Lecture 6

Examples of derived algebraic stacks

In this last lecture, we present examples of derived algebraic stacks.

6.1 The derived moduli space of local systems

We come back to the example that we presented in the first lecture, namely the moduli problem of linear representations of a discrete group. We will now reconsider it from the point of view of derived algebraic geometry. We will try to treat this example in some detail, as we think it is a rather simple, but interesting, example of a derived algebraic stack.

A linear representation of a group G can also be interpreted as a local system on the space BG. We will therefore study the moduli problem from this topological point of view. We fix a finite CW-complex X and we are going to define a derived stack $\mathbb{R}\mathrm{Loc}(X)$ classifying local systems on X. We will see that this stack is an algebraic derived 1-stack and we will describe its higher tangent spaces in terms of cohomology groups of X. When $X = BG$ for a discrete group G, the derived algebraic stack $\mathbb{R}\mathrm{Loc}(X)$ is the *correct moduli space* of linear representations of G.

We start by considering the non-derived algebraic 1-stack **Vect** classifying projective modules of finite type. By definition, **Vect** sends a commutative ring A to the nerve of the groupoid of projective A-modules of finite type. The stack **Vect** is a 1-stack. It is easy to see that **Vect** is an algebraic 1-stack. Indeed, we have a decomposition

$$\mathbf{Vect} \simeq \coprod_n \mathbf{Vect}_n,$$

where $\mathbf{Vect}_n \subset \mathbf{Vect}$ is the substack of projective modules of rank n (recall that a projective A-module of finite type M is of rank n if, for any field K and any

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morphism $A \longrightarrow K$, the K-vector space $M \otimes_A K$ is of dimension n). It is therefore enough to prove that $\mathbf{Vect}_n$ is an algebraic 1-stack. This last statement will itself follow from the identification

$$\mathbf{Vect}_n \simeq [*/Gl_n] = BGl_n,$$

where Gl_n is the affine group scheme sending A to $Gl_n(A)$. In order to prove that $\mathbf{Vect}_n \simeq BGl_n$, we construct a morphism of simplicial presheaves

$$BGl_n \longrightarrow \mathbf{Vect}_n$$

by sending the base point of BGl_n to the trivial projective module of rank n. For a given commutative ring A, the morphism

$$BGl_n(A) \longrightarrow \mathbf{Vect}_n(A)$$

sends the base point to A^n and identifies $Gl_n(A)$ with the automorphism group of A^n. The claim is that the morphism $BGl_n \longrightarrow \mathbf{Vect}_n$ is a local equivalence of simplicial presheaves. As, by construction, this morphism induces isomorphisms on all higher homotopy sheaves, it only remains to show that it induces an isomorphism on the sheaves π_0. But this in turn follows from the fact that $\pi_0(\mathbf{Vect}_n) \simeq *$, because any projective A-module of finite type is locally free for the Zariski topology on $\operatorname{Spec} A$.

The algebraic stack $\mathbf{Vect}$ is now considered as an algebraic derived stack using the inclusion functor $j\colon \mathrm{Ho}(sPr(\mathit{Aff})) \longrightarrow \mathrm{Ho}(d\mathit{Aff}^{\sim})$. We consider a fibrant model $F \in d\mathit{Aff}^{\sim}$ for $j(\mathbf{Vect})$, and we define a new simplicial presheaf

$$\mathbb{R}\mathrm{Loc}(X)\colon d\mathit{Aff}^{\mathrm{op}} \longrightarrow sSet$$

which sends $A \in sComm$ to $\mathrm{Map}(X, |F(A)|)$, the simplicial set of continuous maps from X to $|F(A)|$.

Definition 6.1.1. The derived stack $\mathbb{R}\mathrm{Loc}(X)$ defined above is called the *derived moduli stack of local systems* on X.

