Lecture 7

Inner Kan complexes and normal dendroidal sets

7.1 Inner Kan complexes

In this section, we introduce the notion of inner Kan complexes in the category of dendroidal sets. We begin by recalling the definition of an inner Kan complex in the category of simplicial sets. A horn $\Lambda^k[n]$ in simplicial sets is called an *inner horn* if $0 < k < n$.

Definition 7.1.1. A simplicial set X satisfies the *restricted Kan condition* if every inner horn $f\colon \Lambda^k[n] \longrightarrow X$ has a filler, i.e., there exists a map $g\colon \Delta[n] \longrightarrow X$ such that $f = g \circ j$, where $j\colon \Lambda^k[n] \rightarrowtail \Delta[n]$ denotes the inclusion

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "f"] \arrow[d, tail, "j"'] & X, \\ \Delta[n] \arrow[ur, dotted, "g"'] & \end{tikzcd} \tag{7.1}$$

or, equivalently, if the induced map

$$j^*\colon sSets(\Delta[n], X) \longrightarrow sSets(\Lambda^k[n], X) \tag{7.2}$$

is surjective for every $0 < k < n$.

A simplicial set satisfying the restricted Kan condition is called an *inner Kan simplicial set* (a *quasi-category* in Joyal's terminology). If the filler of (7.1) is unique or, equivalently, if the map (7.2) is a bijection, then X is called a *strict inner Kan simplicial set*.

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The definition of an inner Kan complex for dendroidal sets is similar to the one for simplicial sets, but using the inner horns defined in Lecture 3. Recall from Section 3.2 that if T is any tree, e is any inner edge of T, and ∂_e is the face map in Ω contracting e, then the inner horn $\Lambda^e[T]$ is defined as

$$\Lambda^e[T] = \bigcup_{\beta \neq \partial_e \in \Phi_1(T)} \partial_\beta \Omega[T],$$

where $\Phi_1(T)$ is the set of all faces of T and $\partial_\beta \Omega[T]$ is the β-face of $\Omega[T]$.

The inner horn $\Lambda^e[T]$ is a subobject of the boundary $\partial\Omega[T]$ and extends the notion of inner horn for simplicial sets, namely

$$i_!(\Lambda^k[n]) = \Lambda^k[L_n] \tag{7.3}$$

as a subobject of $i_!(\Delta[n]) = \Omega[L_n]$, where $L_n = i([n])$ is the linear tree with n vertices and $n+1$ edges.

Definition 7.1.2. A dendroidal set X is an *inner Kan complex* if, for every tree T, every inner horn $f\colon \Lambda^e[T] \longrightarrow X$ has a filler, i.e., there exists a map $g\colon \Omega[T] \longrightarrow X$ such that $f = g \circ j$, where $j\colon \Lambda^e[T] \longrightarrow \Omega[T]$ denotes the inclusion

$$\begin{tikzcd} \Lambda^e[T] \arrow[r, "f"] \arrow[d, tail, "j"'] & X, \\ \Omega[T] \arrow[ur, dotted, "g"'] & \end{tikzcd} \tag{7.4}$$

or, equivalently, if the induced map

$$j^*\colon dSets(\Omega[T], X) \longrightarrow dSets(\Lambda^e[T], X) \tag{7.5}$$

is surjective for every tree T and every inner edge e of T.

If the filler of (7.4) is unique or, equivalently, if the map (7.5) is a bijection, then X is called a *strict inner Kan complex*.

A map $f\colon X \longrightarrow Y$ of dendroidal sets is an *inner Kan fibration* if it has the left lifting property with respect to any inner horn inclusion $\Lambda^e[T] \longrightarrow \Omega[T]$, for every tree T and every inner edge e of T. Thus, a dendroidal set X is an inner Kan complex if the map $X \longrightarrow 1$ is an inner Kan fibration, where 1 denotes the terminal object of the category *dSets*.

Proposition 7.1.3. *Let K be any simplicial set and let X be any dendroidal set. Then:*

(i) *The dendroidal set $i_!(K)$ is an inner Kan complex if and only if K is an inner Kan simplicial set.*

(ii) *If X is an inner Kan complex, then the simplicial set $i^*(X)$ is an inner Kan simplicial set.*

Proof. The results follow immediately from the fact that $i_!$ is fully faithful, the adjunction between $i_!$ and i^*, and (7.3). $\square$

A source of strict inner Kan complexes is given by dendroidal nerves of operads (see Example 3.1.4).

Proposition 7.1.4. *Let P be any coloured operad in Sets. Then $N_d(P)$ is a strict inner Kan complex.*

Proof. Any dendrex $x \in N_d(P)_T$ is given by a map $x\colon \Omega[T] \longrightarrow N_d(P)$, which corresponds by the adjunction (3.1) to a map of operads $\Omega(T) \longrightarrow P$. If we choose a planar representative for T, then $\Omega(T)$ is a free operad generated by the operations corresponding to the vertices of the (planar representative of the) tree T. It follows that x is equivalent to a labeling of the (planar representative of the) tree T as follows. The edges of T are labeled by the colours of P and the vertices are labeled by operations in P, where the inputs of such an operation are given by the labels of the incoming edges to the vertex, and the output is the label of the outgoing edge from the vertex. Any inner horn $\Lambda^e[T] \longrightarrow N_d(P)$ completely determines such a labeling of the tree T and thus determines a unique extension $\Omega(T) \longrightarrow P$. $\square$

Remark 7.1.5. The converse of this result is also true, as we will prove in Section 7.3.

