Lecture 3

Algebraic stacks

In the previous lecture we introduced the notion of stacks over some site. We will now consider the more specific case of stacks over the étale site of affine schemes and introduce an important class of stacks called *algebraic stacks*. These are generalizations of schemes and algebraic spaces for which quotients by smooth actions always exist.

Throughout this lecture we will consider the category *Comm* of commutative rings and set $\mathit{Aff} = \mathit{Comm}^{\mathrm{op}}$. For $A \in \mathit{Comm}$, we denote by $\operatorname{Spec} A$ the corresponding object in *Aff* (therefore "Spec" is a formal notation here). We endow *Aff* with the étale topology defined as follows. Recall that a morphism of commutative rings $A \longrightarrow B$ is *étale* if it satisfies the following three conditions:

1. B is flat as an A-module.
2. B is finitely presented as a commutative A-algebra; that is, of the form $A[T_1, \dots, T_n]/(P_1, \dots, P_r)$.
3. B is flat as a $B \otimes_A B$-module.

There exist several equivalent characterizations of étale morphisms (see e.g. [SGA1]); for instance, the third condition can be equivalently replaced by the condition $\Omega^1_{B/A} = 0$, where $\Omega^1_{B/A}$ is the B-module of relative Kähler derivations (corepresenting the functor sending a B-module M to the set of A-linear derivations on B with coefficients in M). Étale morphisms are stable under base change and composition in *Aff*, i.e., by cobase change and composition in *Comm*. Geometrically, an étale morphism $A \longrightarrow B$ should be thought of as a "local isomorphism" of schemes $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$, though here *local* should not be understood in the sense of the Zariski topology.

Now, a family of morphisms $\{A \longrightarrow A_i\}_{i \in I}$ is an étale covering if each morphism $A \longrightarrow A_i$ is étale and if the family of base-change functors

$$- \otimes_A A_i \colon A\text{-}\mathit{Mod} \longrightarrow A_i\text{-}\mathit{Mod}$$

I. Moerdijk and B. Toën, *Simplicial Methods for Operads and Algebraic Geometry*, Advanced Courses in Mathematics - CRM Barcelona, DOI 10.1007/978-3-0348-0052-5_11,

is conservative. This defines a topology on *Aff* by defining that a sieve on $\operatorname{Spec} A$ is a covering sieve if it is generated by an étale covering family.

Finally, a morphism $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is a Zariski open immersion if it is étale and a monomorphism (this is equivalent to imposing that the natural morphism $B \otimes_A B \longrightarrow B$ is an isomorphism, or equivalently that the forgetful functor $B\text{-}\mathit{Mod} \longrightarrow A\text{-}\mathit{Mod}$ is fully faithful).

3.1 Schemes and algebraic n-stacks

We start by the definition of schemes and then define algebraic n-stacks as certain succesive quotients of schemes.

For $\operatorname{Spec} A \in \mathit{Aff}$, we can consider the presheaf represented by $\operatorname{Spec} A$,

$$\operatorname{Spec} A\colon \mathit{Aff}^{\mathrm{op}} = \mathit{Comm} \longrightarrow \mathit{Set},$$

by setting $(\operatorname{Spec} A)(B) = \operatorname{Hom}(A, B)$. A standard result of commutative algebra (faithfully flat descent) states that the presheaf $\operatorname{Spec} A$ is always a sheaf. We thus consider $\operatorname{Spec} A$ as a stack and as an object in $\operatorname{Ho}(\mathit{sPr}(\mathit{Aff}))$. This defines a fully faithful functor

$$\mathit{Aff} \longrightarrow \operatorname{Ho}(\mathit{sPr}(\mathit{Aff})).$$

Any object in $\operatorname{Ho}(\mathit{sPr}(\mathit{Aff}))$ isomorphic to a sheaf of the form $\operatorname{Spec} A$ will be called an *affine scheme*. The full subcategory of $\operatorname{Ho}(\mathit{sPr}(\mathit{Aff}))$ consisting of affine schemes is equivalent to $\mathit{Aff} = \mathit{Comm}^{\mathrm{op}}$, and these two categories will be implicitly identified.

Definition 3.1.1. 1. Let $\operatorname{Spec} A$ be an affine scheme, F a stack and $i\colon F \longrightarrow \operatorname{Spec} A$ a morphism. We say that i is a *Zariski open immersion* (or simply an *open immersion*) if it satisfies the following two conditions:

(a) The stack F is a sheaf (i.e., 0-truncated) and the morphism i is a monomorphism of sheaves.

(b) There exists a family of Zariski open immersions $\{A \longrightarrow A_i\}_i$ such that F is the image of the morphism of sheaves

$$\coprod_i \operatorname{Spec} A_i \longrightarrow \operatorname{Spec} A.$$

2. A morphism of stacks $F \longrightarrow F'$ is a *Zariski open immersion* (or simply an *open immersion*) if, for any affine scheme $\operatorname{Spec} A$ and any morphism $\operatorname{Spec} A \longrightarrow F'$, the induced morphism

$$F \times^h_{F'} \operatorname{Spec} A \longrightarrow \operatorname{Spec} A$$

is a Zariski open immersion in the above sense.

3. A stack F is a *scheme* if there exists a family of affine schemes $\{\operatorname{Spec} A_i\}_i$ and Zariski open immersions $\operatorname{Spec} A_i \longrightarrow F$ such that the induced morphism of sheaves
$$\coprod_i \operatorname{Spec} A_i \longrightarrow F$$
is an epimorphism. Such a family of morphisms $\{\operatorname{Spec} A_i \longrightarrow F\}$ will be called a *Zariski atlas* for F.