We will now describe some basic properties of the derived stack $\mathbb{R}\mathrm{Loc}(X)$. We start by a description of its classical part $h^0(\mathbb{R}\mathrm{Loc}(X))$, which will show that it does classify local systems on X. We will then show that $\mathbb{R}\mathrm{Loc}(X)$ is an algebraic derived stack locally of finite presentation over $\operatorname{Spec} \mathbb{Z}$, and that it can be written as

$$\mathbb{R}\mathrm{Loc}(X) \simeq \coprod_n \mathbb{R}\mathrm{Loc}_n(X)$$

where $\mathbb{R}\mathrm{Loc}_n(X)$ is the part classifiying local systems of rank n and is itself strongly of finite type. Finally, we will compute its tangent spaces in terms of the cohomology of X.

For $A \in Comm$, note that $h^0(\mathbb{R}\mathrm{Loc}(X))(A)$ is by definition the simplicial set $\mathrm{Map}(X, |F(A)|)$. Now, $F(A)$ is a fibrant model for $j(\mathbf{Vect})(A) \simeq \mathbf{Vect}(A)$, and so it is equivalent to the nerve of the groupoid of projective A-modules of finite rank. The simplicial set $\mathrm{Map}(X, |F(A)|)$ is then naturally equivalent to the nerve of the groupoid of functors $\mathrm{Fun}(\Pi_1(X), F(A))$ from the fundamental groupoid of X to $F(A)$. This last groupoid is in turn equivalent to the groupoid of local systems of projective A-modules of finite type on the space X. Thus, we see that $h^0(\mathbb{R}\mathrm{Loc}(X))(A)$ is naturally equivalent to the nerve of the groupoid of local systems of projective A-modules of finite type on the space X. We thus have the following properties:

1. The set $\pi_0(h^0(\mathbb{R}\mathrm{Loc}(X))(A))$ is functorially in bijection with the set of isomorphism classes of local systems of projective A-modules of finite type on X. In particular, when A is a field this is also the set of local systems of finite-dimensional vector spaces over X.

2. For a local system $E \in \pi_0(h^0(\mathbb{R}\mathrm{Loc}(X))(A))$, we have

$$\pi_1(h^0(\mathbb{R}\mathrm{Loc}(X))(A), E) = \mathrm{Aut}(E),$$

   the automorphism group of E as a sheaf of A-modules on X.

3. For all $i > 1$ and all $E \in \pi_0(h^0(\mathbb{R}\mathrm{Loc}(X))(A))$, we have

$$\pi_i(h^0(\mathbb{R}\mathrm{Loc}(X))(A), E) = 0.$$

Let us explain now why the derived stack $\mathbb{R}\mathrm{Loc}(X)$ is algebraic. We start with the trivial case where X is a contractible space. Then, by definition, we have $\mathbb{R}\mathrm{Loc}(X) \simeq \mathbb{R}\mathrm{Loc}(*) \simeq j(\mathbf{Vect})$. As we already know that $j(\mathbf{Vect})$ is an algebraic stack, this implies that $\mathbb{R}\mathrm{Loc}(X)$ is an algebraic derived stack when X is contractible.

The next step is to prove that $\mathbb{R}\mathrm{Loc}(S^n)$ is algebraic for any $n \geq 0$. This can be seen by induction on n. The case $n = 0$ is obvious. Moreover, for any $n > 0$ we have a homotopy pushout of topological spaces

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & S^n, \end{array}$$

where D^n is the n-dimensional ball. This implies the existence of a homotopy pullback diagram of derived stacks

$$\begin{array}{ccc} \mathbb{R}\mathrm{Loc}(S^n) & \longrightarrow & \mathbb{R}\mathrm{Loc}(D^n) \\ \downarrow & & \downarrow \\ \mathbb{R}\mathrm{Loc}(D^n) & \longrightarrow & \mathbb{R}\mathrm{Loc}(S^{n-1}). \end{array}$$

By induction on n and by what we have just seen, the derived stacks $\mathbb{R}\mathrm{Loc}(D^n)$ and $\mathbb{R}\mathrm{Loc}(S^{n-1})$ are algebraic. By the stability of algebraic derived stacks by homotopy pullbacks, we deduce that $\mathbb{R}\mathrm{Loc}(S^n)$ is an algebraic derived stack.

