Chapter VII Reedy model categories

This chapter contains an exposition of the Bousfield-Kan model structure on the category $c\mathbf{S}$ of cosimplicial objects in simplicial sets, also known as cosimplicial spaces. It appears here as the dual of a Reedy model category structure on the category of simplicial objects $s\mathcal{C}$ in a suitable closed model category $\mathcal{C}$. Another standard example of a Reedy structure on a simplicial object category is the Reedy structure on the category of bisimplicial sets, or simplicial objects in simplicial sets — see Section IV.3.

Much of the chapter concerns the general Reedy theory. We preface this development in Section 1 with a discussion of skeleta in a very general context. The main results are Proposition 1.9 and Corollary 1.14; they discuss to what extent a simplicial object in a category $\mathcal{C}$ with enough colimits can be built by "attaching cells". One application is a characterization of cofibrations in the kind of model category considered in Section II.4. See Example 1.15.

In general, if a category $\mathcal{C}$ is a closed model category, then the Reedy structure of the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ has as weak equivalences those morphisms $X \to Y$ in $s\mathcal{C}$ for which each $X_n \to Y_n$ is a weak equivalence for all $n \geq 0$. One of the main auxiliary results is that there is a geometric realization functor

$$|\cdot| : s\mathcal{C} \longrightarrow \mathcal{C}$$

which preserves weak equivalences between Reedy cofibrant objects — see Proposition 3.6. The Reedy model category structure is discussed in detail in Section 2, and the geometric realization functor is the subject of Section 3.

This theory is specialized in Section 4 to the case of cosimplicial spaces $c\mathbf{S}$; that is, to the case of the opposite category to the category of simplicial objects in $\mathbf{S}^{op}$. The resulting model category structure on $c\mathbf{S}$ is the standard one discussed by Bousfield and Kan. In particular, we show in Proposition 4.18 that the cofibrations $A \to B$ defined by the Reedy structure are exactly those maps which are monomorphisms in all levels and induce isomorphisms $H^0A \cong H^0B$ on maximal augmentations.

The material in Section 4, along with that appearing in Section VI.2, is the basis for the construction of the homotopy spectral sequence for a cosimplicial space which is given in Chapter VIII.

1. Decomposition of simplicial objects.

Let $\mathcal{C}$ be a category with all limits and colimits. The purpose of this section is to analyze how simplicial objects are constructed out of smaller components. We will use this inductive argument in later sections.

We begin with skeleta. The category $s\mathcal{C}$ is the functor category $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$. Let $i_n : \mathbf{\Delta}_n \subseteq \mathbf{\Delta}$ be the inclusion of the full subcategory with objects $\mathbf{k}$, $k \leq n$, and let $s_n\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}_n^{\mathrm{op}}}$. There is a restriction function $i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$ which simply forgets the k-simplices, $k > n$. This restriction functor has a left adjoint given by

$$(i_n^* X)_m = \varinjlim_{\mathbf{m}\to\mathbf{k}} X_k = \varinjlim_{\mathbf{m}\downarrow\mathbf{\Delta}_n} X_k \tag{1.1}$$

and the colimit is over morphisms $\mathbf{m} \to \mathbf{k}$ in $\mathbf{\Delta}$ with $k \leq n$. This is an example of a left Kan extension. Every morphism $\mathbf{m} \to \mathbf{k}$ in $\mathbf{\Delta}$ can be factored uniquely as

$$\mathbf{m} \xrightarrow{\phi} \mathbf{k}' \xrightarrow{\psi} \mathbf{k}$$

where ϕ is a surjection and ψ is one-to-one, so the surjections $\mathbf{m} \to \mathbf{k}$, $k \leq n$, can be used to define the colimit of (1.1) and we get

$$(i_n^* X)_m \cong \varinjlim_{\mathbf{m}\downarrow\mathbf{\Delta}_n^+} X_k \tag{1.2}$$

where $\mathbf{\Delta}_+ \subseteq \mathbf{\Delta}$ is the subcategory with the same objects but only surjections as morphisms, and $\mathbf{\Delta}_n^+ = \mathbf{\Delta}_n \cap \mathbf{\Delta}_+$.