Exercise 3.1.2. 1. Show that any Zariski open immersion $F \longrightarrow F'$ is a monomorphism of stacks.

2. Deduce from this fact that a scheme F is always 0-truncated, and thus equivalent to a sheaf.

We now pass to the definition of algebraic stacks. These are stacks obtained by gluing schemes along smooth quotients, and we first need to recall the notion of smooth morphisms of schemes.

Recall that a morphism of commutative rings $A \longrightarrow B$ is *smooth* if it is flat of finite presentation and if moreover B is of finite Tor dimension as a $B \otimes_A B$-module. Smooth morphisms are the algebraic analog of submersions, and there exist equivalent definitions making this analogy more clear (see [SGA1]). Smooth morphisms are stable under composition and base change in *Aff*. The notion of smooth morphisms can be extended to a notion for all schemes in the following way. We say that a morphism of schemes $X \longrightarrow Y$ is *smooth* if there exist Zariski atlases $\{\operatorname{Spec} A_i \longrightarrow X\}$ and $\{\operatorname{Spec} A_j \longrightarrow Y\}$ together with commutative squares
$$\begin{tikzcd} X \arrow[r] & Y \\ \operatorname{Spec} A_i \arrow[u] \arrow[r] & \operatorname{Spec} A_j, \arrow[u] \end{tikzcd}$$
with $\operatorname{Spec} A_i \longrightarrow \operatorname{Spec} A_j$ a smooth morphism —here j depends on i. Again, smooth morphisms of schemes are stable under composition and base change.

We are now ready to define the notion of algebraic stack. The definition is by induction on an algebraicity index n representing the number of successive smooth quotients we take. This index will be forgotten after the definition is achieved.

Definition 3.1.3. 1. A stack F is 0-*algebraic* if it is a scheme.

2. A morphism of stacks $F \longrightarrow F'$ is 0-*algebraic* (or 0-*representable*) if, for any scheme X and any morphism $X \longrightarrow F'$, the stack $F \times^h_{F'} X$ is 0-algebraic (i.e., a scheme).

3. A 0-algebraic morphism of stacks $F \longrightarrow F'$ is smooth if, for any scheme X and any morphism $X \longrightarrow F'$, the morphism of schemes $F \times^h_{F'} X \longrightarrow X$ is smooth.

4. We now let $n > 0$, and assume that the notions of $(n-1)$-algebraic stack, $(n-1)$-algebraic morphism and smooth $(n-1)$-algebraic morphism have been defined.

   (a) A stack F is *n-algebraic* if there exists a scheme X together with a smooth $(n-1)$-algebraic morphism $X \longrightarrow F$ which is an epimorphism. Such a morphism $X \longrightarrow F$ is called a *smooth n-atlas* for F.

   (b) A morphism of stacks $F \longrightarrow F'$ is *n-algebraic* (or *n-representable*) if, for any scheme X and any morphism $X \longrightarrow F'$, the stack $F \times_{F'} X$ is n-algebraic.

   (c) An n-algebraic morphism of stacks $F \longrightarrow F'$ is smooth if, for any scheme X and any morphism $X \longrightarrow F'$, there exists a smooth n-atlas $Y \longrightarrow F \times^h_{F'} X$ such that each morphism $Y \longrightarrow X$ is a smooth morphism of schemes.

5. An *algebraic stack* is a stack which is n-algebraic for some integer n. An *algebraic n-stack* is an algebraic stack which is also an n-stack. An *algebraic space* is an algebraic 0-stack.

6. A morphism of stacks $F \longrightarrow F'$ is *algebraic* (or *representable*) if it is n-algebraic for some n.

7. A morphism of stacks $F \longrightarrow F'$ is *smooth* if it is n-algebraic and smooth for some integer n.

Long, but formal arguments show that algebraic stacks satisfy the following properties:

- Algebraic stacks are stable under finite homotopy limits (i.e., by homotopy pullbacks).
- Algebraic stacks are stable under disjoint union.
- Algebraic morphisms of stacks are stable under composition and base change.
- Algebraic stacks are stable under smooth quotients. Thus, if $F \longrightarrow F'$ is a smooth epimorphism of stacks, then F' is algebraic if and only if F is so.

Exercise 3.1.4. Let F be an algebraic n-stack, $X \in \mathit{Aff}$, and $x\colon X \longrightarrow F$ a morphism of stacks. Show that the sheaf $\pi_n(F, x)$ is representable by an algebraic space, locally of finite type over X.

The standard finiteness properties of schemes can be extended to algebraic stacks in the following way:

- An algebraic morphism $F \longrightarrow F'$ is *locally of finite presentation* if, for any scheme X and any morphism $X \longrightarrow F'$, there exists a smooth atlas $Y \longrightarrow F \times^h_{F'} X$ such that the induced morphism $Y \longrightarrow X$ is locally of finite presentation.

- An algebraic morphism $F \longrightarrow F'$ is *quasi-compact* if, for any affine scheme X and any morphism $X \longrightarrow F'$, there exists a smooth atlas $Y \longrightarrow F \times^h_{F'} X$ with Y an affine scheme.
- An algebraic stack F is *strongly quasi-compact* if, for all n, the induced morphism
$$F \longrightarrow \mathbb{R}\underline{\mathcal{H}om}(\partial\Delta^n, F)$$
is quasi-compact.
- An algebraic stack morphism $F \longrightarrow F'$ is *strongly of finite presentation* if, for any affine scheme X and any morphism $X \longrightarrow F'$, the stack $F \times^h_{F'} X$ is locally of finite presentation and strongly quasi-compact.