Proposition 7.1.6. *Any strict inner Kan complex X is 2-coskeletal.*

Proof. Let X be a strict inner Kan complex and let A be any dendroidal set. Suppose that a map $f\colon \mathrm{Sk}_2 A \longrightarrow \mathrm{Sk}_2 X$ is given. We first show that this map f can be extended to a dendroidal map $\hat{f}\colon A \longrightarrow X$. Assume that f was extended to a map $f_k\colon \mathrm{Sk}_k A \longrightarrow \mathrm{Sk}_k X$ for $k \geq 2$. Let $a \in \mathrm{Sk}_{k+1}(A)$ be a non-degenerate dendrex and suppose that $a \notin \mathrm{Sk}_k(A)$. Hence, $a \in A_T$ and T has exactly $k+1$ vertices. Now, we choose an inner horn $\Lambda^e[T]$, which always exists since $k \geq 2$. The set $\{\beta^* a\}_{\beta \neq \partial_e}$ where β runs over all faces of T defines a horn $\Lambda^e[T] \longrightarrow A$. Since this horn factors through the k-skeleton of A, we obtain, by composition with f_k, a horn $\Lambda^e[T] \longrightarrow X$ in X given by $\{f(\beta^* a)\}_{\beta \neq \alpha}$. If $f_{k+1}(a) \in X_T$ denotes the unique filler of that horn, then we have that $\beta^* f_{k+1}(a) = f(\beta^* a)$ for each $\beta \neq \partial_e$.

To obtain the same property for ∂_e, observe that the dendrices $f(\partial_e^* a)$ and $\partial_e^* f_{k+1}(a)$ both have the same boundary and that they are both of shape S, where S has k vertices. Since $k \geq 2$, we have that S has an inner face, but then it follows that both $f(\partial_e^* a)$ and $\partial_e^* f_{k+1}(a)$ are fillers for the same inner horn in X and hence equal. If we repeat this process for all dendrices in $\mathrm{Sk}_{k+1}(A)$, we get that f_k can be extended to $f_{k+1}\colon \mathrm{Sk}_{k+1}(A) \longrightarrow \mathrm{Sk}_{k+1}(X)$. This holds for all $k \geq 2$, which implies that f can be extended to $\hat{f}\colon A \longrightarrow X$.

In order to prove the uniqueness of $\hat{f}$, assume that g is another extension of f and that it has been shown that $\hat{f}$ and g agree on all dendrices of shape T where T has at most k vertices. Let $a \in X_S$ be a dendrex of shape S, where S has $k+1$ vertices. Then the dendrices $\hat{f}(a)$ and $g(a)$ are dendrices in X that have the same boundary. Since $k \geq 2$, it follows that these dendrices are both fillers for the same inner horn, hence they are the same and thus $\hat{f} = g$. □

7.2 Inner anodyne extensions

In this section, we develop the notion of inner anodyne extensions for dendroidal sets.

Definition 7.2.1. Let $\mathcal{M}$ be a class of monomorphisms in *dSets*. We say that $\mathcal{M}$ is *saturated* if it contains all the isomorphisms and it is closed under pushouts, retracts, arbitrary coproducts, and colimits of sequences (indexed by ordinals).

Given an arbitrary class of monomorphisms $\mathcal{M}$, the *saturated class generated by* $\mathcal{M}$ is the smallest saturated class that contains $\mathcal{M}$, i.e., the intersection of all the saturated classes containing $\mathcal{M}$.

Definition 7.2.2. The class of *inner anodyne extensions* in *dSets* is the saturated class generated by the set of inner horn inclusions. Thus, it is also the class of maps having the left lifting property with respect to the inner Kan fibrations.

The surjectivity property for inner Kan complexes extends to inner anodyne extensions, namely if $u\colon U \longrightarrow V$ is an inner anodyne extension, then the map

$$u^*\colon \mathit{dSets}(V, K) \longrightarrow \mathit{dSets}(U, K)$$

is surjective for any inner Kan complex K. Similarly, u^* is a bijection for any strict inner Kan complex K.

Given any tree T, let $I(T)$ be the subset of the edges of T consisting of only the inner edges. For any nonempty subset $E \subset I(T)$, we denote by $\Lambda^E[T]$ the union of all the faces of $\Omega[T]$ except those obtained by contracting an edge from E, i.e.,

$$\Lambda^E[T] = \bigcup_{\alpha \in \Phi_1(T) \setminus \partial_E} \partial_\alpha \Omega[T],$$

where $\partial_E = \{\partial_e \mid e \in E\}$. Observe that, if $E = \{e\}$, then $\Lambda^E[T] = \Lambda^e[T]$.

Lemma 7.2.3. *For any nonempty $E \subseteq I(T)$, the inclusion $\Lambda^E[T] \rightarrowtail \Omega[T]$ is inner anodyne.*

Proof. We will proceed by induction on the number n of elements of E. If $n = 1$, then $\Lambda^E[T] \rightarrowtail \Omega[T]$ is an inner horn inclusion, thus inner anodyne.

