Lecture 3

Dendroidal sets

In this lecture, we introduce the basic notions and terminology for the category of dendroidal sets.

3.1 Basic definitions and examples

In this section we define the categories of dendroidal sets and planar dendroidal sets as categories of presheaves over Ω and Ω_p. We establish the relation of these categories with the category of simplicial sets and with the category of operads by means of natural adjoint functors between them. Namely, we construct a dendroidal nerve functor from operads to dendroidal sets generalizing the classical nerve construction from small categories to simplicial sets.

Definition 3.1.1. The category *dSets* of *dendroidal sets* is the category of presheaves on Ω. The objects are functors $\Omega^{\mathrm{op}} \longrightarrow \mathit{Sets}$ and the morphisms are given by natural transformations. The category *pdSets* of *planar dendroidal sets* is defined similarly by replacing Ω by Ω_p.

Thus, a dendroidal set X is given by a set $X(T)$, denoted by X_T, for each tree T, together with a map $\alpha^*\colon X_T \longrightarrow X_S$ for each morphism $\alpha\colon S \longrightarrow T$ in Ω. Since X is a functor, $(\mathrm{id})^* = \mathrm{id}$, and if $\alpha\colon S \longrightarrow T$ and $\beta\colon R \longrightarrow S$ are morphisms in Ω, then $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$. The set X_T is called the set of *dendrices of shape* T or the set of T-*dendrices*.

A *morphism of dendroidal sets* $f\colon X \longrightarrow Y$ is given by maps $f\colon X_T \longrightarrow Y_T$ for each tree T, commuting with the structure maps, i.e., if $\alpha\colon S \longrightarrow T$ is any morphism in Ω and $x \in X_T$, then $f(\alpha^* x) = \alpha^* f(x)$.

Given two dendroidal sets Y and X, we say that Y is a *dendroidal subset* of X if $Y_T \subseteq X_T$ for every tree T and the inclusion map $Y \hookrightarrow X$ is a morphism of dendroidal sets.

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Definition 3.1.2. A dendrex $x \in X_T$ is called *degenerate* if there exists a dendrex $y \in X_S$ and a degeneracy $\sigma \colon T \longrightarrow S$ such that $\sigma^*(y) = x$.

There are canonical inclusions and evident restriction functors

$$\begin{tikzcd} \Delta \arrow[r, "u"] \arrow[dr, "i"'] & \Omega_p \arrow[d, "v"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} sSets & pdSets \arrow[l, "u^*"'] \\ & dSets \arrow[ul, "i^*"] \arrow[u, "v^*"'] \end{tikzcd}$$

which all have left and right adjoints

$$\begin{tikzcd} sSets \arrow[r, shift left, "u_!"] \arrow[dr, shift left, "i_!"] & pdSets \arrow[l, shift left, "u^*"] \arrow[d, shift right, "v_!"'] \\ & dSets \arrow[ul, shift left, "i^*"] \arrow[u, shift right, "v^*"'] \end{tikzcd} \qquad \begin{tikzcd} sSets \arrow[r, shift left, "u_*"] \arrow[dr, shift left, "i_*"] & pdSets \arrow[l, shift left, "u^*"] \arrow[d, shift right, "v_*"'] \\ & dSets \arrow[ul, shift left, "i^*"] \arrow[u, shift right, "v^*"'] \end{tikzcd}$$

given by the corresponding Kan extensions. For example, the functor i^* sends a dendroidal set X to the simplicial set

$$i^*(X)_n = X_{i([n])}.$$

Its left adjoint $i_! \colon sSets \longrightarrow dSets$ is 'extension by zero', and sends a simplicial set X to the dendroidal set given by

$$i_!(X)_T = \begin{cases} X_n & \text{if } T \cong i([n]), \\ \emptyset & \text{otherwise.} \end{cases}$$

It follows that $i_!$ is full and faithful and that $i^* i_!$ is the identity functor on simplicial sets.

Example 3.1.3. Let T be a tree. The *standard T-dendrex* is the representable presheaf $\Omega(-, T)$. We will denote it by $\Omega[T]$ (just like $\Delta[n]$ in *sSets*). Explicitly, we have that

$$\Omega[T]_S = \Omega(S, T)$$

for every tree S. The relation $i_!(\Delta[n]) = \Omega[i([n])]$ holds for every n.

By the Yoneda Lemma, each dendrex x of shape T in a dendroidal set X corresponds bijectively to a map of dendroidal sets $\hat{x} \colon \Omega[T] \longrightarrow X$. Note that $\Omega[-]$ is functorial, i.e., if $\alpha \colon S \longrightarrow T$ is a map of dendroidal sets then we have an induced map $\Omega[\alpha] \colon \Omega[S] \longrightarrow \Omega[T]$.

