary data and initial data are more regular. Some of the results stated below can be also obtained by energy methods applied to appropriate homogenizations of the problem (see the next section). However, in order to track precisely the regularity, particularly long time regularity, and the effects of compatibility condition, semigroup representations seem more appropriate. Within this framework, one can take advantage of explicit singular estimates and of exponential decays associated with the semigroup.

In order to proceed with our program, we shall exploit several different representations of the semigroup formula (these methods are known since [22]).

Lemma 2.12. *The following representation holds for all Cauchy data as specified in Theorem* 1.1.

$$W(t) = \begin{pmatrix} D^{\Delta}g(t) \\ D^{\Delta}g_t(t) \end{pmatrix} + e^{At}\begin{pmatrix} w_0 - D^{\Delta}g(0) \\ w_1 - D^{\Delta}g_t(0) \end{pmatrix} \tag{2.25}$$
$$- \int_0^t e^{A(t-s)}\begin{pmatrix} 0 \\ D^{\Delta}g_{tt}(s) \end{pmatrix} ds + \int_0^t e^{A(t-s)}\begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds.$$

Proof. The starting point is (2.23). By integration by parts and using semigroup property valid for analytic semigroups $Ae^{At} = \frac{d}{dt}e^{At}$, for $t > 0$ we obtain the representation stated above. Note that the present representation has no singularity in the kernel, but it requires higher differentiability of the boundary data. □

Remark 2.13. Note that $W = \begin{pmatrix} D^{\Delta}g(t) \\ D^{\Delta}g_t(t) \end{pmatrix} + U^0$ where U^0 satisfies damped wave equation driven by the source $f - D^{\Delta}g_{tt}$ with zero boundary conditions and initial data given by $\begin{pmatrix} w_0 - D^{\Delta}g(0) \\ w_1 - D^{\Delta}g_t(0) \end{pmatrix}$.

In order to obtain information on higher-order time derivatives of solutions, differentiation, in the sense of distribution, of formula in (2.12) leads to the following representation of weak solutions.

Lemma 2.14. *With reference to Cauchy data specified in Theorem* 1.1*, the following representation holds.*

$$W_t(t) = \begin{pmatrix} D^{\Delta} g_t(t) \\ -f(t) \end{pmatrix} + Ae^{At} \begin{pmatrix} w_0 - D^{\Delta} g(0) \\ w_1 - D^{\Delta} g_t(0) \end{pmatrix} - \int_0^t Ae^{A(t-s)} \begin{pmatrix} 0 \\ D^{\Delta} g_{tt}(s) \end{pmatrix} ds + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f_t(s) \end{pmatrix} ds + e^{At} \begin{pmatrix} 0 \\ f(t) \end{pmatrix}. \tag{2.26}$$

The formula in Lemma 2.14 is initially derived in the topology of extended spaces [28, 8]. However, taking advantage of analyticity of the semigroup $Ae^{At} \in \mathcal{L}(H), t > 0$, each term in the formula represents a well-defined element in H for $t > 0$. Semigroup formulas in 2.12 and Lemma 2.14, along with the properties (2.8) of analytic semigroup e^{At}, lead to the following – long time behavior – regularity.

Lemma 2.15. *Let* $f = 0$*, and*

$$g \in L_\infty(H^{1/2}(\Gamma))\,, \quad g_t \in L_\infty(H^{1/2}(\Gamma)), \quad g_{tt} \in L_\infty(H^{-1/2}(\Gamma))$$
$$w_0 \in H^1(\Omega)\,, \quad \gamma w_0 = g(0)\,, \quad w_1 \in H^1(\Omega), \quad \gamma w_1 = g_t(0)$$

we have

$$W \in L_\infty(\mathcal{D}(A^{1-\epsilon}) \oplus L_\infty(H^1(\Omega) \times H^1(\Omega)).$$

In particular $w \in L_\infty(H^1(\Omega)), w_t \in L_\infty(H^1(\Omega))$

Proof. Use directly formula in Lemma 2.12 along with the properties enjoyed by the semigroup e^{At} and listed in (2.6), (2.8). Particular emphasis should be paid to exponential decay at infinity and control of singularity at the origin.

Writing $W(t) = I(t) + II(t) + III(t)$, we obtain the following estimates for each term:

$$|A^{1-\epsilon} III(t)|_H \le \int_0^t \frac{e^{-\omega(t-s)}}{(t-s)^{1-\epsilon}} |g_{tt}(s)|_{H^{-1/2}(\partial\Omega)} ds \le C|g_{tt}|_{L_\infty(H^{-1/2}(\partial\Omega))} \tag{2.27}$$

$$\begin{aligned} |A^{1/2} II(t)|_H &\le |A^{1/2} \begin{pmatrix} w_0 - D^{\Delta} g(0) \\ w_1 - D^{\Delta} g_t(0) \end{pmatrix}|_H \\ &\le C|\mathcal{A}^{1/2}(w_0 - D^{\Delta} g(0))| + |\mathcal{A}^{1/2}(w_1 - D^{\Delta} g_t(0))| \\ &\le C[|w_0|_{H^1(\Omega)}| + |w_1|_{H^1(\Omega)} + |g(0)|_{H^{1/2}(\partial\Omega)} + |g_t(0)|_{H^{1/2}(\partial\Omega)} \\ &\le C_{g,w_0,w_1} \end{aligned} \tag{2.28}$$

where we have used compatibility conditions on the boundary.

The identification in (2.6) applied with $\theta = 1/2$ concludes the estimate for the second term $I(t) \in H^1(\Omega) \times H^1(\Omega)$ as long as $g_t(t) \in H^{1/2}(\partial\Omega), g(t) \in H^{1/2}(\partial\Omega)$. $\square$

Lemma 2.16. *Let $f = 0$ and*

$$g \in L_\infty(H^{3/2})(\partial\Omega))\,, \quad g_t \in L_\infty(H^{1/2}(\partial\Omega))),$$
$$g_{tt} \in L_\infty(H^{-1/2}(\partial\Omega)) \cap L_2(H^{1/2}(\partial\Omega)), g_{ttt} \in L_2(H^{-3/2}(\partial\Omega))$$
$$w_0 \in H^2(\Omega)\,, \quad \gamma w_0 = g(0)\,, \quad w_1 \in H^2(\Omega), \quad \gamma w_1 = g_t(0). \tag{2.29}$$

Then

$$W \in L_\infty(H^2(\Omega) \times H^1(\Omega))$$
$$w_{tt} \in L_\infty(L_2(\Omega)) \cap L_2(H^1(\Omega)).$$

Proof. The proof follows by reading off from formula in Lemma (2.12) with a supplement of the following result:

Proposition 2.17. *Let the operator K be defined as follows:*

$$(Kf)(t) \equiv \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds.$$

Let $Z = (z, z_t) \equiv e^{At} Z_0 + Kf$

Then, there exists constants $C_b > 0$ (independent on $t > 0$) and $\omega_b > 0$ such that

$$|\mathcal{A}z(t)|^2 + |\mathcal{A}^{1/2} z_t(t)|^2 + |z_{tt}(t)|^2 + \int_0^t |\mathcal{A}^{1/2} z_{tt}|^2 ds$$
$$\leq C_b e^{-\omega_b t}[|\mathcal{A}z(0)|^2 + |\mathcal{A}^{1/2} z_t(0)|^2 + |f(0) - b\mathcal{A}z_t(0)|^2]$$
$$+ C_b \int_0^t e^{-\omega_b(t-s)}[|\mathcal{A}^{-1/2} f_t(s)|^2 + |f|^2] ds].$$

The constant $\omega_b = \min[\omega, b^{-1}]$.

Proof. The proof of this proposition follows from energy methods reported in theorem 2.3. The important factor is that the constant C does not depend on t. Here are the details. Noting that

$$Z_t(t) = e^{At} Z_t(0) + (Kf_t)(t)$$

the fourth formula in (2.8) implies

$$|Z_t(t)|_H^2 \leq Ce^{-2\omega t}|Z_t(0)|_H^2 + C\int_0^t e^{-2\omega(t-s)}|\mathcal{A}^{-1/2} f_t(s)|^2 ds. \tag{2.30}$$

Hence

$$|\mathcal{A}^{1/2} z_t(t)|^2 + |z_{tt}(t)|^2 \leq Ce^{-2\omega t}[|\mathcal{A}^{1/2} z_t(0)|^2 + |z_{tt}(0)|^2]$$
$$+ C\int_0^t e^{-2\omega(t-s)}|\mathcal{A}^{-1/2} f_t(s)|^2 ds. \tag{2.31}$$

On the other hand, applying the multiplier $\mathcal{A}z$ to the original equation satisfied by z, gives

$$|\mathcal{A}z(t)|^2 \leq C_b e^{-b^{-1}t}|\mathcal{A}z(0)|^2 + C_b \int_0^t e^{-2b^{-1}(t-s)}[|f(s)|^2 + |z_{tt}(s)|^2]ds. \quad (2.32)$$

Combining the estimates in (2.31) and (2.32) leads, after some calculations, to the final conclusion stated in Proposition 2.17. □

To continue with the proof of Lemma 2.16 we write, as before, $W(t) = I(t) + II(t) + III(t)$, where $W(t)$ is given by Lemma 2.12. Applying the proposition with $f = D^\Delta g_{tt}$ and $Z(0) \equiv \begin{pmatrix} w_0 - D^\Delta g(0) \\ w_1 - D^\Delta g_t(0) \end{pmatrix}$ gives the desired result for term $II + III$. We note the use of compatibility conditions which allows to deduce that $Z(0) \in \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A})$, as needed for application of the proposition.

The estimate for term I is straightforward, as it follows from a priori regularity of the boundary data $g(t)$ and regularity of harmonic extension D^Δ. □

Finally, we shall need regularity of the third time derivative of parabolic extension. This is given in the lemma below.

Lemma 2.18. *Let $f = 0$ and (2.29) hold. Then:*

$$w_{ttt} \in L_2(H^{-1}(\Omega)).$$

Proof. We use the formula in Lemma 2.14 and write

$$W_t(t) = I(t) + II(t) + III(t)\,.$$

In order to read off regularity of w_{ttt} we proceed with evaluation of W_{tt}. This leads to calculation of $\frac{d}{dt}III(t) = III_t$ which can be written as:

$$III_t = Ae^{At}\begin{pmatrix} 0 \\ Dg_{tt}(0) \end{pmatrix} + \int_0^t Ae^{A(t-s)}\begin{pmatrix} 0 \\ Dg_{ttt}(s) \end{pmatrix} ds.$$

The assumptions imposed imply $D^\Delta g_{ttt} \in L_2([\mathcal{D}(\mathcal{A}^{1/2}]')$ hence

$$\begin{pmatrix} 0 \\ D^\Delta g_{ttt}(t) \end{pmatrix} \in L_2([\mathcal{D}(A^{1/2}]').$$

Moreover

$$A^{1/2}\begin{pmatrix} 0 \\ D^\Delta g_{tt}(0) \end{pmatrix} \in L_2([\mathcal{D}(A^{1/2}]').$$

This gives

$$III_t \in L_2([D(A^{1/2})]') = L_2(H_0^1(\Omega) \times H^{-1}(\Omega)).$$

As for II_t, we have

$$\begin{pmatrix} w_0 - D^\Delta g(0) \\ w_1 - D^\Delta g_t(0) \end{pmatrix} \in \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A})$$

since each term is in $\mathcal{A}$ (as in the argument given in previous lemma,) so $Z(0) \in D(A)$. This gives

$$AZ(0) = A \begin{pmatrix} w(0) - Dg(0) \\ w_1 - Dg_t(0) \end{pmatrix} \in H$$

hence

$$II_t = Ae^{At}AZ(0) \in L_2[(D(A^{*1/2})]') \in L_2(H_0^1(\Omega)) \times L_2(H^{-1}(\Omega)))$$

where we have used all the properties of fractional powers and characterization of domains associated with strongly damped model.

The analysis for $I_t = \begin{pmatrix} D^\Delta g_{tt} \\ 0 \end{pmatrix}$ is straightforward. Calculations above testify that the second coordinate of W_{tt} belongs to $L_2(H^{-1}(\Omega))$, as desired. $\square$

2.3.2. "Parabolic extension" of Cauchy data. The aim of this section is to introduce "parabolic extension" into the interior of the cylinder $\Omega \times (0,T)$ of Cauchy data g, u_0, u_1 specified in Theorem 1.1. To this end we shall use harmonic extension D^Δ for the boundary data. This leads to the following problem. Consider

$$\begin{gathered} w_{tt} - c^2\Delta w - b\Delta w_t = f(t) \text{ in } (0,T) \times \Omega \\ w = g \text{ on } \partial\Omega, w(0) = w_0\,, \quad w_t(0) = w_1, \text{ in } \Omega. \end{gathered} \tag{2.33}$$

This will lead to an extension

$$\begin{aligned} &\bar{g} = w \text{ according to (2.33) with} \\ &f = 0\,,\ w_0 = u_0\,,\ w_1 = u_1\,. \end{aligned} \tag{2.34}$$

We first state a result using the harmonic extension (2.16).

Theorem 2.19. *Let*

1. $f \in L_2(L_2(\Omega)) \cap H^1(H^{-1}(\Omega))$
2. $g \in C(H^{3/2}(\partial\Omega))$, $g_t \in C(H^{1/2}(\partial\Omega))$, $g_{tt} \in C(H^{-1/2}(\partial\Omega)) \cap L_2(H^{1/2}(\partial\Omega))$, $g_{ttt} \in L_2(H^{-3/2}(\partial\Omega))$.
3. $g(0) = w_0|_{\partial\Omega}$, $w_0 \in H^2(\Omega)$, $g_t(0) = w_1|_{\partial\Omega}$, $w_1 \in H^2(\Omega)$, $f(0) \in L_2(\Omega)$.

Then $w \in C(H^2(\Omega))$, $w_t \in C(H^1(\Omega))$, $w_{tt} \in C(L_2(\Omega)) \cap L_2(H^1(\Omega))$.

Proof. By introducing new variable $u \equiv w - D^\Delta g$ equation (2.33) can be restated as

$$u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u = f(t) + F(t) \tag{2.35}$$

where $F(t) \equiv -D^\Delta g_{tt}$ and initial data becomes $u(0) = w_0 - D^\Delta g(0)$, $u_t(0) = w_1 - D^\Delta g_t(0)$.

We shall apply the result of Theorem 2.3, see also [9].

For this we verify compatibility conditions:

$$\begin{gathered} w_0 - D^\Delta g(0) \in H^2(\Omega) \cap H_0^1(\Omega)\,, \quad w_1 - D^\Delta g_t(0) \in H_0^1(\Omega) \\ f(0) + F(0) - c^2 w_0 - b\Delta w_1 \in L_2(\Omega) \end{gathered}$$

which are satisfied due to the imposed assumptions. Similarly, the conditions 2. of Theorem 2.3 imposed on f and F are also satisfied by the assumptions. The condition on the forcing term becomes

$$f,\ D^{\Delta}g_{tt}\ \in L_2(L_2(\Omega)) \cap H^1(H^{-1}(\Omega))\,.$$

We shall use regularity of the Dirichlet map (2.17). The above translates into the following regularity of g:

$$g_{tt} \in L_2(H^{-1/2}(\partial\Omega))),\ g_{ttt} \in L_2(H^{-3/2}(\partial\Omega)).$$

With the above regularity, on the strength of Theorem 2.3 we obtain:

$$u \in C(H^2(\Omega) \cap H_0^1(\Omega)), u_t \in C(H_0^1(\Omega)), u_{tt} \in C(L_2(\Omega)) \cap L_2(H_0^1(\Omega)). \tag{2.36}$$

This implies the same regularity for the function w, provided that

$$\begin{aligned} &D^{\Delta}g \in C(H^2(\Omega)), D^{\Delta}g_t \in C(H^1(\Omega)) \\ &D^{\Delta}g_{tt} \in C(L_2(\Omega)) \cap L_2(H^1(\Omega)). \end{aligned}$$

Since $D^{\Delta} \in \mathcal{L}(H^s(\partial\Omega) \rightarrow H^{s+1/2}(\Omega))$ for all real s we obtain the same conclusion with g satisfying

$$\begin{aligned} &g \in C(H^{3/2}(\partial\Omega)), g_t \in C(H^{1/2}(\partial\Omega)) \\ &g_{tt} \in C(H^{-1/2}(\partial\Omega)) \cap L_2(H^{1/2}(\partial\Omega))). \end{aligned} \tag{2.37}$$

□

3. Back to the nonlinear problem

3.1. The Westervelt equation with source term

Consider the Westervelt equation

$$(1-2ku)u_{tt} - c^2\Delta u - b\Delta u_t = 2k(u_t)^2 + q\,, \tag{3.1}$$

with zero Dirichlet boundary conditions and given initial conditions:

$$u = 0 \text{ on } \partial\Omega,\ u(t=0) = u_0\,, \quad u_t(t=0) = u_1\,. \tag{3.2}$$

Here $k = \beta_a/(\rho c^2)$, and q plays the role of a given interior source.

We will show results on local and global well-posedness for this problem before we carry out the proofs of Theorems 1.1, 1.2, 1.3.

3.1.1. Local well-posedness. The fixed point operator $\mathcal{T} : W \subseteq X \rightarrow W$ that we will make use of for using Banach's fixed point theorem, will be defined by $\mathcal{T}(v) \equiv u$ with u solving the following linearization of the original equation

$$(1-2kv)u_{tt} - c^2\Delta u - b\Delta u_t = 2kv_tu_t + q\,, \tag{3.3}$$

with boundary conditions $v = 0$ on $\partial\Omega$, and initial conditions $v(t = 0) = u_0$, $v_t(t = 0) = u_1$. Here,

$$\begin{aligned} W = \big\{ & v \in C((0,T)\times\Omega), ||v||_{L_\infty((0,T)\times\Omega)} \le m, \\ & ||\Delta v||_{L_2(L_2(\Omega))} \le \bar a, ||\nabla v_{tt}||_{L_2(L_2(\Omega))} \le \bar a, ||\nabla v_t||_{C(L_2(\Omega))} \le \bar a, \\ & v(0) = u_0, v_t(0) = u_1 \big\} \end{aligned} \tag{3.4}$$

where $m < \frac{1}{2k}$, $\bar a$ are to be chosen appropriately during the course of the proof. The map $\mathcal{T}$ is well defined, a fact that follows from linear analysis.

The main result of this section is the following:

Theorem 3.1. *Let $b > 0$, $m < \frac{1}{4k}$, T arbitrary. We assume that*

$$E_{u,1}(0) \le \rho\,, \quad \|q_t\|^2_{L_2(H^{-1}(\Omega))} + \|q\|^2_{L_2(L_2(\Omega))} \le \tilde\rho$$

with $\bar a$, ρ, $\tilde\rho$ sufficiently small (but possibly depending on T).

Then there exists a solution $u \in W$ of (3.1), (3.2), which is unique in W and satisfies $\Delta u,\ u_{tt},\ \nabla u_t \in C(0,T;L_2(\Omega))\,, \quad \nabla u_{tt} \in L_2(0,T;L_2(\Omega))\,.$

Proof. In order to show that the map $\mathcal{T}$ is a self-mapping and contraction on W with suitable parameters m, $\bar a$, we make use of Theorem 2.3 with

$$\alpha(t,x) \equiv 1 - 2kv(t,x), \tag{3.5}$$

$$f(t,x) \equiv 2kv_t(t,x)u_t(t,x) + q(t,x). \tag{3.6}$$

Note that Assumption 2.1 is obviously satisfied for $\mathcal{A} = -\Delta$ with homogeneous Dirichlet boundary conditions, due to Poincaré's inequality, H^2 regularity of solutions to the Laplace equation with L_2 right-hand side, and continuity of the embeddings $H^1(\Omega) \to L_6(\Omega)$, $H^2(\Omega) \to C(\Omega)$. Moreover, we have $\mathcal{D}(\mathcal{A}^{1/2}) \subseteq H^1(\Omega)$, $\mathcal{D}(\mathcal{A}) \subseteq H^2(\Omega)$, with $|\mathcal{A}^{1/2}w| = |\nabla w|$, $|\mathcal{A}w| = |\Delta w|$. Additionally we have, $\mathcal{D}(\mathcal{A}) \subset W^1_6(\Omega)\,,$ with $|\nabla w|_{L_6(\Omega)} \le \hat C_1 |\Delta w|$.

Step 1. We will first show that for $v \in W$ the functions α, f according to (3.5) enjoy the regularity required in Theorem 2.3, i.e., $\alpha \in C^1(\mathcal{H})$ and, according to Assumption 2.2, $0 < \underline{\alpha}_0 \le \alpha(t,x) \le \overline{\alpha}_0$, so we deal with the non-degenerate case. Moreover, according to the assumptions of Theorem 2.3 we have to show $f \in L_2(\mathcal{H}) \cap H^1([\mathcal{D}(\mathcal{A}^{1/2})]')$. Note that 3. of the assumptions of Theorem 2.3 follows from $E_{u,1}(0) \le \rho$. We recall that $\mathcal{H} = L_2(\Omega)$. For this purpose we can make use of Proposition 4 in [16] and augment it in a straightforward way by the source term:

Proposition 3.2. *Let $v \in W$ and α, f be defined above by equation (3.5) and assume that $km \le 1/4$. Then*

- $\overline{\alpha}_0 = 3/2 \ge \alpha(t,x) \ge 1/2 = \underline{\alpha}_0$
- $|\alpha_t|_{C(\mathcal{H})} \le 2C_0 k\bar a$
- $\int_0^T |\mathcal{A}^{-1/2} f_t|^2 dt \le 16 C_1^3 C_0 k^2 \bar a^2 (|u_{tt}|^2_{C(\mathcal{H})} + |\mathcal{A}^{1/2} u_t|^2_{L_2(\mathcal{H})}) + 2\|q_t\|^2_{L_2(H^{-1}(\Omega))}$
- $\int_0^t |f|^2 ds \le 8 C_1^3 C_0 k^2 \bar a^2 |u_t|^2_{L_2(H^1)} + 2\|q\|^2_{L_2(L_2(\Omega))}.$

Note that on the strength of Assumption 2.1 and by definition of $\mathcal{A}$ we have

$$|u|^2_{L_\infty((0,T)\times\Omega)} \le C_2^2|\mathcal{A}u|^2_{C(\mathcal{H})}; \qquad |\Delta u|^2_{L_2(\mathcal{H})} \le T|\mathcal{A}u|^2_{C(\mathcal{H})}$$
$$|\nabla u_{tt}|^2_{L_2(\mathcal{H})} \le |\mathcal{A}^{1/2}u_{tt}|^2_{L_2(\mathcal{H})}; \quad |\nabla u_t|^2_{C(\mathcal{H})} \le \frac{2}{c^2}|E_{u_t}|_{C(0,T)}\,. \tag{3.7}$$

Step 2. To obtain the bounds m, $\bar{a}$ required for $u \in W$, we will now make use of estimates (2.10), (2.11) from Theorem 2.3 together with Proposition 3.2.

Applying the estimate (2.10) with $\hat{\epsilon} = \epsilon = \frac{b}{4}$ gives:

$$\begin{aligned}
&E_{u_t}(t) + \frac{b}{2}\int_0^t |\mathcal{A}^{1/2}u_{tt}(s)|^2 ds \\
&\quad\le \left(E_{u_t}(0) + \frac{16C_1^3C_0k^2\bar{a}^2}{b}(|u_{tt}|^2_{C(\mathcal{H})} + |\mathcal{A}u_t|^2_{L_2(\mathcal{H})}) + \frac{2}{b}\|q_t\|^2_{L_2(H^{-1}(\Omega))}\right) \\
&\quad\times e^{C_{b/4}t(2C_0k\bar{a})^4}
\end{aligned} \tag{3.8}$$

which yields

$$\begin{aligned}
&\max\{|E_{u_t}|_{C(0,T)}, \frac{b}{2}|\mathcal{A}^{1/2}u_{tt}(t)|_{L_2(\mathcal{H})}\} \\
&\quad\le \left(E_{u_t}(0) + \frac{2}{b}\|q_t\|^2_{L_2(H^{-1}(\Omega))} + \phi(\bar{a})|E_{u_t}|_{C(0,T)}\right) e^{C_{b/4}T(2C_0k\bar{a})^4}\,,
\end{aligned} \tag{3.9}$$

where

$$\phi(\bar{a}) = \frac{16C_1^3C_0k^2\bar{a}^2}{b}\max\left\{\frac{2}{\underline{\alpha}_0}, \frac{2T}{c^2}\right\}.$$

With $\bar{a}$ sufficiently small so that $\phi(\bar{a}) \le \frac{1}{2}$, and ρ, $\tilde{\rho}$ sufficiently small so that

$$\max\left\{\frac{2}{c^2}, \frac{2}{b}\right\}\left(\rho + \frac{2}{b}\tilde{\rho}\right) e^{C_{b/4}T(2C_0k\bar{a})^4} \le \bar{a}^2\,,$$

we get by (3.7) $|u_{tt}|_{L_2(H^1)} \le \bar{a}$, $|u_t|_{C(H^1)} \le \bar{a}$ as desired, and additionally

$$|u_{tt}|^2_{C(\mathcal{H})} \le \frac{2}{\underline{\alpha}_0}c^2\bar{a}^2\,. \tag{3.10}$$

Applying the estimate (2.11) gives:

$$\begin{aligned}
b|\mathcal{A}u(t)|^2 &\le b|\mathcal{A}u(0)|^2 + \frac{8C_1^3C_0k^2\bar{a}^2}{c^2}|u_t|^2_{L_2(H^1(\Omega 0)} + \frac{1}{c^2}\|q\|^2_{L_2(L_2(\Omega))} \\
&\quad + \frac{\overline{\alpha}_0^2C_0^2}{2c^2}\int_0^t |\mathcal{A}^{1/2}u_{tt}|^2 \le b\rho + \frac{1}{c^2}\tilde{\rho} + \frac{8C_1^3C_0k^2\bar{a}^2}{c^2}T\bar{a}^2 + \frac{\overline{\alpha}_0^2C_0^2}{2c^2}\bar{a}^2\,,
\end{aligned}$$

so that by possibly decreasing ρ, $\tilde{\rho}$, and $\bar{a}$ using (3.7) we can guarantee $|u|_{L_\infty((0,T)\times\Omega)} \le m$, hence the final conclusion $\mathcal{T}W \subset W$ follows.

Step 3. The proof of contractivity of $\mathcal{T}$ is exactly the same as without source term, see [16]. $\square$

3.1.2. Global well-posedness. We shall exploit the barrier's method, typically used in quasilinear hyperbolic problems [5, 32, 34, 23].

Theorem 3.3. *For sufficiently small initial data and source term the solutions are global in time. Moreover, the size of initial data does not depend on time. This is to say: For any given $M > 0$ there exist $\rho > 0$ and $\tilde{\rho} > 0$ such that if*

$$\mathcal{E}(0) \le \rho \tag{3.11}$$

and

$$\|q_t\|^2_{L_2(0,\infty;H^{-1}(\Omega))} + \|q\|^2_{L_2(0,\infty;L_2(\Omega))} \le \tilde{\rho}, \tag{3.12}$$

then

$$\mathcal{E}(t) \le M \text{ for all } t \in \mathbb{R}^+.$$

Proof. Let u be a local solution that exists for sufficiently small data by Theorem 3.1. For this solution, we apply Proposition 2.6 with

$$\alpha \equiv 1 - 2ku\,, \quad f \equiv 2ku_t^2 + q \tag{3.13}$$

(see Proposition 6 in [16]) for which the following estimates can be obtained

$$\int_0^T |\alpha_t(t)|^4 |u_{tt}(t)|^2 dt \le 16 C_0^6 k^4 \int_0^T |\mathcal{A}^{1/2} u_t(t)|^4 |\mathcal{A}^{1/2} u_{tt}(t)|^2 dt$$

$$\int_0^T |f(t)|^2 dt \le 8 C_1^3 C_0^2 k^2 \int_0^T |\mathcal{A}^{1/2} u_t(t)|^4 dt + 2 \int_0^T |q(t)|^2 dt$$

$$\int_0^T |\mathcal{A}^{-1/2} f_t(t)|^2 dt \le 32 C_1^3 C_0^3 k^2 \int_0^T |\mathcal{A}^{1/2} u_{tt}(t)|^2 |\mathcal{A}^{1/2} u_t(t)|^2 dt$$
$$+ 2 \int_0^T |\mathcal{A}^{-1/2} q_t(t)|^2 dt.$$

Therewith, Proposition 2.6 yields

$$\mathcal{E}_0(T) + (\alpha(T) u_{tt}(T), u_{tt}(T)) + \tilde{b} \int_0^T (\mathcal{E}_0(t) + |\mathcal{A}^{1/2} u_{tt}(t)|^2)(1 - \Phi(u, u_t)(t)) dt$$
$$\le C^1 \mathcal{E}(0) + 2 \int_0^T \Big(C^2 |\mathcal{A}^{-1/2} q_t(t)|^2 + C^3 |q(t)|^2 \Big) dt \tag{3.14}$$

where $\tilde{b} = \min\{1, 2/c^2\} \hat{b}$,

$$\mathcal{E}_0(t) = \frac{1}{2} c^2 |\mathcal{A}^{1/2} u_t(t)|^2 + |\mathcal{A} u(t)|^2\,,$$

and

$$\mathcal{E}(t) = \mathcal{E}_0(t) + \frac{1}{2} (\alpha(t) u_{tt}(t), u_{tt}(t))\,,$$

and

$$\Phi(u, u_t)(t) \le \tilde{C} (\mathcal{E}_0(t) + \mathcal{E}_0(t)^2) \tag{3.15}$$

(see the proof of Theorem 3.2 in [16]).

Based on estimate (3.14), we now apply barrier's method, i.e., we assume that there exists some time when degeneration occurs in α or in $(1 - \Phi(u, u_t))$.

Let T_0 be the first such time instance, i.e., the first time when either $\alpha(T_0) = 0$ or $\Phi(u, u_t)(T_0) = 1$, which implies that either

$$2C_2k|\mathcal{A}u(T_0)| \geq 1 \tag{3.16}$$

(see (3.13) and Assumption 2.1) or

$$\tilde{C}(\mathcal{E}_0(T_0) + \mathcal{E}_0(T_0)^2) \geq 1 \tag{3.17}$$

(see (3.15)).

On the other hand, we can take $\rho, \tilde{\rho}$ in (3.11), (3.12) sufficiently small so that with $\hat{M} = C_1\rho + 2\max\{C^2, C^3\}\tilde{\rho}$

$$2C_2k\sqrt{\hat{M}} < 1 \quad \text{ and } \tilde{C}(\hat{M} + \hat{M}^2) < 1\,, \tag{3.18}$$

and apply (3.14), which holds up to $t = T_0$ and yields $|\mathcal{A}u(T_0)|^2 \leq \mathcal{E}_0(T_0) \leq \hat{M}$. Therewith

$$2C_2k|\mathcal{A}u(T_0)| \leq 2C_2k\sqrt{\hat{M}} < 1$$

and

$$\tilde{C}(\mathcal{E}_0(T_0) + \mathcal{E}_0(T_0)^2) \leq \tilde{C}(\hat{M} + \hat{M}^2) < 1$$

a contradiction to (3.16), (3.17).

Note that since $C_1, C_2, C^2, C^3, \tilde{C}$ are intrinsic constants independent of T, for any given M, we can choose $\hat{M}$ such that $\hat{M} \leq M$ and the inequalities (3.18) are satisfied. Then we choose $\rho, \tilde{\rho}$ such that $C_1\rho + 2\max\{C^2, C^3\}\tilde{\rho} \leq \hat{M}$. With these $\rho, \tilde{\rho}$ (independent of T) in (3.11), (3.12) we obtain the global energy estimate $\mathcal{E}(t) \leq \hat{M} \leq M$ for all $t \in \mathbb{R}^+$

□

3.1.3. Decay rates.

Theorem 3.4. *We assume that the initial data and source term are sufficiently small such that* (3.11), (3.12) *holds. Additionally, we assume that the source term decays exponentially to zero*

$$\forall 0 \leq s < T \ : \quad \int_s^T \left(|q_t(t)|^2_{H^{-1}(\Omega)} + |q(t)|^2_{L_2(\Omega))}\right) d\tau \leq C_q e^{-\omega_q s}\,. \tag{3.19}$$

Then the energy decays exponentially fast to zero.

$$\mathcal{E}(t) \leq Ce^{-\omega t}\mathcal{E}(0) \tag{3.20}$$

where $0 < \omega < \min\{\frac{\bar{b}}{C^1}, \omega_q\}$ *with* $\bar{b} = \min\{1, 4/C_0^2\}\min\{1, 2/c^2\}\hat{b}$*, and* $\hat{b}$*,* C^1 *as in* (2.15)*.*

Note that (3.19) follows from

$$\forall t \geq 0 : \quad |q_t(t)|^2_{H^{-1}(\Omega)} + |q(t)|^2_{L_2(\Omega))} \leq \omega_q C_q e^{-\omega_q t}\,.$$

Proof. Using (3.14) with $\rho, \tilde{\rho}$ in (3.11), (3.12) sufficiently small so that

$$2C_2k\sqrt{C_1\rho + 2\tilde{\rho}} \leq 1/2$$

we get $1/2 \leq \alpha(t,x) \leq 3/2$, hence

$$\mathcal{E}(t) \sim \mathcal{E}_0(t) + |u_{tt}|^2\,,\ \mathcal{E}(t) \leq 2\max\{C^2, C^3\}\tilde{\rho}\,.$$

Moreover, with

$$|\mathcal{A}^{1/2}u_{tt}|^2 \geq \frac{1}{C_0}|u_{tt}|^2\,, \tag{3.21}$$

and using the fact that in the proof of (3.14), $t = 0$ can obviously be replaced by $t = s$, we get (see the proof of Theorem 3.3 in [16])

$$\mathcal{E}(T) + \bar{b}\int_s^T \mathcal{E}(\tau)\,d\tau \leq C^1\mathcal{E}(s) + 2C_q\max\{C^2, C^3\}e^{-\omega_q s} \tag{3.22}$$

for any $s < T$.

We consider $\tilde{\mathcal{E}}(\tau) = \mathcal{E}(\tau) + \lambda e^{-\omega_q\tau}$ with $\lambda = 2C_q\max\{C^2, C^3\}/(C^1\omega_q - \bar{b})$ and assume w.l.o.g. that $C^1 \geq 1$ and $\omega_q > \frac{\bar{b}}{C^1}$ so that $\lambda > 0$. (If $\omega_q \leq \frac{\bar{b}}{C^1}$ or $C^1 < 1$, we can replace C^1 in (3.22) by $\tilde{C}^1 > \max\{\frac{\bar{b}}{\omega_q}, 1\} \geq C^1$.)

Therewith, we get $\bar{b}\int_s^T \tilde{\mathcal{E}}(\tau)\,d\tau \leq C^1\tilde{\mathcal{E}}(s)$ so that we can apply a standard semigroup argument (see, [31] or, e.g., Theorem 8.1 in [20]) to obtain $\tilde{\mathcal{E}}(t) \leq \tilde{\mathcal{E}}(0)e^{-\frac{\bar{b}}{C^1}t}$ and therewith (3.20). $\square$

3.2. The Westervelt equation with nonhomogeneous Dirichlet boundary data

We return to the Westervelt equation with zero source term

$$(1 - 2ku)u_{tt} - c^2\Delta u - b\Delta u_t = 2k(u_t)^2\,, \tag{3.23}$$

but nonhomogeneous Dirichlet boundary conditions

$$u = g \text{ on } (0,T) \times \partial\Omega \tag{3.24}$$

and initial conditions

$$u(t=0) \ = \ u_0\,, \quad u_t(t=0) \ = \ u_1\,. \tag{3.25}$$

For this purpose we transform to a problem with homogeneous boundary and initial conditions for u^0 in the decomposition $u = u^0 + \bar{g}$ with $\bar{g}$ being selected as the extension of boundary data given by (2.4).

Substituting u into (3.23) gives

$$\begin{aligned} (1 - 2ku)u^0_{tt} - c^2\Delta u^0 - b\Delta u^0_t &= 2ku_tu^0_t + q \\ u &= 0, \text{ on } (0,T) \times \partial\Omega \\ u^0(0) = 0\,, \quad u^0_t(0) &= 0, \text{ in } \Omega. \end{aligned} \tag{3.26}$$

where the "forcing" q is given by

$$q \equiv 2ku\bar{g}_{tt} + 2ku_t\bar{g}_t.$$

Thus, we are looking for solution $u = u^0 + \bar{g}$ such that u^0 satisfies (3.26) with zero Cauchy (boundary and initial) data. The above leads to a fixed point formulation

$$\mathcal{T}_1 : W \subset X \to W, \text{ with } W \text{ given by (3.4)}$$

and $\mathcal{T}_1(v) \equiv u$, where $u = u^0 + \bar{g}$ with u^0 given by

$$\begin{aligned} (1-2kv)u^0_{tt} - c^2\Delta u^0 - b\Delta u^0_t &= 2kv_t u^0_t + q_v \\ u^0 &= 0 \text{ on } (0,T)\times\partial\Omega \\ u^0(0) = 0\,, \quad u^0_t(0) &= 0\,. \end{aligned} \tag{3.27}$$

and

$$q_v \equiv 2kv\bar{g}_{tt} + 2kv_t\bar{g}_t. \tag{3.28}$$

Since now $v \in W$ does not satisfy homogeneous boundary conditions, we will have to use the following estimates following from Assumption 2.1 and the decomposition $w = (w - D^\Delta\gamma w) + D^\Delta\gamma w$, where $w - D^\Delta\gamma w$ satisfies homogeneous Dirichlet boundary conditions.

$$H^1(\Omega) \subset L_2(\Omega)\,, \quad \text{with } |w| \le \tilde{C}_0(|\nabla w| + |\gamma\tilde{w}|_{H^{1/2}(\partial\Omega)})\,, \tag{3.29}$$

$$\text{and } |\nabla w| \le \tilde{\tilde{C}}_0(|\Delta w| + |\gamma\tilde{w}|_{H^{3/2}(\partial\Omega)})\,, \tag{3.30}$$

$$H^1(\Omega) \subset L_6(\Omega)\,, \quad \text{with } |w|_{L_6(\Omega)} \le \tilde{C}_1(|\nabla w| + |\gamma\tilde{w}|_{H^{1/2}(\partial\Omega)})\,, \tag{3.31}$$

$$\text{and } |\nabla w|_{L_6(\Omega)} \le \tilde{\tilde{C}}_1(|\Delta w| + |\gamma\tilde{w}|_{H^{3/2}(\partial\Omega)})\,, \tag{3.32}$$

$$H^2(\Omega) \subset C(\Omega)\,, \quad \text{with } |w|_{L_\infty(\Omega)} \le \tilde{C}_2(|\Delta w| + |\gamma\tilde{w}|_{H^{3/2}(\partial\Omega)})\,. \tag{3.33}$$

3.2.1. Local well-posedness: Proof of Theorem 1.1. In the fixed point proof of local existence we will fall back to the case of the nonlinear problem with the source and homogeneous boundary data.

To see this, we notice that $u^0 = \mathcal{T}v$ where $\mathcal{T}v$ is defined by (3.3) with q replaced by q_v and zero Cauchy data. This leads to the following description of the action of the map $\mathcal{T}_1$:

$$\mathcal{T}_1 v = \mathcal{T}v + \bar{g}$$

where $\bar{g}$ is given by (2.34) and q_v is given by

$$q_v \equiv 2kv\bar{g}_{tt} + 2kv_t\bar{g}_t. \tag{3.34}$$

To formulate a smallness assumption on the boundary data, we will estimate $\|q_{vt}\|^2_{L_2(H^{-1}(\Omega))} + \|q_v\|^2_{L_2(L_2(\Omega))}$ according to (3.34) by some quantity $C(\bar{g}, m, \bar{a}) = C^D(\bar{g}, m, \bar{a})\}$. Then we will use the extension theorems from Section 2.3 to estimate $C^D(\bar{g}, m, \bar{a})$ from above in terms of appropriate norms of g.

For this, we use the following estimate:

Proposition 3.5.

$$|\mathcal{A}^{-1/2}[v\bar{g}_{ttt}]|(t) \le (b|\nabla\bar{g}_{tt}(t)| + c^2|\nabla\bar{g}_t(t)|)(|v(t)|_{L_\infty(\Omega)} + C_1|\nabla v(t)|_{L_3(\Omega)}).$$

Proof. Since we do not have an estimate of $\bar{g}_{ttt}$, we use the respective PDEs to express two time derivatives via space derivatives, i.e., for any $\phi \in L_2(\Omega)$ with $|\phi| = 1$, we evaluate

$$(\mathcal{A}^{-1/2}(v\bar{g}_{ttt})(t), \phi) = (\bar{g}_{ttt}(t), v(t)\mathcal{A}^{-1/2}\phi) = (c^2\Delta\bar{g}_t + b\Delta\bar{g}_{tt}, v\mathcal{A}^{-1/2}\phi).$$

Since $\mathcal{A}^{-1/2}\phi \in H_0^1(\Omega)$, we obtain

$$\begin{aligned}(\Delta\bar{g}_{tt}, v\mathcal{A}^{-1/2}\phi) &= (\nabla\bar{g}_{tt}, v\nabla\mathcal{A}^{-1/2}\phi + \nabla v\mathcal{A}^{-1/2}\phi) \\ &\le |\nabla\bar{g}_{tt}(t)|(|v(t)|_{L_\infty(\Omega)}|\phi| + |\nabla v(t)|_{L_3(\Omega)}|\mathcal{A}^{-1/2}\phi|_{L_6(\Omega)}).\end{aligned}$$

A similar argument applies to the term $\Delta\bar{g}_t$. □

The following lemma provides the estimate of $||\nabla v||_{C(L_3(\Omega)}$ using the information $||\Delta v||_{L_2(L_2(\Omega))} \le \bar{a}$, $||\nabla v_{tt}||_{L_2(L_2(\Omega))} \le \bar{a}$, $||\nabla v_t||_{C(L_2(\Omega))} \le \bar{a}$, see (3.4).

Lemma 3.6. *For any $\phi \in L_2(L_6(\Omega)) \cap H^1(L_2(\Omega))$ with $\phi(0) \in L_3(\Omega)$ we have $\phi \in C(L_3(\Omega))$ with*

$$||\phi||_{C(L_3(\Omega))} \le \left(3C_4^2\|\phi\|^{1/2}_{H^1(L_2(\Omega))}\|\phi\|^{3/2}_{L_2(L_6(\Omega))}\|\phi_t\|_{L_2(L_2(\Omega))} + |\phi(0)|^3_{L_3(\Omega)}\right)^{1/3},$$

where C_4 is the norm of the continuous embedding $H^{1/4}(0,T) \to L_4(0,T)$

Proof.

$$\begin{aligned}|\phi(t)|^3_{L_3(\Omega)} &= \int_0^t \frac{d}{dt}\int_\Omega |\phi(s,x)|^3 dx\, ds + |\phi(0)|^3_{L_3(\Omega)} \\ &= 3\int_0^t\int_\Omega |\phi(s,x)|^2 \text{sign}(\phi(s,x))\phi_t(s,x)\, dx\, ds + |\phi(0)|^3_{L_3(\Omega)} \\ &\le 3\int_0^t |\phi(s)|^2_{L_4(\Omega)}|\phi_t(s)|_{L_2(\Omega)}\, ds + |\phi(0)|^3_{L_3(\Omega)} \\ &\le 3\|\phi\|^2_{L_4(L_4(\Omega))}\|\phi_t\|_{L_2(L_2(\Omega))} + |\phi(0)|^3_{L_3(\Omega)} \\ &\le 3C_4^2\|\phi\|^2_{H^{1/4}(L_4(\Omega))}\|\phi_t\|_{L_2(L_2(\Omega))} + |\phi(0)|^3_{L_3(\Omega)} \\ &\le 3C_4^2\|\phi\|^{1/2}_{H^1(L_2(\Omega))}\|\phi\|^{3/2}_{L_2(L_6(\Omega))}\|\phi_t\|_{L_2(L_2(\Omega))} + |\phi(0)|^3_{L_3(\Omega)}\end{aligned}$$

where we have used standard interpolation in the last inequality:

$$||\phi||^2_{H^{1/4}(L_4(\Omega))} \le C||\phi||^{1/2}_{H^1(L_2(\Omega))}||\phi||^{3/2}_{L_2(L_6(\Omega))}$$ □

Therewith we can give the following estimates of the $\bar{g}$ terms arising from homogenization:

Proposition 3.7. *For q_v given by (3.34) with $v \in W$ (see (3.4)) the estimate*

$$\|q_{vt}\|^2_{L_2(H^{-1}(\Omega))} + \|q_v\|^2_{L_2(L_2(\Omega))} \le C^*(\bar{g}, m, \bar{a})$$

holds with

$$
\begin{aligned}
C^*(\bar g, m, \bar a) &= C^D(\bar g, m, \bar a) \qquad (3.35)\\
&= 2k(m+\theta(T,m,\bar a,g,u_0))(b\|\nabla \bar g_{tt}\|_{L_2(L_2(\Omega))} + c^2\|\nabla \bar g_t\|_{L_2(L_2(\Omega))})\\
&\quad + 4k\tilde C_1(\bar a + \|g_t\|_{C(H^{1/2}(\partial\Omega))})\|\bar g_{tt}\|_{L_2(L_{3/2}(\Omega))}\\
&\quad + 2k\tilde C_1(\bar a + \|g_{tt}\|_{L_2(H^{1/2}(\partial\Omega))})\|\bar g_t\|_{L_\infty(L_{3/2}(\Omega))}\\
&\quad + 2km\|\bar g_{tt}\|_{L_2(L_2(\Omega))} + 2k\tilde C_1(\bar a + \|g_t\|_{C(H^{1/2}(\partial\Omega))})\|\bar g_t\|_{L_2(L_3(\Omega))}
\end{aligned}
$$

where

$$
\begin{aligned}
\theta(T,m,\bar a,g,u_0)^3 \equiv |\nabla u_0|^3_{L_3(\Omega)} + 3C_4^2\left\{\tilde{\hat C}{}_0^2(\bar a + \|g\|_{L_2(H^{3/2}(\partial\Omega))})^2 + T\bar a^2\right\}^{1/4}\\
\times \tilde{\hat C}{}_1^{3/2}(\bar a + \|g\|_{L_2(H^{3/2}(\partial\Omega))})^{3/2}\sqrt{T}\bar a.
\end{aligned}
\qquad (3.36)
$$

Proof. We use Proposition 3.5 and Lemma 3.6, which together with $\|\psi\|_{L_2(0,T)} \leq T^{1/2}\|\psi\|_{C(0,T)}$ yields

$$||\nabla v||_{C(L_3(\Omega))} \leq \theta(T,m,\bar a,g,u_0)$$

and the estimate $|ab|_{H^{-1}(\Omega)} \leq |ab|_{L_{6/5}(\Omega))} \leq |a|_{L_6(\Omega)}|b|_{L_{3/2}(\Omega)}$ that follows from duality and continuity of the embedding $H^1(\Omega) \to L_6(\Omega)$ to obtain

$$
\begin{aligned}
\|q_{vt}\|_{L_2(H^{-1}(\Omega))} &= 2k\|v\bar g_{ttt} + 2v_t\bar g_{tt} + v_{tt}\bar g_t\|_{L_2(H^{-1}(\Omega))}\\
&\leq 2k(m+\theta(T,m,\bar a,g,u_0))(b\|\nabla \bar g_{tt}\|_{L_2(L_2(\Omega))} + c^2\|\nabla \bar g_t\|_{L_2(L_2(\Omega))})\\
&\quad + 4k\tilde C_1(\bar a + \|g_t\|_{C(H^{1/2}(\partial\Omega))})\|\bar g_{tt}\|_{L_2(L_{3/2}(\Omega))}\\
&\quad + 2k\tilde C_1(\bar a + \|g_{tt}\|_{L_2(H^{1/2}(\partial\Omega))})\|\bar g_t\|_{L_\infty(L_{3/2}(\Omega))}\,,
\end{aligned}
$$

where we have used Proposition 3.5 and Lemma 3.6, as well as

$$
\begin{aligned}
\|q_v\|_{L_2(L_2(\Omega))} &= 2k\|v\bar g_{tt} + v_t\bar g_t\|_{L_2(L_2(\Omega))}\\
&\leq 2km\|\bar g_{tt}\|_{L_2(L_2(\Omega))} + 2k\tilde C_1(\bar a + \|g_t\|_{C(H^{1/2}(\partial\Omega))})\|\bar g_t\|_{L_2(L_3(\Omega))}.
\end{aligned}
\qquad \square
$$

It only remains to combine Proposition 3.7 with the estimates according to Section 2.3, which under condition (1.4) gives

$$
\begin{aligned}
&\|q_{vt}\|^2_{L_2(H^{-1}(\Omega))} + \|q_v\|^2_{L_2(L_2(\Omega))} \qquad (3.37)\\
&\leq \left(\check C(T,m,\bar a,u_0) + \check{\check C}(\|g\|_{L_2(H^{3/2}(\partial\Omega))} + \|g_t\|_{C(H^{1/2}(\partial\Omega))} + \|g_{tt}\|_{L_2(H^{1/2}(\partial\Omega))})\right)
\end{aligned}
$$

with X according to (1.5), and a constant $\check C(T,m,\bar a,u_0)$ independent of g and small for small m, $\bar a$, $|\nabla u_0|_{L_3(\Omega)}$, as well as a constant $\check{\check C}$.

Since when evaluating $\mathcal{T}_1 v$, we add $\bar g$ to u^0, we additionally need smallness of $\|g\|_X$ (cf., (2.37)).

Therewith, along the lines of the proof of Theorem 3.1 (note that the proof of contractivity of $\mathcal{T}$ is exactly the same as in case of homogeneous Dirichlet boundary conditions, see [16]) we arrive at the following result:

Theorem 3.8. *Let $b > 0$, $m < \frac{1}{4k}$, T arbitrary. We assume that $E_{u,0}(0) < \infty$,*

$$E_{u,1}(0) \le \rho\,, \quad \|g\|_X^2 \le \tilde\rho$$

with $\bar a$, ρ, $\tilde\rho$ sufficiently small (but possibly depending on T),

Then there exists a solution $u \in W$ of (3.23), (3.24), (3.25), which is unique in W and satisfies $\Delta u,\ u_{tt},\ \nabla u_t \in C(0,T;L_2(\Omega))\,, \quad \nabla u_{tt} \in L_2(0,T;L_2(\Omega))\,.$

3.2.2. Global well-posedness: Proof of Theorem 1.2. By reducing the situation of nonhomogeneous Dirichlet data to the situation of nonzero source term we obtain a global well-posedness result.

Theorem 3.9. *For sufficiently small initial data and boundary data the solutions are global in time. Moreover, the size of initial data does not depend on time. This is to say: For any given $M > 0$ there exist $\rho > 0$ and $\tilde\rho > 0$ such that if (1.4),*

$$\mathcal{E}(0) \le \rho\,, \tag{3.38}$$

$$\sum_{l=0}^{2} \|\frac{d^l}{dt^l} g\|^2_{L_\infty(R^+,H^{3/2-l})(\partial\Omega))} + \sum_{l=0}^{1} \|\frac{d^{3-l}}{dt^{3-l}} g\|^2_{L_2(R^+,H^{-3/2+2l}(\partial\Omega))} \le \tilde\rho \tag{3.39}$$

then $\mathcal{E}(t) \le M,\ t \in \mathbb{R}^+$.

Proof. We use the extension (2.34) and the fact that $u = u^0 + \bar g$, where u^0 satisfies

$$\begin{aligned} (1-2k(u^0+\bar g))u^0_{tt} - c^2\Delta u^0 - b\Delta u^0_t = 2k(u^0_t)^2 + q^{u^0} \\ u^0 = 0 \text{ on } \partial\Omega \\ u^0(0) = 0\,, \quad u^0_t(0) = 0\,. \end{aligned} \tag{3.40}$$

and

$$q^{u^0} \equiv 2ku^0\bar g_{tt} + 4ku^0_t\bar g_t + 2k\bar g\bar g_{tt} + 2k(\bar g_t)^2\,. \tag{3.41}$$

Similarly to Proposition 3.7 we obtain, using

$$q^{u^0}_t = 2ku^0\bar g_{ttt} + 6ku^0_t\bar g_{tt} + 4ku^0_{tt}\bar g_t + 6k\bar g_t\bar g_{tt} + 2k\bar g\bar g_{ttt}$$

the estimate

$$\begin{aligned} &|q^{u^0}_t(t)|^2_{H^{-1}(\Omega)} + |q^{u^0}(t)|^2_{L_2(\Omega)} \\ &\quad \le 64\Big\{2k(b|\nabla\bar g_{tt}(t)| + c^2|\nabla\bar g_t(t)|)^2(C_2 + C_1\hat C_1|\Omega|^{1/6})^2|\Delta u^0(t)|^2 \\ &\qquad + 6kC_1^2|\nabla u^0_t(t)|^2|\bar g_{tt}(t)|^2_{L_{3/2}(\Omega)} \\ &\qquad + 4kC_1^2|\nabla u^0_{tt}(t)|^2|\bar g_t(t)|^2_{L_{3/2}(\Omega)} \\ &\qquad + 6k\tilde C_1^2(|\nabla\bar g_t(t)| + |g_t(t)|_{H^{1/2}(\partial\Omega)})^2|\bar g_{tt}(t)|^2_{L_{3/2}(\Omega)} \\ &\qquad + 2k(b|\nabla\bar g_{tt}(t)| + c^2|\nabla\bar g_t(t)|)^2(\tilde C_2 + \tilde C_1\tilde{\hat C}_1|\Omega|^{1/6})^2 \\ &\qquad\quad (|\Delta\bar g(t)| + |g(t)|_{H^{3/2}(\partial\Omega)})^2\Big\} \end{aligned}$$

$$
\begin{aligned}
&+ 16\Big\{2kC_2^2|\Delta u^0(t)|^2|\bar{g}_{tt}(t)|^2\\
&\quad + 4kC_1^{3/2}C_0^{1/2}\tilde{C}_1^{3/2}\tilde{C}_0^{1/2}|\nabla u_t^0(t)|^2(|\nabla \bar{g}_t(t)| + |g_t(t)|_{H^{1/2}(\partial\Omega)})^2\\
&\quad + 2k\tilde{C}_2^2(|\Delta \bar{g}(t)| + |g(t)|_{H^{3/2}(\partial\Omega)})^2|\bar{g}_{tt}(t)|^2\\
&\quad + 2k\tilde{C}_1^3\tilde{C}_0(|\nabla \bar{g}_t(t)| + |g_t(t)|_{H^{1/2}(\partial\Omega)})^4\Big\}\\
&\le 256kC_1^2|\nabla u_{tt}^0(t)|^2|\bar{g}_t(t)|^2_{L_{3/2}(\Omega)}\\
&+ Ck\Big(|\Delta u^0(t)|^4 + |\nabla u_t^0(t)|^4\\
&\quad + |\Delta \bar{g}(t)|^4 + |\nabla \bar{g}_t(t)|^4 + |\bar{g}_{tt}(t)|^4\\
&\quad + |g(t)|^4_{H^{3/2}(\partial\Omega)} + |g_t(t)|^4_{H^{1/2}(\partial\Omega)}\\
&\quad + |\nabla \bar{g}_{tt}(t)|^2\Big(|\Delta u^0(t)|^2 + (|\Delta \bar{g}(t)| + |g(t)|_{H^{3/2}(\partial\Omega)})^2\Big)\Big) \qquad (3.42)\\
&\le C\,\Big(\tilde{\rho}|\mathcal{A}^{1/2}u_{tt}(t)|^2 + \mathcal{E}_0(t)^2 + |\Delta \bar{g}(t)|^4 + |\nabla \bar{g}_t(t)|^4 + |\bar{g}_{tt}(t)|^4_{H^1(\Omega)}\\
&\quad + |g(t)|^4_{H^{3/2}(\partial\Omega)} + |g_t(t)|^4_{H^{1/2}(\partial\Omega)}\Big) \qquad (3.43)
\end{aligned}
$$

for some generic constant C, where we have used Lemmas 2.16, 2.18.

Therewith, similarly to (3.14) and again using Lemmas 2.16, 2.18 as well as $\|\phi\|^4_{L_4(0,\infty)} \le \|\phi\|^2_{L_\infty(0,\infty)}\|\phi\|^2_{L_2(0,\infty)}$ we get

$$
\begin{aligned}
&\mathcal{E}_0(T) + (\alpha(T)u_{tt}^0(T), u_{tt}^0(T)) \qquad (3.44)\\
&\quad + (\tilde{b} - C\tilde{\rho})\int_0^T (\mathcal{E}_0(t) + |\mathcal{A}^{1/2}u_{tt}^0(t)|^2)(1 - \Phi(u^0, u_t^0)(t))dt \le C^1\mathcal{E}(0) + C\tilde{\rho}^2
\end{aligned}
$$

with $\mathcal{E}_0$ as in the proof of Theorem (3.3) with u replaced by u^0 and Φ defined by

$$
\begin{aligned}
\Phi(u^0, u_t^0)(t) = C^2\,16C_1^3C_0^3k^2|\mathcal{A}^{1/2}u_t^0|^2 &+ C^3\,4C_1^3C_0^2k^2|\mathcal{A}^{1/2}u_t^0|^2\\
&+ C^4\,16C_0^6k^4|\mathcal{A}^{1/2}u_t^0|^4 + C\mathcal{E}_0(t)
\end{aligned}
$$

see (3.43) and the equation before (50) in [16], hence satisfying (3.15).

Therewith, the rest of the proof can be carried out analogously to the one of Theorem (3.3) to yield global existence of u^0 and therewith of $u = u^0 + \bar{g}$. □

3.2.3. Decay rates: Proof of Theorem 1.3.

Theorem 3.10. *We assume that the initial and boundary data are sufficiently small such that* (3.38)*, and* (3.39) *holds. Additionally, we assume that the boundary data decays exponentially to zero in the sense specified in Theorem* 1.3*, i.e.:*

$$
|g(t)|^2_{H^{3/2}(\partial\Omega)} + |g_t(t)|^2_{H^{1/2}(\partial\Omega)} + |g_{tt}(t)|^2_{H^{1/2}(\partial\Omega)} + |g_{ttt}(t)|^2_{H^{-3/2}(\partial\Omega)} \le C_g e^{-\omega_g t}. \qquad (3.45)
$$

Then the energy decays exponentially fast to zero.

$$
\mathcal{E}(t) + E_{u,0}(t) \le Ce^{-\tilde{\omega}t}
$$

where $0 < \tilde{\omega} < \min\{\frac{\bar{b}}{C^1}, \omega, \frac{1}{b}, \omega_g\}$ *with* $\bar{b} = \min\{1, 4/C_0^2\}(\min\{1, 2/c^2\}\hat{b} - Ck\tilde{\rho})$, $\hat{b}$, C^1 *as in* (2.15), *and* ω *as in* (2.7).

Proof. Again we use the extension (2.34).

In order to proceed along the lines of the proof of Theorem 3.4, we consider (3.40) with (3.41) and estimate (3.42), which similarly to (3.14), (3.44) yields

$$\begin{aligned}
&\mathcal{E}_0(T) + (\alpha(T)u^0_{tt}(T), u^0_{tt}(T)) \\
&\qquad + (\tilde{b} - C\tilde{\rho})\int_s^T (\mathcal{E}_0(t) + |\mathcal{A}^{1/2}u^0_{tt}(t)|^2)(1 - \Phi(u^0, u^0_t)(t))dt \\
&\leq C^1\mathcal{E}(s) + C\int_s^T \Big(|\Delta\bar{g}(t)|^4 + |\nabla\bar{g}_t(t)|^4 + |\bar{g}_{tt}(t)|^4 \\
&\qquad + |g(t)|^4_{H^{3/2}(\partial\Omega)} + |g_t(t)|^4_{H^{1/2}(\partial\Omega)} \\
&\qquad + |\nabla\bar{g}_{tt}(t)|^2\Big(|\Delta u^0(t)|^2 + (|\Delta\bar{g}(t)| + |g(t)|_{H^{3/2}(\partial\Omega)})^2\Big)\Big)\, dt
\end{aligned} \tag{3.46}$$

with $\mathcal{E}_0$ as in the proof of Theorem (3.3) with u replaced by u^0 and Φ satisfying (3.15).

Since by Theorem 3.9 and Lemma 2.16, the term $\Big(|\Delta u^0(t)|^2 + (|\Delta\bar{g}(t)| + |g(t)|_{H^{3/2}(\partial\Omega)})^2$ is in L_∞, it therefore remains to show that (3.45) implies exponential decay

$$\int_s^T \left(|\Delta\bar{g}(t)|^4 + |\nabla\bar{g}_t(t)|^4 + |\bar{g}_{tt}(t)|^4 + |\nabla\bar{g}_{tt}(t)|^2\right) dt \leq Ce^{-\tilde{\omega}t}\,. \tag{3.47}$$

From Lemma 2.12 with $w = \bar{g}$, $w_0 = u_0$, $w_1 = u_1$ we get

$$\begin{pmatrix} \bar{g}(t) - D^\Delta g(t) \\ \bar{g}_t(t) - D^\Delta g_t(t) \end{pmatrix} = e^{At}\begin{pmatrix} u_0 - D^\Delta g(0) \\ u_1 - D^\Delta g_t(0) \end{pmatrix} + K[-D^\Delta g_{tt}](t)$$

which by Proposition 2.17 with $t = 0$ replaced by $t = s$ yields

$$\begin{aligned}
&|\mathcal{A}z(t)|^2 + |\mathcal{A}^{1/2}z_t(t)|^2 + |z_{tt}(t)|^2 + \int_s^t |\mathcal{A}^{1/2}z_{tt}(\tau)|^2 d\tau \\
&\quad \leq C_b e^{-\omega_b(t-s)}[|\mathcal{A}z(s)|^2 + |\mathcal{A}^{1/2}z_t(s)|^2 + |D^\Delta g_{tt}(s) + b\mathcal{A}z_t(s)|^2] \\
&\qquad + C_b\int_s^t e^{-\omega_b(t-\tau)}[|\mathcal{A}^{-1/2}D^\Delta g_{ttt}(\tau)|^2 + |D^\Delta g_{tt}(\tau)|^2]d\tau]
\end{aligned} \tag{3.48}$$

for $z(t) = \bar{g}(t) - D^\Delta g(t)$, where $\omega_b = \min[\omega, b^{-1}]$. To this end, note that in Proposition 2.17 we have $Z(t) = e^{A(t-s)}Z(s) + \int_s^t e^{A(t-\tau)}\begin{pmatrix} 0 \\ f(\tau) \end{pmatrix} d\tau$.

With $s = 0$ in (3.48), using (3.45) and (2.17) this implies

$$2E_{z,1}(t) = |\Delta z(t)|^2 + |\nabla z_t(t)|^2 + |z_{tt}(t)|^2 \leq \tilde{C}_b e^{-\tilde{\omega}t} \tag{3.49}$$

which yields

$$\int_s^T \left(|\Delta z(t)|^4 + |\nabla z_t(t)|^4 + |z_{tt}(t)|^4\right) dt \leq \bar{C}_b e^{-2\tilde{\omega}s}\,. \tag{3.50}$$

Now we use (3.48) with $t = T$ and (3.45) to obtain

$$\int_s^T |\mathcal{A}^{1/2} z_{tt}(\tau)|^2 d\tau \\ \leq C_b e^{-\omega_b(T-s)}[|\mathcal{A}z(s)|^2 + |\mathcal{A}^{1/2}z_t(s)|^2 + |D^\Delta g_{tt}(s) + b\mathcal{A}z_t(s)|^2] + \hat{C}_b e^{-\tilde{\omega}s}.$$

In here, we can use the identity (2.19) that implies $b\mathcal{A}z_t(s) = -\bar{g}_{tt}(s) - c^2\mathcal{A}z(s)$ to get

$$\int_s^T |\mathcal{A}^{1/2} z_{tt}(\tau)|^2 d\tau \\ \leq C_b e^{-\tilde{\omega}(T-s)}[|\mathcal{A}z(s)|^2 + |\mathcal{A}^{1/2}z_t(s)|^2 + |z_{tt}(s) + c^2\mathcal{A}z(s)|^2] + \hat{C}_b e^{-\tilde{\omega}s}.$$

Inserting (3.49) with $t = s$ gives $\int_s^T |\mathcal{A}^{1/2} z_{tt}(\tau)|^2 d\tau \leq \bar{C}_b e^{-\tilde{\omega}s}$ which together with (3.50) and

$$\int_s^T \left(|\underbrace{\Delta D^\Delta g(t)}_{=\mathbf{0}}|^4 + |\nabla D^\Delta g_t(t)|^4 + |D^\Delta g_{tt}(t)|^4 + |\nabla D^\Delta g_{tt}(t)|^2 \right) dt \leq C e^{-\tilde{\omega}s}$$

yields (3.47). So we obtain

$$E_{u^0,1}(t) \leq C e^{-\tilde{\omega}t}.$$

This together with (3.49) yields

$$E_{u,1}(t) \leq 2E_{u^0,1}(t) + 2E_{\bar{g},1}(t) \leq C e^{-\tilde{\omega}t}$$

and therewith the assertion regarding the higher energy $E_{u,1}(t)$. Regarding the lower energy $E_{u,0}(t)$, we evoke (3.29), (3.30), to obtain:

$$E_{u,0}(t) \leq \tilde{C}_0^2(|\nabla u_t(t)| + |g_t(t)_{H^{1/2}(\partial\Omega)})^2 + \tilde{\tilde{C}}_0^2(|\Delta u_t(t)| + |g(t)_{H^{3/2}(\partial\Omega)})^2 \\ \leq 2\max\{\tilde{C}_0, \tilde{\tilde{C}}_0\}(E_{u,1}(t) + |g_t(t)^2_{H^{1/2}(\partial\Omega)} + |g(t)^2_{H^{3/2}(\partial\Omega)}). \qquad \square$$

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On Divergence Form Second-order PDEs with Growing Coefficients in W_p^1 Spaces without Weights

N.V. Krylov

Abstract. We consider second-order divergence form uniformly parabolic and elliptic PDEs with bounded and VMO_x leading coefficients and possibly linearly growing lower-order coefficients. We look for solutions which are summable to the pth power with respect to the usual Lebesgue measure along with their first derivatives with respect to the spatial variables.

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Keywords. Stochastic partial differential equations, Sobolev spaces without weights, growing coefficients, divergence type equations.

1. Introduction

We consider divergence form uniformly parabolic and elliptic second-order PDEs with bounded and VMO_x leading coefficients and possibly linearly growing lower-order coefficients. We look for solutions which are summable to the pth power with respect to the usual Lebesgue measure along with their first derivatives with respect to the spatial variables. In some sense we extend the results of [17], where $p = 2$, to general $p \in (1, \infty)$. However in [17] there is no regularity assumption on the leading coefficients and there are also stochastic terms in the equations.

As in [3] one of the main motivations for studying PDEs with growing first-order coefficients is filtering theory for partially observable diffusion processes.

It is generally believed that introducing weights is the most natural setting for equations with growing coefficients. When the coefficients grow it is quite natural to consider the equations in function spaces with weights that would restrict the set of solutions in such a way that all terms in the equation will be from the same space as the free terms. The present paper seems to be the first one treating

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the unique solvability of these equations with growing lower-order coefficients in the usual Sobolev spaces W^1_p without weights and without imposing any *special* conditions on the relations between the coefficients or on their *derivatives*.

The theory of PDEs and stochastic PDEs in Sobolev spaces *with* weights attracted some attention in the past. We do not use weights and only mention a few papers about stochastic PDEs in $\mathcal{L}_p$-spaces with weights in which one can find further references: [1] (mild solutions, general p), [3], [8], [9], [10] ($p = 2$ in the four last articles).

Many more papers are devoted to the theory of *deterministic* PDEs with growing coefficients in Sobolev spaces with weights. We cite only a few of them sending the reader to the references therein again because neither do we deal with weights nor use the results of these papers. It is also worth saying that our results do not generalize the results of these papers.

In most of them the coefficients are time independent, see [2], [4], [7], [21], part of the result of which are extended in [6] to time-dependent Ornstein-Uhlenbeck operators.

It is worth noting that many issues for deterministic divergence-type equations with time independent growing coefficients in $\mathcal{L}_p$ spaces with arbitrary $p \in (1, \infty)$ *without* weights were also treated previously in the literature. This was done mostly by using the semigroup approach which excludes time dependent coefficients and makes it almost impossible to use the results in the more or less general filtering theory. We briefly mention only a few recent papers sending the reader to them for additional information.

In [19] a strongly continuous in $\mathcal{L}_p$ semigroup is constructed corresponding to elliptic operators with measurable leading coefficients and Lipschitz continuous drift coefficients. In [22] it is assumed that if, for $|x| \to \infty$, the drift coefficients grow, then the zeroth-order coefficient should grow, basically, as the square of the drift. There is also a condition on the divergence of the drift coefficient. In [23] there is no zeroth-order term and the semigroup is constructed under some assumptions one of which translates into the monotonicity of $\pm b(x) - Kx$, for a constant K, if the leading term is the Laplacian. In [5] the drift coefficient is assumed to be globally Lipschitz continuous if the zeroth-order coefficient is constant.

Some conclusions in the above-cited papers are quite similar to ours but the corresponding assumptions are not as general in what concerns the regularity of the coefficients. However, these papers contain a lot of additional important information not touched upon in the present paper (in particular, it is shown in [19] that the corresponding semigroup is not analytic and in [20] that the spectrum of an elliptic operator in $\mathcal{L}_p$ *depends* on p).

The technique, we apply, originated from [18] and [13] and uses special cut-off functions whose support evolves in time in a manner adapted to the drift. As there, we do not make any regularity assumptions on the coefficients in the time variable but unlike [17], where $p = 2$, we use the results of [11] where some regularity on the coefficients in x variable is needed, like, say, the condition that the second-order coefficients be in VMO uniformly with respect to the time variable.

It is worth noting that considering divergence form equations in $\mathcal{L}_p$-spaces is quite useful in the treatment of filtering problems (see, for instance, [15]) especially when the power of summability is taken large and we intend to treat this issue in a subsequent paper.

The article is organized as follows. In Section 2 we describe the problem, Section 3 contains the statements of two main results, Theorem 3.1 on an a priori estimate providing, in particular, uniqueness of solutions and Theorem 3.3 about the existence of solutions. The results about Cauchy's problem and elliptic equations are also given there. Theorem 3.1 is proved in Section 5 after we prepare the necessary tools in Section 4. Theorem 3.3 is proved in the last Section 6.

As usual when we speak of "a constant" we always mean "a finite constant".

The author discussed the article with Hongjie Dong whose comments are greatly appreciated.

2. Setting of the problem

We consider the second-order operator L_t

$$L_t u_t(x) = D_i\big(a_t^{ij}(x)D_j u_t(x) + \mathfrak{b}_t^i(x)u_t(x)\big) + b_t^i(x)D_i u_t(x) - c_t(x)u_t(x),$$

acting on functions $u_t(x)$ defined on $([S,T]\cap\mathbb{R})\times\mathbb{R}^d$ (the summation convention is enforced throughout the article), where S and T are such that $-\infty \le S < T \le \infty$. Naturally,

$$D_i = \frac{\partial}{\partial x^i}$$

Our main concern is proving the unique solvability of the equation

$$\partial_t u_t = L_t u_t - \lambda u_t + D_i f_t^i + f_t^0 \quad t \in [S,T]\cap\mathbb{R}, \tag{2.1}$$

with an appropriate initial condition at $t = S$ if $S > -\infty$, where $\lambda > 0$ is a constant and $\partial_t = \partial/\partial t$. The precise assumptions on the coefficients, free terms, and initial data will be given later. First we introduce appropriate function spaces.

Denote $C_0^\infty = C_0^\infty(\mathbb{R}^d)$, $\mathcal{L}_p = \mathcal{L}_p(\mathbb{R}^d)$, and let $W_p^1 = W_p^1(\mathbb{R}^d)$ be the Sobolev space of functions u of class $\mathcal{L}_p$, such that $Du \in \mathcal{L}_p$, where Du is the gradient of u and $1 < p < \infty$. For $-\infty \le S < T \le \infty$ define

$$\begin{aligned}
\mathbb{L}_p(S,T) &= \mathcal{L}_p((S,T),\mathcal{L}_p), & \mathbb{W}_p^1(S,T) &= \mathcal{L}_p((S,T),W_p^1),\\
\mathbb{L}_p(T) &= \mathbb{L}_p(-\infty,T), & \mathbb{W}_p^1(T) &= \mathbb{W}_p^1(-\infty,T),\\
\mathbb{L}_p &= \mathbb{L}_p(\infty), & \mathbb{W}_p^1 &= \mathbb{W}_p^1(\infty).
\end{aligned}$$

Remember that the elements of $\mathbb{L}_p(S,T)$ need only belong to $\mathcal{L}_p$ on a Borel subset of (S,T) of full measure. We will always assume that these elements are defined everywhere on (S,T) at least as generalized functions on $\mathbb{R}^d$. Similar situation occurs in the case of $\mathbb{W}_p^1(S,T)$.

The following definition is most appropriate for investigating our equations if the coefficients of L are bounded.

Definition 2.1. We introduce the space $\mathcal{W}^1_p(S,T)$, which is the space of functions u_t on $[S,T]\cap\mathbb{R}$ with values in the space of generalized functions on $\mathbb{R}^d$ and having the following properties:

(i) We have $u\in\mathbb{W}^1_p(S,T)$;

(ii) There exist $f^i\in\mathbb{L}_p(S,T)$, $i=0,\dots,d$, such that for any $\phi\in C_0^\infty$ and finite $s,t\in[S,T]$ we have

$$(u_t,\phi)=(u_s,\phi)+\int_s^t\big((f^0_r,\phi)-(f^i_r,D_i\phi)\big)\,dr. \tag{2.2}$$

In particular, for any $\phi\in C_0^\infty$, the function (u_t,ϕ) is continuous on $[S,T]\cap\mathbb{R}$. In case that property (ii) holds, we write

$$\partial_t u_t=D_if^i_t+f^0_t,\quad t\in[S,T]\cap\mathbb{R}.$$

Definition 2.1 allows us to introduce the spaces of initial data

Definition 2.2. Let g be a generalized function. We write $g\in W_p^{1-2/p}$ if there exists a function $v_t\in\mathcal{W}^1_p(0,1)$ such that $\partial_t v_t=\Delta v_t$, $t\in[0,1]$, and $v_0=g$. In such a case we set

$$\|g\|_{W_p^{1-2/p}}=\|v\|_{\mathbb{W}^1_p(0,1)}.$$

Notice that if the function v in Definition 2.2 exists, then it is unique, since the difference of two such functions is a $\mathcal{W}^1_p(0,1)$-solution of the Cauchy problem for the heat equation with zero initial data.

Following Definition 2.1 we understand equation (2.1) as the requirement that for any $\phi\in C_0^\infty$ and finite $s,t\in[S,T]$ we have

$$(u_t,\phi)=(u_s,\phi)+\int_s^t\big[(b^i_rD_iu_r-(c_r+\lambda)u_r+f^0_r,\phi)$$

$$-(a^{ij}_rD_ju_r+\mathfrak{b}^i_ru_r+f^i_r,D_i\phi)\big]\,dr. \tag{2.3}$$

Observe that at this moment it is not clear that the right-hand side makes sense. Also notice that, if the coefficients of L are bounded, then any $u\in\mathcal{W}^1_p(S,T)$ is a solution of (2.1) with appropriate free terms since if (2.2) holds, then (2.1) holds as well with

$$f^i_t-a^{ij}_tD_ju_t-\mathfrak{b}^i_tu_t,\quad i=1,\dots,d,\quad f^0_t+(c_t+\lambda)u_t-b^i_tD_iu_t,$$

in place of f^i_t, $i=1,\dots,d$, and f^0_t, respectively.

We give the definition of solution of (2.1) adopted throughout the article and which in case the coefficients of L are bounded coincides with the one obtained by applying Definition 2.1.

Definition 2.3. Let $f^j\in\mathbb{L}_p(S,T)$, $j=0,\dots,d$ and assume that $S>-\infty$. By a solution of (2.1) with initial condition $u_S\in W_p^{1-2/p}$ we mean a function $u\in\mathbb{W}^1_p(S,T)$ (not $\mathcal{W}^1_p(S,T)$) such that

(i) For any $\phi\in C_0^\infty$ the integral with respect to dr in (2.3) is well defined and is finite for all finite $s,t\in[S,T]$;

(ii) For any $\phi \in C_0^\infty$ equation (2.3) holds for all finite $s, t \in [S, T]$.

In case $S = -\infty$ we drop mentioning initial condition in the above lines.

It is worth mentioning that under our conditions on the coefficients requirement (i) of Definition 2.3 is automatically satisfied (see Corollary 5.5).

3. Main results

For $\rho > 0$ denote $B_\rho(x) = \{y \in \mathbb{R}^d : |x - y| < \rho\}$, $B_\rho = B_\rho(0)$.

Assumption 3.1.

(i) The functions $a_t^{ij}(x)$, $\mathfrak{b}_t^i(x)$, $b_t^i(x)$, and $c_t(x)$ are real valued and Borel measurable and $c \geq 0$.

(ii) There exists a constant $\delta > 0$ such that for all values of arguments and $\xi \in \mathbb{R}^d$

$$a^{ij}\xi^i\xi^j \geq \delta|\xi|^2, \quad |a^{ij}| \leq \delta^{-1}.$$

Also, the constant $\lambda > 0$.

(iii) For any $x \in \mathbb{R}^d$ the function

$$\int_{B_1} (|\mathfrak{b}_t(x+y)| + |b_t(x+y)| + c_t(x+y))\, dy$$

is locally integrable to the p'th power on $\mathbb{R}$, where $p' = p/(p-1)$.

Notice that the matrix $a = (a^{ij})$ need not be symmetric. Also notice that in Assumption 3.1 (iii) the ball B_1 can be replaced with any other ball without changing the set of admissible coefficients $\mathfrak{b}, b, c$.

We take and fix constants $K \geq 0, \rho_0, \rho_1 \in (0, 1]$, and choose a number $q = q(d, p)$ so that

$$q > \min(d, p), \quad q > \min(d, p'), \quad q \geq \max(d, p, p'). \tag{3.1}$$

The following assumptions contain a parameter $\gamma \in (0, 1]$, whose value will be specified later.

Assumption 3.2. For $\mathfrak{b} := (\mathfrak{b}^1, \ldots, \mathfrak{b}^d)$ and $b := (b^1, \ldots, b^d)$ and $(t, x) \in \mathbb{R}^{d+1}$ we have

$$\int_{B_{\rho_1}(x)} \int_{B_{\rho_1}(x)} |\mathfrak{b}_t(y) - \mathfrak{b}_t(z)|^q \, dydz + \int_{B_{\rho_1}(x)} \int_{B_{\rho_1}(x)} |b_t(y) - b_t(z)|^q \, dydz$$
$$+ \int_{B_{\rho_1}(x)} \int_{B_{\rho_1}(x)} |c_t(y) - c_t(z)|^q \, dydz \leq K I_{q>d} + \rho_1^d \gamma,$$

where $I_{q>d} = 1$ if $q > d$ and $I_{q>d} = 0$ if $q = d$.

Obviously, Assumption 3.2 is satisfied if b, $\mathfrak{b}$, and c are independent of x. They also are satisfied with any $q > d$, $\gamma \in (0, 1]$, and $\rho_1 = 1$ on the account of choosing K appropriately if, say,

$$|\mathfrak{b}_t(x) - \mathfrak{b}_t(y)| + |b_t(x) - b_t(y)| + |c_t(x) - c_t(y)| \leq N$$

whenever $|x-y|\leq 1$, where N is a constant. We see that Assumption 3.2 allows b, $\mathfrak{b}$, and c growing linearly in x.

Assumption 3.3. For any $\rho\in(0,\rho_0]$, $s\in\mathbb{R}$, and $i,j=1,\dots,d$ we have

$$\rho^{-2d-2}\int_s^{s+\rho^2}\Big(\sup_{x\in\mathbb{R}^d}\int_{B_\rho(x)}\int_{B_\rho(x)}|a_t^{ij}(y)-a_t^{ij}(z)|\,dydz\Big)\,dt\leq\gamma. \tag{3.2}$$

Obviously, the left-hand side of (3.2) is less than

$$N(d)\sup_{t\in\mathbb{R}}\sup_{|x-y|\leq 2\rho}|a_t^{ij}(x)-a_t^{ij}(y)|,$$

which implies that Assumption 3.3 is satisfied with any $\gamma\in(0,1]$ if, for instance, a is uniformly continuous in x uniformly with respect to t. Recall that if a is independent of t and for any $\gamma>0$ there is a $\rho_0>0$ such that Assumption 3.3 is satisfied, then one says that a is in VMO.

Theorem 3.1. *There exist*

$$\gamma=\gamma(d,\delta,p)\in(0,1],$$

$$N=N(d,\delta,p),\quad \lambda_0=\lambda_0(d,\delta,p,\rho_0,\rho_1,K)\geq 1$$

such that, if the above assumptions are satisfied and $\lambda\geq\lambda_0$ and u is a solution of (2.1) *with zero initial data* (*if $S>-\infty$*) *and some $f^j\in\mathbb{L}_p(S,T)$, then*

$$\lambda\|u\|^2_{\mathbb{L}_p(S,T)}+\|Du\|^2_{\mathbb{L}_p(S,T)}\leq N\big(\sum_{i=1}^d\|f^i\|^2_{\mathbb{L}_p(S,T)}+\lambda^{-1}\|f^0\|^2_{\mathbb{L}_p(S,T)}\big). \tag{3.3}$$

Notice that the main case of Theorem 3.1 is when $S=-\infty$ because if $S>-\infty$ and $u_S=0$, then the function $u_tI_{t\geq S}$ will be a solution of our equation on $(-\infty,T]\cap\mathbb{R}$ with $f^j_t=0$ for $t<S$.

This theorem provides an a priori estimate implying uniqueness of solutions. Observe that the assumption that such a solution exists is quite nontrivial because if $\mathfrak{b}_t(x)\equiv x$, it is not true that $\mathfrak{b}u\in\mathbb{L}_p(S,T)$ for arbitrary $u\in\mathbb{W}^1_p(S,T)$.

It is also worth noting that, as can be easily seen from the proof of Theorem 3.1, one can choose a function $\gamma(d,\delta,p)$ so that it is continuous in (δ,p). The same holds for N and λ_0 from Theorem 3.1.

We have a similar result for nonzero initial data.

Theorem 3.2. *Let $S>-\infty$. In Theorem* 3.1 *replace the assumption that $u_S=0$ with the assumption that $u_S\in W_p^{1-2/p}$. Then its statement remains true if in the right-hand side of* (3.3) *we add the term*

$$N\|u_S\|^2_{W_p^{1-2/p}}.$$

Proof. Take v_t from Definition 2.2 corresponding to $g = u_S$ and set

$$\tilde{u}_t = \begin{cases} u_t & t \geq S, \\ (t - S + 1)v_{S-t} & S \geq t \geq S - 1, \\ 0 & S - 1 \geq t \end{cases}$$

and for $i = 1, \ldots, d$ set

$$\tilde{f}_t^i = \begin{cases} f_t^i & t \geq S, \\ -2(t - S + 1)D_i v_{S-t} & S > t \geq S - 1, \\ 0 & S - 1 \geq t, \end{cases}$$

$$\tilde{f}_t^0 = \begin{cases} f_t^0 & t \geq S, \\ [1 + \lambda(t - S + 1)]v_{S-t} & S > t \geq S - 1, \\ 0 & S - 1 \geq t. \end{cases}$$

We also modify the coefficients of L by multiplying each one of them but a_t^{ij} by $I_{t \geq S}$ and setting

$$\tilde{a}_t^{ij} = \begin{cases} a_t^{ij} & t \geq S, \\ \delta^{ij} & S > t. \end{cases}$$

Here we profit from the fact that no regularity assumption on the dependence of the coefficients on t is imposed. By denoting by $\tilde{L}$ the operator with the modified coefficients we easily see that $\tilde{u}_t$ is a solution (always in the sense of Definition 2.3) of

$$\partial_t \tilde{u}_t = \tilde{L}_t \tilde{u}_t - \lambda \tilde{u}_t + D_i \tilde{f}_t^i + \tilde{f}_t^0, \quad t \leq T.$$

By Theorem 3.1

$$\lambda \|u\|^2_{\mathbb{L}_p(S,T)} + \|Du\|^2_{\mathbb{L}_p(S,T)} \leq N \left(\sum_{i=1}^d \|\tilde{f}^i\|^2_{\mathbb{L}_p(T)} + \lambda^{-1} \|\tilde{f}^0\|^2_{\mathbb{L}_p(T)} \right),$$

where

$$\begin{aligned} \|\tilde{f}^i\|^p_{\mathbb{L}_p(T)} &= \|f^i\|^p_{\mathbb{L}_p(S,T)} + \|\tilde{f}^i\|^p_{\mathbb{L}_p(S-1,S)} \leq \|f^i\|^p_{\mathbb{L}_p(S,T)} + 2^p \|D_i v\|^p_{\mathbb{L}_p(0,1)} \\ &\leq \|f^i\|^p_{\mathbb{L}_p(S,T)} + 2^p \|u_S\|^p_{W_p^{1-2/p}}, \end{aligned}$$

$$\begin{aligned} \|\tilde{f}^0\|^p_{\mathbb{L}_p(T)} &\leq \|f^0\|^p_{\mathbb{L}_p(S,T)} + N(1 + \lambda^p)\|v\|^p_{\mathbb{L}_p(0,1)} \\ &\leq \|f^0\|^p_{\mathbb{L}_p(S,T)} + N(1 + \lambda^p)\|u_S\|^p_{W_p^{1-2/p}}. \end{aligned}$$

Since $\lambda \geq \lambda_0 \geq 1$, we have $1 + \lambda^p \leq 2\lambda^p$ and we get our assertion thus proving the theorem. $\square$

Here is an existence theorem.

Theorem 3.3. *Let the above assumptions be satisfied with γ taken from Theorem* 3.1*. Take $\lambda \geq \lambda_0$, where λ_0 is defined in Theorem* 3.1*. Then for any $f^j \in \mathbb{L}_p(T)$, $j = 0, \ldots, d$, there exists a unique solution of* (2.1) *with $S = -\infty$.*

It turns out that the solution, if it exists, is independent of the space in which we are looking for solutions.

Theorem 3.4. *Let* $1 < p_1 \le p_2 < \infty$ *and let*

$$\gamma = \inf_{p \in [p_1, p_2]} \gamma(d, \delta, p),$$

where $\gamma(d, \delta, p)$ *is taken from Theorem* 3.1. *Suppose that Assumptions* 3.1 *through* 3.3 *are satisfied with so-defined* γ *and with* $p = p_1$ *and* $p = p_2$.

(i) *Let* $-\infty < S < T < \infty$, $f^j \in \mathbb{L}_{p_1}(S,T) \cap \mathbb{L}_{p_2}(S,T)$, $j = 0, \dots, d$, $u_S \in W^{1-2/p_1}_{p_1} \cap W^{1-2/p_2}_{p_2}$, *and let* $u \in \mathbb{W}^1_{p_1}(S,T) \cup \mathbb{W}^1_{p_2}(S,T)$ *be a solution of* (2.1). *Then* $u \in \mathbb{W}^1_{p_1}(S,T) \cap \mathbb{W}^1_{p_2}(S,T)$.

(ii) *Let* $S = -\infty, T = \infty$, $f^j \in \mathbb{L}_{p_1} \cap \mathbb{L}_{p_2}$, $j = 0, \dots, d$, *and let* $u \in \mathbb{W}^1_{p_1} \cup \mathbb{W}^1_{p_2}$ *be a solution of* (2.1) *with*

$$\lambda \ge \sup_{p \in [p_1, p_2]} \lambda_0(d, \delta, p, \rho_0, \rho_1, K), \tag{3.4}$$

where $\lambda_0(d, \delta, p, \rho_0, \rho_1, K)$ *is taken from Theorem* 3.1. *Then* $u \in \mathbb{W}^1_{p_1} \cap \mathbb{W}^1_{p_2}$.

This theorem is proved in Section 6. The following theorem is about Cauchy's problem with nonzero initial data.

Theorem 3.5. *Let* $S > -\infty$ *and take a function* $u_S \in W^{1-2/p}_p$. *Let the above assumptions be satisfied with* γ *taken from Theorem* 3.1. *Take* $\lambda \ge \lambda_0$, *where* λ_0 *is defined in Theorem* 3.1. *Then for any* $f^j \in \mathbb{L}_p(S,T)$, $j = 0, \dots, d$, *there exists a unique solution of* (2.1) *with initial value* u_S.

Proof. As in the proof of Theorem 3.2 we extend our coefficients and f^j_t for $t < S$ and then find a unique solution $\tilde{u}_t$ of

$$\partial_t \tilde{u}_t = \tilde{L}_t \tilde{u}_t - \lambda \tilde{u}_t + D_i \tilde{f}^i_t + \tilde{f}^0_t \quad t \in (-\infty, T] \cap \mathbb{R},$$

By construction $(t - S + 1)v_{S-t}$ satisfies this equation for $t \le S$, so that by uniqueness (Theorem 3.1 with S in place of T) it coincides with $\tilde{u}_t$ for $t \le S$. In particular, $\tilde{u}_S = v_0 = u_S$. Furthermore $\tilde{u}$ satisfies (2.1) since the coefficients of $\tilde{L}_t$ coincide with the corresponding coefficients of L_t for finite $t \in [S,T]$. The theorem is proved.

Remark 3.1. If both S and T are finite, then in the above theorem one can take $\lambda = 0$. To show this take a large $\lambda > 0$ and replace the unknown function u_t with $v_t e^{\lambda t}$. This leads to an equation for v_t with the additional term $-\lambda v_t$ and the free terms multiplied by $e^{-\lambda t}$. The existence of solution v will be then equivalent to the existence of u if S and T are finite.

Remark 3.2. From the above proof and from Theorem 3.4 it follows that the solution, if it exists, is independent of p in the same sense as in Theorem 3.4.

Here is a result for elliptic equations.

Theorem 3.6. *Let the coefficients of L_t be independent of t, so that we can set $L = L_t$ and drop the subscript t elsewhere, let Assumptions* 3.1 (i), (ii) *be satisfied, and let $\mathfrak{b}$, b, and c be locally integrable. Then there exist*

$$\gamma = \gamma(d, \delta, p) \in (0, 1],$$
$$N = N(d, \delta, p), \quad \lambda_0 = \lambda_0(d, \delta, p, \rho_0, \rho_1, K) \geq 1$$

such that, if Assumptions 3.2 *and* 3.3 *are satisfied and $\lambda \geq \lambda_0$ and u is a W^1_p-solution of*

$$Lu - \lambda u + D_i f^i + f^0 = 0 \tag{3.5}$$

in $\mathbb{R}^d$ with some $f^j \in \mathcal{L}_p$, $j = 0, \ldots, d$, then

$$\lambda \|u\|^2_{\mathcal{L}_p} + \|Du\|^2_{\mathcal{L}_p} \leq N\Big(\sum_{i=1}^{d} \|f^i\|^2_{\mathcal{L}_p} + \lambda^{-1}\|f^0\|^2_{\mathcal{L}_p}\Big). \tag{3.6}$$

Furthermore, for any $f^j \in \mathcal{L}_p$, $j = 0, \ldots, d$, and $\lambda \geq \lambda_0$ there exists a unique solution $u \in W^1_p$ of (3.5).

This result is obtained from the previous ones in a standard way (see, for instance, the proof of Theorem 2.1 of [13]). One of remarkable features of (3.6) is that N is independent of $\mathfrak{b}$, b, and c. It is remarkable even if they are constant, when there is no assumptions on them apart from $c \geq 0$. Another point worth noting is that if $\mathfrak{b} = b \equiv 0$, then for the solution u we have $cu \in W^{-1}_p$. However, generally it is not true that $cu \in W^{-1}_p$ for *any* $u \in W^1_p$. For instance $u(x) := (1 + |x|)^{-1} \in W^1_p$ if $p > d$, but if $c(x) = |x|$, then $(1 - \Delta)^{-1/2}(cu)(x) \to 1$ as $|x| \to \infty$ and $(1 - \Delta)^{-1/2}(cu)$ is not integrable to any power $r > 1$. Therefore generally, $(L - \lambda)W^1_p \supset W^{-1}_p$ with proper inclusion, that does not happen if the coefficients of L are bounded.

Remark 3.3. It follows, from the arguments leading to the proof of Theorem 3.6 (see [13]) and from Theorem 3.4, that the solution in Theorem 3.6 is independent of p like in Theorem 3.4 if γ is chosen as in Theorem 3.4 and $\lambda \geq$ RHS of (3.4)$+1$.

4. Differentiating compositions of generalized functions with differentiable functions

Let $\mathcal{D}$ be the space of generalized functions on $\mathbb{R}^d$. We need a formula for the time derivative of $u_t(x + x_t)$, where u_t behaves like a function from $\mathcal{W}^1_p$ and x_t is an $\mathbb{R}^d$-valued differentiable function. The formula is absolutely natural and probably well known. We refer the reader to [16] where such a formula is derived in a much more general setting of stochastic processes. Recall that for any $v \in \mathcal{D}$ and $\phi \in C^\infty_0$ the function $(v, \phi(\cdot - x))$ is infinitely differentiable with respect to x, so that the sup in (4.1) below is measurable.

Definition 4.1. Denote by $\mathfrak{D}(S,T)$ the set of all $\mathcal{D}$-valued functions u (written as $u_t(x)$ in a common abuse of notation) on $[S,T]\cap\mathbb{R}$ such that, for any $\phi\in C_0^\infty$, the function (u_t,ϕ) is measurable. Denote by $\mathfrak{D}^1(S,T)$ the subset of $\mathfrak{D}(S,T)$ consisting of u such that, for any $\phi\in C_0^\infty$, $R\in(0,\infty)$, and finite $t_1,t_2\in[S,T]$ such that $t_1<t_2$ we have

$$\int_{t_1}^{t_2}\sup_{|x|\le R}|(u_t,\phi(\cdot-x))|\,dt<\infty. \tag{4.1}$$

Definition 4.2. Let $f,u\in\mathfrak{D}(S,T)$. We say that the equation

$$\partial_t u_t(x)=f_t(x),\quad t\in[S,T]\cap\mathbb{R}, \tag{4.2}$$

holds *in the sense of distributions* if $f\in\mathfrak{D}^1(S,T)$ and for any $\phi\in C_0^\infty$ for all finite $s,t\in[S,T]$ we have

$$(u_t,\phi)=(u_s,\phi)+\int_s^t(f_r,\phi)\,dr.$$

Let x_t be an $\mathbb{R}^d$-valued function given by

$$x_t=\int_0^t\hat b_s\,ds,$$

where $\hat b_s$ is an $\mathbb{R}^d$-valued locally integrable function on $\mathbb{R}$. Here is the formula.

Theorem 4.3. *Let $f,u\in\mathfrak{D}(S,T)$. Introduce*

$$v_t(x)=u_t(x+x_t)$$

and assume that (4.2) *holds* (*in the sense of distributions*). *Then*

$$\partial_t v_t(x)=f_t(x+x_t)+\hat b^i_tD_iv_t(x),\quad t\in[S,T]\cap\mathbb{R}$$

(*in the sense of distributions*).

Corollary 4.4. *Under the assumptions of Theorem* 4.3 *for any $\eta\in C_0^\infty$ we have*

$$\partial_t[u_t(x)\eta(x-x_t)]=f_t(x)\eta(x-x_t)-u_t(x)\hat b^i_tD_i\eta(x-x_t),\quad t\in[S,T]\cap\mathbb{R}.$$

Indeed, what we claim is that for any $\phi\in C_0^\infty$ and finite $s,t\in[S,T]$

$$((u_t\phi)(\cdot+x_t),\eta)=(u_s\phi,\eta)+\int_s^t\big(\big[f_r\phi+\hat b^i_rD_i(u_r\phi)\big](\cdot+x_r),\eta\big)\,dr.$$

However, to obtain this result it suffices to write down an obvious equation for $u_t\phi$, then use Theorem 4.3 and, finally, use Definition 4.2 to interpret the result.

5. Proof of Theorem 3.1

Throughout this section we suppose that Assumptions 3.1, 3.2, and 3.3 are satisfied (with a $\gamma\in(0,1]$) and start with analyzing the integral in (2.3). Recall that q was introduced before Assumption 3.2.

Lemma 5.1. *Let* $1 \le r < p$ *and*

$$\eta := 1 + \frac{d}{p} - \frac{d}{r} \ge 0 \tag{5.1}$$

with strict inequality if $r = 1$. *Then for any* $U \in \mathcal{L}_r$ *and* $\varepsilon > 0$ *there exist* $V^j \in \mathcal{L}_p$, $j = 0, 1, \ldots, d$, *such that* $U = D_i V^i + V^0$ *and*

$$\sum_{j=1}^{d} \|V^j\|_{\mathcal{L}_p} \le N(d,p,r)\varepsilon^{\eta/(1-\eta)}\|U\|_{\mathcal{L}_r}, \quad \|V^0\|_{\mathcal{L}_p} \le N(d,p,r)\varepsilon^{-1}\|U\|_{\mathcal{L}_r}. \tag{5.2}$$

In particular, for any $w \in W^1_{p'}$

$$|(U,w)| \le N(d,p,r)\|U\|_{\mathcal{L}_r}\|w\|_{W^1_{p'}}.$$

Proof. If the result is true for $\varepsilon = 1$, then for arbitrary $\varepsilon > 0$ it is easily obtained by scaling. Thus let $\varepsilon = 1$ and denote by $R_0(x)$ the kernel of $(1-\Delta)^{-1}$. For $i = 1, \ldots, d$ set $R_i = -D_i R_0$. One knows (see, for instance, Theorem 12.7.1 of [12]) that $R_j(x)$ decrease faster than $|x|^{-n}$ for any $n > 0$ as $|x| \to \infty$ (actually, exponentially fast) and (see, for instance, Theorem 12.7.4 of [12]) that for all $x \ne 0$

$$|R_j(x)| \le \frac{N}{|x|^{d-1}}, \quad j = 0, 1, \ldots, d.$$

Define

$$V^j = R_j * U, \quad j = 0, 1, \ldots, d.$$

If $r = 1$, one obtains (5.2) from Young's inequality since, owing to the strict inequality in (5.1), we have $p < d/(d-1)$, so that $R_j \in \mathcal{L}_p$. If $r > 1$, then for ν defined by

$$\frac{1}{p} = \frac{1}{r} - \frac{\nu}{d}$$

we have $\nu \in (0,1]$, so that

$$|R_j(x)| \le \frac{N}{|x|^{d-\nu}}, \quad j = 0, 1, \ldots, d,$$

and we obtain (5.2) from the Sobolev-Hardy-Littlewood inequality (see, for instance, Lemma 13.8.5 of [12]). After this it only remains to notice that in the sense of generalized functions

$$D_i V^i + V_0 = R_0 * U - \Delta R_0 * U = U.$$

The lemma is proved. □

Observe that by Hölder's inequality for $r = pq/(p+q)$ ($\in [1,p)$ due to $q \ge p'$, see (3.1)) we have

$$\|hv\|_{\mathcal{L}_r} \le \|h\|_{\mathcal{L}_q}\|v\|_{\mathcal{L}_p}.$$

Furthermore, if $r = 1$, then $q = p' > d$ (see (3.1)), $p < d/(d-1)$, and $\eta > 0$. In this way we come to the following.

Corollary 5.2. *Let $h \in \mathcal{L}_q$, $v \in \mathcal{L}_p$, and $w \in W^1_{p'}$. Then for any $\varepsilon > 0$ there exist $V^j \in \mathcal{L}_p$, $j = 0, 1, \dots, d$, such that $hv = D_i V^i + V^0$ and*

$$\sum_{j=1}^{d} \|V^j\|_{\mathcal{L}_p} \le N(d,p)\varepsilon^{(q-d)/d}\|h\|_{\mathcal{L}_q}\|v\|_{\mathcal{L}_p},$$

$$\|V^0\|_{\mathcal{L}_p} \le N(d,p)\varepsilon^{-1}\|h\|_{\mathcal{L}_q}\|v\|_{\mathcal{L}_p}.$$

In particular,

$$|(hv,w)| \le N(d,p)\|h\|_{\mathcal{L}_q}\|v\|_{\mathcal{L}_p}\|w\|_{W^1_{p'}}. \tag{5.3}$$

Lemma 5.3. *Let $h \in \mathcal{L}_q$ and $u \in W^1_p$. Then for any $\varepsilon > 0$ we have*

$$\|hu\|_{\mathcal{L}_p} \le N(d,p)\|h\|_{\mathcal{L}_q}\big(\varepsilon^{(q-d)/d}\|Du\|_{\mathcal{L}_p} + \varepsilon^{-1}\|u\|_{\mathcal{L}_p}\big). \tag{5.4}$$

Proof. As above it suffices to concentrate on $\varepsilon = 1$. In case $q > p$ observe that by Hölder's inequality

$$\|hu\|_{\mathcal{L}_p} \le \|h\|_{\mathcal{L}_q}\|u\|_{\mathcal{L}_s},$$

where $s = pq/(q-p)$. After that it only remains to use embedding theorems (notice that $1 - d/p \ge -d/s$ since $q \ge d$). In the remaining case $q = p$, which happens only if $p > d$ (see (3.1)). In that case the above estimate remains true if we set $s = \infty$. The lemma is proved. □

Before we extract some consequences from the lemma we take a nonnegative $\xi \in C_0^\infty(B_{\rho_1})$ with unit integral and define

$$\bar{b}_s(x) = \int_{B_{\rho_1}} \xi(y) b_s(x-y)\,dy, \quad \bar{\mathfrak{b}}_s(x) = \int_{B_{\rho_1}} \xi(y)\mathfrak{b}_s(x-y)\,dy,$$

$$\bar{c}_s(x) = \int_{B_{\rho_1}} \xi(y) c_s(x-y)\,dy. \tag{5.5}$$

We may assume that $|\xi| \le N(d)\rho_1^{-d}$.

One obtains the first assertion of the following corollary from (5.3) by observing that

$$\begin{aligned}
\|I_{B_{\rho_1}(x_t)}(b_t - \bar{b}_t(x_t))\|^q_{\mathcal{L}_q} &= \int_{B_{\rho_1}(x_t)} |b_t - \bar{b}_t(x_t)|^q\,dx \\
&= \int_{B_{\rho_1}(x_t)} \left| \int_{B_{\rho_1}(x_t)} [b_t(x) - b_t(y)]\xi(x_t - y)\,dy \right|^q dx \\
&\le N \int_{B_{\rho_1}(x_t)} \left| \rho_1^{-d} \int_{B_{\rho_1}(x_t)} |b_t(x) - b_t(y)|\,dy \right|^q dx \\
&\le N\rho_1^{-d} \int_{B_{\rho_1}(x_t)} \int_{B_{\rho_1}(x_t)} |b_t(x) - b_t(y)|^q\,dy\,dx \\
&\le N\rho_1^{-d} K I_{q>d} + N\gamma.
\end{aligned} \tag{5.6}$$

The second assertion follows from estimates like (5.6) and (5.4) where one chooses ε appropriately if $q > d$.

Corollary 5.4. *Let $u \in \mathbb{W}^1_p(S,T)$, let x_s be an $\mathbb{R}^d$-valued measurable function, and let $\eta \in C_0^\infty(B_{\rho_1})$. Set $\eta_s(x) = \eta(x - x_s)$,*

$$K_1 = \sup|\eta| + \sup|D\eta|.$$

Then on (S,T)

(i) *For any $w \in W^1_{p'}$ and $v \in \mathcal{L}_p$*

$$(|b_s - \bar{b}_s(x_s)|\eta_s v, |w|) \le N(d,p,\rho_1,K)\|\eta_s v\|_{\mathcal{L}_p}\|w\|_{W^1_{p'}};$$

(ii) *We have*

$$\begin{aligned}&\|\eta_s|\mathfrak{b}_s - \bar{\mathfrak{b}}_s(x_s)|\,u_s\|_{\mathcal{L}_p} + \|\eta_s|c_s - \bar{c}_s(x_s)|\,u_s\|_{\mathcal{L}_p}\\ &\quad\le N(d,p)\gamma^{1/q}\|\eta_s Du_s\|_{\mathcal{L}_p} + N(d,p,\gamma,\rho_1,K,K_1)\|I_{B_{\rho_1}(x_s)}u_s\|_{\mathcal{L}_p}.\end{aligned}$$

(iii) *Almost everywhere on (S,T) we have*

$$(b^i_s - \bar{b}^i_s(x_s))\eta_s D_i u_s = D_i V^i_s + V^0_s, \tag{5.7}$$

$$\sum_{j=1}^d \|V^j\|_{\mathcal{L}_p} \le N(d,p)\gamma^{1/q}\|\eta_s Du_s\|_{\mathcal{L}_p},$$

$$\|V^0_s\|_{\mathcal{L}_p} \le N(d,p,\gamma,\rho_1,K)\|\eta_s Du_s\|_{\mathcal{L}_p}, \tag{5.8}$$

where V^j_s, $j = 0,\dots,d$, are some measurable $\mathcal{L}_p$-valued functions on (S,T).

To prove (iii) observe that one can find a Borel set $A \subset (S,T)$ of full measure such that $I_A D_i u$, $i = 1,\dots,d$, are well defined as $\mathcal{L}_p$-valued Borel measurable functions. Then (5.7) with $I_A D_i u$ in place of $D_i u$ and (5.8) follow from (5.6), Corollary 5.2, and the fact that the way V^j are constructed uses bounded hence continuous operators and translates the measurability of the data into the measurability of the result. Since we are interested in (5.7) and (5.8) holding only almost everywhere on (S,T), there is no actual need for the replacement.

Corollary 5.5. *Let $u \in \mathbb{W}^1_p(S,T)$, $R \in (0,\infty)$, $\phi \in C_0^\infty(B_R)$, and let finite $S', T' \in (S,T)$ be such that $S' < T'$. Then there is a constant N independent of u and ϕ such that*

$$\int_{S'}^{T'} \left(|(b^i_s D_i u_s,\phi)| + |(\mathfrak{b}^i_s u_s, D_i\phi)| + |(c_s u_s,\phi)|\right) ds \le N\|u\|_{\mathbb{W}^1_p(S,T)}\|\phi\|_{W^1_{p'}}, \tag{5.9}$$

so that requirement (i) *in Definition* 2.3 *can be dropped.*

Proof. By having in mind partitions of unity we convince ourselves that it suffices to prove (5.9) under the assumption that ϕ has support in a ball B of radius ρ_1. Let

x_0 be the center of B and set $x_s \equiv x_0$. Observe that the estimates from Corollary 5.4 imply that

$$
\begin{aligned}
|(\mathfrak{b}^i_s u_s, D_i\phi)| &\le |(\mathfrak{b}^i_s - \bar{\mathfrak{b}}^i_s(x_0))u_s, D_i\phi)| + |\bar{\mathfrak{b}}^i_s(x_0)(u_s, D_i\phi)| \\
&\le N\|u_s\|_{W^1_p}\|\phi\|_{W^1_{p'}} + |\bar{\mathfrak{b}}_s(x_0)|\,\|u_s\|_{W^1_p}\|\phi\|_{W^1_{p'}}.
\end{aligned}
$$

By recalling Assumption 3.1 (iii) and Hölder's inequality we get

$$
\int_{S'}^{T'} |(\mathfrak{b}^i_s u_s, D_i\phi)|\,ds \le N\|u\|_{\mathbb{W}^1_p(S,T)}\|\phi\|_{W^1_{p'}}.
$$

Similarly the integrals of $|(b^i_s D_i u_s, \phi)|$ and $|(c_s u_s, \phi)|$ are estimated and the corollary is proved. □

Since bounded linear operators are continuous we obtain the following.

Corollary 5.6. *Let $\phi \in C_0^\infty$, $T \in (0,\infty)$. Then the operators*

$$
u_\cdot \to \int_0^\cdot (b^i_t D_i u_t, \phi)\,dt, \quad u_\cdot \to \int_0^\cdot (\mathfrak{b}^i_t u_t, D_i\phi)\,dt, \quad u_\cdot \to \int_0^\cdot (c_t u_t, \phi)\,dt
$$

are continuous as operators from $\mathbb{W}^1_p(\infty)$ to $\mathcal{L}_p([-T,T])$.

This result will be used in Section 6.

Before we continue with the proof of Theorem 3.1, we notice that, if $u \in \mathcal{W}^1_p(S,T)$, then as we know (see, for instance, Theorem 2.1 of [14]), the function u_t is a continuous $\mathcal{L}_p$-valued function on $[S,T] \cap \mathbb{R}$.

Now we are ready to prove Theorem 3.1 in a particular case.

Lemma 5.7. *Let $\mathfrak{b}^i$, b^i, and c be independent of x and let $S = -\infty$. Then the assertion of Theorem* 3.1 *holds, naturally, with $\lambda_0 = \lambda_0(d,\delta,p,\rho_0)$ (independent of ρ_1 and K).*

Proof. First let $c \equiv 0$. We want to use Theorem 4.3 to get rid of the first-order terms. Observe that (2.1) reads as

$$
\partial_t u_t = D_i(a^{ij}_t D_j u_t + [\mathfrak{b}^i_t + b^i_t]u_t + f^i_t) + f^0_t - \lambda u_t, \quad t \le T. \tag{5.10}
$$

Recall that from the start (see Definition 2.3) it is assumed that $u \in \mathbb{W}^1_p(T)$. Then one can find a Borel set $A \subset (-\infty,T)$ of full measure such that $I_A f^j$, $j = 0,1,\dots,d$, and $I_A D_i u$, $i = 1,\dots,d$, are well defined as $\mathcal{L}_p$-valued Borel functions satisfying

$$
\int_{-\infty}^{T} I_A\left(\sum_{j=0}^{d}\|f^j_t\|^p_{\mathcal{L}_p} + \|Du_t\|^p_{\mathcal{L}_p}\right)dt < \infty.
$$

Replacing f^j and $D_i u$ in (5.10) with $I_A f^j$ and $I_A D_i u$, respectively, will not affect (5.10). Similarly one can treat the term $h_t = (\mathfrak{b}^i_t + b^i_t)u_t$ for which

$$
\int_{S'}^{T'} \|h_t\|_{\mathcal{L}_p}\,dt < \infty
$$

for each finite $S', T' \in (-\infty, T]$, owing to Assumption 3.1 and the fact that $u \in \mathbb{L}_p(T)$.

After these replacements all terms on the right in (5.10) will be of class $\mathfrak{D}^1(-\infty, T)$ since a is bounded. This allows us to apply Theorem 4.3 and for

$$B^i_t = \int_0^t (\mathfrak{b}^i_s + b^i_s)\,ds, \quad \hat{u}_t(x) = u_t(x - B_t)$$

obtain that

$$\partial_t \hat{u}_t = D_i(\hat{a}^{ij}_t D_j \hat{u}_t) - \lambda \hat{u}_t + D_i \hat{f}^i_t + \hat{f}^0_t, \tag{5.11}$$

where

$$(\hat{a}^{ij}_t, \hat{f}^j_t)(x) = (a^{ij}_t, f^j_t)(x - B_t).$$

Obviously, $\hat{u}$ is in $\mathbb{W}^1_p(T)$ and its norm coincides with that of u. Equation (5.11) shows that $\hat{u} \in \mathcal{W}^1_p(T)$.

By Theorem 4.4 and Remark 2.4 of [11] there exist $\gamma = \gamma(d, \delta, p)$ and $\lambda_0 = \lambda_0(d, \delta, p, \rho_0)$ such that if $\lambda \geq \lambda_0$, then

$$\|D\hat{u}\|_{\mathbb{L}_p(T)} + \lambda^{1/2}\|\hat{u}\|_{\mathbb{L}_p(T)} \leq N\left(\sum_{i=1}^d \|\hat{f}^i\|_{\mathbb{L}_p(T)} + \lambda^{-1/2}\|\hat{f}^0\|_{\mathbb{L}_p(T)}\right). \tag{5.12}$$

Actually, Theorem 4.4 of [11] is proved there only for $T = \infty$, but it is a standard fact that such an estimate implies what we need for any T (cf. the proof of Theorem 6.4.1 of [12]). Since the norms in $\mathcal{L}_p$ and W^1_p are translation invariant, (5.12) implies (3.3) and finishes the proof of the lemma in case $c \equiv 0$.

Our next step is to abandon the condition $c \equiv 0$ but assume that for an $S > -\infty$ we have $u_t = f^j_t = 0$ for $t \leq S$. Observe that without loss of generality we may assume that $T < \infty$. In that case introduce

$$\xi_t = \exp\left(\int_S^t c_s\,ds\right).$$

Then we have $v := \xi u \in \mathbb{W}^1_p(T)$ and

$$\partial_t v_t = D_i(a^{ij}_t D_j v_t + [\mathfrak{b}^i_t + b^i_t]v_t + \xi_t f^i_t) + \xi_t f^0_t - \lambda v_t, \quad t \leq T.$$

By the above result for all $T' \leq T$

$$\int_{-\infty}^{T'} \xi^p_t \|Du_t\|^p_{\mathcal{L}_p}\,dt + \lambda^{p/2}\int_{-\infty}^{T'} \xi^p_t \|u_t\|^p_{\mathcal{L}_p}\,dt$$
$$\leq N_1 \sum_{i=0}^d \int_{-\infty}^{T'} \xi^p_t \|f^i_t\|^p_{\mathcal{L}_p}\,dt + N_1\lambda^{-p/2}\int_{-\infty}^{T'} \xi^p_t \|f^0_t\|^p_{\mathcal{L}_p}\,dt. \tag{5.13}$$

We multiply both part of (5.13) by $pc_{T'}\xi^{-p}_{T'}$ and integrate with respect to T' over (S, T). We use integration by parts observing that both parts vanish at $T' = S$.

Then we obtain

$$\begin{aligned}
&\int_{-\infty}^{T} \|Du_t\|_{\mathcal{L}_p}^p\,dt + \lambda^{p/2}\int_{-\infty}^{T} \|u_t\|_{\mathcal{L}_p}^p\,dt \\
&\quad - \xi_T^{-p}\int_{-\infty}^{T} \xi_t^p\|Du_t\|_{\mathcal{L}_p}^p\,dt - \xi_T^{-p}\lambda^{p/2}\int_{-\infty}^{T} \xi_t^p\|u_t\|_{\mathcal{L}_p}^p\,dt \\
&\le N_1\sum_{i=0}^{d}\int_{-\infty}^{T} \|f_t^i\|_{\mathcal{L}_p}^p\,dt + N_1\lambda^{-p/2}\int_{-\infty}^{T} \|f_t^0\|_{\mathcal{L}_p}^p\,dt \\
&\quad - \xi_T^{-p}N_1\sum_{i=0}^{d}\int_{-\infty}^{T} \xi_t^p\|f_t^i\|_{\mathcal{L}_p}^p\,dt - \xi_T^{-p}N_1\lambda^{-p/2}\int_{-\infty}^{T} \xi_t^p\|f_t^0\|_{\mathcal{L}_p}^p\,dt.
\end{aligned}$$

By adding up this inequality with (5.13) with $T' = T$ multiplied by ξ_T^{-p} we obtain (3.3).

The last step is to avoid assuming that $u_t = 0$ for large negative t. In that case we find a sequence $S_n \to -\infty$ such that $u_{S_n} \to 0$ in W_p^1 and denote by v_t^n the unique solution of class $\mathbb{W}_p^1((0,1)\times\mathbb{R}^d)$ of the heat equation $\partial v_t^n = \Delta v_t^n$ with initial condition u_{S_n}. After that we modify u_t and the coefficients of L_t for $t \le S_n$ as in the proof of Theorem 3.2 by taking there v_t^n and S_n in place of v_t and S, respectively. Then by the above result we obtain

$$\begin{aligned}
\lambda\|u\|_{\mathbb{L}_p(S_n,T)}^2 + \|Du\|_{\mathbb{L}_p(S_n,T)}^2 &\le N\left(\sum_{i=1}^{d}\|\tilde f^i\|_{\mathbb{L}_p(T)}^2 + \lambda^{-1}\|\tilde f^0\|_{\mathbb{L}_p(T)}^2\right), \\
&\le N\left(\sum_{i=1}^{d}\|f^i\|_{\mathbb{L}_p(T)}^2 + \lambda^{-1}\|f^0\|_{\mathbb{L}_p(T)}^2\right) + N(1+\lambda^{-1})\|u_{S_n}\|_{W_p^1}^p.
\end{aligned}$$

By letting $n \to \infty$ we come to (3.3) and the lemma is proved. □

Remark 5.1. In [11] the assumption corresponding to Assumption 3.3 is much weaker since in the corresponding counterpart of (3.2) there is no supremum over $x \in \mathbb{R}^d$. We need our stronger assumption because we need $a_t^{ij}(x - B_t)$ to satisfy the assumption in [11] for any function B_t.

To proceed further we need a construction. Recall that $\bar{\mathfrak{b}}$ and $\bar b$ are introduced in (5.5). From Lemma 4.2 of [13] and Assumption 3.2 it follows that, for $h_t = \bar{\mathfrak{b}}_t, \bar b_t$, it holds that $|D^n h_t| \le \kappa_n$, where $\kappa_n = \kappa_n(n,d,p,\rho_1,K) \ge 1$ and $D^n h_t$ is any derivative of h_t of order $n \ge 1$ with respect to x. By Corollary 4.3 of [13] we have $|h_t(x)| \le K(t)(1+|x|)$, where the function $K(t)$ is locally integrable with respect to t on $\mathbb{R}$. Owing to these properties, for any $(t_0,x_0) \in \mathbb{R}^{d+1}$, the equation

$$x_t = x_0 - \int_{t_0}^{t}(\bar{\mathfrak{b}}_s + \bar b_s)(x_s)\,ds, \quad t \ge t_0,$$

has a unique solution $x_t = x_{t_0,x_0,t}$.

Next, for $i=1,2$ set $\chi^{(i)}(x)$ to be the indicator function of $B_{\rho_1/i}$ and introduce

$$\chi^{(i)}_{t_0,x_0,t}(x)=\chi^{(i)}(x-x_{t_0,x_0,t}).$$

Here is a crucial estimate.

Lemma 5.8. *Suppose that Assumptions* 3.1, 3.2, *and* 3.3 *are satisfied with a* $\gamma\in(0,\gamma(d,p,\delta)]$, *where* $\gamma(d,p,\delta)$ *is taken from Lemma* 5.7. *Take* $(t_0,x_0)\in\mathbb{R}^{d+1}$ *and assume that* $t_0<T$ *and that we are given a function* u *which is a solution of* (2.1) *with* $S=t_0$, *with zero initial condition, some* $f^j\in\mathbb{L}_p(t_0,T)$, *and* $\lambda\geq\lambda_0$, *where* $\lambda_0=\lambda_0(d,\delta,p,\rho_0)$ *is taken from Lemma* 5.7. *Then*

$$\begin{aligned}
&\lambda\|\chi^{(2)}_{t_0,x_0}u\|^2_{\mathbb{L}_p(t_0,T)}+\|\chi^{(2)}_{t_0,x_0}Du\|^2_{\mathbb{L}_p(t_0,T)}\\
&\quad\leq N\sum_{i=1}^d\|\chi^{(1)}_{t_0,x_0}f^i\|^2_{\mathbb{L}_p(t_0,T)}+N\lambda^{-1}\|\chi^{(1)}_{t_0,x_0}f^0\|^2_{\mathbb{L}_p(t_0,T)}\\
&\qquad+N\gamma^{2/q}\|\chi^{(1)}_{t_0,x_0}Du\|^2_{\mathbb{L}_p(t_0,T)}+N^*\lambda^{-1}\|\chi^{(1)}_{t_0,x_0}Du\|^2_{\mathbb{L}_p(t_0,T)}\\
&\qquad+N^*\|\chi^{(1)}_{t_0,x_0}u\|^2_{\mathbb{L}_p(t_0,T)}+N^*\lambda^{-1}\sum_{i=1}^d\|\chi^{(1)}_{t_0,x_0}f^i\|^2_{\mathbb{L}_p(t_0,T)},
\end{aligned}\tag{5.14}$$

where and below in the proof by N *we denote generic constants depending only on* d,δ, *and* p *and by* N^* *constants depending only on the same objects,* γ, ρ_1, *and* K.

Proof. Shifting the origin allows us to assume that $t_0=0$ and $x_0=0$. With this stipulations we will drop the subscripts t_0,x_0.

Fix a $\zeta\in C_0^\infty$ with support in B_{ρ_1} and such that $\zeta=1$ on $B_{\rho_1/2}$ and $0\leq\zeta\leq1$. Set $x_t=x_{0,0,t}$,

$$\hat{\mathfrak{b}}_t=\bar{\mathfrak{b}}_t(x_t),\quad \hat{b}_t=\bar{b}_t(x_t),\quad \hat{c}_t=\bar{c}_t(x_t)$$

$$\eta_t(x)=\zeta(x-x_t),\quad v_t(x)=u_t(x)\eta_t(x).$$

The most important property of η_t is that

$$\partial_t\eta_t=(\hat{\mathfrak{b}}^i_t+\hat{b}^i_t)D_i\eta_t.$$

Also observe for the later that we may assume that

$$\chi^{(2)}_t\leq\eta_t\leq\chi^{(1)}_t,\quad |D\eta_t|\leq N\rho_1^{-1}\chi^{(1)}_t,\tag{5.15}$$

where $\chi^{(i)}_t=\chi^{(i)}_{0,0,t}$ and $N=N(d)$.

By Corollary 4.4 (also see the argument before (5.11)) we obtain that for finite $t\in[0,T]$

$$\begin{aligned}
\partial_t v_t&=D_i(\eta_t a^{ij}_tD_ju_t+\mathfrak{b}^i_tv_t)-(a^{ij}_tD_ju_t+\mathfrak{b}^i_tu_t)D_i\eta_t\\
&\quad+b^i_t\eta_tD_iu_t-(c_t+\lambda)v_t+D_i(f^i_t\eta_t)-f^i_tD_i\eta_t+f^0_t\eta_t+(\hat{\mathfrak{b}}^i_t+\hat{b}^i_t)u_tD_i\eta_t.
\end{aligned}$$

We transform this further by noticing that

$$\eta_ta^{ij}_tD_ju_t=a^{ij}_tD_jv_t-a^{ij}_tu_tD_j\eta_t.$$

To deal with the term $b^i_t\eta_t D_i u_t$ we use Corollary 5.4 and find the corresponding functions V^j_t. Then simple arithmetics show that

$$\partial_t v_t = D_i\big(a^{ij}_t D_j v_t + \hat{\mathfrak{b}}^i_t v_t\big) - (\hat{c}_t + \lambda)v_t + \hat{b}^i_t D_i v_t + D_i \hat{f}^i_t + \hat{f}^0_t,$$

where

$$\hat{f}^0_t = f^0_t\eta_t - f^i_t D_i\eta_t - a^{ij}_t(D_j u_t)D_i\eta_t + (\hat{\mathfrak{b}}^i_t - \mathfrak{b}^i_t)u_t D_i\eta_t + V^0_t + (\hat{c}_t - c_t)u_t\eta_t,$$
$$\hat{f}^i_t = f^i_t\eta_t - a^{ij}_t u_t D_j\eta_t + (\mathfrak{b}^i_t - \hat{\mathfrak{b}}^i_t)u_t\eta_t + V^i_t, \quad i = 1,\dots,d.$$

It we extend u_t and f^j_t as zero for $t < 0$, then it will be seen from Lemma 5.7 that for $\lambda \geq \lambda_0$

$$\lambda\|v\|^2_{\mathbb{L}_p(0,T)} + \|Dv\|^2_{\mathbb{L}_p(0,T)} \leq N\sum_{i=1}^d \|\hat{f}^i\|^2_{\mathbb{L}_p(0,T)} + N\lambda^{-1}\|\hat{f}^0\|^2_{\mathbb{L}_p(0,T)}. \tag{5.16}$$

Recall that here and below by N we denote generic constants depending only on d, δ, and p.

Now we start estimating the right-hand side of (5.16). First we deal with $\hat{f}^i_t$. Recall (5.15) and use Corollary 5.4 to get

$$\|(\mathfrak{b}^i_t - \hat{\mathfrak{b}}^i_t)u_t\eta_t\|^2_{\mathcal{L}_p} \leq N\gamma^{2/q}\|\chi^{(1)}_t Du_t\|^2_{\mathcal{L}_p} + N^*\|\chi^{(1)}_t u_t\|^2_{\mathcal{L}_p} \tag{5.17}$$

(we remind the reader that by N^* we denote generic constants depending only on d, δ, p, γ, ρ_1, and K). By adding that

$$\|a^{ij}uD_j\eta\|^2_{\mathbb{L}_p(0,T)} \leq N^*\|\chi^{(1)}_\cdot u\|^2_{\mathbb{L}_p(0,T)},$$

we derive from (5.8) and (5.17) that

$$\begin{aligned}\sum_{i=1}^d \|\hat{f}^i\|^2_{\mathbb{L}_p(0,T)} &\leq N\sum_{i=1}^d \|\chi^{(1)}_\cdot f^i\|^2_{\mathbb{L}_p(0,T)} \\ &\quad + N\gamma^{2/q}\|\chi^{(1)}_\cdot Du\|^2_{\mathbb{L}_p(0,T)} + N^*\|\chi^{(1)}_\cdot u\|^2_{\mathbb{L}_p(0,T)}.\end{aligned} \tag{5.18}$$

While estimating $\hat{f}^0$ we use (5.8) again and observe that we can deal with $(\hat{\mathfrak{b}}^i_t - \mathfrak{b}^i_t)u_t D_i\eta_t$ and $(c_t - \hat{c}_t)u_t\eta_t$ as in (5.17) this time without paying too much attention to the dependence of our constants on γ, ρ_1, and K and obtain that

$$\begin{aligned}&\|(\hat{\mathfrak{b}}^i - \mathfrak{b}^i)uD_i\eta\|^2_{\mathbb{L}_p(0,T)} + \|(c - \hat{c})u\eta\|^2_{\mathbb{L}_p(0,T)} \\ &\leq N^*(\|\chi^{(1)}_\cdot Du\|^2_{\mathbb{L}_p(0,T)} + \|\chi^{(1)}_\cdot u\|^2_{\mathbb{L}_p(0,T)}).\end{aligned}$$

By estimating also roughly the remaining terms in $\hat{f}^0$ and combining this with (5.18) and (5.16), we see that the left-hand side of (5.16) is less than the right-hand side of (5.14). However,

$$|\chi^{(2)}_t Du_t| \leq |\eta_t Du_t| \leq |Dv_t| + |u_t D\eta_t| \leq |Dv_t| + N\rho_1^{-1}|u_t\chi^{(1)}_t|$$

which easily leads to (5.14). The lemma is proved.

Next, from the result giving “local” in space estimates we derive global in space estimates but for functions having, roughly speaking, small “past” support in the time variable. In the following lemma κ_1 is the number introduced before Lemma 5.8.

Lemma 5.9. *Suppose that Assumptions* 3.1, 3.2, *and* 3.3 *are satisfied with a* $\gamma \in (0,\gamma(d,p,\delta)]$, *where* $\gamma(d,p,\delta)$ *is taken from Lemma* 5.7. *Assume that* u *is a solution of* (2.1) *with* $S=-\infty$, *some* $f^j \in \mathbb{L}_p(T)$, *and* $\lambda \geq \lambda_0$, *where* $\lambda_0=\lambda_0(d,\delta,p,\rho_0)$ *is taken from Lemma* 5.7. *Take a finite* $t_0 \leq T$ *and assume that* $u_t=0$ *if* $t\leq t_0$. *Then for* $I_{t_0} := I_{(t_0,T')}$, *where* $T'=(t_0+\kappa_1^{-1})\wedge T$, *we have*

$$\begin{aligned}\lambda^{p/2}\|I_{t_0}u\|^p_{\mathbb{L}_p}+\|I_{t_0}Du\|^p_{\mathbb{L}_p} &\leq N\sum_{i=1}^d \|I_{t_0}f^i\|^p_{\mathbb{L}_p}+N\lambda^{-p/2}\|I_{t_0}f^0\|^p_{\mathbb{L}_p} \\ &\quad + N\gamma^{p/q}\|I_{t_0}Du\|^p_{\mathbb{L}_p}+N^*\lambda^{-p/2}\|I_{t_0}Du\|^p_{\mathbb{L}_p} \\ &\quad + N^*\|I_{t_0}u\|^p_{\mathbb{L}_p}+N^*\lambda^{-p/2}\sum_{i=1}^d\|I_{t_0}f^i\|^p_{\mathbb{L}_p},\end{aligned} \tag{5.19}$$

where and below in the proof by N *we denote generic constants depending only on* d,δ, *and* p *and by* N^* *constants depending only on the same objects,* γ, ρ_1, *and* K.

Proof. Take $x_0\in\mathbb{R}^d$ and use the notation introduced before in Lemma 5.8. By this lemma with T' in place of T we have

$$\begin{aligned}&\lambda^{p/2}\|I_{t_0}\chi^{(2)}_{t_0,x_0}u\|^p_{\mathbb{L}_p}+\|I_{t_0}\chi^{(2)}_{t_0,x_0}Du\|^p_{\mathbb{L}_p} \\ &\leq N\sum_{i=1}^d\|I_{t_0}\chi^{(1)}_{t_0,x_0}f^i\|^p_{\mathbb{L}_p}+N\lambda^{-p/2}\|I_{t_0}\chi^{(1)}_{t_0,x_0}f^0\|^p_{\mathbb{L}_p} \\ &\quad + N\gamma^{p/q}\|I_{t_0}\chi^{(1)}_{t_0,x_0}Du\|^p_{\mathbb{L}_p}+N^*\lambda^{-p/2}\|I_{t_0}\chi^{(1)}_{t_0,x_0}Du\|^p_{\mathbb{L}_p} \\ &\quad + N^*\|I_{t_0}\chi^{(1)}_{t_0,x_0}u\|^p_{\mathbb{L}_p}+N^*\lambda^{-p/2}\sum_{i=1}^d\|I_{t_0}\chi^{(1)}_{t_0,x_0}f^i\|^p_{\mathbb{L}_p}.\end{aligned} \tag{5.20}$$

One knows that for each $t\geq t_0$, the mapping $x_0\to x_{t_0,x_0,t}$ is a diffeomorphism with Jacobian determinant given by

$$\left|\frac{\partial x_{t_0,x_0,t}}{\partial x_0}\right| = \exp\Big(-\int_{t_0}^t\sum_{i=1}^d D_i[\bar{\mathfrak{b}}^i_s+\bar{b}^i_s](x_{t_0,x_0,s})\,ds\Big).$$

By the way the constant κ_1 is introduced, we have

$$e^{-N\kappa_1(t-t_0)}\leq\left|\frac{\partial x_{t_0,x_0,t}}{\partial x_0}\right|\leq e^{N\kappa_1(t-t_0)},$$

where N depends only on d. Therefore, for any nonnegative Lebesgue measurable function $w(x)$ it holds that

$$e^{-N\kappa_1(t-t_0)}\int_{\mathbb{R}^d}w(y)\,dy\leq\int_{\mathbb{R}^d}w(x_{t_0,x_0,t})\,dx_0\leq e^{N\kappa_1(t-t_0)}\int_{\mathbb{R}^d}w(y)\,dy.$$

In particular, since

$$\int_{\mathbb{R}^d} |\chi^{(i)}_{t_0,x_0,t}(x)|^p\,dx_0 = \int_{\mathbb{R}^d} |\chi^{(i)}(x - x_{t_0,x_0,t})|^p\,dx_0,$$

we have

$$e^{-N\kappa_1(t-t_0)} = N_i^* e^{-N\kappa_1(t-t_0)} \int_{\mathbb{R}^d} |\chi^{(i)}(x-y)|^p\,dy$$
$$\le N_i^* \int_{\mathbb{R}^d} |\chi^{(i)}_{t_0,x_0,t}(x)|^p\,dx_0 \le N_i^* e^{N\kappa_1(t-t_0)} \int_{\mathbb{R}^d} |\chi^{(i)}(x-y)|^p\,dy = e^{N\kappa_1(t-t_0)},$$

where $N_i^* = |B_1|^{-1}\rho_1^{-d} i^d$ and $|B_1|$ is the volume of B_1. It follows that

$$\int_{\mathbb{R}^d} |\chi^{(1)}_{t_0,x_0,t}(x)|^p\,dx_0 \le (N_1^*)^{-1} e^{N\kappa_1(t-t_0)},$$
$$(N_2^*)^{-1} e^{-N\kappa_1(t-t_0)} \le \int_{\mathbb{R}^d} |\chi^{(2)}_{t_0,x_0,t}(x)|^p\,dx_0.$$

Furthermore, since $u_t = 0$ if $t \le t_0$ and $T' \le t_0 + \kappa_1^{-1}$, in evaluating the norms in (5.20) we need not integrate with respect to t such that $\kappa_1(t-t_0) \ge 1$ or $\kappa_1(t-t_0) \le 0$, so that for all t really involved we have

$$\int_{\mathbb{R}^d} |\chi^{(1)}_{t_0,x_0,t}(x)|^2\,dx_0 \le (N_1^*)^{-1} e^N, \quad (N_2^*)^{-1} e^{-N} \le \int_{\mathbb{R}^d} |\chi^{(2)}_{t_0,x_0,t}(x)|^2\,dx_0.$$

After this observation it only remains to integrate (5.20) through with respect to x_0 and use the fact that $N_1^* = 2^{-d} N_2^*$. The lemma is proved. □

Proof of Theorem 3.1. Obviously we may assume that $S = -\infty$. Then first we show how to choose an appropriate $\gamma = \gamma(d,\delta,p) \in (0,1]$. For one, we take it smaller than the one from Lemma 5.7. Then call N_0 the constant factor of $\gamma^{p/q}\|I_{t_0} Du\|^p_{\mathbb{L}_p}$ in (5.19). We know that $N_0 = N_0(d,\delta,p)$ and we choose $\gamma \in (0,1]$ so that $N_0\gamma^{p/q} \le 1/2$. Then under the conditions of Lemma 5.9 we have

$$\lambda^{p/2}\|I_{t_0}u\|^p_{\mathbb{L}_p} + \|I_{t_0}Du\|^p_{\mathbb{L}_p} \le N\sum_{i=1}^d \|I_{t_0}f^i\|^p_{\mathbb{L}_p} + N\lambda^{-p/2}\|I_{t_0}f^0\|^p_{\mathbb{L}_p}$$
$$+ N^*\lambda^{-p/2}\|I_{t_0}Du\|^p_{\mathbb{L}_p} + N^*\|I_{t_0}u\|^p_{\mathbb{L}_p} + N^*\lambda^{-p/2}\sum_{i=1}^d \|I_{t_0}f^i\|^p_{\mathbb{L}_p}. \tag{5.21}$$

After γ has been fixed we recall that $\kappa_1 = \kappa_1(d,p,\rho_1,K)$ and take a $\zeta \in C_0^\infty(\mathbb{R})$ with support in $(0,\kappa_1^{-1})$ such that

$$\int_{-\infty}^{\infty} \zeta^p(t)\,dt = 1. \tag{5.22}$$

For $s \in \mathbb{R}$ define $\zeta^s_t = \zeta(t-s)$, $u^s_t(x) = u_t(x)\zeta^s_t$. Obviously $u^s_t = 0$ if $t \le s \wedge T$. Therefore, we can apply (5.21) to u^s_t with $t_0 = s \wedge T$ observing that

$$\partial_t u^s_t = D_i(a^{ij}_t D_j u^s_t + \mathfrak{b}^i_t u^s_t) + b^i_t u^s_t - (c_t + \lambda)u^s_t + D_i(\zeta^s_t f^i_t) + \zeta^s_t f^0_t + (\zeta^s_t)' u_t.$$

Then from (5.21) for $\lambda \geq \lambda_0$, where $\lambda_0 = \lambda_0(d, \delta, p, \rho_0)$ is taken from Lemma 5.7, we obtain

$$
\begin{aligned}
&\lambda^{p/2} \|I_{s\wedge T}\zeta^s u\|^p_{\mathbb{L}_p} + \|I_{s\wedge T}\zeta^s Du\|^p_{\mathbb{L}_p} \\
&\quad \leq N \sum_{i=1}^d \|I_{s\wedge T}\zeta^s f^i\|^p_{\mathbb{L}_p} + N\lambda^{-p/2}\|I_{s\wedge T}\zeta^s f^0\|^p_{\mathbb{L}_p} + N\|I_{s\wedge T}(\zeta^s)' u\|^p_{\mathbb{L}_p} \\
&\qquad + N^*\lambda^{-p/2}\|I_{s\wedge T}\zeta^s Du\|^p_{\mathbb{L}_p} + N^*\|I_{s\wedge T}\zeta^s u\|^p_{\mathbb{L}_p} + N^*\lambda^{-p/2}\sum_{i=1}^d \|I_{s\wedge T}\zeta^s f^i\|^p_{\mathbb{L}_p}.
\end{aligned} \tag{5.23}
$$

We integrate through (5.23) with respect to $s \in \mathbb{R}$, observe that

$$
I_{s\wedge T < t < [(s\wedge T)+\kappa_1^{-1}]\wedge T} = I_{t<T} I_{s\wedge T < t < (s\wedge T)+\kappa_1^{-1}} = I_{t<T} I_{s<t<s+\kappa_1^{-1}},
$$

and that (5.22) yields

$$
\begin{aligned}
\int_{-\infty}^{\infty} I_{s\wedge T}(t)(\zeta^s_t)^p\,ds &= \int_{-\infty}^{\infty} I_{s\wedge T < t < [(s\wedge T)+\kappa_1^{-1}]\wedge T}\zeta^p(t-s)\,ds \\
&= I_{t<T}\int_{t-\kappa_1^{-1}}^{t} \zeta^p(t-s)\,ds = I_{t<T}.
\end{aligned}
$$

We also notice that, since κ_1 depends only on d, p, ρ_1, K, we have

$$
\int_{-\infty}^{\infty} |\zeta'(s)|^p\,ds = N^*.
$$

Then we conclude

$$
\begin{aligned}
&\lambda^{p/2}\|u\|^p_{\mathbb{L}_p(T)} + \|Du\|^p_{\mathbb{L}_p(T)} \\
&\quad \leq N_1 \sum_{i=1}^d \|f^i\|^p_{\mathbb{L}_p(T)} + N_1\lambda^{-p/2}\|f^0\|^p_{\mathbb{L}_p(T)} \\
&\qquad + N_1^*\lambda^{-p/2}\|Du\|^p_{\mathbb{L}_p(T)} + N_1^*\|u\|^p_{\mathbb{L}_p(T)} + N_1^*\lambda^{-p/2}\sum_{i=1}^d \|f^i\|^p_{\mathbb{L}_p(T)}.
\end{aligned}
$$

Without losing generality we assume that $N_1 \geq 1$ and we show how to choose $\lambda_0 = \lambda_0(d, \delta, p, \rho_0, \rho_1, K) \geq 1$. Above we assumed that $\lambda \geq \lambda_0(d, \delta, p, \rho_0)$, where $\lambda_0(d, \delta, p, \rho_0)$ is taken from Lemma 5.7. Therefore, we take

$$
\lambda_0 = \lambda_0(d, \delta, p, \rho_0, \rho_1, K) \geq \lambda_0(d, \delta, p, \rho_0)
$$

such that $\lambda_0^{p/2} \geq 2N_1^*$. Then we obviously come to (3.3) (with $S = -\infty$). The theorem is proved. $\square$

6. Proof of Theorems 3.3 and 3.4

We need two auxiliary results.

Lemma 6.1. *For any* $\tau, R \in (0,\infty)$*, we have*

$$\int_{-\tau}^{\tau}\int_{B_R}(|\mathfrak{b}_s(x)|^{p'}+|b_s(x)|^{p'}+c_s^{p'}(x))\,dxds<\infty. \tag{6.1}$$

Proof. Obviously it suffices to prove (6.1) with $B_{\rho_1}(x_0)$ in place of B_R for any x_0. In that case, for instance, (notice that $q \geq p'$, see (3.1))

$$\int_{B_{\rho_1}(x_0)}|\mathfrak{b}_s(x)|^{p'}\,dx \leq N\Big(\int_{B_{\rho_1}(x_0)}|\mathfrak{b}_s(x)-\bar{\mathfrak{b}}_s(x_0)|^q\,dx\Big)^{p'/q}+N|\bar{\mathfrak{b}}_s(x_0)|^{p'}.$$

According to (5.6)

$$\int_{B_{\rho_1}(x_0)}|\mathfrak{b}_s(x)|^{p'}\,dx \leq N+N|\bar{\mathfrak{b}}_s(x_0)|^{p'}$$

and in what concerns $\mathfrak{b}$ it only remains to use Assumption 3.1 (iii). Similarly, b_s and c_s are treated. The lemma is proved. □

The solution of our equation will be obtained as the weak limit of the solutions of equations with cut-off coefficients. Therefore, the following result is appropriate. By the way, observe that usual way of proving the existence of solutions based on a priori estimates and the method of continuity cannot work in our setting mainly because of what is said after Theorem 3.6.

Lemma 6.2. *Let* $\phi \in C_0^\infty$*,* $\tau \in (0,\infty)$*. Let* $u^m, u \in \mathbb{W}_p^1$*,* $m = 1,2,\dots$*, be such that* $u^m \to u$ *weakly in* $\mathbb{W}_p^1$*. For* $m = 1,2,\dots$ *define* $\chi_m(t) = (-m)\vee t \wedge m$*,* $\mathfrak{b}_{mt}^i = \chi_m(\mathfrak{b}_t^i)$*,* $b_{mt}^i = \chi_m(b_t^i)$*, and* $c_{mt} = \chi_m(c_t)$*. Then the functions*

$$\int_0^t (b_{ms}^i D_i u_s^m,\phi)\,ds, \quad \int_0^t (\mathfrak{b}_{ms}^i u_s^m, D_i\phi)\,ds, \quad \int_0^t (c_{ms}u_s^m,\phi)\,ds \tag{6.2}$$

converge weakly in the space $\mathcal{L}_p([-\tau,\tau])$ *as* $m\to\infty$ *to*

$$\int_0^t (b_s^i D_i u_s,\phi)\,ds, \quad \int_0^t (\mathfrak{b}_s^i u_s, D_i\phi)\,ds, \quad \int_0^t (c_s u_s,\phi)\,ds, \tag{6.3}$$

respectively.

