Lecture 6

Boardman–Vogt resolution and homotopy coherent nerve

In this lecture, we describe a generalization of the W-construction of Boardman and Vogt for coloured operads in any monoidal category with a suitable notion of interval. This generalized W-construction is then used to define the homotopy coherent dendroidal nerve of a given coloured operad P. The homotopy coherent nerve will play a fundamental role in the definition of homotopy P-algebras and weak higher categories.

6.1 The classical W-construction

In this section we recall the Boardman–Vogt resolution of operads in the category of topological spaces (see [BV73, Ch. III]).

Let *Top* denote the category of compactly generated topological spaces. The W-construction is a functor

$$W\colon \mathit{Oper}_C(\mathit{Top}) \longrightarrow \mathit{Oper}_C(\mathit{Top}),$$

where $\mathit{Oper}_C(\mathit{Top})$ denotes the category of C-coloured operads in *Top*, together with a natural transformation $\gamma\colon W \longrightarrow \mathrm{id}$ (i.e., W is an augmented endofunctor in $\mathit{Oper}_C(\mathit{Top})$). To each topological coloured operad P there is an associated topological coloured operad $W(P)$ and a natural map $\gamma_P\colon W(P) \longrightarrow P$.

If we think of the maps of coloured operads $P \longrightarrow \mathit{Top}$ (considering *Top* as a coloured operad with the cartesian product) as describing P-algebras in *Top*, then the maps $W(P) \longrightarrow \mathit{Top}$ will describe *homotopy P-algebras* in *Top*. The augmentation induces a map

$$\mathit{Oper}_C(\mathit{Top})(P, \mathit{Top}) \xrightarrow{\gamma_P^*} \mathit{Oper}_C(\mathit{Top})(W(P), \mathit{Top})$$

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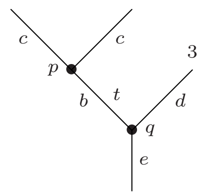
which views any P-algebra as a homotopy P-algebra.

In the case of non-symmetric operads, the functor W can be explicitly described as follows. Let P be a non-symmetric C-coloured operad in *Top* and let $H = [0,1]$ be the unit interval. The colours of $W(P)$ are the same as those of P and the space of operations $W(P)(c_1,\ldots,c_n;c)$ is a quotient of a space of labelled planar trees. We consider, for $c_1,\ldots,c_n,c$ in C, the topological space $A(c_1,\ldots,c_n;c)$ of planar trees whose edges are labelled by elements of C and, in particular, for every such tree the input edges are labelled by the given $c_1,\ldots,c_n$ and the output edge is labelled by c. We assign to each of the inner edges of these trees a length $t \in H$. Each vertex with input edges labelled by $b_1,\ldots,b_m \in C$ (in the planar order) and output edge labelled by $b \in C$ is labelled by an element of $P(b_1,\ldots,b_m;b)$.

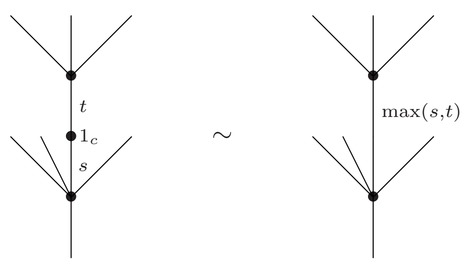
Example 6.1.1. The following tree is an element of $A(c,c,d;e)$:



where $p \in P(c,c;b)$, $q \in P(b,d;e)$, and $t \in [0,1]$.

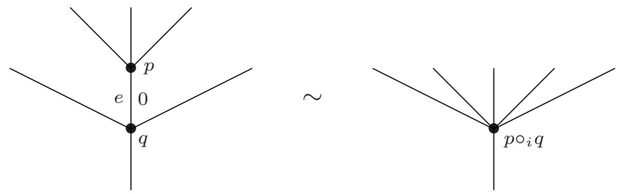
There is a canonical topology in $A(c_1,\ldots,c_n;c)$ induced by the topology of P and the standard topology of the unit interval. The space $W(P)(c_1,\ldots,c_n;c)$ is the quotient space of $A(c_1,\ldots,c_n;c)$ obtained by the following two relations:

(i) If a tree has a unary vertex v labelled by an identity, then we identify such tree with the tree obtained by removing this vertex and identifying the input edge x of v with its output edge y. The length assigned to the new edge is the maximum of the lengths of the edges x and y, or it has no length if the new edge is outer.