Example 3.1.4. The functor $\Omega \longrightarrow$ *Oper* which sends a tree T to the coloured operad $\Omega(T)$ induces an adjunction

$$\tau_d : dSets \rightleftarrows Oper : N_d. \tag{3.1}$$

The functor N_d is called the *dendroidal nerve*. Explicitly, for any operad P the dendroidal nerve of P is the dendroidal set

$$N_d(P)_T = Oper(\Omega(T), P).$$

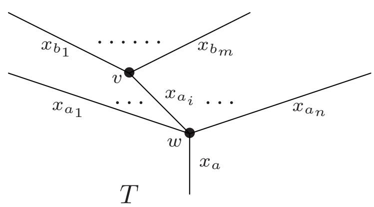
The dendroidal nerve functor is fully faithful and $N_d(\Omega(T)) = \Omega[T]$ for every T in Ω. It extends the usual nerve functor from categories to simplicial sets. If $\mathcal{E}$ is any monoidal category and $\underline{\mathcal{E}}$ is the associated coloured operad (see Example 1.3.5), then

$$i^*(N_d(\underline{\mathcal{E}})) = N(\mathcal{E}).$$

For a dendroidal set X, we refer to the left adjoint $\tau_d(X)$ as the *operad generated by* X. It can be explicitly described as follows. For any dendroidal set X, the set of colours $\mathrm{col}(\tau_d(X))$ is equal to X_η. The operations of the operad are generated by the elements of X_{C_n}, where C_n is the n-th corolla, with the following relations:

(i) $s(x_a) = \mathrm{id}_{x_a} \in \tau_d(X)(x_a; x_a)$ if $x_a \in X_\eta$ and $s = \sigma^*$, where σ is the degeneracy $\sigma\colon C_1 \longrightarrow \eta$.

(ii) If T is a tree of the form



T

and $x \in X_T$, then $d_w(x) \circ_{x_{a_i}} d_v(x) = d_{x_{a_i}}(x)$, where

$$\begin{aligned} d_w(x) &\in \tau_d(X)(x_{a_1}, \ldots, x_{a_n}; x_a), \\ d_v(x) &\in \tau_d(X)(x_{b_1}, \ldots, x_{b_m}; x_{a_i}), \\ d_{a_{x_i}}(x) &\in \tau_d(X)(x_{a_1}, \ldots, x_{a_{i-1}}, x_{b_1}, \ldots, x_{b_m}, x_{a_{i+1}}, \ldots, x_{a_n}; x_a), \end{aligned}$$

and $d_w = \partial_w^*$ is induced by the face map associated to removing the root vertex; $d_v = \partial_v^*$ is induced by the outer face map by cutting the upper part of the tree; and $d_{x_{a_i}} = \partial_{x_{a_i}}^*$ is induced by the inner face map by contracting the edge labeled x_{a_i}.

For example, $\tau_d(\Omega[T]) = \Omega(T)$ for every tree T in Ω.

The functor τ_d also extends the functor $\tau\colon sSets \longrightarrow Cat$ left adjoint to the simplicial nerve, i.e., $\tau(X) = j^*\tau_d(i_!(X))$ for every simplicial set X. In particular, there is a diagram of adjoint functors

$$\begin{tikzcd} sSets \arrow[r, shift left, "i_!"] \arrow[d, shift right, "\tau"'] & dSets \arrow[l, shift left, "i^*"] \arrow[d, shift right, "\tau_d"'] \\ Cat \arrow[r, shift left, "j_!"] \arrow[u, shift right, "N"'] & Oper \arrow[l, shift left, "j^*"] \arrow[u, shift right, "N_d"'] \end{tikzcd}$$

with left adjoints on top or to the left. Moreover, the following commutativity relations hold up to natural isomorphisms:

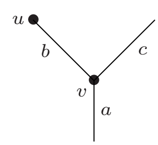
$$\tau N = \mathrm{id}, \quad \tau_d N_d = \mathrm{id}, \quad i^* i_! = \mathrm{id}, \quad j^* j_! = \mathrm{id},$$

and

$$j_!\,\tau = \tau_d\, i_!, \quad N j^* = i^* N_d, \quad i_!\, N = N_d\, j_!.$$

There is also a column in the middle of the square relating planar dendroidal sets with non-Σ operads.

Remark 3.1.5. Not everything commutes in the above diagram. The canonical map $\tau i^*(X) \longrightarrow j^*\tau_d(X)$ is not an isomorphism in general. This can be viewed, for example, by taking the representable dendroidal set $\Omega[T]$, where T is the tree with three edges, one binary vertex and one nullary vertex:



Let $X = \partial_u\Omega[T] \cup \partial_v\Omega[T] \subseteq \partial\Omega[T]$ be the union of the outer faces. Then $i^*(X) = 0$. But $\tau_d(X) = \Omega(T)$, so $j^*\tau_d(X) \neq 0$.

Later we shall have to use that the Yoneda embedding $\Omega \longrightarrow dSets$, mapping a tree T to the representable dendroidal set $\Omega[T]$, preserves pushouts of the form given in Lemma 2.3.3. We state this explicitly as follows.

Proposition 3.1.6. *Let the diagram*

$$\begin{tikzcd} R \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] & S \arrow[d, two heads] \\ T \arrow[r, two heads] & P \end{tikzcd}$$

be a pushout square of surjections in Ω*. Then this pushout square is* absolute*, i.e., preserved by any functor. In particular, the induced square*

$$\begin{tikzcd} \Omega[R] \arrow[r, two heads] \arrow[d, two heads] & \Omega[S] \arrow[d, two heads] \\ \Omega[T] \arrow[r, two heads] & \Omega[P] \end{tikzcd}$$

is a pushout square in dSets.

Note that any surjection in Ω has a section, hence remains an epimorphism after applying the Yoneda embedding (or any other functor).

The proof is based on the well-known fact that *split coequalizers* are absolute [Mac71, Ch. VI, §6]. We recall that a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "k"] \arrow[r, shift right, "l"'] & B \arrow[r, "p"] & C \end{tikzcd}$$

is *split* if there exist maps $t\colon C \longrightarrow B$ and $s\colon B \longrightarrow A$ such that $pt = \mathrm{id}_C$, $ks = \mathrm{id}_B$ and $tp = ls$. Any such split diagram is a coequalizer.