Note that, when $n = 0$, we have $\mathbb{R}\underline{\mathcal{H}om}(\partial\Delta^n, F) \simeq F \times F$, and the condition of strongly quasi-compactness implies in particular that the diagonal morphism $F \longrightarrow F \times F$ is quasi-compact. In general, being strongly quasi-compact involves quasi-compactness conditions for all the "higher diagonals".

Exercise 3.1.5. Let X be an affine scheme and G be a sheaf of groups on Aff/X. We form the classifying stack $K(G, 1) \in \mathrm{Ho}(\mathit{sPr}(\mathit{Aff})/X)$, and consider it in $\mathrm{Ho}(\mathit{sPr}(\mathit{Aff}))$.

1. Show that, if $K(G, 1)$ is an algebraic stack, then G is represented by an algebraic space locally of finite type.
2. Conversely, if G is representable by an algebraic space which is smooth over X, then $K(G, 1)$ is an algebraic stack.
3. Assume that $K(G, 1)$ is algebraic. Show that $K(G, 1)$ is quasi-compact. Show that $K(G, 1)$ is strongly quasi-compact if and only if G is quasi-compact.

3.2 Some examples

Classifying stacks: Suppose that G is a sheaf of groups over some affine scheme X, and assume that G is an algebraic space, flat and of finite presentation over X. We can form $K(G, 1) \in \mathrm{Ho}(\mathit{sPr}(\mathit{Aff}))$, the classifying stack of the group G, as explained in §2.2. The stack $K(G, 1)$ is however not exactly the right object to consider, at least when G is not smooth over X. Indeed, for Y an affine scheme over X, $[Y, K(G, 1)]$ classifies G-torsors over Y which are locally trivial for the étale topology on Y. This is a rather unnatural condition, as there exist G-torsors, locally trivial for the flat topology on Y, which are not étale locally trivial (for instance, when $X = \operatorname{Spec} k$ is a perfect field of characteristic p, the Frobenius map $\mathrm{Fr}\colon \mathbb{G}_m \longrightarrow \mathbb{G}_m$ is a μ_p-torsor over $\mathbb{G}_m$ which is not étale locally trivial). To remedy this, we introduce a slight modification of the classifying stack $K(G, 1)$ by changing the topology in the following way. We consider the simplicial presheaf $BG\colon X \longmapsto B(G(X))$, viewed as an object in $\mathit{sPr}_{\mathrm{ffqc}}(\mathit{Aff})$, the model category of simplicial presheaves on the site of affine schemes endowed with the faithfully flat and quasi-compact topology ("ffqc" for short). Note that étale coverings are ffqc coverings, and therefore we have a natural full embedding

$$\mathrm{Ho}(sPr_{\mathrm{ffqc}}(Aff)) \subset \mathrm{Ho}(sPr(Aff)),$$

where the objects in $\mathrm{Ho}(sPr_{\mathrm{ffqc}}(Aff))$ are stacks satisfying the more restrictive descent condition for ffqc hypercoverings. We consider the simplicial presheaf $BG \in sPr(Aff)$, and denote by $K_{fl}(G,1) \in \mathrm{Ho}(sPr_{\mathrm{ffqc}}(Aff)) \subset \mathrm{Ho}(sPr(Aff))$ a fibrant replacement of BG in the model category of stacks for the ffqc topology. It is a non-trivial statement that $K_{fl}(G,1)$ is an algebraic stack (see for instance [La-Mo, Proposition 10.13.1]). Moreover, the natural morphism $K_{fl}(G,1) \longrightarrow X$ is smooth. Indeed, we choose a smooth and surjective morphism $Y \longrightarrow K_{fl}(G,1)$, with Y an affine scheme. The composition $Y \longrightarrow X$ is clearly a flat surjective morphism of finite presentation. We let $X' = Y \times^h_{K_{fl}(G,1)} X$, and consider the diagram of stacks

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[dr, "u"'] & Y \arrow[r] \arrow[d, "q"] & K_{fl}(G,1) \arrow[dl] \\ & X. & \end{tikzcd}$$

In this diagram, v is a flat surjective morphism of finite presentation, because it is the base change of the trivial section $X \longrightarrow K_{fl}(G,1)$, which is flat, surjective and of finite presentation. Moreover, u is a smooth morphism, because it is the base change of the smooth atlas $Y \longrightarrow K_{fl}(G,1)$. We conclude that the morphism q is also smooth.

Higher classifying stacks: Assume now that A is a sheaf of abelian groups over an affine scheme X which is an algebraic space, flat and of finite presentation over X. We form the simplicial presheaf $B^n(A) = B(B^{n-1}(A))$, by iterating the classifying space construction. We denote by $K_{fl}(A,n) \in \mathrm{Ho}(sPr_{\mathrm{ffqc}}(Aff)) \subset \mathrm{Ho}(sPr(Aff))$ a fibrant model for $B^n(A)$ with respect to the ffqc topology. It is again true that $K_{fl}(A,n)$ is an algebraic n-stack when $n > 1$. Indeed, $K(A,n)$ is the quotient of X by the trivial action of the group stack $K(A,n-1)$. As this group stack is algebraic and smooth for $n > 1$, the quotient stack is again an algebraic stack.