(ii) If there is a tree with an internal edge e with zero length, then we identify it with the tree obtained by contracting the edge e, using the corresponding $\circ_i$

operation of the coloured operad P.



The collection $W(P)(c_1, \ldots, c_n; c)$ for $c_1, \ldots, c_n, c \in C$ forms a C-coloured operad. The unit for each colour c is the tree $|$ coloured by c. Composition is given by grafting, assigning length 1 to the newly arisen internal edges.

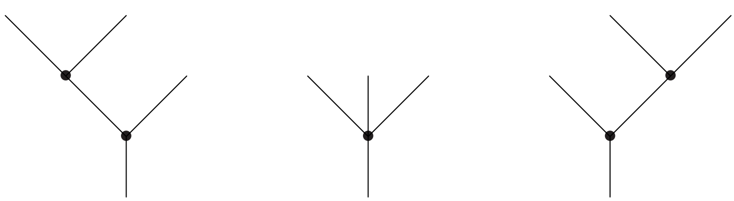
Remark 6.1.2. There is a W-construction for symmetric operads defined in a similar way (see [BM07, §3]). The forgetful functor from symmetric operads to non-symmetric operads has a left adjoint which identifies the category of non-symmetric operads with a full coreflective subcategory of the category of symmetric operads. If

$$\Sigma : \mathit{Oper}_{\not\Sigma}(\mathit{Top}) \rightleftarrows \mathit{Oper}(\mathit{Top}) : U$$

is the free-forgetful adjunction relating non-symmetric operads and symmetric operads, then $W(\Sigma P) = \Sigma(WP)$.

Example 6.1.3. Let P be the non-symmetric coloured operad with only one colour and $P(c, \overset{(n)}{\ldots}, c; c) = P(n)$ consisting of a single n-ary operation for $n \geq 1$ and $P(\ ; c) = P(0) = \emptyset$.

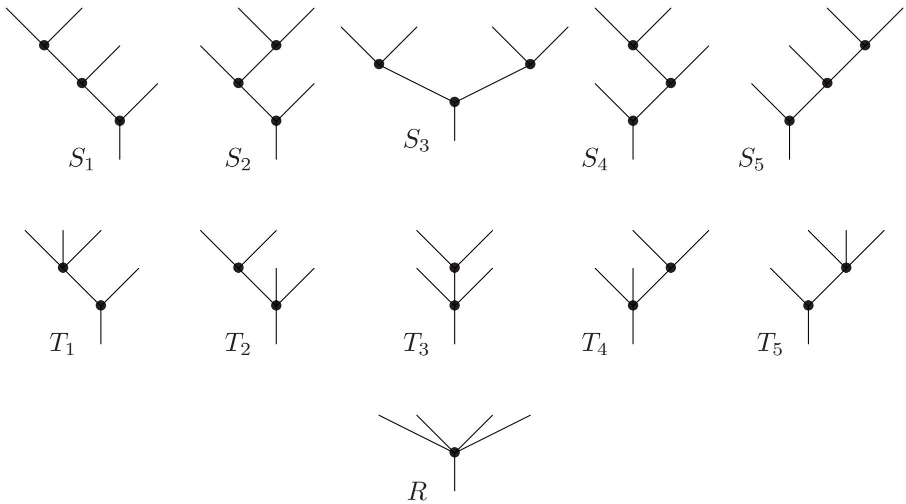
The operad $W(P)$ will again have only one colour. Since every unary vertex in a labelled tree in $W(P)(n)$ can only be labelled by the identity, it is enough (by relation (i) in the definition of $W(P)$) to consider only those trees in $W(P)(n)$ without unary vertices. We call these trees *regular trees*. Thus, if $n = 1, 2$, then $W(P)(n)$ is a one-point space. For the case $n = 3$, we need to consider all possible regular trees with three inputs. There are three such trees:



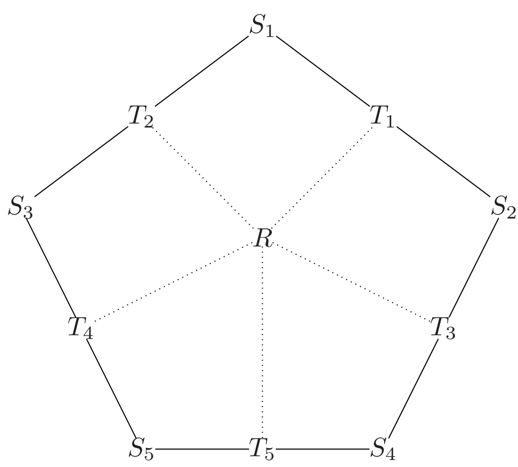
The tree in the middle contributes one point to the space $W(P)(3)$ while each other tree contributes a copy of the interval $[0, 1]$ (both have only one inner edge). There is one identification to be made when the length of the inner edge is zero, in which case the corresponding tree is identified with the tree in the middle. The space $W(P)(3)$ is then the disjoint union of two copies of $[0, 1]$ where we identify the ends named 0 to a single point. What we get is that $W(P)(3) = [-1, 1]$.

It is convenient to keep track of the trees corresponding to each point in the interval $[-1,1]$. The point 0 corresponds to the middle tree in the picture, while a point $t \in (0,1]$ (resp. $t \in [-1,0)$) corresponds to the tree on the right (resp. left) where the length of the inner edge is t.

The case $n = 4$ is a bit more involved. There are eleven regular trees with four inputs, five of them with one internal edge and five of them with two internal edges. Here is the complete list of them:



Each of the trees S_i contributes a copy of the square $[0,1] \times [0,1]$ and each of the trees T_i contributes a copy of the interval $[0,1]$. There are several identifications when the lengths of the internal edges are zero. The space $W(P)(4)$ consists of five copies of $[0,1] \times [0,1]$ glued together by means of these identifications. Thus, $W(P)(4)$ is a pentagon, that we can picture as follows:



Each point in the pentagon corresponds to an element of $W(P)(4)$. The center corresponds to the tree R, the vertices correspond to the trees S_i, and the middle points of the edges correspond to the trees T_i when the length of the internal edge is 1. Moving from the boundary towards the center shrinks the length of the inner edges of the trees from 1 to 0.

One can compute in this way $W(P)(n)$ for every n. In fact, $W(P)(n)$ is a subdivision of the n-th Stasheff polytope for every n.

Example 6.1.4. Let *Top* be the category of compactly generated topological spaces with the interval given by the unit interval $[0,1]$. Let $\mathcal{C}$ be a small category considered as a discrete topological category, i.e., $\mathcal{C}(A,B)$ is viewed as a discrete topological space for every A and B in $\mathcal{C}$. Thus, we can view $\mathcal{C}$ as a coloured operad in *Top*, where the colours are the objects of $\mathcal{C}$ and where all operations are unary.

Then $W(\mathcal{C})$ will be again a topological operad with only unary operations, i.e., a topological category. The objects (colours) of $W(\mathcal{C})$ are the same as those of $\mathcal{C}$. The morphisms $W(\mathcal{C})(A,B)$ are represented by sequences of morphisms in $\mathcal{C}$

$$A = C_0 \xrightarrow{f_1} \overset{t_1}{C_1} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \overset{t_{n-1}}{C_{n-1}} \xrightarrow{f_n} C_n = B$$

with 'waiting times' $t_i \in [0,1]$ for $1 \le i \le n-1$. If $t_i = 0$, such a sequence is identified with

$$A = C_0 \xrightarrow{f_1} \overset{t_1}{C_1} \xrightarrow{f_2} \cdots \longrightarrow \overset{t_{i-1}}{C_{i-1}} \xrightarrow{f_{i+1}f_i} \overset{t_{i+1}}{C_{i+1}} \longrightarrow \cdots \xrightarrow{f_{n-1}} \overset{t_{n-1}}{C_{n-1}} \xrightarrow{f_n} C_n = B.$$

If $f_i = \mathrm{id}$, then the sequence is identified with

$$A = C_0 \xrightarrow{f_1} \overset{t_1}{C_1} \xrightarrow{f_2} \cdots \longrightarrow \overset{s}{C_{i-1}} \xrightarrow{f_{i+1}} \overset{t_{i+1}}{C_{i+1}} \longrightarrow \cdots \xrightarrow{f_{n-1}} \overset{t_{n-1}}{C_{n-1}} \xrightarrow{f_n} C_n = B,$$

where $s = \max(t_{i-1}, t_i)$.