Assume that the result holds for $n < k$ and suppose that E has k elements. Let e be any element of E and let $F = E \setminus \{e\}$. Then the map $\Lambda^E[T] \longrightarrow \Omega[T]$ factors as

$$\begin{tikzcd}
\Lambda^E[T] \arrow[r] \arrow[dr] & \Lambda^F[T] \arrow[d] \\
& \Omega[T],
\end{tikzcd}$$

where the vertical map in the diagram is inner anodyne by the induction hypothesis, since F has $k-1$ elements. We can express the horizontal map as the following pushout:

$$\begin{tikzcd}
\Lambda^F[T/e] \arrow[r] \arrow[d] & \Lambda^E[T] \arrow[d] \\
\Omega[T/e] \arrow[r] & \Lambda^F[T].
\end{tikzcd}$$

Now, the map on the left is inner anodyne (again by the induction hypothesis), hence so is the map on the right, since inner anodyne extensions are closed under pushouts. Therefore $\Lambda^E[T] \longrightarrow \Omega[T]$ is a composition of two inner anodyne extensions and thus it is inner anodyne too. □

The above lemma implies that $\Lambda^I[T] \longrightarrow \Omega[T]$ is inner anodyne, where $\Lambda^I[T]$ is an abbreviation for $\Lambda^{I(T)}[T]$.

We now consider how dendrices in an inner Kan complex can be grafted. Recall that for any two trees T and S, and l a leaf of T, we denote by $T \circ_l S$ the tree obtained by grafting S onto T by identifying l with the root of S. Both S and T naturally embed as subfaces of $T \circ_l S$, which induces the obvious inclusions $\Omega[S] \rightarrowtail \Omega[T \circ_l S]$ and $\Omega[T] \rightarrowtail \Omega[T \circ_l S]$, the pushout of which we denote by $\Omega[T] \cup_l \Omega[S] \longrightarrow \Omega[T \circ_l S]$.

Lemma 7.2.4. *For any two trees T and S and any leaf l of T, the inclusion*

$$\Omega[T] \cup_l \Omega[S] \longrightarrow \Omega[T \circ_l S]$$

is an inner anodyne extension.

Proof. We may assume that $T \neq | \neq S$, otherwise the result is obvious. We proceed by induction on the sum n of the number of vertices of T and S. Let $R = T \circ_l S$. If $n = 2$, then $\Omega[T] \cup_l \Omega[S] \longrightarrow \Omega[T \circ_l S]$ is a horn inclusion, and thus inner anodyne.

Assume that the result holds for $2 \leq n < k$ and that the sum of the number of vertices of T and S is k. Recall that $\Lambda^I[R]$ is the union of all the outer faces of $\Omega[R]$. Observe that the map $\Omega[T] \cup_l \Omega[S] \longrightarrow \Omega[R]$ factors as

$$\begin{tikzcd}
\Omega[T] \cup_l \Omega[S] \arrow[r] \arrow[dr] & \Lambda^I[R] \arrow[d] \\
& \Omega[R]
\end{tikzcd}$$

and that the vertical map is inner anodyne by Lemma 7.2.3. We will show that the horizontal map is inner anodyne by expressing it as a pushout of an inner anodyne extension.

For the purpose of this proof, let us say that an *external cluster* of a tree is a vertex v with the property that one of the edges adjacent to it is inner while all the rest are outer. Let $\mathrm{Cl}(T)$ (resp. $\mathrm{Cl}(S)$) denote the set of all external clusters of T (resp. of S) which do not contain l (resp. the root of S). For each vertex $v \in \mathrm{Cl}(T)$, the face ∂_v of $\Omega[R]$ corresponding to v is isomorphic to $\Omega[(T/v) \circ_l S]$ and the map $\Omega[T/v] \cup_l \Omega[S] \longrightarrow \Omega[(T/v) \circ_l S]$ is inner anodyne by the induction hypothesis. In a similar way, for every $w \in \mathrm{Cl}(S)$ the face ∂_w of $\Omega[R]$ that corresponds to w is isomorphic to $\Omega[T \circ_l (S/w)]$ and the map $\Omega[T] \cup_l \Omega[S/w] \longrightarrow \Omega[T \circ_l (S/w)]$ is inner anodyne, again by the induction hypothesis. The following diagram is a pushout:

$$\begin{tikzcd}
\left(\coprod_{c\in \mathrm{Cl}(T)} (\Omega[T/c] \cup_l \Omega[S])\right) \coprod \left(\coprod_{c\in \mathrm{Cl}(T)} (\Omega[T] \cup_l \Omega[S/c])\right) \arrow[r] \arrow[d] & \Omega[T] \cup_l \Omega[S] \arrow[d] \\
\left(\coprod_{c\in \mathrm{Cl}(T)} (\Omega[(T/c) \circ_l S])\right) \coprod \left(\coprod_{c\in \mathrm{Cl}(T)} (\Omega[T \circ_l (S/c)])\right) \arrow[r] & \Lambda^I[R],
\end{tikzcd}$$

