Lecture 2

Trees as operads

In this lecture, we introduce convenient categories of trees that will be used for the definition of dendroidal sets. These categories are generalizations of the simplicial category Δ used to define simplicial sets. First we consider the case of planar trees and then the more general case of non-planar trees.

2.1 A formalism of trees

A *tree* is a non-empty connected finite graph with no loops. A vertex in a graph is called *outer* if it has only one edge attached to it. All the trees we will consider are *rooted trees*, i.e., equipped with a distinguished outer vertex called the *output* and a (possibly empty) set of outer vertices (not containing the output vertex) called the set of *inputs*.

When drawing trees, we will delete the output and input vertices from the picture. From now on, the term 'vertex' in a tree will always refer to a remaining vertex. Given a tree T, we denote by $V(T)$ the set of vertices of T and by $E(T)$ the set of edges of T.

The edges attached to the deleted input vertices are called *input edges* or *leaves*; the edge attached to the deleted output vertex is called *output edge* or *root*. The rest of the edges are called *inner edges*. The root induces an obvious direction in the tree, 'from the leaves towards the root'. If v is a vertex of a finite rooted tree, we denote by $\mathrm{out}(v)$ the unique outgoing edge and by $\mathrm{in}(v)$ the set of incoming edges (note that $\mathrm{in}(v)$ can be empty). The cardinality of $\mathrm{in}(v)$ is called the *valence* of v, the element of $\mathrm{out}(v)$ is the *output* of v, and the elements of $\mathrm{in}(v)$ are the *inputs* of v.

As an example, consider the following picture of a tree:

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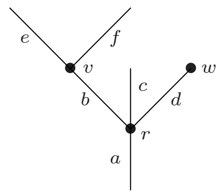
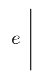


The output vertex at the edge a and the input vertices at e, f and c have been deleted. This tree has three vertices r, v and w of respective valences 3, 2, and 0. It also has three input edges or leaves, namely e, f and c. The edges b and d are inner edges and the edge a is the root. A tree with no vertices



whose input edge (which we denote by e) coincides with its output edge will be denoted by η_e, or simply by η.

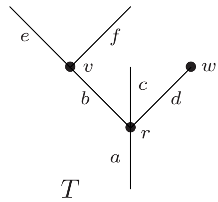
Definition 2.1.1. A *planar rooted tree* is a rooted tree T together with a linear ordering of $\mathrm{in}(v)$ for each vertex v of T.

The ordering of $\mathrm{in}(v)$ for each vertex is equivalent to drawing the tree on the plane. When we draw a tree we will always put the root at the bottom. One drawback of drawing a tree on the plane is that it immediately becomes a planar tree; we thus may have many different 'pictures' for the same tree. For example, the two trees

are two different planar representations of the same tree.

2.2 Planar trees

Let T be a planar rooted tree. Any such tree generates a non-Σ operad, which we denote by $\Omega_p(T)$. The set of colours of $\Omega_p(T)$ is the set $E(T)$ of edges of T, and the operations are generated by the vertices of the tree. More explicitly, each vertex v with input edges $e_1, \ldots, e_n$ and output edge e defines an operation $v \in \Omega_p(T)(e_1, \ldots, e_n; e)$. The other operations are the unit operations and the operations obtained by compositions. This operad has the property that, for all $e_1, \ldots, e_n, e$, the set of operations $\Omega_p(T)(e_1, \ldots, e_n; e)$ contains at most one element. For example, consider the same tree T pictured before:



The operad $\Omega_p(T)$ has six colours a, b, c, d, e, and f. Then $v \in \Omega_p(T)(e, f; b)$, $w \in \Omega_p(T)(\,;d)$, and $r \in \Omega_p(b, c, d; a)$ are the generators, while the other operations are the units $1_a, 1_b, \ldots, 1_f$ and the operations obtained by compositions, namely $r \circ_1 v \in \Omega_p(T)(e, f, c, d; a)$, $r \circ_3 w \in \Omega_p(T)(b, c; a)$, and

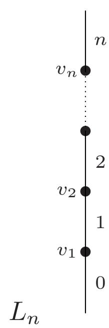
$$r(v, 1_c, w) = (r \circ_1 v) \circ_4 w = (r \circ_3 w) \circ_1 v \in \Omega_p(T)(e, f, c; a).$$

This is a complete description of the operad $\Omega_p(T)$.