Proof. By Corollary 5.6 and by the fact that (strongly) continuous operators are weakly continuous we obtain that

$$\int_0^t (b_s^i D_i u_s^m,\phi)\,ds \to \int_0^t (b_s^i D_i u_s,\phi)\,ds$$

as $m\to\infty$ weakly in the space $\mathcal{L}_p([-\tau,\tau])$. Therefore, in what concerns the first function in (6.2), it suffices to show that

$$\int_0^t (D_i u_s^m,(b_s^i - b_{ms}^i)\phi)\,ds \to 0$$

weakly in $\mathcal{L}_p([-\tau,\tau])$. In other words, it suffices to show that for any $\xi\in\mathcal{L}_{p'}([-\tau,\tau])$

$$\int_{-\tau}^{\tau}\xi_t\big(\int_0^t(D_iu^m_s,(b^i_s-b^i_{ms})\phi)\,ds\big)\,dt\to 0.$$

This relation is rewritten as

$$\int_{-\tau}^{\tau}(D_iu^m_s,\eta_s(b^i_s-b^i_{ms})\phi)\,ds\to 0, \tag{6.4}$$

where

$$\eta_s:=\int_s^{\tau\,\mathrm{sgn}\,s}\xi_t\,dt$$

is bounded on $[-\tau,\tau]$. However, by the dominated convergence theorem and Lemma 6.1, we have $\eta_s(b^i_s-b^i_{ms})\phi\to 0$ as $m\to\infty$ strongly in $\mathbb{L}_{p'}(-\tau,\tau)$ and by assumption $Du^m\to Du$ weakly in $\mathbb{L}_p(-\tau,\tau)$. This implies (6.4). Similarly, one proves our assertion about the remaining functions in (6.2). The lemma is proved. □

Proof of Theorem 3.3. Owing to Theorem 3.1 implying that the solution on $(-\infty,T]\cap\mathbb{R}$ is unique, without loss of generality we may assume that $T=\infty$. Define $\mathfrak{b}_{mt}$, b_{mt}, and c_{mt} as in Lemma 6.2 and consider equation (2.1) with $\mathfrak{b}_{mt}$, b_{mt}, and c_{mt} in place of $\mathfrak{b}_t$, b_t, and c_t, respectively. Obviously, $\mathfrak{b}_{mt}$, b_{mt}, and c_{mt} satisfy Assumption 3.2 with the same γ and K as $\mathfrak{b}_t$, b_t, and c_t do. By Theorem 3.1 and the method of continuity for $\lambda\geq\lambda_0(d,\delta,p,\rho_0,\rho_1,K)$ there exists a unique solution u^m of the modified equation on $\mathbb{R}$.

By Theorem 3.1 we also have

$$\|u^m\|_{\mathbb{L}_p}+\|Du^m\|_{\mathbb{L}_p}\leq N,$$

where N is independent of m. Hence the sequence of functions u^m is bounded in the space $\mathbb{W}^1_p$ and consequently has a weak limit point $u\in\mathbb{W}^1_p$. For simplicity of presentation we assume that the whole sequence u^m converges weakly to u. Take a $\phi\in C_0^\infty$. Then by Lemma 6.2 the functions (6.2) converge to (6.3) weakly in $\mathcal{L}_p([-\tau,\tau])$ as $m\to\infty$ for any τ. Obviously, the same is true for $(u^m_t,\phi)\to(u_t,\phi)$ and the remaining terms entering the equation for u^m_t. Hence, by passing to the weak limit in the equation for u^m_t we see that for any $\phi\in C_0^\infty$ equation (2.3) holds for almost any $s,t\in\mathbb{R}$.

Now notice that, for each $t\in\mathbb{R}$, owing to Corollary 5.5 the equation

$$\begin{aligned}(\hat u_t,\phi):=\int_0^1(u_s,\phi)\,ds+\int_0^1\Big(\int_s^t\big[(b^i_rD_iu_r-(c_r+\lambda)u_r+f^0_r,\phi)\\ -(a^{ij}_rD_ju_r+\mathfrak{b}^i_ru_r+f^i_r,D_i\phi)\big]\,dr\Big)\,ds\end{aligned} \tag{6.5}$$

defines a distribution. Furthermore, by the above for any $\phi\in C_0^\infty$ we have $(u_t,\phi)=(\hat u_t,\phi)$ (a.e.). A standard argument shows that for almost all $t\in\mathbb{R}$, $(u_t,\phi)=(\hat u_t,\phi)$ for any $\phi\in C_0^\infty$, that is $u_t=\hat u_t$ (a.e.) and $\hat u_t\in\mathbb{W}^1_p$. In particular, we see that we can replace u_r in (6.5) with $\hat u_r$. Finally, for any $t_1,t_2\in\mathbb{R}$

$$
\begin{aligned}
&(\hat u_{t_2},\phi)-(\hat u_{t_1},\phi)\\
&=\int_0^1\Big(\int_{t_1}^{t_2}\big[(b^i_rD_i\hat u_r-(c_r+\lambda)\hat u_r+f^0_r,\phi)-(a^{ij}_rD_j\hat u_r+\mathfrak{b}^i_r\hat u_r+f^i_r,D_i\phi)\big]\,dr\Big)\,ds\\
&=\int_{t_1}^{t_2}\big[(b^i_rD_i\hat u_r-(c_r+\lambda)\hat u_r+f^0_r,\phi)-(a^{ij}_rD_j\hat u_r+\mathfrak{b}^i_r\hat u_r+f^i_r,D_i\phi)\big]\,dr
\end{aligned}
$$

and the theorem is proved. □

Proof of Theorem 3.4. (i) One reduces the general case to the one that $u_S=0$ as in the proof of Theorem 3.2. Also, obviously, one can assume that λ is as large as we like, say satisfying (3.4), since S and T are finite. By continuing $u_t(x)$ as zero for $t\le S$ we see that we may assume that $S=\infty$. If we set $f^j_t=0$ for $t\ge T$ and use Theorem 3.3 about the existence of solutions on $(-\infty,\infty)$ along with Theorem 3.1, which guarantees uniqueness of solutions on $(-\infty,T]$, then we see that we only need to prove assertion (ii) of the theorem.

(ii) In the above proof of Theorem 3.3 we have constructed the unique solutions of our equations as the weak limits of the solutions of equations with cut-off coefficients. Therefore, if we knew that the result is true for equations with bounded coefficients, then we would obtain it in our general case as well.

Thus it only remains to concentrate on equations with bounded coefficients. Existence an uniqueness theorems also show that it suffices to prove that, if u is the solution corresponding to $p=p_2$, then $u\in\mathbb{W}^1_{p_1}$.

Take a $\zeta\in C_0^\infty(\mathbb{R}^{d+1})$ such that $\zeta(0)=1$, set $\zeta^n_t(x)=\zeta(t/n,x/n)$, and notice that $u^n_t:=u_t\zeta^n_t$ satisfies

$$\partial_tu^n_t=L_tu^n_t-\lambda u^n_t+D_if^i_{nt}+f^0_{nt},$$

where

$$
\begin{aligned}
f^i_{nt}&=f^i_t\zeta^n_t-u_ta^{ij}_tD_j\zeta^n_t,\quad i\ge1,\\
f^0_{nt}&=f^0_t\zeta^n_t-f^i_tD_i\zeta^n_t-(a^{ij}_tD_ju_t+a^i_tu_t)D_i\zeta^n_t-b^i_tu_tD_i\zeta^n_t+u_t\partial_t\zeta^n_t.
\end{aligned}
$$

Since u^n_t has compact support and $p_1\le p_2$, it holds that $u^n\in\mathbb{W}^1_p$ for any $p\in[1,p_2]$ and by Theorem 3.1 for $p\in[p_1,p_2]$ we have

$$\|u^n\|_{\mathbb{W}^1_p}\le N\sum_{i=0}^d\|f^i_n\|_{\mathbb{L}_p}. \tag{6.6}$$

One knows that

$$\|f^i\|_{\mathbb{L}_p}\le N(\|f^i\|_{\mathbb{L}_{p_1}}+\|f^i\|_{\mathbb{L}_{p_2}}),$$

so that by Hölder's inequality

$$\|f^i_n\|_{\mathbb{L}_p}\le N+N\|uD\zeta^n\|_{\mathbb{L}_p}\le N+\|u\|_{\mathbb{L}_{p_2}}\|D\zeta^n\|_{\mathbb{L}_q},$$

with constants N independent of n, where

$$q = \frac{pp_2}{p_2 - p}.$$

Similar estimates are available for other terms in the right-hand side of (6.6). Since

$$\|\partial_t \zeta^n\|_{\mathbb{L}_q} + \|D\zeta^n\|_{\mathbb{L}_q} = Nn^{-1+(p_2-p)(d+1)/(p_2p)} \to 0$$

as $n \to \infty$ if

$$\frac{1}{p} - \frac{1}{p_2} < \frac{1}{d+1}, \tag{6.7}$$

estimate (6.6) implies that $u \in \mathbb{W}^1_p$.

Thus knowing that $u \in \mathbb{W}^1_{p_2}$ allowed us to conclude that $u \in \mathbb{W}^1_p$ as long as $p \in [p_1, p_2]$ and (6.7) holds. We can now replace p_2 with a smaller p and keep going in the same way each time increasing $1/p$ by the same amount until p reaches p_1. Then we get that $u \in \mathbb{W}^1_{p_1}$. The theorem is proved □

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Global Properties of Transition Kernels Associated with Second-order Elliptic Operators

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Dedicated to Prof. H. Amann on the occasion of his 70th birthday

Abstract. We study global regularity properties of transitions kernels associated with second-order differential operators in $\mathbf{R}^N$ with unbounded drift and potential terms. Under suitable conditions, we prove Sobolev regularity of transition kernels and pointwise upper bounds. As an application, we obtain sufficient conditions implying the differentiability of the associated semigroup on the space of bounded and continuous functions on $\mathbf{R}^N$.

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1. Introduction

Given a second-order elliptic partial differential operator with real coefficients

$$A = \sum_{i,j=1}^{N} D_i \left(a_{ij} D_j\right) + \sum_{i=1}^{N} F_i D_i - V = A_0 + F \cdot D - V, \tag{1.1}$$

where $A_0 = \sum_{i,j=1}^{N} D_i \left(a_{ij} D_j\right)$, we consider the parabolic problem

$$\begin{cases} u_t(x,t) = Au(x,t), & x \in \mathbf{R}^N,\ t > 0, \\ u(x,t) = f(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.2}$$

where $f \in C_b(\mathbf{R}^N)$.

We assume the following conditions on the coefficients of A which will be kept in the whole paper without further mentioning.

(H) $a_{ij} = a_{ji}$, $F_i : \mathbf{R}^N \to \mathbf{R}$, $V : \mathbf{R}^N \to [0,+\infty)$, with $a_{ij} \in C^{1+\alpha}(\mathbf{R}^N)$, V, $F_i \in C^{\alpha}_{\mathrm{loc}}(\mathbf{R}^N)$ for some $0<\alpha<1$ and

$$\lambda|\xi|^2 \le \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for every $x,\xi \in \mathbf{R}^N$ and suitable $0<\lambda\le\Lambda$.

Notice that neither the drift $F=(F_1,\dots,F_N)$ or the potential V are assumed to be bounded in $\mathbf{R}^N$.

Problem (1.2) has always a bounded solution but, in general, there is no uniqueness. However, if f is nonnegative, it is not difficult to show that (1.2) has a minimal solution u among all non negative solutions. Taking such a solution u one constructs a semigroup of positive contractions $T(\cdot)$ on $C_b(\mathbf{R}^N)$ such that

$$u(x,t) = T(t)f(x), \quad t>0,\, x\in\mathbf{R}^N$$

solves (1.2). Furthermore, the semigroup can be represented in the form

$$T(t)f(x) = \int_{\mathbf{R}^N} p(x,y,t)f(y)\,dy, \quad t>0,\, x\in\mathbf{R}^N,$$

for $f\in C_b(\mathbf{R}^N)$. Here p is a positive function and for almost every $y\in\mathbf{R}^N$, it belongs to $C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\mathbf{R}^N\times(0,\infty))$ as a function of (x,t) and solves the equation $\partial_t p = Ap$, $t>0$. We refer to Section 2, [8, Chapter 1] and [11] (in the case $V=0$) for a review of these results as well as for conditions ensuring uniqueness for (1.2).

Now, we fix $x\in\mathbf{R}^N$ and consider p as a function of (y,t). Then p satisfies

$$\partial_t p = A^* p, \quad t>0, \tag{1.3}$$

in the following sense (see [10, Lemma 2.1]): Let $0\le t_1<t_2$ and $\varphi\in C^{2,1}(Q(t_1,t_2))$ (see below for the notation) be such that $\varphi(\cdot,t)$ has compact support for every $t\in[t_1,t_2]$. Then

$$\begin{aligned}\int_{Q(t_1,t_2)} &(\partial_t\varphi(y,t)+A\varphi(y,t))\,p(x,y,t)\,dy\,dt \\ &= \int_{\mathbf{R}^N}\left(p(x,y,t_2)\varphi(y,t_2)-p(x,y,t_1)\varphi(y,t_1)\right)\,dy.\end{aligned}\tag{1.4}$$

The aim of this paper is to study global regularity properties of the kernel p as a function of $(y,t)\in\mathbf{R}^N\times(a,T)$ for $0<a<T$.

We prove that $p(x,\cdot,\cdot)$ belongs to $W^{1,0}_k(\mathbf{R}^N\times(a,T))$ (see below for the notation) provided that

$$\int_{a_0}^T\int_{\mathbf{R}^N}\left(V(y)^k+|F(y)|^k\right)p(x,y,t)\,dy\,dt<\infty, \quad \forall k>1$$

for fixed $x \in \mathbf{R}^N$ and $0 < a_0 < a$. This generalizes [10, Corollary 3.1 and Lemma 3.1] and in some sense Theorem 4.1 in [2]. Assuming that certain Lyapunov functions (exponentials or powers) are integrable with respect to $p(x,y,t)\,dy$ for $(x,t) \in \mathbf{R}^N \times (a,T)$, pointwise upper bounds for p are obtained. If in addition $V \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^N)$, $F \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^N, \mathbf{R}^N)$ are such that DV, DF are dominated by some exponential functions, then $p \in W^{2,1}_k(\mathbf{R}^N \times (a,T))$ for all $k > 1$. As a consequence, we obtain also upper bounds for $|D_y p|$. In the case where F and V and their corresponding derivatives up to the second order satisfy growth conditions of exponential type, upper bounds are also obtained for $|D_{yy}p|$ and $|\partial_t p|$. As a consequence, we deduce that the semigroup $T(\cdot)$ is differentiable on $C_b(\mathbf{R}^N)$ for $t > 0$.

In the case where $V = 0$, regularity and pointwise estimates for p can be found in [10], [14] and for the solution of (1.3) with a L^1-function as the initial datum we refer to [3], [4].

Other bounds for the transition densities p are obtained in [1], using time dependent Lyapunov functions techniques.

Notation. $B_R(x)$ denotes the open ball of $\mathbf{R}^N$ of radius R and center x. If $x = 0$ we simply write B_R. For $0 \le a < b$, we use $Q(a,b)$ for $\mathbf{R}^N \times (a,b)$ and Q_T for $Q(0,T)$ (here the intervals can be either open or closed). We write $C = C(a_1, \dots, a_n)$ to point out that the constant C depends on the quantities $a_1, \dots, a_n$. To simplify the notation, we understand the dependence on the dimension N and on quantities determined by the matrix (a_{ij}) as the ellipticity constant or the modulus of continuity of the coefficients.

If $u : \mathbf{R}^N \times J \to \mathbf{R}$, where $J \subset [0,\infty[$ is an interval, we use the following notation:

$$\partial_t u = \frac{\partial u}{\partial t}, \; D_i u = \frac{\partial u}{\partial x_i}, \; D_{ij} u = D_i D_j u$$
$$Du = (D_1 u, \dots, D_N u), \; D^2 u = (D_{ij} u)$$

and

$$|Du|^2 = \sum_{i=1}^N |D_i u|^2, \qquad |D^2 u|^2 = \sum_{i,j=1}^N |D_{ij} u|^2.$$

Let us come to notation for function spaces. $C^j_b(\mathbf{R}^N)$ is the space of j times differentiable functions in $\mathbf{R}^N$, with bounded derivatives up to the order j. $C^\infty_c(\mathbf{R}^N)$ is the space of test functions. $C^\alpha(\mathbf{R}^N)$ denotes the space of all bounded and α-Hölder continuous functions on $\mathbf{R}^N$.

For $1 \le k \le \infty$, $j \in \mathbf{N}$, $W^j_k(\mathbf{R}^N)$ denotes the classical Sobolev space of all L^k-functions having weak derivatives in $L^k(\mathbf{R}^N)$ up to the order j. Its usual norm is denoted by $\|\cdot\|_{j,k}$ and by $\|\cdot\|_k$ when $j = 0$.

Let us now define some spaces of functions of two variables (following basically the notation of [7]). $C_0(Q(a,b))$ is the Banach space of continuous functions u defined in $Q(a,b)$ such that $\lim_{|x|\to\infty} u(x,t) = 0$ uniformly with respect to $t \in$

$[a,b]$. $C^{2,1}(Q(a,b))$ is the space of all bounded functions u such that $\partial_t u$, Du and $D_{ij}u$ are bounded and continuous in $Q(a,b)$. For $0<\alpha\le 1$ we denote by $C^{2+\alpha,1+\alpha/2}(Q(a,b))$ the space of all bounded functions u such that $\partial_t u$, Du and $D_{ij}u$ are bounded and α-Hölder continuous in $Q(a,b)$ with respect to the parabolic distance $d((x,t),(y,s)) := |x-y|+|t-s|^{\frac12}$. Local Hölder spaces are defined, as usual, requiring that the Hölder condition holds in every compact subset.

We shall also use parabolic Sobolev spaces. We denote by $W_k^{r,s}(Q(a,b))$ the space of functions $u\in L^k(Q(a,b))$ having weak space derivatives $D_x^\alpha u\in L^k(Q(a,b))$ for $|\alpha|\le r$ and weak time derivatives $\partial_t^\beta u\in L^k(Q(a,b))$ for $\beta\le s$, equipped with the norm

$$\|u\|_{W_k^{r,s}(Q(a,b))} := \|u\|_{L^k(Q(a,b))} + \sum_{|\alpha|\le r}\|D_x^\alpha u\|_{L^k(Q(a,b))} + \sum_{|\beta|\le s}\|\partial_t^\beta u\|_{L^k(Q(a,b))}.$$

$\mathcal{H}^{k,1}(Q_T)$ denotes the space of all functions $u\in W_k^{1,0}(Q_T)$ with $\partial_t u\in (W_{k'}^{1,0}(Q_T))'$, the dual space of $W_{k'}^{1,0}(Q_T)$, endowed with the norm

$$\|u\|_{\mathcal{H}^{k,1}(Q_T)} := \|\partial_t u\|_{(W_{k'}^{1,0}(Q_T))'} + \|u\|_{W_k^{1,0}(Q_T)}$$

where $\frac1k+\frac1{k'}=1$. Finally, for $k>2$, $\mathcal{V}^k(Q_T)$ is the space of all functions $u\in W_k^{1,0}(Q_T)$ such that there exists $C>0$ for which

$$\left|\int_{Q_T} u\partial_t\phi\,dx\,dt\right| \le C\left(\|\phi\|_{L^{\frac{k}{k-2}}(Q_T)} + \|D\phi\|_{L^{\frac{k}{k-1}}(Q_T)}\right)$$

for every $\phi\in C_c^{2,1}(Q(a,b))$. Notice that $\frac{k}{k-1}=k'$, $\frac{k}{k-2}=\left(\frac k2\right)'$. $\mathcal{V}^k(Q_T)$ is a Banach space when endowed with the norm

$$\|u\|_{\mathcal{V}^k(Q_T)} = \|u\|_{W_k^{1,0}(Q_T)} + \|\partial_t u\|_{\frac k2,k;Q_T},$$

where $\|\partial_t u\|_{\frac k2,k;Q_T}$ is the best constant C such that the above estimate holds.

The space $\mathcal{H}^{k,1}(Q_T)$ was introduced and studied by Krylov [6]. All properties of the spaces $\mathcal{H}^{k,1}(Q_T)$ and $\mathcal{V}^k(Q_T)$ needed here, can be found in [10, Appendix].

In the whole paper the transition density p will be considered as a function of (y,t) for arbitrary but fixed $x\in\mathbf{R}^N$. The writing $\|p\|$ therefore stands for any norm of p as function of (y,t), for a fixed x.

2. Local regularity and integrability of transition densities

As a first step we recall some local regularity results for the kernel p associated with the minimal semigroup

$$T(t)f(x) = \int_{\mathbf{R}^N} p(x,y,t)f(y)\,dy,$$

i.e., the semigroup which defines the minimal bounded positive solutions of equation (1.2) when $f\ge 0$.

Regularity properties of the kernels p with respect to the variables (y,t) are known even under weaker conditions than our hypothesis (H), see [2]. We combine the results of [2] with the Schauder estimates to obtain regularity of p with respect to all the variables (x,y,t). The proof is similar to the one of Proposition 2.1 in [10].

Proposition 2.1. *Under assumption* (H) *the kernel* $p = p(x,y,t)$ *is a positive continuous function in* $\mathbf{R}^N \times \mathbf{R}^N \times (0,\infty)$ *which enjoys the following properties.*

(i) *For every* $x \in \mathbf{R}^N$, $1 < s < \infty$, *the function* $p(x,\cdot,\cdot)$ *belongs to* $\mathcal{H}^{s,1}_{\rm loc}(\mathbf{R}^N \times (0,\infty))$. *In particular* $p, D_y p \in L^s_{\rm loc}(\mathbf{R}^N \times (0,\infty))$ *and* $p(x,\cdot,\cdot)$ *is continuous.*

(ii) *For every* $y \in \mathbf{R}^N$ *the function* $p(\cdot,y,\cdot)$ *belongs to* $C^{2+\alpha,1+\alpha/2}_{\rm loc}(\mathbf{R}^N \times (0,\infty))$ *and solves the equation* $\partial_t p = Ap$, $t > 0$. *Moreover*

$$\sup_{|y|\le R} \|p(\cdot,y,\cdot)\|_{C^{2+\alpha,1+\alpha/2}(B_R\times[\varepsilon,T])} < \infty$$

for every $0 < \varepsilon < T$ *and* $R > 0$.

(iii) *If, in addition,* $F \in C^1(\mathbf{R}^N)$, *then* $p(x,\cdot,\cdot) \in W^{2,1}_{s,{\rm loc}}(Q_T)$ *for every* $x \in \mathbf{R}^N$, $1 < s < \infty$, *and satisfies the equation* $\partial_t p = A^*_y p$, *where*

$$A^* = A_0 - F\cdot D - (V + \operatorname{div} F)$$

is the formal adjoint of A.

The uniqueness of the bounded solution of (1.2) does not hold in general, but it is ensured by the existence of a *Lyapunov function* (cf. [10, Proposition 2.2]), that is a $C^{2+\alpha}_{\rm loc}$-function $W : \mathbf{R}^N \to [0,\infty)$ such that $\lim_{|x|\to\infty} W(x) = +\infty$ and $AW \le \lambda W$ for some $\lambda > 0$. Lyapunov functions are easily found imposing suitable conditions on the coefficients of A. For instance, $W(x) = |x|^2$ is a Lyapunov function for A provided that $\sum_i a_{ii}(x) + F(x)\cdot x - |x|^2 V(x) \le C|x|^2$ for some $C > 0$. The following result can be proved as in [10, Proposition 2.2].

Proposition 2.2. *Let* W *be a Lyapunov function for* A *and let* $u, v \in C_b(\mathbf{R}^N \times [0,T]) \cap C^{2,1}(\mathbf{R}^N \times (0,T])$ *solve* (1.2). *Then* $u = v$.

Now we turn our attention to integrability properties of p and show how they can be deduced from the existence of suitable Lyapunov functions. In the proof of Proposition 2.4 below we need to approximate the semigroup $(T(t))_{t\ge 0}$ with semigroups generated by uniformly elliptic operators. This is done in the next lemma.

Lemma 2.3. *Assume that* A *has a Lyapunov function* W. *Take* $\eta \in C^\infty_c(\mathbf{R})$ *with* $\eta(s) = 1$ *for* $|s| \le 1$, $\eta(s) = 0$ *for* $|s| \ge 2$, *and define* $\eta_n(x) = \eta\left(\left|\frac{x}{n}\right|\right)$, $F_n = \eta_n F$, $V_n := \eta_n V$ *and* $A_n = A_0 + F_n\cdot D - V_n$. *Consider the analytic semigroup* $(T_n(t))_{t\ge 0}$ *generated by* A_n *in* $C_b(\mathbf{R}^N)$. *Then, for every* $f \in C^{2+\alpha}(\mathbf{R}^N)$ *there exists a sequence* (n_k) *such that* $T_{n_k}(\cdot)f(\cdot) \to T(\cdot)f(\cdot)$ *in* $C^{2,1}(\mathbf{R}^N \times [0,T])$.

Proof. Let $u_n(x,t) = T_n(t)f(x)$, $u(x,t) = T(t)f(x)$ and fix a radius $\varrho > 0$. If $n > \varrho + 1$ the Schauder estimates for the operator A (see, e.g., [5, Theorem 8.1.1]) yield

$$\|u_n\|_{C^{2+\alpha,1+\alpha/2}(B_\varrho\times[0,T])} \le C_\varrho \|f\|_{C^{2+\alpha}(\mathbf{R}^N)}.$$

By a standard diagonal argument we find a subsequence (n_k) such that u_{n_k} converges to a function u in $C^{2,1}(\mathbf{R}^N \times (0,\infty))$. Since $\partial_t u_{n_k} - Au_{n_k} = 0$ in $B_\varrho \times [0,T]$ for $n_k > \varrho$ we have $\partial_t u - Au = 0$ in $\mathbf{R}^N \times [0,T]$. Moreover, $u(x,0) = f(x)$ and $|u(x,t)| \le \|f\|_\infty$, since this is true for u_n. By Proposition 2.2 we infer that $u(x,t) = T(t)f(x)$. $\square$

The integrability of Lyapunov functions with respect to the measures $p(x,y,t)\,dy$ is given by the following result, which is an extension of [12, Lemma 3.9], where the case $V=0$ is considered.

Proposition 2.4. *A Lyapunov function W is integrable with respect to the measures $p(x,y,t)dy$. Setting*

$$\zeta(x,t) = \int_{\mathbf{R}^N} p(x,y,t)W(y)\,dy, \tag{2.1}$$

the inequality $\zeta(t,x) \le e^{\lambda t}W(x)$ holds. Moreover, $|AW|$ is integrable with respect to $p(x,\cdot,t)$, $\zeta \in C^{2,1}(\mathbf{R}^N \times (0,\infty)) \cap C(\mathbf{R}^N \times [0,\infty))$ and $D_t\zeta(x,t) \le \int_{\mathbf{R}^N} p(x,y,t)AW(y)\,dy$.

Proof. For $\alpha \ge 0$, set $W_\alpha := W \wedge \alpha$ and $\zeta_\alpha(x,t) := \int_{\mathbf{R}^N} p(x,y,t)W_\alpha(y)\,dy$.

Let us consider, for every $0<\varepsilon<1$, $\psi_\varepsilon \in C^\infty(\mathbf{R})$ such that $\psi_\varepsilon(t) = t$ for $t \le \alpha$, ψ_ε constant in $[\alpha+\varepsilon,\infty)$, $\psi_\varepsilon' \ge 0$, and $\psi_\varepsilon'' \le 0$. Since $\psi_\varepsilon'' \le 0$ one deduces that

$$t\psi_\varepsilon'(t) \le \psi_\varepsilon(t), \quad \forall t \ge 0. \tag{2.2}$$

Now we approximate A with $A_n := A_0 + F_n \cdot \nabla - V_n$ and $(T(t))_{t\ge0}$ with $(T_n(t))_{t\ge0}$ as in Lemma 2.3. Denoting by $p_n(x,y,t)$ the kernel of $(T_n(t))_{t\ge0}$, since $\psi_\varepsilon \circ W \in C_b^{2+\alpha}(\mathbf{R}^N)$ we have

$$\partial_t T_n(t)(\psi_\varepsilon \circ W)(x) = \int_{\mathbf{R}^N} p_n(x,y,t)A_n(\psi_\varepsilon \circ W)(y)\,dy.$$

On the other hand, by (2.2), we obtain

$$\begin{aligned} A_n(\psi_\varepsilon \circ W)(x) &= \psi_\varepsilon'(W(x))A_nW(x) + V_n(x)\left[\psi_\varepsilon'(W(x))W(x) - \psi_\varepsilon(W(x))\right] \\ &\qquad + \psi_\varepsilon''(W(x))\sum_{i,j=1}^N a_{ij}(x)D_iW(x)D_jW(x) \\ &\le \psi_\varepsilon'(W(x))A_nW(x). \end{aligned}$$

Thus,

$$\partial_t T_n(t)(\psi_\varepsilon \circ W)(x) \le \int_{\mathbf{R}^N} p_n(x,y,t)\psi_\varepsilon'(W(y))A_nW(y)\,dy$$

and also

$$\partial_t T_n(t)(\psi_\varepsilon \circ W)(x) \le \int_{\mathbf{R}^N} p_n(x,y,t)\psi_\varepsilon'(W(y))AW(y)\,dy$$

if n is sufficiently large since, for fixed ε, the function $\psi_\varepsilon' \circ W$ has compact support. Letting $n \to \infty$ and using Lemma 2.3 (possibly passing to a subsequence) we deduce

$$\partial_t T(t)(\psi_\varepsilon \circ W)(x) \leq \int_{\mathbf{R}^N} p(x,y,t)\psi_\varepsilon'(W(y))AW(y)\,dy. \tag{2.3}$$

Next we observe that $\psi_\varepsilon \circ W \leq \alpha + 1$, $\psi_\varepsilon'(t) \to \chi_{(-\infty,\alpha]}(t)$, and $\psi_\varepsilon \circ W \to W_\alpha$ pointwise as $\varepsilon \to 0$. From [8, Proposition 2.2.9] we deduce that $T(t)(\psi_\varepsilon \circ W) \to T(t)W_\alpha$ in $C^{2,1}(\mathbf{R}^N \times (0,\infty))$. So, letting $\varepsilon \to 0$ in (2.3) and using dominated convergence in the right-hand side (all the integrals can be taken on the compact set $\{W \leq \alpha+1\}$, where AW is bounded) we get

$$D_t\zeta_\alpha(x,t) \leq \int_{\{W\leq\alpha\}} p(x,y,t)AW(y)\,dy. \tag{2.4}$$

To conclude we proceed as in the proof of [12, Lemma 3.9]. From (2.4) we obtain

$$D_t\zeta_\alpha(x,t) \leq \lambda\zeta_\alpha(x,t) \tag{2.5}$$

and hence, by Gronwall's lemma, $\zeta_\alpha(x,t) \leq e^{\lambda t}W_\alpha(x)$. Letting $\alpha \to \infty$ we obtain $\zeta(x,t) \leq e^{\lambda t}W(x)$ and then W is summable with respect to the measure $p(x,\cdot,t)$. The inequality $0 \leq \zeta_\alpha \leq \zeta$ and the interior Schauder estimates show that the family (ζ_α) is relatively compact in $C^{2,1}(\mathbf{R}^N \times (0,\infty))$. Since $\zeta_\alpha \to \zeta$ pointwise as $\alpha \to +\infty$, it follows that $\zeta \in C^{2,1}(\mathbf{R}^N \times (0,\infty))$. Moreover, the inequality $\zeta_\alpha(x,t) \leq \zeta(x,t) \leq e^{\lambda t}W(x)$ implies that $\zeta(\cdot,t) \to W(\cdot)$ as $t \to 0^+$, uniformly on compact sets. Set $E = \{x \in \mathbf{R}^N : AW(x) \geq 0\}$. Clearly

$$\int_E p(x,y,t)AW(y)\,dy \leq \lambda\int_E p(x,y,t)W(y)\,dy \leq \lambda\zeta(x,t) < \infty. \tag{2.6}$$

Moreover, letting $\alpha \to +\infty$ in (2.3), we obtain that

$$D_t\zeta(x,t) \leq \liminf_{\alpha\to+\infty}\int_{\{W\leq\alpha\}} p(x,y,t)AW(y)\,dy.$$

This fact and (2.6) imply that $|AW|$ is summable with respect to $p(x,\cdot,t)$ and that the above $\liminf$ is a limit, so that the proof is complete. □

Assuming that AW tends to $-\infty$ faster than $-W$ one obtains, by Proposition 2.4, that the function ζ in (2.1) is bounded with respect to the space variables, see [12, Theorem 3.10] for the case $V = 0$.

Proposition 2.5. *Assume that the Lyapunov function W satisfies the inequality $AW \leq -g(W)$ where $g : [0,\infty) \to \mathbf{R}$ is a differentiable convex function such that $g(0) \leq 0$, $\lim_{s\to+\infty} g(s) = +\infty$ and $1/g$ is integrable in a neighbourhood of $+\infty$. Then for every $a > 0$ the function ζ defined in (2.1) is bounded in $\mathbf{R}^N \times [a,\infty)$. Moreover, the semigroup $(T(t))_{t\geq0}$ is compact in $C_b(\mathbf{R}^N)$.*

Proof. Observe that $g(s) \leq sg'(s)$, since g is convex with $g(0) \leq 0$. Let us prove that

$$\int_{\mathbf{R}^N} p(x,y,t) g(W(y))\, dy \geq g(\zeta(x,t)). \tag{2.7}$$

For, fix x and t and set $s_0 = \zeta(x,t)$. Then, for all $y \in \mathbf{R}^N$ we have

$$g(W(y)) \geq g(s_0) + g'(s_0)(W(y) - s_0)$$

and therefore, multiplying by $p(x,y,t)$ and integrating

$$\begin{aligned}&\int_{\mathbf{R}^N} p(x,y,t) g(W(y))\, dy \\ &\quad \geq g(s_0) \int_{\mathbf{R}^N} p(x,y,t)\, dy + g'(s_0)s_0 \Big(1 - \int_{\mathbf{R}^N} p(x,y,t)\, dy\Big) \geq g(s_0).\end{aligned}$$

From Proposition 2.4 and (2.7) we deduce

$$D_t \zeta(x,t) \leq \int_{\mathbf{R}^N} p(x,y,t) AW(y)\, dy \leq -\int_{\mathbf{R}^N} p(x,y,t) g(W(y))\, dy \leq -g(\zeta(x,t))$$

and therefore $\zeta(x,t) \leq z(x,t)$, where z is the solution of the ordinary Cauchy problem

$$\begin{cases} z' = -g(z) \\ z(x,0) = W(x). \end{cases}$$

Let ℓ denote the greatest zero of g. Then $z(x,t) \leq \ell$ if $W(x) \leq \ell$. On the other hand, if $W(x) > \ell$, then z is decreasing and satisfies

$$t = \int_{z(x,t)}^{W(x)} \frac{ds}{g(s)} \leq \int_{z(x,t)}^{\infty} \frac{ds}{g(s)}. \tag{2.8}$$

This inequality easily yields, for every $a > 0$, a constant $C(a)$ such that $z(x,t) \leq C(a)$ for every $t \geq a$ and $x \in \mathbf{R}^N$.

The compactness of $(T(t))_{t\geq 0}$ in $C_b(\mathbf{R}^N)$ can be proved as in [12, Theorem 3.10]. □

Remark 2.6. If $\int_{\mathbf{R}^N} p(x,y,t)\, dy = 1$ (as is the case if $V = 0$) then (2.7) follows from Jensen's inequality and the condition $g(0) \leq 0$ is not needed.

Let us state a condition under which certain exponentials or polynomials are Lyapunov functions. Using the same procedure as for the case $V = 0$ (see [10, Proposition 2.5 and 2.6]) we obtain the following results.

Proposition 2.7. *Let Λ be the maximum eigenvalue of (a_{ij}) as in* (H). *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta} \left(F(x) \cdot \frac{x}{|x|} - \frac{V(x)}{\delta\beta|x|^{\beta-1}} \right) < -c, \tag{2.9}$$

$0 < c < \infty$, for some $c, \delta > 0, \beta > 1$ such that $\delta < (\beta\Lambda)^{-1}c$. Then $W(x) = \exp\{\delta|x|^\beta\}$ is a Lyapunov function. Moreover, if $\beta > 2$, there exist positive constants c_1, c_2 such that

$$\zeta(x,t) \le c_1 \exp\left(c_2 t^{-\beta/(\beta-2)}\right) \tag{2.10}$$

for $x \in \mathbf{R}^N,\ t > 0$.

Proposition 2.8. *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|} - \frac{|x|}{2\alpha}V(x)\right) < 0, \tag{2.11}$$

for some $\alpha > 0,\ \beta > 2$. Then $W(x) = \left(1+|x|^2\right)^\alpha$ is a Lyapunov function and there exists a positive constant c such that

$$\zeta(x,t) \le ct^{-(2\alpha)/(\beta-2)} \tag{2.12}$$

for $x \in \mathbf{R}^N,\ 0 < t \le 1$.

Remark 2.9. Proposition 2.7 will be used to check the integrability of $|F|^k$ and V^k with respect to p, assuming that $|F|, V$ grow at infinity not faster than $\exp\{|x|^\gamma\}$ for some $\gamma < \beta$.

3. Uniform and pointwise bounds on transition densities

In this section we fix $T > 0$ and consider p as a function of $(y,t) \in \mathbf{R}^N \times (0,T)$ for arbitrary, but fixed, $x \in \mathbf{R}^N$. Further, fix $0 < a_0 < a < b < b_0 \le T$ and assume for definiteness $b_0 - b \ge a - a_0$. Setting

$$\Gamma(k,x,a_0,b_0) := \left(\int_{Q(a_0,b_0)} (1+|F(y)|^k + V(y)^k)p(x,y,t)\,dy\,dt\right)^{\frac{1}{k}}, \tag{3.1}$$

the proofs of Proposition 3.1, Lemma 3.1 and Proposition 3.2 in [10] remain valid for the case $V \neq 0$. So, we obtain that

$$p \in \mathcal{H}^{s,1}(Q(a,b)) \quad \text{for all } s \in (1,k),$$

provided that $\Gamma(k,x,a_0,b_0) < \infty$ for some $k > N+2$. Hence, by the embedding theorem for $\mathcal{H}^{s,1}$, $s > N+2$, (see [10, Theorem 7.1]), we have

Theorem 3.1. *If $\Gamma(k,x,a_0,b_0) < \infty$ for some $k > N+2$, then p belongs to $L^\infty(Q(a,b))$.*

To obtain uniform and pointwise bounds on p we introduce the functions

$$\Gamma_1(k,x,a_0,b_0) = \left(\int_{Q(a_0,b_0)} (1+|F(y)|^k)p(x,y,t)\,dy\,dt\right)^{\frac{1}{k}}, \tag{3.2}$$

$$\Gamma_2(k,x,a_0,b_0) = \left(\int_{Q(a_0,b_0)} V^{\frac{k}{2}}(y)p(x,y,t)\,dy\,dt\right)^{\frac{2}{k}}. \tag{3.3}$$

Clearly $\Gamma_1(k,x,a_0,b_0)+\Gamma_2(k,x,a_0,b_0) \le C\Gamma(k,x,a_0,b_0)$. The following result shows that only the assumption $\Gamma_1(k,x,a_0,b_0), \Gamma_2(k,x,a_0,b_0) < \infty$ for some $k > N+2$ is needed to obtain the boundedness of p.

Theorem 3.2. *If $\Gamma_1(k,x,a_0,b_0), \Gamma_2(k,x,a_0,b_0) < \infty$ for some $k > N+2$ then*

$$\|p\|_{L^\infty(Q(a,b))} \le C\left(\Gamma_1^k(k,x,a_0,b_0) + \Gamma_2^{\frac{k}{2}}(k,x,a_0,b_0) + \frac{b_0-a_0}{(a-a_0)^{\frac{k}{2}}}\right). \tag{3.4}$$

Proof. **Step 1.** Assume first that $\Gamma(k,x,a_0,b_0) < \infty$ so that $p \in L^\infty(Q(a,b))$ for every $a_0 < a < b < b_0$ by Theorem 3.1 and consider $q = \eta^{\frac{k}{2}} p \in L^\infty(Q_T)$ where η is a smooth function with compact support in (a_0,b_0) such that $0 \le \eta \le 1$, $\eta(t) = 1$ for $a \le t \le b$. Clearly $q \in L^\infty(Q_T)$.

Let $\varphi \in C^{2,1}(Q_T)$ be such that $\varphi(\cdot,t)$ has compact support for every t. From (1.4) we obtain

$$\left|\int_{Q_T} q(\partial_t\varphi + A_0\varphi)\,dy\,dt\right| = \left|\int_{Q_T} (qF\cdot D\varphi - Vq\varphi + \frac{k}{2}p\varphi\eta^{\frac{k-2}{2}}\partial_t\eta)\,dy\,dt\right|.$$

Next we note that

$$\|p\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}(b_0-a_0)^{\frac{2}{k}}$$

and that

$$\|Fq\|_{L^k(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\Gamma_1(k,x,a_0,b_0)$$
$$\|Vq\|_{L^{\frac{k}{2}}(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\Gamma_2(k,x,a_0,b_0).$$

Since also

$$\|q\|_{L^k(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}(b_0-a_0)^{\frac{1}{k}},$$
$$\|q\|_{L^{\frac{k}{2}}(Q_T)} \le \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}(b_0-a_0)^{\frac{2}{k}},$$

Theorem 7.3 in [10] now implies that

$$\|q\|_{L^\infty(Q_T)} \le C\Big(\|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\Gamma_1(k,x,a_0,b_0) + \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\Big(\Gamma_2(k,x,a_0,b_0) + \frac{(b_0-a_0)^{\frac{2}{k}}}{a-a_0}\Big)\Big)$$

and hence, after a simple calculation,

$$\|q\|_{L^\infty(Q_T)} \le C\left(\Gamma_1^k(k,x,a_0,b_0) + \Gamma_2^{\frac{k}{2}}(k,x,a_0,b_0) + \frac{b_0-a_0}{(a-a_0)^{\frac{k}{2}}}\right)$$

and (3.4) follows.

Step 2. Let us now consider the general case. Fix a smooth function $\theta \in C_c^\infty(\mathbf{R})$ such that $\theta(s) = 1$ for $|s| \le 1$, $\theta(s) = 0$ for $|s| \ge 2$ and define $\theta_n(x) = \theta\left(\frac{|x|}{n}\right)$, $V_n = V\theta_n$. We consider the minimal semigroup $(U_n(t))_{t\ge 0}$ generated in $C_b(\mathbf{R}^N)$ by the operator $A_n = A_0 + F\cdot D - V_n$. Since $V_n \le V$ the procedure for constructing the minimal semigroup recalled in Section 2 and the maximum principle yield $U_n(t)f \le T(t)f$ for every $f \in C_b(\mathbf{R}^N)$. If p_n denotes the kernel of U_n the above inequality is equivalent to $p_n(x,y,t) \le p_(x,y,t)$. To show that p_n converges pointwise to p we consider the analytic semigroup $(T_n(t))_{t\ge 0}$ generated by A on $C_b(B_n)$, under Dirichlet boundary conditions (B_n is the ball of centre 0 and radius n). Since $V_n = V$ in B_n, the maximum principle gives $T_n(t)f \le U_n(t)f \le T(t)f$ in B_n for every $f \in C_b(\mathbf{R}^N)$, $f \ge 0$. Then $r_n(x,y,t) \le p_n(x,y,t) \le p(x,y,t)$ for $x, y \in B_\rho$ with $\rho < n$, where r_n is the kernel of T_n in B_n. Letting $n \to \infty$ we see that $p_n \to p$ pointwise, since this is true for r_n, see [11, Theorem 4.4].
The proof now easily follows by approximation from Step 1. Let $\Gamma_i^n(k,x,a_0,b_0)$ be the functions defined in (3.1), (3.2), (3.3) relative to p_n. Since $p_n \le p$ and $V_n \le V$, it follows that $\Gamma_i^n(k,x,a_0,b_0) \le \Gamma_i(k,x,a_0,b_0)$. Moreover, $\Gamma^n(k,x,a_0,b_0) < \infty$ for every n, since V_n is bounded. Then we obtain from Step 1

$$\|p_n\|_{L^\infty(Q(a,b))} \le C\left(\Gamma_1^{k}(k,x,a_0,b_0) + \Gamma_2^{\frac{k}{2}}(k,x,a_0,b_0) + \frac{b_0 - a_0}{(a-a_0)^{\frac{k}{2}}}\right)$$

and the statement follows letting $n \to \infty$. □

Now we apply similar techniques to obtain pointwise bounds.

We consider the following assumption depending on the weight function ω which, in our examples, will be a polynomial or an exponential.

(H1) W_1, W_2 are Lyapunov functions for A, $W_1 \le W_2$ and there exists $1 \le \omega \in C^2(\mathbf{R}^N)$ such that

(i) $\omega \le cW_1, \quad |D\omega| \le c\omega^{\frac{k-1}{k}} W_1^{\frac{1}{k}}, \quad |D^2\omega| \le c\omega^{\frac{k-2}{k}} W_1^{\frac{2}{k}}$

(ii) $\omega V^{\frac{k}{2}} \le cW_2$ and $\omega|F|^k \le cW_2$

for some $k > N+2$ and a constant $c > 0$.

We denote by ζ_1, ζ_2 the functions defined by (2.1) and associated with W_1, W_2, respectively.

By Proposition 2.4 we know that (H1) implies $\Gamma_i(k,x,a_0,b_0) < \infty$ for $i = 1,2$. In particular, since $k > N+2$, Theorem 3.2 shows that $p(x,\cdot,\cdot) \in L^\infty(Q(a,b))$ for every $x \in \mathbf{R}^N$.

The use of different Lyapunov functions allows us to obtain more precise estimates in the theorem below and its corollaries.

The proof of the following result is similar to the one of [10, Theorem 4.1]. For reader's convenience we give the details.

Theorem 3.3. *Assume* (H1). *Then, there exists a constant* $C>0$ *such that*

$$0<\omega(y)p(x,y,t)\le C\left(\int_{a_0}^{b_0}\zeta_2(x,t)\,dt+\frac{1}{(a-a_0)^{\frac{k}{2}}}\int_{a_0}^{b_0}\zeta_1(x,t)\,dt\right) \tag{3.5}$$

for all $x,\ y\in\mathbf{R}^N, a\le t\le b$.

Proof. **Step 1.** Assume first that ω is bounded. As in the proof of Theorem 3.2 we choose a smooth function $\eta(t)$ such that $\eta(t)=1$ for $a\le t\le b$ and $\eta(t)=0$ for $t\le a_0$ and $t\ge b_0$, $0\le\eta'\le\frac{2}{a-a_0}$. We consider $\psi\in C^{2,1}(Q_T)$ such that $\psi(\cdot,T)=0$ and such that $\psi(\cdot,t)$ has compact support for all t. Setting $q=\eta^{\frac{k}{2}}p$ and taking $\varphi(y,t)=\eta^{\frac{k}{2}}(t)\omega(y)\psi(y,t)$, from (1.4) we obtain

$$\int_{Q_T}\omega q\left(-\partial_t\psi-A_0\psi\right)dydt=\int_{Q_T}\Big[q\Big(\psi A_0\omega+2\sum_{i,j=1}^N a_{ij}D_i\omega D_j\psi+ \tag{3.6}$$
$$\omega F\cdot D\psi+\psi F\cdot D\omega-V\omega\psi\Big)+\frac{k}{2}p\omega\psi\eta^{\frac{k-2}{2}}\partial_t\eta\Big]\,dy\,dt.$$

Since ω is bounded, then $\omega q\in L^1(Q_T)\cap L^\infty(Q_T)$, by Theorem 3.2 and then [10, Theorem 7.3] yields

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)}\le C\Big(&\|\omega q\|_{L^k(Q_T)}+\|\omega q\|_{L^{\frac{k}{2}}(Q_T)}+\|qD^2\omega\|_{L^{\frac{k}{2}}(Q_T)}+\|qD\omega\|_{L^k(Q_T)}\\&+\|\omega qF\|_{L^k(Q_T)}+\|qFD\omega\|_{L^{\frac{k}{2}}(Q_T)}+\|qV\omega\|_{L^{\frac{k}{2}}(Q_T)}\\&+\frac{1}{a-a_0}\|p\omega\eta^{\frac{k-2}{2}}|_{L^{\frac{k}{2}}(Q_T)}\Big).\end{aligned}$$

Next observe that

$$\|\omega q\|_{L^k(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\|\omega q\|_{L^1(Q_T)}^{\frac{1}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\left(\int_{a_0}^{b_0}\zeta_1\,dt\right)^{\frac{1}{k}},$$

$$\|\omega q\|_{L^{\frac{k}{2}}(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\|\omega q\|_{L^1(Q_T)}^{\frac{2}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\zeta_1\,dt\right)^{\frac{2}{k}},$$

and that, by (H1)(ii),

$$\|\omega qF\|_{L^k(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\|\omega qF^k\|_{L^1(Q_T)}^{\frac{1}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}\left(\int_{a_0}^{b_0}\zeta_2\,dt\right)^{\frac{1}{k}},$$

$$\|\omega qV\|_{L^{\frac{k}{2}}(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\|\omega qV^{\frac{k}{2}}\|_{L^1(Q_T)}^{\frac{2}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\zeta_2\,dt\right)^{\frac{2}{k}}.$$

Moreover, as in the proof of Theorem 3.2 one has

$$\|\omega p\eta^{\frac{k-2}{2}}\|_{L^{\frac{k}{2}}(Q_T)}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\|\omega p\|_{L^1(Q(a_0,b_0))}^{\frac{2}{k}}\le\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}\left(\int_{a_0}^{b_0}\zeta_1\,dt\right)^{\frac{2}{k}}.$$

Next we combine (H1)(i) and (H1)(ii) to estimate the remaining terms

$$\|D\omega q F\|_{L^{\frac{k}{2}}(Q_T)} \le \left(\int_{Q_T} q^{\frac{k}{2}}\omega^{\frac{k-2}{2}} W_2\right)^{\frac{2}{k}} \le \|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{2}} \left(\int_{a_0}^{b_0} \zeta_2\, dt\right)^{\frac{2}{k}}$$

and, similarly,

$$\|D^2\omega q\|_{L^{\frac{k}{2}}(Q_T)} \le \|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left(\int_{a_0}^{b_0} \zeta_1\, dt\right)^{\frac{2}{k}}$$

$$\|D\omega q\|_{L^k(Q_T)} \le \|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}} \left(\int_{a_0}^{b_0} \zeta_1\, dt\right)^{\frac{1}{k}}.$$

Collecting similar terms and recalling that $W_1 \le W_2$ we obtain

$$\begin{aligned}\|\omega q\|_{L^\infty(Q_T)} \le C\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}} \left(\int_{a_0}^{b_0} \zeta_2\, dt\right)^{\frac{1}{k}} \\ + C\|\omega q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left(\left(\int_{a_0}^{b_0} \zeta_2\, dt\right)^{\frac{2}{k}} + \frac{1}{a-a_0}\left(\int_{a_0}^{b_0} \zeta_1\, dt\right)^{\frac{2}{k}}\right)\end{aligned}$$

hence, after simple computations,

$$\|\omega q\|_{L^\infty(Q_T)} \le C\left(\int_{a_0}^{b_0} \zeta_2\, dt + \frac{1}{(a-a_0)^{\frac{k}{2}}}\int_{a_0}^{b_0} \zeta_1\, dt\right)$$

and (3.5) follows for a bounded ω.

Step 2. If ω is not bounded, we consider $\omega_\varepsilon = \frac{\omega}{1+\varepsilon\omega}$. A straightforward computation shows that ω_ε satisfies $(H1)$ with a constant C independent of ε. Therefore, from Step 1 we obtain

$$0 < \omega_\varepsilon(y)p(x,y,t) \le C\left(\int_{a_0}^{b_0} \zeta_2(x,t)\, dt + \frac{1}{(a-a_0)^{\frac{k}{2}}}\int_{a_0}^{b_0} \zeta_1(x,t)\, dt\right), \tag{3.7}$$

with C independent of ε and, letting $\varepsilon \to 0$, the statement is proved. □

Corollary 3.4. *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|} - \frac{V(x)}{\delta\beta|x|^{\beta-1}}\right) < -c, \qquad 0 < c < \infty \tag{3.8}$$

for some $\delta > 0, c > 0, \beta > 2$ such that $\delta < (\beta\Lambda)^{-1}c$, where Λ is the maximum eigenvalue of (a_{ij}), and that $V(x) + |F(x)| \le c_1 e^{c_2|x|^{\beta-\varepsilon}}$ for some $\varepsilon, c_1, c_2 > 0$. Then

$$0 < p(x,y,t) \le c_3 \exp\left(c_4 t^{-\frac{\beta}{\beta-2}}\right)\exp\left(-\delta|y|^\beta\right)$$

for $x, y \in \mathbf{R}^N$, $0 < t \le T$, for suitable $c_3, c_4 > 0$.

Proof. We take $\omega(y) = e^{\delta|y|^\beta}$, $W_1(y) = W_2(y) = e^{\gamma|y|^\beta}$ for $\delta < \gamma < (\beta\Lambda)^{-1}c$ and use Theorem 3.3 with $a = t$ and $a - a_0 = b_0 - b = b - a = \frac{t}{2}$. The thesis then follows using Proposition 2.7. □

Example.

(i) The above corollary applies with any $\gamma < (\beta\Lambda)^{-1}c$ and without any restriction on $V \geq 0$ when

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|}\right) < -c, \qquad 0 < c < \infty,$$

for some $\beta > 2$ and $|F(x)| \leq c_1 e^{c_2|x|^{\beta-\varepsilon}}$ for some $\varepsilon, c_1, c_2 > 0$. This is obvious if $V = 0$ and, in the general case, it follows by observing that the kernel p is pointwise dominated, by the maximum principle, by the corresponding kernel of the operator with $V = 0$.

(ii) Let us consider the Schrödinger operator $\Delta - a^2|x|^s$ with $a > 0, s > 2$. Then Corollary 3.4 applies with $\beta = 1 + \frac{s}{2}$ and any $\delta < \frac{2a}{s+2}$. This yields

$$0 < p(x,y,t) \leq c_3 \exp\left(c_4 t^{-\frac{s+2}{s-2}}\right)\exp\left(-\delta|y|^{\frac{s+2}{2}}\right) := c(t)\phi(y).$$

Using the symmetry of p and the semigroup law (see [9, Example 3.13]), we obtain

$$p(x,y,t) \leq c_3 \exp\left(c_4 t^{-\frac{s+2}{s-2}}\right)\exp\left(-\delta|x|^{\frac{s+2}{2}}\right)\exp\left(-\delta|y|^{\frac{s+2}{2}}\right).$$

This estimate was obtained in [9, Example 3.13].

(iii) Let us generalize the previous situation to the case of the operators

$$A = \Delta - |x|^r\frac{x}{|x|}\cdot D - |x|^s$$

with $r > 1$. We distinguish three cases.

(a) If $s < 2r$, then $\beta = r + 1$ and δ can be any positive number less than $\frac{1}{r+1}$. Therefore

$$0 < p(x,y,t) \leq c_1 \exp\left(c_2 t^{-\frac{r+1}{r-1}}\right)\exp\left(-\delta|y|^{r+1}\right).$$

(b) If $s = 2r$, then $\beta = r + 1$ as before but now δ must be less than $\frac{1+\sqrt{5}}{2(r+1)}$.

(c) If $s > 2r$, then $\beta = 1 + \frac{s}{2}$ and $\delta < \frac{2}{s+2}$. Then we get, as in (ii)

$$0 < p(x,y,t) \leq c_1 \exp\left(c_2 t^{-\frac{s+2}{s-2}}\right)\exp\left(-\delta|y|^{\frac{s+2}{2}}\right) := c(t)\phi(y). \tag{3.9}$$

In this case one can also obtain estimates with respect to x proceeding as in (ii). We consider the formal adjoint $A^* = \Delta + |x|^r\frac{x}{|x|}\cdot D + (N + r - 1)|x|^{r-1} - |x|^s$. The associated minimal semigroup has the kernel $p^*(x,y,t) = p(y,x,t)$ which satisfies (3.9), by the same argument as above. This yields $p(t,x,y) \leq c(t)\phi(x)$ and, proceeding as in (ii),

$$p(x,y,t) \leq c_1 \exp\left(c_2 t^{-\frac{s+2}{s-2}}\right)\exp\left(-\delta|x|^{\frac{s+2}{2}}\right)\exp\left(-\delta|y|^{\frac{s+2}{2}}\right).$$

Under conditions similar to those of Corollary 3.4, the estimate of p can be improved with respect to the time variable, loosing the exponential decay in y.

Corollary 3.5. *Assume that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|}-\frac{|x|}{2\alpha}V(x)\right)<0, \tag{3.10}$$

for some $\alpha>0$ *and* $\beta>2$. *If* $|F(x)|+\sqrt{V(x)}\le c(1+|x|^2)^{\gamma_1}$ *and* $\omega(x):=(1+|x|^2)^{\gamma_2}$ *with* $0<k\gamma_1+\gamma_2\le\alpha$, $\gamma_1\ge\frac{\beta-2}{4}$ *and* $k>N+2$, *then there exists a constant* $C>0$ *such that*

$$0<p(x,y,t)\le\frac{C}{t^{\sigma}}(1+|y|^2)^{-\gamma_2},$$

for all x, $y\in\mathbf{R}^N$, $0<t\le 1$ *where*

$$\sigma=\frac{2}{\beta-2}\left((k-2)\gamma_1+\gamma_2\right).$$

Proof. Observe that $W_r(x)=(1+|x|^2)^r$ is a Lyapunov function for every $0<r\le\alpha$. If $\zeta_r(x,t)$ is the corresponding function defined in (2.1), then Proposition 2.8 yields

$$\zeta_r(x,t)\le c_r t^{\frac{-2r}{\beta-2}}$$

for $x\in\mathbf{R}^N$ and $0<t\le1$. We set $a=t$ and $a-a_0=b_0-b=b-a=\frac{t^s}{2}$ where $s\ge1$ will be chosen later and we apply Theorem 3.3 with $\omega(x)=W_1(x)=\left(1+|x|^2\right)^{\gamma_2}$ and $W_2(x)=\left(1+|x|^2\right)^{k\gamma_1+\gamma_2}$. Thus we obtain

$$p(x,y,t)\le C\left(t^{-\frac{2(k\gamma_1+\gamma_2)}{\beta-2}+s}+t^{-\frac{2\gamma_2}{\beta-2}-s\frac{k}{2}+s}\right)(1+|y|^2)^{-\gamma_2}.$$

Minimizing over s we get $s=\frac{4\gamma_1}{\beta-2}$ and the thesis follows. □

Example. Let us consider again the operators

$$A=\Delta-|x|^r\frac{x}{|x|}\cdot D-|x|^s$$

with $r>1$. Again we distinguish three cases.

(a) If $s+1\le r$, then $\beta=r+1$ and $\gamma_1=\frac{r}{2}$. It is easily seen that (3.10) holds for every $\alpha>0$ and hence

$$p(x,y,t)\le Ct^{-(k-2)\frac{r}{r-1}-\frac{2\gamma_1}{r-1}}\left(1+|y|^2\right)^{-\gamma_2}$$

for every $\gamma_2\ge0$, $0<t\le1$, $y\in\mathbf{R}^N$.

(b) If $r<s+1$, then (3.10) holds for $\beta=s+2$ and every $\alpha>0$. So, we have to distinguish two cases.

(i) If $s\le 2r$, then $\gamma_1=\frac{r}{2}$ and

$$p(x,y,t)\le Ct^{-(k-2)\frac{r}{s}-\frac{2\gamma_1}{s}}\left(1+|y|^2\right)^{-\gamma_2},$$

(ii) If $s > 2r$, then $\gamma_1 = \frac{s}{4}$ and

$$p(x,y,t) \le Ct^{-\frac{k-2}{2}-\frac{2\gamma_1}{s}}\left(1+|y|^2\right)^{-\gamma_2},$$

for every $\gamma_2 \ge 0$, $0 < t \le 1$, $y \in \mathbf{R}^N$.

Remark 3.6. The results of this section generalize Theorem 4.1 and its corollaries in [10] and also the results obtained in [9] in the case of exponential decay but not for polynomial decay, where the results in [9] are more precise.

4. Regularity properties

In this section we obtain the differentiability of the transition semigroup $T(\cdot)$ associated with the transition kernels p in $C_b(\mathbf{R}^N)$ in the case where the coefficients F and V are of exponential type.

We assume here that $a_{ij} \in C_b^2(\mathbf{R}^N)$, $V \in C^1(\mathbf{R}^N)$ and $F \in C^2(\mathbf{R}^N)$. All results of this section can be proved exactly by the same arguments as in [10, Section 5 and Section 6].

Theorem 4.1. *Suppose that there exist constants $\beta > 2$, $c > 0$ such that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|} - \frac{V(x)}{\delta\beta|x|^{\beta-1}}\right) < -c.$$

Assume moreover that

$$V(x) + |DV(x)| + |F(x)| + |DF(x)| + |D^2F(x)| \le c_1 \exp(c_2|x|^{\beta-\varepsilon})$$

for some ε, $c_1, c_2 > 0$. Then the following estimates hold

(i) $0 < p(x,y,t) \le c_3 \exp\{c_4 t^{-\frac{\beta}{\beta-2}}\}\exp\{-\gamma|y|^\beta\}$
(ii) $|D_y p(x,y,t)| \le c_3 \exp\{c_4 t^{-\frac{\beta}{\beta-2}}\}\exp\{-\gamma|y|^\beta\}$
(iii) $|D_y^2 p(x,y,t)| \le c_3 \exp\{c_4 t^{-\frac{\beta}{\beta-2}}\}\exp\{-\gamma|y|^\beta\}$
(iv) $|\partial_t p(x,y,t)| \le c_3 \exp\{c_4 t^{-\frac{\beta}{\beta-2}}\}\exp\{-\gamma|y|^\beta\}$

for suitable c_3, c_4, $\gamma > 0$ and for all $0 < t \le T$ and $x, y \in \mathbf{R}^N$.

Remark 4.2. (a) Assuming only that there exist constants $\beta > 2$, $c > 0$ such that

$$\limsup_{|x|\to\infty} |x|^{1-\beta}\left(F(x)\cdot\frac{x}{|x|} - \frac{V(x)}{\delta\beta|x|^{\beta-1}}\right) < -c,$$

and $V(x) + |F(x)| \le C\exp(|x|^\gamma)$ for some $C > 0$ and $\gamma < \beta$, the functions $p\log^2 p$ and $p\log p$ are integrable in $Q(a,b)$ and in $\mathbf{R}^N$ for fixed $t \in [a,b]$

respectively and

$$\begin{aligned}
&\int_{Q(a,b)} \frac{|D_y p(x,y,t)|^2}{p(x,y,t)}\,dy\,dt \\
&\quad \le \frac{1}{\lambda^2}\int_{Q(a,b)} (|F(y)|^2 + V^2(y))p(x,y,t)\,dy\,dt \\
&\qquad + \int_{Q(a,b)} p(x,y,t)\log^2 p(x,y,t)\,dy\,dt \\
&\qquad + \frac{2}{\lambda}\int_{\mathbf{R}^N} \Big[p(x,y,t) - p(x,y,t)\log p(x,y,t)\Big]_{t=a}^{t=b} dy < \infty.
\end{aligned}$$

In particular, $p^{\frac{1}{2}}$ belongs to $W_2^{1,0}(Q(a,b))$ (see [10, Theorem 5.1]). This implies in particular that $p \in W_k^{2,1}(Q(a,b))$ provided that also DF is of exponential type for some $k > N+2$ (see [10, Theorem 5.2]).

(b) From Theorem 3.3 and (a) (cf. [10, Theorem 5.3]) one can observe that the assumption $a_{ij} \in C_b^2(\mathbf{R}^N)$ is not needed for (i) and (ii).

As a consequence we obtain the differentiability of $T(\cdot)$ in $C_b(\mathbf{R}^N)$.

Theorem 4.3. *Under the assumptions of Theorem* 4.1*, the transition semigroup* $T(\cdot)$ *is differentiable on* $C_b(\mathbf{R}^N)$ *for* $t > 0$.

Example. Let $a \in \mathbf{R}$. From Theorem 4.3 we deduce that the operator

$$A = \Delta - |x|^r x \cdot D - a^2|x|^s$$

with $r > 0$ and $s \ge 0$ generates a differentiable semigroup in $C_b(\mathbf{R}^N)$. This result is known for $a = 0$ (see [13, Proposition 4.4]).

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Metric-induced Morphogenesis and Non-Euclidean Elasticity: Scaling Laws and Thin Film Models

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Abstract. The purpose of this paper is to report on recent developments concerning the analysis and the rigorous derivation of thin film models for structures exhibiting residual stress at free equilibria. This phenomenon has been observed in different contexts: growing leaves, torn plastic sheets and specifically engineered polymer gels. The study of wavy patterns in these contexts suggest that the sheet endeavors to reach a non-attainable equilibrium and hence assumes a non-zero stress rest configuration.

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1. Elastic energy of a growing tissue and non-Euclidean elasticity

This paper concerns elastic structures which exhibit non-zero strain at free equilibria. Many growing tissues (leaves, flowers or marine invertebrates) attain complicated configurations during their free growth. Recent work has focused on some of the related questions by using variants of thin plate theory [1, 5, 4, 23]. However, the theories used are not all identical and some of them arbitrarily ignore certain terms and boundary conditions without prior justification. This suggests that it might be useful to rigorously derive an asymptotic theory for the shape of a residually strained thin lamina to clarify the role of the assumptions used while shedding light on the errors associated with the use of the approximate theory that results. Recently, such rigorous derivations were carried out [8, 17, 19, 21] in the context of standard nonlinear elasticity for thin plates and shells.

The purpose of this paper is to present these results in a concise manner, departing from the 3d incompatible elasticity theory conjectured to explain the mechanism for the spontaneous formation of non-Euclidean metrics. Namely, recall that a smooth Riemannian metric on a simply connected domain can be realized as the pull-back metric of an orientation preserving deformation if and only if the associated Riemann curvature tensor vanishes identically. When this condition fails, one seeks a deformation yielding the closest metric realization. It is conjectured that the same principle plays a role in the developmental processes of naturally growing tissues, where the process of growth provides a mechanism for the spontaneous formation of non-Euclidean metrics and consequently leads to complicated morphogenesis of the thin film exhibiting waves, ruffles and non-zero residual stress.

Below, we set up a variational model describing this phenomenon by introducing the non-Euclidean version of the nonlinear elasticity functional, and establish its Γ-convergence under a proper scaling. Heuristically, a sequence of functionals F_n is said to Γ-converge to a limit functional F if the minimizers of F_n, if converging, have a minimizer of F as a limit.

Consider a sequence of thin 3d films $\Omega^h = \Omega \times (-h/2, h/2)$, viewed as the reference configurations of thin elastic tissues. Here, $\Omega \subset \mathbb{R}^2$ is an open, bounded and simply connected set which we refer to as the mid-plate of thin films under consideration. Each Ω^h is now assumed to undergo a growth process, described instantaneously by a (given) smooth tensor:

$$a^h = [a^h_{ij}] : \Omega^h \longrightarrow \mathbb{R}^{3\times 3} \quad \text{such that} \quad \det a^h(x) > 0.$$

According to the formalism in [25], the multiplicative decomposition

$$\nabla u = F a^h \tag{1.1}$$

is postulated for the gradient of any deformation $u : \Omega^h \longrightarrow \mathbb{R}^3$. The tensor $F = \nabla u (a^h)^{-1}$ corresponds to the elastic part of u, and accounts for the reorganization of Ω^h in response to the growth tensor a^h. The validity of decomposition (1.1) into an elastic and inelastic part requires that it is possible to separate out a reference configuration, and thus this formalism is most relevant for the description of processes such as plasticity, swelling and shrinkage in thin films, or plant morphogenesis.

The elastic energy of u depends now only on F:

$$I^h_W(u) = \frac{1}{h}\int_{\Omega^h} W(F)\,\mathrm{d}x = \frac{1}{h}\int_{\Omega^h} W(\nabla u (a^h)^{-1})\,\mathrm{d}x, \qquad \forall u \in W^{1,2}(\Omega^h, \mathbb{R}^3). \tag{1.2}$$

We remark that although our results are valid for thin laminae that might be residually strained by a variety of means, we only consider the one-way coupling of growth to shape and ignore the feedback from shape back to growth (plasticity, swelling, shrinkage etc.). However, it seems fairly easy to include this coupling once the basic coupling mechanisms are known.

In (1.2), the energy density $W : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}_+$ is a nonlinear function, assumed to be $\mathcal{C}^2$ in a neighborhood of $SO(3)$ and assumed to satisfy the following conditions of normalization, frame indifference and nondegeneracy:

$$\begin{aligned}\exists c>0 \quad \forall F\in\mathbb{R}^{3\times 3} \quad \forall R\in SO(3) \quad & W(R)=0, \quad W(RF)=W(F),\\ & W(F)\geq c\ \mathrm{dist}^2(F,SO(3)).\end{aligned} \tag{1.3}$$

The reason for using a nonlinear elasticity model (rather than the more familiar linear elasticity) is that, as our analysis shows, the resulting deformations u^h when $h\to 0$, are expected to be of order $\mathcal{O}(1)$, even though their gradients are locally $\mathcal{O}(h)$ close to rigid rotations.

We now compare the above approach with the 'target metric' formalism [5, 20]. On each Ω^h one assumes to be given a smooth Riemannian metric $g^h=[g^h_{ij}]$. A deformation u of Ω^h is then an orientation preserving realization of g^h, when

$$(\nabla u)^T\nabla u = g^h \ \text{ and } \ \det\nabla u>0,$$

or equivalently, by the polar decomposition theorem,

$$\nabla u(x)\in\mathcal{F}^h(x)=\Big\{R\sqrt{g^h}(x);\ R\in SO(3)\Big\} \qquad \text{a.e. in } \Omega^h. \tag{1.4}$$

It is hence instructive to study the following energy, bounding from below $I^h_W(u)$:

$$\tilde I^h_{\mathrm{dist}}(u)=\frac{1}{h}\int_{\Omega^h}\mathrm{dist}^2(\nabla u(x),\mathcal{F}^h(x))\ \mathrm{d}x \qquad \forall u\in W^{1,2}(\Omega^h,\mathbb{R}^3). \tag{1.5}$$

The functional $\tilde I^h_{\mathrm{dist}}$ measures the average pointwise deviation of the deformation u from being an orientation preserving realization of g^h. Note that $\tilde I^h_{\mathrm{dist}}$ is comparable in magnitude with I^h_W, for $W=\mathrm{dist}^2(\cdot,SO(3))$. Also, observe that the intrinsic metric of the material is transformed by a^h to the target metric $g^h=(a^h)^Ta^h$ and, for isotropic W, only the symmetric positive definite part of a^h given by $\sqrt{g^h}$ plays a role in determining the deformed shape.

2. The residual stress and a result on its scaling

Note that one could define the energy as the difference between the pull-back metric of a deformation u and the given metric: $I^h_{\mathrm{str}}(u)=\int|(\nabla u)^T\nabla u-g^h|^2\ \mathrm{d}x$. However, such 'stretching' functional is not appropriate from the variational point of view, because there always exists $u\in W^{1,\infty}$ such that $I^h_{\mathrm{str}}(u)=0$. Further, if the Riemann curvature tensor R^h associated to g^h does not vanish identically, say $R^h_{ijkl}(x)\neq 0$, then u has a 'folding structure' [9]; it cannot be orientation preserving (or reversing) in any open neighborhood of x.

As proved in [20], the functionals I^h_W, $\tilde I^h_W$ below and $\tilde I^h_{\mathrm{dist}}$ have strictly positive infima for non-flat g^h, which points to the existence of non-zero stress at free equilibria (in the absence of external forces or boundary conditions):

Theorem 2.1. *For each fixed h, the following two conditions are equivalent:*

(i) *The Riemann curvature tensor $R^h_{ijkl}\not\equiv 0$,*

(ii) $\inf\big\{\tilde I^h_{\mathrm{dist}}(u);\ u\in W^{1,2}(\Omega^h,\mathbb{R}^3)\big\}>0$.

Several interesting questions further arise in the study of the proposed energy functionals. A first one is to determine the scaling of the infimum energy in terms of the vanishing thickness $h \to 0$. Another is to find the limiting zero-thickness theories under the obtained scaling laws.

In [20], we considered the case where g^h is given by a tangential Riemannian metric $[g_{\alpha\beta}]$ on Ω, and is independent of the thickness variable:

$$g^h = g(x', x_3) = \left[\begin{array}{cc} \left[g_{\alpha\beta}(x')\right] & \begin{array}{c} 0 \\ 0 \end{array} \\ 0 \quad 0 & 1 \end{array}\right] \qquad \forall x' \in \Omega, \quad x_3 \in (-h/2, h/2). \tag{2.1}$$

The above particular choice of the metric is motivated by the results of [14]. The experiment presented therein consisted in fabricating programmed flat disks of gels having a non-constant monomer concentration which induces a 'differential shrinking factor'. A disk was then activated in a temperature raised above a critical threshold, wherein the gel shrunk with a factor proportional to its concentration. This process defined a new target metric on the disk, of the form (2.1) and radially symmetric. Consequently, the metric induced a 3d configuration in the initially planar plate; one of the most remarkable features of this deformation is the onset of some transversal oscillations (wavy patterns), which broke the radial symmetry.

Following our point of view, note that if $[g_{\alpha\beta}]$ in (2.1) has non-zero Gaussian curvature $\kappa_{[g_{\alpha\beta}]}$, then each $R^h \not\equiv 0$. In [20], we observed the following:

Theorem 2.2. *$[g_{\alpha\beta}]$ has an isometric immersion $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ if and only if*

$$h^{-2} \inf \tilde{I}^h_{\mathrm{dist}} \le C$$

(for a uniform constant C). Also, $\kappa_{[g_{\alpha\beta}]} \not\equiv 0$ if and only if, with a uniform positive constant c:

$$h^{-2} \inf \tilde{I}^h_{\mathrm{dist}} \ge c > 0.$$

The existence (or lack thereof) of local or global isometric immersions of a given 2d Riemannian manifold into $\mathbb{R}^3$ is a longstanding problem in differential geometry, its main feature being finding the optimal regularity. By a classical result of Kuiper [15], a $\mathcal{C}^1$ isometric embedding into $\mathbb{R}^3$ can be obtained by means of convex integration (see also [9]). This regularity is far from $W^{2,2}$, where information about the second derivatives is also available. On the other hand, a smooth isometry exists for some special cases, e.g., for smooth metrics with uniformly positive or negative Gaussian curvatures on bounded domains in $\mathbb{R}^2$ (see [11], Theorems 9.0.1 and 10.0.2). Counterexamples to such theories are largely unexplored. By [13], there exists an analytic metric $[g_{\alpha\beta}]$ with nonnegative Gaussian curvature on a 2d sphere, with no $\mathcal{C}^3$ isometric embedding. However such metric always admits a $\mathcal{C}^{1,1}$ embedding (see [10] and [12]). For a related example see also [24].

3. The prestrained Kirchhoff model

Consider now a class of more general 3d non-Euclidean elasticity functionals:

$$\tilde{I}_W^h(u) = \int_{\Omega^h} W(x', \nabla u(x))\, \mathrm{d}x, \tag{3.1}$$

where the inhomogeneous stored energy density $W : \Omega \times \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}_+$ satisfies the conditions given below of frame invariance, normalization, growth and regularity, as in (1.3): with respect to the energy well $\mathcal{F}^h$ given in (1.4), relative to $g^h = g$ as in (2.1). Note that $\mathcal{F}^h(x) = \mathcal{F}(x')$ is independent of h and of x_3.

(i) $W(x', RF) = W(x', F)$ for all $R \in SO(3)$,
(ii) $W(x', \sqrt{g}(x')) = 0$,
(iii) $W(x', F) \geq c \operatorname{dist}^2(F, \mathcal{F}(x'))$, with some uniform constant $c > 0$,
(iv) W has regularity $\mathcal{C}^2$ in some neighborhood of the set $\{(x', F);\ x' \in \Omega, F \in \mathcal{F}(x')\}$.

Properties (i)–(iii) are assumed to hold for all $x \in \Omega$ and all $F \in \mathbb{R}^{3\times 3}$.

The following two results provide a description of the limiting behavior of the energies $\tilde{I}_W^h$ as $h \to 0$. Namely, we prove that any sequence of deformations u^h with $\tilde{I}_W^h(u^h) \leq Ch^2$, converges to a $W^{2,2}$ regular isometric immersion y of the metric $[g_{\alpha\beta}]$. Conversely, every y with these properties can be recovered as a limit of u^h whose energy scales like h^2. The Γ-limit [3] of the energies is a curvature functional on the space of all $W^{2,2}$ realizations y of $[g_{\alpha\beta}]$ in $\mathbb{R}^3$:

$$\frac{1}{h^2}\tilde{I}_W^h \xrightarrow{\Gamma} \mathcal{I}_2(y) \quad \text{where} \quad \mathcal{I}_2(y) = \frac{1}{24}\int_\Omega \tilde{\mathcal{Q}}_2(x')\Big(\sqrt{[g_{\alpha\beta}]}^{-1}(\nabla y)^T \nabla \vec{n}\Big)\, \mathrm{d}x'. \tag{3.2}$$

Here $\vec{n}$ is the unit normal to the image surface $y(\Omega)$, while $\tilde{\mathcal{Q}}_2(x')$ are the following quadratic forms, nondegenerate and positive definite on the symmetric 2×2 tensors:

$$\tilde{\mathcal{Q}}_3(x')(F) = \nabla^2 W(x', \cdot)_{|\sqrt{g}(x')}(F, F),$$
$$\tilde{\mathcal{Q}}_2(x')(F_{2\times 2}) = \min\{\tilde{\mathcal{Q}}_3(x')(\tilde{F});\ \tilde{F}_{2\times 2} = F_{2\times 2}\}.$$

We use the following notational convention: for a matrix F, its $n \times m$ principle minor is denoted by $F_{n\times m}$ and the superscript ^{T} refers to the transpose of a matrix or an operator.

Theorem 3.1. *Assume that a given sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfies*

$$\tilde{I}_W^h(u^h) \leq Ch^2, \tag{3.3}$$

where $C > 0$ is a uniform constant. Then, for some sequence of constants $c^h \in \mathbb{R}^3$, the following holds for the renormalized deformations $y^h(x', x_3) = u^h(x', hx_3) - c^h \in W^{1,2}(\Omega^1, \mathbb{R}^3)$:

(i) *y^h converge, up to a subsequence, in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to $y(x', x_3) = y(x')$ and $y \in W^{2,2}(\Omega, \mathbb{R}^3)$.*

(ii) *The matrix field* $Q(x')$ *with columns* $Q(x') = \Big[\partial_1 y(x'), \partial_2 y(x'), \vec{n}(x')\Big] \in \mathcal{F}(x')$*, for a.e.* $x' \in \Omega$*. Here,*

$$\vec{n} = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|} \tag{3.4}$$

is the (well-defined) normal to the image surface $y(\Omega)$*. Consequently,* y *realizes the mid-plate metric:* $(\nabla y)^T \nabla y = [g_{\alpha\beta}]$*.*

(iii) *We have the lower bound:* $\liminf_{h\to 0} \frac{1}{h^2} \tilde{I}^h_W(u^h) \geq \mathcal{I}_2(y)$*, where* $\mathcal{I}_2$ *is given in* (3.2)*.*

We further prove that the lower bound in (iii) above is optimal, in the following sense. Let $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ be a Sobolev regular isometric immersion of the given mid-plate metric, that is $(\nabla y)^T \nabla y = [g_{\alpha\beta}]$. The normal vector $\vec{n} \in W^{1,2}(\Omega, \mathbb{R}^3)$ is then given by (3.4) and it is well defined because $|\partial_1 y \times \partial_2 y| = (\det g)^{1/2} > 0$. We have:

Theorem 3.2. *For every isometric immersion* $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ *of* $[g_{\alpha\beta}]$*, there exists a sequence of recovery deformations* $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ *such that the assertion* (i) *of Theorem* 3.1 *holds, together with*

$$\lim_{h\to 0} \frac{1}{h^2} \tilde{I}^h_W(u^h) = \mathcal{I}_2(y).$$

Assume now a slightly more general case of plates with slowly varying thickness, that is when

$$\Omega^h = \{(x', x_3);\ x \in \Omega,\ -hq_1(x') < x_3 < hq_2(x')\}$$

with some positive $\mathcal{C}^1$ functions $q_1, q_2 : \Omega \longrightarrow (0, \infty)$. In this setting, the same results as in Theorem 3.1 and 3.2 have been re-proved in [22], with the limiting functional

$$\mathcal{I}_2^{q_1,q_2}(y) = \frac{1}{24} \int_\Omega \big(q_1(x') + q_2(x')\big)^3 \tilde{\mathcal{Q}}_2(x') \Big(\sqrt{[g_{\alpha\beta}]}^{\,-1} (\nabla y)^T \nabla \vec{n}\Big)\, \mathrm{d}x'.$$

For classical elasticity ($g^h = \mathrm{Id}$) of shells with mid-surface of arbitrary geometry and slowly oscillating boundaries as above, the analysis has been previously carried out in [18].

An important reference in the context of Theorems 3.1 and 3.2 (for flat films) is [26], containing the derivation of Kirchhoff plate theory for heterogeneous multilayers from 3d nonlinear energies given through an inhomogeneous density in $\int W(x_3/h, \nabla u)$.

4. A rigidity estimate

As a crucial ingredient of the proof of compactness in Theorem 3.1, we present a generalization of the nonlinear rigidity estimate obtained [7] in the Euclidean setting, extended to the non-Euclidean metrics in [20]. Note that in case $g^h =$

Id, the infimum of $I^h_{\rm dist}$ in (1.5) is naturally 0 and is attained only by the rigid motions. In [7], the authors proved an optimal estimate of the deviation in $W^{1,2}$ of a deformation u (on Ω^h), from rigid motions, in terms of the energy $I^h_{\rm dist}(u)$. In our setting, since there is no realization of $I^h_{\rm dist}(u) = 0$ if the Riemann curvature of the metric g^h is non-zero, we choose to estimate the deviation of the deformation from a linear map at the expense of an extra term, proportional to the gradient of the metric.

Theorem 4.1. *Let $\mathcal{U}$ be an open, bounded subset of $\mathbb{R}^n$ and let g be a smooth (up to the boundary) metric on $\mathcal{U}$. For every $u \in W^{1,2}(\mathcal{U}, \mathbb{R}^n)$ there exists $Q \in \mathbb{R}^{n\times n}$ such that:*

$$\int_{\mathcal{U}} |\nabla u(x) - Q|^2 \ \mathrm{d}x \leq C\left(\int_{\mathcal{U}} \mathrm{dist}^2\Big(\nabla u, SO(n)\sqrt{g}(x)\Big) \ \mathrm{d}x + \|\nabla g\|^2_{L^\infty}(\mathrm{diam}\ \mathcal{U})^2|\mathcal{U}|\right),$$

where the constant C depends on $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, and on the domain $\mathcal{U}$. The dependence on $\mathcal{U}$ is uniform for a family of domains which are bilipschitz equivalent with controlled Lipschitz constants.

For an embeddable metric g (i.e., whose $R_{ijkl} \equiv 0$) a related result has been obtained in [2]; namely an estimate of the deviation of (orientation preserving) deformation u from the realizations of g in terms of the L^1 stretching energy $\int |(\nabla u)^T\nabla u - g|$.

5. A hierarchy of scalings

Given a sequence of growth tensors a^h (say, each close to Id) defined on Ω^h, the general objective is now to analyze the behavior of the minimizers of the corresponding energies I^h_W as $h \to 0$. By Theorem 2.1, the infimum: $m_h = \inf\Big\{I^h_W(u);\ u \in W^{1,2}(\Omega^h, \mathbb{R}^3)\Big\}$ must be strictly positive whenever the Riemann tensor of the metric $g^h = (a^h)^T a^h$ does not vanish identically on Ω^h. This condition for g^h, under suitable scaling properties, can be translated into a first-order curvature condition (5.1) below. In a first step (Theorem 5.1) we established [16] a lower bound on m_h in terms of a power law: $m_h \geq ch^\beta$, for all values of β greater than a critical β_0 in (5.2). This critical exponent depends on the asymptotic behavior of the perturbation $a^h - \mathrm{Id}$ in terms of the thickness h. Under existence conditions for certain classes of isometries, it can be shown that actually $m_h \sim h^{\beta_0}$.

Theorem 5.1. *For a given sequence of growth tensors a^h define their variations:*

$$\mathrm{Var}(a^h) = \|\nabla_{\tan}(a^h{}_{|\Omega})\|_{L^\infty(\Omega)} + \|\partial_3 a^h\|_{L^\infty(\Omega^h)}$$

together with the scaling in h:

$$\omega_1 = \sup\left\{\omega;\ \lim_{h\to 0}\frac{1}{h^\omega}\,\mathrm{Var}(a^h) = 0\right\}.$$

Assume that: $\|a^h\|_{L^\infty(\Omega^h)} + \|(a^h)^{-1}\|_{L^\infty(\Omega^h)} \leq C$ and $\omega_1 > 0$.

Further, assume that for some $\omega_0 \geq 0$, there exists the limit

$$\epsilon_g(x') = \lim_{h\to 0} \frac{1}{h^{\omega_0}} \fint_{-h/2}^{h/2} a^h(x',t) - \mathrm{Id}\ \mathrm{d}t \quad \textit{in } L^2(\Omega, \mathbb{R}^{3\times 3}),$$

which moreover satisfies

$$\mathrm{curl}^T \mathrm{curl}\ (\epsilon_g)_{2\times 2} \not\equiv 0, \tag{5.1}$$

and that $\omega_0 < \min\{2\omega_1, \omega_1 + 1\}$.

Then, for every β with

$$\beta > \beta_0 = \max\{\omega_0 + 2, 2\omega_0\}, \tag{5.2}$$

there holds: $\limsup_{h\to 0} \frac{1}{h^\beta} \inf I_0^h = +\infty$.

We expect it should be possible to rigorously derive a hierarchy of prestrained limiting theories, differentiated by the embeddability properties of the target metrics, encoded in the scalings of (the components of) their Riemann curvature tensors. This is in the same spirit as the different scalings of external forces leading to a hierarchy of nonlinear elastic plate theories, displayed by Friesecke, James and Müller in[8]. For shells, which are thin films with mid-surface or arbitrary (non-flat) geometry, an infinite hierarchy of models was proposed, by means of asymptotic expansion in [21], and it remains in agreement with all the rigorously obtained results [6, 17, 18, 19].

6. The prestrained von Kármán model

Towards studying the dynamical growth problem (that is, incorporating the feedback from the minimizing shape u^h at the prior time-step, to growth tensor a^h at the current time-step) in [16] we considered the growth tensor

$$a^h(x', x_3) = \mathrm{Id} + h^2 \epsilon_g(x') + h x_3 \gamma_g(x'), \tag{6.1}$$

with given matrix fields $\epsilon_g, \gamma_g : \overline{\Omega} \longrightarrow \mathbb{R}^{3\times 3}$. Note that the assumptions of Theorem 5.1 do not hold, since in the present case $\omega_0 = 2\omega_1 = \omega_1 + 1 = 2$.

We proved that, in this setting $\inf I_W^h \leq Ch^4$, while the lower bound $h^{-4} \inf I_W^h \geq c > 0$ is equivalent to

$$\mathrm{curl}((\gamma_g)_{2\times 2}) \not\equiv 0 \quad \text{or} \quad 2\mathrm{curl}^T \mathrm{curl}(\epsilon_g)_{2\times 2} + \det(\gamma_g)_{2\times 2} \not\equiv 0, \tag{6.2}$$

which are the (negated) linearized Gauss-Codazzi equations corresponding to the metric $I = \mathrm{Id} + h^2(\epsilon_g)_{2\times 2}$ and the second fundamental form $II = \frac{1}{2}h(\gamma_g)_{2\times 2}$ on Ω. Equivalently, the above conditions guarantee that the highest-order terms in the expansion of the Riemann curvature tensor components R_{1213}, R_{2321} and R_{1212} of $g^h = (a^h)^T a^h$ do not vanish. Also, either of them implies that $\inf \mathcal{I}_4 > 0$ (see definition below), which yields the lower bound on $\inf I_W^h$.

The Γ-limit of the rescaled energies is, in turn, expressed in terms of the out-of-plane displacement $v \in W^{2,2}(\Omega, \mathbb{R})$ and in-plane displacement $w \in W^{1,2}(\Omega, \mathbb{R}^2)$:

$$\begin{aligned} &\frac{1}{h^4} I_W^h \xrightarrow{\Gamma} \mathcal{I}_4 \quad \text{where} \\ &\mathcal{I}_4(w,v) = \frac{1}{24}\int_\Omega \mathcal{Q}_2\big(\nabla^2 v + \frac{1}{2}(\gamma_g)_{2\times 2}\big) \\ &\qquad\qquad + \frac{1}{2}\int_\Omega \mathcal{Q}_2\big(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v\otimes\nabla v - \frac{1}{2}(\epsilon_g)_{2\times 2}\big), \end{aligned} \tag{6.3}$$

with the quadratic nondegenerate form $\mathcal{Q}_2$, acting on matrices $F \in \mathbb{R}^{2\times 2}$:

$$\mathcal{Q}_2(F) = \min\{\mathcal{Q}_3(\tilde F);\ \tilde F \in \mathbb{R}^{3\times 3}, \tilde F_{2\times 2} = F\} \text{ and } \mathcal{Q}_3(\tilde F) = D^2W(\mathrm{Id})(\tilde F\otimes \tilde F).$$

The two terms in (6.3) measure: the first order in h change of II, and the second order change in I, under the deformation $id + hve_3 + h^2 w$ of Ω. Moreover, any sequence of deformations u^h with $I_W^h(u^h) \leq Ch^4$ is, asymptotically, of this form.

More precisely, we proved in [16]:

Theorem 6.1. *Let the growth tensor a^h be as in* (6.1)*. Assume that the energies of a sequence of deformations $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfy*

$$I_W^h(u^h) \leq Ch^4, \tag{6.4}$$

where W fulfills (1.3)*. Then there exist proper rotations $\bar R^h \in SO(3)$ and translations $c^h \in \mathbb{R}^3$ such that, for the normalized deformations*

$$y^h(x', x_3) = (\bar R^h)^T u^h(x', hx_3) - c^h : \Omega^1 \longrightarrow \mathbb{R}^3,$$

the following holds:

(i) *$y^h(x', x_3)$ converge in $W^{1,2}(\Omega^1, \mathbb{R}^3)$ to x'.*

(ii) *The scaled displacements*

$$V^h(x') = \frac{1}{h}\int_{-1/2}^{1/2} y^h(x', t) - x' \,\mathrm{d}t \tag{6.5}$$

converge (up to a subsequence) in $W^{1,2}(\Omega, \mathbb{R}^3)$ to the vector field of the form $(0,0,v)^T$, with the only non-zero out-of-plane scalar component: $v \in W^{2,2}(\Omega, \mathbb{R})$.

(iii) *The scaled in-plane displacements $h^{-1}V^h_{\tan}$ converge (up to a subsequence) weakly in $W^{1,2}(\Omega, \mathbb{R}^2)$ to an in-plane displacement field $w \in W^{1,2}(\Omega, \mathbb{R}^2)$.*

(iv) *Recalling the definition* (6.3)*, there holds*

$$\liminf_{h\to 0} \frac{1}{h^4} I_W^h(u^h) \geq \mathcal{I}_4(w,v).$$

The limsup part of the Γ-convergence statement in Theorem 6.2 establishes that for any pair of displacements (w, v) in suitable limit spaces, one can construct a sequence of 3d deformations of thin plates Ω^h which approximately yield the energy $\mathcal{I}_4(w, v)$. The form of such *recovery sequence* delivers an insight on how to reconstruct the 3d deformations out of the data on the mid-plate Ω. In particular, comparing the present von Kármán growth model with the classical model ([8], Section 6.1) we observe a novel warping effect in the non-tangential growth.

Theorem 6.2. *Assume the setting of Theorem* 6.1. *For every* $w \in W^{1,2}(\Omega, \mathbb{R}^3)$ *and every* $v \in W^{2,2}(\Omega, \mathbb{R})$, *there exists a sequence of deformations* $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ *such that the following holds:*

(i) *The sequence* $y^h(x', x_3) = u^h(x', hx_3)$ *converges in* $W^{1,2}(\Omega^1, \mathbb{R}^3)$ *to* x'.

(ii) $V^h(x') = h^{-1}\int_{-h/2}^{h/2} (u^h(x', t) - x')\,\mathrm{d}t$ *converges in* $W^{1,2}(\Omega, \mathbb{R}^3)$ *to* $(0, 0, v)^T$.

(iii) $h^{-1}V^h_{\tan}$ *converges in* $W^{1,2}(\Omega, \mathbb{R}^2)$ *to* w.

(iv) *Recalling the definition* (6.3) *one has:*

$$\lim_{h\to 0} \frac{1}{h^4} I^h_W(u^h) = \mathcal{I}_4(w, v).$$

The main consequences of the Γ-convergence results above are as follows:

Corollary 6.3. *Assume the setting of Theorem* 6.1. *Then:*

(i) *There exist uniform constants* $C, c \geq 0$ *such that, for every* h,

$$c \leq \frac{1}{h^4} \inf I^h_W \leq C. \tag{6.6}$$

If moreover (6.2) *holds then one may have* $c > 0$.

(ii) *There exists at least one minimizing sequence* $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3$ *for* I^h_W:

$$\lim_{h\to 0} \left(\frac{1}{h^4} I^h_W(u^h) - \frac{1}{h^4} \inf I^h_W \right) = 0. \tag{6.7}$$

For any such sequence the convergences (i), (ii) *and* (iii) *of Theorem* 6.1 *hold and the limit* (w, v) *is a minimizer of* $\mathcal{I}_4$.

(iii) *For any minimizer* (w, v) *of* $\mathcal{I}_4$, *there exists a minimizing sequence* u^h, *satisfying* (6.7) *together with* (i), (ii), (iii) *and* (iv) *of Theorem* 6.2.

7. The prestrained von Kármán equations

For elastic energy W satisfying (1.3) which is additionally isotropic,

$$\forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \qquad W(FR) = W(F), \tag{7.1}$$

one can see [8] that the quadratic forms in $\mathcal{I}_4$ are given explicitly as

$$\begin{aligned} \mathcal{Q}_3(F) &= 2\mu|\mathrm{sym}\ F|^2 + \lambda|\mathrm{tr}\ F|^2, \\ \mathcal{Q}_2(F_{2\times 2}) &= 2\mu|\mathrm{sym}\ F_{2\times 2}|^2 + \frac{2\mu\lambda}{2\mu+\lambda}|\mathrm{tr}\ F_{2\times 2}|^2, \end{aligned} \tag{7.2}$$

for all $F \in \mathbb{R}^{3\times 3}$. Here, tr stands for the trace of a quadratic matrix, and μ and λ are the Lamé constants, satisfying: $\mu \geq 0$, $3\lambda + \mu \geq 0$.

Under these conditions, the Euler-Lagrange equations of the limiting functional $\mathcal{I}_4$ are equivalent, under a change of variables which replaces the in-plane displacement w by the Airy stress potential Φ, to the new system proposed in [23]:

$$\Delta^2\Phi = -S(K_G + \lambda_g), \qquad B\Delta^2 v = [v, \Phi] - B\Omega_g,$$

where $S = \mu(3\lambda+2\mu)/(\lambda+\mu)$ is the Young's modulus, K_G the Gaussian curvature, $B = S/(12(1-\nu^2))$ the bending stiffness, and $\nu = \lambda/(2(\lambda+\mu))$ the Poisson ratio given in terms of the Lamé constants λ and μ. The corrections due to the prestrain are

$$\lambda_g = \mathrm{curl}^T\mathrm{curl}\ (\epsilon_g)_{2\times 2}, \qquad \Omega_g = \mathrm{div}^T\mathrm{div}\left((\gamma_g)_{2\times 2} + \nu\ \mathrm{cof}\ (\gamma_g)_{2\times 2}\right).$$

More precisely:

Theorem 7.1. *Assume* (1.3) *and* (7.1). *Then the leading order displacements in a thin tissue which tries to adapt itself to an internally imposed metric* $g^h = (a^h)^T a^h$ *with* a^h *as in* (6.1) *satisfy:*

$$\begin{aligned}
\Delta^2\Phi &= -S\Big(\det\nabla^2 v + \mathrm{curl}^T\mathrm{curl}(\epsilon_g)_{2\times 2}\Big),\\
B\Delta^2 v &= \mathrm{cof}\nabla^2\Phi : \nabla^2 v - B\ \mathrm{div}^T\mathrm{div}\Big((\gamma_g)_{2\times 2} + \nu\ \mathrm{cof}(\gamma_g)_{2\times 2}\Big),
\end{aligned}$$

together with the (*free*) *boundary conditions on* $\partial\Omega$*:*

$$\begin{aligned}
&\Phi = \partial_{\vec n}\Phi = 0,\\
&\tilde\Psi : (\vec n\otimes\vec n) + \nu\ \tilde\Psi : (\tau\otimes\tau) = 0,\\
&(1-\nu)\partial_\tau\Big(\tilde\Psi : (\vec n\otimes\tau)\Big) + \mathrm{div}\ \Big(\tilde\Psi + \nu\ \mathrm{cof}\tilde\Psi\Big)\,\vec n = 0.
\end{aligned}$$

Here $\vec n$ *denotes the normal,* τ *the tangent to* $\partial\Omega$*, while*

$$\tilde\Psi = \nabla^2 v + \mathrm{sym}(\gamma_g)_{2\times 2}.$$

The in-plane displacement w *can be recovered from the Airy stress potential* Φ *and the out-of-plane displacement* v*, uniquely up to rigid motions, by means of*

$$\begin{aligned}
\mathrm{cof}\nabla^2\Phi = 2\mu&\Big(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v\otimes\nabla v - \mathrm{sym}(\epsilon_g)_{2\times 2}\Big)\\
&+ \frac{2\mu\lambda}{2\mu+\lambda}\Big(\mathrm{div}\ w + \frac{1}{2}|\nabla v|^2 - \mathrm{tr}(\epsilon_g)_{2\times 2}\Big)\mathrm{Id}.
\end{aligned}$$

Notice that in the particular case when $(\mathrm{sym}\kappa_g)_{2\times 2} = 0$ on $\partial\Omega$, the two last boundary conditions become:

$$\begin{aligned}
&\partial^2_{\vec n\vec n}v + \nu\Big(\partial^2_{\tau\tau}v - K\partial_{\vec n}v\Big) = 0,\\
&(2-\nu)\partial_\tau\partial_{\vec n}\partial_\tau v + \partial^3_{\vec n\vec n\vec n}v + K\Big(\Delta v + 2\partial^2_{\vec n\vec n}v\Big) = 0,
\end{aligned}$$

where K stands for the (scalar) curvature of $\partial\Omega$, so that $\partial_\tau\tau = K\vec n$. If additionally $\partial\Omega$ is polygonal, then the above equations simplify to equations (5) in [23].

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Compactness and Asymptotic Behavior in Nonautonomous Linear Parabolic Equations with Unbounded Coefficients in $\mathbb{R}^d$

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Dedicated to Herbert Amann

Abstract. We consider a class of second-order linear nonautonomous parabolic equations in $\mathbb{R}^d$ with time periodic unbounded coefficients. We give sufficient conditions for the evolution operator $G(t, s)$ be compact in $C_b(\mathbb{R}^d)$ for $t > s$, and describe the asymptotic behavior of $G(t, s)f$ as $t - s \to \infty$ in terms of a family of measures μ_s, $s \in \mathbb{R}$, solution of the associated Fokker-Planck equation.

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1. Introduction

Linear nonautonomous parabolic equations in $\mathbb{R}^d$ are a classical subject in the mathematical literature. Most papers and books about regular solutions are devoted to the case of bounded coefficients (e.g., [9, 5], but the list is very long), and recently the interest towards unbounded coefficients grew up. The standard motivations to the study of unbounded coefficients are on one side the well known connections with stochastic ODEs with unbounded nonlinearities, and on the other side the changes of variables that transform bounded into unbounded coefficients, occurring in different mathematical models. However, only for a few equations with unbounded coefficients it is possible to recover the familiar results about the bounded coefficients case. Many of them exhibit very different, and at first glance

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surprising, aspects. Therefore, a third motivation is the interest in new phenomena in PDEs.

This paper deals with one of these new phenomena, giving sufficient conditions in order that the evolution operator $G(t,s)$ associated to a class of second-order parabolic equations is a compact contraction in $C_b(\mathbb{R}^d)$ for $t>s$. Precisely, Cauchy problems such as

$$u_t(t,x) = \mathcal{A}(t)u(t,\cdot)(x), \quad t>s,\ x\in\mathbb{R}^d, \tag{1.1}$$

$$u(s,x) = \varphi(x), \quad x\in\mathbb{R}^d, \tag{1.2}$$

will be considered, where the elliptic operators $\mathcal{A}(t)$ are defined by

$$\begin{aligned}(\mathcal{A}(t)\varphi)(x) &:= \sum_{i,j=1}^{d} q_{ij}(t,x)D_{ij}\varphi(x) + \sum_{i=1}^{d} b_i(t,x)D_i\varphi(x)\\ &:= \mathrm{Tr}\left(Q(t,x)D^2\varphi(x)\right) + \langle b(t,x), \nabla\varphi(x)\rangle,\end{aligned} \tag{1.3}$$

and the (smooth enough) coefficients q_{ij}, b_i are allowed to be unbounded. If φ is smooth and it has compact support, a classical bounded solution to (1.1)–(1.2) is readily constructed, as the limit as $R\to\infty$ of the solutions u_R of Cauchy-Dirichlet problems in the balls $B(0,R)$. However, classical bounded solutions need not be unique. Under assumptions that guarantee positivity preserving in (1.1)–(1.2) (and hence, uniqueness of its bounded classical solution), a basic study of the evolution operator $G(t,s)$ for (1.1) in $C_b(\mathbb{R}^d)$ is in the paper [8]. The evolution operator turns out to be Markovian, since it has the representation

$$G(t,s)\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)p_{t,s,x}(dy), \quad t>s,\ x\in\mathbb{R}^d,\ \varphi\in C_b(\mathbb{R}^d),$$

where the probability measures $p_{t,s,x}$ are given by $p_{t,s,x}(dy) = g(t,s,x,y)dy$ for a positive function g.

It is easy to see that if a Markovian $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$, then it does not preserve $C_0(\mathbb{R}^d)$, the space of the continuous functions vanishing as $|x|\to\infty$, and it cannot be extended to a bounded operator in $L^p(\mathbb{R}^d, dx)$ for $1\le p<\infty$. Therefore, much of the theory developed for bounded coefficients fails.

When a parabolic problem is not well posed in L^p spaces with respect to the Lebesgue measure, it is natural to look for other measures μ, and in particular to weighted Lebesgue measures, such that $G(t,s)$ acts in $L^p(\mathbb{R}^d,\mu)$. This is well understood in the autonomous case $\mathcal{A}(t)\equiv\mathcal{A}$, where the dynamics is held by a semigroup $T(t)$ and $G(t,s) = T(t-s)$. Then, an important role is played by *invariant measures*, that are Borel probability measures μ such that

$$\int_{\mathbb{R}^d} T(t)\varphi\, d\mu = \int_{\mathbb{R}^d} \varphi\, d\mu, \quad \varphi\in C_b(\mathbb{R}^d).$$

If a Markov semigroup has an invariant measure μ, it can be extended in a standard way to a contraction semigroup in all the spaces $L^p(\mathbb{R}^d,\mu)$, $1\le p<\infty$. Under broad assumptions the invariant measure is unique, and it is strongly related with the asymptotic behavior of $T(t)$, since $\lim_{t\to\infty} T(t)\varphi = \int_{\mathbb{R}^d}\varphi\, d\mu$, locally uniformly if $\varphi \in C_b(\mathbb{R}^d)$ and in $L^p(\mathbb{R}^d, \mu)$ if $\varphi \in L^p(\mathbb{R}^d, \mu)$, $1 < p < \infty$. In the nonautonomous case the role of the invariant measure is played by families of measures $\{\mu_s : \; s \in \mathbb{R}\}$, called *evolution systems of measures*, that satisfy

$$\int_{\mathbb{R}^d} G(t,s)\varphi \, d\mu_t = \int_{\mathbb{R}^d} \varphi \, d\mu_s, \quad t > s, \; \varphi \in C_b(\mathbb{R}^d). \tag{1.4}$$

If (1.4) is satisfied, the function $s \mapsto \mu_s$ satisfies (at least, formally) the Fokker-Planck equation

$$D_s\mu_s + \mathcal{A}(s)^*\mu_s = 0, \quad s \in \mathbb{R},$$

which is a parabolic equation for measures without any initial, or final, condition. Therefore it is natural to have infinitely many solutions, and to look for uniqueness of special solutions. For instance, in the autonomous case the unique stationary solution is the invariant measure, in the periodic case $\mathcal{A}(t) = \mathcal{A}(t+T)$ under reasonable assumptions there is a unique T-periodic solution, etc. Arguing as in the autonomous case, it is easy to see that if (1.4) holds then $G(t,s)$ may be extended to a contraction from $L^p(\mathbb{R}^d, \mu_s)$ to $L^p(\mathbb{R}^d, \mu_t)$ for $t > s$. Therefore, it is natural to investigate asymptotic behavior of $G(t,s)$ not only in $C_b(\mathbb{R}^d)$ but also in these L^p spaces.

A basic study of the evolution operator for parabolic equations with (smooth enough) unbounded coefficients is in [8]. In its sequel [10] we studied asymptotic behavior of $G(t,s)$ in the case of time-periodic coefficients.

In this paper sufficient conditions will be given for the evolution operator $G(t,s)$ be compact in $C_b(\mathbb{R}^d)$. Then, compactness will be used to obtain asymptotic behavior results in the case of time-periodic coefficients. Indeed, compactness implies that there exists a unique T-periodic evolution system of measures $\{\mu_s : \; s \in \mathbb{R}\}$, and that denoting by $m_s\varphi$ the mean value

$$m_s\varphi := \int_{\mathbb{R}^d} \varphi(x)\mu_s(dx), \quad s \in \mathbb{R}, \; \varphi \in C_b(\mathbb{R}^d), \tag{1.5}$$

there is $\omega < 0$ such that for each $\varepsilon > 0$ we have

$$\|G(t,s)\varphi - m_s\varphi\|_\infty \leq M_\varepsilon e^{(\omega+\varepsilon)(t-s)}\|\varphi\|_\infty, \quad t > s, \; \varphi \in C_b(\mathbb{R}^d), \tag{1.6}$$

for some $M_\varepsilon > 0$. As a consequence, for every $p \in (1,\infty)$ and $\varepsilon > 0$ we get

$$\|G(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} \leq M e^{(\omega+\varepsilon)(t-s)}\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}, \quad t > s, \; \varphi \in L^p(\mathbb{R}^d, \mu_s), \tag{1.7}$$

for some $M = M(p,\varepsilon) > 0$. Note that while the constant M may depend on p, the exponential rate of decay is independent of p. These results complement the asymptotic behavior results of [10], where (1.7) was obtained under different assumptions.

2. Preliminaries: the evolution operator $G(t,s)$

We use standard notations. $C_b(\mathbb{R}^d)$ is the space of the bounded continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$, endowed with the sup norm. $C_0(\mathbb{R}^d)$ is the space of the continuous functions that vanish at infinity. For $k \in \mathbb{N} \cup \{0\}$, $C^k_c(\mathbb{R}^d)$ is the space of k times differentiable functions with compact support. For $0 < \alpha < 1$, $a < b$ and $R > 0$, $C^{\alpha/2,\alpha}([a,b] \times B(0,R))$ and $C^{1+\alpha/2,2+\alpha}([a,b] \times B(0,R))$ are the usual parabolic Hölder spaces in the set $[a,b] \times B(0,R)$; $C^{\alpha/2,\alpha}_{\rm loc}(\mathbb{R}^{1+d})$ and $C^{1+\alpha/2,2+\alpha}_{\rm loc}(\mathbb{R}^{1+d})$ are the subspaces of $C_b(\mathbb{R}^{1+d})$ consisting of functions whose restrictions to $[a,b] \times B(0,R)$ belong to $C^{\alpha/2,\alpha}([a,b] \times B(0,R))$ and to $C^{1+\alpha/2,2+\alpha}([a,b] \times B(0,R))$, respectively, for every $a < b$ and $R > 0$.

In this section we recall some results from [8] about the evolution operator for parabolic equations with unbounded coefficients. They were proved under standard regularity and ellipticity assumptions, and nonstandard qualitative assumptions.

Hypothesis 2.1.

(i) *The coefficients q_{ij}, b_i $(i,j = 1,\dots,d)$ belong to $C^{\alpha/2,\alpha}_{\rm loc}(\mathbb{R}^{1+d})$ for some $\alpha \in (0,1)$.*

(ii) *For every $(s,x) \in \mathbb{R}^{1+d}$, the matrix $Q(s,x)$ is symmetric and there exists a function $\eta : \mathbb{R}^{1+d} \to \mathbb{R}$ such that $0 < \eta_0 := \inf_{\mathbb{R}^{1+d}} \eta$ and*

$$\langle Q(s,x)\xi, \xi\rangle \geq \eta(s,x)|\xi|^2, \qquad \xi \in \mathbb{R}^d,\ (s,x) \in \mathbb{R}^{1+d}.$$

(iii) *There exist a positive function $W \in C^2(\mathbb{R}^d)$ and a number $\lambda \in \mathbb{R}$ such that*

$$\lim_{|x|\to\infty} W(x) = \infty \qquad \text{and} \qquad \sup_{s\in\mathbb{R},\, x\in\mathbb{R}^d} (\mathcal{A}(s)W)(x) - \lambda W(x) < 0.$$

Assumptions (i) and (ii) imply that for every $s \in \mathbb{R}$ and $\varphi \in C_b(\mathbb{R}^d)$, the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}(t)u(t,x), & t > s,\ x \in \mathbb{R}^d, \\ u(s,x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.1}$$

has a bounded classical solution. Assumption (iii) implies that the bounded classical solution to (2.1) is unique (in fact, a maximum principle that yields uniqueness is proved in [8] under a slightly weaker assumption). The evolution operator $G(t,s)$ is defined by

$$G(t,s)\varphi = u(t,\cdot), \quad t \geq s \in \mathbb{R}, \tag{2.2}$$

where u is the unique bounded solution to (2.1). Some of the properties of $G(t,s)$ are in next theorem.

Theorem 2.2. *Let Hypothesis 2.1 hold. Define $\Lambda := \{(t,s,x) \in \mathbb{R}^{2+d} : t > s,\ x \in \mathbb{R}^d\}$. Then:*

(i) *for every $\varphi \in C_b(\mathbb{R}^d)$, the function $(t,s,x) \mapsto G(t,s)\varphi(x)$ is continuous in $\overline{\Lambda}$. For each $s \in \mathbb{R}$, $(t,x) \mapsto G(t,s)\varphi(x)$ belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,\infty) \times \mathbb{R}^d)$;*

(ii) *for every $\varphi \in C^2_c(\mathbb{R}^d)$, the function $(t,s,x) \mapsto G(t,s)\varphi(x)$ is continuously differentiable with respect to s in $\overline{\Lambda}$ and $D_s G(t,s)\varphi(x) = -G(t,s)\mathcal{A}(s)\varphi(x)$ for any $(t,s,x) \in \overline{\Lambda}$;*

(iii) *for each* $(t,s,x) \in \Lambda$ *there exists a Borel probability measure* $p_{t,s,x}$ *in* $\mathbb{R}^d$ *such that*

$$G(t,s)\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)p_{t,s,x}(dy), \quad \varphi \in C_b(\mathbb{R}^d). \tag{2.3}$$

Moreover, $p_{t,s,x}(dy) = g(t,s,x,y)dy$ *for a positive function* g*.*

(iv) $G(t,s)$ *is strong Feller; extending it to* $L^\infty(\mathbb{R}^d, dx)$ *through formula* (2.3)*, it maps* $L^\infty(\mathbb{R}^d, dx)$ *(and, in particular,* $B_b(\mathbb{R}^d)$*) into* $C_b(\mathbb{R}^d)$ *for* $t > s$*, and*

$$\|G(t,s)\varphi\|_\infty \le \|\varphi\|_\infty, \quad \varphi \in L^\infty(\mathbb{R}^d, dx),\ t > s.$$

(v) *If* (φ_n) *is a bounded sequence in* $C_b(\mathbb{R}^d)$ *that converges uniformly to* φ *in each compact set* $K \subset \mathbb{R}^d$*, then for each* $s \in \mathbb{R}$ *and* $T > 0$*,* $G(\cdot,s)\varphi_n$ *converges to* $G(\cdot,s)\varphi$ *uniformly in* $[s,s+T] \times K$*, for each compact set* $K \subset \mathbb{R}^d$*.*

(vi) *For every* $s \in \mathbb{R}$ *and* $R > 0$*,* $0 < \varepsilon < T$ *there is* $C = C(s,\varepsilon,T,R) > 0$ *such that*

$$\sup_{s+\varepsilon \le t \le s+T} \|G(t,s)\varphi\|_{C^2(B(0,R))} \le C\|\varphi\|_\infty, \quad \varphi \in C_b(\mathbb{R}^d).$$

Statements (i) to (v) are explicitly mentioned in [8] (Thm. 2.1, Prop. 2.4, Cor. 2.5, Prop. 3.1, Lemma 3.2). Statement (vi) is hidden in the proof of Theorem 2.1, where $G(t,s)\varphi$ is obtained by an approximation procedure, in three steps: first, for $\varphi \in C_c^{2+\alpha}(\mathbb{R}^d)$, then for $\varphi \in C_0(\mathbb{R}^d)$, and then for $\varphi \in C_b(\mathbb{R}^d)$. At each step, we have interior Schauder estimates for a sequence u_n that approaches $G(t,s)\varphi$, namely for $s \in \mathbb{R}$ and $R > 0$, $0 < \varepsilon < T$ there is $C = C(s,\varepsilon,T,R) > 0$ such that

$$\|u_n\|_{C^{1+\alpha/2,2+\alpha}([s+\varepsilon,s+T]\times B(0,R))} \le C\|\varphi\|_\infty, \quad n \in \mathbb{N},$$

and u_n converges to $G(t,s)\varphi$ locally uniformly. This yields (vi).

To get evolution system of measures we have to strengthen assumption 2.1(iii). The following theorem is proved in [8].

Theorem 2.3. *Under Hypotheses* 2.1*, assume in addition that there exist a positive function* $W \in C^2(\mathbb{R}^d)$ *and numbers* a*,* $c > 0$ *such that*

$$\lim_{|x|\to\infty} W(x) = +\infty \quad \text{and} \quad (\mathcal{A}(s)W)(x) \le a - cW(x), \quad (s,x) \in \mathbb{R}^{1+d}. \tag{2.4}$$

Then there exists a tight[1] *evolution system of measures* $\{\mu_s : s \in \mathbb{R}\}$ *for* $G(t,s)$*. Moreover,*

$$G(t,s)W(x) := \int_{\mathbb{R}^d} W(y)p_{t,s,x}(dy) \le W(x) + \frac{a}{c}, \quad t > s,\ x \in \mathbb{R}^d, \tag{2.5}$$

and

$$\int_{\mathbb{R}^d} W(y)\mu_t(dy) \le \min W + \frac{a}{c}, \quad t \in \mathbb{R}. \tag{2.6}$$

[1]i.e., $\forall \varepsilon > 0\ \exists R = R(\varepsilon) > 0$ such that $\mu_s(B(0,R)) \ge 1 - \varepsilon$, for all $s \in \mathbb{R}$.

3. Compactness in $C_b(\mathbb{R}^d)$

A necessary and sufficient condition for $G(t,s)$ be compact in $C_b(\mathbb{R}^d)$ for $t>s$ is very similar to the corresponding condition in the autonomous case ([13]).

Proposition 3.1. *Under Hypothesis* 2.1 *the following statements are equivalent:*

(a) *for any $t>s$, $G(t,s): C_b(\mathbb{R}^d)\to C_b(\mathbb{R}^d)$ is compact.*
(b) *for any $t>s$ the family of measures $\{p_{t,s,x}(dy):\ x\in\mathbb{R}^d\}$ is tight, i.e., for every $\varepsilon>0$ there exists $R=R(t,s,\varepsilon)>0$ such that*

$$p_{t,s,x}(B(0,R))\geq 1-\varepsilon,\quad x\in\mathbb{R}^d.$$

Proof. We follow the proof given in [13, Prop. 3.6] for the autonomous case.

Let statement (a) hold. For every $R>0$ let $\varphi_R:\mathbb{R}^d\to\mathbb{R}$ be a continuous function such that $\mathbb{1}_{B(0,R)}\leq\varphi_R\leq\mathbb{1}_{B(0,R+1)}$. Since $\|\varphi_R\|_\infty\leq 1$ and $G(t,s)$ is compact, there is a sequence $G(t,s)\varphi_{R_n}$ that converges uniformly in the whole $\mathbb{R}^d$ to a limit function g. Since φ_R goes to $\mathbb{1}$ as $R\to+\infty$, uniformly on each compact set and $\|\varphi_R\|_\infty\leq 1$ for every R, by Theorem 2.2(v) $\lim_{R\to\infty}G(t,s)\varphi_R=G(t,s)\mathbb{1}=\mathbb{1}$ uniformly on each compact set. Then, $g\equiv 1$ and $\lim_{R\to+\infty}\|G(t,s)\varphi_R-\mathbb{1}\|_\infty=0$. Therefore, fixed any $\varepsilon>0$, we have

$$p_{t,s,x}(B(0,R))=(G(t,s)\mathbb{1}_{B(0,R)})(x)\geq(G(t,s)\varphi_R)(x)\geq 1-\varepsilon,\quad x\in\mathbb{R}^d,$$

for R large enough.

Let now statement (b) hold. For $t>s$ fix $r\in(s,t)$ and recall that (see formula (2.3))

$$(G(t,s)\varphi)(x)=(G(t,r)G(r,s)\varphi)(x)=\int_{\mathbb{R}^d}(G(r,s)\varphi)(y)p_{t,r,x}(dy),$$

for any $\varphi\in C_b(\mathbb{R}^d)$ and any $x\in\mathbb{R}^d$. For every $R>0$ set

$$(G_R\varphi)(x)=\int_{B(0,R)}(G(r,s)\varphi)(y)p_{t,r,x}(dy),\quad x\in\mathbb{R}^d.$$

Each $G_R: C_b(\mathbb{R}^d)\to C_b(\mathbb{R}^d)$ is a compact operator, since it may be written as $G_R=\mathcal{S}\circ\mathcal{R}\circ G(r,s)$ where $G(r,s):C_b(\mathbb{R}^d)\to C_b(\mathbb{R}^d)$ is continuous, $\mathcal{R}:C_b(\mathbb{R}^d)\to C(B(0,R))$ is the restriction operator, and $\mathcal{S}:C(B(0,R))\to C_b(\mathbb{R}^d)$ is defined by

$$\mathcal{S}\psi(x)=\int_{B(0,R)}\psi(y)p_{t,r,x}(dy)=(G(t,r)\widetilde{\psi})(x),\qquad x\in B(0,R),$$

where $\widetilde{\psi}(x)$ is the null extension of ψ to the whole $\mathbb{R}^d$. Now, $\mathcal{R}\circ G(s,r):C_b(\mathbb{R}^d)\to C(B(0,R))$ is compact by Theorem 2.2(v), and $\mathcal{S}$ is continuous from $C(B(0,R))$ to $C_b(\mathbb{R}^d)$ because $G(t,r)$ is strong Feller by Theorem 2.2(iv).

Moreover, $G_R\to G(t,s)$ in $\mathcal{L}(C_b(\mathbb{R}^d))$, as $R\to+\infty$. Indeed, for $\varepsilon>0$ there is $R_0>0$ such that $p_{t,r,x}(B(0,R))\geq 1-\varepsilon$ for each $x\in\mathbb{R}^d$ and $R\geq R_0$, and consequently

$$|(G(t,s)\varphi)(x) - (G_R\varphi)(x)| \le \|G(r,s)\varphi\|_\infty \int_{\mathbb{R}^d\setminus B(0,R)} p_{t,r,x}(dy) \le \varepsilon\|\varphi\|_\infty,$$

for $R \ge R_0$ and for each $x \in \mathbb{R}^d$.

Being limit of compact operators, $G(t,s)$ is compact. □

Remark 3.2. Some remarks are in order.

(i) An insight in the proof shows that if $G(t,s)$ is compact for some $t > s$, then the family $\{p_{t,s,x}(dy) : \ x \in \mathbb{R}^d\}$ is tight; conversely if for some $r > s$ the family $\{p_{r,s,x}(dy) : \ x \in \mathbb{R}^d\}$ is tight then $G(t,s)$ is compact for each $t > r$.

(ii) If for some $r > s$ the family $\{p_{r,s,x}(dy) : \ x \in \mathbb{R}^d\}$ is tight, then the family $\{p_{t,s,x}(dy) : \ t \ge r, \ x \in \mathbb{R}^d\}$ is tight. Indeed, for every $R > 0$ we have

$$\begin{aligned} p_{t,s,x}(\mathbb{R}^d \setminus B(0,R)) &= (G(t,s)\mathbb{1}_{\mathbb{R}^d\setminus B(0,R)})(x) \\ &= (G(t,r)G(r,s)\mathbb{1}_{\mathbb{R}^d\setminus B(0,R)})(x) \\ &\le \|G(r,s)\mathbb{1}_{\mathbb{R}^d\setminus B(0,R)}\|_\infty, \end{aligned}$$

so that, if $p_{r,s,x}(\mathbb{R}^d \setminus B(0,R)) = G(r,s)\mathbb{1}_{\mathbb{R}^d\setminus B(0,R)}(x) \le \varepsilon$ for every x, also $p_{t,s,x}(\mathbb{R}^d \setminus B(0,R)) \le \varepsilon$ for every x.

(iii) As in the autonomous case ([13]), if $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$, it does not preserve $L^p(\mathbb{R}^d, dx)$ for any $p \in [1,+\infty)$ and it does not preserve $C_0(\mathbb{R}^d)$. Indeed, let $R > 0$ be so large that $p_{t,s,x}(B(0,R)) \ge 1/2$ for every $x \in \mathbb{R}^d$, and let $\varphi \in C_c(\mathbb{R}^d)$ be such that $\varphi \ge \mathbb{1}_{B(0,R)}$. Then,

$$(G(t,s)\varphi)(x) \ge (G(t,s)\mathbb{1}_{B(0,R)})(x) = p_{t,s,x}(B(0,R)) \ge \frac{1}{2},$$

for every x, so that $G(t,s)\varphi$ does not belong to any space $L^p(\mathbb{R}^d, dx)$ and to $C_0(\mathbb{R}^d)$.

(iv) A similar argument shows that $\inf G(t,s)\varphi > 0$ for each $t > s$ and $\varphi \in C_b(\mathbb{R}^d) \setminus \{0\}$, $\varphi \ge 0$. Indeed, if $\varphi(x) > 0$ for each x, and $R > 0$ is as before, then $(G(t,s)\varphi)(x) \ge \delta(G(t,s)\mathbb{1}_{B(0,R)})(x) \ge \delta/2$, with $\delta = \min_{|x|\le R}\varphi(x) > 0$. If $\varphi(x) \ge 0$ for each x, it is sufficient to recall that $G(t,s)\varphi = G(t,(s+t)/2)G((s+t)/2,s)\varphi$ and that $G((s+t)/2,s)\varphi(x) > 0$ for each x by Theorem 2.2 (iii).

However, to check the tightness condition of Proposition 3.1 is not obvious, since the measures $p_{t,s,x}$ are not explicit, in general. In the case of time-depending Ornstein-Uhlenbeck operators (e.g., [1]),

$$(\mathcal{A}(t)\varphi)(x) = \frac{1}{2}\mathrm{Tr}\left(Q(t)D^2_x\varphi(x)\right) + \langle A(t)x + f(t), \nabla\varphi(x)\rangle, \quad x \in \mathbb{R}^d,$$

the measures $p_{t,s,x}$ are explicit Gaussian measures and it is possible to see that the tightness condition does not hold. Alternatively, one can check that $G(t,s)$ maps $L^p(\mathbb{R}^d, dx)$ into itself for every $p \in (1,\infty)$ and therefore it cannot be compact in $C_b(\mathbb{R}^d)$.

If the assumptions of Theorem 2.3 hold, estimate (2.5) implies that the family $\{p_{t,s,x} : t > s,\ x \in B(0,r)\}$ is tight for every $r > 0$. However, this is not enough for compactness. To obtain compactness we have to strengthen condition (2.4) on the auxiliary function W.

Theorem 3.3. *Let Hypotheses* 2.1 *hold. Assume in addition that there exist a* C^2 *function* $W : \mathbb{R}^d \mapsto \mathbb{R}$*, such that* $\lim_{|x|\to\infty} W(x) = +\infty$*, a number* $R > 0$ *and a convex increasing function* $g : [0,+\infty) \to \mathbb{R}$ *such that* $1/g$ *is in* $L^1(a,+\infty)$ *for large* a*, and*

$$(\mathcal{A}(s)W)(x) \le -g(W(x)), \qquad s \in \mathbb{R},\ |x| \ge R. \tag{3.1}$$

Then, for every $\delta > 0$ *there is* $C = C(\delta) > 0$ *such that* $(G(t,s)W)(x) \le C$ *for every* $x \in \mathbb{R}^d$ *and* $s \le t-\delta$*. Consequently, the family of probabilities* $\{p_{t,s,x}(dy) : s \le t-\delta,\ x \in \mathbb{R}^d\}$ *is tight, and* $G(t,s)$ *is compact in* $C_b(\mathbb{R}^d)$ *for* $t > s$*.*

Proof. As a first step we show that

$$(G(t,s)W)(x) - (G(t,r)W)(x) \ge -\int_r^s (G(t,\sigma)\mathcal{A}(\sigma)W)(x)d\sigma, \quad r < s < t,\ x \in \mathbb{R}^d. \tag{3.2}$$

Let $\varphi \in C^\infty(\mathbb{R})$ be a nonincreasing function such that $\varphi \equiv 1$ in $(-\infty,0]$, $\varphi \equiv 0$ in $[1,+\infty)$, and define $\psi_n(t) = \int_0^t \varphi(s-n)ds$ for each $n \in \mathbb{N}$. The functions ψ_n are smooth and enjoy the following properties:

- $\psi_n(t) = t$ for $t \in [0,n]$,
- $\psi_n(t) \equiv$ const. for $t \ge n+1$,
- $0 \le \psi_n' \le 1$ and $\psi_n'' \le 0$,
- for every $t \ge 0$, the sequence $(\psi_n'(t))$ is increasing.

Then, the function $W_n := \psi_n \circ W$ belongs to $C_b^2(\mathbb{R}^d)$ and it is constant outside a compact set. By Theorem 2.2(ii), applied to $W_n - c$, we have

$$\begin{aligned}
(G(t,s)W_n)(x) - (G(t,r)W_n)(x) &= -\int_r^s (G(t,\sigma)\mathcal{A}(\sigma)W_n)(x)\,d\sigma \\
&= -\int_r^s G(t,\sigma)\left\{\psi_n'(W)(\mathcal{A}(\sigma)W) + \psi_n''(W)\langle Q\nabla W, \nabla W\rangle\right\}(x)d\sigma \\
&\ge -\int_r^s G(t,\sigma)(\psi_n'(W)\mathcal{A}(\sigma)W)(x)d\sigma \\
&= -\int_r^s d\sigma \int_{E_\sigma} \psi_n'(W(y))(\mathcal{A}(\sigma)W)(y)p_{t,\sigma,x}(dy) \\
&\quad -\int_r^s d\sigma \int_{\mathbb{R}^d\setminus E_\sigma} \psi_n'(W(y))(\mathcal{A}(\sigma)W)(y)p_{t,\sigma,x}(dy),
\end{aligned}$$

where $E_\sigma = \{x \in \mathbb{R}^d : (\mathcal{A}(\sigma)W)(x) > 0\}$. Letting $n \to +\infty$, the left-hand side goes to $(G(t,s)W)(x) - (G(t,r)W)(x)$. Concerning the right-hand side, both integrals converge by monotone convergence. We have to prove that their limits are finite. The first term converges to $-\int_r^s d\sigma \int_{E_\sigma} (\mathcal{A}(\sigma)W)(y)p_{t,\sigma,x}(dy)$, which is finite since the sets E_σ are equibounded in $\mathbb{R}^d$ (recall that the function $\mathcal{A}(\sigma)W$ tends to $-\infty$ as $|x| \to +\infty$, uniformly with respect to $\sigma \in [r,s]$). The second term may be estimated by

$$
\begin{aligned}
&-\int_r^s d\sigma \int_{\mathbb{R}^d\setminus E_\sigma} \psi_n'(W(y))(\mathcal{A}(\sigma)W)(y)p_{t,\sigma,x}(dy)\\
&\le \int_r^s d\sigma \int_{E_\sigma} \psi_n'(W(y))(\mathcal{A}(\sigma)W)(y)p_{t,\sigma,x}(dy) + (G(t,s)W_n)(x) - (G(t,r)W_n)(x).
\end{aligned}
$$

Letting $n \to +\infty$, we obtain that $\int_r^s d\sigma \int_{\mathbb{R}^d\setminus E_\sigma}(\mathcal{A}(\sigma)W)(y)p_{t,\sigma,x}(dy)$ is finite.

Summing up, the function $\sigma \mapsto (G(t,\sigma)(\mathcal{A}(\sigma)W))(x)$ is in $L^1(r,s)$ and (3.2) follows.

Possibly replacing g by $\widetilde{g} = g - C$ for a suitable constant C, we may assume that $(\mathcal{A}(s)W)(x) \le -g(W(x))$ for every $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

Fix $x \in \mathbb{R}^d$, $t \in \mathbb{R}$, and set

$$\beta(s) := (G(t,t-s)W)(x), \quad s \ge 0.$$

Then β is measurable, since it is the limit of the sequence of continuous functions $s \mapsto G(t,t-s)W_n(x)$. Inequality (3.2) implies

$$\beta(b) - \beta(a) \le -\int_{t-b}^{t-a} (G(t,\sigma)g(W))(x)d\sigma, \qquad a < b,$$

and, since g is convex,

$$
\begin{aligned}
(G(t,\sigma)g(W))(x) &= \int_{\mathbb{R}^d} g(W(y))p_{t,\sigma,x}(dy)\\
&\ge g\left(\int_{\mathbb{R}^d} W(y)p_{t,\sigma,x}(dy)\right) = g((G(t,\sigma)W)(x))
\end{aligned}
$$

so that

$$
\begin{aligned}
\beta(b) - \beta(a) &\le -\int_{t-b}^{t-a} g((G(t,\sigma)W)(x))d\sigma\\
&= -\int_{t-b}^{t-a} g(\beta(t-\sigma))d\sigma \qquad (3.3)\\
&= -\int_a^b g(\beta(\sigma))d\sigma,
\end{aligned}
$$

for any $a < b$. Then, for every $s \ge 0$, $\beta(s) \le \zeta(s)$, where ζ is the solution of the Cauchy problem

$$
\begin{cases}
\zeta'(s) = -g(\zeta(s)), \quad s \ge 0,\\
\zeta(0) = W(x).
\end{cases}
$$

Indeed, assume by contradiction that there exists $s_0 > 0$ such that $\beta(s_0) > \zeta(s_0)$, and denote by I the largest interval containing s_0 such that $\beta(s) > \zeta(s)$ for each $s \in I$.

Inequality (3.3) implies that $\beta(b)-\beta(a)\le -m(b-a)$ for $b>a$, with $m:=\min g$. In other words, the function $s\mapsto \beta(s)+ms$ is decreasing. This implies that I contains some left neighborhood of s_0. Indeed, since $s\mapsto\beta(s)+ms$ is decreasing, then

$$\lim_{s\to s_0^-}\beta(s)+ms\ge\beta(s_0)+ms_0>\zeta(s_0)+ms_0=\lim_{s\to s_0^-}\zeta(s)+ms$$

so that $\lim_{s\to s_0^-}(\beta(s)-\zeta(s))>0$, which yields $\beta>\zeta$ in a left neighborhood of s_0.

Let $a=\inf I$. Then $a<s_0$, and there is a sequence $(s_n)\uparrow a$ such that $\beta(s_n)\le\zeta(s_n)$, so that $\beta(a)+ma\le\lim_{n\to\infty}\beta(s_n)+ms_n\le\zeta(a)+ma$, that is $\beta(a)\le\zeta(a)$. On the other hand, for each $s\in I$ we have

$$\beta(s)-\beta(a)\le\int_a^s -g(\beta(\sigma))d\sigma,\quad \zeta(s)-\zeta(a)=\int_a^s -g(\zeta(\sigma))d\sigma,$$

so that

$$\beta(s)-\zeta(s)\le\int_a^s\left[-g(\beta(\sigma))+g(\zeta(\sigma))\right]d\sigma,\quad s\in I.$$

Since $\beta(\sigma)>\zeta(\sigma)$ for every $\sigma\in I$ and g is increasing, the integral in the right-hand side is nonpositive, a contradiction. Therefore, $\beta(s)\le\zeta(s)$ for every $s\ge 0$.

By standard arguments about ODE's, for every $\delta>0$ there is $C=C(\delta)$ independent on the initial datum $W(x)$ such that $\zeta(s)\le C$ for every $s\ge\delta$. Therefore,

$$\beta(s)=(G(t,t-s)W)(x)\le C,\quad s\ge\delta,$$

with C independent of t. This implies that for every $\delta>0$ the family of probabilities $p_{t,s,x}(dy)$ with $s\le t-\delta$ and $x\in\mathbb{R}^d$ is tight, because for every $R>0$ we have

$$\begin{aligned}
p_{t,s,x}(\mathbb{R}^d\setminus B(0,R))&=\int_{\mathbb{R}^d\setminus B(0,R)}p_{t,s,x}(dy)\\
&\le\frac{1}{\inf\{W(y):\ |y|\ge R\}}\int_{\mathbb{R}^d\setminus B(0,R)}W(y)p_{t,s,x}(dy)\\
&\le\frac{1}{\inf\{W(y):\ |y|\ge R\}}(G(t,s)W)(x)\le\frac{C}{\inf\{W(y):\ |y|\ge R\}}
\end{aligned}$$

and $\inf\{W(y):\ |y|\ge R\}$ goes to $+\infty$ as $r\to+\infty$. So, condition (b) of Proposition 3.1 is satisfied. □

Example (As in the autonomous case). If there is $R>0$ such that

$$\mathrm{Tr}\,Q(s,x)+\langle b(s,x),x\rangle-\frac{2}{|x|^2}\langle Q(s,x)x,x\rangle\le -c|x|^2(\log|x|)^\gamma,\quad s\in\mathbb{R},\ |x|\ge R,$$

with $c>0$ and $\gamma>1$, then the condition (3.1) is satisfied by any W such that $W(x)=\log|x|$ for $|x|\ge R$, with $g(s)=cs^\gamma$. If the regularity and ellipticity assumptions 2.1(i)(ii) hold, Theorem 3.3 implies that the evolution operator $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for $t>s$.

4. Compactness and asymptotic behavior

In this section we derive asymptotic behavior results from compactness of $G(t,s)$ in $C_b(\mathbb{R}^d)$.

Throughout the section we assume that Hypothesis 2.1 holds, and that the coefficients q_{ij} and b_i, $i,j=1,\dots,d$ are periodic in time, with period $T>0$. Then the asymptotic behavior of $G(t,s)$ is driven by the spectral properties of the operators

$$V(s) := G(s+T,s), \quad s\in\mathbb{R}.$$

This is well known in the case of evolution operators associated to families $A(t)$ of generators of analytic semigroups, see, e.g., [7, §7.2], [11, Ch. 6], [6]. Most of the arguments are independent of analyticity and will be adapted to our situation.

To begin with, since each $V(s)$ is a contraction in $C_b(\mathbb{R}^d)$, its spectrum is contained in the unit circle. Its spectral radius is 1, since 1 is an eigenvalue. The nonzero eigenvalues of $V(s)$ are independent of s, since the equality $G(t,s)V(s) = V(t)G(t,s)$ implies that for each eigenfunction φ of $V(s)$, $G(t,s)\varphi\neq 0$ is an eigenfunction of $V(t)$ with the same eigenvalue, for $t>s$.

If $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for $t>s$, then $\sigma(V(s))\setminus\{0\}$ consists of isolated eigenvalues, hence it is independent of s. Therefore,

$$\sup\{|\lambda| :\ \lambda\in\sigma(V(s)),\ |\lambda|<1,\ s\in\mathbb{R}\} := r < 1. \tag{4.1}$$

Denoting by $Q(s)$ the spectral projection

$$Q(s) = \frac{1}{2\pi i}\int_{\partial B(0,a)} (\lambda I - V(s))^{-1} d\lambda, \quad s\in\mathbb{R},$$

with any $a\in(r,1)$, it is not difficult to see that for every $\varepsilon>0$ there is $M_\varepsilon>0$ such that

$$\|G(t,s)Q(s)\varphi\|_\infty \le M_\varepsilon e^{(t-s)(\log r(s)+\varepsilon)/T}\|\varphi\|_\infty, \quad t>s,\ \varphi\in C_b(\mathbb{R}^d). \tag{4.2}$$

(The proof may be obtained from the proof of (4.4) in Proposition 4.4, replacing the L^p spaces considered there by $C_b(\mathbb{R}^d)$.)

In the proof of the next proposition we use an important corollary of the Krein-Rutman Theorem, whose proof may be found in, e.g., [3, Ch. 1].

Theorem 4.1. *Let K be a cone with nonempty interior part $\mathring{K}$ in a Banach space X, and let $L : X\mapsto X$ be a linear compact operator such that $L\varphi\in\mathring{K}$ for each $\varphi\in K\setminus\{0\}$. Then the spectral radius r of L is a simple eigenvalue of L, and all the other eigenvalues have modulus $<r$.*

Proposition 4.2. *If $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for $t>s$, then 1 is a simple eigenvalue of $V(s)$ for each s, and it is the unique eigenvalue on the unit circle. The spectral projection $P(s) = I-Q(s)$ is given by*

$$P(s)\varphi(x) = \int_{\mathbb{R}^d}\varphi(y)\mu_s(dy), \quad \varphi\in C_b(\mathbb{R}^d),\ s\in\mathbb{R}, x\in\mathbb{R}^d,$$

where $\{\mu_s :\ s\in\mathbb{R}\}$ is a T-periodic evolution system of measures.

Proof. Let $K = \{\varphi \in C_b(\mathbb{R}^d) : \forall x \in \mathbb{R}^d,\ \varphi(x) \geq 0\}$ be the cone of the nonnegative functions in $C_b(\mathbb{R}^d)$. By Remark 3.2(iv), if $\varphi \in C_b(\mathbb{R}^d) \setminus \{0\}$ is such that $\varphi(x) \geq 0$ for each x, then $\inf V(s)\varphi = \inf G(s+T,s)\varphi(x) > 0$. In other words, $V(s)$ maps $K \setminus \{0\}$ into the interior part of K. Theorem 4.1 implies that the spectral radius 1 of $V(s)$ is a simple eigenvalue, and it is the unique eigenvalue of $V(s)$ on the unit circle. The associated spectral projection $P(s) = I - Q(s)$, with $Q(s)$ defined above, may be expressed as

$$P(s)\varphi = m_s\varphi\,\mathbb{1}, \quad \varphi \in C_b(\mathbb{R}^d),$$

for some m_s in the dual space of $C_b(\mathbb{R}^d)$. To prove that $m_s\varphi = \int_{\mathbb{R}^d}\varphi(y)\mu_s(dy)$ for some measure μ_s we use the Stone–Daniell Theorem (e.g., [4, Thm. 4.5.2]): it is enough to check that $m_s\varphi \geq 0$ if $\varphi \geq 0$, and that for each sequence $(\varphi_n) \subset C_b(\mathbb{R}^d)$ such that $(\varphi_n(x))$ is decreasing and converges to 0 for each $x \in \mathbb{R}^d$, we have $\lim_{n\to\infty} m_s\varphi_n = 0$. In this case, μ_s is a probability measure for every s, because $P(s)\mathbb{1} = \mathbb{1}$.

By the general spectral theory, $P(s) = \lim_{\lambda\to 1^-} V_\lambda$, where $V_\lambda := (\lambda - 1)(\lambda I - V(s))^{-1}$. In its turn, $(\lambda I - V(s))^{-1} = \sum_{k=0}^{\infty} V(s)^k/\lambda^{k+1}$ maps nonnegative functions into nonnegative functions because $V(s)$ does. Therefore, $m_s\varphi\,\mathbb{1} = P(s)\varphi \geq 0$ for each $\varphi \geq 0$.

Let now $\varphi_n \downarrow 0$. We claim that $V(s)\varphi_n$ converges to 0 uniformly. Indeed, since the measures $\{p_{s+T,s,x} :\ x \in \mathbb{R}^d\}$ are tight, for each $\varepsilon > 0$ there is $R > 0$ such that $\int_{\mathbb{R}^d\setminus B(0,R)} p_{s+T,s,x}(dy) \leq \varepsilon$, for each $x \in \mathbb{R}^d$. On the other hand, φ_n converges to 0 uniformly on $B(0,R)$ by the Dini Monotone Convergence Theorem, so that for n large, say $n \geq n_0$, we have $\varphi_n(y) \leq \varepsilon$, for $|y| \leq R$. Therefore, for $n \geq n_0$ we have

$$\begin{aligned} 0 \leq V(s)\varphi_n(x) &= \int_{B(0,R)} \varphi_n(y)p_{s+T,s,x}(dy) + \int_{\mathbb{R}^d\setminus B(0,R)} \varphi_n(y)p_{s+T,s,x}(dy) \\ &\leq \varepsilon + \|\varphi_n\|_\infty\varepsilon \end{aligned}$$

for each $x \in \mathbb{R}^d$. Since $V(s)\varphi_n$ converges to 0 uniformly, then $P(s)V(s)\varphi_n = V(s)P(s)\varphi_n$ converges to 0 uniformly. But $V(s)$ is the identity on the range of $P(s)$. Then, $P(s)\varphi_n$ converges uniformly to 0, which implies that $\lim_{n\to\infty} m_s\varphi_n = 0$.

Let us prove that $\{\mu_s :\ s \in \mathbb{R}\}$ is a T-periodic evolution system of measures. Since $s \mapsto P(s)$ is T-periodic, then $\mu_s = \mu_{s+T}$ for each $s \in \mathbb{R}$. Moreover, since $V(t)G(t,s) = G(t,s)V(s)$, then $P(t)G(t,s)\varphi = G(t,s)P(s)\varphi$, for each $\varphi \in C_b(\mathbb{R}^d)$. This means

$$\int_{\mathbb{R}^d} G(t,s)\varphi\, d\mu_t\mathbb{1} = G(t,s)\bigg(\int_{\mathbb{R}^d}\varphi\, d\mu_s\,\mathbb{1}\bigg), \quad \varphi \in C_b(\mathbb{R}^d),$$

and since $G(t,s)\mathbb{1} = \mathbb{1}$, then

$$\int_{\mathbb{R}^d} G(t,s)\varphi\, d\mu_t = \int_{\mathbb{R}^d}\varphi\, d\mu_s, \quad \varphi \in C_b(\mathbb{R}^d),$$

so that $\{\mu_s :\ s \in \mathbb{R}\}$ is an evolution system of measures. $\square$

Corollary 4.3. *Assume that $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for $t>s$. Then:*

(i) *There exists a unique T-periodic evolution system of measures $\{\mu_s:\ s\in\mathbb{R}\}$;*

(ii) *Setting $\omega_0=\log r/T$, where r is defined in (4.1), for each $\omega>\omega_0$ there exists $M=M(\omega)>0$ such that*

$$\|G(t,s)\varphi-\int_{\mathbb{R}^d}\varphi\,d\mu_s\|_\infty\le Me^{\omega(t-s)}\|\varphi\|_\infty,\quad t>s,\ \varphi\in C_b(\mathbb{R}^d),\tag{4.3}$$

while for $\omega<\omega_0$ there is no M such that (4.3) holds.

