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Commutant Lifting

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In fond memory of Paul

Abstract. This article is the story of how the author had the good fortune to be able to prove the primordial version of the commutant lifting theorem. The phrase "good fortune" is used advisedly. The story begins with the intersection of two lives, Paul's and the author's.

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Paul Halmos's paper [6], one of his first two in pure operator theory, spawned three major developments: the theory of subnormal operators, the theory of hyponormal operators, and the theory of unitary dilations and operator models. Several of the articles in this volume, including this one, concern these developments; they illustrate how Paul's original ideas in [6] grew into major branches of operator theory. The present article belongs to the realm of unitary dilations.

The commutant lifting theorem of Béla Sz.-Nagy and Ciprian Foiaş is a centerpiece of the theory of unitary dilations, in large part because of its intimate connection with interpolation problems, including many arising in engineering. This article describes my own involvement with commutant lifting. It is a personal history that I hope conveys a picture of how research often gets done, in particular, how fortunate happenstance can play a decisive role.

My first piece of good fortune, as far as this story goes, was to be a mathematics graduate student at the University of Michigan when Paul Halmos arrived in Ann Arbor in the fall of 1961. At that time I had passed the Ph.D. oral exams but was unsure of my mathematical direction, except to feel it should be some kind of analysis. I had taken the basic functional analysis course the preceding fall but felt I lacked a good grasp of the subject. Paul was to teach the same course in fall 1961; I decided to sit in.

Prior to encountering Paul in person I was aware that he was well known and that he had written a book on measure theory; that was basically the extent of my knowledge. Paul's entry in the first class meeting, some 48 years ago, stands out in my memory. It was an electrifying moment – Paul had a commanding classroom presence. His course concentrated on Hilbert space, taught by his version of the Moore method, with an abundance of problems for the students to work on. Through the course I was encouraged to ask Paul to direct my dissertation, and he agreed. As a topic he suggested I look at invariant subspaces of normal operators.

Paul's basic idea in [6] is to use normal operators to gain insight into the structure of more general Hilbert space operators. The idea is a natural one; thanks to the spectral theorem, the structure of normal operators is well understood, at least at an abstract level.

Paul proved in [6] that every Hilbert space contraction has, in the terminology of the paper, a unitary dilation: given a contraction T acting on a Hilbert space H, there is a Hilbert space H' containing H as a subspace, and a unitary operator U on H', such that one obtains the action of T on a vector in H by applying U to the vector followed by the orthogonal projection of H' onto H; in Paul's terminology, T is the compression of U to H. A few years after [6] appeared Béla Sz.-Nagy [23] improved Paul's result by showing that, with H and T as above, one can take the containing Hibert space H' and the unitary operator U on H' in such a way that T^n is the compression to H of U^n for every positive integer n. As was quickly recognized, Sz.-Nagy's improvement is a substantial one. For example, Sz.-Nagy's result has as a simple corollary the inequality of J. von Neumann: if T is a Hilbert space contraction and p is a polynomial, then the norm of $p(T)$ is bounded by the supremum norm of p on the unit circle – an early success of the Halmos idea to use normal operators (in this case, unitary operators) to study more general operators. A dilation of the type Sz.-Nagy constructed was for a time referred to as a strong dilation, or as a power dilation; it is now just called a dilation.

After becoming Paul's student I was swept into several confluent mathematical currents. The Sz.-Nagy–Foiaş operator model theory, the creation of a remarkable collaboration spanning over 20 years, was in its relatively early stages, and was of course of great interest to Paul. Exciting new connections between abstract analysis and complex analysis were emerging, leading in particular to the subject of function algebras. These connections often involved Hardy spaces, which thus gained enhanced prominence. Kenneth Hoffman's book [10] embodied and propelled this ferment. (It is the only mathematics book I have studied nearly cover to cover.)

As my dissertation was slated to be about invariant subspaces of normal operators, I learned as much as I could about normal operators, picking up in the process the basics of vector-valued function theory. In the end, the dissertation focused on a particular normal operator whose analysis involved Hardy spaces in an annulus [17]. Simultaneously with working on my dissertation, I tried to understand the Sz.-Nagy–Foiaş theory.