Definition 2.2.1. The *category of planar rooted trees* Ω_p is the full subcategory of the category of non-Σ coloured operads whose objects are $\Omega_p(T)$ for any tree T.

We can view Ω_p as the category whose objects are planar rooted trees. The set of morphisms from a tree S to a tree T is given by the set of non-Σ coloured operad maps from $\Omega_p(S)$ to $\Omega_p(T)$. Observe that any morphism $S \longrightarrow T$ in Ω_p is completely determined by its effect on the colours (i.e., edges).

The category Ω_p extends the simplicial category Δ. Indeed, any $n \geq 0$ defines a linear tree



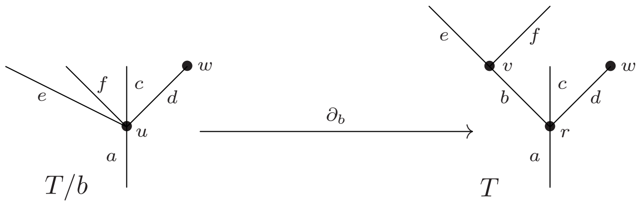
with $n+1$ edges and n vertices $v_1, \ldots, v_n$. We denote this tree by $[n]$ or L_n. Any order-preserving map $\{0, \ldots, n\} \longrightarrow \{0, \ldots, m\}$ defines an arrow $[n] \longrightarrow [m]$ in the category Ω_p. In this way, we obtain an embedding

$$\Delta \xrightarrow{\ u\ } \Omega_p.$$

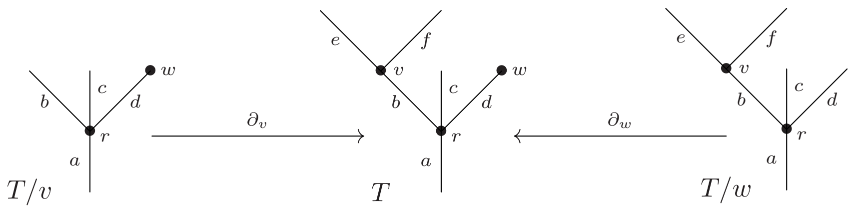
This embedding is fully faithful. Moreover, it describes Δ as a sieve (or ideal) in Ω_p, in the sense that for any arrow $S \longrightarrow T$ in Ω_p, if T is linear then so is S. In the next sections we give a more explicit description of the morphisms in Ω_p.

2.2.1 Face maps

Let T be a planar rooted tree and b an inner edge in T. Let us denote by T/b the tree obtained from T by contracting b. Then there is a natural map $\partial_b \colon T/b \longrightarrow T$ in Ω_p, called the *inner face* map associated with b. This map is the inclusion on both the colours and the generating operations of $\Omega_p(T/b)$, except for the operation u, which is sent to $r \circ_b v$. Here r and v are the two vertices in T at the two ends of b, and u is the corresponding vertex in T/b, as in the picture:



Now let T be a planar rooted tree and v a vertex of T with exactly one inner edge attached to it. Let T/v be the tree obtained from T by removing the vertex v and all the outer edges. There is a face map associated to this operation, denoted $\partial_v \colon T/v \longrightarrow T$, which is the inclusion both on the colours and on the generating operations of $\Omega_p(T/v)$. These types of face maps are called the *outer faces* of T. The following are two outer face maps:



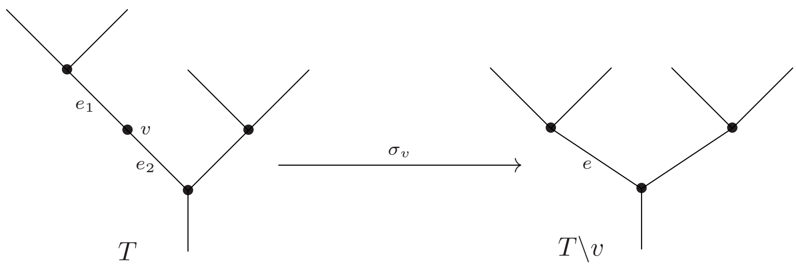
Note that the possibility of removing the root vertex of T is included in this definition. This situation can happen only if the root vertex is attached to exactly one inner edge, thus not every tree T has an outer face induced by its root. There is another particular situation which requires special attention, namely the inclusion of the tree with no vertices η into a tree with one vertex, called a *corolla*. In this case we get $n+1$ face maps if the corolla has n leaves. The operad $\Omega_p(\eta)$ consists of only one colour and the identity operation on it. Then a map of operads $\Omega_p(\eta) \longrightarrow \Omega_p(T)$ is just a choice of an edge of T.

