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On a Theorem of Camps and Dicks

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Abstract. We provide a short, intuitive proof of a theorem of Camps and Dicks [1].

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1. The theorem

Below, rings are associative with 1, but possibly noncommutative. Modules are unital. We also make use of the well-known fact that a ring R is semi-simple if and only if every maximal left ideal is a summand.

Theorem 1. *Let R and S be rings and let ${}_RM_S$ be an R-S-bimodule. If M_S has finite uniform dimension and for $r \in R$ the equality $\operatorname{ann}_M(r) = (0)$ implies $r \in U(R)$ then R is semi-local.*

Proof. Let $\overline{R} = R/J(R)$, and let ${}_{\overline{R}}A$ be a maximal submodule of ${}_{\overline{R}}\overline{R}$. We wish to show that A is a direct summand. Since M_S has finite uniform dimension there exists an element $b \in R$, $\overline{b} \notin A$, such that $\operatorname{ann}_M(b) \subseteq M_S$ has maximal uniform dimension (with respect to the restriction $\overline{b} \notin A$).

Let $x \in R$ be such that $\overline{xb} \in A$. Notice the containment $\operatorname{ann}_M(b - bxb) \supseteq \operatorname{ann}_M(b) \oplus \operatorname{ann}_M(1 - xb)$ (in fact, equality holds, although we do not need that information). But $\overline{b - bxb} \notin A$, so by the maximality condition on b we conclude $\operatorname{ann}_M(1-xb) = (0)$. Therefore $1-xb \in U(R)$. Repeating the argument, we see that $1-yxb \in U(R)$ for all $y \in R$, so $xb \in J(R)$. We have thus shown that $A \cap \overline{Rb} = (\overline{0})$. By maximality of A we have $A \oplus \overline{Rb} = \overline{R}$, finishing the proof. □

Corollary 2. *If S is a ring and M_S is an Artinian right S-module then $R = \operatorname{End}(M_S)$ is a semi-local ring.*

Notice that in the proof of Theorem 1, we could weaken the condition "$r \in U(R)$" to "r is left invertible." We also remark that in the original proof given by Camps and Dicks in [1], they showed that R is semi-local if and only if there exists an integer $n \geq 0$ and a function $d : R \to \{1, 2, \ldots, n\}$ satisfying $d(b - bxb) = d(b) + d(1 - xb)$, and if $d(a) = 0$ then $a \in U(R)$. One can recover this fact by letting $d(a)$ denote the composition length of the right annihilator of $\overline{a} \in \overline{R}$ and following the ideas in the proof of Theorem 1.

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References

[1] Rosa Camps and Warren Dicks, On semilocal rings, *Israel J. Math.* **81** (1993), no. 1–2, 203–211.

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