I received my degree in the spring of 1963. From Ann Arbor I went to the Institute for Advanced Study, where I spent a year as an NSF postdoc before joining the Berkeley mathematics faculty in the fall of 1964 (just in time to witness the Free Speech Movement). I don't remember exactly when I started thinking about commutant lifting; most likely it happened at the Institute. Let me back up a little.

Every contraction has a unitary dilation, but not a unique one: given any unitary dilation, one can inflate it by tacking on a unitary direct summand. However, there is always a unitary dilation that is minimal, i.e., not producible by inflation of a smaller one, and this minimal unitary dilation is unique to within unitary equivalence. The simplest unitary operator that can be a minimal unitary dilation of an operator besides itself is the bilateral shift on L^2 of the unit circle, the operator W on L^2 of multiplication by the coordinate function. In trying to understand the Sz.-Nagy–Foiaş theory better, I asked myself which operators (other than W itself) can have W as a minimal unitary dilation. Otherwise put, the question asks for a classification of those proper subspaces K of L^2 with the property that W is the minimal unitary dilation of its compression to K. Any such subspace, one can show, is either a nonreducing invariant subspace of W, or a nonreducing invariant subspace of W^*, or the orthogonal complement of a nonreducing invariant subspace of W in a larger one. The invariant subspace structure of W is given by the theorem of Arne Beurling [3] and its extension by Henry Helson and David Lowdenslager [7]. One concludes that the operators in question, besides W itself, are, to within unitary equivalence, the unilateral shift S (the restriction of W to the Hardy space H^2), the adjoint S^* of S, and the compressions of S to the proper invariant subspaces of S^*.

By Beurling's theorem, the general proper, nontrivial, invariant subspace of S is the subspace uH^2 with u a nonconstant inner function. The corresponding orthogonal complement $K_u^2 = H^2 \ominus uH^2$ is the general proper, nontrivial, invariant subspace of S^*. The compression of S to K_u^2 will be denoted by S_u. The operators S_u, along with S and S^*, are the simplest Sz.-Nagy–Foiaş model operators.

Early in their program Sz.-Nagy and Foiaş defined an H^∞ functional calculus for completely nonunitary contractions, among which are the operators S_u. For φ a function in H^∞, the operator $\varphi(S_u)$ is the compression to K_u^2 of the operator on H^2 of multiplication by φ. The operator $\varphi(S_u)$ depends only on the coset of φ in the quotient algebra H^∞/uH^∞. One thereby gets an injection of H^∞/uH^∞ onto a certain operator algebra on K_u^2 whose members commute with S_u.

At some point, either when I was still in Ann Arbor or during my year at the Institute, I read James Moeller's paper [12], in which he determines the spectra of the operators S_u. Moeller's analysis shows that if the point λ is not in the spectrum of S_u, the operator $(S_u - \lambda I)^{-1}$ is an H^∞ function of S_u. This made me wonder whether every operator commuting with S_u might not be an H^∞ function of S_u.

In pondering this question, an obvious way for one to start is to look at the case where u is a finite Blaschke product, in other words, where K_u^2 has finite dimension. In this case a positive answer lies near the surface, but more is true thanks to the solutions of the classical interpolation problems of Carathéodory–Fejér [4] and Nevanlinna–Pick [14], [16], to which I was led by my question. Those solutions tell you that if u is a finite Blaschke product, then every operator on K_u^2 that commutes with S_u is an H^∞ function of S_u for an H^∞ function whose supremum norm equals the operator norm. Knowing this, one can generalize to the case of an infinite Blaschke product by means of a limit argument. Once I realized that, I was dead sure the same result holds for general u. But here I was stuck for quite a while; the behavior of inner functions that contain singular factors is subtler than that of those that do not. A step in the right direction, it seemed, would be to prove for general u that the injection of H^∞/uH^∞ into $\mathcal{B}(K_u^2)$ (the algebra of operators on K_u^2) preserves norms, something the classical interpolation theory gives you for the Blaschke case. But on that I was stuck as well.