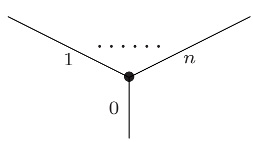
where the left vertical map is the coproduct of all the inner anodyne extensions mentioned above, thus inner anodyne. This implies that the vertical map on the right is inner anodyne too. $\square$

7.3 Homotopy in an inner Kan complex

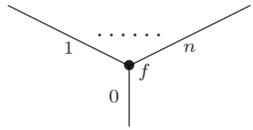
In this section, we study a notion of homotopy inside dendroidal sets. Two dendrices are said to be homotopic if one is a composition of the other with a degenerate dendrex. We will show that this homotopy theory within a dendroidal set is well behaved if the dendroidal set is an inner Kan complex. In that case, the resulting homotopy relation is an equivalence relation. From this it follows that to every inner Kan complex X we can associate an operad $\mathrm{Ho}(X)$, which we call the *homotopy operad* associated to X. The aim of this section is to prove a converse of Proposition 7.1.4, namely

Theorem 7.3.1. *A dendroidal set X is a strict inner Kan complex if and only if X is the dendroidal nerve of an operad.*

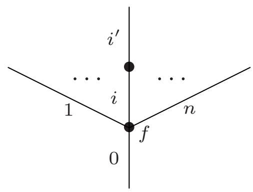
For each $n \geq 0$, let C_n be the n-th corolla:



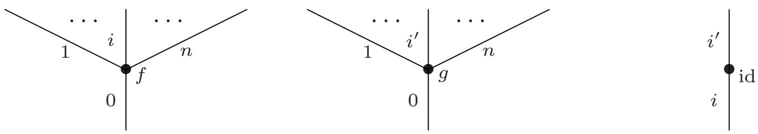
For each $0 \le i \le n$, we denote by $i\colon \eta \longrightarrow C_n$ the outer face map in Ω that sends the unique edge of η to the edge i in C_n. An element $f \in X_{C_n}$ will be denoted by



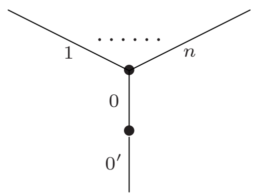
Definition 7.3.2. Let X be an inner Kan complex and let $f, g \in X_{C_n}$ and $n \ge 0$. For $1 \le i \le n$, we say that f *is homotopic to* g *along the edge* i, and denote it by $f \sim_i g$, if there is a dendrex H of shape



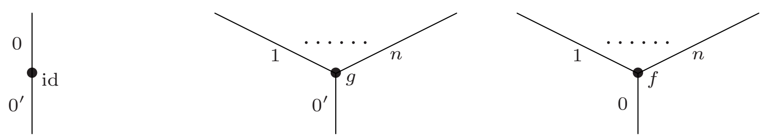
whose three faces are:



where the 'id' in the third tree is a degeneracy of i. In the same way, we will say that f is homotopic to g along the edge 0, and denote it by $f \sim_0 g$, if there is a dendrex of shape



whose three faces are:



When $f \sim_i g$ for some $0 \le i \le n$, we will refer to the corresponding H as a *homotopy from* f *to* g *along* i.

Remark 7.3.3. Notice that, in a strict inner Kan complex X, the homotopy relation just defined is the identity relation.

Proposition 7.3.4. *Let X be an inner Kan complex. The relations $\sim_i$ on the set X_{C_n} are equivalence relations for each $0 \leq i \leq n$, and these equivalence relations all coincide.*

Proof. For a detailed proof, see [MW09, Proposition 6.3 and Lemma 6.4]. □

Due to this proposition, we will use the notation $f \sim g$ instead of $f \sim_i g$. Given an inner Kan complex X and $x_1, \ldots, x_n, x \in X_\eta$, we denote by

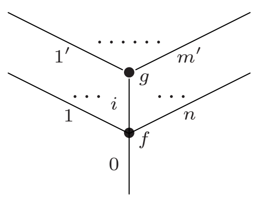
$$X(x_1, \ldots, x_n; x) \subseteq X_{C_n}$$

the set of all dendrices f such that $0^*(f) = x$ and $i^*(f) = x_i$ for $1 \leq i \leq n$. We can define a coloured collection $\mathrm{Ho}(X)$ as follows. The set of colours is the set X_η, and given objects $x_1, \ldots, x_n, x \in X_\eta$, we define

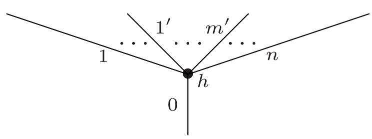
$$\mathrm{Ho}(x_1, \ldots, x_n; x) = X(x_1, \ldots, x_n; x_0)/\sim$$

where $\sim$ is the equivalence relation on X_{C_n} given by Proposition 7.3.4. In order to put an operad structure on the collection $\mathrm{Ho}(X)$, we need to define the composition operations $\circ_i$.

Definition 7.3.5. Let X be an inner Kan complex and let $f \in X_{C_n}$ and $g \in X_{C_m}$ be two dendrices in X. A dendrex $h \in X_{C_{n+m-1}}$ is a $\circ_i$-*composition* of f and g, denoted by $h \sim f \circ_i g$, if there is a dendrex λ in X,



with inner face



The dendrex λ is called a *witness* for the composition.