We will study this example in the following particular case. Let $\mathcal{C} = [n]$ be the linear tree viewed as a discrete topological category. An $[n]$-algebra in *Top* consists of a sequence of spaces $X_0, \dots, X_n$ and maps $f_{ji} \colon X_i \longrightarrow X_j$ for $i \le j$ such that $f_{ii} = \mathrm{id}$ and

$$f_{kj} \circ f_{ji} = f_{ki} \tag{6.1}$$

if $i \le j \le k$.

The topological category $W([n])$ has objects $0, 1, \dots, n$, and a morphism $i \longrightarrow j$ in $W([n])$ is a sequence of 'times' $t_{i+1}, \dots, t_{j-1}$ where $t_k \in [0,1]$. In other words, $W([n])(i,j)$ is the cube $[0,1]^{j-i-1}$ if $i+1 \le j$; a point if $i = j$; and the empty set if $i > j$. Composition on $W([n])$ is given by juxtaposing two such sequences putting an extra time 1 in the middle, i.e.,

$$(t_{i+1}, \dots, t_{j-1}) \colon i \longrightarrow j \quad \text{and} \quad (t_{j+1}, \dots, t_{k-1}) \colon j \longrightarrow k$$

compose into

$$(t_{i+1},\ldots,t_{j-1},1,t_{j+1},\ldots,t_{k-1})\colon i\longrightarrow k.$$

A $W([n])$-algebra is then a sequence of spaces $X_0,\ldots,X_n$ and maps f_{ji} as before, but for which condition (6.1) holds only up to specified coherent higher homotopies.

Remark 6.1.5. For any non-symmetric C-coloured operad P in *Top*, consider all planar trees T with input edges $c_1,\ldots,c_n$ and output edge c, such that each vertex of T with input edges $b_1,\ldots,b_m$ and output edge b is labelled by an element of $P(b_1,\ldots,b_m;b)$. Then

$$A(c_1,\ldots,c_n;c)\cong\coprod_T H^T,$$

where the coproduct is taken over all such trees T and $H^T=H^{\times k}$, where $H=[0,1]$ and k is the number of inner edges of T.

The remaining identifications to construct $W(P)(c_1,\ldots,c_n;c)$ are completely determined by the combinatorics of the trees T. This observation is the key to generalizing the W-construction to coloured operads in other monoidal categories.

6.2 The generalized W-construction

In this section, we generalize the W-construction to coloured operads in monoidal categories. For this, one needs a suitable replacement of the unit interval $[0,1]$ used above to give lengths to the inner edges of the trees.

Definition 6.2.1. Let $\mathcal{E}$ be a monoidal category with tensor product $\otimes$ and unit I. An *interval* in $\mathcal{E}$ is an object H equipped with two 'points', i.e., maps $0,1\colon I\rightrightarrows H$, an augmentation $\varepsilon\colon H\longrightarrow I$ satisfying $\varepsilon\circ 0=\mathrm{id}=\varepsilon\circ 1$, and a binary operation (playing the role of the maximum)

$$\vee\colon H\otimes H\longrightarrow H$$

which is associative, and for which 0 is unital and 1 is absorbing, i.e.,

$$0\vee x=x=x\vee 0\quad\text{and}\quad 1\vee x=1=x\vee 1$$

for any $x\colon I\longrightarrow H$.

Example 6.2.2. The unit interval $[0,1]$ is an interval in *Top*. One can choose as $\vee$ the maximum operation or the 'reversed' multiplication, i.e., $s\vee t$ is defined by the identity $(1-s\vee t)=(1-s)(1-t)$.

The groupoid $J=(0\overset{\sim}{\longleftrightarrow}1)$ is an interval in *Cat*. Another possible interval for *Cat* is the two-object category $I=(0\longrightarrow 1)$.

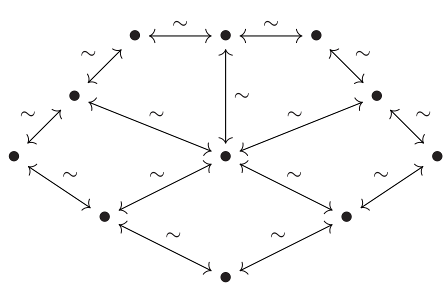
For any interval H and any coloured operad P in $\mathcal{E}$, there is a new coloured operad $W_H(P)$ (on the same colours as P) together with a natural map of operads $W_H(P)\longrightarrow P$. The operad $W_H(P)$ is constructed as $W(P)$ in the case of topological spaces, now glueing objects of the form $H^{\otimes k}$ instead of cubes $[0,1]^k$ (see Remark 6.1.5). The functor W_H is called the W-construction in $\mathcal{E}$ associated to the interval H. As was pointed out in the case of topological operads, the W-construction can be defined in a similar way for symmetric coloured operads in $\mathcal{E}$ (cf. Remark 6.1.2). We refer the reader to [BM06] and [BM07] for more details on the generalized W-construction.