Proof. Let $\{\mu_s:\ s\in\mathbb{R}\}$ be the T-periodic evolution system of measures given by Proposition 4.2. Since $P(s)\varphi=\int_{\mathbb{R}^d}\varphi\,d\mu_s\mathbb{1}$, estimate (4.2) implies (4.3). Since there exist eigenvalues of $V(s)$ with modulus r, (4.3) cannot hold for $\omega<\omega_0$. Indeed, if $V(s)\varphi=r\varphi$ then $P(s)\varphi=0$ and $G(s+nT,s)\varphi=r^n\varphi=e^{\omega_0 nT}\varphi$ for each $n\in\mathbb{N}$, so that $\|G(s+nT,s)\varphi-m_s\varphi\|_\infty=\|G(s+nT,s)\varphi\|_\infty=e^{\omega_0 nT}\|\varphi\|_\infty$.

If $\{\nu_s:\ s\in\mathbb{R}\}$ is another T-periodic evolution system of measures, fix $t\in\mathbb{R}$ and $\varphi\in C_b(\mathbb{R}^d)$. Since $G(t,s)\varphi-\int_{\mathbb{R}^d}\varphi\,d\mu_s$ goes to zero uniformly as $s\to-\infty$, then

$$0=\lim_{s\to-\infty}\int_{\mathbb{R}^d}\left(G(t,s)\varphi-\int_{\mathbb{R}^d}\varphi\,d\mu_s\right)d\nu_t=\lim_{s\to-\infty}\left(\int_{\mathbb{R}^d}\varphi\,d\nu_s-\int_{\mathbb{R}^d}\varphi\,d\mu_s\right).$$

Since $s\mapsto\int_{\mathbb{R}^d}\varphi\,d\nu_s-\int_{\mathbb{R}^d}\varphi\,d\mu_s$ is T-periodic and goes to 0 as $s\to-\infty$, it vanishes in $\mathbb{R}$. By the arbitrariness of φ, $\nu_s=\mu_s$ for every $s\in\mathbb{R}$. □

Once we have an evolution system of measures $\{\mu_s:\ s\in\mathbb{R}\}$, $G(t,s)$ is extendable to a contraction (still denoted by $G(t,s)$) from $L^p(\mathbb{R}^d,\mu_s)$ to $L^p(\mathbb{R}^d,\mu_t)$ for $t>s$. Compactness and asymptotic behavior results in $C_b(\mathbb{R}^d)$ imply compactness and asymptotic behavior results in such L^p spaces, as the next proposition shows.

Proposition 4.4. *Let $G(t,s)$ be compact in $C_b(\mathbb{R}^d)$ for $t>s$. Then for every $p\in(1,\infty)$, $G(t,s):L^p(\mathbb{R}^d,\mu_s)\mapsto L^p(\mathbb{R}^d,\mu_t)$ is compact for $t>s$. Moreover, for every $\omega\in(\omega_0,0)$ and $p\in(1,\infty)$ there exist $M=M(\omega,p)>0$ such that*

$$\|G(t,s)\varphi-\int_{\mathbb{R}^d}\varphi\,d\mu_s\|_{L^p(\mathbb{R}^d,\mu_t)}\le Me^{\omega(t-s)}\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)},\quad t>s,\ \varphi\in L^p(\mathbb{R}^d,\mu_s),\tag{4.4}$$

and for every $\omega<\omega_0$ there is no M such that (4.4) holds. Here ω_0 is given by Corollary 4.3(ii).

Proof. Let us prove that $G(t,s):L^p(\mathbb{R}^d,\mu_s)\mapsto L^p(\mathbb{R}^d,\mu_t)$ is compact for $t>s$. We have $G(t,s)=G(t,(t+s)/2)G((t+s)/2,s)$ where $G((t+s)/2,s)$ is bounded from $L^\infty(\mathbb{R}^d)$ to $C_b(\mathbb{R}^d)$ by Theorem 2.2(iv), and $G(t,(t+s)/2)$ is compact in $C_b(\mathbb{R}^d)$. Therefore, $G(t,s)$ is compact in $L^\infty(\mathbb{R}^d)$.

Now, if μ_1 and μ_2 are probability measures and a linear operator is bounded from $L^1(\mathbb{R}^d,\mu_1)$ to $L^1(\mathbb{R}^d,\mu_2)$ and compact from $L^\infty(\mathbb{R}^d,\mu_1)$ to $L^\infty(\mathbb{R}^d,\mu_2)$, then it is compact from $L^p(\mathbb{R}^d,\mu_1)$ to $L^p(\mathbb{R}^d,\mu_2)$, for every $p\in(1,\infty)$ (the proof is the same as in [13, Prop. 4.6], where only one probability measure was considered).

Let us prove (4.4). Since $L^\infty(\mathbb{R}^d,\mu_t) = L^\infty(\mathbb{R}^d,\mu_s) = L^\infty(\mathbb{R}^d,dx)$ by [8, Prop. 5.2], interpolating (4.3) and $\|G(t,s)\varphi - m_s\varphi\|_{L^1(\mathbb{R}^d,\mu_t)} \le 2\|\varphi\|_{L^1(\mathbb{R}^d,\mu_s)}$ we obtain that $\|G(t,s)-m_s\|_{\mathcal{L}(L^p(\mathbb{R}^d,\mu_s),L^p(\mathbb{R}^d,\mu_t))}$ decays exponentially as $(t-s)\to\infty$. However, the decay rate that we obtain by interpolation depends on p. To prove (4.4) it is enough to show that the spectrum of the operators $V(s)$ in $L^p(\mathbb{R}^d,\mu_s)$ does not depend on s, and coincides with the spectrum in $C_b(\mathbb{R}^d)$. Since $V(s) = G(s+T,s)$, then $V(s)$ is compact in $L^p(\mathbb{R}^d,\mu_s)$. Therefore, its L^p spectrum (except zero) consists of eigenvalues, that are independent of s. They are independent of p too, as well as the associated spectral projections, by [2, Cor. 1.6.2].

The statement follows now as in the case of evolution operators in a fixed Banach space as in the mentioned references [7, 11, 6]. Note however that our Banach spaces $L^p(\mathbb{R}^d,\mu_s)$ vary with s, so that the classical theory cannot be used verbatim. For the reader's convenience we give the proof below.

Let $t-s = \sigma + kT$, with $k\in\mathbb{N}$ and $\sigma\in[0,T)$. We have

$$\begin{aligned}\|G(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} &= \|G(t,t-\sigma)V(s)^k(I-P(s))\varphi\|_{L^p(\mathbb{R}^d,\mu_t)}\\ &\le \|V(s)^k(I-P(s))\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}\\ &= \|[V(s)(I-P(s))]^k\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}.\end{aligned}$$

For $\omega>\omega_0$ let $\varepsilon>0$ be such that $\log(r+\varepsilon)\le\omega$, and let $k(s)\in\mathbb{N}$ be such that $\|[V(s)(I-P(s))]^k\varphi\|_{L^p(\mathbb{R}^d,\mu_s)} \le (r+\varepsilon)^k$ for each $k>k(s)$. Therefore, if the integer part $[(t-s)/T]$ is larger than $k(s)$ we have

$$\begin{aligned}\|G(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} &\le (r+\varepsilon)^k\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}\\ &\le e^{(t-s)\omega}\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)},\end{aligned}$$

for each $\varphi\in L^p(\mathbb{R}^d,\mu_s)$. Using the obvious inequality

$$\|G(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} \le 2\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)} \quad \text{for} \quad [(t-s)/T]\le k(s),$$

we arrive at

$$\|G(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} \le M_s e^{(t-s)\omega}\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}, \quad \varphi\in L^p(\mathbb{R}^d,\mu_s)$$

for some $M_s>0$. It remains to show that M_s can be taken independent of s. Since V is T-periodic, we may take $k(s)=k(s+T)$ and hence $M_s = M_{s+T}$ for every $s\in\mathbb{R}$. Therefore it is enough to show that M_s can be taken independent of s for $s\in[0,T)$. For $0\le s<T$ and $t\ge T$ we have $m_T G(T,s)\varphi = \int_{\mathbb{R}^d} G(T,s)\varphi\,d\mu_T = m_s\varphi$, hence

$$\begin{aligned}\|G(t,s)\varphi - m_s\varphi\|_{L^p(\mathbb{R}^d,\mu_t)} &= \|(G(t,T)-m_T)G(T,s)\varphi\|_{L^p(\mathbb{R}^d,\mu_t)}\\ &\le M_T e^{\omega(t-T)}\|G(T,s)\varphi\|_{L^p(\mathbb{R}^d,\mu_T)}\\ &\le M_T e^{|\omega|T}e^{\omega(t-s)}\|\varphi\|_{L^p(\mathbb{R}^d,\mu_s)}\end{aligned}$$

So, we can take $M_s = M_T e^{|\omega|T}$ for $0\le s<T$. (4.4) follows. □

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Gradient Estimates and Domain Identification for Analytic Ornstein-Uhlenbeck Operators

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Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. Let P be the Ornstein-Uhlenbeck semigroup associated with the stochastic Cauchy problem

$$dU(t) = AU(t)\,dt + dW_H(t),$$

where A is the generator of a C_0-semigroup S on a Banach space E, H is a Hilbert subspace of E, and W_H is an H-cylindrical Brownian motion. Assuming that S restricts to a C_0-semigroup on H, we obtain L^p-bounds for $D_H P(t)$. We show that if P is analytic, then the invariance assumption is fulfilled. As an application we determine the L^p-domain of the generator of P explicitly in the case where S restricts to a C_0-semigroup on H which is similar to an analytic contraction semigroup. The results are applied to the 1D stochastic heat equation driven by additive space-time white noise.

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1. Introduction

Consider the stochastic Cauchy problem

$$\begin{aligned} dU(t) &= AU(t)\,dt + dW_H(t), \quad t \geqslant 0, \\ U(0) &= x. \end{aligned} \tag{SCP}$$

Here A generates a C_0-semigroup $S = (S(t))_{t\geqslant 0}$ on a real Banach space E, H is a real Hilbert subspace continuously embedded in E, W_H is an H-cylindrical Brownian motion on a probability space $(\Omega, \mathscr{F}\, P)$, and $x \in E$. A *weak solution* is

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a measurable adapted E-valued process $U^x = (U^x(t))_{t\geqslant 0}$ such that $t \mapsto U^x(t)$ is integrable almost surely and for all $t \geqslant 0$ and $x^* \in \mathsf{D}(A^*)$ one has

$$\langle U^x(t), x^*\rangle = \langle x, x^*\rangle + \int_0^t \langle U^x(s), A^*x^*\rangle\, ds + W_H(t)i^*x^* \quad \text{almost surely.}$$

Here $i : H \hookrightarrow E$ is the inclusion mapping. A necessary and sufficient condition for the existence of a weak solution is that the operators $I_t : L^2(0,t;H) \to E$,

$$I_t g := \int_0^t S(s)ig(s)\, ds,$$

are γ-radonifying for all $t \geqslant 0$. If this is the case, then $s \mapsto S(t-s)i$ is stochastically integrable on $(0,t)$ with respect to W_H and the process U^x is given by

$$U^x(t) = S(t)x + \int_0^t S(t-s)i\, dW_H(s), \quad t \geqslant 0.$$

For more information and an explanation of the terminology we refer to [30].

Assuming the existence of the solution U^x, on the Banach space $C_{\mathrm{b}}(E)$ of all bounded continuous functions $f : E \to \mathbb{R}$ one defines the *Ornstein-Uhlenbeck semigroup* $P = (P(t))_{t\geqslant 0}$ by

$$P(t)f(x) := \mathbb{E}f(U^x(t)), \quad t \geqslant 0,\ x \in E. \tag{1.1}$$

The operators $P(t)$ are linear contractions on $C_{\mathrm{b}}(E)$ and satisfy $P(0) = I$ and $P(s)P(t) = P(s+t)$ for all $s,t \geqslant 0$. For all $f \in C_{\mathrm{b}}(E)$ the mapping $(t,x) \mapsto P(t)f(x)$ is continuous, uniformly on compact subsets of $[0,\infty) \times E$.

If the operator $I_\infty : L^2(0,\infty;H) \to E$ defined by

$$I_\infty g := \int_0^\infty S(t)ig(t)\, dt$$

is γ-radonifying, then the problem (SCP) admits a unique invariant measure μ_∞. This measure is a centred Gaussian Radon measure on E, and its covariance operator equals $I_\infty I_\infty^*$. Throughout this paper we shall assume that this measure exists; if (SCP) has a solution, then this assumption is for instance fulfilled if S is uniformly exponentially stable. The reproducing kernel Hilbert space associated with μ_∞ is denoted by H_∞. The inclusion mapping $H_\infty \hookrightarrow E$ is denoted by i_∞. Recall that $Q_\infty := i_\infty i_\infty^* = I_\infty I_\infty^*$. It is well known that S restricts to a C_0-contraction semigroup on H_∞ [5] (the proof for Hilbert spaces E extends without change to Banach spaces E), which we shall denote by S_∞.

By a standard application of Jensen's inequality, the semigroup P has a unique extension to a C_0-contraction semigroup to the spaces $L^p(E,\mu_\infty)$, $1 \leqslant p < \infty$. By slight abuse of notation we shall denote this semigroup by P again. Its infinitesimal generator will be denoted by L. In order to give an explicit expression for L it is useful to introduce, for integers $k,l \geqslant 0$, the space $\mathscr{F}C_{\mathrm{b}}^{k,l}(E)$ consisting of all functions $f \in C_{\mathrm{b}}(E)$ of the form

$$f(x) = \varphi(\langle x, x_1^*\rangle, \ldots, \langle x, x_N^*\rangle)$$

with $f \in C_{\mathrm{b}}^k(\mathbb{R}^N)$ and $x_1^*, \dots, x_N^* \in \mathsf{D}(A^{*l})$. With this notation one has that $\mathscr{F}C_{\mathrm{b}}^{2,1}(E)$ is a core for L, and on this core one has

$$Lf(x) = \frac{1}{2} \operatorname{tr} D_H^2 f(x) + \langle x, A^* Df(x) \rangle.$$

Here,

$$D_H f(x) = \sum_{n=1}^N \frac{\partial \varphi}{\partial x_n}(\langle x, x_1^* \rangle, \dots, \langle x, x_N^* \rangle) \otimes i^* x_n^*,$$

$$Df(x) = \sum_{n=1}^N \frac{\partial \varphi}{\partial x_n}(\langle x, x_1^* \rangle, \dots, \langle x, x_N^* \rangle) \otimes x_n^*,$$

denote the Fréchet derivatives into the directions of H and E, respectively.

2. Gradient estimates: the H-invariant case

Our first result gives a pointwise gradient bound for P under the assumption that S restricts to a C_0-semigroup on H which will be denoted by S_H. As has been shown in [17, Corollary 5.6], under this assumption the operator D_H is closable as a densely defined operator from $L^p(E, \mu_\infty)$ to $L^p(E, \mu; H)$ for all $1 \leqslant p < \infty$. The domain of its closure is denoted by $\mathsf{D}_p(D_H)$.

Proposition 2.1 (Pointwise gradient bounds). *If S restricts to a C_0-semigroup on H, then for all $1 < p < \infty$ there exists a constant $C \geqslant 0$ such that for all $t > 0$ and $f \in \mathscr{F}C_{\mathrm{b}}^{1,0}(E)$ we have*

$$\sqrt{t} |D_H P(t) f(x)| \leqslant C \kappa(t) (P(t) |f|^p(x))^{1/p},$$

where $\kappa(t) := \sup_{s \in [0,t]} \|S_H(s)\|_{\mathscr{L}(H)}$.

Proof. The proof follows the lines of [25, Theorem 8.10] and is inspired by the proof of [10, Theorem 6.2.2], where the null controllable case was considered.

The distribution μ_t of the random variable $U^0(t)$ is a centred Gaussian Radon measure on E. Let H_t denote its RKHS and let $i_t : H_t \hookrightarrow E$ be the inclusion mapping. As is well known and easy to prove, cf. [9, Appendix B] one has

$$H_t = \Big\{ \int_0^t S(t-s) i g(s) \, ds : \; g \in L^2(0, t; H) \Big\}$$

with

$$\|h\|_{H_t} = \inf \Big\{ \|g\|_{L^2(0,t;H)} : \; h = \int_0^t S(t-s) i g(s) \, ds \Big\}.$$

The mapping

$$\phi^{\mu_t} : i_t^* x^* \mapsto \langle \cdot, x^* \rangle, \quad x^* \in E^*,$$

defines an isometry from H_t onto a closed subspace of $L^2(E, \mu_t)$. For $h \in H_t$ we shall write $\phi_h^{\mu_t}(x) := (\phi^{\mu_t} h)(x)$.

Fix $h \in H$. Since S restricts to a C_0-semigroup S_H on H we may consider the function $g \in L^2(0,t;H)$ given by $g(s) = \frac{1}{t}S(s)h$. From the identity $S(t)h = \int_0^t S(t-s)g(s)\,ds$ we deduce that $S(t)h \in H_t$ and

$$\|S(t)h\|_{H_t}^2 \leqslant \|g\|_{L^2(0,t;H)}^2 = \frac{1}{t^2}\int_0^t \|S(s)h\|_H^2\,ds \leqslant \frac{1}{t}\kappa(t)^2\|h\|_H^2. \tag{2.1}$$

Fix a function $f \in \mathscr{F}C_{\mathrm{b}}^{1,0}(E)$, that is, $f(x) = \varphi(\langle x, x_1^*\rangle, \dots, \langle x, x_N^*\rangle)$ with $\varphi \in C_{\mathrm{b}}^1(\mathbb{R}^N)$ and $x_1^*, \dots, x_N^* \in E^*$. It is easily checked that for all $t > 0$ we have $P(t)f \in \mathscr{F}C_{\mathrm{b}}^{1,0}(E)$; in particular this implies that $P(t)f \in \mathsf{D}_p(D_H)$. By the Cameron-Martin formula [3],

$$\begin{aligned}\frac{1}{\varepsilon}\big(P(t)f(x+\varepsilon h) - P(t)f(x)\big) &= \frac{1}{\varepsilon}\int_E \big(f(S(t)(x+\varepsilon h)+y) - f(S(t)x+y)\big)\,d\mu_t(y)\\ &= \int_E \frac{1}{\varepsilon}(E_{\varepsilon S(t)h} - 1)f(S(t)x+y)\,d\mu_t(y),\end{aligned}$$

where for $h \in H_t$ we write

$$E_h(x) := \exp(\phi_h^{\mu_t}(x) - \tfrac{1}{2}\|h\|_{H_t}^2).$$

It is easy to see that for each $h \in H_t$ the family $\big(\frac{1}{\varepsilon}(E_{\varepsilon h} - 1)\big)_{0<\varepsilon<1}$ is uniformly bounded in $L^2(E,\mu_t)$, and therefore uniformly integrable in $L^1(E,\mu_t)$. Passage to the limit $\varepsilon \downarrow 0$ in the previous identity now gives

$$[D_H P(t)f(x), h] = \int_E f(S(t)x+y)\phi_{S(t)h}^{\mu_t}(y)\,d\mu_t(y).$$

By Hölder's inequality with $\frac{1}{r} + \frac{1}{q} = 1$ and the Kahane-Khintchine inequality, which can be applied since $\phi_{S(t)h}^{\mu_t}$ is a Gaussian random variable,

$$\begin{aligned}&|[D_H P(t)f(x), h]|\\ &\quad\leqslant \Big(\int_E |f(S(t)x+y)|^r\,d\mu_t(y)\Big)^{\frac{1}{r}}\Big(\int_E |\phi_{S(t)h}^{\mu_t}(y)|^q\,d\mu_t(y)\Big)^{\frac{1}{q}}\\ &\quad\leqslant K_q\Big(\int_E |f(S(t)x+y)|^r\,d\mu_t(y)\Big)^{\frac{1}{r}}\Big(\int_E |\phi_{S(t)h}^{\mu_t}(y)|^2\,d\mu_t(y)\Big)^{\frac{1}{2}}\\ &\quad= K_q(P(t)|f|^r(x))^{\frac{1}{r}}\|S(t)h\|_{H_t}.\end{aligned}$$

Using (2.1) we find that

$$\big|\sqrt{t}[D_H P(t)f(x), h]\big| \leqslant K_q\kappa(t)(P(t)|f|^r(x))^{\frac{1}{r}}\|h\|_H,$$

and by taking the supremum over all $h \in H$ of norm 1 we obtain the desired estimate. $\square$

Corollary 2.2. *If S restricts to a C_0-semigroup on H, then for all $1 < p < \infty$ the operators $D_H P(t)$, $t > 0$, extend uniquely to bounded operators from $L^p(E,\mu_\infty)$ to $L^p(E,\mu_\infty;H)$, and there exists a constant $C \geqslant 0$ such that for any $t > 0$,*

$$\sqrt{t}\|D_H P(t)\|_{\mathscr{L}(L^p(E,\mu_\infty),L^p(E,\mu_\infty;H))} \leqslant C\kappa(t).$$

Proof. Integrating the inequality of the proposition and using the fact that μ_∞ is an invariant measure for P we obtain

$$\begin{aligned}\|\sqrt{t}D_H P(t)f\|^p_{L^p(E,\mu_\infty)} &\leqslant C^p\kappa(t)^p \int_E P(t)|f|^p(x)\,d\mu_\infty(x)\\ &= C^p\kappa(t)^p \int_E |f|^p(x)\,d\mu_\infty(x) = C^p\kappa(t)^p\|f\|^p_{L^p(E,\mu_\infty)}. \qquad \square\end{aligned}$$

3. Gradient estimates: the analytic case

Analyticity of the semigroup P on $L^p(E,\mu_\infty)$ has been investigated by several authors [15, 16, 18, 24]. The following result of [18] is our starting point. Recall that in the definition of an *analytic C_0-contraction semigroup*, contractivity is required on an open sector containing the positive real axis.

Proposition 3.1. *For any $1 < p < \infty$ the following assertions are equivalent:*
(1) *P is an analytic C_0-semigroup on $L^p(E,\mu_\infty)$;*
(2) *P is an analytic C_0-contraction semigroup on $L^p(E,\mu_\infty)$;*
(3) *S restricts to an analytic C_0-contraction semigroup on H_∞;*
(4) *$Q_\infty A^*$ acts as a bounded operator in H.*

A more precise formulation of (4) is that there should exist a bounded operator $B : H \to H$ such that $iBi^*x^* = Q_\infty A^*x^*$ for all $x^* \in E^*$. The identity $Q_\infty A^* + AQ_\infty = -ii^*$ implies that $B + B^* = -I$.

In what follows we shall simply say that 'P is analytic' to express that the equivalent conditions of the proposition are satisfied for some (and hence for all) $1 < p < \infty$.

The next result has been shown in [24] for $p = 2$ and was extended to $1 < p < \infty$ in [25].

Proposition 3.2. *If P is analytic, then $\mathscr{F}C^{2,1}_{\mathrm{b}}(E)$ is a core for the generator L of P in $L^p(E,\mu_\infty)$, and on this core L is given by*

$$L = D_H^* B D_H.$$

Our first aim is to show that analyticity of P implies that H is S-invariant. For self-adjoint P this was proved in [7, 18].

Theorem 3.3. *If P is analytic, then S restricts to a bounded analytic C_0-semigroup S_H on H.*

Proof. Consider the linear mapping

$$V : i_\infty^* x^* \mapsto i^*x^*, \quad x^* \in E^*. \tag{3.1}$$

It is shown in [17] that $i_\infty^* x^* = 0$ implies $i^*x^* = 0$, so that this mapping is well defined, and that the closability of D_H implies the closability of V as a densely defined operator from H_∞ to H. With slight abuse of notation we denote its closure by V again and let $\mathsf{D}(V)$ the domain of the closure.

By [1, Proposition 7.1], the operator $-VV^*B$ is sectorial of angle $< \frac{\pi}{2}$, and therefore $G := VV^*B$ generates a bounded analytic C_0-semigroup $(T(t))_{t\geq 0}$ on H. To prove the theorem, by uniqueness of analytic continuation and duality it suffices to show that $T(t) \circ i^* = i^* \circ S^*(t)$ for all $t \geqslant 0$.

For all $x^* \in \mathsf{D}(A^*)$ we have $Bi^*x^* \in \mathsf{D}(V^*)$ and $V^*Bi^*x^* = i_\infty^* A^*x^*$. Indeed, for $y^* \in E^*$ one has

$$[Bi^*x^*, Vi_\infty^* y^*] = \langle i_\infty^* A^* x^*, i_\infty^* y^* \rangle,$$

which implies the claim. By applying the operator V to this identity we obtain $i^*x^* \in \mathsf{D}(G)$ and $G\, i^*x^* = i^*A^*x^*$, from which it follows that $T(t)i^*x^* = i^*S^*(t)x^*$. This proves the theorem, with $S_H = T^*$. □

This result should be compared with [18, Theorem 9.2], where it is shown that if S restricts to an analytic C_0-semigroup on H which is contractive in some equivalent Hilbert space norm, then P is analytic on $L^p(E, \mu_\infty)$.

Under the assumption that P is analytic on $L^p(E, \mu_\infty)$, the gradient estimates of the previous section can be improved as follows. Recall that a collection of bounded operators $\mathscr{T}$ between Banach spaces X and Y is said to be *R-bounded* if there exists a constant C such that for any finite subset $T_1, \ldots, T_n \subset \mathscr{T}$ and any $x_1, \ldots, x_n \in X$ we have

$$\mathbb{E}\Big\| \sum_{j=1}^n r_j T_j x_j \Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{j=1}^n r_j x_j \Big\|^2,$$

where $(r_j)_{j\geqslant 1}$ is an independent collection of Rademacher random variables. The notion of R-boundedness plays an important role in recent advances in the theory of evolution equations (see [12, 21]).

Theorem 3.4. *If P is analytic, then for all $1 < p < \infty$ the set*

$$\{\sqrt{t} D_H P(t) : \ t > 0\}$$

is R-bounded in $\mathscr{L}(L^p(E, \mu_\infty), L^p(E, \mu_\infty; H))$ and we have the square function estimate

$$\Big\| \Big(\int_0^t \| D_H P(t) f \|_H^2 \, dt \Big)^{1/2} \Big\|_{L^p(E,\mu_\infty)} \lesssim \|f\|_{L^p(E,\mu_\infty)}$$

with implied constant independent of $f \in L^p(E, \mu_\infty)$.

Proof. By Proposition 3.2 and Theorem 3.3, the theorem is a special case of [25, Theorem 2.2]. □

The above result plays a crucial role in our recent paper [25] in which L^p-domain characterisations for the operator L and its square root have been obtained. Before stating the result, let us informally sketch how Theorem 3.4 enters the argument. In order to prove a domain characterisation for the operator L, we first aim to obtain two-sided estimates for $\|\sqrt{-L} f\|_{L^p(E,\mu_\infty)}$ in terms of suitable Sobolev norms. For this purpose we consider a variant of an operator theoretic framework introduced in [2] in the analysis of the famous Kato square root problem. The idea behind this framework is that the second-order operator L can be naturally studied through the first-order Hodge-Dirac-type operator

$$\Pi = \begin{bmatrix} 0 & -D_H^* B \\ D_H & 0 \end{bmatrix} \text{ on } L^p(E, \mu_\infty) \oplus L^p(E, \mu_\infty; H).$$

This operator is bisectorial and its square is the sectorial operator given by

$$-\Pi^2 = \begin{bmatrix} D_V^* B D_V & 0 \\ 0 & D_V D_V^* B \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix},$$

where $\underline{L} := D_V D_V^* B$. The approach in [25] consists of proving estimates for $\sqrt{-L}f$ along the lines of the following formal calculation:

$$\|D_H f\|_p = \|\Pi(f,0)\|_p \leqslant \|\Pi/\sqrt{\Pi^2}\|_p \, \|\sqrt{\Pi^2}(f,0)\|_p = \|\Pi/\sqrt{\Pi^2}\|_p \, \|\sqrt{L}f\|_p.$$

Oversimplifying things considerably, the proof consists of turning this calculation into rigorous mathematics. This can be done once we know that the operator $\Pi/\sqrt{\Pi^2}$ is bounded. Since the function $z \mapsto z/\sqrt{z^2}$ is a bounded analytic function on each bisector around the real axis, it suffices to show that Π has a bounded H^∞-functional calculus. This in turn will follow if we show that

1. the resolvent set $\{(it - \Pi)^{-1}\}_{t \in \mathbb{R}\setminus\{0\}}$ is R-bounded;
2. the operator Π^2 admits a bounded functional calculus.

To prove (1), we observe that

$$(I - it\Pi)^{-1} = \begin{bmatrix} (1 - t^2 L)^{-1} & -it(I - t^2 L)^{-1} D_H^* B \\ it D_H (I - t^2 L)^{-1} & (I - t^2 \underline{L})^{-1} \end{bmatrix}, \quad t \in \mathbb{R} \setminus \{0\}.$$

It suffices to prove R-boundedness for each of the entries separately. The diagonal entries can be dealt with using abstract results on R-boundedness for positive contraction semigroups on L^p-spaces. The R-boundedness for the off-diagonal entries can be derived using Theorem 3.4.

To prove (2) we use the fact that the semigroup generated by $\underline{L}$ equals $P \otimes S_H^*$ on the range of the gradient D_H. Here S_H denotes the restriction of the semigroup S to H (see Theorem 3.3). Therefore (2) follows, provided that the negative generator $-A_H$ of S_H has a bounded H^∞-calculus. This reduces the original question about $\sqrt{-L}$ to a question about the operator A_H, which is defined directly in terms of the data H and A of the problem. The latter question should be thought of as expressing the compatibility of the drift (represented by the operator A) and the noise (represented by the Hilbert space H). This compatibility condition is not automatically satisfied. In fact, by a result of Le Merdy [22], $-A_H$ admits a bounded H^∞-functional calculus on H if and only if S_H is an analytic C_0-*contraction* semigroup on H with respect to some equivalent Hilbert space norm. Such needs not always be the case, as is shown by well-known examples [26].

The following result summarises the informal discussion above and provides an additional equivalent condition in terms of the operator A_∞. In this result we let $\mathsf{D}_p(D_H^2)$ denote the second-order Sobolev space associated with the operator D_H.

Theorem 3.5. *Let $1 < p < \infty$. If P is analytic on $L^p(E, \mu_\infty)$, then the following assertions are equivalent:*

(1) $\mathsf{D}_p(\sqrt{-L}) = \mathsf{D}_p(D_H)$ *with norm equivalence*

$$\|\sqrt{-L}f\|_{L^p(E,\mu_\infty)} \eqsim \|D_H f\|_{L^p(E,\mu_\infty;H)};$$

(2) $\mathsf{D}(\sqrt{-A_\infty}) = \mathsf{D}(V)$ *with norm equivalence*

$$\|\sqrt{-A_\infty}h\|_{H_\infty} \eqsim \|Vh\|_H;$$

(3) $-A_H$ *admits a bounded H^∞-functional calculus on H.*

If these equivalent conditions are satisfied we have

$$\mathsf{D}_p(L) = \mathsf{D}_p(D_H^2) \cap \mathsf{D}_p(A_\infty^* D),$$

where D is the Malliavin derivative in the direction of H_∞.

Proof. By Proposition 3.2 and Theorem 3.3, the theorem is a special case of [25, Theorems 2.1, 2.2] provided we replace A_∞ by A_∞^* in (2). The equivalence of (2) for A_∞ and A_∞^*, however, is well known (see also [25, Lemma 10.2]). □

The problem of identifying the domains of $\sqrt{-L}$ and L has a long and interesting history. We finish this paper by presenting three known special cases of Theorem 3.5. In each case, it is easy to verify that (3) is satisfied.

Example 1. For the classical Ornstein-Uhlenbeck operator, which corresponds to $H = E = \mathbb{R}^d$ and $A = -I$, conditions (2) and (3) of Theorem 3.5 are trivially fulfilled and (1) reduces to the classical Meyer inequalities of Malliavin calculus. For a discussion of Meyer's inequalities we refer to the book of Nualart [31].

Example 2. Meyer's inequalities were extended to infinite dimensions by Shigekawa [32], and Chojnowska-Michalik and Goldys [6, 7], who considered the case where E is a Hilbert space and A_H is self-adjoint. Both authors deduce the generalised Meyer inequalities from square functions estimates. The identification of $\mathsf{D}_p(L)$ in the self-adjoint case is due to Chojnowska-Michalik and Goldys [6, 7], who extended the case $p = 2$ obtained earlier by Da Prato [8].

So far, these examples were concerned with the selfadjoint case.

Example 3. A non-selfadjoint extension of Meyer's inequalities has been given for the case $E = \mathbb{R}^d$ by Metafune, Prüss, Rhandi, and Schnaubelt [27] under the non-degeneracy assumption $H = \mathbb{R}^d$. In this situation the semigroup P is analytic on $L^p(\mu_\infty)$ [15], see also [16, 18]; no symmetry assumptions need to be imposed on A. The S-invariance of H and the fact that the generator of $S = S_H$ admits a bounded H^∞-calculus are trivial. Therefore, (3) is satisfied again. Note that the domain characterisation reduces to $\mathsf{D}_p(L) = \mathsf{D}_p(D^2)$, where D is the derivative on $\mathbb{R}^d$. The techniques used in [27] to prove (1) are very different, involving diagonalisation arguments and the non-commuting Dore-Venni theorem. The identification of $\mathsf{D}_p(L) = \mathsf{D}_p(D^2)$ for $p = 2$ had been obtained previously by Lunardi [23].

Our final corollary extends the characterisations of $\mathsf{D}_p(L)$ contained in Examples 2 and 3 and lifts the non-degeneracy assumption on H in Example 3.

Corollary 3.6. *If S restricts to an analytic C_0-semigroup on H which is contractive with respect to some equivalent Hilbert space norm, then for all $1 < p < \infty$ we have*

$$\mathsf{D}_p(L) = \mathsf{D}_p(D_H^2) \cap \mathsf{D}_p(A_\infty^* D),$$

where D is the Malliavin derivative in the direction of H_∞.

Proof. As has already been mentioned in the discussion preceding Theorem 3.4, the assumptions imply that P is analytic. Moreover, since the restricted semigroup S_H is similar to an analytic contraction semigroup, its negative generator $-A_H$ admits a bounded H^∞-calculus, and the result follows from Theorem 3.5. □

Let us finally mention that the results in [25] have been proved for a more general class of elliptic operators on Wiener spaces (cf. Section 3 of that paper). In this setting the data consist of

- an arbitrary Gaussian measure μ on a separable Banach space E with reproducing kernel Hilbert space $\mathscr{H}$;
- an analytic C_0-contraction semigroup $\mathscr{S}$ on $\mathscr{H}$ with generator $\mathscr{A}$.

Given these data, the semigroup $\mathscr{P}$ is defined on $L^2(E,\mu)$ by second quantisation of the semigroup $\mathscr{S}$. Roughly speaking, this means that one uses the Wiener-Itô isometry to identify $L^2(E,\mu)$ with the symmetric Fock space over $\mathscr{H}$, i.e., the direct sum of symmetric tensor powers of $\mathscr{H}$. The semigroup $\mathscr{P}$ is then defined by applying $\mathscr{S}$ to each factor

$$\mathscr{P}(t) \sum_{\sigma\in S_n} (h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}) := \sum_{\sigma\in S_n} \mathscr{S}(t)h_{\sigma(1)} \otimes \cdots \otimes \mathscr{S}(t)h_{\sigma(n)},$$

where S_n is the permutation group on $\{1,\dots,n\}$. For the details of this construction we refer to [19]. Equivalently, the semigroup $\mathscr{P}$ can be defined via the following generalisation of the classical Mehler formula,

$$\mathscr{P}(t)f(x) = \int_E f(\mathscr{S}(t)x + \sqrt{I - \mathscr{S}^*(t)\mathscr{S}(t)}y)\, d\mu(y),$$

which makes sense by virtue of the fact that any bounded linear operator on $\mathscr{H}$ admits a unique measurable linear extension to E [3]. The generator $\mathscr{L}$ of the semigroup $\mathscr{P}$ is the elliptic operator formally given by

$$\mathscr{L} = D^* \mathscr{A} D,$$

where D denotes the Malliavin derivative associated with μ and its adjoint D^* is the associated divergence operator. The application to Ornstein-Uhlenbeck operators described in this paper is obtained by taking $\mu \sim \mu_\infty$ and $\mathscr{A} \sim A_\infty^*$ (cf. [5, 28]).

4. An example

In this section we present an example of a Hilbert space E, a continuously embedded Hilbert subspace $H \hookrightarrow E$, and a C_0-semigroup generator A on E such that:

- the semigroup S generated by A fails to be analytic;
- the stochastic Cauchy problem

$$dU(t) = AU(t)\,dt + dW_H(t)$$

  admits a unique invariant measure, which we denote by μ_∞.
- the associated Ornstein-Uhlenbeck semigroup P is analytic on $L^2(E, \mu_\infty)$.

Thus, although analyticity of P implies analyticity of S_H (Theorem 3.3), it does not imply analyticity of S.

Let $E = L^2(\mathbb{R}_+, e^{-x}\,dx)$ be the space of all measurable functions f on $\mathbb{R}_+$ such that

$$\|f\| := \Big(\int_0^\infty |f(x)|^2\, e^{-x} dx\Big)^{\frac{1}{2}} < \infty.$$

The rescaled left translation semigroup S,

$$S(t)f(x) := e^{-t} f(x+t), \quad f \in E,\ t > 0,\ x > 0,$$

is strongly continuous and contractive on E, and satisfies $\|S(t)\| = e^{-t/2}$. Let $H = H^2(\mathbb{C}_+)$ be the Hardy space of analytic functions g on the open right half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} :\ \mathrm{Re}\, z > 0\}$ such that

$$\|g\|_H := \sup_{x>0} \Big(\int_{-\infty}^\infty |g(x+iy)|^2\, dy\Big)^{\frac{1}{2}} < \infty.$$

Since $\lim_{x\to+\infty} g(x) = 0$ for all $g \in H$, the restriction mapping $i : g \mapsto g|_{\mathbb{R}_+}$ is well defined as a bounded operator from H to E. By uniqueness of analytic continuation, this mapping is injective. Since i factors through $L^\infty(\mathbb{R}_+, e^{-x}\,dx)$, i is Hilbert-Schmidt [29, Corollary 5.21]. As a consequence (see, e.g., [9, Chapter 11]), the Cauchy problem $dU(t) = AU(t)\,dt + dW_H(t)$ admits a unique invariant measure μ_∞.

The rescaled left translation semigroup S_H,

$$S_H(t)g(z) := e^{-t} g(z+t), \quad f \in H,\ t \geqslant 0,\ \mathrm{Re}\, z > 0,$$

is strongly continuous on H, it extends to an analytic contraction semigroup of angle $\frac{1}{2}\pi$, and satisfies $\|S_H(t)\|_H = e^{-t/2}$. Clearly, for all $t \geqslant 0$ we have $S(t) \circ i = i \circ S_H(t)$. By these observations combined with [18, Theorem 9.2], the associated Ornstein-Uhlenbeck semigroup P is analytic.

5. Application to the stochastic heat equation

In this final section we shall apply our results to the following stochastic PDE with additive space-time white noise:

$$\begin{aligned}
\frac{\partial u}{\partial t}(t,y) &= \frac{\partial^2 u}{\partial y^2}(t,y) + \frac{\partial^2 W}{\partial t\,\partial y}(t,y), && t \geqslant 0,\ y \in [0,1], \\
u(t,0) &= u(t,1) = 0, && t \geqslant 0, \\
u(0,y) &= 0, && y \in [0,1].
\end{aligned} \tag{5.1}$$

This equation can be cast into the abstract form (SCP) by taking $H = E = L^2(0,1)$ and A the Dirichlet Laplacian Δ on E. The resulting equation

$$\begin{aligned}
dU(t) &= AU(t)\,dt + dW(t), \\
U(0) &= 0,
\end{aligned}$$

where now W denotes an H-cylindrical Brownian motion, has a unique solution U given by

$$U(t) = \int_0^t S(t-s)\,dW(s), \quad t \geqslant 0,$$

where S denotes the heat semigroup on E generated by A. Let μ_∞ denote the unique invariant measure on E associated with U, and let H_∞ denote its reproducing kernel Hilbert space. Let $i_\infty : H_\infty \hookrightarrow E$ denote the canonical embedding and let $i : H \to E$ be the identity mapping. By [17, Theorem 3.5, Corollary 5.6] the densely defined operator $V : i_\infty^* x^* \mapsto i^* x^*$ defined in (3.1) is closable from H_∞ to H.

Let L be the generator of the Ornstein-Uhlenbeck semigroup P on $L^p(E,\mu_\infty)$ associated with U. Since P is analytic, the results of Sections 2 and 3 can be applied. Noting that Δ is selfadjoint on H, condition (3) of Theorem 3.5 is satisfied and therefore

$$\mathsf{D}_p(\sqrt{-L}) = \mathsf{D}_p(D) \quad (1 < p < \infty)$$

where $D = D_H = D_E$ denotes the Fréchet derivative on $L^p(E,\mu_\infty)$.

One can go a step further by noting that the problem (5.1) is well posed even on the space

$$\widetilde{E} := C_0[0,1] = \{f \in C[0,1] :\ f(0) = f(1) = 0\},$$

in the sense that the random variables $U(t)$ are $\widetilde{E}$-valued almost surely and that U admits has a modification $\widetilde{U}$ with continuous (in fact, even Hölder continuous) trajectories in $\widetilde{E}$. Moreover, the invariant measure μ_∞ is supported on $\widetilde{E}$. In analogy to (1.1) this allows us to define an "Ornstein-Uhlenbeck semigroup" $\widetilde{P}$ on $L^p(\widetilde{E},\mu_\infty)$ associated with $\widetilde{U}$ by

$$\widetilde{P}(t)f(x) := \mathbb{E} f(\widetilde{U}^x(t)), \quad t \geqslant 0,\ x \in \widetilde{E},$$

where $\widetilde{U}^x(t) = \widetilde{S}(t)x + \widetilde{U}(t)$ and $\widetilde{S}$ is the heat semigroup on $\widetilde{E}$. It is important to observe that we are not in the framework considered in the previous sections, due to the fact that $H = L^2(0,1)$ is not continuously embedded in $\tilde{E}$. Let $\widetilde{L}$ denote the generator of $\widetilde{P}$. Under the natural identification

$$L^p(\widetilde{E}, \mu_\infty) = L^p(E, \mu_\infty)$$

(using that the underlying measure spaces are identical up to a set of measure zero), we have $\widetilde{P}(t) = P(t)$ and $\widetilde{L} = L$, so that

$$\mathsf{D}_p(\sqrt{-\widetilde{L}}) = \mathsf{D}_p(\sqrt{-L}) = \mathsf{D}_p(D) \quad (1 < p < \infty). \tag{5.2}$$

This representation may seem somewhat unsatisfactory, as the right-hand side refers explicitly to the ambient space E in which $\widetilde{E}$ is embedded. An intrinsic representation of $\mathsf{D}_p(\sqrt{-\widetilde{L}})$ can be obtained as follows. For functions $F : \widetilde{E} \to \mathbb{R}$ of the form

$$F(f) = \phi\Big(\int_0^1 f g_1\, dt, \ldots, \int_0^1 f g_N\, dt\Big), \quad f \in \widetilde{E},$$

with $\phi \in C_{\rm b}^2(\mathbb{R}^N)$ and $g_1, \ldots, g_N \in H$, we define $\widetilde{D}F : \widetilde{E} \to H$ by

$$\widetilde{D}F(f) = \sum_{n=1}^N \frac{\partial \phi}{\partial y_n}\Big(\int_0^1 f g_1\, dt, \ldots, \int_0^1 f g_N\, dt\Big) g_n, \quad f \in \widetilde{E}.$$

This operator is closable in $L^p(\widetilde{E}, \mu_\infty)$ for all $1 \leqslant p < \infty$. On $L^2(\widetilde{E}, \mu_\infty)$ we have the representation

$$\widetilde{L} = \widetilde{D}^* \widetilde{D}.$$

As a result we can apply [25, Theorem 2.1] directly to the operator V and obtain that

$$\mathsf{D}_p(\sqrt{-\widetilde{L}}) = \mathsf{D}_p(\widetilde{D}) \quad (1 < p < \infty). \tag{5.3}$$

This answers a question raised by Zdzisław Brzeźniak (personal communication). To make the link between the formulas (5.2) and (5.3) note that, under the identification $L^p(\widetilde{E}, \mu_\infty) = L^p(E, \mu_\infty)$, one also has $\mathsf{D}_p(\widetilde{D}) = \mathsf{D}_p(D)$.

Remark 5.1. It is possible to give explicit representations for the space H_∞ and the operator V. To begin with, the covariance operator Q_∞ of μ_∞ is given by

$$Q_\infty f = \int_0^\infty S(t) S^*(t) f\, dt = \int_0^\infty S(2t) f\, dt = \tfrac{1}{2}\Delta^{-1} f, \quad f \in E.$$

It follows that the reproducing kernel Hilbert space H_∞ associated with μ_∞ equals

$$H_\infty = \mathsf{R}(\sqrt{Q_\infty}) = \mathsf{D}(\sqrt{-\Delta}) = H_0^1(0,1).$$

Noting that $Q_\infty = i_\infty \circ i_\infty^*$, we see that the operator $V : i_\infty^* x^* \mapsto i^* x^*$ is given by

$$\mathsf{D}(V) = H^2(0,1) \cap H_0^1(0,1),$$
$$V f = 2\Delta f, \quad f \in \mathsf{D}(V).$$

Remark 5.2. Formulas for $\mathsf{D}_p(\widetilde{L})$ analogous to (5.2) and (5.3) can be deduced from Theorem 3.5 and [25, Theorem 2.2] in a similar way.

The Ornstein-Uhlenbeck operators L and $\widetilde{L}$ considered above are symmetric on $L^2(E,\mu_\infty)$, and therefore the domain identifications for their square roots could essentially be obtained from the results of [6, 32]. The above argument, however, can be applied to a large class of second order elliptic differential operators A on $L^2(0,1)$ (but explicit representations as in Remark 5.1 are only possible when A is selfadjoint).

In fact, under mild assumptions on the coefficients and under various types of boundary conditions, such operators A have a bounded H^∞-calculus on $H = E = L^2(0,1)$ (see [11, 14, 20] and there references therein). By the result of Le Merdy [22] mentioned earlier, this implies that the analytic semigroup S generated by A is contractive in some equivalent Hilbertian norm. Hence, by [18, Theorem 9.2], the associated Ornstein-Uhlenbeck semigroup is analytic. Typically, under Dirichlet boundary conditions, S is uniformly exponentially stable. This implies (see [9]) that the solution U of (SCP) admits a unique invariant measure. Finally, the analyticity of S typically implies space-time Hölder regularity of U (see [4, 13]), so that the corresponding stochastic PDE is well posed in $\widetilde{E} = C_0[0,1]$. We plan to provide more details in a forthcoming publication.

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$\mathcal{R}$-sectoriality of Cylindrical Boundary Value Problems

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Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. We prove $\mathcal{R}$-sectoriality or, equivalently, L^p-maximal regularity for a class of operators on cylindrical domains of the form $\mathbb{R}^{n-k} \times V$, where $V \subset \mathbb{R}^k$ is a domain with compact boundary, $\mathbb{R}^k$, or a half-space. Instead of extensive localization procedures, we present an elegant approach via operator-valued multiplier theory which takes advantage of the cylindrical shape of both, the domain and the operator.

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1. Introduction

This note considers the vector-valued L^p-approach to boundary value problems of the type

$$\begin{array}{rcll} \partial_t u + A(x,D)u & = & f & \text{in } \mathbb{R}_+ \times \Omega, \\ B_j(x,D)u & = & 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \quad (j = 1, \dots, m), \\ u|_{t=0} & = & u_0 & \text{in } \Omega, \end{array} \tag{1.1}$$

on cylindrical domains $\Omega \subset \mathbb{R}^n$ of the form

$$\Omega = \mathbb{R}^{n-k} \times V, \tag{1.2}$$

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where V is a standard domain (see Definition 2.1) in $\mathbb{R}^k$. Here

$$A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha$$

is a differential operator in Ω of order $2m$ for $m \in \mathbb{N}$ and

$$B_j(x,D) = \sum_{|\beta|\leq m_j} b_\beta(x) D^\beta, \quad m_j < 2m, \quad j = 1, \dots, m,$$

are operators acting on the boundary.

Under the assumption that (1.1) is *parameter-elliptic* and *cylindrical*, we will prove $\mathcal{R}$-sectoriality for the operator A of the corresponding Cauchy problem. Recall that $\mathcal{R}$-sectoriality is equivalent to maximal regularity, cf. [21, Theorem 4.2]. Maximal regularity, in turn, is a powerful tool for the treatment of related nonlinear problems.

Roughly speaking, the assumption 'cylindrical' implies that A resolves into two parts

$$A = A_1 + A_2$$

such that A_1 acts merely on $\mathbb{R}^{n-k}$ and A_2 acts merely on V (see Definition 2.2). Note that many standard systems, such as the heat equation with Dirichlet or Neumann boundary conditions, are of this form. Also note that many physical problems naturally lead to equations in cylindrical domains. We refer to the textbook [10] for an introduction to such type problems. Therefore, boundary value problems of this type are certainly of independent interest. On the other hand, they also naturally appear during localization procedures of boundary value problems on general domains. For instance, if a system of equations via localization is reduced to a half-space or a layer problem, then one is usually faced to a problem in the domain

$$\Omega = \mathbb{R}^{n-1} \times V,$$

where $V = (0,d)$ and $d \in (0,\infty]$. Such reduced problems are often of the above type.

Of course, also problem (1.1) could be treated by a localization procedure employing an infinite partition of the unity (note that the boundary is non-compact). However, such procedures are generally extensive and take quite some pages of exhausting calculations and estimations. For this reason, here we pursue a different strategy. In fact, we essentially take advantage of the cylindrical structure of the domain and the operator and employ operator-valued multiplier theory. Roughly speaking, by this method $\mathcal{R}$-sectoriality of (1.1) in Ω is reduced to the corresponding result on the cross-section V, for which it is well known (see, e.g., [11]). This approach reveals a much shorter and more elegant way to prove the important maximal regularity for boundary value problems of type (1.1) on cylindrical domains of the form (1.2). The chosen approach also demonstrates the strength of operator-valued multiplier theory and its significance in the treatment of partial differential equations in general.

We remark that the idea of such a splitting of the variables and operators is already performed by Guidotti in [15] and [16]. In these papers the author constructs semiclassical fundamental solutions for a class of elliptic operators on cylindrical domains. This proves to be a strong tool for the treatment of related elliptic and parabolic ([15] and [16]), as well as of hyperbolic ([16]) problems. In particular, this approach leads to semiclassical representation formulas for solutions of related elliptic and parabolic boundary value problems. Based on these formulas and on a multiplier result of Amann [6] the author derives a couple of interesting results for these problems in a Besov space setting. In particular, the given applications include asymptotic behavior in the large, singular perturbations, exact boundary conditions on artificial boundaries, and the validity of maximum principles. Very recently in [13] the wellposedness of a class of parabolic boundary value problems in a vector-valued Hölder space setting is proved, when $\Omega = [0, L] \times V$, the first part is given by $A_1 = a(x_n)\partial_n^{2m}$, $x_n \in [0, L]$, and when A_2 is uniformly elliptic.

In contrast to [15], [16], and [13], here we present the L^p-approach to cylindrical boundary value problems. Therefore the notion of $\mathcal{R}$-boundedness comes into play, which is not required in the framework of Besov or Hölder spaces. Also note that in [15] and [16] $A_1 = -\Delta$ is assumed, with a remark on possible generalizations. Here we explicitly consider a wider class of first parts A_1 including higher-order operators with variable coefficients. Moreover, with a Banach space E, we consider E-valued solutions and allow the coefficients of the second part A_2 to be operator-valued. Applications for equations with operator-valued coefficients are, for instance, given by coagulation-fragmentation systems (cf. [8]), spectral problems of parametrized differential operators in hydrodynamics (cf. [12]), or (homogeneous) systems in general. Albeit in this note we concentrate on the proof of maximal regularity for problems of type (1.1), we remark that further applications similar to the ones given in [15] and [16] also in the L^p-framework considered here are possible.

Note that E-valued boundary value problems in standard domains, such as $\mathbb{R}^n$, a half-space, and domains with a compact boundary were extensively studied in [11]. There a bounded H^∞-calculus and hence maximal regularity for the operator of the associated Cauchy problem is proved in the situation when E is of class $\mathcal{HT}$. The results obtained in the paper at hand also extend the maximal regularity results proved in [11] to a class of domains with non-compact boundary. For classical papers on scalar-valued boundary value problems we refer to [14], [1], [2], and [20] in the elliptic case and to [4] and [3] in the parameter-elliptic case. (For a more comprehensive list see also [11].) For an approach to a class of elliptic differential operators with Dirichlet boundary conditions in uniform C^2-domains we refer to [17] and [9]. We want to remark that all cited results above are based on standard localization procedures for the domain, contrary to the approach presented in this paper. Here we only localize a certain part of the coefficients but not the domain.

This paper is structured as follows. In Section 2 we define the notion of a cylindrical boundary value problem and give the precise statement of our main result. In Section 3 we recall the notion of parameter-ellipticity and of $\mathcal{R}$-bounded operator families. The proof of our main result Theorem 2.3 then is given in Section 4. The main steps are split in three subsections. In Subsection 4.1 we treat the corresponding operator-valued model problem, that is, here we assume (partly) constant coefficients. By a perturbation argument, in Subsection 4.2 we extend the $\mathcal{R}$-sectoriality of the model problem to slightly varying coefficients. The general case then is handled in Subsection 4.3. The statement of the main result is restricted to the case that the two parts A_1 and A_2 are of the same order. However, the same proof works for mixed-order systems. This will be briefly outlined in Section 5.

2. Main result

We proceed with the precise statement of our main result.

Definition 2.1. Let $k \in \mathbb{N}$. We call $V \subset \mathbb{R}^k$ a standard domain, if it is $\mathbb{R}^k$, the half-space $\mathbb{R}^k_+ = \{x = (x_1, \dots, x_k) \in \mathbb{R}^k : \ x_k > 0\}$, or if it has compact boundary.

Let F be a Banach space and let $\Omega := \mathbb{R}^{n-k} \times V \subset \mathbb{R}^n$, where V is a standard domain in $\mathbb{R}^k$. For $x \in \Omega$ we write $x = (x^1, x^2) \in \mathbb{R}^{n-k} \times V$, whenever we want to refer to the cylindrical geometry of Ω. Accordingly, we write $\alpha = (\alpha^1, \alpha^2) \in \mathbb{N}_0^{n-k} \times \mathbb{N}_0^k$ for a multiindex $\alpha \in \mathbb{N}_0^n$. In the sequel we consider the vector-valued boundary value problem

$$\begin{aligned} \lambda u + A(x, D)u &= f \text{ in } \Omega, \\ B_j(x, D)u &= 0 \text{ on } \partial\Omega \quad (j = 1, \dots, m), \end{aligned} \tag{2.1}$$

with $A(x, D) = \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha$, $m \in \mathbb{N}$, a differential operator in the interior and operators $B_j(x, D) = \sum_{|\beta| < 2m} b_\beta(x) D^\beta$ on the boundary. Vector-valued in this context means that u is F-valued, hence derivatives have to be understood in appropriate F-valued spaces. Accordingly the coefficients of $A(\cdot, D)$ and $B_j(\cdot, D)$ are operator-valued, that is $\mathcal{L}(F)$-valued. In particular, we will consider the following class of operators.

Definition 2.2. The boundary value problem (2.1) is called *cylindrical* if the operator $A(\cdot, D)$ is represented as

$$\begin{aligned} A(x, D) &= A_1(x^1, D) + A_2(x^2, D) \\ &:= \sum_{|\alpha^1| \le 2m} a^1_{\alpha^1}(x^1) D^{(\alpha^1, 0)} + \sum_{|\alpha^2| \le 2m} a^2_{\alpha^2}(x^2) D^{(0, \alpha^2)} \end{aligned}$$

and the boundary operator is given as

$$B_j(x, D) = B_{2,j}(x^2, D) := \sum_{|\beta^2| \le m_j} b^2_{j, \beta^2}(x^2) D^{(0, \beta^2)} \quad (m_j < 2m, \ j = 1, \dots, m).$$

Thus the differential operators $A(x, D)$ and $B_j(x, D)$ resolve completely into parts of which each one acts just on $\mathbb{R}^{n-k}$ or just on V.

As the $L^p(\Omega, F)$-realization of the boundary value problem

$$(A, B) := (A(\cdot, D), B_1(\cdot, D), \dots, B_m(\cdot, D))$$

given by (2.1) we define for $1 < p < \infty$,

$$\begin{aligned} D(A) &:= \{u \in W^{2m,p}(\Omega, F);\ B_j(\cdot, D)u = 0 \quad (j = 1, \dots, m)\} \\ Au &:= A(\cdot, D)u, \quad u \in D(A). \end{aligned}$$

From now on the cross-section V is assumed to be a standard domain with C^{2m}-boundary. Furthermore, the following smoothness assumptions on the coefficients may hold:

$$\left.\begin{aligned} &a^1_{\alpha^1} \in C(\mathbb{R}^{n-k}, \mathbb{C}) \text{ for } |\alpha^1| = 2m, \quad a^1_{\alpha^1}(\infty) := \lim_{|x^1| \to \infty} a^1_{\alpha^1}(x^1) \text{ exists}, \\ &a^2_{\alpha^2} \in C(\overline{V}, \mathcal{L}(F)) \text{ for } |\alpha^2| = 2m, \quad a^2_{\alpha^2}(\infty) := \lim_{|x^2| \to \infty} a^2_{\alpha^2}(x^2) \text{ exists}, \\ &a^1_{\alpha^1} \in [L^\infty + L^{r_\nu}](\mathbb{R}^{n-k}, \mathbb{C}) \text{ for } |\alpha^1| = \nu < 2m,\ r_\nu \geq p,\ \frac{2m-\nu}{n-k} > \frac{1}{r_\nu}, \\ &a^2_{\alpha^2} \in [L^\infty + L^{r_\nu}](V, \mathcal{L}(F)) \text{ for } |\alpha^2| = \nu < 2m,\ r_\nu \geq p,\ \frac{2m-\nu}{k} > \frac{1}{r_\nu}, \\ &b^2_{j,\beta^2} \in C^{2m-m_j}(\partial V, \mathcal{L}(F)) \quad (j = 1, \dots, m;\ |\beta^2| \leq m_j). \end{aligned}\right\} \quad (2.2)$$

Our main result reads as follows. For a rigorous definition of maximal regularity, $\mathcal{R}$-sectoriality, and parameter-ellipticity we refer to the subsequent section (i.p. Definitions 3.2, 3.4, and 3.10).

Theorem 2.3. *Let $1 < p < \infty$, let F be a Banach space of class $\mathcal{HT}$ enjoying property (α), and let $\Omega := \mathbb{R}^{n-k} \times V \subset \mathbb{R}^n$, where V is a standard domain of class C^{2m} in $\mathbb{R}^k$. For the boundary value problem* (2.1) *we assume that*

- (i) *it is cylindrical,*
- (ii) *the coefficients of $A(\cdot, D)$ and $B_j(\cdot, D)$, $j = 1, \dots, m$, satisfy* (2.2)*,*
- (iii) *it is parameter-elliptic in Ω of angle $\varphi_{(A,B)} \in [0, \pi/2)$,*
- (iv) *the boundary value problem $(A^\#(\infty, D), B_1(\cdot, D), \dots, B_m(\cdot, D))$ with the limit operator $A^\#(\infty, D) := \sum_{|\alpha|=2m} a_\alpha(\infty) D^\alpha$ is parameter-elliptic in Ω with angle less or equal to $\varphi_{(A,B)}$.*

Then for each $\phi > \varphi_{(A,B)}$ there exists $\delta = \delta(\phi) \geq 0$ such that $A + \delta$ is $\mathcal{R}$-sectorial in $L^p(\Omega, F)$ with $\phi^{\mathcal{RS}}_{A+\delta} \leq \phi$ and we have

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\alpha (\lambda + A + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ \ell \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n,\ 0 \leq \ell + |\alpha| \leq 2m\}) < \infty. \quad (2.3)$$

By [21, Theorem 4.2] we obtain

Corollary 2.4. *Let the assumptions of Theorem* 2.3 *be given. Then the operator $A + \delta$ has maximal regularity on $L^p(\Omega, F)$.*

Example. It is not difficult to verify that problem (2.1) with $A = -\Delta$ the negative Laplacian in Ω subject to Dirichlet or Neumann boundary conditions satisfies the assumptions of Theorem 2.3.

3. $\mathcal{R}$-sectoriality and parameter-ellipticity

Throughout this article X, Y, E, and F denote Banach spaces. Given any closed operator A acting on a Banach space we denote by $D(A), \ker(A)$, and $R(A)$ domain of definition, kernel, and range of the operator and by $\rho(A)$ and $\sigma(A)$ its resolvent set and spectrum respectively. The symbol $\mathcal{L}(X,Y)$ stands for the Banach space of all bounded linear operators from X to Y equipped with operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. As an abbreviation we set $\mathcal{L}(X) := \mathcal{L}(X,X)$.

For $p \in [1,\infty)$ and a domain $G \subset \mathbb{R}^n$, $L^p(G,F)$ denotes the F-valued Lebesgue space of all p-Bochner-integrable functions, i.e., of functions $f : G \to F$ satisfying

$$\|f\|_{L^p(G,F)} := \left(\int_G \|f(x)\|_F^p dx \right)^{\frac{1}{p}} < \infty.$$

We also write $L^\infty(G,F)$ for the space consisting of all functions f satisfying $\|f\|_\infty := \operatorname{ess\,sup}_{x\in G} \|f(x)\|_F < \infty$. The F-valued Sobolev space of order $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is denoted by $W^{m,p}(G,F)$, that is the space of all $f \in L^p(G,F)$ whose F-valued distributional derivatives up to order m are functions in $L^p(G,F)$ again. Its norm is given by

$$\|f\|_{W^{m,p}(G,F)} := \left(\sum_{|\alpha|\le m} \|D^\alpha f\|_{L^p(G,F)}^p \right)^{\frac{1}{p}},$$

where $\alpha \in \mathbb{N}_0^n$ is a multiindex. We write $\|\cdot\|_p := \|\cdot\|_{L^p(G,F)}$ and $\|\cdot\|_{p,m} := \|\cdot\|_{W^{m,p}(G,F)}$, if no confusion seems likely. Finally, for $m \in \mathbb{N}_0 \cup \{\infty\}$, $C^m(G,F)$ denotes the space of all m-times continuously differentiable functions. For general facts on vector-valued function spaces we refer to the nice booklet of Amann, [7].

Definition 3.1. A closed linear operator A in a Banach space X is called *sectorial*, if

1. $\overline{D(A)} = X$, $\ker(A) = \{0\}$, $\overline{R(A)} = X$,
2. $(-\infty, 0) \subset \rho(A)$ and there is some $C > 0$ such that $\|t(t+A)^{-1}\|_{\mathcal{L}(X)} \le C$ for all $t > 0$.

In this case it is well known, see, e.g., [11], that there exists a $\phi \in [0,\pi)$ such that the uniform estimate in 2. extends to all

$$\lambda \in \Sigma_{\pi-\phi} := \{z \in \mathbb{C}\backslash\{0\};\ |\arg(z)| < \pi - \phi\}.$$

The number

$$\phi_A := \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi},\ \sup_{\lambda\in\Sigma_{\pi-\phi}} \|\lambda(\lambda+A)^{-1}\|_{\mathcal{L}(X)} < \infty\}$$

is called *spectral angle* of A. The class of sectorial operators is denoted by $S(X)$.

Observe that $\sigma(A) \subset \overline{\Sigma}_{\phi_A}$. In case $\phi_A < \frac{\pi}{2}$, the operator $-A$ generates a bounded holomorphic C_0-semigroup on X. For a suitable treatment of related nonlinear problems, however, the generation of a holomorphic semigroup might not be enough. Then the stronger property of maximal regularity is required which is defined as follows.

Definition 3.2. Let $1 \leq p \leq \infty$, let X be a Banach space, and let $A : D(A) \to X$ be closed and densely defined. Then A is said to have (L^p-) *maximal regularity*, if for each $f \in L^p(\mathbb{R}_+, X)$ there is a unique solution $u : \mathbb{R}_+ \to D(A)$ of the Cauchy problem

$$\begin{cases} u' + Au &= f \quad \text{in } \mathbb{R}_+, \\ u(0) &= 0, \end{cases}$$

satisfying the estimate

$$\|u'\|_{L^p(\mathbb{R}_+,X)} + \|Au\|_{L^p(\mathbb{R}_+,X)} \leq C\|f\|_{L^p(\mathbb{R}_+,X)}$$

with a $C > 0$ independent of $f \in L^p(\mathbb{R}_+, X)$.

If the Banach space X is of class $\mathcal{HT}$ (see Definition 3.6), by [21, Theorem 4.2] it is well known that the property of having maximal regularity is equivalent to the $\mathcal{R}$-sectoriality of an operator A. This concept is based on the notion of $\mathcal{R}$-bounded operator families, which we introduce next. We refer to [11] and [18] for a comprehensive introduction to the notion of $\mathcal{R}$-bounded operator families and restrict ourselves here to the definition.

Definition 3.3. A family $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-*bounded*, if there exist a $C > 0$ and a $p \in [1, \infty)$ such that for all $N \in \mathbb{N}, T_j \in \mathcal{T}, x_j \in X$ and all independent symmetric $\{-1, 1\}$-valued random variables ε_j on a probability space $(G, \mathcal{M}, P)$ for $j = 1, \ldots, N$, we have that

$$\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|_{L^p(G,Y)} \leq C \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|_{L^p(G,X)}. \tag{3.1}$$

The smallest $C > 0$ such that (3.1) is satisfied is called $\mathcal{R}$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}(\mathcal{T})$.

Definition 3.4. A closed operator A in X satisfying condition 1. of Definition 3.1 is called $\mathcal{R}$-*sectorial*, if there exist an angle $\phi \in [0, \pi)$ and a constant $C_\phi > 0$ such that

$$\mathcal{R}(\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}) \leq C_\phi. \tag{3.2}$$

The class of $\mathcal{R}$-sectorial operators is denoted by $\mathcal{RS}(X)$ and we call $\phi_A^{\mathcal{R}S}$ given as the infimum over all angles ϕ such that (3.2) holds the $\mathcal{R}$-*angle* of A.

We remark that in general $\mathcal{R}$-boundedness is stronger than the uniform boundedness with respect to the operator norm. Therefore $\mathcal{R}$-sectoriality always implies the sectoriality of an operator A and we have

$$\phi_A \leq \phi_A^{\mathcal{R}S}.$$

We will use the following two results on $\mathcal{R}$-boundedness frequently in subsequent proofs. The first one shows that $\mathcal{R}$-bounds behave as uniform bounds concerning sums and products. This follows as a direct consequence of the definition of $\mathcal{R}$-boundedness. The second one is known as the contraction principle of Kahane. A proof can be found in [18] or [11].

Lemma 3.5.

a) *Let X,Y, and Z be Banach spaces and let $\mathcal{T}, \mathcal{S} \subset \mathcal{L}(X,Y)$ as well as $\mathcal{U} \subset \mathcal{L}(Y,Z)$ be $\mathcal{R}$-bounded. Then $\mathcal{T}+\mathcal{S} \subset \mathcal{L}(X,Y)$ and $\mathcal{U}\mathcal{T} \subset \mathcal{L}(X,Z)$ are $\mathcal{R}$-bounded as well and we have*

$$\mathcal{R}(\mathcal{T}+\mathcal{S}) \leq \mathcal{R}(\mathcal{S}) + \mathcal{R}(\mathcal{T}), \quad \mathcal{R}(\mathcal{U}\mathcal{T}) \leq \mathcal{R}(\mathcal{U})\mathcal{R}(\mathcal{T}).$$

Furthermore, if $\overline{\mathcal{T}}$ denotes the closure of $\mathcal{T}$ with respect to the strong operator topology, then we have $\mathcal{R}(\overline{\mathcal{T}}) = \mathcal{R}(\mathcal{T})$.

b) [contraction principle of Kahane]
Let $p \in [1,\infty)$. Then for all $N \in \mathbb{N}, x_j \in X, \varepsilon_j$ as above, and for all $a_j, b_j \in \mathbb{C}$ with $|a_j| \leq |b_j|$ for $j = 1,\dots,N$,

$$\left\| \sum_{j=1}^{N} a_j \varepsilon_j x_j \right\|_{L^p(G,X)} \leq 2 \left\| \sum_{j=1}^{N} b_j \varepsilon_j x_j \right\|_{L^p(G,X)} \tag{3.3}$$

holds.

Let E be a Banach space and let $\mathcal{S}(\mathbb{R}^n, E)$ denote the Schwartz space of all rapidly decreasing E-valued functions and let $\mathcal{S}'(\mathbb{R}^n, E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n, \mathbb{C}), E)$. Then the E-valued Fourier transform

$$\mathcal{F}\varphi(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(x)dx$$

defines an isomorphism of the space $\mathcal{S}(\mathbb{R}^n, E)$ which extends by duality to the larger space $\mathcal{S}'(\mathbb{R}^n, E)$. Given two Banach spaces E_1, E_2 and any operator-valued function $m \in L^\infty(\mathbb{R}^n, \mathcal{L}(E_1, E_2))$, we may define the operator

$$T_m : \mathcal{S}(\mathbb{R}^n, E_1) \to \mathcal{S}'(\mathbb{R}^n, E_2); \quad T_m\varphi := \mathcal{F}^{-1}m\mathcal{F}\varphi.$$

We say m defines an operator-valued Fourier multiplier, if T_m extends to a bounded operator

$$T_m : \ L^p(\mathbb{R}^n, E_1) \to L^p(\mathbb{R}^n, E_2).$$

In order to state the operator-valued multiplier result our approach is based on, two further notions from Banach space geometry are required.

Definition 3.6.

a) The Hilbert transform $H : \mathcal{S}(\mathbb{R}, E) \to \mathcal{S}'(\mathbb{R}, E)$ is given by $Hf := \mathcal{F}^{-1}m\mathcal{F}f$ where $m(\xi) := \frac{i\xi}{|\xi|}$. The Banach space E is of class $\mathcal{HT}$ or, equivalently, a UMD space, if there exists a $q \in (1,\infty)$ such that H extends to a bounded operator on $L^q(\mathbb{R}, E)$. In other words, $m_E := m \cdot id_E$ is an operator-valued (one variable) Fourier multiplier.

b) A Banach space E is said to have property (α), if there exists a $C > 0$ such that for all $n \in \mathbb{N}, \alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \leq 1$, all $x_{ij} \in E$, and all independent symmetric $\{-1,1\}$-valued random variables ε_i^1 on a probability space $(G_1, \mathcal{M}_1, P_1)$ and ε_j^2 on a probability space $(G_2, \mathcal{M}_2, P_2)$ for $i, j = 1, \ldots, N$, we have that

$$\int_{G_1} \int_{G_2} \left\| \sum_{i,j=1}^{N} \varepsilon_i^1(u)\varepsilon_j^2(v)\alpha_{ij}x_{ij} \right\|_E dudv$$
$$\leq C \int_{G_1} \int_{G_2} \left\| \sum_{i,j=1}^{N} \varepsilon_i^1(u)\varepsilon_j^2(v)x_{ij} \right\|_E dudv.$$

By Plancherel's theorem Hilbert spaces are of class $\mathcal{HT}$. Besides that, it is well known that the spaces $L^p(G, F)$ are of class $\mathcal{HT}$ provided that $1 < p < \infty$ and that F is of class $\mathcal{HT}$. Moreover, $\mathbb{C}^n$ and the spaces $L^p(G, F)$ enjoy property (α) for $1 \leq p < \infty$, if F does so (cf. [18]).

We are now in position to state the mentioned operator-valued Fourier multiplier theorem. For a proof we refer to [18, Proposition 5.2].

Proposition 3.7. *Let E_1, E_2 be Banach spaces of class $\mathcal{HT}$ with property (α), $1 < p < \infty$, and set $X_i := L^p(\mathbb{R}^n, E_i)$, $i = 1, 2$. Given any set Λ, let $m_\lambda \in C^n(\mathbb{R}^n \backslash \{0\}, \mathcal{L}(E_1, E_2))$ for $\lambda \in \Lambda$ and assume that*

$$\mathcal{R}(\{\xi^\alpha D^\alpha m_\lambda(\xi);\ \xi \in \mathbb{R}^n \backslash \{0\},\ \lambda \in \Lambda, \alpha \in \{0,1\}^n\}) \leq C_m < \infty.$$

Then for all $\lambda \in \Lambda$ we have

$$T_\lambda := \mathcal{F}^{-1} m_\lambda \mathcal{F} \in \mathcal{L}(X_1, X_2)$$

and that

$$\mathcal{R}(\{T_\lambda;\ \lambda \in \Lambda\}) \leq C(n, p, E_1, E_2) C_m < \infty.$$

Next we recall the notion of parameter-ellipticity from [11]. Let F be a Banach space, $G \subset \mathbb{R}^n$ be a domain, and set

$$A(x, D) := \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha,$$

where $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$, and $a_\alpha : G \to \mathcal{L}(F)$. For $\lambda \in \mathbb{C}$ and boundary operators

$$B_j(x, D) := \sum_{|\beta| \leq m_j} b_{j,\beta}(x) D^\beta,$$

where $m_j < 2m$, $\beta \in \mathbb{N}_0^n$, and $b_{j,\beta} : \partial G \to \mathcal{L}(F)$ for $j = 1, \ldots, m$, we consider the boundary value problem

$$\begin{aligned} \lambda u + A(x, D)u &= f \quad \text{in } G, \\ B_j(x, D)u &= 0 \quad \text{on } \partial G \quad (j = 1, \ldots, m). \end{aligned} \tag{3.4}$$

Definition 3.8. Let F be a Banach space, $G \subset \mathbb{R}^n$, $m \in \mathbb{N}$, and $a_\alpha \in \mathcal{L}(F)$. The $\mathcal{L}(F)$-valued homogeneous polynomial

$$a(\xi) := \sum_{|\alpha|=2m} a_\alpha \xi^\alpha \quad (\xi \in \mathbb{R}^n)$$

is called *parameter-elliptic*, if there exists an angle $\phi \in [0, \pi)$ such that the spectrum $\sigma(a(\xi))$ of $a(\xi)$ in $\mathcal{L}(F)$ satisfies

$$\sigma(a(\xi)) \subset \Sigma_\phi \quad (\xi \in \mathbb{R}^n,\ |\xi| = 1). \tag{3.5}$$

Then

$$\varphi := \inf\{\phi : \ (3.5) \text{ holds}\}$$

is called *angle of ellipticity* of a.
A differential operator $A(x, D) := \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$ with coefficients $a_\alpha : G \to \mathcal{L}(F)$ is called parameter-elliptic in G with angle of ellipticity φ, if the principal part of its symbol

$$a^\#(x, \xi) := \sum_{|\alpha|=2m} a_\alpha(x) \xi^\alpha$$

is parameter-elliptic with this angle of ellipticity for all $x \in \overline{G}$.

Definition 3.9. Let F be a Banach space and let $G \subset \mathbb{R}^n$ be a C^1-domain. Let $a_\alpha : G \to \mathcal{L}(F)$ and $b_{j,\beta} : \partial G \to \mathcal{L}(F)$. Set $B_j^\#(x, D) := \sum_{|\beta|=m_j} b_{\beta,j}(x) D^\beta$ and let $A^\#(x, D) := \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha$ be parameter-elliptic in G of angle of ellipticity $\varphi \in [0, \pi)$. For each $x_0 \in \partial G$ we write the boundary value problem in local coordinates about x_0. The boundary value problem (3.4) is said to satisfy the *Lopatinskii-Shapiro condition*, if for each $\phi > \varphi$ the ODE on $\mathbb{R}_+$

$$\begin{aligned} (\lambda + A^\#(x_0, \xi', D_{x_n}))v(x_n) &= 0, \quad && x_n > 0, \\ B_j^\#(x_0, \xi', D_{x_n})v(0) &= h_j, && j = 1, \dots, m, \\ v(x_n) &\to 0, && x_n \to \infty, \end{aligned}$$

has a unique solution $v \in C((0, \infty), F)$ for each $(h_1, \dots, h_m)^T \in F^m$ and each $\lambda \in \overline{\Sigma}_{\pi-\phi}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'| + |\lambda| \neq 0$.

We refer to [22] for an introduction to the Lopatinskii-Shapiro condition for scalar-valued boundary value problems and to [11] for an extensive treatment of the F-valued case. Parameter-ellipticity of a boundary value problem now reads as follows.

Definition 3.10. The boundary value problem (A, B) given through (3.4) is called parameter-elliptic in G of angle $\varphi \in [0, \pi)$, if $A(\cdot, D)$ is parameter-elliptic in G of angle $\varphi \in [0, \pi)$ and if the Lopatinskii-Shapiro condition holds. To indicate that φ is the angle of ellipticity of the boundary value problem (A, B) we use the subscript notation $\varphi_{(A,B)}$.

4. Proof of the main result

We denote by

$$\begin{aligned} D(A_2) &:= \{u \in W^{2m,p}(V,F);\ B_{2,j}(\cdot,D)u = 0\ (j=1,\dots,m)\} \\ A_2 u &:= A_2(\cdot,D)u, \quad u \in D(A_2), \end{aligned}$$

the $L^p(V,F)$-realization of the induced boundary value problem

$$\begin{aligned} \lambda u + A_2(x^2,D)u &= f \quad \text{in } V, \\ B_{2,j}(x^2,D)u &= 0 \quad \text{on } \partial V \quad (j=1,\dots,m), \end{aligned} \tag{4.1}$$

on the cross-section V of Ω. As the original boundary value problem (2.1) is assumed to be parameter-elliptic with ellipticity angle $\varphi_{(A,B)} \in [0,\pi/2)$, it is easy to see that the same is valid for the boundary value problem (4.1) and that the corresponding angle $\varphi_{(A_2,B_2)}$ is no larger than $\varphi_{(A,B)}$. By employing finite open coverings of $\overline{V}$, in [11] the following result is proved.

Proposition 4.1. *Let $V \subset \mathbb{R}^k$ be a standard domain of class C^{2m}. Given the assumptions* (2.2) *on the coefficients a^2 and b^2, for each $\phi > \varphi_{(A_2,B_2)}$ there exists a $\delta_2 = \delta_2(\phi) \geq 0$ such that $A_2 + \delta_2 \in \mathcal{RS}(L^p(V,F))$ with $\phi^{\mathcal{RS}}_{A_2+\delta_2} \leq \phi$. Moreover, we have*

$$\mathcal{R}(\{\lambda^{1-\frac{|\gamma|}{2m}} D^\gamma (\lambda + A_2 + \delta_2)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \leq |\gamma| \leq 2m\}) < \infty. \tag{4.2}$$

Remark 4.2. In [11] just the case $k \geq 2$ is treated, whereas the case $k=1$ is well known.

From the definition it is clear that the coefficients of the cylindrical parts A_1 and A_2 of A only depend on x^1 or x^2, respectively. For the sake of simplicity we therefore drop the special indications for x, if no confusion seems likely. To be precise we write

$$A_i(x^i,D) = A_i(x,D) = \sum_{|\alpha| \leq 2m} a^i_\alpha(x) D^\alpha$$

for $i=1,2$, where

$$a^1_\alpha(x) = \begin{cases} 0 & , \quad \alpha_2 \neq 0, \\ a^1_{\alpha^1}(x^1), & \quad \alpha_2 = 0, \end{cases}$$

$$a^2_\alpha(x) = \begin{cases} 0 & , \quad \alpha_1 \neq 0, \\ a^2_{\alpha^2}(x^2), & \quad \alpha_1 = 0. \end{cases}$$

Further we set $E := L^p(\Omega,F)$ and $X := L^p(\mathbb{R}^{n-k},E) \cong L^p(\Omega,F)$. Given an operator $T : D(T) \subset E \to E$, its canonical extension is defined by

$$\begin{aligned} D(\tilde{T}) &:= L^p(\mathbb{R}^{n-k}, D(T)) \\ (\tilde{T}u)(x) &:= T(u(x)), \quad u \in D(\tilde{T}),\ x \in \mathbb{R}^{n-k}. \end{aligned}$$

4.1. Constant coefficients a^1_α

In the first step we consider the model problem for the cylindrical boundary value problem (2.1), i.e., we assume $A_1(x,D)$ on $\mathbb{R}^{n-k}$ to be given as homogeneous differential operator

$$A_1(D) := \sum_{|\alpha|=2m} a^1_\alpha D^\alpha$$

with constant coefficients $a^1_\alpha \in \mathbb{C}$. Let A_1 denote its realization in X with domain $D(A_1) := W^{2m,p}(\mathbb{R}^{n-k}, E)$. We set

$$A_0(\cdot, D) := A_1(D) + A_2(\cdot, D)$$

and

$$A_0 := A_1 + \tilde{A}_2, \quad D(A_0) := D(A_1) \cap D(\tilde{A}_2).$$

Note that no further restrictions on $A_2(x,D)$ have to be assumed.

Since it will always be clear from the context what we mean, from now on we do not distinguish between $\tilde{A}_2$ and A_2. In other words, from this point on we drop again the tilde notation and just write A_2 for simplicity.

Let $\phi > \varphi_{(A_0,B)}$, $\lambda \in \Sigma_{\pi-\phi}$ and $u \in \mathcal{S}(\mathbb{R}^{n-k}, D(A_2)) \subset D(A_0)$. Applying E-valued Fourier transform $\mathcal{F}$ to $f := (\lambda + A_0 + \delta_2)u$ gives us

$$(\lambda + a_1(\cdot) + A_2 + \delta_2)\mathcal{F}u = \mathcal{F}f.$$

Hence we formally have

$$u = \mathcal{F}^{-1} m^0_\lambda \mathcal{F} f,$$

where m^0_λ is given by the operator-valued symbol

$$m^0_\lambda(\xi) := (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}, \quad \xi \in \mathbb{R}^{n-k}.$$

Note that $m^0_\lambda \in C^\infty(\mathbb{R}^{n-k}, \mathcal{L}(E))$ is well defined if

$$-(\lambda + a_1(\xi)) \in \rho(A_2 + \delta_2) \quad (\xi \in \mathbb{R}^{n-k}).$$

In view of $\varphi_{(A_2,B_2)} \leq \varphi_{(A_0,B)}$ and Proposition 4.1 this is obviously satisfied in case that $\lambda + a_1(\xi) \in \Sigma_{\pi-\phi}$. This, however, follows directly from the parameter-ellipticity of $A_1(D)$, which is obtained as an immediate consequence of the parameter-ellipticity of (A_0, B), and since the ellipticity angle φ_{A_1} of A_1 fulfills $\varphi_{A_1} \leq \varphi_{(A_0,B)} < \phi$.

In order to obtain

$$(\lambda + A_0 + \delta_2)^{-1} = \mathcal{F}^{-1} m^0_\lambda \mathcal{F} f \in \mathcal{L}(X),$$

the idea is to apply the operator-valued multiplier result of Proposition 3.7 to m^0_λ. For this purpose, we next establish suitable representation formulas for derivatives of m^0_λ.

Lemma 4.3. *Let $\phi > \varphi_{(A_0,B)}$. Given $\alpha \in \{0,1\}^{n-k}$, let*

$$\mathcal{Z}_\alpha := \left\{ \mathcal{W} = (\omega^1, \ldots, \omega^r) \in (\{0,1\}^{n-k})^r;\ r \leq n-k,\ \omega^j \neq 0, \sum_{j=1}^r \omega^j = \alpha \right\}$$

denote the set of all additive decompositions of α into $r = r_{\mathcal{W}}$ many positive multiindices. Then, with $C_{\mathcal{W}} := (-1)^r r!$, the formula

$$\xi^\alpha D^\alpha m^0_\lambda(\xi) = (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \cdot \sum_{\mathcal{W}\in\mathcal{Z}_\alpha} C_{\mathcal{W}} \left(\prod_{j=1}^r \xi^{\omega^j} D^{\omega^j} a_1(\xi)\right) (\lambda + a_1(\xi) + A_2 + \delta_2)^{-r}$$

holds for all $\lambda \in \Sigma_{\pi-\phi}$ and $\xi \in \mathbb{R}^{n-k}$.

Proof. Let $|\alpha| = 1$. Then there exists $i \in \{1, \ldots, n-k\}$ such that $\alpha = e_i$. In this case $\mathcal{Z}_\alpha = \{(\alpha)\}$ and we get immediately

$$\begin{aligned}\xi_i D_i m^0_\lambda(\xi) &= -\xi_i (D_i a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta)^{-2} \\ &= (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}(-1)\xi^\alpha (D^\alpha a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta)^{-1}.\end{aligned}$$

Now assume the statement to be true for $\alpha \in \{0,1\}^{n-k}$ with $|\alpha| < n-k$. Then for $l \in \{1, \ldots, n-k\}$ such that $\alpha_l = 0$ we obtain

$$\begin{aligned}&\xi_l \xi^\alpha D_l D^\alpha m^0_\lambda(\xi) \\ &= \xi_l D_l \sum_{\mathcal{W}\in\mathcal{Z}_\alpha} C_{\mathcal{W}} \left(\prod_{j=1}^{r_{\mathcal{W}}} \xi^{\omega^j} D^{\omega^j} a_1(\xi)\right)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-(1+r_{\mathcal{W}})} \\ &= \xi_l \sum_{\mathcal{W}\in\mathcal{Z}_\alpha} C_{\mathcal{W}} \\ &\quad \cdot \Bigg[\sum_{i=1}^{r_{\mathcal{W}}} \xi^{\omega^i} (D_l D^{\omega^i} a_1)(\xi) \left(\prod_{j\neq i} \xi^{\omega^j} D^{\omega^j} a_1(\xi)\right)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-(1+r_{\mathcal{W}})} \\ &\quad + \left(\prod_j \xi^{\omega^j} D^{\omega^j} a_1(\xi)\right)(-(1+r_{\mathcal{W}}))(D_l a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-(2+r_{\mathcal{W}})}\Bigg] \\ &= (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \\ &\quad \cdot \sum_{\mathcal{W}\in\mathcal{Z}_{\alpha+e_l}} C_{\mathcal{W}} \left(\prod_{j=1}^{r_{\mathcal{W}}} \xi^{\omega^j} D^{\omega^j} a_1(\xi)\right)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-r_{\mathcal{W}}}. \qquad \square\end{aligned}$$

In the sequel we denote by $(\beta, \gamma) \in \mathbb{N}_0^{n-k} \times \mathbb{N}_0^k$ a multiindex such that β is the part corresponding to the variables $x^1 \in \mathbb{R}^{n-k}$ and γ corresponding to the variables $x^2 \in V$. In order to obtain the general estimate (2.3) for the full operator A, we also have to consider the more involved symbols

$$m_\lambda(\xi) := \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma m^0_\lambda(\xi) = \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}$$

for $\lambda \in \Sigma_{\pi-\phi}$, $\xi \in \mathbb{R}^{n-k}$, and $|\beta| + |\gamma| \le 2m$.

Lemma 4.4. *Let $\phi > \varphi_{(A_0,B)}$. For $\alpha \in \{0,1\}^{n-k}$ we have*

$$\xi^\alpha D^\alpha m_\lambda(\xi) = \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \cdot \sum_{\alpha' \le \alpha} C_{\alpha',\beta} \sum_{\mathcal{W} \in \mathcal{Z}_{\alpha-\alpha'}} C_{\mathcal{W}} \prod_{j=1}^{r} \left(\xi^{\omega^j} (D^{\omega^j} a_1)(\xi) (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \right),$$

for all $\lambda \in \Sigma_{\pi-\phi}$, $\xi \in \mathbb{R}^{n-k}$, and $(\beta, \gamma) \in \mathbb{N}_0^{n-k} \times \mathbb{N}_0^k$ such that $|\beta| + |\gamma| \le 2m$, and with certain constants $C_{\alpha',\beta} \in \mathbb{Z}$.

Proof. We first show

$$\xi^\alpha D^\alpha m_\lambda(\xi) = \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta \sum_{\alpha' \le \alpha} (\prod_{i;\alpha'_i \ne 0} \beta_i) \xi^{\alpha-\alpha'} D^{\alpha-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}. \tag{4.3}$$

Let $\alpha = e_i$ for some $i \in \{1, \dots, n-k\}$. Then

$$\xi_i D_i m_\lambda(\xi) = \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta \left(\beta_i D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} + \xi_i D_i D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \right)$$

already proves (4.3) for the case $|\alpha| = 1$. Assume the statement to be true for $\alpha \in \{0,1\}^{n-k}$ with $|\alpha| < n-k$. For $l \in \{1, \dots, n-k\}$ such that $\alpha_l = 0$ we have

$$\begin{aligned}
&\xi_l \xi^\alpha D_l D^\alpha m_\lambda(\xi) \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi_l D_l \sum_{\alpha' \le \alpha} (\prod_{i;\alpha'_i \ne 0} \beta_i) \xi^\beta \xi^{\alpha-\alpha'} D^{\alpha-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \Bigg(\sum_{\alpha' \le \alpha} (\prod_{i;\alpha'_i \ne 0} \beta_i) \beta_l \xi^\beta \xi^{\alpha-\alpha'} D^{\alpha-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \\
&\qquad + \sum_{\alpha' \le \alpha} (\prod_{i;\alpha'_i \ne 0} \beta_i) \xi^\beta \xi^{\alpha+e_l-\alpha'} D^{\alpha+e_l-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \Bigg) \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta \\
&\quad \Bigg(\sum_{\alpha' \le \alpha+e_l; \alpha'_l = 1} (\prod_{i;\alpha'_i \ne 0} \beta_i) \xi^{\alpha+e_l-\alpha'} D^{\alpha+e_l-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \\
&\qquad + \sum_{\alpha' \le \alpha+e_l; \alpha'_l = 0} (\prod_{i;\alpha'_i \ne 0} \beta_i) \xi^{\alpha+e_l-\alpha'} D^{\alpha+e_l-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \Bigg) \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta \sum_{\alpha' \le \alpha+e_l} (\prod_{i;\alpha'_i \ne 0} \beta_i) \xi^{\alpha-\alpha'} D^{\alpha-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}.
\end{aligned}$$

This proves (4.3). Setting $C_{\alpha',\beta} := \prod_{i;\alpha'_i\neq 0} \beta_i$ and applying Lemma 4.3 now yields

$$\begin{aligned}
\xi^\alpha D^\alpha m_\lambda(\xi) &= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma \sum_{\alpha'\le\alpha} C_{\alpha',\beta} \\
&\quad \cdot \sum_{\mathcal{W}\in\mathcal{Z}_{\alpha-\alpha'}} C_{\mathcal{W}} \left(\prod_{j=1}^{r} \xi^{\omega^j} D^{\omega^j} a_1(\xi) \right) D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-(r+1)} \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \sum_{\alpha'\le\alpha} C_{\alpha',\beta} \\
&\quad \cdot \sum_{\mathcal{W}\in\mathcal{Z}_{\alpha-\alpha'}} C_{\mathcal{W}} \prod_{j=1}^{r} \left(\xi^{\omega^j} (D^{\omega^j} a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \right).
\end{aligned}$$

This proves the assertion. □

With the above formulas at hand we can prove $\mathcal{R}$-sectoriality for the model problem.

Proposition 4.5. *For each $\phi > \varphi_{(A_0,B)}$ we have $A_0 + \delta_2 \in \mathcal{RS}(X)$ with $\delta_2 = \delta_2(\phi)$ as in Proposition 4.1. Moreover, $\phi^{\mathcal{RS}}_{A_0+\delta_2} \le \phi$ and it holds that*

$$\mathcal{R}(\{\lambda^{1-\frac{|\beta|+|\gamma|}{2m}} D^\beta D^\gamma (\lambda + A_0 + \delta_2)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le |\beta| + |\gamma| \le 2m\}) < \infty. \quad (4.4)$$

Furthermore, the domain of A_0 is given as

$$D(A_0) = L^p(\mathbb{R}^{n-k}, D(A_2)) \cap \bigcap_{j=1}^{2m} W^{j,p}(\mathbb{R}^{n-k}, W^{2m-j,p}(V,F)).$$

Proof. Let $\phi > \varphi_{(A_0,B)}$. We show that m_λ fulfills the assumptions of the multiplier result Proposition 3.7, i.e., that

$$\mathcal{R}(\{\xi^\alpha D^\alpha m_\lambda(\xi);\ \xi \in \mathbb{R}^{n-k},\ \lambda \in \Sigma_{\pi-\phi},\ \alpha \in \{0,1\}^{n-k}\}) < \infty.$$

As $\mathcal{R}$-boundedness by virtue of Lemma 3.5 is preserved under summation and composition of $\mathcal{R}$-bounded operator families, it suffices by Lemma 4.4 to prove that

$$\begin{aligned}
\mathcal{R}(\{\lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1};& \\
\xi \in \mathbb{R}^{n-k},\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le |\beta| + |\gamma| \le 2m\}) &< \infty
\end{aligned}$$

and that

$$\begin{aligned}
\mathcal{R}(\{\xi^\alpha (D^\alpha a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-1};& \\
\xi \in \mathbb{R}^{n-k},\ \lambda \in \Sigma_{\pi-\phi},\ \alpha \in \{0,1\}^{n-k}\}) &< \infty.
\end{aligned}$$

Thanks to (4.2) this follows by the contraction principle of Kahane if we can show that both

$$\kappa_1(\lambda,\xi) := \frac{\lambda^{1-\frac{|\beta|+|\gamma|}{2m}}\xi^\beta}{(\lambda+a_1(\xi))^{1-\frac{|\gamma|}{2m}}} \quad \text{and} \quad \kappa_2(\lambda,\xi) := \frac{\xi^\alpha D^\alpha a_1(\xi)}{\lambda+a_1(\xi)}$$

are uniformly bounded in $(\lambda,\xi)\in\Sigma_{\pi-\phi}\times\mathbb{R}^{n-k}$. To see this, we first observe that

$$\kappa_1(s^{2m}\lambda, s\xi) = \kappa_1(\lambda,\xi) \quad (s>0),$$

hence that κ_1 is quasi-homogeneous of degree zero. We set

$$K := \{(\lambda,\xi)\in\overline{\Sigma}_{\pi-\phi}\times\mathbb{R}^{n-k}:\ |\lambda|+|\xi|^{2m}=1\}. \tag{4.5}$$

By the ellipticity condition, we obtain $a_1(\xi)\in\overline{\Sigma}_{\varphi_{A_1}}$ for all $\xi\in\mathbb{R}^{n-k}\setminus\{0\}$. Since $\varphi_{A_1}<\phi$, it therefore easily follows that

$$\lambda + a_1(\xi)\neq 0 \quad \text{on } K.$$

Consequently, κ_1 is a continuous function on the compact set K and we obtain

$$|\kappa_1(\lambda,\xi)|\le M \quad ((\lambda,\xi)\in K).$$

By the quasi-homogeneity of κ_1 this implies

$$|\kappa_1(s^{2m}\lambda,s\xi)|\le M \quad ((\lambda,\xi)\in K,\ s>0).$$

We have

$$|s^{2m}\lambda|+|s\xi|^{2m} = s^{2m}(|\lambda|+|\xi|^{2m}).$$

Thus, if we set $s=(|\lambda|+|\xi|^{2m})^{-1/2m}$ we deduce

$$(s^{2m}\lambda,s\xi)\in K \quad ((\lambda,\xi)\in\Sigma_{\pi-\phi}\times\mathbb{R}^{n-k})$$

and therefore that

$$|\kappa_1(\lambda,\xi)| = |\kappa_1(s^{2m}\lambda,s\xi)|\le M \quad ((\lambda,\xi)\in\Sigma_{\pi-\phi}\times\mathbb{R}^{n-k}).$$

The uniform boundedness of κ_2 can be proved in exactly the same way. By applying Proposition 3.7, relation (4.4) follows.

In particular, we have

$$(\lambda+A_0+\delta_2)^{-1} = \mathcal{F}^{-1}m_\lambda^0\mathcal{F}f\in\mathcal{L}(X)$$

and

$$D(A_0)\subset\bigcap_{j=1}^{2m}W^{j,p}(\mathbb{R}^{n-k},W^{2m-j,p}(V,F)).$$

Furthermore, we can represent the resolvent applied to $f\in\mathcal{S}(\mathbb{R}^{n-k},E)$ as a Bochner integral via

$$(\lambda+A_0+\delta_2)^{-1}f(x^1) = \frac{1}{(2\pi)^{(n-k)/2}}\int_{\mathbb{R}^{n-k}} e^{ix^1\cdot\xi}(\lambda+a_1(\xi)+A_2+\delta_2)^{-1}\mathcal{F}f(\xi)d\xi.$$

Since taking the trace acts as a bounded operator on E, it commutes with the integral sign. This yields

$$B_{2,j}(\lambda+A_0+\delta_2)^{-1}f = 0 \quad (f\in\mathcal{S}(\mathbb{R}^{n-k},E)).$$

Employing a density argument we conclude that

$$D(A_0) = L^p(\mathbb{R}^{n-k}, D(A_2)) \cap \bigcap_{j=1}^{2m} W^{j,p}(\mathbb{R}^{n-k}, W^{2m-j,p}(V,F)).$$

Assuming that $(A_0 + \delta_2)u = 0$ for $u \in D(A_0)$ next implies that

$$(a_1(\xi) + A_2 + \delta_2)\mathcal{F}u(\xi) = 0 \quad (\xi \in \mathbb{R}^{n-k}).$$

Since $A_2 + \delta_2$ is sectorial and a_1 parameter-elliptic this yields $\mathcal{F}u = 0$, hence $u = 0$. By permanence properties for sectorial operators, i.e., in this case for $A_2 + \delta_2$, we obtain that the same is true for the dual operator of $A_0 + \delta_2$. This implies that $A_0 + \delta_2$ is injective and has dense range. Hence we have proved that $A_0 + \delta_2 \in \mathcal{R}S(X)$. □

4.2. Slightly varying coefficients a^1_α

By a perturbation argument in this paragraph we generalize the $\mathcal{R}$-sectoriality for constant coefficients to the case of slightly varying coefficients of A_1. To this end, we will employ the following perturbation result which is based on a standard Neumann series argument.

Lemma 4.6. *Let R be a linear operator in X such that $D(A_0) \subset D(R)$ and let δ_2 be given as in Proposition* 4.5*. Assume that there are $\eta > 0$ and $\delta > \delta_2$ such that*

$$\|Rx\|_X \le \eta \|(A_0 + \delta)x\|_X \quad (x \in D(A)).$$

Then $A_0 + R + \delta \in \mathcal{R}S(X)$, $\phi^{\mathcal{R}S}_{A_0+R+\delta} \le \phi^{\mathcal{R}S}_{A_0+\delta_2}$, and for every $\phi > \varphi_{(A_0,B)}$ we have

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A_0 + R + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le \ell + |\beta| + |\gamma| \le 2m\}) < \infty, \quad (4.6)$$

whenever $\eta < \mathcal{R}(\{(A_0 + \delta)(\lambda + A_0 + \delta)^{-1}\})^{-1}$.

Proof. As

$$\begin{aligned}\|R(\lambda + A_0 + \delta)^{-1}\|_{\mathcal{L}(X)} &\le \eta \|(A_0 + \delta)(\lambda + A_0 + \delta)^{-1}\|_{\mathcal{L}(X)} \\ &\le \eta \mathcal{R}(\{(A_0 + \delta)(\lambda + A_0 + \delta)^{-1}\}) \\ &< 1\end{aligned}$$

by assumption, we see that

$$\lambda + A_0 + R + \delta = \left(1 + R(\lambda + A_0 + \delta)^{-1}\right)(\lambda + A_0 + \delta)$$

is invertible. This implies

$$\begin{aligned}&\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A_0 + R + \delta)^{-1} \\ &= \lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A_0 + \delta)^{-1} \sum_{j=0}^{\infty} (-R(\lambda + A_0 + \delta)^{-1})^j.\end{aligned}$$

By assumption we have $\delta_0 := \delta - \delta_2 > 0$. The fact that

$$|\lambda + \delta_0| \ge c_\phi \delta_0 \quad (\lambda \in \Sigma_{\pi-\phi})$$

for some $c_\phi > 0$ yields the existence of a $M_\phi > 0$ such that

$$\frac{|\lambda^{\ell/2m}|}{|(\lambda+\delta_0)^{1-(|\beta|+|\gamma|)/2m}|} \leq M_\phi \quad (\lambda \in \Sigma_{\pi-\phi}).$$

Thanks to the contraction principle of Kahane and Proposition 4.5 we deduce

$$\begin{aligned}
&\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A_0 + \delta)^{-1}\}) \\
&\quad \leq\; C\mathcal{R}(\{(\lambda+\delta_0)^{1-\frac{|\beta|+|\gamma|}{2m}} D^\beta D^\gamma ((\lambda+\delta_0) + A_0 + \delta_2)^{-1}\}) \;\leq\; C.
\end{aligned}$$

Lemma 3.5(a) then yields

$$\begin{aligned}
&\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A_0 + \delta)^{-1}(-R(\lambda+A_0+\delta)^{-1})^j\}) \\
&\quad \leq\; \mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A_0 + \delta)^{-1}\})\mathcal{R}(\{(R(\lambda+A_0+\delta)^{-1})^j\}) \\
&\quad \leq\; C\eta^j \mathcal{R}(\{(A_0+\delta)(\lambda+A_0+\delta)^{-1})\})^j \;\leq\; C\nu^j \quad (j \in \mathbb{N}_0)
\end{aligned}$$

with $\nu := \eta\mathcal{R}(\{(A_0+\delta)(\lambda+A_0+\delta)^{-1}\}) < 1$. Employing again Lemma 3.5(a), in particular the fact that the $\mathcal{R}$-bound is preserved when taking the closure in the strong operator topology, the assertion follows. □

Corollary 4.7. *Let $R(x^1, D) := \sum_{|\alpha^1|=2m} r_{\alpha^1}(x^1) D^{(\alpha^1,0)}$ be given such that the condition $\sum\limits_{|\alpha^1|=2m} \|r_{\alpha^1}\|_\infty < \eta$ is satisfied. Set*

$$A^{va}(x, D) := A_0(x^2, D) + R(x^1, D), \quad x \in \Omega, \tag{4.7}$$

and denote its X-realization by A^{va} defined on $D(A^{va}) = D(A_0)$. Then there exists a $\delta > 0$ such that $A^{va} + \delta \in \mathcal{RS}(X)$ with $\phi^{\mathcal{RS}}_{A^{va}+\delta} \leq \phi^{\mathcal{RS}}_{A_0+\delta_2}$ provided that η is sufficiently small. In this case for $\phi > \varphi_{(A_0,B)}$ we have

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A^{va} + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \leq l + |\beta| + |\gamma| \leq 2m\}) < \infty. \tag{4.8}$$

Proof. By Proposition 4.5, in particular by relation (4.4), there exists a $C > 0$ such that

$$\|D^{(\alpha^1,0)}(A_0+\delta)^{-1}\|_{\mathcal{L}(X)} \leq C \quad (\alpha^1 \in \mathbb{N}_0^{n-k},\ |\alpha^1| = 2m)$$

for each $\delta > \delta_2$. For a fixed $\delta > \delta_2$ this implies

$$\begin{aligned}
\|Ru\|_p \;&\leq\; \sum_{|\alpha^1|=2m} \|r_{\alpha^1}\|_\infty \|D^{(\alpha^1,0)}(A_0+\delta)^{-1}(A_0+\delta)u\|_p \\
&\leq\; C\eta\|(A_0+\delta)u\|_p \quad (u \in D(A_0)).
\end{aligned}$$

Thus, if we assume that $\eta < 1/C\mathcal{R}(\{(A_0+\delta)(\lambda+A_0+\delta)^{-1}\})$, the assertion follows from Lemma 4.6. □

4.3. Variable coefficients a^1_α

In the next lemma we establish estimates that will turn out to be crucial for the localization procedure.

Lemma 4.8. *Let* $1 < p < \infty$, $(\beta^1, 0) \in \mathbb{N}_0^{n-k} \times \mathbb{N}_0^k$, $|(\beta^1, 0)| = \nu < 2m$, *and* $r_\nu \geq p$ *such that* $2m - \nu > \frac{n-k}{r_\nu}$. *Let* $b \in [L^\infty + L^{r_\nu}](\mathbb{R}^{n-k})$, A^{va} *be the operator as defined in* (4.7), *and assume that* $\phi > \varphi_{(A,B)}$.

(a) *For every* $\varepsilon > 0$ *there exists* $C(\varepsilon) > 0$ *such that*

$$\|bD^{(\beta^1,0)}u\|_p \leq \varepsilon \|u\|_{p,2m} + C(\varepsilon)\|u\|_p \quad (u \in W^{2m,p}(\mathbb{R}^{n-k}, E)).$$

(b) *For every* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\mathcal{R}(\{bD^{(\beta^1,0)}(\lambda + A^{va} + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi}\}) \leq \varepsilon.$$

Proof. (a) Let $\varepsilon > 0$ be arbitrary. For simplicity we set $\beta = (\beta^1, 0)$. For $b \in L^\infty(\mathbb{R}^{n-k})$ we obtain by Hölder's inequality and vector-valued complex interpolation (see, e.g., [5]) that

$$\|bD^\beta u\|_p \leq \|b\|_\infty \|u\|_{p,\nu} \leq C\|b\|_\infty \|u\|_{p,2m}^{\frac{\nu}{2m}} \|u\|_p^{1-\frac{\nu}{2m}} \quad (u \in W^{2m,p}(\mathbb{R}^{n-k}, E)).$$

With the help of Young's inequality we then can achieve that

$$\|bD^\beta u\|_p \leq \varepsilon\|u\|_{p,2m} + C(\varepsilon)\|u\|_p \quad (u \in W^{2m,p}(\mathbb{R}^{n-k}, E)).$$

Now let $b \in L^{r_\nu}(\mathbb{R}^{n-k})$, $r := \frac{r_\nu}{p}$, and $\frac{1}{r} + \frac{1}{r'} = 1$. Then Hölder's inequality and the vector-valued version of the Gagliardo-Nirenberg inequality (see [19]) imply

$$\|bD^\beta u\|_p \leq C\|b\|_{pr}\|D^\beta u\|_{pr'} \leq C\|b\|_{r_\nu}\|u\|_{p,2m}^\tau \|u\|_p^{1-\tau},$$

where $\tau = \frac{n-k}{r_\nu(2m-\nu)} \in (0,1)$ by our assumption on r_ν. Again an application of Young's inequality yields

$$\|bD^\beta u\|_p \leq \varepsilon\|u\|_{p,2m} + C(\varepsilon)\|u\|_p \quad (u \in W^{2m,p}(\mathbb{R}^{n-k}, E)).$$

(b) Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a family of independent symmetric $\{-1,1\}$-valued random variables on a probability space $([0,1], \mathcal{M}, P)$, $\lambda_j \in \Sigma_{\pi-\phi}$, and $f_j \in X$. For $b \in L^\infty(\mathbb{R}^{n-k})$, $\delta_0 > 0$, and arbitrary $t \in [0,1]$ we have

$$\begin{aligned}\Big\|\sum_{j=1}^N \varepsilon_j(t) bD^\beta(\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j\Big\|_p \\ \leq \|b\|_\infty \Big\|\sum_{j=1}^N \varepsilon_j(t) D^\beta(\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j\Big\|_p.\end{aligned}$$

Note that there is a $c_\phi > 0$ such that

$$|\lambda + \delta_0| \geq c_\phi \delta_0 \quad (\lambda \in \Sigma_{\pi-\phi},\ \delta_0 > 0).$$

Taking L^p-norm with respect to t and applying the contraction principle of Kahane therefore yields

$$
\begin{aligned}
&\Big\|\sum_{j=1}^N \varepsilon_j(\cdot) b D^\beta(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_{L^p([0,1],X)}\\
&\quad\le C\|b\|_\infty\Big\|\sum_{j=1}^N \varepsilon_j(\cdot)\Big(\frac{\lambda_j+\delta_0}{\delta_0}\Big)^{1-\frac{|\beta|}{2m}} D^\beta(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_{L^p([0,1],X)}.
\end{aligned}
$$

Thanks to (4.8) this implies

$$
\begin{aligned}
&\Big\|\sum_{j=1}^N \varepsilon_j(\cdot) b D^\beta(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_{L^p([0,1],X)}\\
&\quad\le C\|b\|_\infty \delta_0^{-(1-\frac{|\beta|}{2m})}\Big\|\sum_{j=1}^N \varepsilon_j(\cdot) f_j\Big\|_{L^p([0,1],X)}.
\end{aligned}
$$

Thus for $\delta_0 > (C\|b\|_\infty/\varepsilon)^{1/(1-|\beta|/2m)}$ the assertion follows.

In case that $b\in L^{r_\nu}(\mathbb{R}^{n-k})$, Hölder's inequality and the Gagliardo-Nirenberg inequality imply for $\tau(2m-\nu)=\frac{n-k}{r_\nu}$ and arbitrary $t\in[0,1]$ that

$$
\begin{aligned}
&\Big\|\sum_{j=1}^N \varepsilon_j(t) b D^\beta(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_p\\
&\quad\le \|b\|_{pr}\Big\|\sum_{j=1}^N \varepsilon_j(t) D^\beta(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_{pr'}\\
&\quad\le C\|b\|_{r_\nu}\Big(\sum_{|\alpha|=2m}\Big\|\sum_{j=1}^N \varepsilon_j(t) D^\alpha(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_p^p\Big)^{\tau/p}\\
&\qquad\cdot\Big\|\sum_{j=1}^N \varepsilon_j(t)(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_p^{1-\tau}.
\end{aligned}
$$

Taking L^p-norm with respect to t and applying once more Hölder's inequality we deduce

$$
\begin{aligned}
&\Big\|\sum_{j=1}^N \varepsilon_j(\cdot) b D^\beta(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_{L^p([0,1],X)}\\
&\quad\le C\|b\|_{r_\nu}\Big(\sum_{|\alpha|=2m}\Big\|\sum_{j=1}^N \varepsilon_j(\cdot) D^\alpha(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_{L^p([0,1],X)}^p\Big)^{\tau/p}\\
&\qquad\cdot\Big\|\sum_{j=1}^N \varepsilon_j(\cdot)(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\Big\|_{L^p([0,1],X)}^{1-\tau}.
\end{aligned}
$$

The contraction principle of Kahane then gives us

$$\begin{aligned}
&\Big\| \sum_{j=1}^{N} \varepsilon_j(\cdot) b D^{\beta} (\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j \Big\|_{L^p([0,1],X)} \\
&\quad \le C\|b\|_{r_\nu} \Big(\sum_{|\alpha|=2m} \Big\| \sum_{j=1}^{N} \varepsilon_j(\cdot) D^{\alpha} (\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j \Big\|^p_{L^p([0,1],X)} \Big)^{\tau/p} \\
&\qquad \cdot \Big\| \sum_{j=1}^{N} \varepsilon_j(\cdot) \frac{\lambda_j + \delta_0}{\delta_0} (\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j \Big\|^{1-\tau}_{L^p([0,1],X)}.
\end{aligned}$$

Taking into account (4.8) we arrive at

$$\begin{aligned}
&\Big\| \sum_{j=1}^{N} \varepsilon_j(\cdot) b D^{\beta} (\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j \Big\|_{L^p([0,1],X)} \\
&\quad \le C\|b\|_{r_\nu} \delta_0^{\tau-1} \Big\| \sum_{j=1}^{N} \varepsilon_j(\cdot) f_j \Big\|_{L^p([0,1],X)}.
\end{aligned}$$

Choosing $\delta_0 > (C\|b\|_{r_\nu}/\varepsilon)^{1/(1-\tau)}$ proves the lemma. $\square$

Proof of Theorem 2.3. We denote by

$$A_1^{\#}(x, D) := \sum_{|\alpha|=2m} a^1_{\alpha}(x) D^{\alpha}$$

the principal part of $A_1(x, D)$ and by $A_1^{\#}$ its realization in X with domain $D(A_1^{\#}) = W^{2m,p}(\mathbb{R}^{n-k}, E)$. Recall that $A_1^{\#}(x, D) = A_1^{\#}(x^1, D)$ does not depend on $x^2 \in V$. Freezing the coefficients at some arbitrary $x_0^1 \in \mathbb{R}^{n-k} \cup \{\infty\}$, Proposition 4.5 applies to $A_1(D) := A_1^{\#}(x_0^1, D)$.

So, we first choose a large ball $B_{r_0}(0) \subset \mathbb{R}^{n-k}$ with a fixed radius $r_0 > 0$ such that

$$|a^1_{\alpha^1}(x^1) - a^1_{\alpha^1}(\infty)| \le \eta/M_\alpha, \quad \text{for all } |x^1| \ge r_0,\ |\alpha^1| = 2m,$$

and set $U_0 := \mathbb{R}^{n-k} \setminus \overline{B}_{r_0}(0)$. Here $M_\alpha = \big|\{\alpha^1 \in \mathbb{N}_0^{n-k};\ |\alpha^1| = 2m,\ a_{\alpha^1} \neq 0\}\big|$ and $\eta = \eta(\infty)$ is the constant given in Corollary 4.7 for the principal part of the 'limiting operator' $A_1^{\#}(\infty, D) = \sum_{|\alpha|=2m} a^1_\alpha(\infty) D^\alpha$. For every $x_0^1 \in \overline{B}_{r_0}(0)$ let $\eta = \eta(x_0^1)$ be the constant given in Corollary 4.7 for the 'frozen coefficients operator' $A_1(D) := A_1^{\#}(x_0^1, D)$. By our continuity assumptions on the coefficients then there exists a radius $r = r(x_0^1)$ such that

$$|a^1_{\alpha^1}(x^1) - a^1_{\alpha^1}(x_0^1)| \le \eta(x_0^1)/M_\alpha, \quad \text{for all } |x^1 - x_0^1| \le r(x_0^1),\ |\alpha^1| = 2m.$$

Obviously the collection $\{B_{r(x_0^1)}(x_0^1):\ x_0^1 \in \overline{B}_{r_0}(0)\}$ represents an open covering of $\overline{B}_{r_0}(0)$. Thus, by compactness we have

$$\overline{B}_{r_0}(0) \subseteq \bigcup_{j=1}^{N} B_{r(x_j^1)}(x_j^1)$$

for a certain finite set $(x_j^1)_{j=1}^N$.

For simplicity we set $x_j := (x_j^1, 0)$, $r_j := r(x_j^1)$, and $U_j := B_{r_j}(x_j^1)$ for $j = 1, \ldots, N$, as well as $x_0^1 := \infty$. For each $j = 0, \ldots, N$ we define coefficients of $A_1^{\#}(x, D)$-localizations

$$A_j^{1,\mathrm{loc}}(x, D) := \sum_{|\alpha|=2m} a_{j,\alpha}^1(x) D^\alpha$$

by reflection, i.e., we set

$$a_{0,\alpha}^1(x) = \begin{cases} a_\alpha^1(x) \qquad , & x^1 \notin \overline{B}_{r_0}(0), \\ a_\alpha^1(\frac{r_0^2}{|x|^2}x), & x^1 \in \overline{B}_{r_0}(0), \end{cases}$$

and

$$a_{j,\alpha}^1(x) = \begin{cases} a_\alpha^1(x) \qquad\qquad\qquad\qquad , & x^1 \in \overline{B}_{r_j}(x_j^1), \\ a_\alpha^1(x_j + \frac{r_j^2}{|x-x_j|^2}(x - x_j)), & x^1 \notin \overline{B}_{r_j}(x_j^1). \end{cases}$$

Then by definition we have

$$\sum_{|\alpha^1|=2m} |a_{j,\alpha^1}^1(x) - a_{\alpha^1}^1(x_j)| \le \eta(x_j^1)$$

for $x = (x^1, 0) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ and $j = 0, \ldots, N$, that is, $A_j^{1,\mathrm{loc}}(x, D)$ is a small variation of $A^{\#}(x_j^1, D) := A_1^{\#}(x_j^1, D) + A_2(x, D)$. Hence Corollary 4.7 applies to

$$A_j^{\mathrm{loc}} := A_j^{1,\mathrm{loc}} + A_2.$$

In other words, for each $\phi > \varphi_{(A,B)}$ there exists $\delta = \delta(\phi) > 0$ such that $A_j^{\mathrm{loc}} + \delta \in \mathcal{RS}(X)$ and we have

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A_j^{\mathrm{loc}} + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le \ell + |\beta| + |\gamma| \le 2m\}) \le C_\phi < \infty \tag{4.9}$$

for $j = 0, \ldots, N$.

Next we choose a partition of unity $(\varphi_j)_{j=0}^N \subset C^\infty(\mathbb{R}^{n-k})$ of $\mathbb{R}^{n-k}$ subordinate to the open covering $(U_j)_{j=0}^N$ such that $0 \le \varphi_j \le 1$. In addition, we fix $\psi_j \in C^\infty(\mathbb{R}^{n-k})$ such that $\psi_j \equiv 1$ on supp φ_j and supp $\psi_j \subset U_j$. We set $\mathcal{B}(x, D) := A(x, D) - A^{\#}(x, D)$ and pick $\lambda \in \Sigma_{\pi-\phi}$. Then

$$\lambda u + A(\cdot, D)u = f$$

holds if and only if

$$\lambda u + A^{\#}(\cdot, D)u = f - \mathcal{B}(\cdot, D)u.$$

Multiplying the line above by φ_j we obtain

$$\lambda\varphi_j u + A^{\#}(\cdot, D)\varphi_j u = \varphi_j f + [A^{\#}(\cdot, D), \varphi_j]u - \varphi_j\mathcal{B}(\cdot, D)u,$$

where the commutators

$$[A^{\#}(\cdot, D), \varphi_j] := A^{\#}(\cdot, D)\varphi_j - \varphi_j A^{\#}(\cdot, D) = [A_1^{\#}(\cdot, D), \varphi_j]$$

do only depend on $A_1^{\#}(\cdot, D)$. Applying the resolvent of A_j^{loc} to the localized equations we deduce

$$\varphi_j u = (\lambda + A_j^{\mathrm{loc}} + \delta)^{-1}\varphi_j f + (\lambda + A_j^{\mathrm{loc}} + \delta)^{-1}([A^{\#}(\cdot, D), \varphi_j]u - \varphi_j\mathcal{B}(\cdot, D)u).$$

By multiplying with ψ_j and by summing up over j we gain the representation

$$u = \sum_{j=0}^{N}\psi_j(\lambda + A_j^{\mathrm{loc}} + \delta)^{-1}\varphi_j f + \sum_{j=0}^{N}\psi_j(\lambda + A_j^{\mathrm{loc}} + \delta)^{-1}([A^{\#}(\cdot, D), \varphi_j]u - \varphi_j\mathcal{B}(\cdot, D))u.$$

Hence we obtain

$$(I - \sum_{j=0}^{N}\psi_j(\lambda + A_j^{\mathrm{loc}} + \delta)^{-1}\mathcal{C}_j(\cdot, D))u = \sum_{j=0}^{N}\psi_j(\lambda + A_j^{\mathrm{loc}} + \delta)^{-1}\varphi_j f,$$

where

$$\mathcal{C}_j(\cdot, D) := [A_1^{\#}(\cdot, D), \varphi_j] - \varphi_j\mathcal{B}(\cdot, D)$$

is a differential operator in X of lower order whose coefficients fulfill the assumptions of Lemma 4.8. We set

$$R_0(\lambda, \delta) := \sum_{j=0}^{N}\psi_j(\lambda + A_j^{\mathrm{loc}} + \delta)^{-1}\varphi_j \tag{4.10}$$

and

$$R_1(\lambda, \delta) := \sum_{j=0}^{N}\psi_j(\lambda + A_j^{\mathrm{loc}} + \delta)^{-1}\mathcal{C}_j(\cdot, D).$$

Relation (4.9) and Lemma 4.8(a) now imply that

$$\begin{aligned}
&\|R_1(\lambda, \delta' + \delta_0)u\|_{W^{2m,p}(\Omega,F)} + \delta_0\|R_1(\lambda, \delta' + \delta_0)u\|_p \\
&\quad\le C\left(\|R_1(\lambda + \delta_0, \delta')u\|_{W^{2m,p}(\Omega,F)} + |\lambda + \delta_0|\|R_1(\lambda + \delta_0, \delta')u\|_p\right) \\
&\quad\le C\|\mathcal{C}_j(\cdot, D)u\|_p \\
&\quad\le C\left(\varepsilon\|u\|_{W^{2m,p}(\mathbb{R}^{n-k},E)} + C(\varepsilon)\|u\|_p\right) \\
&\quad\le \frac{1}{2}\left(\|u\|_{W^{2m,p}(\mathbb{R}^{n-k},E)} + \delta_0\|u\|_p\right) \\
&\quad\le \frac{1}{2}\left(\|u\|_{W^{2m,p}(\Omega,F)} + \delta_0\|u\|_p\right) \quad (\lambda \in \Sigma_{\pi-\phi})
\end{aligned}$$

for some $\delta' > 0$ and provided that $\delta_0 > 0$ is sufficiently large. Setting $\delta := \delta' + \delta_0$ we see that then

$$L_\lambda := (I - R_1(\lambda, \delta))^{-1}R_0(\lambda, \delta) : L^p(\mathbb{R}^{n-k}, E) \to D(A)$$

is a left inverse of $\lambda + A + \delta$ which admits an estimate

$$|\lambda| \|L_\lambda f\|_p \leq C \|f\|_p \quad (\lambda \in \Sigma_{\pi-\phi}).$$

Thus, if we can prove that there exists a right inverse as well, we obtain $A + \delta \in S(X)$ and $\phi_{A+\delta} \leq \phi$.

To this end, let $f \in X$ be arbitrary. Then

$$\begin{aligned}
(\lambda + A(\cdot, D) + \delta) R_0(\lambda, \delta) f &= (\lambda + A^{\#}(\cdot, D) + \delta) R_0(\lambda, \delta) f + \mathcal{B}(\cdot, D) R_0(\lambda, \delta) f \\
&= (\lambda + A^{\#}(\cdot, D) + \delta) \sum_{j=0}^{N} \psi_j (\lambda + A_j^{\mathrm{loc}} + \delta)^{-1} \varphi_j f \\
&\quad + \mathcal{B}(\cdot, D) \sum_{j=0}^{N} \psi_j (\lambda + A_j^{\mathrm{loc}} + \delta)^{-1} \varphi_j f \\
&= \sum_{j=0}^{N} \psi_j (\lambda + A^{\#}(\cdot, D) + \delta)(\lambda + A_j^{\mathrm{loc}} + \delta)^{-1} \varphi_j f \\
&\quad + \sum_{j=0}^{N} \mathcal{D}(\cdot, D)(\lambda + A_j^{\mathrm{loc}} + \delta)^{-1} \varphi_j f,
\end{aligned}$$

where

$$\mathcal{D}(\cdot, D) := [A_1^{\#}(\cdot, D), \psi_j] + \mathcal{B}(\cdot, D) \psi_j$$

is again a differential operator in X of lower order whose coefficients fulfill the assumptions of Lemma 4.8. Since supp $\psi_j \subset U_j$ and $\psi \equiv 1$ on supp φ_j, we obtain

$$(\lambda + A(\cdot, D) + \delta) R_0(\lambda, \delta) f = f + R_2(\lambda, \delta) f$$

with

$$R_2(\lambda, \delta) := \sum_{j=0}^{N} \mathcal{D}(\cdot, D)(\lambda + A_j^{\mathrm{loc}} + \delta)^{-1} \varphi_j.$$

Lemma 4.8(b) implies $\|R_2(\lambda, \delta)\|_{\mathcal{L}(X)} \leq 1/2$ for large enough $\delta > 0$. Consequently, $R_\lambda := R_0(\lambda, \delta)(I + R_2(\lambda, \delta))^{-1}$ is a right inverse of $\lambda + A + \delta$.

With the help of the Leibniz rule and the contraction principle of Kahane, from representation (4.10) and relation (4.9) we obtain that

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma R_0(\lambda, \delta)\}) \leq C(N+1).$$

In view of Lemma 4.8(b) and Lemma 3.5 the representation

$$(\lambda + A + \delta)^{-1} = R_0(\lambda, \delta) \sum_{i=0}^{\infty} R_2(\lambda, \delta)^i$$

as a Neumann series finally gives us

$$\begin{aligned}
&\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le \ell + |\beta| + |\gamma| \le 2m\}) \\
&\quad \le \mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma R_0(\lambda,\delta)\})\mathcal{R}(\{\sum_{i=0}^{\infty} R_2(\lambda,\delta)^i\}) \\
&\quad \le (N+1)C \sum_{i=0}^{\infty} (N+1)^i (C\varepsilon)^i = \frac{(N+1)C}{1-(N+1)C\varepsilon} < \infty.
\end{aligned}$$

Hence the proof of Theorem 2.3 is complete. □

5. Mixed orders

All parts of the proof can easily be adjusted to the situation when the differential operators $A_1(\cdot, D)$ and $A_2(\cdot, D)$ have different orders, say $2m_1$ and $2m_2$ respectively. Then a cylindrical boundary value problem is given as

$$\begin{aligned}
\lambda u + A(x,D)u &= f \quad \text{in } \Omega, \\
B_j(x,D)u &= 0 \quad \text{on } \partial\Omega \quad (j = 1, \dots, m),
\end{aligned} \tag{5.1}$$

with

$$\begin{aligned}
A(x,D) &= A_1(x^1, D) + A_2(x^2, D) \\
&:= \sum_{|\alpha^1| \le 2m_1} a^1_{\alpha^1}(x^1) D^{(\alpha^1, 0)} + \sum_{|\alpha^2| \le 2m_2} a^2_{\alpha^2}(x^2) D^{(0,\alpha^2)}
\end{aligned}$$

and

$$\begin{aligned}
B_j(x,D) &= B_{2,j}(x^2, D) \\
&:= \sum_{|\beta^2| \le m_{2,j}} b^2_{j,\beta^2}(x^2) D^{(0,\beta^2)} \quad (m_{2,j} < 2m_2,\ j = 1, \dots, m_2).
\end{aligned}$$

However, then the notion of parameter-ellipticity for the entire cylindrical boundary value problem is no longer appropriate. Instead we assume the differential operator $A_1(\cdot, D)$ to be parameter-elliptic in $\mathbb{R}^{n-k}$ as well as the boundary value problem

$$\begin{aligned}
\lambda u + A_2(x,D)u &= f \quad \text{in } V, \\
B_{2,j}(x,D)u &= 0 \quad \text{on } \partial V \quad (j = 1, \dots, m),
\end{aligned} \tag{5.2}$$

to be parameter-elliptic in the cross-section V of Ω with a joint angle of parameter-ellipticity $\varphi \in [0, \pi/2)$. The exact same proof as the one of Theorem 2.3 can be used to show the following result.

Theorem 5.1. *Given the assumptions of Theorem* 2.3*, let* $A_1(\cdot, D)$ *in* $\mathbb{R}^{n-k}$ *as well as the boundary value problem* (5.2) *in* V *be parameter-elliptic with a joint angle of parameter-ellipticity* $\varphi \in [0, \pi/2)$*. For* $\Omega = \mathbb{R}^{n-k} \times V$ *we define the* $L^p(\Omega, F)$*-*

realization of the cylindrical boundary value problem (5.1) *by*

$$\begin{aligned} D(A) = \Big\{ & u \in L^p(\Omega, F);\ D^\alpha u \in L^p(\Omega, F) \\ & \textit{for } \frac{|\alpha^1|}{2m_1} + \frac{|\alpha^2|}{2m_2} \le 1 \textit{ and } B_j(\cdot, D)u = 0 \quad (j = 1, \ldots, m) \Big\} \\ Au = {} & A(\cdot, D)u, \quad u \in D(A). \end{aligned}$$

Then for each $\phi > \varphi$ *there exists* $\delta = \delta(\phi) > 0$ *such that* $A + \delta \in \mathcal{RS}(L^p(\Omega, F))$ *with* $\phi^{\mathcal{RS}}_{A+\delta} \le \phi$*. Moreover, for* $\alpha = (\alpha^1, \alpha^2) \in \mathbb{N}_0^{n-k} \times \mathbb{N}_0^k$ *we have*

$$\mathcal{R}\left(\left\{\lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})} D^\alpha (\lambda + A + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le \frac{|\alpha^1|}{2m_1} + \frac{|\alpha^2|}{2m_2} \le 1\right\}\right) < \infty.$$

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Analytic Solutions for the Two-phase Navier-Stokes Equations with Surface Tension and Gravity

Jan Prüss and Gieri Simonett

Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. We consider the motion of two superposed immiscible, viscous, incompressible, capillary fluids that are separated by a sharp interface which needs to be determined as part of the problem. Allowing for gravity to act on the fluids, we prove local well-posedness of the problem. In particular, we obtain well-posedness for the case where the heavy fluid lies on top of the light one, that is, for the case where the Rayleigh-Taylor instability is present. Additionally we show that solutions become real analytic instantaneously.

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1. Introduction and main results

We consider a free boundary problem describing the motion of two immiscible, viscous, incompressible capillary fluids, *fluid*$_1$ and *fluid*$_2$, occupying the regions

$$\Omega_i(t) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : (-1)^i(y - h(t,x)) > 0,\ t \geq 0\}, \quad i = 1,2.$$

The fluids, thus, are separated by the interface

$$\Gamma(t) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y = h(t,x) : x \in \mathbb{R}^n,\ t \geq 0\},$$

called the free boundary, which needs to be determined as part of the problem. The motion of the fluids is governed by the incompressible Navier-Stokes equations where surface tension on the free boundary is included. In addition, we also allow

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for gravity to act on the fluids. The governing equations then are given by the system

$$\left\{\begin{aligned} \rho\big(\partial_t u + (u|\nabla)u\big) - \mu\Delta u + \nabla q &= 0 && \text{in} \quad \Omega(t)\\ \operatorname{div} u &= 0 && \text{in} \quad \Omega(t)\\ -[\![S(u,q)\nu]\!] &= \sigma\kappa\nu + [\![\rho]\!]\gamma_a y\nu && \text{on} \quad \Gamma(t)\\ [\![u]\!] &= 0 && \text{on} \quad \Gamma(t)\\ V &= (u|\nu) && \text{on} \quad \Gamma(t)\\ u(0) &= u_0 && \text{in} \quad \Omega_0\\ \Gamma(0) &= \Gamma_0\,. \end{aligned}\right. \tag{1.1}$$

Here ρ and μ are given by

$$\rho = \rho_1\chi_{\Omega_1(t)} + \rho_2\chi_{\Omega_2(t)}, \quad \mu = \mu_1\chi_{\Omega_1(t)} + \mu_2\chi_{\Omega_2(t)},$$

with χ the indicator function, where the constants ρ_i and μ_i denote the densities and viscosities of the respective fluids. The constant $\sigma > 0$ denotes the surface tension, and γ_a is the acceleration of gravity. Moreover, $S(u,q)$ is the stress tensor defined by

$$S(u,q) = \mu_i\big(\nabla u + (\nabla u)^{\mathsf{T}}\big) - qI \quad \text{in} \quad \Omega_i(t),$$

where $q = \tilde{q} + \rho\gamma_a y$ denotes the modified pressure incorporating the potential of the gravity force, and

$$[\![v]\!] = (v_{|_{\Omega_2(t)}} - v_{|_{\Omega_1(t)}}\big)|_{\Gamma(t)}$$

denotes the jump of the quantity v, defined on the respective domains $\Omega_i(t)$, across the interface $\Gamma(t)$. Finally, $\kappa = \kappa(t,\cdot)$ is the mean curvature of the free boundary $\Gamma(t)$, $\nu = \nu(t,\cdot)$ is the unit normal field on $\Gamma(t)$, and $V = V(t,\cdot)$ is the normal velocity of $\Gamma(t)$. Here we use the convention that $\nu(t,\cdot)$ points from $\Omega_1(t)$ into $\Omega_2(t)$, and that $\kappa(t,x)$ is negative when $\Omega_1(t)$ is convex in a neighborhood of $x \in \Gamma(t)$.

Given is the initial position $\Gamma_0 = \operatorname{graph}(h_0)$ of the interface, and the initial velocity

$$u_0 : \Omega_0 \to \mathbb{R}^{n+1}, \quad \Omega_0 := \Omega_1(0) \cup \Omega_2(0).$$

The unknowns are the velocity field $u(t,\cdot) : \Omega(t) \to \mathbb{R}^{n+1}$, the pressure field $q(t,\cdot) : \Omega(t) \to \mathbb{R}$, and the free boundary $\Gamma(t)$, where $\Omega(t) := \Omega_1(t) \cup \Omega_2(t)$.

Our main result shows that problem (1.1) admits a unique local smooth solution, provided that $\|\nabla h_0\|_\infty := \sup_{x\in\mathbb{R}^n} |\nabla h_0(x)|$ is sufficiently small.

Theorem 1.1. *Let $p > n+3$. Then given $\beta > 0$, there exists $\eta = \eta(\beta) > 0$ such that for all initial values*

$$(u_0, h_0) \in W_p^{2-2/p}(\Omega_0, \mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n), \quad [\![u_0]\!] = 0,$$

satisfying the compatibility conditions

$$[\![\mu D(u_0)\nu_0 - \mu(\nu_0|D(u_0)\nu_0)\nu_0]\!] = 0, \quad \operatorname{div} u_0 = 0 \ \textit{ on } \ \Omega_0,$$

with $D(u_0) := (\nabla u_0 + (\nabla u_0)^{\mathsf{T}})$*, and the smallness-boundedness condition*

$$\|\nabla h_0\|_\infty \leq \eta, \qquad \|u_0\|_\infty \leq \beta,$$

there is $t_0 = t_0(u_0, h_0) > 0$ *such that problem* (1.1) *admits a classical solution* (u, q, Γ) *on* $(0, t_0)$*. The solution is unique in the function class described in Theorem* 4.2*. In addition,* $\Gamma(t)$ *is a graph over* $\mathbb{R}^n$ *given by a function* $h(t)$ *and* $\mathcal{M} = \bigcup_{t\in(0,t_0)} \big(\{t\} \times \Gamma(t)\big)$ *is a real analytic manifold, and with*

$$\mathcal{O} := \{(t, x, y) :\ t \in (0, t_0),\ x \in \mathbb{R}^n, y \neq h(t, x)\},$$

the function $(u, q) : \mathcal{O} \to \mathbb{R}^{n+2}$ *is real analytic.*

Remarks 1.2. (a) More precise statements for the transformed problem will be given in Section 4. Due to the restriction $p > n + 3$ we obtain

$$h \in C(J; BUC^2(\mathbb{R}^n)) \cap C^1(J; BUC^1(\mathbb{R}^n)),$$

with $J = [0, t_0]$, where BUC means bounded and uniformly continuous. In particular, the normal of $\Omega_1(t)$, the normal velocity of $\Gamma(t)$, and the mean curvature of $\Gamma(t)$ are well defined and continuous, so that (1.1) makes sense pointwise. For u we obtain

$$u \in BUC(J \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1}), \quad \nabla u \in BUC(\mathcal{O}, \mathbb{R}^{(n+1)^2}).$$

Also interesting is the fact that the surface pressure jump is analytic on $\mathcal{M}$ as well.

(b) It is possible to relax the assumption $p > n + 3$. In fact, $p > (n + 3)/2$ turns out to be sufficient. In order to keep the arguments simple, we impose here the stronger condition $p > n + 3$.

(c) It is well known that the situation where gravity is acting on two superposed immiscible fluids – with the heavier fluid lying above a fluid of lesser density – leads to an instability, the Rayleigh-Taylor instability. In this case, small disturbances of the equilibrium situation $(u, h) = (0, 0)$ can cause instabilities, where the heavy fluid moves down under the influence of gravity, and the light material is displaced upwards, leading to vortices. Our results show that problem (1.1) is also well posed in this case, provided $\|\nabla h_0\|_\infty$ is small enough, yielding smooth solutions for a short time. In the forthcoming publication [29] we will give a rigorous proof showing that the equilibrium solution $(u, h) = (0, 0)$ is L_p-unstable. To the best of our knowledge these are the first rigorous results concerning the Navier-Stokes equations subject to the Rayleigh-Taylor instability.

(d) If $\gamma_a = 0$ then it is shown in [28] that problem (1.1) admits a solution with the same regularity properties on an arbitrary fixed time interval $[0, t_0]$, provided that $\|u_0\|_{W_p^{2-2/p}(\Omega_0)}$ and $\|h_0\|_{W_p^{3-2/p}(\mathbb{R}^n)}$ are sufficiently small (depending on t_0).

(e) We point out that in Theorem 1.1 we only need a smallness condition on the sup-norm of ∇h_0 (relative to the vertical component of the velocity). In case of a more general geometry, this condition can always be achieved by a judicious choice of a reference manifold.

The motion of a layer of viscous, incompressible fluid in an ocean of infinite extent, bounded below by a solid surface and above by a free surface which includes the effects of surface tension and gravity (in which case Ω_0 is a strip, bounded above by Γ_0 and below by a fixed surface Γ_b) has been considered by Allain [1], Beale [7], Beale and Nishida [8], Tani [35], by Tani and Tanaka [36], and by Shibata and Shimizu [32]. If the initial state and the initial velocity are close to equilibrium, global existence of solutions is proved in [7] for $\sigma > 0$, and in [36] for $\sigma \geq 0$, and the asymptotic decay rate for $t \to \infty$ is studied in [8]. We also refer to [9], where in addition the presence of a surfactant on the free boundary and in one of the bulk phases is considered.

In case that $\Omega_1(t)$ is a bounded domain, $\gamma_a = 0$, and $\Omega_2(t) = \emptyset$, one obtains the *one-phase* Navier-Stokes equations with surface tension, describing the motion of an isolated volume of fluid. For an overview of the existing literature in this case we refer to the recent publications [28, 31, 32, 33].

Results concerning the *two-phase problem* (1.1) with $\gamma_a = 0$ in the $3D$-case are obtained in [11, 12, 13, 34]. In more detail, Densiova [12] establishes existence and uniqueness of solutions (of the transformed problem in Lagrangian coordinates) with $v \in W_2^{s,s/2}$ for $s \in (5/2, 3)$ in case that one of the domains is bounded. Tanaka [34] considers the two-phase Navier-Stokes equations with thermo-capillary convection in bounded domains, and he obtains existence and uniqueness of solutions with $(v, \theta) \in W_2^{s,s/2}$ for $s \in (7/2, 4)$, with θ denoting the temperature.

In order to prove our main result we transform problem (1.1) into a problem on a fixed domain. The transformation is expressed in terms of the unknown height function h describing the free boundary. Our analysis proceeds with establishing maximal regularity results for an associated linear problem. relying on the powerful theory of maximal regularity, in particular on the H^∞-calculus for sectorial operators, the Dore-Venni theorem, and the Kalton-Weis theorem, see for instance [2, 14, 16, 22, 23, 26, 30].

Based on the linear estimates we can solve the nonlinear problem by the contraction mapping principle. Analyticity of solutions is obtained as in [28] by the implicit function theorem in conjunction with a scaling argument, relying on an idea that goes back to Angenent [4, 5] and Masuda [24]; see also [17, 18, 20].

The plan for this paper is as follows. Section 2 contains the transformation of the problem to a half-space and the determination of the proper underlying linear problem. In Section 3 we analyze this linearization and prove the crucial maximal regularity result in an L_p-setting. Section 4 is then devoted to the nonlinear problem and contains the proof of our main result. Finally we collect and prove in an appendix some of the technical results used in order to estimate the nonlinear terms.

2. The transformed problem

The nonlinear problem (1.1) can be transformed to a problem on a fixed domain by means of the transformations

$$\begin{aligned} v(t,x,y) &:= (u_1,\dots,u_n)(t,x,y+h(t,x)),\\ w(t,x,y) &:= u_{n+1}(t,x,y+h(t,x)),\\ \pi(t,x,y) &:= q(t,x,y+h(t,x)), \end{aligned}$$

where $t \in J = [0,a]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $y \neq 0$. With a slight abuse of notation we will in the sequel denote the transformed velocity again by u, that is, we set $u = (v,w)$. With this notation we obtain the transformed problem

$$\left\{\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= F(u,\pi,h) && \text{in } \dot{\mathbb{R}}^{n+1}\\ \operatorname{div} u &= F_d(u,h) && \text{in } \dot{\mathbb{R}}^{n+1}\\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] &= G_v(u,[\![\pi]\!],h) && \text{on } \mathbb{R}^n\\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] - \sigma\Delta h - [\![\rho]\!]\gamma_a h &= G_w(u,h) && \text{on } \mathbb{R}^n\\ [\![u]\!] &= 0 && \text{on } \mathbb{R}^n\\ \partial_t h - \gamma w &= -(\gamma v|\nabla h) && \text{on } \mathbb{R}^n\\ u(0) = u_0,\ h(0) &= h_0, \end{aligned}\right. \tag{2.1}$$

for $t > 0$, where $\dot{\mathbb{R}}^{n+1} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \neq 0\}$.

The nonlinear functions have been computed in [28] and are given by:

$$\begin{aligned} F_v(v,w,\pi,h) &= \mu\{-2(\nabla h|\nabla_x)\partial_y v + |\nabla h|^2\partial_y^2 v - \Delta h\partial_y v\} + \partial_y\pi\nabla h\\ &\quad + \rho\{-(v|\nabla_x)v + (\nabla h|v)\partial_y v - w\partial_y v\} + \rho\partial_t h\partial_y v,\\ F_w(v,w,h) &= \mu\{-2(\nabla h|\nabla_x)\partial_y w + |\nabla h|^2\partial_y^2 w - \Delta h\partial_y w\}\\ &\quad + \rho\{-(v|\nabla_x)w + (\nabla h|v)\partial_y w - w\partial_y w\} + \rho\partial_t h\partial_y w,\\ F_d(v,h) &= (\nabla h|\partial_y v) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} G_v(v,w,[\![\pi]\!],h) =&\ -[\![\mu(\nabla_x v + (\nabla_x v)^{\mathsf{T}})]\!]\nabla h + |\nabla h|^2[\![\mu\partial_y v]\!] + (\nabla h|\,[\![\mu\partial_y v]\!])\nabla h\\ &\ -[\![\mu\partial_y w]\!]\nabla h + \{[\![\pi]\!] - \sigma(\Delta h - G_\kappa(h))\}\nabla h,\\ G_w(v,w,h) =&\ -(\nabla h|\,[\![\mu\nabla_x w]\!]) - (\nabla h|\,[\![\mu\partial_y v]\!]) + |\nabla h|^2[\![\mu\partial_y w]\!] - \sigma G_\kappa(h) \end{aligned} \tag{2.3}$$

with

$$G_\kappa(h) = \frac{|\nabla h|^2\Delta h}{(1+\sqrt{1+|\nabla h|^2})\sqrt{1+|\nabla h|^2}} + \frac{(\nabla h|\nabla^2 h\nabla h)}{(1+|\nabla h|^2)^{3/2}}, \tag{2.4}$$

where $\nabla^2 h$ denotes the Hessian matrix of all second-order derivatives of h.