Lemma 3.1.7. *Consider a square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "u"] \\ Z \arrow[r, "v"] & P, \end{tikzcd} \tag{3.2}$$

a section $s\colon Z \longrightarrow X$ *of* g*, and the induced diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "fsg"'] & Y \arrow[r, "u"] & P. \end{tikzcd} \tag{3.3}$$

(i) *The diagram* (3.2) *is a pushout if and only if* (3.3) *is a coequalizer.*

(ii) *If there are sections* $j\colon Y \longrightarrow X$ *of* f *and* $t\colon P \longrightarrow Y$ *of* u *satisfying the identity* $tu = (fsg)j$*, then* (3.3) *is a split coequalizer with 'splitting'*

$$X \xleftarrow{j} Y \xleftarrow{t} P.$$

In particular, the pushout (3.2) *is absolute if such sections* s*,* t *and* j *exist.*

Proof. Part (ii) is clear from the definition, and part (i) is an elementary diagram chase. By way of example, we prove that (3.2) is a pushout if (3.3) is a coequalizer. Take another object W, and arrows $\varphi\colon Y \longrightarrow W$ and $\psi\colon Z \longrightarrow W$ with $\varphi f = \psi g$. We look for a unique $\chi\colon P \longrightarrow W$ with $\chi u = \varphi$ and $\chi v = \psi$. Now $\varphi fsg = \psi gsg = \psi g = \varphi f$, so φ factors uniquely through the coequalizer u in (3.3) as $\varphi = \chi u$. Then also $\psi = \chi v$; indeed, $\chi v = \chi vgs = \chi ufs = \varphi fs = \psi gs = \psi$. $\square$

Lemma 3.1.8. *In any pushout square of degeneracies*

$$\begin{tikzcd} R \arrow[r, "\sigma_v"] \arrow[d, "\sigma_w"'] & S \arrow[d, "\sigma_w"] \\ T \arrow[r, "\sigma_v"] & P \end{tikzcd} \tag{3.4}$$

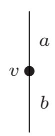
in Ω*, there exist sections* s*,* t *and* j

$$\begin{tikzcd} R & S \arrow[l, "j"'] \\ T \arrow[u, "s"'] & P \arrow[u, "t"] \end{tikzcd}$$

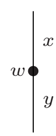
(of σ_w, σ_w *and* σ_v *respectively) satisfying the equation*

$$t\sigma_w = \sigma_v s \sigma_w j \colon S \longrightarrow S.$$

Proof. Let v and w be the vertices



and



in R, with $v \neq w$. Sections of the maps in the pushout square (3.4) correspond to set-theoretic sections of sets of edges

$$\begin{tikzcd} E(R) \arrow[r] \arrow[d] & E(S) \arrow[d] \\ E(T) \arrow[r] & E(P). \end{tikzcd}$$

Since these sections are uniquely determined outside the edges a, b, x, y, it really comes down to finding sections in the following pushout diagram of sets:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U/X \arrow[d] \\ U/A \arrow[r] & V, \end{tikzcd} \tag{3.5}$$

where $A = \{a, b\}$, $X = \{x, y\}$, and $U = A \cup X$. We can distinguish two cases:

If A and X are disjoint, then the diagram (3.5) looks like

$$\begin{tikzcd} A + X \arrow[r, "f"] \arrow[d, "g"'] & A + \{1\} \arrow[d, "u"] \\ \{0\} + X \arrow[r, "v"] & \{0,1\} \end{tikzcd}$$

and we can take any sections s, j and t,

$$\begin{tikzcd} A + X & A + \{1\} \arrow[l, "j"'] \\ \{0\} + X \arrow[u, "s"] & \{0,1\}, \arrow[u, "t"'] \end{tikzcd}$$

with $s(0) = t(0)$. Then $fsgj = tu$, as required.

If A and X are not disjoint, say $x = b$, then the diagram (3.5) looks like

$$\begin{tikzcd} \{a, b = x, y\} \arrow[r, "f"] \arrow[d, "g"'] & \{a, b\} \arrow[d, "u"] \\ \{x, y\} \arrow[r, "v"] & \{0\} \end{tikzcd}$$

with $fa = a$, $fb = fy = b$ and $ga = gb$, $gy = y$. Then one can take sections s, j and t,

$$\begin{tikzcd} \{a, b, y\} & \{a, b\} \arrow[l, "j"'] \\ \{x, y\} \arrow[u, "s"] & \{0\}, \arrow[u, "t"'] \end{tikzcd}$$

with $s(x) = b$, $t(0) = b$, $j(b) = b$, to obtain the identity $fsgj = tu$ again. □

Proof of Proposition 3.1.6. It suffices (as in Lemma 2.3.3) to consider the case where the surjections f and g are degeneracies

$$\sigma_v \colon R \longrightarrow S = R \backslash v \quad \text{and} \quad \sigma_w \colon R \longrightarrow T = R \backslash w.$$

The proposition is evidently true in the case $v = w$. If $v \neq w$, then Lemma 3.1.8 completes the proof. □

3.2 Faces, boundaries and horns

In this section we define face maps, boundaries and horns in the context of dendroidal sets.