Example 6.2.3. Let *Cat* be the category of (small) categories with the groupoid interval $J = (0 \overset{\sim}{\longleftrightarrow} 1)$. Let I denote the unit for the cartesian product of categories, i.e., I is the category with one object and one (identity) morphism. Consider a non-symmetric operad P in *Cat* with $P(n) = I$ corresponding to one n-ary operation for every $n \geq 1$, and $P(0) = \emptyset$.

The operad $W_J(P)$ is a one-colour operad and, as in Example 6.1.3, the term $W_J(P)(n)$ is described by using regular trees (i.e., trees with no unary vertices) with n leaves. If $n = 1, 2$, then $W_J(P)(n)$ is the one-object category I. For $n = 3$, there are three regular trees, two with one inner edge (each contributing a copy of J) and one without inner edges (contributing a copy of I). In the W-construction, we identify the object named 0 in every copy of J with the unique object of I. Hence, $W_J(P)(3)$ is a category with three objects and a unique isomorphism between any two objects. We can picture it as

$$1 \overset{\sim}{\longleftrightarrow} 0 \overset{\sim}{\longleftrightarrow} 1.$$

Similarly, in the case $n = 4$, there are eleven regular trees with four inputs (see Example 6.1.3), five of them with two internal edges and five of them with one internal edge. Each of the trees with two internal edges contributes a copy of $J \times J$ to $W_J(P)(4)$. Using the identifications given by the W-construction, one can show that $W_J(P)(4)$ consists of a category with eleven objects and a unique isomorphism between any two objects, and we can picture it as



6.3 The homotopy coherent nerve

In this section, we use the generalized W-construction to define, for every coloured operad P in a symmetric monoidal category, a dendroidal set $hcN_d(P)$ called the *homotopy coherent nerve* of P. This dendroidal set is similar to the one obtained via the dendroidal nerve construction, but with homotopies built into it.

Let $\mathcal{E}$ be a symmetric monoidal category with an interval H. For each tree T in Ω we can consider the operad $\Omega(T)$ as a discrete operad in $\mathcal{E}$. This is done by applying to the operad $\Omega(T)$ in *Sets* the strong monoidal functor $\mathit{Sets} \longrightarrow \mathcal{E}$ sending any set X to $\coprod_{x\in X} I$, where I is the unit of $\mathcal{E}$. Then we have a functor

$$\Omega \longrightarrow \mathit{Oper}(\mathcal{E})$$

that assigns to each tree T the operad $W_H(\Omega(T))$. By Kan extension, this functor induces a pair of adjoint functors

$$hc\tau_d : \mathit{dSets} \rightleftarrows \mathit{Oper}(\mathcal{E}) : hcN_d. \tag{6.2}$$

Definition 6.3.1. The functor $hcN_d\colon \mathit{Oper}(\mathcal{E}) \longrightarrow \mathit{dSets}$ is called the *homotopy coherent nerve* functor.

More explicitly, for every tree T in Ω and any coloured operad P in $\mathcal{E}$, the homotopy coherent nerve is given by

$$hcN_d(P)_T = \mathit{Oper}(W_H(\Omega(T)), P).$$

To have a better understanding of the functor hcN_d it will be useful to have a description of the operad $W_H(\Omega(T))$. We do it in the case $\mathcal{E} = \mathit{Top}$, although a similar description applies to any monoidal category $\mathcal{E}$ with an interval.