Before studying solvability results for problem (2.1) let us first introduce suitable function spaces. Let $\Omega \subseteq \mathbb{R}^m$ be open and X be an arbitrary Banach space. By $L_p(\Omega;X)$ and $H^s_p(\Omega;X)$, for $1 \le p \le \infty$, $s \in \mathbb{R}$, we denote the X-valued Lebesgue and the Bessel potential spaces of order s, respectively. We will also frequently make use of the fractional Sobolev-Slobodeckij spaces $W^s_p(\Omega;X)$, $1 \le p < \infty$, $s \in \mathbb{R}\setminus\mathbb{Z}$, with norm

$$\|g\|_{W^s_p(\Omega;X)} = \|g\|_{W^{[s]}_p(\Omega;X)} + \sum_{|\alpha|=[s]} \left(\int_\Omega \int_\Omega \frac{\|\partial^\alpha g(x) - \partial^\alpha g(y)\|^p_X}{|x-y|^{m+(s-[s])p}}\, dx\, dy \right)^{1/p}, \quad (2.5)$$

where $[s]$ denotes the largest integer smaller than s. Let $a \in (0,\infty]$ and $J = [0,a]$. We set

$${}_0W^s_p(J;X) := \begin{cases} \{g \in W^s_p(J;X) : g(0) = g'(0) = \cdots = g^{(k)}(0) = 0\}, \\ \text{if} \quad k + \frac{1}{p} < s < k+1+\frac{1}{p},\ k \in \mathbb{N}\cup\{0\}, \\ W^s_p(J;X), \quad \text{if} \quad s < \frac{1}{p}. \end{cases}$$

The spaces ${}_0H^s_p(J;X)$ are defined analogously. Here we remind that $H^k_p = W^k_p$ for $k \in \mathbb{Z}$ and $1 < p < \infty$, and that $W^s_p = B^s_{pp}$ for $s \in \mathbb{R}\setminus\mathbb{Z}$.
For $\Omega \subset \mathbb{R}^m$ open and $1 \le p < \infty$, the homogeneous Sobolev spaces $\dot H^1_p(\Omega)$ of order 1 are defined as

$$\begin{aligned} \dot H^1_p(\Omega) &:= (\{g \in L_{1,\mathrm{loc}}(\Omega) : \|\nabla g\|_{L_p(\Omega)} < \infty\}, \|\cdot\|_{\dot H^1_p(\Omega)}) \\ \|g\|_{\dot H^1_p(\Omega)} &:= \left(\sum_{j=1}^m \|\partial_j g\|^p_{L_p(\Omega)}\right)^{1/p}. \end{aligned} \quad (2.6)$$

Then $\dot H^1_p(\Omega)$ is a Banach space, provided we factor out the constant functions and equip the resulting space with the corresponding quotient norm, see for instance [21, Lemma II.5.1]. We will in the sequel always consider the quotient space topology without change of notation. In case that Ω is locally Lipschitz, it is known that $\dot H^1_p(\Omega) \subset H^1_{p,\mathrm{loc}}(\overline{\Omega})$, see [21, Remark II.5.1], and consequently, any function in $\dot H^1_p(\Omega)$ has a well-defined trace on $\partial\Omega$.

For $s \in \mathbb{R}$ and $1 < p < \infty$ we also consider the homogeneous Bessel-potential spaces $\dot H^s_p(\mathbb{R}^n)$ of order s, defined by

$$\begin{aligned} \dot H^s_p(\mathbb{R}^n) &:= (\{g \in \mathcal{S}'(\mathbb{R}^n) : \dot I^s g \in L_p(\mathbb{R}^n)\}, \|\cdot\|_{\dot H^s_p(\mathbb{R}^n)}), \\ \|g\|_{\dot H^s_p(\mathbb{R}^n)} &:= \|\dot I^s g\|_{L_p(\mathbb{R}^n)}, \end{aligned} \quad (2.7)$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of all tempered distributions, and $\dot I^s$ is the Riesz potential given by

$$\dot I^s g := (-\Delta)^{s/2} g := \mathcal{F}^{-1}(|\xi|^s \mathcal{F} g), \quad g \in \mathcal{S}'(\mathbb{R}^n).$$

By factoring out all polynomials, $\dot H^s_p(\mathbb{R}^n)$ becomes a Banach space with the natural quotient norm. For $s \in \mathbb{R}\setminus\mathbb{Z}$, the homogeneous Sobolev-Slobodeckij spaces $\dot W^s_p(\mathbb{R}^n)$ of fractional order can be obtained by real interpolation as

$$\dot W^s_p(\mathbb{R}^n) := (\dot H^k_p(\mathbb{R}^n), \dot H^{k+1}_p(\mathbb{R}^n))_{s-k,p}, \quad k < s < k+1,$$

where $(\cdot,\cdot)_{\theta,p}$ is the real interpolation method. It follows that

$$\dot{I}^s \in \mathrm{Isom}(\dot{H}_p^{t+s}(\mathbb{R}^n), \dot{H}_p^t(\mathbb{R}^n)) \cap \mathrm{Isom}(\dot{W}_p^{t+s}(\mathbb{R}^n), \dot{W}_p^t(\mathbb{R}^n)), \quad s,t \in \mathbb{R}, \tag{2.8}$$

with $\dot{W}_p^k = \dot{H}_p^k$ for $k \in \mathbb{Z}$. We refer to [6, Section 6.3] and [37, Section 5] for more information on homogeneous functions spaces. In particular, it follows from parts (ii) and (iii) in [37, Theorem 5.2.3.1] that the definitions (2.6) and (2.7) are consistent if $\Omega = \mathbb{R}^n$, $s = 1$, and $1 < p < \infty$. We note in passing that

$$\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|g(x)-g(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\Big)^{1/p}, \quad \Big(\int_0^\infty t^{(1-s)p}\|\frac{d}{dt}P(t)g\|_{L_p(\mathbb{R}^n)}^p \frac{dt}{t}\Big)^{1/p} \tag{2.9}$$

define equivalent norms on $\dot{W}_p^s(\mathbb{R}^n)$ for $0 < s < 1$, where $P(\cdot)$ denotes the Poisson semigroup, see [37, Theorem 5.2.3.2 and Remark 5.2.3.4]. Moreover,

$$\gamma_\pm \in \mathcal{L}(\dot{W}_p^1(\mathbb{R}_\pm^{n+1}), \dot{W}_p^{1-1/p}(\mathbb{R}^n)), \tag{2.10}$$

where $\gamma_\pm$ denotes the trace operators, see for instance [21, Theorem II.8.2].

3. The linearized two-phase Stokes problem with free boundary

It turns out that, unfortunately, the nonlinear term $(\gamma v|\nabla h)$ occurring in (2.1) cannot be made small in the norm of $\mathbb{F}_4(a)$, defined below in (4.2), by merely taking $\|\nabla h\|_\infty$ small. This can, however, be achieved for the modified term $(b - \gamma v|\nabla h)$, provided b is properly chosen so that $b(0) = \gamma v_0$. As a consequence, we now need to consider the modified linear problem

$$\left\{\begin{aligned}
\rho\partial_t u - \mu\Delta u + \nabla\pi &= f && \text{in} \quad \dot{\mathbb{R}}^{n+1}\\
\operatorname{div} u &= f_d && \text{in} \quad \dot{\mathbb{R}}^{n+1}\\
-[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] &= g_v && \text{on} \quad \mathbb{R}^n\\
-2[\![\mu\partial_y w]\!] + [\![\pi]\!] &= g_w + \sigma\Delta h + [\![\rho]\!]\gamma_a h && \text{on} \quad \mathbb{R}^n\\
[\![u]\!] &= 0 && \text{on} \quad \mathbb{R}^n\\
\partial_t h - \gamma w + (b(t,x)|\nabla)h &= g_h && \text{on} \quad \mathbb{R}^n\\
u(0) = u_0,\ h(0) &= h_0.
\end{aligned}\right. \tag{3.1}$$

Here we mention that the simpler case where $b = 0$ and $\gamma_a = 0$ was studied in [28, Theorem 5.1]. We obtain the following maximal regularity result.

Theorem 3.1. *Let $p > n+3$ be fixed, and assume that ρ_j and μ_j are positive constants for $j = 1,2$, and set $J = [0,a]$. Suppose*

$$b_0 \in \mathbb{R}^n, \quad b_1 \in W_p^{1-1/2p}(J; L_p(\mathbb{R}^n,\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n,\mathbb{R}^n)),$$

and set $b(\cdot) = b_0 + b_1(\cdot)$. *Then the Stokes problem with free boundary* (3.1) *admits a unique solution* (u, π, h) *with regularity*

$$\begin{aligned}
&u \in H^1_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})),\\
&\pi \in L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^{n+1})),\\
&[\![\pi]\!] \in W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n)),\\
&h \in W^{2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap H^1_p(J; W^{2-1/p}_p(\mathbb{R}^n)) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n))
\end{aligned} \tag{3.2}$$

if and only if the data $(f, f_d, g, g_h, u_0, h_0)$ *satisfy the following regularity and compatibility conditions:*

(a) $f \in L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))$,

(b) $f_d \in H^1_p(J; \dot{H}^{-1}_p(\mathbb{R}^{n+1})) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1}))$,

(c) $g = (g_v, g_w) \in W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n, \mathbb{R}^{n+1}))$,

(d) $g_h \in W^{1-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n))$,

(e) $u_0 \in W^{2-2/p}_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})$, $h_0 \in W^{3-2/p}_p(\mathbb{R}^n)$,

(f) $\operatorname{div} u_0 = f_d(0)$ *in* $\dot{\mathbb{R}}^{n+1}$ *and* $[\![u_0]\!] = 0$ *on* $\mathbb{R}^n$ *if* $p > 3/2$,

(g) $-[\![\mu\partial_y v_0]\!] - [\![\mu\nabla_x w_0]\!] = g_v(0)$ *on* $\mathbb{R}^n$ *if* $p > 3$.

The solution map $[(f, f_d, g, g_h, u_0, h_0) \mapsto (u, \pi, h)]$ *is continuous between the corresponding spaces.*

If $b_1 \equiv 0$ *then the result is true for all* $p \in (1, \infty)$, $p \neq 3/2, 3$.

Proof. (i) Since $\mathbb{F}_4(a)$, defined by

$$\mathbb{F}_4(a) := W^{1-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n)),$$

is a multiplication algebra for $p > n+3$, the operator $[h \mapsto (b|\nabla)h]$ maps the space

$$\mathbb{E}_4(a) := W^{2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap H^1_p(J; W^{2-1/p}_p(\mathbb{R}^n)) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n))$$

continuously into $\mathbb{F}_4(a)$ with bound $|b_0| + C_a\|b_1\|_{\mathbb{F}_4(a)}$, see Lemma 5.5(a).
As in the proof of [28, Theorem 5.1] it suffices to consider the reduced problem

$$\left\{\begin{array}{rll}
\rho\partial_t u - \mu\Delta u + \nabla\pi = 0 & \text{in} & \dot{\mathbb{R}}^{n+1}\\
\operatorname{div} u = 0 & \text{in} & \dot{\mathbb{R}}^{n+1}\\
-[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = 0 & \text{on} & \mathbb{R}^n\\
-2[\![\mu\partial_y w]\!] + [\![\pi]\!] = \sigma\Delta h + [\![\rho]\!]\gamma_a h & \text{on} & \mathbb{R}^n\\
[\![u]\!] = 0 & \text{on} & \mathbb{R}^n\\
\partial_t h - \gamma w + (b(t,x)|\nabla)h = \tilde{g}_h & \text{on} & \mathbb{R}^n\\
u(0) = 0,\ h(0) = 0, & &
\end{array}\right. \tag{3.3}$$

where the function $\tilde{g}_h \in {}_0\mathbb{F}_4(a)$ is defined in a similar way as in formula (5.5) in [28]. This can be accomplished by choosing $h_1 := h_{1,b} \in \mathbb{E}_4(a)$ such that

$$h_1(0) = h_0, \quad \partial_t h_1(0) = g_h(0) + \gamma w_0 - (b(0)|\nabla h_0),$$

and then setting $\tilde{g}_h := \tilde{g}_{h,b} := g_h + \gamma w_1 - (b|\nabla h_1) - \partial_t h_1$, where w_1 has the same meaning as in step (i) of the proof of [28, Theorem 5.1].

(ii) We first consider the reduced problem (3.3) for the case where $b \equiv b_0$ is constant. The corresponding boundary symbol $s_{b_0}(\lambda, \xi)$ is given by

$$s_{b_0}(\lambda, \xi) = \lambda + \big(\sigma|\xi| - [\![\rho]\!]\gamma_a/|\xi|\big)k(z) + i(b_0|\xi), \tag{3.4}$$

where we use the same notation as in the proof of [28, Theorem 5.1]. Here we remind that k has the following properties: k is holomorphic in $\mathbb{C} \setminus \mathbb{R}_-$ and

$$k(0) = \frac{1}{2(\mu_1 + \mu_2)}, \quad zk(z) \to \frac{1}{\rho_1 + \rho_2} \quad \text{for} \quad |z| \to \infty, \tag{3.5}$$

uniformly in $z \in \bar{\Sigma}_\vartheta$ for $\vartheta \in [0, \pi)$ fixed. In particular there is a constant $N = N(\vartheta)$ such that

$$|k(z)| + |zk(z)| \le N, \quad z \in \Sigma_\vartheta. \tag{3.6}$$

In the following we fix $\beta > 0$. For further analysis it will be convenient to introduce the related extended symbol

$$\tilde{s}(\lambda, \tau, \zeta) := \lambda + \sigma\tau k(z) + i\tau\zeta - [\![\rho]\!]\gamma_a k(z)/\tau, \tag{3.7}$$

where $(\lambda, \tau) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta$ with η sufficiently small, $z := \lambda/\tau^2$, and $\zeta \in U_{\beta,\delta}$ with $U_{\beta,\delta} := \{\zeta \in \mathbb{C} : |\mathrm{Re}\,\zeta| < \beta + 1, \ |\mathrm{Im}\,\zeta| < \delta\}$ and $\delta \in (0, 1]$. Clearly $\tilde{s}(\lambda, |\xi|, (b_0|\xi/|\xi|)) = s_{b_0}(\lambda, \xi)$ for $(\lambda, \xi) \in \Sigma_\eta \times \dot{\mathbb{R}}^n$.

We are going to show that for every fixed $\beta > 0$ there are positive constants λ_0, δ, $\eta = \eta(\beta)$, and $c_j = c_j(\beta, \lambda_0, \delta, \eta)$ such that

$$c_0\big[|\lambda| + |\tau|\big] \le |\tilde{s}(\lambda, \tau, \zeta)| \le c_1\big[|\lambda| + |\tau|\big], \tag{3.8}$$

for all $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$ with $|\lambda| \ge \lambda_0$. The upper estimate is easy to obtain: fixing $\vartheta \in (\pi/2, \pi)$ and $\lambda_0 > 0$, it follows from (3.6) and the identity $k(z)/\tau = zk(z)\tau/\lambda$ that

$$|\tilde{s}(\lambda, \tau, \zeta)| \le |\lambda| + \big(\sigma N + (\beta + 2) + |[\![\rho]\!]|\gamma_a N/\lambda_0\big)|\tau| \le c_1\big[|\lambda| + |\tau|\big] \tag{3.9}$$

for all $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$, where $|\lambda| \ge \lambda_0$ and $\eta \in (0, \eta_0)$ with $\eta_0 := (\vartheta - \pi/2)/3$.

In order to obtain a lower estimate we proceed as follows. Suppose first that $\beta, \lambda_0 > 0$ are fixed and η_0 is as above. Then we obtain

$$\begin{aligned} |\tilde{s}(\lambda, \tau, \zeta)| &\ge |\lambda| - \big(\sigma N + (\beta + 2) + |[\![\rho]\!]|\gamma_a N/\lambda_0\big)|\tau| \\ &\ge (1/2)|\lambda| + (m/4)|\tau| = c_0(\beta, \lambda_0)[|\lambda| + |\tau|], \end{aligned} \tag{3.10}$$

provided $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$, $\eta \in (0, \eta_0)$, and $|\lambda| \ge \lambda_0$ as well as $|\lambda| \ge m|\tau|$ with

$$(m/4) \ge \sigma N + (\beta + 2) + |[\![\rho]\!]|\gamma_a N/\lambda_0.$$

Next we will derive an estimate from below in case that $|\lambda| \le M|\tau|^2$ with M a positive constant. From (3.5) follows that there are constants $H, L, R > 0$, depending on M, such that

$$L \le \mathrm{Re}\,(\sigma k(z)) \le R, \quad |\mathrm{Im}\,(\sigma k(z))| \le H, \tag{3.11}$$

whenever $(\lambda,\tau)\in\Sigma_{\pi/2+\eta}\times\Sigma_\eta$, for $\eta\in(0,\eta_0)$ and $|\lambda|\le M|\tau|^2$, where $z=\lambda/\tau^2$. By choosing δ small enough we obtain from (3.11) and the definition of $U_{\beta,\delta}$

$$0<L-\delta\le \mathrm{Re}\,(\sigma k(z)+i\zeta)\le R+\delta,\quad |\mathrm{Im}\,(\sigma k(z)+i\zeta)|\le H+(\beta+1)$$

provided $(\lambda,\tau,\zeta)\in\Sigma_{\pi/2+\eta}\times\Sigma_\eta\times U_{\beta,\delta}$, $\eta\in(0,\eta_0)$ and $|\lambda|\le M|\tau|^2$, where $z=\lambda/\tau^2$. By choosing η small enough we conclude that there is $\alpha=\alpha(M,\beta,\delta,\eta)\in(0,\pi/2)$ such that

$$\tau(\sigma k(z)+i\zeta)\in\Sigma_\alpha \tag{3.12}$$

whenever $(\lambda,\tau,\zeta)\in\Sigma_{\pi/2+\eta}\times\Sigma_\eta\times U_{\beta,\delta}$ and $|z|\le M$ with $z=\lambda/\tau^2$. We can additionally assume that η is chosen so that $\psi:=\pi/2-\alpha-\eta>0$. This implies

$$\begin{aligned}|\tilde{s}(\lambda,\tau,\zeta)|&\ge c(\psi)\big[|\lambda|+|\tau|\,|\sigma k(z)+i\zeta|\big]-|\tau||[\![\rho]\!]|\gamma_a N/\lambda_1\\&\ge c(\psi)\min(1,L-\delta)\big[|\lambda|+|\tau|\big]-|\tau||[\![\rho]\!]|\gamma_a N/\lambda_1\\&\ge c_0(M,\beta,\lambda_1)\big[|\lambda|+|\tau|\big],\end{aligned}\tag{3.13}$$

provided $(\lambda,\tau,\zeta)\in\Sigma_{\pi/2+\eta}\times\Sigma_\eta\times U_{\beta,\delta}$, $|\lambda|\le M|\tau|^2$ and $|\lambda|\ge\lambda_1$, where λ_1 is chosen big enough.

Noting that the curves $|\lambda|=m|\tau|$ and $|\lambda|=M|\tau|^2$ intersect at $(m/M,m^2/M)$ we obtain (3.8) by choosing $\lambda_0:=\max(\lambda_1,m^2/M)$.

(iii) In the following, we fix $\beta>0$ and we assume that $b_0\in\mathbb{R}^n$ with $|b_0|\le\beta$. Let then S_{b_0} be the operator corresponding to the symbol s_{b_0}. It is clear that S_{b_0} is bounded from ${}_0\mathbb{E}_4(a)$ to ${}_0\mathbb{F}_4(a)=:X$ and it remains to prove that it is boundedly invertible. For this we use the $\mathcal{H}^\infty$-calculus and similar arguments as in [27, Section 4] and [28, Section 5]. First we note that D_n admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus in X with angle 0; this follows from [14, Theorem 4.11]. Therefore by the estimates obtained in (3.8), the operator family

$$\{(\lambda+D_n^{1/2})\tilde{s}^{-1}(\lambda,D_n^{1/2},\zeta):(\lambda,\zeta)\in\Sigma_{\pi/2+\eta}\times U_{\beta,\delta},\ |\lambda|\ge\lambda_0\}$$

is $\mathcal{R}$-bounded. Since $G=\partial_t$ is in $\mathcal{H}^\infty(X)$ with angle $\pi/2$, the theorem of Kalton and Weis [22, Theorem 4.4] implies that the operator family

$$\{(G+D_n^{1/2})\tilde{s}^{-1}(G,D_n^{1/2},\zeta):\zeta\in U_{\beta,\delta}\}$$

is bounded and holomorphic on $U_{\beta,\delta}$. Finally, we employ the Dunford calculus for the bounded linear operator $R_{b_0}:=(b_0|R)$, where R denotes the Riesz operator with symbol $\xi/|\xi|$, $\xi\in\dot{\mathbb{R}}^n$. The operator R_{b_0} is bounded and its spectrum is $\sigma(R_{b_0})=[-|b_0|,|b_0|]$, as, e.g., the Mikhlin theorem shows. Since the operator family

$$\{(G+D_n^{1/2})\tilde{s}^{-1}(G,D_n^{1/2},\zeta):\zeta\in U_{\beta,\delta}\}$$

is bounded and holomorphic in a neighborhood of $\sigma(R_{b_0})$, the classical Dunford calculus shows that the operator

$$(G+D_n^{1/2})\tilde{s}^{-1}(G,D_n^{1/2},R_{b_0})$$

is bounded in X, uniformly for all $b_0\in\mathbb{R}^n$ with $|b_0|\le\beta$. This shows that $S_{b_0}:{}_0\mathbb{E}_4(a)\to{}_0\mathbb{F}_4(a)$ is boundedly invertible, uniformly for all $b_0\in\mathbb{R}^n$ with $|b_0|\le\beta$.

We emphasize that the bound for the operator $S_{b_0}^{-1} : {}_0\mathbb{F}_4(a) \to {}_0\mathbb{E}_4(a)$ depends only on the parameters ρ_j, μ_j, σ, γ_a, p and β, for $|b_0| \le \beta$.

(iv) By means of a perturbation argument the result for constant b can be extended to variable $b = b_0 + b_1(t,x)$. In fact, given $\beta > 0$ there exists a number $\eta > 0$ such that the solution operator S_b^{-1} exists and is bounded uniformly, provided $|b_0| \le \beta$ and $\|b_1\|_\infty + \|b_1\|_{\mathbb{F}_4(a)} \le 2\eta$. This follows easily from the estimate

$$\|(b_1|\nabla h)\|_{{}_0\mathbb{F}_4(a)} \le c_0(\|b_1\|_\infty + \|b_1\|_{\mathbb{F}_4(a)})\|h\|_{{}_0\mathbb{E}_4(a)},$$

see Lemma 5.5(c).

(v) In the general case we use a localization technique, similar to [3, Section 9]. For this purpose we first decompose J into subintervals $J_k = [k\delta, (k+1)\delta]$ of length $\delta > 0$ and solve the problem successively on these subintervals. Since $b \in BUC(J; C_0(\mathbb{R}^n, \mathbb{R}^n))$, i.e., b is bounded and uniformly continuous, given any $\eta > 0$ we may choose $\delta > 0$ and $\varepsilon > 0$ such that

$$|b(t,x) - b(s,y)| \le \eta \quad \text{for all} \quad (t,x), (s,y) \in J \times \mathbb{R}^n$$

with $|t-s| \le \delta$ and $|x-y|_\infty \le \varepsilon$. Let $\{U_j := x_j + (\varepsilon/2)Q : j \in \mathbb{N}\}$ be an enumeration of the open covering $\{(\varepsilon/2)(z/2+Q) : z \in \mathbb{Z}^n\}$ of $\mathbb{R}^n$, where $Q = (-1,1)^n$. Clearly,

$$|b(t,x) - b(s,y)| \le \eta, \quad s,t \in J_k, \quad x,y \in U_j. \tag{3.14}$$

Let ϕ be a smooth cut-off function with support contained in $(\varepsilon/2)Q$ such that $\phi \equiv 1$ on $(\varepsilon/4)Q$. Define

$$\phi_j := (\tau_{x_j}\phi)\Big(\sum_{k\in\mathbb{N}} (\tau_{x_k}\phi)^2\Big)^{-1/2}, \quad j \in \mathbb{N},$$

where $(\tau_{x_j}\phi)(x) := \phi(x - x_j)$. Consequently, ϕ_j is a smooth cut-off function with $\mathrm{supp}(\phi_j) \subset U_j$ and $\sum_j \phi_j^2 \equiv 1$. For a function space $\mathfrak{F}(J;\mathbb{R}^n) \subset L_p(J; L_p(\mathbb{R}^n))$ we define

$$\begin{aligned} r(h_j) &:= \sum_j \phi_j h_j, \qquad && (h_j) \in \mathfrak{F}(J;\mathbb{R}^n)^{\mathbb{N}}, \\ r^c h &:= (\phi_j h), && h \in \mathfrak{F}(J;\mathbb{R}^n). \end{aligned}$$

Similarly as in [3, Section 9] one shows that

$$r \in \mathcal{L}(\ell_p(\mathfrak{F}(J;\mathbb{R}^n)), \mathfrak{F}(J;\mathbb{R}^n)), \ r^c \in \mathcal{L}(\mathfrak{F}(J;\mathbb{R}^n), \ell_p(\mathfrak{F}(J;\mathbb{R}^n))), \quad rr^c = I, \tag{3.15}$$

for $\mathfrak{F}(J;\mathbb{R}^n) \in \{\mathbb{F}_4(a), \mathbb{E}_4(a)\}$.

Let θ be a smooth cut-off function with $\mathrm{supp}(\theta) \subset (\varepsilon/2)Q$ such that $\theta \equiv 1$ on $\mathrm{supp}(\phi)$ and let $\theta_j := \tau_{x_j}\theta$. Define

$$b_{j,k}(t,x) := \theta_j(x)\left(b(t,x) - b(k\delta, x_j)\right), \quad (t,x) \in J \times \mathbb{R}^n.$$

It follows that

$$\|b_{j,k}\|_{BC(J_k\times\mathbb{R}^n)} + \|b_{j,k}\|_{\mathbb{F}_4(J_k)} \le c_0\eta, \quad k = 0,\dots,m, \quad j \in \mathbb{N}, \tag{3.16}$$

provided δ is chosen small enough, where BC stands for bounded and continuous. Indeed, the estimates for $\|b_{j,k}\|_{BC(J_k,\mathbb{R}^n)}$ follow immediately from (3.14), while the estimates for $\|b_{j,k}\|_{\mathbb{F}_4(J_k)}$ can be shown by approximating b by functions that have better time regularity and by carefully estimating the products $\|\theta_j(b-b(k\delta,x_j))\|_{\mathbb{F}_4(J_k)}$.

We now concentrate on the first interval $J_0=[0,\delta]$. Let $L\in\mathcal{L}({}_0\mathbb{E}_4(a),{}_0\mathbb{F}_4(a))$ denote the operator with symbol $\sigma\tau k(z)$, i.e.,

$$L:=(\sigma D_n^{1/2}-[\![\rho]\!]\gamma_a D_n^{-1/2})k(GD_n^{-1}):=L_1+L_2.$$

If follows from (3.16) and step (iii) that the operator

$$S_j:=G+L+(b(0,x_j)+b_{j,0}|\nabla):{}_0\mathbb{E}_4(\delta)\to{}_0\mathbb{F}_4(\delta)$$

is invertible. Moreover, there is a constant C_0, depending only on $\sup_j|b(0,x_j)|$ – and therefore only on $\|b\|_{BC(J\times\mathbb{R}^n)}$ – such that $\|S_j^{-1}\|_{\mathcal{L}({}_0\mathbb{F}_4(\delta),{}_0\mathbb{E}_4(\delta))}\le C_0,\ j\in\mathbb{N}$.

(vi) Suppose that for a given $g\in{}_0\mathbb{F}_4(\delta)$ we have a solution $h\in{}_0\mathbb{E}_4(\delta)$ of

$$Gh+Lh+(b|\nabla)h=g.$$

Multiplying this equation by ϕ_j, using that $b\,\partial^\alpha\phi_j=(b(0,x_j)+b_{j,0})\,\partial^\alpha\phi_j$ and $rr^c=I$ this yields

$$S_j\phi_jh-[L,\phi_j]h-(b|\nabla\phi_j)h=(S_j-[L,\phi_j]r-(b|\nabla\phi_j)r)\,r^ch=r^cg,$$

where $[\cdot,\cdot]$ denotes the commutator. We now interpret this equation as an equation in $\ell_p({}_0\mathbb{F}_4(\delta))$. It follows from step (iv) that $(S_j)\in\mathrm{Isom}(\ell_p({}_0\mathbb{E}_4(\delta)),\ell_p({}_0\mathbb{F}_4(\delta)))$ and

$$\|(S_j^{-1})\|_{\mathcal{L}(\ell_p({}_0\mathbb{F}_4(\delta)),\ell_p({}_0\mathbb{E}_4(\delta)))}\le C_0. \tag{3.17}$$

We shall show below in step (vi) that the commutators satisfy

$$([L,\phi_j]+(b|\nabla\phi_j))\in\mathcal{L}({}_0\mathbb{F}(a),\ell_p({}_0\mathbb{F}_4(a))). \tag{3.18}$$

Assuming this property, it follows from (3.15) that

$$\|(([L,\phi_j]+(b|\nabla\phi_j))r(h_j))\|_{\ell_p({}_0\mathbb{F}_4(\delta))}\le C\|(h_j)\|_{\ell_p({}_0\mathbb{F}_4(\delta))}\le C\delta^\alpha\|(h_j)\|_{\ell_p({}_0\mathbb{E}_4(\delta))}$$

for some α depending only on p and n. Therefore, choosing δ small enough we can conclude that $(S_j-([L,\phi_j]+(b|\nabla\phi_j))r)\in\mathrm{Isom}(\ell_p({}_0\mathbb{E}_4(\delta)),\ell_p({}_0\mathbb{F}_4(\delta)))$ with

$$\|\,(S_j-([L,\phi_j]+(b|\nabla\phi_j))r)^{-1}\,\|\le 2C_0.$$

Let $T_b:=r\,(S_j-([L,\phi_j]+(b|\nabla\phi_j))r)^{-1}\,r^c$. Then $T_b\in\mathcal{L}({}_0\mathbb{F}_4(\delta),{}_0\mathbb{E}_4(\delta))$ is a left inverse of $S_b:=G+L+(b|\nabla)$. Hence

$$\|h\|_{{}_0\mathbb{E}_4(\delta)}=\|T_bS_bh\|_{{}_0\mathbb{E}_4(\delta)}\le 2C_0\|r\|\,\|r^c\|\,\|S_bh\|_{{}_0\mathbb{E}_4(\delta)},\quad h\in{}_0\mathbb{E}_4(\delta). \tag{3.19}$$

Replacing b by ρb, $\rho\in[0,1]$, we have a continuous family $\{S_{\rho b}\}$ of operators $S_{\rho b}$ which all satisfy the a priori estimate (3.19) uniformly in $\rho\in[0,1]$. Since S_0 is an isomorphism, we can infer from a homotopy argument that S_b is an isomorphism as well. Repeating successively these arguments for the intervals J_k, including the reduction from step (i), proves the assertion of the corollary.

(vii) We still have to verify the estimate in (3.18). Since the covering $\{U_j:j\in\mathbb{N}\}$ has finite multiplicity, one obtains

$$\|((\partial^\alpha\phi_j)g)\|_{\ell_p({}_0\mathbb{F}_4(a))}\le C(\alpha)\|g\|_{{}_0\mathbb{F}_4(a)},\quad g\in{}_0\mathbb{F}_4(a). \tag{3.20}$$

This together with Proposition 5.5(b) shows that

$$\|(b|\nabla\phi_j)h\|_{\ell_p({}_0\mathbb{F}_4(a)} \le C\|bh\|_{{}_0\mathbb{F}_4(a)} \le C_0(\|b\|_\infty + \|b\|_{\mathbb{F}_4(a)})\|h\|_{{}_0\mathbb{F}_4(a)}.$$

The estimates for the commutators $[L,\phi_j]$ are more involved. The operator $A = GD_n^{-1}$ with canonical domain is sectorial and admits a bounded $\mathcal{H}^\infty$-calculus with angle $\pi/2$ in ${}_0H^s_p(J;K^r_p(\mathbb{R}^n))$, for $K \in \{H,W\}$, and also in ${}_0W^s_p(J;K^r_p(\mathbb{R}^n))$ by real interpolation. Hence fixing $\theta \in (0,\pi/2)$, the following resolvent estimate holds in these spaces:

$$\|z(z-A)^{-1}\| \le M, \qquad \text{for all } z \in -\Sigma_\theta.$$

The function $k(z)$ is holomorphic in $\mathbb{C}\setminus(-\infty,-2\delta_0]$ for some $\delta_0>0$ and behaves like $1/z$ as $|z|\to\infty$. Choose the contour

$$\Gamma = (\infty,\delta_0]e^{i\psi} \cup \delta_0 e^{i[\psi,2\pi-\psi]} \cup [\delta_0,\infty)e^{-i\psi},$$

where $\pi > \psi > \pi-\theta$. Then we have the Dunford integral

$$k(A) = \frac{1}{2\pi i}\int_\Gamma k(z)(z-A)^{-1}dz,$$

which is absolutely convergent. This shows that $k(A)$ is bounded, as is $Ak(A)$ thanks to $A \in \mathcal{H}^\infty$, thus $A^{1/2}k(A)$ is bounded as well. Therefore the identity $k(A)D_n^{-1/2} = G^{-1/2}A^{1/2}k(A)$ shows that L_2 is bounded since $G^{-1/2}$ is, and (3.18) follows for $[L_2,\phi_j]$. For the commutator $[L_1,\phi_j]$ we obtain

$$[L_1,\phi_j] = \sigma[k(A)D_n^{1/2},\phi_j] = \sigma[k(A),\phi_j]D_n^{1/2} + \sigma k(A)[D_n^{1/2},\phi_j].$$

Using the Dunford integral for $k(A)$ this yields

$$[k(A),\phi_j] = \frac{1}{2\pi i}\int_\Gamma k(z)[(z-A)^{-1},\phi_j]dz = \frac{1}{2\pi i}\int_\Gamma k(z)(z-A)^{-1}[A,\phi_j](z-A)^{-1}dz,$$

hence with

$$\begin{aligned}[A,\phi_j] &= GD_n^{-1}[\phi_j,D_n]D_n^{-1} = A(\Delta\phi_j + 2(\nabla\phi_j|\nabla))D_n^{-1}\\ &= A(\Delta\phi_j D_n^{-1} + 2i(\nabla\phi_j|R)D_n^{-1/2}),\end{aligned}$$

we have

$$[k(A),\phi_j]D_n^{1/2} = \frac{1}{2\pi i}\int_\Gamma k(z)A(z-A)^{-1}\{-\Delta\phi_j G^{-1/2}A^{1/2} + 2i(\nabla\phi_j|R)\}(z-A)^{-1}dz.$$

Let $h \in {}_0\mathbb{F}_4$ be given. Then we obtain from

$$\|k(z)A(z-A)^{-1}\|_{\mathcal{L}({}_0\mathbb{F}_4)} \le C/|z|, \quad \|A^{1/2}(z-A)^{-1}\|_{\mathcal{L}({}_0\mathbb{F}_4)} \le C/|z|^{1/2}, \quad z\in\Gamma,$$

from (3.20), and from Minkowski's inequality for integrals

$$\begin{aligned}&\Big\|\Big(\int_\Gamma k(z)A(z-A)^{-1}\Delta\phi_j G^{-1/2}A^{1/2}(z-A)^{-1}h\,dz\Big)\Big\|_{\ell_p({}_0\mathbb{F}_4)}\\ &\le C\int_\Gamma \frac{1}{|z|}\|(\Delta\phi_j G^{-1/2}A^{1/2}(z-A)^{-1}h)\|_{\ell_p({}_0\mathbb{F}_4)}\,|dz|\\ &\le C\int_\Gamma \frac{1}{|z|^{3/2}}\|h\|_{{}_0\mathbb{F}_4}\,|dz| \le C\|h\|_{{}_0\mathbb{F}_4}\end{aligned}$$

where we also used that $G^{-1/2}$ is bounded on compact intervals. In the same way we can estimate the second term in the integral representation of $[k(A),\phi_j]D^{1/2}$, this time using the fact that R is bounded.

To estimate the commutators $[D_n^{1/2},\phi_j]$ in ${}_0\mathbb{F}_4$ note that

$$\begin{aligned}(D_n)^{1/2} = D_n(D_n)^{-1/2} &= \frac{1}{\sqrt{\pi}} D_n \int_0^\infty e^{-D_n t} t^{-\frac12}\,dt\\ &= \frac{1}{\sqrt{\pi}}\left(D_n\int_0^1 e^{-D_n t}t^{-\frac12}\,dt + D_n\int_1^\infty e^{-D_n t}t^{-\frac12}\,dt\right)\\ &=: \frac{1}{\sqrt{\pi}}(T_1+T_2),\end{aligned}$$

with $e^{-D_n t}$ denoting the bounded analytic semigroup generated by the Laplacian in $H^s_p(\mathbb{R}^n)$ which extends by real interpolation to $W^s_p(\mathbb{R}^n)$, and then canonically to ${}_0\mathbb{F}_4$. Thus by (3.20) there is a constant $C>0$ such that for $h\in {}_0\mathbb{F}_4$ we have

$$\begin{aligned}\|(\phi_j T_2 h)\|_{\ell_p({}_0\mathbb{F}_4)} &= \left\|(\phi_j\int_1^\infty D_n e^{-D_n t}t^{-\frac12}h\,dt)\right\|_{\ell_p({}_0\mathbb{F}_4)}\\ &\le C\left\|\int_1^\infty D_n e^{-D_n t}t^{-\frac12}h\,dt\right\|_{{}_0\mathbb{F}_4}\\ &\le C\int_1^\infty t^{-\frac32}\,dt\,\|h\|_{{}_0\mathbb{F}_4}\le C\|h\|_{{}_0\mathbb{F}_4},\\ \|(T_2\phi_j h)\|_{\ell_p({}_0\mathbb{F}_4)} &= \left\|\int_1^\infty D_n e^{-D_n t}t^{-\frac12}(\phi_j h)\,dt\right\|_{\ell_p({}_0\mathbb{F}_4)}\\ &\le C\|(\phi_j h)\|_{\ell_p({}_0\mathbb{F}_4)}\le C\|h\|_{{}_0\mathbb{F}_4}.\end{aligned}$$

Hence $\|([\phi_j,T_2]h)\|_{\ell_p({}_0\mathbb{F}_4)}\le C\|h\|_{{}_0\mathbb{F}_4}$.

We consider next the commutator $[T_1,\phi_j]$. Let $k_t(x)=(2\pi t)^{-n/2}\exp(-|x|^2/4t)$ denote the Gaussian kernel. Then for fixed $t>0$, the operator $D_n e^{-D_n t}$ is the convolution with kernel $-\Delta k_t(x)$, which is of class C^∞. It is not difficult to see that there are constants $C,c>0$ such that

$$|\Delta k_t(x)|\le Ct^{-(n+2)/2}\,e^{-c|x|^2/t},\quad x\in\mathbb{R}^n,\ t>0. \tag{3.21}$$

Choosing a cut-off function $\chi\in C^\infty(\mathbb{R}^n)$ with $\chi\equiv 1$ in $B_\rho(0)$, $\operatorname{supp}(\chi)\subset B_{2\rho}(0)$ and $0\le\chi\le 1$ elsewhere, we set

$$-\Delta k_t(x) = -(1-\chi(x))\Delta k_t(x)-\chi(x)\Delta k_t(x) =: k_{3,t}(x)+k_{4,t}(x),\quad x\in\mathbb{R}^n,\ t>0,$$

and we denote by T_l the convolution operators with kernels $\int_0^1 k_{l,t}t^{-1/2}dt$, $l=3,4$. For the kernel of T_3 we obtain from (3.21) the estimate

$$\begin{aligned}|\int_0^1 k_{3,t}(x)t^{-1/2}\,dt| &\le C\int_0^1 e^{-c|x|^2/t}t^{-(n+3)/2}\,dt\\ &\le Ce^{-c_1|x|^2}\int_1^\infty e^{-c_2|x|^2 s}s^{(n-1)/2}ds\le Ce^{-c_1|x|^2},\end{aligned}$$

as $k_{3,t}(x) = 0$ for $|x| \le \rho$. Thus this kernel is in $L_1(\mathbb{R}^n)$ and hence we may estimate the commutator $[T_3, \phi_j]$ in the same way as $[T_2, \phi_j]$.

For the remaining commutator $[T_4, \phi_j]$ note that

$$\partial^\alpha [T_4, \phi_j] = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} [T_4, \partial^\beta \phi_j] \partial^{\alpha - \beta}.$$

This shows that it is enough to estimate the commutator $[T_4, \phi_j]$ in $L_p(\mathbb{R}^n)$, as it then extends to $H_p^m(\mathbb{R}^n)$ and by interpolation to $W_p^s(\mathbb{R}^n)$, and then canonically to ${}_0\mathbb{F}_4$. Next we observe that for $x, y \in \mathbb{R}^n$

$$\partial^\alpha \phi_j(y) - \partial^\alpha \phi_j(x) = \partial^\alpha \phi_j'(x)(y - x) + r_{j,\alpha}(x, y),$$

where $|r_{j,\alpha}(x, y)| \le C|x - y|^2$, with some constant C independent of j and $|\alpha| \le 2$. Therefore

$$\begin{aligned}
[T_4, \phi_j]h(x) &= \int_0^1 \int_{\mathbb{R}^n} (\phi_j(y) - \phi_j(x)) k_{4,t}(x - y) h(y)\, dy\, t^{-\frac{1}{2}}\, dt \\
&= -\phi_j'(x) \int_{\mathbb{R}^n} \int_0^1 (y - x) k_{4,t}(x - y) t^{-\frac{1}{2}}\, dt\, h(y)\, dy + \\
&\qquad + \int_{\mathbb{R}^n} \int_0^1 r_j(x, y) k_{4,t}(x - y) t^{-\frac{1}{2}}\, dt\, h(y)\, dy \\
&=: T_{5,j} h(x) + T_{6,j} h(x).
\end{aligned}$$

We observe that the support of the kernel $k_{4,t}$ is contained in $B_{2\rho}(0)$, and consequently we may replace h by $\psi_j h$, where ψ_j is a cut-off function which equals 1 on $\mathrm{supp}(\phi_j) + B_{2\rho}(0)$. In the following we fix a smooth cut-off function ψ which equals 1 on $\mathrm{supp}(\phi) + B_{2\rho}(0)$ and then set $\psi_j := \tau_{x_j} \psi$. We then have

$$\|(T_{l,j} h)\|_{\ell_p(L_p)} = \|(T_{l,j} \psi_j h)\|_{\ell_p(L_p)} \le \sup_k \|T_{l,k}\|_{\mathcal{L}(L_p)} \|(\psi_j h)\|_{\ell_p(L_p)} \le C\|h\|_{L_p},$$

provided we can show that the operators $T_{l,k}$ are L_p-bounded with bound independent of $k \in \mathbb{N}$ for $l = 5, 6$.

The operators $T_{l,j}$ satisfy

$$\begin{aligned}
T_{5,j} h &= \phi_j'(q * h) &&\text{with}\quad q(x) = \chi(x) \int_0^1 x \Delta k_t(x) t^{-\frac{1}{2}}\, dt,\ x \in \mathbb{R}^n \\
|T_{6,j} h| &\le r * |h| &&\text{with}\quad r(x) = C\chi(x) \int_0^1 |x|^2 |\Delta k_t(x)| t^{-\frac{1}{2}}\, dt,\ x \in \mathbb{R}^n.
\end{aligned}$$

The Fourier transform of q is given by $\widehat{q}(\xi) = C\widehat{\chi} * \int_0^1 \nabla_\xi (|\xi|^2 e^{-t|\xi|^2}) t^{-1/2} dt$ and we verify that

$$\sup_{\alpha \le (1,\ldots,1)} \sup_{\xi \in \mathbb{R}^n} |\xi|^{|\alpha|} |\partial^\alpha \widehat{q}(\xi)| \le M$$

for some $M < \infty$. It thus follows from Mikhlin's multiplier theorem that

$$\|T_{5,j} h\|_{L_p} \le C\|\phi_j'\|_\infty \|h\|_{L_p} \le C\|h\|_{L_p}.$$

Finally, in order to estimate $T_{6,j}$ we infer from (3.21) that

$$r(x) \leq C \int_0^1 |x|^2 e^{-c|x|^2/t}\, t^{-\frac{n+3}{2}}\, dt \leq C e^{-c_1|x|^2} |x|^{-(n-1)} \int_1^\infty e^{-c_2 s} s^{(n-1)/2} ds$$

for $x \in \mathbb{R}^n$. It follows that $r \in L_1(\mathbb{R}^n)$ which implies by Young's inequality $\|T_{6,j}h\|_p \leq C\|h\|_p$ with a uniform constant C. □

Remarks 3.2. (a) We mention that the proof for the estimate of $[D_n^{1/2}, \phi_j]$ follows the ideas of [15, Lemma 6.4].

(b) If $\rho_2 \leq \rho_1$, i.e., the light fluid lies above the heavy one, then the estimate (3.8) can be improved in the following sense: for every $\beta > 0$ and $\lambda_0 > 0$ there are positive constants δ, $\eta = \eta(\beta)$ and $c_j = c_j(\beta, \lambda_0, \delta, \eta)$ such that

$$c_0\big[|\lambda| + |\tau|\big] \leq \tilde{s}(\lambda, \tau, \zeta) \leq c_1\big[|\lambda| + |\tau|\big] \tag{3.22}$$

for all $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$ and $|\lambda| \geq \lambda_0$. For this we observe that estimates (3.9) and (3.10) certainly also hold in case that $\rho_2 \leq \rho_1$. On the other hand, given $M > 0$ we conclude as in (3.11) that $L \leq \mathrm{Re}\,((\rho_1 - \rho_2)\gamma_a k(z)) \leq R$ and $|\mathrm{Im}\,((\rho_1 - \rho_2)\gamma_a k(z))| \leq H$ for $|z| \leq M$, with appropriate positive constants L, R, H. This shows that there exists $\alpha = \alpha(M, \eta) \in (0, \pi/2)$ such that

$$(\rho_1 - \rho_2)\gamma_a k(z)/\tau \in \Sigma_\alpha, \quad (\lambda, \tau) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta, \quad |z| \leq M \tag{3.23}$$

with $\eta \in (0, \eta_0)$ chosen small enough, where we can assume that α coincides with the angle in (3.12). Combining (3.12) and (3.23) yields

$$\begin{aligned}
|\tilde{s}(\lambda, \tau, \zeta)| &\geq c(\psi)\big[|\lambda| + |\tau(\sigma k(z) + i\zeta) + (\rho_1 - \rho_2)\gamma_a k(z)/\tau|\big] \\
&\geq c(\psi)c(\alpha)\big[|\lambda| + |\tau(\sigma k(z) + i\zeta)| + |(\rho_1 - \rho_2)\gamma_a k(z)/\tau|\big] \\
&\geq c_0(M, \beta, \delta, \eta)\big[|\lambda| + |\tau|\big]
\end{aligned}$$

provided $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$ and $|\lambda| \leq M|\tau|^2$. Noting again that the curves $|\lambda| = m|\tau|$ and $|\lambda| = M|\tau|^2$ intersect at $(m/M, m^2/M)$ we obtain (3.22) by choosing M big enough.

(c) If $\rho_2 \leq \rho_1$ we can conclude from the lower estimate in (3.22) that the function $\tilde{s}$ does not have zeros in $\overline{\Sigma}_{\pi/2} \times \mathbb{R}_+ \times [-\beta, \beta]$. This holds in particular true for the symbol $s(\lambda, \tau) := \tilde{s}(\lambda, \tau, 0)$, indicating that there are no instabilities in case that the light fluid lies on top of the heavy one.

(d) If $\rho_2 > \rho_1$ then it is shown in [29] that the symbol s has for each $\tau \in (0, \tau_*)$ with $\tau_* := ((\rho_2 - \rho_1)\gamma_a/\sigma)^{1/2}$ a zero $\lambda = \lambda(\tau) > 0$, pertinent to the Rayleigh-Taylor instability.

(e) Further mapping properties of the boundary symbol $s(\lambda, \tau) := \tilde{s}(\lambda, \tau, 0)$ and the associated operator S in case that $\gamma_a = 0$ have been derived in [27]. In particular, we have investigated the singularities and zeros of s, and we have studied the mapping properties of S in case of low and high frequencies, respectively.

4. The nonlinear problem

In this section we prove existence and uniqueness of solutions for the nonlinear problem (2.1), and we show additionally that solutions immediately regularize and are real analytic in space and time. In order to facilitate this task, we first introduce some notation. We set

$$\begin{aligned}
\mathbb{E}_1(a) &:= \{u \in H^1_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})) : [\![u]\!] = 0\},\\
\mathbb{E}_2(a) &:= L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^{n+1})),\\
\mathbb{E}_3(a) &:= W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n)),\\
\mathbb{E}_4(a) &:= W^{2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap H^1_p(J; W^{2-1/p}_p(\mathbb{R}^n))\\
&\qquad \cap W^{1/2-1/2p}_p(J; H^2_p(\mathbb{R}^n)) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n)),\\
\mathbb{E}(a) &:= \{(u, \pi, q, h) \in \mathbb{E}_1(a) \times \mathbb{E}_2(a) \times \mathbb{E}_3(a) \times \mathbb{E}_4(a) : [\![\pi]\!] = q\}.
\end{aligned} \tag{4.1}$$

The space $\mathbb{E}(a)$ is given the natural norm

$$\|(u, \pi, q, h)\|_{\mathbb{E}(a)} = \|u\|_{\mathbb{E}_1(a)} + \|\pi\|_{\mathbb{E}_2(a)} + \|q\|_{\mathbb{E}_3(a)} + \|h\|_{\mathbb{E}_4(a)}$$

which turns it into a Banach space. Moreover, we set

$$\begin{aligned}
\mathbb{F}_1(a) &:= L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})),\\
\mathbb{F}_2(a) &:= H^1_p(J; H^{-1}_p(\mathbb{R}^{n+1})) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1})),\\
\mathbb{F}_3(a) &:= W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n, \mathbb{R}^{n+1})),\\
\mathbb{F}_4(a) &:= W^{1-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n)),\\
\mathbb{F}(a) &:= \mathbb{F}_1(a) \times \mathbb{F}_2(a) \times \mathbb{F}_3(a) \times \mathbb{F}_4(a).
\end{aligned} \tag{4.2}$$

The generic elements of $\mathbb{F}(a)$ are the functions (f, f_d, g, g_h).

Let $b \in \mathbb{F}_4(a)^n$ be a given function. Then we define the nonlinear mapping

$$N_b(u, \pi, q, h) := \big(F(u, \pi, h), F_d(u, h), G(u, q, h), (b - \gamma v|\nabla h)\big) \tag{4.3}$$

for $(u, \pi, q, h) \in \mathbb{E}(a)$, where, as before, $u = (v, w)$, $F = (F_v, F_w)$ and $G = (G_v, G_w)$. We will now study the mapping properties of N_b and we will derive estimates for the Fréchet derivative of N_b. In the following, the notion C^ω means real analytic.

Proposition 4.1. *Suppose $p > n + 3$ and $b \in \mathbb{F}_4(a)^n$. Then*

$$N_b \in C^\omega(\mathbb{E}(a), \mathbb{F}(a)), \quad a > 0. \tag{4.4}$$

Let $DN_b(u, \pi, q, h)$ denote the Fréchet derivative of N_b at $(u, \pi, q, h) \in \mathbb{E}(a)$. Then $DN_b(u, \pi, q, h) \in \mathcal{L}({}_0\mathbb{E}(a), {}_0\mathbb{F}(a))$, and for any number $a_0 > 0$ there is a positive *constant* $M_0 = M_0(a_0, p)$ *such that*

$$\begin{aligned}
&\|DN_b(u,\pi,q,h)\|_{\mathcal{L}({}_0\mathbb{E}(a),\,{}_0\mathbb{F}(a))}\\
&\le M_0\big[\|b-\gamma v\|_{BC(J;BC)\cap\mathbb{F}_4(a)} + \|(u,\pi,q,h)\|_{\mathbb{E}(a)}\big]\\
&+ M_0\big[\big(\|\nabla h\|_{BC(J;BC^1)} + \|h\|_{\mathbb{E}_4(a)} + \|u\|_{BC(J;BC)}\big)\|u\|_{\mathbb{E}_1(a)}\big]\\
&+ M_0\big[P(\|\nabla h\|_{BC(J;BC)})\|\nabla h\|_{BC(J;BC)} + Q\big(\|\nabla h\|_{BC(J;BC^1)}, \|h\|_{\mathbb{E}_4(a)}\big)\|h\|_{\mathbb{E}_4(a)}\big]
\end{aligned}$$

for all $(u,\pi,q,h) \in \mathbb{E}(a)$ *and all* $a \in (0, a_0]$*. Here,* P *and* Q *are fixed polynomials with coefficients equal to one.*

Proof. The proof of the proposition is relegated to the end of the appendix. □

Given $h_0 \in W_p^{3-2/p}(\mathbb{R}^n)$ we define

$$\Theta_{h_0}(x,y) := (x, y + h_0(x)), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}. \tag{4.5}$$

Letting $\Omega_{h_0,i} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : (-1)^i(y - (h_0(x)) > 0\}$ and $\Omega_{h_0} := \Omega_{h_0,1} \cup \Omega_{h_0,2}$ we obtain from Sobolev's embedding theorem that

$$\Theta_{h_0} \in \mathrm{Diff}^2(\dot{\mathbb{R}}^{n+1}, \Omega_{h_0}) \cap \mathrm{Diff}^2(\mathbb{R}^{n+1}_-, \Omega_{h_0,1}) \cap \mathrm{Diff}^2(\mathbb{R}^{n+1}_+, \Omega_{h_0,2}),$$

i.e., Θ_{h_0} yields a C^2-diffeomorphism between the indicated domains. The inverse transformation obviously is given by $\Theta_{h_0}^{-1}(x,y) = (x, y - h_0(x))$. It then follows from the chain rule and the transformation rule for integrals that

$$\Theta_{h_0}^* \in \mathrm{Isom}(H_p^k(\dot{\mathbb{R}}^{n+1}), H_p^k(\Omega_{h_0})), \quad [\Theta_{h_0}^*]^{-1} = \Theta_*^{h_0}, \quad k = 0,1,2,$$

where we use the notation

$$\begin{aligned}
\Theta_{h_0}^* u &:= u \circ \Theta_{h_0}, \quad u : \Omega_{h_0} \to \mathbb{R}^m,\\
\Theta_*^{h_0} v &:= v \circ \Theta_{h_0}^{-1}, \quad v : \dot{\mathbb{R}}^{n+1} \to \mathbb{R}^m.
\end{aligned}$$

We are now ready to prove our main result of this section.

Theorem 4.2. (*Existence of solutions for the nonlinear problem* (2.1)).

(a) *For every* $\beta > 0$ *there exists a constant* $\eta = \eta(\beta) > 0$ *such that for all initial values*

$$(u_0, h_0) \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n) \quad \textit{with} \quad [\![u_0]\!] = 0,$$

satisfying the compatibility conditions

$$[\![\mu D(\Theta_*^{h_0}u_0)\nu_0 - \mu(\nu_0|D(\Theta_*^{h_0}u_0)\nu_0)\nu_0]\!] = 0, \quad div(\Theta_*^{h_0}u_0) = 0, \tag{4.6}$$

and the smallness-boundedness condition

$$\|\nabla h_0\|_\infty \le \eta, \quad \|u_0\|_\infty \le \beta, \tag{4.7}$$

there is a number $t_0 = t_0(u_0, h_0)$ *such that the nonlinear problem* (2.1) *admits a unique solution* $(u, \pi, [\![\pi]\!], h) \in \mathbb{E}_1(t_0)$.

(b) *The solution has the additional regularity properties*

$$(u,\pi) \in C^\omega((0,t_0) \times \dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+2}), \quad [\![\pi]\!], h \in C^\omega((0,t_0) \times \mathbb{R}^n). \tag{4.8}$$

In particular, $\mathcal{M} = \bigcup_{t\in(0,t_0)} \big(\{t\} \times \Gamma(t)\big)$ *is a real analytic manifold.*

Proof. The proof of this result proceeds in a similar way as the proof of Theorem 6.3 in [28].

For a given function $b \in \mathbb{F}_4(a)^n$ we consider the nonlinear problem

$$\left\{\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= F(u,\pi,h) && \text{in} \quad \dot{\mathbb{R}}^{n+1} \\ \operatorname{div} u &= F_d(u,h) && \text{in} \quad \dot{\mathbb{R}}^{n+1} \\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] &= G_v(u,[\![\pi]\!],h) && \text{on} \quad \mathbb{R}^n \\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] - \sigma\Delta h &= G_w(u,h) && \text{on} \quad \mathbb{R}^n \\ [\![u]\!] &= 0 && \text{on} \quad \mathbb{R}^n \\ \partial_t h - \gamma w + (b|\nabla h) &= (b-\gamma v|\nabla h) && \text{on} \quad \mathbb{R}^n \\ u(0) = u_0,\ h(0) &= h_0, \end{aligned}\right. \tag{4.9}$$

which clearly is equivalent to (2.1).

In order to economize our notation we set $z := (u,\pi,q,h)$ for $(u,\pi,q,h) \in \mathbb{E}(a)$. With this notation, the nonlinear problem (2.1) can be restated as

$$L_b z = N_b(z), \quad (u(0),h(0)) = (u_0,h_0), \tag{4.10}$$

where L_b denotes the linear operator on the left-hand side of (4.9), and N_b correspondingly denotes the nonlinear mapping on the right-hand site of (4.9).

It is convenient to first introduce an auxiliary function $z^* = z_b^* \in \mathbb{E}(a)$ which resolves the compatibility conditions and the initial conditions in (4.10), and then to solve the resulting reduced problem

$$L_b z = N_b(z+z^*) - L_b z^* =: K_b(z), \quad z \in {}_0\mathbb{E}(a), \tag{4.11}$$

by means of a fixed point argument.

(i) Suppose that (u_0,h_0) satisfies the (first) compatibility condition in (4.6), and let

$$[\![\pi_0]\!] := \theta_{h_0}^*\{[\![\mu(\nu_0|D(\Theta_*^{h_0}u_0)\nu_0)]\!] + \sigma\kappa\},$$

where $\theta_{h_0} := \Theta_{h_0}|_{\mathbb{R}^{n+1}\times\{0\}}$. Here we observe that $\theta_{h_0}^*[\![\omega]\!] = [\![\Theta_{h_0}^*\omega]\!]$ for any function $\omega : \Omega_{h_0} \to \mathbb{R}^m$ which has one-sided limits. It is then clear from the definition in (2.3)–(2.4) that the following compatibility conditions hold:

$$\begin{aligned} -[\![\mu\partial_y v_0]\!] - [\![\mu\nabla_x w_0]\!] &= G_v(u_0,[\![\pi_0]\!],h_0) && \text{on} \quad \mathbb{R}^n \\ -2[\![\mu\partial_y w_0]\!] + [\![\pi_0]\!] - \sigma\Delta h_0 &= G_w(u_0,h_0) && \text{on} \quad \mathbb{R}^n \end{aligned} \tag{4.12}$$

where, as before, $u_0 = (v_0,w_0)$. Next we introduce special functions $(0,f_d^*,g^*,g_h^*) \in \mathbb{F}(a)$ which resolve the necessary compatibility conditions. First we set

$$c^*(t) := \left\{\begin{aligned} &\mathcal{R}_+ e^{-tD_{n+1}}\mathcal{E}_+(v_0|\nabla h_0) && \text{in} \quad \mathbb{R}^{n+1}_+, \\ &\mathcal{R}_- e^{-tD_{n+1}}\mathcal{E}_-(v_0|\nabla h_0) && \text{in} \quad \mathbb{R}^{n+1}_-, \end{aligned}\right. \tag{4.13}$$

where $\mathcal{E}_\pm \in \mathcal{L}(W_p^{2-2/p}(\mathbb{R}^{n+1}_\pm), W_p^{2-2/p}(\mathbb{R}^{n+1}))$ is an appropriate extension operator and $\mathcal{R}_\pm$ is the restriction operator. Due to $(v_0|\nabla h_0) \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1})$ we obtain

$$c^* \in H^1_p(J; L_p(\mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1})).$$

Consequently,

$$f^*_d := \partial_y c^* \in \mathbb{F}_2(a) \ \text{ and } \ f^*_d(0) = F_d(v_0, h_0). \tag{4.14}$$

Next we set

$$g^*(t) := e^{-D_n t} G(u_0, [\![\pi_0]\!], h_0). \quad g^*_h(t) := e^{-D_n t}(b(0) - \gamma v_0 | \nabla h_0). \tag{4.15}$$

It then follows from (4.14) and [19, Lemma 8.2] that $(0, f^*_d, g^*, g^*_h) \in \mathbb{F}(a)$. (4.12) and the second condition in (4.6) show that the necessary compatibility conditions of Theorem 3.1 are satisfied and we can conclude that the linear problem

$$L_b z^* = (0, f^*_d, g^*, g^*_h), \quad (u^*(0), h^*(0)) = (u_0, h_0), \tag{4.16}$$

has a unique solution $z^* = z^*_b \in \mathbb{E}(a)$. With the auxiliary function z^* now determined, we can focus on the reduced equation (4.11), which can be converted into the fixed point equation

$$z = L_b^{-1} K_b(z), \quad z \in {}_0\mathbb{E}(a). \tag{4.17}$$

Due to the choice of (f^*_d, g^*, g^*_h) we have $K_b(z) \in {}_0\mathbb{F}(a)$ for any $z \in {}_0\mathbb{E}(a)$, and it follows from Proposition 4.1 that

$$K_b \in C^\omega({}_0\mathbb{E}(a), {}_0\mathbb{F}(a)).$$

Consequently, $L_b^{-1} K_b : {}_0\mathbb{E}(a) \to {}_0\mathbb{E}(a)$ is well defined and smooth.

(ii) An inspection of the proof of Theorem 3.1 shows that given $\beta > 0$ we can find a positive number $\delta_0 = \delta_0(b)$ such that

$$L_b^{-1} \in \mathcal{L}({}_0\mathbb{F}(a), {}_0\mathbb{E}(a)), \quad \|L_b^{-1}\|_{\mathcal{L}({}_0\mathbb{F}(a), {}_0\mathbb{E}(a))} \le M, \quad a \in [0, \delta_0], \tag{4.18}$$

whenever $b \in \mathbb{F}_4(a)^n$ and $\|b\|_{BC[0,a], BC(\mathbb{R}^n))} \le \beta$. It should be pointed out that the bound M is universal for all functions $b \in \mathbb{F}_4(a)^n$ with $\|b\|_\infty \le \beta$, whereas the number $\delta_0 = \delta(b)$ may depend on b.

(iii) We will now fix a pair of initial values $(u_0, h_0) \in W^{2-2/p}_p(\dot{\mathbb{R}}^{n+1}) \times W^{3-2/p}_p(\mathbb{R}^n)$ satisfying (4.6) and (4.7) with

$$\eta := 1/(16 M_0 M), \tag{4.19}$$

where the constants M_0 and M are given in (4.1) and (4.18), respectively. We choose

$$b(t) := e^{-D_n t} \gamma v_0, \quad t \ge 0. \tag{4.20}$$

Then $b \in \mathbb{F}_4(a)^n$ and $\|b\|_{BC([0,a]; BC(\mathbb{R}^n))} \le \|\gamma v_0\|_{BC(\mathbb{R}^n)} \le \beta$ for any $a > 0$, as $\{e^{-D_n t} : t \ge 0\}$ is a contraction semigroup on $BUC(\mathbb{R}^n)$. Hence the estimate (4.18) holds true for this (and any other choice) of initial values. It should be pointed out once more that the bound M is universal for all initial values u_0 with $\|v_0\|_\infty \le \beta$ – and hence for $b(t) := e^{-D_n t} \gamma v_0$ – whereas the number δ_0 may depend on γv_0.

We note in passing that $g_h^* = 0$ for this particular choice of the function b. Without loss of generality we can assume that $M_0, M \geq 1$. We shall show that $L_b^{-1}K_b$ is a contraction on a properly defined subset of ${}_0\mathbb{E}(a)$ for $a \in (0, \delta_0]$ chosen sufficiently small. For $r > 0$ and $a \in (0, \delta_0]$ we set

$$ {}_0\mathbb{B}_{\mathbb{E}(a)}(z^*, r) := \{z \in \mathbb{E}_1(a) : z - z^* \in {}_0\mathbb{E}(a),\ \|z - z^*\|_{\mathbb{E}(a)} < r\}. $$

We remark that a and r are independent parameters that can be chosen as we please. Let then $r_0 > 0$ be fixed. It is not difficult to see that there exists a number $R_0 = R_0(u_0, h_0, \delta_0, r_0)$ such that

$$ \begin{aligned} &\|\nabla(h + h^*)\|_{BC(J;BC^1)} + \|h + h^*\|_{\mathbb{E}_4(a)} + \|u + u^*\|_{BC(J;BC)} \\ &\qquad\qquad + Q\big(\|\nabla(h + h^*)\|_{BC(J;BC^1)}, \|h + h^*\|_{\mathbb{E}_4(a)}\big) \leq R_0 \end{aligned} $$

for all $u \in {}_0\mathbb{B}_{\mathbb{E}_1(a)}(0, r)$ and $h \in {}_0\mathbb{B}_{\mathbb{E}_4(a)}(0, r)$, with $a \in (0, \delta_0]$ and $r \in (0, r_0]$ arbitrary, where $z^* = (u^*, \pi^*, q^*, h^*)$ is the solution of equation (4.16) and where Q is defined in Proposition 4.1. Let $M_1 := M_0(1 + R_0)$. It then follows from Proposition 4.1 and (4.18) that

$$ \begin{aligned} &\|D(L_b^{-1}K_b)(z)\|_{{}_0\mathbb{E}(a)} \\ &\quad \leq M_1 M\big[\|b - \gamma(v + v^*)\|_{BC(J;BC) \cap \mathbb{F}_4(a)} + \|z + z^*\|_{\mathbb{E}(a)}\big] \\ &\qquad + M_0 M\big[P\big(\|\nabla(h + h^*)\|_{BC(J;BC)}\big)\|\nabla(h + h^*)\|_{BC(J;BC)}\big] \end{aligned} \tag{4.21} $$

for all $z \in {}_0\mathbb{B}_{\mathbb{E}(a)}(0, r)$ and $a \in (0, \delta_0]$.

(iv) For (u_0, h_0) fixed, the norm of z^* in $\mathbb{E}(a)$ (which involves various integral expressions evaluated over the interval $(0, a)$) can be made as small as we like by choosing $a \in (0, \delta_0]$ small. Let then $a_1 \in (0, \delta_0]$ be fixed so that

$$ \begin{aligned} &\|\nabla h^*\|_{BC([0,a_1],BC)} \leq 2\eta, \\ &M_1 M\big(\|b - \gamma v^*\|_{BC([0,a_1];BC) \cap \mathbb{F}_4(a_1)} + \|z^*\|_{\mathbb{E}(a_1)}\big) \leq 1/8. \end{aligned} \tag{4.22} $$

Since $(\nabla h^*, b - \gamma v^*) \in {}_0BC([0, \delta_0], BC(\mathbb{R}^n))$ and $\|\nabla h^*(0)\|_\infty = \|\nabla h_0\|_\infty \leq \eta$, the estimates in (4.22) certainly hold for a_1 sufficiently small.
In a next step we choose $2r_1 \in (0, r_0]$ so that

$$ \begin{aligned} &\|\nabla h\|_{{}_0BC([0,a_1],BC(\mathbb{R}^n))} \leq \eta, \\ &M_1 M\big(\|\gamma v\|_{{}_0BC([0,a_1];BC) \cap {}_0\mathbb{F}_4(a_1)} + \|z\|_{{}_0\mathbb{E}(a_1)}\big) \leq 1/8, \end{aligned} \tag{4.23} $$

for all $h \in {}_0\mathbb{B}_{\mathbb{E}(a_1)}(0, 2r_1)$, $v \in {}_0\mathbb{B}_{\mathbb{E}(a_1)}(0, 2r_1)$, and $z \in {}_0\mathbb{B}_{\mathbb{E}(a_1)}(0, 2r_1)$. It follows from Proposition 5.1 that (4.23) can indeed be achieved. Combining (4.19)–(4.23) gives

$$ \|D(L_b^{-1}K_b)(z)\|_{{}_0\mathbb{E}(a)} \leq 1/2, \quad z \in {}_0\mathbb{B}_{\mathbb{E}(a_1)}(0, 2r_1) \tag{4.24} $$

showing that $L_b^{-1}K_b : {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1) \to {}_0\mathbb{E}(a_1)$ is a contraction, where ${}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1)$ denotes the closed ball in ${}_0\mathbb{E}(a_1)$ with center at 0 and radius r_1.

It remains so show that $L_b^{-1}K_b$ maps ${}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1)$ into itself. From (4.24) and the mean value theorem follows

$$\begin{aligned}\|L_b^{-1}K_b(z)\|_{{}_0\mathbb{E}(a_1)} &\le \|L_b^{-1}K_b(z) - L_b^{-1}K_b(0)\|_{{}_0\mathbb{E}(a_1)} + \|L_b^{-1}K_b(0)\|_{{}_0\mathbb{E}(a_1)}\\ &\le r_1/2 + \|L_b^{-1}K_b(0)\|_{{}_0\mathbb{E}(a_1)}, \quad z \in {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1).\end{aligned}$$

Here we observe that the norm of $L_b^{-1}K_b(0) = L_b^{-1}(K(z^*) - (0, f_d^*, g^*, g_h^*))$ in ${}_0\mathbb{E}(a_1)$ can be made as small as we wish by choosing a_1 small enough. We may assume that a_1 was already chosen so that $\|L_b^{-1}K_b(0)\|_{{}_0\mathbb{E}(a_1)} \le r_1/2$.

(v) We have shown in (iv) that the mapping

$$L_b^{-1}K_b : {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1) \to {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1)$$

is a contraction. By the contraction mapping theorem $L_b^{-1}K_b$ has a unique fixed point $\hat{z} \in {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1) \subset {}_0\mathbb{E}(a_1)$ and it follows immediately from (4.10)–(4.11) that $\hat{z}+z^*$ is the (unique) solution of the nonlinear problem (2.1) in ${}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(z^*, r_1)$. Setting $t_0 = a_1$ gives the assertion in part (a) of the Theorem.

(vi) The proof that (u, π, q, h) is analytic in space and time proceeds exactly in the same way as in steps (vi)–(vii) of the proof of Theorem 6.3 in [28], with the only difference that here $g_h^* = g_{h,\lambda,\nu}^* = 0$, and that the operator D_ν in formula (6.30) of [28] is to be replaced by $D_{\lambda,\nu}$, defined by

$$\mathcal{D}_{\lambda,\nu}h := (\lambda b_{\lambda,\nu} - \nu|\nabla h), \qquad b_{\lambda,\nu}(t,x) := b(\lambda t, x + t\nu). \tag{4.25}$$

We note that $D_{1,0} = (b|\nabla\cdot)$. In the same way as in [25, Lemma 8.2] one obtains that

$$[(\lambda,\nu) \mapsto b_{\lambda,\nu}] : (1-\delta, 1+\delta) \times \mathbb{R}^n \to \mathbb{F}_4(a) \tag{4.26}$$

is real analytic. The remaining arguments are now the same as in [28], and this completes the proof of Theorem 4.2. □

Proof of Theorem 1.1. Clearly, the compatibility conditions of Theorem 1.1 are satisfied if and only if (4.6) is satisfied. Moreover, the smallness-boundedness condition of Theorem 1.1 is equivalent to (4.7), where we have slightly abused notation by using the same symbol for u_0 and its transformed version $\Theta_{h_0}^* u_0$.

Theorem 4.2 yields a unique solution $(v, w, \pi, [\pi], h) \in \mathbb{E}(t_0)$ which satisfies the additional regularity properties listed in part (b) of the theorem. Setting

$$(u, q)(t, x, y) = (v, w, \pi)(t, x, y - h(t, x)), \quad (t, x, y) \in \mathcal{O},$$

we then conclude that $(u, q) \in C^\omega(\mathcal{O}, \mathbb{R}^{n+2})$ and $[q] \in C^\omega(\mathcal{M})$. The regularity properties listed in Remark 1.2(a) are implied by Proposition 5.1(a),(c). Finally, since $\pi(t, x, y)$ is defined for every $(t, x, y) \in \mathcal{O}$, we can conclude that

$$q(t, \cdot) \in \dot{H}_p^1(\Omega(t)) \subset UC(\Omega(t))$$

for every $t \in (0, t_0)$. □

5. Appendix

In this section we state and prove some technical results that were used above.

Proposition 5.1. *Suppose $p > n+3$. Then the following embeddings hold:*

(a) $\mathbb{E}_1(a) \hookrightarrow BC(J; W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})) \hookrightarrow BC(J; BC^1(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}))$ *and there is a constant $C_0 = C_0(p)$ such that*
$$\|u\|_{{}_0BC(J;W_p^{2-2/p})} + \|u\|_{{}_0BC(J;BC^1)} \le C_0\|u\|_{{}_0\mathbb{E}_1(a)}$$
for all $u \in {}_0\mathbb{E}_1(a)$ and all $a \in (0,\infty)$.

(b) $\mathbb{E}_3(a) \hookrightarrow BC(J; BC(\mathbb{R}^n))$ *and there exists a constant $C_0 = C_0(p)$ such that*
$$\|g\|_{{}_0BC(J;BC)} \le C_0\|g\|_{{}_0\mathbb{E}_3(a)}$$
for all $g \in {}_0\mathbb{E}_3(a)$ and all $a \in (0,\infty)$.

(c) $\mathbb{F}_4(a) \hookrightarrow BC(J; W_p^1(\mathbb{R}^n)) \cap BC(J; BC^1(\mathbb{R}^n))$ *and there exists a constant $C_0 = C_0(p)$ such that*
$$\|g\|_{{}_0BC(J;W_p^1)} + \|g\|_{{}_0BC(J;BC^1)} \le C_0\|g\|_{{}_0\mathbb{F}_4(a)}$$
for all $g \in {}_0\mathbb{F}_4(a)$ and all $a \in (0,\infty)$.

(d) $\mathbb{E}_4(a) \hookrightarrow BC^1(J; BC^1(\mathbb{R}^n)) \cap BC(J; BC^2(\mathbb{R}^n))$ *and there exists a constant $C_0 = C_0(p)$ such that*
$$\|h\|_{{}_0BC^1(J;BC^1)} + \|h\|_{{}_0BC(J;BC^2)} \le C_0\|h\|_{{}_0\mathbb{E}_4(a)}$$
for all $h \in {}_0\mathbb{E}_4(a)$ and all $a \in (0,\infty)$.

(e) $\partial_j \in \mathcal{L}(\mathbb{E}_4(a), \mathbb{E}_3(a)) \cap \mathcal{L}(\mathbb{E}_4(a), \mathbb{F}_4(a))$ *for $j = 1, \ldots, n$. Moreover, for every given $a_0 > 0$ there is a constant $C_0 = C_0(a_0, p)$ such that*
$$\|\partial_j h\|_{\mathbb{E}_3(a)} + \|\partial_j h\|_{\mathbb{F}_4(a)} \le C_0\|h\|_{\mathbb{E}_4(a)}$$
for all $h \in \mathbb{E}_4(a)$ and all $a \in (0, a_0]$.

Proof. We refer to [25, Proposition 6.2] for a proof of (a)–(b). The assertion in (c) can established in the same way, using that $\mathbb{F}_4(a) \hookrightarrow BC(J; W_p^{2-3/p}(\mathbb{R}^n))$, see [19, Remark 5.3(d)]. In order to show that the embedding constant in (d) does not depend on $a \in (0, a_0]$ we define an extension operator in the following way: for $h \in {}_0BC^1([0,a]; X)$, with X an arbitrary Banach space, we first set $\tilde{h}(t) := 0$ for $t \le 0$, so that $\tilde{h} \in BC^1((-\infty, a]; X)$, and then define
$$(\mathcal{E}h)(t) := \begin{cases} h(t) & \text{if} \quad 0 \le t \le a, \\ 3\tilde{h}(2a-t) - 2\tilde{h}(3a-2t) & \text{if} \quad a \le t. \end{cases} \tag{5.1}$$
A moment of reflection shows that $\mathcal{E}h \in {}_0BC^1([0,\infty); X)$, and that $\mathcal{E}h$ is an extension of h. It is evident that the norm of $\mathcal{E} : {}_0BC^1([0,a]; X) \to {}_0BC^1([0,\infty); X)$ is independent of $a \in [0, a_0]$. The assertion follows now by the same arguments as in the proof of [25, Proposition 6.2].

Let $a_0 > 0$ be fixed. In order to establish part (e) it suffices to show that there is a constant $C_0 = C(a_0, p, r)$ such that

$$\|g\|_{W_p^r([0,a];X)} \leq C_0 \|g\|_{H_p^1([0,a];X)}, \quad a \in (0, a], \tag{5.2}$$

where X is an arbitrary Banach space and $r \in [0, 1]$. This follows from Hardy's inequality as follows: for $r \in (0, 1)$ fixed we have

$$\begin{aligned}
\frac{1}{2}\langle g\rangle^p_{W_p^r([0,a];X)} &= \int_0^a \int_s^a \frac{\|g(t)-g(s)\|_X^p}{(t-s)^{1+rp}}\, dt\, ds \\
&= \int_0^a \int_0^{a-s} \frac{\|g(s+\tau)-g(s)\|_X^p}{\tau^{1+rp}}\, d\tau\, ds \\
&\leq \int_0^a \int_0^{a-s} \frac{1}{\tau^{1+rp}} \left(\int_0^\tau \|\partial g(s+\sigma)\|_X\, d\sigma\right)^p d\tau\, ds \\
&\leq c(r,p) \int_0^a \int_0^{a-s} \frac{1}{\tau^{1-(1-r)p}} \|\partial g(s+\tau)\|_X^p\, d\tau\, ds \\
&= c(r,p) \int_0^a \frac{1}{\tau^{1-(1-r)p}} \int_0^{a-\tau} \|\partial g(s+\tau)\|_X^p\, ds\, d\tau \\
&\leq c(r,p) \int_0^a \frac{1}{\tau^{1-(1-r)p}}\, d\tau\, \|\partial g\|^p_{L_p([0,a];X)}
\end{aligned}$$

where ∂g is the derivative of g, and this readily yields (5.2). □

Our next result will be important in order to derive estimates for the nonlinearities in (2.1).

Lemma 5.2. *Suppose $p > n + 3$. Let $a_0 \in (0, \infty)$ be given. Then*

(a) $\mathbb{E}_3(a)$ *is a multiplication algebra and we have the following estimate*

$$\|g_1 g_2\|_{\mathbb{E}_3(a)} \leq (\|g_1\|_\infty + \|g_1\|_{\mathbb{E}_3(a)})(\|g_2\|_\infty + \|g_2\|_{\mathbb{E}_3(a)}) \tag{5.3}$$

for all $(g_1, g_2) \in \mathbb{E}_3(a) \times \mathbb{E}_3(a)$ and all $a > 0$.

(b) *There exists a constant $C_0 = C_0(a_0, p)$ such that*

$$\|g_1 g_2\|_{{}_0\mathbb{E}_3(a)} \leq C_0(\|g_1\|_\infty + \|g_1\|_{\mathbb{E}_3(a)})\|g_2\|_{{}_0\mathbb{E}_3(a)} \tag{5.4}$$

for all $(g_1, g_2) \in \mathbb{E}_3(a) \times {}_0\mathbb{E}_3(a)$ and all $a \in (0, a_0]$.

(c) *There exists a constant $C_0 = C_0(a_0, p)$ such that*

$$\|g\partial_j h\|_{{}_0\mathbb{E}_3(a)} \leq C_0 \|g\|_{\mathbb{E}_3(a)} \|h\|_{{}_0\mathbb{E}_4(a)}, \quad j = 1, \ldots, n, \tag{5.5}$$

for all $(g, h) \in \mathbb{E}_3(a) \times {}_0\mathbb{E}_4(a)$ and $a \in (0, a_0]$.

(d) *Suppose $(g, \psi) \in \mathbb{E}_3(a) \times \mathbb{E}_3(a)$ and let $\beta(t,x) := \sqrt{1 + \psi^2(t,x)}$. Then $\dfrac{g}{\beta^k} \in$ $\mathbb{E}_3(a)$ for $k \in \mathbb{N}$ and the following estimate holds*

$$\left\|\frac{g}{\beta^k}\right\|_{\mathbb{E}_3(a)} \leq (1 + \|\psi\|_{\mathbb{E}_3(a)})^k (\|g\|_\infty + \|g\|_{\mathbb{E}_3(a)}). \tag{5.6}$$

Proof. The assertions in (a)–(b) follow from (the proof of) Proposition 6.6.(ii) and (iv) in [25].

(c) To economize our notation we set $r = 1/2 - 1/2p$ and $\theta = 1 - 1/p$. Suppose that $(g, h) \in \mathbb{E}_3(a) \times {}_0\mathbb{E}_4(a)$. We first observe that

$$\begin{aligned}\|g\partial_j h\|_{W_p^r(J;L_p)} &\le \big(\|g\|_{L_p(J;L_p)} + \langle g\rangle_{W_p^r(J;L_p)}\big)\|\partial_j h\|_{0BC(J;L_\infty)} \\ &\quad + \left(\int_0^a\int_0^a \|g(s)(\partial_j h(t) - \partial_j h(s))\|_{L_p}^p \,\frac{dt\,ds}{|t-s|^{1+rp}}\right)^{1/p}.\end{aligned}$$

Using Hölder's inequality, and the fact that $(1 - r - 1/p) = r > 0$, we obtain the estimate

$$\begin{aligned}&\int_0^a\int_0^a \|g(s)(\partial_j h(t) - \partial_j h(s))\|_{L_p}^p \,\frac{dt\,ds}{|t-s|^{1+rp}} \\ &\le \int_0^a\int_0^a \|g(s)\|_{L_\infty}^p \left(\left|\int_s^t \|\partial_t\partial_j h(\tau)\|_{L_p} d\tau\right|\right)^p \frac{dt\,ds}{|t-s|^{1+rp}} \\ &\le \int_0^a \|g(s)\|_{L_\infty}^p \left(\int_0^a \frac{dt}{|t-s|^{1-(1-r-1/p)p}}\right) ds \int_0^a \|\partial_t\partial_j h(\tau)\|_{L_p}^p d\tau \\ &\le C_0(a_0, p)\|g\|_{L_p(J;L_\infty)}^p \|\partial_t\partial_j h\|_{L_p(J;L_p)}^p\end{aligned} \tag{5.7}$$

for $a \in (0, a_0]$. Hence we conclude that

$$\begin{aligned}\|g\partial_j h\|_{0W_p^r(J;L_p)} &\le C_0\|g\|_{W_p^r(J;L_p)}\big(\|\partial_j h\|_{0BC(J;L_\infty)} + \|\partial_t\partial_j h\|_{L_p(J;L_p)}\big) \\ &\le C_0\|g\|_{\mathbb{E}_3(a)}\|h\|_{0\mathbb{E}_4(a)}\end{aligned} \tag{5.8}$$

uniformly in $a \in (0, a_0]$. It is easy to verify that

$$\begin{aligned}\|g\partial_j h\|_{L_p(J;W_p^\theta)} &\le \|g\|_{L_p(J;W_p^\theta)}\|\partial_j h\|_{0BC(J;L_\infty)} + \|g\|_{L_p(J;L_\infty)}\|\partial_j h\|_{0BC(J;W_p^\theta)} \\ &\le C_0\|g\|_{\mathbb{E}_3(a)}\|h\|_{0\mathbb{E}_4(a)}.\end{aligned} \tag{5.9}$$

Combining the estimates (5.8)–(5.9) yields (5.5).

(d) As in the proof of Proposition 6.6.(v) in [25] we obtain

$$\begin{aligned}\|g/\beta\|_{W_p^r(J;L_p)} &\le \|1/\beta\|_\infty(\|g\|_{L_p(J;L_p)} + \langle g\rangle_{W_p^r(J;L_p)}) + \|g\|_\infty\langle 1/\beta\rangle_{W_p^r(J;L_p)} \\ &\le (1 + \langle 1/\beta\rangle_{W_p^r(J;L_p)})(\|g\|_\infty + \|g\|_{W_p^r(J;L_p)}).\end{aligned}$$

Thus it remains to estimate the term $\langle 1/\beta\rangle_{W_p^r(J;L_p)}$. Using that $\beta^2(t,x) - \beta^2(s,x) = \psi^2(t,x) - \psi^2(s,x)$ one easily verifies that

$$\left|\frac{1}{\beta(s,x)} - \frac{1}{\beta(t,x)}\right| = \left|\frac{\beta^2(t,x) - \beta^2(s,x)}{\beta(s,x)\beta(t,x)(\beta(t,x) + \beta(s,x))}\right| \le |\psi(t,x) - \psi(s,x)|$$

and this yields $\langle 1/\beta\rangle_{W_p^r(J;L_p)} \le \langle\psi\rangle_{W_p^r(J;L_p)}$. Consequently,

$$\|g/\beta\|_{W_p^r(J;L_p)} \le (1 + \|\psi\|_{W_p^r(J:L_p)})(\|g\|_\infty + \|g\|_{W_p^r(J;L_p)}).$$

A similar argument shows that

$$\|g/\beta\|_{L_p(J;W_p^\theta)} \leq (1 + \|\psi\|_{L_p(J;W_p^\theta)})(\|g\|_\infty + \|g\|_{L_p(J;W_p^\theta)}).$$

Combining the last two estimates gives (5.6) for $k = 1$. The general case then follows by induction. □

Corollary 5.3. *Suppose $p > n+3$. Let $a_0 \in (0,\infty)$ and $k \in \mathbb{N}$ with $k \geq 1$ be given.*

(a) *There exists a constant $C_0 = C_0(a_0, p, k)$ such that*

$$\|(g_1 \cdots g_k)\bar{g}\|_{{}_0\mathbb{E}_3(a)} \leq C_0 \prod_{i=1}^{k} \big(\|g_i\|_\infty + \|g_i\|_{\mathbb{E}_3(a)}\big)\|\bar{g}\|_{{}_0\mathbb{E}_3(a)}$$

for all functions $g_i \in \mathbb{E}_3(a)$, $1 \leq i \leq k$, $\bar{g} \in {}_0\mathbb{E}_3(a)$, and all $a \in (0, a_0]$.

(b) *There exists a constant $C_0 = C_0(a_0, p, k)$ such that*

$$\begin{aligned}
&\|g(\partial_{\ell_1} h \cdots \partial_{\ell_k} h \partial_j \bar{h})\|_{{}_0\mathbb{E}_3(a)} \\
&\leq C_0\big(\|\nabla h\|_\infty^k + \|h\|_{\mathbb{E}_4(a)}\|\nabla h\|_\infty^{k-1} + \|\nabla h\|^k_{BC(J;W_p^{1-1/p})}\big)\|g\|_{\mathbb{E}_3(a)}\|\bar{h}\|_{{}_0\mathbb{E}_4(a)} \\
&\leq C_0\big(\|\nabla h\|^k_{BC(J;BC^1)} + \|h\|_{\mathbb{E}_4(a)}\|\nabla h\|_\infty^{k-1}\big)\|g\|_{\mathbb{E}_3(a)}\|\bar{h}\|_{{}_0\mathbb{E}_4(a)}
\end{aligned}$$

for $h \in \mathbb{E}_4(a)$, $\bar{h} \in {}_0\mathbb{E}_4(a)$, $a \in (0, a_0]$, $1 \leq j \leq n$, and $\ell_i \in \{1, \ldots n\}$ with $i = 1, \ldots, k$.

Proof. (a) follows from (5.4) by iteration.
(b) The first line in (5.8) shows that

$$\begin{aligned}
&\|g(\partial_{\ell_1} h \cdots \partial_{\ell_k} h \partial_j \bar{h})\|_{W_p^r(J;L_p)} \\
&\leq C_0\|g\|_{W_p^r(J;L_p)}(\|\partial_{\ell_1} h \cdots \partial_{\ell_k} h \partial_j \bar{h}\|_{{}_0BC(J;L_\infty)} + \|\partial_t(\partial_{\ell_1} h \cdots \partial_{\ell_k} h \partial_j \bar{h})\|_{L_p(J;L_\infty)}).
\end{aligned}$$

Next we note that

$$\|\partial_{\ell_1} h \cdots \partial_{\ell_k} h(\partial_t\partial_j \bar{h})\|_{L_p(J;L_\infty)} \leq \|\nabla h\|_\infty^k \|\partial_t \partial_j \bar{h}\|_{L_p(J;L_\infty)},$$

and

$$\|\partial_{\ell_1} h \cdots (\partial_t \partial_{\ell_i} h) \cdots \partial_{\ell_k} h \partial_j \bar{h}\|_{L_p(J;L_\infty)} \leq \|\partial_t \partial_{\ell_i} h\|_{L_p(J;L_\infty)}\|\nabla h\|_\infty^{k-1}\|\partial_j \bar{h}\|_{{}_0BC(J;L_\infty)}.$$

Proposition 6.1(d) now implies the assertion for $\|\cdot\|_{W_p^r(J;L_p)}$. On the other hand we have by (5.9) for $\theta = 1 - 1/p$

$$\begin{aligned}
&\|g(\partial_{\ell_1} h \cdots \partial_{\ell_k} h \partial_j \bar{h})\|_{L_p(J;W_p^\theta)} \\
&\leq \|g\|_{L_p(J;W_p^\theta)}\|\partial_{\ell_1} h \cdots \partial_{\ell_k} h \partial_j \bar{h}\|_\infty + \|g\|_{L_p(J;L_\infty)}\|\partial_{\ell_1} h \cdots \partial_{\ell_k} h \partial_j \bar{h}\|_{{}_0BC(J;W_p^\theta)} \\
&\leq C_0\|g\|_{\mathbb{E}_3(a)}\big(\|\nabla h\|_\infty^k + \|\nabla h\|^k_{BC(J;W_p^{1-1/p})}\big)\|h\|_{{}_0\mathbb{E}_4(a)}
\end{aligned}$$

since $W_p^\theta(\mathbb{R}^n)$ is a multiplication algebra. The last inequality then follows from Sobolev's embedding theorem. □

Remark 5.4. It can be shown that the estimate in (5.5) can be improved as follows: For every $a_0 \in (0,\infty)$ there is a constant $C_0 = C_0(a_0,p) > 0$ and a constant $\theta = \theta(p) > 0$ such that

$$\|g\partial_j h\|_{{}_0\mathbb{E}_3(a)} \le C_0\, a^\theta \|g\|_{\mathbb{E}_3(a)} \|h\|_{{}_0\mathbb{E}_4(a)}$$

holds for all $(g,h) \in \mathbb{E}_3(a) \times {}_0\mathbb{E}_4(a)$ and $a \in (0,a_0]$. In the same way, the constant C_0 in Corollary 5.3(b) can be replaced by $C_0 a^\theta$.

Lemma 5.5. *Suppose $p > n+3$. Let $a_0 \in (0,\infty)$ be given. Then*

(a) $\mathbb{F}_4(a)$ *is a multiplication algebra and we have the estimate*

$$\|g_1 g_2\|_{\mathbb{F}_4(a)} \le C_a \|g_1\|_{\mathbb{F}_4(a)} \|g_2\|_{\mathbb{F}_4(a)}$$

for all $(g_1,g_2) \in \mathbb{F}_4(a) \times \mathbb{F}_4(a)$, where the constant C_a depends on a.

(b) *There exists a constant $C_0 = C_0(a_0,p)$ such that*

$$\|g_1 g_2\|_{{}_0\mathbb{F}_4(a)} \le C_0(\|g_1\|_\infty + \|g_1\|_{\mathbb{F}_4(a)}) \|g_2\|_{{}_0\mathbb{F}_4(a)} \tag{5.10}$$

for all $(g_1,g_2) \in \mathbb{F}_4(a) \times {}_0\mathbb{F}_4(a)$ and all $a \in (0,a_0]$.

(c) *There exists a constant $C_0 = C_0(a_0,p)$ such that*

$$\|g\partial_j h\|_{{}_0\mathbb{F}_4(a)} \le C_0(\|g\|_\infty + \|g\|_{\mathbb{F}_4(a)}) \|h\|_{{}_0\mathbb{E}_4(a)}, \quad j=1,\dots,n, \tag{5.11}$$

for all $(g,h) \in \mathbb{F}_4(a) \times {}_0\mathbb{E}_4(a)$ and $a \in (0,a_0]$.

Proof. Here we equip $\mathbb{F}_4(a)$ with the (equivalent) norm

$$\|g\|_{\mathbb{F}_4(a)} = \|g\|_{W_p^{1-1/2p}(J;L_p)} + \sum_{i=1}^n \|\partial_i g\|_{L_p(J;W_p^{1-1/p})}\,. \tag{5.12}$$

(a) This follows from Proposition 5.1(c) by similar arguments as in the proof of Proposition 6.6(ii) and (iv) in [25].

(b) It follows from part (a) and Proposition 5.1(c) that

$$\|g_1 g_2\|_{{}_0W_p^r(J;L_p)} \le C_0(\|g_1\|_\infty + \|g_1\|_{W_p^r(J;L_p)}) \|g_2\|_{{}_0\mathbb{F}_4(a)}, \quad (g_1,g_2) \in \mathbb{F}_4(a) \times {}_0\mathbb{F}_4(a)$$

where $r = 1-1/2p$. Next, observe that again by Proposition 5.1(c)

$$\begin{aligned}
&\|(\partial_i g_1) g_2\|_{L_p(J;W_p^\theta)} \\
&\quad \le \|\partial_i g_1\|_{L_p(J;W_p^\theta)} \|g_2\|_{{}_0BC(J;L_\infty)} + \|\partial_i g_1\|_{L_p(J;L_\infty)} \|g_2\|_{{}_0BC(J;W_p^\theta)} \\
&\quad \le C_0 \|g_1\|_{\mathbb{F}_4(a)} \|g_2\|_{{}_0\mathbb{F}_4(a)}
\end{aligned}$$

where $\theta = 1-1/p$. Moreover,

$$\begin{aligned}
\|g_1 \partial_i g_2\|_{L_p(J;W_p^\theta)} &\le \|g_1\|_{L_p(J;W_p^\theta)} \|\partial_i g_2\|_{{}_0BC(J;L_\infty)} + \|g_1\|_\infty \|\partial_i g_2\|_{L_p(J;W_p^\theta)} \\
&\le C_0(\|g_1\|_\infty + \|g_1\|_{\mathbb{F}_4(a)}) \|g_2\|_{{}_0\mathbb{F}_4(a)}.
\end{aligned}$$

The estimates above in conjunction with (5.12) yields (5.10).

(c) follows from (b) by setting $g_2 = \partial_j h$ and from Proposition 5.1(e), which certainly is also true for ${}_0\mathbb{E}_4(a)$. $\square$

Proof of Proposition 4.1. It follows as in the proof of [28, Proposition 6.2] that $N_b \in C^\omega(\mathbb{E}(a), \mathbb{F}(a))$, and moreover, that $DH_b(z) \in \mathcal{L}({}_0\mathbb{E}(a), {}_0\mathbb{F}(a))$ for $z \in \mathbb{E}(a)$. It thus remains to prove the estimates stated in the proposition.

Without always writing this explicitly, all the estimates derived below will be uniform in $a \in (0, a_0]$, for $a_0 > 0$ fixed. Moreover, all estimates will be uniform for $(\bar{u}, \bar{\pi}, \bar{q}, \bar{h}) \in {}_0\mathbb{E}(a)$.

(i) Without changing notation we consider here the extension of h from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ defined by $h(t, x, y) = h(t, x)$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ and $t \in J$. With this interpretation we have

$$\|\partial h\|_{\infty, J\times\mathbb{R}^{n+1}} = \|\partial h\|_{\infty, J\times\mathbb{R}^n}, \quad h \in \mathbb{E}(a), \quad \partial \in \{\partial_j, \Delta, \partial_t\}, \tag{5.13}$$

where $\|\cdot\|_{\infty,U}$ denotes the sup-norm for the set $U \subset J \times \mathbb{R}^{n+1}$. Next we observe that

$$\begin{aligned} BC(J; BC(\mathbb{R}^{n+1})) \cdot L_p(J; L_p(\mathbb{R}^{n+1})) &\hookrightarrow L_p(J; L_p(\mathbb{R}^{n+1})), \\ BC(J; L_p(\mathbb{R}^{n+1})) \cdot L_p(J; BC(\mathbb{R}^{n+1})) &\hookrightarrow L_p(J; L_p(\mathbb{R}^{n+1})), \\ BC(J; BC(\mathbb{R}^{n+1})) \cdot BC(J; BC(\mathbb{R}^{n+1})) &\hookrightarrow BC(J; BC(\mathbb{R}^{n+1})), \end{aligned} \tag{5.14}$$

that is, multiplication is continuous and bilinear in the indicated function spaces (with norm equal to 1).

Let us first consider the term $F_1(u, h) := |\nabla h|^2 \partial_y^2 u$ appearing in the definition of F. Its Fréchet derivative at (u, h) is given by

$$DF_1(u, h)[\bar{u}, \bar{h}] = |\nabla h|^2 \partial_y^2 \bar{u} + 2(\nabla h | \nabla \bar{h}) \partial_y^2 u.$$

Suppose $(\bar{u}, \bar{h}) \in {}_0\mathbb{E}_1(a) \times {}_0\mathbb{E}_4(a)$. From (5.13), the first and third line in (5.14) and Proposition 5.1(d) follows the estimate

$$\|DF_1(u, h)[\bar{u}, \bar{h}]\|_{{}_0\mathbb{F}(a)} \le C_0 \|\nabla h\|_\infty (\|\nabla h\|_\infty + \|u\|_{\mathbb{E}_1(a)})(\|\bar{u}\|_{{}_0\mathbb{E}_1(a)} + \|\bar{h}\|_{{}_0\mathbb{E}_4(a)})$$

for all $(u, h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$. It is important to note that the constant C_0 does not depend on the length of the interval $J = (0, a)$ for $a \in (0, a_0]$.

Next, let us take a closer look at the term $F_2(u, h) := \Delta h \partial_y u$ in the definition of F. The Fréchet derivative is $DF_2(u, h)[\bar{u}, \bar{h}] = \Delta h \partial_y \bar{u} + \Delta \bar{h} \partial_y u$. We infer from (5.13), the first and second line in (5.14), and Proposition 5.1 that

$$\begin{aligned} \|DF_2(u, h)[\bar{u}, \bar{h}]\|_{{}_0\mathbb{F}(a)} &\le (\|\Delta h\|_{L_p(J;L_\infty)} + \|\partial_y u\|_{L_p(J;L_p)}) \cdot \\ &\qquad (\|\partial_y \bar{u}\|_{{}_0BC(J;L_p)} + \|\Delta \bar{h}\|_{{}_0BC(J;L_\infty)}) \\ &\le C_0 (\|h\|_{\mathbb{E}_4(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar{u}\|_{{}_0\mathbb{E}_1(a)} + \|\bar{h}\|_{{}_0\mathbb{E}_4(a)}) \end{aligned}$$

for all $(u, h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$.

The derivative of $F_3(u, h) := (u | \nabla h) \partial_y u$, where $\nabla h := (\nabla h, 0)$, is given by

$$DF_3(u, h)[\bar{u}, \bar{h}] = (\bar{u} | \nabla h) \partial_y u + (u | \nabla h) \partial_y \bar{u} + (u | \nabla \bar{h}) \partial_y u$$

and it follows from (5.13)–(5.14) and Proposition 5.1(a), (d) that there is a constant $C_0 > 0$ such that

$$\|DF_3(u,h)[\bar{u},\bar{h}]\|_{_0\mathbb{F}(a)} \le C_0(\|\nabla h\|_\infty + \|u\|_\infty)\|u\|_{\mathbb{E}_1(a)}(\|\bar{u}\|_{_0\mathbb{E}_1(a)} + \|\bar{h}\|_{_0\mathbb{E}_4(a)})$$

for all $(u,h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$.

Let us also consider the term $F_4(u,h) := \partial_t h \partial_y u$. Observing that

$$DF_4(u,h)[\bar{u},\bar{h}] = \partial_t h \partial_y \bar{u} + \partial_t \bar{h} \partial_y u,$$

that $\partial_t : \mathbb{E}_4(a) \to \mathbb{F}_4(a)$ is linear and continuous and

$$\mathbb{F}_4(a) \hookrightarrow L_p(J; BC^1(\mathbb{R}^n)) \cap BC(J; BC^1(\mathbb{R}^n)) \tag{5.15}$$

we conclude from (5.13)–(5.15) and Proposition 5.1(a), (c) that there is a constant $C_0 = C_0(a_0)$ such that

$$\begin{aligned}\|DF_4(u,h)[\bar{u},\bar{h}]\|_{_0\mathbb{F}(a)} &\le (\|\partial_t h\|_{L_p(J;L_\infty)} + \|\partial_y u\|_{L_p(J;L_p)}) \\ &\qquad (\|\partial_y \bar{u}\|_{_0BC(J;L_p)} + \|\partial_t \bar{h}\|_{_0BC(J;L_\infty)}) \\ &\le C_0(\|h\|_{\mathbb{E}_4(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar{u}\|_{_0\mathbb{E}_1(a)} + \|\bar{h}\|_{_0\mathbb{E}_4(a)})\end{aligned}$$

for all $(u,h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$.

The derivative of $F_5(\pi,h) := \partial_y \pi \nabla h$ is given by

$$DF_5(\pi,h)[\bar{\pi},\bar{h}] = \partial_y \bar{\pi} \nabla h + \partial_y \pi \nabla \bar{h}.$$

It follows from (5.13)–(5.14) and Proposition 5.1(d) that there exists C_0 such that

$$\begin{aligned}\|DF_5(\pi,h)[\bar{\pi},\bar{h}]\|_{_0\mathbb{F}(a)} &\le (\|\nabla h\|_\infty + \|\partial_y \pi\|_{L_p(J;L_p)}) \\ &\qquad (\|\partial_y \bar{\pi}\|_{L_p(J;L_p)} + \|\nabla \bar{h}\|_{_0BC(J;L_\infty)}) \\ &\le C_0(\|\nabla h\|_\infty + \|\pi\|_{\mathbb{E}_2(a)})(\|\bar{\pi}\|_{_0\mathbb{E}_2(a)} + \|\bar{h}\|_{_0\mathbb{E}_4(a)})\end{aligned}$$

for all $(\pi,h) \in \mathbb{E}_2(a) \times \mathbb{E}_4(a)$. The remaining terms in the definition of F can be analyzed in the same way. Summarizing we have shown that there is a constant C_0 such that

$$\begin{aligned}&\|DF(u,\pi,h)[\bar{u},\bar{\pi},\bar{h}]\|_{_0\mathbb{F}_1(a)} \\ &\qquad \le C_0\big[\|\nabla h\|_\infty + \|\nabla h\|_\infty^2 + \|(u,\pi,h)\|_{\mathbb{E}(a)} \\ &\qquad\qquad + (\|\nabla h\|_\infty + \|u\|_\infty)\|u\|_{\mathbb{E}_1(a)}\big]\|(\bar{u},\bar{\pi},\bar{h})\|_{_0\mathbb{E}(a)}\end{aligned} \tag{5.16}$$

for all $(u,\pi,h) \in \mathbb{E}(a)$ and all $a \in (0,a_0]$.

(ii) We will now consider the nonlinear function $F_d(u,h) = (\nabla h | \partial_y v)$, stemming from the transformed divergence. Since $h(x,y) := h(x)$ does not depend on y we have

$$F_d(u,h) = (\nabla h | \partial_y u) = \partial_y (\nabla h | u). \tag{5.17}$$

We note that

$$\partial_y \in \mathcal{L}\big(H_p^1(J; L_p(\mathbb{R}^{n+1})), H_p^1(J; H_p^{-1}(\mathbb{R}^{n+1}))\big). \tag{5.18}$$

The norm of this linear mapping does not depend on the length of the interval $J = [0, a]$. It is easy to see that multiplication is continuous in the following function spaces:

$$\begin{aligned} &H^1_p(J; BC(\mathbb{R}^{n+1})) \cdot H^1_p(J; L_p(\mathbb{R}^{n+1})) \hookrightarrow H^1_p(J; L_p(\mathbb{R}^{n+1})) \\ &BC(J; BC^1(\dot{\mathbb{R}}^{n+1})) \cdot L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1})) \hookrightarrow L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1})). \end{aligned} \tag{5.19}$$

The derivative of F_d at $(u, h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$ is given by

$$DF_d(u, h)[\bar{u}, \bar{h}] = (\nabla h|\partial_y \bar{u}) + (\nabla \bar{h}|\partial_y u) = \partial_y((\nabla h|\bar{u}) + (\nabla \bar{h}|u)).$$

We want to derive a uniform estimate for $DF_d(u, h)[\bar{u}, \bar{h}]$ which does not depend on the length of the interval $J = [0, a]$. We conclude from (5.13)–(5.15) that

$$\begin{aligned} &\|(\nabla h|\bar{u})\|_{{}_0H^1_p(J;L_p)} \sim \|(\nabla h|\bar{u})\|_{L_p(J;L_p)} + \|(\partial_t \nabla h|\bar{u})\|_{L_p(J;L_p)} + \|(\nabla h|\partial_t \bar{u})\|_{L_p(J;L_p)} \\ &\le \|\nabla h\|_\infty \|\bar{u}\|_{L_p(J;L_p)} + \|\partial_t \nabla h\|_{L_p(J;L_\infty)} \|\bar{u}\|_{{}_0BC(J;L_p)} + \|\nabla h\|_\infty \|\partial_t \bar{u}\|_{L_p(J;L_p)} \\ &\le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)}) \|\bar{u}\|_{{}_0H^1_p(J;L_p)}. \end{aligned}$$

Similar arguments also yield $\|(\nabla \bar{h}|u)\|_{{}_0H^1_p(J;L_p)} \le C_0 \|u\|_{H^1_p(J;L_p)} \|\bar{h}\|_{{}_0\mathbb{E}_4(a)}$. These estimates in combination with (5.18) show that there is a constant C_0 such that

$$\begin{aligned} &\|(\nabla h|\partial_y \bar{u}) + (\partial_y u|\nabla \bar{h})\|_{{}_0H^1_p(J;H^{-1}_p)} \\ &\quad \le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar{u}\|_{{}_0\mathbb{E}_1(a)} + \|\bar{h}\|_{{}_0\mathbb{E}_4(a)}) \end{aligned}$$

for all $(u, h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$, where C_0 is uniform in $a \in (0, a_0]$. Observing that

$$\|(\nabla h|\partial_y \bar{u})\|_{L_p(J;L_p)} + \Sigma_{j=1}^{n+1} \|(\partial_j \nabla h|\partial_y \bar{u}) + (\nabla h|\partial_j \partial_y \bar{u})\|_{L_p(J;L_p)}$$

defines an equivalent norm for $\|(\nabla h|\partial_y \bar{u})\|_{L_p(J;H^1_p)}$, we infer once more from (5.13)–(5.14) and Proposition 5.1 that

$$\begin{aligned} &\|(\nabla h|\partial_y \bar{u})\|_{L_p(J;H^1_p)} \le C_0(\|h\|_{\mathbb{E}_4(a)} + \|\nabla h\|_\infty) \|\bar{u}\|_{{}_0\mathbb{E}_1(a)} \\ &\|(\nabla \bar{h}|\partial_y u)\|_{L_p(J;H^1_p)} \le C_0 \|u\|_{L_p(J;H^2_p)} \|\bar{h}\|_{{}_0\mathbb{E}_4(a)}. \end{aligned}$$

Summarizing we have shown that there exists a constant C_0 such that

$$\begin{aligned} &\|DF_d(u, h)[\bar{u}, \bar{h}]\|_{{}_0\mathbb{F}_2(a)} \\ &\qquad \le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar{u}\|_{{}_0\mathbb{E}_1(a)} + \|\bar{h}\|_{{}_0\mathbb{E}_4(a)}) \end{aligned} \tag{5.20}$$

for all $(u, h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$ and $a \in (0, a_0]$.

(iii) We remind that

$$[\![\mu \partial_i \,\cdot]\!] \in \mathcal{L}\big(H^1_p(J; L_p(\dot{\mathbb{R}}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1})), \mathbb{E}_3(a)\big) \tag{5.21}$$

where $[\![\mu \partial_i u]\!]$ denotes the jump of the quantity $\mu \partial_i u$ with u a generic function from $\dot{\mathbb{R}}^{n+1} \to \mathbb{R}$, and where $\partial_i = \partial_{x_i}$ for $i = 1, \ldots, n$ and $\partial_{n+1} = \partial_y$.

The mapping $G(u, q, h)$ is made up of terms of the form

$$[\![\mu \partial_i u_k]\!] \partial_j h, \quad [\![\mu \partial_i u_k]\!] \partial_j h \partial_l h, \quad q \partial_j h, \quad \Delta h \partial_j h, \quad G_\kappa(h), \quad G_\kappa(h) \partial_j h$$

where u_k denotes the kth component of a function $u \in \mathbb{E}_1(a)$. It follows from Lemma 5.2(a) and (5.21) that the mappings

$$(u,h) \mapsto [\![\mu\partial_i u_k]\!]\partial_j h,\ [\![\mu\partial_i u_k]\!]\partial_j h\partial_l h : \mathbb{E}_1(a)\times\mathbb{E}_4(a)\to\mathbb{E}_3(a),$$
$$(q,h)\mapsto q\partial_j h : \mathbb{E}_3(a)\times\mathbb{E}_4(a)\to\mathbb{E}_3(a),\quad h\mapsto \Delta h\partial_j h : \mathbb{E}_4(a)\to\mathbb{E}_3(a)$$

are multilinear and continuous.

Let us now take a closer look at the term $G_1(u,h) := [\![\mu\partial_i u_k]\!]\partial_j h$. Its Fréchet derivative is given by

$$DG_1(u,h)[\bar u,\bar h] = \partial_j h[\![\mu\partial_i\bar u_k]\!] + [\![\mu\partial_i u_k]\!]\partial_j\bar h.$$

Setting $g_1 = \partial_j h$ and $g_2 := [\![\mu\partial_i\bar u_k]\!]$ we obtain from (5.4) and (5.21) the estimate

$$\|\partial_j h[\![\mu\partial_i\bar u_k]\!]\|_{{}_0\mathbb{E}_3(a)} \le C_0(\|\nabla h\|_\infty + \|\nabla h\|_{\mathbb{E}_3(a)})\|\bar u\|_{{}_0\mathbb{E}_1(a)}.$$

On the other hand, setting $g := [\![\mu\partial_i u_k]\!]$ we conclude from (5.5) and (5.21) that

$$\|[\![\mu\partial_i u_k]\!]\partial_j\bar h\|_{{}_0\mathbb{E}_3(a)} \le C_0\|u\|_{\mathbb{E}_1(a)}\|\bar h\|_{{}_0\mathbb{E}_4(a)}.$$

Consequently,

$$\begin{aligned}&\|DG_1(u,h)[\bar u,\bar h]\|_{{}_0\mathbb{E}_3(a)}\\ &\qquad\le C_0(\|\nabla h\|_\infty + \|\nabla h\|_{\mathbb{E}_3(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar u\|_{{}_0\mathbb{E}_1(a)} + \|\bar h\|_{{}_0\mathbb{E}_4(a)})\end{aligned} \tag{5.22}$$

for all $(u,h)\in\mathbb{E}_1(a)\times\mathbb{E}_4(a)$, and all $a\in(0,a_0]$.

Given $(u,h)\in\mathbb{E}_1(a)\times\mathbb{E}_4(a)$ let $G_2(u,h) := [\![\mu\partial_i u_k]\!]\partial_j h\partial_l h$. The Fréchet derivative of G_2 is given by

$$DG_2(u,h)[\bar u,\bar h] = \partial_j h\partial_l h[\![\mu\partial_i\bar u_k]\!] + [\![\mu\partial_i u_k]\!]\partial_j h\partial_j\bar h + [\![\mu\partial_i u_k]\!]\partial_l h\partial_j\bar h.$$

From Corollary 5.3(a),(b) and (5.21) follows that there is a constant C_0 such that

$$\begin{aligned}\|DG_2(u,h)[\bar u,\bar h]\|_{{}_0\mathbb{E}_3(a)} &\le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)})^2\|\bar u\|_{{}_0\mathbb{E}_1(a)}\\ &\quad + C_0\big(\|\nabla h\|_{BC(J;BC^1)} + \|h\|_{\mathbb{E}_4(a)}\big)\|u\|_{\mathbb{E}_1(a)}\|\bar h\|_{{}_0\mathbb{E}_4(a)})\end{aligned} \tag{5.23}$$

for all $(u,h)\in\mathbb{E}_1(a)\times\mathbb{E}_4(a)$ and all $a\in(0,a_0]$.

The terms $G_3(q,h) := q\partial_j h$ and $G_4(h) := \Delta h\partial_j h$ can be analyzed in the same way as the term G_1, yielding the following estimates

$$\begin{aligned}&\|DG_3(q,h)[\bar q,\bar h]\|_{{}_0\mathbb{E}_3(a)}\\ &\qquad\le C_0(\|\nabla h\|_\infty + \|\nabla h\|_{\mathbb{E}_3(a)} + \|q\|_{\mathbb{E}_3(a)})(\|\bar q\|_{{}_0\mathbb{E}_3(a)} + \|\bar h\|_{{}_0\mathbb{E}_4(a)})\end{aligned} \tag{5.24}$$

as well as

$$\|DG_4(h)\bar h\|_{{}_0\mathbb{E}_3(a)} \le C_0(\|\nabla h\|_\infty + \|\nabla h\|_{\mathbb{E}_3(a)} + \|\nabla^2 h\|_{\mathbb{E}_3(a)})\|\bar h\|_{{}_0\mathbb{E}_4(a)}. \tag{5.25}$$

Let us now consider the term

$$G_5(h) = \frac{1}{(1+\beta)\beta}(\partial_j h)^2\Delta h,\quad \beta(t,x) := \sqrt{1+|\nabla h(t,x)|^2},$$

appearing in the definition of G_κ. The Fréchet derivative of G_5 at h is given by

$$\begin{aligned} DG_5(h)\bar h = &-\Big(\frac{1}{(1+\beta)^2\beta^2}+\frac{1}{(1+\beta)\beta^3}\Big)(\partial_j h)^2\Delta h\partial_k h\partial_k \bar h \\ &+\frac{1}{(1+\beta)\beta}\big(2\partial_j h\Delta h\partial_j\bar h+(\partial_j h)^2\Delta\bar h\big). \end{aligned}$$

Before continuing, we note that the term $1/(1+\beta)$ can be treated in exactly the same way as $1/\beta$, as a short inspection of the proof of Lemma 5.2(d) shows. It follows then from Corollary 5.3(a)–(b) and from (5.6) that there is a constant C_0 such that

$$\|DG_5(h)\bar h\|_{{}_0\mathbb{E}_3(a)}\le C_0\big[P(\|\nabla h\|_\infty)+Q(\|\nabla h\|_{BC(J;BC^1)},\|h\|_{\mathbb{E}_4(a)})\big]\,\|\bar h\|_{{}_0\mathbb{E}_4(a)} \tag{5.26}$$

for all $h\in\mathbb{E}_4(a)$ and all $a\in(0,a]$, where P and Q are polynomials with coefficients equal to one and vanishing zero-order terms. Analogous arguments can be used for the remaining terms $(\nabla h|\nabla^2 h\nabla h)/\beta^3$ and $G_\kappa(h)\partial_j h$ appearing in G, yielding the same estimate as in (5.26).

(iv) It remains to consider the nonlinear term $H_b(v,h):=(b-\gamma v|\nabla h)$. The Fréchet derivative is given by $DH_b(v,h)[\bar v,\bar h]=-(\nabla h|\gamma\bar v)+(b-\gamma v|\nabla\bar h)$. From Lemma 5.5(b) with $g_1=\partial_j h$ and $g_2=\gamma\bar v_k$, where $\bar v_k$ denotes the kth component of $\bar v$, follows $\|(\nabla h|\gamma\bar v)\|_{{}_0\mathbb{F}_4(a)}\le C_0(\|\nabla h\|_\infty+\|h\|_{\mathbb{E}_4(a)})\|\bar v\|_{{}_0\mathbb{E}_1(a)}$. Lemma 5.5(c) with $g=(b-\gamma v)_k$ and $h=\bar h$ implies

$$\|(b-\gamma v|\nabla\bar h)\|_{{}_0\mathbb{F}_4(a)}\le C_0(\|b-\gamma v\|_\infty+\|b-\gamma v\|_{\mathbb{F}_4(a)})\|\bar h\|_{{}_0\mathbb{E}_4(a)}.$$

We have, thus, shown that

$$\|DH_b(v,h)\|\le C_0(\|\nabla h\|_\infty+\|h\|_{\mathbb{E}_4(a)}+\|b-\gamma v\|_{BC(J;BC)\cap\mathbb{F}_4(a)}). \tag{5.27}$$

Combining the estimates in (5.16), (5.20) and (5.22)–(5.27) yields the assertions of Proposition 4.1. □

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On Conserved Penrose-Fife Type Models

Jan Prüss and Mathias Wilke

Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. In this paper we investigate quasilinear parabolic systems of conserved Penrose-Fife type. We show maximal L_p-regularity for this problem with inhomogeneous boundary data. Furthermore we prove global existence of a solution, under the assumption that the absolute temperature is bounded from below and above. Moreover, we apply the Lojasiewicz-Simon inequality to establish the convergence of solutions to a steady state as time tends to infinity.

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1. Introduction and the model

We are interested in the conserved Penrose-Fife type equations

$$\begin{aligned} \partial_t\psi = \Delta\mu, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \quad t \in J,\ x \in \Omega, \\ \partial_t\left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = 0, \quad t \in J,\ x \in \Omega, \end{aligned} \tag{1.1}$$

where $\vartheta = 1/\theta$ and θ denotes the absolute temperature of the system, ψ is the order parameter and $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^4$. The function Φ' is the derivative of the physical potential, which characterizes the different phases of the system. A typical example is the *double well* potential $\Phi(s) = (s^2 - 1)^2$ with the two distinct minima $s = \pm 1$. Typically, the nonlinear function λ is a polynomial of second order.

For an explanation of (1.1) we will follow the lines of Alt & Pawlow [2] (see also Brokate & Sprekels [4, Section 4.4]). We start with the rescaled Landau-

Ginzburg functional (total Helmholtz free energy)

$$\mathcal{F}(\psi,\theta) = \int_\Omega \left(\frac{\gamma(\theta)}{2\theta}|\nabla\psi|^2 + \frac{f(\psi,\theta)}{\theta}\right)\,dx,$$

where the free energy density $F(\psi,\theta) := \frac{\gamma(\theta)}{2}|\nabla\psi|^2 + f(\psi,\theta)$ is rescaled by $1/\theta$. The reduced chemical potential μ is given by the variational derivative of $\mathcal{F}$ with respect to ψ, i.e.,

$$\mu = \frac{\delta\mathcal{F}}{\delta\psi}(\psi,\theta) = \frac{1}{\theta}\left(-\gamma(\theta)\Delta\psi + \frac{\partial f(\psi,\theta)}{\partial\psi}\right).$$

Assuming that ψ is a conserved quantity, we have the conservation law

$$\partial_t\psi + \mathrm{div} j = 0.$$

Here j is the flux of the order parameter ψ, for which we choose the well-accepted constitutive law $j = -\nabla\mu$, i.e., the phase transition is driven by the chemical potential μ (see [4, (4.4)]). The kinetic equation for ψ thus reads

$$\partial_t\psi = \Delta\mu, \quad \mu = \frac{1}{\theta}\left(-\gamma(\theta)\Delta\psi + \frac{\partial f(\psi,\theta)}{\partial\psi}\right).$$

If the volume of the system is preserved, the internal energy e is given by the variational derivative

$$e = \frac{\delta\mathcal{F}(\psi,\theta)}{\delta(1/\theta)}.$$

This yields the expression

$$e(\psi,\theta) = f(\psi,\theta) - \theta\frac{\partial f(\psi,\theta)}{\partial\theta} + \frac{1}{2}\left(\gamma(\theta) - \theta\frac{\partial\gamma(\theta)}{\partial\theta}\right)|\nabla\psi|^2.$$

It can be readily checked that the Gibbs relation

$$e(\psi,\theta) = F(\psi,\theta) - \theta\frac{\partial F(\psi,\theta)}{\partial\theta}.$$

holds. If we assume that no mechanical stresses are active, the internal energy e satisfies the conservation law

$$\partial_t e + \mathrm{div} q = 0,$$

where q denotes the heat flux of the system. Following Alt & Pawlow [2], we assume that $q = \nabla\left(\frac{1}{\theta}\right)$, so that the kinetic equation for e reads

$$\partial_t e + \Delta\left(\frac{1}{\theta}\right) = 0.$$

Let us now assume that $\gamma(\theta) = \theta$ and $f(\psi,\theta) = \theta\Phi(\psi) - \lambda(\psi) - \theta\log\theta$. In this case we obtain $e = \theta - \lambda(\psi)$ and

$$\mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\frac{1}{\theta},$$

hence system (1.1) for $\vartheta = 1/\theta$ and $b(s) = -1/s$, $s > 0$. Suppose $(j|\nu) = (q|\nu) = 0$ on $\partial\Omega$ with $\nu = \nu(x)$ being the outer unit normal in $x \in \partial\Omega$. This yields the boundary conditions $\partial_\nu \mu = 0$ and $\partial_\nu \vartheta = 0$ for the chemical potential μ and the function ϑ, respectively. Since (1.1) is of fourth order with respect to the function ψ we need an additional boundary condition. An appropriate and classical one from a variational point of view is $\partial_\nu \psi = 0$. Finally, this yields the initial-boundary value problem

$$\begin{aligned} \partial_t \psi - \Delta\mu = f_1, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t \in J,\ x \in \Omega, \\ \partial_t \left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = f_2, &\quad t \in J,\ x \in \Omega, \\ \partial_\nu \mu = g_1,\ \partial_\nu \psi = g_2,\ \partial_\nu \vartheta = g_3, &\quad t \in J,\ x \in \partial\Omega, \\ \psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega. \end{aligned} \tag{1.2}$$

The functions $f_j, g_j, \psi_0, \vartheta_0, \Phi, \lambda$ and b are given. Note that if θ has only a small deviation from a constant value $\theta_* > 0$, then the term $1/\theta$ can be linearized around θ_* and (1.2) turns into the nonisothermal Cahn-Hilliard equation for the order parameter ψ and the relative temperature $\theta - \theta_*$, provided $b(s) = -1/s$.

In the case of the Penrose-Fife equations, Brokate & Sprekels [4] and Zheng [20] proved global well-posedness in an L_2-setting if the spatial dimension is equal to 1. Sprekels & Zheng showed global well-posedness of the non-conserved equations (that is $\partial_t \psi = -\mu$) in higher space dimensions in [18], a similar result can be found in the article of Laurencot [11]. Concerning asymptotic behavior we refer to the articles of Kubo, Ito & Kenmochi [10], Shen & Zheng [17], Feireisl & Schimperna [8] and Rocca & Schimperna [14]. The last two authors studied well-posedness and qualitative behavior of solutions to the non-conserved Penrose-Fife equations. To be precise, they proved that each solution converges to a steady state, as time tends to infinity. Shen & Zheng [17] established the existence of attractors for the non-conserved equations, whereas Kubo, Ito & Kenmochi [10] studied the non-conserved as well as the conserved Penrose-Fife equations. Beside the proof of global well-posedness in the sense of weak solutions they also showed the existence of a global attractor. Finally, we want to mention that the physical potential Φ may also be of logarithmic type, such that $\Phi'(s)$ has singularities at $s = \pm 1$. This forces the order parameter to stay in the physically reasonable interval $(-1, 1)$, provided that the initial value $\psi(0) = \psi_0 \in (-1, 1)$. In general, such a result cannot be obtained in the case of the double well potential, since there is no maximum principle available for the fourth-order equation $(1.2)_1$. For a result on global existence, uniqueness and asymptotic behaviour of solutions of the *Cahn-Hilliard equation* in case of a logarithmic potential, we refer the reader, e.g., to Abels & Wilke [1] and the references cited therein. However, in this paper we will only deal with smooth potentials.

In the following sections we will prove well-posedness of (1.2) for solutions in the maximal L_p-regularity classes

$$\begin{aligned} \psi &\in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)), \\ \vartheta &\in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)), \end{aligned}$$

where $J = [0, T]$, $T > 0$. In Section 2 we investigate a linearized version of (1.2) and prove maximal L_p-regularity. Section 3 is devoted to local well-posedness of (1.2). To this end we apply the contraction mapping principle. In Section 4, we show that the solution exists globally in time, under the unproven assumption that the inverse of the absolute temperature is uniformly bounded from below and above. Actually, it is a formidable task to derive such bounds for ϑ. As far as we know, it has been proven only in some particular cases for the *non-conserved* system, i.e., if $(1.2)_1$ is replaced by

$$\partial_t\psi - \Delta\psi = \lambda'(\psi)\vartheta - \Phi'(\psi),$$

a semilinear heat equation for the order parameter ψ. For more details, we refer the reader to the references [9] and [16].

Finally, in Section 5, we study the asymptotic behavior of the solution to (1.2) as $t \to \infty$. The Lojasiewicz-Simon inequality will play an important role in the analysis.

2. The linear problem

In this section we deal with a linearized version of (1.2).

$$\begin{aligned} \partial_t u + \Delta^2 u + \Delta(\eta_1 v) &= f_1, \quad t \in J,\ x \in \Omega,\\ \partial_t v - a_0\Delta v + \eta_2\partial_t u &= f_2, \quad t \in J,\ x \in \Omega,\\ \partial_\nu \Delta u + \partial_\nu(\eta_1 v) &= g_1, \quad t \in J,\ x \in \partial\Omega,\\ \partial_\nu u = g_2,\ \partial_\nu v &= g_3, \quad t \in J,\ x \in \partial\Omega,\\ u(0) = u_0,\ v(0) &= v_0, \quad t = 0,\ x \in \Omega. \end{aligned} \tag{2.1}$$

Here $\eta_1 = \eta_1(x), \eta_2 = \eta_2(x), a_0 = a_0(x)$ are given functions such that

$$\eta_1 \in B_{pp}^{4-4/p}(\Omega),\ \eta_2 \in B_{pp}^{2-2/p}(\Omega) \quad \text{and} \quad a_0 \in C(\overline{\Omega}). \tag{2.2}$$

We assume furthermore that $a_0(x) \geq \sigma > 0$ for all $x \in \overline{\Omega}$ and some constant $\sigma > 0$. Hence equation $(2.1)_2$ does not degenerate. We are interested in solutions

$$u \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)) =: E_1(T)$$

and

$$v \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)) =: E_2(T)$$

of (2.1). By the well-known trace theorems (cf. [3, Theorem 4.10.2])

$$E_1(T) \hookrightarrow C(J; B_{pp}^{4-4/p}(\Omega)) \quad \text{and} \quad E_2(T) \hookrightarrow C(J; B_{pp}^{2-2/p}(\Omega)), \tag{2.3}$$

we necessarily have $u_0 \in B_{pp}^{4-4/p}(\Omega) =: X_\gamma^1$, $v_0 \in B_{pp}^{2-2/p}(\Omega) =: X_\gamma^2$ and the compatibility conditions

$$\partial_\nu\Delta u_0 + \partial_\nu(\eta_1 v_0) = g_1|_{t=0}, \quad \partial_\nu u_0 = g_2|_{t=0}, \quad \text{as well as} \quad \partial_\nu v_0 = g_3|_{t=0},$$

whenever $p > 5$, $p > 5/3$ and $p > 3$, respectively (cf. [6, Theorem 2.1]). In the sequel we will assume that $p > (n+2)/2$ and $p \geq 2$. This yields the embeddings

$$B_{pp}^{4-4/p}(\Omega) \hookrightarrow H_p^2(\Omega) \cap C^1(\bar{\Omega}) \text{ and } B_{pp}^{2-2/p}(\Omega) \hookrightarrow H_p^1(\Omega) \cap C(\bar{\Omega}).$$

We are going to prove the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ a bounded domain with boundary $\partial\Omega \in C^4$ and let $p > (n+2)/2$, $p \geq 2$, $p \neq 3, 5$. Assume in addition that $\eta_1 \in B_{pp}^{4-4/p}(\Omega)$, $\eta_2 \in B_{pp}^{2-2/p}(\Omega)$ and $a_0 \in C(\bar{\Omega})$, $a_0(x) \geq \sigma > 0$ for all $x \in \bar{\Omega}$. Then the linear problem* (2.1) *admits a unique solution*

$$(u, v) \in H_p^1(J_0; L_p(\Omega)^2) \cap L_p(J_0; (H_p^4(\Omega) \times H_p^2(\Omega))),$$

if and only if the data are subject to the following conditions.

1. $f_1, f_2 \in L_p(J_0; L_p(\Omega)) = X(J_0)$,
2. $g_1 \in W_p^{1/4-1/4p}(J_0; L_p(\partial\Omega)) \cap L_p(J_0; W_p^{1-1/p}(\partial\Omega)) = Y_1(J_0)$,
3. $g_2 \in W_p^{3/4-1/4p}(J_0; L_p(\partial\Omega)) \cap L_p(J_0; W_p^{3-1/p}(\partial\Omega)) = Y_2(J_0)$,
4. $g_3 \in W_p^{1/2-1/2p}(J_0; L_p(\partial\Omega)) \cap L_p(J_0; W_p^{1-1/p}(\partial\Omega)) = Y_3(J_0)$,
5. $u_0 \in B_{pp}^{4-4/p}(\Omega) = X_\gamma^1$, $v_0 \in B_{pp}^{2-2/p}(\Omega) = X_\gamma^2$,
6. $\partial_\nu \Delta u_0 + \partial_\nu(\eta_1 v_0) = g_1|_{t=0}$, $p > 5$,
7. $\partial_\nu u_0 = g_2|_{t=0}$, $\partial_\nu v_0 = g_3|_{t=0}$, $p > 3$.

Proof. Suppose that the function $u \in E_1(T)$ in (2.1) is already known. Then in a first step we will solve the linear heat equation

$$\partial_t v - a_0 \Delta v = f_2 - \eta_2 \partial_t u, \tag{2.4}$$

subject to the boundary and initial conditions $\partial_\nu v = g_3$ and $v(0) = v_0$. By the properties of the function a_0 we may apply [6, Theorem 2.1] to obtain a unique solution $v \in E_2(T)$ of (2.4), provided that $f_2 \in L_p(J \times \Omega)$, $v_0 \in B_{pp}^{2-2/p}(\Omega)$,

$$g_3 \in W_p^{1/2-1/2p}(J; L_p(\partial\Omega)) \cap L_p(J; W_p^{1-1/p}(\partial\Omega)) =: Y_3(J),$$

and the compatibility condition $\partial_\nu v_0 = g_3|_{t=0}$ if $p > 3$ is valid. The solution may then be represented by the variation of parameters formula

$$v(t) = v_1(t) - \int_0^t e^{-A(t-s)} \eta_2 \partial_t u(s)\, ds, \tag{2.5}$$

where A denotes the L_p-realization of the differential operator $\mathcal{A}(x) = -a_0(x)\Delta_N$, Δ_N means the Neumann-Laplacian and e^{-At} stands for the bounded analytic semigroup, which is generated by $-A$ in $L_p(\Omega)$. Furthermore the function $v_1 \in E_2(T)$ solves the linear problem

$$\partial_t v_1 - a_0 \Delta v_1 = f_2, \quad \partial_\nu v_1 = g_3, \quad v_1(0) = v_0.$$

We fix a function $w^* \in E_1(T)$ such that $w^*|_{t=0} = u_0$ and make use of (2.5) and the fact that $(u - w^*)|_{t=0} = 0$ to obtain

$$v(t) = v_1(t) + v_2(t) - (\partial_t + A)^{-1} \eta_2 \partial_t (u - w^*)$$

with $v_2(t) := -\int_0^t e^{-A(t-s)}\eta_2\partial_t w^*$. Set $v^* = v_1 + v_2 \in E_2(T)$ and

$$F(u) = -(\partial_t + A)^{-1}\eta_2\partial_t(u - w^*).$$

Then we may reduce (2.1) to the problem

$$\begin{aligned} \partial_t u + \Delta^2 u &= \Delta G(u) + f_1, \quad t \in J,\ x \in \Omega,\\ \partial_\nu \Delta u &= \partial_\nu G(u) + g_1, \quad t \in J,\ x \in \partial\Omega,\\ \partial_\nu u &= g_2, \quad t \in J,\ x \in \partial\Omega,\\ u(0) &= u_0, \quad t = 0,\ x \in \Omega, \end{aligned} \tag{2.6}$$

where $G(u) := -\eta_1(F(u) + v^*)$. For a given $T \in (0, T_0]$ we set

$$_0E_1(T) = \{u \in E_1(T) : u|_{t=0} = 0\}$$

and

$$\begin{aligned} E_0(T) &:= X(T) \times Y_1(T) \times Y_2(T)\\ {}_0E_0(T) &:= \{(f, g, h) \in E_0(T) : g|_{t=0} = h|_{t=0} = 0\}, \end{aligned}$$

where $X(T) := L_p((0,T) \times \Omega)$,

$$Y_1(T) := W_p^{1/4-1/4p}(0, T; L_p(\partial\Omega)) \cap L_p(0, T; W_p^{1-1/p}(\partial\Omega)),$$

and

$$Y_2(T) := W_p^{3/4-1/4p}(0, T; L_p(\partial\Omega)) \cap L_p(0, T; W_p^{3-1/p}(\partial\Omega)).$$

The spaces $E_1(T)$ and $E_0(T)$ are endowed with the canonical norms $|\cdot|_1$ and $|\cdot|_0$, respectively. We introduce the new function $\tilde{u} := u - w^* \in {}_0E_1(T)$ and we set

$$\tilde{F}(\tilde{u}) := -(\partial_t + A)^{-1}\eta_2\partial_t\tilde{u}$$

as well as $\tilde{G}(\tilde{u}) := -\eta_1\tilde{F}(\tilde{u})$. If $u \in E_1(T)$ is a solution of (2.6), then the function $\tilde{u} \in {}_0E_1(T)$ solves the problem

$$\begin{aligned} \partial_t \tilde{u} + \Delta^2 \tilde{u} &= \Delta \tilde{G}(\tilde{u}) + \tilde{f}_1, \quad t \in J,\ x \in \Omega,\\ \partial_\nu \Delta \tilde{u} &= \partial_\nu \tilde{G}(\tilde{u}) + \tilde{g}_1, \quad t \in J,\ x \in \partial\Omega,\\ \partial_\nu \tilde{u} &= \tilde{g}_2, \quad t \in J,\ x \in \partial\Omega,\\ \tilde{u}(0) &= 0, \quad t = 0,\ x \in \Omega, \end{aligned} \tag{2.7}$$

with the modified data

$$\begin{aligned} \tilde{f}_1 &:= f_1 - \Delta(\eta_1 v^*) - \partial_t w^* - \Delta^2 w^* \in X(T),\\ \tilde{g}_1 &:= g_1 - \partial_\nu(\eta v^*) - \partial_\nu \Delta w^* \in {}_0Y_1(T), \end{aligned}$$

and

$$\tilde{g}_2 := g_2 - \partial_\nu w^* \in {}_0Y_2(T).$$

Observe that by construction we have $\tilde{g}_1|_{t=0} = 0$ and $\tilde{g}_2|_{t=0} = 0$ if $p > 5$ and $p > 5/3$, respectively.

Let us estimate the term $\Delta\tilde{G}(u)$ in $L_p(J;L_p(\Omega))$, where $u \in {}_0E_1(T)$. We compute

$$\begin{aligned}|\Delta\tilde{G}(u)|_{L_p(J;L_p(\Omega))} &\le |\tilde{F}(u)\Delta\eta_1|_{L_p(J;L_p(\Omega))}\\ &\qquad + 2|(\nabla\tilde{F}(u)|\nabla\eta_1)|_{L_p(J;L_p(\Omega))} + |\eta_1\Delta\tilde{F}(u)|_{L_p(J;L_p(\Omega))}.\end{aligned}$$

Since $\eta_1 \in B^{4-4/p}_{pp}(\Omega)$ does not depend on the variable t, we obtain

$$\begin{aligned}|\tilde{F}(u)\Delta\eta_1|_{L_p(J;L_p(\Omega))} &\le |\Delta\eta_1|_{L_p(\Omega)}|\tilde{F}(u)|_{L_p(J;L_\infty(\Omega))},\\ |(\nabla\tilde{F}(u)|\nabla\eta_1)|_{L_p(J;L_p(\Omega))} &\le |\nabla\eta_1|_{L_\infty(\Omega)}|\nabla\tilde{F}(u)|_{L_p(J;L_p(\Omega))},\end{aligned}$$

and

$$|\eta_1\Delta\tilde{F}(u)|_{L_p(J;L_p(\Omega))} \le |\eta_1|_{L_\infty(\Omega)}|\Delta\tilde{F}(u)|_{L_p(J;L_p(\Omega))}.$$

Therefore we have to estimate $\tilde{F}(u)$ for each $u \in {}_0E_1(T)$ in the topology of the spaces $L_p(J;L_\infty(\Omega))$ and $L_p(J;H^2_p(\Omega))$. Let $u \in {}_0E_1$ and recall that $\tilde{F}(u)$ is defined by $\tilde{F}(u) = -(\partial_t + A)^{-1}\eta_2\partial_t u$. The operator $(\partial_t + A)^{-1}$ is a bounded linear operator from $L_p(J;L_p(\Omega))$ to ${}_0H^1_p(J;L_p(\Omega)) \cap L_p(J;H^2_p(\Omega)) = {}_0E_2(T)$. Moreover, by the trace theorem and by Sobolev embedding, it holds that

$${}_0H^1_p(J;L_p(\Omega)) \cap L_p(J;H^2_p(\Omega)) \hookrightarrow C(J;B^{2-2/p}_{pp}(\Omega)) \hookrightarrow C(J;C(\bar{\Omega})).$$

Note that the bound of $(\partial_t + A)^{-1}$ as well as the embedding constant do not depend on the length of the interval $J = [0,T] \subset [0,T_0] = J_0$, since the time trace at $t = 0$ vanishes. With these facts, we obtain

$$\begin{aligned}|(\partial_t + A)^{-1}\eta_2\partial_t u|_{L_p(J;L_\infty(\Omega))} &\le T^{1/p}|(\partial_t + A)^{-1}\eta_2\partial_t u|_{L_\infty(J;L_\infty(\Omega))}\\ &\le T^{1/p}C|(\partial_t + A)^{-1}\eta_2\partial_t u|_{E_2(T)}\\ &\le T^{1/p}C|\eta_2\partial_t u|_{L_p(J;L_p(\Omega))}\\ &\le T^{1/p}C|\eta_2|_{L_\infty(\Omega)}|u|_{E_1(T)}.\end{aligned}$$

To estimate $\tilde{F}(u)$ in $L_p(J;H^2_p(\Omega))$ we need another representation of $\tilde{F}(u)$. To be precise, we rewrite $\tilde{F}(u)$ as follows

$$\tilde{F}(u) = -(\partial_t + A)^{-1}\eta_2\partial_t u = -\partial_t^{1/2}(\partial_t + A)^{-1}\partial_t^{1/2}(\eta_2 u).$$

This is possible, since $u \in {}_0E_1(T)$. Now observe that for each $u \in {}_0E_1$ it holds that $\eta_2 u \in {}_0H^{3/4}_p(J;H^1_p(\Omega))$. This can be seen as follows. First of all, it suffices to show that $\eta_2 u \in L_p(J;H^1_p(\Omega))$, since η_2 does not depend on the variable t. But

$$\begin{aligned}|\eta_2 u|_{L_p(J;H^1_p(\Omega))} &\le |\eta_2\nabla u|_{L_p(J;L_p(\Omega))} + |u\nabla\eta_2|_{L_p(J;L_p(\Omega))}\\ &\le C\left(|\eta_2|_{L_\infty(\Omega)}|u|_{E_1(T)} + |u|_{L_p(J;L_\infty(\Omega))}|\eta_2|_{H^1_p(\Omega)}\right)\\ &\le C|u|_{E_1(T)}|\eta_2|_{B^{2-2/p}_{pp}(\Omega)},\end{aligned}$$

and this yields the claim, since

$$u \in {}_0H^1_p(J;L_p(\Omega)) \cap L_p(J;H^4_p(\Omega)) \hookrightarrow {}_0H^{3/4}_p(J;H^1_p(\Omega)),$$

by the mixed derivative theorem. It follows readily that

$$\partial_t^{1/2}(\eta_2 u) \in {}_0H_p^{1/4}(J; H_p^1(\Omega))$$

and

$$(\partial_t + A)^{-1}(I + A)^{1/2}\partial_t^{1/2}(\eta_2 u) \in {}_0H_p^{5/4}(J; L_p(\Omega)) \cap {}_0H_p^{1/4}(J; H_p^2(\Omega)).$$

Since the operator $(I+A)^{1/2}$ with domain $D((I+A)^{1/2}) = H_p^1(\Omega)$ commutes with the operator $(\partial_t + A)^{-1}$, this yields

$$(\partial_t + A)^{-1}\partial_t^{1/2}(\eta_2 u) \in {}_0H_p^{5/4}(J; H_p^1(\Omega)) \cap {}_0H_p^{1/4}(J; H_p^3(\Omega))$$

for each fixed $u \in {}_0E_1(T)$. By the mixed derivative theorem we obtain furthermore

$${}_0H_p^{5/4}(J; H_p^1(\Omega)) \cap {}_0H_p^{1/4}(J; H_p^3(\Omega)) \hookrightarrow {}_0H_p^{3/4}(J; H_p^2(\Omega)).$$

Therefore

$$\tilde{F}(u) = -\partial_t^{1/2}(\partial_t + A)^{-1}\partial_t^{1/2}(\eta_2 u) \in {}_0H_p^{1/4}(J; H_p^2(\Omega)),$$

and there exists a constant $C > 0$ being independent of $T > 0$ and $u \in {}_0E_1(T)$ such that

$$|\tilde{F}(u)|_{H_p^{1/4}(J;H_p^2(\Omega))} \le C|u|_{E_1(T)},$$

for each $u \in {}_0E_1(T)$. In particular this yields the estimate

$$\begin{aligned} |\tilde{F}(u)|_{L_p(J;H_p^2(\Omega))} &\le T^{1/2p}|\tilde{F}(u)|_{L_{2p}(J;H_p^2(\Omega))} \\ &\le T^{1/2p}|\tilde{F}(u)|_{H_p^{1/4}(J;H_p^2(\Omega))} \le T^{1/2p}C|u|_{E_1(T)}, \end{aligned}$$

by Hölder's inequality and $C > 0$ does not depend on the length T of the interval J. We have thus shown that

$$|\Delta\tilde{G}(u)|_{L_p(J;L_p(\Omega))} \le \mu_1(T)C|u|_{E_1(T)},$$

where we have set $\mu_1(T) := T^{1/2p}(1+T^{1/2p})$. Observe that $\mu_1(T) \to 0_+$ as $T \to 0_+$. The next step consists of estimating the term $\partial_\nu\tilde{G}(u)$ in ${}_0W_p^{1/4-1/4p}(J; L_p(\partial\Omega)) \cap L_p(J; W_p^{1-1/p}(\partial\Omega))$. To this end, we recall the trace map

$${}_0H_p^{1/2}(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)) \hookrightarrow {}_0W_p^{1/4-1/4p}(J; L_p(\partial\Omega)) \cap L_p(J; W_p^{1-1/p}(\partial\Omega))$$

for the Neumann derivative on $\partial\Omega$. Therefore, by the results above, it remains to estimate $\tilde{G}(u)$ in ${}_0H_p^{1/2}(J; L_p(\Omega))$. By the complex interpolation method we have

$$|w|_{H_p^{1/2}(J;L_p(\Omega))} \le C|w|_{L_p(J;L_p(\Omega))}^{1/2}|w|_{H_p^1(J;L_p(\Omega))}^{1/2}$$

for each $w \in {}_0H_p^1(J; L_p(\Omega))$, and $C > 0$ does not depend on $T > 0$. Using Hölder's inequality, this yields

$$\begin{aligned} |w|_{H_p^{1/2}(J;L_p(\Omega))} &\le T^{1/4p}C|w|_{L_{2p}(J;L_p(\Omega))}^{1/2}|w|_{H_p^1(J;L_p(\Omega))}^{1/2} \\ &\le T^{1/4p}C|w|_{H_p^1(J;L_p(\Omega))}. \end{aligned}$$

Finally we obtain the estimate

$$|\tilde{G}(u)|_{H_p^{1/2}(J;L_p(\Omega))} \le T^{1/2p}|\eta_1|_{L_\infty(\Omega)}C|u|_{\mathbb{E}_1(T)},$$

which in turn implies

$$|\partial_\nu \tilde{G}(u)|_{Y_1(J)} \le |\tilde{G}(u)|_{H_p^{1/2}(J;L_p(\Omega))} + |\tilde{G(u)}|_{L_p(J;H_p^2(\Omega))} \le \mu_2(T)C|u|_{E_1(T)},$$

where $\mu_2(T) := T^{1/4p}(1+T^{1/4p})$ and $\mu_2(T) \to 0_+$ as $T \to 0_+$. Define two operators $L, B : {}_0E_1(T) \to {}_0E_0(T)$ by means of

$$Lu := \begin{bmatrix} \partial_t u + \Delta^2 u \\ \partial_\nu \Delta u \\ \partial_\nu u \end{bmatrix} \quad \text{and} \quad Bu := \begin{bmatrix} \Delta \tilde{G}(u) \\ \partial_\nu \tilde{G}(u) \\ 0 \end{bmatrix}.$$

With these definitions, we may rewrite (2.7) in the abstract form

$$Lu = Bu + f, \quad f := (\tilde{f}_1, \tilde{g}_1, \tilde{g}_2) \in {}_0E_0(T).$$

By [6, Theorem 2.1], the operator L is bijective with bounded inverse L^{-1}, hence $u \in {}_0E_1(T)$ is a solution of (2.7) if and only if $(I - L^{-1}B)u = L^{-1}f$. Observe that $L^{-1}B$ is a bounded linear operator from ${}_0E_1(T)$ to ${}_0E_1(T)$ and

$$|L^{-1}Bu|_{E_1(T)} \le |L^{-1}|_{\mathcal{B}(E_0(T),E_1(T))}|Bu|_{E_0(T)} \le (\mu_1(T) + \mu_2(T))C|u|_{E_1(T)}.$$

Here the constant $C > 0$ as well as the bound of L^{-1} are independent of $T > 0$. This shows that choosing $T > 0$ sufficiently small, we may apply a Neumann series argument to conclude that (2.7) has a unique solution $u \in {}_0E_1(T)$ on a possibly small time interval $J = [0, T]$. Since the linear system (2.7) is invariant with respect to time shifts, we may set $J = J_0$. □

3. Local well-posedness

In this section we will use the following setting. For $T_0 > 0$, to be fixed later, and a given $T \in (0, T_0]$ we define

$$\mathbb{E}_1(T) := E_1(T) \times E_2(T), \qquad {}_0\mathbb{E}_1(T) := \{(u, v) \in \mathbb{E}_1(T) : (u, v)|_{t=0} = 0\}$$

and

$$\mathbb{E}_0(T) := X(T) \times X(T) \times Y_1(T) \times Y_2(T) \times Y_3(T),$$

as well as

$${}_0\mathbb{E}_0(T) := \{(f_1, f_2, g_1, g_2, g_3) \in \mathbb{E}_0(T) : g_1|_{t=0} = g_2|_{t=0} = g_3|_{t=0} = 0\},$$

with canonical norms $|\cdot|_1$ and $|\cdot|_0$, respectively. The aim of this section is to find a local solution $(\psi, \vartheta) \in \mathbb{E}_1(T)$ of the quasilinear system

$$\begin{aligned} \partial_t\psi - \Delta\mu = f_1, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \quad & t \in J,\ x \in \Omega, \\ \partial_t\left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = f_2, \quad & t \in J,\ x \in \Omega, \\ \partial_\nu\mu = g_1,\ \partial_\nu\psi = g_2,\ \partial_\nu\vartheta = g_3, \quad & t \in J,\ x \in \partial\Omega, \\ \psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, \quad & t = 0,\ x \in \Omega. \end{aligned} \tag{3.1}$$

To this end, we will apply Banach's fixed point theorem. For this purpose let $p > (n+2)/2$, $p \geq 2$, $f_1, f_2 \in X(T_0)$, $g_j \in Y_j(0,T_0)$, $j = 1,2,3$, $\psi_0 \in X_\gamma^1$ and $\vartheta_0 \in X_\gamma^2$ be given such that the compatibility conditions

$$\partial_\nu \Delta\psi_0 - \partial_\nu \Phi'(\psi_0) + \partial_\nu(\lambda'(\psi_0)\vartheta_0) = -g_1|_{t=0},\ \partial_\nu\psi_0 = g_2|_{t=0} \quad \text{and} \quad \partial_\nu\vartheta_0 = g_3|_{t=0}$$

are satisfied, whenever $p > 5$, $p > 5/3$ and $p > 3$, respectively. In the sequel we will assume that $\lambda, \phi \in C^{4-}(\mathbb{R})$, $b \in C^{3-}(0,\infty)$ and $b'(s) > 0$ for all $s > 0$. Note that by the Sobolev embedding theorem we have $\vartheta_0 \in C(\bar\Omega)$ as well as $b'(\vartheta_0) \in C(\bar\Omega)$. Since ϑ represents the inverse of the absolute temperature of the system, it is reasonable to assume $\vartheta_0(x) > 0$ for all $x \in \bar\Omega$. Therefore, there exists a constant $\sigma > 0$ such that $\vartheta_0(x), b'(\vartheta_0(x)) \geq \sigma > 0$ for all $x \in \bar\Omega$. We define $a_0(x) := 1/b'(\vartheta_0(x))$, $\eta_1(x) = \lambda'(\psi_0(x))$ and $\eta_2(x) = a_0(x)\eta_1(x)$. By assumption, it holds that $a_0 \in B_{pp}^{2-2/p}(\Omega)$, $\eta_1 \in B_{pp}^{4-4/p}(\Omega)$ and $\eta_2 \in B_{pp}^{2-2/p}(\Omega)$, cf. [15, Section 4.6 & Section 5.3.4].