Definition 3.2.1. Let T be an object of Ω and $\alpha\colon S \longrightarrow T$ a face map in Ω. The *α-face* of $\Omega[T]$ is the dendroidal subset of $\Omega[T]$ given by the image of the map $\Omega[\alpha]\colon \Omega[S] \longrightarrow \Omega[T]$. We denote it by $\partial_\alpha \Omega[T]$. We write $\Phi_1(T)$ for the set of all faces of T.

Thus we have that

$$\partial_\alpha \Omega[T]_R = \{R \xrightarrow{\beta} S \xrightarrow{\alpha} T \text{ where } \beta \in \Omega[S]_R\}.$$

When the face map α is an inner face obtained by contracting an inner edge e, we denote ∂_α by ∂_e.

Definition 3.2.2. Let T be an object of Ω. The *boundary* of $\Omega[T]$ is the dendroidal subset $\partial\Omega[T]$ of $\Omega[T]$ obtained as the union of all possible faces of $\Omega[T]$. Namely,

$$\partial\Omega[T] = \bigcup_{\alpha \in \Phi_1(T)} \partial_\alpha \Omega[T].$$

If we take the union of all the faces except for one, we have the definition of a horn.

Definition 3.2.3. Let T be an object of Ω and $\alpha \in \Phi_1(T)$ a face of T. The *α-horn* of $\Omega[T]$ is the dendroidal subset $\Lambda^\alpha[T]$ of $\Omega[T]$ obtained as the union over all faces of T except α. That is,

$$\Lambda^\alpha[T] = \bigcup_{\beta \neq \alpha \in \Phi_1(T)} \partial_\beta \Omega[T].$$

As before, if α is an inner face map contracting an edge e, then we denote $\Lambda^\alpha[T]$ by $\Lambda^e[T]$. The horns of the form $\Lambda^e[T]$ are called *inner horns*. The other horns are called *outer horns*.

A *horn* in a dendroidal set X is given by a map of dendroidal sets

$$\Lambda^\alpha[T] \longrightarrow X.$$

This horn is *inner* if $\Lambda^\alpha[T]$ is an inner horn and it is *outer* if $\Lambda^\alpha[T]$ is an outer horn.

The definitions of faces, boundaries and horns in dendroidal sets naturally extend the corresponding ones for simplicial sets. For example, if $\Lambda^k[n] \subseteq \Delta[n]$ denotes the simplicial k-horn, then the dendroidal set

$$i_!(\Lambda^k[n]) \subseteq i_!(\Delta[n]) = \Omega[L_n],$$

where L_n denotes the linear tree with n vertices and $n+1$ edges, is a horn in the dendroidal sense. Moreover, the horn $\Lambda^k[n]$ is an inner horn (i.e., $0 < k < n$) if and only if $i_!(\Lambda^k[n])$ is an inner horn.

Boundaries and horns can also be described in term of colimits. This extends, in the case of simplicial sets, the presentation of the boundary $\partial\Delta[n]$ and the horn $\Lambda^k[n]$ as a colimit of standard simplices.

Let $T_1 \longrightarrow T_2 \longrightarrow \cdots \longrightarrow T_n$ be a sequence of n face maps in Ω. The composition of these maps is called a *subface of codimension n* of T_n. Note that subfaces of codimension 1 are precisely the face maps. It follows from the dendroidal identities in Section 2.2.3 that every subface of a tree of codimension 2 decomposes in exactly two different ways as a composition of faces. Let $\Phi_2(T)$ be the set of all subfaces of codimension 2 of T. Thus, for each $\beta\colon S \longrightarrow T$ in $\Phi_2(T)$ there are exactly two maps $\beta_1\colon S \longrightarrow T_1$ and $\beta_2\colon S \longrightarrow T_2$ through which β factors. Using β_1 and β_2, we can define two maps γ_1 and γ_2,

$$\coprod_{(S\to T)\in\Phi_2(T)} \Omega[S] \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} \coprod_{(R\to T)\in\Phi_1(T)} \Omega[R],$$

where the components of γ_i are the compositions

$$\Omega[S] \xrightarrow{\Omega[\beta_i]} \Omega[T_i] \longrightarrow \Omega[R]$$

for each $\beta\colon S \longrightarrow T$ in $\Phi_2(T)$ and $i = 1, 2$.

Lemma 3.2.4. *Let T be an object of Ω. Then the boundary $\partial\Omega[T]$ can be obtained as the coequalizer*

$$\coprod_{(S\to T)\in\Phi_2(T)} \Omega[S] \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} \coprod_{(R\to T)\in\Phi_1(T)} \Omega[R] \longrightarrow \partial\Omega[T].$$

Proof. The universal property is verified by using the definition of $\partial\Omega[T]$ and the fact that every subface of codimension 2 decomposes exactly in two different ways as a composition of faces. □

Corollary 3.2.5. *A map of dendroidal sets $\partial\Omega[T] \longrightarrow X$ corresponds exactly to a sequence of dendrices $\{x_R\}_{(R\to T)\in\Phi_1(T)}$ that agree on common faces, i.e., if $\beta\colon S \to T$ is a subface of codimension 2 which factors as*

$$\begin{tikzcd} S \arrow[r, "\beta_1"] \arrow[d, "\beta_2"'] \arrow[dr, "\beta"] & T_1 \arrow[d, "\alpha_1"] \\ T_2 \arrow[r, "\alpha_2"'] & T, \end{tikzcd}$$

then $\beta_1^*(x_{T_1}) = \beta_2^*(x_{T_2})$. □

If α is a face of T, the α-horn $\Lambda^\alpha[T]$ can be computed using the same coequalizer as before, but excluding the face α.