Remark 7.3.6. Notice that, for $1 \leq i \leq n$, we have by definition that $H\colon f \sim_i g$ if and only if H is a witness for the composition $g \sim f \circ_i \mathrm{id}$. Similarly, for $i = 0$ we have that $H\colon f \sim_0 g$ if and only if H is a witness for the composition $g \sim \mathrm{id} \circ f$.

If X is an inner Kan complex and $f \sim f'$ and $g \sim g'$, then, if $h \sim f \circ_i g$ and $h' \sim f' \circ_i g'$, we have that $h \sim h'$ (see [MW09, Lemma 6.9]). Hence composition is well defined on homotopy classes.

Proposition 7.3.7. *There is a unique structure of a symmetric coloured operad on* $\mathrm{Ho}(X)$ *for which the canonical map of collections* $\mathrm{Sk}_1(X) \longrightarrow \mathrm{Ho}(X)$ *extends to a map of dendroidal sets* $X \longrightarrow N_d(\mathrm{Ho}(X))$. *The latter map is an isomorphism whenever* X *is a strict inner Kan complex.*

Proof. Given $[f] \in \mathrm{Ho}(X)(x_1, \dots, x_n; x)$ and $[g] \in \mathrm{Ho}(X)(y_1, \dots, y_m; x_i)$, the assignment

$$[f] \circ_i [g] = [f \circ_i g]$$

is well defined by Remark 7.3.6. This gives the $\circ_i$ operations in the coloured operad $\mathrm{Ho}(X)$. The actions of the symmetric group Σ_n are defined in the following way. For any element $\sigma \in \Sigma_n$, let $\sigma\colon C_n \longrightarrow C_n$ be the induced map in Ω that permutes the edges of the n-th corolla. The map $\sigma^*\colon X_{C_n} \longrightarrow X_{C_n}$ restricts to a function

$$\sigma^*\colon X(x_1, \dots, x_n; x) \longrightarrow X(x_{\sigma(1)}, \dots, x_{\sigma(n)}; x)$$

that respects the homotopy relation. Hence, we get a map

$$\sigma^*\colon \mathrm{Ho}(X)(x_1, \dots, x_n; x) \longrightarrow \mathrm{Ho}(X)(x_{\sigma(1)}, \dots, x_{\sigma(n)}; x).$$

It is straightforward that these structure maps provide the coloured collection $\mathrm{Ho}(X)$ with an operad structure.

To prove that the quotient map $q\colon \mathrm{Sk}_1(X) \longrightarrow \mathrm{Ho}(X)$ extends to a map $q\colon X \longrightarrow N_d(\mathrm{Ho}(X))$ of dendroidal sets, it is enough to give its values for dendrices $x \in X_T$ where T is a tree with two vertices, since $N_d(\mathrm{Ho}(X))$ is 2-coskeletal by Proposition 7.1.6. So, let T be a tree with two vertices and e be the inner edge of this tree. Then the map

$$\Lambda^e[T] \rightarrowtail \Omega[T] \xrightarrow{x} X$$

factors through $\mathrm{Sk}_1(X)$, so its composition $\Lambda^e[T] \longrightarrow N_d(\mathrm{Ho}(X))$ with q has a unique extension by Proposition 7.1.4. We take this extension to be $q(x)\colon \Omega[T] \longrightarrow N_d(\mathrm{Ho}(X))$ and this defines the map $q\colon \mathrm{Sk}_2(X) \longrightarrow \mathrm{Sk}_2(N_d(\mathrm{Ho}(X)))$, and hence all of $q\colon X \longrightarrow N_d(\mathrm{Ho}(X))$ by 2-coskeletality.

If X is itself a strict inner Kan complex, then the homotopy relation is the identity relation, so $\mathrm{Sk}_1(X) \longrightarrow \mathrm{Ho}(X)$ is the identity map. Since X and $N_d(\mathrm{Ho}(X))$ are strict inner Kan complexes, it follows that $q\colon X \longrightarrow N_d(\mathrm{Ho}(X))$ is an isomorphism. □

The following result together with Proposition 7.3.7 provide the proof of Theorem 7.3.1.

Proposition 7.3.8. *The natural map* $\tau_d(X) \longrightarrow \mathrm{Ho}(X)$ *is an isomorphism of operads for every inner Kan complex* X.

Proof. It is enough to prove that the map $q\colon X \longrightarrow N_d(\mathrm{Ho}(X))$ of Proposition 7.3.7 has the universal property of the unit of the adjunction. This means that, for any operad P and any map $\varphi\colon X \longrightarrow N_d(P)$, there is a unique map of operads $\psi\colon \mathrm{Ho}(X) \longrightarrow P$ such that $N_d(\psi)q = \varphi$. Observe that φ induces a map $\mathrm{Ho}(X) \longrightarrow \mathrm{Ho}(N_d(P))$ such that the diagram

$$\begin{array}{ccc} \mathrm{Sk}_1(X) & \xrightarrow{\varphi} & \mathrm{Sk}_1 N_d(P) \\ \downarrow q & & \downarrow q_P \\ \mathrm{Ho}(X) & \xrightarrow{\mathrm{Ho}(\varphi)} & \mathrm{Ho}(N_d(P)) \end{array}$$

commutes. Now, $\mathrm{Ho}(N_d(P)) = P$ and q_P is the identity, as we can see from the proof of Proposition 7.3.7. Hence, $\mathrm{Ho}(\varphi)$ defines a map $\psi\colon \mathrm{Ho}(X) \longrightarrow P$ of coloured collections. In fact, one can see that ψ is a map of operads, and the uniqueness follows from the surjectivity of q. $\square$