Thanks to Theorem 2.1 we may define a pair of functions $(u^*, v^*) \in \mathbb{E}_1(T_0)$ as the solution of the problem

$$\begin{aligned}
\partial_t u^* + \Delta^2 u^* + \Delta(\eta_1 v^*) &= f_1, & t \in [0,T_0],\ x \in \Omega,\\
\partial_t v^* - a_0\Delta v^* + \eta_2\partial_t u^* &= a_0 f_2, & t \in [0,T_0],\ x \in \Omega,\\
\partial_\nu\Delta u^* + \partial_\nu(\eta_1 v^*) &= -g_1 - e^{-B^2 t}g_0, & t \in [0,T_0],\ x \in \partial\Omega,\\
\partial_\nu u^* &= g_2, & t \in [0,T_0],\ x \in \partial\Omega,\\
\partial_\nu v^* &= g_3, & t \in [0,T_0],\ x \in \partial\Omega,\\
u^*(0) = \psi_0,\ v^*(0) &= \vartheta_0, & t = 0,\ x \in \Omega,
\end{aligned} \tag{3.2}$$

where $B = -\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator on $\partial\Omega$ and $e^{-B^2 t}$ is the analytic semigroup which is generated by $-B^2$. Furthermore $g_0 = 0$ if $p < 5$ and $g_0 = -g_1|_{t=0} - (\partial_\nu\Delta\psi_0 + \partial_\nu(\eta_1\vartheta_0))$ if $p > 5$.

Define a linear operator $\mathbb{L} : {}_0\mathbb{E}_1(T_0) \to {}_0\mathbb{E}_0(T_0)$ by

$$\mathbb{L}(u,v) = \begin{bmatrix} \partial_t u + \Delta^2 u + \eta_1\Delta v \\ \partial_t v - a_0\Delta v + \eta_2\partial_t u \\ \partial_\nu\Delta u + \partial_\nu(\eta_1 v) \\ \partial_\nu u \\ \partial_\nu v \end{bmatrix}.$$

Then, by Theorem 2.1, the operator $\mathbb{L} : {}_0\mathbb{E}_1(T_0) \to {}_0\mathbb{E}_0(T_0)$ is bounded and bijective, hence an isomorphism with bounded inverse $\mathbb{L}^{-1}$. For all $(u,v) \in {}_0\mathbb{E}_1(T)$ we set

$$\begin{aligned}
G_1(u,v) &= (\lambda'(\psi_0) - \lambda'(u))v + \Phi'(u),\\
G_2(u,v) &= (a_0\lambda'(\psi_0) - a(v)\lambda'(u))\partial_t u - (a_0 - a(v))\Delta v - (a_0 - a(v))f_2,
\end{aligned}$$

where $a(v(t,x)) = 1/b'(v(t,x))$ and $a_0 = a(\vartheta_0)$. Lastly we define a nonlinear mapping $G : \mathbb{E}_1(T) \times {}_0\mathbb{E}_1(T) \to {}_0\mathbb{E}_0(T)$ by

$$G((u^*,v^*);(u,v)) = \begin{bmatrix} \Delta G_1(u+u^*, v+v^*) \\ G_2(u+u^*, v+v^*) \\ \partial_\nu G_1(u+u^*, v+v^*) - \tilde{g}_0 \\ 0 \\ 0 \end{bmatrix},$$

where $\tilde{g}_0 = 0$ if $p < 5$ and $\tilde{g}_0 = e^{-B^2 t}\partial_\nu G_1(\psi_0, \vartheta_0)$ if $p > 5$. Then it is easy to see that $\psi = u + u^* \in E_1(T)$ and $\vartheta = v + v^* \in E_2(T)$ is a solution of (1.2) if and only if

$$\mathbb{L}(u,v) = G((u^*,v^*);(u,v))$$

or equivalently

$$(u,v) = \mathbb{L}^{-1}G((u^*,v^*);(u,v)).$$

In order to apply the contraction mapping principle we consider a ball $\mathbb{B}_R = \mathbb{B}_R^1 \times \mathbb{B}_R^2 \subset {}_0\mathbb{E}_1(T)$, where $R \in (0,1]$. Furthermore we define a mapping $\mathcal{T} : \mathbb{B}_R \to {}_0\mathbb{E}_1(T)$ by $\mathcal{T}(u,v) = \mathbb{L}^{-1}G((u^*,v^*);(u,v))$. We shall prove that $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$ and that $\mathcal{T}$ defines a strict contraction on $\mathbb{B}_R$. To this end we define the shifted ball $\mathbb{B}_R(u^*,v^*) = \mathbb{B}_R^1(u^*) \times \mathbb{B}_R^2(v^*) \subset \mathbb{E}_1(T)$ by

$$\mathbb{B}_R(u^*,v^*) = \{(u,v) \in \mathbb{E}_1(T) : (u,v) = (\tilde{u},\tilde{v}) + (u^*,v^*),\ (\tilde{u},\tilde{v}) \in \mathbb{B}_R\}.$$

To ensure that the mapping G_2 is well defined, we choose $T_0 > 0$ and $R > 0$ sufficiently small. This yields that all functions $v \in \mathbb{B}_R^2(v^*)$ have only a small deviation from the initial value ϑ_0. To see this, write

$$|\vartheta_0(x) - v(t,x)| \le |\vartheta_0(x) - v^*(t,x)| + |v^*(t,x) - v(t,x)| \le \mu(T) + R,$$

for all functions $v \in \mathbb{B}_R^2(v^*)$, where $\mu = \mu(T)$ is defined by

$$\mu(T) = \max_{(t,x)\in[0,T]\times\Omega} |v^*(t,x) - \vartheta_0(x)|.$$

Observe that $\mu(T) \to 0$ as $T \to 0$, by the continuity of v^* and ϑ_0. This in turn implies that $v(t,x) \ge \sigma/2 > 0$ and $b'(v(t,x)) \ge \sigma/2 > 0$ for $(t,x) \in [0,T] \times \bar{\Omega}$ and all $v \in \mathbb{B}_R^2(v^*)$, with $T_0 > 0$, $R > 0$ being sufficiently small. Moreover, for all $v, \bar{v} \in \mathbb{B}_R^2(v^*)$ we obtain the estimates

$$|a(\vartheta_0(x)) - a(v(t,x))| \le C|\vartheta_0(x) - v(t,x)| \tag{3.3}$$

and

$$|a(\bar{v}(t,x)) - a(v(t,x))| \le C|\bar{v}(t,x) - v(t,x)|, \tag{3.4}$$

valid for all $(t,x) \in [0,T] \times \bar{\Omega}$, with some constant $C > 0$, since b' is locally Lipschitz continuous.

The next proposition provides all the facts to show the desired properties of the operator $\mathcal{T}$.

Proposition 3.1. *Let $n \in \mathbb{N}$ and $p > (n+2)/2$, $p \geq 2$, $b \in C^{2-}(0,\infty)$, $b'(s) > 0$ for all $s > 0$, $\lambda, \Phi \in C^{4-}(\mathbb{R})$ and $\vartheta_0(x) > 0$ for all $x \in \bar{\Omega}$. Then there exists a constant $C > 0$, independent of T, and functions $\mu_j = \mu_j(T)$ with $\mu_j(T) \to 0$ as $T \to 0$, such that for all $(u,v), (\bar{u}, \bar{v}) \in \mathbb{B}_R(u^*, v^*)$ the following statements hold.*

1. $|\Delta G_1(u,v) - \Delta G_1(\bar{u},\bar{v})|_{X(T)} \leq (\mu_1(T) + R)|(u,v) - (\bar{u},\bar{v})|_{\mathbb{E}_1(T)}$,
2. $|G_2(u,v) - G_2(\bar{u},\bar{v})|_{X(T)} \leq C(\mu_2(T) + R)|(u,v) - (\bar{u},\bar{v})|_{\mathbb{E}_1(T)}$,
3. $|\partial_\nu G_1(u,v) - \partial_\nu G_1(\bar{u},\bar{v})|_{Y_1(T)} \leq C(\mu_3(T) + R)|(u,v) - (\bar{u},\bar{v})|_{\mathbb{E}_1(T)}$.

The proof is given in the Appendix.

It is now easy to verify the self-mapping property of $\mathcal{T}$. Let $(u,v) \in \mathbb{B}_R$. By Proposition 3.1 there exists a function $\mu = \mu(T)$ with $\mu(T) \to 0$ as $T \to 0$ such that

$$\begin{aligned}
|\mathcal{T}(u,v)|_1 &= |\mathbb{L}^{-1} G((u^*,v^*),(u,v))|_1 \leq |\mathbb{L}^{-1}||G((u^*,v^*),(u,v))|_0 \\
&\leq C(|G((u^*,v^*),(u,v)) - G((u^*,v^*),(0,0))|_0 + |G((u^*,v^*),(0,0))|_0) \\
&\leq C(|\Delta G_1(u+u^*, v+v^*) - \Delta G_1(u^*,v^*)|_{X(T)} \\
&\qquad + |G_2(u+u^*, v+v^*) - G_2(u^*,v^*)|_{X(T)} \\
&\qquad + |\partial_\nu G_1(u+u^*, v+v^*) - \partial_\nu G_1(u^*,v^*)|_{Y_1(T)} \\
&\qquad + |G((u^*,v^*),(0,0))|_0) \\
&\leq C(\mu(T) + R)|(u,v)|_1 + |G((u^*,v^*),(0,0))|_0 \\
&\leq C(\mu(T) + R)R + |G((u^*,v^*),(0,0))|_0.
\end{aligned}$$

Hence we see that $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$ if T and R are sufficiently small, since $G((u^*,v^*),(0,0))$ is a fixed function. Furthermore for all $(u,v),(\bar{u},\bar{v}) \in \mathbb{B}_R$ we have

$$\begin{aligned}
|\mathcal{T}(u,v) - \mathcal{T}(\bar{u},\bar{v})|_1 &= |\mathbb{L}^{-1}(G((u^*,v^*),(u,v)) - G((u^*,v^*),(\bar{u},\bar{v})))|_1 \\
&\leq |\mathbb{L}^{-1}||G((u^*,v^*),(u,v)) - G((u^*,v^*),(\bar{u},\bar{v}))|_0 \\
&\leq C(|\Delta G_1(u+u^*, v+v^*) - \Delta G_1(\bar{u}+u^*, \bar{v}+v^*)|_{X(T)} \\
&\qquad + |\partial_\nu G_1(u+u^*, v+v^*) - \partial_\nu G_1(\bar{u}+u^*, \bar{v}+v^*)|_{Y_1(T)} \\
&\qquad + |G_2(u+u^*, v+v^*) - G_2(\bar{u}+u^*, \bar{v}+v^*)|_{X(T)}) \\
&\leq C(\mu(T) + R)|(u,v) - (\bar{u},\bar{v})|_1.
\end{aligned}$$

Thus $\mathcal{T}$ is a strict contraction on $\mathbb{B}_R$, if T and R are again small enough. Therefore we may apply the contraction mapping principle to obtain a unique fixed point $(\tilde{u},\tilde{v}) \in \mathbb{B}_R$ of $\mathcal{T}$. In other words the pair $(\psi,\vartheta) = (\tilde{u}+u^*, \tilde{v}+v^*) \in \mathbb{E}_1(T)$ is the unique local solution of (1.2). We summarize the preceding calculations in

Theorem 3.2. *Let $n \in \mathbb{N}$, $p > (n+2)/2$, $p \geq 2$, $p \neq 3, 5$, $b \in C^{3-}(0,\infty)$, $b'(s) > 0$ for all $s > 0$ and let $\lambda, \Phi \in C^{4-}(\mathbb{R})$. Then there exists an interval $J = [0,T] \subset [0,T_0] = J_0$ and a unique solution (ψ,ϑ) of (1.2) on J, with*

$$\psi \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega))$$

and

$$\vartheta \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega)), \quad \vartheta(t,x) > 0 \text{ for all } (t,x) \in J \times \bar{\Omega},$$

provided the data are subject to the following conditions.

1. $f_1, f_2 \in L_p(J_0 \times \Omega)$,
2. $g_1 \in W_p^{1/4-1/4p}(J_0; L_p(\partial\Omega)) \cap L_p(J_0; W_p^{1-1/p}(\partial\Omega))$,
3. $g_2 \in W_p^{3/4-1/4p}(J_0; L_p(\partial\Omega)) \cap L_p(J_0; W_p^{3-1/p}(\partial\Omega))$,
4. $g_3 \in W_p^{1/2-1/2p}(J_0; L_p(\partial\Omega)) \cap L_p(J_0; W_p^{1-1/p}(\partial\Omega))$,
5. $\psi_0 \in B_{pp}^{4-4/p}(\Omega)$, $\vartheta_0 \in B_{pp}^{2-2/p}(\Omega)$,
6. $\partial_\nu \Delta\psi_0 - \partial_\nu \Phi'(\psi_0) + \partial_\nu(\lambda'(\psi_0)\vartheta_0) = -g_1|_{t=0}$, *if* $p > 5$,
7. $\partial_\nu \psi_0 = g_2|_{t=0}$, $\partial_\nu \vartheta_0 = g_3|_{t=0}$, *if* $p > 3$,
8. $\vartheta_0(x) > 0$ *for all* $x \in \bar{\Omega}$.

The solution depends continuously on the given data and if the data are independent of t, *the map* $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ *defines a local semiflow on the natural (nonlinear) phase manifold*

$$\mathcal{M}_p := \{(\psi_0, \vartheta_0) \in B_{pp}^{4-4/p}(\Omega) \times B_{pp}^{2-2/p}(\Omega) : \psi_0 \text{ and } \vartheta_0 \text{ satisfy } 6. - 8.\}.$$

4. Global well-posedness

In this section we will investigate the global existence of the solution to the conserved Penrose-Fife type system

$$\begin{aligned} \partial_t\psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t > 0,\ x \in \Omega, \\ \partial_t\left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = 0, &\quad t > 0,\ x \in \Omega, \\ \partial_\nu\mu = 0,\ \partial_\nu\psi = 0,\ \partial_\nu\vartheta = 0, &\quad t > 0,\ x \in \partial\Omega, \\ \psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega, \end{aligned} \tag{4.1}$$

with respect to time if the spatial dimension n is less or equal to 3. Note that the boundary conditions are equivalent to $\partial_\nu\vartheta = \partial_\nu\psi = \partial_\nu\Delta\psi = 0$. A successive application of Theorem 3.2 yields a maximal interval of existence $J_{\max} = [0, T_{\max})$ for the solution $(\psi, \vartheta) \in E_1(T) \times E_2(T)$ of (4.1), where $T \in (0, T_{\max})$. In the sequel we will make use of the following assumptions.

(H1) $\Phi \in C^{4-}(\mathbb{R})$ and there exist some constants $c_j > 0$, $\gamma \geq 0$ such that

$$\Phi(s) \geq -\frac{\eta}{2}s^2 - c_1,\ |\Phi'''(s)| \leq c_2(1 + |s|^\gamma),$$

for all $s \in \mathbb{R}$, where $\eta < \lambda_1$ with λ_1 being the smallest nontrivial eigenvalue of the negative Laplacian on Ω with Neumann boundary conditions and $\gamma < 3$ if $n = 3$.

(H2) $\lambda \in C^{4-}(\mathbb{R})$ and $\lambda'', \lambda''' \in L_\infty(\mathbb{R})$. In particular, there is a constant $c > 0$ such that $|\lambda'(s)| \leq c(1 + |s|)$ for all $s \in \mathbb{R}$.

(H3) $b \in C^{3-}((0,\infty))$, $b'(s) > 0$ on $(0,\infty)$ and there is a constant $\kappa > 1$ such that

$$\frac{1}{\kappa} \leq \vartheta(t,x) \leq \kappa$$

on $J_{\max} \times \Omega$. In particular, there exists $\sigma > 1$ such that

$$\frac{1}{\sigma} \leq b'(\vartheta(t,x)) \leq \sigma,$$

on $J_{\max} \times \Omega$.

Remark: Condition (H1) is certainly fulfilled, if Φ is a polynomial of degree $2m$, $m < 3$.

We prove global well-posedness with respect to time by contradiction. For this purpose, assume that $T_{\max} < \infty$. Multiply $\partial_t\psi = \Delta\mu$ by μ and integrate by parts to the result

$$\frac{d}{dt}\left(\frac{1}{2}|\nabla\psi|_2^2 + \int_\Omega \Phi(\psi)\ dx\right) + |\nabla\mu|_2^2 - \int_\Omega \lambda'(\psi)\vartheta\partial_t\psi\ dx = 0. \tag{4.2}$$

Next we multiply $(4.1)_2$ by ϑ and integrate by parts. This yields

$$\int_\Omega \vartheta b'(\vartheta)\partial_t\vartheta\ dx + |\nabla\vartheta|_2^2 + \int_\Omega \lambda'(\psi)\vartheta\partial_t\psi\ dx = 0. \tag{4.3}$$

Set $\beta'(s) = sb'(s)$ and add (4.2) to (4.3) to obtain the equation

$$\frac{d}{dt}\Big(\frac{1}{2}|\nabla\psi|_2^2 + \int_\Omega \Phi(\psi)\ dx + \int_\Omega \beta(\vartheta)\ dx\Big) + |\nabla\mu|_2^2 + |\nabla\vartheta|_2^2 = 0. \tag{4.4}$$

Integrating (4.4) with respect to t, we obtain

$$E(\psi(t),\vartheta(t)) + \int_0^t \left(|\nabla\mu(s)|_2^2 + |\nabla\vartheta(s)|_2^2\right)\ dt = E(\psi_0,\vartheta_0), \tag{4.5}$$

for all $t \in J_{\max}$, where

$$E(u,v) := \frac{1}{2}|\nabla u|_2^2 + \int_\Omega \Phi(u)\ dx + \int_\Omega \beta(v)\ dx.$$

It follows from (H1) and the Poincaré-Wirtinger inequality that

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega |\nabla\psi(t)|^2\ dx + \frac{1-\varepsilon}{2}\int_\Omega |\nabla\psi(t)|^2\ dx + \int_\Omega \Phi(\psi(t))\ dx \\
&\quad \geq \frac{\varepsilon}{2}\int_\Omega |\nabla\psi(t)|^2\ dx + \frac{(1-\varepsilon)\lambda_1 - \eta}{2}|\psi(t)|_2^2 - c_1|\Omega| - \frac{\lambda_1}{2|\Omega|}\left(\int_\Omega \psi_0\ dx\right),
\end{aligned}$$

since by equation $\partial_t\psi = \Delta\mu$ and the boundary condition $\partial_\nu\mu = 0$, it holds that

$$\int_\Omega \psi(t,x)\ dx \equiv \int_\Omega \psi_0(x)\ dx, \quad t \in J_{\max}.$$

Hence for a sufficiently small $\varepsilon > 0$ we obtain the a priori estimates

$$\psi \in L_\infty(J_{\max}; H_2^1(\Omega)) \quad \text{and} \quad |\nabla\mu|, |\nabla\vartheta| \in L_2(J_{\max}; L_2(\Omega)), \tag{4.6}$$

since $\beta(\vartheta(t,x))$ is uniformly bounded on $J_{\max}\times\Omega$, by (H3). However, things are more involved for higher-order estimates. Here we have the following result.

Proposition 4.1. *Let $n \le 3$, $p > (n+2)/2$, $p \ge 2$ and let (ψ,ϑ) be the maximal solution of* (4.1) *with initial value $\psi_0 \in B^{4-4/p}_{pp}(\Omega)$ and $\vartheta_0 \in B^{2-2/p}_{pp}(\Omega)$. Suppose furthermore $b \in C^{3-}(0,\infty)$, $b'(s) > 0$ for all $s > 0$, $\lambda, \Phi \in C^{4-}(\mathbb{R})$ and let* (H1)–(H3) *hold.*

Then $\psi \in L_\infty(J_{\max}\times\Omega)$ and $\vartheta \in H^1_2(J_{\max};L_2(\Omega)) \cap L_\infty(J_{\max};H^1_2(\Omega))$. Moreover, it holds that $\partial_t\psi \in L_r(J_{\max}\times\Omega)$, where $r := \min\{p, 2(n+4)/n\}$.

Proof. The proof is given in the Appendix. □

Define the new function $u = b(\vartheta)$. Then u satisfies the nonautonomous linear differential equation in divergence form

$$\partial_t u - \operatorname{div}(a(t,x)\nabla u) = f, \tag{4.7}$$

subject to the boundary and initial conditions $\partial_\nu u = 0$ and $u(0) = b(\vartheta_0) =: u_0$, where $a(t,x) := 1/b'(\vartheta(t,x))$ and $f := -\lambda'(\psi)\partial_t\psi$. With (H3), the regularity of ϑ from Proposition 4.1 carries over to the function u; in particular $u_0 \in B^{2-2/p}_{pp}(\Omega)$. This yields, that u is a *weak solution* of (4.7) in the sense of Lieberman [12] & DiBenedetto [7], and u is bounded by (H3).

Furthermore, by (H3)

$$0 < \frac{1}{\sigma} \le a(t,x) \le \sigma < \infty,$$

for all $(t,x) \in J_{\max}\times\Omega$. Note that by Proposition 4.1 it holds that $f = -\lambda'(\psi)\partial_t\psi \in L_r(J_{\max}\times\Omega)$, $r := \min\{p, 2(n+4)/n\}$. Consider the case $r = 2(n+4)/n$. Then it can be readily checked that

$$\frac{n+2}{2} < \frac{2(n+4)}{n} = r$$

provided $n \le 5$. It follows from Lieberman [12] & DiBenedetto [7] that there exists a real number $\alpha \in (0,1/2)$ such that $u \in C^{\alpha,2\alpha}(\overline{\Omega_{T_{\max}}})$, provided $f \in L_p(J_{\max}\times\Omega)$ and $p > (n+2)/2$. Here $C^{\alpha,2\alpha}(\overline{\Omega_{T_{\max}}})$ is defined as

$$C^{\alpha,2\alpha}(\overline{\Omega_{T_{\max}}}) := \{v \in C(\overline{\Omega_{T_{\max}}}) : \sup_{(t,x),(s,y)\in\overline{\Omega_{T_{\max}}}} \frac{|v(t,x)-v(s,y)|}{|t-s|^\alpha + |x-y|^{2\alpha}} < \infty\}.$$

and we have set $\Omega_{T_{\max}} = (0,T_{\max})\times\Omega$. The properties of the function b then yield that $\vartheta = b^{-1}(u) \in C^{\alpha,2\alpha}(\overline{\Omega_{T_{\max}}})$. In a next step we solve the initial-boundary value problem

$$\begin{aligned} \partial_t\vartheta - a(t,x)\Delta\vartheta &= g, && t \in J_{\max},\ x \in \Omega,\\ \partial_\nu\vartheta &= 0, && t \in J_{\max},\ x \in \partial\Omega,\\ \vartheta(0) &= \vartheta_0, && t = 0,\ x \in \Omega, \end{aligned} \tag{4.8}$$

with $g := -a(t,x)\lambda'(\psi)\partial_t\psi \in L_r(J_{\max}\times\Omega)$ and $r = 2(n+4)/n > (n+2)/2$. By [6, Theorem 2.1] we obtain

$$\vartheta \in H^1_r(J_{\max}; L_r(\Omega)) \cap L_r(J_{\max}; H^2_r(\Omega)),$$

of (4.8), since

$$\vartheta_0 \in B^{2-2/p}_{pp}(\Omega) \hookrightarrow B^{2-2/r}_{rr}(\Omega), \quad p \geq r.$$

At this point we use equation (6.8) from the proof of Proposition 4.1 to conclude $\partial_t\psi \in L_s(J_{\max}\times\Omega)$, with $s = \min\{p,q\}$ where q is restricted by

$$\frac{1}{q} \geq \frac{1}{r} - \frac{2}{n+4}.$$

For the case $r = 2(n+4)/n$, this yields

$$\frac{1}{q} \geq \frac{n-4}{2(n+4)},$$

i.e., q may be arbitrarily large in case $n \leq 3$ and we may set $s = p$. Now we solve (4.8) again, this time with $g \in L_p(J_{\max}\times\Omega)$, to obtain

$$\vartheta \in H^1_p(J_{\max}; L_p(\Omega)) \cap L_p(J_{\max}; H^2_p(\Omega))$$

and therefore $\vartheta(T_{\max}) \in B^{2-2/p}_{pp}(\Omega)$ is well defined. Next, consider the equation

$$\partial_t\psi + \Delta^2\psi = \Delta\Phi'(\psi) - \Delta(\lambda'(\psi)\vartheta),$$

subject to the initial and boundary conditions $\psi(0) = \psi_0$ and $\partial_\nu\psi = \partial_\nu\Delta\psi = 0$. By maximal L_p-regularity there exists a constant $M = M(J_{\max}) > 0$ such that

$$|\psi|_{E_1(T)} \leq M(1 + |\Delta\Phi'(\psi)|_{X(T)} + |\Delta(\lambda'(\psi)\vartheta)|_{X(T)}). \tag{4.9}$$

for each $T \in J_{\max}$. Since $\vartheta \in E_2(T_{\max})$ we may apply [13, Lemma 4.1] to the result

$$|\Delta\Phi'(\psi)|_{X(T)} + |\Delta(\lambda'(\psi)\vartheta)|_{X(T)} \leq C(1 + |\psi|^\delta_{E_1(T)}), \tag{4.10}$$

with some $\delta \in (0,1)$ and $C > 0$ being independent of $T \in J_{\max}$. Combining (4.9) with (4.10), we obtain the estimate

$$|\psi|_{E_1(T)} \leq C(1 + |\psi|^\delta_{E_1(T)}),$$

which in turn yields that $|\psi|_{E_1(T)}$ is bounded as $T \nearrow T_{\max}$, since $\delta \in (0,1)$. Therefore the value $\psi(T_{\max}) \in B^{4-4/p}_{pp}(\Omega)$ is well defined and we may continue the solution (ψ,ϑ) beyond the point $T_{\max}$, contradicting the assumption that $J_{\max} = [0, T_{\max})$ is the maximal interval of existence.

We summarize these considerations in

Theorem 4.2. *Let $n \leq 3$, $p > (n+2)/2$, $p \geq 2$ and $p \neq 3, 5$. Assume that* (H1)–(H3) *hold. Then for each $T_0 > 0$ there exists a unique solution*

$$\psi \in H^1_p(J_0; L_p(\Omega)) \cap L_p(J_0; H^4_p(\Omega)) = E_1(T_0)$$

and

$$\vartheta \in H^1_p(J_0; L_p(\Omega)) \cap L_p(J_0; H^2_p(\Omega)) = E_2(T_0),$$

of (1.2)*, provided the data are subject to the following conditions:*

1. $\psi_0 \in B_{pp}^{4-4/p}(\Omega)$*,* $\vartheta_0 \in B_{pp}^{2-2/p}(\Omega)$*;*
2. $\partial_\nu \Delta\psi_0 = 0$*, if* $p > 5$*,* $\partial_\nu \psi_0 = 0$*;*
3. $\partial_\nu \vartheta_0 = 0$*, if* $p > 3$*,* $\vartheta_0(x) > 0$ *for all* $x \in \bar{\Omega}$*.*

The solution depends continuously on the given data and the map $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ *defines a semiflow on the natural phase manifold*

$$\mathcal{M}_p := \{(\psi_0, \vartheta_0) \in B_{pp}^{4-4/p}(\Omega) \times B_{pp}^{2-2/p}(\Omega) : \psi_0 \text{ and } \vartheta_0 \text{ satisfy } 2. \ \&\ 3.\}.$$

5. Asymptotic behavior

Let $n \leq 3$. In the following we will investigate the asymptotic behavior of global solutions of the homogeneous system

$$\begin{aligned} \partial_t \psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t > 0,\ x \in \Omega, \\ \partial_t \left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = 0, &\quad t > 0,\ x \in \Omega, \\ \partial_\nu \mu = 0, &\quad t > 0,\ x \in \partial\Omega, \\ \partial_\nu \psi = 0, &\quad t > 0,\ x \in \partial\Omega, \\ \partial_\nu \vartheta = 0, &\quad t > 0,\ x \in \partial\Omega, \\ \psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega, \end{aligned} \tag{5.1}$$

as $t \to \infty$. To this end let $(\psi_0, \vartheta_0) \in \mathcal{M}_p$, $p > (n+2)/2$, $p \geq 2$ and denote by $(\psi(t), \vartheta(t))$ the unique global solution of (5.1). In the sequel we will make use of the following assumptions.

(H4) $b \in C^{3-}((0,\infty))$, $b'(s) > 0$ on $(0,\infty)$ and there is a constant $\kappa > 1$ such that

$$\frac{1}{\kappa} \leq \vartheta(t,x) \leq \kappa$$

on $J_{\max} \times \Omega$. In particular, there exists $\sigma > 1$ such that

$$\frac{1}{\sigma} \leq b'(\vartheta(t,x)) \leq \sigma,$$

on $J_{\max} \times \Omega$.

(H5) The functions Φ, λ and b are real analytic on $\mathbb{R}$.

We remark that assumption (H4) is identical to (H3) for a global solution. We stated it here for the sake of readability.

Note that the boundary conditions $(5.1)_{3,5}$ yield

$$\int_\Omega \psi(t,x)\ dx \equiv \int_\Omega \psi_0(x)\ dx,$$

and

$$\int_\Omega (b(\vartheta(t,x)) + \lambda(\psi(t,x)))\ dx \equiv \int_\Omega (b(\vartheta_0(x)) + \lambda(\psi_0(x)))\ dx.$$

Replacing ψ by $\tilde{\psi} = \psi - c$, where $c := \frac{1}{|\Omega|}\int_\Omega \psi_0(x)\,dx$ we see that $\int_\Omega \tilde{\psi}\,dx \equiv 0$, if $\Phi(s)$ and $\lambda(s)$ are replaced by $\tilde{\Phi}(s) = \Phi(s+c)$ and $\tilde{\lambda}(s) = \lambda(s+c)$, respectively. Similarly we can achieve that

$$\int_\Omega (b(\vartheta(t,x)) + \lambda(\psi(t,x)))\,dx \equiv 0,$$

by a shift of λ, to be precise $\bar{\lambda}(s) := \lambda(s) - d$, where

$$d := \frac{1}{|\Omega|}\int_\Omega (b(\vartheta_0(x)) + \lambda(\psi_0(x)))\,dx.$$

With these modifications of the data we obtain the constraints

$$\int_\Omega \psi(t,x)\,dx \equiv 0 \quad \text{and} \quad \int_\Omega (b(\vartheta(t,x)) + \lambda(\psi(t,x)))\,dx \equiv 0. \tag{5.2}$$

Recall from Section 4 the energy functional

$$E(u,v) = \frac{1}{2}|\nabla u|_2^2 + \int_\Omega \Phi(u)\,dx + \int_\Omega \beta(v)\,dx,$$

defined on the energy space $V = V_1 \times V_2$, where

$$V_1 := \left\{u \in H_2^1(\Omega) : \int_\Omega u\,dx = 0\right\}, \qquad V_2 := H_2^r(\Omega),\ r \in (n/4, 1).$$

and V is equipped with the canonical norm $|(u,v)|_V := |u|_{H_2^1(\Omega)} + |v|_{H_2^r(\Omega)}$. It is convenient to embed V into a Hilbert space $H = H_1 \times H_2$ where

$$H_1 := \left\{u \in L_2(\Omega) : \int_\Omega u\,dx = 0\right\} \quad \text{and} \quad H_2 := L_2(\Omega).$$

Proposition 5.1. *Let* $(\psi,\vartheta) \in E_1 \times E_2$ *be a global solution of* (5.1) *and assume* (H1)–(H4). *Then*

1. $\psi \in L_\infty(\mathbb{R}_+; H_p^{2s}(\Omega)),\ s \in [0,1),\ p \in (1,\infty),\ \partial_t\psi \in L_2(\mathbb{R}_+ \times \Omega)$*;*
2. $\vartheta \in L_\infty(\mathbb{R}_+; H_2^1(\Omega)),\ \partial_t\vartheta \in L_2(\mathbb{R}_+ \times \Omega)$.

In particular the orbits $\psi(\mathbb{R}_+)$ *and* $\vartheta(\mathbb{R}_+)$ *are relatively compact in* $H_2^1(\Omega)$ *and* $H_2^r(\Omega)$*, respectively, where* $r \in [0,1)$.

Proof. Assertions 1 & 2 follow directly from (H1)–(H4) and the proof of Proposition 4.1, which is given in the Appendix. Indeed, one may replace the interval $J_{\max}$ by $\mathbb{R}_+$, since the operator $-A^2 = -\Delta_N^2$ generates an exponentially stable, analytic semigroup e^{-A^2t} in the space

$$\mathbb{X}_p := \{u \in L_p(\Omega) : \int_\Omega u\,dx = 0\}$$

with domain

$$D(A^2) = \{u \in H_p^4(\Omega) \cap \mathbb{X}_p : \partial_\nu u = \partial_\nu \Delta u = 0 \text{ on } \partial\Omega\}. \qquad \square$$

By Assumption (H4), there exists some bounded interval $J_\vartheta \subset \mathbb{R}_+$ with $\vartheta(t,x) \in J_\vartheta$ for all $t \geq 0,\ x \in \Omega$. Therefore we may modify the nonlinearities b and β outside J_ϑ in such a way that $b, \beta \in C_b^{3-}(\mathbb{R})$.

Unfortunately the energy functional E is not yet the right one for our purpose, since we have to include the nonlinear constraint

$$\int_\Omega (\lambda(\psi) + b(\vartheta))\ dx = 0,$$

into our considerations. The linear constraint $\int_\Omega \psi\ dx = 0$ is part of the definition of the space H_1. For the nonlinear constraint we use a functional of Lagrangian type which is given by

$$L(u, v) = E(u, v) - \overline{v} F(u, v),$$

defined on V, where $F(u, v) := \int_\Omega (\lambda(u) + b(v))\ dx$ and $\bar{w} = \frac{1}{|\Omega|} \int_\Omega w\ dx$ for a function $w \in L_1(\Omega)$. Concerning the differentiability of L we have the following result.

Proposition 5.2. *Under the conditions* (H1)–(H4)*, the functional L is twice continuously Fréchet differentiable on V and the derivatives are given by*

$$\langle L'(u, v), (h, k)\rangle_{V^*,V} \\ = \langle E'(u, v), (h, k)\rangle_{V^*,V} - \overline{k} F(u, v) - \overline{v} \langle F'(u, v), (h, k)\rangle_{V^*,V} \quad (5.3)$$

and

$$\langle L''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V} = \langle E''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V} \\ - \overline{k_1} \langle F'(u, v), (h_2, k_2)\rangle_{V^*,V} - \overline{k_2} \langle F'(u, v), (h_1, k_1)\rangle_{V^*,V} \\ - \overline{v} \langle F''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V}, \quad (5.4)$$

where $(h, k), (h_j, k_j) \in V,\ j = 1, 2,$ *and*

$$\langle E'(u, v), (h, k)\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h\ dx + \int_\Omega \Phi'(u) h\ dx + \int_\Omega \beta'(v) k\ dx,$$

$$\langle E''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V} \\ = \int_\Omega \nabla h_1 \nabla h_2\ dx + \int_\Omega \Phi''(u) h_1 h_2\ dx + \int_\Omega \beta''(v) k_1 k_2\ dx,$$

$$\langle F'(u, v), (h, k)\rangle_{V^*,V} = \int_\Omega \lambda'(u) h\ dx + \int_\Omega b'(v) k\ dx$$

and

$$\langle F''(u, v)(h_1, k_1), (h_2, k_2)\rangle_{V^*,V} = \int_\Omega \lambda''(u) h_1 h_2\ dx + \int_\Omega b''(v) k_1 k_2\ dx.$$

Proof. We only consider the first derivative, the second one is treated in a similar way. Since the bilinear form

$$a(u, v) := \int_\Omega \nabla u(x) \nabla v(x)\ dx \quad (5.5)$$

defined on $V_1 \times V_1$ is bounded and symmetric, the first term in E is twice continuously Fréchet differentiable. For the functional

$$G_1(u) := \int_\Omega \Phi(u)\, dx, \quad u \in V_1,$$

we argue as follows. With $u, h \in V_1$ it holds that

$$\begin{aligned}
&\Phi(u(x)+h(x)) - \Phi(u(x)) - \Phi'(u(x))h(x)\\
&\qquad = \int_0^1 \frac{d}{dt}\, \Phi(u(x)+th(x))\, dt - \int_0^1 \Phi'(u(x))h(x)\, dt\\
&\qquad = \int_0^1 \Big(\Phi'(u(x)+th(x)) - \Phi'(u(x))\Big) h(x)\, dt\\
&\qquad = \int_0^1 \int_0^t \frac{d}{ds}\, \Phi'(u(x)+sh(x))h(x)\, ds\, dt\\
&\qquad = \int_0^1 \int_0^t \Phi''(u(x)+sh(x))h(x)^2\, ds\, dt\\
&\qquad = \int_0^1 \Phi''(u(x)+sh(x))h(x)^2(1-s)\, ds.
\end{aligned}$$

From the growth condition (H1), Hölder's inequality and the Sobolev embedding theorem it follows that

$$\begin{aligned}
&\Big| \int_\Omega \Big(\Phi(u(x)+h(x)) - \Phi(u(x)) - \Phi'(u(x))h(x)\Big)\, dx \Big|\\
&\qquad \le C \int_\Omega (1 + |u(x)|^4 + |h(x)|^4)|h(x)|^2\, dx\\
&\qquad \le C(1 + |u|_6^4 + |h|_6^4)|h|_6^2\\
&\qquad \le C(1 + |u|_{V_1}^4 + |h|_{V_1}^4)|h|_{V_1}^2.
\end{aligned}$$

This proves that G_1 is Fréchet differentiable and also $G_1'(u) = \Phi'(u) \in L_{6/5}(\Omega) \hookrightarrow V_1^*$. The next step is the proof of the continuity of $G_1' : V_1 \to V_1^*$. We make again use of (H1), the Hölder inequality and the Sobolev embedding theorem to obtain

$$\begin{aligned}
|G_1'(u) - G_1'(\bar u)|_{V_1^*} &\le C \left(\int_\Omega |\Phi'(u(x)) - \Phi'(\bar u(x))|^{\frac{6}{5}}\, dx \right)^{\frac{5}{6}}\\
&\le C \left(\int_\Omega \int_0^1 |\Phi''(tu(x) + (1-t)\bar u(x))|^{\frac{6}{5}} |u(x) - \bar u(x)|^{\frac{6}{5}}\, dt\, dx \right)^{\frac{5}{6}}\\
&\le C \left(\int_\Omega (1 + |u(x)|^{\frac{24}{5}} + |\bar u(x)|^{\frac{24}{5}}) |u(x) - \bar u(x)|^{\frac{6}{5}}\, dx \right)^{\frac{5}{6}}\\
&\le C \left(\int_\Omega (1 + |u(x)|^6 + |\bar u(x)|^6)\, dx \right)^{\frac{2}{3}} \left(\int_\Omega |u(x) - \bar u(x)|^6 \right)^{\frac{1}{6}}\\
&\le C(1 + |u|_{V_1}^4 + |\bar u|_{V_1}^4)|u - \bar u|_{V_1}.
\end{aligned}$$

Actually this proves that G_1' is even locally Lipschitz continuous on V_1. The Fréchet differentiability of G_1' and the continuity of G_1'' can be proved in an analogue way. The fundamental theorem of differential calculus and the Sobolev embedding theorem yield the estimate

$$|\Phi'(u+h) - \Phi'(u) - \Phi''(u)h|_{V_1^*} \le C\left(\int_\Omega \int_0^1 |\Phi'''(u(x)+sh(x))|^{\frac{6}{5}} |h(x)|^{\frac{12}{5}}\ ds\ dx\right)^{\frac{5}{6}}.$$

We apply Assumption (H1) and Hölder's inequality to the result

$$\begin{aligned}|\Phi'(u+h) - \Phi'(u) - \Phi''(u)h|_{V_1^*} &\le C\left(\int_\Omega (1+|u(x)|^{\frac{18}{5}} + |h(x)|^{\frac{18}{5}})|h(x)|^{\frac{12}{5}}\ dx\right)^{\frac{5}{6}} \\ &\le C\left(\int_\Omega (1+|u(x)|^6 + |h(x)|^6)\ dx\right)^{\frac{1}{2}} \left(\int_\Omega |h(x)|^6\ dx\right)^{\frac{1}{3}} \\ &= C(1+|u|_{V_1}^3 + |h|_{V_1}^3)|h|_{V_1}^2.\end{aligned}$$

Hence the Fréchet derivative is given by the multiplication operator $G_1''(u)$ defined by $G_1''(u)v = \Phi''(u)v$ for all $v \in V_1$ and $\Phi''(u) \in L_{3/2}(\Omega)$. We will omit the proof of continuity of G_1''. The way to show the C^2-property of the functional

$$G_2(u) := \int_\Omega \lambda(u(x))\ dx, \quad u \in V_1,$$

is identical to the one above, by Assumption (H2). Concerning the C^2-differentiability of the functionals

$$G_3(v) := \int_\Omega \beta(v(x))\ dx \quad \text{and} \quad G_4(v) := \int_\Omega b(v(x))\ dx, \quad v \in V_2,$$

one may adopt the proof for G_1 and G_2. In fact, this time it is easier, since β and b are assumed to be elements of the space $C_b^{3-}(\mathbb{R})$, however one needs the assumption $r \in (n/4, 1)$. We will skip the details. Finally the product rule of differentiation yields that L is twice continuously Fréchet differentiable on $V_1 \times V_2$. $\square$

The corresponding stationary system to (5.1) will be of importance for the forthcoming calculations. Setting all time-derivatives in (5.1) equal to 0 yields

$$\Delta\mu = 0 \quad \text{and} \quad \Delta\vartheta = 0,$$

subject to the boundary conditions $\partial_\nu\mu = \partial_\nu\vartheta = 0$. Thus we have $\mu \equiv \mu_\infty = \text{const}$, $\vartheta \equiv \vartheta_\infty = \text{const}$ and there remains the nonlinear elliptic problem of second order

$$\begin{cases} -\Delta\psi_\infty + \Phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty = \mu_\infty, \quad x \in \Omega, \\ \partial_\nu\psi_\infty = 0, \quad x \in \partial\Omega, \end{cases} \tag{5.6}$$

with the constraints (5.2) for the unknowns ψ_∞ and ϑ_∞. The following proposition collects some properties of the functional L and the ω-limit set

$$\omega(\psi,\vartheta) := \{(\varphi,\theta) \in V_1 \times V_2 : \exists\ (t_n) \nearrow \infty \text{ s.t.} \\ (\psi(t_n),\vartheta(t_n)) \to (\varphi,\theta) \text{ in } V_1 \times V_2\}.$$

Proposition 5.3. *Under Hypotheses* (H1)–(H4) *the following assertions are true.*

1. *The ω-limit set is nonempty, connected and compact.*
2. *Each point $(\psi_\infty,\vartheta_\infty) \in \omega(\psi,\vartheta)$ is a strong solution of the stationary problem* (5.6)*, where $\vartheta_\infty, \mu_\infty = \mathrm{const}$ and $(\psi_\infty,\vartheta_\infty)$ satisfies the constraints* (5.2) *for the unknowns $\vartheta_\infty, \mu_\infty$.*
3. *The functional L is constant on $\omega(\psi,\vartheta)$ and each point $(\psi_\infty,\vartheta_\infty) \in \omega(\psi,\vartheta)$ is a critical point of L, i.e., $L'(\psi_\infty,\vartheta_\infty) = 0$ in V^*.*

Proof. The fact that $\omega(\psi,\vartheta)$ is nonempty, connected and compact follows from Proposition 5.1 and some well-known facts in the theory of dynamical systems.

Now we turn to 2. Let $(\psi_\infty,\vartheta_\infty) \in \omega(\psi,\vartheta)$. Then there exists a sequence $(t_n) \nearrow +\infty$ such that $(\psi(t_n),\vartheta(t_n)) \to (\psi_\infty,\vartheta_\infty)$ in V as $n \to \infty$. Since $\partial_t\psi, \partial_t\vartheta \in L_2(\mathbb{R}_+ \times \Omega)$ it follows that $\psi(t_n+s) \to \psi_\infty$ and $\vartheta(t_n+s) \to \vartheta_\infty$ in $L_2(\Omega)$ for all $s \in [0,1]$ and by relative compactness also in V. This can be seen as follows.

$$\begin{aligned}|\psi(t_n+s) - \psi_\infty|_2 &\le |\psi(t_n+s) - \psi(t_n)|_2 + |\psi(t_n) - \psi_\infty|_2 \\ &\le \int_{t_n}^{t_n+s} |\partial_t\psi(t)|_2\ dt + |\psi(t_n) - \psi_\infty|_2 \\ &\le s^{1/2}\left(\int_{t_n}^{t_n+s} |\partial_t\psi(t)|_2^2\ dt\right)^{1/2} + |\psi(t_n) - \psi_\infty|_2.\end{aligned}$$

Then, for $t_n \to \infty$ this yields $\psi(t_n+s) \to \psi_\infty$ for all $s \in [0,1]$. The proof for ϑ is the same. Integrating (4.4) with $f_1 = f_2 = 0$ from t_n to t_n+1 we obtain

$$E(\psi(t_n+1),\vartheta(t_n+1)) - E(\psi(t_n),\vartheta(t_n)) \\ + \int_0^1 \int_\Omega \left(|\nabla\mu(t_n+s,x)|^2 + |\nabla\vartheta(t_n+s,x)|^2\right)\ dx\ ds = 0.$$

Letting $t_n \to +\infty$ yields

$$|\nabla\mu(t_n+\cdot,\cdot)|, |\nabla\vartheta(t_n+\cdot,\cdot)| \to 0 \quad \text{in } L_2([0,1]\times\Omega).$$

This in turn yields a subsequence (t_{n_k}) such that $\nabla\mu(t_{n_k}+s), \nabla\vartheta(t_{n_k}+s) \to 0$ in $L_2(\Omega;\mathbb{R}^n)$ for a.e. $s \in [0,1]$. Hence $\nabla\vartheta_\infty = 0$, since the gradient is a closed operator in $L_2(\Omega;\mathbb{R}^n)$. This in turn yields that ϑ_∞ is a constant.

Furthermore the Poincaré-Wirtinger inequality implies that

$$
\begin{aligned}
&|\mu(t_{n_k}+s^*)-\mu(t_{n_l}+s^*)|_2 \\
&\le C_p\biggl(|\nabla\mu(t_{n_k}+s^*)-\nabla\mu(t_{n_l}+s^*)|_2+\int_\Omega|\Phi'(\psi(t_{n_k}+s^*))-\Phi'(\psi(t_{n_l}+s^*))|\,dx \\
&\qquad +\int_\Omega|\lambda'(\psi(t_{n_k}+s^*))\vartheta(t_{n_k}+s^*)-\lambda'(\psi(t_{n_l}+s^*))\vartheta(t_{n_l}+s^*)|\,dx\biggr),
\end{aligned}
$$

for some $s^*\in[0,1]$. Taking the limit $k,l\to\infty$ we see that $\mu(t_{n_k}+s^*)$ is a Cauchy sequence in $L_2(\Omega)$, hence it admits a limit, which we denote by μ_∞. In the same manner as for ϑ_∞ we therefore obtain $\nabla\mu_\infty=0$, hence μ_∞ is a constant. Observe that the relation

$$
\mu_\infty=\frac{1}{|\Omega|}\left(\int_\Omega(\Phi'(\psi_\infty)-\lambda'(\psi_\infty)\vartheta_\infty)\ dx\right)
$$

is valid. Multiplying $(5.1)_1$ by a function $\varphi\in H_2^1(\Omega)$ and integrating by parts we obtain

$$
\begin{aligned}
(\mu(t_{n_k}+s^*),\varphi)_2 &= (\nabla\psi(t_{n_k}+s^*),\nabla\varphi)_2 \\
&\quad+(\Phi'(\psi(t_{n_k}+s^*)),\varphi)_2-(\lambda'(\psi(t_{n_k}+s^*))\vartheta(t_{n_k}+s^*),\varphi)_2.
\end{aligned}
$$

As $t_{n_k}\to\infty$ it follows that

$$
(\mu_\infty,\varphi)_2=(\nabla\psi_\infty,\nabla\varphi)_2+(\Phi'(\psi_\infty),\varphi)_2-\vartheta_\infty(\lambda'(\psi_\infty),\varphi)_2. \tag{5.7}
$$

By the Lax-Milgram theorem the bounded, symmetric and elliptic form

$$
a(u,v):=\int_\Omega\nabla u\nabla v\ dx,
$$

defined on the space $V_1\times V_1$ induces a bounded operator $A:V_1\to V_1^*$ with nonempty resolvent, such that

$$
a(u,v)=\langle Au,v\rangle_{V_1^*,V_1},
$$

for all $(u,v)\in V_1\times V_1$. It is well known that the domain of the part A_p of the operator A in

$$
\mathbb{X}_p=\{u\in L_p(\Omega):\int_\Omega u\ dx=0\}
$$

is given by

$$
D(A_p)=\{u\in\mathbb{X}_p\cap H_p^2(\Omega),\ \partial_\nu u=0\}.
$$

Going back to (5.7) we obtain from (H1) and (H2) that $\psi_\infty\in D(A_q)$, where $q=6/(\beta+2)$. Since $q>6/5$ we may apply a bootstrap argument to conclude $\psi_\infty\in D(A_2)$. Integrating (5.7) by parts, assertion 2 follows.

In order to prove 3., we make use of (5.3) to obtain

$$\begin{aligned}
\langle L'(\psi_\infty,\vartheta_\infty),(h,k)\rangle_{V^*,V} \\
&= \langle E'(\psi_\infty,\vartheta_\infty),(h,k)\rangle_{V^*,V} - \vartheta_\infty\langle F'(\psi_\infty,\vartheta_\infty),(h,k)\rangle_{V^*,V} \\
&= \int_\Omega (-\Delta\psi_\infty + \Phi'(\psi_\infty))h\; dx + \int_\Omega \beta'(\vartheta_\infty)k\; dx \\
&\qquad\qquad - \vartheta_\infty \int_\Omega (\lambda'(\psi_\infty)h + b'(\vartheta_\infty)k)\; dx \\
&= \int_\Omega \mu_\infty h\; dx = 0,
\end{aligned}$$

for all $(h,k)\in V$, since μ_∞ and ϑ_∞ are constant. A continuity argument finally yields the last statement of the proposition. □

The following result is crucial for the proof of convergence.

Proposition 5.4 (Lojasiewicz-Simon inequality). *Let* $(\psi_\infty,\vartheta_\infty)\in\omega(\psi,\vartheta)$ *and assume* (H1)–(H5). *Then there exist constants* $s\in(0,\frac{1}{2}], C,\delta>0$ *such that*

$$|L(u,v)-L(\psi_\infty,\vartheta_\infty)|^{1-s} \le C|L'(u,v)|_{V^*},$$

whenever $|(u,v)-(\psi_\infty,\vartheta_\infty)|_V\le\delta$.

Proof. We show first that $\dim N(L''(\psi_\infty,\vartheta_\infty))<\infty$. By (5.4) we obtain

$$\begin{aligned}
\langle L''(\psi_\infty,\vartheta_\infty)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} \\
&= \int_\Omega \nabla h_1\nabla h_2\; dx + \int_\Omega \Phi''(\psi_\infty)h_1h_2\; dx + \int_\Omega \beta''(\vartheta_\infty)k_1k_2\; dx \\
&\quad - \overline{k_1}\int_\Omega (\lambda'(\psi_\infty)h_2 + b'(\vartheta_\infty)k_2)\; dx \\
&\quad - \overline{k_2}\int_\Omega (\lambda'(\psi_\infty)h_1 + b'(\vartheta_\infty)k_1)\; dx \\
&\quad - \overline{\vartheta_\infty}\int_\Omega (\lambda''(\psi_\infty)h_1h_2 + b''(\vartheta_\infty)k_1k_2)\; dx.
\end{aligned}$$

Since $\beta''(s)=b'(s)+sb''(s)$ and $\vartheta_\infty\equiv$ const we have

$$\begin{aligned}
\langle L''(\psi_\infty,\vartheta_\infty)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} \\
&= \int_\Omega \nabla h_1\nabla h_2\; dx + \int_\Omega \left(\Phi''(\psi_\infty)h_1 - \overline{k_1}\lambda'(\psi_\infty) - \vartheta_\infty\lambda''(\psi_\infty)h_1\right)h_2\; dx \\
&\quad + \int_\Omega (b'(\vartheta_\infty)(k_1-2\overline{k_1}) - \overline{\lambda'(\psi_\infty)h_1})k_2\; dx
\end{aligned}$$

for all $(h_j,k_j)\in V$. If $(h_1,k_1)\in N(L''(\psi_\infty,\vartheta_\infty))$, it follows that

$$b'(\vartheta_\infty)(k_1-2\overline{k_1}) - \overline{\lambda'(\psi_\infty)h_1} = 0.$$

It is obvious that a solution k_1 to this equation must be constant, hence it is given by

$$k_1 = -(b'(\vartheta_\infty))^{-1}\overline{\lambda'(\psi_\infty)h_1}, \tag{5.8}$$

where we also made use of (H4). Concerning h_1 we have

$$\langle Ah_1, h_2\rangle_{V_1^*,V_1} = \int_\Omega (k_1\lambda'(\psi_\infty) + \vartheta_\infty\lambda''(\psi_\infty)h_1 - \Phi''(\psi_\infty)h_1)h_2\, dx, \tag{5.9}$$

since k_1 is constant. By Proposition 5.3 it holds that $\psi_\infty \in D(A_2) \hookrightarrow L_\infty(\Omega)$, hence $Ah_1 \in H_1$, which means that $h_1 \in D(A_2)$ and from (5.9) we obtain

$$A_2h_1 + P(\Phi''(\psi_\infty)h_1 - \vartheta_\infty\lambda''(\psi_\infty)h_1 - k_1\lambda'(\psi_\infty)) = 0,$$

where P denotes the projection $P : H_2 \to H_1$, defined by $Pu = u - \overline{u}$. It is an easy consequence of the embedding $D(A_2) \hookrightarrow L_\infty(\Omega)$ that the linear operator $B : H_1 \to H_1$ given by

$$Bh_1 = P(\Phi''(\psi_\infty)h_1 - \vartheta_\infty\lambda''(\psi_\infty)h_1 - k_1\lambda'(\psi_\infty))$$

is bounded, where k_1 is given by (5.8). Furthermore the operator A_2 defined in the proof of Proposition 5.3 is invertible, hence $A_2^{-1}B : H_1 \to D(A_2)$ is a compact operator by compact embedding and this in turn yields that $(I + A_2^{-1}B)$ is a Fredholm operator. In particular it holds that $\dim N(I + A_2^{-1}B) < \infty$, whence $N(L''(\psi_\infty, \vartheta_\infty))$ is finite dimensional and moreover

$$N(L''(\psi_\infty, \vartheta_\infty)) \subset D(A_2) \times (H_2^r(\Omega) \cap L_\infty(\Omega)) \hookrightarrow L_\infty(\Omega) \times L_\infty(\Omega).$$

By Hypothesis (H5), the restriction of L' to the space $D(A_2) \times (H_2^r(\Omega) \cap L_\infty(\Omega))$ is analytic in a neighbourhood of $(\psi_\infty, \theta_\infty)$. For the definition of analyticity in Banach spaces we refer to [5, Section 3]. Now the claim follows from [5, Theorem 3.10 & Corollary 3.11]. □

Let us now state the main result of this section.

Theorem 5.5. *Assume* (H1)–(H5) *and let* (ψ, ϑ) *be a global solution of* (5.1). *Then the limits*

$$\lim_{t\to\infty} \psi(t) =: \psi_\infty, \quad \textit{and} \quad \lim_{t\to\infty} \vartheta(t) =: \vartheta_\infty = \text{const}$$

exist in $H_2^1(\Omega)$ *and* $H_2^r(\Omega)$, $r \in (0,1)$, *respectively, and* $(\psi_\infty, \vartheta_\infty)$ *is a strong solution of the stationary problem* (5.6).

Proof. Since by Proposition 5.3 the ω-limit set is compact, we may cover it by a union of *finitely* many balls with center $(\varphi_i, \theta_i) \in \omega(\psi, \vartheta)$ and radius $\delta_i > 0$, $i = 1, \ldots, N$. Since $L(u, v) \equiv L_\infty$ on $\omega(\psi, \vartheta)$ and each (φ_i, θ_i) is a critical point of L, there are *uniform* constants $s \in (0, \frac{1}{2}]$, $C > 0$ and an open set $U \supset \omega(\psi, \vartheta)$, such that

$$|L(u, v) - L_\infty|^{1-s} \le C|L'(u, v)|_{V^*}, \tag{5.10}$$

for all $(u, v) \in U$. Define $H : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$H(t) := (L(\psi(t), \vartheta(t)) - L_\infty)^s.$$

The function H is nonincreasing and $\lim_{t\to\infty} H(t) = 0$, since $L(\psi(t),\vartheta(t)) = E(\psi(t),\vartheta(t))$ and since E is a strict Lyapunov functional for (5.1), which follows from (4.4). Furthermore we have $\lim_{t\to\infty} \operatorname{dist}((\psi(t),\vartheta(t)),\omega(\psi,\vartheta)) = 0$, i.e., there exists $t^* \geq 0$, such that $(\psi(t),\vartheta(t)) \in U$, for all $t \geq t^*$. Next, we compute and estimate the time derivative of H. By (4.4) and Proposition 5.4 we obtain

$$\begin{aligned} -\frac{d}{dt}\, H(t) &= s\left(-\frac{d}{dt}\, L(\psi(t),\vartheta(t))\right)|L(\psi(t),\vartheta(t)) - L_\infty|^{s-1} \\ &\geq C\frac{|\nabla\mu(t)|_2^2 + |\nabla\vartheta(t)|_2^2}{|L'(\psi(t),\vartheta(t))|_{V^*}} \end{aligned} \tag{5.11}$$

So have to estimate the term $|L'(\psi(t),\vartheta(t))|_{V^*}$. For convenience we will write $\psi = \psi(t)$ and $\vartheta = \vartheta(t)$. From (5.3) we obtain with $\bar{h} = 0$

$$\begin{aligned} &\langle L'(\psi,\vartheta),(h,k)\rangle_{V^*,V} \\ &\quad= \int_\Omega(-\Delta\psi + \Phi'(\psi))h\; dx + \int_\Omega \vartheta b'(\vartheta)k\; dx - \overline{\vartheta}\int_\Omega(\lambda'(\psi)h + b'(\vartheta)k)\; dx \\ &\quad= \int_\Omega(\mu-\overline{\mu})h\; dx + \int_\Omega(\vartheta-\overline{\vartheta})\lambda'(\psi)h\; dx + \int_\Omega(\vartheta-\overline{\vartheta})b'(\vartheta)k\; dx \end{aligned} \tag{5.12}$$

An application of the Hölder and Poincaré-Wirtinger inequality yields the estimates

$$\left|\int_\Omega(\vartheta-\overline{\vartheta})\lambda'(\psi)h\; dx\right| \leq |\lambda'(\psi)|_\infty|\vartheta-\overline{\vartheta}|_2|h|_2 \leq c|\nabla\vartheta|_2|h|_2, \tag{5.13}$$

$$\left|\int_\Omega(\vartheta-\overline{\vartheta})b'(\vartheta)k\; dx\right| \leq |b'(\vartheta)|_\infty|\vartheta-\overline{\vartheta}|_2|k|_2 \leq c|\nabla\vartheta|_2|k|_2 \tag{5.14}$$

and

$$\left|\int_\Omega(\mu-\overline{\mu})h\; dx\right| \leq c|\nabla\mu|_2|h|_2, \tag{5.15}$$

whence we obtain

$$|L'(\psi(t),\vartheta(t))|_{V^*} \leq C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2),$$

by taking the supremum over all functions $(h,k) \in V$ with norm less than 1 in (5.12)–(5.15). This in connection with (5.11) yields

$$-\frac{d}{dt}H(t) \geq C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2),$$

hence $|\nabla\mu|, |\nabla\vartheta| \in L_1([t^*,\infty), L_2(\Omega))$. Using the equation $\partial_t\psi = \Delta\mu$ we see that $\partial_t\psi \in L_1([t^*,\infty), H_2^1(\Omega)^*)$, hence the limit

$$\lim_{t\to\infty}\psi(t) =: \psi_\infty$$

exists in $H_2^1(\Omega)^*$ and even in $H_2^1(\Omega)$ thanks to Proposition 5.1. From equation $(5.1)_2$ it follows that $\partial_t e \in L_1([t^*,\infty); H_2^1(\Omega)^*)$, where $e := b(\vartheta) + \lambda(\psi)$, i.e., the limit $\lim_{t\to\infty} e(t)$ exists in $H_2^1(\Omega)^*$. This in turn yields that the limit

$$\lim_{t\to\infty} b(\vartheta(t)) =: b_\infty$$

exists in $L_2(\Omega)$, by relative compactness, cf. Proposition 5.1. By the monotonicity assumption (H3) we obtain $\vartheta(t) = b^{-1}(b(\vartheta(t)))$ and thus the limit of $\vartheta(t)$ as t tends to infinity exists in $L_2(\Omega)$. From the relative compactness of the orbit $\vartheta(\mathbb{R}_+)$ it follows that the limit

$$\lim_{t\to\infty} \vartheta(t) =: \vartheta_\infty$$

also exists in $H_2^r(\Omega)$, $r \in [0,1)$. Finally Proposition 5.3 yields the last statement of the theorem. $\square$

6. Appendix

Proof of Proposition 3.1 Let $(u,v), (\bar u, \bar v) \in \mathbb{B}_R(u^*, v^*)$. By the Sobolev embedding it holds that $u, \bar u$ and $v, \bar v$ are uniformly bounded in $C^1(\overline{\Omega})$ and $C(\overline{\Omega})$, respectively. Furthermore, we will use the following inequality, which has been proven in [19, Lemma 6.2.3].

$$|f(w)-f(\bar w)|_{H_p^s(L_p)} \le \mu(T)(|w-\bar w|_{H_p^{s_0}(L_p)}+|w-\bar w|_{\infty,\infty}), \quad 0<s<s_0<1, \tag{6.1}$$

valid for every $f \in C^{2-}(\mathbb{R})$ and all $w, \bar w \in \mathbb{B}_R^1(u^*) \cup \mathbb{B}_R^2(v^*)$. Here $\mu = \mu(T)$ denotes a function, with the property $\mu(T) \to 0$ as $T \to 0$. The proof consists of several steps

(i) By Hölder's inequality it holds that

$$\begin{aligned}
&|\Delta\Phi'(u) - \Delta\Phi'(\bar u)|_{X(T)}\\
&\quad \le |\Delta u \Phi''(u) - \Delta \bar u \Phi''(\bar u)|_{X(T)} + ||\nabla u|^2 \Phi'''(u) - |\nabla \bar u|^2 \Phi'''(\bar u)|_{X(T)}\\
&\quad \le |\Delta u|_{rp,rp} |\Phi''(u) - \Phi''(\bar u)|_{r'p,r'p} + |\Delta u - \Delta \bar u|_{rp,rp} |\Phi''(\bar u)|_{r'p,r'p}\\
&\qquad + T^{1/p}\left(|\nabla u|_{\infty,\infty}^2 |\Phi'''(u) - \Phi'''(\bar u)|_{\infty,\infty} + |\nabla u - \nabla \bar u|_{\infty,\infty} |\Phi'''(\bar u)|_{\infty,\infty}\right)\\
&\quad \le T^{1/r'p}\left(|\Delta u|_{rp,rp} |\Phi''(u) - \Phi''(\bar u)|_{\infty,\infty} + |\Delta u - \Delta \bar u|_{rp,rp} |\Phi''(\bar u)|_{\infty,\infty}\right)\\
&\qquad + T^{1/p}\left(|\nabla u|_{\infty,\infty}^2 |\Phi'''(u) - \Phi'''(\bar u)|_{\infty,\infty} + |\nabla u - \nabla \bar u|_{\infty,\infty} |\Phi'''(\bar u)|_{\infty,\infty}\right),
\end{aligned}$$

since $u, \bar u \in C(J; C^1(\overline{\Omega}))$. We have

$$\Delta w \in H_p^{\theta_2/2}(J; H_p^{2(1-\theta_2)}(\Omega)) \hookrightarrow L_{rp}(J \times \Omega), \quad \theta_2 \in [0,1],$$

for every function $w \in E_1(T)$, since $r > 1$ may be chosen close to 1. Therefore we obtain

$$|\Delta\Phi'(u) - \Delta\Phi'(\bar u)|_{X(T)} \le \mu(T)\,(R + |u^*|_1)\,|u - \bar u|_1,$$

due to the assumption $\Phi \in C^{4-}(\mathbb{R})$.

(ii) Consider the term $(\lambda'(\psi_0)-\lambda'(u))\Delta v-(\lambda'(\psi_0)-\lambda'(\bar u))\Delta\bar v$.

$$\begin{aligned}
&|(\lambda'(\psi_0)-\lambda'(u))\Delta v-(\lambda'(\psi_0)-\lambda'(\bar u))\Delta\bar v|_{X(T)}\\
&\qquad\leq |(\lambda'(\psi_0)-\lambda'(u))\Delta(v-\bar v)|_{X(T)}+|(\lambda'(u)-\lambda'(\bar u))\Delta\bar v|_{X(T)}\\
&\qquad\leq |\psi_0-u|_{\infty,\infty}|v-\bar v|_{E_2(T)}+|u-\bar u|_{\infty,\infty}|\bar v|_{E_2(T)}\\
&\qquad\leq (|\psi_0-u^*|_{\infty,\infty}+|u^*-u|_{\infty,\infty})|v-\bar v|_{E_2(T)}\\
&\qquad\qquad+|u-\bar u|_{E_1(T)}(|\bar v-v^*|_{E_2(T)}+|v^*|_{E_2(T)})\\
&\qquad\leq C(\mu(T)+R)|(u,v)-(\bar u,\bar v)|_1,
\end{aligned}$$

since $\lambda\in C^{4-}(\mathbb{R})$. Next, we consider the term $\nabla(\lambda'(\psi_0)-\lambda'(u))\nabla v-\nabla(\lambda'(\psi_0)-\lambda'(\bar u))\nabla\bar v$. We obtain

$$\begin{aligned}
&|\nabla(\lambda'(\psi_0)-\lambda'(u))\nabla v-\nabla(\lambda'(\psi_0)-\lambda'(\bar u))\nabla\bar v|_{X(T)}\\
&\qquad\leq |\nabla(\lambda'(\psi_0)-\lambda'(u))|_\infty|\nabla(v-\bar v)|_{X(T)}+|\nabla(\lambda'(u)-\lambda'(\bar u))|_\infty|\nabla\bar v|_{X(T)}.
\end{aligned}$$

Since

$$\nabla(\lambda'(\psi_0)-\lambda'(u))=\nabla\psi_0(\lambda''(\psi_0)-\lambda''(u))+\lambda''(u)(\nabla\psi_0-\nabla u),$$

and the same for $\nabla(\lambda'(u)-\lambda'(\bar u))$, we may argue as above, to conclude

$$\begin{aligned}
&|\nabla(\lambda'(\psi_0)-\lambda'(u))|_{\infty,\infty}|\nabla(v-\bar v)|_{X(T)}+|\nabla(\lambda'(u)-\lambda'(\bar u))|_{\infty,\infty}|\nabla\bar v|_{X(T)}\\
&\qquad\qquad\leq(\mu(T)+R)|(u,v)-(\bar u,\bar v)|_1.
\end{aligned}$$

Finally, we estimate the remaining part with Hölder's inequality to the result

$$\begin{aligned}
&|v\Delta(\lambda'(\psi_0)-\lambda'(u))-\bar v\Delta(\lambda'(\psi_0)-\lambda'(\bar u))|_{X(T)}\\
&\quad\leq |v-\bar v|_{\infty,\infty}|\Delta(\lambda'(\psi_0)-\lambda'(u))|_{X(T)}+|\bar v|_{r'p,r'p}|\Delta(\lambda'(u)-\lambda'(\bar u))|_{rp,rp},
\end{aligned}\tag{6.2}$$

where $1/r+1/r'=1$. For the first part, we obtain

$$\begin{aligned}
&|\Delta(\lambda'(\psi_0)-\lambda'(u))|_{X(T)}\\
&\quad\leq |\Delta\psi_0|_p|\lambda''(\psi_0)-\lambda''(u)|_{\infty,\infty}+|\Delta\psi_0-\Delta u|_p|\lambda''(u)|_{\infty,\infty}\\
&\qquad\quad+|\nabla\psi_0|^2_{\infty,\infty}|\lambda'''(\psi_0)-\lambda'''(u)|_{\infty,\infty}+|\lambda'''(u)|_{\infty,\infty}|\nabla\psi_0-\nabla u|_{\infty,\infty}\\
&\quad\leq C(|\psi_0-u|_{\infty,\infty}+|\nabla\psi_0-\nabla u|_{\infty,\infty}+|\Delta\psi_0-\Delta u|_{p,p})\\
&\quad\leq C(\mu(T)+R),
\end{aligned}$$

since $\psi_0\in H^2_p(\Omega)\cap C^1(\overline{\Omega})$ and $\lambda\in C^{4-}(\mathbb{R})$. For the second term in (6.2) we obtain

$$\begin{aligned}
&|\Delta(\lambda'(u)-\lambda'(\bar u))|_{rp,rp}\\
&\qquad\leq |\Delta u|_{rp,rp}|\lambda''(u)-\lambda''(\bar u)|_{\infty,\infty}+|\lambda''(\bar u)|_{\infty,\infty}|\Delta u-\Delta\bar u|_{rp,rp}\\
&\qquad\qquad+|\nabla u|^2_{\infty,\infty}|\lambda'''(u)-\lambda'''(\bar u)|_{\infty,\infty}+|\lambda'''(\bar u)|_{\infty,\infty}|\nabla u-\nabla\bar u|_{\infty,\infty}\\
&\qquad\leq C|u-\bar u|_{E_1(T)},
\end{aligned}$$

since $u,\bar u\in C(J;C^1(\overline{\Omega}))$ and $r>1$ can be chosen close enough to 1, due to the fact that $\bar v\in C(J;C(\overline{\Omega}))$.

Finally, we observe

$$|\bar{v}|_{r'p,r'p} \le |\bar{v} - v^*|_{r'p,r'p} + |v^*|_{r'p,r'p} \le \mu(T) + R.$$

(iii) For simplicity we set $f(u,v) = a_0\lambda'(\psi_0) - a(v)\lambda'(u)$. Then we compute

$$\begin{aligned}
&|f(u,v)\partial_t u - f(\bar{u},\bar{v})\partial_t \bar{u}|_{X(T)} \\
&\qquad \le |\partial_t u(f(u,v) - f(\bar{u},\bar{v}))|_{X(T)} + |f(\bar{u},\bar{v})(\partial_t u - \partial_t \bar{u})|_{X(T)} \qquad (6.3)\\
&\qquad \le (|\partial_t u - \partial_t u^*|_{X(T)} + |\partial_t u^*|_{X(T)})|f(u,v) - f(\bar{u},\bar{v})|_{\infty,\infty} \\
&\qquad\qquad + |f(\bar{u},\bar{v})|_{\infty,\infty}|\partial_t u - \partial_t \bar{u}|_{X(T)} \\
&\qquad \le C(\mu_3(T) + R)|f(u,v) - f(\bar{u},\bar{v})|_{\infty,\infty} \\
&\qquad\qquad + |f(\bar{u},\bar{v})|_{\infty,\infty}|\partial_t u - \partial_t \bar{u}|_{X(T)}.
\end{aligned}$$

Next we estimate

$$\begin{aligned}
&|f(u,v) - f(\bar{u},\bar{v})|_{\infty,\infty} \\
&\qquad \le |a(v)(\lambda'(u) - \lambda'(\bar{u}))|_{\infty,\infty} + |\lambda'(\bar{u})(a(v) - a(\bar{v}))|_{\infty,\infty} \\
&\qquad \le |a(v)|_{\infty,\infty}|\lambda'(u) - \lambda'(\bar{u})|_{\infty,\infty} + |\lambda'(\bar{u})|_{\infty,\infty}|a(v) - a(\bar{v})|_{\infty,\infty} \\
&\qquad \le C(|u - \bar{u}|_{\infty,\infty} + |v - \bar{v}|_{\infty,\infty}) \le C|(u,v) - (\bar{u},\bar{v})|_1.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
|f(\bar{u},\bar{v})|_{\infty,\infty} &\le |a_0|_{\infty,\infty}|\lambda'(\psi_0) - \lambda'(\bar{u})|_{\infty,\infty} + |\lambda'(\bar{u})|_{\infty,\infty}|a_0 - a(\bar{v})|_{\infty,\infty} \\
&\le C(|\psi_0 - \bar{u}|_{\infty,\infty} + |\vartheta_0 - \bar{v}|_{\infty,\infty}) \\
&\le C(|\psi_0 - u^*|_{\infty,\infty} + |u^* - \bar{u}|_{\infty,\infty} + |\vartheta_0 - v^*|_{\infty,\infty} + |v^* - \bar{v}|_{\infty,\infty}) \\
&\le C(\mu(T) + R).
\end{aligned}$$

The estimate of $(a_0 - a(v))\Delta v - (a_0 - a(\bar{v}))\Delta\bar{v}$ in $L_p(J; L_p(\Omega))$ can be carried out in a similar way.

(iv) We compute

$$\begin{aligned}
|(a(v) - a(\bar{v})f_2|_{X(T)} &\le |a(v) - a(\bar{v})|_{\infty,\infty}|f_2|_{X(T)} \le |v - \bar{v}|_{\infty,\infty}|f_2|_{X(T)} \\
&\le \mu(T)|v - \bar{v}|_{E_2(T)} \le \mu(T)|(u,v) - (\bar{u},\bar{v})|_1,
\end{aligned}$$

since $f_2 \in X(T)$ is a fixed function, hence $|f_2|_{X(T)} \to 0$ as $T \to 0$.

(v) By trace theory, we obtain

$$\begin{aligned}
&|\partial_\nu(\Phi'(u) - \Phi'(\bar{u}))|_{Y_1(T)} \\
&\qquad \le C|\Phi'(u) - \Phi'(\bar{u})|_{H_p^{1/2}(J;L_p(\Omega))} + |\Phi'(u) - \Phi'(\bar{u})|_{L_p(J;H_p^2(\Omega))}.
\end{aligned}$$

The second norm has already been estimated in (i), so it remains to estimate $\Phi'(u) - \Phi'(\bar{u})$ in $H_p^{1/2}(J; L_p(\Omega))$. Here we will use (6.1), to obtain

$$\begin{aligned}
|\Phi'(u) - \Phi'(\bar{u})|_{H_p^{1/2}(L_p)} &\le \mu(T)(|u - \bar{u}|_{H_p^{s_0}(L_p)} + |u - \bar{u}|_{\infty,\infty}) \\
&\le \mu(T)C|u - \bar{u}|_{E_1(T)} \le \mu(T)C|(u,v) - (\bar{u},\bar{v})|_1,
\end{aligned}$$

since $s_0 < 1$.

(vi) We may apply (ii) and trace theory, to conclude that it suffices to estimate

$$\begin{aligned}(\lambda'(\psi_0)-\lambda'(u))v-(\lambda'(\psi_0)-\lambda'(\bar u))\bar v\\ =(\lambda'(\psi_0)-\lambda'(u))(v-\bar v)-(\lambda'(u)-\lambda'(\bar u))\bar v\end{aligned}$$

in $H_p^{1/2}(J;L_p(\Omega))$. This yields

$$\begin{aligned}&|(\lambda'(\psi_0)-\lambda'(u))(v-\bar v)|_{H_p^{1/2}(L_p)}\\ &\quad\le|\lambda'(\psi_0)-\lambda'(u)|_{H_p^{1/2}(L_p)}|v-\bar v|_{\infty,\infty}+|\lambda'(\psi_0)-\lambda'(u)|_{\infty,\infty}|v-\bar v|_{H_p^{1/2}(L_p)}\\ &\quad\le(|\lambda'(\psi_0)-\lambda'(u^*)|_{H_p^{1/2}(L_p)}+|\lambda'(u^*)-\lambda'(u)|_{H_p^{1/2}(L_p)})|v-\bar v|_{E_2(T)}\\ &\qquad\quad+(|\psi_0-u^*|_{\infty,\infty}+|u^*-u|_{\infty,\infty})|v-\bar v|_{E_2(T)}\\ &\quad\le\Big(|\lambda'(\psi_0)-\lambda'(u^*)|_{H_p^{1/2}(L_p)}+\mu(T)R+(\mu(T)+R)\Big)\,|v-\bar v|_{E_2(T)}.\end{aligned}$$

Clearly $\lambda'(\psi_0)-\lambda'(u^*)\in {}_0H_p^{1/2}(J;L_p(\Omega))$, since ψ_0 does not depend on t and since $\lambda\in C^{4-}(\mathbb{R})$. Therefore it holds that

$$|\lambda'(\psi_0)-\lambda'(u^*)|_{H_p^{1/2}(L_p)}\to 0$$

as $T\to 0$. The second part $(\lambda'(u)-\lambda'(\bar u))\bar v$ can be treated as follows.

$$\begin{aligned}&|(\lambda'(u)-\lambda'(\bar u))\bar v|_{H_p^{1/2}(L_p)}\\ &\quad\le|\lambda'(u)-\lambda'(\bar u)|_{H_p^{1/2}(L_p)}|\bar v|_{\infty,\infty}+|\lambda'(u)-\lambda'(\bar u)|_{\infty,\infty}|\bar v|_{H_p^{1/2}(L_p)}\\ &\quad\le C(\mu(T)+R+\mu(T))|u-\bar u|_{E_1(T)},\end{aligned}$$

where we applied again (6.1). This completes the proof of the proposition.

Proof of Proposition 4.1 Let $J_{\max}^\delta:=[\delta,T_{\max}]$ for some small $\delta>0$. Setting $A^2=\Delta_N^2$ with domain

$$D(A^2)=\{u\in H_p^4(\Omega):\partial_\nu u=\partial_\nu\Delta u=0\text{ on }\partial\Omega\},$$

the solution $\psi(t)$ of equation $(4.1)_1$ may be represented by the variation of parameters formula

$$\psi(t)=e^{-A^2t}\psi_0+\int_0^t Ae^{-A^2(t-s)}\Big(\lambda'(\psi(s))\vartheta(s)-\Phi'(\psi(s))\Big)\,ds,\quad t\in J_{\max},\tag{6.4}$$

where e^{-A^2t} denotes the analytic semigroup, generated by $-A^2=-\Delta_N^2$ in $L_p(\Omega)$. By (H1), (H2) and (4.6) it holds that

$$\Phi'(\psi)\in L_\infty(J_{\max};L_{q_0}(\Omega))\quad\text{and}\quad\lambda'(\psi)\in L_\infty(J_{\max};L_6(\Omega)),$$

with $q_0=6/(\gamma+2)$. We then apply A^r, $r\in(0,1)$, to (6.4) and make use of semigroup theory to obtain

$$\psi\in L_\infty(J_{\max}^\delta;H_{q_0}^{2r}(\Omega)),\tag{6.5}$$

valid for all $r \in (0,1)$, since $q_0 < 6$. It follows from (6.5) that $\psi \in L_\infty(J^\delta_{\max}; L_{p_1}(\Omega))$ if $2r - 3/q_0 \geq -3/p_1$, and

$$\Phi'(\psi) \in L_\infty(J^\delta_{\max}; L_{q_1}(\Omega)) \quad \text{as well as} \quad \lambda'(\psi) \in L_\infty(J^\delta_{\max}; L_{p_1}(\Omega)),$$

with $q_1 = p_1/(\gamma+2)$. Hence we have this time

$$\psi \in L_\infty(J^\delta_{\max}; H^{2r}_{q_1}(\Omega)), \quad r \in (0,1).$$

Iteratively we obtain a sequence $(p_n)_{n\in\mathbb{N}_0}$ such that

$$2r - \frac{3}{q_n} \geq -\frac{3}{p_{n+1}}, \quad n \in \mathbb{N}_0$$

with $q_n = p_n/(\gamma+2)$ and $p_0 = 6$. Thus the sequence $(p_n)_{n\in\mathbb{N}_0}$ may be recursively estimated by

$$\frac{1}{p_{n+1}} \geq \frac{\gamma+2}{p_n} - \frac{2r}{3},$$

for all $n \in \mathbb{N}_0$ and $r \in (0,1)$. From this definition it is not difficult to obtain the following estimate for $1/p_{n+1}$.

$$\begin{aligned} \frac{1}{p_{n+1}} &\geq \frac{(\gamma+2)^{n+1}}{p_0} - \frac{2r}{3}\sum_{k=0}^{n}(\gamma+2)^k \\ &= \frac{(\gamma+2)^{n+1}}{p_0} - \frac{2r}{3}\left(\frac{(\gamma+2)^{n+1}-1}{\gamma_1+1}\right) \\ &= (\gamma+2)^{n+1}\left(\frac{1}{p_0} - \frac{2r}{3\gamma+3}\right) + \frac{2r}{3\gamma+3}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{6.6}$$

By the assumption (H1) on γ we see that the term in brackets is negative if $r \in (0,1)$ is sufficiently close to 1 and therefore, after finitely many steps the entire right side of (6.6) is negative as well, whence we may choose p_n arbitrarily large or we may even set $p_n = \infty$ for $n \geq N$ and a certain $N \in \mathbb{N}_0$. In other words this means that for those $r \in (0,1)$ we have

$$\psi \in L_\infty(J^\delta_{\max}; H^{2r}_p(\Omega)), \tag{6.7}$$

for all $p \in [1,\infty]$. It is important, that we can achieve this result in *finitely* many steps!

Next we will derive an estimate for $\partial_t\psi$. For all forthcoming calculations we will use the abbreviation $\psi = \psi(t)$ and $\vartheta = \vartheta(t)$. Since we only have estimates on the interval $J^\delta_{\max}$, we will use the following solution formula.

$$\psi(t) = e^{-A^2(t-\delta)}\psi_\delta + \int_0^{t-\delta} Ae^{-A^2 s}\Big(\lambda'(\psi)\vartheta - \Phi'(\psi)\Big)(t-s)\,ds, \quad t \in J^\delta_{\max}$$

where $\psi_\delta := \psi(\delta)$. Differentiating with respect to t, we obtain

$$\begin{aligned} \partial_t\psi(t) = A\int_0^{t-\delta} e^{-A^2 s}(\lambda''(\psi)\vartheta\partial_t\psi + \lambda'(\psi)\partial_t\vartheta - \Phi''(\psi)\partial_t\psi)(t-s)\,ds \\ + F(t,\psi_\delta,\vartheta_\delta), \end{aligned} \tag{6.8}$$

for all $t \geq \delta$ and with

$$F(t,\psi_\delta,\vartheta_\delta) := Ae^{-A^2(t-\delta)}(\lambda'(\psi_\delta)\vartheta_\delta - \Phi'(\psi_\delta)) - A^2 e^{-A^2(t-\delta)}\psi_\delta.$$

Let us discuss the function F in detail. By the trace theorem we have $\psi_\delta \in B^{4-4/p}_{pp}(\Omega)$ and $\vartheta_\delta \in B^{2-2/p}_{pp}(\Omega)$. Since we assume $p > (n+2)/2$, it holds that $\psi_\delta, \vartheta_\delta \in L_\infty(\Omega)$. Furthermore, the semigroup $e^{-A^2 t}$ is analytic. Therefore there exist some constants $C > 0$ and $\omega \in \mathbb{R}$ such that

$$|F(t,\psi_\delta,\vartheta_\delta)|_{L_p(\Omega)} \leq C\left(\frac{1}{(t-\delta)^{1/2}} + \frac{1}{t-\delta}\right)e^{\omega t},$$

for all $t > \delta$. This in turn implies that

$$F(\cdot,\psi_\delta,\vartheta_\delta) \in L_p(J^{\delta'}_{\max} \times \Omega)$$

for all $p \in (1,\infty)$, where $0 < \delta < \delta' < T_{\max}$. We will now use equations $(5.1)_{1,2}$ to rewrite the integrand in (6.8) in the following way.

$$\begin{aligned}
&(\lambda''(\psi)\vartheta - \Phi''(\psi))\partial_t\psi + \lambda'(\psi)\partial_t\vartheta \\
&\qquad = (\lambda''(\psi)\vartheta - \Phi''(\psi))\Delta\mu + \frac{\lambda'(\psi)}{b'(\vartheta)}\Delta\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)}\Delta\mu \\
&\qquad = \operatorname{div}\left[\left(\lambda''(\psi)\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)} - \Phi''(\psi)\right)\nabla\mu\right] + \operatorname{div}\left[\frac{\lambda'(\psi)}{b'(\vartheta)}\nabla\vartheta\right] \qquad (6.9)\\
&\qquad - \nabla\left(\lambda''(\psi)\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)} - \Phi''(\psi)\right)\cdot\nabla\mu - \nabla\frac{\lambda'(\psi)}{b'(\vartheta)}\cdot\nabla\vartheta.
\end{aligned}$$

Thus we obtain a decomposition of the following form

$$\begin{aligned}
&(\lambda''(\psi)\vartheta - \Phi''(\psi))\partial_t\psi + \lambda'(\psi)\partial_t\vartheta \\
&\qquad = \operatorname{div}(f_\mu\nabla\mu + f_\vartheta\nabla\vartheta) + g_\mu\nabla\mu + g_\vartheta\nabla\vartheta + h_\mu\nabla\vartheta\nabla\mu + h_\vartheta|\nabla\vartheta|^2,
\end{aligned}$$

with

$$\begin{aligned}
f_\mu &:= \lambda''(\psi)\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)} - \Phi''(\psi), & f_\vartheta &:= \frac{\lambda'(\psi)}{b'(\vartheta)},\\
g_\mu &:= -\left(\lambda'''(\psi)\vartheta - 2\frac{\lambda'(\psi)\lambda''(\psi)}{b'(\vartheta)} - \Phi''(\psi)\right)\nabla\psi, & g_\vartheta &:= -\frac{\lambda''(\psi)}{b'(\vartheta)}\nabla\psi,\\
h_\mu &:= \lambda''(\psi) - \frac{b''(\vartheta)\lambda'(\psi)^2}{b'(\vartheta)^2}, & h_\vartheta &:= \frac{b''(\vartheta)\lambda'(\psi)}{b'(\vartheta)^2}.
\end{aligned}$$

By Assumption (H3) and the first part of the proof it holds that $f_j, g_j, h_j \in L_\infty(J^\delta_{\max} \times \Omega)$ for each $j \in \{\mu,\vartheta\}$ and this in turn yields that

$$\begin{aligned}
\operatorname{div}(f_\mu\nabla\mu + f_\vartheta\nabla\vartheta) &\in L_2(J^\delta_{\max}; H^1_2(\Omega)^*),\\
g_\mu\cdot\nabla\mu + g_\vartheta\cdot\nabla\vartheta &\in L_2(J^\delta_{\max} \times \Omega),\\
h_\mu\nabla\vartheta\cdot\nabla\mu + h_\vartheta|\nabla\vartheta|^2 &\in L_1(J^\delta_{\max} \times \Omega),
\end{aligned}$$

where we also made use of (4.6). Setting

$$T_1 = Ae^{-A^2t} * \operatorname{div}(f_\mu \nabla\mu + f_\vartheta \nabla\vartheta), \quad T_2 = Ae^{-A^2t} * (g_\mu \cdot \nabla\mu + g_\vartheta \cdot \nabla\vartheta)$$

and

$$T_3 = Ae^{-A^2t} * (h_\mu \nabla\vartheta \cdot \nabla\mu + h_\vartheta |\nabla\vartheta|^2),$$

we may rewrite (6.8) as

$$\partial_t\psi = T_1 + T_2 + T_3 + F(t, \psi_0, \vartheta_0).$$

Going back to (6.8) we obtain

$$\begin{aligned}
&T_1 \in H_2^{1/4}(J^\delta_{\max}; L_2(\Omega)) \cap L_2(J^\delta_{\max}; H_2^1(\Omega)) \hookrightarrow L_2(J^\delta_{\max} \times \Omega),\\
&T_2 \in H_2^{1/2}(J^\delta_{\max}; L_2(\Omega)) \cap L_2(J^\delta_{\max}; H_2^2(\Omega)) \hookrightarrow L_2(J^\delta_{\max} \times \Omega), \quad \text{and}\\
&F(\cdot, \psi_\delta, \vartheta_\delta) \in L_2(J^{\delta'}_{\max} \times \Omega).
\end{aligned}$$

Observe that we do not have full regularity for T_3 since A has no maximal regularity in $L_1(\Omega)$, but nevertheless we obtain

$$T_3 \in H_1^{1/2-}(J^\delta_{\max}; L_1(\Omega)) \cap L_1(J^\delta_{\max}; H_1^{2-}(\Omega)).$$

Here we used the notation $H_p^{s-} := H_p^{s-\varepsilon}$ and $\varepsilon > 0$ is sufficiently small. An application of the mixed derivative theorem then yields

$$H_1^{1/2-}(J^\delta_{\max}; L_1(\Omega)) \cap L_1(J^\delta_{\max}; H_1^{2-}(\Omega)) \hookrightarrow L_p(J^\delta_{\max}; L_2(\Omega)),$$

if $p \in (1, 8/7)$, whence

$$\partial_t\psi \in L_2(J^{\delta'}_{\max} \times \Omega) + L_p(J^{\delta'}_{\max}; L_2(\Omega))$$

for some $1 < p < 8/7$. Now we go back to (6.9) where we replace this time only $\partial_t\vartheta$ by the differential equation $(5.1)_2$ to obtain

$$\begin{aligned}
(\lambda''(\psi)\vartheta &- \Phi''(\psi))\partial_t\psi + \lambda'(\psi)\partial_t\vartheta\\
&= \left(\lambda''(\psi)\vartheta - \Phi''(\psi) - \frac{\lambda'(\psi)^2}{b'(\vartheta)}\right)\partial_t\psi\\
&\quad + \operatorname{div}\left[\frac{\lambda'(\psi)}{b'(\vartheta)}\nabla\vartheta\right] - \frac{\lambda''(\psi)}{b'(\vartheta)}\nabla\psi \cdot \nabla\vartheta + \frac{\lambda'(\psi)b''(\vartheta)}{b'(\vartheta)^2}|\nabla\vartheta|^2\\
&= f\partial_t\psi + \operatorname{div}[g\nabla\vartheta] + h \cdot \nabla\vartheta + k|\nabla\vartheta|^2.
\end{aligned}$$

Rewrite (6.8) in the following way

$$\partial_t\psi = S_1 + S_2 + S_3 + S_4 + F(t, \psi_0, \vartheta_0), \tag{6.10}$$

where the functions S_j are defined in the same manner as T_j. Since $f, g, h \in L_\infty(J^\delta_{\max} \times \Omega)$ it follows again from regularity theory that

$$\begin{aligned} S_1 &\in H_2^{1/2}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_2(J^{\delta'}_{\max}; H_2^2(\Omega)) \\ &\qquad + H_p^{1/2}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_p(J^{\delta'}_{\max}; H_2^2(\Omega)), \\ S_2 &\in H_2^{1/4}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_2(J^{\delta'}_{\max}; H_2^1(\Omega)), \\ S_3 &\in H_2^{1/2}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_2(J^{\delta'}_{\max}; H_2^2(\Omega)), \end{aligned}$$

and it can be readily verified that

$$H_p^{1/2}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_p(J^{\delta'}_{\max}; H_2^2(\Omega)) \hookrightarrow L_2(J^{\delta'}_{\max} \times \Omega),$$

whenever $p \in [1, 2]$. Now we turn our attention to the term $S_4 = Ae^{-A^2 t} * k|\nabla\vartheta|^2$. First we observe that by the mixed derivative theorem the embedding

$$Z_q := H_q^{1/2-}(J^{\delta'}_{\max}; L_1(\Omega)) \cap L_q(J^{\delta'}_{\max}; H_1^{2-}(\Omega)) \hookrightarrow L_2(J^{\delta'}_{\max} \times \Omega)$$

is valid, provided that $q \in (8/5, 2]$. Hence it holds that

$$|S_4|_{2,2} \le C|S_4|_{Z_q} \le C|k|\nabla\vartheta|^2|_{q,1} \le C|\nabla\vartheta|^2_{2q,2},$$

with some constant $C > 0$. Taking the norm of $\partial_t\psi$ in $L_2(J^{\delta'}_{\max} \times \Omega)$ we obtain from (6.10)

$$|\partial_t\psi|_{2,2} \le C\left(\sum_{j=1}^{3} |S_j|_{2,2} + |\nabla\vartheta|^2_{2q,2} + |F(\cdot, \psi_\delta, \vartheta_\delta)|_{2,2}\right).$$

The Gagliardo-Nirenberg inequality in connection with (4.6) yields the estimate

$$|\nabla\vartheta|^2_{2q,2} \le c|\nabla\vartheta|^{2a}_{2,2}|\nabla\vartheta|^{2(1-a)}_{\infty,2} \le c|\nabla\vartheta|^{2(1-a)}_{\infty,2},$$

provided that $a = 1/q$. Multiply $(4.1)_2$ by $\partial_t\vartheta$ and integrate by parts to the result

$$\int_\Omega b'(\vartheta(t,x))|\partial_t\vartheta(t,x)|^2\,dx + \frac{1}{2}\frac{d}{dt}|\nabla\vartheta(t)|_2^2 = -\int_\Omega \lambda'(\psi(t,x))\partial_t\psi(t,x)\partial_t\vartheta(t,x)\,dx.$$

Making use of (H3) and Young's inequality we obtain

$$C_1|\partial_t\vartheta|^2_{2,2} + \frac{1}{2}|\nabla\vartheta(t)|^2_2 \le C_2(|\partial_t\psi|^2_{2,2} + |\nabla\vartheta_0|^2_2), \tag{6.11}$$

after integrating w.r.t. t. This in turn yields the estimate

$$|\nabla\vartheta|^2_{2q,2} \le c|\nabla\vartheta|^{2(1-a)}_{\infty,2} \le c(1 + |\partial_t\psi|^{2(1-a)}_{2,2}).$$

In order to gain something from this inequality we require that $2(1-a) < 1$, i.e., q is restricted by $1 < q < 2$. Finally, if we choose $q \in (8/5, 2)$ and use the uniform boundedness of the L_2 norms of $S_j,\ j \in \{1, 2, 3\}$ we obtain

$$|\partial_t\psi|_{2,2} \le C(1 + |\partial_t\psi|^{2(1-a)}_{2,2}).$$

Since by construction $2(1-a) < 1$, it follows that the L_2-norm of $\partial_t\psi$ is bounded on $J^{\delta'}_{\max} \times \Omega$. In particular, this yields the statement for ϑ by equation (6.11).

Now we go back to (6.8) with δ replaced by δ'. By Assumption (H5), by the bounds $\partial_t\vartheta, \partial_t\psi \in L_2(J^{\delta'}_{\max}; L_2(\Omega))$ and by the first part of the proof we obtain

$$\lambda''(\psi)\vartheta\partial_t\psi + \lambda'(\psi)\partial_t\vartheta - \Phi''(\psi)\partial_t\psi \in L_2(J^{\delta'}_{\max}; L_2(\Omega)).$$

Since the operator $A^2 = \Delta^2$ with domain

$$D(A^2) = \{u \in H^4_p(\Omega) : \partial_\nu u = \partial_\nu \Delta u = 0\}$$

has the property of maximal L_p-regularity (cf. [6, Theorem 2.1]), we obtain from (6.8)

$$\partial_t\psi - F(\cdot, \psi_{\delta'}, \vartheta_{\delta'}) \in H^{1/2}_2(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_2(J^{\delta'}_{\max}; H^2_2(\Omega)) \hookrightarrow L_r(J^{\delta'}_{\max}; L_r(\Omega)),$$

and the last embedding is valid for all $r \le 2(n+4)/n$. By the properties of the function F it follows

$$\partial_t\psi \in L_r(J^{\delta''}_{\max}; L_r(\Omega)),$$

for all $r \le 2(n+4)/n$ and some $0 < \delta'' < T_{\max}$. To obtain an estimate for the whole interval $J_{\max}$, we use the fact that we already have a local strong solution, i.e., $\partial_t\psi \in L_p(0, \delta''; L_p(\Omega))$, $p > (n+2)/2$. The proof is complete.

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The Asymptotic Profile of Solutions of a Class of Singular Parabolic Equations

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Abstract. We consider a class of singular parabolic problems with Dirichlet boundary conditions. We use the Rayleigh quotient and recent Harnack estimates to derive estimates from above and from below for the solution. Moreover we study the asymptotic behaviour when the solution is approaching the extinction time.

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1. Introduction

Let $u(x,t)$ be the weak solution of the following initial boundary value problem:

$$u_t = \operatorname{div} \mathbf{A}(x,t,\nabla u), \quad (x,t) \in \Omega \times (t > 0), \tag{1.1}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (t > 0), \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $u_0 \in L^1(\Omega)$ and $\int_\Omega u_0(x)dx > 0$. The functions $\mathbf{A} := (A_1, \ldots, A_N)$ are regular and are assumed to satisfy the following structure conditions:

$$\mathbf{A}(x,t,\nabla u) \cdot \nabla u \geq c_0 |\nabla u|^p, \tag{1.4}$$

$$|\mathbf{A}(x,t,\nabla u)| \leq c_1 |\nabla u|^{p-1}, \tag{1.5}$$

with $\frac{2N}{N+1} < p < 2$ and c_0, c_1 given positive constants.

A function $u \in C_{\mathrm{loc}}(\mathbb{R}^+; L^2_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(\mathbb{R}^+; W^{1,p}_{\mathrm{loc}}(\Omega))$ is a weak solution of (1.1)–(1.3), if for any compact subset K of Ω and for every subinterval $[t_1, t_2] \in \mathbb{R}^+$,

$$\int_K u\ \phi\ dx|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K (-u\ \phi_t + \mathbf{A}(x,t,\nabla u) \cdot \nabla\phi)\ dx\ dt = 0, \tag{1.6}$$

for all $\phi \in W^{1,2}_{\rm loc}(\mathbb{R}^+; L^2(K)) \cap L^p_{\rm loc}(\mathbb{R}^+; W^{1,p}_0(K))$, where ϕ is a bounded testing function. We use this definition of solution because u_t may have a modest degree of regularity and in general has meaning only in the sense of distributions. For more details see [2], Chap. II, Remark 1.1.

In the last few years, several papers were devoted to the study of the asymptotic behaviour of solutions to the porous media and the p-Laplace equations. We refer the reader to the recent monograph by Vazquez ([12]) and to the references therein. In almost all these references the Authors use elliptic results to study the asymptotic behaviour of the solutions. If, on one hand, this makes the proof simple and very elegant, on the other hand it appears that this method is not flexible and cannot be applied for more general operators. In recent papers ([8]), ([9]) the Authors followed an alternative approach introduced by Berryman-Holland ([1]) and used in the context of the asymptotic behaviour of solutions to degenerate parabolic equations in [3], [7] and [10]. This approach is more parabolic than the previous one, namely, relying on the properties of the evolution equations, it is possible to study the asymptotic behaviour of the solutions and derive the elliptic properties of the asymptotic limit as a by-product.

This new method is applied to study the case of equations with time dependent coefficients for the degenerate case.

We recall to the reader that in the singular case the phenomenon of the extinction of the solution in finite time occurs. This fact compels us to employ different techniques and mathematical tools with respect to the degenerate case. This generalization is based on recent techniques developed in [5], that allow us to avoid the use of comparison functions as in [3] and [10]. With respect to the results proved in [8] and [9], here we use the Rayleigh quotient.

Remark 1.1. For the sake of simplicity, in this paper we consider only the case of initial boundary value problem (1.1)–(1.3), under the structure assumptions (1.4)–(1.5). It is possible to prove similar results in the case of porous-medium like equations, or even doubly nonlinear equations and for mixed boundary conditions (using the techniques introduced in [10]).

Remark 1.2. As we take the initial datum in $L^1(\Omega)$ we are compelled to limit ourselves to the case $\frac{2N}{N+1} < p < 2$. Actually, under this threshold, solutions of a Cauchy-Dirichlet problem with initial datum in L^1 could be unbounded (see, for instance, [2]).

The structure of the paper is as follows: in Section 2 we prove some preliminary results, that will be useful in the sequel. In Section 3, first we state some estimates from above, valid for the solution of the initial value problem (1.1)–(1.3) and then we prove proper estimates from below. In Section 4 we study the behaviour of solution up to the boundary. Finally in Section 5 we are able to prove the results concerning the asymptotic behaviour of the solution.

2. Notation and preliminary results

We recall some notation and some results we will need to prove the main results.

Let $\mathcal{A}$ be a domain of $\mathbb{R}^N$ and let $|\mathcal{A}|$ denote the Lebesgue measure of the set $\mathcal{A}$. For $\rho > 0$, let $B_\rho(x) \subset \mathbb{R}^N$ be the ball centered at x of radius ρ, $B_\rho = B_\rho(0)$ and $\Omega_\rho(x)$ be equal to $B_\rho(x) \cap \Omega$.

We introduce the set

$$Q_{\rho,\tau}(x_0, t_0) := B_\rho(x_0) \times (t_0, t_0 + \tau),$$

with $Q_{\rho,\tau} \subset \Omega \times (t > 0)$ and

$$\Omega_{\rho,\tau}(x_0, t_0) := \Omega_\rho(x_0) \times (t_0, t_0 + \tau).$$

We recall now some results proved in [5] to which we refer the reader for the proof:

Theorem 2.1 (L^1_{loc}-L^∞_{loc} Harnack-type estimates). *Let u be a non-negative, weak solution to (1.1)–(1.5). Assume p is in the super-critical range $\frac{2N}{N+1} < p < 2$. There exists a positive constant γ depending only upon the data, such that for all cylinders*

$$\Omega_{2\rho}(y) \times [t, t+s], \tag{2.1}$$

$$\sup_{\Omega_\rho(y)\times[t+\frac{s}{2}, t+s]} u \le \frac{\gamma}{(s)^{\frac{N}{\lambda}}} \left(\int_{\Omega_{2\rho}(y)} u(x,t)dx \right)^{\frac{p}{\lambda}} + \gamma \left(\frac{s}{\rho^p} \right)^{\frac{1}{2-p}} \tag{2.2}$$

where

$$\lambda \overset{def}{=} N(p-2) + p. \tag{2.3}$$

Theorem 2.2 (An L^1_{loc} Form of the Harnack Inequality for all $1 < p < 2$). *Let u be a non-negative, weak solution to (1.1)–(1.5). Assume $1 < p < 2$ and consider a ball $B_\rho(y)$ such that $B_{2\rho}(y) \subset \Omega$. Then there exists a positive constant γ depending only upon the data, such that for all cylinders $B_{2\rho}(y) \times [s,t]$,*

$$\sup_{s<\tau<t} \int_{B_\rho(y)} u(x,\tau)dx \le \gamma \inf_{s<\tau<t} \int_{B_{2\rho}(y)} u(x,\tau)dx + \gamma \left(\frac{t-s}{\rho^\lambda} \right)^{\frac{1}{2-p}} \tag{2.4}$$

where $\lambda = N(p-2)+p$. The constant $\gamma = \gamma(p) \to \infty$ as either $p \to 2$ or as $p \to 1$.

Theorem 2.3 (Intrinsic Harnack estimate). *Let u be a non-negative, weak solution to (1.1)–(1.5). Assume p is in the super-critical range $p_* = \frac{2N}{N+1} < p < 2$. There exist positive constants δ_* and c, depending only upon the data, such that for all $P_o \in \Omega \times (0,\infty)$ and all cylinders of the type $Q_{8\rho}(P_o) \subset \Omega \times (0,\infty)$,*

$$c\, u(x_o, t_o) \le \inf_{B_\rho(x_o)} u(\cdot, t) \tag{2.5}$$

for all times

$$t_o - \delta_*[u(P_o)]^{2-p}\rho^p \le t \le t_o + \delta_*[u(P_o)]^{2-p}\rho^p. \tag{2.6}$$

The constants c and δ_* tend to zero as either $p \to 2$ or as $p \to p_*$.

We point out that this inequality is simultaneously a "forward and backward in time" Harnack estimate as well as a Harnack estimate of elliptic type. Inequalities of this type would be false for non-negative solutions of the heat equation. This is reflected in (2.10)–(2.11), as the constants c and δ_* tend to zero as $p \to 2$. As proved in [5], it turns out that these inequalities lose meaning also as p tends to the critical value p_*

Remark 2.4. Let us stress that all the above results hold in the context of local non-negative solutions of singular parabolic equations. For our purpose and for the sake of simplicity, we stated them only for non-negative solutions of a Cauchy-Dirichlet problem.

Let us quote now a well-known regularity result for singular parabolic equations (for the proof, see [2], Theorems 1.1 and 1.2, pages 77–78).

Theorem 2.5 (Regularity result). *Let u be a local weak solution to* (1.1)–(1.3) *in a domain Ω and assume that the structure conditions* (1.4) *and* (1.5) *hold. Assume $1 < p < 2$. Then for each closed set K strictly contained in Ω, u is uniformly Hölder continuous in K with the Hölder continuity constant that depends in a quantitative way on the data. The same result holds for a solution of a Cauchy-Dirichlet problem. More precisely, assume that u is a local weak solution to* (1.1)–(1.5)*. Assume $1 < p < 2$. Then for each $\varepsilon > 0$, u is uniformly Hölder continuous in $\Omega \times (\varepsilon, \infty)$ with the constant of Hölder continuity that depends in a quantitative way on the data.*

Remark 2.6. All the results previously stated hold in the case that the function $\mathbf{A} = (A_1, \dots, A_n)$ is assumed to be only measurable. The following result requires differentiability of the coefficients.

We consider now some auxiliary results to be used later. First we introduce the Rayleigh quotient

$$E(u(t)) = \frac{\int_\Omega \mathbf{A}(x,t,\nabla u) \cdot \nabla u \; dx}{(\int_\Omega u^2 dx)^{p/2}} \,. \tag{2.7}$$

In addition, set $\mathbf{s}(x,t) := \nabla u(x,t)$, that is, $s_i = u_{x_i}, i = 1, \dots, N$.

Let us denote by $\mathbf{A}_t$ the derivatives of $\mathbf{A}(x,t,\mathbf{s})$ with respect to t and with

$$\mathbf{A}_\mathbf{s} := \frac{\partial(A_1, \dots, A_N)}{\partial(u_{x_1}, \dots, u_{x_N})}.$$

Now we prove the following

Theorem 2.7 (Rayleigh quotient). *Let u be a non-negative, weak solution to* (1.1)–(1.3) *and assume that the structure conditions* (1.4) *and* (1.5) *hold with p in the supercritical range $p_* = \frac{2N}{N+1} < p < 2$. Let T^* be the extinction time. Assume*

$$the\ functions\ A_i(x,t,\nabla u)\ to\ be\ differentiable, \tag{2.8}$$

$$\int_\Omega \mathbf{A}_t(x,t,\nabla u) \cdot \nabla u \; dx \le 0, \tag{2.9}$$

and

$$\int_\Omega \operatorname{div}(\mathbf{A}(x,t,\nabla u))\ \operatorname{div}(\mathbf{A_s}(x,t,\nabla u)\nabla u)\ dx \qquad (2.10)$$
$$\geq (p-1)\int_\Omega |\operatorname{div}\mathbf{A}(x,t,\nabla u)|^2\ dx.$$

Then

$$||u||_{L^2} \leq [(2-p)E(u(0))(T^*-t)]^{\frac{1}{2-p}}. \qquad (2.11)$$

Remark 2.8. Condition (2.10) is naturally verified by equations like the p-Laplacian.

Proof. Following [4], Prop. 13, we prove that $E(u(t))$ under restrictions (2.8)–(2.10) is non-increasing in time.

From the equation (1.1) we have

$$\frac{1}{2}\frac{d}{dt}||u||^2_{L^2} = -\int_\Omega \mathbf{A}(x,t,\nabla u)\cdot\nabla u\ dx. \qquad (2.12)$$

On the other hand, by applying the divergence theorem and Hölder inequality we get

$$\int_\Omega \mathbf{A}(x,t,\nabla u)\cdot\nabla u\ dx = -\int_\Omega u\ \operatorname{div}\mathbf{A}\ dx \leq ||u||_{L^2}\left(\int_\Omega |\operatorname{div}\mathbf{A}|^2\ dx\right)^{\frac{1}{2}}, \qquad (2.13)$$

from which we obtain

$$\int_\Omega |\operatorname{div}\mathbf{A}|^2\ dx \geq \frac{\left(\int_\Omega \mathbf{A}(x,t,\nabla u)\cdot\nabla u\ dx\right)^2}{\int_\Omega u^2 dx}. \qquad (2.14)$$

Moreover, by using once more (1.1), we have

$$\frac{d}{dt}\int_\Omega \mathbf{A}(x,t,\nabla u)\cdot\nabla u\ dx = -\int_\Omega |\operatorname{div}\mathbf{A}|^2\ dx + \int_\Omega (\mathbf{A}_t\ \nabla u + \mathbf{A_s}\nabla u_t\nabla u)\ dx. \qquad (2.15)$$

By inserting (2.9) and (2.10) in (2.15) and then using (2.14) we obtain

$$\frac{d}{dt}\int_\Omega \mathbf{A}(x,t,\nabla u)\cdot\nabla u\ dx \leq -p\int_\Omega |\operatorname{div}\mathbf{A}|^2\ dx \leq -p\ \frac{\left(\int_\Omega \mathbf{A}\cdot\nabla u\ dx\right)^2}{\int_\Omega u^2 dx}. \qquad (2.16)$$

From (2.12) and (2.16) we deduce

$$\frac{\frac{d}{dt}\int_\Omega \mathbf{A}(x,t,\nabla u)\cdot\nabla u\ dx}{\int_\Omega \mathbf{A}(x,t,\nabla u)\cdot\nabla u\ dx} \leq -p\ \frac{\int_\Omega \mathbf{A}(x,t,\nabla u)\cdot\nabla u\ dx}{\int_\Omega u^2\ dx} = \frac{p}{2}\ \frac{\frac{d}{dt}\int_\Omega u^2\ dx}{\int_\Omega u^2 dx} \qquad (2.17)$$

and (2.17) implies that $E(u(t))$ is non-increasing in time. To prove (2.11) we remark that inserting the Rayleigh quotient in (2.12) we get

$$\frac{d}{dt}||u(t)||_2^{2-p} = -(2-p)E(u(t)). \qquad (2.18)$$

From (2.18) one easily deduces (2.11). □

Remark 2.9. If we assume $\mathbf{A}(x,t,\nabla u)$ depending on u, Theorem 2.7 holds by replacing (2.10) with

$$\int_\Omega \operatorname{div} \mathbf{A}(x,t,u,\nabla u)\ [\operatorname{div}(\mathbf{A_s}\nabla u) - \mathbf{A}_u]\ dx \geq (p-1)\int_\Omega |\operatorname{div}\mathbf{A}|^2\ dx.$$

3. Estimate from above and below

We show in this section that the solution of (1.1)–(1.3) is bounded both from above and from below under structure conditions (1.4)–(1.5) and (2.8)–(2.10).

3.1. L^2-estimate from below

In this section we prove the estimates from above that we will need in the proof of the main result.

Theorem 3.1. *Let u be a non-negative, weak solution to* (1.1)–(1.3)*. Assume* (1.4)*,* (1.5) *and* (2.8)–(2.10) *hold. Assume p is in the supercritical range $p_* = \frac{2N}{N+1} < p < 2$. Then there is a finite time T^*, depending only upon N, p and u_0, such that $u(\cdot,t) = 0$ for all $t \geq T^*$. Moreover there exists a constant $\gamma_2 > 0$ depending only upon N and p such that, for each $0 < t < T^*$,*

$$\gamma_2\ ||u||_{L^2}^{2-p}|\Omega|^{\frac{N(p-2)+2p}{2N}} \geq (T^*-t). \tag{3.1}$$

Proof. We follow the same pattern of [4], [6] and [10], to which we refer for more details.

First of all note that by Theorem 2.1 for each $t > 0$, $u(\cdot,t)$ belongs to L^∞ and its L^∞-norm is controlled by the L^1-norm of the initial datum. As the measure of Ω is finite, we have that, for any $t > 0$, $u(\cdot,t)$ belongs to L^2 and its norm is controlled by the L^1-norm of the initial datum.

From the definition of weak solution, choosing as test function $\phi = u$ and integrating in the space variables, it follows that

$$\frac{1}{2}\frac{d}{dt}||u||_{L^2}^2 = -\int_\Omega \mathbf{A}(x,t,\nabla u)\cdot\nabla u\ dx. \tag{3.2}$$

By the structure condition (1.4), we get

$$\frac{1}{2}\frac{d}{dt}||u||_{L^2}^2 \leq -\ c_0\int_\Omega |\nabla u|^p dx. \tag{3.3}$$

By the Hölder and Sobolev inequalities

$$\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}} \leq \gamma|\Omega|^{\frac{N(p-2)+2p}{N}}\int_\Omega |\nabla u|^p dx, \tag{3.4}$$

where the constant γ does not depend upon $|\Omega|, p, N$. Plugging (3.4) into (3.3), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \leq -\frac{1}{\gamma|\Omega|^{\frac{N(p-2)+2p}{N}}}\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}}. \tag{3.5}$$

For any $t > 0$, the function $\psi := \int_\Omega u^2(x,t)dx$ satisfies the following ordinary differential inequality:

$$\dot{\psi} + \frac{2}{\gamma|\Omega|^{\frac{N(p-2)+2p}{N}}}\psi^{p/2} \leq 0 \tag{3.6}$$

with $\psi(\varepsilon) < \infty$ for any $\varepsilon > 0$ (the solution belongs to L^2 for any positive time). Therefore, by starting from a positive time and by integrating (3.6), one can deduce the existence of a finite extinction time T^*. $\square$

In an analogous way, it is possible to prove (3.1) by the ordinary differential inequality (3.6).

3.2. L^2-estimates from above

In an almost straightforward way from Theorem 2.2 we have

Theorem 3.2. *Let u be a non-negative, weak solution to* (1.1)–(1.3). *Assume* (1.4), (1.5) *and* (2.8)–(2.10) *hold. Assume p is in the super-critical range $p_* = \frac{2N}{N+1} < p < 2$. Then there is a constant γ_3 (depending only upon N and p) such that, for any $t_o < T^*$,*

$$\left(\int_\Omega u^2(x,t_o)dx\right)^{\frac{1}{2}} \leq \gamma_3\,(T^* - t_o)^{\frac{1}{2-p}}\,. \tag{3.7}$$

Proof. Arguing as in [4], page 72, we have that

$$\frac{d}{dt}||u||_{L^2}^{2-p} = -(2-p)E(u(t)). \tag{3.8}$$

By Theorem 2.7 we have that $E(u(t))$ is non-increasing in time and choosing a time t_0, we have for any $t_0 < t < T^*$,

$$||u||_{L^2} \leq [(2-p)E(u(t_0))(T^* - t_0)]^{\frac{1}{2-p}}. \tag{3.9}$$

We need now an estimate of $E(u(t_0)) = \dfrac{\int_\Omega \mathbf{A}\cdot\nabla u dx}{\int_\Omega u^2 dx}$ from above. By Theorem 3.1 and (1.5) we obtain

$$E(u(t_0)) \leq \tilde{\gamma}\int_\Omega |\mathbf{A}(x,t,\nabla u)\cdot\nabla u|\;dx \leq \tilde{\gamma}\;c_1\int_\Omega |\nabla u|^p dx. \tag{3.10}$$

In order to get an estimate from above $\int_\Omega |\nabla u|^p\;dx$ we follow [8], Theorem 4.1, step 2.

By L^∞ estimate (2.2), Theorem 2.1, we have that for any t such that $\frac{T^*}{2} < t < T^*$, $u(t) \in L^\infty(\Omega)$ and then $u \in L^2(\Omega)$, since Ω is bounded. Now we have to prove that there exists a time t_0 such that $\int_\Omega |\nabla u|^p dx$ is bounded. Indeed starting from (1.6) with $\phi = u$, we get

$$\frac{1}{2}\int_\Omega u^2(x,T^*)dx - \frac{1}{2}\int_\Omega u^2\left(x,\frac{T^*}{2}\right)dx \leq -\,c_0\int_{\frac{T^*}{2}}^{T^*}\int_\Omega |\nabla u|^p dxdt, \tag{3.11}$$

which yields

$$c_0 \int_{\frac{T^*}{2}}^{T^*} \int_\Omega |\nabla u|^p dxdt \le \frac{1}{2} \int_\Omega u^2\left(x, \frac{T^*}{2}\right) dx. \tag{3.12}$$

This implies that there exists a time level $t_0 \in [\frac{T^*}{2}, T^*]$ where

$$c_0 \int_\Omega |\nabla u|^p dx \le \frac{1}{T^*} \int_\Omega u^2\left(x, \frac{T^*}{2}\right) dx. \tag{3.13}$$

So we can estimate $E(u(t_0))$ and therefore $||u||_{L^2}$. □

3.3. L^∞-estimates from above

By Theorem 2.1 and by inequality (3.7) we get the L^∞-estimates from above.

Theorem 3.3. *Let u be a non-negative, weak solution to* (1.1)–(1.3). *Assume* (1.4), (1.5) *and* (2.8)–(2.10) *hold. Assume p is in the super-critical range $p_* = \frac{2N}{N+1} < p < 2$. Then there is a constant γ_4 (depending only upon N and p) such that, for any $t_o < T^*$,*

$$\sup_\Omega u(x, t_o) \le \gamma_4 \left(T^* - t_o\right)^{\frac{1}{2-p}}. \tag{3.14}$$

Proof. Inequality (3.14) follows from (2.2) choosing $t = T^* - 2t_o$, $t+s = T^*$, $\rho = d_\Omega$, where d_Ω is the diameter of Ω and estimating $\int_\Omega u(x, t_o)dx$ with $\gamma\left(T^* - t_o\right)^{\frac{1}{2-p}}$ by using (3.7). □

3.4. L^∞ interior estimates from below

From Theorems 3.1 and 3.3 we can deduce estimates in the interior of Ω. More precisely

Theorem 3.4. *Let u be a non-negative, weak solution to* (1.1)–(1.3). *Assume* (1.4), (1.5) *and* (2.8)–(2.10) *hold. Assume p is in the super-critical range $p_* = \frac{2N}{N+1} < p < 2$. There exists a positive number d, depending only upon the geometry of Ω, p and N, such that for any $\frac{T^*}{2} < t_o < T^*$, there is a point $x_o \in \Omega$ with* $\mathrm{dist}(x_o, \partial\Omega) > d$ *such that*

$$u(x_o, t_o) \ge \gamma_5 (T^* - t_o)^{\frac{1}{2-p}}, \tag{3.15}$$

where γ_5 is a positive constant depending only upon the data.

Proof. By (3.1) we have

$$\sup_\Omega u(x, t_o) \ge |\Omega|^{-\frac{1}{2}} ||u||_{L^2} \ge \gamma_6 (T^* - t_o)^{\frac{1}{(2-p)}} \tag{3.16}$$

and then by (3.14)

$$||u||_{L^2} \ge \gamma_7 \sup_\Omega u(x, t_o), \tag{3.17}$$

with $\gamma_7 = \gamma_7(n, p, \Omega)$.

Define the set A as the set of the points $x \in \Omega$ where

$$u(x, t_o) \ge \frac{\gamma_7}{\sqrt{2|\Omega|}} \sup_\Omega u(x, t_o).$$

Let $\gamma_8 = \dfrac{\gamma_7}{\sqrt{2\ |\Omega|}}\ \sup_{\Omega} u(x,t_o)$. By (3.17) we have

$$|\mathsf{A}| \geq \frac{1}{2}\gamma_7^2. \tag{3.18}$$

Since the set A has strictly positive measure and Ω is a bounded regular set, then there exists a positive constant d such that there exists a point $x_o \in \mathsf{A}$ with $\mathrm{dist}(x_o, \mathsf{A}) > d$. Then (3.15) follows from the definition of γ_8 and (3.16). □

The following statement is a direct consequence of Theorem 3.4 and of the intrinsic Harnack estimates of Theorem 2.3.

Theorem 3.5. *Let u be a non-negative, weak solution to* (1.1)–(1.3). *Assume* (1.4), (1.5) *and* (2.8)–(2.10) *hold. Assume that p is in the super-critical range $p_* = \frac{2N}{N+1} < p < 2$. For any positive number d there is a constant γ_9 depending only upon the geometry of Ω, d, p and N, such that, for any $\frac{T^*}{2} < t_o < T^*$, for any point $x_o \in \Omega$ with* $\mathrm{dist}(x_o, \partial\Omega) > d$,

$$u(x_o, t_o) \geq \gamma_9 (T^* - t_o)^{\frac{1}{2-p}}. \tag{3.19}$$

4. Estimates at the boundary from above and from below

Thanks to the previous estimates, we can extend some results up to the boundary. Let $d(x)$ be the distance from the point x to the boundary $\partial\Omega$.

Theorem 4.1. *Let u be a non-negative, weak solution to* (1.1)–(1.3). *Assume* (1.4), (1.5) *and* (2.8)–(2.10) *hold. Assume p is in the super-critical range $p_* = \frac{2N}{N+1} < p < 2$. There exist two constants $\beta \in (0,1)$ and γ_{10}, such that for each $x \in \Omega$ and for each $\frac{T^*}{2} < t < T^*$,*

$$u(x,t)\ \geq (T^* - t)^{\frac{1}{2-p}}\ \gamma_{10}\ d(x)^{\beta}. \tag{4.1}$$

Proof. Denote by $K_n = \{x \in \Omega \text{ such that } d(x) \geq 2^{-n}\}$. As the boundary is Lipschitz-continuous, there exists n_0 such that for each $n \geq n_0$ the distance between K_n and any point $x \in K_{n+1}$ is less than or equal to $2^{-(n+1)}$. By Theorem 3.5 for any $x_o \in K_{n_o}$ and for any $\frac{T^*}{2} < t < T^*$,

$$u(x_o, t) \geq \gamma_9\ (T^* - t)^{\frac{1}{2-p}}. \tag{4.2}$$

By Theorem 2.3 we have that for each $x \in K_{n_0+1}$,

$$u(x,t) \geq c\ u(x_o,t) \geq c\ \gamma_9\ (T^* - t)^{\frac{1}{2-p}}.$$

By induction for any $x \in K_{n_0+n}$ one gets that

$$u(x,t) \geq c^n \gamma_9\ (T^* - t)^{\frac{1}{2-p}}. \tag{4.3}$$

As we are working with $2^{-n-1} \leq d(x) \leq 2^{-n}$, inequality (4.3) easily implies the statement of the theorem. □

Theorem 4.2. *Let u be a non-negative, weak solution to* (1.1)–(1.3)*. Assume* (1.4), (1.5) *and* (2.8)–(2.10) *hold. Assume that p is in the super-critical range $p_* = \frac{2N}{N+1} < p < 2$. There exist two constants $\eta \in (0,1)$ and γ, such that for each $x \in \Omega$ and for each $\frac{T^*}{2} < t < T^*$,*

$$\gamma\ d(x)^\eta \geq u(x,t)\ (T^* - t)^{\frac{1}{2-p}}. \tag{4.4}$$

Proof. Let x_o be a point of $\partial\Omega$. As the boundary is Lipschitz continuous, there is a bi-Lipschitz continuous diffeomorphism $\mathcal{T}$ that maps a neighborhood of x_o in the hemisphere $B^+(0,1) = \{x \in B(0,1)$ such that $x_N > 0\}$. The function $v(y,t) = u(\mathcal{T}^{-1}y,t)$ is a solution of

$$v_t = \operatorname{div} \mathbf{A_1}(y,t,\nabla v), \quad (y,t) \in B^+(0,1) \times (t > 0), \tag{4.5}$$

$$v(y,t) = 0, \quad t > 0 \quad y \in B(0,1) \cap \{y \in \mathbb{R}^N \text{ such that } y_N = 0\}, \tag{4.6}$$

where

$$\mathbf{A_1}(y,t,\nabla v) = \mathbf{A}(\mathcal{T}^{-1}y, t, J(\mathcal{T}^{-1})(y)\nabla u(\mathcal{T}^{-1}y,t)|J(\mathcal{T}^{-1})(y)|).$$

$J(\mathcal{T}^{-1})(y)$ is the Jacobian matrix and $|J(\mathcal{T}^{-1})(y)|$ is the Jacobian determinant. Let us extend v in $B(0,1)$. Denote by $\bar{y}$ the first $N-1$ components of the vector y. Define $w(y,t) = v(y,t)$ if $y_n \geq 0$ and $w(y,t) = -v(\bar{y}, -y_N), t)$ if $y_n \leq 0$. The function w is a solution of

$$w_t = \operatorname{div} \mathbf{A_2}(y,t,\nabla w), \quad (w,t) \in B(0,1) \times (t > 0), \tag{4.7}$$

where $\mathbf{A_2}$ is equal to $\mathbf{A_1}$ if $y_N \geq 0$; when $y_N \leq 0$,

$$\begin{aligned}
&\mathbf{A_2}(y;t;D_1 w, \ldots, D_{N-1}w, D_N w) \\
&\quad = -\mathbf{A_2}\Big((\bar{y}, -y_N); t, D_1 w((\bar{y}, -y_N), t), \ldots, D_{N-1}w((\bar{y}, -y_N), t), \\
&\qquad - D_N w((\bar{y}, -y_N), t)\Big).
\end{aligned}$$

As $\mathbf{A_2}$ satisfies the structure conditions (1.4)–(1.5) by Theorem 2.5, the function w is Hölder continuous and this implies (4.4). □

5. Asymptotic behaviour

In this section we investigate the behavior of the solution of (1.1)–(1.3) when it is approaching the extinction time. We work as in [3], [10] and [4] and we set $t = T^* - T^* e^{-\tau}$ and

$$w(x,\tau) = \frac{u(x, T^* - T^* e^{-\tau})}{(T^* e^{-\tau})^{\frac{1}{2-p}}}\,. \tag{5.1}$$

The function $w(x,\tau)$ is a non-negative solution of the equation

$$w_\tau = \operatorname{div}\ \tilde{\mathbf{A}}(x,\tau,\nabla w) + \frac{1}{2-p} w, \tag{5.2}$$

where

$$\tilde{\mathbf{A}}(x,\tau,\nabla w) := (T^*e^{-\tau})^{-\frac{p-1}{2-p}}\ \mathbf{A}(x, T^* - T^*e^{-\tau}, (T^*e^{-\tau})^{\frac{1}{2-p}}\nabla w) \tag{5.3}$$

and

$$w(x,0) = u_0(x)\ (T^*)^{-\frac{1}{2-p}}.$$

Note that $\tilde{\mathbf{A}}$ satisfies the structure conditions (1.4)–(1.5). From the results of the previous section we have:

Theorem 5.1. *For any positive time t_1, for any closed set K strictly contained in Ω there are strictly positive constants $C_1 - C_5$, depending only upon the data, t_1 and K, such that for any $t \geq t_1$,*

- *for any $x \in \Omega$,*
$$w(x,t) \leq C_1; \tag{5.4}$$
- *for any $x \in K$,*
$$w(x,t) \geq C_2; \tag{5.5}$$
- *for any $x \in \Omega$,*
$$C_3 d(x)^\beta \leq w(x,t) \leq C_4 d(x)^\eta, \tag{5.6}$$
where $d(x)$ the distance from the point x to the boundary $\partial\Omega$;
- *w is uniformly α-Hölder continuous with the Hölder continuity constant that depends only upon the data and*
$$||w||_{C^{\alpha,\frac{\alpha}{p}}(\Omega,[t_1,\infty])} \leq C_5.$$

By Theorem 5.1, $w(t)$ is equi-Hölder continuous, therefore, up to a subsequence, there is a function $v \in C^\alpha(\Omega)$ such that $w \to v$ in C^α.

If we want the function v to be the solution of a suitable partial differential equation, we have to assume some hypotheses on the coefficient $\tilde{\mathbf{A}}(x,\tau,\tilde{\mathbf{s}})$ in (5.2), where $\tilde{\mathbf{s}}(x,t) := \nabla w(x,t)$, that is, $\tilde{s}_i = w_{x_i}, i = 1,\ldots,N$.

$$\tilde{\mathbf{A}}(x,\tau,\tilde{\mathbf{s}}) \text{ is a } C^0 \text{ function with respect to time,} \tag{5.7}$$

$$\exists \text{ a function } H(x,\tau,\tilde{\mathbf{s}}) \text{ such that } \frac{\partial H}{\partial \tilde{s}_i} = \tilde{A}_i, \tag{5.8}$$

$$\text{and } \frac{\partial H}{\partial \tau} \leq 0, \tag{5.9}$$

$$\exists \quad \text{a positive constant } C_6 : \int_\Omega H(x,\tau,\nabla w)\,dx \geq C_6. \tag{5.10}$$

Note that the assumption of continuity on $\mathbf{A}(x,t,\mathbf{s})$ implies that

$$\exists \lim_{\tau\to+\infty} \tilde{\mathbf{A}}(x,\tau,\tilde{\mathbf{s}}) = \mathbf{A}_\infty(x,\tilde{\mathbf{s}}).$$

Theorem 5.2. *Assume that hypotheses* (1.4) *and* (1.5) *hold. Then the function v belongs to $W_0^{1,p} \cap L^2(\Omega)$ and it is a non-trivial solution of*

$$\operatorname{div}(\mathbf{A}_\infty(x,\nabla v)) = -\frac{1}{2-p}v. \tag{5.11}$$

Proof. The functional

$$F(x,\tau,\nabla w(x,\tau)) = \int_\Omega H(x,\tau,\nabla w)\,dx - \frac{1}{2(2-p)}\int_\Omega w^2(x,\tau)\,dx$$

is monotone decreasing in time. In fact

$$\begin{aligned}\frac{d}{d\tau}\int_\Omega H\,dx &= \int_\Omega \left[\frac{\partial H}{\partial \tilde{s}_i}\nabla_i w_\tau + \frac{\partial H}{\partial \tau}\right]dx\\ &\le \int_\Omega \frac{\partial H}{\partial \tilde{s}_i}\nabla_i w_\tau\,dx\\ &= -\int_\Omega (\operatorname{div}\tilde{\mathbf{A}})\;w_\tau\,dx\\ &= -\int_\Omega (w_\tau)^2\,dx + \frac{1}{2-p}\int_\Omega w w_\tau\,dx.\end{aligned}$$

Then

$$\frac{dF}{d\tau} \le -\int_\Omega (w_\tau)^2 \;\le\; 0.$$

As the functional F is bounded from below, this implies that, up to a subsequence, $\dfrac{dF}{d\tau}$ (and therefore $\int_\Omega w_\tau^2(x)dx$) converges to zero.

Then w converges to its limit $v \in W_0^{1,p} \cap L^2(\Omega)$ and v is the solution of (5.11). □

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Linearized Stability for Nonlinear Partial Differential Delay Equations

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Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. The object of the paper are partial differential delay equations of the form $\dot{u}(t) + Bu(t) \ni F(u_t)$, $t \geq 0$, $u_0 = \varphi$, with $B \subset X \times X$ ω-accretive in a Banach space X. We extend the principle of linearized stability around an equilibrium from the semilinear case, with B linear, to the fully nonlinear case, with B having a linear 'resolvent-differential' at the equilibrium.

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Keywords. Nonlinear partial differential delay equations; accretive operators; linearized stability; subtangential conditions.

1. Introduction

The object of study of this paper is the following partial functional differential equation with delay:

$$\text{(PFDE)} \quad \begin{cases} \dot{u}(t) + Bu(t) \ni F(u_t), & t \geq 0 \\ u_{|I} = \varphi \in \hat{E}. \end{cases}$$

$B \subset X \times X$ is a (generally) nonlinear and multivalued operator in a Banach (state) space X, with $(B + \omega I)$ accretive, some $\omega \in \mathbb{R}$, and, for given $I = [-R, 0]$, $R > 0$ (finite delay), or $I = (-\infty, 0]$ (infinite delay), and $t \geq 0$, $u_t : I \to X$ is the history of u up to $t : u_t(s) = u(t + s)$, $s \in I$, and $\varphi : I \to X$ is a given initial history out of a subset $\hat{E}$ of a space E of continuous functions from I to X. Moreover, F is a given history-responsive operator with domain $D(F) \subset E$ and range in X.

Our objective is a solution to the following problem of *linearized stability* for (PFDE): Assume that there exists an equilibrium solution $\varphi_e \in E$ to (PFDE), let

$x_e = \varphi_e(0)$, and assume that there exist 'linearizations' $\tilde{B}$ of B at x_e, and $\tilde{F}$ of F at φ_e such that the solution semigroup (in E) to the 'linearization'

$$(\text{PFDE})_{\text{lin}} \quad \begin{cases} \dot{u}(t) + \tilde{B}u(t) \ni \tilde{F}(u_t), & t \geq 0 \\ u_{|I} = \varphi \in E \end{cases}$$

of (PFDE) is exponentially stable. Is it then true that the equilibrium φ_e is locally exponentially stable with respect to the nonlinear problem (PFDE)?

The existing positive results in this direction generally are for the semilinear case, with $B : D(B) \subset X \to X$ linear and single-valued, and either the infinitesimal generator of a C_0-semigroup of bounded linear operators on X, or a Hille-Yosida operator, and under the assumption that F be globally Fréchet-differentiable with $F' : E \to B(E, X)$ being locally Lipschitz, or just Fréchet-differentiable at φ_e (c.f. [1] [15, Thm. 6.1], [19, 20, 27]). For more general $B \subset X \times X$ linear, and F not globally, but possibly defined only on 'thin' subsets of E, see [25, Thm. 4.2]. For general, possibly nonlinear $B \subset X \times X$, and under local range conditions on both B and F (possibly defined only on 'thin' subsets of E), a positive answer has been given in [24, Thm. 4.1].

The purpose of this paper is to extend all of these results in two directions: (a) the fully nonlinear case with $B \subset X \times X$ nonlinear, ω-accretive, and linearly 'resolvent-differentiable' at x_e (Definition 2.2), and (b) the local case of F possibly being defined only on a 'thin' subset of E, and not necessarily locally Lipschitz, and under a subtangential condition on B and F as in [25].

As has been demonstrated by the applications in [25, Section 5], the generality in having F only defined on 'thin' subsets (possibly not even containing an interior point), and not necessarily locally Lipschitz, is crucial for population models set up in the (natural) state space $L^1(\Omega)$, Ω an open subset of $\mathbb{R}^n$.

For existence results for (PFDE) in general, mostly in the global context, we refer here to [4, 5, 6, 7, 12, 13, 14, 16, 21, 23, 26, 28, 29, 30]; for a much more complete list, the reader is referred to the survey [22]. The monograph [31] surveys the semilinear case with B the generator of a C_0-semigroup.

Notation and terminology. Throughout this paper, all Banach spaces are assumed to be over the reals. For X a Banach space, and $\lambda \in \mathbb{R} \setminus \{0\}$, define $[\cdot, \cdot]_\lambda : X \times X \to \mathbb{R}$ by

$$[x, y]_\lambda = \frac{\|x + \lambda y\| - \|x\|}{\lambda}.$$

Then, as, for fixed $x, y \in X$, the function $\{\lambda \to [x, y]_\lambda\}$ is nondecreasing for $\lambda > 0$, one can define the bracket $[x, y]$ (the right-hand Gâteaux-derivative of the norm at x in the direction of y) by

$$[x, y] = \lim_{\lambda \searrow 0} [x, y]_\lambda = \inf_{\lambda > 0} [x, y]_\lambda.$$

(cf. [3, 18]).

$B(X)$ denotes the space of all bounded linear operators from X into X. Given a subset D of X, *cl* D will denote its closure in X. Recall that a subset $C \subset X \times X$

is said to be *accretive* in X if for each $\lambda > 0$, and each pair $(x_i, y_i) \in C, i \in \{1,2\}$, we have $\|(x_1 + \lambda y_1) - (x_2 + \lambda y_2)\| \geq \|x_1 - x_2\|$, and ω-*accretive* in X for some $\omega \in \mathbb{R}$, if $(C + \omega I)$ is accretive. If, in addition, $R(I + \lambda C) = X$ for all $\lambda > 0$ with $\lambda\omega < 1$, then C is said to be m$-\omega$-accretive. If $C \subset X \times X$ is ω-accretive, then, for any $\lambda > 0$ with $\lambda\omega < 1$, $J_\lambda^C = (I + \lambda C)^{-1}$ denotes the resolvent of C.