Lemma 3.2.6. *Let T be an object of Ω and α a face of T. Then the horn $\Lambda^\alpha[T]$ is the coequalizer*

$$\coprod_{(S\to T)\in\Phi_2(T)} \Omega[S] \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} \coprod_{(R\to T)\neq\alpha\in\Phi_1(T)} \Omega[R] \longrightarrow \Lambda^\alpha[T].$$

Proof. The proof is analogous to that of Lemma 3.2.4. □

Corollary 3.2.7. *Let α be a face map in T. A horn $\Lambda^\alpha[T] \longrightarrow X$ in X corresponds exactly to a sequence of dendrices $\{x_R\}_{(R\to T)\neq\alpha\in\Phi_1(T)}$ that agree on common faces, i.e., if $\beta\colon S \longrightarrow T$ is a subface of codimension 2 which factors as*

$$\begin{tikzcd} S \arrow[r, "\beta_1"] \arrow[d, "\beta_2"'] \arrow[dr, "\beta"] & T_1 \arrow[d, "\alpha_1"] \\ T_2 \arrow[r, "\alpha_2"'] & T, \end{tikzcd}$$

where $\alpha_1, \alpha_2 \neq \alpha$, then $\beta_1^*(x_{T_1}) = \beta_2^*(x_{T_2})$. □

Finally, we will use the following terminology for dendrices in a dendroidal set. Let $\alpha\colon S \longrightarrow T$ be a map in Ω, let X be a dendroidal set, and let $t \in X_T$ be a T-dendrex. Consider the S-dendrex given by $\alpha^*(t)$. Then:

(i) $\alpha^*(t)$ is a *face* (resp. *inner face*, *outer face*) of t if α is a face (resp. inner face, outer face) of T.

(ii) $\alpha^*(t)$ is a *subface* of t if α is a subface of T.

(iii) $\alpha^*(t)$ is *isomorphic* to t if α is an isomorphism.

(iv) $\alpha^*(t)$ is a *degeneracy* of t is α is a composition of degeneracies.

3.3 Skeleta and coskeleta

Let $\Omega^{\leq n}$ denote the full subcategory of Ω consisting of trees with n or less vertices. Similarly, one can define the full subcategory $\Delta^{\leq n}$ as the full subcategory of Δ with objects $[k]$ where $0 \leq k \leq n$. There is a commutative diagram

$$\begin{tikzcd} \Delta^{\leq n} \arrow[r, "j"] \arrow[d, "i_n"'] & \Omega^{\leq n} \arrow[d, "i_n"] \\ \Delta \arrow[r, "i"'] & \Omega, \end{tikzcd}$$

where i_n denotes the fully faithful inclusion functor. The functors i_n induce functors i_n^* between the corresponding categories of presheaves and thus we have a commutative diagram

$$\begin{tikzcd} sSets^{\leq n} & dSets^{\leq n} \arrow[l, "j^*"'] \\ sSets \arrow[u, "i_n^*"] & dSets \arrow[l, "i^*"] \arrow[u, "i_n^*"'] \end{tikzcd} \tag{3.6}$$

consisting of the inverse image functor of a pullback of presheaf toposes, together with the corresponding left adjoints $i_{n!}$, $j_!$ and $i_!$, and right adjoints i_{n*}, j_* and i_*. Moreover, all α_* and $\alpha_!$ are full and faithful ($\alpha = i_n, j, i$).

Definition 3.3.1. Let X be a dendroidal set. The *n-th skeleton* of X is defined as $\mathrm{Sk}_n(X) = i_{n!}{i_n}^*(X)$ and the *n-th coskeleton* of X as $\mathrm{coSk}_n(X) = i_{n*}{i_n}^*(X)$. (One can define with a similar formula the n-th skeleton of a simplicial set.)

There are natural morphisms

$$\mathrm{Sk}_n(X) \longrightarrow X \longrightarrow \mathrm{coSk}_n(X)$$

given by the counit and the unit of the corresponding adjunctions, for every dendroidal set X. There are also inclusions $\mathrm{Sk}_n(X) \subseteq \mathrm{Sk}_{n+1}(X)$ for every $n \geq 0$. It follows that $X = \cup_{n=0}^{\infty} \mathrm{Sk}_n(X)$, and this presentation of X is called the *skeletal filtration* of X. One defines similarly the *coskeletal filtration* of a dendroidal set.

Recall that the functor $i_n^* \colon dSets \longrightarrow dSets^{\leq n}$ is defined on representables as

$$i_n^*(\Omega[T]) = \Omega(i_n(-), T)$$

for every T in $\Omega^{\leq n}$. Its left adjoint $i_{n!}$ is defined on representables as

$$i_{n!}(\Omega^{\leq n}[T]) = \Omega(-, i_n(T))$$

for every T in Ω. The fact that i_n is fully faithful implies that ${i_n}^*i_{n!}(\Omega[T]) = \Omega[T]$ for every object T in Ω. Hence, ${i_n}^*i_{n!}(X) = X$ for all X, since every dendroidal set is a canonical colimit of representables. Using the adjunction one can check that also ${i_n}^*i_{n*}(X) = X$ for all X. It follows that $i_{n!}$ and i_{n*} are both fully faithful.