Proof of Theorem 7.3.1. One direction was already proved in Proposition 7.1.4. For the other one, suppose that X is a strict inner Kan complex. Then Proposition 7.3.7 and Proposition 7.3.8 imply that $X \cong N_d(\tau_d(X))$. $\square$

7.4 Homotopy coherent nerves are inner Kan

Throughout this section, $\mathcal{E}$ will denote a monoidal model category with cofibrant unit I. We will also assume that $\mathcal{E}$ is equipped with an *interval* in the sense of [BM06], which we denote by H. Recall from Definition 6.2.1 that such an interval is given by an object H of $\mathcal{E}$ together with maps

$$I \underset{1}{\overset{0}{\rightrightarrows}} H \xrightarrow{\epsilon} I \qquad \text{and} \qquad H \otimes H \xrightarrow{\vee} H$$

satisfying certain conditions. This means in particular that H is an interval in Quillen's sense ([Qui67]), so 0 and 1 together define a cofibration $I \coprod I \longrightarrow H$, and ϵ is a weak equivalence. In Section 6.2 we explained how such an interval H allows one to construct for each coloured operad P in $\mathcal{E}$ a Boardman–Vogt resolution $W_H(P) \longrightarrow P$. Each operad in *Sets* can be viewed as an operad in $\mathcal{E}$ via the strong monoidal functor *Sets* $\longrightarrow \mathcal{E}$, as in Example 1.3.6, and hence has such a Boardman–Vogt resolution. When we apply this construction to the operads $\Omega(T)$, we obtain the *homotopy coherent dendroidal nerve* $hcN_d(P)$ of any operad P in $\mathcal{E}$, defined as the dendroidal set given by

$$hcN_d(P)_T = \mathit{Oper}(W_H(\Omega(T)), P).$$

The goal of this section is to prove that the homotopy coherent nerve is an inner Kan complex.

Theorem 7.4.1. *Let P be a C-coloured operad in $\mathcal{E}$ such that, for every $(n+1)$-tuple $(c_1, \ldots, c_n, c)$ of colours of P, the object $P(c_1, \ldots, c_n; c)$ is fibrant in $\mathcal{E}$. Then $hcN_d(P)$ is an inner Kan complex.*

As observed in Section 6.3, our construction of the dendroidal homotopy coherent nerve specializes to that of the homotopy coherent nerve of an $\mathcal{E}$-enriched category. In the case where $\mathcal{E}$ is the category of topological spaces or simplicial sets, one recovers the classical definition of Cordier and Porter ([CP86]). It follows as a particular case of Theorem 7.4.1 that the homotopy coherent nerve of an $\mathcal{E}$-enriched category with fibrant Hom objects is a quasi-category in the sense of Joyal. This was proved in [CP86] when $\mathcal{E}$ is the category of simplicial sets.

Recall from Example 6.3.2, in the case $\mathcal{E} = \mathit{Top}$, the description of the operads $W_H(\Omega(T))$ involved in the definition of the homotopy coherent nerve. We have a similar description for these operads in the case of a general monoidal model category $\mathcal{E}$. First of all, recall from (1.1) the symmetrization functor

$$\Sigma\colon \mathit{Oper}_{\not\Sigma}(\mathcal{E}) \longrightarrow \mathit{Oper}(\mathcal{E}),$$

which is left adjoint to the forgetful functor from symmetric operads to non-symmetric ones. If T is a tree in Ω, then any planar representative $\bar{T}$ of T naturally describes a non-Σ operad $\Omega(\bar{T})$ such that $\Omega(T) = \Sigma(\Omega(\bar{T}))$. It follows that

$$W_H(\Omega(T)) = \Sigma(W_H(\Omega(\bar{T}))),$$

since the W-construction commutes with symmetrization.

The coloured operad $W_H(\Omega(\bar{T}))$ can be described explicitly (Example 6.3.2). The colours of $W_H(\Omega(\bar{T}))$ are the colours of $\Omega(\bar{T})$, i.e., the edges of T. Let $\sigma = (e_1, \ldots, e_n; e)$ be an $(n+1)$-tuple of colours of $\Omega(\bar{T})$. If $\Omega(\bar{T})(\sigma) = \emptyset$, then $W_H(\Omega(\bar{T}))(\sigma) = 0$. If $\Omega(\bar{T})(\sigma) \neq \emptyset$, then there is a subtree T_σ of T (and a corresponding planar subtree $\bar{T}_\sigma$ of $\bar{T}$) whose leaves are $e_1, \ldots, e_n$ and whose root is e. Thus we have that

$$W_H(\Omega(\bar{T}))(e_1, \ldots, e_n; e) = \bigotimes_{f \in \mathrm{inn}(\sigma)} H,$$

where $\mathrm{inn}(\sigma)$ is the set of *inner* edges of T_σ (or of $\bar{T}_\sigma$). This last tensor product can be interpreted as the 'space' of assignments of lengths to inner edges in $\bar{T}_\sigma$; it is the unit if $\mathrm{inn}(\sigma)$ is empty.