We are now ready to show that $\mathbb{R}\mathrm{Loc}(X)$ is algebraic. We write X_k to denote the k-th skeleton of X. Since X is a finite CW-complex, there is an n such that $X = X_n$. Moreover, for any k there exists a homotopy pushout diagram of topological spaces

$$\begin{array}{ccc} \coprod S^{k-1} & \longrightarrow & \coprod D^k \\ \downarrow & & \downarrow \\ X_{k-1} & \longrightarrow & X_k, \end{array}$$

where the disjoint unions are finite. This implies that we have a homotopy pullback square of derived stacks

$$\begin{array}{ccc} \mathbb{R}\mathrm{Loc}(X_k) & \longrightarrow & \mathbb{R}\mathrm{Loc}(X_{k-1}) \\ \downarrow & & \downarrow \\ \prod^h \mathbb{R}\mathrm{Loc}(D^k) & \longrightarrow & \prod^h \mathbb{R}\mathrm{Loc}(S^{k-1}). \end{array}$$

By the stability of algebraic derived stacks by finite homotopy limits, we deduce that $\mathbb{R}\mathrm{Loc}(X_k)$ is algebraic by induction on k (the case $k = 0$ being clear, as $\mathbb{R}\mathrm{Loc}(X_0)$ is a finite product of $\mathbb{R}\mathrm{Loc}(*)$).

To finish the study of this example, we will compute the higher tangent spaces of the derived stack $\mathbb{R}\mathrm{Loc}(X)$. We let A be a commutative algebra and consider the natural morphism

$$\mathbb{R}\mathrm{Loc}(*)(A \oplus A[i]) \longrightarrow \mathbb{R}\mathrm{Loc}(*)(A).$$

This morphism has a natural section and its homotopy fiber at an A-module E is equivalent to $K(\mathrm{End}(E), i+1)$. It is therefore naturally equivalent to

$$[K(\mathrm{End}(-), i+1)/\mathrm{Vect}(A)] \longrightarrow N(\mathrm{Vect}(A)),$$

where $\mathrm{Vect}(A)$ is the groupoid of projective A-modules of finite type, $N(\mathrm{Vect}(A))$ is its nerve, and $[K(\mathrm{End}(-), i+1)/\mathrm{Vect}(A)]$ is the homotopy colimit of the simplicial presheaf $\mathrm{Vect}(A) \longrightarrow sSet$ sending E to $K(\mathrm{End}(E), i+1)$ —this is a general fact: for any simplicial presheaf $F\colon I \longrightarrow sSet$ we have a natural morphism $\mathrm{Hocolim}_I F \longrightarrow N(I) \simeq \mathrm{Hocolim}_I(*)$. We consider the geometric realization of this morphism to get a map of topological spaces

$$|[K(\mathrm{End}(-), i+1)/\mathrm{Vect}(A)]| \longrightarrow |N(\mathrm{Vect}(A))|,$$

which is equivalent to the geometric realization of

$$\mathbb{R}\mathrm{Loc}(*)(A \oplus A[i]) \longrightarrow \mathbb{R}\mathrm{Loc}(*)(A).$$

We take the image of this morphism by $\mathrm{Map}(X, -)$ to get

$$\mathbb{R}\mathrm{Loc}(X)(A \oplus A[i]) \simeq \mathrm{Map}(X, \mathbb{R}\mathrm{Loc}(*)(A \oplus A[i])) \longrightarrow \mathrm{Map}(X, \mathbb{R}\mathrm{Loc}(*)(A)) \simeq \mathbb{R}\mathrm{Loc}(X)(A).$$

This implies that the morphism

$$\mathbb{R}\mathrm{Loc}(*)(A \oplus A[i]) \longrightarrow \mathbb{R}\mathrm{Loc}(*)(A)$$

is equivalent to the morphism

$$\mathrm{Map}(X, |[K(\mathrm{End}(-), i+1)/\mathrm{Vect}(A)]|) \longrightarrow \mathrm{Map}(X, |N(\mathrm{Vect}(A))|).$$