Notions of solutions. Given an initial-history space E of functions from I to X, $T > 0$, and $B \subset X \times X$ ω-accretive for some $\omega \in \mathbb{R}$, a function $u : I \cup [0,T] \to X$ will be called a *mild* solution to (PFDE) if (i) $u_{|I} = \varphi$, and (ii) $u_{|[0,T]}$ is continuous and is a mild solution to the initial value problem

$$\text{(CP)} \quad \begin{cases} \dot{u}(t) + Bu(t) \ni f(t), & 0 \leq t \leq T, \\ u(0) = \varphi(0) \end{cases}$$

with $f(t) = F(u_t)$. In turn, given $f \in L^1(0,T;X)$, a continuous function $u : [0,T] \to X$ with $u(0) = \varphi(0)$ is a *mild* solution to (CP) if, given any $\epsilon > 0$, there exists an ϵ-discrete-scheme approximation $D_B(t_0, \ldots, t_N; u_0, \ldots, u_N; f_1, \ldots, f_n)$ consisting of an ϵ-partition of the interval $[0,T]$,

$$0 \leq t_0 < t_1 < \cdots < t_N \leq T,\ t_0 < \epsilon,\ T - t_N < \epsilon,\ t_i - t_{i-1} < \epsilon, i \in \{1, \ldots, N\}, \tag{1.1}$$

and elements $\{u_0, u_1, \ldots, u_N\}$, and $\{f_1, f_2, \ldots, f_N\}$ in X such that

$$\begin{aligned} & u_i \in D(B), i \in \{1, \ldots, N\}, \\ & \frac{u_i - u_{i-1}}{t_i - t_{i-1}} + Bu_i \ni f_i,\ i \in \{1, \ldots, N\}, \\ & \sum_1^N \int_{t_{i-1}}^{t_i} \|f_i - f(\tau)\|\, d\tau < \epsilon,\ \|u_0 - u(0)\| < \epsilon, \text{and} \end{aligned} \tag{1.2}$$

such that, if the step function $u_\epsilon : [t_0, t_N] \to X$ is defined by

$$u_\epsilon(t) = \begin{cases} u_0 & t = t_0, \\ u_i & t \in (t_{i-1}, t_i],\ i \in \{1, \ldots, N\}, \end{cases}$$

then $\|u_\epsilon(t) - u(t)\| < \epsilon$ uniformly over $t \in [t_0, t_N]$.

For all these notions and the general theory of accretive sets and evolution equations, the reader is referred to [3, 18].

2. Linearized stability for (PFDE)

We shall place our results in the context of general local subtangential conditions on B and F for the existence of solutions to (PFDE), as well as in the context of a general class of initial-history spaces E as in [25].

2.A The initial history space. Given a Banach (state) space X, and letting $I = (-\infty, 0]$ (infinite delay), or $I = [-R, 0]$ for some $R > 0$ (finite delay), the initial history space is assumed to be a Banach space $(E, \|\cdot\|)$ of continuous functions $\varphi : I \to X$ with the following properties:

(E1) (a) For all $\varphi \in E$, $\|\varphi(0)\| \leq \|\varphi\|$.
(b) For all $x \in X$, $\tilde{x} \in E$, where $\tilde{x}(s) \equiv x$, $s \in I$, and
(c) there exists $C_E \geq 1$ such that $\|\tilde{x}\| \leq C_E \|x\|$ for all $x \in X$.
(d) For $\varphi, (\varphi_n)_n$ in E, if $\|\varphi_n - \varphi\| \to 0$, then (1) $\|\varphi_n(s) - \varphi(s)\| \to 0$ for all $s \in I$, and (2) $\int_\alpha^\beta \varphi_n(s)ds \to \int_\alpha^\beta \varphi(s)ds$ for all $\alpha, \beta \in I$, $\alpha < \beta$.

(E2) If $\lambda > 0, x \in X, \psi \in E$, and $\varphi^\psi_{\lambda,x} \in C^1(I;X)$ is the solution to $\varphi - \lambda\varphi' = \psi$, with $\varphi(0) = x$, then $\varphi^\psi_{\lambda,x} \in E$, and $\left\|\varphi^\psi_{\lambda,x}\right\| \leq \max\{\|x\|, \|\psi\|\}$.

(E3) (a) If $x : I \cup [0,\infty) \to X$ is continuous, and $x_{|I} \in E$, then
(i) $x_t \in E$ for all $t \geq 0$, and
(ii) the map $\{t \to x_t\}$ is continuous from $\mathbb{R}^+$ into E.
(b) There exist $M_0 \geq 1$, and a locally bounded function $M_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that, given $x : I \cup [0,\infty) \to X$ as in (a) above,
$\|x_t\| \leq M_0 \|x_0\| + M_1(t) \max_{0 \leq s \leq t} \|x(s)\|$ for all $t \geq 0$.

(Concerning these axioms for E, compare [9, 11].)

Examples. In the finite-delay case, usually, the initial-history space will be $E = C([-R,0];X)$ with sup-norm. For the infinite-delay case $I = (-\infty, 0]$, the most prominent initial-history spaces are weighted sup-norm spaces of the type $E_v = \{\varphi \in C(\mathbb{R}^-, X) : v\varphi \in BUC(\mathbb{R}^-;X)\}$, with norm $\|\varphi\|_v := \sup\{v(s)\|\varphi(s)\| \mid s \in \mathbb{R}^-\}$, where the (weight-) function $v : \mathbb{R}^- \to (0,1]$ has the following properties:

(v1) v is continuous, nondecreasing, and $v(0) = 1$;
(v2) $\lim_{u\to 0^-} v(s+u)/v(s) = 1$ uniformly over $s \in \mathbb{R}^-$.

Typical such weight functions are $v(s) \equiv 1$ (with, in this case, $E_v = BUC(\mathbb{R}^-, X)$ with sup-norm), $v(s) = e^{\mu s}$, or $v(s) = (1+|s|\ )^{-\mu}$, $\mu > 0$ (spaces of 'fading memory type'). (The Banach spaces E_v are sometimes called UC_g-spaces, $v = 1/g$, and have been considered by various authors, cf. [11].)

Aside from E_v, also the following subspaces fulfill axioms **(E1)**–**(E3)**:

(a) $E_{v_l} = \{\varphi \in E_v \mid \lim_{s\to-\infty} v(s)\varphi(s) \text{ exists}\}$, and
(b) $E_{v_0} = \{\varphi \in E_v \mid \lim_{s\to-\infty} v(s)\varphi(s) = 0\}$, in case $\lim_{s\to-\infty} v(s) = 0$.

2.B The framework for (PFDE). Given an initial-history space E as in **2.A**, we start from the following assumptions:

(A1) $B \subset X \times X$ is ω-accretive for some $\omega \in \mathbb{R}$.
(A2) $\hat{X} \subset X$ is a closed subset of X with $\hat{X} \cap D(B) \neq \emptyset$.
(A3) (a) $\hat{E}_0 = \{\varphi \in E \mid \varphi(0) \in cl\,(D(B) \cap \hat{X})\}$.
(b) $\hat{E} \subset \hat{E}_0$ is a closed subset of $\hat{E}_0$.
(c) If $x \in \hat{X} \cap D(B), \psi \in \hat{E}$, and $\lambda > 0$ is sufficiently small, then $\varphi^\psi_{\lambda,x} \in \hat{E}$, where $\varphi^\psi_{\lambda,x}$ is the solution $\varphi \in E$ to $\varphi - \lambda\varphi' = \psi$, $\varphi(0) = x$.
(A4) The operator $F : \hat{E}_0 \to X$ is such that
(a) $F : \hat{E}_0 \to X$ is continuous;
(b) there exists $L_F > 0$ such that, if $\varphi_1, \varphi_2 \in \hat{E}_0$ with $\varphi_1(0) = \varphi_2(0)$, then $\|F\varphi_1 - F\varphi_2\| \leq L_F \|\varphi_1 - \varphi_2\|$, and

(c) there exists $M_F > 0$ such that, if $\varphi_1, \varphi_2 \in \hat{E}_0$ with $\|\varphi_1 - \varphi_2\| = \|\varphi_1(0) - \varphi_2(0)\|$, then $[\varphi_1(0) - \varphi_2(0), F\varphi_1 - F\varphi_2] \le M_F \|\varphi_1 - \varphi_2\|$.

2.C Solution theory for (PFDE). As for the existence of solutions to (PFDE), we recall the main existence result of [25, Thm. 2.1]:

Theorem 2.1. *Given the assumptions* **(A1)**–**(A4)**, *if*

$$\text{(STC)} \quad \liminf_{\lambda \to 0^+} \frac{1}{\lambda} \operatorname{dist}(\psi(0) + \lambda F(\psi); (I + \lambda B)(D(B) \cap \hat{X})) = 0 \quad \textit{for all} \quad \psi \in \hat{E},$$

then we have:

(a) *For all $\varphi \in \hat{E}$ there exists a global mild solution u_φ to*

$$\text{(PFDE)} \quad \begin{cases} \dot{u}(t) + Bu(t) \ni F(u_t), & t \ge 0, \\ u_0 = \varphi \end{cases}$$

with $(u_\varphi)_t \in \hat{E}$ for all $t \ge 0$. In particular, $u_\varphi(t) \in \hat{X}$ for all $t \ge 0$.

(b) *If, in addition, either $F\mid_{\hat{E}} : \hat{E} \to X$ is locally Lipschitz-continuous on $\hat{E}$, or there exists $M_0 > 0$ such that $[\varphi(0) - \psi(0), F\varphi - F\psi] \le M_0 \|\varphi - \psi\|$ for all $\varphi, \psi \in \hat{E}$, then, for any $\varphi \in \hat{E}$, the mild solution u_φ as in* (a) *is unique amongst all mild solutions u to* (PFDE) *with the property that $u_t \in \hat{E}$ for all $t \ge 0$.*

For our result on linearized stability, we need the following approach to constructing the mild solution u_φ :

Associate with (PFDE) the operator A in E defined by

$$\begin{cases} D(A) = \{\varphi \in \hat{E}_0 \mid \varphi' \in E, \varphi(0) \in D(B), \varphi'(0) \in F\varphi - B\varphi(0)\} \\ A\varphi := -\varphi' , \varphi \in D(A). \end{cases} \tag{2.1}$$

Given the assumptions of Theorem 2.1, the following assertions hold ([25]):

(S1) The operator A is γ-accretive in E with $\gamma = max\{0, \omega + M_F\}$.

(S2) For every $\varphi \in \hat{E}$, there exists a unique mild solution $\Phi_\varphi : \mathbb{R}^+ \to E$ to the Cauchy problem in E corresponding to A,

$$\text{(CP)}(A; \varphi; 0) \quad \begin{cases} \dot{\Phi}(t) + A\Phi(t) = 0, & t \ge 0, \\ \Phi(0) = \varphi. \end{cases}$$

The solution semigroup $(S(t))_{t \ge 0}$ generated by $-A$ via $S(t)\varphi := \Phi_\varphi(t)$ leaves $\hat{E}$ invariant.

(S3) If $\varphi \in \hat{E}$, and $u_\varphi : I \cup \mathbb{R}^+ \to X$ is defined by

$$u_\varphi(t) = \begin{cases} \varphi(t) & t \in I \\ (S(t)\varphi)(0) & t \ge 0, \end{cases} \tag{2.2}$$

then $S(t)\varphi = (u_\varphi)_t$ for $t \ge 0$ (i.e., $(S(t))_{t \ge 0}$ acts as a translation), and the function u_φ is a global mild solution to (PFDE).

2.D Resolvent differentiability. The following concept of differentiating a (possibly) multivalued operator will be basic for our result.

Definition 2.2. *For $C \subset Z \times Z$ an α-accretive operator in a Banach space Z, and $z_0 \in Z$ with $z_0 \in R(I + \lambda C)$ for all $\lambda > 0$, $\lambda\alpha < 1$, an $\tilde{\alpha}$-accretive operator $\tilde{C} \subset Z \times Z$ with $(R(I + \lambda C) - z_0) \subset R(I + \lambda\tilde{C})$ for all $\lambda > 0$ small enough, is called a resolvent-differential of C at z_0, if the following holds:*

$$\text{(RD)} \begin{cases} \text{There exists a function } \eta : \bigcup\{\{\lambda\} \times R(I+\lambda C) \mid 0 < \lambda < \lambda_0\} \to \mathbb{R}^+, \\ \text{some } \lambda_0 > 0,\ \lambda_0\alpha < 1, \text{ with } \lim_{(\lambda,z)\to(0,\hat{z})} \eta(\lambda, z) = 0, \text{ such that for} \\ \text{every } \epsilon > 0, \text{ there exist } \delta > 0, \text{ and } \lambda_1 > 0,\ \lambda_1 \le \lambda_0, \text{ such that, if} \\ z \in R(I + \lambda C), \text{and} \quad \|z - z_0\| < \delta, \text{then} \\ \qquad \left\| J_\lambda^C z - J_\lambda^C z_0 - J_\lambda^{\tilde{C}}(z - z_0)\right\| \le\ \epsilon\lambda \|z - z_0\| \ +\ \lambda\eta(\lambda, z) \\ \text{for all } 0 < \lambda \le \lambda_1. \end{cases}$$

Remark 2.3. The notion of resolvent-differentiability has originally been introduced in [24]. As discussed in [24, Remarks 2.2], with regard to ω-accretive operators, this notion seems to be the natural extension of Fréchet-differentiability from 'single-valued' to 'multivalued'. In particular, while the principle of linearized stability for the Cauchy problem $\dot{u}(t) + Au(t) \ni 0$ holds for A single-valued and Fréchet-differentiable at the equilibrium, it carries over to (α-accretive) multivalued A if A has a linear resolvent-differential $\tilde{A}$ at the equilibrium ([24, Thm. 2.1]). More to the point, if $A : D(A) \subset X \to X$ is (single-valued and) α-accretive, with $R(I + \lambda A) \supset D(A)$ for $\lambda > 0$ small enough, $x_0 \in D(A)$, and A has an F-differential $\tilde{A} \in B(X)$ at x_0 relative to $D(A)$ (in the sense of **(L)** (3) below), then $\tilde{A}$ is a resolvent-differential of A at x_0 (with $\eta(\lambda, x) = M \left\|J_\lambda^A x_0 - x_0\right\|$, for some constant $M > 0$); see [24, Remark 2.3].

In order to formulate the linearization principle for (PFDE), we start from the following additional assumptions.

(L) (1) There exists an equilibrium solution $\varphi_e \in D(A) \cap \hat{E}$ of the solution semigroup $(S(t))_{t\ge0}$ for (PFDE) such that $x_e := \varphi_e(0) \in R(I + \lambda B)$ for all $\lambda > 0$, $\lambda\omega < 1$.

(2) There exists a linear and m$-\tilde{\omega}$-accretive resolvent-differential $\tilde{B} \subset X \times X$ of B at x_e, some $\tilde{\omega} \in \mathbb{R}$.

(3) There exists a bounded linear operator $\tilde{F} : E \to X$ that is a $D(A)$-Fréchet-derivative of F at φ_e, i.e., given any $\epsilon > 0$, there exists $\delta > 0$ such that, if $\varphi \in D(A)$, and $\|\varphi - \varphi_e\| < \delta$, then

$$\left\|F\varphi - F\varphi_e - \tilde{F}(\varphi - \varphi_e)\right\| \le \epsilon \|\varphi - \varphi_e\| .$$

With assumption **(L)** in place, we consider in E the solution operator $\tilde{A}$

$$\begin{cases} D(\tilde{A}) = \{\varphi \in E \mid \varphi' \in E, \varphi(0) \in D(\tilde{B}), \varphi'(0) \in \tilde{F}(\varphi) - \tilde{B}\varphi(0)\} \\ \tilde{A}\varphi := -\varphi' , \varphi \in D(\tilde{A}), \end{cases}$$

associated with $(\text{PFDE})_{\text{lin}}$.

We are now able to formulate the main result on linearized stability for (PFDE).

Theorem 2.4. *Given the assumptions* **(L)** *and those of Theorem* 2.1*, if there exists* $\tilde{\gamma} > 0$ *such that the linearized operator* $(\tilde{A} - \tilde{\gamma} I) \subset E \times E$ *is accretive, then the initial history problem* (*PFDE*) *is locally exponentially stable at the equilibrium* φ_e *in the following sense: given any* $0 < \gamma_1 < \tilde{\gamma}$, *there exists* $\delta > 0$ *such that, for all* $\varphi \in \hat{E}$ *with* $\|\varphi - \varphi_e\| < \delta$,

$$\left\|u_\varphi(t) - x_e\right\| \leq \left\|(u_\varphi)_t - \varphi_e\right\| \leq e^{-\gamma_1 t} \|\varphi - \varphi_e\| \quad \textit{for all } t \geq 0,$$

where $u_\varphi : \mathbb{R}^+ \to X$ *is the mild solution to* (*PFDE*) *as in Theorem* 2.1.

Remarks 2.5. 1. The main improvement of Theorem 2.4 over existing results is the general fully nonlinear and local context of assumptions **(A1)**–**(A4)**, as opposed to having B m$-\omega$-accretive, and F globally defined, in tandem with the facts (a) that B need not be linear, but just linearly resolvent-differentiable at x_e, and (b) that F need not be Fréchet-differentiable, but only $D(A)$-differentiable at φ_e as in assumption **(L)**, (3). In this regard, notice that, if $B \subset X \times X$ is m$-\omega$-accretive, and F is globally defined and Lipschitz, then **(A1)**–**(A4)** and condition (STC) are fulfilled automatically for $\hat{X} = X$, and $\hat{E} = \hat{E}_0$.

2. In particular, we note that in the global case $\hat{X} = X$, $\hat{E} = \hat{E}_0$, and $F : E \to X$ globally Lipschitz, versions of Theorem 2.4 have been proved (a) in the finite-delay case for $B : D(B) \subset X \to X$ linear and single-valued, and generating a C_0-semigroup, under the assumption that F be globally Fréchet-differentiable with $F' : E \to B(E, X)$ being locally Lipschitz ([15, Thm. 6.1], [19, 20]), and (b) in the infinite-delay case for $B : D(B) \subset X \to X$ (single-valued and) a Hille-Yosida operator, and F Fréchet-differentiable at the equilibrium ([1]). (Note that Hille-Yosida operators are – single-valued, linear and – m$-\omega$-accretive for an equivalent norm on X.)

3. Under local range flow-invariance conditions for $\hat{X} \subset X$, and $\hat{E} \subset E$, more restrictive than condition (STC) (compare [25, Lemma 2.7]), versions of Theorem 2.4 have been proved in [27, Thm. 2.4] for $B : D(B) \subset X \to X$ linear, and the infinitesimal generator of a C_0-semigroup of bounded linear operators on X, and in [24, Thm. 4.1] for linearly resolvent-differentiable $B \subset X \times X$ ω-accretive. In both of these instances, the corresponding proofs were based on the range condition $R(I + \lambda A) \supset \hat{E}$ for $\lambda > 0$ small enough, so that, in constructing the solution semigroup in E, we were able to work with the resolvents of A. This marks a substantial difference to the proof for the more general situation of Theorem 2.4, where, instead, we are forced to consider general ϵ-discrete-scheme approximations (see Section 3).

Finally, we note that the special version of Theorem 2.4 for $B = \tilde{B}$ has been given in [25, Thm. 4.2], by means of a much more direct method of proof than the one for the nonlinear case in Section 3 below.

4. As Theorem 2.4 touches upon exponential asymptotic stability of the solution semigroup to $(\mathrm{PFDE})_{\mathrm{lin}}$, we note that there are instances where this can be read directly from the relevant parameters of $\tilde{B}$ and $\tilde{F}$, cf. [25, Prop. 4.4].

Examples 1. Given a bounded open domain $\Omega \subset \mathbb{R}^N$, $1 \leq N \leq 3$, of class C^2, we let $X := C_0(\Omega) := \{u \in C(\bar{\Omega}) \mid u_{|\partial\Omega} = 0\}$ with sup-norm, and consider the Dirichlet-Laplacian in X,

$$\begin{cases} D(\Delta_0) = \{u \in C_0(\Omega) \mid \Delta u \in C_0(\Omega)\} \\ \Delta_0 u := \Delta u \in \mathcal{D}(\Omega)', \end{cases} \tag{2.3}$$

and note that, given any $d > 0$, $(-d\Delta_0)$ is m$-\omega_d$-accretive for some $\omega_d < 0$ in X (cf. [2, Section 6.1]).

2. Given real numbers $\alpha \leq 0, \beta \geq 0$, $[\alpha, \beta]$ denotes the order interval of all $u \in X$ such that $\alpha \leq u(\omega) \leq \beta$ for all $\omega \in \bar{\Omega}$.

For a continuous and monotonically non-decreasing function $\beta : \mathbb{R} \to \mathbb{R}$, with $\beta(0) = 0$, we denote by $\tilde{\beta} : X \to X$ its realization in $X = C_0(\Omega)$, given by $(\tilde{\beta}u)(\omega) = \beta(u(\omega))$, $\omega \in \bar{\Omega}$, and consider the following nonlinearly perturbed model of a diffusive population with delay in the birth process:

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) + (-d\Delta_0 + \tilde{\beta})u(x,t) \\ \qquad\qquad = au(x,t)\left[1 - bu(x,t) - \int_{-1}^{0} u(x, t + r(s))d\eta(s)\right] \\ u_{|[-R,0]} = \varphi \in C([-R,0]; C_0(\Omega)) \end{cases} \tag{2.4}$$

where $a, b, d > 0, \eta$ is a positive bounded regular Borel measure on $[-1, 0]$ with $(\|\eta\| > 0$ and$)$ $b + \|\eta\| = 1$, and $r : [-1, 0] \to [-R, 0]$ is a continuous delay function. (For the special case of $\beta \equiv 0$, this equation serves as a model for the density of red blood cells in an animal, cf. [10, 19, 32].)

Proposition 2.6.

(a) *For all $\varphi \in E := C([-R,0]; C_0(\Omega))$ with $\varphi(s) \geq 0$ for all $s \in [-R, 0]$, there exists a unique global mild solution u_φ to* (2.4) *with*

$$u_\varphi(t) \in [0, \max\{1/b, \|\varphi(0)\|\}]$$

for all $t \geq 0$.

(b) *If $\beta : \mathbb{R} \to \mathbb{R}$ is differentiable at $0 \in \mathbb{R}$, with $\beta'(0) = \rho$, and $a < -\omega_d + \rho$, then the zero equilibrium of* (2.4) *is locally exponentially stable.*

(Note that, by the definition of the function β, $\rho \geq 0$.)

Remarks 2.7. 1. The assertions of Proposition 2.6 actually hold for $-d\Delta_0$ being replaced by any m$-\omega$-accretive linear operator $D \subset C_0(\Omega) \times C_0(\Omega)$, and the operator $\tilde{\beta}$ being replaced by any (single-valued) m-accretive operator $C : D(C) \subset X \to X$ with $D(C) \supset D(D)$, $C(0) = 0$, and such that $B := D + C$ is m$-\alpha$-accretive, some $\alpha \in \mathbb{R}$, and such that

(i) the resolvents of $B = D + C$ are order-preserving, and there exists

(ii) a $D(C)$-Fréchet derivative $\tilde{C} \in B(X)$ of C at $0 \in X$: for every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in D(D)$, $\|x\| < \delta \Rightarrow \left\|Cx - \tilde{C}x\right\| < \epsilon \|x\|$,

(and, clearly, for proposition (b), such that $\tilde{B} = D + \tilde{C}$ is $\tilde{\omega}$-accretive with $\tilde{\omega} < 0$, and such that $a < -\tilde{\omega}$).

2. With the operators $B = (-d\Delta_0 + \tilde{\beta})$, or, more generally, of the form $B = D + C$ as specified just above, assertions analogous to those of Proposition 2.6 hold as well for the following variant of model (2.4)

$$\begin{cases} \dot{u}(t) + Bu(t) \ni u(t)\Big[1 + au(t) - b(u(t))^2 \\ \qquad\qquad -(1 + a - b)\int_{-R}^{0} f(s)u(t+s)ds\Big], \quad t \geq 0 \\ u_{|[-R,0]} = \varphi, \end{cases} \tag{2.5}$$

as well as a variant of (2.4) with temporal averages over the past being replaced by spatio-temporal averages, such as

$$\begin{cases} \dot{u}(t) + Bu(t) \ni au(t)\Big[1 - bu(t) \\ \qquad\qquad - \int_{-R}^{t}\int_{\Omega} g(\cdot - y, t - s)u(s)(y)dyds\Big], \quad t \geq 0 \\ u_{|(-R,0]} = \varphi \end{cases} \tag{2.6}$$

For a discussion of the models (2.4)–(2.6) for the state space $C(\bar{\Omega})$, and for $B = -\Delta$, with either Dirichlet or Neumann boundary conditions, a (partial) list of references is [8, 10, 17, 19, 20, 32]. In the $L^1(\Omega)$-context (with Ω not necessarily bounded), and the operator B being any m-completely accretive operator in $L^1(\Omega)$, existence and flow-invariance of solutions to models (2.4)–(2.6) have been treated in [25, Section 5].

Proof of Proposition 2.6 *and Remark* 2.7. As for proposition (a), note first that the operator $B := (-d\Delta_0 + \tilde{\beta})$ is m-accretive in X (as it is an additive perturbation of an m-accretive operator by a single-valued continuous and s-accretive operator, [3, Prop. 2.23, Thm. 16.4]). Moreover, the resolvents J_λ^B are order-preserving, so that $J_\lambda^B[0,\alpha] \subset [0,\alpha]$ for any $\alpha > 0$. The proof of (a) now follows from [25, Thm. 2.5] along the lines of the corresponding proof of [25, Prop. 5.1 (a)] (with appropriate changes due to the change from the L^1- to the sup-norm).

As for proposition (b), we note (without proof here) the following general fact:

Assume that, given a Banach space X,

(i) $D \subset X \times X$ *is* m$-\omega$*-accretive linear,*

(ii) $C : D(C) \subset X \to X$, *with* $D(C) \supset D(D)$ *is such that* $B := D + C$ *is* α*-accretive, some* $\alpha \in \mathbb{R}$, *with* $R(I + \lambda B) \supset D(B) = D(D)$, *for* $\lambda > 0$ *small enough,*

(iii) $x_0 \in D(B)$, *and there exists*

(iv) $\tilde{C} \in B(X)$ *such that for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $x \in D(D)$, $\|x - x_0\| < \delta$ *implies* $\left\|Cx - Cx_0 - \tilde{C}(x - x_0)\right\| < \epsilon \|x - x_0\|$, *then* $\tilde{B} := D + \tilde{C}$ *is a resolvent-differential of* B *at* x_0 (*in the sense of Definition* 2.2).

In particular, the operator $(-d\Delta_0 + \rho I)$ is a resolvent-differential of the operator $B := (-d\Delta_0 + \tilde{\beta})$ at $0 \in C_0(\Omega)$.

Next, note that, (for any $\alpha > 0$, and) for $F : \hat{E}_{0,\alpha} = \{\varphi \in C([-R,0]; C_0(\Omega)) \mid \varphi(0) \in [0,\alpha]\} \to C_0(\Omega)$, $F\varphi = a\varphi(0)[1 - b\varphi(0) - \int_{-1}^{0} \varphi(r(s))d\eta(s)]$, $\tilde{F}\varphi = a\varphi(0)$ is a $D(A)$-Fréchet-derivative of F at $\varphi_e \equiv 0$ (with A the solution operator of (2.4) as in **2.C** above).

Proposition (b) follows by combining the foregoing two facts with Theorem 2.4, the assumption $a < -\omega_d + \rho$, and the fact that, for finite delay, the solution semigroup to (PFDE) is exponentially stable if, in the setting of **2.B** above, $(\omega + M_F) < 0$ ([25, Prop. 4.4]). □

3. Proof of Theorem 2.4

Step 1: Given $0 < \gamma_1 < \tilde{\gamma}$ as in Theorem 2.4, let $\epsilon_1 = \tilde{\gamma} - \gamma_1$. According to assumptions **(L)** (2) and (3), given $\epsilon = \frac{\epsilon_1}{4}$, there exist $\delta > 0$, and $\lambda_0 > 0$, $\lambda_0\omega < 1$, such that

$$\varphi \in \hat{E},\ \|\varphi - \varphi_e\| < 4\delta \Longrightarrow \left\| F\varphi - F\varphi_e - \tilde{F}(\varphi - \varphi_e) \right\| < \epsilon \|\varphi - \varphi_e\|, \tag{3.1}$$

and

$$\begin{gathered} 0 < \lambda < \lambda_0,\ x \in R(I + \lambda B),\ \|x - x_e\| < 4\delta \Longrightarrow \\ \left\| J_\lambda^B x - J_\lambda^B x_e - J_\lambda^{\tilde{B}}(x - x_e) \right\| \leq \epsilon\lambda \|x - x_e\| + \lambda\eta(\lambda; x). \end{gathered} \tag{3.2}$$

(For (3.1) to hold for $\varphi \in \hat{E}$, we have used assumption **(L)** (3), in conjunction with the fact that, by [25, proof of Thm. 2.1, *Step* 3], *cl* $D(A) = \hat{E}_0$.)

Let $\varphi \in \hat{E}$ such that $\|\varphi - \varphi_e\| < \delta$.

Choose $T_0 > 0$ such that $e^{\gamma T_0} < 2$, and let $M_\varphi := \sup\{\|y\| \mid y \in F(S[0, T_0 + 1]\varphi)\}$ (with $(S(t))_{t \geq 0}$ the solution semigroup associated to A as in **(S2)** above).

Let $\eta_0 := \min\{\lambda_0, (3(\left\|\tilde{F}\right\| + |\tilde{\omega}|\ ) + 1)^{-1}, \delta(\max\{(3 + M_\varphi), (\|F\varphi_e\| + 1)\})^{-1}\}$, and choose a strictly decreasing nullsequence $(\eta_n)_n$ of positive reals $\eta_n < \eta_0$.

As, according to **(S2)**, there exists a unique global mild solution to (CP)$(A; \varphi; 0)$, there exist for $T := T_0 + 1$, and all $n \in \mathbb{N}$, η_n-discrete-scheme approximations $D_A(0 = t_0^n, t_1^n, \ldots, t_{\tilde{N}_n}^n; \varphi = \varphi_0^n, \varphi_1^n, \ldots, \varphi_{\tilde{N}_n}^n; \psi_1^n, \ldots, \psi_{\tilde{N}_n}^n)$ to (CP)$(A; \varphi; 0)$, consisting of an η_n-partition of the interval $[0, T]$,

$$0 = t_0^n < t_1^n < \cdots < t_{\tilde{N}_n}^n \leq T,\ t_i^n - t_{i-1}^n < \eta_n, i \in \{1, \ldots, \tilde{N}_n\}, T - t_{\tilde{N}_n}^n < \eta_n, \tag{3.3}$$

and elements $\{\varphi_0^n = \varphi, \varphi_1^n, \ldots, \varphi_{\tilde{N}_n}^n\}$, and $\{\psi_1^n, \ldots, \psi_{\tilde{N}_n}^n\}$ in E such that

$$\begin{aligned}
&\varphi_i^n \in D(A), i \in \{1, \ldots, \tilde{N}_n\}, \\
&\frac{\varphi_i^n - \varphi_{i-1}^n}{t_i^n - t_{i-1}^n} + A\varphi_i^n = \psi_i^n,\ i \in \{1, \ldots, \tilde{N}_n\}, \\
&\sum_1^{\tilde{N}_n} (t_i^n - t_{i-1}^n)\, \|\psi_i^n\| < \eta_n, \quad \text{and}
\end{aligned} \tag{3.4}$$

such that, if the step function $\Phi_n : [0, t_{\tilde{N}_n}^n] \to E$ is defined by

$$\Phi_n(t) = \begin{cases} \varphi & t = 0, \\ \varphi_i^n & t \in (t_{i-1}^n, t_i^n],\ i \in \{1, \ldots, \tilde{N}_n\}, \end{cases} \qquad \text{then}$$

$$\|\Phi_n(t) - S(t)\varphi\| < \eta_n \qquad \text{uniformly over} \quad t \in [0, t_{\tilde{N}_n}^n]. \tag{3.5}$$

Furthermore, with this choice, we have

$$\lim_{n\to\infty} \|F(\Phi_n(t)) - F(S(t)\varphi)\| = 0 \qquad \text{uniformly over} \quad t \in \left[0, T_0 + \frac{1}{2}\right]. \tag{3.6}$$

(For all this, compare *Step* 3 of the proof of [25, Thm. 2.1].)

Step 2: Given $n \in \mathbb{N}$, let $N_n \in \mathbb{N}$, $N_n \le \tilde{N}_n$, such that $t_{N_n-1}^n \le T_0 \le t_{N_n}^n$, and let $h_i^n := t_i^n - t_{i-1}^n$, $i \in \{1, \ldots, N_n\}$, and choose $n_0 \in \mathbb{N}$ such that $e^{\gamma(T_0 + \eta_{n_0})} < 2$, and, according to (3.6),

$$\|F(\Phi_n(t_i^n)) - F(S(t_i^n)\varphi)\| < 1 \quad \text{for all } n \ge n_0, \text{ and all } i \in \{1, \ldots, N_n\}. \tag{3.7}$$

From now on, we restrict ourselves to $n \ge n_0$, and, for the time being, suppress the upper index n for the above discrete-scheme approximations.

Our next goal is an estimate on $\|\varphi_i - \varphi_e\|$, $i \in \{1, \ldots, N_n\}$:
With, for $\lambda > 0$, $J_\lambda = (I + \lambda A)^{-1}$, and $\tilde{J}_\lambda = (I + \lambda\tilde{A})^{-1}$ denoting the resolvents of A and, respectively, of $\tilde{A}$, we first note that, from (3.4), $\varphi_i = J_{h_i}(\varphi_{i-1} + h_i\psi_i)$, $i \in \{1, \ldots, N_n\}$. Thus,

$$\begin{aligned}
\varphi_i - \varphi_e = J_{h_i}(\varphi_{i-1} + h_i\psi_i) - \varphi_e - \tilde{J}_{h_i}(\varphi_{i-1} + h_i\psi_i - \varphi_e) \\
+ \tilde{J}_{h_i}(\varphi_{i-1} + h_i\psi_i - \varphi_e),\ i \in \{1, \ldots, N_n\}.
\end{aligned} \tag{3.8}$$

(Here, we use that, according to [26, Thm. 2.1] (with $\hat{X} = X$ and $\hat{E} = E$), $R(I + \lambda\tilde{A}) = E$ for all $\lambda > 0$ small enough.)

As for the first term on the right of (3.8), given $\lambda > 0$ small enough, and $\psi \in R(I + \lambda A)$, from the definitions of A and $\tilde{A}$, we have

$$\begin{aligned}\left\| J_\lambda \psi - \varphi_e - \tilde{J}_\lambda(\psi - \varphi_e) \right\| &= \left\| (J_\lambda \psi)(0) - x_e - (\tilde{J}_\lambda(\psi - \varphi_e))(0) \right\| \\ (J_\lambda \psi)(0) &= J_\lambda^B(\psi(0) + \lambda F(J_\lambda \psi)) \\ (\tilde{J}_\lambda(\psi - \varphi_e))(0) &= J_\lambda^{\tilde{B}}((\psi - \varphi_e)(0) + \lambda \tilde{F}(\tilde{J}_\lambda(\psi - \varphi_e)) \\ x_e &= J_\lambda^B(x_e + \lambda F \varphi_e).\end{aligned}$$

Thus,

$$\begin{aligned}&\left\| J_\lambda \psi - \varphi_e - \tilde{J}_\lambda(\psi - \varphi_e) \right\| = \left\| (J_\lambda \psi)(0) - x_e - (\tilde{J}_\lambda(\psi - \varphi_e))(0) \right\| \\ &\quad \leq \left\| J_\lambda^B(\psi(0) + \lambda F(J_\lambda \psi)) - J_\lambda^B x_e - J_\lambda^{\tilde{B}}(\psi(0) - x_e + \lambda F(J_\lambda \psi)) \right\| \\ &\qquad + \left\| J_\lambda^B(x_e + \lambda F \varphi_e) - J_\lambda^B x_e - J_\lambda^{\tilde{B}}(\lambda F \varphi_e) \right\| \\ &\qquad + \lambda \left\| J_\lambda^{\tilde{B}}(F(J_\lambda \psi) - F\varphi_e - \tilde{F}(J_\lambda \psi - \varphi_e)) \right\| \\ &\qquad + \lambda \left\| J_\lambda^{\tilde{B}} \tilde{F}(J_\lambda \psi - \varphi_e - \tilde{J}_\lambda(\psi - \varphi_e)) \right\|,\end{aligned}$$

so that

$$\left(1 - \lambda \frac{\|\tilde{F}\|}{1 - \lambda\tilde{\omega}}\right) \left\| J_\lambda \psi - \varphi_e - \tilde{J}_\lambda(\psi - \varphi_e) \right\| \tag{3.9}$$

$$\leq \left\| J_\lambda^B(\psi(0) + \lambda F(J_\lambda \psi)) - J_\lambda^B x_e - J_\lambda^{\tilde{B}}(\psi(0) - x_e + \lambda F(J_\lambda \psi)) \right\| \tag{3.10}$$

$$+ \left\| J_\lambda^B(x_e + \lambda F \varphi_e) - J_\lambda^B x_e - J_\lambda^{\tilde{B}}(\lambda F \varphi_e) \right\| \tag{3.11}$$

$$+ \frac{\lambda}{1 - \lambda\tilde{\omega}} \left\| F(J_\lambda \psi) - F\varphi_e - \tilde{F}(J_\lambda \psi - \varphi_e) \right\| \tag{3.12}$$

Specializing to $\lambda = h_i$, and $\psi = \varphi_{i-1} + h_i \psi_i$ (recall: $h_i = h_i^n$, $\varphi_i = \varphi_i^n$, and $\psi_i = \psi_i^n$, and $n \geq n_0$), we now estimate the terms on the right of (3.9) by means of the differentiability assumptions (3.1) and (3.2):

The term (3.10):

$$\begin{aligned}&\left\| \varphi_{i-1}(0) - x_e + h_i \psi_i(0) + h_i F(\varphi_i) \right\| \\ &\quad \leq \left\| \varphi_{i-1} - \varphi_e \right\| + h_i(\|\psi_i\| + \|F(\varphi_i)\|) \\ &\quad \leq \|\Phi_n(t_{i-1}) - S(t_{i-1})\varphi\| + \|S(t_{i-1})\varphi - S(t_{i-1})\varphi_e\| + h_i \|\psi_i\| \\ &\qquad + h_i(\|F(\Phi_n(t_i)) - F(S(t_i)\varphi)\| + \|F(S(t_i)\varphi)\|) \\ &\quad \leq \eta_n + e^{\gamma T_0} \|\varphi - \varphi_e\| + \eta_n + h_i(1 + M_\varphi) < 3\delta\end{aligned}$$

(according to (3.4)–(3.7), and the choice of $\eta_0 \geq \eta_n$). Thus, from (3.2),

$$\begin{aligned}&\left\|J^B_{h_i}(\varphi_{i-1}(0)+h_iF\varphi_i)-J^B_{h_i}x_e-J^{\tilde{B}}_{h_i}(\varphi_{i-1}(0)-x_e+h_i\psi_i(0)+h_iF\varphi_i)\right\| \quad (3.13)\\ &\leq \epsilon h_i(\left\|\varphi_{i-1}-\varphi_e\right\|+h_i\left\|\psi_i\right\|+h_i\left\|F\varphi_i\right\|)+h_i\eta(h_i;\varphi_{i-1}(0)+h_i\psi_i(0)+h_iF\varphi_i)\end{aligned}$$

The term (3.11): As $h_i\left\|F\varphi_e\right\| < \eta_0\left\|F\varphi_e\right\| < \delta$, from (3.2),

$$\left\|J^B_{h_i}(x_e+h_iF\varphi_e)-J^B_{h_i}x_e-J^{\tilde{B}}_{h_i}(h_iF\varphi_e)\right\| \tag{3.14}$$

$$\leq \quad \epsilon h_i^2\left\|F\varphi_e\right\|+h_i\eta(h_i;x_e+h_iF\varphi_e). \tag{3.15}$$

The term (3.12): As $\left\|\varphi_i-\varphi_e\right\| \leq \left\|\Phi_n(t_i)-S(t_i)\varphi\right\|+\left\|S(t_i)\varphi-S(t_i)\varphi_e\right\| < \eta_n + e^{\gamma(T_0+\eta_n)}\left\|\varphi-\varphi_e\right\| < 3\delta$, (3.1) reveals that

$$\left\|F(\varphi_i)-F\varphi_e-\tilde{F}(\varphi_i-\varphi_e)\right\| \leq \epsilon\left\|\varphi_i-\varphi_e\right\|. \tag{3.16}$$

Noting that, according to the choice of η_0, $0 < (1-h_i\tilde{\omega})(1-h_i(\tilde{\omega}+\left\|\tilde{F}\right\|))^{-1} \leq 2$, and $(1-h_i(\tilde{\omega}+\left\|\tilde{F}\right\|))^{-1} \leq \frac{3}{2}$, and using the estimates (3.9)–(3.16), we conclude from (3.8) that, for all $n \geq n_0$, and all $i \in \{1,\ldots,N_n\}$,

$$\begin{aligned}\left\|\varphi_i-\varphi_e\right\| \leq\ &\left\|\varphi_{i-1}-\varphi_e\right\|\left(\frac{1}{1+h_i\tilde{\gamma}}+2\epsilon h_i\right)+h_i\left\|\psi_i\right\|\Bigg(\left(\frac{1}{1+h_i\tilde{\gamma}}+2\epsilon h_i\right)\\ &+2\epsilon h_i\left\|\varphi_i-\varphi_e\right\|+2\epsilon h_i\eta_n(\left\|F\varphi_i\right\|+\left\|F\varphi_e\right\|) \qquad (3.17)\\ &+2h_i(\eta(h_i;\varphi_{i-1}(0)+h_i\psi_i(0)+h_iF\varphi_i)+\eta(h_i;x_e+h_iF\varphi_e)\Bigg).\end{aligned}$$

With regard to this inequality, we note the following estimates:
First, as $F\varphi_i = F(\Phi_n(t_i))$, we have

$$\left\|F\varphi_i\right\| \leq \left\|F(\Phi_n(t_i))-F(S(t_i)\varphi)\right\|+\left\|F(S(t_i)\varphi)\right\| < 1+M_\varphi \tag{3.18}$$

for all $n \geq n_0$, $i \in \{1,\ldots,N_n\}$.

$$\begin{aligned}&\left(\frac{1}{1+h_i\tilde{\gamma}}+2\epsilon h_i\right)\left\|\varphi_{i-1}-\varphi_e\right\|+2\epsilon h_i\left\|\varphi_i-\varphi_e\right\| = \left\|\varphi_{i-1}-\varphi_e\right\| \qquad (3.19)\\ &+h_i\left(2\epsilon-\frac{\tilde{\gamma}}{1+h_i\tilde{\gamma}}\right)\left(\left\|\varphi_{i-1}-\varphi_e\right\|-\left\|\varphi_i-\varphi_e\right\|\right)+h_i\left(4\epsilon-\frac{\tilde{\gamma}}{1+h_i\tilde{\gamma}}\right)\left\|\varphi_i-\varphi_e\right\|\Big)\\ &\leq \left\|\varphi_{i-1}-\varphi_e\right\|+Ch_i\left\|\varphi_i-\varphi_{i-1}\right\|+h_i\left(\epsilon_1-\frac{\tilde{\gamma}}{1+\eta_n\tilde{\gamma}}\right)\left\|\varphi_i-\varphi_e\right\|\end{aligned}$$

for a positive constant C (independent of $n \geq n_0$, and $i \in \{1,\ldots,N_n\}$).

$$\begin{aligned}\left\|\varphi_i-\varphi_{i-1}\right\| &= \left\|\Phi_n(t_i)-\Phi_n(t_{i-1})\right\| \qquad (3.20)\\ &\leq \left\|\Phi_n(t_i)-S(t_i)\varphi\right\|+\left\|S(t_i)\varphi-S(t_{i-1})\varphi\right\|+\left\|S(t_{i-1})\varphi-\Phi_n(t_{i-1})\right\|\\ &\leq 2\eta_n+\rho_\varphi(\eta_n),\end{aligned}$$

where, for $\eta > 0$, $\rho_\varphi(\eta) := \sup\{\|S(t)\varphi - S(s)\varphi\| \mid s,t \in [0,T_0+1], |t-s| \le \eta\}$. (Note that, as $S(\cdot)\varphi$ is uniformly continuous on $[0,T_0+1]$, $\rho_\varphi(\eta_n) \to 0$ as $n \to \infty$.) Invoking these estimates in (3.17), leads to

$$\begin{aligned}\|\varphi_i - \varphi_e\| - \|\varphi_{i-1} - \varphi_e\| \qquad\qquad (3.21)\\ \le h_i\left(\epsilon_1 - \frac{\tilde\gamma}{1+\eta_n\tilde\gamma}\right)\|\varphi_i - \varphi_e\| + C_1 h_i\|\psi_i\| + C_2 h_i(\eta_n + \rho_\varphi(\eta_n))\\ + 2h_i(\eta(h_i;\varphi_{i-1}(0) + h_i\psi_i(0) + h_iF\varphi_i) + \eta(h_i; x_e + h_iF\varphi_e))\end{aligned}$$

for all $n \ge n_0$, $i \in \{1,\dots,N_n\}$, and suitable positive constants C_1 and C_2.

Step 3: Given $0 < s < t < T_0$, choose $n_1 \ge n_0$ large enough so that $\eta_{n_1} < \min\{s, \frac{1}{2}(t-s), T_0 - t\}$. Then, reinvoking the upper indices, for every $n \ge n_1$ there exist indices $2 \le l \le k \le N_n$ such that $s \in (t^n_{l-1}, t^n_l]$, and $t \in (t^n_{k-1}, t^n_k]$. Summing (3.21) from $i = (l+1)$ to $i = k$ (and taking into account the properties (3.4)–(3.6) of the discrete-scheme approximations) leads to

$$\begin{aligned}\|\Phi_n(t) - \varphi_e\| - \|\Phi_n(s) - \varphi_e\| = \|\varphi^n_k - \varphi_e\| - \|\varphi^n_l - \varphi_e\| \qquad (3.22)\\ \le \left(\epsilon_1 - \frac{\tilde\gamma}{1+\eta_n\tilde\gamma}\right)\sum_{l+1}^{k} h^n_i\|\varphi^n_i - \varphi_e\| + C_1\eta_n + C_2(T_0+1)(\eta_n + \rho_\varphi(\eta_n))\\ + 2\sum_{l+1}^{k} h^n_i(\eta(h^n_i;\varphi^n_{i-1}(0) + h^n_i\psi^n_i(0) + h^n_iF\varphi^n_i) + \eta(h^n_i; x_e + h^n_iF\varphi_e))\end{aligned}$$

At this point, first note that $h^n_i\|\varphi^n_i - \varphi_e\| = \int_{t^n_{i-1}}^{t^n_i}\|\Phi_n(\tau) - \varphi_e\|\,d\tau$, so that, by the choice of ϵ_1,

$$\begin{aligned}\left(\epsilon_1 - \frac{\tilde\gamma}{1+\eta_n\tilde\gamma}\right)\sum_{l+1}^{k} h^n_i\|\varphi^n_i - \varphi_e\| \qquad (3.23)\\ = \left(\epsilon_1 - \frac{\tilde\gamma}{1+\eta_n\tilde\gamma}\right)\int_{t^n_l}^{t^n_k}\|\Phi_n(\tau) - \varphi_e\|\,d\tau \longrightarrow -\gamma_1\int_s^t\|S(\tau)\varphi - \varphi_e\|\,d\tau\end{aligned}$$

as $n \to \infty$.

Next, with regard to the last term on the right of (3.22), we will show that

$$\lim_{n\to\infty}(\eta(h^n_i;\varphi^n_{i-1}(0) + h^n_i\psi^n_i(0) + h^n_iF\varphi^n_i) + \eta(h^n_i; x_e + h^n_iF\varphi_e)) = 0 \qquad (3.24)$$

uniformly over $i \in \{1,\dots,N_n\}$.

As $\sum_{l+1}^k h^n_i \le (T_0+1)$, this will show that the last term on the right of (3.22) tends to zero as $n \to \infty$.

Proof of (3.24): The result for the second term follows directly from the assumptions on the function η, and the fact that $0 \le h^n_i \le \eta_n \to 0$, and $\|x_e + h^n_iF\varphi_e - x_e\| \le \eta_n\|F\varphi_e\| \to 0$ as $n \to \infty$ uniformly over $i \in \{1,\dots,N_n\}$.

As for the first term, assume that $(\eta(h^n_i;\varphi^n_{i-1}(0) + h^n_i\psi^n_i(0) + h^n_iF\varphi^n_i))_n$ does not converge to zero uniformly over $i \in \{1,\dots,N_n\}$. Then there exist $\beta > 0$, and sequences $(n_k)_k \subset \mathbb{N}$, $n_k \to \infty$, and $(i_k)_k$, $i_k \in \{1, \ldots, N_{n_k}\}$, such that, with $\lambda_k := h_{i_k}^{n_k}$, and $z_k := \varphi_{i_k-1}^{n_k}(0) + h_{i_k}^{n_k}\psi_{i_k-1}^{n_k}(0) + h_{i_k}^{n_k}F\varphi_{i_k-1}^{n_k}$,

$$\eta(\lambda_k; z_k) \geq \beta \qquad \text{for all } k \in \mathbb{N}. \tag{3.25}$$

We now notice that, for all $n \in \mathbb{N}$, and all $i \in \{1, \ldots, N_n\}$, and with u_φ denoting the mild solution to (PFDE) according to **(S3)** above,

$$\begin{aligned} &\left\|\varphi_{i-1}^n(0) + h_i^n\psi_i^n(0) + h_i^n F\varphi_i^n - u_\varphi(t_{i-1}^n)\right\| \\ \leq \quad &\left\|\Phi_n(t_{i-1}^n)(0) - (S(t_{i-1}^n)\varphi)(0)\right\| + h_i^n \left\|\psi_i^n\right\| + h_i^n \left\|F\varphi_i^n\right\| \leq \eta_n(2 + M_\varphi). \end{aligned}$$

Thus, $\{\varphi_{i-1}^n(0)+h_i^n\psi_i^n(0)+h_i^nF\varphi_i^n \mid i \in \{1, \ldots, N_n\}\} \subset u_\varphi[0, T_0]+\eta_n(2+M_\varphi)B_X$ (with B_X denoting the closed unit ball of X).

In particular, $z_k = x_k + y_k$, $k \in \mathbb{N}$, for sequences $(x_k)_k \subset u_\varphi[0, T_0]$, and $(y_k)_k \subset \eta_{n_k}(2 + M_\varphi)B_X$. As $u_\varphi[0, T_0]$ is compact, there exist $x_0 \in X$, and a subsequence $(x_{k_l})_l$ of $(x_k)_k$ such that $x_{k_l} \to x_0$. Altogether, we have $\lambda_{k_l} \to 0$, and $z_{k_l} \to x_0$, so that, by the assumptions on the function η, $\eta(\lambda_{k_l}; z_{k_l}) \to 0$. This contradiction to (3.25) completes the proof of (3.24).

With (3.23) and (3.24) in place, we conclude from (3.22), by letting $n \to \infty$,

$$\|S(t)\varphi - \varphi_e\| - \|S(s)\varphi - \varphi_e\| \leq -\gamma_1 \int_s^t \|S(\tau)\varphi - \varphi_e\| \, d\tau \text{ for all } 0 < s \leq t < T_0.$$

By continuity in s, t on either side, this holds for all $0 \leq s \leq t \leq T_0$. Altogether, we thus conclude that

$$\|S(t)\varphi - \varphi_e\| \leq e^{-\gamma_1 t} \|\varphi - \varphi_e\| \text{ for all } 0 \leq t \leq T_0,\ \varphi \in \hat{E}, \|\varphi - \varphi_e\| < \delta. \tag{3.26}$$

At this point, let $\varphi \in \hat{E}, \|\varphi - \varphi_e\| < \delta$, and let $T_1 = \sup\{t > 0 \mid \|S(s)\varphi - \varphi_e\| \leq e^{-\gamma_1 s}\|\varphi - \varphi_e\| \text{ for all } 0 \leq s \leq t\}$. From (3.26), we know that $T_1 \geq T_0$. Assuming that $T_1 < \infty$, there exists $T_1 < t_1 \leq T_1 + T_0$ such that $\|S(t_1)\varphi - \varphi_e\| > e^{-\gamma_1 t_1}\|\varphi - \varphi_e\|$. However, $\|S(T_1)\varphi - \varphi_e\| \leq e^{-\gamma_1 T_1}\|\varphi - \varphi_e\| < \delta$, so that, as $0 \leq t_1 - T_1 \leq T_0$, according to (3.26), $\|S(t_1)\varphi - \varphi_e\| = \|S(t_1 - T_1)S(T_1)\varphi - \varphi_e\| \leq e^{-\gamma_1(t_1 - T_1)}\|S(T_1)\varphi - \varphi_e\| \leq e^{-\gamma_1 t_1}\|\varphi - \varphi_e\|$.
This contradiction serves to complete the proof of Theorem 2.4. □

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Stochastic Equations with Boundary Noise

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Abstract. We study the wellposedness and pathwise regularity of semilinear non-autonomous parabolic evolution equations with boundary and interior noise in an L^p setting. We obtain existence and uniqueness of mild and weak solutions. The boundary noise term is reformulated as a perturbation of a stochastic evolution equation with values in extrapolation spaces.

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Keywords. Parabolic stochastic evolution equation, multiplicative boundary noise, non-autonomous equations, mild solution, variational solution, extrapolation.

1. Introduction

In this paper we investigate the wellposedness and pathwise regularity of semilinear non-autonomous parabolic evolution equations with boundary noise. A model example which fits in the class of problems we study is given by

$$\begin{aligned} \frac{\partial u}{\partial t}(t,s) &= \mathcal{A}(t,s,D)u(t,s) && \text{on } (0,T]\times S, \\ \mathcal{B}(t,s,D)u(t,s) &= c(t,u(t,s))\frac{\partial w}{\partial t}(t,s) && \text{on } (0,T]\times \partial S, \\ u(0,s) &= u_0(s), && \text{on } S. \end{aligned} \tag{1.1}$$

Here $S\subset\mathbb{R}^d$ is a bounded domain with C^2 boundary, $\mathcal{A}(t,\cdot,D)=\operatorname{div}(a(t,\cdot)\nabla)$ for uniformly positive definite, symmetric matrices $a(t,s)$ with the conormal boundary operator $\mathcal{B}(t,s,D)$, $c(t,\xi)$ is Lipschitz in $\xi\in\mathbb{C}$, $(w(t))_{t\geq 0}$ is a Brownian motion for an filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and with values in $L^r(\partial S)$ for some $r\geq 2$, and u_0 is an $\mathcal{F}_0$-measurable initial value. Actually, we also allow for lower-order terms, interior noise, nonlocal nonlinearities, and more general stochastic terms, see Section 4.

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As a first step one has to give a precise meaning to the formal boundary condition in (1.1). We present two solution concepts for (1.1) in Section 4, namely a mild and a weak one, which are shown to be equivalent. Our analysis is then based on the mild version of (1.1), which fits into the general framework of [30] where parabolic non-autonomous evolution equations in Banach spaces were treated. The results in [30] rely on the stochastic integration theory in certain classes of Banach spaces (see [8, 22, 24]). In order to use [30], the inhomogeneous boundary term is reformulated as an additive perturbation of a stochastic evolution equation corresponding to homogeneous boundary conditions. This perturbation maps into a so-called extrapolation space for the realization $A(t)$ of $\mathcal{A}(t,\cdot,D)$ in $L^p(S)$ with the boundary condition $\mathcal{B}(t,\cdot,D)u = 0$ (where $p \in [2,r]$). Such an approach was developed for deterministic problems by Amann in, e.g., [5] and [6]. We partly use somewhat different techniques taken from [19], see also the references therein. For this reformulation, one further needs the solution map of a corresponding elliptic boundary value problem with boundary data in $L^r(\partial S)$ which is the range space of the Brownian motion. Here we heavily rely on the theory presented in [5], see also the references therein. We observe that in [5] a large class of elliptic systems was studied. Accordingly, we could in fact allow for systems in (1.1), but we decided to restrict ourselves to the scalar case in order to simplify the presentation.

We establish in Theorem 4.3 the existence and uniqueness of a mild solution u to (1.1). Such a solution is a process $u : [0,T] \times \Omega \to L^p(S)$ where (Ω, P) is the probability space for the Brownian motion. We further show that for a.e. fixed $\omega \in \Omega$ the path $t \mapsto u(t,\omega)$ is (Hölder) continuous with values in suitable interpolation spaces between $L^p(S)$ and the domain of $A(t)$, provided that u_0 belongs to a corresponding interpolation space a.s. As a consequence, the paths of u belong to $C([0,T], L^q(S))$ for all $q < dp/(d-1)$. At this point, we make use of the additional regularity provided by the L^p approach to stochastic evolution equations.

In [21] an autonomous version of (1.1) has been studied in a Hilbert space situation (i.e., $r = p = 2$) employing related techniques. However, in this paper only regularity in the mean and no pathwise regularity has been treated. In [13, §13.3], Da Prato and Zabczyk have also investigated boundary noise of Neumann type. They deal with a specific situation where $a(t) = I$, the domain is a cube and the noise acts on one face which allows more detailed results. See also [3], [12], [14] and [28] for further contributions to problems with boundary noise. As explained in Remark 4.9 we cannot treat Dirichlet type boundary conditions due to our methods. In one space dimension Dirichlet boundary noise has been considered in [4] in weighted L^p-spaces by completely different techniques, see also [12].

In the next section, we first recall the necessary material about parabolic deterministic evolution equations and about stochastic integration. Then we study an abstract stochastic evolution equation related to (1.1) in Section 3. Finally, in the last section we treat a more general version of (1.1) and discuss various examples concerning the stochastic terms.

2. Preliminaries

We write $a \lesssim_K b$ if there exists a constant c only depending on K such that $a \le cb$. The relation $a \eqsim_K b$ expresses that $a \lesssim_K b$ and $b \lesssim_K a$. If it is clear what is meant, we just write $a \lesssim b$ for convenience. Throughout, X denotes a Banach space, X^* its dual, and $\mathcal{B}(X, Y)$ the space of linear bounded operators from X into another Banach space Y. If the spaces are real, everything below should be understood for the complexification of the objects under consideration. The complex interpolation space for an interpolation couple (X_1, X_2) of order $\eta \in (0, 1)$ is designated by $[X_1, X_2]_\eta$. We refer to [29] for the relevant definitions and basic properties.

2.1. Parabolic evolution families

We briefly discuss the approach to non-autonomous parabolic evolution equations developed by Acquistapace and Terreni, [2]. For $w \in \mathbb{R}$ and $\phi \in [0, \pi]$, set $\Sigma(\phi, w) = \{\lambda \in \mathbb{C} : |\arg(z - w)| \le \phi\}$. A family $(A(t), D(A(t)))_{t\in[0,T]}$ satisfies the hypothesis (AT) if the following two conditions hold, where $T > 0$ is given.

(AT1) $A(t)$ are densely defined, closed linear operators on a Banach space X and there are constants $w \in \mathbb{R}$, $K \ge 0$, and $\phi \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(\phi, w) \subset \varrho(A(t))$ and

$$\|R(\lambda, A(t))\| \le \frac{K}{1 + |\lambda - w|}$$

holds for all $\lambda \in \Sigma(\phi, w)$ and $t \in [0, T]$.

(AT2) There are constants $L \ge 0$ and $\mu, \nu \in (0, 1]$ such that $\mu + \nu > 1$ and

$$\|A_w(t)R(\lambda, A_w(t))(A_w(t)^{-1} - A_w(s)^{-1})\| \le L|t - s|^\mu(|\lambda| + 1)^{-\nu}$$

holds for all $\lambda \in \Sigma(\phi, 0)$ and $s, t \in [0, T]$, where $A_w(t) = A(t) - w$.

Condition (A1) just means sectoriality with angle $\phi > \pi/2$ and uniform constants, whereas (A2) says that the resolvents satisfy a Hölder condition in stronger norms. In fact, Acquistapace and Terreni have studied a somewhat weaker version of (AT2) and allowed for non dense domains. Later on, we work on reflexive Banach spaces, where sectorial operators are automatically densely defined so that we have included the density assumption in (AT1) for simplicity. The conditions (AT) and several variants of them have intensively been studied in the literature, where also many examples can be found, see, e.g., [1, 2, 6, 26, 31]. If (AT1) holds and the domains $D(A(t))$ are constant in time, then the Hölder continuity of $A(\cdot)$ in $\mathcal{B}(D(A(0)), X)$ with exponent η implies (AT2) with $\mu = \eta$ and $\nu = 1$ (see [2, Section 7]).

Let $\eta \in (0, 1)$, $\theta \in [0, 1]$, and $t \in [0, T]$. Assume that (AT1) holds. The fractional power $(-A_w(t))^{-\theta} \in \mathcal{B}(X)$ is defined by

$$(-A_w(t))^{-\theta} = \frac{1}{2\pi i} \int_\Gamma (w - \lambda)^{-\theta} R(\lambda, A(t))\, d\lambda,$$

where the contour $\Gamma = \{\lambda : \arg(\lambda - w) = \pm\phi\}$ is orientated counter clockwise. The operator $(w - A(t))^{\theta}$ is defined as the inverse of $(w - A(t))^{-\theta}$. We will also use the complex interpolation space

$$X_\eta^t = [X, D(A(t))]_\eta \,.$$

Moreover, the *extrapolation space* $X_{-\theta}^t$ is the completion of X with respect to the norm $\|x\|_{X_{-\theta}^t} = \|(-A_w(t))^{-\theta}x\|$. Let $A_{-1}(t) : X \to X_{-1}^t$ be the unique continuous extension of $A(t)$ which is sectorial of the same type. Then $(w - A_{-1}(t))^{\alpha} : X_{-\theta}^t \to X_{-\theta-\alpha}^t$ is an isomorphism, where $0 \leq \theta \leq \alpha + \theta \leq 1$. If X is reflexive, then one can identify the dual space $(X_{-1}^t)^*$ with $D(A(t)^*)$ endowed with its graph norm and the adjoint operator $A_{-1}(t)^*$ with $A(t)^* \in \mathcal{B}(D(A(t)^*), X^*)$. We mostly write $A(t)$ instead of $A_{-1}(t)$. See, e.g., [6, 19] for more details.

Under condition (AT), we consider the non-autonomous Cauchy problem

$$\begin{aligned} u'(t) &= A(t)u(t), \qquad t \in [s, T], \\ u(s) &= x, \end{aligned} \tag{2.1}$$

for given $x \in X$ and $s \in [0, T)$. A function u is a *classical solution* of (2.1) if $u \in C([s, T]; X) \cap C^1((s, T]; X)$, $u(t) \in D(A(t))$ for all $t \in (s, T]$, $u(s) = x$, and $\frac{du}{dt}(t) = A(t)u(t)$ for all $t \in (s, T]$. The solution operators of (2.1) give rise to the following definition. A family of bounded operators $(P(t, s))_{0 \leq s \leq t \leq T}$ on X is called a *strongly continuous evolution family* if

1. $P(s, s) = I$ for all $s \in [0, T]$,
2. $P(t, s) = P(t, r)P(r, s)$ for all $0 \leq s \leq r \leq t \leq T$,
3. the map $\{(\tau, \sigma) \in [0, T]^2 : \sigma \leq \tau\} \ni (t, s) \to P(t, s)$ is strongly continuous.

The next theorem says that the operators $A(t)$, $0 \leq t \leq T$, 'generate' an evolution family having parabolic regularity. It is a consequence of [1, Theorem 2.3], see also [2, 6, 26, 31].

Theorem 2.1. *If condition* (AT) *holds, then there exists a unique strongly continuous evolution family* $(P(t, s))_{0 \leq s \leq t \leq T}$ *such that* $u = P(\cdot, s)x$ *is the unique classical solution of* (2.1) *for every* $x \in X$ *and* $s \in [0, T)$. *Moreover,* $(P(t, s))_{0 \leq s \leq t \leq T}$ *is continuous in* $\mathcal{B}(X)$ *on* $0 \leq s < t \leq T$ *and there exists a constant* $C > 0$ *such that*

$$\|P(t, s)x\|_{X_\alpha^t} \leq C(t - s)^{\beta - \alpha}\|x\|_{X_\beta^s} \tag{2.2}$$

for all $0 \leq \beta \leq \alpha \leq 1$ *and* $0 \leq s < t \leq T$.

We further recall from [32, Theorem 2.1] that there is a constant $C > 0$ such that

$$\|P(t, s)(w - A(s))^{\theta}x\| \leq C(\mu - \theta)^{-1}(t - s)^{-\theta}\|x\| \tag{2.3}$$

for all $0 \leq s < t \leq T$, $\theta \in (0, \mu)$ and $x \in D((w - A(s))^{\theta})$. Clearly, (2.3) allows to extend $P(t, s)$ to a bounded operator $P_{-\theta}(t, s) : X_{-\theta}^s \to X$ satisfying

$$\|P_{-\theta}(t, s)(w - A_{-1}(s))^{\theta}\| \leq C(\mu - \theta)^{-1}(t - s)^{-\theta} \tag{2.4}$$

for all $0 \leq s < t \leq T$ and $\theta \in (0, \mu)$. Again, we mostly omit the index $-\theta$.

2.2. Stochastic integration

Let H be a separable Hilbert space with scalar product $[\cdot,\cdot]$, X be a Banach space, and (S,Σ,μ) be a measure space. A function $\phi : S \to X$ is called *strongly measurable* if it is the pointwise limit of a sequence of simple functions. Let X_1 and X_2 be Banach spaces. An operator-valued function $\Phi : S \to \mathcal{B}(X_1, X_2)$ will be called X_1-*strongly measurable* if the X_2-valued function Φx is strongly measurable for all $x \in X_1$.

Throughout this paper $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(\gamma_n)_{n\geq 1}$ is a *Gaussian sequence*; i.e., a sequence of independent, standard, real-valued Gaussian random variables defined on $(\Omega,\mathcal{F},\mathbb{P})$. An operator $R \in \mathcal{B}(H,X)$ is said to be a γ-*radonifying operator* if there exists an orthonormal basis $(h_n)_{n\geq 1}$ of H such that $\sum_{n\geq 1} \gamma_n Rh_n$ converges in $L^2(\Omega;X)$, see [7, 17]. In this case we define

$$\|R\|_{\gamma(H,X)} := \Big(\mathbb{E}\Big\|\sum_{n\geq 1} \gamma_n\, Rh_n\Big\|^2\Big)^{\frac{1}{2}}.$$

This number does not depend on the sequence $(\gamma_n)_{n\geq 1}$ and the basis $(h_n)_{n\geq 1}$, and defines a norm on the space $\gamma(H,X)$ of all γ-radonifying operators from H into X. Endowed with this norm, $\gamma(H,X)$ is a Banach space, and it holds $\|R\| \leq \|R\|_{\gamma(H,X)}$. Moreover, $\gamma(\mathcal{H},X)$ is an operator ideal in the sense that if $S_1 \in \mathcal{B}(\tilde{\mathcal{H}},\mathcal{H})$ and $S_2 \in \mathcal{B}(X,\tilde{X})$, then $R \in \gamma(\mathcal{H},X)$ implies $S_2RS_1 \in \gamma(\tilde{\mathcal{H}},\tilde{X})$ and

$$\|S_2RS_1\|_{\gamma(\tilde{\mathcal{H}},\tilde{X})} \leq \|S_2\|\|R\|_{\gamma(\mathcal{H},X)}\|S_1\|. \tag{2.5}$$

If X is a Hilbert space, then $\gamma(H,X) = \mathcal{C}^2(H,X)$ isometrically, where $\mathcal{C}^2(H,X)$ is the space of Hilbert-Schmidt operators. Also for $X = L^p$ there is a convenient characterization of $R \in \gamma(H,L^p)$ given in [10, Theorem 2.3]. We use a slightly different formulation taken from [23, Lemma 2.1].

Lemma 2.2. *Let (S,Σ,μ) be a σ-finite measure space and let $1 \leq p < \infty$. For an operator $R \in \mathcal{B}(H,L^p(S))$ the following assertions are equivalent.*

1. *$R \in \gamma(H,L^p(S))$.*
2. *There exists a function $g \in L^p(S)$ such that for all $h \in H$ we have $|Rh| \leq \|h\|_H \cdot g$ μ-almost everywhere.*

Moreover, in this situation we have

$$\|R\|_{\gamma(H,L^p(S))} \lesssim_p \|g\|_{L^p(S)}. \tag{2.6}$$

A Banach space X is said to have *type* 2 if there exists a constant $C \geq 0$ such that for all finite subsets $\{x_1,\dots,x_N\}$ of X we have

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^2\Big)^{\frac{1}{2}} \leq C\Big(\sum_{n=1}^N \|x_n\|^2\Big)^{\frac{1}{2}}.$$

Hilbert spaces and L^p-spaces with $p \in [2,\infty)$ have type 2. We refer to [17] for details. We will also need UMD Banach spaces. The definition of a UMD space will be omitted, but we recall that every UMD space is reflexive. We refer to [11] for an overview on the subject. Important examples of UMD spaces are the reflexive scale of L^p, Sobolev, Bessel-potential and Besov spaces.

A detailed stochastic integration theory for operator-valued processes $\Phi : [0,T] \times \Omega \to \mathcal{B}(H,X)$, where X is a UMD space, has been developed in [22]. The full generality of this theory is not needed here, since we can work with UMD spaces X of type 2 which allow for a somewhat simpler theory. Instead of of being a UMD space with type 2, one can also assume that X is a space of martingale type 2 (cf. [8, 24]).

A family $W_H = (W_H(t))_{t\in\mathbb{R}_+}$ of bounded linear operators from H to $L^2(\Omega)$ is called an *H-cylindrical Brownian motion* if

(i) $\{W_H(t_j)h_k : j = 1,\dots,J;\ k = 1,\dots,K\}$ is a Gaussian vector for all choices of $t_j \geq 0$ and $h_k \in H$, and $\{W_H(t)h : t \geq 0\}$ is a standard scalar Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ for each $h \in H$;

(ii) $\mathbb{E}(W_H(s)g \cdot W_H(t)h) = (s \wedge t)\,[g,h]_H$ for all $s,t \in \mathbb{R}_+$ and $g,h \in H$.

Now let X be a UMD Banach space with type 2. For an H-strongly measurable and adapted $\Phi : [0,T]\times\Omega \to \gamma(H,X)$ which belongs to $L^2((0,T)\times\Omega;\gamma(H,X))$ one can define the stochastic integral $\int_0^T \Phi(s)\,dW_H(s)$ as a limit of integrals of adapted step processes, and there is a constant C not depending on Φ such that

$$\mathbb{E}\Big\|\int_0^T \Phi(s)\,dW_H(s)\Big\|^2 \leq C^2\|\Phi\|^2_{L^2((0,T)\times\Omega;\gamma(H,X))},$$

cf. [8], [22], and the references therein. By a localization argument one may extend the class of integrable processes to all H-strongly measurable and adapted $\Phi : [0,T] \times \Omega \to \gamma(H,X)$ which are contained in $L^2(0,T;\gamma(H,X))$ a.s. Below we use in particular the next result (see [8] and [22, Corollary 3.10]).

Proposition 2.3. *Let X be a UMD space with type 2 and W_H be a H-cylindrical Brownian motion. Let $\Phi : [0,T] \times \Omega \to \gamma(H,X)$ be H-strongly measurable and adapted. If $\Phi \in L^2(0,T;\gamma(H,X))$ a.s., then Φ is stochastically integrable with respect to W_H and for all $p \in (1,\infty)$ it holds*

$$\Big(\mathbb{E}\sup_{t\in[0,T]}\Big\|\int_0^t \Phi(s)\,dW_H(s)\Big\|^p\Big)^{\frac{1}{p}} \lesssim_{X,p} \|\Phi\|_{L^p(\Omega;L^2(0,T;\gamma(H,X)))}.$$

In the setting of Proposition 2.3 we also have, for $x^* \in X^*$,

$$\Big\langle \int_0^T \Phi(s)\,dW_H(s), x^*\Big\rangle = \int_0^T \Phi(s)^*x^*\,dW_H(s) \quad \text{a.s.,} \tag{2.7}$$

cf. [22, Theorem 5.9].

3. The abstract stochastic evolution equation

Let H_1 and H_2 be separable Hilbert spaces, and let X and Y be Banach spaces. On X we consider the stochastic evolution equation

$$\begin{cases} dU(t) = (A(t)U(t) + F(t,U(t)) + \Lambda_G(t)G(t,U(t)))\,dt \\ \qquad\qquad + B(t,U(t))\,dW_{H_1}(t) + \Lambda_C(t)C(t,U(t))\,dW_{H_2}(t),\ t\in[0,T], \\ U(0) = u_0. \end{cases} \tag{SE}$$

Here $(A(t))_{t\in[0,T]}$ is a family of closed operators on X satisfying (AT). The processes W_{H_1} and W_{H_2} are independent cylindrical Brownian motions with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. The initial value is a strongly $\mathcal{F}_0$-measurable mapping $u_0 : \Omega \to X$. We assume that the mappings $\Lambda_G(t) : Y^t \to X^t_{-\theta_G}$ and $\Lambda_C(t) : Y^t \to X^t_{-\theta_C}$ are linear and bounded, where the numbers $\theta_G, \theta_C \in [0,1]$ are specified below. In Section 4, the operators $\Lambda_G(t)$ and $\Lambda_C(t)$ are used to treat inhomogeneous boundary conditions. Concerning $A(t)$, we make the following hypothesis.

(H1) Assume that $(A(t))_{t\in[0,T]}$ and $(A(t)^*)_{t\in[0,T]}$ satisfy (AT) and that there exists an $\eta_0 \in (0,1]$ and a family of Banach spaces $(\widetilde{X}_\eta)_{\eta\in[0,\eta_0]}$ such that

$$\widetilde{X}_{\eta_0} \hookrightarrow \widetilde{X}_{\eta_1} \hookrightarrow \widetilde{X}_{\eta_2} \hookrightarrow \widetilde{X}_0 = X \quad \text{for all } \ \eta_0 > \eta_1 > \eta_2 > 0,$$

and each $\widetilde{X}_\eta$ is a UMD space with type 2. Moreover, it holds

$$[X, D(A(t))]_\eta \hookrightarrow \widetilde{X}_\eta \quad \text{for all } \ \eta \in [0,\eta_0],$$

where the embeddings are bounded uniformly in $t \in [0,T]$.

Assumption (H1) has been employed in [30] to deduce space time regularity results for equations of the form (SE), where spaces such as $\widetilde{X}_\eta$ have been used to get rid of the time dependence of interpolation spaces; see also [20, (H2)]. We have included an assumption on $(A(t)^*)_{t\in[0,T]}$ for the treatment of variational solutions. This could be done in a more general way as well, but for us the above setting suffices. Assumption (H1) can be verified in many applications, see, e.g., Section 4.

Let $a \in [0,\eta_0)$. The nonlinear terms F, G, B and C in (SE) map as follows:

$$F : [0,T]\times\Omega\times\widetilde{X}_a \to X, \qquad G(t) : \Omega\times\widetilde{X}_a \to Y^t,$$
$$B(t) : \Omega\times\widetilde{X}_a \to \gamma(H, X^t_{-1}), \qquad C(t) : \Omega\times\widetilde{X}_a \to \gamma(H_2, Y^t),$$

for each $t \in [0,T]$, where Y^t are Banach spaces. We put $G(t)(\omega,x) = G(t,\omega,x)$, $B(t)(\omega,x) = B(t,\omega,x)$ and $C(t)(\omega,x) = C(t,\omega,x)$ for $(t,\omega,x) \in [0,T]\times\Omega\times X$. Assuming (H1) and $a \in [0,\eta_0)$, we state our main hypotheses on F, G, B and C.

(H2) For all $x \in \widetilde{X}_a$, the map $(t,\omega) \mapsto F(t,\omega,x)$ is strongly measurable and adapted. The function F has linear growth and is Lipschitz continuous in space uniformly in $[0,T]\times\Omega$; that is, there are constants L_F and C_F such that for all $t \in [0,T]$, $\omega \in \Omega$ and $x, y \in \widetilde{X}_a$ we have

$$\begin{aligned} \|F(t,\omega,x) - F(t,\omega,y)\|_X &\le L_F\|x-y\|_{\widetilde{X}_a}, \\ \|F(t,\omega,x)\|_X &\le C_F(1+\|x\|_{\widetilde{X}_a}). \end{aligned}$$

(H3) For all $x \in \widetilde{X}_a$, the map $(t,\omega) \mapsto (-A_w(t))^{-\theta_G}\Lambda_G(t)G(t,\omega,x) \in X$ is strongly measurable and adapted. The function $(-A_w)^{-\theta_G}\Lambda_G G$ has linear growth and is Lipschitz continuous in space uniformly in $[0,T]\times\Omega$; i.e., there are constants L_G and C_G such that for all $t \in [0,T]$, $\omega \in \Omega$ and $x, y \in \widetilde{X}_a$ we have

$$\begin{aligned}\|(-A_w(t))^{-\theta_G}\Lambda_G(t)(G(t,\omega,x) - G(t,\omega,y))\|_X &\le L_G\|x-y\|_{\widetilde{X}_a},\\ \|(-A_w(t))^{-\theta_G}\Lambda_G(t)G(t,\omega,x)\|_X &\le C_G(1+\|x\|_{\widetilde{X}_a}).\end{aligned}$$

(H4) Let $\theta_B \in [0,\mu)$ satisfy $a+\theta_B < \frac{1}{2}$. For all $x \in \widetilde{X}_a$, the map $(t,\omega) \mapsto (-A_w(t))^{-\theta}B(t,\omega,x) \in \gamma(H_1,X)$ is strongly measurable and adapted. The function $(-A_w)^{-\theta_B}B$ has linear growth and is Lipschitz continuous in space uniformly in $[0,T]\times\Omega$; that is, there are constants L_B and C_B such that for all $t \in [0,T]$, $\omega \in \Omega$ and $x, y \in \widetilde{X}_a$ we have

$$\begin{aligned}\|(-A_w(t))^{-\theta_B}(B(t,\omega,x) - B(t,\omega,y))\|_{\gamma(H_1,X)} &\le L_B\|x-y\|_{\widetilde{X}_a},\\ \|(-A_w(t))^{-\theta_B}B(t,\omega,x)\|_{\gamma(H_1,X)} &\le C_B(1+\|x\|_{\widetilde{X}_a}).\end{aligned}$$

(H5) Let $\theta_C \in [0,\mu)$ satisfy $a+\theta_C < \frac{1}{2}$. For all $x \in \widetilde{X}_a$, the mapping $(t,\omega) \mapsto (-A_w(t))^{-\theta_C}\Lambda_C(t)C(t,\omega,x) \in \gamma(H_2,X)$ is strongly measurable and adapted. The function $(-A_w)^{-\theta_C}\Lambda_C C$ has linear growth and is Lipschitz continuous in space uniformly in $[0,T]\times\Omega$; that is, there are constants L_G and C_G such that for all $t \in [0,T]$, $\omega \in \Omega$ and $x, y \in \widetilde{X}_a$ we have

$$\begin{aligned}\|(-A_w(t))^{-\theta_C}\Lambda_C(t)(C(t,\omega,x) - C(t,\omega,y))\|_{\gamma(H_2,X)} &\le L_C\|x-y\|_{\widetilde{X}_a},\\ \|(-A_w(t))^{-\theta_C}\Lambda_C(t)C(t,\omega,x)\|_{\gamma(H_2,X)} &\le C_C(1+\|x\|_{\widetilde{X}_a}).\end{aligned}$$

We introduce our first solution concept.

Definition 3.1. *Assume that* (H1)–(H5) *hold for some* $\theta_G, \theta_B, \theta_C \ge 0$ *and* $a \in [0,\eta_0)$. *Let* $r \in (2,\infty)$ *satisfy* $\min\{1-\theta_G, \frac{1}{2}-\theta_B, \frac{1}{2}-\theta_C\} > \frac{1}{r}$. *We call an* $\widetilde{X}_a$*-valued process* $(U(t))_{t\in[0,T]}$ *a* mild solution *of* (SE) *if*

(i) $U : [0,T]\times\Omega \to \widetilde{X}_a$ *is strongly measurable and adapted, and we have* $U \in L^r(0,T;\widetilde{X}_a)$ *almost surely,*

(ii) *for all* $t \in [0,T]$, *we have*

$$U(t) = P(t,0)u_0 + P*F(\cdot,U)(t) + P*\Lambda_G G(\cdot,U)(t) + P\diamond_1 B(\cdot,U)(t) + P\diamond_2\Lambda_C C(\cdot,U)(t)$$

in X *almost surely.*

Here we have used the abbreviations

$$P*\phi(t) = \int_0^t P(t,s)\phi(s)\,ds, \qquad P\diamond_k \Phi(t) = \int_0^t P(t,s)\Phi(s)\,dW_{H_k}(s), \quad k=1,2,$$

whenever the integrals are well defined. Under our hypotheses both $P*F(\cdot,U)(t)$ and $P*\Lambda_G G(\cdot,U)(t)$ are in fact well defined in X. Indeed, for the first one this is clear from (H2). For the second one we may write

$$P(t,s)\Lambda_G(s)G(s,U(s)) = P(t,s)(-A_w(s))^{\theta_G}(-A_w(s))^{-\theta_G}\Lambda_G(s)G(s,U(s))$$

It then follows from (2.4), Hölder's inequality, and (H3) that

$$\begin{aligned}\int_0^t \|P(t,s)\Lambda_G(s)G(s,U(s))\|_X\,ds \\ &\lesssim \int_0^t (t-s)^{-\theta_G}\|(-A_w(s))^{-\theta_G}\Lambda_G(s)G(s,U(s))\|_X\,ds \\ &\lesssim 1+\|U\|_{L^r(0,T;\widetilde{X}_a)},\end{aligned}$$

using that $1-\theta_G > \frac{1}{r}$. Similarly one can show that $P\diamond_1 B(\cdot,U)(t)$ and $P\diamond_2 \Lambda_C C(\cdot,U)(t)$ are well defined in X, taking into account Proposition 2.3: Estimate (2.4), Hölder's inequality and (H4) imply that

$$\begin{aligned}\int_0^t \|P(t,s)B(s,U(s))\|^2_{\gamma(H_1,X)}\,ds \\ &\lesssim \int_0^t (t-s)^{-2\theta_B}\|(-A_w(s))^{-\theta_B}B(s,U(s))\|^2_{\gamma(H_1,X)}\,ds \\ &\lesssim 1+\|U\|^2_{L^r(0,T;\widetilde{X}_a)}\end{aligned}$$

since $\frac{1}{2}-\theta_B > \frac{1}{r}$. In the same way it can be proved that the integral with respect to W_{H_2} is well defined.

We also recall the definition of a variational solution from [30]. To that purpose, for $t\in[0,T]$, we set

$$\begin{aligned}\Gamma_t = \big\{\varphi\in C^1([0,t];X^*)\ :\ &\varphi(s)\in D(A(s)^*) \text{ for all } s\in[0,t] \\ &\text{and } [s\mapsto A(s)^*\varphi(s)]\in C([0,t];X^*)\big\}.\end{aligned}\tag{3.1}$$

Definition 3.2. *Assume that* (H1)–(H5) *hold with* $a\in[0,\eta_0)$. *An* $\widetilde{X}_a$*-valued process* $(U(t))_{t\in[0,T]}$ *is called a* variational solution *of* (SE) *if*

(i) *U belongs to $L^2(0,T;\widetilde{X}_a)$ a.s. and U is strongly measurable and adapted,*
(ii) *for all $t\in[0,T]$ and all $\varphi\in\Gamma_t$, almost surely we have*

$$\begin{aligned}\langle U(t),\varphi(t)\rangle - \langle u_0,\varphi(0)\rangle = \int_0^t &[\langle U(s),\varphi'(s)\rangle + \langle U(s),A(s)^*\varphi(s)\rangle \\ &+ \langle F(s,U(s)),\varphi(s)\rangle + \langle \Lambda_G(s)G(s,U(s)),\varphi(s)\rangle]\,ds \\ &+ \int_0^t B(s,U(s))^*\varphi(s)\,dW_{H_1}(s) \\ &+ \int_0^t (\Lambda_C(s)C(s,U(s)))^*\varphi(s)\,dW_{H_2}(s).\end{aligned}\tag{3.2}$$

The integrand $B(s,U(s))^*\varphi(s)$ in (3.2) should be read as

$$((-A_w(s))^{-\theta_B}B(s,U(s)))^*(-A_w(s)^*)^{\theta_B}\varphi(s).$$

It follows from (H4) that the function $s\mapsto((-A_w(s))^{-\theta_B}B(s,U(s)))^*$ is X^*-strongly measurable. Moreover, the map

$$s\mapsto(-A_w(s)^*)^{\theta_B}\varphi(s) = (-A_w(s)^*)^{-1+\theta_B}(-A_w(s)^*)\varphi(s)$$

belongs to $C([0,t];X^*)$ by the Hölder continuity of $s \mapsto (-A_w(s))^{-1+\theta_B}$ (cf. [26, (2.10) and (2.11)]) and the assumption on φ. Using (H4), we thus obtain that the integrand is contained in $L^2(0,T;H_1)$ a.s. As a result, the first stochastic integral in (3.2) is well defined. The other integrands have to be interpreted similarly.

The next result shows that both solution concepts are equivalent in our setting. It follows from Proposition 5.4 and Remark 5.3 in [30] in the same way as Theorem 3.4 below. (Remark 5.3 can be used since X is reflexive as a UMD space.)

Proposition 3.3. *Assume that* (H1)–(H5) *hold for some* $\theta_G, \theta_B, \theta_C \geq 0$ *and* $a \in [0, \eta_0)$. *Let* $r \in (2,\infty)$ *satisfy* $\max\{\theta_C, \theta_B\} < \frac{1}{2} - \frac{1}{r}$ *and* $\theta_G < 1 - \frac{1}{r}$. *Let* $U : [0,T] \times \Omega \to \widetilde{X}_a$ *be a strongly measurable and adapted process such that* U *belongs to* $L^r(0,T;\widetilde{X}_a)$ *a.s. Then* U *is a mild solution of* (SE) *if and only if* U *is a variational solution of* (SE).

We can now state the main existence and regularity result for (SE).

Theorem 3.4. *Assume that* (H1)–(H5) *hold for some* $\theta_G, \theta_B, \theta_C \geq 0$ *and* $a \in [0, \eta_0)$. *Let* $u_0 : \Omega \to \widetilde{X}_a^0$ *be strongly* $\mathcal{F}_0$ *measurable. Then the following assertions hold.*

(1) *There is a unique mild solution* U *of* (SE) *with paths in* $C([0,T];\widetilde{X}_a)$ *a.s.*
(2) *For every* $\delta, \lambda > 0$ *with*
$$\delta + a + \lambda < \min\{1 - \theta_G, \tfrac{1}{2} - \theta_B, \tfrac{1}{2} - \theta_C, \eta_0\}$$
there exists a version of U *such that* $U - P(\cdot,0)u_0$ *in* $C^\lambda([0,T];\widetilde{X}_{\delta+a})$ *a.s.*
(3) *If* $\delta, \lambda > 0$ *are as in* (2) *and if* $u_0 \in \widetilde{X}_{a+\delta+\lambda}$ *a.s., then* U *has a version with paths in* $C^\lambda([0,T];\widetilde{X}_{\delta+a})$ *a.s.*

Proof. Assertions (1) and (2) can be reduced to the case
$$\begin{cases} dU(t) = (A(t)U(t) + \tilde{F}(t,U(t)) + \tilde{B}(t,U(t))\, dW_H(t), \ t \in [0,T], \\ U(0) = u_0. \end{cases}$$
taking $\tilde{F} = F + \Lambda_G G$ and $\tilde{B} = (B, \Lambda_C C)$ and $H = H_1 \times H_2$. The theorem now follows from [30, Theorem 6.3]. In view of (2), for assertion (3) we only have to show that $P(\cdot,0)u_0$ has the required regularity, which is proved in [30, Lemma 2.3]. We note that, in order to apply the above results from [30] here, one has to replace in [30] the real interpolation spaces of type $(\eta,2)$ by complex interpolation spaces of exponent η. This can be done using the arguments given in [30]. $\square$

4. Boundary noise

Let $S \subseteq \mathbb{R}^d$ be a bounded domain with C^2-boundary and outer unit normal vector of $n(s)$. On S we consider the stochastic equation with boundary noise

$$\begin{aligned}\frac{\partial u}{\partial t}(t,s) &= \mathcal{A}(t,s,D)u(t,s) + f(t,s,u(t,s)) && (4.1)\\ &\quad + b(t,s,u(t,s))\,\frac{\partial w_1}{\partial t}(t,s), && s\in S,\ t\in(0,T],\\ \mathcal{B}(t,s,D)u(t,s) &= G(t,u(t,\cdot))(s) + \tilde{C}(t,u(t,\cdot))(s)\frac{\partial w_2}{\partial t}(t,s)\,, && s\in\partial S,\ t\in(0,T],\\ u(0,s) &= u_0(s), && s\in S.\end{aligned}$$

Here w_k are Brownian motions as specified below, and we use the differential operators

$$\mathcal{A}(t,s,D) = \sum_{i,j=1}^{d} D_i\big(a_{ij}(t,s)D_j\big) + a_0(t,s), \quad \mathcal{B}(t,s,D) = \sum_{i,j=1}^{d} a_{ij}(t,s)n_i(s)D_j.$$

For simplicity we only consider the case of a scalar equation, but systems could be treated in the same way, cf., e.g., [5, 16].

(A1) We assume that the coefficients of $\mathcal{A}$ and $\mathcal{B}$ are real and satisfy

$$\begin{aligned}&a_{ij} \in C^{\mu}([0,T];C(\overline{S})),\ a_{ij}(t,\cdot)\in C^1(\overline{S}),\ D_k a_{ij}\in C([0,T]\times\overline{S}),\\ &a_0 \in C^{\mu}([0,T],L^d(S))\cap C([0,T];C(\overline{S}))\end{aligned}$$

for a constant $\mu \in (\frac{1}{2},1]$ and all $i,j,k=1,\dots,d$ and $t\in[0,T]$. Further, let (a_{ij}) be symmetric and assume that there is a $\kappa>0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(t,s)\xi_i\xi_j \geq \kappa|\xi|^2 \qquad \text{for all } \ s\in\overline{S},\ t\in[0,T],\ \xi\in\mathbb{R}^d. \tag{4.2}$$

In the following we reformulate the problem (4.1) as (SE) thereby giving (4.1) a precise sense. Set $X = L^p(S)$ for some $p\in(1,\infty)$. Let $\alpha\in[0,2]$ satisfy $\alpha-\frac{1}{p}\neq 1$. We introduce the space

$$H^{\alpha,p}_{\mathcal{B}(t)}(S) = \begin{cases}\{f\in H^{\alpha,p}(S):\ \mathcal{B}(t,\cdot,D)f=0\}, & \alpha-\frac{1}{p}>1,\\ H^{\alpha,p}(S), & \alpha-\frac{1}{p}<1,\end{cases}$$

where $H^{\alpha,p}(S)$ denotes the usual Bessel-potential space (see [29]). We also set

$$\widetilde{X}_\eta = H^{2\eta,p}(S) \qquad \text{for all } \ \eta\geq 0.$$

We further define $A(t): D(A(t))\to X$ by $A(t)x = \mathcal{A}(t,\cdot,D)x$ and

$$D(A(t)) = \{x\in H^{2,p}(S) : \mathcal{B}(t,\cdot,D)x = 0\} = H^{2,p}_{\mathcal{B}(t)}(S).$$

Lemma 4.1. *Let $X = L^p(S)$ and $p \in (1,\infty)$. Assume that* (A1) *is satisfied. The following assertions hold.*

(1) *The operators $A(t)$, $t \in [0,T]$, satisfy (AT) and the graph norms of $A(t)$ are uniformly equivalent with $\|\cdot\|_{H^{2,p}(S)}$. In particular, $(A(t))_{t\in[0,T]}$ generates a unique strongly continuous evolution family $(P(t,s))_{0\le s\le t\le T}$ on X.*

(2) *We have $X^t_\theta = H^{2\theta,p}_{\mathcal{B}(t)}(S)$ for all $\theta \in (0,1)$ with $2\theta - \frac{1}{p} \neq 1$, as well as $X^t_\eta = \widetilde{X}_\eta = H^{2\eta,p}(S)$ for all $\eta \in [0, \frac{1}{2} + \frac{1}{2p})$, in the sense of isomorphic Banach spaces. The norms of these isomorphisms are bounded uniformly for $t \in [0,T]$.*

(3) *Let $p \in [2,\infty)$. Then condition* (H1) *holds with $\eta_0 = 1/2$.*

Proof. (1): See [1] and [31]. Note that $A(t)^*$ on $L^{p'}(S) = X^*$ is given by $A^*(t)\varphi = \mathcal{A}(t,\cdot,D)\varphi$ with $D(A(t)^*) = H^{2,p'}_{\mathcal{B}(t)}(S)$, and thus also $(A(t)^*)_{0\le t\le T}$ satisfies (AT).

(2): Let $\theta \in (0,1)$ and $p \in (1,\infty)$ satisfy $2\theta - \frac{1}{p} \neq 1$. Then Theorem 5.2 and Remark 5.3(c) in [5] show that

$$X^t_\theta = [L^p(S), D(A(t))]_\theta = [L^p(S), H^{2,p}_{\mathcal{B}(t)}(S)]_\theta = H^{2\theta,p}_{\mathcal{B}(t)}(S) \tag{4.3}$$

isomorphically, see also [27, Theorem 4.1] and [29, Theorem 1.15.3]. Inspecting the proofs given in [27] one sees that the isomorphisms in (4.3) are bounded uniformly in $t \in [0,T]$. Similarly, if $2\theta - \frac{1}{p} < 1$, then $X^t_\theta = H^{2\theta,p}_{\mathcal{B}(t)}(S) = H^{2\theta,p}(S) = \widetilde{X}_\theta$.

(3): This is clear from (1), (2) and the definitions. Note that the spaces $\widetilde{X}^t_\eta$ are UMD spaces with type 2 because they are isomorphic to closed subspaces of L^p-spaces with $p \in [2,\infty)$. ☐

Remark 4.2. Let the constant $w \ge 0$ be given by (AT). In problem (4.1) we replace $\mathcal{A}$ and f by $\mathcal{A} - w$ and $f + w$, respectively, without changing the notation. This modification does not affect the assumptions (A1) and (A2), and from now we can thus take $w = 0$ in (AT).

Next, we apply Theorem 9.2 and Remark 9.3(e) of [5] in order to construct the operators $\Lambda_C(t)$ and $\Lambda_D(t)$. In [5] it is assumed that $\partial S \in C^\infty$. However, the results from [5] used below remain valid under our assumption that $\partial S \in C^2$, due to Remark 7.3 of [5] combined with Theorem 2.3 of [15].

Let $t \in [0,T]$. In view of our main Theorem 4.3 we consider only $p \ge 2$ and $\alpha \in (1, 1+\frac{1}{p})$ though some of the results stated below can be generalized to other exponents. Let

$$Y = \partial W^{\alpha,p}(S) := W^{\alpha-1-1/p,p}(\partial S)$$

be the Slobodeckii space of negative order on the boundary which is defined via duality, e.g., in (5.16) of [5]. Let $y \in Y$. Theorem 9.2 and Remark 9.3(e) of [5] give a unique weak solution $x \in H^{\alpha,p}(S)$ of the elliptic problem

$$\begin{aligned} \mathcal{A}(t,\cdot,D)x &= 0 \qquad \text{on } S,\\ \mathcal{B}(t,\cdot,D)x &= y \qquad \text{on } \partial S. \end{aligned}$$

(Weak solutions are defined by means of test functions $v \in H^{2-\alpha,p'}(S)$, see [5, (9.4)].) We set $N(t)y := x$. Formula (9.15) of [5] implies that the 'Neumann map' $N(t)$ belongs $\mathcal{B}(\partial W^{\alpha,p}(S), H^{\alpha,p}(S))$ and that the map $N(\cdot) : [0,T] \to \mathcal{B}(\partial W^{\alpha,p}(S), H^{\alpha,p}(S))$ is continuous.

Concerning the other terms in the first line of (4.1) and the noise terms, we make the following hypotheses.

(A2) The functions $f, b : [0,T] \times \Omega \times S \times \mathbb{R} \to \mathbb{R}$ are jointly measurable, adapted to $(\mathcal{F}_t)_{t\geq 0}$, and Lipschitz functions and of linear growth in the third variable, uniformly in the other variables.

(A3) For $k = 1, 2$, the process w_k can be written in the form $i_k W_{H_k}$, where $i_1 \in \gamma(H_1, L^r(S))$ for some $r \in [1,\infty)$ and $i_2 \in \gamma(H_2, L^s(\partial S))$ for some $s \in [1,\infty)$, and W_{H_1} and W_{H_2} are independent H_k-cylindrical Brownian motions with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Supposing that (A2) holds, we define $F : [0,T] \times \Omega \times X \to X$ by setting $F(t,\omega,x)(s) = f(t,\omega,s,x(s))$. Then F satisfies (H2). We further define the function $B(t,\omega,x)h$ on S for $(t,\omega,x) \in [0,T] \times \Omega \times X$ and $h \in H_1$ by means of

$$B(t,\omega,x)h = b(t,\omega,\cdot,x(\cdot))\, i_1 h \tag{4.4}$$

In Examples 4.7 and 4.8 we give conditions on w_1 and θ_B such that $(-A)^{-\theta_B} B$ maps $[0,T] \times \Omega \times X$ into $\gamma(H_1, X)$ and (H4) holds.

Assumption (A3) has to be interpreted in the sense that

$$w_k(t,s) = \sum_{n\geq 1} (i_k h_n^k)(s) W_{H_k}(t) h_n^k, \qquad t \in \mathbb{R}_+, s \in S,\ k = 1, 2,$$

where $(h_n^k)_{n\geq 1}$ is an orthonormal basis for H_k, and the sum converges in $L^r(S)$ if $k = 1$ and in $L^s(\partial S)$ if $k = 2$. We note that then $(w_k(t,\cdot))_{t\geq 0}$ is a Brownian motion with values in $L^r(S)$ and $L^s(\partial S)$, respectively. Conversely, if $(w_k(t,\cdot))_{t\geq 0}$, $k = 1, 2$, are independent Brownian motions with values in $L^r(S)$ and $L^s(\partial S)$, then we can always construct i_k and W_{H_k} as above, cf. Example 4.6 below.

We recall that $H^{\alpha,p}(S) = X^t_{\frac{\alpha}{2}}$ for $t \in [0,T]$ and $\alpha \in (1, 1 + \frac{1}{p})$ by Lemma 4.1(2). Moreover, the operator $A(t)$ has bounded imaginary powers in X (uniformly in $t \in [0,T]$), see, e.g., Example 4.7.3(d) and Section 4.7 in [6]. It then follows that

$$H^{\alpha,p}(S) = X^t_{\frac{\alpha}{2}} = D((-A(t))^{\frac{\alpha}{2}}) \tag{4.5}$$

with uniformly equivalent norms for $t \in [0,T]$, see, e.g., [29, Theorem 1.15.3]. Therefore, the extrapolated operator $A_{-1}(t)$ maps $H^{\alpha,p}(S)$ into $X^t_{\frac{\alpha}{2}-1}$, and hence

$$\Lambda(t) = \Lambda_G(t) = \Lambda_C(t) := -A_{-1}(t)N(t) \in \mathcal{B}(Y, X^t_{\frac{\alpha}{2}-1})$$

with uniformly bounded norms for $t \in [0,T]$. Let $\theta \in [1 - \frac{\alpha}{2}, 1]$. As above, we further obtain $X^t_{\frac{\alpha}{2}-1+\theta} = H^{\alpha-2+2\theta,p}(S) \hookrightarrow X$, so that

$$(-A(t))^{-\theta}\Lambda(t) \in \mathcal{B}(Y, H^{\alpha-2+2\theta,p}(S)) \tag{4.6}$$

with uniformly bounded norms for $t \in [0,T]$.

In order to relate the boundary noise term in (4.1) with (SE), we set

$$(C(t,\omega,x)h)(s) = \tilde{C}(t,\omega,x)(s)(i_2 h)(s)$$

for $h \in H_2$. We aim at the mapping property $C(t,\omega,x) : H_2 \to Y = \partial W^{\alpha,p}(S)$ since it will enable us to verify the hypothesis (H5). In fact, if $C(t,\omega,x) \in \mathcal{B}(H_2, Y)$ then $(-A(t))^{-\theta}\Lambda(t)C(t,\omega,x)$ maps H_2 continuously into $H^{\alpha-2+2\theta,p}(S) \hookrightarrow X$ if $\theta \in [1-\frac{\alpha}{2}, 1]$. In Examples 4.4 and 4.6 we give conditions on $\tilde{C}$ and i_2 implying (H5) for C. The deterministic boundary term G can be treated in a similar way.

We want to present a variational formulation of (4.1), starting with an informal discussion. Let $\varphi \in \Gamma_t$, where Γ_t is given by (3.1). Then $\varphi(r) \in D(A(r)^*) = W^{2,p'}_{\mathcal{B}(r)}(S)$. Formally, multiplying (4.1) by φ, integrating over $[0,t] \times S$, integrating by parts and interchanging the order of integration, we obtain that, almost surely,

$$\begin{aligned}
&\int_S [u(t,s)\varphi(t)(s) - u_0(s)\varphi(0)(s)]\, ds \qquad\qquad (4.7)\\
&\quad= \int_0^t \int_S u(r,s)[\mathcal{A}(r,\cdot,D)\varphi(r) + \varphi'(r)](s)\, ds\, dr\\
&\qquad + \int_0^t \int_S f(r,s,u(r,s))\varphi(r)(s)\, ds\, dr\\
&\qquad + \int_S \int_0^t b(r,s,u(r,s))\varphi(r)(s)\, dw_1(r,s)\, ds + T_1.
\end{aligned}$$

In the boundary term T_1 the part with $\nabla\varphi(r)$ disappears since $\varphi(r) \in D(A(r)^*)$, and the other term is given by

$$\begin{aligned}
T_1 &= \int_{\partial S} \int_0^t \mathcal{B}(r,\cdot,D)u(r,\cdot)\mathrm{tr}(\varphi(r))\, dr\, d\sigma\\
&= \int_{\partial S} \int_0^t G(r,u(r,\cdot))\mathrm{tr}(\varphi(r))\, dr\, d\sigma + \int_{\partial S} \int_0^t \tilde{C}(r,u(r,\cdot))\mathrm{tr}(\varphi(r)) dw_2(r,\cdot)\, d\sigma
\end{aligned}$$

where tr denotes the trace operator on $W^{2,p'}_{\mathcal{B}(r)}(S)$.

We now start from the equation (4.7) and rewrite it using (2.7) and the notation introduced above. Setting $u(t,s) =: U(t)(s)$, equality (4.7) becomes

$$\begin{aligned}
\langle U(t), \varphi(t)\rangle - \langle u_0, \varphi(0)\rangle &= \int_0^t \langle U(r), (\mathcal{A}(r,\cdot,D)\varphi(r) + \varphi'(r)\rangle\, dr + T_1 \qquad (4.8)\\
&\quad + \int_0^t \langle F(r,U(r)), \varphi(r)\rangle\, dr + \int_0^t B(r,U(r))^*\varphi(r)\, dW_{H_1}(r),
\end{aligned}$$

and the boundary term yields

$$T_1 = \int_0^t \langle G(r,U(r)), \mathrm{tr}(\varphi(r))\rangle\, dr + \int_0^t C(r,U(r))^*\mathrm{tr}(\varphi(r))\, dW_{H_2}(r).$$

Here the brackets denote the duality pairing on $L^p(S)$ and $L^p(\partial S)$, respectively. We claim that for all $x \in W^{2,p'}_{\mathcal{B}(t)}(S)$ it holds

$$\operatorname{tr}(x) = \Lambda(t)^* x = (-A_{-1}(t)N(t))^* x.$$

Indeed, let $\alpha \in (1, 1+\frac{1}{p})$, $x \in W^{2,p'}_{\mathcal{B}(t)}(S) = D(A^*(t))$ and $y \in Y = \partial W^{\alpha,p}(S)$. Then we have $N(t)y \in H^{\alpha,p}(S)$ and $a(t)\nabla x \cdot n = 0$ on ∂S. Observe that $\Lambda(t)^*$ maps $D(A(t)^*)$ into Y^*. Integrating by parts and using formula (9.4) of [5], we obtain

$$\begin{aligned}\langle y, \Lambda(t)^* x\rangle_Y &= \langle \Lambda(t)y, x\rangle_{X^t_{-1}} = -\langle N(t)y, A(t)^* x\rangle_X \\ &= -\int_S N(t)y[\nabla \cdot (a(t)\nabla x) + a_0(t)x]\, ds \\ &= 0 + \int_S [(a(t)\nabla N(t)y) \cdot \nabla x + a_0(t)(N(t)y)x]\, ds = \langle y, \operatorname{tr}(x)\rangle_Y,\end{aligned}$$

which proves the claim. Therefore, T_1 becomes

$$T_1 = \int_0^t \langle \Lambda(t)G(r, U(r)), \varphi(r)\rangle\, dr + \int_0^t (\Lambda(t)C(r, U(r)))^* \varphi(r)\, dW_{H_2}(r).$$

Combining this expression with (4.8) we arrive at the definition of a variational solution to the stochastic evolution equation (4.1), as introduced in Definition 3.2. The above calculations thus motivate the following definitions. We say u is a *variational* (resp. *mild*) *solution* to (4.1) if $U(t)(s) = u(t,s)$ is a variational (resp. mild) solution to (SE) with the above definitions of $A(t)$, F, Λ_G, G, B, Λ_C, C and W_{H_k}. We can now state out main result.

Theorem 4.3. *Let* $p \in [2, \infty)$, $X = L^p(S)$, $\alpha \in (1, 1+\frac{1}{p})$, $\theta_B \in [0, \frac{1}{2})$, $\theta_C \in (1-\frac{\alpha}{2}, \frac{1}{2})$ *and* $\theta_G \in (1-\frac{\alpha}{2}, 1)$. *Assume that* (A1)–(A3) *and* (H3)–(H5) *hold, where* C, G, B, Λ_C *and* Λ_G *are defined above. Let* $u_0 : \Omega \to X$ *be strongly* $\mathcal{F}_0$*-measurable. Then the following assertions are true.*

1. *There exists a unique variational and mild solution* u *of* (4.1) *with paths in* $C([0,T]; X)$ *a.s.*
2. *For every* $\delta, \lambda > 0$ *with* $\delta + \lambda < \min\{1-\theta_G, \frac{1}{2}-\theta_B, \frac{1}{2}-\theta_C\}$ *there exists a version of* u *such that* $u - P(\cdot, 0)u_0$ *in* $C^\lambda([0,T]; \widetilde{X}_\delta)$ *a.s.*
3. *If* $\delta, \lambda > 0$ *are as in* (2) *and if* $u_0 \in \widetilde{X}_{\delta+\lambda}$ *a.s., then* u *has a version with paths in* $C^\lambda([0,T]; \widetilde{X}_\delta)$.

Note that we need $\frac{1}{2} - \theta_C < \frac{\alpha}{2} - \frac{1}{2} < \frac{1}{2p}$. Thus, if $\frac{1}{2} - \theta_B \geq \frac{1}{2p}$, $1 - \theta_G \geq \frac{1}{2p}$ and the other assumptions in Theorem 4.3 hold, then we can take $\lambda, \delta \geq 0$ with $\delta + \lambda < \frac{1}{2p}$ and deduce that $u - P(\cdot, 0)u_0$ belongs to $C^\lambda([0,T]; \widetilde{X}_\delta)$ a.s. If we also have $u_0 \in H^{\frac{1}{p},p}(S)$, then we obtain a solution u of (4.1) with paths in $C([0,T]; H^{2\delta,p}(S))$ for all $\delta < \frac{1}{2p}$. In this case Sobolev's embedding (see [29, Theorem 4.6.1]) implies that

$$u \in C([0,T]; L^q(S)) \quad \text{for all} \quad \begin{cases} q < \frac{dp}{d-1} & \text{if } d \geq 2, \\ q < \infty, & \text{if } d = 1. \end{cases}$$

Proof of Theorem 4.3. The existence and uniqueness of a mild solution with the asserted regularity follows from Theorem 3.4 and the above observations. The equivalence with the variational solution is a consequence of Proposition 3.3. $\square$

We now discuss several examples under which (H4) and (H5) hold. The hypothesis (H3) can be treated in the same way. We start with some observations concerning Gaussian random variables ξ with values in a Banach space Z, see, e.g., [7], [9] and the references therein. The covariance $Q \in \mathcal{B}(Z^*, Z)$ of ξ is given by $Qx^* = \mathbb{E}(\langle \xi, x^*\rangle \xi)$ for $x^* \in Z^*$. One introduces an inner product $[\cdot,\cdot]$ on the range of Q by setting

$$[Qx^*, Qy^*] := \langle Qx^*, y^*\rangle = \mathbb{E}(\langle \xi, x^*\rangle \langle \xi, y^*\rangle) \tag{4.9}$$

for $x^*, y^* \in Z^*$, and we define $\|Qx^*\|_H^2 = [Qx^*, Qx^*]$. The *reproducing kernel Hilbert space* H of ξ is the completion of QZ^* with respect to $\|\cdot\|_H$. Then the identity on QZ^* can be extended to a continuous embedding $i : H \hookrightarrow E$, and it holds $Q = ii^*$. On the other hand, the random variables $w_k(t,\cdot)$ in (A3) are Gaussian with covariance $Q_k = t\, i_k i_k^*$ for all $t \geq 0$ and $k = 1, 2$.

Example 4.4. Let (A3) hold with $H_2 = L^2(\partial S)$. Assume that covariance operator $Q_2 \in \mathcal{B}(L^2(\partial S))$ of w_2 is compact. Then there exist numbers $(\lambda_n)_{n\geq 1}$ in $\mathbb{R}_+$ and an orthonormal system $(e_n)_{n\geq 1}$ in $L^2(\partial S)$ such that

$$Q_2 = \sum_{n\geq 1} \lambda_n e_n \otimes e_n.$$

Assume that

$$\sum_{n\geq 1} \lambda_n \|e_n\|_\infty^2 < \infty.$$

We observe that the operator i_2 is given by $i_2 = \sum_{n\geq 1} \sqrt{\lambda_n} e_n \otimes e_n$ and belongs to $\mathcal{B}(L^2(\partial S), L^\infty(\partial S))$. Let $p \in [2,\infty)$. Assume that $\tilde{C} : [0,T] \times \Omega \times L^p(S) \to L^p(\partial S)$ is strongly measurable and adapted, as well as Lipschitz and of linear growth in the third variable uniformly in $[0,T] \times \Omega$. Then (H5) holds for $C = \tilde{C} i_2$ with $a = 0$ and every $\theta_C \in (1 - \frac{\alpha}{2}, \frac{1}{2})$, where $\alpha \in (1, 1 + \frac{1}{p})$.

Proof. Lemma 2.2 implies that $i_2 \in \gamma(H_2, L^p(\partial S))$. Fix $t \in [0,T]$, $\omega \in \Omega$ and $x, y \in X = L^p(S)$. Denote $K = \|i_2\|_{\mathcal{B}(H_2, L^\infty(\partial S))}$. The embedding $L^p(\partial S) \hookrightarrow Y = \partial W^{\alpha,p}(S)$ and (2.5) yield

$$\|C(t,\omega,x) - C(t,\omega,y)\|_{\gamma(H_2,Y)} \lesssim_{p,\alpha} \|C(t,\omega,x) - C(t,\omega,y)\|_{\gamma(H_2,L^p(\partial S))}. \tag{4.10}$$

Furthermore, for $h \in H_2$ and $s \in S$ we have

$$\begin{aligned} |((C(t,\omega,x) - C(t,\omega,y))h)(s)| &= |\tilde{C}(t,\omega,x)(s) - \tilde{C}(t,\omega,y)(s)|\, |i_2 h(s)| \\ &\leq K |\tilde{C}(t,\omega,x)(s) - \tilde{C}(t,\omega,y)(s)|\, \|h\|_{H_2}. \end{aligned} \tag{4.11}$$

Lemma 2.2 and the assumptions of the example then imply that

$$\begin{aligned} \|C(t,\omega,x) - C(t,\omega,y)\|_{\gamma(H_2,L^p(\partial S))} &\lesssim_p K \|\tilde{C}(t,\omega,x) - \tilde{C}(t,\omega,y)\|_{L^p(\partial S)} \\ &\leq K L_{\tilde{C}} \|x - y\|_{L^p(S)}. \end{aligned}$$

Using (4.6), we can now deduce the first part of (H5). The second part is shown in a similar way. □

Remark 4.5. Note that in Example 4.4 the noise could be a bit more irregular since in (4.10) one can still regain some integrability by choosing α and θ_C appropriately.

Example 4.6. Let $q \in (p, \infty]$ and $s \in [p, \infty)$ satisfy $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$. We assume that w_2 is an $L^s(\partial S)$-valued Brownian motion. Let H_2 be the reproducing kernel Hilbert space of the Gaussian random variable $w_2(1) = w_2(1, \cdot)$ with covariance Q and i_2 be the embedding of H_2 into $L^s(\partial S)$. Then we have $i_2 \in \gamma(H_2, L^s(\partial S))$ (cf. [7], [9] and the references therein for details). It is easy to check that $t^{-1/2}w_2(t)$ also has the covariance Q for $t > 0$. Due to Proposition 2.6.1 in [18] we thus obtain

$$t^{-1/2}w_2(t) = \sum_{n\geq 1} \langle t^{-1/2}w_2(t), x_n^* \rangle Q x_n^*$$

in X a.s. for every orthonormal basis $(Qx_n^*)_{n\geq 1}$ of H_2. Therefore

$$w_2(t) = \sum_{n\geq 1} \langle w_2(t), x_n^* \rangle Q x_n^*$$

converges in X a.s. We now define $W_{H_2}(t) : QL^{s'}(\partial S) \to L^2(\Omega)$ by setting

$$W_{H_2}(t)Qx^* = \sum_{n\geq 1} \langle w_2(t), x_n^* \rangle \langle Qx_n^*, x^* \rangle = \langle w_2(t), x^* \rangle$$

for each $x^* \in L^{s'}(\partial S)$ and $t \geq 0$. Then we deduce $\|W_{H_2}(t)Qx^*\|_2^2 = \langle Qx^*, x^* \rangle = \|Qx^*\|_{H_2}^2$ from (4.9), and thus W_{H_2} extends to a bounded operator from H_2 into $L^2(\Omega)$. It is easy to check that W_{H_2} is the required cylindrical Brownian motion with $w_2 = i_2 W_{H_2}$; i.e., (A3) holds for $k = 2$. Assume that $\tilde{C} : [0, T] \times \Omega \times X :\to L^q(\partial S)$ is strongly measurable and adapted, as well as Lipschitz and of linear growth in the third variable uniformly in $[0, T] \times \Omega$. Then (H5) holds for $C = \tilde{C} i_2$, where we take $a = 0$, $\theta_C \in (1 - \frac{\alpha}{2}, \frac{1}{2})$ and $\alpha \in (1, 1 + \frac{1}{p})$.

Proof. Fix $t \in [0, T]$, $\omega \in \Omega$ and $x, y \in X = L^p(S)$. We argue as in the previous example, but in (4.11) we consider $\tilde{C}(t, \omega, x) - \tilde{C}(t, \omega, y)$ as an multiplication operator from $L^s(\partial S)$ to $L^p(\partial S)$. Using Hölder's inequality and (2.5), we thus obtain

$$\begin{aligned}
\|C(t, \omega, x) - C(t, \omega, y)\|_{\gamma(H_2, Y)} &\lesssim_{p,\alpha} \|C(t, \omega, x) - C(t, \omega, y)\|_{\gamma(H_2, L^p(\partial S))} \\
&\leq \|\tilde{C}(t, \omega, x) - \tilde{C}(t, \omega, y)\|_{L^q(\partial S)} \|i_2\|_{\gamma(H_2, L^s(\partial S))} \\
&\leq L_{\tilde{C}} \|x - y\|_{L^p(\partial S)} \|i_2\|_{\gamma(H_2, L^s(\partial S))}.
\end{aligned}$$

The first part of (H5) now follows in view of (4.6). The second part can be proved in the same way. □

We now come to condition (H4).

Example 4.7. Assume that (A1)–(A3) hold with $r \in (d, \infty)$. Then (H4) is satisfied for all $\theta_B \in (\frac{d}{2r}, \frac{1}{2})$.

Proof. Let $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $\theta_B \in (\frac{d}{2r}, \frac{1}{2})$. As in Example 5.5 of [25] one can show

$$L^q(S) \hookrightarrow X^t_{-\theta_B}, \tag{4.12}$$

where the embedding is uniformly bounded for $t \in [0,T]$. Fix $t \in [0,T]$, $\omega \in \Omega$ and $x, y \in X = L^p(S)$. Arguing as in the previous example, by means of (4.12), (4.4), Hölder's inequality, (A2) and (2.5) we can estimate

$$\begin{aligned}
&\|(-A(t))^{-\theta_B}(B(t,\omega,x) - B(t,\omega,y))\|_{\gamma(H_1,X)} \\
&\qquad \lesssim_{\theta_B,p,r,n} \|B(t,\omega,x) - B(t,\omega,y)\|_{\gamma(H_1,L^q(S)))} \\
&\qquad \le \|b(t,\omega,x) - b(t,\omega,y)\|_{L^p(S)} \|i_1\|_{\gamma(H_2,L^r(S))} \\
&\qquad \le L_b \|x-y\|_{L^p(S)} \|i_1\|_{\gamma(H_2,L^r(S))}.
\end{aligned}$$

This proves the first part of (H4). The second part is obtained in a similar way. $\square$

Finally, we consider the white noise situation in the case $d = 1$.

Example 4.8. Let $d = 1$ and $p > 2$ and assume that (A1)–(A3) hold with $i_1 = I$. Then (H4) is satisfied for all $\theta_B \in (\frac{1}{2p} + \frac{1}{4}, \frac{1}{2})$.

Proof. Let $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$ and $\theta_B \in (\frac{1}{2p} + \frac{1}{4}, \frac{1}{2})$. Fix $t \in [0,T]$, $\omega \in \Omega$ and $x, y \in X = L^p(S)$. Observe that $(-A(t))^{-\theta_B}$ can be extended to $L^q(S)$ where it coincides with the fractional power of the corresponding realization $A_q(t)$ of $\mathcal{A}(t,\cdot,D)$ on $L^q(S)$ with the boundary condition $\mathcal{B}(t,\cdot,D)v = 0$. We further obtain

$$D((-A_q(t))^{\theta_B}) \hookrightarrow (L^q(S), H^{2,q}(S))_{\theta_B,\infty} \hookrightarrow [L^q(S), H^{2,q}(S)]_\vartheta = H^{2\vartheta,q}(S).$$

for $\vartheta \in (\frac{1}{2p} + \frac{1}{4}, \theta_B)$ with uniform embedding constants, see Sections 1.10.3 and 1.15.2 of [29] and (4.3). Sobolev's embedding then yields that $D((-A_q(t))^{\theta_B}) \hookrightarrow C(\overline{S})$. Using also Hölder's inequality, we thus obtain

$$\begin{aligned}
&|[((-A(t))^{-\theta_B}(B(t,\omega,x) - B(t,\omega,y))h)](s)| \\
&\qquad \lesssim_{\theta_B,p} \|(B(t,\omega,x) - B(t,\omega,y))h\|_{L^q(S)} \\
&\qquad \le \|b(t,\omega,x) - b(t,\omega,y)\|_{L^p(S)} \|h\|_{L^2(S)} \\
&\qquad \le L_b \|x-y\|_{L^p(S)} \|h\|_{L^2(S)}
\end{aligned}$$

for all $s \in S$. Now we can apply Lemma 2.2 to obtain that

$$\|(-A(t))^{-\theta_B}(B(t,\omega,x) - B(t,\omega,y))\|_{\gamma(H_1,X)} \lesssim_{\theta_B,p,n} L_b \|x-y\|_{L^p(S)}.$$

The other condition (H4) can be verified in the same way. $\square$

In the next remark we explain why one cannot consider Dirichlet boundary conditions with the above methods. This problem was not stated clearly in [21]. In the one-dimensional case with $S = \mathbb{R}_+$, a version of (4.1) with Dirichlet boundary conditions has been treated in [4] using completely other methods and working on a weighted L^p space on $\mathbb{R}_+$.

Remark 4.9. Since we are looking for a solution in $X = L^p(S)$, we have to require that $\alpha - 2 + 2\theta_C \geq 0$, see (4.6). The restriction $\theta_C < \frac{1}{2}$ in Theorem 3.4 then leads to $1 - \frac{\alpha}{2} \leq \theta_C < \frac{1}{2}$, so that $\alpha > 1$. On the other hand, in the case of Dirichlet boundary conditions one has $\partial W^{\alpha,p}(S) = W^{\alpha-\frac{1}{p},p}(\partial S)$ and the Neumann map $N(t)$ has to be replaced by the Dirichlet map $D(t) \in \mathcal{B}(\partial W^{\alpha,p}(S), W^{\alpha,p}(S))$, where $D(t)y := x \in W^{\alpha,p}(S)$ is the solution of the elliptic problem

$$\begin{aligned} \mathcal{A}(t,\cdot,D)x &= 0 \qquad \text{on } S, \\ x &= y \qquad \text{on } \partial S \end{aligned}$$

for a given $y \in \partial W^{\alpha,p}(S)$. To achieve that $\Lambda_C(t) := -A_{-1}(t)D(t)$ maps into $X^t_{-\theta_C}$, we need that $H^{\alpha,p}(S) = H^{\alpha,p}_{\mathcal{B}(t)}(S)$, and hence $\alpha - \frac{1}{p} < 0$ in the Dirichlet case; which contradicts $\alpha > 1$ and $p \geq 1$.

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A Note on Necessary Conditions for Blow-up of Energy Solutions to the Navier-Stokes Equations

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Dedicated to Herbert Amann

Abstract. In the present note, we address the question about behavior of L_3-norm of the velocity field as time t approaches blow-up time T. It is known that the upper limit of the above norm must be equal to infinity. We show that, for blow-ups of type I, the lower limit of L_3-norm equals to infinity as well.

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1. Motivation

Consider the Cauchy problem for the classical 3D-Navier-Stokes system

$$\left.\begin{array}{c}\partial_t v + v\cdot\nabla\, v - \nu\,\Delta\, v = -\nabla\, q\\ \operatorname{div} v = 0\end{array}\right\}\qquad \text{in } Q_+, \tag{1.1}$$

$$v|_{t=0} = a \in C^\infty_{0,0}(\mathbb{R}^3). \tag{1.2}$$

Here, v and q stand for the velocity field and for the pressure field, respectively, $Q_+ = \mathbb{R}^3\times]0,+\infty[$, and

$$C^\infty_{0,0}(\mathbb{R}^3) = \{a\in C^\infty_0(\mathbb{R}^3):\ \operatorname{div} a = 0 \text{ in } \mathbb{R}^3\}.$$

In what follows, we always assume that $\nu = 1$.

It is well known due to J. Leray, see [5], the Cauchy problem (1.1), (1.2) has at least one solution called the weak Leray-Hopf solution. To give its modern definition, let us introduce standard energy spaces H and V. H is the closure of the set $C^\infty_{0,0}(\mathbb{R}^3)$ in $L_2(\mathbb{R}^3)$ and V is the closure of the same set with respect to the norm generated by the Dirichlet integral.

Definition 1.1. A velocity field $v \in L_\infty(0,+\infty;H)\cap L_2(0,+\infty;V)$ is called a weak Leray-Hopf solution to the Cauchy problem (1.1), (1.2) if the following conditions hold:

$$\int\limits_{Q_+}(v\cdot\partial_t w+v\otimes v:\nabla w-\nabla v:\nabla w)\,dxdt=0 \tag{1.3}$$

for any $w\in C^\infty_0(Q_+)$ with $\operatorname{div} w=0$ in Q_+; the function

$$t\mapsto\int\limits_{\mathbb{R}^3}v(x,t)\cdot w(x)dx \tag{1.4}$$

is continuous on $[0,+\infty[$ for all $w\in L_2(\mathbb{R}^3)$;

$$\|v(\cdot,t)-a(\cdot)\|_2\to 0 \tag{1.5}$$

as $t\to+0$;

$$\|v(\cdot,t)\|_2^2+2\int_0^t\int\limits_{\mathbb{R}^3}|\nabla v|^2dxdt'\leq\|a\|_2^2 \tag{1.6}$$

for all $t\in[0,+\infty[$.

The definition does not contain the pressure field at all. However, using the linear theory, we can introduce so-called associated pressure $q(\cdot,t)$, which, for all $t>0$, is the Newtonian potential of $v_{i,j}(\cdot,t)v_{j,i}(\cdot,t)$ and satisfies the pressure equation

$$-\Delta\, q(\cdot,t)=v_{i,j}(\cdot,t)v_{j,i}(\cdot,t)=\operatorname{div}\operatorname{div}v(\cdot,t)\otimes v(\cdot,t) \tag{1.7}$$

in $\mathbb{R}^3$. Since v is known to belong $L_{\frac{10}{3}}(Q_+)$, pressure q is in $L_{\frac{5}{3}}(Q_+)$. Moreover, the Navier-Stokes system is satisfied in the sense of distributions and even a.e. in Q_+. We refer to the paper [3] for details.

Uniqueness of weak Leray-Hopf solutions is still unknown. However, there is a simple but deep connection between smoothness and uniqueness. It has been pointed out by J. Leray in his celebrated paper [5] and reads: any smooth solution to (1.1), (1.2) is unique in the class of weak Leray-Hopf solutions. The problem of smoothness of weak Leray-Hopf solutions is actually one of the seven Millennium problems.

In the paper, we deal with certain necessary conditions for possible blow-ups of solutions to the Cauchy problem (1.1), (1.2). Suppose that $T>0$ is the first moment of time when singularities occur. Then, as it has been shown by J. Leray, given $3<s\leq+\infty$, there exists a constant c_s such that

$$\|v(\cdot,t)\|_{s,\mathbb{R}^3}\geq\frac{c_s}{(T-t)^{\frac{s-3}{2s}}} \tag{1.8}$$

for $T/2\leq t<T$.

However, in the marginal case $s = 3$, we have a weaker result

$$\limsup_{t\to T-0} \|v(\cdot,t)\|_3 = +\infty, \tag{1.9}$$

which has been established in [2]. Apparently, a natural question can be raised whether the statement

$$\lim_{t\to T-0} \|v(\cdot,t)\|_3 = +\infty \tag{1.10}$$

is true or not. In [9], there has been proved a weaker version of (1.10), namely,

$$\lim_{t\to T-0} \frac{1}{T-t}\int\limits_t^T \|v(\cdot,\tau)\|_3^3 d\tau = +\infty. \tag{1.11}$$

The aim of the present paper is to show validity of (1.10) provided the blow-up of type I takes place, i.e.,

$$\|v(\cdot,t)\|_\infty \le \frac{C_\infty}{\sqrt{T-t}} \tag{1.12}$$

for any $T/2 \le t < T$ and for some positive constant C_∞. Our main result can be formulated as follows.

Theorem 1.1. *Let T be a blow-up time and let, for some $3 < s \le +\infty$, there exist a positive constant C_s such that*

$$\|v(\cdot,t)\|_s \le \frac{C_s}{(T-t)^{\frac{s-3}{2s}}} \tag{1.13}$$

for any $T/2 \le t < T$. Then (1.10) *holds.*

Let us outline our proof of Theorem 1.1. Firstly, we reduce the general case to a particular one $s = +\infty$ showing that if (1.13) is true for some $3 < s < +\infty$, then it is true for $s = +\infty$ as well. Secondly, assuming that (1.10) is violated, i.e., a sequence t_k tending to T exists such that

$$\sup_k \|v(\cdot,t_k)\|_3 = M < +\infty, \tag{1.14}$$

we may use a blow-up machinery and construct a non-trivial ancient solution defined in $\mathbb{R}^3\times]-\infty,0[$ with the following properties. It vanishes at time $t = 0$ and its L_3-norm is finite say at time $t = -1$. In order to apply backward uniqueness results, proved in [2], we need to check that the above ancient solution has a certain behavior at infinity with respect to spatial variables. This can be done with the help of the conception of so-called local energy solutions to the Cauchy problem, see [4] and also [7].

Finally, we would like to make the following remark regarding to condition (1.13). It is interesting to figure out whether condition (1.14) itself implies regularity. It is worthy to note that the important consequence of (1.14) is that

$$v(\cdot,T) \in L_3(\mathbb{R}^3). \tag{1.15}$$

Then any reasonable blow up procedure at a singular point gives an ancient solution vanishing at time $t = 0$ and a hope to apply backward uniqueness. Actually, condition (1.13) is one of others that ensure the existence of an ancient solution with such a property. In principle, we could use the universal scaling as in [10], which provides the existence of the limit of scaled functions with no additional assumption. But in such a case, we would loose possibility to use backward uniqueness results because it is unknown whether or not the corresponding ancient solution is zero at $t = 0$. This also raises the interesting question why we are still not able to prove smoothness of $L_{3,\infty}$-solution with the help of the universal scaling.

2. Some auxiliary things

In the paper, we are going to use the following notion. $B(x_0, R)$ stands for a spatial ball centered at a point x_0 and having radius R, $B(R) = B(0, R)$, and $B = B(1)$. By $Q(z_0, R)$, where $z_0 = (x_0, t_0)$ is a space-time point, we denote a parabolic ball $B(x_0, R)\times]t_0 - R^2, t_0[$, and $Q(R) = Q(0, R)$, $Q = Q(1)$. All constants depending on non-essential parameters will be denoted simply by c.

Lemma 2.1. *Suppose that* (1.13) *holds for some* $3 < s < +\infty$. *Then it is true for* $s = +\infty$.

Proof. From (1.7) and (1.13) it follows that $q(\cdot, t) \in L_{\frac{s}{2}}(\mathbb{R}^3)$ for $T/2 < t < T$.

Fix $\varepsilon > 0$ and $z_0 = (x_0, t_0)$ with $t_0 < T$ arbitrarily. Applying (1.13) and Hölder inequality, we find

$$\begin{aligned}\frac{1}{R^2}\int\limits_{Q(z_0,R)}(|v|^3+|q|^{\frac{3}{2}})dz \quad &\leq c(s)\frac{1}{R^2}\Big(\int\limits_{Q(z_0,R)}(|v|^s+|q|^{\frac{s}{2}})dz\Big)^{\frac{3}{s}}R^{5(1-\frac{3}{s})}\\ &\leq c(s)R^{5(1-\frac{3}{s})-2}\Big(\int\limits_{t_0-R^2}^{t_0}\frac{C_s^s}{(T-t)^{\frac{s-3}{2}}}dt\Big)^{\frac{3}{s}}\leq c(s)C_s^3R^{3-\frac{15}{s}}R^{2\frac{3}{s}}\frac{1}{(T-t_0)^{3\frac{s-3}{2s}}}\\ &\leq c(s)C_s^3\Big(\frac{R}{\sqrt{T-t_0}}\Big)^{3\frac{s-3}{2s}}.\end{aligned}$$

We let $R = \sqrt{\gamma(T - t_0)}$ and pick up $0 < \gamma < 1$ so that $c(s)C_s^3\gamma^{3\frac{s-3}{2s}} \leq \varepsilon/2$. Now, we apply the local regularity theory for suitable weak solutions to the Navier-Stokes equations, developed in [1], [6], [3], and [2]. It reads that if $\varepsilon \leq \varepsilon_0$, where ε_0 is a universal constant, then

$$|v(z_0)| \leq \frac{c}{R} = \frac{c}{\sqrt{\gamma(T-t_0)}}$$

for all $z_0 = (x_0, t_0)$ with $x_0 \in \mathbb{R}^3$ and $T/2 < t_0 < T$ and for some universal constant c. □

So, we need prove Theorem 1.1 in a particular case $s = +\infty$ only.

3. Ancient solution

By assumptions of Theorem 1.1, there must be singular points at $t = T$. We take any of them say $x_0 \in \mathbb{R}^3$. Then local regularity theory gives the following inequality

$$\frac{1}{R^2} \int\limits_{Q((x_0,T),R)} (|v|^3 + |q|^{\frac{3}{2}})dz \geq \varepsilon_0 > 0 \tag{3.1}$$

for $0 < R < R_0 = \frac{1}{3}\min\{1, \sqrt{T}\}$ with universal constant ε_0. Without loss of generality, we may assume that $x_0 = 0$.

Proceeding in the same way as in [10], we can find that condition (1.13) implies the following bound

$$\sup_{0<R\leq R_0} \Big\{ \frac{1}{R^2} \int\limits_{Q((0,T),R)} (|v|^3 + |q|^{\frac{3}{2}})dz + \frac{1}{R} \int\limits_{Q((0,T),R)} |\nabla v|^2 dz + \frac{1}{R} \sup_{T-R^2<t<T} \int\limits_{B(R)} |v(x,t)|^2 dx \Big\} = M_1 < +\infty. \tag{3.2}$$

Next, we may scale our functions v and q essentially in the same way as it has been done in [9], namely,

$$u^{(k)}(y,s) = R_k v(R_k y, T + R_k^2 s), \qquad p^{(k)}(y,s) = R_k^2 q(R_k y, T + R_k^2 s)$$

for $y \in B(R_0/R_k)$ and for $s \in]-(R_0/R_k)^2, 0[$, where $R_k = \sqrt{T - t_k}$.

Now, let us see what happens if $k \to +\infty$. This is more or less well-understood procedure and the reader can find details in [2], [9]–[11]. As a result, we have two measurable functions u and p defined on $Q_- = \mathbb{R}^3 \times]-\infty, 0[$ with the following properties:

$$\begin{aligned} u^{(k)} &\to u && \text{in} \quad L_3(Q(a)), \\ \nabla u^{(k)} &\rightharpoonup \nabla u && \text{in} \quad L_2(Q(a)), \\ p^{(k)} &\rightharpoonup p && \text{in} \quad L_{\frac{3}{2}}(Q(a)), \\ u^{(k)} &\to u && \text{in} \quad C([-a^2, 0]; L_{\frac{9}{8}}(B(a))) \end{aligned} \tag{3.3}$$

for any $a > 0$. The pair u and p satisfies the Navier-Stokes equations in Q_- in the sense distributions. We call it an ancient solution to the Navier-Stokes equations. Moreover, since inequalities (1.12) and (3.2) are invariant with respect to the Navier-Stokes scaling, we can show that

$$\sup_{0<a<+\infty} \Big\{ \frac{1}{a^2} \int\limits_{Q(a)} (|u|^3 + |p|^{\frac{3}{2}})de + \frac{1}{a} \int\limits_{Q(a)} |\nabla u|^2 de + \frac{1}{a} \sup_{-a^2<s<0} \int\limits_{B(a)} |u(y,s)|^2 dy \Big\} \leq M_1 < +\infty \tag{3.4}$$

and

$$|u(y,s)| \leq \frac{C_\infty}{\sqrt{-s}} \tag{3.5}$$

for all $e = (y,s) \in Q_-$.

The important consequence of (1.15) and the last line in (3.3), is the following fact

$$u(\cdot, 0) = 0 \tag{3.6}$$

in $\mathbb{R}^3$, see [9] in a similar situation.

Now, our goal is to show that the above ancient solution is non-trivial. Unfortunately, we cannot get this by direct passing to the limit in the formula

$$\frac{1}{a^2} \int\limits_{Q(a)} (|u^{(k)}|^3 + |p^{(k)}|^{\frac{3}{2}}) de = \frac{1}{a^2 R_k^2} \int\limits_{Q(aR_k)} (|v|^3 + |q|^{\frac{3}{2}}) dz \geq \varepsilon_0 > 0 \tag{3.7}$$

for $aR_k < 3/4$. The reason is simple: there is no hope to prove strong convergence of the pressure. However, we still have local strong convergence of $u^{(k)}$ so that

$$\frac{1}{a^2} \int\limits_{Q(a)} |u^{(k)}|^3 de \to \frac{1}{a^2} \int\limits_{Q(a)} |u|^3 de \tag{3.8}$$

for any $0 < a \leq 3/4$.

To prove that our ancient solution is non-trivial, let us first note that according to (3.2)

$$\frac{1}{a^2} \int\limits_{Q(a)} (|u^{(k)}|^3 + |p^{(k)}|^{\frac{3}{2}}) de \leq M_1 \tag{3.9}$$

for sufficient large k and for all $a \in]0, 3/4]$.

The second observation is quite typical when treating the pressure. In the ball $B(3/4)$, the pressure can be split into two parts

$$p^{(k)} = p_1^{(k)} + p_2^{(k)},$$

where the first term is defined by the variational identity

$$\int\limits_{B(3/4)} p_1^{(k)}(y,s) \Delta\, \varphi(y) dy = - \int\limits_{B(3/4)} u^{(k)}(y,s) \otimes u^{(k)}(y,s) : \nabla^2 \varphi(y) dy$$

being valid for any $\varphi \in W^2_{\frac{3}{2}}(B(3/4))$ with $\varphi = 0$ on $\partial B(3/4)$. It is not difficult to show that the first counter-part of the pressure satisfies the estimate

$$\|p_1^{(k)}(\cdot, s)\|_{\frac{3}{2}, B(3/4)} \leq c \|u^{(k)}(\cdot, s)\|^2_{3, B(3/4)} \tag{3.10}$$

for all $-\infty < s < 0$ while the second one is a harmonic function in $B(3/4)$ for the same s. Since $p_2^{(k)}(\cdot, s)$ is harmonic, we have

$$\sup_{y\in B(1/2)} |p_2^{(k)}(y,s)|^{\frac{3}{2}} \leq c \int\limits_{B(3/4)} |p_2^{(k)}(y,s)|^{\frac{3}{2}} dy \tag{3.11}$$
$$\leq c \int\limits_{B(3/4)} |p^{(k)}(y,s)|^{\frac{3}{2}} dy + c \int\limits_{B(3/4)} |u^{(k)}(y,s)|^3 dy.$$

Then, for any $0 < a < 1/2$,

$$\varepsilon_0 \leq \frac{1}{a^2} \int\limits_{Q(a)} (|u^{(k)}|^3 + |p^{(k)}|^{\frac{3}{2}}) de \leq c\frac{1}{a^2} \int\limits_{Q(a)} (|u^{(k)}|^3 + |p_1^{(k)}|^{\frac{3}{2}} + |p_2^{(k)}|^{\frac{3}{2}}) de$$
$$\leq c\frac{1}{a^2} \int\limits_{Q(a)} (|u^{(k)}|^3 + |p_1^{(k)}|^{\frac{3}{2}}) de + ca^3 \frac{1}{a^2} \int\limits_{-a^2}^{0} \sup_{y\in B(1/2)} |p_2^{(k)}(y,s)|^{\frac{3}{2}} ds.$$

Combining (3.9)–(3.11), we find

$$\varepsilon_0 \leq c\frac{1}{a^2} \int\limits_{Q(3/4)} |u^{(k)}|^3 de + ca \int\limits_{-a^2}^{0} ds \int\limits_{B(3/4)} \Big(|p^{(k)}(y,s)|^{\frac{3}{2}} + |u^{(k)}(y,s)|^3\Big) dy$$
$$\leq c\frac{1}{a^2} \int\limits_{Q(3/4)} |u^{(k)}|^3 de + ca \int\limits_{Q(3/4)} \Big(|p^{(k)}|^{\frac{3}{2}} + |u^{(k)}|^3\Big) de$$
$$\leq c\frac{1}{a^2} \int\limits_{Q(3/4)} |u^{(k)}|^3 de + cM_1 a$$

for the same a. Passing to the limit and choosing sufficiently small a, we show that

$$0 < c\varepsilon_0 a^2 \leq \int\limits_{Q(3/4)} |u|^3 de \tag{3.12}$$

for some positive $0 < a < 1/2$. So, our ancient solution u is non-trivial.

If would show that for some positive R_*

$$|u| + |\nabla u| \in L_\infty((\mathbb{R}^3 \setminus B(R_*))\times] - (5/6)^2, 0[),$$

we could use arguments from [2] and conclude that, by (3.6), $\nabla \wedge u \equiv 0$ in $\mathbb{R}^3\times] - (3/4)^2, 0[$ which, together with the incompressibility condition, means that $u(\cdot, t)$ is harmonic in $\mathbb{R}^3$. And it is bounded there. So, u must be a function of t only. But estimate (3.4) says that such a function must be zero in $] - (3/4)^2, 0[$. The latter contradicts (3.12).