Definition 3.3.2. A dendroidal set X is called *n-coskeletal* if $X = \mathrm{coSk}_n(X)$.

Proposition 3.3.3. *A dendroidal set X is n-coskeletal if, for every dendroidal set Y, each map $\mathrm{Sk}_n(Y) \longrightarrow X$ extends uniquely along $\mathrm{Sk}_n(Y) \longrightarrow Y$ to a map $Y \longrightarrow X$.*

Proof. Since X is n-coskeletal, $X = \mathrm{coSk}_n(X)$. By an adjointness argument, there is a bijection between the sets of maps $dSets(Y, X)$ and $dSets(\mathrm{Sk}_n(Y), X)$. $\square$

If we make the definition of the Kan extension $i_{n!}$ explicit, we find that, for any tree R in Ω,

$$\mathrm{Sk}_n(X)_R = \varinjlim_{(T,\alpha)} \Omega(R, T),$$

where the colimit ranges over all trees T with at most n vertices and all $\alpha \in X_T$, i.e., maps $\alpha\colon \Omega[T] \longrightarrow X$. In other words, $\mathrm{Sk}_n(X)_R$ consists of equivalence classes of pairs (α, u), with $u\colon R \longrightarrow T$ in Ω and $\alpha\colon \Omega[T] \longrightarrow X$ in *dSets*, and $|V(T)| \leq n$. The equivalence relation on such pairs is generated by

$$(\alpha v, u) \sim (\alpha, vu)$$

where $R \xrightarrow{u} T' \xrightarrow{v} T$ and $\Omega[T] \xrightarrow{\alpha} X$. The counit maps the equivalence class of (α, u) to $u^*(\alpha)$, or, in another notation, it maps (α, u) to the composition $\alpha \circ u$,

$$\Omega[R] \xrightarrow{u} \Omega[T] \xrightarrow{\alpha} X.$$

Lemma 3.3.4. *For each $n \geq 0$, the counit of the adjunction $\mathrm{Sk}_n(X) \longrightarrow X$ is a monomorphism for every dendroidal set X.*

Proof. To show that $\mathrm{Sk}_n(X) \longrightarrow X$ is injective, we need to prove that, if a diagram of the form

$$\begin{tikzcd}
\Omega[S] \arrow[r, "\alpha"] & X \\
\Omega[R] \arrow[u, "u"] \arrow[r, "v"] & \Omega[T] \arrow[u, "\beta"']
\end{tikzcd}$$

in *dSets* commutes, where S and T have at most n vertices, then $(\alpha, u) \sim (\beta, v)$. To see this, factor $u = if$ and $v = jg$ as in Lemma 2.3.2, and take the pushout P in Ω,

$$\begin{tikzcd}
S & & \\
S' \arrow[u, tail, "i"] \arrow[r, two heads, "h"] & P & \\
R \arrow[u, two heads, "f"] \arrow[r, two heads, "g"] & T' \arrow[u, two heads, "k"'] \arrow[r, tail, "j"] & T.
\end{tikzcd}$$

Then the functor $\Omega[-]$ preserves this pushout by Proposition 3.1.6, so we obtain a diagram in *dSets* of the form

$$\begin{tikzcd}
\Omega[S] \arrow[rr, "\alpha"] & & X \\
\Omega[S'] \arrow[u, tail, "i"] \arrow[r, two heads, "h"] & \Omega[P] \arrow[ur, dotted, "\gamma"'] & \\
\Omega[R] \arrow[u, two heads, "f"] \arrow[r, two heads, "g"] & \Omega[T'] \arrow[u, two heads, "k"'] \arrow[r, tail, "j"] & \Omega[T]. \arrow[uu, "\beta"']
\end{tikzcd}$$

Now S', P and T' have at most n vertices since S and T do, so we can use the equivalence relation defining $\mathrm{Sk}_n(X)$, and find

$$(\alpha, u) = (\alpha, if) \sim (\alpha i, f) = (\gamma h, f) \sim (\gamma, hf),$$

and, in exactly the same way, $(\beta, v) \sim (\gamma, kg)$. Since $hf = kg$, this shows that $(\alpha, u) \sim (\beta, v)$ and proves the lemma. □

The following proposition relates the skeleton and coskeleton constructions between the category of simplicial sets and the category of dendroidal sets.

Proposition 3.3.5. *The following relations hold:*

(i) $i^*(\mathrm{Sk}_n(X)) = \mathrm{Sk}_n(i^*(X))$ *and* $i^*(\mathrm{coSk}_n(X)) = \mathrm{coSk}_n(i^*(X))$ *for every dendroidal set X and every $n \geq 0$.*

(ii) $i_!(\mathrm{Sk}_n(X)) = \mathrm{Sk}_n(i_!(X))$ *and* $i_*(\mathrm{coSk}_n(X)) = \mathrm{coSk}_n(i_*(X))$ *for every simplicial set X and every $n \geq 0$.*

Proof. The proof is straightforward by using the following commutativity relations between the functors involved:

$$i^* i_{n!} = i_{n!} j^*, \quad j_* i_n^* = i_n^* i_*, \quad i_n^* i_! = j_! i_n^*, \quad i^* i_{n*} = i_{n*} j^*,$$

which follow from the fact that (3.6) is a pullback of presheaf toposes. □

3.4 Normal monomorphisms

Recall from Definition 3.1.2 that a dendrex $t \in X_T$ is called degenerate if $t = \sigma(s)$ where σ is a composition of degeneracies and s is another dendrex. Any dendrex $t \in X_T$ where T is a tree with no unary vertices is non-degenerate.