A morphism $X \longrightarrow |N(\mathrm{Vect}(A))|$ corresponds to a local system E of projective A-modules of finite type on X. The homotopy fiber of the above morphism at E is then equivalent to the simplicial set of homotopy lifts of $X \longrightarrow |N(\mathrm{Vect}(A))|$ to a morphism $X \longrightarrow |[K(\mathrm{End}(-), i+1)/\mathrm{Vect}(A)]|$. This simplicial set is in turn naturally equivalent to $DK(C^*(X, \mathrm{End}(E))[i+1])$, the simplicial set obtained from the complex $C^*(X, \mathrm{End}(E))[i+1]$ by the Dold–Kan construction. Here $C^*(X, \mathrm{End}(E))$ denotes the complex of cohomology of X with coefficients in the local system $\mathrm{End}(E)$. We therefore have the following formula for the higher tangent complexes:

$$T^i_E\mathbb{R}\mathrm{Loc}(X) \simeq H^0(C^*(X, \mathrm{End}(E))[i+1]) \simeq H^{i+1}(X, \mathrm{End}(E)).$$

More generally, it is possible to prove that there is an isomorphism in $D(A)$

$$\mathbb{T}_E\mathbb{R}\mathrm{Loc}(X) \simeq C^*(X, \mathrm{End}(E))[1].$$

6.2 The derived moduli of maps

As for non-derived stacks, the homotopy category of derived stacks $\mathrm{Ho}(dAff^{\sim})$ is cartesian closed. The corresponding internal Hom will be denoted by $\mathbb{R}\underline{\mathcal{H}om}$. Note that, even though we use the same notations for the internal Homs of stacks and derived stacks, the inclusion functor

$$j : \mathrm{Ho}(sPr(Aff)) \longrightarrow \mathrm{Ho}(dAff^{\sim})$$

does not commute with them. However, we always have

$$h^0(\mathbb{R}\underline{\mathcal{H}om}(F, F')) \simeq \mathbb{R}\underline{\mathcal{H}om}(h^0(F), h^0(F'))$$

for all derived stacks F and F'. The situation is therefore very similar to the case of homotopy pullbacks.

We have just seen an example of a derived stack constructed as an internal Hom between two stacks. Indeed, if we use again the notations of the last example, we have

$$\mathbb{R}\mathrm{Loc}(X) \simeq \mathbb{R}\underline{\mathcal{H}om}(K, \mathbf{Vect}),$$

where $K = S_*(X)$ is the singular simplicial set of X.

We now consider another example. Let X and Y be two schemes, and assume that X is flat and proper (say over $\mathrm{Spec}\, k$ for some base ring k), and that Y is smooth over k. It is possible to prove that the derived stack

$$\mathbb{R}\underline{\mathcal{H}om}_{dAff/\mathrm{Spec}\, k}(X, Y)$$

is a derived scheme which is homotopically finitely presented over $\mathrm{Spec}\, k$. We will not sketch the argument here, as it is out of the scope of these lectures, and we refer to [HAGII] for more details. The derived scheme $\mathbb{R}\underline{\mathcal{H}om}(X, Y)$ is called the *derived moduli space of maps* from X to Y. Its classical part $h^0(\mathbb{R}\underline{\mathcal{H}om}(X, Y))$ is the usual moduli scheme of maps from X to Y, and for such a map we have

$$\mathbb{T}_f \mathbb{R}\underline{\mathcal{H}om}_{dAff/\mathrm{Spec}\, k}(X, Y) \simeq C^*(X, f^*(\mathbb{T}_Y)),$$

where all these tangent complexes are relative to $\mathrm{Spec}\, k$.

We mention here that these derived mapping spaces of maps can also be used in order to construct the so-called derived moduli of stable maps to an algebraic variety, by letting X vary in the moduli space of stable curves. We refer to [To1] for more details about this construction, and for some explanations of how Gromov–Witten theory can be extracted from this derived stack of stable maps.