Groupoid quotients: We describe here the standard way to construct algebraic stacks using quotients by smooth groupoid actions. We start with a simplicial object in $sPr(C)$,

$$F_* \colon \Delta^{\mathrm{op}} \longrightarrow sPr(Aff).$$

We say that F_* is a *Segal groupoid* if it satisfies the following two conditions:

1. For any $n > 1$, the natural morphism

$$F_n \longrightarrow F_1 \times^h_{F_0} F_1 \times^h_{F_0} \cdots \times^h_{F_0} F_1,$$

induced by the morphism $[1] \longrightarrow [n]$ sending 0 to i and 1 to $i+1$ (for $0 \leq i < n$) is an isomorphism of stacks.

2. The natural morphism
$$F_2 \longrightarrow F_1 \times^h_{F_0} F_1$$
induced by the morphism $[1] \longrightarrow [2]$ sending 0 to 0 and 1 to 1 or 2 is an isomorphism of stacks.

Exercise 3.2.1. Let F_* be a Segal monoid object in $sPr(\mathit{Aff})$, and suppose that $F_n(X)$ is a set for all n and all X. Show that F_* is the nerve of a presheaf of groupoids on *Aff*.

We now assume that F_* is a Segal groupoid and moreover that all the face morphisms $F_1 \longrightarrow F_0$ are smooth morphisms between algebraic stacks. We consider the homotopy colimit of the diagram $[n] \longmapsto F_n$, and denote it by $|F_*| \in \mathrm{Ho}(sPr(\mathit{Aff}))$. The stack $|F_*|$ is called the *quotient stack* of the Segal groupoid F_*. It can been proved that $|F_*|$ is again an algebraic stack. Moreover, if each F_i is an algebraic n-stack, then $|F_*|$ is an algebraic $(n+1)$-stack. This is a formal way to produce higher algebraic stacks starting, say, from schemes, but this is often not the way stacks arise in practice.

An important very special case of the quotient stack construction is the case of a smooth group scheme G acting on a scheme X. In this case we form the groupoid object $B(X,G)$ whose value in degree n is $X \times G^n$, and whose transition morphisms are given by the action of G on X. This is a groupoid object in schemes and thus can be considered as a groupoid object in sheaves, and therefore as a very special kind of Segal groupoid. The quotient stack of this Segal groupoid is denoted by $[X/G]$ and is called the quotient stack of X by G. It is an algebraic 1-stack for which a natural smooth atlas is the natural projection $X \longrightarrow [X/G]$. It can be characterized by a universal property: morphisms of stacks $[X/G] \longrightarrow F$ are in one-to-one correspondence with morphisms of G-equivariant stacks $X \longrightarrow F$ (here we need to use a model category G-$sPr(\mathit{Aff})$ of G-equivariant simplicial presheaves in order to have the correct homotopy category of G-equivariant stacks).

Simplicial presentation: Algebraic stacks can also be characterized as the simplicial presheaves represented by a certain kind of simplicial schemes. For this, we let X_* be a simplicial object in the category of schemes. For any finite simplicial set K (finite here means generated by a finite number of cells), we can form X_*^K, which is the scheme of morphisms from K to X_*. It is, by definition, the equalizer of the two natural morphisms
$$\prod_{[n]} X_n^{K_n} \rightrightarrows \prod_{[p]\longrightarrow[q]} X_p^{K_q}.$$
This equalizer exists as a scheme when K is finite (because it then only involves finite limits).

A simplicial scheme X_* is then called a *weak smooth groupoid* if, for any $0 \leq k \leq n$, the natural morphism

$$X_n = X_*^{\Delta^n} \longrightarrow X_*^{\Lambda^{n,k}}$$

is a smooth and surjective morphism of schemes (surjective here has to be understood pointwise, but as the morphism is smooth this is equivalent to saying that it induces an epimorphism on the corresponding sheaves). A weak smooth groupoid X_* is moreover *n-truncated* if, for any $k > n+1$, the natural morphism

$$X_k = X_*^{\Delta^k} \longrightarrow X_*^{\partial\Delta^k}$$

is an isomorphism.

It is then possible to prove that a stack F is an algebraic n-stack if there exists an n-truncated weak smooth groupoid X_* and an isomorphism in $\mathrm{Ho}(sPr(\mathit{Aff}))$ $F \simeq X_*$. We refer to [Pr] for details.

Some famous algebraic 1-stacks: We review here two famous examples of algebraic 1-stacks, namely the stack of smooth and proper curves and the stack of vector bundles on a curve. We refer to [La-Mo] for more details.

For $X \in \mathit{Aff}$ an affine scheme, we let $\mathcal{M}_g(X)$ be the full subgroupoid of sheaves F on Aff/X such that the corresponding morphism of sheaves $F \longrightarrow X$ is representable by a smooth and proper curve of genus g over X (i.e., F is itself a scheme, and the morphism $F \longrightarrow X$ is smooth, proper, with geometric fibers being connected curves of genus g). For $Y \longrightarrow X$ in Aff, we have a restriction functor from sheaves on Aff/X to sheaves on Aff/Y, and this defines a natural functor of groupoids

$$\mathcal{M}_g(X) \longrightarrow \mathcal{M}_g(Y).$$

This defines a presheaf of groupoids on Aff, and taking the nerve of these groupoids gives a simplicial presheaf denoted by $\mathcal{M}_g$. The stack $\mathcal{M}_g$ is called the *stack of smooth curves of genus g*. It is such that, for $X \in \mathit{Aff}$, $\mathcal{M}_g(X)$ is a 1-truncated simplicial set whose π_0 is the set of isomorphism classes of smooth proper curves of genus g over X, and whose π_1 at a given curve is its automorphism group. It is a well-known theorem that $\mathcal{M}_g$ is an algebraic 1-stack which is smooth and of finite presentation over $\mathrm{Spec}\,\mathbb{Z}$. This stack is even Deligne–Mumford, that is, the diagonal morphism $\mathcal{M}_g \longrightarrow \mathcal{M}_g \times \mathcal{M}_g$ is unramified (i.e., locally a closed immersion for the étale topology). Equivalently, this means that there exists an atlas $X \longrightarrow \mathcal{M}_g$ which is étale rather than only smooth.