The composition product in the coloured operad $W_H(\Omega(\bar{T}))$ is given in terms of the $\circ_i$-operations. If $\sigma = (e_1, \ldots, e_n; e)$ and $\rho = (f_1, \ldots, f_m; e_i)$ are two $(n+1)$-tuples of colours, then the composition map

$$\begin{array}{c} \Omega(\bar{T})(e_1, \ldots, e_n; e_0) \otimes \Omega(\bar{T})(f_1, \ldots, f_m; e) \\ \Big\downarrow{\scriptstyle \circ_i} \\ \Omega(\bar{T})(e_1, \ldots, e_{i-1}, f_1, \ldots, f_m, e_{i+1}, \ldots, e_n; e) \end{array} \tag{7.6}$$

is defined as follows. The trees $\bar{T}_\sigma$ and $\bar{T}_\rho$ are grafted along e_i to form the tree $\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho$, that is again a planar subtree of $\bar{T}$. In fact, $\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho = \bar{T}_{\sigma \circ_i \rho}$, where

$\sigma \circ_i \rho = (e_1, \ldots, e_{i-1}, f_1, \ldots, f_m, e_{i+1}, \ldots, e_n; e_0)$. For the sets of inner edges we have

$$\mathrm{inn}(\sigma \circ_i \rho) = \mathrm{inn}(\sigma) \cup \mathrm{inn}(\rho) \cup \{e_i\}.$$

The composition product in (7.6) is the map

$$\begin{tikzcd}
H^{\otimes \mathrm{inn}(\sigma)} \otimes H^{\otimes \mathrm{inn}(\rho)} \arrow[r, dotted] \arrow[d, "\cong"] & H^{\otimes \mathrm{inn}(\sigma \circ_i \rho)} \arrow[d, "\cong"] \\
H^{\otimes \mathrm{inn}(\sigma) \cup \mathrm{inn}(\rho)} \otimes I \arrow[r, "\mathrm{id} \otimes 1"] & H^{\otimes \mathrm{inn}(\sigma) \cup \mathrm{inn}(\rho)} \otimes H,
\end{tikzcd}$$

where $1 \colon I \longrightarrow H$ is one of the endpoints of the interval H.

This description of the operad $W_H(\Omega(\bar{T}))$ is functorial in the planar tree T. In particular, for an inner edge e of T, the tree T/e inherits a planar structure $\overline{T/e}$ from $\bar{T}$, and $W_H(\Omega(\overline{T/e})) \longrightarrow W_H(\Omega(\bar{T}))$ is the natural map assigning length 0 to the edge e.

Proof of Theorem 7.4.1. Let T be a tree in Ω and a an inner edge of T. We need to find an extension to the following diagram:

$$\begin{tikzcd}
\Lambda^a[T] \arrow[r, "\varphi"] \arrow[d, tail] & hcN_d(P). \\
\Omega[T] \arrow[ur, dotted] &
\end{tikzcd}$$

Let $\bar{T}$ be a planar representative of T. An extension $\psi \colon \Omega[T] \longrightarrow hcN_d(P)$ corresponds, by adjointness, to a morphism of non-Σ operads

$$\hat{\psi} \colon W_H(\Omega(\bar{T})) \longrightarrow P.$$

For each face map $S \longrightarrow T$, the tree S inherits a planar structure $\bar{S}$ from $\bar{T}$, and the given map $\varphi \colon \Lambda^a[T] \longrightarrow hcN_d(P)$ corresponds, again by adjointness, to a map of operads in $\mathcal{E}$,

$$\hat{\varphi} \colon W_H(\Lambda^a[T]) \longrightarrow P,$$

where we view $W_H(\Lambda^a[T])$ as the colimit of operads in $\mathcal{E}$,

$$W_H(\Lambda^a[T]) = \mathrm{colim}\; W(\Omega(\bar{S})) \tag{7.7}$$

over all the faces of T except the one contracting a. In other words, φ corresponds to a compatible family of maps

$$\hat{\varphi}_S \colon W_H(\Omega(\bar{S})) \longrightarrow P.$$

We will show that there exists an operad map $\hat{\psi}$ extending $\hat{\varphi}_S$ for all faces $S \neq T/a$. Note that the colours of $\Omega(\bar{T})$ are the same as those of the colimit in (7.7), so we have a map $\psi_0 = \varphi_0$ on colours:

$$\psi_0 \colon E(T) \longrightarrow \{\text{Colours of } P\}.$$

If $\sigma = (e_1, \ldots, e_n; e)$ is an $(n+1)$-tuple of edges of T such that $W_H(\Omega(\bar{T})) \neq \emptyset$, and $T_\sigma \subseteq T$ (with $T_\sigma \neq T$), then T_σ is contained in an outer face S of T. Hence $W_H(\Omega(\bar{T}))(\sigma) = W_H(\Omega(\bar{T}_\sigma))(\sigma) = W_H(\Omega(\bar{S}))(\sigma)$, and we have a map

$$\hat{\varphi}_S(\sigma) \colon W_H(\Omega(\bar{T}))(\sigma) \longrightarrow P(\sigma),$$