We will use the term *face map* to refer to an inner or outer face map.

2.2.2 Degeneracy maps

There is one more type of map that can be associated with a vertex v of valence one in T as follows. Let $T\backslash v$ be the tree obtained from T by removing the vertex v and merging the two edges incident to it into one edge e. Then there is a map

$\sigma_v \colon T \longrightarrow T\backslash v$ in Ω_p called the *degeneracy map* associated with v, which sends the colours e_1 and e_2 of $\Omega_p(T)$ to e, sends the generating operation v to id_e, and is the identity for the other colours and operations. It can be pictured like this:



Face maps and degeneracy maps generate the whole category Ω_p. The following lemma is the generalization to Ω_p of the well-known fact that in Δ each arrow can be written as a composition of degeneracy maps followed by face maps. For the proof of this fact we refer the reader to Lemma 2.3.2, where we prove a similar statement in the category of non-planar trees.

Lemma 2.2.2. *Any arrow $f \colon A \longrightarrow B$ in Ω_p decomposes (up to isomorphism) as*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{\sigma} & \uparrow{\scriptstyle\delta} \\ & & C \end{array}$$

where $\sigma \colon A \longrightarrow C$ is a composition of degeneracy maps and $\delta \colon C \longrightarrow B$ is a composition of face maps. □

2.2.3 Dendroidal identities

In this section we are going to make explicit the relations between the generating maps (faces and degeneracies) of Ω_p. The identities that we obtain generalize the simplicial ones in the category Δ.

Elementary face relations

Let $\partial_a \colon T/a \longrightarrow T$ and $\partial_b \colon T/b \longrightarrow T$ be distinct inner faces of T. It follows that the inner faces $\partial_a \colon (T/b)/a \longrightarrow T/b$ and $\partial_b \colon (T/a)/b \longrightarrow T/a$ exist, we have $(T/a)/b = (T/b)/a$, and the following diagram commutes:

$$\begin{array}{ccc} (T/a)/b & \xrightarrow{\partial_b} & T/a \\ {\scriptstyle\partial_a}\downarrow & & \downarrow{\scriptstyle\partial_a} \\ T/b & \xrightarrow{\partial_b} & T. \end{array}$$

Let $\partial_v\colon T/v \longrightarrow T$ and $\partial_w\colon T/w \longrightarrow T$ be distinct outer faces of T, and assume that T has at least three vertices. Then the outer faces $\partial_w\colon (T/v)/w \longrightarrow T/v$ and $\partial_v\colon (T/w)/v \longrightarrow T/w$ also exist, $(T/v)/w = (T/w)/v$, and the following diagram commutes:

$$\begin{array}{ccc} (T/v)/w & \xrightarrow{\partial_w} & T/v \\ {\scriptstyle \partial_v}\downarrow & & \downarrow{\scriptstyle \partial_v} \\ T/w & \xrightarrow{\partial_w} & T. \end{array}$$

In case that T has only two vertices, there is a similar commutative diagram involving the inclusion of η into the n-th corolla.

The last remaining case is when we compose an inner face with an outer one in any order. There are several possibilities and in all of them we suppose that $\partial_v\colon T/v \longrightarrow T$ is an outer face and $\partial_e\colon T/e \longrightarrow T$ is an inner face.

- If in T the edge e is not adjacent to the vertex v, then the outer face $\partial_v\colon (T/e)/v \longrightarrow T/e$ and the inner face $\partial_e\colon (T/v)/e \longrightarrow T/v$ exist, $(T/e)/v = (T/v)/e$, and the following diagram commutes:

$$\begin{array}{ccc} (T/v)/e & \xrightarrow{\partial_e} & T/v \\ {\scriptstyle \partial_v}\downarrow & & \downarrow{\scriptstyle \partial_v} \\ T/e & \xrightarrow{\partial_e} & T. \end{array}$$

- Suppose that in T the inner edge e is adjacent to the vertex v and denote the other adjacent vertex to e by w. Observe that v and w contribute a vertex $v \circ_e w$ or $w \circ_e v$ to T/e. Let us denote this vertex by z. Then the outer face $\partial_z\colon (T/e)/z \longrightarrow T/e$ exists if and only if the outer face $\partial_w\colon (T/v)/w \longrightarrow T/v$ exists, and in this case $(T/e)/z = (T/v)/w$. Moreover, the following diagram commutes:

$$\begin{array}{ccccc} (T/v)/w & = & (T/e)/z & \xrightarrow{\partial_z} & T/e \\ {\scriptstyle \partial_w}\downarrow & & & & \downarrow{\scriptstyle \partial_e} \\ T/v & & \xrightarrow{\qquad\partial_v\qquad} & & T. \end{array}$$

  It follows that we can write $\partial_v\partial_w = \partial_e\partial_z$, where $z = v \circ_e w$ if v is 'closer' to the root of T or $z = w \circ_e v$ if w is 'closer' to the root of T.