If $X \in s\mathcal{C}$ we define the n^{th} *skeleton* of X by the formula

$$\operatorname{sk}_n X = i_n^* i_{n*} X. \tag{1.3}$$

There are natural maps $\operatorname{sk}_m X \to \operatorname{sk}_n X$, $m \leq n$, and $\operatorname{sk}_n X \to X$. The morphism $1_{\mathbf{m}}$ is an initial object in the category $\mathbf{m} \downarrow \mathbf{\Delta}_n$ if $m \leq n$, so there is an isomorphism $(\operatorname{sk}_n X)_m \cong X_m$ in that range. It follows that there is a natural isomorphism

$$\varinjlim_n \operatorname{sk}_n X \xrightarrow{\cong} X.$$

Here is an example. Because $\mathcal{C}$ has limits and colimits, $s\mathcal{C}$ has a canonical structure as a simplicial category. (See Section II.2). In particular if $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, then

$$(X \otimes K)_n = \bigsqcup_{K_n} X_n.$$

It is now a straightforward exercise to prove

Proposition 1.4. *If $X \in s\mathcal{C}$ is constant and $K \in \mathbf{S}$, then there are natural isomorphisms*

$$\operatorname{sk}_n X \cong X \text{ and } X \otimes \operatorname{sk}_n K \cong \operatorname{sk}_n(X \otimes K).$$

To explain how $\operatorname{sk}_n X$ is built from $\operatorname{sk}_{n-1} X$ we define the n^{th} *latching object* $L_n X$ of X by the formula

$$\begin{aligned} L_n X &= (\operatorname{sk}_{n-1} X)_n \\ &\cong \varinjlim_{\mathbf{n}\downarrow\mathbf{\Delta}_{n-1}^+} X_k. \end{aligned} \tag{1.5}$$

If $Z \in \mathcal{C}$, we may regard Z as a constant object in $s\mathcal{C}$ and there is an adjoint isomorphism

$$\hom_{\mathcal{C}}(Z, X_n) \cong \hom_{s\mathcal{C}}(Z \otimes \Delta^n, X)$$

for all $n \geq 0$. This immediately supplies maps in $s\mathcal{C}$

$$L_n X \otimes \Delta^n \to \operatorname{sk}_{n-1} X$$

and

$$X_n \otimes \Delta^n \to \operatorname{sk}_n X.$$

Furthermore, by Proposition 1.4, $\operatorname{sk}_{n-1}(X_n \otimes \Delta^n) = X_n \otimes \operatorname{sk}_{n-1} \Delta^n = X_n \otimes \partial\Delta^n$ and we obtain a diagram

$$\begin{tikzcd}
L_n X \otimes \partial\Delta^n \arrow[r] \arrow[d] & L_n X \otimes \Delta^n \arrow[d] \\
X_n \otimes \partial\Delta^n \arrow[r] & \operatorname{sk}_{n-1} X.
\end{tikzcd} \tag{1.6}$$

PROPOSITION 1.7. *For all $X \in s\mathcal{C}$ there is a natural pushout diagram, $n \geq 0$,*

$$\begin{tikzcd}
X_n \otimes \partial\Delta^n \cup_{L_n X \otimes \partial\Delta^n} L_n X \otimes \Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} X \arrow[d] \\
X_n \otimes \Delta^n \arrow[r] & \operatorname{sk}_n X
\end{tikzcd}$$

PROOF: In light of Proposition 1.4, if we apply $i_n^* i_{n*} = \operatorname{sk}_n(\cdot)$ to this diagram we obtain an isomorphic diagram. Since $i_n^* : s_n\mathcal{C} \to s\mathcal{C}$ is a left adjoint, we need only show this is a pushout diagram in degrees less than or equal to n.

In degrees $m < n$, the map $(L_n X \otimes \partial\Delta^n)_m \to (L_n X \otimes \Delta^n)_m$ is an isomorphism, so

$$(X_n \otimes \partial\Delta^n \cup_{L_n X \otimes \partial\Delta^n} L_n X \otimes \Delta^n)_m \to (X_n \otimes \Delta^m)_m$$

is an isomorphism, and the assertion is that $(\operatorname{sk}_{n-1} X)_m \cong (\operatorname{sk}_n X)_m$, which is true since both are isomorphic to X_m.

In degree n, the left vertical map is isomorphic to

$$\Big(\bigsqcup_{(\partial\Delta^n)_n} X_n\Big) \sqcup L_n X \to \Big(\bigsqcup_{(\partial\Delta^n)_n} X_n\Big) \sqcup X_n$$

and the right vertical map is isomorphic to the natural map $L_n X \to X_n$, by definition of $L_n X$. This is enough to show that the diagram is a pushout in degree n. $\square$

REMARK 1.8. There is another, more explicit, description of the latching objects L_nX, which can be summarized as follows:

(1) By convention, $L_0X = \emptyset$, where $\emptyset$ denotes the initial object of the category $\mathcal{C}$.
(2) There is an isomorphism $L_1X \cong X_0$, and the canonical map $L_1X \to X_1$ can be identified with the degeneracy map $s_0 : X_0 \to X_1$.
(3) For $n > 1$, the object L_nX is defined by the coequalizer

$$\bigsqcup_{0\le i<j\le n-1} X_{n-2} \rightrightarrows \bigsqcup_{i=0}^{n-1} X_{n-1} \to L_nX$$

where for $i < j$ the restrictions of the two displayed maps to X_{n-2} are given by the composites

$$X_{n-2} \xrightarrow{s_i} X_{n-1} \xrightarrow{in_j} \bigsqcup_{i=0}^{n-1} X_{n-1}$$

and

$$X_{n-2} \xrightarrow{s_{j-1}} X_{n-1} \xrightarrow{in_i} \bigsqcup_{i=0}^{n-1} X_{n-1}$$

(this definition corresponds to the simplicial identity $s_js_i = s_is_{j-1}$). The canonical map $s : L_nX \to X_n$ is induced by the degeneracies $s_i : X_{n-1} \to X_n$.