given by $\hat{\varphi}_S \colon W_H(\Omega(\bar{S})) \longrightarrow P$. Thus, the only part of the map of operads $\hat{\psi} \colon W_H(\Omega(\bar{T})) \longrightarrow P$ not determined by φ is the one when $T_\tau = T$, where $\tau = (e_1, \ldots, e_n; e)$ and $e_1, \ldots, e_n$ are all the input edges of $\bar{T}$ (in the planar order) and e is the output edge. In this case, $\hat{\psi}(\tau)$ has to be a map

$$\hat{\psi} \colon W_H(\Omega(\bar{T}))(\tau) = H^{\otimes i(\tau)} \longrightarrow P(\tau)$$

such that $\hat{\psi}(\sigma) = \hat{\varphi}_S(\sigma)$ if $\sigma \neq \tau$, and together with these $\hat{\psi}(\sigma)$ respects operad composition. The first condition determines $\hat{\psi}(\tau)$ on the subobject of $H^{\otimes i(\tau)}$ which is given by a value 0 on one of the tensor factors marked by an edge e_i *other than* the given a. The second condition determines $\hat{\psi}(\tau)$ on the subobject of $H^{\otimes i(\tau)}$ which is given by a value 1 on one of the factors. Thus, if we write 1 for the map $I \rightarrowtail H$ and $\partial H \rightarrowtail H$ for the map $I \coprod I \longrightarrow H$, and define $\partial H^{\otimes k} \rightarrowtail H^{\otimes k}$ by the Leibniz rule (i.e., $\partial(A \otimes B) = \partial(A) \otimes B \cup A \otimes \partial(B)$), then the problem of finding $\hat{\psi}(\tau)$ is the same as finding an extension to the diagram

$$\begin{tikzcd}
\partial(H^{\otimes \mathrm{inn}(\sigma) \setminus \{a\}} \otimes H) \cup H^{\otimes \mathrm{inn}(\sigma) \setminus \{a\}} \otimes I \arrow[r] \arrow[d, tail] & P(\tau) \\
H^{\otimes \mathrm{inn}(\sigma) \setminus \{a\}} \otimes H \arrow[r, "\cong"] & H^{\otimes \mathrm{inn}(\sigma)}. \arrow[u, dotted, "\hat{\psi}(\sigma)"']
\end{tikzcd}$$

This extension exists because $P(\tau)$ is fibrant by assumption, and the left-hand map is a trivial cofibration by the pushout-product axiom for monoidal model categories. □

7.5 The exponential property

Recall from Theorem 4.2.2 that the category of dendroidal sets is a closed symmetric monoidal category. The main result of this section is that the internal hom of this monoidal structure $\mathrm{Hom}_{dSets}(D, Y)$ is an inner Kan complex if D is normal and Y is inner Kan. It is a consequence of the following result from [MW09], which we quote here without proof:

Theorem 7.5.1 ([MW09, Propostion 9.2]). *Let S and T be any two trees in Ω. Then the natural map*

$$\partial\Omega[S]\otimes\Omega[T] \bigcup_{\partial\Omega[S]\otimes\Lambda^e[T]} \Omega[S]\otimes\Lambda^e[T] \rightarrowtail \Omega[S]\otimes\Omega[T]$$

is inner anodyne, where e is any inner edge of T. □

It follows by standard arguments with saturated classes that, if $A \rightarrowtail B$ is a normal monomorphism and $C \rightarrowtail D$ is inner anodyne, then

$$A\otimes D \bigcup_{A\otimes C} B\otimes C \rightarrowtail B\otimes D \tag{7.8}$$

is again inner anodyne. Using the Hom-$\otimes$ adjunction, one draws the standard conclusions, such as that if $Y \longrightarrow X$ is an inner Kan fibration and $C \rightarrowtail D$ is inner anodyne, then

$$\mathrm{Hom}(D,Y) \longrightarrow \mathrm{Hom}(C,Y)\times_{\mathrm{Hom}(C,X)}\mathrm{Hom}(D,X) \tag{7.9}$$

has the right lifting property with respect to normal monomorphisms. If $C \rightarrowtail D$ is just normal, then (7.9) is an inner Kan fibration. In particular, taking $C=\emptyset$ and $X=1$ (the terminal object of *dSets*), we obtain the following.

Theorem 7.5.2 ([MW09, Theorem 9.1]). *Let Y and D be dendroidal sets and assume that D is a normal dendroidal set. If Y is a (strict) inner Kan complex, then so is* $\mathrm{Hom}_{dSets}(D,Y)$. □

The result given by Theorem 7.5.1 is also true in *pdSets*. However, for the tensor product to be defined, one has to assume that either S or T is linear (see Remark 4.2.4). The general statement analogous to (7.8) for *pdSets* takes the following form. Let $K \rightarrowtail L$ be a monomorphism between simplicial sets and let $C \rightarrowtail D$ be a monomorphism in *pdSets*. Then

$$u_!(K)\otimes D \bigcup_{u_!(K)\otimes C} u_!(L)\otimes C \rightarrowtail u_!(L)\otimes D$$

is inner anodyne whenever $K \rightarrowtail L$ or $C \rightarrowtail D$ is.