Elementary degeneracy relations

Let $\sigma_v\colon T \longrightarrow T\backslash v$ and $\sigma_w\colon T \longrightarrow T\backslash w$ be two degeneracies of T. Then the degeneracies $\sigma_v\colon T\backslash w \longrightarrow (T\backslash w)\backslash v$ and $\sigma_w\colon T\backslash v \longrightarrow (T\backslash v)\backslash w$ exist, we have

$(T\backslash v)\backslash w = (T\backslash w)\backslash v$, and the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\sigma_v} & T\backslash v \\ {\scriptstyle \sigma_w}\downarrow & & \downarrow{\scriptstyle \sigma_w} \\ T\backslash w & \xrightarrow{\sigma_v} & (T\backslash v)\backslash w. \end{array}$$

Combined relations

Let $\sigma_v \colon T \longrightarrow T\backslash v$ be a degeneracy and $\partial \colon T' \longrightarrow T$ be a face map such that $\sigma_v \colon T' \longrightarrow T'\backslash v$ makes sense (i.e., T' still contains v and its two adjacent edges as a subtree). Then there exists an induced face map $\partial \colon T'\backslash v \longrightarrow T\backslash v$ determined by the same vertex or edge as $\partial \colon T' \longrightarrow T$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\sigma_v} & T\backslash v \\ {\scriptstyle \partial}\uparrow & & \uparrow{\scriptstyle \partial} \\ T' & \xrightarrow{\sigma_v} & T'\backslash v. \end{array}$$

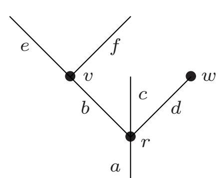
Let $\sigma_v \colon T \longrightarrow T\backslash v$ be a degeneracy and $\partial \colon T' \longrightarrow T$ be a face map induced by one of the adjacent edges to v or the removal of v, if that is possible. It follows that $T' = T\backslash v$ and the composition

$$T\backslash v \xrightarrow{\partial} T \xrightarrow{\sigma_v} T\backslash v$$

is the identity map $\mathrm{id}_{T\backslash v}$.

2.3 Non-planar trees

Any non-planar tree T generates a (symmetric) coloured operad $\Omega(T)$. Similarly as in the case of planar trees, the set of colours of $\Omega(T)$ is the set of edges $E(T)$ of T. The operations are generated by the vertices of the tree, and the symmetric group on n letters Σ_n acts on each operation with n inputs by permuting the order of its inputs. Each vertex v of the tree with output edge e and a numbering of its input edges $e_1, \ldots, e_n$ defines an operation $v \in \Omega(e_1, \ldots, e_n; e)$. The other operations are the unit operations and the operations obtained by compositions and the action of the symmetric group. For example, consider the tree



T

The operad $\Omega(T)$ has six colours a, b, c, d, e, and f. The generating operations are the same as the generating operations of $\Omega_p(T)$. All the operations of $\Omega_p(T)$ are operations of $\Omega(T)$, but there are more operations in $\Omega(T)$ obtained by the action of the symmetric group. For example if σ is the transposition of two elements of Σ_2, we have an operation $v \circ \sigma \in \Omega(f, e; b)$. Similarly if σ is the transposition of Σ_3 that interchanges the first and third elements, then there is an operation $r \circ \sigma \in \Omega(d, c, b; a)$.

More formally, if T is any tree, then $\Omega(T) = \Sigma(\Omega_p(\overline{T}))$, where $\overline{T}$ is a planar representative of T. In fact, a choice of a planar structure on T is precisely a choice of generators for $\Omega(T)$.

Definition 2.3.1. The *category of rooted trees* Ω is the full subcategory of the category of coloured operads whose objects are $\Omega(T)$ for any tree T.