Good fortune accompanied me to Berkeley, where I became a colleague of Henry Helson. Some mathematicians, like me when I was younger, tend to keep to themselves the problems they are trying to solve; others, like Henry, are driven to talk with others about the problems, sharing their sometimes tentative ideas. In the Academic Year 1965–1966 Henry was working on a problem in prediction theory related to earlier work he had done with Gabor Szegö [9]. He would regularly drop by my office to discuss the problem. Back then prediction theory was a mystery to me, and I failed to understand very much of what Henry was saying. I did a lot of nodding, interrupted by an occasional comment or question. This was going on one Friday afternoon in fall 1965 when Henry brought up a proof he had found of Zeev Nehari's theorem on boundedness of Hankel operators [13]. Henry's proof, which is much slicker than the original one, uses ideas from his paper with Szegö, namely, a duality argument facilitated by a factorization result of Frigyes Riesz. (Riesz's result states that a nonzero function f in the Hardy space H^1 can be factored as $f = f_1 f_2$, where f_1 and f_2 are in H^2, and $|f_1|^2 = |f_2|^2 = |f|$ almost everywhere on the unit circle.)

The following day it suddenly struck me that Henry's technique was exactly what I needed to show that the injection $H^\infty/uH^\infty \to \mathcal{B}(K_u^2)$ preserves norms in the general case, and also that it is weak-star-topology $\to$ weak-operator-topology continuous. Once I knew that I was able to combine it with what I already knew to prove the theorem I had long sought, which states: *Every operator T on K_u^2 that commutes with S_u equals $\varphi(S_u)$ for a function φ in H^∞ whose supremum norm equals $\|T\|$*. The proof was completed over the weekend. It uses some vector-valued function theory, including the vector generalization of Beurling's theorem due to Peter Lax [11], and ideas from an earlier paper of mine [18].

After proving the theorem I spent quite a while exploring some of its implications. I wrote up my results in the summer of 1966; the paper containing them [19] was published in May of 1967.

In [19] I was not brave enough to conjecture that my theorem generalizes to arbitrary unitary dilations (although I did prove a rather restrictive vector-valued generalization). It did not take long for Sz.-Nagy and Foiaş to produce the generalization, their famous commutant lifting theorem. Their paper [24] contains an informative discussion of the theorem.

Perhaps I should have looked more deeply into Hankel operators after Henry showed me his proof of Nehari's theorem, but I did not; my thoughts were elsewhere. Sometime in the 1970s Douglas Clark observed that my theorem is a fairly simple corollary of Nehari's. Clark did not publish his observation; it appears, though, in the notes for a course he gave at the University of Georgia. A bit later Nikolai Nikolski independently made the same observation. The derivation of my theorem from Nehari's can be found in Nikolski's book [15] (pp. 180ff.).

Following Sz.-Nagy and Foiaş's original proof of the commutant lifting theorem, several alternative proofs were found. My favorite, because it brings us back to Hankel operators, is due to Rodrigo Arocena [2].

In 1968 Vadim Adamyan, Damir Arov and Mark Kreĭn published the first [1] of a series of papers on Hankel operators. Originally they seemingly were unaware of Nehari's paper; a reference to Nehari was added to their paper in proof. Among other things, Adamyan–Arov–Kreĭn found a proof of Nehari's theorem along the same lines as the familiar operator theory approach to the Hamburger moment problem.

Nehari's theorem is a special case of the commutant lifting theorem. Arocena realized that the Adamyan–Arov–Kreĭn technique can be juiced up to give a proof of the full theorem. An exposition of Arocena's proof can be found in my article [21].

Ever since Sz.-Nagy and Foiaş proved their theorem, commutant lifting has played a central role in operator theory. A picture of the scope of the idea of commutant lifting, and of its engineering connections, can be found in the book of Foiaş and Arthur Frazho [5].

My own romance with commutant lifting seems to have come full circle. My recent paper [22] contains a version of commutant lifting for unbounded operators: If T is a closed densely defined operator on K_u^2 that commutes with S_u, then $T = \varphi(S_u)$ for a function φ in the Nevanlinna class ($\varphi = \psi/\chi$, where ψ and χ are in H^∞ and χ is not the zero function). Is there a general theorem in the theory of unitary dilations that contains this result?

Small footnote: I eventually understood enough about Henry's problem in prediction theory to contribute to its solution. The result is our joint paper [8] and my subsequent paper [20]. My work on [8] took place while Henry was on leave in France during the Academic Year 1966–1967, and our communication took place via airmail.

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