Another very important and famous example of an algebraic 1-stack is the stack of G-bundles on some smooth projective curve C (say, over some base field k). Let G be a smooth affine algebraic group over k. We start by considering the stack BG, which is a stack over $\mathrm{Spec}\,k$. It is the quotient stack $[\mathrm{Spec}\,k/G]$ for the trivial action of G on $\mathrm{Spec}\,k$. As G is a smooth algebraic group, this stack is an algebraic

1-stack. When C is a smooth and proper curve over $\operatorname{Spec} k$, we can consider the stack of morphisms (of stacks over $\operatorname{Spec} k$)

$$\mathrm{Bun}_G(C) = \mathbb{R}\underline{\mathcal{H}om}_{\mathit{Aff}/\operatorname{Spec} k}(C, BG),$$

which by definition is the stack of principal G-bundles on C. By definition, for $X \in \mathit{Aff}$, $\mathrm{Bun}_G(C)(X)$ is a 1-truncated simplicial set whose π_0 is the set of isomorphism classes of principal G-bundles on C and whose π_1 at a given bundle is its automorphism group. It is also a well-known theorem that the stack $\mathrm{Bun}_G(C)$ is an algebraic 1-stack, which is smooth and locally of finite presentation over $\operatorname{Spec} k$. However, this stack is not quasi-compact and is only a countable union of quasi-compact open substacks.

Higher linear stacks: Let $X = \operatorname{Spec} A$ be an affine scheme and E be a positively graded cochain complex of A-modules. We assume that E is perfect, i.e., it is quasi-isomorphic to a bounded complex of projective A-modules of finite type. We define a stack $\mathbb{V}(E)$ over X in the following way. For every commutative A-algebra B, we set

$$\mathbb{V}(E)(B) = \mathrm{Map}(E, B),$$

where Map denotes the mapping space of the model category of complexes of A-modules. More explicitly, $\mathbb{V}(E)(B)$ is the simplicial set whose set of n-simplicies is the set $\mathrm{Hom}(Q(E) \otimes_A C_*(\Delta^n, A), B)$. Here $Q(E)$ is a cofibrant resolution of E in the model category of complexes of A-modules (for the projective model structure, for which equivalences are quasi-isomorphisms and fibrations are epimorphisms), $C_*(\Delta^n, A)$ is the homology complex of the simplicial set Δ^n with coefficients in A, and the Hom is taken in the category of complexes of A-modules. In other words, $\mathbb{V}(E)(B)$ is the simplicial set obtained from the complex $\underline{\mathrm{Hom}}^*(Q(E), B)$ by the Dold–Kan correspondence. When B varies in the category of commutative A-algebras, this defines a simplicial presheaf $\mathbb{V}(E)$ together with a morphism $\mathbb{V}(E) \longrightarrow X = \operatorname{Spec} A$. For every commutative A-algebra B, we have

$$\pi_i(\mathbb{V}(E)(B)) \simeq \mathrm{Ext}^{-i}(E, B).$$

It can be shown that the stack $\mathbb{V}(E)$ is an algebraic n-stack strongly of finite presentation over X, where n is such that $H^i(E) = 0$ for all $i > n$, and that $\mathbb{V}(E)$ is smooth if and only if the Tor amplitude of E is non-negative (i.e., E is quasi-isomorphic to a complex of projective A-modules of finite type which is moreover concentrated in non-negative degrees). For this, we can first assume that E is a bounded complex of projective modules of finite type. We then set $K = E^{\leq 0}$, the part of E which is concentrated in non-positive degrees, and we have a natural morphism of complexes $E \longrightarrow K$. This morphism induces a morphism of stacks

$$\mathbb{V}(K) \longrightarrow \mathbb{V}(E).$$

By definition, $\mathbb{V}(K)$ is naturally equivalent to the affine scheme $\operatorname{Spec} A[H^0(K)]$, where $A[H^0(K)]$ denotes the free commutative A-algebra generated by the A-module $H^0(K)$. It is well known that $\mathbb{V}(H^0(k))$ is smooth over $\operatorname{Spec} A$ if and only if $H^0(K)$ is projective and of finite type. This is equivalent to saying that E has non-negative Tor amplitude. The only thing to check is then that the natural morphism

$$\mathbb{V}(K) \longrightarrow \mathbb{V}(E)$$

is $(n-1)$-algebraic and smooth. But this follows by induction on n, as this morphism is locally on $\mathbb{V}(E)$ of the form $Y \times \mathbb{V}(L) \longrightarrow Y$, for L the homotopy cofiber (i.e., the cone) of the morphism $E \longrightarrow K$. This homotopy cofiber is itself quasi-isomorphic to $E^{>0}[1]$, and thus is a perfect complex of non-negative Tor amplitude with $H^i(L) = 0$ for $i > n-1$.

Exercise 3.2.2. Let $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[T]$ and let E be the complex of $\mathbb{Z}[T]$-modules given by

$$0 \longrightarrow \mathbb{Z}[T] \xrightarrow{\times T} \mathbb{Z}[T] \longrightarrow 0,$$

concentrated in degrees 1 and 2. Show that $\mathbb{V}(E)$ is an algebraic 2-stack such that the sheaf $\pi_1(\mathbb{V}(E))$ is not representable by any affine scheme (it is in fact not representable by any algebraic space).