We can view Ω as the category whose objects are rooted trees. The set of morphisms from a tree S to a tree T is given by the set of coloured operad maps from $\Omega(S)$ to $\Omega(T)$. Note that any morphism $S \longrightarrow T$ in Ω is completely determined by its effect on the colours (i.e., edges).

The morphisms in Ω are generated by faces and degeneracies (as in the planar case) and also by (non-planar) isomorphisms.

Lemma 2.3.2. *Any arrow $f\colon S \longrightarrow T$ in Ω decomposes as*

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \big\downarrow{\scriptstyle\sigma} & & \big\uparrow{\scriptstyle\delta} \\ S' & \xrightarrow{\varphi} & T' \end{array}$$

where $\sigma\colon S \longrightarrow S'$ is a composition of degeneracy maps, $\varphi\colon S' \longrightarrow T'$ is an isomorphism, and $\delta\colon T' \longrightarrow T$ is a composition of face maps.

Proof. We proceed by induction on the sum of the number of vertices of S and T. If T and S have no vertices, then $T = S = \eta$ and f is the identity. Note that, without loss of generality, we can assume that f sends the root of S to the root of T; otherwise we can factor it as a map $S \longrightarrow T'$ that preserves the root followed by a map $T' \longrightarrow T$ that is a composition of outer faces. Also, we can assume that f is an epimorphism on the leaves since, if this is not the case, f factors as $S \longrightarrow T/v \xrightarrow{\partial_v} T$, where v is the vertex below the leaf in T that is not in the image of f.

If a and b are edges of S such that $f(a) = f(b)$, then a and b must be on the same (linear) branch of S and f sends intermediate vertices to identities.

Since f is a map of coloured operads, we can factor it in a unique way as a surjection followed by an injection on the colours. This corresponds to a factorization in Ω,

$$S \xrightarrow{\psi} S' \xrightarrow{\xi} T,$$

where ψ is a composition of degeneracies and ξ is bijective on leaves, sends the root of S' to the root of T, and is injective on the colours (by the previous observations).

If ξ is surjective on colours, then ξ is an isomorphism. If ξ is not surjective, then there is an edge e in T not in the image of ξ. Since e is an internal edge (not a leaf), ξ factors as

$$S' \xrightarrow{\xi'} T/e \xrightarrow{\partial_e} T.$$

Now we continue by induction on the map ξ'. □

In general, limits and colimits do not exist in the category Ω; for example, Ω lacks sums and products. However, certain pushouts do exist in Ω, as expressed in the following lemma:

Lemma 2.3.3. *Let $f\colon R \twoheadrightarrow S$ and $g\colon R \twoheadrightarrow T$ be two surjective maps in Ω. Then the pushout*

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ {\scriptstyle g}\downarrow & & \vdots \\ T & \cdots\to & P \end{array}$$

exists in Ω.

Proof. The maps f and g can each be written as a composition of an isomorphism and a sequence of degeneracy maps by Lemma 2.3.2. Since pushout squares can be pasted together to get larger pushout squares, it thus suffices to prove the lemma in the case where f and g are degeneracy maps given by unary vertices v and w in R, i.e., $f\colon R \longrightarrow S$ is $\sigma_v\colon R \longrightarrow R\backslash v$ and $g\colon R \longrightarrow T$ is $\sigma_w\colon R \longrightarrow R\backslash w$. If $v = w$, then the following diagram is a pushout:

$$\begin{array}{ccc} R & \xrightarrow{\sigma_v} & R\backslash v \\ {\scriptstyle \sigma_v}\downarrow & & \| \\ R\backslash v & = & R\backslash v. \end{array}$$

If $v \neq w$, then the commutative square

$$\begin{array}{ccc} R & \xrightarrow{\sigma_v} & R\backslash v \\ {\scriptstyle \sigma_w}\downarrow & & \downarrow{\scriptstyle \sigma_w} \\ R\backslash w & \xrightarrow{\sigma_v} & (R\backslash v)\backslash w = (R\backslash w)\backslash v \end{array}$$

is also a pushout, as one easily checks. □

2.3.1 Dendroidal identities with isomorphisms

The dendroidal identities for the category Ω are the same as for the category Ω_p plus some more relations involving the isomorphisms in Ω. As an example, we give the following relation, that involves inner faces and isomorphisms. Let T be a tree with an inner edge a and let $f\colon T \longrightarrow T'$ be a (non-planar) isomorphism. Then the trees T/a and T'/b exist, where $b = f(a)$, the map f restricts to an isomorphism $f\colon T/a \longrightarrow T'/b$, and the following diagram commutes:

$$\begin{array}{ccc} T/a & \xrightarrow{\partial_a} & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ T'/b & \xrightarrow{\partial_b} & T'. \end{array}$$

Similar relations hold for outer faces and degeneracies.