Lemma 3.4.1. *Any dendrex $x \in X_T$ is the restriction $\sigma^*(y)$ of a non-degenerate dendrex $y \in X_R$ along a surjection $\sigma \colon T \twoheadrightarrow R$ in Ω. Moreover, given x, the map σ and the dendrex y are unique up to isomorphism.*

Proof. By the Yoneda Lemma, x corresponds to a map $\hat{x} \colon \Omega[T] \longrightarrow X$. Consider, among all possible factorizations

$$\Omega[T] \xrightarrow{\sigma} \Omega[R] \xrightarrow{\hat{y}} X$$

of $\hat{x}$, those where R has a minimal number of vertices, so that y is necessarily non-degenerate. It suffices to show that any two such 'minimal' factorizations are isomorphic. But, given another one,

$$\Omega[T] \xrightarrow{\sigma'} \Omega[R] \xrightarrow{\hat{y}'} X,$$

we can form the pushout

$$\begin{tikzcd} T \arrow[r, two heads, "\sigma'"] \arrow[d, two heads, "\sigma"'] & R' \arrow[d, two heads, "\tau'"] \\ R \arrow[r, two heads, "\tau"] & P \end{tikzcd}$$

in Ω by Lemma 2.3.3. Since $\Omega[-]$ of this pushout is a pushout in dendroidal sets (by Proposition 3.1.6) and since $\hat{y}\sigma = \hat{x} = \hat{y}'\sigma'$ by assumption, we find $\hat{z}\colon \Omega[P] \longrightarrow X$ with $\hat{z}\tau = \hat{y}$ and $\hat{z}\tau' = \hat{y}'$. But then, by minimality of R and R', both τ and τ' must be isomorphisms. Thus, we have the following diagram:

$$\begin{tikzcd} \Omega[T] \arrow[r, "\sigma'"] \arrow[d, "\sigma"'] & \Omega[R'] \arrow[d, "\hat{y}'"] \\ \Omega[R] \arrow[r, "\hat{y}"] \arrow[ur, "\sim"] & X, \end{tikzcd}$$

and $(\sigma, \hat{y})$ and $(\sigma', \hat{y}')$ are isomorphic. $\square$

For $n > 0$, we can consider the following commutative diagram:

$$\begin{tikzcd} \coprod_{(T,t)} \partial\Omega[T] \arrow[r] \arrow[d] & \mathrm{Sk}_{n-1}(X) \arrow[d] \\ \coprod_{(T,t)} \Omega[T] \arrow[r] & \mathrm{Sk}_n(X), \end{tikzcd} \tag{3.7}$$

where the coproducts are taken over all isomorphism classes of pairs (T, t) in the category of elements of X such that T has n vertices and $t \in X_T$ is non-degenerate. For the case $n = 0$, note that $\mathrm{Sk}_0(X) = \coprod_{x \in X_\eta} \Omega[\eta]$.

Definition 3.4.2. A monomorphism $X \rightarrowtail Y$ in *dSets* is called *normal* if, for every tree T in Ω, every non-degenerate element $y \in Y_T$ which is not in the image of X_T has a trivial stabilizer $\mathrm{Aut}(T)_y \subseteq \mathrm{Aut}(T)$, where $\mathrm{Aut}(-)$ denotes the automorphism group of the corresponding tree. An object X is called *normal* if the map $\emptyset \rightarrowtail X$ is a normal monomorphism.

Example 3.4.3. For every tree T in Ω, the representable dendroidal set $\Omega[T]$ is normal. If $\sigma\colon X \rightarrowtail Y$ is a monomorphism and Y is normal, then σ is a normal monomorphism.

The skeletal filtration of a dendroidal set is called *normal* if the diagram (3.7) is a pushout. This property gives a characterization of normal dendroidal sets:

Proposition 3.4.4. *A dendroidal set is normal if and only if it admits a normal skeletal filtration.*

Proof. If a dendroidal set X admits a normal skeletal filtration, it is easy to see that X is normal, and the proof is left to the reader.

We prove in detail the other direction. Let us fix a representing element (T, β) in each isomorphism class $[(T, \beta)]$ of non-degenerate dendrices $\beta \in X_T$, i.e., $\hat{\beta} \colon \Omega[T] \longrightarrow X$. We begin by observing that, in the category of sets, a pullback diagram of the form

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, tail, "n"'] & C \arrow[d, tail, "m"] \\ B \arrow[r, two heads, "p"] & D, \end{tikzcd}$$

with p an epimorphism and m a monomorphism as indicated, is a pushout if and only if, as subsets of $B \times B$, the set $B \times_D B$ is contained in the union of the diagonal $B \to B \times B$ and $A \times_C A \rightarrowtail B \times_D B$. Since pullbacks and pushouts in presheaf categories are computed pointwise, the same observation applies to pushout diagrams in the category of dendroidal sets. Let us now check that the diagram

$$\begin{tikzcd} \partial P = \coprod \partial\Omega[T] \arrow[r] \arrow[d, tail] & \mathrm{Sk}_{n-1}(X) \arrow[d, tail] \arrow[dr, tail] & \\ P = \coprod \Omega[T] \arrow[r] & \mathrm{Sk}_n(X) \arrow[r, tail] & X \end{tikzcd}$$