The algebraic 2-stack of abelian categories: This is a non-trivial example of an algebraic 2-stack. The material is taken from [An]. For a commutative ring A, we consider the following category *Ab*(A). Its objects are abelian A-linear categories which are equivalent to the category R-*Mod* of left R-modules for some associative A-algebra R which is projective and of finite type as an A-module. The morphisms in *Ab*(A) are the A-linear equivalences of categories. For a morphism of commutative rings $A \longrightarrow B$, we have a functor

$$Ab(A) \longrightarrow Ab(B)$$

sending an abelian category $\mathcal{C}$ to $\mathcal{C}^{B/A}$, the category of B-modules in $\mathcal{C}$. Precisely $\mathcal{C}^{B/A}$ can be taken to be the category of all A-linear functors from BB, the A-linear category with a unique object and B as its A-algebra of endomorphisms, to $\mathcal{C}$. This defines a presheaf of categories $A \longmapsto Ab(A)$ on *Aff*. Taking the nerves of these categories, we obtain a simplicial presheaf $\mathbf{Ab} \in sPr(Aff)$. The simplicial presheaf $\mathbf{Ab}$ is not a stack, but we still consider it as an object in $\operatorname{Ho}(sPr(Aff))$. The main result of [An] states that $\mathbf{Ab}$ is an algebraic 2-stack which is locally of finite presentation.

The algebraic n-stack of $[n, 0]$-perfect complexes: For a commutative ring A, we consider a category $P(A)$ defined as follows. Its objects are the cofibrant complexes of A-modules (for the projective model structure) which are perfect (i.e., quasi-isomorphic to a bounded complex of projective modules of finite type). The morphisms in $P(A)$ are the quasi-isomorphisms of complexes of A-modules. For a morphism of commutative rings $A \longrightarrow B$, we have a base-change functor

$$- \otimes_A B \colon P(A) \longrightarrow P(B).$$

This does not however define, stricly speaking, a presheaf of categories, as the base-change functors are only compatible with composition up to a natural isomorphism. In other words, $A \longmapsto P(A)$ is only a weak functor from *Comm* to the 2-category of categories. Fortunately, there exists a standard procedure to replace any weak functor by an equivalent strict functor: it consists in replacing P by the presheaf of cartesian sections of the Grothendieck construction $\int P \longrightarrow Comm$ (see [SGA1]). Thus, we define a new category $P'(A)$ whose objects consist of the following data:

1. For any commutative A-algebra B, an object $E_B \in P(B)$.
2. For any commutative A-algebra B and any commutative B-algebra C, an isomorphism in $P'(C)$,

$$\phi_{B,C} \colon E_B \otimes_B C \simeq E_C.$$

We require moreover that, for any commutative A-algebra B, any commutative B-algebra C, and any commutative C-algebra D, the two possible isomorphisms

$$\phi_{C,D} \circ (\phi_{B,C} \otimes_C D) \colon (E_B \otimes_B C) \otimes_C D \simeq E_B \otimes_B D \longrightarrow E_D$$

$$\phi_{B,D} \colon E_B \otimes_B D \longrightarrow E_D$$

are equal. The morphisms in $P'(A)$ are simply taken to be families of morphisms $E_B \longrightarrow E'_B$ which commute with the collections $\phi_{B,C}$ and $\phi'_{B,C}$.

With these definitions, $A \longmapsto P'(A)$ is a functor $Comm \longrightarrow Cat$, and there is moreover an equivalence of lax functors $P' \longrightarrow P$. We compose the functor P' with the nerve construction and get a simplicial presheaf **Perf** on *Aff*. It can be proved that the simplicial presheaf **Perf** is a stack in the sense of Definition 2.1.3 (1). This is not an obvious result (see for instance [H-S] for a proof), and can be reduced to the well-known flat cohomological descent for quasi-coherent complexes. It can also be proved that, for $X = \operatorname{Spec} A \in Aff$, the simplicial set $\mathbf{Perf}(X)$ satisfies the following properties:

1. The set $\pi_0(\mathbf{Perf}(X))$ is in a natural bijection with the set of quasi-isomorphism classes of perfect complexes of A-modules.
2. For $x \in \mathbf{Perf}(X)$ corresponding to a perfect complex E, we have

$$\pi_1(\mathbf{Perf}(X), x) \simeq \operatorname{Aut}(E),$$

where the automorphism group is taken in the derived category $D(A)$ of the ring A.

3. For $x \in \mathbf{Perf}(X)$ corresponding to a perfect complex E, we have

$$\pi_i(\mathbf{Perf}(X), x) \simeq \mathrm{Ext}^{1-i}(E, E)$$

for any $i > 1$. Again, these Ext groups are computed in the triangulated category $D(A)$.

For any $n \geq 0$ and $a \leq b$ with $b - a = n$, we can define a subsimplicial presheaf $\mathbf{Perf}^{[a,b]} \subset \mathbf{Perf}$ which consists of all perfect complexes of Tor amplitude contained in the interval $[a, b]$ (i.e., complexes quasi-isomorphic to a complex of projective modules of finite type concentrated in degrees $[a, b]$). It can be proved that the substacks $\mathbf{Perf}^{[a,b]}$ form an open covering of $\mathbf{Perf}$. Moreover, $\mathbf{Perf}^{[a,b]}$ is an algebraic $(n+1)$-stack which is locally of finite presentation. This way, even though $\mathbf{Perf}$ is not, strictly speaking, an algebraic stack (because it is not an n-stack for any n), it is an increasing union of open algebraic substacks. We say that $\mathbf{Perf}$ is *locally algebraic*. The fact that $\mathbf{Perf}^{[a,b]}$ is an algebraic $(n+1)$-stack is not easy either. We refer to [To-Va] for a complete proof.