Example 6.3.2. Let $\mathcal{E} = \mathit{Top}$ be the category of compactly generated topological spaces with the interval $H = [0, 1]$. Let T be any tree in Ω. The colours of the operad $W(\Omega(T))$ are the same as those of $\Omega(T)$, i.e., the edges of T. If $\sigma = (e_1, \ldots, e_n, e)$ are edges of T such that there is a subtree T_σ of T with $e_1, \ldots, e_n$ as input edges and e as output edge, then

$$W(\Omega(T))(e_1, \ldots, e_n; e) = H^{\#\mathrm{inn}(T_\sigma)}$$

where $\mathrm{inn}(T_\sigma)$ is the set of inner edges of T_σ. In fact, $W(\Omega(T))(e_1, \ldots, e_n; e)$ is a $\#\mathrm{inn}(T_\sigma)$-dimensional cube representing the space of assignments of lengths $t_i \in [0, 1]$ to the internal edges of T_σ. It is the one-point space if $\mathrm{inn}(T_\sigma)$ is the empty set.

If there is no subtree in T with $e_1, \ldots, e_n$ as input edges and e as output edge, then

$$W(\Omega(T))(e_1, \ldots, e_n; e) = \emptyset.$$

The composition operations in $W(\Omega(T))$ are given in terms of the $\circ_i$ operations as follows. If $\sigma = (e_1, \ldots, e_n; e)$ and $\sigma' = (d_1, \ldots, d_m; e_i)$ represent two subtrees of T, then the composition map

$$\begin{array}{c} W(\Omega(T))(e_1, \ldots, e_n; e) \times W(\Omega(T))(d_1, \ldots, d_m; e_i) \\ \Big\downarrow \circ_i \\ W(\Omega(T))(e_1, \ldots, e_{i-1}, d_1, \ldots, d_m, e_{i+1}, \ldots, e_n; e) \end{array}$$

is defined by grafting the trees T_σ and $T_{\sigma'}$ along the edge e_i to form another subtree $T_{\sigma\circ_i\sigma'}$. This new tree $T_{\sigma\circ_i\sigma'}$ has as internal edges the ones of T_σ and the ones of $T_{\sigma'}$ plus a new one e_i which is assigned length 1.

The left adjoint $hc\tau_d$ is closely related to the W-construction. Let P be a coloured operad in *Sets* and let $P_{\mathcal{E}}$ be the corresponding operad in $\mathcal{E}$ obtained via the functor $Sets \longrightarrow \mathcal{E}$. Then we have the following proposition:

Proposition 6.3.3. *Let $\mathcal{E}$ be a symmetric monoidal category with an interval H and let P be any coloured operad in Sets. Then there is a natural isomorphism of operads*

$$hc\tau_d(N_d(P)) \cong W_H(P_{\mathcal{E}}).$$

Proof. The dendroidal set $N_d(P)$ is a colimit of representables $\Omega[T]$ over all morphisms $\Omega[T] \longrightarrow N_d(P)$. Since, by adjunction,

$$dSets(\Omega[T], N_d(P)) \cong Oper(\tau_d(\Omega[T]), P) \cong Oper(\Omega(T), P),$$

it follows that

$$N_d(P) = \varinjlim_{\Omega(T)\to P} \Omega[T].$$

Using the fact that $hc\tau_d$ preserves colimits and that $hc\tau_d(\Omega[T]) \cong W_H(\Omega(T))$, we have that

$$hc\tau_d(N_d(P)) = \varinjlim_{\Omega(T)\to P} W_H(\Omega(T)).$$

The required isomorphism follows now by direct inspection of the explicit construction of $W_H(P_{\mathcal{E}})$ given in [BM06]. □

An immediate consequence of this result is that there is a natural bijection

$$dSets(N_d(P), hcN_d(Q)) \cong Oper(W_H(P_{\mathcal{E}}), Q).$$

More generally, we have the following theorem:

Theorem 6.3.4. *Let $\mathcal{E}$ be a symmetric monoidal category with an interval H. Then there is a natural isomorphism*

$$\mathrm{Hom}_{dSets}(N_d(P), hcN_d(Q))_T \cong Oper(W_H(P_{\mathcal{E}} \otimes_{BV} \Omega(T)), Q)$$

for every tree T in Ω and coloured operads P and Q in sets.

Proof. By the definition of the internal hom in dendroidal sets, we have that

$$\mathrm{Hom}_{dSets}(N_d(P), hcN_d(Q))_T = dSets(N_d(P) \otimes \Omega[T], hcN_d(Q)).$$

The required natural isomorphism follows now from the adjunction (6.2) and Proposition 6.3.3, using the fact that

$$N_d(P) \otimes \Omega[T] \cong N_d(P \otimes_{BV} \Omega(T))$$

for every operad P and every tree T in Ω. □