4. Spatial decay for ancient solutions

We know that

$$\|u^{(k)}(\cdot,-1)\|_3 \leq M$$

and thus by (3.3)

$$\|u(\cdot,-1)\|_3 \leq M. \tag{4.1}$$

Now, let us consider the following Cauchy problem

$$\left.\begin{array}{c}\partial_t w + w\cdot\nabla\, w - \Delta\, w = -\nabla\, r\\ \operatorname{div} w = 0\end{array}\right\} \quad \text{in } \widetilde{Q} = \mathbb{R}^3\times]-1,1[, \tag{4.2}$$

$$w(\cdot,-1) = u(\cdot,-1). \tag{4.3}$$

We would like to construct a solution to problem (4.2), (4.3) satisfying the local energy inequality. To this end, let us recall notation and some facts from [7].

$$L_{m,\mathrm{unif}} = \{u \in L_{m,\mathrm{loc}} \;:\; \|u\|_{L_{m,\mathrm{unif}}} = \sup_{x_0\in\mathbb{R}^3}\Big(\int\limits_{B(x_0,1)} |u(x)|^m dx\Big)^{1/m} < +\infty\},$$

$$E_m = \{u \in L_{m,\mathrm{unif}} \;:\; \int\limits_{B(x_0,1)} |u(x)|^m dx \to 0 \quad \text{as} \quad |x_0|\to+\infty\},$$

$$\overset{\circ}{E}_m = \{u \in E_m \;:\; \operatorname{div} u = 0 \quad \text{in} \quad \mathbb{R}^3\}.$$

Apparently,

$$u(\cdot,-1) \in \overset{\circ}{E}_2\ . \tag{4.4}$$

Definition 4.1. A pair of functions w and r defined in the space-time cylinder $\widetilde{Q}$ is called a local energy weak Leray-Hopf solution or simply local energy solution to the Cauchy problem (4.2), (4.3) if the following conditions are satisfied:

$$w \in L_\infty(-1,1;L_{2,\mathrm{unif}}), \qquad \sup_{x_0\in\mathbb{R}^3}\int_{-1}^{1}\int_{B(x_0,1)} |\nabla w|^2 dz < +\infty,$$

$$r \in L_{\frac32}(-1,1;L_{\frac32,\mathrm{loc}}(\mathbb{R}^3)); \tag{4.5}$$

$$w \text{ and } r \text{ meet (4.2) in the sense of distributions;} \tag{4.6}$$

$$\text{the function } t\mapsto \int_{\mathbb{R}^3} w(x,t)\cdot\widetilde{w}(x)\,dx \text{is continuous on } [-1,1] \tag{4.7}$$

for any compactly supported function $\widetilde{w}\in L_2(\mathbb{R}^3)$; for any compact K,

$$\|w(\cdot,t)-u(\cdot,-1)\|_{L_2(K)} \to 0 \quad as \quad t\to -1+0; \tag{4.8}$$

$$\int_{\mathbb{R}^3} \varphi|w(x,t)|^2\,dx + 2\int_{-1}^{t}\int\limits_{\mathbb{R}^3}\varphi|\nabla w|^2\,dxdt$$

$$\leq \int_{-1}^{t}\int_{\mathbb{R}^3}\Big(|w|^2(\partial_t\varphi+\Delta\varphi) + w\cdot\nabla\varphi(|w|^2+2r)\Big)\,dxdt \tag{4.9}$$

for a.a. $t \in]-1,1[$ and for all nonnegative functions $\varphi \in C_0^\infty(\mathbb{R}^3\times]-1,2[)$; for any $x_0 \in \mathbb{R}^3$, there exists a function $c_{x_0} \in L_{\frac{3}{2}}(-1,1)$ such that

$$r_{x_0}(x,t) \equiv r(x,t) - c_{x_0}(t) = r^1_{x_0}(x,t) + r^2_{x_0}(x,t), \tag{4.10}$$

for $(x,t) \in B(x_0,3/2)\times]-1,1[$, where

$$r^1_{x_0}(x,t) = -\frac{1}{3}|w(x,t)|^2 + \frac{1}{4\pi}\int\limits_{B(x_0,2)} K(x-y) : w(y,t)\otimes w(y,t)\,dy,$$

$$r^2_{x_0}(x,t) = \frac{1}{4\pi}\int\limits_{\mathbb{R}^3\setminus B(x_0,2)} (K(x-y) - K(x_0-y)) : w(y,t)\otimes w(y,t)\,dy$$

and $K(x) = \nabla^2(1/|x|)$.

We have, see [4] and also [7].

Proposition 4.1. *Under assumption* (4.4)*, there exists at least one local energy solution to problem* (4.2)*,* (4.3)*.*

To describe spatial decay of local energy solution, we need additional notation

$$\alpha_w(t) = \|w(\cdot,t)\|^2_{L_{2,\mathrm{unif}}}, \qquad \beta_w(t) = \sup_{x_0\in\mathbb{R}^3}\int\limits_{-1}^{t}\int\limits_{B(x_0,1)} |\nabla w|^2 dxdt',$$

$$\gamma_w(t) = \sup_{x_0\in\mathbb{R}^3}\int\limits_{-1}^{t}\int\limits_{B(x_0,1)} |w|^3 dxdt', \quad \delta_r(t) = \sup_{x_0\in\mathbb{R}^3}\int\limits_{-1}^{t}\int\limits_{B(x_0,3/2)} |r_{x_0}|^{\frac{3}{2}} dxdt'.$$

One of the most important properties of local energy solutions is a kind of uniform local boundedness of the energy, i.e.,

$$\sup_{-1\le t\le 1}\alpha_w(t) + \beta_w(1) + \gamma_w^{\frac{2}{3}}(1) \le A < +\infty. \tag{4.11}$$

Next, fix a smooth cut-off function χ so that $\chi(x) = 0$ if $x \in B$, $\chi(x) = 1$ if $x \notin B(2)$, and then let $\chi_R(x) = \chi(x/R)$. Hence, one can define

$$\alpha^R_w(t) = \|\chi_R w(\cdot,t)\|^2_{L_{2,\mathrm{unif}}}, \qquad \beta^R_w(t) = \sup_{x_0\in\mathbb{R}^3}\int\limits_{-1}^{t}\int\limits_{B(x_0,1)} |\chi_R\nabla w|^2 dxdt',$$

$$\gamma^R_w(t) = \sup_{x_0\in\mathbb{R}^3}\int\limits_{-1}^{t}\int\limits_{B(x_0,1)} |\chi_R w|^3 dxdt', \quad \delta^R_r(t) = \sup_{x_0\in\mathbb{R}^3}\int\limits_{-1}^{t}\int\limits_{B(x_0,3/2)} |\chi_R r_{x_0}|^{\frac{3}{2}} dxdt'.$$

As it was shown in [7], the following decay estimate is true.

Lemma 4.2. *Assume that the pair w and r is a local energy solution to* (4.2), (4.3). *Then*

$$\sup_{-1\leq t\leq} \alpha^R_w(t)+\beta^R_w(1)+(\gamma^R_w)^{\frac{2}{3}}(1)+(\delta^R_r)^{\frac{4}{3}}(1)$$
$$\leq C(A)\Big[\|\chi_R u(\cdot,-1)\|^2_{L_{2,\mathrm{unif}}}+1/R^{\frac{2}{3}}\Big]. \tag{4.12}$$

Since any local energy solution to the Cauchy problem (4.2), (4.3) is also a suitable weak solution to the Navier-Stokes equations, one can apply the local regularity theory to them and deduce from Lemma 4.2 that there exists a positive number R_* such that

$$|w(z)|+|\nabla\, w(z)|\leq A_1 \tag{4.13}$$

for all $z=(x,t)\in \Big(\mathbb{R}^3\setminus B(R_*)\Big)\times[-(5/6)^2,1]$.

If we would show that

$$u\equiv w \tag{4.14}$$

on $\mathbb{R}^3\times[0,1[$, this would make it possible to apply backward uniqueness results (actually, to vorticity equations) and conclude that $u=0$ on $\mathbb{R}^3\times[-(3/4)^2,0]$ which contradicts (3.12). So, the rest of the paper is devoted to a proof of (4.14).

Our first observation in this direction is that u is C^∞-function in Q_-. This follows from (3.5). Detail discussion on differentiability properties of bounded ancient solutions can be found in [8] and [10]. In addition, the pressure $p(\cdot,t)$ is a BMO-solution to the pressure equations

$$-\Delta p(\cdot,t)=\mathrm{divdiv}\, u(\cdot,t)\otimes u(\cdot,t)$$

in R^3.

Using a suitable cut-off function in time and differentiability properties of w and u, we can get the following three relations:

$$\int\limits_{\mathbb{R}^3}\varphi(x)w(x,\tau)\cdot u(\cdot,t)dx\Big|^{\tau=t}_{\tau=-1}$$
$$=\int\limits_{-1}^{t}\int\limits_{\mathbb{R}^3}(w\otimes w-\nabla\, w):(\varphi\nabla\, u+u\otimes\nabla\,\varphi)dxd\tau$$
$$+\int\limits_{-1}^{t}\int\limits_{\mathbb{R}^3}ru\cdot\nabla\,\varphi dxd\tau+\int\limits_{-1}^{t}\int\limits_{\mathbb{R}^3}\phi w\cdot\partial_t u dxd\tau;$$

$$\int\limits_{\mathbb{R}^3}\varphi(x)|w(x,\tau)|^2dx\Big|^{\tau=t}_{\tau=-1}+2\int\limits_{-1}^{t}\int\limits_{\mathbb{R}^3}\varphi|\nabla\, w|^2dxd\tau$$
$$\leq\int\limits_{-1}^{t}\int\limits_{\mathbb{R}^3}\Big(|w|^2\Delta\,\varphi+w\cdot\nabla\,\varphi(|w|^2+2r)\Big)dxd\tau;$$

$$\int_{\mathbb{R}^3} \varphi(x)|u(x,\tau)|^2 dx\Big|_{\tau=-1}^{\tau=t} + 2\int_{-1}^{t}\int_{\mathbb{R}^3} \varphi|\nabla u|^2 dx d\tau$$
$$= \int_{-1}^{t}\int_{\mathbb{R}^3} \Big(|u|^2\Delta\varphi + u\cdot\nabla\varphi(|u|^2+2p)\Big) dx d\tau$$

for any $0 \le \varphi \in C_0^\infty(\mathbb{R}^3)$. Letting $\overline{u} = w - u$ and $\overline{p} = r - p$, we can find from them the main inequality

$$\int_{\mathbb{R}^3} \varphi(x)|\overline{u}(x,t)|^2 dx + 2\int_{-1}^{t}\int_{\mathbb{R}^3} \varphi|\nabla \overline{u}|^2 dx d\tau \tag{4.15}$$
$$\le \int_{-1}^{t}\int_{\mathbb{R}^3} \Big(|\overline{u}|^2\Delta\varphi + \overline{u}\cdot\nabla\varphi(|\overline{u}|^2+2\overline{p}) + u\cdot\nabla\varphi|\overline{u}|^2 - 2\varphi\nabla u : \overline{u}\otimes\overline{u}\Big) dx d\tau$$

for a.a. t in $]-1,0[$.

Next, for u and $\overline{u}$, we may introduce the analogous quantities

$$\alpha_u(t) = \|u(\cdot,t)\|^2_{L_{2,\mathrm{unif}}}, \qquad \alpha_{\overline{u}}(t) = \|\overline{u}(\cdot,t)\|^2_{L_{2,\mathrm{unif}}}$$
$$\beta_u(t) = \sup_{x_0\in\mathbb{R}^3}\int_{-1}^{t}\int_{B(x_0,1)} |\nabla u|^2 dx dt', \qquad \beta_{\overline{u}}(t) = \sup_{x_0\in\mathbb{R}^3}\int_{-1}^{t}\int_{B(x_0,1)} |\nabla \overline{u}|^2 dx dt',$$
$$\gamma_u(t) = \sup_{x_0\in\mathbb{R}^3}\int_{-1}^{t}\int_{B(x_0,1)} |u|^3 dx dt', \qquad \gamma_{\overline{u}}(t) = \sup_{x_0\in\mathbb{R}^3}\int_{-1}^{t}\int_{B(x_0,1)} |\overline{u}|^3 dx dt',$$
$$\delta_p(t) = \sup_{x_0\in\mathbb{R}^3}\int_{-1}^{t}\int_{B(x_0,3/2)} |p_{x_0}|^{\frac32} dx dt', \qquad \delta_{\overline{p}}(t) = \sup_{x_0\in\mathbb{R}^3}\int_{-1}^{t}\int_{B(x_0,3/2)} |\overline{p}_{x_0}|^{\frac32} dx dt'$$

where

$$p_{x_0}(x,t) \equiv p(x,t) - p^0_{x_0}(t) = p^1_{x_0}(x,t) + p^2_{x_0}(x,t),$$
$$p^1_{x_0}(x,t) = -\frac13|u(x,t)|^2 + \frac{1}{4\pi}\int_{B(x_0,2)} K(x-y) : u(y,t)\otimes u(y,t)\,dy,$$
$$p^2_{x_0}(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B(x_0,2)} (K(x-y) - K(x_0-y)) : u(y,t)\otimes u(y,t)\,dy,$$
$$\overline{p}_{x_0}(x,t) \equiv \overline{p}(x,t) - \overline{p}^0_{x_0}(t) = \overline{p}^1_{x_0}(x,t) + \overline{p}^2_{x_0}(x,t),$$
$$\overline{p}^1_{x_0}(x,t) = r^1_{x_0}(x,t) - p^1_{x_0}(x,t),$$
$$\overline{p}^2_{x_0}(x,t) = r^2_{x_0}(x,t) - p^2_{x_0}(x,t).$$

By (3.5) and by our definitions,

$$\alpha_u(t) \leq \frac{c}{(-t)}, \quad \beta_u(t) \leq \frac{c}{(-t)^2}, \quad \gamma_u(t) + \delta_p(t) \leq \frac{c}{(-t)^{\frac{3}{2}}} \tag{4.16}$$

for all $-1 \leq t < 0$. Indeed, the first bound follows directly from (3.5). To get the second one, we need to use (3.5), BMO-estimate of the pressure via velocity field, and then local regularity theory in the same way as in the proof of Lemma 2.1. It is useful to note that the above arguments imply the estimate $|\nabla u(x,t)| \leq c/(-t)$ for all $(x,t) \in Q_-$. As to the third bound, the second term is estimated with the help of the singular integral theory, (3.5), and definitions of $p^1_{x_0}$ and $p^2_{x_0}$.

We fix $x_0 \in \mathbb{R}^3$ and a smooth non-negative function φ such that $\varphi \equiv 1$ in B and $\operatorname{spt}\varphi \subset B(3/2)$ and let $\varphi_{x_0}(x) = \varphi(x - x_0)$. Considering (4.15) with such a cut-off function φ_{x_0}, taking into account (4.11) and (4.16), and arguing for example as in [7], we can find the inequality

$$\begin{aligned}\alpha_{\overline{u}}(t_0) + \beta_{\overline{u}}(t_0) \leq c\Big[\int\limits_{-1}^{t_0} \alpha_{\overline{u}}(t)dt + \gamma_{\overline{u}}(t_0) + \sup_{x_0\in\mathbb{R}^3}\int\limits_{-1}^{t_0}\int\limits_{B(x_0,3/2)} |\overline{p}_{x_0}||\overline{u}|dxdt \\ + \frac{c}{\sqrt{-t_0}}\int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt + \frac{c}{(-t_0)}\int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt\Big]\end{aligned} \tag{4.17}$$

for all $-1 \leq t_0 < 0$.

Next, we can re-write the well-known (in the theory of the Navier-Stokes equations) multiplicative inequality in terms of quantities introduced above

$$\gamma_{\overline{u}}(t_0) \leq c\Big(\int\limits_{-1}^{t_0} \alpha^{3}_{\overline{u}}(t)dt\Big)^{\frac{1}{4}}\Big(\beta_{\overline{u}}(t_0) + \int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt\Big)^{\frac{3}{4}}.$$

To simplify the latter, we first make use of (4.11) and (4.16) in the following way

$$\alpha_{\overline{u}}(t_0) \leq c(\alpha_w(t_0) + \alpha_u(t_0)) \leq c\Big(A + \frac{c}{(-t_0)}\Big) \leq \frac{C(A)}{(-t_0)}$$

for all $-1 \leq t_0 < 0$. And thus

$$\gamma_{\overline{u}}(t_0) \leq \frac{C(A)}{\sqrt{-t_0}}\Big(\int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt\Big)^{\frac{1}{4}}\Big(\beta_{\overline{u}}(t_0) + \int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt\Big)^{\frac{3}{4}} \tag{4.18}$$

It remains to estimate the third term on the right-hand side of (4.17)

$$I = \sup_{x_0\in\mathbb{R}^3}\int\limits_{-1}^{t_0}\int\limits_{B(x_0,3/2)} |\overline{p}_{x_0}||\overline{u}|dxdt \leq I_1 + I_2, \tag{4.19}$$

where

$$I_1 = \sup_{x_0 \in \mathbb{R}^3} \int\limits_{-1}^{t_0} \int\limits_{B(x_0,3/2)} |\overline{p}^1_{x_0}||\overline{u}| dxdt \leq I_1' + I_1'',$$

$$I_2 = \sup_{x_0 \in \mathbb{R}^3} \int\limits_{-1}^{t_0} \int\limits_{B(x_0,3/2)} |\overline{p}^2_{x_0}||\overline{u}| dxdt,$$

and

$$I_1' = c \sup_{x_0 \in \mathbb{R}^3} \int\limits_{-1}^{t_0} \int\limits_{B(x_0,3/2)} \Big||w|^2 - |u|^2\Big| |\overline{u}| dxdt,$$

$$I_1'' = c \sup_{x_0 \in \mathbb{R}^3} \int\limits_{-1}^{t_0} \int\limits_{B(x_0,3/2)} \Big| \int\limits_{B(x_0,2)} K(x-y) : \Big(w(y,t) \otimes w(y,t) - u(y,t) \otimes u(y,t)\Big) dy \Big| |\overline{u}(x,t)| dxdt.$$

I_1' is evaluated easily, namely,

$$I_1' \leq c \sup_{x_0 \in \mathbb{R}^3} \int\limits_{-1}^{t_0} \int\limits_{B(x_0,3/2)} |\overline{u}|^2 (|\overline{u}| + 2|u|) dxdt \leq c\gamma_{\overline{u}}(t_0) + \frac{c}{\sqrt{-t_0}} \int\limits_{-1}^{t_0} \alpha_{\overline{u}}(t) dt. \quad (4.20)$$

To estimate I_1'', we exploit the same idea and $L_{\frac{3}{2}}$- and L_2-estimates for singular integrals

$$\begin{aligned} I_1'' &\leq c \sup_{x_0 \in \mathbb{R}^3} \int\limits_{-1}^{t_0} \int\limits_{B(x_0,3/2)} |\overline{u}(x,t)| \Big\{ \Big| \int\limits_{B(x_0,2)} K(x-y) : \overline{u}(y,t) \otimes \overline{u}(y,t) dy \Big| \\ &\quad + \Big| \int\limits_{B(x_0,2)} K(x-y) : \Big(\overline{u}(y,t) \otimes u(y,t) + u(y,t) \otimes \overline{u}(y,t)\Big) dy \Big| \Big\} dxdt \\ &\leq c\gamma_{\overline{u}}(t_0) + \frac{c}{\sqrt{-t_0}} \int\limits_{-1}^{t_0} \alpha_{\overline{u}}(t) dt. \end{aligned}$$

So, by (4.20), we have

$$I_1 \leq c\gamma_{\overline{u}}(t_0) + \frac{c}{\sqrt{-t_0}} \int\limits_{-1}^{t_0} \alpha_{\overline{u}}(t) dt. \quad (4.21)$$

In order to find upper bound for I_2, we simply repeat arguments of Lemma 2.1 in [7] with $R=1$ there. This gives us the following estimate

$$
\begin{aligned}
|p^2_{x_0}(x,t)| \leq c \int\limits_{\mathbb{R}^3\setminus B(x_0,2)} & \Big|K(x-y)-K(x_0-y)\Big|\Big|w(y,t)\otimes w(y,t) \\
& - u(y,t)\otimes u(y,t)\Big|dx \\
\leq{} & c\|w(\cdot,t)\otimes w(\cdot,t)-u(\cdot,t)\otimes u(\cdot,t)\|_{L_{1,\mathrm{unif}}}
\end{aligned}
$$

being valid for any $x\in B(x_0,32/)$ and thus

$$
I_2 \leq c \sup_{x_0\in\mathbb{R}^3}\int\limits_{-1}^{t_0} \|w(\cdot,t)\otimes w(\cdot,t)-u(\cdot,t)\otimes u(\cdot,t)\|_{L_{1,\mathrm{unif}}} \int\limits_{B(x_0,3/2)} |\overline{u}(x,t)|dx.
$$

Furthermore, by (4.11),

$$
\begin{aligned}
&\|w(\cdot,t)\otimes w(\cdot,t)-u(\cdot,t)\otimes u(\cdot,t)\|_{L_{1,\mathrm{unif}}} \\
&\quad = \sup_{x_0\in\mathbb{R}^3}\int\limits_{B(x_0,1)} |\overline{u}(y,t)\otimes w(y,t)+u(y,t)\otimes\overline{u}(y,t)|dx \\
&\quad \leq \alpha^{\frac12}_{\overline{u}}(t)\alpha^{\frac12}_{w}(t)+\alpha^{\frac12}_{\overline{u}}(t)\alpha^{\frac12}_{u}(t) \leq \frac{C(A)}{\sqrt{-t}}\alpha^{\frac12}_{\overline{u}}(t).
\end{aligned}
$$

So,

$$
I_2 \leq \frac{C(A)}{\sqrt{-t_0}}\int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt
$$

and, by (4.18) and (4.21), we have

$$
\begin{aligned}
I &\leq c\gamma_{\overline{u}}(t_0)+\frac{C(A)}{\sqrt{-t_0}}\int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt \\
&\leq \frac{C(A)}{\sqrt{-t_0}}\Big[\Big(\int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt\Big)^{\frac14}\Big(\beta_{\overline{u}}(t_0)+\int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt\Big)^{\frac34}+\int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt\Big].
\end{aligned}
\tag{4.22}
$$

Combining (4.17) and (4.22) and applying Young inequality, we arrive at the final estimate

$$
\alpha_{\overline{u}}(t_0) \leq C(A,\delta)\int\limits_{-1}^{t_0}\alpha_{\overline{u}}(t)dt
$$

which is valid for all $-1\leq t_0\leq\delta<0$. The latter says that $\alpha_{\overline{u}}(t)=0$ in $[-1,0[$ and, hence, $u(\cdot,t)=w(\cdot,t)$ for the same t.

This completes the proof of Theorem 1.1. □

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Local Solvability of Free Boundary Problems for the Two-phase Navier-Stokes Equations with Surface Tension in the Whole Space

Senjo Shimizu

Dedicated to Professor Herbert Amann on the occasion of his 70th birthday.

Abstract. We consider the free boundary problem of the two-phase Navier-Stokes equation with surface tension and gravity in the whole space. We prove a local-in-time unique existence theorem in the space $W^{2,1}_{q,p}$ with $2 < p < \infty$ and $n < q < \infty$ for any initial data satisfying certain compatibility conditions. Our theorem is proved by the standard fixed point argument based on the maximal L_p-L_q regularity theorem for the corresponding linearized equations.

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Keywords. Navier-Stokes equations, free boundary problems, surface tension, gravity force, local solvability.

1. Introduction and results

In this paper, we show the results for a local-in-time existence theorem of the free boundary problems related to the motion of a viscous, incompressible fluid for the two-phase Navier-Stokes equations in the whole space $\mathbb{R}^n$ $(n \geq 2)$. The effect of surface tension on a free boundary and the effect of gravity force are included.

The free interface $\Gamma(t)$ is given only at initial time $t = 0$ as

$$\Gamma(0) = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n = \alpha(x'),\ x' \in \mathbb{R}^{n-1}\}$$

for $\alpha(x') \in W_q^{3-1/q}(\mathbb{R}^{n-1})$, while that at $t > 0$ remains to be determined. We set

$$\Omega(t) = \Omega_+(t) \cup \Omega_-(t) = \mathbb{R}^n \setminus \Gamma(t).$$

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The initial domain $\Omega(0) = \Omega_+(0) \cup \Omega_-(0)$ occupied by the fluid is defined by

$$\Omega_+(t) = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > \alpha(x'),\ x' \in \mathbb{R}^{n-1}\},$$
$$\Omega_-(t) = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n < \alpha(x'),\ x' \in \mathbb{R}^{n-1}\},$$

under the assumption $\|\nabla'\alpha\|_{L_\infty(\mathbb{R}^{n-1})} \le K_0$ for some K_0 with $0 < K_0 \le 1$. Such a domain is called a bent space.

Let the initial velocity v_0 and the initial domain $\Omega(0)$ be given. Our problem is to find the domain $\Omega_\pm(t)$, the velocity vector field $v(x,t) = {}^T(v_1, \ldots, v_n)$, and the scalar pressure $\theta(x,t)$, $x \in \Omega(t)$ satisfying the Navier-Stokes equations:

$$\begin{aligned}
&\rho(\partial_t v + (v\cdot\nabla)v) - \operatorname{Div} S(v,\theta) = 0 && \text{in } \Omega(t),\ t>0,\\
&\operatorname{div} v = 0 && \text{in } \Omega(t),\ t>0,\\
&[\![S(v,\theta)\nu_t]\!] = \sigma\mathcal{H}\nu_t + [\![\rho]\!]c_g x_n \nu_t && \text{on } \Gamma(t),\ t>0,\\
&V = v\cdot\nu_t && \text{on } \Gamma(t),\ t>0,\\
&v|_{t=0} = v_0 && \text{in } \Omega(0).
\end{aligned} \tag{1.1}$$

Here ρ and μ are given by

$$\rho = \begin{cases} \rho_+ & \text{in } \Omega_+(t) \\ \rho_- & \text{in } \Omega_-(t) \end{cases}, \qquad \mu = \begin{cases} \mu_+ & \text{in } \Omega_+(t) \\ \mu_- & \text{in } \Omega_-(t) \end{cases},$$

where positive constants $\rho_\pm$ and $\mu_\pm$ denote the densities and viscosities, respectively, which may have a jump across $\Gamma(t)$. For given functions $v_\pm(t)$ defined on $\Omega_\pm(t)$, we set

$$v(x,t) = \begin{cases} v_+(x,t) & x \in \Omega_+(t), \\ v_-(x,t) & x \in \Omega_-(t). \end{cases}$$

On the other hand, given function $v(t)$ defined on $\Omega(t)$, $v_\pm(t)$ denotes the restriction of $v(t)$ to $\Omega_\pm(t)$.

$$[\![v(t)]\!] = (v_+(t) - v_-(t))|_{\Gamma(t)}$$

denotes the jump of v on the interface. Positive constants σ and c_g denote the coefficient of surface tension and the acceleration of gravity, respectively. ν_t is the unit outward normal to $\Gamma(t)$ of $\Omega_+(t)$, $S(v,\theta) = \mu D(v) - \theta I$ is the stress tensor, $D(v) = (D_{ij}(v)) = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ is the deformation tensor, $\mathcal{H}$ is the mean curvature given by $\mathcal{H}\nu_t = \Delta_{\Gamma(t)}x$, where $\Delta_{\Gamma(t)}$ is the Laplace-Beltrami operator on $\Gamma(t)$, and V is the velocity of the evolution of $\Gamma(t)$ in the direction of ν_t. For differentiation, we use the symbol: $\operatorname{div} v = \sum_{j=1}^n \partial_j v_j$,

$$(v\cdot\nabla)v = {}^T\Big(\sum_{j=1}^n v_j\partial_j v_1, \ldots, \sum_{j=1}^n v_j\partial_j v_n\Big), \quad \operatorname{Div} S = {}^T\Big(\sum_{j=1}^n \partial_j S_{1j}, \ldots, \sum_{j=1}^n \partial_j S_{nj}\Big),$$

where $\partial_j = \partial/\partial x_j$, ${}^TM$ denotes the transposed M, and $S = (S_{ij})$ ($n\times n$ matrix).

Now, we shall discuss some known results of the unique existence theorem of problem (1.1). Let $W^{2,1}_{q,p}(\Omega\times(0,T)) = L_p((0,T), W^2_q(\Omega)) \cap W^1_p((0,T), L_q(\Omega))$, and for simplicity we write $W^{2,1}_{q,p} = W^{2,1}_{q,p}(\Omega\times(0,T))$ for some $T>0$ and $W^{2,1}_p =$

$W^{2,1}_{p,p}$. We state only the case for the incompressible Navier-Stokes equations for unbounded domain and the case where the surface tension is taken into account.

For one-phase ocean problems, Allain [2] proved local unique solvability when $n = 2$. Tani [25] proved the local unique solvability in $W_2^{\alpha,\frac{\alpha}{2}}$ with $\alpha \in (5/2, 3)$ when $n = 3$. Beale [5] proved the global unique solvability in $W_2^{\alpha,\frac{\alpha}{2}}$ with $\alpha \in (3, 7/2)$ when $n = 3$, provided that the initial data are sufficiently small. Beale and Nishida [6] obtained the asymptotic power-like in time decay of the global solutions. Tani and Tanaka [26] proved the global solvability in $W_2^{\alpha,\frac{\alpha}{2}}$ with $\alpha \in (5/2, 3)$ when $n = 3$, provided that initial data are sufficiently small.

For two-phase problems, Denisova [9, 10] proved the local unique solvability for arbitrary initial data in $W_2^{\alpha,\frac{\alpha}{2}}$ with $\alpha \in (5/2, 3)$ when $n = 3$. Denisova and Solonnikov [11, 12] proved the local unique solvability for arbitrary initial data in the Hölder spaces when $n = 3$. Tanaka [27] obtained the global unique solvability in $W_2^{\alpha,\frac{\alpha}{2}}$ with $\alpha \in (5/2, 3)$ for small initial data when $n = 3$. Abels [1] considered varifold and measure-valued varifold solutions for singular free interfaces. All of these results except for [1] were obtained in the L_2 framework or in the Hölder space setting.

Prüss and Simonett [17, 18, 19] proved local unique solvability in $W^{2,1}_p$ ($p > n+2$) for two-phase free boundary problems with surface tension and gravity. They proved the maximal regularity theorem relying on the $\mathcal{H}^\infty$-calculus for linearized problems and using the Hanzawa transform connecting a free boundary with a fixed boundary.

In this paper, we prove a local unique existence theorem of (1.1) in the space $W^{2,1}_{q,p}$ ($2 < p < \infty$ and $n < q < \infty$) for any initial data that satisfy certain compatibility conditions and for an initial domain as a bent space. Our approach is completely different from Prüss and Simonett [17, 18, 19]; indeed, we prove a maximal regularity theorem relying on the operator-valued Fourier multiplier theorem from Weis [28] and Denk, Hieber and Prüss [13] for linearized problems and using the Lagrangean transform connecting a free boundary with a fixed boundary. We believe that solving the problem (1.1) in the space $W^{2,1}_{q,p}$ is important not only from the viewpoint of a lower regularity condition for the initial data but also from the viewpoint of the scaling argument. When we consider the scaling for positive parameter λ,

$$v(x,t) \to v_\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t),$$
$$\theta(x,t) \to \theta_\lambda(x,t) = \lambda^2 \theta(\lambda x, \lambda^2 t),$$

which maintains the problem (1.1) invariant. Since it holds that

$$\|v_\lambda\|_{L_p((0,\infty),L_q(\mathbb{R}^n))} = \lambda^{1-\frac{2}{p}-\frac{n}{q}} \|v\|_{L_p((0,\infty),L_q(\mathbb{R}^n))},$$
$$\frac{2}{p} + \frac{n}{q} = 1 \tag{1.2}$$

is the critical scale under the scaling.

Before stating our main results, we first discuss the formulation of the problem (1.1) by the Lagrange coordinate, instead of the Euler coordinate. Aside from the dynamical boundary condition, a further kinematic condition for $\Gamma(t)$ is satisfied that gives $\Gamma(t)$ as a set of points $x = x(\xi, t)$, $\xi \in \Gamma(0)$, where $x(\xi, t)$ is the solution of the Cauchy problem:

$$\frac{dx}{dt} = v(x, t), \quad x|_{t=0} = \xi. \tag{1.3}$$

This expresses the fact that the free surface $\Gamma(t)$ exists for all $t > 0$ of the same fluid particles, which do not leave it and are not incident on it from $\Omega(t)$.

From here, we write $\Omega = \Omega(0)$ and $\Gamma = \Gamma(0)$. The problem (1.1) can therefore be written as an initial boundary value problem in the given region Ω if we go over from Euler coordinates $x \in \Omega(t)$ to the Lagrange coordinates $\xi \in \Omega$ connected with x by (1.3). If a velocity vector field $u(\xi, t) = {}^T(u_1, \ldots, u_n)$ is known as a function of the Lagrange coordinates ξ, then this connection can be written in the form:

$$x = \xi + \int_0^t u(\xi, \tau)\, d\tau \equiv X_u(\xi, t). \tag{1.4}$$

Passing to the Lagrange coordinates in (1.1) and setting $\theta(X_u(\xi, t), t) = \pi(\xi, t)$, we obtain

$$\begin{aligned}
&\rho \partial_t u - \operatorname{Div} S(u, \pi) = \operatorname{Div} Q(u) + R(u)\nabla\pi && \text{in } \Omega,\ t > 0,\\
&\operatorname{div} u = E(u) = \operatorname{div}\tilde{E}(u) && \text{in } \Omega,\ t > 0,\\
&[\![(S(u, \pi) + Q(u))\nu_{tu}]\!] - \sigma\mathcal{H}\nu_{tu} - [\![\rho]\!] c_g X_{u,n}\nu_{tu} = 0 && \text{on } \Gamma,\ t > 0,\\
&[\![u]\!] = 0 && \text{on } \Gamma,\ t > 0,\\
&u|_{t=0} = u_0(\xi) && \text{in } \Omega,
\end{aligned} \tag{1.5}$$

where $u_0(\xi) = v_0(x)$ and $X_{u,n}$ stands for the n-th component of X_u. Moreover, ν_{tu} stands for the unit outer normal to $\Gamma(t)$ given by $\nu_{tu} = {}^T A^{-1}\nu / |{}^T A^{-1}\nu|$, where ν denotes the unit outer normal to Γ of Ω_+, namely

$$\nu = (\sqrt{\mathfrak{g}_\alpha})^{-1}(\nabla'\alpha, -1), \quad \mathfrak{g}_\alpha = 1 + |\nabla'\alpha|^2.$$

A is the matrix whose element $\{a_{jk}\}$ is the Jacobian of (1.4):

$$a_{jk} = \frac{\partial x_j}{\partial \xi_k} = \delta_{jk} + \int_0^t \frac{\partial u_j}{\partial \xi_k}\, d\tau,$$

and $Q(u)$, $R(\pi)$, $E(u)$ and $\tilde{E}(u)$ are nonlinear terms of the following form:

$$\begin{aligned}
Q(u) &= \mu V_1\left(\int_0^t \nabla u\, d\tau\right)\nabla u, \quad R(u) = V_2\left(\int_0^t \nabla u\, d\tau\right),\\
E(u) &= V_3\left(\int_0^t \nabla u\, d\tau\right)\nabla u, \quad \tilde{E}(u) = V_4\left(\int_0^t \nabla u\, d\tau\right)u,
\end{aligned} \tag{1.6}$$

with some polynomials $V_j(\cdot)$ of $\int_0^t \nabla u\, d\tau$, $j = 1, 2, 3, 4$, such as $V_j(0) = 0$ (cf. Appendix in [20]).

We discuss a local in time unique existence theorem for the problem (1.5) instead of the problem (1.1). In order to state our main results precisely, we introduce the function spaces. For the differentiations of scalar θ and n-vector of function $u = (u_1, \dots, u_n)$, we use the following symbol:

$$\nabla\theta = (\partial_1\theta, \dots, \partial_n\theta), \qquad \nabla^2\theta = (\partial_i\partial_j\theta \mid i, j = 1, \dots n),$$
$$\nabla u = (\partial_i u_j \mid i, j = 1, \dots, n), \quad \nabla^2 u = (\partial_i\partial_j u_k \mid i, j, k = 1, \dots, n).$$

Let $L_q(G)$ and $W_q^m(G)$ denote the usual Lebesgue space and Sobolev space on a given domain G, while $\|\cdot\|_{L_q(G)}$ and $\|\cdot\|_{W_q^m(G)}$ denote their norms, respectively. $v \in W_q^m(\dot{\mathbb{R}}^n)$ means that $v_\pm \in W_q^m(\mathbb{R}^n_\pm)$ while $\|v\|_{W_q^m(\dot{\mathbb{R}}^n)} = \|v\|_{W_q^m(\mathbb{R}^n_+)} + \|v\|_{W_q^m(\mathbb{R}^n_-)}$. Note that $v \in W_q^1(\mathbb{R}^n)$ is equivalent to $v \in W_q^1(\dot{\mathbb{R}}^n)$ and $[\![v]\!] = 0$. Given Banach space X with norm $\|\cdot\|_X$, X^n denotes the n-product space of X, while the norm of X^n is denoted by $\|\cdot\|_X$ for simplicity, that is

$$X^n = \{f = (f_1, \dots, f_n) \mid f_i \in X\}, \;\; \|f\|_X = \sum_{j=1}^n \|f_j\|_X \;\text{ for } f = (f_1, \dots, f_n) \in X^n.$$

Set

$$\hat{W}_q^1(G) = \{\theta \in L_{q,\mathrm{loc}}(G) \mid \nabla\theta \in L_q(G)^n\}.$$

Let $\hat{W}_q^{-1}(G)$ denote the dual space of $\hat{W}_{q'}^1(G)$, where $1/q + 1/q' = 1$. For $\theta \in \hat{W}_q^{-1}(G) \cap L_q(G)$, we have

$$\|\theta\|_{\hat{W}_q^{-1}(G)} = \sup\left\{\Big|\int_G \theta\varphi\,dx\Big| \mid \varphi \in \hat{W}_{q'}^1(G), \;\; \|\nabla\varphi\|_{L_{q'}(G)} = 1\right\}.$$

For $1 \le p \le \infty$, $L_p(\mathbb{R}, X)$ and $W_p^m(\mathbb{R}, X)$ denote the usual Lebesgue space and Sobolev space of X-valued functions defined on the whole line $\mathbb{R}$, and $\|\cdot\|_{L_p(\mathbb{R},X)}$ and $\|\cdot\|_{W_p^m(\mathbb{R},X)}$ denote their norms, respectively. Set

$$\begin{aligned}
L_{p,\gamma_0}(\mathbb{R}, X) &= \{f : \mathbb{R} \to X \mid e^{-\gamma t} f(t) \in L_p(\mathbb{R}, X) \;\text{ for any } \gamma \ge \gamma_0\},\\
L_{p,\gamma_0,(0)}(\mathbb{R}, X) &= \{f \in L_{p,\gamma_0}(\mathbb{R}, X) \mid f(t) = 0 \;\text{ for } t < 0\},\\
W^m_{p,\gamma_0}(\mathbb{R}, X) &= \{f \in L_{p,\gamma_0}(\mathbb{R}, X) \mid e^{-\gamma t}\partial_t^j f(t) \in L_p(\mathbb{R}, X)\\
&\qquad\qquad\qquad\qquad \text{for } j = 1, \dots, m \text{ and } \gamma \ge \gamma_0\},\\
W^m_{p,\gamma_0,(0)}(\mathbb{R}, X) &= W^m_{p,\gamma_0}(\mathbb{R}, X) \cap L_{p,\gamma_0,(0)}(\mathbb{R}, X),\\
L_{p,(0)}(\mathbb{R}, X) &= L_{p,0,(0)}(\mathbb{R}, X), \quad W^m_{p,(0)}(\mathbb{R}, X) = W^m_{p,0,(0)}(\mathbb{R}, X).
\end{aligned}$$

Let $\mathcal{L}$ and $\mathcal{L}_\lambda^{-1}$ denote the Laplace transform and its inverse transform, that is

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^\infty e^{-\lambda t} f(t)\,dt, \quad \mathcal{L}_\lambda^{-1}[g](t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{\lambda t} g(\lambda)\,d\tau, \tag{1.7}$$

where $\lambda = \gamma + i\tau$. Given $s \in \mathbb{R}$ and X-valued function $f(t)$, we set

$$\Lambda_\gamma^s f(t) = \mathcal{L}_\lambda^{-1}[|\lambda|^s \mathcal{L}[f](\lambda)](t).$$

We introduce the Bessel potential space of X-valued functions of order s as follows:

$$
\begin{aligned}
H^s_{p,\gamma_0}(\mathbb{R},X) &= \{f:\mathbb{R}\to X \mid e^{-\gamma t}\Lambda^s_\gamma f(t)\in L_p(\mathbb{R},X) \text{ for any } \gamma\ge\gamma_0\},\\
H^s_{p,\gamma_0,(0)}(\mathbb{R},X) &= \{f\in H^s_{p,\gamma_0}(\mathbb{R},X)\mid f(t)=0 \text{ for } t<0\},\\
H^s_{p,(0)}(\mathbb{R},X) &= H^s_{p,0,(0)}(\mathbb{R},X).
\end{aligned}
$$

If X is a UMD space and $1<p<\infty$, then replacing the Fourier multiplier theorem of S.G. Mihlin [15, 16] by that of J. Bourgain [7] in the paper due to A.P. Calderón [8] about the Bessel potential space, we see that $H^s_{p,\gamma}(\mathbb{R},X)$ is continuously imbedded into $H^r_{p,\gamma}(\mathbb{R},X)$ when $s>r\ge 0$ and $H^s_{p,\gamma}(\mathbb{R},X)=W^s_{p,\gamma}(\mathbb{R},X)$ when s is a non-negative integer. Moreover, the Sobolev imbedding theorem holds, that is if $1<p<q<\infty$, $s>r\ge0$ and $s-r=1/p-1/q$, then $H^s_{p,\gamma}(\mathbb{R},X)$ is continuously imbedded into $H^r_{q,\gamma}(\mathbb{R},X)$, and if $0<s-1/p<1$, then every function in $H^s_{p,\gamma}$ coincides almost everywhere with a Lipschitz continuous function of order $s-1/p$. For notational simplicity, we write

$$
\begin{aligned}
W^{2,1}_{q,p,\gamma_0}(G\times\mathbb{R}) &= L_{p,\gamma_0}(\mathbb{R},W^2_q(G))\cap W^1_{p,\gamma_0}(\mathbb{R},L_q(G)),\\
W^{2,1}_{q,p,\gamma_0,(0)}(G\times\mathbb{R}) &= L_{p,\gamma_0,(0)}(\mathbb{R},W^2_q(G))\cap W^1_{p,\gamma_0,(0)}(\mathbb{R},L_q(G)),\\
H^{1,1/2}_{q,p,\gamma_0}(G\times\mathbb{R}) &= L_{p,\gamma_0}(\mathbb{R},W^1_q(G))\cap H^{\frac12}_{p,\gamma_0}(\mathbb{R},L_q(G)),\\
H^{1,1/2}_{q,p,\gamma_0,(0)}(G\times\mathbb{R}) &= L_{p,\gamma_0,(0)}(\mathbb{R},W^1_q(G))\cap H^{\frac12}_{p,\gamma_0,(0)}(\mathbb{R},L_q(G)),\\
W^{2,1}_{q,p,(0)}(G\times\mathbb{R}) &= W^{2,1}_{q,p,0,(0)}(G\times\mathbb{R}),\quad H^{1,1/2}_{q,p,(0)}(G\times\mathbb{R}) = H^{1,1/2}_{q,p,0,(0)}(G\times\mathbb{R}).
\end{aligned}
$$

The Bessel potential space $H^{\frac12}_{p,\gamma_0}(\mathbb{R},W^1_q(G))$ is obtained by complex interpolation:

$$
H^{\frac12}_{p,\gamma_0}(\mathbb{R},W^1_q(G)) = [L_{p,\gamma_0}(\mathbb{R},W^2_q(G)),W^1_{p,\gamma_0}(\mathbb{R},L_q(G))]_{1/2}
$$

where $[\cdot,\cdot]_s$ denotes the complex interpolation functor (cf. Shibata-Shimizu [21]), and therefore,

$$
H^{\frac12}_{p,\gamma_0}(\mathbb{R},W^1_q(G)) \supset W^{2,1}_{q,p,\gamma_0}(G\times\mathbb{R}). \tag{1.8}
$$

The following theorem is the main result in this paper.

Theorem 1.1. *Let $\alpha\in W^{3-1/q}_q(\mathbb{R}^{n-1})$, $2<p<\infty$ and $n<q<\infty$. Assume that there exists constant K_0 with $0<K_0\le1$ such that $\|\nabla'\alpha\|_{L_\infty(\mathbb{R}^{n-1})}\le K_0$. Then for any initial data $u_0\in(L_q(\Omega),W^2_q(\Omega))_{1-1/p,p}=B^{2(1-1/p)}_{q,p}(\Omega)$ which satisfies the compatibility conditions:*

$$
\operatorname{div}u_0=0 \quad \text{in } \Omega,\quad [\![D(u_0)\nu-(D(u_0)\nu,\nu)\nu]\!]=0 \quad \text{on } \Gamma,\quad [\![u_0]\!]=0 \quad \text{on } \Gamma, \tag{1.9}
$$

there exists $T>0$ depending on $\|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}$ and $\|\alpha\|_{W^{3-1/q}_q(\mathbb{R}^{n-1})}$ such that the problem (1.5) admits a unique solution

$$
(u,\pi)\in W^{2,1}_{q,p}(\Omega\times(0,T))\times L_p((0,T),\hat W^1_q(\Omega)).
$$

Here, $(\cdot,\cdot)_{1-1/p,p}$ denotes the real interpolation functor.

To prove Theorem 1.1, we use the contraction mapping principle based on the L_p-L_q maximal regularity theorem of the following linearized problem:

$$\begin{aligned}
&\rho\partial_t u - \operatorname{Div} S(u,\pi) = \rho f && \text{in } \Omega,\ t>0,\\
&\operatorname{div} u = f_d && \text{in } \Omega,\ t>0,\\
&\partial_t\eta - \nu\cdot u = d && \text{on } \Gamma,\ t>0,\\
&[\![S(u,\pi)\nu]\!] - (\sigma\Delta_\Gamma + [\![\rho]\!]c_g)\eta\,\nu = [\![h]\!] && \text{on } \Gamma,\ t>0,\\
&[\![u]\!] = 0 && \text{on } \Gamma,\ t>0,\\
&u|_{t=0} = 0 && \text{in } \Omega,\\
&\eta|_{t=0} = 0 && \text{on } \Gamma.
\end{aligned} \tag{1.10}$$

Theorem 1.2. *Let $1<p,q<\infty$, $n<r<\infty$ and $q\le r$. Assume that $\|\alpha\|_{W_r^{3-1/r}(\mathbb{R}^{n-1})}\le M$. Then there exist K_0 with $0<K_0\le 1$ and $\gamma_0>1$ depending on M, p, q and n such that if $\|\nabla'\alpha\|_{L_\infty(\mathbb{R}^{n-1})}\le K_0$, then the following assertion holds: For f, f_d, d and h of* (1.10) *satisfying the conditions*

$$\begin{gathered}
f\in L_{p,\gamma_0,(0)}(\mathbb{R},L_q(\Omega))^n,\quad f_d\in L_{p,\gamma_0,(0)}(\mathbb{R},W_q^1(\Omega))\cap W^1_{p,\gamma_0,(0)}(\mathbb{R},L_q(\Omega)),\\
d\in L_{p,\gamma_0,(0)}(\mathbb{R},W_q^{2-1/q}(\mathbb{R}^{n-1})),\quad h\in H^{1,1/2}_{q,p,\gamma_0,(0)}(\Omega\times\mathbb{R})^n,
\end{gathered}$$

the problem (1.10) *admits a unique solution (u,π,η) such that*

$$\begin{gathered}
u\in W^{2,1}_{q,p,\gamma_0,(0)}(\Omega\times\mathbb{R})^n,\quad \pi\in L_{p,\gamma_0,(0)}(\mathbb{R},\hat W_q^1(\Omega)),\\
\eta\in W^1_{p,\gamma_0,(0)}(\mathbb{R},W_q^{2-1/q}(\mathbb{R}^{n-1}))\cap L_{p,\gamma_0,(0)}(\mathbb{R},W_q^{3-1/q}(\mathbb{R}^{n-1})).
\end{gathered}$$

Moreover there exists an extension of the pressure $\bar\pi$ such that $[\![\bar\pi]\!]=[\![\pi]\!]$ and $\bar\pi\in H^{1,1/2}_{q,p,\gamma_0,(0)}(\Omega\times\mathbb{R})$. The solution satisfies the estimate

$$\begin{aligned}
&\|e^{-\gamma t}u\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R})}+\|e^{-\gamma t}\nabla\pi\|_{L_p(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}\bar\pi\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}\\
&\qquad\qquad+\|e^{-\gamma t}(\partial_t\eta,\nabla'\eta)\|_{L_p(\mathbb{R},W_q^{2-1/q}(\mathbb{R}^{n-1}))}\\
&\le C\big(\,\|e^{-\gamma t}f\|_{L_p(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}d\|_{L_p(\mathbb{R},W_q^{2-1/q}(\mathbb{R}^{n-1}))}\\
&\qquad+\|e^{-\gamma t}f_d\|_{L_p(\mathbb{R},W_q^1(\Omega))}+\|e^{-\gamma t}\partial_t\bar f_d\|_{L_p(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}h\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}\big)
\end{aligned}$$

for all $\gamma\ge\gamma_0$, where positive constant C depends on M, γ_0, p, q, and n.

In §2 and §3 we prove Theorem 1.2. In §2 we consider the Neumann problem in a bent space and in §3 consider a problem with surface tension and gravity in a bent space. In §4, we reduce the boundary condition to a linearized problem. §5 is devoted to initial flow. In §6, we solve the nonlinear problem by the contraction mapping principle based on Theorem 1.2.

2. Analysis in a bent space for the Neumann problem

In this section, we consider the Neumann problem in a bent space:

$$\begin{aligned}
&\rho\partial_t u - \operatorname{Div} S(u,\pi) = \rho f && \text{in } \Omega, \quad t>0,\\
&\operatorname{div} u = f_d && \text{in } \Omega, \quad t>0,\\
&[\![S(u,\pi)\nu]\!] = [\![h]\!] && \text{on } \Gamma, \quad t>0,\\
&[\![u]\!] = 0 && \text{on } \Gamma, \quad t>0,\\
&u|_{t=0} = 0 && \text{in } \Omega.
\end{aligned} \tag{2.1}$$

We set

$$\begin{aligned}
\mathcal{I}_{p,q,\gamma_0}(\Omega\times\mathbb{R}) &= \{(f,f_d,h) \in L_{p,\gamma_0,(0)}(\mathbb{R},L_q(\Omega))^n \times (L_{p,\gamma_0,(0)}(\mathbb{R},W_q^1(\Omega))\\
&\qquad \cap W^1_{p,\gamma_0,(0)}(\mathbb{R},\hat W_q^{-1}(\mathbb{R}^n))) \times H^{1,1/2}_{q,p,\gamma_0,(0)}(\Omega\times\mathbb{R})^n\},\\
\mathcal{D}_{p,q,\gamma_0}(\Omega\times\mathbb{R}) &= \{(u,\pi,\bar\pi) \in W^{2,1}_{q,p,\gamma_0,(0)}(\Omega\times\mathbb{R})^n \times L_{p,\gamma_0,(0)}(\mathbb{R},\hat W_q^1(\Omega))\\
&\qquad \times H^{1,1/2}_{q,p,\gamma_0,(0)}(\Omega\times\mathbb{R}) \mid [\![\bar\pi]\!] = [\![\pi]\!]\}.
\end{aligned}$$

The following theorem is the main result in this section.

Theorem 2.1. *Let $1<p,q<\infty$, $n<r<\infty$ and $q\le r$. Assume that $\|\alpha\|_{W_r^{2-1/r}(\mathbb{R}^{n-1})} \le M$. Then there exist constants K_0 with $0<K_0\le 1$ and $\gamma_0>0$ depending on M, p, q and n such that if $\|\nabla'\alpha\|_{L_\infty(\mathbb{R}^{n-1})} \le K_0$, then the following assertion holds: For $(f,f_d,h) \in \mathcal{I}_{p,q,\gamma_0}(\Omega\times\mathbb{R})$, the problem* (2.1) *admits a unique solution $(u,\pi,\bar\pi) \in \mathcal{D}_{p,q,\gamma_0}(\Omega\times\mathbb{R})$ satisfying an estimate:*

$$\begin{aligned}
&\|e^{-\gamma t}u\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R})} + \|e^{-\gamma t}\nabla\pi\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\bar\pi\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}\\
&\le C\big(\|e^{-\gamma t}f\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}f_d\|_{L_p(\mathbb{R},W_q^1(\Omega))}\\
&\qquad + \|e^{-\gamma t}(\partial_t f_d,\gamma f_d)\|_{L_p(\mathbb{R},\hat W_q^{-1}(\mathbb{R}^n))} + \|e^{-\gamma t}h\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}\big)
\end{aligned}$$

for any $\gamma\ge\gamma_0$, where C is a positive constant depending on M, p, q and n.

Let

$$\begin{aligned}
\dot{\mathbb{R}}^n &= \mathbb{R}^n \setminus \mathbb{R}^n_0 = \mathbb{R}^n_+ \cup \mathbb{R}^n_-,\\
\mathbb{R}^n_0 &= \{x=(x',x_n)\in\mathbb{R}^n \mid x_n=0,\ x'\in\mathbb{R}^{n-1}\},\\
\mathbb{R}^n_\pm &= \{x=(x',x_n)\in\mathbb{R}^n \mid \pm x_n>0,\ x'\in\mathbb{R}^{n-1}\}.
\end{aligned}$$

In order to prove Theorem 2.1, we based our argument on the maximal L_p-L_q regularity result of the Neumann problem with the plainer interface $\mathbb{R}^n_0$:

$$\begin{aligned}
&\rho\partial_t u - \operatorname{Div} S(u,p) = \rho f, && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\operatorname{div} u = f_d && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&[\![S(u,p)\nu_0]\!] = [\![h]\!] && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![u]\!] = 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&u|_{t=0} = 0 && \text{in } \dot{\mathbb{R}}^n,
\end{aligned} \tag{2.2}$$

where $\nu_0 = {}^T(0,\dots,0,-1)$. We set

$$\begin{aligned}\mathcal{I}_{p,q}(\dot{\mathbb{R}}^n\times\mathbb{R}) = \{(f,f_d,h) \in L_{p,(0)}(\mathbb{R},L_q(\dot{\mathbb{R}}^n))^n \times (L_{p,(0)}(\mathbb{R},W_q^1(\dot{\mathbb{R}}^n))\\ \cap W^1_{p,(0)}(\mathbb{R},\hat{W}_q^{-1}(\mathbb{R}^n))) \times H^{1,1/2}_{q,p,(0)}(\dot{\mathbb{R}}^n\times\mathbb{R})^n\},\\ \mathcal{D}_{p,q}(\dot{\mathbb{R}}^n\times\mathbb{R}) = \{(u,\pi,\bar\pi) \in W^{2,1}_{q,p,(0)}(\dot{\mathbb{R}}^n\times\mathbb{R})^n \times L_{p,(0)}(\mathbb{R},\hat{W}_q^1(\dot{\mathbb{R}}^n))\\ \times H^{1,1/2}_{q,p,(0)}(\dot{\mathbb{R}}^n\times\mathbb{R}) \mid [\![\bar\pi]\!] = [\![\pi]\!]\}.\end{aligned}$$

We obtained the following result (cf. Theorem 1.2 in [23]).

Theorem 2.2. *Let $1<p,q<\infty$. For $(f,f_d,h)\in\mathcal{I}_{p,q}(\dot{\mathbb{R}}^n\times\mathbb{R})$, the problem* (2.2) *admits a unique solution $(u,\pi,\bar\pi)\in\mathcal{D}_{p,q}(\dot{\mathbb{R}}^n\times\mathbb{R})$ satisfying the estimate:*

$$\begin{aligned}&\|e^{-\gamma t}u\|_{W^{2,1}_{q,p}(\dot{\mathbb{R}}^n\times\mathbb{R})} + \|e^{-\gamma t}\nabla\pi\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\\ &\qquad + \|e^{-\gamma t}\bar\pi\|_{H^{1,1/2}_{q,p}(\dot{\mathbb{R}}^n\times\mathbb{R})} + \|\gamma e^{-\gamma t}u\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\\ \le& C(\|e^{-\gamma t}f\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}f_d\|_{L_p(\mathbb{R},W^1_q(\dot{\mathbb{R}}^n))}\\ &\quad + \|e^{-\gamma t}(\partial_t f_d,\gamma f_d)\|_{L_p(\mathbb{R},\hat{W}_q^{-1}(\mathbb{R}^n))} + \|e^{-\gamma t}h\|_{H^{1,1/2}_{q,p}(\dot{\mathbb{R}}^n\times\mathbb{R})})\end{aligned}$$

for any $\gamma\ge\gamma_0$, where C is a positive constant depending on p, q and n.

For a function $\alpha(x')$ defined on $x'\in\mathbb{R}^{n-1}$, we have an extension $A_\pm(x)$.

Lemma 2.3. *Let $n<r<\infty$ and $0<\epsilon<1$. For $\alpha\in W_r^{2-1/r}(\mathbb{R}^{n-1})$, there exists $A_\pm(x)\in W_r^2(\mathbb{R}^n_\pm)$ which satisfies $A_\pm|_{x_n=\pm0}=\alpha$, $\|\partial_nA_\pm\|_{L_\infty(\mathbb{R}^n_\pm)}\le\epsilon$ and*

$$\begin{aligned}&\|A_\pm\|_{L_r(\mathbb{R}^n_\pm)} \le C\|\alpha\|_{L_r(\mathbb{R}^{n-1})},\\ &\|\nabla^kA_\pm\|_{L_r(\mathbb{R}^n_\pm)} \le C\|\alpha\|_{W_r^{k-1/r}(\mathbb{R}^{n-1})}, \quad k=1,2.\end{aligned} \tag{2.3}$$

Proof. For $\pm x_n>0$, if we set

$$A_\pm(x) = 2\mathcal{F}_{\xi'}[e^{\mp\sqrt{1+|\xi'|^2}x_n}\hat\alpha(\xi')](x') - \mathcal{F}_{\xi'}[e^{\mp2\sqrt{1+|\xi'|^2}x_n}\hat\alpha(\xi')](x'),$$

then $A_\pm$ satisfies $A_\pm|_{x_n=\pm0}=\alpha$, $\partial_nA_\pm|_{x_n=\pm0}=0$. For $A_\pm(x',sx_n)$, since we can take $s>0$ so small as

$$|\partial_nA_\pm(x',sx_n)| \le s\|\partial_nA_\pm\|_{L_\infty(\mathbb{R}^n_\pm)} \le s\|A_\pm\|_{W_r^2(\mathbb{R}^n_\pm)} \le \epsilon,$$

it holds that $\|\partial_nA_\pm\|_{L_\infty(\mathbb{R}^n_\pm)}\le\epsilon$. For the estimate (2.3), see, e.g., §2, Theorem 8.2 in [14]. □

In what follows, we prove Theorem 2.1. For $A_\pm(y)$ defined in Lemma 2.3, we set $A(y)=A_+(y)$, $y_n>0$, $A(y)=A_-(y)$, $y_n<0$. Let us consider the map $x\in\Omega\to\dot{\mathbb{R}}^n\ni y$ defined by the formula: $x={}^T(y',y_n+A(y))$ with $y'=(y_1,\dots,y_n)$. If we set $f(y_n)=y_n+A(\cdot,y_n)$, then by Lemma 2.3,

$$\partial_nf = 1+\partial_nA \ge 1-\|\partial_nA\|_{L_\infty(\dot{\mathbb{R}}^n)} \ge \frac{1}{2},$$

it follows that $f(y_n)$ is a strictly monotone increasing function. Therefore there exists an inverse function $y_n = f^{-1}(\cdot, x_n)$ which satisfies

$$y_n > 0 \quad \Rightarrow \quad x_n > A_+(y', 0) = \alpha(x'),$$
$$y_n < 0 \quad \Rightarrow \quad x_n < A_-(y', 0) = \alpha(x').$$

This shows that Ω and $\dot{\mathbb{R}}^n$ are homeomorphic.

By using

$$\frac{\partial y_n}{\partial x_j} = -\frac{\partial_j A}{1+\partial_n A}, \quad \frac{\partial y_n}{\partial x_n} = \frac{1}{1+\partial_n A} = 1 - \frac{\partial_n A}{1+\partial_n A},$$

by the chain rule we have

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} - \frac{\partial_j A}{1+\partial_n A}\frac{\partial}{\partial y_n}, \quad \frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n} - \frac{\partial_n A}{1+\partial_n A}\frac{\partial}{\partial y_n}. \tag{2.4}$$

We set $y = \Phi(x) = {}^T(x', x_n - A(x', f^{-1}(x)))$. For $(u, \pi, \bar{\pi}) \in \mathcal{D}_{p,q,\gamma_0}(\Omega \times \mathbb{R})$ which satisfy (2.1), we set

$$v(y) = u \circ \Phi^{-1}(y) = u(x), \quad \theta(y) = \pi \circ \Phi^{-1}(y) = \pi(x).$$

Then by (2.4), (2.1) is equivalent to the equation:

$$\begin{aligned}
&\rho\partial_t v - \operatorname{Div} S(v,\theta) = \rho\tilde{f} + F(v,\theta) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\operatorname{div} v = (1+\partial_n A)\tilde{f}_d + F_d(v) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&[\![S(v,\theta)\nu_0]\!] = [\![H^0 + H(v)]\!] && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![v]\!] = 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&v|_{t=0} = 0 && \text{in } \dot{\mathbb{R}}^n,
\end{aligned} \tag{2.5}$$

where $\tilde{f} = f \circ \Phi^{-1}$, $\tilde{f}_d = f_d \circ \Phi^{-1}$, and $F = {}^T(F_1, \ldots, F_n)$, F_d, $H^0 = {}^T(H^0_1, \ldots, H^0_n)$ and $H = {}^T(H_1, \ldots, H_n)$ are given by

$$\begin{aligned}
F_{\pm i}(v,\theta) &= -\mu_\pm \sum_{\ell=1}^n \partial_\ell \left(\frac{\partial_\ell A}{1+\partial_n A}\partial_n v_i + \frac{\partial_i A}{1+\partial_n A}\partial_n v_\ell\right)\\
&- \sum_{\ell=1}^n \frac{\partial_\ell A}{1+\partial_n A}\partial_n \left\{\mu_\pm\left(\partial_\ell v_i + \partial_i v_\ell - \frac{\partial_\ell A}{1+\partial_n A}\partial_n v_i - \frac{\partial_i A}{1+\partial_n A}\partial_n v_\ell\right) - \delta_{i\ell}\theta\right\},\\
&\qquad\qquad i = 1, \ldots, n,
\end{aligned}$$

$$\begin{aligned}
&F_d(v) = \sum_{j=1}^n \partial_n(\partial_j A v_j) - \sum_{j=1}^n \partial_j(\partial_n A v_j),\\
&H^0_{\pm i} = \sqrt{\mathfrak{g}_\alpha}(\tilde{h}_{\pm i} + \partial_n A \tilde{h}_{\pm n}), \quad i = 1, \ldots, n-1, \quad \tilde{h}_i = h_i \circ \Phi^{-1},\ \tilde{h}_n = h_n \circ \Phi^{-1},\\
&H^0_{\pm n} = \sqrt{\mathfrak{g}_\alpha}\tilde{h}_{\pm n},\\
&H_{\pm i}(v) = 2\mu_\pm \partial_i A \partial_n v_n - B_{\pm i}(v) - \partial_i A B_{\pm n}(v), \quad i = 1, \ldots, n-1,\\
&H_{\pm n}(v) = -B_{\pm n}(v),
\end{aligned}$$

$$
\begin{aligned}
B_{\pm i}(v) =& \mu_\pm\left(\frac{\partial_n A}{1+\partial_n A}\partial_n v_i + \frac{\partial_i A}{1+\partial_n A}\partial_i v_n\right) \\
&+ \mu_\pm \sum_{\ell=1}^{n-1} \partial_\ell A\left(\partial_\ell v_i + \partial_i v_\ell - \frac{\partial_\ell A}{1+\partial_n A}\partial_n v_i - \frac{\partial_i A}{1+\partial_n A}\partial_n v_\ell\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,\ldots,n-1, \\
B_{\pm n}(v) =& 2\mu_\pm \frac{\partial_n A}{1+\partial_n A}\partial_n v_n \\
&+ \mu_\pm \sum_{\ell=1}^{n-1} \partial_\ell A\left(\partial_\ell v_n + \partial_n v_\ell - \frac{\partial_\ell A}{1+\partial_n A}\partial_\ell v_n - \frac{\partial_n A}{1+\partial_n A}\partial_n v_\ell\right). \qquad (2.6)
\end{aligned}
$$

By using Theorem 2.2, we solve (2.5) by the contraction mapping principle. Given $(v, \theta, \bar\theta) \in \mathcal{D}_{p,q}(\dot{\mathbb{R}}^n \times \mathbb{R})$, let $(w, \kappa, \bar\kappa)$ be a solution to the equation:

$$
\begin{aligned}
&\rho\partial_t w - \mathrm{Div}\, S(w,\kappa) = \rho\tilde f + F(v,\theta) && \text{in } \dot{\mathbb{R}}^n,\ t>0 \\
&\mathrm{div}\, w = (1+\partial_n A)\tilde f_d + F_d(v) && \text{in } \dot{\mathbb{R}}^n,\ t>0 \\
&[\![S(w,\kappa)\nu_0]\!] = [\![H^0 + H(v)]\!] && \text{on } \mathbb{R}^n_0,\ t>0 \\
&[\![w]\!] = 0 && \text{on } \mathbb{R}^n_0,\ t>0. \\
&w|_{t=0} = 0 && \text{in } \mathbb{R}^n_0. \qquad (2.7)
\end{aligned}
$$

We remark that if $f_d \in W^1_{p,(0)}(\mathbb{R}, \hat W^{-1}_q(\mathbb{R}^n))$, then

$$(1+\partial_n A)\tilde f_d \in W^1_{p,(0)}(\mathbb{R}, \hat W^{-1}_q(\mathbb{R}^n)). \qquad (2.8)$$

Indeed, for $\varphi \in \hat W^1_{q'}(\Omega)$ we set $\psi(y) = \varphi\circ\Phi^{-1}(y) = \varphi(x)$. We have

$$\int_\Omega f_d(x)\varphi(x)\,dx = \int_{\dot{\mathbb{R}}^n} \tilde f_d(y)\psi(y)(1+\partial_n A)\,dy$$

because of the Jacobian $\det\frac{\partial x}{\partial y} = 1+\partial_n A$. Therefore it holds that

$$
\begin{aligned}
|((1+\partial_n A)\tilde f_d, \psi)_{\dot{\mathbb{R}}^n}| = |(f_d,\varphi)_\Omega| &\le \|f_d\|_{W^{-1}_q(\mathbb{R}^n)}\|\nabla\varphi\|_{L_{q'}(\mathbb{R}^n)} \\
&\le C\|f_d\|_{W^{-1}_q(\mathbb{R}^n)}\|\nabla\psi\|_{L_{q'}(\mathbb{R}^n)},
\end{aligned}
$$

which shows

$$\|(1+\partial_n A)\tilde f_d\|_{W^1_p(\mathbb{R},\hat W^{-1}_q(\mathbb{R}^n))} \le C\|f_d\|_{W^1_p(\mathbb{R},\hat W^{-1}_q(\mathbb{R}^n))}. \qquad (2.9)$$

Also we have

$$\|(1+\partial_n A)\tilde f_d\|_{L_p(\mathbb{R},W^1_q(\dot{\mathbb{R}}^n))} \le C\|f_d\|_{L_p(\mathbb{R},W^1_q(\Omega))}. \qquad (2.10)$$

For notational simplicity we set

$$K_1 = \|\nabla A\|_{L_\infty(\dot{\mathbb{R}}^n)}, \quad K_2 = \|A\|_{W^2_r(\dot{\mathbb{R}}^n)}.$$

We may assume that $0 < K_1 < 1$ a priori, and therefore $K_1^\ell \le K_1$ for $\ell \ge 1$. For $a \in L_r(\dot{\mathbb{R}}^n)$ and $b \in L_{p,(0)}(\mathbb{R}, W^1_q(\dot{\mathbb{R}}^n))$, we have

$$\|ab\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \le C\|a\|_{L_r(\dot{\mathbb{R}}^n)}\|b\|_{L_p(\mathbb{R},W^1_q(\dot{\mathbb{R}}^n))}. \qquad (2.11)$$

In fact, by the assumption $q \le r$, choosing s in such a way that $1/q = 1/r + 1/s$, noting that $n(1/q - 1/s) = n/r < 1$ and using the Hölder inequality and the Sobolev imbedding theorem we have

$$\|ab(t)\|_{L_q(\dot{\mathbb{R}}^n)} \le \|a\|_{L_r(\dot{\mathbb{R}}^n)} \|b(t)\|_{L_s(\dot{\mathbb{R}}^n)} \le C\|a\|_{L_r(\dot{\mathbb{R}}^n)} \|b(t)\|_{W_q^1(\dot{\mathbb{R}}^n)}.$$

Applying the Hölder inequality and the Sobolev imbedding theorem to $F(v,\theta)$ in (2.6) and using (2.11), we see that

$$\begin{aligned} &\|e^{-\gamma t} F(v,\theta)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \\ &\quad \le C_1 K_1 \|e^{-\gamma t}(\partial_t v, \nabla^2 v, \nabla\theta)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + C_2 K_2 \|e^{-\gamma t} v\|_{L_p(\mathbb{R},W_q^1(\dot{\mathbb{R}}^n))}. \end{aligned} \tag{2.12}$$

Here and hereafter, C_1 and C_2 denote generic constants independent of K_1 and K_2. By the interpolation inequality, it holds that

$$\|e^{-\gamma t}\nabla v\|_{L_q(\dot{\mathbb{R}}^n)} \le \epsilon \|e^{-\gamma t}\nabla^2 v\|_{L_q(\dot{\mathbb{R}}^n)} + \frac{1}{4\epsilon}\|e^{-\gamma t} v\|_{L_q(\dot{\mathbb{R}}^n)},$$

which implies

$$\|e^{-\gamma t}\nabla v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \le \epsilon \|e^{-\gamma t}\nabla^2 v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + \frac{1}{4\epsilon\gamma}\|\gamma e^{-\gamma t} v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}. \tag{2.13}$$

Combining (2.12) and (2.13), we have

$$\begin{aligned} &\|e^{-\gamma t} F(v,\theta)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \le C_1 K_1 \|e^{-\gamma t}(\partial_t v, \nabla^2 v, \nabla\theta)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \\ &\quad + C_2 K_2 \left(\epsilon \|e^{-\gamma t}\nabla^2 v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + \frac{C(\epsilon)+1}{\gamma}\|\gamma e^{-\gamma t} v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\right). \end{aligned} \tag{2.14}$$

The function $F_d(v)$ is also expressed by $F_d(v) = \operatorname{div} \tilde{F}_d(v)$. By (2.11) and (2.13) we obtain

$$\|e^{-\gamma t}\partial_t \tilde{F}_d(v)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \le C_1 K_1 \|e^{-\gamma t}\partial_t v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}, \tag{2.15}$$

$$\begin{aligned} &\|e^{-\gamma t}\nabla F_d(v)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \le C_1 K_1 \|e^{-\gamma t}\nabla^2 v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \\ &\quad + C_2 K_2 \left(\epsilon \|e^{-\gamma t} v\|_{L_p(\mathbb{R},W_q^1(\dot{\mathbb{R}}^n))} + \frac{C(\epsilon)+1}{\gamma}\|\gamma e^{-\gamma t} v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\right). \end{aligned} \tag{2.16}$$

Applying (2.11) to $B(v)$ in (2.6), we see that

$$\begin{aligned} &\|e^{-\gamma t}\langle D_t\rangle^{\frac12} B(v)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}\nabla B(v)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \\ &\quad \le C_1 K_1 \|e^{-\gamma t}(\partial_t v, \langle D_t\rangle^{\frac12}\nabla v, \nabla^2 v)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \\ &\qquad + C_2 K_2 \left(\|e^{-\gamma t}\langle D_t\rangle^{\frac12} v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t} v\|_{L_p(\mathbb{R},W_q^1(\dot{\mathbb{R}}^n))}\right). \end{aligned}$$

By using the relation (cf. Proposition 2.6 in [21])

$$\begin{aligned} &\|e^{-\gamma t}\langle D_t\rangle^{\frac12} u\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \\ &\qquad \le CR^{-\frac12}\|e^{-\gamma t}\partial_t u\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + CR^{\frac12}\|\gamma e^{-\gamma t} u\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \end{aligned} \tag{2.17}$$

for every $u \in H^{\frac{1}{2}}_{p,(0)}(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))$, $1 \le R < \infty$ and (2.13), we obtain

$$\begin{aligned}
&\|e^{-\gamma t}\langle D_t\rangle^{\frac{1}{2}}B(v)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}\nabla B(v)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\\
&\quad\le C_1K_1\|e^{-\gamma t}(\partial_t v, \langle D_t\rangle^{\frac{1}{2}}\nabla v, \nabla^2 v)\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\\
&\qquad + C_2K_2\Big(R^{-\frac{1}{2}}\|e^{-\gamma t}\partial_t v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + \epsilon\|e^{-\gamma t}\nabla^2 v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\\
&\qquad + \frac{C(\epsilon)+1}{\gamma}\|\gamma e^{-\gamma t}v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\Big).
\end{aligned} \tag{2.18}$$

Obviously we have

$$\|\langle D_t\rangle^{\frac{1}{2}}H^0\|_{L_q(\dot{\mathbb{R}}^n)} + \|\nabla H^0\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} \le C\|(\langle D_t\rangle^{\frac{1}{2}}\tilde{h}, \nabla\tilde{h})\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}. \tag{2.19}$$

For notational simplicity we set

$$\begin{aligned}
&[(v,\theta,\bar{\theta})]_{p,q,\gamma} = \|e^{-\gamma t}\, v\|_{W^{2,1}_{q,p}(\dot{\mathbb{R}}^n\times\mathbb{R})}\\
&\qquad + \|e^{-\gamma t}\,\nabla\theta\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}\,\bar{\theta}\|_{H^{1,1/2}_{q,p}(\dot{\mathbb{R}}^n\times\mathbb{R})} + \|\gamma e^{-\gamma t}\, v\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}
\end{aligned}$$

for $(v,\theta,\bar{\theta}) \in \mathcal{D}_{p,q}(\dot{\mathbb{R}}^n\times\mathbb{R})$ and for any $\gamma \ge 0$. Applying Theorem 2.2 to (2.7) and using (2.9), (2.10), (2.14)–(2.16), (2.18) and (2.19), we have

$$\begin{aligned}
&[(w,\kappa,\bar{\kappa})]_{p,q,\gamma}\\
&\le (C_1K_1 + C_2K_2\epsilon + C_2K_2R^{-\frac{1}{2}} + C_2K_2(R^{\frac{1}{2}} + C(\epsilon)+1)\gamma^{-1})[(v,\theta,\bar{\theta})]_{p,q,\gamma} + C\mathcal{I}_\gamma,
\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}
\mathcal{I}_\gamma &= \|e^{-\gamma t}\tilde{f}\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}(1+\partial_n A)\tilde{f}_d\|_{W^1_p(\mathbb{R},\hat{W}^{-1}_q(\mathbb{R}^n))}\\
&\quad + \|e^{-\gamma t}(1+\partial_n A)\tilde{f}_d\|_{L_p(\mathbb{R},W^1_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}(\langle D_t\rangle^{\frac{1}{2}}f_d, \nabla\tilde{h})\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}
\end{aligned}$$

for any $\gamma \ge 0$. We define the map Φ on $\mathcal{D}_{p,q}(\dot{\mathbb{R}}^n\times\mathbb{R})$ by $\Phi(v,\theta,\bar{\theta}) = (w,\kappa,\bar{\kappa})$. Since the equation (2.7) is linear, from (2.20) we have

$$\begin{aligned}
&[\Phi(v_1,\theta_1,\bar{\theta}_1) - \Phi(v_2,\theta_2,\bar{\theta}_2)]_{p,q,\gamma}\\
&\le (C_1K_1 + C_2K_2\epsilon + C_2K_2R^{-\frac{1}{2}} + C_2K_2(R^{\frac{1}{2}} + C(\epsilon)+1)\gamma_0^{-1})\\
&\quad\times [(v_1,\theta_1,\bar{\theta}_1) - (v_2,\theta_2,\bar{\theta}_2)]_{p,q,\gamma}
\end{aligned} \tag{2.21}$$

for any $\gamma > \gamma_0$. After choosing K_1, ϵ and R in such a way that

$$C_1K_1 \le 1/8, \quad C_2K_2\epsilon \le 1/8, \quad C_2K_2R^{-\frac{1}{2}} \le 1/8,$$

we take γ_0 in such a way that

$$C_2K_2(R^{\frac{1}{2}} + C(\epsilon)+1)\gamma_0^{-1} \le 1/8.$$

Then (2.21) shows that Φ is a contraction map on $\mathcal{D}_{p,q}(\dot{\mathbb{R}}^n\times\mathbb{R})$, and therefore by the fixed point theorem of S. Banach, we see that the map Φ admits a unique fixed point $(v, \theta, \bar\theta) \in \mathcal{D}_{p,q}(\dot{\mathbb{R}}^n \times \mathbb{R})$, which solves the equation (2.5). Moreover from (2.20) with $(w, \kappa, \bar\kappa) = (v, \theta, \bar\theta)$ and $C_1K_1 + C_2K_2\epsilon + C_2K_2R^{-\frac{1}{2}} + C_2K_2(R^{\frac{1}{2}} + C(\epsilon) + 1)\gamma_0^{-1} \le 1/2$, we have $[(v, \theta, \bar\theta)]_{p,q,\gamma} \le 2C\mathcal{I}_\gamma$ for $\gamma \ge \gamma_0$, from which Theorem 2.1 follows immediately.

3. Analysis in a bent space for a problem with surface tension and gravity

In this section we consider the problem with surface tension and gravity in a bent space:

$$
\begin{aligned}
&\rho\partial_t u - \operatorname{Div} S(u,\pi) = 0 && \text{in } \Omega,\ t > 0,\\
&\operatorname{div} u = 0 && \text{in } \Omega,\ t > 0,\\
&\partial_t \eta - \nu \cdot u = d && \text{on } \Gamma,\ t > 0,\\
&[\![S(u,\pi)\nu]\!] - (\sigma\Delta_\Gamma + [\![\rho]\!]c_g)\eta\,\nu = 0 && \text{on } \Gamma,\ t > 0,\\
&[\![u]\!] = 0 && \text{on } \Gamma,\ t > 0,\\
&u|_{t=0} = 0 \quad \text{in } \Omega, \quad \eta|_{t=0} = 0 \quad \text{on } \Gamma.
\end{aligned}
\tag{3.1}
$$

We set

$$
\begin{aligned}
\mathcal{E}_{p,q,\gamma_0}(\Omega \times \mathbb{R}) = \{ &(u, \pi, \bar\pi, \eta) \in W^{2,1}_{q,p,\gamma_0,(0)}(\Omega \times \mathbb{R})^n \times L_{p,\gamma_0,(0)}(\mathbb{R}, \hat W^1_q(\Omega))\\
&\times H^{1,1/2}_{q,p,\gamma_0,(0)}(\Omega \times \mathbb{R}) \times (W^1_{p,\gamma_0,(0)}(\mathbb{R}, W^{2-1/q}_q(\Gamma)) \cap L_{p,\gamma_0,(0)}(\mathbb{R}, W^{3-1/q}_q(\Gamma)))\\
&\mid [\![\bar\pi]\!] = [\![\pi]\!] \}.
\end{aligned}
$$

The following theorem is the main result in this section.

Theorem 3.1. *Let $1 < p, q < \infty$, $n < r < \infty$ and $q \le r$. Assume that $\|\alpha\|_{W^{3-1/r}_r(\mathbb{R}^{n-1})} \le M$. Then there exists constant K_0 with $0 < K_0 \le 1$ and $\gamma_0 > 1$ depending on M, p, q and n such that if $\|\nabla'\alpha\|_{L_\infty(\mathbb{R}^{n-1})} \le K_0$, then the following assertion holds: For $d \in L_{p,\gamma_0,(0)}(\mathbb{R}, W^{2-1/q}_q(\Gamma))$, the problem (3.1) admits a unique solution $(u, \pi, \bar\pi, \eta) \in \mathcal{E}_{p,q,\gamma_0}(\Omega \times \mathbb{R})$ satisfying the estimate:*

$$
\begin{aligned}
\|e^{-\gamma t} u\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R})} &+ \|e^{-\gamma t}\nabla\pi\|_{L_p(\mathbb{R}, L_q(\Omega))}\\
&+ \|e^{-\gamma t}\bar\pi\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}\\
&+ \|e^{-\gamma t}(\partial_t\eta, \nabla'\eta)\|_{L_p(\mathbb{R}, W^{2-1/q}_q(\mathbb{R}^{n-1}))}\\
&\le C\|e^{-\gamma t} d\|_{L_p(\mathbb{R}, W^{2-1/q}_q(\mathbb{R}^{n-1}))}
\end{aligned}
$$

for any $\gamma \ge \gamma_0$, where C is a constant depending on M, γ_0, p, q and n.

A proof of Theorem 1.2

Combining Theorem 2.1 and Theorem 3.1, we obtain Theorem 1.2.

In order to prove Theorem 3.1, we based our argument on the maximal L_p-L_q regularity result of the problem with surface tension and gravity with plainer interface in $\mathbb{R}^n$:

$$\begin{aligned}
&\rho\partial_t u - \operatorname{Div} S(u,\pi) = 0 && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\operatorname{div} u = 0 && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\partial_t\eta - \nu_0\cdot u = d && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![S(u,\pi)\nu_0]\!] - (\sigma\Delta' + [\![\rho]\!]c_g)\eta\,\nu_0 = [\![g]\!] && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![u]\!] = 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&u|_{t=0} = 0 \quad \text{in } \dot{\mathbb{R}}^n, \quad \eta|_{t=0} = 0 \quad \text{on } \mathbb{R}^{n-1},
\end{aligned}\tag{3.2}$$

where $\nu_0 = {}^T(0,\dots,0,-1)$. We set

$$\begin{aligned}
\mathcal{E}_{p,q,\gamma_0}(\dot{\mathbb{R}}^n\times\mathbb{R}) = \{&(u,\pi,\bar\pi,\eta)\in W^{2,1}_{q,p,\gamma_0,(0)}(\dot{\mathbb{R}}^n\times\mathbb{R})^n\times L_{p,\gamma_0,(0)}(\mathbb{R},\hat W^1_q(\dot{\mathbb{R}}^n))\\
&\times H^{1,1/2}_{q,p,\gamma_0,(0)}(\dot{\mathbb{R}}^n\times\mathbb{R})\times(W^1_p(\mathbb{R},W^{2-1/q}_q(\mathbb{R}^{n-1}))\cap L_p(\mathbb{R},W^{3-1/q}_q(\mathbb{R}^{n-1})))\\
&\mid [\![\bar\pi]\!] = [\![\pi]\!]\}.
\end{aligned}$$

We obtained the following result (cf. Theorem 1.4 in [23]).

Theorem 3.2. *Let $1<p,q<\infty$. There exists a constant $\gamma_0>1$ depending on p, q and n such that the following assertion holds: For $d\in L_{p,\gamma_0,(0)}(\mathbb{R},W^{2-1/q}_q(\mathbb{R}^{n-1}))$ and $g\in H^{1,1/2}_{q,p,\gamma_0(0)}(\dot{\mathbb{R}}^n\times\mathbb{R})^n$, the problem (3.2) admits a unique solution $(u,\pi,\bar\pi,\eta)\in\mathcal{E}_{p,q,\gamma_0}(\dot{\mathbb{R}}^n\times\mathbb{R})$ satisfying the estimate:*

$$\begin{aligned}
\|e^{-\gamma t}u\|_{W^{2,1}_{q,p}(\dot{\mathbb{R}}^n\times\mathbb{R})} &+ \|e^{-\gamma t}\nabla\pi\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\\
&+ \|e^{-\gamma t}\bar\pi\|_{H^{1,1/2}_{q,p}(\dot{\mathbb{R}}^n\times\mathbb{R})}\\
&+ \|\gamma e^{-\gamma t}u\|_{L_p(\mathbb{R},L_q(\dot{\mathbb{R}}^n))}\\
&+ \|e^{-\gamma t}(\partial_t\eta,\nabla'\eta)\|_{L_p(\mathbb{R},W^{2-1/q}_q(\mathbb{R}^{n-1}))}\\
\le C\big(&\|e^{-\gamma t}d\|_{L_p(\mathbb{R},W^{2-1/q}_q(\mathbb{R}^{n-1}))} + \|e^{-\gamma t}g\|_{H^{1,1/2}_{q,p}(\dot{\mathbb{R}}^n\times\mathbb{R})}\big)
\end{aligned}$$

for any $\gamma\ge\gamma_0$, where C is a constant depending on M, γ_0, p, q and n.

In what follows, we prove Theorem 3.1. We use the same notation as in §2. A Laplace-Beltrami operator on Γ is defined by

$$\Delta_\Gamma\eta = \frac{1}{\sqrt{\mathfrak{g}}}\sum_{j,k=1}^{n-1}\partial_j(\sqrt{\mathfrak{g}}\mathfrak{g}^{jk}\partial_k\eta),\ \mathfrak{g}^{jk} = \frac{(1+|\nabla'\alpha|^2)\delta_{jk} - \partial_j\alpha\partial_k\alpha}{\mathfrak{g}},\ \mathfrak{g} = 1+|\nabla'\alpha|^2.$$

We derive an equivalent equation to (3.1) in $\dot{\mathbb{R}}^n$. If we set

$$v(y) = u \circ \Phi^{-1}(y) = u(x), \quad \theta(y) = p \circ \Phi^{-1}(y) = p(x),$$

then by (2.4) we have

$$\begin{aligned}
&\rho\partial_t v - \operatorname{Div} S(v,\theta) = F(v,\theta) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\operatorname{div} v = F_d(v) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\partial_t\eta + v_n = d + D(v) && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![S(v,\theta)\nu_0]\!] - (\sigma\Delta' + [\![\rho]\!]c_g)\eta\,\nu_0 = [\![H(v)]\!] + T(\eta)\nu_0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![v]\!] = 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&v|_{t=0} = 0 \quad \text{in } \dot{\mathbb{R}}^n, \quad \eta|_{t=0} = 0 \quad \text{on } \mathbb{R}^{n-1}, &&
\end{aligned} \tag{3.3}$$

where $F(v,\theta)$, $F_d(v)$ and $H(v)$ are same as in (2.6), $D(v)$ and the nth component of $T(\eta)$ expressed by $T_n(\eta)$ are given by

$$\begin{aligned}
D(v) &= (\sqrt{\mathfrak{g}})^{-1}(\nabla'\alpha, \sqrt{1+|\nabla'\alpha|^2} - 1)\cdot v,\\
T_n(\eta) &= -\sigma\mathfrak{g}^{-1}|\nabla'\alpha|^2\Delta'\eta - \sigma\mathfrak{g}^{-2}\sum_{j,k=1}^{n-1}\partial_j\alpha\partial_k\alpha\partial_j\partial_k\eta.
\end{aligned} \tag{3.4}$$

We seek the solution of (3.3) by setting $v = w + z$ and $\theta = \kappa + \tau$ such that (w,κ) satisfies the Neumann problem:

$$\begin{aligned}
&\rho\partial_t w - \operatorname{Div} S(w,\kappa) = F(w,\kappa) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\operatorname{div} w = F_d(w) && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&[\![S(w,\kappa)\nu_0]\!] = [\![H(w)]\!] && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![w]\!] = 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&w|_{t=0} = 0 && \text{in } \dot{\mathbb{R}}^n,
\end{aligned} \tag{3.5}$$

and (z,τ,η) satisfies the equation:

$$\begin{aligned}
&\rho\partial_t z - \operatorname{Div} S(z,\tau) = 0 && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\operatorname{div} z = 0 && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\partial_t\eta + z_n = d + D(z) && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![S(z,\tau)\nu_0]\!] - (\sigma\Delta' + [\![\rho]\!]c_g)\eta\,\nu_0 = T(\eta)\nu_0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![z]\!] = 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&z|_{t=0} = 0 \quad \text{in } \dot{\mathbb{R}}^n, \quad \eta|_{t=0} = 0 \quad \text{in } \mathbb{R}^{n-1}. &&
\end{aligned} \tag{3.6}$$

Since we know the unique solvability of (3.5) because (3.5) is the same problem as (2.5), we solve (3.6) by the contradiction mapping principle based on Theorem 3.2. Given $(v,\theta,\bar{\theta},\eta) \in \mathcal{E}_{p,q,\gamma_0}(\dot{\mathbb{R}}^n\times\mathbb{R})$, let $(w,\kappa,\bar{\kappa},\zeta) \in \mathcal{E}_{p,q,\gamma_0}(\dot{\mathbb{R}}^n\times\mathbb{R})$ be a unique solution to the equation:

$$
\begin{aligned}
&\rho\partial_t w - \operatorname{Div} S(w,\kappa) = 0 && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\operatorname{div} w = 0 && \text{in } \dot{\mathbb{R}}^n,\ t>0,\\
&\partial_t\eta + w_n = d + D(v) && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![S(w,\kappa)\nu_0]\!] - (\sigma\Delta' + [\![\rho]\!]c_g)\zeta\,\nu_0 = T(\eta)\nu_0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&[\![w]\!] = 0 && \text{on } \mathbb{R}^n_0,\ t>0,\\
&w|_{t=0} = 0 \quad \text{in } \mathbb{R}^n_0, \quad \eta|_{t=0} = 0 \quad \text{in } \mathbb{R}^{n-1}.
\end{aligned}
\tag{3.7}
$$

The next lemma is proved in the same way as Lemma 2.3.

Lemma 3.3. *Let $1<r<\infty$, $0<\epsilon<1$ and $\alpha \in W^{3-1/r}_r(\mathbb{R}^{n-1})$. Then there exists $A_\pm(x)\in W^3_r(\mathbb{R}^n_\pm)$ which satisfies $A_\pm|_{x_n=\pm 0}=\alpha$, $\|\partial_n A_\pm\|_{L_\infty(\mathbb{R}^n_\pm)} \le \epsilon$ and*

$$
\begin{aligned}
&\|A_\pm\|_{L_r(\mathbb{R}^n_\pm)} \le C\|\alpha\|_{L_r(\mathbb{R}^{n-1})},\\
&\|\nabla^k A_\pm\|_{L_r(\mathbb{R}^n_\pm)} \le C\|\alpha\|_{W^{k-1/r}_r(\mathbb{R}^{n-1})}, \quad k=1,2,3.
\end{aligned}
\tag{3.8}
$$

Let $\phi(x_n)$ be a function in $C^\infty(\mathbb{R})$ such that $\phi(x_n)=1$ when $x_n<1$ and $\phi(x_n)=0$ when $x_n>2$. We extend η to the function $Y_\pm$ defined on $\mathbb{R}^n_\pm$ by using the formula

$$
\begin{aligned}
&Y_+(x,t) = \phi(x_n)\mathcal{L}^{-1}_\lambda\mathcal{F}^{-1}_{\xi'}[e^{-(1+|\xi'|^2)^{1/2}x_n}\mathcal{L}\mathcal{F}_{\xi'}[\eta](\xi',\lambda)](x',t) \quad && x_n>0,\\
&Y_-(x,t) = 0 && x_n<0,
\end{aligned}
$$

where $\mathcal{F}_{x'}$ and $\mathcal{F}^{-1}_{\xi'}$ denote the Fourier transform and its inversion transform with respect to x', and $\mathcal{L}$ and $\mathcal{L}^{-1}_\lambda$ denote the Laplace transform and its inversion transform defined by (1.7). We know that

$$
[\![Y]\!] = \eta, \quad \|Y\|_{W^\ell_q(\dot{\mathbb{R}}^n)} \le C\|\eta\|_{W^{\ell-1/q}_q(\mathbb{R}^{n-1})}, \quad \ell=1,2,3. \tag{3.9}
$$

For the estimate of $D(v)$ and $T(\eta)$ in (3.7), we use extended functions A and Y instead of α and η:

$$
\begin{aligned}
&D(v) = (\sqrt{\mathfrak{g}_A})^{-1}(\nabla' A, \sqrt{1+|\nabla' A|^2}-1)\cdot v, \quad \mathfrak{g}_A = 1+|\nabla' A|^2,\\
&T_n(\eta) = -\sigma\mathfrak{g}_A^{-1}|\nabla' A|^2\Delta' Y - \sigma\mathfrak{g}_A^{-2}\sum_{j,k=1}^{n-1}\partial_j A\partial_k A\partial_j\partial_k Y.
\end{aligned}
$$

We set

$$
K_1 = \|\nabla A\|_{L_\infty(\dot{\mathbb{R}}^n)}, \quad K_2 = \|A\|_{W^3_r(\dot{\mathbb{R}}^n)}.
$$

By the Sobolev imbedding theorem under the assumption $n<r<\infty$ we have

$$
\|A\|_{W^2_\infty(\dot{\mathbb{R}}^n)} \le CK_2. \tag{3.10}
$$

We assume that $0 < K_1 < 1$ a priori, and therefore $K_1^\ell \leq K_1$ for $\ell \geq 1$. Using (2.11) and (3.10) we have

$$\begin{aligned}
&\|e^{-\gamma t} D(v)\|_{L_p(\mathbb{R}, W_q^2(\dot{\mathbb{R}}^n))} \\
&\quad \leq C_1 K_1 \|e^{-\gamma t} \nabla^2 v\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} + C_2 K_2 \|e^{-\gamma t} v\|_{L_p(\mathbb{R}, W_q^1(\dot{\mathbb{R}}^n))},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\|e^{-\gamma t}(\langle D_t\rangle^{\frac{1}{2}} T(Y), \nabla T(Y))\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} \\
&\quad \leq C_1 K_1 \|e^{-\gamma t}(\langle D_t\rangle^{\frac{1}{2}} Y, \nabla' Y)\|_{L_p(\mathbb{R}, W_q^2(\mathbb{R}^{n-1}))} + C_2 K_2 \|e^{-\gamma t} Y\|_{L_p(\mathbb{R}, W_q^2(\mathbb{R}^{n-1}))}
\end{aligned} \tag{3.12}$$

for $\gamma \geq \gamma_0$. For notational simplicity we set

$$\begin{aligned}
&[(v, \theta, \bar{\theta}, \eta)]_{p,q,\gamma_0} \\
&\quad = \|e^{-\gamma t}\, v\|_{W_{q,p}^{2,1}(\dot{\mathbb{R}}^n \times \mathbb{R})} + \|e^{-\gamma t}\, \nabla \theta\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}\, \bar{\theta}\|_{H_{q,p}^{1,1/2}(\dot{\mathbb{R}}^n \times \mathbb{R})} \\
&\qquad + \|\gamma e^{-\gamma t}\, v\|_{L_p(\mathbb{R}, L_q(\dot{\mathbb{R}}^n))} + \|e^{-\gamma t}(\partial_t \eta, \nabla \eta)\|_{L_p(\mathbb{R}, W_q^{2-1/q}(\dot{\mathbb{R}}^{n-1}))}
\end{aligned}$$

for $(v, \theta, \bar{\theta}, \eta) \in \mathcal{E}_{p,q,\gamma_0}(\dot{\mathbb{R}}^n \times \mathbb{R})$ and $\gamma \geq \gamma_0$. Applying Theorem 3.2 to (3.7) and using (3.11) and (3.12), for any $\gamma \geq \gamma_0$ with $\gamma_0 \geq 1$ we have

$$\begin{aligned}
[(w, \kappa, \bar{\kappa}, \zeta)]_{p,q,\gamma_0} &\leq (C_1 K_1 + C_2 K_2 \epsilon + C_2 K_2 (C(\epsilon) + 1)\gamma_0^{-1})[(v, \theta, \bar{\theta}, \eta)]_{p,q,\gamma_0} \\
&\quad + C\|d\|_{L_p(\mathbb{R}, W_q^{2-1/q}(\mathbb{R}^{n-1}))}.
\end{aligned} \tag{3.13}$$

We define the map Φ on $\mathcal{E}_{p,q,\gamma_0}(\dot{\mathbb{R}}^n \times \mathbb{R})$ by $\Phi(v, \theta, \bar{\theta}, \eta) = (w, \kappa, \bar{\kappa}, \zeta)$. Since the equation (3.7) is linear, from (3.13) we have

$$\begin{aligned}
&[\Phi(v_1, \theta_1, \bar{\theta}_1, \eta_1) - \Phi(v_2, \theta_2, \bar{\theta}_2, \eta_2)]_{p,q,\gamma_0} \\
&\quad \leq (C_1 K_1 + C_2 K_2 \epsilon + C_2 K_2 (C(\epsilon) + 1)\gamma_0^{-1})[(v_1, \theta_1, \bar{\theta}_1, \eta_1) - (v_2, \theta_2, \bar{\theta}_2, \eta_2)]_{p,q,\gamma_0}
\end{aligned} \tag{3.14}$$

for any $\gamma \geq \gamma_0$. After choosing K_1 and ϵ in such a way that

$$C_1 K_1 \leq 1/8, \quad C_2 K_2 \epsilon \leq 1/8,$$

we take γ_0 in such a way that

$$C_2 K_2 (C(\epsilon) + 1)\gamma_0^{-1} \leq 1/4.$$

Then (3.14) shows that Φ is a contraction map on $\mathcal{E}_{p,q,\gamma_0}(\dot{\mathbb{R}}^n \times \mathbb{R})$, and therefore by the fixed point theorem of S. Banach, we see that the map Φ admits a unique fixed point $(v, \theta, \bar{\theta}, \eta) \in \mathcal{E}_{p,q,\gamma_0}(\dot{\mathbb{R}}^n \times \mathbb{R})$, which solves the equation (3.6). Moreover from (3.13) with $(w, \kappa, \bar{\kappa}, \zeta) = (v, \theta, \bar{\theta}, \eta)$ and $C_1 K_1 + C_2 K_2 \epsilon + C_2 K_2 (C(\epsilon) + 1)\gamma_0^{-1} \leq 1/2$, we have $[(v, \theta, \bar{\theta}, \eta)]_{p,q,\gamma_0} \leq 2C\|e^{-\gamma t} d\|_{L_p(\mathbb{R}, W_q^{2-1/q}(\mathbb{R}^{n-1}))}$, from which Theorem 3.1 follows immediately.

4. Reduction of the boundary condition to linearized problems

In this section, we shall discuss the reduction of the boundary condition

$$[\![(S(u,\pi)+Q(u))\nu_{tu}]\!]-\sigma\mathcal{H}\nu_{tu}-[\![\rho]\!]c_g X_{u,n}\nu_{tu}=0, \tag{4.1}$$

which is the first key step in our proof of Theorem 1.1. Let $\mathbf{\Pi}_t$ and $\mathbf{\Pi}$ be projections to tangent hyperplanes of $\Gamma(t)$ and Γ, which are defined by

$$\mathbf{\Pi}_t d = d-(d,\nu_{tu})\nu_{tu}, \quad \mathbf{\Pi} d = d-(d,\nu)\nu \tag{4.2}$$

for an arbitrary vector field d defined on $\Gamma(t)$ and Γ, respectively. We know the following fact (cf. Solonnikov [24] and Shibata-Shimizu [22, Appendix]).

Lemma 4.1. *If $\nu_t\cdot\nu\neq 0$, then for arbitrary vector d, $d=0$ is equivalent to*

$$\mathbf{\Pi}\,\mathbf{\Pi}_t d=0, \quad \nu\cdot d=0. \tag{4.3}$$

We apply Lemma 4.1 for (4.1). Since we obtain

$$[\![\mathbf{\Pi}_t(\mu D(u)+Q(u))\nu_{tu}]\!]=0 \tag{4.4}$$

by applying $\mathbf{\Pi}_t$ to the left-hand side of (4.1), the first equation of (4.3) for (4.1) is given by

$$[\![\mathbf{\Pi}\mu D(u)\nu]\!]=-[\![\mathbf{\Pi}(\mathbf{\Pi}_t-\mathbf{\Pi})(\mu D(u)\nu_{tu})+\mathbf{\Pi}(\mu D(u)(\nu_{tu}-\nu))+\mathbf{\Pi}\,\mathbf{\Pi}_t(Q(u)\nu_{tu})]\!], \tag{4.5}$$

where we have used $\mathbf{\Pi}\,\mathbf{\Pi}=\mathbf{\Pi}$.

On the other hand, we consider the inner product of the boundary condition with ν. Using the fact that $\mathcal{H}\nu_{tu}=\Delta_{\Gamma(t)}X_u$ and substituting (1.4) for (4.1), we obtain

$$\begin{aligned}&[\![\nu\cdot(S(u,\pi)+Q(u))\nu_{tu}]\!]-\sigma\nu\cdot(\Delta_{\Gamma(t)}-\Delta_\Gamma)\Big(\xi+\int_0^t u(\xi,\tau)\,d\tau\Big)\\ &\quad-\sigma\nu\cdot\Delta_\Gamma\Big(\xi+\int_0^t u(\xi,\tau)\,d\tau\Big)-[\![\rho]\!]c_g\nu\cdot\Big(\xi_n+\int_0^t u_n(\xi,\tau)\,d\tau\Big)\nu_{tu}=0.\end{aligned} \tag{4.6}$$

Taking a commutator between Δ_Γ and $\nu\cdot$, we have

$$\nu\cdot\Delta_\Gamma\int_0^t u\,d\tau=\Delta_\Gamma\Big(\int_0^t \nu\cdot u\,d\tau\Big)-(\Delta_\Gamma\nu)\cdot\int_0^t u\,d\tau-2\Big(\nabla_\Gamma\cdot\int_0^t u\,d\tau\Big)\nabla_\Gamma\cdot\nu. \tag{4.7}$$

By (4.6) and (4.7), we obtain

$$\begin{aligned}&[\![\nu\cdot S(u,\pi)\nu]\!]-\sigma\Delta_\Gamma\int_0^t\nu\cdot u\,d\tau\\ &+\sigma\Big\{\Delta_\Gamma\nu\cdot\int_0^t u\,d\tau+\nu\cdot(\Delta_\Gamma-\Delta_{\Gamma(t)})\int_0^t u\,d\tau+\nu\cdot(\Delta_\Gamma-\Delta_{\Gamma(t)})\xi\Big\}\\ &\quad=[\![\nu\cdot S(u,\pi)(\nu-\nu_{tu})-\nu\cdot Q(u)\nu_{tu}]\!]+\sigma\mathcal{H}_0(\Gamma)+[\![\rho]\!]c_g\xi_n\\ &\qquad+[\![\rho]\!]c_g\int_0^t u_n\,d\tau\,\nu\cdot\nu_{tu}+[\![\rho]\!]c_g\xi_n\,\nu\cdot(\nu_{tu}-\nu)-2\sigma\Big(\nabla_\Gamma\cdot\int_0^t u\,d\tau\Big)\nabla_\Gamma\cdot\nu,\end{aligned} \tag{4.8}$$

where we have used $\nu\cdot\Delta_\Gamma\xi=\mathcal{H}_0(\Gamma)$. We denote the terms in the bracket of the left-hand side of (4.8) by $F(u)$, that is

$$F(u)=\Delta_\Gamma\nu\cdot\int_0^t u\,d\tau+\nu\cdot(\Delta_\Gamma-\Delta_{\Gamma(t)})\int_0^t u\,d\tau+\nu\cdot(\Delta_\Gamma-\Delta_{\Gamma(t)})\xi.$$

In (4.8), since $\Delta_{\Gamma(t)}$ and Δ_Γ contain the second-order tangential derivatives of X_u, in order to avoid the loss of regularity we apply the inverse operator $(m-\Delta_\Gamma)^{-1}$ with sufficiently large number m to $F(u)$. Namely, we proceed as follows:

$$\begin{aligned}
&[\![\nu\cdot S(u,\pi)\nu]\!]+\sigma(m-\Delta_\Gamma)\Big(\int_0^t\nu\cdot u\,d\tau+(m-\Delta_\Gamma)^{-1}F(u)\Big)-\sigma m\int_0^t\nu\cdot u\,d\tau\\
&\quad=[\![\nu\cdot S(u,\pi)(\nu-\nu_{tu})-\nu\cdot Q(u)\nu_{tu}]\!]+\sigma\mathcal{H}_0(\Gamma)+[\![\rho]\!]c_g\xi_n\\
&\qquad+[\![\rho]\!]c_g\int_0^t u_n\,d\tau\,\nu\cdot\nu_{tu}+[\![\rho]\!]c_g\xi_n\,\nu\cdot(\nu_{tu}-\nu)-2\sigma\Big(\nabla_\Gamma\cdot\int_0^t u\,d\tau\Big)\nabla_\Gamma\cdot\nu.
\end{aligned}\tag{4.9}$$

We introduce a function η by the formula

$$\eta=\int_0^t\nu\cdot u\,d\tau+(m-\Delta_\Gamma)^{-1}F(u)\quad\text{on }\Gamma.\tag{4.10}$$

Then, from (4.9) and (4.10), we obtain the two equations on the boundary Γ as follows:

$$\begin{aligned}
&[\![\nu\cdot S(u,\pi)\nu]\!]+\sigma(m-\Delta_\Gamma)\eta\\
&\quad=[\![\nu\cdot S(u,\pi)(\nu-\nu_{tu})-\nu\cdot Q(u)\nu_{tu}]\!]+\sigma\mathcal{H}_0(\Gamma)+[\![\rho]\!]c_g\xi_n\\
&\qquad+[\![\rho]\!]c_g\int_0^t u_n\,d\tau\,\nu\cdot\nu_{tu}+[\![\rho]\!]c_g\xi_n\,\nu\cdot(\nu_{tu}-\nu)-2\sigma\Big(\nabla_\Gamma\cdot\int_0^t u\,d\tau\Big)\nabla_\Gamma\cdot\nu\\
&\qquad+\sigma m\int_0^t\nu\cdot u\,d\tau,
\end{aligned}\tag{4.11}$$

$$\partial_t\eta-\nu\cdot u=(m-\Delta_\Gamma)^{-1}\dot F(u),\tag{4.12}$$

where $\dot F(u)$ denotes the derivative of $F(u)$ with respect to t.

Finally we arrive at the equivalent equation to (1.5) as follows:

$$\begin{aligned}
&\rho\partial_t u-\operatorname{Div}S(u,\pi)=\operatorname{Div}Q(u)+R(u)\nabla\pi && \text{in }\Omega,\ t>0,\\
&\operatorname{div}u=E(u)=\operatorname{div}\tilde E(u) && \text{in }\Omega,\ t>0,\\
&\partial_t\eta-\nu\cdot u=G(u) && \text{on }\Gamma,\ t>0,\\
&[\![\mathbf{\Pi}\mu D(u)\nu]\!]=[\![H_t(u)]\!] && \text{on }\Gamma,\ t>0,\\
&[\![\nu\cdot S(u,\pi)\nu]\!]+\sigma(m-\Delta_\Gamma)\eta=[\![H_n(u,\pi)]\!]+\sigma\mathcal{H}_0(\Gamma)+[\![\rho]\!]c_g\xi_n && \text{on }\Gamma,\ t>0,\\
&[\![u]\!]=0 && \text{on }\Gamma,\ t>0,\\
&u|_{t=0}=u_0(\xi)\text{ in }\Omega,\quad \eta|_{t=0}=0\text{ on }\Gamma,
\end{aligned}\tag{4.13}$$

where

$$
\begin{aligned}
&G(u) = (m - \Delta_\Gamma)^{-1}\dot F(u),\\
&\dot F(u) = \Delta_\Gamma \nu \cdot u - \nu \cdot \dot\Delta_{\Gamma(t)} \int_0^t u\,d\tau + \nu \cdot (\Delta_\Gamma - \Delta_{\Gamma(t)})u - \nu \cdot \dot\Delta_{\Gamma(t)}\xi,\\
&[\![H_t(u)]\!] = -[\![\mathbf{\Pi}(\mathbf{\Pi}_t - \mathbf{\Pi})(\mu D(u)\nu_{tu}) + \mathbf{\Pi}(\mu D(u)(\nu_{tu} - \nu)) + \mathbf{\Pi}\,\mathbf{\Pi}_t(Q(u)\nu_{tu})]\!],\\
&[\![H_n(u,\pi)]\!] = [\![\nu \cdot S(u,\pi)(\nu - \nu_{tu}) - \nu \cdot (Q(u)\nu_{tu})]\!]\\
&\qquad\qquad + [\![\rho]\!]c_g \int_0^t u_n\,d\tau\,\nu\cdot\nu_{tu} + [\![\rho]\!]c_g\xi_n\,\nu\cdot(\nu_{tu} - \nu)\\
&\qquad\qquad - 2\sigma\Big(\nabla_\Gamma \cdot \int_0^t u\,d\tau\Big)\nabla_\Gamma\cdot\nu + \sigma m \int_0^t \nu\cdot u\,d\tau,
\end{aligned}
$$

and $Q(u)$, $R(u)$, $E(u)$ and $\tilde E(u)$ are nonlinear terms defined by (1.6).

5. Initial flow

In this section, we shall discuss the initial flow to reduce the problem (4.13) to the case where $u_0(\xi) = 0$ and $\sigma\mathcal{H}_0(\Gamma) + [\![\rho]\!]c_g\xi_n = 0$. We study the problem in two steps.

Step 1. Let (u_1, π_1) be a solution to the problem:

$$
\begin{aligned}
&\rho\lambda u_1 - \mathrm{Div}\,S(u_1,\pi_1) = 0 && \text{in } \Omega,\\
&\mathrm{div}\,u_1 = 0 && \text{in } \Omega,\\
&[\![S(u_1,\pi_1)\nu]\!] = (\sigma\mathcal{H}_0(\Gamma) + [\![\rho]\!]c_g\xi_n)\nu && \text{on } \Gamma,\\
&[\![u_1]\!] = 0 && \text{on } \Gamma.
\end{aligned}
\tag{5.1}
$$

If positive number λ is large enough, then we know that (5.1) admits a unique solution

$$(u_1, \pi_1) \in W^2_q(\Omega) \times \hat W^1_q(\Omega)$$

which satisfies the estimate

$$
\begin{aligned}
|\lambda|\|u_1\|_{L_q(\Omega)} &+ \|\nabla^2 u_1\|_{L_q(\Omega)} + \|\nabla\pi_1\|_{L_q(\Omega)}\\
&\le C_1(\sigma\|\mathcal{H}_0(\Gamma)\|_{W^{1-1/q}_q(\Gamma)} + c_g\|\alpha\|_{W^{1-1/q}_q(\Gamma)}).
\end{aligned}
\tag{5.2}
$$

Step 2. We consider the linear time-dependent problem in the time interval $(0, 2)$:

$$
\begin{aligned}
&\rho\partial_t u_2 - \mathrm{Div}\,S(u_2,\pi_2) = -\rho\lambda u_1 && \text{in } \Omega\times(0,2),\\
&\mathrm{div}\,u_2 = 0 && \text{in } \Omega\times(0,2),\\
&[\![S(u_2,\pi_2)\nu]\!] = 0 && \text{on } \Gamma\times(0,2),\\
&[\![u_2]\!] = 0 && \text{on } \Gamma\times(0,2),\\
&u_2|_{t=0} = u_0(\xi) - u_1(\xi) \text{ in } \Omega.
\end{aligned}
\tag{5.3}
$$

In order to treat (5.3), we discuss an analytic semigroup to the initial boundary value problem

$$\begin{aligned} &\rho\partial_t u - \mathrm{Div}\, S(u,\pi) = 0 && \mathrm{div}\, u = 0 && \text{in } \Omega,\ t>0,\\ &[\![S(u,\pi)\nu]\!] = 0 && [\![u]\!] = 0 && \text{on } \Gamma,\ t>0,\\ &u|_{t=0} = u_0 && && \text{in } \Omega. \end{aligned} \tag{5.4}$$

A corresponding resolvent problem to (5.4) is

$$\begin{aligned} &\lambda u - \frac{1}{\rho}\mathrm{Div}\, S(u,\pi) = u_0 && \mathrm{div}\, u = 0 && \text{in } \Omega,\\ &[\![S(u,\pi)\nu]\!] = 0 && [\![u]\!] = 0 && \text{on } \Gamma. \end{aligned} \tag{5.5}$$

Let us introduce the Helmholtz decomposition

$$L_q(\Omega)^n = J_q(\Omega) \oplus G_q(\Omega) \tag{5.6}$$

for $1 < q < \infty$, where $\oplus$ is the direct sum and

$$\begin{aligned} J_q(\Omega) &= \overline{\{u \in C_0^\infty(\Omega)^n \mid \mathrm{div}\, u = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L_q(\Omega)}},\\ G_q(\Omega) &= \{\nabla\pi \mid \pi \in \hat{W}_q^1(\Omega),\quad [\![\pi]\!] = 0\}. \end{aligned}$$

Let us define the solution operator K from $W_q^2(\Omega)^n$ into $\hat{W}_q^1(\Omega)$ by the formula $K(u) = \pi$:

$$\begin{aligned} &\Delta\pi = 0 && \text{in } \Omega,\\ &[\![\pi]\!] = [\![\nu\cdot\mu D(u)\nu - \mathrm{div}\, u]\!] && \text{on } \Gamma,\\ &[\![\rho^{-1}\partial_\nu\pi]\!] = [\![\rho^{-1}\nu\cdot(\mu\mathrm{Div}\, D(u) - \nabla\mathrm{div}\, u)]\!] && \text{on } \Gamma. \end{aligned}$$

We introduce an operator A_q and the domain $\mathcal{D}(A_q)$:

$$\begin{aligned} A_q u &= -\mathrm{Div}\, S(u, K(u)) \quad \text{for} \ \ u \in \mathcal{D}(A_q),\\ \mathcal{D}(A_q) &= \{u \in J_q(\Omega)\cap W_q^2(\Omega)^n \mid [\![S(u,K(u))\nu]\!] = 0,\ [\![u]\!] = 0\}. \end{aligned}$$

By a similar argument as §3 in [21] we obtain the following.

Theorem 5.1.

(1) *The operator A_q generates an analytic semigroup $\{e^{-tA_q}\}_{t\ge 0}$ on $J_q(\Omega)$.*

(2) *If $u_0 \in (J_q(\Omega), \mathcal{D}(A_q))_{1-1/p,p}$, then (5.4) admits a unique solution $(u,\pi,\bar\pi) \in \mathcal{D}_{p,q,\gamma_0}(\Omega\times\mathbb{R})$ which satisfies the estimate*

$$\begin{aligned} &\|e^{-\gamma t}\, u\|_{W_{q,p}^{2,1}(\Omega\times\mathbb{R})} + \|e^{-\gamma t}\,\nabla\pi\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\,\bar\pi\|_{H_{q,p}^{1,1/2}(\Omega\times\mathbb{R})}\\ &\quad \le C\|u_0\|_{(J_q(\Omega),\mathcal{D}(A_q))_{1-1/p,p}} \end{aligned}$$

for any $\gamma \ge \gamma_0$.

If the initial data $u_0 \in (L_q(\Omega), W_q^2(\Omega))_{1-1/p,p} = B_{q,p}^{2(1-1/p)}(\Omega)$ satisfies the compatibility condition (1.9), then by Theorem 5.1 we know that the problem (5.3) admits a unique solution

$$u_2 \in W_{q,p}^{2,1}(\Omega \times (0,2)), \quad \pi_2 \in L_p((0,2), \hat{W}_q^1(\Omega)).$$

Moreover there exists $\bar{\pi}_2 \in H_{q,p}^{1,1/2}(\Omega \times (0,2))$ such that $[\![\bar{\pi}]\!] = [\![\pi_2]\!]$. ($u_2$, π_2, $\bar{\pi}_2$) satisfies the estimate

$$\begin{aligned}\|u_2\|_{W_{q,p}^{2,1}(\Omega\times(0,2))} + \|\pi_2\|_{L_p((0,2),\hat{W}_q^1(\Omega))} + \|\bar{\pi}_2\|_{H_{q,p}^{1,1/2}(\Omega\times(0,2))} \\ \le C_2\|u_0 + u_1\|_{B_{q,p}^{2(1-1/p)}(\Omega))}. \quad (5.7)\end{aligned}$$

If we set $z = u_1 + u_2$ and $\tau = \pi_1 + \pi_2$, then (z, τ) satisfies the time-dependent linear equation in the time interval $(0, 2)$:

$$\begin{aligned}
&\rho\partial_t z - \operatorname{Div} S(z,\tau) = 0 && \text{in } \Omega \times (0,2),\\
&\operatorname{div} z = 0 && \text{in } \Omega \times (0,2),\\
&[\![S(z,\tau)\nu]\!] = (\sigma\mathcal{H}_0(\Gamma) + [\![\rho]\!]c_g\xi_n)\nu && \text{on } \Gamma \times (0,2),\\
&[\![z]\!] = 0 && \text{on } \Gamma \times (0,2),\\
&z|_{t=0} = u_0 && \text{in } \Omega. \qquad (5.8)
\end{aligned}$$

z is our initial flow.

Now, we look for a solution (u, π) of the equation (4.13) of the form: $u = z+w$ and $\pi = \tau + \kappa$ in the time interval $(0, T)$ with $0 < T \le 1$. Setting $\bar{\tau} = \pi_1 + \bar{\pi}_2$, we see that w, κ and η should satisfy the equations:

$$\begin{aligned}
&\rho\partial_t w - \operatorname{Div} S(w,\kappa) = \operatorname{Div} Q(z+w) + R(z+w)\nabla(\tau+\kappa) && \text{in } \Omega \times (0,T),\\
&\operatorname{div} w = E(z+w) = \operatorname{div}\tilde{E}(z+w) && \text{in } \Omega \times (0,T),\\
&\partial_t\eta - \nu\cdot w = G(z+w) + \nu\cdot z && \text{on } \Gamma \times (0,T),\\
&[\![\mathbf{\Pi}\mu\, D(w)\nu]\!] = [\![H_t(z+w)]\!] && \text{on } \Gamma \times (0,T),\\
&[\![\nu\cdot S(w,\kappa)\nu]\!] + \sigma(m - \Delta_\Gamma)\eta = [\![H_n(z+w, \bar{\tau}+\kappa)]\!] && \text{on } \Gamma \times (0,T),\\
&[\![w]\!] = 0 && \text{on } \Gamma \times (0,T),\\
&w|_{t=0} = 0 \text{ in } \Omega, \quad \eta|_{t=0} = 0 \text{ on } \Gamma. && \qquad (5.9)
\end{aligned}$$

6. The nonlinear problem

In this section we solve (5.9), namely we shall prove the following theorem.

Theorem 6.1. *Let $2 < p < \infty$ and $n < q < \infty$. Let (u_1, π_1) be a solution of* (5.1) *and (u_2, π_2, η_2) be a solution of* (5.3)*. Then there exists $T > 0$ depending on $\|u_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}$ and $\|\alpha\|_{W_q^{3-1/q}(\mathbb{R}^{n-1})}$ such that* (5.9) *admits a unique solution*

$$w \in W_{q,p}^{2,1}(\Omega \times (0,T))^n, \quad \kappa \in L_p((0,T), \hat{W}_q^1(\Omega)),$$
$$\eta \in W_p^1((0,T), W_q^{2-1/q}(\Gamma)) \cap L_p((0,T), W_q^{3-1/q}(\Gamma)).$$

We prove Theorem 6.1 by using the maximal L_p-L_q regularity theorem. If $e^{-\gamma_0 t}u \in L_p(\mathbb{R}, L_q(\Omega))$ for $\gamma_0 > 1$, then $u \in L_p((0,T), L_q(\Omega))$ for any T with $0 < T < \infty$, because it holds that

$$\begin{aligned}\int_0^T \|u\|_{L_q(\Omega)}^p\,dt &= \int_0^T \|e^{p\gamma_0 t}e^{-\gamma_0 t}u\|_{L_q(\Omega)}^p\,dt\\ &\le e^{p\gamma_0 T}\int_0^\infty \|e^{-\gamma_0 t}u\|_{L_q(\Omega)}^p\,dt\\ &\le C\|e^{-\gamma_0 t}u\|_{L_p(\mathbb{R},L_q(\Omega))}.\end{aligned}$$

Therefore as a corollary of Theorem 1.2 with $q = r$, it holds that the maximal regularity of (1.10) is local in time.

Theorem 6.2. *Let $1 < p, q < \infty$. If the right-hand members f, f_d, d and h of* (1.10) *satisfy the conditions*

$$\begin{aligned}&f \in L_p((0,T), L_q(\Omega))^n,\ \ f_d \in L_p((0,T), W_q^1(\Omega)) \cap W_p^1((0,T), \hat{W}_q^{-1}(\mathbb{R}^n)),\\ &\qquad d \in L_p((0,T), W_q^{2-1/q}(\Gamma)),\ \ h \in H_{q,p}^{1,1/2}(\Omega \times (0,T))^n\end{aligned}$$

and compatibility conditions

$$\bar{f}_d|_{t=0} = 0,\ \ h|_{t=0} = 0,$$

then (1.10) *admits a unique solution* (u, π, η)*:*

$$\begin{aligned}&u \in W_{q,p}^{2,1}(\Omega \times (0,T)),\quad \pi \in L_p((0,T), \hat{W}_q^1(\Omega)),\\ &\eta \in W_p^1((0,T), W_q^{2-1/q}(\Gamma)) \cap L_p((0,T), W_q^{3-1/q}(\Gamma)).\end{aligned}$$

Moreover there exists $[\![\bar{\pi}]\!] = [\![\pi]\!]$ such that $\bar{\pi} \in H_{q,p}^{1,1/2}(\Omega \times (0,T))$. The solutions satisfy the estimate

$$\begin{aligned}&\|u\|_{W_{q,p}^{2,1}(\Omega\times(0,T))} + \|\nabla\pi\|_{L_p((0,T),L_q(\Omega))} + \|\bar{\pi}\|_{H_{q,p}^{1,1/2}(\Omega\times(0,T))}\\ &\quad + \|\partial_t\eta\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))} + \|\eta\|_{L_p((0,T),W_q^{3-1/q}(\Gamma))}\\ &\le C\Big(\ \|f\|_{L_p((0,T),L_q(\Omega))} + \|d\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))}\\ &\quad + \|f_d\|_{L_p((0,T),W_q^1(\Omega))} + \|\partial_t f_d\|_{L_p((0,T),\hat{W}_q^{-1}(\mathbb{R}^n))} + \|h\|_{H_{q,p}^{1,1/2}(\Omega\times(0,T))}\Big).\end{aligned} \tag{6.1}$$

In what follows we construct a solution of (5.9) by the contraction mapping principle based on Theorem 6.2.

Step 1. We set

$$M = \max(C_1, C_2)\left\{\sigma\|\mathcal{H}_0(\Gamma)\|_{W_q^{1-1/q}(\Gamma)} + [\![\rho]\!]c_g\|\alpha\|_{W_q^{1-1/q}(\Gamma)} + \|u_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}\right\}. \tag{6.2}$$

By (5.2) and (5.7) we define the underlying space $\mathcal{I}_{R,T}$ by

$$
\begin{aligned}
\mathcal{I}_{R,T} = \{ & (v,\theta,\bar\theta,\eta) \mid \\
& v \in W^{2,1}_{q,p}(\Omega\times(0,T))^n,\ \theta \in L_p((0,T),\hat W^1_q(\Omega)),\ \bar\theta \in H^{1,\frac12}_{q,p,0}(\Omega\times(0,T)),\\
& \partial_t\eta \in L_p((0,T),W^{2-1/q}_q(\Gamma)),\quad \eta\in L_p((0,T),W^{3-1/q}_q(\Gamma)),\\
& \theta=\bar\theta \ \text{ on } \Gamma\times(0,T), && \text{(As.1)}\\
& v(\xi,0)=0 \ \text{ in } \Omega,\quad \eta(\xi',0)=0 \ \text{ on } \Gamma, && \text{(As.2)}\\
& \|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\theta\|_{L_p((0,T),W^1_q(\Omega))} + \|\bar\theta\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))}\\
& \quad + \|\partial_t\eta\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))} + \|\eta\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))} \le R\}, && \text{(As.3)}
\end{aligned}
$$

where T is a positive number and $R\ge 1$ is a large number determined later. Since we consider the local in time solvability of (5.9), we may assume that $0<T\le 1$ in the course of our proof of Theorem 6.1.

Given $(v,\theta,\bar\theta,\eta,)\in\mathcal{I}_{R,T}$, let (V,Θ,Y) be a solution to the linear equation

$$
\begin{aligned}
&\rho\partial_t V - \operatorname{Div} S(V,\Theta)\\
&= \operatorname{Div} Q(u_1+u_2+v) + R(u_1+u_2+v)\nabla(\pi_1+\pi_2+\theta) && \text{in } \Omega,\ t>0,\\
&\operatorname{div} V = E(u_1+u_2+v) = \operatorname{div}\tilde E(u_1+u_2+v) && \text{in } \Omega,\ t>0,\\
&\partial_t Y - \nu\cdot V = G(u_1+u_2+v) + \nu\cdot(u_1+u_2) && \text{on } \Gamma,\ t>0,\\
&[\![\mathbf{\Pi}\mu D(V)\nu]\!] = [\![H_t(u_1+u_2+v)]\!] && \text{on } \Gamma,\ t>0,\\
&[\![\nu\cdot S(V,\Theta)\nu]\!] + \sigma(m-\Delta_\Gamma)Y = [\![H_n(u_1+u_2+v,\pi_1+\bar\pi_2+\bar\theta)]\!] && \text{on } \Gamma,\ t>0,\\
&[\![V]\!]=0 && \text{on } \Gamma,\ t>0,\\
&V|_{t=0}=0 \text{ in } \Omega,\quad Y|_{t=0}=0 \text{ on } \Gamma. && (6.3)
\end{aligned}
$$

Our task is to show that if we define the map $\Phi(v,\theta,\bar\theta,\eta)=(V,\Theta,\bar\Theta,Y)$, then Φ is a contraction map from $I_{R,T}$ into itself. To do so we check the condition on the right-hand members in (6.3), to apply Theorem 6.2 to (6.3). All the constants independent of R and T are denoted by C in what follows. To estimate L_∞ norm of functions and the multiplication of several functions we use the following fact:

$$
\begin{aligned}
& W^1_q(\Omega)\subset L_\infty(\Omega),\quad \|f\|_{L_\infty(\Omega)}\le C\|f\|_{W^1_q(\Omega)} \quad \text{for } f\in W^1_q(\Omega),\\
& \Big\| f\prod_{j=1}^N g_j\Big\|_{L_q(\Omega)} \le C\|f\|_{L_q(\Omega)}\prod_{j=1}^N\|g_j\|_{W^1_q(\Omega)} \quad \text{for } f\in L_q(\Omega),\ g_j\in W^1_q(\Omega),\\
& \Big\|\prod_{j=1}^N f_j\Big\|_{W^1_q(\Omega)} \le C\prod_{j=1}^N \|f_j\|_{W^1_q(\Omega)} \quad \text{for } f_j\in W^1_q(\Omega)\ (j=1,\dots,N), \qquad (6.4)
\end{aligned}
$$

which follows from the Sobolev imbedding theorem and the assumption: $n<q<\infty$. Here and hereafter, p' denotes the dual exponent of p, that is $p'=p/(p-1)$.

Lemma 6.3 (Lemma 2.2 in [20]**).** (1) *Let* $0 < T \le 1$ *and set*

$$W^{2,1}_{q,p,0}(\Omega \times (0,T)) = \{v \in W^{2,1}_{q,p}(\Omega \times (0,T)) : v(\xi, 0) = 0\}.$$

Then, there exists a bounded linear operator $\mathbb{E} : W^{2,1}_{q,p,0}(\Omega \times (0,T)) \to W^{2,1}_{q,p}(\Omega \times \mathbb{R})$ *such that* $\mathbb{E}v = v$ *on* $\Omega \times (0,T)$, $\mathbb{E}v = 0$ *for* $t \notin [0, 2T]$ *and*

$$\|\mathbb{E}v\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R})} \le C\|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))}$$

for any $v \in W^{2,1}_{q,p,0}(\Omega \times (0,T))$, *where* C *is independent of* T.
(2) *There exists a bounded linear operator* $\mathbb{E}_2 : W^{2,1}_{q,p}(\Omega \times (0,2)) \to W^{2,1}_{q,p}(\Omega \times \mathbb{R})$ *such that* $\mathbb{E}_2 u_2 = u_2$ *on* $\Omega \times (0,1)$, $\mathbb{E}_2 u_2 = 0$ *for* $t \notin [-2, 2]$ *and*

$$\|\mathbb{E}_2 u_2\|_{W^{2,1}_{q,p}(\Omega\times\mathbb{R})} \le C\|u_2\|_{W^{2,1}_{q,p}(\Omega\times(0,2))}$$

for any $u_2 \in W^{2,1}_{q,p}(\Omega \times (0,2))$.

Lemma 6.4. *For* $0 < t \le T < 1$ *we obtain*

(1) $$\Big\|\int_0^t (u_1 + u_2 + v)\, d\tau\Big\|_{L_\infty((0,T),W^2_q(\Omega))} \le T^{1/p'}(2M + R), \tag{6.5}$$

(2) $$\Big\|\int_0^t (u_1 + u_2 + v)\, d\tau\Big\|_{L_p((0,T),W^2_q(\Omega))} \le T(2M + R), \tag{6.6}$$

(3) $$\Big\|\int_0^t (u_1 + \mathbb{E}_2 u_2 + \mathbb{E}v)\, d\tau\Big\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \le T^{\frac12}(2M + R), \tag{6.7}$$

(4) $$\|u_1 + u_2 + v\|_{L_\infty((0,T),W^1_q(\Omega))} \le C(2M + T^{\frac12 - \frac1p} R), \tag{6.8}$$

(5) $$\|u_1 + u_2 + v\|_{L_p((0,T),W^1_q(\Omega))} \le CT^{\frac1p}(2M + R). \tag{6.9}$$

Proof. The first inequality follows from

$$\begin{aligned}
\Big\|\int_0^t \nabla(u_1 + u_2 + v)\, d\tau\Big\|_{W^1_q(\Omega)} &\le \int_0^t \|u_1 + u_2 + v\|_{W^2_q(\Omega)}\, dt \\
&\le \|u_1\|_{W^2_q(\Omega)} \int_0^t dt + \Big(\int_0^t dt\Big)^{1/p'} \big(\|u_2\|_{L_p((0,T),W^2_q(\Omega))} + \|v\|_{L_p((0,T),W^2_q(\Omega))}\big) \\
&\le t^{1/p'}(2M + R),
\end{aligned}$$

where we have used (5.2), (5.7) and (As.3). The second inequality is easily derived. By using the complex interpolation relation

$$H^{1/2}_p(\mathbb{R}, L_q(\Omega)) = [L_p(\mathbb{R}, L_q(\Omega)), W^1_p(\mathbb{R}, L_q(\Omega))]_{1/2},$$

between

$$\Big\|\int_0^t (u_1 + \mathbb{E}_2 u_2 + \mathbb{E}v)\, d\tau\Big\|_{W^1_p(\mathbb{R},L_q(\Omega))} \le \|u_1 + \mathbb{E}_2 u_2 + \mathbb{E}v\|_{W^1_p(\mathbb{R},L_q(\Omega))} \tag{6.10}$$

and

$$\Big\|\int_0^t (u_1 + \mathbb{E}_2 u_2 + \mathbb{E}v)\, d\tau\Big\|_{L_p(\mathbb{R},L_q(\Omega))} \le T\|u_1 + \mathbb{E}_2 u_2 + \mathbb{E}v\|_{L_p(\mathbb{R},L_q(\Omega))}, \tag{6.11}$$

we obtain

$$\left\|\int_0^t (u_1 + \mathbb{E}_2 u_2 + \mathbb{E}v)\,d\tau\right\|_{H_p^{\frac{1}{2}}(\mathbb{R},L_q(\Omega))} \le T^{\frac{1}{2}}\|u_1 + \mathbb{E}_2 u_2 + \mathbb{E}v\|_{H_p^{\frac{1}{2}}(\mathbb{R},L_q(\Omega))} \le T^{\frac{1}{2}}(2M + R). \tag{6.12}$$

Combining (6.6) and (6.12) we have the inequality (3). To obtain the fourth inequality, we use the relation

$$H_p^{\frac{1}{2}}((0,T), W_q^1(\Omega)) \subset \mathrm{Lip}^{\frac{1}{2}-\frac{1}{p}}([0,T], W_q^1(\Omega)), \tag{6.13}$$

and we have

$$\|u_2(t)\|_{W_q^1(\Omega)} \le \|u_0\|_{W_q^1(\Omega)} + C\|u_2\|_{H_p^{\frac{1}{2}}((0,2),W_q^1(\Omega))} t^{\frac{1}{2}-\frac{1}{p}} \tag{6.14}$$

$$\|v(t)\|_{W_q^1(\Omega)} \le C\|v\|_{H_p^{\frac{1}{2}}((0,1),W_q^1(\Omega))} t^{\frac{1}{2}-\frac{1}{p}} \tag{6.15}$$

for $0 < t < 1$. For the last inequality, we use (6.14) and (6.15). □

Remark 6.5. In order to simplify the estimate of polynomials of $\int_0^t \nabla(u_1+u_2+v)\,d\tau$, we choose $T > 0$ so small that

$$\left\|\int_0^t \nabla(u_1 + u_2 + v)\,d\tau\right\|_{W_q^1(\Omega)} \le 1 \tag{6.16}$$

for $0 < t \le T$.

Step 2 (Estimate of G)**.** Nonlinear terms Q, R, E and $\tilde{E}$ are the same terms as that of a free boundary problem without surface tension and gravity estimated as in §2 in [20]. Therefore in this paper we estimate nonlinear terms G, H_t and H_n. We consider the term

$$G(u) = (m - \Delta_\Gamma)^{-1}\dot{F}(u),$$

where $\dot{F}(u)$ denotes the derivative of $F(u)$ with respect to t, and given by

$$\dot{F}(u) = \Delta_\Gamma \nu \cdot u - \nu \cdot \dot{\Delta}_{\Gamma(t)} \int_0^t u\,d\tau + \nu \cdot (\Delta_\Gamma - \Delta_{\Gamma(t)})u - \nu \cdot \dot{\Delta}_{\Gamma(t)}\xi.$$

By (6.2), we have

$$\|\alpha\|_{W_q^{3-1/q}(\Gamma)} \le M. \tag{6.17}$$

Since $\nu = (\nabla'\alpha, -1)/\sqrt{1 + |\nabla'\alpha|^2}$, we have

$$\|\nu\|_{W_q^{2-1/q}(\Gamma)} \le M. \tag{6.18}$$

The radius vectors of surfaces Γ and $\Gamma(t)$ for (1.1) are equal to $r_0 = (\xi', \alpha(\xi'))$ and

$$r_t = \bigg(\xi_1 + \int_0^t u_1(\xi', \alpha(\xi'), \tau)\, d\tau,\ \ldots,\ \xi_{n-1} + \int_0^t u_{n-1}(\xi', \alpha(\xi'), \tau)\, d\tau, \\ \alpha(\xi') + \int_0^t u_n(\xi', \alpha(\xi'), \tau)\, d\tau\bigg).$$

Δ_Γ and $\Delta_{\Gamma(t)}$ are defined by

$$\Delta_\Gamma = \frac{1}{\sqrt{\mathfrak{g}}} \sum_{j,k=1}^{n-1} \frac{\partial}{\partial \xi_j} \frac{\hat{\mathfrak{g}}_{jk}}{\sqrt{\mathfrak{g}}} \frac{\partial}{\partial \xi_k}, \quad \Delta_{\Gamma(t)} = \frac{1}{\sqrt{\mathfrak{g}}_t} \sum_{j,k=1}^{n-1} \frac{\partial}{\partial \xi_j} \frac{\hat{\mathfrak{g}}_{tjk}}{\sqrt{\mathfrak{g}}_t} \frac{\partial}{\partial \xi_k} \tag{6.19}$$

where

$$\mathfrak{g}_{jk} = \frac{\partial r_0}{\partial \xi_j} \cdot \frac{\partial r_0}{\partial \xi_k}, \quad \mathfrak{g}_{tjk} = \frac{\partial r_t}{\partial \xi_j} \cdot \frac{\partial r_t}{\partial \xi_k}.$$

We set G as a matrix whose jk elements are denoted by $\mathfrak{g}_{jk}$, and set

$$\mathfrak{g} = \det G = \det(\mathfrak{g}_{jk}) = 1 + |\nabla'\alpha|^2.$$

$\hat{\mathfrak{g}}_{jk}$ are elements of a cofactor matrix of G and

$$G^{-1} = \frac{\hat{\mathfrak{g}}_{jk}}{\mathfrak{g}} = \frac{(1 + |\nabla'\alpha|^2)\delta_{jk} - \partial_j\alpha\partial_k\alpha}{\mathfrak{g}}.$$

We set G_t as a matrix whose jk elements are denoted by $\mathfrak{g}_{tjk}$. $\hat{\mathfrak{g}}_{tjk}$ are elements of a cofactor matrix of G_t,

$$\mathfrak{g}_t = \det G_t = \det(\mathfrak{g}_{tjk}), \quad G_t^{-1} = \frac{\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}.$$

If we set

$$\beta_{ij} = \int_0^t \frac{\partial u_i}{\partial \xi_j}(\xi', \alpha(\xi'), \tau)\, d\tau,$$

then we have

$$\begin{aligned} \frac{\partial r_{ti}}{\partial \xi_j} &= \delta_{ij} + \beta_{ij} + \partial_j\alpha\beta_{in}, \qquad i = 1, \ldots, n-1, \\ \frac{\partial r_{tn}}{\partial \xi_j} &= \partial_j\alpha + \beta_{nj} + \partial_j\alpha\beta_{nn}, \end{aligned}$$

and

$$\begin{aligned} \mathfrak{g}_{tjk} = \delta_{jk} + \beta_{jk} + \beta_{kj} + \sum_{\ell=1}^n \beta_{\ell j}\beta_{\ell k} + \partial_k\alpha\left(\beta_{jn} + \beta_{nj} + \sum_{\ell=1}^n \beta_{\ell j}\beta_{\ell n}\right) \\ + \partial_j\alpha(\beta_{kn} + \beta_{nk} + \sum_{\ell=1}^n \beta_{\ell n}\beta_{\ell k}) + \partial_j\alpha\partial_k\alpha\left(1 + 2\beta_{nn} + \sum_{\ell=1}^n \beta_{\ell n}^2\right). \end{aligned}$$

Therefore $\mathfrak{g}_{tjk}$, $\hat{\mathfrak{g}}_{tjk}$ and $\mathfrak{g}_t$ are expressed by

$$\mathfrak{g}_{tjk} = \mathfrak{g}_{jk} + J_1(\nabla'\alpha)K_1\left(\int_0^t \nabla'(u_1+u_2+v)\,d\tau\right), \tag{6.20}$$

$$\hat{\mathfrak{g}}_{tjk} = \hat{\mathfrak{g}}_{jk} + J_2(\nabla'\alpha)K_2\left(\int_0^t \nabla'(u_1+u_2+v)\,d\tau\right), \tag{6.21}$$

$$\mathfrak{g}_t = \mathfrak{g} + J_3(\nabla'\alpha)K_3\left(\int_0^t \nabla'(u_1+u_2+v)\,d\tau\right), \tag{6.22}$$

where $J_1(\cdot)$, $J_2(\cdot)$ and $J_3(\cdot)$ are polynomials of $\nabla\alpha$, and $K_1(\cdot)$, $K_2(\cdot)$ and $K_3(\cdot)$ are polynomials of $\int_0^t \nabla(u_1+u_2+v)\,d\tau$, respectively, such as $K_j(0)=0$.

Proposition 6.6 (cf. Amann [4]**).** *Let $1<q<\infty$. For every $f\in W_q^{-1/q}(\Gamma)$, there exists $m>>1$ such that*

$$\|(m-\Delta_\Gamma)^{-1}f\|_{W_q^{2-1/q}(\Gamma)} \le C_{q,\Gamma}\|f\|_{W_q^{-1/q}(\Gamma)} \tag{6.23}$$

holds.