Exercise 3.2.3. 1. Show how to define a stack $\mathbf{MPerf}$ of morphisms between perfect complexes, whose value at $X \in \mathit{Aff}$ is equivalent to the nerve of the category of quasi-isomorphisms in the category of morphisms between perfect complexes over X.

2. Show that the morphism *source* and *target* define an algebraic morphism of stacks

$$\pi\colon \mathbf{MPerf} \longrightarrow \mathbf{Perf} \times \mathbf{Perf}.$$

(Here you will need the following result of homotopical algebra: If M is a model category and $\mathrm{Mor}(M)$ denotes the model category of morphisms, then the homotopy fiber of the source and target map $N(w\mathrm{Mor}(M)) \longrightarrow N(wM) \times N(wM)$, taken at a point (x, y), is naturally equivalent to the mapping space $\mathrm{Map}(x, y)$.)

3. Show that the morphism π is locally smooth near any point corresponding to a morphism $E \longrightarrow E'$ of perfect complexes such that $\mathrm{Ext}^i(E, E') = 0$ for all $i > 0$.

3.3 Coarse moduli spaces and homotopy sheaves

The purpose of this part is to show that algebraic n-stacks strongly of finite presentation can be approximated by schemes by means of some *dévissage*. The existence of this approximation has several important consequences about the behaviour of algebraic n-stacks, such as the existence of virtual coarse moduli spaces or homotopy group schemes. Conceptually, the results of this part show that algebraic n-stacks are not that far from being schemes or algebraic spaces, and that for many purposes they behave like *convergent series of schemes*.

Convention: Throughout this part, all algebraic n-stacks will be strongly of finite presentation over some affine base scheme $\mathrm{Spec}\, k$ (for k some commutative ring).

The key notion is that of *total gerbe*, whose precise definition is as follows.

Definition 3.3.1. Let F be an algebraic n-stack. We say that F is a *total (n-)gerbe* if for all $i > 0$ the natural projection

$$I_F^{(i)} = \mathbb{R}\underline{\mathrm{Hom}}(S^i, F) \longrightarrow F$$

is a flat morphism.

In the previous definition, $I_F^{(i)}$ is called the *i-th inertia stack* of F. Note that, when F is a 1-stack, $I_F^{(1)}$ is equivalent to the inertia stack of F in the usual sense. In particular, for an algebraic 1-stack, being a total gerbe in the sense of Definition 3.3.1 is equivalent to the fact that the projection morphism

$$F \times^h_{F\times F} F \longrightarrow F$$

is flat, and thus equivalent to the usual notion of gerbes for algebraic 1-stacks (see [La-Mo, Definition 3.15]).

Proposition 3.3.2. *Let F be an algebraic n-stack which is a total gerbe. Then the following conditions are satisfied:*

1. *If $M(F)$ is the sheaf associated to $\pi_0(F)$ for the flat (ffqc) topology, then $M(F)$ is represented by an algebraic space and the morphism $F \longrightarrow M(F)$ is flat and of finite presentation.*
2. *For any $X \in Aff$ and any morphism $x\colon X \longrightarrow F$, $\pi_i^{fl}(F, x)$, the sheaf on X associated to $\pi_i(F, x)$ with respect to the ffqc topology, is an algebraic space, flat, and of finite presentation over X.*

Proof: Condition 1 follows from a well-known theorem of Artin, ensuring representability by algebraic spaces of quotients of schemes by flat equivalence relations. The argument goes as follows. We choose a smooth atlas $X \longrightarrow F$ with X an affine scheme, and we let $X_1 = X \times^h_F X$. We define $R \subset X \times X$, the sub-ffqc-sheaf image of $X_1 \longrightarrow X \times X$, which defines an equivalence relation on X. Clearly, $M(F)$ is isomorphic to the quotient ffqc-sheaf $(X/R)^{fl}$. We now prove the following two properties:

1. The sheaf R is an algebraic space.
2. The two projections $R \longrightarrow X$ are smooth.

In order to prove property 1, we consider the natural projection $X_1 \longrightarrow R$. Let $x, y\colon Y \longrightarrow R \subset X \times X$ be morphisms with Y affine. Then $X_1 \times^h_R Y$ is equivalent to $\Omega_{x,y}F \simeq Y \times^h_F Y$, the stack of paths from x to y. As the objects x and y are locally (for the flat topology) equivalent on Y because $(x, y) \in R$, the stack $\Omega_{x,y}F \simeq Y \times^h_F Y$ is algebraic and locally (for the flat topology on Y) equivalent to the loop stack $\Omega_x F$, defined by the homotopy cartesian square

$$\begin{array}{ccc} \Omega_x F & \longrightarrow & I_F^{(1)} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & F. \end{array}$$

By hypothesis on F, we deduce that $X_1 \times^h_R Y \longrightarrow Y$ is flat, surjective and of finite presentation. As this is true for any $Y \longrightarrow R$, we have that the morphism of stacks $X_1 \longrightarrow R$ is surjective, flat and finitely presented. If $U \longrightarrow X_1$ is a smooth atlas, we have that the sheaf R is isomorphic to the quotient ffqc-sheaf

$$R \simeq \mathrm{Colim}\,(U \times_{X \times X} U \rightrightarrows U)\,,$$

and, by what we have just seen, the projections $U \times_{X \times X} U \longrightarrow U$ are flat and finitely presented morphisms of affine schemes. By [La-Mo, Corollary 10.4], we have that R is an algebraic space.