is a pushout. The square is clearly a pullback and we know that $\mathrm{Sk}_n(X) \longrightarrow X$ is a monomorphism for each n, by Lemma 3.3.4. Hence it is enough to prove that $P \times_X P \rightarrowtail P \times P$ is contained in the union of the diagonal $P \longrightarrow P \times P$ and $\partial P \times_X \partial P \rightarrowtail P \times P$. To this end, fix one representative (T, α) in each isomorphism class $[(T, \alpha)]$ (these classes index the coproduct in the diagram). For a tree R, an element $\xi \in (P \times_X P)_R$ is a commutative square

$$\begin{tikzcd} \Omega[S] \arrow[r, "\alpha"] & X \\ \Omega[R] \arrow[u, "u"] \arrow[r, "v"] & \Omega[T], \arrow[u, "\beta"'] \end{tikzcd}$$

where (S, α) and (T, β) are representatives as above; in particular, $\alpha \in X_S$ and $\beta \in X_T$ are non-degenerate. If neither $u \colon R \longrightarrow S$ nor $v \colon R \longrightarrow T$ is surjective, then u and v factor through $\partial\Omega[S]$ and $\partial\Omega[T]$, so the element $\xi \in (P \times_X P)_R$ in fact lies in $(\partial P \times_X \partial P)_R$. Hence we may assume that one of u and v is surjective; say u is. Now factor $v = gj$ as an epimorphism followed by a monomorphism, and form the pushout

$$\begin{tikzcd} S \arrow[r, two heads, "h"] & P & \\ R \arrow[u, two heads, "u"] \arrow[r, two heads, "g"] & T' \arrow[u, two heads, "k"'] \arrow[r, tail, "j"] & T. \end{tikzcd}$$

We use Proposition 3.1.6 to find a diagram

$$\begin{tikzcd} \Omega[S] \arrow[r, two heads, "h"] & \Omega[P] \arrow[r, "\gamma"] & X \\ \Omega[R] \arrow[u, two heads, "u"] \arrow[r, two heads, "g"] & \Omega[T'] \arrow[u, two heads, "k"'] \arrow[r, tail, "j"] & \Omega[T] \arrow[u, "\beta"'] \end{tikzcd}$$

with $\gamma h = \alpha$. But α is non-degenerate, so h must be an isomorphism. But then we have maps

$$T \xleftarrow{\;j\;} T' \xrightarrow{h^{-1}k} S,$$

where S and T have exactly n vertices. So j must be an isomorphism, as must be k. Thus, writing $\theta = h^{-1}k$, we have

$$\begin{tikzcd} \Omega[S] \arrow[r, "\alpha"] & X \\ \Omega[R] \arrow[u, two heads, "u"] \arrow[r, two heads, "v"] & \Omega[T]. \arrow[u, "\beta"'] \arrow[ul, "\theta"', "\sim"] \end{tikzcd}$$

In other words, (S, α) and (T, β) lie in the same isomorphism class, hence they must be equal by our choice of representatives. But then $\theta = \mathrm{id}$, since X is assumed normal. We conclude that $u = v$ as well, so the element $\xi \in P \times_X P$ represented by (α, u) and (β, v) lies in fact in the diagonal. This completes the proof. $\square$

Lemma 3.4.5. *Let $p\colon Y \longrightarrow X$ be a map of dendroidal sets, and assume that X is normal. Then Y is normal as well.*

Proof. First recall that any normal object has a nice skeletal filtration, i.e., can be built up by attaching cells x by pushouts of the form

$$\begin{tikzcd} \partial\Omega[T] \arrow[r] \arrow[d, tail] & A \arrow[d, tail] \\ \Omega[T] \arrow[r, "x"] & B. \end{tikzcd}$$

Such an attached cell x always comes with a map $\Omega[T] \xrightarrow{x} B$ which is 'injective on its interior' (just like for usual cell complexes). Indeed, by looking at pushouts of this form in *Sets* as before, one sees that the kernel pair $\Omega[T]\times_B\Omega[T] \subseteq \Omega[T]\times\Omega[T]$ of the map x is the union of the diagonal $\Omega[T]$ and $\partial\Omega[T] \times_A \partial\Omega[T]$. This simple observation implies that, for any non-degenerate dendrex $x \in X(T)$ in a normal dendroidal set X, the corresponding map $\Omega[T] \xrightarrow{x} X$ has the cancellation property with respect to epimorphisms in Ω:

$$\text{if } \Omega[R] \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \Omega[T] \xrightarrow{x} X \text{ and } x\beta = x\gamma, \text{ then } \beta = \gamma.$$

To prove the lemma, let $y \in Y(T)$ be non-degenerate and suppose that $\alpha \in \mathrm{Aut}(T)$ fixes y, with $\alpha \neq 1$. Then α also fixes py, so py must be degenerate by the assumption that X is normal; say that $py = \rho^* x$, where $\rho\colon T \twoheadrightarrow R$ and $x \in X(R)$ is non-degenerate,

$$\begin{tikzcd} T \arrow[r, "\alpha", "\sim"'] & T \arrow[r, "y"] \arrow[d, two heads, "\rho"] & Y \arrow[d, "p"] \\ & R \arrow[r, "x"'] & X. \end{tikzcd}$$

Then $x(\rho\alpha) = x\rho$ because $y\alpha = y$; hence, $\rho\alpha = \rho$ by the cancellation property. This means that α is an automorphism of T which permutes the edges on each of the fibers of ρ. But these fibers are linear, so α must be the identity. $\square$