Proposition 6.7. *Let $1<q<\infty$.*
(1) *For every $f\in W_q^{-1/q}(\Gamma)$ and $g\in W_q^1(\Omega)$ we have*

$$\|fg\|_{W_q^{-1/q}(\Gamma)} \le C_{q,\Gamma}\|f\|_{W_q^{-1/q}(\Gamma)}\|g\|_{W_q^1(\Omega)}. \tag{6.24}$$

(2) *For $f\in W_q^{-1/q}(\Gamma)$ and $g_j\in W_q^{1-1/q}(\Gamma)$ $(j=1,\dots,N)$, we have*

$$\left\|f\prod_{j=1}^N g_j\right\|_{W_q^{-1/q}(\Gamma)} \le C\|f\|_{W_q^{-1/q}(\Gamma)}\prod_{j=1}^N\|g_j\|_{W_q^{1-1/q}(\Gamma)}.$$

(3) *For $f_j\in W_q^{1-1/q}(\Gamma)$ $(j=1,\dots,N)$, we have*

$$\left\|\prod_{j=1}^N g_j\right\|_{W_q^{1-1/q}(\Gamma)} \le C\prod_{j=1}^N\|f_j\|_{W_q^{1-1/q}(\Gamma)}.$$

Proof. Let $1/q+1/q'=1$. For $\varphi\in W_{q'}^{1-1/q'}(\Gamma)$ there exists $\psi\in W_{q'}^1(\Omega)$ such that

$$\psi|_\Gamma=\varphi, \qquad \|\psi\|_{W_{q'}^1(\Omega)} \le C\|\varphi\|_{W_{q'}^{1-1/q'}(\Gamma)}.$$

It holds that

$$\|g\varphi\|_{W_{q'}^{1-1/q'}(\Gamma)} \le C\|g\psi\|_{W_{q'}^1(\Omega)} \le C\left(\|\nabla g\psi\|_{L_{q'}(\Omega)} + \|g\|_{L_\infty(\Omega)}\|\psi\|_{W_{q'}^1(\Omega)}\right). \tag{6.25}$$

Let $1/q'=1/q+1/r$. Since $n(\frac{1}{q'}-\frac{1}{r})<1$, by the embedding relation, $W_{q'}^1(\Omega)\subset L_r(\Omega)$ holds. Therefore we have

$$\begin{aligned}\|\nabla g\psi\|_{L_{q'}(\Omega)} &\le C\|g\|_{W_q^1(\Omega)}\|\psi\|_{L_r(\Omega)}\\ &\le C\|g\|_{W_q^1(\Omega)}\|\psi\|_{W_{q'}^1(\Omega)} \le C\|g\|_{W_q^1(\Omega)}\|\varphi\|_{W_{q'}^{1-\frac{1}{q'}}(\Gamma)}.\end{aligned} \tag{6.26}$$

Combining (6.25) with (6.26), we obtain

$$\left|\int_\Gamma fg\varphi\,d\sigma\right| \le C\|f\|_{W_q^{-\frac{1}{q}}(\Gamma)}\|g\varphi\|_{W_{q'}^{1-\frac{1}{q'}}(\Gamma)} \le C\|f\|_{W_q^{-\frac{1}{q}}(\Gamma)}\|g\|_{W_q^1(\Omega)}\|\varphi\|_{W_{q'}^{1-\frac{1}{q'}}(\Gamma)},$$

which implies that

$$\|fg\|_{W_q^{-\frac{1}{q}}(\Gamma)} = \sup_{\varphi\in C_0^\infty(\Gamma),\ \|\varphi\|_{W_{q'}^{1-\frac{1}{q'}}(\Gamma)}=1}\left|\int_\Gamma fg\varphi\,d\sigma\right| \le C\|f\|_{W_q^{-\frac{1}{q}}(\Gamma)}\|g\|_{W_q^1(\Omega)}.$$

Assertions (2) and (3) are corollaries of (1). □

First we consider the term: $(m-\Delta_\Gamma)^{-1}\Delta_\Gamma\nu\cdot(u_1+u_2+v)$. By Propositions 6.6 and 6.7, we have

$$\begin{aligned}&\|(m-\Delta_\Gamma)^{-1}\Delta_\Gamma\nu\cdot(u_1+u_2+v)\|_{W_q^{2-1/q}(\Gamma)}\\ &\quad\le C_{q,\Gamma}\|\Delta_\Gamma\nu\cdot(u_1+u_2+v)\|_{W_q^{-1/q}(\Gamma)}\\ &\quad\le C_{q,\Gamma}\|\Delta_\Gamma\nu\|_{W_q^{-1/q}(\Gamma)}\|u_1+u_2+v\|_{W_q^1(\Omega)}.\end{aligned}$$

By (6.18) and (6.9),

$$\begin{aligned}&\|(m-\Delta_\Gamma)^{-1}\Delta_\Gamma\nu\cdot(u_1+u_2+v)\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))}\\ &\quad\le C_{q,\Gamma}\|\Delta_\Gamma\nu\|_{W_q^{-1/q}(\Gamma)}\|u_1+u_2+v\|_{L_p((0,T),W_q^1(\Omega))}\\ &\quad\le CM(2M+R)T^{\frac{1}{p}}.\end{aligned}\tag{6.27}$$

Next we consider the term: $(m-\Delta_\Gamma)^{-1}\nu\cdot\dot\Delta_{\Gamma(t)}\int_0^t(u_1+u_2+v)\,d\tau$. $\dot\Delta_{\Gamma(t)}$ is given by

$$\dot\Delta_{\Gamma(t)} = \sum_{j,k=1}^{n-1}\partial_t\Big(\frac{\hat{\mathfrak g}_{tjk}}{\mathfrak g_t}\Big)\frac{\partial^2}{\partial\xi_j\partial\xi_k} + \sum_{k=1}^{n-1}\left[\sum_{j=1}^{n-1}\partial_t\Big(\frac{\partial_j\hat{\mathfrak g}_{tjk}}{\mathfrak g_t} - \frac{\hat{\mathfrak g}_{tjk}\partial_j\mathfrak g_t}{2\mathfrak g_t^2}\Big)\right]\frac{\partial}{\partial\xi_k}.\tag{6.28}$$

Since

$$\begin{aligned}&\|\mathfrak g_t\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))} \le CM,\\ &\|\partial_t\hat{\mathfrak g}_{tjk}\|_{L_p((0,T),W_q^{-1/q}(\Gamma))}\\ &\quad\le C\|\nabla'\alpha\|_{W_q^{1-1/q}(\Gamma)}\|\nabla'(u_1+u_2+v)\|_{L_p((0,T),W_q^1(\Omega))} \le C(2M+R)M,\\ &\|\partial_t\hat{\mathfrak g}_{tjk}\nabla'\mathfrak g_t\|_{L_\infty((0,T),W_q^{-1/q}(\Gamma))}\\ &\quad\le C\|\nabla'\alpha\|_{W_q^{1-1/q}(\Gamma)}\|\nabla'(u_1+u_2+v)\|_{L_p((0,T),W_q^1(\Omega))}\\ &\qquad\times\|\nabla'\mathfrak g+\nabla'(J_3K_3)\|_{L_\infty((0,T),W_q^{1-1/q}(\Gamma))} \le C(2M+R)M^2,\end{aligned}$$

we have

$$\Big\|\frac{\partial_t\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}\Big\|_{L_p((0,T),W_q^{1-1/q}(\Gamma))} \tag{6.29}$$

$$\le C(\|\partial_t\hat{\mathfrak{g}}_{tjk}\|_{L_p((0,T),W_q^{1-1/q}(\Gamma))}+\|(\partial_t\hat{\mathfrak{g}}_{tjk})\nabla'\mathfrak{g}_t\|_{L_p((0,T),W_q^{-1/q}(\Gamma))})$$

$$\le C(2M+R)M^2, \tag{6.30}$$

$$\Big\|\partial_t\Big(\frac{\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}\Big)\Big\|_{L_p((0,T),W_q^{1-1/q}(\Gamma))}\le C(2M+R)M^2, \tag{6.31}$$

$$\Big\|\partial_t\Big(\frac{\partial_j\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}\Big),\ \partial_t\Big(\frac{\hat{\mathfrak{g}}_{tjk}\partial_j\mathfrak{g}_t}{2\mathfrak{g}_t^2}\Big)\Big\|_{L_p((0,T),W_q^{-1/q}(\Gamma))}\le C(2M+R)M^2. \tag{6.32}$$

Therefore we obtain from (6.18), (6.9), (6.31) and (6.32),

$$\begin{aligned}
&\Big\|(m-\Delta_\Gamma)^{-1}\nu\cdot\dot{\Delta}_{\Gamma(t)}\int_0^t(u_1+u_2+v)\,d\tau\Big\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))}\\
&\le C_{q,\Gamma}\Big\|\nu\cdot\dot{\Delta}_{\Gamma(t)}\int_0^t(u_1+u_2+v)\,d\tau\Big\|_{L_p((0,T),W_q^{-1/q}(\Gamma))}\\
&\le C_{q,\Gamma}\|\nu\|_{W_q^{1-1/q}(\Gamma))}\Big\|\partial_t\Big(\frac{\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}\Big)\Big\|_{L_p((0,T),W_q^{1-1/q}(\Gamma))}\\
&\qquad\times\Big\|\int_0^t\nabla'^2(u_1+u_2+v)\,d\tau\Big\|_{L_\infty((0,T),L_q(\Omega))}\\
&\quad+C_{q,\Gamma}\|\nu\|_{W_q^{1-1/q}(\Gamma)}\Bigg(\Big\|\partial_t\Big(\frac{\partial_j\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}\Big)\Big\|_{L_p((0,T),W_q^{-1/q}(\Gamma))}\\
&\quad+\Big\|\partial_t\Big(\frac{\hat{\mathfrak{g}}_{tjk}\partial_j\mathfrak{g}_t}{2\mathfrak{g}_t^2}\Big)\Big\|_{L_p((0,T),W_q^{1-1/q}(\Gamma))}\Bigg)\Big\|\int_0^t\nabla'(u_1+u_2+v)\,d\tau\Big\|_{L_\infty((0,T),W_q^1(\Omega))}\\
&\le C_{q,\Gamma}T^{1/p'}(2M+R)^2M^3.
\end{aligned} \tag{6.33}$$

Next we consider the term: $(m-\Delta_\Gamma)^{-1}\nu\cdot(\Delta_\Gamma-\Delta_{\Gamma(t)})(u_1+u_2+v)$. By (6.19),

$$\begin{aligned}
&(\Delta_\Gamma-\Delta_{\Gamma(t)})u\\
&=\sum_{j,k=1}^{n-1}\Big(\frac{\hat{\mathfrak{g}}_{jk}}{\mathfrak{g}}-\frac{\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}\Big)\frac{\partial^2u}{\partial\xi_j\partial\xi_k}\\
&\qquad+\sum_{k=1}^{n-1}\Bigg[\sum_{j=1}^{n-1}\Big(\frac{\partial_j\hat{\mathfrak{g}}_{jk}}{\mathfrak{g}}-\frac{\partial_j\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}\Big)-\Big(\frac{\hat{\mathfrak{g}}_{jk}\partial_j\mathfrak{g}}{2\mathfrak{g}^2}-\frac{\hat{\mathfrak{g}}_{tjk}\partial_j\mathfrak{g}_t}{2\mathfrak{g}_t^2}\Big)\Bigg]\frac{\partial u}{\partial\xi_k}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{n-1} \frac{(\mathfrak{g}_t - \mathfrak{g})\hat{\mathfrak{g}}_{jk} + \mathfrak{g}(\hat{\mathfrak{g}}_{jk} - \hat{\mathfrak{g}}_{tjk})}{\mathfrak{g}\mathfrak{g}_t} \frac{\partial^2 u}{\partial \xi_j \partial \xi_k} \\
&\quad + \sum_{j,k=1}^{n-1} \frac{(\mathfrak{g}_t - \mathfrak{g})\partial_j \hat{\mathfrak{g}}_{jk} + \mathfrak{g}(\partial_j \hat{\mathfrak{g}}_{jk} - \partial_j \hat{\mathfrak{g}}_{tjk})}{\mathfrak{g}\mathfrak{g}_t} \frac{\partial u}{\partial \xi_k} \\
&\quad + \sum_{j,k=1}^{n-1} \frac{(\mathfrak{g}_t^2 - \mathfrak{g}^2)\hat{\mathfrak{g}}_{jk}\partial_j \mathfrak{g} + \mathfrak{g}^2(\hat{\mathfrak{g}}_{jk} - \hat{\mathfrak{g}}_{tjk})\partial_j \mathfrak{g} + \mathfrak{g}^2 \hat{\mathfrak{g}}_{tjk}(\partial_j \mathfrak{g} - \partial_j \mathfrak{g}_t)}{2\mathfrak{g}\mathfrak{g}_t} \frac{\partial u}{\partial \xi_k}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\|\mathfrak{g}_t - \mathfrak{g}\|_{W_q^{1-1/q}(\Gamma)} \leq C \Big(\|\nabla'^2 \alpha\|_{W_q^{-1/q}(\Gamma)} \| \int_0^t \nabla'(u_1 + u_2 + v)\, d\tau \|_{W_q^1(\Omega)} \\
&\qquad + \|\nabla' \alpha\|_{W_q^{1-1/q}(\Gamma)} \| \int_0^t \nabla'^2 (u_1 + u_2 + v)\, d\tau \|_{L_q(\Omega)} \Big) \leq C T^{1/p'} (2M + R) M, \\
&\|\mathfrak{g}_t - \mathfrak{g}\|_{L_\infty((0,T), W_q^{1-1/q}(\Gamma))} \leq C T^{1/p'} (2M + R) M, \\
&\|\hat{\mathfrak{g}}_{tjk} - \hat{\mathfrak{g}}_{jk}\|_{L_\infty((0,T), W_q^{1-1/q}(\Gamma))} \leq C T^{1/p'} (2M + R) M, \\
&\|\mathfrak{g}, \hat{\mathfrak{g}}_{jk}, \hat{\mathfrak{g}}_{tjk}\|_{L_\infty((0,T), W_q^{1-1/q}(\Gamma))} \leq CM,
\end{aligned}
$$

and for example

$$
\begin{aligned}
&\|(\mathfrak{g}_t - \mathfrak{g})\hat{\mathfrak{g}}_{jk} \nabla'^2 (u_1 + u_2 + v)\|_{L_p((0,T), W_q^{-1/q}(\Gamma))} \\
&\quad \leq C \|\mathfrak{g}_t - \mathfrak{g}\|_{L_\infty((0,T), W_q^{1-1/q}(\Gamma))} \\
&\qquad \times \|\hat{\mathfrak{g}}_{jk}\|_{L_\infty((0,T), W_q^{1-1/q}(\Gamma))} \|\nabla'^2 (u_1 + u_2 + v)\|_{L_p((0,T), L_q(\Omega))} \\
&\quad \leq C T^{1/p'} (2M + R)^2 M,
\end{aligned}
$$

we have

$$
\|(\Delta_\Gamma - \Delta_{\Gamma(t)})(u_1 + u_2 + v)\|_{L_p((0,T), W_q^{-1/q}(\Gamma))} \leq C T^{1/p'} (2M + R)^2 M^2.
$$

Therefore we obtain

$$
\begin{aligned}
&\|(m - \Delta_\Gamma)^{-1} \nu \cdot (\Delta_\Gamma - \Delta_{\Gamma(t)})(u_1 + u_2 + v)\|_{L_p((0,T), W_q^{2-1/q}(\Gamma))} \\
&\quad \leq C \|\nu\|_{W_q^{1-1/q}(\Gamma)} \|(\Delta_\Gamma - \Delta_{\Gamma(t)})(u_1 + u_2 + v)\|_{L_p((0,T), W_q^{-1/q}(\Gamma))} \\
&\quad \leq C T^{1/p'} (2M + R)^2 M^3. \qquad (6.34)
\end{aligned}
$$

Finally we consider the term: $(m - \Delta_\Gamma)^{-1} \nu \cdot \dot{\Delta}_\Gamma(t)\xi$. By a direct calculation with

$$
\dot{\Delta}_{\Gamma(t)} \xi_\ell = \sum_{j=1}^{n-1} \Big(\frac{\partial_j \dot{\hat{\mathfrak{g}}}_{tj\ell}}{\mathfrak{g}_t} - \frac{\partial_j \hat{\mathfrak{g}}_{tj\ell} \dot{\mathfrak{g}}_t}{\mathfrak{g}_t^2} - \frac{\dot{\hat{\mathfrak{g}}}_{tj\ell} \partial_j \mathfrak{g}_t}{2\mathfrak{g}_t^2} - \frac{\hat{\mathfrak{g}}_{tj\ell} \partial_j \dot{\mathfrak{g}}_t}{2\mathfrak{g}_t^2} + \frac{\hat{\mathfrak{g}}_{tj\ell} \partial_j \mathfrak{g}_t \dot{\mathfrak{g}}_t}{\mathfrak{g}_t^3} \Big),
$$

$$\dot{\Delta}_{\Gamma(t)}\xi_n = \sum_{j,k=1}^{n-1}\Big(\frac{\dot{\hat{\mathfrak{g}}}_{tjk}}{\mathfrak{g}_t} - \frac{\hat{\mathfrak{g}}_{tjk}\dot{\mathfrak{g}}_t}{\mathfrak{g}_t^2}\Big)\frac{\partial^2\alpha}{\partial\xi_j\partial\xi_k}$$
$$+ \sum_{j,k=1}^{n-1}\Big(\frac{\partial_j\dot{\hat{\mathfrak{g}}}_{tjk}}{\mathfrak{g}_t} - \frac{\partial_j\hat{\mathfrak{g}}_{tjk}\dot{\mathfrak{g}}_t}{\mathfrak{g}_t^2} - \frac{\dot{\hat{\mathfrak{g}}}_{tjk}\partial_j\mathfrak{g}_t}{2\mathfrak{g}_t^2} - \frac{\hat{\mathfrak{g}}_{tjk}\partial_j\dot{\mathfrak{g}}_t}{2\mathfrak{g}_t^2} + \frac{\hat{\mathfrak{g}}_{tjk}\partial_j\mathfrak{g}_t\dot{\mathfrak{g}}_t}{\mathfrak{g}_t^3}\Big)\frac{\partial\alpha}{\partial\xi_k},$$

it holds that

$$\nu\cdot\dot{\Delta}_\Gamma(t)\xi = \frac{1}{\sqrt{1+|\nabla'\alpha|^2}}\sum_{j,k=1}^{n-1}\left(\frac{\partial_t\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t} - \frac{\hat{\mathfrak{g}}_{tjk}\partial_t\mathfrak{g}_t}{\mathfrak{g}_t^2}\right)\frac{\partial^2\alpha}{\partial\xi_j\partial\xi_k} \tag{6.35}$$

(cf. [24]). Since, from (6.9),

$$\begin{aligned}
\left\|\frac{\partial_t\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}\right\|_{L_p((0,T),W_q^{-1/q}(\Gamma))} &\le C\|\nabla'\alpha\|_{W_q^{-1/q}(\Gamma))}\|\nabla'(u_1+u_2+v)\|_{L_p((0,T),L_q(\Omega))}\\
&\le CT^{\frac{1}{p}}(2M+R)M,\\
\left\|\frac{\hat{\mathfrak{g}}_{tjk}\partial_t\hat{\mathfrak{g}}_t}{\mathfrak{g}_t^2}\right\|_{L_p((0,T),W_q^{-1/q}(\Gamma))} &\le CT^{\frac{1}{p}}(2M+R)M^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\left\|(m-\Delta_\Gamma)^{-1}\nu\cdot\dot{\Delta}_\Gamma(t)\xi\right\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))}\\
&\le C\left(\left\|\frac{\partial_t\hat{\mathfrak{g}}_{tjk}}{\mathfrak{g}_t}\right\|_{L_p((0,T),W_q^{-1/q}(\Gamma))} + \left\|\frac{\hat{\mathfrak{g}}_{tjk}\partial_t\mathfrak{g}_t}{\mathfrak{g}_t^2}\right\|_{L_p((0,T),W_q^{-1/q}(\Gamma))}\right)\|\alpha\|^2_{W_q^{3-1/q}(\Gamma)}\\
&\le C((2M+R)t^{\frac{1}{p}})M^4.
\end{aligned} \tag{6.36}$$

Therefore from (6.27), (6.33), (6.34) and (6.36), we obtain

$$\begin{aligned}
&G(u_1+u_2+v)+\nu\cdot(u_1+u_2)\in L_p((0,T),W_q^{2-1/q}(\Gamma)),\\
&\|G(u_1+u_2+v)+\nu\cdot(u_1+u_2)\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))}\\
&\qquad\le C\{T^{\frac{1}{p}}(2M+R)M^4 + T^{1/p'}(2M+R)^2M^3\}.
\end{aligned} \tag{6.37}$$

Step 3 (Estimate of H_t and H_n)**.** Next we estimate the right-hand side of the boundary condition H_t and H_n. From the definition of $H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})$, we have to extend not only u_2 and v but also $\int_0^t\nabla(u_1+u_2+v)\,d\tau$ to the whole time $\mathbb{R}$. To do this, we use Lemma 6.3 and the following lemma.

Lemma 6.8 (Lemma 2.4 in [20]**).** *Let* $2<p<\infty$ *and* $n<q<\infty$*. Set*

$$\hat{W}^{1,1}_{q,p}(\dot{\Omega}\times I) = \{f\in W^{1,1}_{q,\infty}(\dot{\Omega}\times I) : \partial_t f\in L_p(I,W^1_q(\dot{\Omega}))\}.$$

Let $0<T\le 1$*. Then, there exist linear operators* $\mathbb{F}_1 : W^1_q(\Omega)\to\hat{W}^{1,1}_{q,p}(\Omega\times\mathbb{R})$, $\mathbb{F}_2 : W^{2,1}_{q,p}(\Omega\times(0,2))\to\hat{W}^{1,1}_{q,p}(\Omega\times\mathbb{R})$, $\mathbb{F}_3 : W^{2,1}_{q,p,0}(\Omega\times(0,T))\to\hat{W}^{1,1}_{q,p}(\Omega\times\mathbb{R})$,

such that

$$
\begin{aligned}
&\mathbb{F}_j z(\xi,t) = \int_0^t \nabla z(\xi,\tau)\,d\tau, \quad 0 \le t \le T,\\
&\mathbb{F}_j z = 0 \quad \textit{for } t \notin [0,2T],\\
&\|\mathbb{F}_1 z\|_{L_\infty(\mathbb{R},W^1_q(\Omega))} \le CT\|z\|_{W^1_q(\Omega)},\\
&\|\partial_t(\mathbb{F}_1 z)\|_{L_\infty(\mathbb{R},L_q(\Omega))} \le C\|z\|_{W^1_q(\Omega)},\\
&\|\partial_t(\mathbb{F}_1 z)\|_{L_p(\mathbb{R},W^1_q(\Omega))} \le C\|z\|_{W^1_q(\Omega)},\\
&\|\mathbb{F}_j z\|_{L_\infty(\mathbb{R},W^1_q(\Omega))} \le CT^{1/p'}\|z\|_{W^{2,1}_{q,p}(\Omega\times I_j)} \quad \textit{for } j=2,3,\\
&\|\partial_t(\mathbb{F}_j z)\|_{L_\infty(\mathbb{R},L_q(\Omega))} \le C\|z\|_{W^{2,1}_{q,p}(\Omega\times I_j)} \quad \textit{for } j=2,3,\\
&\|\partial_t(\mathbb{F}_j z)\|_{L_p(\mathbb{R},W^1_q(\Omega))} \le C\|z\|_{W^{2,1}_{q,p}(\Omega\times I_j)} \quad \textit{for } j=2,3,
\end{aligned}
$$

where $I_2=(0,2)$, $I_3=(0,T)$, *and* C *is independent of* z *and* T.

We recall that

$$
[\![H_t(u)]\!] = -[\![\mathbf{\Pi}(\mathbf{\Pi}_t-\mathbf{\Pi})(\mu D(u)\nu_{tu}) + \mathbf{\Pi}(\mu D(u)(\nu_{tu}-\nu)) + \mathbf{\Pi}\,\mathbf{\Pi}_t(Q(u)\nu_{tu})]\!],
$$

where $Q(u)$ is defined in (1.6). We consider the term $H_t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)$ instead of $H_t(u_1+u_2+v)$ with time integral $\mathbb{F}_1u_1+\mathbb{F}_2u_2+\mathbb{F}_3v$, because we have to estimate the norm of $H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})$. Since

$$
\begin{aligned}
&({}^tA^{-1})_{jk}\\
&= \delta_{jk} + B_{jk}\left(\int_0^t \nabla'_k u_1\,d\tau,\dots,\int_0^t \nabla'_k u_{j-1}\,d\tau,\int_0^t \nabla'_k u_{j+1}\,d\tau,\dots,\int_0^t \nabla'_k u_n\,d\tau\right),
\end{aligned}
\tag{6.38}
$$

where B_{jk} is some polynomial such as $B_{ij}(0,\dots,0)=0$ (cf. (A.5) Appendix in [20]), we can write

$$
{}^tA^{-1}\nu = \nu + B\left(\int_0^t \nabla u\,d\tau\right)\nu,
$$

where (j,k) component of B is given by B_{jk} in (6.38). Therefore we obtain

$$
\nu_{tu} = \frac{{}^tA^{-1}\nu}{|{}^tA^{-1}\nu|} = \frac{\nu + B(\int_0^t \nabla u\,d\tau)\nu}{|\nu + B(\int_0^t \nabla u\,d\tau)\nu|}, \tag{6.39}
$$

$$
\nu_{tu} - \nu = \frac{B\nu}{|(I+B)\nu|} - \nu\frac{\nu\cdot B\nu + B\nu\cdot(I+B)\nu}{|\nu||(I+B)\nu|(|(I+B)\nu|+|\nu|)}, \tag{6.40}
$$

$$
\partial_t\nu_{tu} = \frac{\partial_t B\,\nu}{|(I+B)\nu|} - \frac{(I+B)\nu(I+B)\nu)\cdot\partial_t B\nu}{|(I+B)\nu|^3}. \tag{6.41}
$$

By (6.4) and (6.5), we obtain

$$
\left\|B(\mathbb{F}_1u_1+\mathbb{F}_2u_2+\mathbb{F}_3v)\right\|_{L_\infty(\mathbb{R},W^1_q(\Omega))} \le CT^{1/p'}(2M+R), \tag{6.42}
$$

and by (6.4), (6.40)–(6.42) and (6.18), we obtain

$$\|\tilde{\nu}_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}\|_{L_\infty(\mathbb{R},W^1_q(\Omega))} \leq CM, \tag{6.43}$$

$$\begin{aligned}\|\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)} - \nu\|_{L_\infty(\mathbb{R},W^1_q(\Omega))} &\leq \|B\|_{L_\infty(\mathbb{R},W^1_q(\Omega))}\|\nu\|_{W^1_q(\Omega)}\\ &\leq CT^{1/p'}(2M+R)M. \end{aligned}\tag{6.44}$$

Here and hereafter ν_{tu} and ν denote again the extension of ν_{tu} and ν, respectively, on Γ to Ω. By

$$\begin{aligned}\Big\|\partial_t B(\mathbb{F}_1u_1+\mathbb{F}_2u_2+\mathbb{F}_3v)\Big\|_{L_\infty((0,T),L_q(\Omega))} &\leq C\|u_1+\mathbb{E}_2u_2+\mathbb{E}v\|_{L_\infty(\mathbb{R},W^1_q(\Omega))}\\ &\leq C(2M+R), \end{aligned}\tag{6.45}$$

$$\begin{aligned}\Big\|\partial_t B(\mathbb{F}_1u_1+\mathbb{F}_2u_2+\mathbb{F}_3v)\Big\|_{L_p((0,T),W^1_q(\Omega))} &\leq C\|u_1+\mathbb{E}_2u_2+\mathbb{E}v\|_{L_p(\mathbb{R},W^2_q(\Omega))}\\ &\leq C(2M+R), \end{aligned}\tag{6.46}$$

and (6.4), (6.16), (As.3), (5.1), (5.3), we have

$$\|\partial_t\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}\|_{L_\infty(\mathbb{R},L_q(\Omega))} \leq C\|u_1+\mathbb{E}_2u_2+\mathbb{E}v\|_{L_\infty(\mathbb{R},W^1_q(\Omega))} \leq C(2M+R), \tag{6.47}$$

$$\|\partial_t\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}\|_{L_p(\mathbb{R},W^1_q(\Omega))} \leq C\|u_1+\mathbb{E}_2u_2+\mathbb{E}v\|_{L_p(\mathbb{R},W^2_q(\Omega))} \leq C(2M+R). \tag{6.48}$$

To estimate the norm of $H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})$, we use the following lemma.

Lemma 6.9 (Lemma 2.6 in [20]). *Let $1 < p < \infty$, $n < q < \infty$ and $0 < T \leq 1$. Let $\hat{W}^{1,1}_{q,p}(\Omega\times\mathbb{R})$ be the same space as in Lemma 6.8. If $f \in \hat{W}^{1,1}_{q,p}(\Omega\times\mathbb{R})$, $g \in H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})$ and f vanishes when $t \notin [0,2T]$, then we have*

$$\|fg\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leq$$
$$C_{p,q}[\|f\|_{L_\infty(\mathbb{R},W^1_q(\Omega))}+T^{(q-n)/(pq)}\|f_t\|^{(1-n/(2q))}_{L_\infty(\mathbb{R},L_q(\Omega))}\|f_t\|^{n/(2q)}_{L_p(\mathbb{R},W^1_q(\Omega))}]\|g\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}.$$

Proposition 2.8 in [20] and Lemma 6.3 yield the following lemma.

Lemma 6.10. *Let $1 < p,q < \infty$. For $u_2 \in W^{2,1}_{q,p}(\Omega\times(0,2))$ and $v \in W^{2,1}_{q,p}(\Omega\times(0,T))$, there exists a constant $C_{p,q} > 0$ such that*

$$\begin{aligned}\|\mathbb{E}_2u_2\|_{H^{1/2}_p(\mathbb{R},W^1_q(\Omega))} &\leq C_{p,q}\|u_2\|_{W^{2,1}_{q,p}(\Omega\times(0,2))},\\ \|\mathbb{E}v\|_{H^{1/2}_p(\mathbb{R},W^1_q(\Omega))} &\leq C_{p,q}\|v\|_{W^{2,1}_{q,p}(\Omega\times(0,T))}.\end{aligned}$$

By (4.2), it holds that

$$(\mathbf{\Pi}-\mathbf{\Pi}_t)d = (d,\nu_{tu})\nu_{tu} - (d,\nu)\nu = (d,\nu_{tu})(\nu_{tu}-\nu) + (d,\nu_{tu}-\nu)\nu. \tag{6.49}$$

Setting $f = \nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}$ and $g = D(u_1+\mathbb{E}_2u_2+\mathbb{E}v)$, and applying Lemma 6.9, we have from (6.44), (6.47), (6.48) and Lemma 6.10,

$$\|D(u_1+\mathbb{E}_2u_2+\mathbb{E}v)\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \leq C(2M+R)M. \tag{6.50}$$

Setting $f = \nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)} - \nu$ and $g = (D(u_1 + \mathbb{E}_2u_2 + \mathbb{E}v)\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}, \nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)})$, and applying Lemma 6.9, we have from Lemma 6.10,

$$\begin{aligned}&\|(D(u_1 + \mathbb{E}_2u_2 + \mathbb{E}v)\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}, \nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)})\\ &\times (\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)} - \nu)\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \le C(T^{\frac{1}{p'}}M + T^{\frac{q-n}{pq}})(2M + R)^2M^2. \end{aligned}\tag{6.51}$$

Combining (6.49), (6.50) and (6.51) we have

$$\begin{aligned}&\|\mathbf{\Pi}(\mathbf{\Pi} - \mathbf{\Pi}_t)\mu D(u_1 + \mathbb{E}_2u_2 + \mathbb{E}v)\nu_{tu_1+\mathbb{E}_2u_2+\mathbb{E}v}\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}\\ &\qquad\le C(T^{\frac{1}{p'}}M + T^{\frac{q-n}{pq}})(2M + R)^2M^3. \end{aligned}\tag{6.52}$$

In a similar manner we obtain

$$\begin{aligned}&\|\mathbf{\Pi}\mu D(u_1 + \mathbb{E}_2u_2 + \mathbb{E}v)(\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)} - \nu)\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}\\ &\qquad\le C(T^{\frac{1}{p'}}M + T^{\frac{q-n}{pq}})(2M + R)^2M^2, \qquad (6.53)\\ &\|\mathbf{\Pi}\,\mathbf{\Pi}_tQ(u_1 + \mathbb{E}_2u_2 + \mathbb{E}v)\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})}\\ &\qquad\le C(T^{\frac{1}{p'}} + T^{\frac{q-n}{pq}})(2M + R)^2M^3. \qquad (6.54)\end{aligned}$$

Therefore by (6.52), (6.53), and (6.54) we obtain

$$\begin{aligned}&H_t(u_1 + \mathbb{E}_2u_2 + \mathbb{E}v) \in H^{1,1/2}_{q,p}(\Omega\times\mathbb{R}),\\ &\|H_t(u_1 + \mathbb{E}_2u_2 + \mathbb{E}v)\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \le C(T^{\frac{1}{p'}}M + T^{\frac{q-n}{pq}})(2M + R)^2M^3. \end{aligned}\tag{6.55}$$

From the definition of $H^{1,1/2}_{q,p}(\Omega\times(0,T))$, we see that

$$\begin{aligned}&H_t(u_1 + u_2 + v) \in H^{1,1/2}_{q,p}(\Omega\times(0,T)),\\ &\|H_t(u_1 + u_2 + v)\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))} \le C(T^{\frac{1}{p'}}M + T^{\frac{q-n}{pq}})(2M + R)^2M^3. \end{aligned}\tag{6.56}$$

Finally we consider the term H_n. We recall that

$$\begin{aligned}[\![H_n(u,\pi)]\!] &= [\![\nu\cdot S(u,\pi)(\nu - \nu_{tu}) - \nu\cdot Q(u)\nu_{tu}]\!]\\ &\quad + [\![\rho]\!]c_g\int_0^t u_n\,d\tau\,\nu\cdot\nu_{tu} + [\![\rho]\!]c_g\xi_n\,\nu\cdot(\nu_{tu} - \nu)\\ &\quad - 2\sigma\Big(\nabla_\Gamma\cdot\int_0^t u\,d\tau\Big)\nabla_\Gamma\cdot\nu + \sigma m\int_0^t \nu\cdot u\,d\tau.\end{aligned}$$

We consider the term $H_n(u_1 + \mathbb{E}_2u_2 + \mathbb{E}v, \pi_1 + \bar\pi_2 + \bar\theta)$ instead of $H_n(u_1 + u_2 + v, \pi_1 + \pi_2 + \theta)$ with time integral $\mathbb{F}_1u_1 + \mathbb{F}_2u_2 + \mathbb{F}_3v$, because we have to estimate the norm of $H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})$. Similar to the way we estimated $H_t(u_1 + \mathbb{E}_2u_2 + \mathbb{E}v)$, we have

$$\|\nu\cdot(S(u_1+\mathbb{E}_2u_2+\mathbb{E}v,\pi_1+\bar\pi_2+\bar\theta)(\nu-\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)})\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \le C(T^{\frac{1}{p'}}M+T^{\frac{q-n}{pq}})(2M+R)^2M^2, \quad (6.57)$$

$$\|\nu\cdot Q(u_1+\mathbb{E}_2u_2+\mathbb{E}v)\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} \le C(T^{\frac{1}{p'}}+T^{\frac{q-n}{pq}})(2M+R)^2M^2. \quad (6.58)$$

Let $\chi(t)$ be a function in $C_0^\infty(\mathbb{R})$ such that $\chi(t)=1$ for $|t|\le 1$ and $\chi(t)=0$ for $|t|\ge 2$. Setting $f=\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}-\nu$ and $g=\chi(t)\xi_n\nu$, and applying Lemma 6.9, we have

$$\begin{aligned}\|(\nu_{t(u_1+\mathbb{E}_2u_2+\mathbb{E}v)}-\nu)\chi(t)\xi_n\nu\|_{H^{1,1/2}_{q,p}(\Omega\times\mathbb{R})} &\le C(T^{1/p'}M+T^{\frac{q-n}{pq}})(2M+R)\|\chi(t)\xi_n\nu\|_{L_p(\mathbb{R},W^1_q(\Omega))}\\ &\le C(T^{1/p'}M+T^{\frac{q-n}{pq}})(2M+R)M^2.\end{aligned}$$

By the definition of $H^{1,1/2}_{q,p}(\Omega\times(0,T))$, we see that

$$\|(\nu_{t(u_1+u_2+v)}-\nu)\xi_n\nu\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))} \le C(T^{1/p'}M+T^{\frac{q-n}{pq}})(2M+R)M^2. \quad (6.59)$$

By (6.7) we have

$$\left\|(\nabla_\Gamma\cdot\nu)\nabla_\Gamma\cdot\int_0^t u\,d\tau\right\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))} \le C(T^{1/p'}M+T^{\frac{q-n}{pq}})(2M+R)M^2. \quad (6.60)$$

Therefore from (6.57), (6.58), (6.7), (6.59) and (6.60), we obtain

$$H_n(u_1+u_2+v,\pi_1+\pi_2+\theta)\in H^{1,1/2}_{q,p}(\Omega\times(0,T)),$$

$$\|H_n(u_1+u_2+v,\pi_1+\pi_2+\theta)\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))} \le C(T^{1/p'}M+T^{\frac{q-n}{pq}})(2M+R)M^2. \quad (6.61)$$

Step 4. Combining (6.37), (6.56), (6.61), we have

$$\begin{aligned}\|V\|_{W^{2,1}_{q,p}(\Omega\times(0,T))}&+\|\Theta\|_{L_p((0,T),\hat W^1_q(\Omega))}+\|\bar\Theta\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))}\\ &+\|\partial_tY\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}+\|Y\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))}\\ &\le C\{T^{\frac{1}{p}}(2M+R)M^4+(T^{\frac{1}{p'}}M+T^{\frac{q-n}{pq}})(2M+R)^2M^3\}. \quad (6.62)\end{aligned}$$

Here we also use

$$\begin{aligned}\|\mathrm{Div}\,Q(u_1+u_2+v),\ R(u_1+u_2+v)\nabla(\pi_1+\pi_2+\theta)\|_{L_p((0,T),L_q(\Omega))} &\le CT^{\frac{1}{p'}}(2M+R)^2,\\ \|E(u_1+u_2+v)\|_{L_p((0,T),W^1_q(\Omega))} &\le CT^{\frac{1}{p'}}(2M+R)^2,\\ \|\tilde E(u_1+u_2+v)\|_{W^1_p((0,T),L_q(\Omega))} &C(T^{\frac{1}{p'}}+T^{\frac{q-n}{pq}})(2M+R)^2\end{aligned}$$

(cf. §2 in [20]). For given M defined by (6.2), we choose $R > 0$ and $T > 0$ such that the right-hand member of (6.62) is less than R, then by (6.62) we have

$$\|V\|_{W^{2,1}_{q,p}(\Omega\times(0,T))} + \|\Theta\|_{L_p((0,T),\hat W^1_q(\Omega))} + \|\bar\Theta\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))} + \|\partial_t Y\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))} + \|Y\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))} \le R. \tag{6.63}$$

Moreover from the definition of $E(u_1+u_2+v)$, $\tilde E(u_1+u_2+v)$, $H_t(u_1+u_2+v)$ and $H_n(u_1+u_2+v,\pi_1+\pi_2+\theta)$, we see that

$$\begin{aligned} E(u_1+u_2+v)|_{t=0} &= \tilde E(u_1+u_2+v)|_{t=0} = 0,\\ H_t(u_1+u_2+v)|_{t=0} &= H_n(u_1+u_2+v,\pi_1+\pi_2+\theta)|_{t=0} = 0. \end{aligned}$$

Therefore for the map $\Phi(v,\theta,\bar\theta,\eta) = (V,\Theta,\bar\Theta,Y)$ it maps $I_{R,T}$ into itself.

Now we show that Φ is a contraction map. To do this, given $(v_i,\theta_i,\bar\theta_i,\eta_i) \in I_{R,T}$ $(i=1,2)$, we set

$$(V_i,\Theta_i,\bar\Theta_i,Y_i) = \Phi(v_i,\theta_i,\bar\theta_i,\eta_i), \quad i = 1,2.$$

From (6.3), we see that $V_1 - V_2$, $\Theta_1-\Theta_2$ and $Y_1 - Y_2$ satisfy the linear equation

$$\begin{aligned}
&\partial_t(V_1-V_2) - \mathrm{Div}\, S(V_1-V_2,\Theta_1-\Theta_2) = \mathrm{Div}\, Q(u_1+u_2+v_1)\\
&\quad - \mathrm{Div}\, Q(u_1+u_2+v_2) + R(\pi_1+\pi_2+\theta_1) - R(\pi_1+\pi_2+\theta_2) && \text{in } \Omega,\ t>0,\\
&\mathrm{div}\,(V_1-V_2) = E(u_1+u_2+v_1) - E(u_1+u_2+v_2)\\
&\qquad\qquad = \mathrm{div}\tilde E(u_1+u_2+v_1) - \mathrm{div}\tilde E(u_1+u_2+v_2) && \text{in } \Omega,\ t>0,\\
&\partial_t(Y_1-Y_2) - \nu\cdot(V_1-V_2) = G(u_1+u_2+v_1) - G(u_1+u_2+v_2) && \text{on } \Gamma,\ t>0,\\
&[\![\mathbf{\Pi} D(V_1-V_2)\nu]\!] = [\![H_t(u_1+u_2+v_1) - H_t(u_1+u_2+v_2)]\!] && \text{on } \Gamma,\ t>0,\\
&[\![\nu\cdot S(V_1-V_2,\Theta_1-\Theta_2)\nu]\!] + \sigma(m-\Delta_\Gamma)(Y_1-Y_2)\\
&\quad = [\![H_n(u_1+u_2+v_1,\pi_1+\bar\pi_2+\bar\theta_1)\\
&\qquad - H_n(u_1+u_2+v_2,\pi_1+\bar\pi_2+\bar\theta_2)]\!] && \text{on } \Gamma,\ t>0,\\
&[\![V_1-V_2]\!] = 0 && \text{on } \Gamma,\ t>0,\\
&(V_1-V_2)|_{t=0} = 0 \text{ in } \Omega, \quad (Y_1-Y_2)|_{t=0} = 0 \text{ on } \Gamma.
\end{aligned} \tag{6.64}$$

Applying the same argument as in the proof of (6.63), we can show that

$$\begin{aligned}
&\|V_1-V_2\|_{W^{2,1}_{q,p}((0,T)\times\Omega)} + \|\Theta_1-\Theta_2\|_{L_p((0,T),\hat W^1_q(\Omega))}\\
&\qquad + \|\bar\Theta_1-\bar\Theta_2\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))} + \|Y_1-Y_2\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}\\
&\le C\{T^{\frac1p}M^4 + (T^{\frac{1}{p'}} + T^{\frac{q-n}{pq}})(2M+R)M^3\}\\
&\quad \times \Big[\|v_1-v_2\|_{W^{2,1}_{q,p}((0,T)\times\Omega)} + \|\theta_1-\theta_2\|_{L_p((0,T),\hat W^1_q(\Omega))}\\
&\qquad + \|\bar\theta_1-\bar\theta_2\|_{H^{1,1/2}_{q,p}(\Omega\times(0,T))} + \|\eta_1-\eta_2\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))}\Big].
\end{aligned} \tag{6.65}$$

Choosing $T > 0$ so small that

$$C\{T^{\frac{1}{p}}M^4 + (T^{\frac{1}{p'}} + T^{\frac{q-n}{pq}})(2M+R)M^3\} \leq \frac{1}{2}$$

in (6.65), we see that Φ is a contraction map on $I_{R,T}$. Therefore Φ has the fixed point $(V, \Theta, \bar{\Theta}, Y)$ which solves (6.3). This completes the proof of Theorem 6.1.

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Inversion of the Lagrange Theorem in the Problem of Stability of Rotating Viscous Incompressible Liquid

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Dedicated to Professor H. Amann on the occasion of his jubilee

Abstract. The paper contains analysis of the spectrum of a linear problem arising in the study of the stability of a finite isolated mass of uniformly rotating viscous incompressible self-gravitating liquid. It is assumed that the capillary forces on the free boundary of the liquid are not taken into account. It is proved that when the second variation of the energy functional can take negative values, then the spectrum of the problem contains finite number of points with positive real parts, which means instability of the rotating liquid in a linear approximation. The proof relies on the theorem on the invariant subspaces of dissipative operators in the Hilbert space with an indefinite metrics.

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1. Formulation of main result

This paper is devoted to the problem of stability of a finite mass of a viscous incompressible liquid of a unit density uniformly rotating about a fixed axis (x_3-axis). The liquid is subject to the forces of self-gravitation; the surface tension on the free boundary is not taken into account. The velocity and the pressure of the rotating liquid are given by

$$\boldsymbol{V}(x) = \omega(\boldsymbol{e}_3 \times \boldsymbol{x}), \quad P(x) = \frac{\omega^2}{2}|x'|^2 + p_0, \tag{1.1}$$

where ω is the angular velocity of rotation, $\boldsymbol{e}_3 = (0, 0, 1)$, $x' = (x_1, x_2, 0)$, $p_0 = const$. The domain $\mathcal{F}$ occupied by the liquid (the equilibrium figure) is defined by the equation

$$\frac{\omega^2}{2}|x'|^2 + \kappa\mathcal{U}(x) + p_0 = 0, \quad x \in \mathcal{G} = \partial\mathcal{F}, \tag{1.2}$$

where $\mathcal{U}(x) = \int_{\mathcal{F}} \frac{dy}{|x-y|}$.

In what follows we assume that $\mathcal{F}$ is a given bounded domain with a smooth boundary defined by (1.2). The barycenter of $\mathcal{F}$ is located on the axis of rotation, moreover, we assume that

$$\int_{\mathcal{F}} x_i dx = 0, \quad i = 1, 2, 3.$$

For simplicity we assume that $\mathcal{F}$ is rotationally symmetric with respect to the x_3-axis. The case of non-symmetric $\mathcal{F}$ is considered below in Sec. 4.

The functions (1.1) represent a solution of the free boundary problem governing the evolution of an isolated liquid mass bounded only by a free surface:

$$\begin{gathered}
\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0,\\
\nabla\cdot\boldsymbol{v}(x,t) = 0, \quad x \in \Omega_t, \ \ t > 0,\\
T(\boldsymbol{v},p)\boldsymbol{n} = \kappa U(x,t)\boldsymbol{n}(x),\\
V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x \in \Gamma_t = \partial\Omega_t,\\
\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_0,
\end{gathered} \tag{1.3}$$

where Ω_t is a domain occupied by the liquid, unknown for $t > 0$ and given for $t = 0$, Γ_t is the boundary of Ω_t, $\boldsymbol{n}$ is the exterior normal to Γ_t, $T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$ is the stress tensor, $S(\boldsymbol{v}) = \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j}\right)_{j,k=1,2,3}$ is the doubled rate-of-strain tensor, $\boldsymbol{n}$ is the exterior normal to Γ_t, V_n is the velocity of evolution of Γ_t in the normal direction,

$$U(x,t) = \int_{\Omega_t} \frac{dy}{|x-y|}$$

is the Newtonian potential. The solution of (1.3) obeys the conservation laws

$$\begin{gathered}
|\Omega_t| = |\Omega_0|,\\
\int_{\Omega_t} \boldsymbol{v}(x,t)dx = \int_{\Omega_0} \boldsymbol{v}_0(x)dx,\\
\int_{\Omega_t} \boldsymbol{v}(x,t)\cdot\boldsymbol{\eta}_i(x)dx = \int_{\Omega_0} \boldsymbol{v}_0(x)\cdot\boldsymbol{\eta}_i(x)dx, \quad i = 1, 2, 3,
\end{gathered}$$

where $\boldsymbol{\eta}_i(x) = \boldsymbol{e}_i \times \boldsymbol{x}$, $\boldsymbol{e}_i = (\delta_{ij})_{j=1,2,3}$.

It is customary to consider the free boundary problem for the perturbations of $\boldsymbol{V}$ and P written in the coordinate system rotating about the x_3-axis with the same angular velocity ω. It has the form

$$\begin{gathered}
\boldsymbol{w}_t + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}) - \nu\nabla^2\boldsymbol{w} + \nabla s = 0,\\
\nabla\cdot\boldsymbol{w}(y,t) = 0, \qquad y \in \Omega'_t, \quad t > 0,\\
T(\boldsymbol{w},s)\boldsymbol{n}' = (\frac{\omega^2}{2}|y'|^2 + \kappa U'(y,t) + p_0)\boldsymbol{n}',
\end{gathered} \tag{1.4}$$

$$V_n' = \boldsymbol{w}\cdot\boldsymbol{n}', \quad y\in\Gamma_t',$$
$$\boldsymbol{w}(y,0) = \boldsymbol{v}_0(y) - \boldsymbol{V}(y) \equiv \boldsymbol{w}_0(y), \quad y\in\Omega_0,$$

where $\Omega_t' = \mathcal{Z}(\omega t)\Omega_t$,

$$\mathcal{Z}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\boldsymbol{w}_0$ is a small perturbation of $\boldsymbol{V}$, Ω_0 is a given domain close to $\mathcal{F}$.

To the solution (1.2) of (1.3) corresponds the zero solution of (1.4).

We assume that

$$|\Omega_0| = |\mathcal{F}|, \quad \int_{\Omega_0} \boldsymbol{v}_0(x)dx = \int_{\mathcal{F}} \boldsymbol{V}(x)dx = 0,$$
$$\int_{\Omega_0} x_i dx = 0, \quad \int_{\Omega_0} \boldsymbol{v}_0(x)\cdot\boldsymbol{\eta}_i(x)dx = \int_{\mathcal{F}} \boldsymbol{V}(x)\cdot\boldsymbol{\eta}_i(x)dx, \quad i=1,2,3, \tag{1.5}$$

which implies

$$|\Omega_t'| = |\mathcal{F}|, \quad \int_{\Omega_t'} x_i dx = 0, \quad i=1,2,3,$$
$$\int_{\Omega_t'} \boldsymbol{w}(x,t)dx = 0,$$
$$\int_{\Omega_t'} \boldsymbol{w}(x,t)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\Omega_t'} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx = \omega\int_{\mathcal{F}} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx, \quad i=1,2,3. \tag{1.6}$$

If Γ_t' is close to $\mathcal{G}$, then it can be given by the relation

$$x = y + \boldsymbol{N}(y)\rho(y,t), \quad y\in\mathcal{G}, \tag{1.7}$$

where $\boldsymbol{N}$ is the exterior normal to $\mathcal{G}$ and ρ is a small function. The conditions (1.5) are equivalent to

$$\int_{\mathcal{G}} \varphi(y,\rho)dS = 0, \quad \int_{\mathcal{G}} \psi_i(y,\rho)dS = 0, \quad i=1,2,3, \tag{1.8}$$

where

$$\varphi(y,\rho) = \rho - \frac{\rho^2}{2}\mathcal{H}(y) + \frac{\rho^3}{3}\mathcal{K}(y),$$
$$\psi_i(y,\rho) = \varphi(y,\rho)y_i + N_i(y)\Big(\frac{\rho^2}{2} - \frac{\rho^3}{3}\mathcal{H}(y) + \frac{\rho^4}{4}\mathcal{K}(y)\Big),$$

$\mathcal{H}$ and $\mathcal{K}$ are the doubled mean curvature and the Gaussian curvature of $\mathcal{G}$, respectively. The kinematic boundary condition $V_n = \boldsymbol{w}\cdot\boldsymbol{n}$ can be written in the form

$$\rho_t(y,t) = \frac{\boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x)}{\boldsymbol{N}(y)\cdot\boldsymbol{n}(x)}, \quad x = y + \boldsymbol{N}(y)\rho(y,t). \tag{1.9}$$

Linearization of (1.4), (1.6), (1.8) with respect to $\boldsymbol{w}, q, \rho$ leads to

$$\begin{gathered} \boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla \cdot \boldsymbol{v}(x,t) = 0, \quad x \in \mathcal{F}, \ \ t > 0, \\ T(\boldsymbol{v}, p)\boldsymbol{N} + \boldsymbol{N}B_0\rho = 0, \\ \rho_t = \boldsymbol{N}(x) \cdot \boldsymbol{v}(y,t) \quad x \in \mathcal{G} = \partial\mathcal{F}, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \mathcal{F}, \qquad \rho(x,0) = \rho_0(x), \quad x \in \mathcal{G}, \end{gathered} \tag{1.10}$$

$$\int_{\mathcal{G}} \rho(y,t)dS = 0, \quad \int_{\mathcal{G}} \rho(y,t)y_i dS = 0, \tag{1.11}$$

$$\begin{gathered} \int_{\mathcal{F}} \boldsymbol{v}(y,t)dy = 0, \\ \int_{\mathcal{F}} \boldsymbol{v}(y,t) \cdot \boldsymbol{\eta}_i(y)dy + \omega \int_{\mathcal{G}} \rho(y,t)\boldsymbol{\eta}_3(y) \cdot \boldsymbol{\eta}_i(y)dS = 0, \\ i = 1,2,3, \end{gathered} \tag{1.12}$$

where $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$, $p(x,t)$, $x \in \mathcal{F}$, and $\rho(x,t)$, $x \in \mathcal{G}$, are unknown functions,

$$\begin{gathered} B_0\rho(x,t) = b(x)\rho - \kappa \int_{\mathcal{G}} \frac{\rho(y,t)dS}{|x-y|}, \\ b(x) = -\omega^2 \boldsymbol{x}' \cdot \boldsymbol{N}(x) - \kappa \frac{\partial \mathcal{U}(x)}{\partial N}. \end{gathered}$$

We assume that $b(x) \geq b_0 > 0$.

Both conditions (1.11) and (1.12) hold for arbitrary $t > 0$, provided they are satisfied for the initial data $\boldsymbol{v}_0$ and ρ_0.

The behavior of the solutions of (1.10)–(1.12) for large t is determined by the spectrum of the corresponding stationary operator. Therefore we consider the spectral problem

$$\begin{gathered} \lambda\boldsymbol{v} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla \cdot \boldsymbol{v}(x,t) = 0, \quad x \in \mathcal{F}, \ \ t > 0, \\ T(\boldsymbol{v}, p)\boldsymbol{N} + \boldsymbol{N}B_0\rho = 0, \\ \lambda\rho = \boldsymbol{N}(x) \cdot \boldsymbol{v}(y), \quad x \in \mathcal{G}, \end{gathered} \tag{1.13}$$

in the space of complex-valued functions defined by the conditions (1.11) and (1.12), i.e.,

$$\int_{\mathcal{G}} \rho(y)dS = 0, \quad \int_{\mathcal{G}} \rho(y)y_i dS = 0, \tag{1.14}$$

$$\begin{gathered} \int_{\mathcal{F}} \boldsymbol{v}(y)dy = 0, \\ \int_{\mathcal{F}} \boldsymbol{v}(y) \cdot \boldsymbol{\eta}_i(y)dy + \omega \int_{\mathcal{G}} \rho(y)\boldsymbol{\eta}_3(y) \cdot \boldsymbol{\eta}_i(y)dS = 0, \\ i = 1,2,3, \end{gathered} \tag{1.15}$$

Our objective is to prove the following theorem.

Theorem 1. *Let*

$$B\rho = B_0\rho + \frac{\omega^2|x'|^2}{\int_{\mathcal{F}}|z'|^2dz}\int_{\mathcal{G}}\rho|y'|^2dS. \tag{1.16}$$

If

$$\mathcal{S} = \int_{\mathcal{F}}(x_1^2 - x_3^2)dx = \int_{\mathcal{F}}(x_2^2 - x_3^2)dx > 0 \tag{1.17}$$

and if the quadratic form

$$\int_{\mathcal{G}}\rho B\rho dS \tag{1.18}$$

$$= \int_{\mathcal{G}} b(x)\rho^2 dS - \kappa\int_{\mathcal{G}}\int_{\mathcal{G}}\frac{\rho(x)\rho(y)dS_x dS_y}{|x-y|} + \frac{\omega^2}{\|\boldsymbol{\eta}_3\|^2_{L_2(\mathcal{F})}}\Big(\int_{\mathcal{G}}\rho(x)|x'|^2dS\Big)^2$$

takes negative values for some $\rho(y)$ satisfying (1.14)*, then the problem* (1.10) *has a finite number of the points of spectrum with the positive real part.*

The quadratic form (1.18) is the second variation of the energy functional. Therefore the statements similar to that of Theorem 1 are referred to as the inversion of the Lagrange theorem. On the basis of Theorem 1 the instability of the zero solution of a nonlinear problem (1.10)–(1.12) can be proved [1].

When the form (1.18) is positive definite in the space (1.14), then the uniformly rotating liquid is stable [2].

The results of the present paper are not quite new. Theorem 1 is proved in [3, 4]; the proof given here is more complete and simple. As in [3, 4], it is based on the theorem on invariant subspaces of dissipative operators in the Hilbert space with indefinite metrics [5]. We follow the ideas of the paper [6] where this theorem is applied to the problem of motion of a top with cavities filled with a viscous incompressible liquid. Another proof of existence of solutions of the evolution problem (1.10)–(1.12) with the exponential growth for large $t > 0$ based on the construction of the appropriate Lyapunov function is given in [7]. The method of construction of such function is proposed in [8].

2. Auxiliary propositions

We start with some definitions and auxiliary relations. We introduce the following spaces:

$J \subset L_2(\mathcal{F})$: the space of all the divergence free (in a weak sense) vector fields from $L_2(\mathcal{F})$,

$\widetilde{J}$: the subspace of vector fields $\boldsymbol{u} \in J$ such that

$$\int_{\mathcal{F}}\boldsymbol{u}(x)dx = 0, \quad \int_{\mathcal{F}}\boldsymbol{u}(x)\cdot\boldsymbol{\eta}_3(x)dx = 0, \tag{2.1}$$

$J_\perp$: the subspace of vector fields $\boldsymbol{u} \in J$ such that

$$\int_{\mathcal{F}} \boldsymbol{u}(x) \cdot \boldsymbol{\eta}(x) dx = 0,$$

where $\boldsymbol{\eta}(x) = \boldsymbol{a} + \boldsymbol{b} \times x$ is an arbitrary vector of rigid motion,

H: the subspace of functions from $L_2(\mathcal{G})$ satisfying the orthogonality conditions (1.14),

H_1: the subspace of functions from $L_2(\mathcal{G})$ satisfying the orthogonality conditions (1.14) and

$$\int_{\mathcal{G}} \rho(y) y_3 y_\alpha dS = 0, \quad \alpha = 1, 2. \tag{2.2}$$

We set $(f, g) = \int_{\mathcal{G}} f(x) g(x) dS$ and we denote by P_J, $\widetilde{P}$ and $P_\perp$ orthogonal (in $L_2(\mathcal{F})$) projections on J, $\widetilde{J}$ and $J_\perp$, respectively. We also introduce orthogonal in $L_2(\mathcal{G})$ projections P and P_1 on H and H_1.

Proposition 1. *Arbitrary vector field of rigid motion $\boldsymbol{\eta}(x) = \boldsymbol{a} + \boldsymbol{b} \times x$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors, satisfies the equation*

$$B_0(\boldsymbol{\eta} \cdot N) = -\omega^2 \boldsymbol{\eta}(x) \cdot x', \quad x \in \mathcal{G}. \tag{2.3}$$

Proof. We take arbitrary small smooth function $r(x)$ and consider the integral

$$I(r) = \int_\Gamma \Big(\frac{\omega^2}{2} |x'|^2 + \kappa U(x) + p_0 \Big) \boldsymbol{\eta}(x) \cdot \boldsymbol{n} dS,$$

where $U(x) = \int_\Omega |x - y|^{-1} dy$ and Ω is a domain whose boundary Γ is given by the equation

$$x = y + \boldsymbol{N}(y) r(y), \quad y \in \mathcal{G}.$$

It can be shown that only the term containing ω^2 is different from zero; indeed, we have $\int_\Gamma \boldsymbol{\eta} \cdot \boldsymbol{n} dS = \int_{\mathcal{F}} \nabla \cdot \boldsymbol{\eta}(x) dx = 0$ and

$$\int_\Gamma U(x) \boldsymbol{n}(x) dS = \int_\Omega \nabla U(x) dx = \int_\Omega \int_\Omega \frac{y - x}{|y - x|^3} dx dy = 0,$$

$$\begin{aligned}
\int_\Gamma U(x) \boldsymbol{\eta}_i(x) \cdot \boldsymbol{n}(x) dS &= \int_\Omega \nabla U(x) \cdot \boldsymbol{\eta}_i(x) dx \\
&= \int_\Omega \int_\Omega \frac{y - x}{|y - x|^3} \cdot \boldsymbol{\eta}_i(x - y) dx dy + \int_\Omega \int_\Omega \frac{y - x}{|y - x|^3} \cdot \boldsymbol{\eta}_i(y) dx dy \\
&= - \int_\Omega \nabla U(y) \cdot \boldsymbol{\eta}_i(y) dy,
\end{aligned}$$

from which we can conclude that

$$\int_\Gamma U(x) \boldsymbol{\eta}_i(x) \cdot \boldsymbol{n}(x) dS = 0.$$

Hence

$$I[r] = \frac{\omega^2}{2} \int_\Gamma |x'|^2 \boldsymbol{\eta}(x) \cdot \boldsymbol{n}(x) dS = \omega^2 \int_\Omega \boldsymbol{\eta}(x) \cdot x' dx \tag{2.4}$$

and $I[0] = 0$. Now we compute the first variation of both parts of this equation with respect to r, making use of the formula

$$\delta U = \kappa \frac{\partial \mathcal{U}(x)}{\partial N} + \kappa \int_{\mathcal{G}} \frac{\rho(y,t)dS}{|x-y|},$$

(the proof can be found in [9]). In view of (1.2), we have

$$-\int_{\mathcal{G}} B_0 r \boldsymbol{\eta}(x) \cdot \boldsymbol{N}(x) dS = \omega^2 \int_{\mathcal{G}} r(x) \boldsymbol{\eta}(x) \cdot x' dS,$$

i.e.,

$$\int_{\mathcal{G}} r(x) B_0 (\boldsymbol{\eta}(x) \cdot \boldsymbol{N}(x)) dS = -\omega^2 \int_{\mathcal{G}} r(x) \boldsymbol{\eta}(x) \cdot x' dS.$$

Since $r(x)$ is arbitrary, this equation is equivalent to (2.3). The proposition is proved.

From

$$\int_{\mathcal{G}} |x'|^2 \boldsymbol{\eta}(x) \cdot \boldsymbol{N}(x) dS = 2 \int_{\mathcal{F}} \boldsymbol{\eta}(x) \cdot x' dx = 0$$

it follows that also

$$B(\boldsymbol{\eta} \cdot \boldsymbol{N}) = -\omega^2 \boldsymbol{\eta}(x) \cdot x'. \tag{2.5}$$

Direct computations show that

$$\int_{\mathcal{G}} \boldsymbol{\eta}_1(x) \cdot \boldsymbol{N}(x) x_2 x_3 dS = \mathcal{S}, \quad \int_{\mathcal{G}} \boldsymbol{\eta}_2(x) \cdot \boldsymbol{N}(x) x_1 x_3 dS = -\mathcal{S},$$

$$\int_{\mathcal{G}} \boldsymbol{\eta}_1(x) \cdot \boldsymbol{N}(x) x_1 x_3 dS = \int_{\mathcal{G}} \boldsymbol{\eta}_2(x) \cdot \boldsymbol{N}(x) x_2 x_3 dS = 0. \tag{2.6}$$

Indeed,

$$\begin{aligned}\int_{\mathcal{G}} \boldsymbol{\eta}_1(x) \cdot \boldsymbol{N}(x) x_2 x_3 dS &= \int_{\mathcal{F}} \boldsymbol{\eta}_1(x) \cdot \nabla(x_2 x_3) dx \\ &= \int_{\mathcal{G}} (\boldsymbol{e}_3 x_2 - \boldsymbol{e}_2 x_3)(\boldsymbol{e}_3 x_2 + \boldsymbol{e}_2 x_3) dx = \int_{\mathcal{F}} (x_2^2 - x_3^2) dx,\end{aligned}$$

and other equations are verified in the same way.

As a consequence, we obtain

$$\int_{\mathcal{G}} \boldsymbol{\eta}_1 \cdot \boldsymbol{N} B(\boldsymbol{\eta}_1 \cdot \boldsymbol{N}) dS = \omega^2 \int_{\mathcal{G}} x_3 x_2 \boldsymbol{\eta}_1 \cdot \boldsymbol{N} dS = \omega^2 \mathcal{S},$$

which shows that (1.17) is necessary for the positivity of the quadratic form (1.18).

Making use of (2.6), we can easily prove

Proposition 2. *An arbitrary $\rho \in L_2(\mathcal{G})$ can be represented in the form*

$$\rho(x) = \rho_1(x) + \rho_2(x) \tag{2.7}$$

where

$$\rho_1(x) = \mathcal{S}^{-1}(\boldsymbol{\eta}_1(x)\cdot \boldsymbol{N}(x)I_2(\rho) - \boldsymbol{\eta}_2(x)\cdot \boldsymbol{N}(x)I_1(\rho)),$$
$$I_\alpha(\rho) = \int_{\mathcal{G}} \rho(x)x_3x_\alpha dS, \quad \alpha = 1,2,$$

and ρ_2 *satisfies the orthogonality conditions* (2.2). *If* $\rho \in H$*, then* $\rho_2 \in H_1$. *If* $\rho = \boldsymbol{\eta}_\beta \cdot \boldsymbol{N}$, $\beta = 1,2$, *then* $\rho_2 = 0$.

The equation (2.7) defines a non-orthogonal projection Q on the space (2.2):

$$\rho_2 = Q\rho = \rho - \mathcal{S}^{-1}\Big(\boldsymbol{\eta}_1(x)\cdot \boldsymbol{N}(x)I_2(\rho) - \boldsymbol{\eta}_2(x)\cdot \boldsymbol{N}(x)I_1(\rho)\Big).$$

In view of (2.5), we have $(B\rho_1,\rho_2) = 0$, which implies

$$(\widehat{B}\rho,\rho) = (B\rho,\rho) = (\widehat{B}\rho_1,\rho_1) + (\widehat{B}\rho_2,\rho_2),$$

if $\int_{\mathcal{G}} \rho dS = 0$ ($\widehat{B}$ is defined in (3.4)). Since

$$(B\rho_1,\rho_1) = \omega^2\mathcal{S}^{-1}\sum_{\alpha=1}^{2} I_\alpha^2(\rho) \geq 0,$$

we have

$$(\widehat{B}Q\rho, Q\rho) \leq (\widehat{B}\rho,\rho). \tag{2.8}$$

3. On the problem (1.13)–(1.15)

We pass to the analysis of the spectral problem (1.13)–(1.15). By (1.15),

$$\boldsymbol{v}(x) = \boldsymbol{v}^\perp(x) + \sum_{i=1}^{3} d_i(\rho)\boldsymbol{\eta}_i(x),$$

where $\boldsymbol{v}^\perp = P_\perp \boldsymbol{v}$ and

$$d_i(\rho) = -\frac{\omega}{\|\boldsymbol{\eta}_i\|^2_{L_2(\mathcal{F})}}\int_{\mathcal{G}} \rho(x)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dS.$$

We introduce new velocity and pressure according to the formulas

$$\boldsymbol{u} = \boldsymbol{v} - d_3(\rho)\boldsymbol{\eta}_3(x), \quad q = p - \omega|x'|^2 d_3(\rho) + \frac{1}{|\mathcal{G}|}\int_{\mathcal{G}} B\rho dS$$

and convert (1.13)–(1.15) to

$$\begin{gathered}
\lambda\boldsymbol{u} + 2\omega(\boldsymbol{e}_3\times\boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla q = \frac{\omega\boldsymbol{\eta}_3(x)}{\|\boldsymbol{\eta}_3\|^2_{L_2(\mathcal{F})}}\int_{\mathcal{G}} \boldsymbol{u}\cdot\boldsymbol{N}|x'|^2 dS,\\
\nabla\cdot\boldsymbol{u}(x) = 0, \quad x\in\mathcal{F},\\
T(\boldsymbol{v},q)\boldsymbol{N} + \boldsymbol{N}\widehat{B}\rho = 0,\\
\lambda\rho = \boldsymbol{N}(x)\cdot\boldsymbol{u}(x) \quad x\in\mathcal{G},
\end{gathered} \tag{3.1}$$

$$\int_{\mathcal{G}} \rho(y)dS = 0, \quad \int_{\mathcal{G}} \rho(y)y_i dS = 0, \tag{3.2}$$

$$\int_{\mathcal{F}} \boldsymbol{u}(y)dy = 0, \quad \int_{\mathcal{F}} \boldsymbol{u}(y)\cdot\boldsymbol{\eta}_3(y)dy = 0,$$

$$\int_{\mathcal{F}} \boldsymbol{u}(y,t)\cdot\boldsymbol{\eta}_j(y)dy + \omega\int_{\mathcal{G}} \rho(y,t)\boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_j(y)dS = 0, \quad j=1,2, \tag{3.3}$$

where

$$\widehat{B}\rho = B\rho - \frac{1}{|\mathcal{G}|}\int_{\mathcal{G}} B\rho dS. \tag{3.4}$$

Since

$$\int_{\mathcal{F}} (\boldsymbol{e}_3\times\boldsymbol{u})\cdot\boldsymbol{\eta}_3(x)dx = \int_{\mathcal{F}} (\boldsymbol{e}_2u_1 - \boldsymbol{e}_1u_2)\cdot\boldsymbol{\eta}_3(x)dx = \frac{1}{2}\int_{\mathcal{G}} \boldsymbol{u}\cdot\boldsymbol{N}|x'|^2dS,$$

one can show by a simple calculation that (3.1) is equivalent to

$$\begin{gathered} \lambda\boldsymbol{u} + 2\omega\widetilde{P}(\boldsymbol{e}_3\times\boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla r = 0, \\ \nabla\cdot\boldsymbol{u} = 0, \quad x\in\mathcal{F}, \\ S(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{N}(\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N})|_{\mathcal{G}} = 0, \\ \lambda\rho = \boldsymbol{N}(x)\cdot\boldsymbol{u}(x) \quad x\in\mathcal{G}, \end{gathered} \tag{3.5}$$

where r is a solution of the Dirichlet problem

$$\nabla^2 r(x) = 0, \quad x\in\mathcal{F}, \quad r(x) = \nu\boldsymbol{N}\cdot S(u)\boldsymbol{N} + \widehat{B}\rho, \quad x\in\mathcal{G}. \tag{3.6}$$

The pressure is excluded, and we can write (3.5), (3.6), (3.2), (3.3) in the following abstract form:

$$\lambda U = \mathcal{A}U, \tag{3.7}$$

where $U = (\boldsymbol{u},\rho)^T$, $\mathcal{A} = (A_{ij})_{i,j=1,2}$,

$$A_{11}\boldsymbol{u} = \nu\nabla^2\boldsymbol{u} - \nabla r_1 - 2\omega\widetilde{P}(\boldsymbol{e}_3\times\boldsymbol{u}), \quad A_{12}\rho = -\nabla r_2,$$

$$A_{21}\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{N}, \quad A_{22}\rho = 0,$$

$$\nabla^2 r_1 = 0, \quad \nabla^2 r_2 = 0, \quad x\in\mathcal{F},$$

$$r_1 = \nu\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}, \quad r_2 = \widehat{B}\rho, \quad x\in\mathcal{G}.$$

As the domain of $\mathcal{A}$, $D(\mathcal{A})$, we take the set $U = (\boldsymbol{u},\rho)^T$ with $\boldsymbol{u}\in W_2^2(\mathcal{F})$ and $\rho\in W_2^{1/2}(\mathcal{G})$, satisfying (3.2), (3.3) and the boundary condition

$$S(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{N}(\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N})|_{\mathcal{G}} = 0.$$

We also consider a modified problem

$$\begin{gathered} \lambda\boldsymbol{u} + 2\omega\widetilde{P}(\boldsymbol{e}_3\times\boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla r = 0, \\ \nabla\cdot\boldsymbol{u} = 0, \quad x\in\mathcal{F}, \\ S(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{N}(\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N})|_{\mathcal{G}} = 0, \\ \lambda\rho = \boldsymbol{N}(x)\cdot\boldsymbol{u}(x) - Q_0(\boldsymbol{N}\cdot\boldsymbol{u}), \quad x\in\mathcal{G}, \end{gathered} \tag{3.8}$$

where Q_0 is the orthogonal projection on the finite-dimensional space $\mathrm{Ker}\,|_H \widehat{B}$ that consists of the elements of H satisfying the equation $\widehat{B}\rho = 0$. This problem can be written in the form similar to (3.7):

$$\lambda U = \mathcal{A}' U,$$

where $\mathcal{A}'$ is the 2×2 matrix operator with

$$\begin{aligned} &A'_{11}\boldsymbol{u} = A_{11}\boldsymbol{u}, && A'_{12}\rho = A_{12}\rho,\\ &A'_{21}\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{N} - Q_0(\boldsymbol{u}\cdot\boldsymbol{N}), && A'_{22}\rho = 0. \end{aligned}$$

The domain of $\mathcal{A}'$ consists of all the elements of $D(\mathcal{A})$ satisfying the additional orthogonality condition

$$\int_{\mathcal{G}} \rho(x)\varphi(x)dS = 0, \quad \forall\varphi \in \mathrm{Ker}|_H \widehat{B}. \tag{3.9}$$

The following proposition is proved by a direct calculation (see [3]).

Proposition 3. *If* $U = (\boldsymbol{u},\rho)^T \in D(\mathcal{A}')$, *then* $(\boldsymbol{f}, g)^T = \mathcal{A}'U$ *satisfies the conditions*

$$\int_{\mathcal{G}} g dS = 0, \quad \int_{\mathcal{G}} g\varphi dS = 0, \quad \int_{\mathcal{G}} g x_i dS = 0, \quad i = 1,2,3,$$

$$\int_{\mathcal{F}} \boldsymbol{f}(x)dx = 0, \quad \int_{\mathcal{F}} \boldsymbol{f}(x)\cdot\boldsymbol{\eta}_3(x)dx = 0, \tag{3.10}$$

$$\int_{\mathcal{F}} \boldsymbol{f}(x)\cdot\boldsymbol{\eta}_\alpha(x)dx + \omega\int_{\mathcal{G}} g(x)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_\alpha(x)dS = 0, \quad \alpha = 1,2,$$

where φ *is an arbitrary element of* $\mathrm{Ker}\,|_H\widehat{B}$.

If $\boldsymbol{f}$ *and* g *satisfy* (3.10) *and* $\lambda \neq 0, \pm i\omega$, *then the solution* $\boldsymbol{u} \in W_2^2(\mathcal{F})$, $\rho \in W_2^{1/2}(\mathcal{G})$ *satisfies* (3.2), (3.3), (3.9).

The next proposition characterizes the spectrum of $\mathcal{A}'$.

Proposition 4. *The spectrum of* $\mathcal{A}'$ *consists of a countable number of eigenvalues with the accumulation points* $\lambda = \infty$ *and* $\lambda = 0$. *There are no eigenvalues in the domain* $\mathrm{Re}\lambda \gg 1$ *of the complex plane* λ *and on the imaginary axis.*

Proof. We transform the problem (3.1)–(3.3) once more. We use the relations

$$\begin{aligned} 2\omega\int_{\mathcal{F}} (\boldsymbol{e}_3\times\boldsymbol{u})\cdot\boldsymbol{\eta}_1 dx &= \omega\int_{\mathcal{G}} \boldsymbol{u}\cdot\boldsymbol{N}\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_1 dS - \omega\int_{\mathcal{F}} \boldsymbol{u}\cdot\boldsymbol{\eta}_2 dx\\ &= \omega\int_{\mathcal{G}} \boldsymbol{u}\cdot\boldsymbol{N}\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_1 dS - \omega\mathcal{S}_0 d_2(\rho),\\ 2\omega\int_{\mathcal{F}} (\boldsymbol{e}_3\times\boldsymbol{u})\cdot\boldsymbol{\eta}_2 dx &= \omega\int_{\mathcal{G}} \boldsymbol{u}\cdot\boldsymbol{N}\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_2 dS + \omega\int_{\mathcal{F}} \boldsymbol{u}\cdot\boldsymbol{\eta}_1 dx\\ &= \omega\int_{\mathcal{G}} \boldsymbol{u}\cdot\boldsymbol{N}\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_2 dS + \omega\mathcal{S}_0 d_1(\rho), \end{aligned}$$

where $\mathcal{S}_0 = \int_{\mathcal{F}}(x_1^2 + x_3^2)dx = \int_{\mathcal{F}}(x_2^2 + x_3^2)dx$. There relations enable us to write the first equation in (3.5) in the form

$$-\nu\nabla^2 \boldsymbol{u}^\perp + \nabla r = -\lambda \boldsymbol{u}^\perp - 2\omega P_\perp(\boldsymbol{e}_3 \times \boldsymbol{u}) + \omega\Big(\boldsymbol{\eta}_1 d_2(\rho) - \boldsymbol{\eta}_2 d_1(\rho)\Big). \tag{3.11}$$

Following [10] and [11], Ch. 8, we introduce the operator $\mathbf{T}_1$ that makes correspond the solution $\boldsymbol{w}^\perp \in W_2^2(\mathcal{F}) \cap J_\perp$ of the problem

$$\begin{aligned} &-\nu\nabla^2 \boldsymbol{w}^\perp + \nabla q = \boldsymbol{f}(x), \quad x \in \mathcal{F}, \\ &T(\boldsymbol{w}^\perp, q)\boldsymbol{N}|_{\mathcal{G}} = 0 \end{aligned}$$

to the vector field $\boldsymbol{f} \in J_\perp$, and the operator $\mathbf{T}_2$ defined by $\mathbf{T}_2\psi = \boldsymbol{v}^\perp$ where $\boldsymbol{v}^\perp \in W_2^2(\mathcal{F}) \cap J_\perp$ is a solution of the problem

$$\begin{aligned} &-\nu\nabla^2 \boldsymbol{v}^\perp + \nabla p = \boldsymbol{l}_0(\psi) + \sum_{i=1}^3 l_i(\psi)\boldsymbol{\eta}_i(x), \quad x \in \mathcal{F}, \\ &T(\boldsymbol{v}^\perp, p)\boldsymbol{N}|_{\mathcal{G}} = \psi(x)\boldsymbol{N}(x) \end{aligned} \tag{3.12}$$

with

$$\boldsymbol{l}_0(\psi) = -\frac{1}{|\mathcal{F}|}\int_{\mathcal{G}} \psi(x)\boldsymbol{N}(x)dS, \quad l_i(\psi) = -\frac{1}{\|\boldsymbol{\eta}_i\|^2_{L_2(\mathcal{F})}}\int_{\mathcal{G}} \psi\boldsymbol{\eta}_i \cdot \boldsymbol{N}dS,$$

and ψ is an element of $W_2^{1/2}(\mathcal{G})$ satisfying the condition $\int_{\mathcal{G}} \psi dS = 0$. The operator $\mathbf{T}_2$ can be extended to $W_2^{-1/2}(\mathcal{G})$, and then its range is contained in $W_2^1(\mathcal{F}) \cap J_\perp$, and $\boldsymbol{v}^\perp = \boldsymbol{T}_2\psi$ is a weak solution of (3.12) satisfying the integral identity

$$\begin{aligned} \frac{\nu}{2}\int_{\mathcal{F}} S(\boldsymbol{v}^\perp) : S(\boldsymbol{\varphi})dx = &\sum_{i=1}^3 \int_{\mathcal{F}} l_i(\psi)\boldsymbol{\eta}_i(x) \cdot \boldsymbol{\varphi}(x)dx \\ &+ \int_{\mathcal{G}} \psi(x)\boldsymbol{N}(x) \cdot \boldsymbol{\varphi}(x)dS, \quad \forall \boldsymbol{\varphi} \in W_2^1(\mathcal{F}) \cap J_\perp \end{aligned}$$

(we note that the necessary compatibility conditions

$$\boldsymbol{l}_0(\psi) \cdot \int_{\mathcal{F}} \boldsymbol{\eta}(x)dx + \sum_{i=1}^3 l_i(\psi)\int_{\mathcal{F}} \boldsymbol{\eta}_i(x) \cdot \boldsymbol{\eta}(x)dx + \int_{\mathcal{G}} \psi(x)\boldsymbol{N}(x) \cdot \boldsymbol{\eta}(x)dS = 0,$$

$\forall \boldsymbol{\eta}(x) = \boldsymbol{a} + \boldsymbol{b} \times x$, are satisfied).

It is clear that

$$\begin{aligned} &\|\mathbf{T}_1 \boldsymbol{f}\|_{W_2^2(\mathcal{F})} \le c\|\boldsymbol{f}\|_{L_2(\mathcal{F})}, \\ &\|\mathbf{T}_2\psi\|_{W_2^1(\mathcal{F})} \le c\|\psi\|_{W_2^{-1/2}(\mathcal{G})}, \end{aligned}$$

From (3.11) and from the boundary condition $T(\boldsymbol{u}^\perp, r)\boldsymbol{N} = -\widehat{B}(\rho)\boldsymbol{N}$ we can conclude that

$$\begin{aligned} \boldsymbol{u}^\perp &= -\mathbf{T}_1(\lambda\boldsymbol{u}^\perp + 2\omega P_\perp(\boldsymbol{e}_3 \times \boldsymbol{u})) - \mathbf{T}_2(\widehat{B}\rho) \\ &= -\mathbf{T}_1(\lambda\boldsymbol{u}^\perp + 2\omega P_\perp(\boldsymbol{e}_3 \times \boldsymbol{u})) - \frac{1}{\lambda}\mathbf{T}_2(\widehat{B}(\boldsymbol{u} \cdot \boldsymbol{N})). \end{aligned} \tag{3.13}$$

Finally, we add the equation

$$\begin{aligned}\boldsymbol{\eta}_1 d_1(\rho)+\boldsymbol{\eta}_2 d_2(\rho) &= \frac{\omega\boldsymbol{\eta}_1}{\mathcal{S}_0}\int_{\mathcal{G}}\rho(x)x_3x_1dS+\frac{\omega\boldsymbol{\eta}_2}{\mathcal{S}_0}\int_{\mathcal{G}}\rho(x)x_3x_2dS\\ &= -\frac{\boldsymbol{\eta}_1(x)}{\omega\mathcal{S}_0}\int_{\mathcal{G}}\widehat{B}\rho\boldsymbol{\eta}_2\cdot\boldsymbol{N}dS+\frac{\boldsymbol{\eta}_2(x)}{\omega\mathcal{S}_0}\int_{\mathcal{G}}\widehat{B}\rho\boldsymbol{\eta}_1\cdot\boldsymbol{N}dS\\ &= -\frac{\boldsymbol{\eta}_1(x)}{\lambda\omega\mathcal{S}_0}\int_{\mathcal{G}}\widehat{B}(\boldsymbol{u}\cdot\boldsymbol{N})\boldsymbol{\eta}_2\cdot\boldsymbol{N}dS+\frac{\boldsymbol{\eta}_2(x)}{\lambda\omega\mathcal{S}_0}\int_{\mathcal{G}}\widehat{B}(\boldsymbol{u}\cdot\boldsymbol{N})\boldsymbol{\eta}_1\cdot\boldsymbol{N}dS\end{aligned}$$

to (3.13) and obtain

$$\boldsymbol{u}=-\lambda\mathbf{T}_1P_\perp\boldsymbol{u}-2\omega\mathbf{T}_1P_\perp(\boldsymbol{e}_3\times\boldsymbol{u})-\frac{1}{\lambda}\mathbf{T}_3\widehat{B}(\boldsymbol{u}\cdot\boldsymbol{N})\equiv\mathcal{L}(\lambda)\boldsymbol{u}, \tag{3.14}$$

where

$$\mathbf{T}_3\psi=\mathbf{T}_2\psi+\frac{\boldsymbol{\eta}_1(x)}{\omega\mathcal{S}_0}\int_{\mathcal{G}}\psi\boldsymbol{\eta}_2\cdot\boldsymbol{N}dS-\frac{\boldsymbol{\eta}_2(x)}{\omega\mathcal{S}_0}\int_{\mathcal{G}}\psi\boldsymbol{\eta}_1\cdot\boldsymbol{N}dS.$$

Since $\boldsymbol{u}\cdot\boldsymbol{N}\in W_2^{-1/2}(\mathcal{G})$ for arbitrary $\boldsymbol{u}\in J$, the operator $\mathcal{L}(\lambda)$ is well defined in $\widetilde{J}$ and is completely continuous, moreover, $\mathcal{L}(\lambda)$ is a holomorphic function of λ everywhere except for $\lambda=0$ and $\lambda=\infty$.

We have shown that (3.5), (3.6) imply the equation (3.14) for $\boldsymbol{u}\in\widetilde{J}$; now we prove that (3.5), (3.6) follow from (3.14). Indeed, if (3.14) holds, then

$$\begin{aligned}\boldsymbol{u}^\perp &= -\mathbf{T}_1(\lambda\boldsymbol{u}^\perp+2\omega P_\perp(\boldsymbol{e}_3\times\boldsymbol{u}))-\frac{1}{\lambda}\mathbf{T}_2(\widehat{B}(\boldsymbol{u}\cdot\boldsymbol{N})),\\ \int_{\mathcal{F}}\boldsymbol{u}(x)\cdot\boldsymbol{\eta}_1(x)dx &= -\frac{1}{\lambda\omega}\int_{\mathcal{G}}\widehat{B}(\boldsymbol{u}\cdot\boldsymbol{N})\boldsymbol{\eta}_2\cdot\boldsymbol{N}dS\\ &= \frac{\omega}{\lambda}\int_{\mathcal{G}}(\boldsymbol{u}(x)\cdot\boldsymbol{N}(x)-Q_0(\boldsymbol{u}\cdot\boldsymbol{N}))x_3x_1dS,\\ \int_{\mathcal{F}}\boldsymbol{u}(x)\cdot\boldsymbol{\eta}_2(x)dx &= \frac{\omega}{\lambda}\int_{\mathcal{G}}(\boldsymbol{u}(x)\cdot\boldsymbol{N}(x)-Q_0(\boldsymbol{u}\cdot\boldsymbol{N}))x_3x_2dS.\end{aligned}$$

From the smoothing properties of $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ mentioned above it follows that the equation (3.11) is satisfied with $\rho=\lambda^{-1}(I-Q_0)\boldsymbol{u}\cdot\boldsymbol{N}$ and as a consequence (3.2), (3.3), (3.8), (3.9) hold.

We are in a position to apply the theorem on the holomorphic operator function to $\mathcal{L}(\lambda)$. One of the assumptions of this theorem is that (3.14) implies $\boldsymbol{u}=0$ for a certain $\lambda\neq 0$. We verify this assumption for λ in the half-plane $\mathrm{Re}\lambda\geq L\gg 1$. To this end, we multiply the first equation in (3.8) by $\boldsymbol{u}$ and integrate the product over $\mathcal{F}$. We integrate by parts and use the boundary conditions, which leads to

$$\lambda\|\boldsymbol{u}\|^2+2\omega\int_{\mathcal{F}}(u_1\bar{u}_2-\bar{u}_1u_2)dx+\frac{1}{\lambda}\int_{\mathcal{G}}\widehat{B}(\boldsymbol{u}\cdot\boldsymbol{N})\bar{\boldsymbol{u}}\cdot\boldsymbol{N}dS+\frac{\nu}{2}\|S(\boldsymbol{u}^\perp)\|^2=0, \tag{3.15}$$

where $\|\cdot\|$ is the norm in $L_2(\mathcal{F})$. Moreover, from

$$\lambda\rho=(I-Q_0)\boldsymbol{u}\cdot\boldsymbol{N}$$

it follows that

$$\lambda d_\alpha(\rho) = d_\alpha(\boldsymbol{u}\cdot\boldsymbol{N}) = d_\alpha(\boldsymbol{u}^\perp\cdot\boldsymbol{N}) + \sum_{\beta=1}^{2} d_\beta(\rho) d_\alpha(\boldsymbol{\eta}_\beta\cdot\boldsymbol{N}), \quad \alpha = 1,2.$$

Hence for large Reλ we have $\sum_{\alpha=1}^{2}|d_\alpha(\rho)| \le c\sum_{\beta=1}^{2}|d_\beta(\boldsymbol{u}^\perp\cdot\boldsymbol{N})|$ and

$$\begin{aligned}\Big|\int_{\mathcal{G}} \widehat{B}(\boldsymbol{u}\cdot\boldsymbol{N})\bar{\boldsymbol{u}}\cdot\boldsymbol{N}dS\Big| &= \Big|\int_{\mathcal{G}} B(\boldsymbol{u}\cdot\boldsymbol{N})\bar{\boldsymbol{u}}\cdot\boldsymbol{N}dS\Big| \\ &= \Big|\int_{\mathcal{G}} B(\boldsymbol{u}^\perp\cdot\boldsymbol{N} + \sum_{\alpha=1}^{2} d_\alpha(\rho)\boldsymbol{\eta}_\alpha\cdot\boldsymbol{N})(\bar{\boldsymbol{u}}^\perp\cdot\boldsymbol{N} + \sum_{\alpha=1}^{2}\bar{d}_\alpha(\rho)\boldsymbol{\eta}_\alpha\cdot\boldsymbol{N})dS\Big| \\ &\le c\int_{\mathcal{G}} |\boldsymbol{u}^\perp\cdot\boldsymbol{N}|^2 dS. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we can conclude that

$$\mathrm{Re}\lambda\|\boldsymbol{u}\|^2 + \frac{\nu}{2}\|S(\boldsymbol{u}^\perp)\|^2 \le c\mathrm{Re}\lambda^{-1}\int_{\mathcal{G}} |\boldsymbol{u}^\perp\cdot\boldsymbol{N}|^2 dS,$$

and, as a consequence, that $\boldsymbol{u} = 0$ and $\rho = 0$, if Reλ is sufficiently large.

According to the theorem on the holomorphic operator function [12], the spectrum of the pencil $I - \mathcal{L}(\lambda)$ consists of a countable number of eigenvalues with the accumulation points at zero and infinity. It can be verified that every $\lambda \ne 0$ that does not coincide with these eigenvalues belongs to the resolvent set of the operator $\mathcal{A}'$. Let $\boldsymbol{u} \in \widetilde{J}$ be the solution of the equation

$$\boldsymbol{u} = \mathcal{L}(\lambda)\boldsymbol{u} + \boldsymbol{h}, \quad \text{where} \quad \boldsymbol{h} = -\frac{1}{\lambda}\boldsymbol{T}_3\widehat{B}g + \boldsymbol{T}_1 P_\perp \boldsymbol{f} \in \widetilde{J}$$

with arbitrary $\boldsymbol{f} \in \widetilde{J}$, $g \in W_2^{3/2}(\mathcal{G})$ satisfying (3.10). By repeating the above arguments it is not difficult to show that $U = (\boldsymbol{u}, \rho)^T$ with $\rho = \lambda^{-1}(\boldsymbol{u}\cdot\boldsymbol{N} + g)$ is a solution of

$$\lambda U = \mathcal{A}'U + F, \quad F = (\boldsymbol{f}, g)^T.$$

From the estimate

$$\|\boldsymbol{u}\| \le c\|\boldsymbol{h}\|$$

and from the properties of $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ it follows that

$$\|\boldsymbol{u}\|_{W_2^2(\mathcal{F})} + \|\rho\|_{W_2^{1/2}(\mathcal{G})} \le c\Big(\|\boldsymbol{f}\|_{L_2(\mathcal{F})} + \|g\|_{W_2^{3/2}(\mathcal{G})}\Big).$$

This completes the proof of the first statement of the proposition.

Now we pass to the proof of the second statement. Let $\lambda \ne 0$ be an imaginary eigenvalue of $\mathcal{A}'$. We multiply the first equation in (3.8) by $\boldsymbol{u}(x)$ and integrate over $\mathcal{F}$. Then we integrate by parts and use the boundary conditions. This leads to

$$\lambda\|\boldsymbol{u}\|^2_{L_2(\mathcal{F})} + 2\omega\int_{\mathcal{F}}(u_1\bar{u}_2 - u_2\bar{u}_1)dx - \int_{\mathcal{G}}\widehat{B}(\rho)\bar{\boldsymbol{u}}\cdot\boldsymbol{N}dS + \frac{\nu}{2}\|S(\boldsymbol{u}^\perp)\|^2_{L_2(\mathcal{F})} = 0. \tag{3.17}$$

Since

$$\int_{\mathcal{G}} \widehat{B}(\rho)\bar{\boldsymbol{u}}\cdot\boldsymbol{N}dS = \int_{\mathcal{G}} \widehat{B}(\rho)(\bar{\boldsymbol{u}}\cdot\boldsymbol{N} - Q_0(\bar{\boldsymbol{u}}\cdot\boldsymbol{N}))dS = \frac{1}{\bar{\lambda}}\int_{\mathcal{G}} B\rho\bar{\rho}dS,$$

the first three terms in (3.17) are imaginary, hence $\|S(\boldsymbol{u}^\perp)\|^2_{L_2(\mathcal{F})} = 0$, $\boldsymbol{u}^\perp(x) = 0$ and $\boldsymbol{u}(x) = \sum_{\alpha=1}^2 d_\alpha(\rho)\boldsymbol{\eta}_\alpha(x)$. The first equation in (3.1) reduces to

$$\lambda\sum_{\alpha=1}^2 d_\alpha\boldsymbol{\eta}_\alpha + 2\omega(\boldsymbol{e}_3\times\sum_{\alpha=1}^2 d_\alpha\boldsymbol{\eta}_\alpha) + \nabla q$$
$$= \boldsymbol{\eta}_3(x)\frac{\omega}{\|\boldsymbol{\eta}_2\|^2_{L_2(\mathcal{F})}}\int_{\mathcal{G}}\sum_{\alpha=1}^2 d_\alpha\boldsymbol{\eta}_\alpha(y)\cdot\boldsymbol{N}(y)|y'|^2dS = 0.$$

Applying the operator rot to this equation one arrives at

$$2\lambda\sum_{\alpha=1}^2 d_\alpha\boldsymbol{e}_\alpha = 2\omega(d_2\boldsymbol{e}_1 - d_1\boldsymbol{e}_2),$$

i.e., to

$$\lambda d_1(\rho) = \omega d_2(\rho), \quad \lambda d_2(\rho) = -\omega d_1(\rho). \tag{3.18}$$

Moreover, taking the relations $d_\alpha(Q_0\boldsymbol{\eta}_\beta\cdot\boldsymbol{N}) = 0$ and (2.6) into account, we obtain

$$\lambda d_1(\rho) = d_1(\sum_{\alpha=1}^2 d_\alpha(\rho)\boldsymbol{\eta}_\alpha\cdot\boldsymbol{N}) = -\frac{\omega\mathcal{S}}{\mathcal{S}_0}d_2(\rho), \quad \lambda d_2(\rho) = \frac{\omega\mathcal{S}}{\mathcal{S}_0}d_1(\rho). \tag{3.19}$$

From (3.18), (3.19) it follows that $d_1 = d_2 = 0$, and, as a consequence, $\boldsymbol{u} = 0$, $q = 0$, $\rho = 0$.

Finally, if $\boldsymbol{u}, \rho$ is a solution of (3.8) with $\lambda = 0$, then the same arguments yield $\boldsymbol{u}^\perp = 0$ and $\boldsymbol{u} = \sum_{\alpha=1}^2 d_\alpha(\rho)\boldsymbol{\eta}_\alpha(x)$. Multiplying the equation $(I - Q_0)\boldsymbol{u}\cdot\boldsymbol{N}|_{\mathcal{G}} = 0$ by x_3x_β and integrating, we arrive at $\int_{\mathcal{G}}\boldsymbol{u}\cdot\boldsymbol{N}x_3x_\beta dS = 0$, i.e., at

$$\sum_{\alpha=1}^2 d_\alpha(\rho)\int_{\mathcal{G}}\boldsymbol{\eta}_\alpha(x)\cdot\boldsymbol{N}(x)x_3x_\beta dS = 0, \quad \beta = 1, 2.$$

This implies $d_1 = d_2 = 0$, hence $\nabla q = 0$, $q = q_0 =$ const. From the boundary condition $-q_0 + B(\rho) = 0$ it follows that $\widehat{B}(\rho) = 0$. In view of (3.9), this implies $\rho = 0$. The proposition is proved.

Proof of Theorem 1. We go back to Proposition 2. By (2.8), the quadratic form $(\widehat{B}r, r)$ can take negative values for some $r \in H_1$. We introduce the space $H_2 = H_1 \ominus \mathrm{Ker}\widehat{B}$ and the projection P_2 on this space. We note that $B_2 = P_2\widehat{B}P_2 = bI + K$, where $b(x) > 0$ and K is an integral operator with a weakly singular kernel, hence B_2 has a finite number of negative eigenvalues (see [14], Ch. 10). We denote by H_- the space spanned by the corresponding eigenfunctions of B_2. For $r \in H_-$, $r \neq 0$, we have $(B_2r, r) < 0$ and $r \in H_2 \ominus H_- \equiv H_+$, $r \neq 0$, satisfy the inequality $(B_2r, r) > 0$. Hence $H_2 = H_+ \oplus H_-$ is a Pontryagin space with the indefinite scalar product $(B_2r, \rho) = (\widehat{B}r, \rho)$.

Next, we introduce the space Y of the elements $U = (\boldsymbol{u}, \rho)^T$, with $\boldsymbol{u} \in \widetilde{J}(\mathcal{F})$ and $\rho \in H \ominus \mathrm{Ker}\widehat{B}$ satisfying the condition (3.3), i.e., by (2.7), $\rho = \Sigma(\boldsymbol{u}) + r$, where

$$\Sigma(\boldsymbol{u}) = \frac{1}{\omega\mathcal{S}}\Big(P_0\boldsymbol{\eta}_1(x)\cdot\boldsymbol{N}(x)\int_{\mathcal{F}}\boldsymbol{u}(z)\cdot\boldsymbol{\eta}_2(z)dz - P_0\boldsymbol{\eta}_2(x)\cdot\boldsymbol{N}(x)\int_{\mathcal{F}}\boldsymbol{u}(z)\cdot\boldsymbol{\eta}_1(z)dz\Big),$$

$r \in H_2$, $P_0 = I - Q_0$. We also set $\mathcal{J}_0 = \begin{pmatrix} I & 0 \\ 0 & \widehat{B} \end{pmatrix}$ and $\mathcal{J} = \Pi\mathcal{J}_0\Pi$ where Π is the orthogonal projection (in $L_2(\mathcal{F}) \times L_2(\mathcal{G})$) on the space Y. Let $[U, V]$ be a scalar product in $L_2(\mathcal{F}) \times L_2(\mathcal{G})$. For arbitrary $U \in Y$ we have

$$\begin{aligned}[\mathcal{J}U, U] = [\mathcal{J}_0U, U] &= \|\boldsymbol{u}\|^2_{L_2(\mathcal{F})} + (B\Sigma\boldsymbol{u}, \Sigma\boldsymbol{u}) + (BQ\rho, Q\rho) \\ &= \|\boldsymbol{u}\|^2_{L_2(\mathcal{F})} + \mathcal{S}^{-1}\sum_{\alpha=1}^{2}\Big|\int_{\mathcal{F}}\boldsymbol{u}\cdot\boldsymbol{\eta}_\alpha dx\Big|^2 + (B_2r, r).\end{aligned}$$

The spaces Y_+ and Y_- of the elements $U_+ = (\boldsymbol{u}, \Sigma(\boldsymbol{u}) + r_+)^T$ and $U_- = (0, r_-)$ possess the properties $[\mathcal{J}(U_+, U_+)] > 0$, $[\mathcal{J}(U_-, U_-)] < 0$ for non-zero U_+ and U_-, respectively, moreover, they are $\mathcal{J}$-orthogonal: $[\mathcal{J}U_+, U_-] = 0$. Finally, the operator $\mathcal{J}$ is bounded, self-adjoint and invertible in Y: if $\mathcal{J}U = 0$, then $[\mathcal{J}U, V] = 0$, $\forall V \in Y$; taking $V = U_+ - U_-$ we obtain

$$\begin{aligned}0 &= [\mathcal{J}_0(U_+ + U_-), (U_+ - U_-)] \\ &= \|\boldsymbol{u}\|^2_{L_2(\mathcal{F})} + \mathcal{S}^{-1}\sum_{\alpha=1}^{2}\Big|\int_{\mathcal{F}}\boldsymbol{u}\cdot\boldsymbol{\eta}_\alpha dx\Big|^2 + (B_2r_+, r_+) - (B_2r_-, r_-),\end{aligned}$$

which implies $U = 0$. Hence Y is the Pontryagin space and $[\mathcal{J}U, V]$ is an indefinite scalar product in Y.

Now we consider the expression $[\mathcal{J}\mathcal{A}'U, U]$ for $U \in Y$.

In view of Proposition 3, $\mathcal{A}'U \in Y$, hence

$$\begin{aligned}&[\mathcal{J}\mathcal{A}'U, U] = [\mathcal{J}_0\mathcal{A}'U, U] = [\mathcal{A}'U, \mathcal{J}_0U] \\ &= \int_{\mathcal{F}}(\nabla^2\boldsymbol{u} - \nabla(r_1 + r_2) - 2\omega(\boldsymbol{e}_3\times\boldsymbol{u}))\cdot\boldsymbol{u}dx + \int_{\mathcal{G}}\boldsymbol{u}\cdot\boldsymbol{N}\widehat{B}(\bar{\rho})dS \\ &= -\frac{\nu}{2}\int_{\mathcal{F}}|S(\boldsymbol{u})|^2dx + 2\omega\int_{\mathcal{F}}(u_2\bar{u}_1 - u_1\bar{u}_2)dx + \int_{\mathcal{G}}(\boldsymbol{u}\cdot\boldsymbol{N}\widehat{B}\bar{\rho} - \widehat{B}\rho\bar{\boldsymbol{u}}\cdot\boldsymbol{N})dS.\end{aligned}$$

Hence $\mathrm{Re}[\mathcal{J}\mathcal{A}'U, U] \le 0$, which means that $-i\mathcal{A}'$ is a $\mathcal{J}$-dissipative operator in Y. By the Pontryagin-M. Krein-Langer-Azizov theorem [5], $\mathcal{A}'$ has a finite-dimensional invariant subspace L and the eigenvalues of $\mathcal{A}'|_L$ have non-negative real parts. We have seen above that the operator $\mathcal{A}'$ can not have imaginary eigenvalues (including $\lambda = 0$). Hence these eigenvalues have positive real parts.

If $\lambda \ne 0$ and $U' = (\boldsymbol{u}, \rho')^T$ satisfies the equation $\mathcal{A}'U' = \lambda U'$ then $U = (\boldsymbol{u}, \rho)^T$ with $\rho = \rho' + \lambda^{-1}Q_0(\boldsymbol{u}\cdot\boldsymbol{N})$ satisfies (3.7) with the same λ. This completes the proof of Theorem 1.

4. The case of non-symmetric $\mathcal{F}$

In this case the equation (1.2) defines one-parameter family of equilibrium figures $\mathcal{F}_\theta$ obtained by rotation of the angle θ of one of them, $\mathcal{F}_0$, around the x_3-axis. In what follows we mean by $\mathcal{F}$ arbitrary such figure. As shown in [13], a linearized problem (1.10) takes the form

$$\begin{gathered}
\boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = 0,\\
\nabla \cdot \boldsymbol{v}(x,t) = 0, \quad x \in \mathcal{F}, \ \ t > 0,\\
T(\boldsymbol{v},p)\boldsymbol{N} + \boldsymbol{N}B_0\rho = 0, \qquad (4.1)\\
\rho_t = \boldsymbol{N}(x)\cdot\boldsymbol{v}(y,t) - \frac{h(x)}{\int_{\mathcal{G}} h^2(z)dS}\int_{\mathcal{G}} h(y)\boldsymbol{v}(y,t)\cdot\boldsymbol{N}(y)dS, \quad x \in \mathcal{G},\\
\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \mathcal{F}, \qquad \rho(x,0) = \rho_0(x), \quad x \in \mathcal{G},
\end{gathered}$$

where

$$h(x) = \boldsymbol{\eta}_3(x)\cdot\boldsymbol{N}(x).$$

In the case of axially symmetric $\mathcal{F}$ we have $h(x) = 0$, and the term with the integral in the boundary condition drops out. The orthogonality conditions (1.11), (1.12) should be supplemented with

$$\int_{\mathcal{G}} \rho(x,t)h(x)dS = 0.$$

By (2.5), $Bh = 0$.

The operator $\mathcal{A}$ is defined as above but the definition of its domain $D(\mathcal{A})$ contains the additional orthogonality condition

$$\int_{\mathcal{G}} \rho(x)h(x)dS = 0. \qquad (4.2)$$

Theorem 1 takes the following form:

Theorem $1'$**.** *Let*

$$\min_{|\theta|\le\pi}\int_{\mathcal{F}}((x_1\cos\theta + x_2\sin\theta)^2 - x_3^2)dx \ge c_0 > 0. \qquad (4.3)$$

If the quadratic form (1.18) *takes negative values for some ρ satisfying the conditions* (1.14), (4.2)*, then the operator $\mathcal{A}$ has a finite number of eigenvalues with a positive real part.*

Since $h \in \mathrm{Ker}\widehat{B}$, the proof of this theorem reduces to the analysis of the spectrum of the operator $\mathcal{A}'$, that is carried out as above. We prove the analogue of Proposition 2.

Proposition $2'$**.** *Arbitrary $\rho \in L_2(\mathcal{G})$ can be represented in the form* (2.7)*, where ρ_2 satisfies the conditions* (2.2) *and*

$$\rho_1(x) = a_1\boldsymbol{\eta}_1(x)\cdot\boldsymbol{N}(x) + a_2\boldsymbol{\eta}_2(x)\cdot\boldsymbol{N}(x).$$

The constants a_1, a_2 *are found from the algebraic system*

$$\sum_{\beta=1}^{2} \int_{\mathcal{G}} y_3 y_\alpha \boldsymbol{\eta}_\beta(y) \cdot \boldsymbol{N}(y) dS a_\beta = \int_{\mathcal{G}} \rho(y) y_3 y_\alpha dS, \quad \alpha = 1, 2. \tag{4.4}$$

If $\rho \in H$, *then* $\rho_2 \in H_1$. *If* $\rho = \boldsymbol{\eta}_\beta(x) \cdot \boldsymbol{N}(x)$, $\beta = 1, 2$, *then* $\rho_2 = 0$.

Proof. We turn to the equations (2.6):

$$(\boldsymbol{\eta}_1 \cdot \boldsymbol{N}, x_3 x_2) = \int_{\mathcal{F}} (x_2^2 - x_3^2) dx \equiv B$$

$$(\boldsymbol{\eta}_2 \cdot \boldsymbol{N}, x_3 x_1) = \int_{\mathcal{F}} (x_3^2 - x_1^2) dx \equiv -A,$$

$$(\boldsymbol{\eta}_1 \cdot \boldsymbol{N}, x_3 x_1) = -(\boldsymbol{\eta}_2 \cdot \boldsymbol{N}, x_3 x_2) = \int_{\mathcal{F}} x_1 x_2 dx \equiv C.$$

It follows that

$$\Big((B(\boldsymbol{\eta}_\alpha \cdot \boldsymbol{N}), \boldsymbol{\eta}_\beta \cdot \boldsymbol{N})\Big)_{\alpha,\beta=1,2} = \omega^2 \begin{pmatrix} B & -C \\ -C & A \end{pmatrix}. \tag{4.5}$$

This matrix is positive definite, because, by (1.17),

$$B\xi_1^2 - 2C\xi_2\xi_2 + A\xi_2^2 = \int_{\mathcal{F}} \Big((x_2\xi_1 - x_1\xi_2)^2 - x_3^2(\xi_1^2 + \xi_2^2)\Big) dx \geq c_0(\xi_1^2 + \xi_2^2).$$

Hence its determinant is positive. The determinant of the matrix

$$\Big((y_3 y_\alpha, \boldsymbol{\eta}_\beta \cdot \boldsymbol{N}\Big)_{\alpha,\beta=1,2} = \omega^2 \begin{pmatrix} C & -A \\ B & -C \end{pmatrix}$$

is negative, and the system (4.4) for a_1, a_2 is uniquely solvable. The proposition is proved.

As above, we have

$$(B\rho, \rho) = (B\rho_1, \rho_1) + (B\rho_2, \rho_2).$$

By the positivity of the matrix (4.5),

$$(B\rho_1, \rho_1) = \sum_{\alpha,\beta=1}^{2} (B\boldsymbol{\eta}_\alpha \cdot \boldsymbol{N}, \boldsymbol{\eta}_\beta \cdot \boldsymbol{N}) a_\alpha a_\beta \geq c(a_1^2 + a_2^2) \geq c(I_1^2(\rho) + I_2^2(\rho)).$$

It is also clear that

$$(B\rho_1, \rho_1) \leq c(I_1^2(\rho) + I_2^2(\rho)).$$

Further arguments related to the analysis of the spectrum of $\mathcal{A}'$ are essentially the same as in Sec. 3. If $\mathcal{A}'U' = \lambda U'$ where $\lambda \neq 0$ and $U' = (\boldsymbol{u}, \rho')^T$, then $\mathcal{A}U = \lambda U$ with $U = (\boldsymbol{u}, \rho)^T$,

$$\rho = \rho' + \lambda^{-1}\Big(Q_0(\boldsymbol{u} \cdot \boldsymbol{N}) - \frac{h}{\int_{\mathcal{G}} h^2 dS} \int_{\mathcal{G}} \boldsymbol{u} \cdot \boldsymbol{N} h dS\Big).$$

This proves Theorem 1′.

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Questions of Stability for a Parabolic-hyperbolic System

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Abstract. We consider a model describing a situation in which a population follows the density gradient of a nutrient that is produced at a spatially inhomogeneous rate of production and subject to diffusion. We show the stability of the equilibrium solution.

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1. Introduction

In the paper [3] Chen, Friedman and Hu consider the partial differential equations

$$\begin{aligned} u_t + \operatorname{div}(u\nabla\phi) &= G(u,\phi), \\ \phi_t - \Delta\phi &= F(u,\phi), \end{aligned} \tag{1.1}$$

or, equivalently,

$$\begin{aligned} u_t + \nabla u\nabla\phi &= G(u,\phi) - u\Delta\phi, \\ \phi_t - \Delta\phi &= F(u,\phi). \end{aligned} \tag{1.2}$$

on a bounded domain Ω with the boundary condition

$$\frac{\partial\phi}{\partial n} = 0 \tag{1.3}$$

on $\partial\Omega$. These equations provide a model for a system in which certain cells of density u move through the region Ω in the direction of the gradient of the density ϕ of the nutrient nourishing them. Their rate of reproduction also may depend on the cell density and the density of the nutrient. Models of this kind are often referred to as Keller-Segel models. For a review of this topic see, e.g., [6] and [7].

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In most cases they take the form of a parabolic system of equations, while in our case the first equation in (1.2) is, if we assume ϕ to be given for the moment, a hyperbolic equation. This is also, e.g., true in [4], but in contrast to [3] the second equation there is elliptic.

The characteristic curves for the hyperbolic equation are parallel to $\partial\Omega$ owing to condition (1.3), thus no boundary condition for u is needed. Examples of F and G discussed in [3] are

$$G(u,\phi) = \mu k(u)\phi - \gamma u,$$
$$F(u,\phi) = \delta - k(u)\phi$$

with positive numbers γ, δ, μ and a function $k(u)$, in particular

$$k(u) = \frac{u}{1+u}.$$

Under certain assumptions Chen, Friedman and Hu prove the existence of a long-term solution and the local stability of certain equilibrium states of the system. These equilibria are such that u and ϕ equal constant numbers u_* and ϕ_*, respectively, in Ω. Constant equilibrium solutions (u_*, ϕ_*) must fulfill the equations

$$G(u_*, \phi_*) = 0, F(u_*, \phi_*) = 0, \tag{1.4}$$

thus we have two equations for two unknowns. In this paper we will only consider the stability of equilibria, therefore the global assumptions about F and G made in [3] are of no significance to us. Defining

$$\begin{bmatrix} \frac{\partial G}{\partial u} & \frac{\partial G}{\partial \phi} \\ \frac{\partial F}{\partial u} & \frac{\partial F}{\partial \phi} \end{bmatrix} (u_*, \phi_*) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = J,$$

Chen, Friedman and Hu assume the following two conditions for the stability of the equilibria.

1. All eigenvalues of J have negative real part, or $ad - cb > 0$, $a + d < 0$,
2. $a + cu_* < 0$.

This means that, assuming Condition 2, any constant solution of the system of partial differential equations is stable with respect to spatially non-constant perturbations if it is stable with respect to spatially constant perturbations. Now it would be interesting to allow F and G, in particular the production rates for the nutrient, to depend on the position $x \in \overline{\Omega}$. Equations (1.4) would then become

$$G(u_*, \phi_*, x) = 0, F(u_*, \phi_*, x) = 0. \tag{1.5}$$

In most cases these equations only have solutions depending on x, and then they are no longer the proper equations for an equilibrium, and one has to consider the equations

$$\begin{aligned} \nabla u_* \nabla \phi_* &= G(u_*, \phi_*, x) - u_* \Delta \phi_*, \\ -\Delta \phi_* &= F(u_*, \phi_*, x) \end{aligned} \tag{1.6}$$

instead. Unless ϕ_* is constant, the cells would still move around in such an equilibrium state. The case of constant ϕ_* is of interest, though, as one would expect it to make sense to have equilibria without any movement with a population density varying with the density of nutrient production. In case $k(u) = u$, i.e.,

$$G(u, \phi, x) = \mu u\phi - \gamma u, F(u, \phi, x) = \delta(x) - u\phi. \tag{1.7}$$

there are such equilibria with a variable u_* and a constant ϕ_*, as $F = G = 0$ is equivalent to

$$\begin{aligned} u_*\phi_* &= \delta(x), \\ \mu u_*\phi_* &= \gamma u_*, \end{aligned}$$

and

$$\begin{aligned} \phi_* &= \frac{\gamma}{\mu}, \\ u_*(x) &= \frac{\mu}{\gamma}\delta(x), \end{aligned}$$

while for other k there may not be such equilibria.

For $k(u) = u$ the Jacobian matrix is

$$\begin{bmatrix} \frac{\partial G}{\partial u} & \frac{\partial G}{\partial \phi} \\ \frac{\partial F}{\partial u} & \frac{\partial F}{\partial \phi} \end{bmatrix}(u_*, \phi_*) = \begin{bmatrix} \mu\phi_* - \gamma & \mu u_* \\ -\phi_* & -u_* \end{bmatrix} = \begin{bmatrix} 0 & \frac{\mu^2}{\gamma}\delta(x) \\ -\frac{\gamma}{\mu} & -\frac{\mu}{\gamma}\delta(x) \end{bmatrix},$$

its trace is $-\frac{\mu}{\gamma}\delta(x)$, its determinant is $\mu\delta(x)$ and $a + cu_* = -\delta(x)$. This means that if $\delta(x) > 0$ everywhere, Conditions 1 and 2 are always fulfilled pointwise.

For the remainder of the paper we assume that the functions F and G belong to C^4 as functions of all their variables, and that $\partial\Omega$ belongs to C^4 as well, that $G(u_*(x), \phi_*, x) = F(u_*(x), \phi_*, x) = 0$ for a constant ϕ_* and a strictly positive function $u_* \in C^4\left(\overline{\Omega}\right)$, and that Condition 2 is fulfilled pointwise. The formal linearization of equations (1.2) at such an equilibrium is the system

$$\begin{aligned} v_t &= -\nabla u_*\nabla\varphi + G_u(u_*, \phi_*, x)v + G_\phi(u_*, \phi_*, x)\varphi - u_*\Delta\varphi, \\ \varphi_t - \Delta\varphi &= F_u(u_*, \phi_*, x)v + F_\phi(u_*, \phi_*, x)\varphi \end{aligned} \tag{1.8}$$

for perturbations φ of ϕ_* and v of u_*. Before we begin stating our result, we express equations (1.2) in terms of the variables $v = u - u_*$ and $\varphi = \phi - \phi_*$. We have

$$\begin{aligned} v_t + \nabla v\nabla\varphi &= -\nabla u_*\nabla\varphi + \widehat{G}(v, \varphi, x) - (v + u_*)\Delta\varphi, \\ \varphi_t - \Delta\varphi &= \widehat{F}(v, \varphi, x) \end{aligned} \tag{1.9}$$

with $\widehat{F}(v, \varphi, x) = F(v + u_*, \varphi + \phi_*, x)$ and $\widehat{G}(v, \varphi, x) = G(v + u_*, \varphi + \phi_*, x)$.

Let us also rewrite the system (1.8) in the form $U_t = LU$ with $U = \begin{bmatrix} v & \varphi \end{bmatrix}^T$,

$$LU = \begin{bmatrix} -\nabla u_*\nabla\varphi + G_u(u_*, \phi_*, x)v + G_\phi(u_*, \phi_*, x)\varphi - u_*\Delta\varphi \\ \Delta\varphi + F_u(u_*, \phi_*, x)v + F_\phi(u_*, \phi_*, x)\varphi \end{bmatrix},$$

introducing the notations

$$D(L) = \left\{U = \begin{bmatrix} v & \varphi \end{bmatrix}^T \mid \varphi(x) \in C^3\left(\overline{\Omega}\right), v(x) \in C^1\left(\overline{\Omega}\right), \frac{\partial\varphi}{\partial n} = 0\right\}$$

and

$$\sigma_e = \{z \in \mathbb{C} \mid \text{there is a } U \in D(L) \setminus \{0\} \text{ with } LU = zU\}$$

for the set of eigenvalues of L.

We prove that spectral stability implies dynamic stability, i.e., that the absence of eigenvalues of the formal linearization with non-negative real part implies the non-linear stability of the equilibrium. In addition, for constant equilibria and for the functions F and G given in (1.7) we reproduce the result of Chen, Friedman and Hu that Conditions 1 and 2 imply stability. We work in Sobolev spaces W_p^k with $p > 3$. To be precise, we prove

Theorem 1.1. *Assume that*

$$\sigma_e \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\} = \emptyset. \tag{1.10}$$

Then there is a number $\eta > 0$ *such that if* $\varphi_0 \in W_p^3$ *and* $v_0 \in W_p^1$ *and* $\|\varphi_0\|_{W_p^3} + \|v_0\|_{W_p^1} \leq \eta$*, then there exists a unique solution* (v, φ) *of equations* (1.9) *for* $t \in [0, \infty)$ *such that* $v \in C^0\left([0,\infty), W_p^1(\Omega)\right) \cap C^1\left([0,\infty), L^p(\Omega)\right)$, $\varphi \in C^0\left([0,\infty), W_p^3(\Omega)\right) \cap C^1\left([0,\infty), W_p^1(\Omega)\right)$, $\frac{\partial \varphi}{\partial n} = 0$ *and* $\varphi(0) = \varphi_0$, $v(0) = v_0$*. Also* $\|\varphi(t)\|_{W_p^3} + \|v(t)\|_{W_p^1} \to 0$ *as* $t \to \infty$*.*

Condition (1.10) *is implied by Conditions* 1 *and* 2 *if* F *and* G *are either independent of* x *or of the form given in equations* (1.7)*.*

The system (1.2) is similar to the equations for viscous compressible fluids in that it combines a hyperbolic equation with a parabolic one. Therefore it seems reasonable to transform the equations into Lagrange coordinates produced by the vector field $\nabla\phi$. Doing this, we encounter the problem that if we control k spatial derivatives of ϕ, we only control $k-1$ derivatives of the Lagrange coordinates and therefore $k-2$ of those of the transform of the normal n, leading to a priori estimates of only $k-1$ derivatives of ϕ. Thus we have a loss of regularity, which does not occur in the original coordinates. To avoid this we will transform the dependent variables in our problem.

2. Notation

Let $\Omega \subset \mathbb{R}^n$ be a domain. By $L^p(\Omega)$ we denote for $p \in (1,\infty)$ the set of all measurable real (or complex, the distinction will usually not cause any difficulty) functions for which the pth power of their absolute value is integrable, by $W_p^k(\Omega)$ for integer $k > 0$ and again $p \in (1,\infty)$ the subset of $L^p(\Omega)$ of all functions having distributional derivatives up to order k belonging to $L^p(\Omega)$ also. If the set is the domain Ω mentioned above, it is usually omitted. If B is any Banach space, we denote by $C^k(\Omega, B)$ the set of all k-times continuously differentiable functions from Ω to B, by $C^k\left(\overline{\Omega}, B\right)$ the subset of functions for which all these derivatives have a continuous extension to $\overline{\Omega}$.

3. Local existence

We will first show that for given initial values a solution exists for a time interval of a length going to infinity as the size of the initial values goes to zero. A similar result in Hölder spaces was already proved in [3]. There is not much genuinely new in this section, the point is just to state precisely what we need here and give an outline of a proof. We confine ourselves to initial values such that $\varphi_0 \in W_p^3$, $\frac{\partial \varphi_0}{\partial n} = 0$ and $v_0 \in W_p^1$.

The operator Δ with the boundary conditions $\frac{\partial \varphi}{\partial n} = 0$ generates an analytic semigroup on the space $L^p(\Omega)$ with the domain of definition

$$D(\Delta) = \left\{ \varphi \in L^p(\Omega) \mid \varphi \in W_p^2(\Omega), \frac{\partial \varphi}{\partial n} = 0 \right\}.$$

We will prove the existence of a solution by a fixed point argument in the space

$$\begin{aligned} B_2 = \{ & (v,\varphi) \mid v \in C^0\left([0,T], W_p^1(\Omega)\right) \cap C^1\left([0,T], L^p(\Omega)\right), \\ & \varphi \in C^0\left([0,T], W_p^3(\Omega)\right) \cap C^1\left([0,T], W_p^1(\Omega)\right) \} \end{aligned}$$

with the norm

$$\|(v,\varphi)\|_2 = \max_{0\le t\le T} \left(\|v(t)\|_{W_p^1} + \|v_t(t)\|_{L^p} + \|\varphi(t)\|_{W_p^3} + \|\varphi_t(t)\|_{W_p^1} \right).$$

The result is the following.

Theorem 3.1. *There is a positive number C such that if*

$$\eta = \|\varphi_0\|_{W_p^3} + \|v_0\|_{W_p^1}$$

and $\frac{\partial \varphi_0}{\partial n} = 0$, then a unique solution (v,φ) of the system (1.9) exists on any interval $[0,T]$ with $T \le -C - C^{-1}\log(\eta)$ if this number is positive. In addition then

$$\|(v,\varphi)\|_2 \le Ce^{CT}\eta \le 1$$

and if $T \ge 1$, then also

$$\|\varphi(T)\|_{W_p^3} \le C\left[\|(\varphi,v)\|_{C^0\left([T-1,T],W_p^1\right)} + \|(\varphi_t,v_t)\|_{L^p([T-1,T]\times\Omega)} \right].$$

Proof. To reformulate our problem as a fixed-point equation let us assume $(\widehat{v},\widehat{\varphi}) \in B_2$ and $\|(\widehat{v},\widehat{\varphi})\|_2 \le 1$. Then we define (v,φ) as the solution of the system of equations

$$\begin{aligned} & v_t + \nabla v \nabla \widehat{\varphi} = -\nabla u_* \nabla \widehat{\varphi} + \widehat{G}(\widehat{v},\widehat{\varphi},x) - (\widehat{v}+u_*)\Delta\widehat{\varphi} = \widehat{g}, \\ & \varphi_t - \Delta\varphi = \widehat{F}(\widehat{v},\widehat{\varphi},x) \end{aligned} \tag{3.1}$$

with the initial values $\varphi(0) = \varphi_0, v(0) = v_0$. We can differentiate the second equation with respect to time and obtain for $f = \varphi_t$ that

$$f_t - \Delta f = \widehat{F}_v(\widehat{v},\widehat{\varphi},x)\,\widehat{v}_t + \widehat{F}_\varphi(\widehat{v},\widehat{\varphi},x)\,\widehat{\varphi}_t,$$

which implies by standard estimates for semigroups – a fairly abstract and general version is contained in Theorem 5.2.1. in [1] – that

$$\|f(t)\|_{W^1_p} \leq C\left(\|\varphi_0\|_{W^3_p} + \|v_0\|_{W^1_p}\right) + Ct^{1/2}\,\|(\widehat{v},\widehat{\varphi})\|_2$$

and therefore

$$\|\varphi(t)\|_{W^3_p} + \|\varphi_t(t)\|_{W^1_p} \leq C\left(\|\varphi_0\|_{W^3_p} + \|v_0\|_{W^1_p}\right) + Ct^{1/2}\,\|(\widehat{v},\widehat{\varphi})\|_2$$

by standard elliptic estimates. (See, among many others, [2].) Introducing characteristic coordinates by

$$\frac{dT_t}{dt} = \nabla\varphi\left(T_t(x), t\right), T_0(x) = x,$$

we obtain

$$\|T_t\|_{W^2_p} + \left\|T_t^{-1}\right\|_{W^2_p} \leq C(T)\,\|(\widehat{v},\widehat{\varphi})\|_2\,. \tag{3.2}$$

Therefore also

$$\|\widehat{g}\circ T_t\|_{W^1_p} \leq C(T)\,\|(\widehat{v},\widehat{\varphi})\|_2$$

(for the definition of $\widehat{g}$ see equations (3.1)) and

$$\|v_t(t)\|_{L^p} + \|v(t)\|_{W^1_p} \leq C\,\|v_0\|_{W^1_p} + TC(T)\,\|(\widehat{v},\widehat{\varphi})\|_2$$

with an increasing function $C(T)$. We therefore have

$$\|(v,\varphi)\|_2 \leq C_1\left(\|\varphi_0\|_{W^3_p} + \|v_0\|_{W^1_p}\right) + T^{1/2}C(T)\,\|(\widehat{v},\widehat{\varphi})\|_2\,,$$

where $C(T)$ now denotes a different increasing function. If we make $T = T_0$ so small that $T_0^{1/2}C(T_0) \leq 1/2$ and then restrict the initial values by the inequality

$$\|\varphi_0\|_{W^3_p} + \|v_0\|_{W^1_p} \leq \frac{1}{2C_1} = \eta_1,$$

the mapping taking $(\widehat{v},\widehat{\varphi})$ to (v,φ) maps the ball $\|(v,\varphi)\|_2 \leq 1$ to itself. It is also easy to see that at least sufficiently high powers of the mapping are contractions in lower-order norms, and thus it has a unique fixed point. Also with $C_2 = 2C_1$ we have

$$\|\varphi(t)\|_{W^3_p} + \|v(t)\|_{W^1_p} \leq C_2\left(\|\varphi_0\|_{W^3_p} + \|v_0\|_{W^1_p}\right)$$

for $t \in [0, T_0]$. If the initial value was small enough we can repeat this process, and we then obtain the existence of a solution on the interval $[0, T_0 k]$ if $\eta C_2^{k-1} \leq \eta_1$ and the inequality

$$\|\varphi(t)\|_{W^3_p} + \|v(t)\|_{W^1_p} \leq C_2^k\left(\|\varphi_0\|_{W^3_p} + \|v_0\|_{W^1_p}\right)$$

for $0 \leq t \leq T_0 k$. From this we can easily derive our claims with the exception of the last inequality. To obtain that inequality, observe that the right-hand side of the equation for φ belongs to $C^{1/2}\left([T-1,T], L^p\right)$ and this norm can be estimated by the right-hand side above. Using Theorem 2.5.3, Section III in [1], we then have that $\|\varphi\|_{C^{1/2}\left([T-1/2,T],W^2_p\right)}$ is bounded. Then we can differentiate the equation with respect to t and use the usual parabolic estimates to get $\|\varphi_t\|_{W^{2,1}_p([T-1/3,T]\times\Omega)}$ and therefore $\|\varphi_t(t)\|_{W_p^{2-2/p}}$ for $t \in [T-1/4, T]$ is bounded. (See, e.g., Theorem 7.20 in [9].) Using standard elliptic estimates, we can then obtain our claims. □

4. Long term existence and stability

Throughout the following calculations we assume that $\|(v,\varphi)\|_2 \leq 1$. In this section we will prove a priori estimates for small solutions of the equations (1.2) by considering them as solutions of the inhomogeneous version of the linearized equations (1.8). They are not particularly hard to solve, but in switching to Lagrange coordinates, as was already indicated, if we do not look at the non-linear problem from the right angle, we have a loss of regularity. In order to avoid this problem, we transform the equation, using the second equation in (1.9) to remove $u_*\Delta\varphi$ from the first. Then we have

$$\begin{aligned} v_t + \nabla v \nabla \varphi &= -\nabla u_* \nabla \varphi + \widehat{G}(v,\varphi,x) - u_* \left(\varphi_t - \widehat{F}(v,\varphi,x)\right) - v\Delta\varphi, \\ \varphi_t - \Delta\varphi &= \widehat{F}(v,\varphi,x). \end{aligned} \tag{4.1}$$

This leads to the equations

$$\begin{aligned} &(v+u_*\varphi)_t + \nabla(v+u_*\varphi)\nabla\varphi \\ &\quad = \nabla(u_*\varphi)\nabla\varphi - \nabla u_*\nabla\varphi + \widehat{G}(v,\varphi,x) + u_*\widehat{F}(v,\varphi,x) - v\Delta\varphi, \\ &\varphi_t - \Delta\varphi = \widehat{F}(v,\varphi,x), \end{aligned}$$

suggesting the definition $w = v + u_*\varphi$. With these new variables the equilibrium in question is still located at $w = 0, \varphi = 0$, and for (w,φ) we have the equations

$$\begin{aligned} &w_t + \nabla w \nabla \varphi \\ &\quad = -\nabla u_* \nabla\varphi + u_* |\nabla\varphi|^2 + \varphi \nabla u_* \nabla \varphi \\ &\qquad + \widehat{G}(w-u_*\varphi,\varphi,x) + u_*\widehat{F}(w-u_*\varphi,\varphi,x) - (w-u_*\varphi)\Delta\varphi, \\ &\varphi_t - \Delta\varphi = \widehat{F}(w-u_*\varphi,\varphi,x). \end{aligned} \tag{4.2}$$

Letting

$$\mathsf{G}(P,w,\varphi,x) = u_*|P|^2 + \varphi\nabla u_* P + \widehat{G}(w-u_*\varphi,\varphi,x) + u_*\widehat{F}(w-u_*\varphi,\varphi,x)$$

and

$$\mathsf{F}(w,\varphi,x) = \widehat{F}(w-u_*\varphi,\varphi,x)$$

we obtain

$$\begin{aligned} w_t + \nabla w\nabla\varphi &= -\nabla u_*\nabla\varphi + \mathsf{G}(\nabla\varphi,w,\varphi,x) - (w-u_*\varphi)\Delta\varphi, \\ \varphi_t + |\nabla\varphi|^2 - \Delta\varphi &= \mathsf{F}(w,\varphi,x) + |\nabla\varphi|^2. \end{aligned} \tag{4.3}$$

Let us define

$$\begin{aligned} A(x) &= \mathsf{G}_w(0,0,0,x), B(x) = \mathsf{G}_\varphi(0,0,0,x), \\ C(x) &= \mathsf{F}_w(0,0,x), D(x) \quad = \mathsf{F}_\varphi(0,0,x) \end{aligned}$$

and

$$R_1 = u_* |\nabla\varphi|^2 + \varphi\nabla u_*\nabla\varphi - (w - u_*\varphi)\,\Delta\varphi + \mathsf{G}(0, w, \varphi, x) - A(x)\,w - B(x)\,\varphi,$$
$$R_2 = \mathsf{F}(w, \varphi, x) - C(x)\,w - D(x)\,\varphi + |\nabla\varphi|^2 .$$

Then

$$\|R_1\|_{W_p^1} \le C\left(\|w\|_{W_p^1}^2 + \|\varphi\|_{W_p^2}^2 + \|\Delta\varphi\|_{W_p^1}^2\right),$$
$$\|R_2\|_{L^p} \le C\left(\|\varphi\|_{W_{2p}^1}^2 + \|w\|_{L^{2p}}^2\right)$$

and

$$\begin{aligned} w_t + \nabla w\nabla\varphi &= -\nabla u_*\nabla\varphi + A(x)\,w + B(x)\,\varphi + R_1, \\ \varphi_t + \nabla\varphi\nabla\varphi - \Delta\varphi &= C(x)\,w + D(x)\,\varphi + R_2. \end{aligned} \tag{4.4}$$

Now we introduce Lagrange coordinates by

$$T_t'(y) = \nabla\varphi(T_t(y), t), T_0(y) = y.$$

Then let $\widetilde{w}(y,t) = w(T_t(y), t)$ and

$$\nabla_T (f \circ T_t) = (\nabla f) \circ T_t, L_t (f \circ T_t) = \Delta f \circ T_t.$$

We refrain from stating the exact form of these differential operators, although they will be needed later in the proof, but we leave these calculations to the reader. Then, in these new coordinates, equation (1.9) is equivalent to

$$\begin{aligned} \widetilde{w}_t &= -\nabla_T \widetilde{u}_* \nabla_T \widetilde{\varphi} + \widetilde{A}\widetilde{w} + \widetilde{B}\widetilde{\varphi} + \widetilde{R}_1, \\ \widetilde{\varphi}_t - L_t\widetilde{\varphi} &= \widetilde{C}(x)\,\widetilde{w} + \widetilde{D}(x)\,\widetilde{\varphi} + \widetilde{R}_2. \end{aligned} \tag{4.5}$$

This is why we consider the evolution equation

$$\begin{aligned} \widetilde{w}_t &= -\nabla_T \widetilde{u}_* \nabla_T \widetilde{\varphi} + \widetilde{A}\widetilde{w} + \widetilde{B}\widetilde{\varphi}, \\ \widetilde{\varphi}_t - L_t\widetilde{\varphi} &= \widetilde{D}\widetilde{\varphi} + \widetilde{C}\widetilde{w} \end{aligned} \tag{4.6}$$

with the boundary values

$$\widetilde{n} \cdot \nabla_T \widetilde{\varphi} = 0.$$

We will find out that this is now accessible to arguments considering this as a perturbation of the equation

$$\begin{aligned} \widetilde{w}_t &= -\nabla u_* \nabla\widetilde{\varphi} + A\widetilde{w} + B\widetilde{\varphi}, \\ \widetilde{\varphi}_t - \Delta\widetilde{\varphi} &= D\widetilde{\varphi} + C\widetilde{w} \end{aligned} \tag{4.7}$$

with boundary condition $\frac{\partial\widetilde{\varphi}}{\partial n} = 0$. We now omit the $\widetilde{}$ in the notation. We want to show we can use the theory of analytic semigroups by studying the operator

$$\mathbb{A}\begin{bmatrix} w \\ \varphi \end{bmatrix} = \begin{bmatrix} -\nabla u_*\nabla\varphi + Aw + B\varphi \\ \Delta\varphi + D\varphi + Cw \end{bmatrix}.$$

Then $\mathbb{A} : D(\mathbb{A}) \to B$ with

$$B = \left\{ U = \begin{bmatrix} w \\ \varphi \end{bmatrix} \mid w \in W_p^1, \varphi \in L^p \right\}$$

and

$$D(\mathbb{A}) = \left\{ \begin{bmatrix} w \\ \varphi \end{bmatrix} \in B \mid \varphi \in W_p^2, \frac{\partial \varphi}{\partial n} = 0 \right\}.$$

Then equation (4.7) becomes $U_t = \mathbb{A}U$ and we have the following result.

Lemma 4.1. *Assume that no z with $\mathrm{Re}(z) \geq 0$ is an eigenvalue of $\mathbb{A}$. Then this half-plane belongs to the resolvent set of $\mathbb{A}$ and there exists a number C such that*

$$\left\| (zI - \mathbb{A})^{-1} \right\|_{L(B)} \leq C (1 + |z|)^{-1}$$

for z with $\mathrm{Re}(z) \geq 0$. There also is an $\varepsilon > 0$ such that

$$\|\exp(t\mathbb{A})\|_{L(B)} \leq Ce^{-\varepsilon t}.$$

Proof. First we prove the resolvent estimate. If

$$zU - \mathbb{A}U = F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^T,$$

then

$$\begin{aligned} (z - A) w &= -\nabla u_* \nabla \varphi + B\varphi + F_1, \\ z\varphi - \Delta\varphi &= D\varphi + Cw + F_2 \end{aligned} \tag{4.8}$$

We have

$$\begin{aligned} A = \frac{\partial \mathsf{G}}{\partial w}(0,0,0,x) &= \frac{\partial \widehat{G}}{\partial v}(0,0,x) + u_* \frac{\partial \widehat{F}}{\partial v}(0,0,x) \\ &= \frac{\partial G}{\partial u}(u_*, \phi_*, x) + u_* \frac{\partial F}{\partial u}(u_*, \phi_*, x) = a + cu_* < 0, \end{aligned}$$

thus

$$A = a + cu_* \tag{4.9}$$

and $\max\limits_{x \in \overline{\Omega}} A(x) < 0$. From the first equation in (4.8) for $\mathrm{Re}(z) \geq 0$

$$w = (z - A)^{-1} (-\nabla u_* \nabla \varphi + B\varphi + F_1) \tag{4.10}$$

and

$$z\varphi - \Delta\varphi = D\varphi + C(-\nabla u_* \nabla \varphi + B\varphi + F_1)(z - A)^{-1} + F_2.$$

From standard elliptic estimates we then obtain

$$\|\varphi\|_{W_p^2} + |z| \|\varphi\|_{L^p} \leq C \left(\|\varphi\|_{W_p^1} + \|F\|_{L^p} \right).$$

From equation (4.10) then

$$\|w\|_{W_p^1} (1 + |z|) \leq C \left(\|F\|_B + \|\varphi\|_{W_p^1} \right).$$

This proves

$$\left\| (zI - \mathbb{A})^{-1} \right\|_{L(B)} \leq C |z|^{-1}$$

for large z. By standard compactness arguments one also has that if the only solution of $\mathbb{A}U = zU$ is zero for all z with $\operatorname{Re}(z) \geq 0$, then the resolvent set contains all such z and

$$\left\| (zI - \mathbb{A})^{-1} \right\|_{L(B)} \leq C\,(1 + |z|)^{-1}$$

for such z. The existence of solutions for very large z is obvious in view of the above considerations, and then it carries over to smaller z owing to the estimates we proved. The remainder is a consequence of, e.g., Theorem 2.1, Part II in [5]. □

Lemma 4.2. *For* $U = \begin{bmatrix} \varphi & v + u_* \varphi \end{bmatrix}^T$ *the equation* $\mathbb{A}U = zU$ *is equivalent to*

$$zv = -\nabla u_* \nabla \varphi + av + b\varphi - u_* \Delta \varphi$$

and

$$z\varphi = \Delta \varphi + d\varphi + cv.$$

Proof. Using equations (4.8) and (4.9) and remembering the definition of A, B, C, and D in equations (4.4), this is an easy calculation. □

Now we want to obtain the information we need about the long-term behavior by considering the problem as a perturbation of this linearization. To do this we also need to consider the equation

$$\begin{gathered} w_t = -\nabla u_* \nabla \varphi + Aw + B\varphi + f_1, \\ \varphi_t - \Delta \varphi = D\varphi + Cw + f_2, \quad \frac{\partial \varphi}{\partial n} = g \text{ on } \partial\Omega. \end{gathered} \tag{4.11}$$

It is easy to show the following result.

Lemma 4.3. *Assume* f_1 *belongs to* $L^p\left((0,T), W_p^1\right)$, f_2 *to* $L^p\left((0,T) \times \Omega\right)$ *and* g *to* $W^{1-1/p,1/2-1/2p}\left((0,T) \times \partial\Omega\right)$ *as well as* $g(0) = 0$. *Then there is exactly one pair of functions* $\varphi \in W_p^{2,1}, w \in C^0\left([0,T], W_p^1\right)$ *with* $w_t \in L^p\left((0,T) \times \Omega\right)$ *fulfilling equation* (4.11) *and* $\varphi(0) = 0$, $w(0) = 0$ *as well as the estimate*

$$\begin{aligned} &\|\varphi\|_{W_p^{2,1}} + \|w\|_{C^0\left([0,T], W_p^1\right)} + \|w_t\|_{L^p((0,T)\times\Omega)} \\ &\quad \leq C(T)\left(\|f_2\|_{L^p((0,T)\times\Omega)} + \|g\|_{W_p^{1-1/p,1/2-1/2p}((0,T)\times\partial\Omega)} + \|f_1\|_{L^p\left((0,T), W_p^1\right)}\right). \end{aligned}$$

Proof. For sufficiently short time intervals we can treat this system as a perturbation of the single equations $w_t = f_1$ and $\varphi_t - \Delta\varphi = f_2, \frac{\partial \varphi}{\partial n} = g$. Then we can continue step by step, also using the information about the operator $\mathbb{A}$ we already derived. There we can use Theorem 5.4 in [8]. A proof can also be done in a similar way to that of Theorem 13 in [10]. □

Now the solution φ, v constructed in Theorem 3.1 has the property that

$$\|(v, \varphi)\|_2 \leq C(T)\left(\|v(0)\|_{W_p^1} + \|\varphi(0)\|_{W_p^3}\right).$$

We also have

$$\|T_t - id_{\overline{\Omega}}\|_{W_p^2} + \left\|T_t^{-1} - id_{\overline{\Omega}}\right\|_{W_p^2} \leq C(T)\left(\|v(0)\|_{W_p^1} + \|\varphi(0)\|_{W_p^3}\right).$$

Thus

$$\left\|\widetilde{R}_1(t)\right\|_{W_p^1} + \left\|\widetilde{R}_2(t)\right\|_{L^p} \leq C(T)\left(\|v(0)\|_{W_p^1} + \|\varphi(0)\|_{W_p^3}\right)^2,$$

it is also easy to see that the remaining error terms one has to incorporate into f_1 and f_2 to make $\widetilde{w}$ and $\widetilde{\varphi}$ solutions of (4.11) can be estimated in the same way. If we write $\widetilde{w}$ and $\widetilde{\varphi}$ in the form $\widetilde{w} = w_1 + w_2$ and $\widetilde{\varphi} = \varphi_1 + \varphi_2$ where (w_1, φ_1) solve equation (4.11) with $f_1 = f_2 = 0$ and $(w_1, \varphi_1)(0) = (w_0, \varphi_0)$, while (w_1, φ_1) solve equation (4.11) with the given f_1 and f_2 and $(w_2, \varphi_2)(0) = 0$, then combining Lemmas 4.1, 4.2 and 4.3 we have

$$\begin{aligned}&\|\widetilde{\varphi}\|_{W_p^{2,1}((T-1,T)\times\Omega)} + \|\widetilde{w}\|_{C^0\left([T-1,T],W_p^1\right)} + \|\widetilde{w}_t\|_{L^p((T-1,,T)\times\Omega)}\\ &\quad\leq C(T)\left(\|v_0\|_{W_p^1} + \|\varphi_0\|_{W_p^3}\right)^2 + Ce^{-\varepsilon T}\left(\|\varphi_0\|_{W_p^3} + \|v_0\|_{W_p^1}\right).\end{aligned}$$

Then we can use the last inequality in Theorem 3.1, which one can easily see is still true if we replace v by w, to obtain

$$\|\varphi(T)\|_{W_p^3} + \|v(T)\|_{W_p^1} \leq C(T)\left(\|\varphi_0\|_{W_p^3} + \|v_0\|_{W_p^1}\right)^2 + Ce^{-\varepsilon T}\left(\|\varphi_0\|_{W_p^3} + \|v_0\|_{W_p^1}\right).$$

For T so large that $Ce^{-\varepsilon T} < 1/2$ and initial values small enough to assure the existence of a solution on $[0, T]$ we then have

$$\|\varphi(T)\|_{W_p^3} + \|v(T)\|_{W_p^1} \leq \frac{1}{2}\left(\|\varphi_0\|_{W_p^3} + \|v_0\|_{W_p^1}\right) + C\left(\|\varphi_0\|_{W_p^3} + \|v_0\|_{W_p^1}\right)^2.$$

For sufficiently small initial values then

$$\|\varphi(T)\|_{W_p^3} + \|v(T)\|_{W_p^1} \leq \frac{3}{4}\left(\|\varphi_0\|_{W_p^3} + \|v_0\|_{W_p^1}\right).$$

This allows us to continue the solution indefinitely and implies that it converges to zero as claimed. This concludes the proof of the first part of Theorem 1.1.

5. Excluding eigenvalues

For $\mathrm{Re}(z) \geq 0$ we have to conclude $v = 0, \varphi = 0$ from

$$\begin{bmatrix} zv \\ z\varphi - \Delta\varphi \end{bmatrix} = \begin{bmatrix} -\nabla u_*\nabla\varphi + av + b\varphi - u_*\Delta\varphi \\ cv + d\varphi \end{bmatrix}$$

and $\frac{\partial\varphi}{\partial n} = 0$, where we have suppressed the variable x throughout. Solving the lower equation for $\Delta\varphi$ we obtain

$$\Delta\varphi = (z - d)\varphi - cv.$$

We can use this to replace $\Delta\varphi$ in the upper equation, which results in

$$zv = -\nabla u_*\nabla\varphi + av + b\varphi - u_*((z - d)\varphi - cv).$$

We assumed $-a - u_*c > 0$, thus $-a - u_*c + z \neq 0$, and we can solve for v to obtain

$$v = \frac{b + u_*d - u_*z}{z - a - u_*c}\varphi - \frac{\nabla u_*\nabla\varphi}{z - a - u_*c}.$$

Replacing v in the second equation gives us

$$\Delta\varphi = \frac{ad - bc - z\,(a+d) + z^2}{z - a - u_* c}\varphi + \frac{c\nabla u_*}{z - a - u_*}\nabla\varphi.$$

To duplicate the conditions of Chen, Friedman and Hu, we assume F and G do not explicitly depend on x and neither does u_*, and $D = ad - bc > 0, \tau = a + d < 0$ and $\Gamma = a + u_* c < 0$. If $\varphi \neq 0$ it has to be an eigenfunction of $-\Delta$ with Neumann boundary conditions for an eigenvalue $\lambda_k \geq 0$, thus

$$\frac{D - \tau z + z^2}{z - \Gamma} = -\lambda_k$$

and

$$-\lambda_k \Gamma + D + (\lambda_k - \tau)\, z + z^2 = 0.$$

This is an equation which cannot have any roots with a non-negative real part, as $-\lambda_k\Gamma + D > 0$ and $\lambda_k - \tau > 0$.

For variable equilibria, considering the case with a variable δ we have

$$\Gamma = -\delta, D = \mu\delta, \tau = -\frac{\mu}{\gamma}\delta$$

and therefore

$$\Delta\varphi = \frac{\mu\delta + z\frac{\mu}{\gamma}\delta + z^2}{\delta + z}\varphi - \frac{\nabla\delta}{\delta + z}\nabla\varphi$$

and

$$\operatorname{div}\left((\delta + z)\,\nabla\varphi\right) = (\delta + z)\,\Delta\varphi + \nabla\delta\nabla\varphi = \left(\mu\delta + z\frac{\mu}{\gamma}\delta + z^2\right)\varphi.$$

If we multiply the equation by $\overline{\varphi}$ and integrate over Ω, we obtain after an integration by parts that

$$-\int\limits_{\Omega} (\delta + z)\,|\nabla\varphi|^2\,dx = \int\limits_{\Omega} \left(\mu\delta + z\frac{\mu}{\gamma}\delta + z^2\right)|\varphi|^2\,dx.$$

Therefore

$$\int\limits_{\Omega} \mu\delta\,|\varphi|^2 + \delta\,|\nabla\varphi|^2\,dx + z\int\limits_{\Omega} \frac{\mu}{\gamma}\delta\,|\varphi|^2 + |\nabla\varphi|^2\,dx + z^2\int\limits_{\Omega} |\varphi|^2\,dx = 0.$$

If $\varphi \neq 0$, we have a quadratic equation with only positive coefficients which again cannot have a root in the half-plane described by $\operatorname{Re}(z) \geq 0$.

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