We now consider property 2. For this, we consider the diagram

$$U \longrightarrow R \longrightarrow X.$$

The first morphism is a flat and finitely presented cover, and the composition of the two morphisms equals the composition $U \longrightarrow X_1 \longrightarrow X$, and is thus smooth. Hence $R \longrightarrow X$ is locally (for the flat finitely presented topology on R) a smooth morphism, and therefore it is smooth. This finishes the proof of the first part of the proposition, as $X \longrightarrow M(F)$ is now a smooth atlas, showing that $M(F)$ is an algebraic space.

To prove the second statement of the proposition, we will use (1) applied to certain stacks of iterated loops. We let $x\colon X \longrightarrow F$ and consider the loop stack $\Omega_x F$ of F at x, defined by

$$\Omega_x F = X \times^h_F X.$$

In the same way, we have the iterated loop stacks

$$\Omega_x^{(i)} F = \Omega_x(\Omega_x^{(i-1)} F).$$

Note that we have homotopy cartesian squares

$$\begin{array}{ccc} \Omega_x^{(i)} F & \longrightarrow & I_F^{(i)} \\ \downarrow & & \downarrow \\ X & \longrightarrow & F, \end{array}$$

showing that $\Omega_x^{(i)}F \longrightarrow X$ is flat for any i. Moreover, for any $Y \in \mathit{Aff}$ and any $s\colon Y \longrightarrow \Omega_x^{(i)}F$, we have a homotopy cartesian square

$$\begin{tikzcd} \Omega_s^{(j)}\Omega_x^{(i)}F \arrow[r] \arrow[d] & I^{(j)}_{\Omega_x^{(i)}F} \arrow[d] \\ Y \arrow[r] & \Omega_x^{(i)}F. \end{tikzcd}$$

Now, as $\Omega_x^{(i)}F$ is a group object over X, we have isomorphisms of stacks over Y,

$$\Omega_s^{(j)}\Omega_x^{(i)}F \simeq \Omega_x^{(j+i)}F \times_X Y$$

obtained by translating along the section s. Therefore, we have that

$$I^{(j)}_{\Omega_x^{(i)}F} \longrightarrow \Omega_x^{(i)}F$$

is flat for any i and any j. We can therefore apply (1) to the stacks $\Omega_x^{(i)}F$. As we have

$$M(\Omega_x^{(i)}F) \simeq \pi_i^{fl}(F,x),$$

this gives that the sheaves $\pi_i^{fl}(F,x)$ are algebraic spaces. Moreover, the morphism $\Omega_x^{(i)}F \longrightarrow \pi_i^{fl}(F,x)$ is flat, surjective and of finite presentation, showing that so is $\pi_i^{fl}(F,x)$ as an algebraic space over X. □

Exercise 3.3.3. 1. Let $f\colon F \longrightarrow F'$ be a morphism of finite presentation between algebraic stacks strongly of finite presentation over some affine scheme. Assume that F' is reduced. Show that there exists a non-empty open substack $U \subset F'$ such that the base-change morphism $F \times^h_{F'} U \longrightarrow U$ is flat (use smooth atlases and the generic flatness theorem statement that the result is true when F and F' are affine schemes).

2. Deduce from (1) that, if F is a reduced algebraic stack strongly of finite presentation over some affine scheme, then F has a non-empty open substack $U \subset F$ which is a total gerbe in the sense of Definition 3.3.1.

The previous exercise, together with Proposition 3.3.2, has the following important consequence:

Corollary 3.3.4. *Let F be an algebraic stack strongly of finite presentation over some affine scheme X. There exists a finite sequence of closed substacks*

$$\emptyset \subset F_r \subset F_{r-1} \subset \cdots \subset F_1 \subset F_0 = F$$

such that each $F_i - F_{i+1}$ is a total gerbe. We can moreover choose the F_i with the following properties:

1. *For all i, the ffqc-sheaf $M(F_i - F_{i+1})$ is a scheme of finite type over X.*
2. *For all i, all affine schemes Y, all morphisms $y\colon Y \longrightarrow (F_i - F_{i+1})$, and all $j > 0$, the ffqc-sheaf $\pi_j(F_i - F_{i+1}, y)$ is a flat algebraic space of finite presentation over Y.*

In other words, any algebraic stack F strongly finitely presented over some affine scheme gives rise to several schemes $M(F_i - F_{i+1})$, which are stratified pieces of the non-existing coarse moduli space for F. Over each of these schemes, locally for the étale topology, we have the flat groups $\pi_j(F_i - F_{i+1})$. Therefore, up to a stratification, the stack F behaves very much like a homotopy type whose homotopy groups would be represented by schemes (or algebraic spaces).

Exercise 3.3.5. Recall that an algebraic stack is *Deligne–Mumford* if it possesses an étale atlas (rather than simply smooth).

1. Let F be an algebraic stack which is étale over an affine scheme X. Prove that F is Deligne–Mumford and that F is a total gerbe. Show also that the projection $F \longrightarrow M(F)$ is an étale morphism.
2. Let F be a Deligne–Mumford stack and $p\colon F \longrightarrow t_{\leq 1}(F)$ be its 1-truncation. Show that $t_{\leq 1}(F)$ is itself a Deligne–Mumford 1-stack and that p is étale.