Claims (2) and (3) follow from the description of $L_nX = \mathrm{sk}_{n-1} X_n$ given in (1.2) — this is an exercise for the reader.

Morphisms also have skeletal filtrations. If $f : A \to X$ is a morphism in $s\mathcal{C}$, define $\mathrm{sk}_n^A X$ by setting $\mathrm{sk}_{-1}^A = A$ and, for $n \ge 0$, defining sk_n^A by the pushout diagram

$$\begin{tikzcd}
\mathrm{sk}_n A \arrow[r] \arrow[d, "\mathrm{sk}_n f"'] & A \arrow[d] \\
\mathrm{sk}_n X \arrow[r] & \mathrm{sk}_n^A X.
\end{tikzcd}$$

The analog of Proposition 1.7 is the next result, which is proved in an identical manner. Let $L_n(f) = (\mathrm{sk}_{n-1}^A X)_n = A_n \cup_{L_nA} L_nX$.

PROPOSITION 1.9. *For all morphisms $A \to X$ in $s\mathcal{C}$ there is a pushout diagram*

$$\begin{tikzcd}
X_n \otimes \partial\Delta^n \cup_{L_n(f)\otimes\partial\Delta^n} L_n(f) \otimes \Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}^A X \arrow[d] \\
X_n \otimes \Delta^n \arrow[r] & \mathrm{sk}_n^A X.
\end{tikzcd}$$

□

There is a situation under which the pushout diagrams of Proposition 1.7 and 1.9 simplify considerably. This we now explain.

If I is a small category, let I^δ be the discrete category with the same objects as I, but no non-identity morphism. The left adjoint r^* to the restriction functor $r_* : \mathcal{C}^I \to \mathcal{C}^{I^\delta}$ has a very simple form

$$(r^*Z)_j = \bigsqcup_{i \to j} Z_i.$$

We call such a diagram *I-free*. More generally, a morphism $f : A \to X$ in $\mathcal{C}^I$ is *I-free* if there is an I-free object $X' \in \mathcal{C}^I$ and an isomorphism under A of f with the inclusion of the summand $A \to A \sqcup X'$. This implies that there is an object $\{Z_i\} \in \mathcal{C}^{I^\delta}$ so that

$$X_j \cong A_j \sqcup (\bigsqcup_{i \to j} Z_i).$$

Notice an object X is I-free if and only if the morphism $\phi \to X$ from the initial object is I-free.

DEFINITION 1.10. An object $X \in s\mathcal{C}$ is *degeneracy free* if the underlying degeneracy diagram is free. That is, if $\mathbf{\Delta}_+ \subseteq \mathbf{\Delta}$ is the subcategory with same objects but only surjective morphisms, then X regarded as an object in $\mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$ is $\mathbf{\Delta}_+^{\mathrm{op}}$-free. A morphism $A \to X$ is degeneracy free if, when regarded as a morphism in $\mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$, is $\mathbf{\Delta}_+^{\mathrm{op}}$ free.

If X is degeneracy free, then there is a sequence $\{Z_n\}_{n \geq 0}$ of objects in $\mathcal{C}$ so that

$$X_n \cong \bigsqcup_{\phi : \mathbf{n} \to \mathbf{m}} Z_m$$

where ϕ runs over the epimorphisms in $\mathbf{\Delta}$. A degeneracy free map $A \to X$ yields a similar decomposition:

$$X_n \cong A_n \sqcup \bigsqcup_{\phi : \mathbf{n} \to \mathbf{m}} Z_m. \tag{1.11}$$

We say $A \to X$ is degeneracy free on $\{Z_m\}$.

The following result says that degeneracy free maps are closed under a variety of operations.

LEMMA 1.12.
1) Let $f_\alpha : A_\alpha \to X_\alpha$ be a set of maps so that f_α is degeneracy free on $\{Z_n\}$. Then $\bigsqcup_\alpha f_\alpha$ is free on $\{\bigsqcup_\alpha Z_n^\alpha\}$.