2.3.2 Isomorphisms along faces and degeneracies

For any tree T in Ω, let $P(T)$ be the set of planar structures of T. Note that $P(T) \neq \emptyset$ for every tree T. Thus, the category Ω is equivalent to the category Ω' whose objects are planar trees, i.e., pairs (T, p) where T is an object of Ω and $p \in P(T)$, and whose morphisms are given by

$$\Omega'((T, p), (T', p')) = \Omega(T, T').$$

A morphism $\varphi\colon (T, p) \longrightarrow (T', p')$ in Ω' is called *planar* if, when we pull back the planar structure p' on T' to one on T along φ, then it coincides with p. Using this equivalent formulation of Ω, the category Ω_p is then the subcategory of Ω consisting of the same objects and planar maps only, i.e., compositions of faces and degeneracies. In Ω_p, the only automorphisms are identities.

If $\delta\colon T \rightarrowtail S$ is a composition of faces and $\alpha\colon S \longrightarrow S'$ is an isomorphism, there is a factorization

$$\begin{array}{ccc} T & \overset{\delta}{\rightarrowtail} & S \\ {\scriptstyle \alpha'}\,\vdots\,\wr & & \wr\downarrow{\scriptstyle \alpha} \\ T' & \overset{\delta'}{\rightarrowtail} & S', \end{array}$$

where δ' is again a composition of faces and α' is an isomorphism. This factorization is unique if one fixes some conventions, e.g., one takes the objects of Ω to be *planar* trees, and takes faces and degeneracies to be planar maps. Similarly, isomorphisms can be pushed forward and pulled back along a composition of degeneracies. Let $\sigma\colon T \longrightarrow S$ be a composition of degeneracies and $\alpha\colon S \longrightarrow S'$ and

$\beta\colon T \longrightarrow T'$ be two isomorphisms. Then there are factorizations

$$\begin{array}{ccc} T & \xrightarrow{\ \sigma\ } & S \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ T' & \xrightarrow{\ \sigma'\ } & S' \end{array} \qquad\qquad \begin{array}{ccc} T & \xrightarrow{\ \sigma\ } & S \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \beta'} \\ T' & \xrightarrow{\ \sigma''\ } & S' \end{array}$$

where α' and β' are isomorphisms and σ' and σ'' are compositions of degeneracies.

Thus, any arrow in Ω can be written in the form $\delta\sigma\alpha$ or $\delta\alpha\sigma$ with δ a composition of faces, σ a composition of degeneracies, and α an isomorphism.

2.3.3 The presheaf of planar structures

Let $P\colon \Omega^{\mathrm{op}} \longrightarrow \mathit{Sets}$ be the presheaf on Ω that sends each tree to its set of planar structures. Observe that $P(T)$ is a torsor under $\mathrm{Aut}(T)$ for every tree T, where $\mathrm{Aut}(T)$ denotes the set of automorphisms of T. Recall that the category of elements Ω/P is the category whose objects are pairs (T, x) with $x \in P(T)$. A morphism between two objects (T, x) and (S, y) is given by a morphism $f\colon T \longrightarrow S$ in Ω such that $P(f)(y) = x$. Hence, $\Omega/P = \Omega_p$ and we have a projection $v\colon \Omega_p \longrightarrow \Omega$. There is a commutative triangle

$$\begin{array}{ccc} \Delta & \xrightarrow{\ u\ } & \Omega_p \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle v} \\ & & \Omega, \end{array}$$

where i is the fully faithful embedding of Δ into Ω which sends the object $[n]$ in Δ to the linear tree L_n with n vertices and $n+1$ edges for every $n \geq 0$.

2.3.4 Relation with the simplicial category

We have seen that both the categories Ω and Ω_p extend the category Δ, by viewing the objects of Δ as linear trees. In fact, it is possible to obtain Δ as a comma category of Ω or of Ω_p as follows.

Let η be the tree in Ω consisting of no vertices and one edge, and let η_p be the planar representative of η in Ω_p. If T is any tree in Ω, then $\Omega(T, \eta)$ consists of only one morphism if T is a linear tree, or it is the empty set otherwise. The same happens for Ω_p and η_p. Thus, $\Omega/\eta = \Omega_p/\eta_p = \Delta$.