2) Suppose $f : A \to X$ is degeneracy free on $\{Z_n\}$ and

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r] & Y \end{tikzcd}$$

is a pushout diagram. Then g is degeneracy free on $\{Z_n\}$.
3) Let $A_0 \to A_1 \to A_2 \to \cdots$ be a sequence of morphisms so that $A_{j-1} \to A_j$ is degeneracy free on $\{Z_n^j\}$. Then $A_0 \to \varinjlim A_j$ is degeneracy free on $\{\bigsqcup_j Z_n^j\}$.
4) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that preserves coproducts. If $f : A \to X$ in $s\mathcal{C}$ is degeneracy free on $\{Z_n\}$, then $Ff : FA \to FX$ is degeneracy free on $\{FZ_n\}$.

LEMMA 1.13. *A morphism $f : A \to X$ is degeneracy free if and only if there are objects Z_n and maps $Z_n \to X_n$ so that the induced map*

$$(A_n \cup_{L_n A} L_n X) \sqcup Z_n \to X_n$$

is an isomorphism.

PROOF: If f is degeneracy free on $\{Z_n\}$, the decomposition follows from the formulas (1.5) and (1.11). To prove the converse, fix the given isomorphisms. Then Proposition 1.9 implies there is a pushout diagram

$$\begin{tikzcd} Z_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1}^A X \arrow[d] \\ Z_n \otimes \Delta^n \arrow[r] & \operatorname{sk}_n^A X \end{tikzcd}$$

in $\mathcal{C}^{\mathbf{\Delta}_+^{op}}$.

However, the morphism of simplicial sets $\partial\Delta^n \to \Delta^n$ is degeneracy free on the canonical n-simplex in Δ_n^n; hence, $\operatorname{sk}_{n-1}^A X \to \operatorname{sk}_n^A X$ is degeneracy free on Z_n in degree n. The result now follows from Lemma 1.12.3; indeed $A \to X$ is degeneracy free on $\{Z_n\}$. $\square$

Here is a consequence of the proof of Lemma 1.13:

COROLLARY 1.14. *Suppose $A \to X$ is degeneracy-free on $\{Z_n\}$ in $s\mathcal{C}$. Then for all $n \geq 0$ there is a pushout diagram,*

$$\begin{tikzcd} Z_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1}^A X \arrow[d] \\ Z_n \otimes \Delta^n \arrow[r] & \operatorname{sk}_n^A X. \end{tikzcd}$$

Notice that one can interpret this result as saying $\operatorname{sk}_n^A X$ is obtained from $\operatorname{sk}_{n-1}^A X$ by attaching n-cells.

EXAMPLE 1.15. Lemmas 1.12 and 1.13 provide any number of examples of degeneracy free morphisms. For example, a cofibration in simplicial sets is degeneracy free, by Lemma 1.13. Also, consider a category $\mathcal{C}$ equipped with a functor $G : \mathcal{C} \to \mathbf{Sets}$ with a left adjoint F and satisfying the hypotheses of Theorem II.4.1. Define a morphism $f : A \to X$ in $s\mathcal{C}$ to be *free* (this terminology is from Quillen) if there is a sequence of sets $\{Z_n\}$ so that f is degeneracy free on $\{FZ_n\}$. Then a morphism in $s\mathcal{C}$ is a cofibration if and only if it is a retract of a free map. To see this, note that the small object argument of Lemma II.4.2, coupled with Lemma 1.12 factors any morphism $A \to B$ as

$$A \xrightarrow{j} X \xrightarrow{q} Y$$

where j is a free map and a cofibration and q is a trivial fibration. Thus any cofibration is a retract of a free map. Conversely, Corollary 1.14 and Proposition II.3.4 imply any free map is a cofibration. Similar remarks apply to the model categories supplied in Theorems II.5.8 and II.6.8. In the latter case one must generalize Lemma 1.12.3 to longer colimits.

The notion of coskeleta is dual to the notion of skeleta. The theory is analogous, and we give only an outline.

The restriction functor $i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$ has right adjoint $i_n^!$ with

$$(i_n^! X)_m = \varprojlim_{\mathbf{k}\to\mathbf{m}} X_k,$$

with the limit over all morphism $\mathbf{k} \to \mathbf{m}$ in $\mathbf{\Delta}$ with $k \leq n$. Equally, one can take the limit over morphisms $\mathbf{k} \to \mathbf{m}$ which are injections. The composite gives the n^{th} coskeleton functor:

$$\operatorname{cosk}_n X = i_n^! i_{n*} X. \tag{1.16}$$

More generally, if $f : X \to B$ is a morphism in $s\mathcal{C}$, let $\operatorname{cosk}_{-1}^B X = B$ and let $\operatorname{cosk}_n^B X$ be defined by the pullback

$$\begin{tikzcd} \operatorname{cosk}_n^B X \arrow[r] \arrow[d] & \operatorname{cosk}_n X \arrow[d] \\ B \arrow[r] & \operatorname{cosk}_n B. \end{tikzcd} \tag{1.17}$$

Then there are maps $\operatorname{cosk}_n^B X \to \operatorname{cosk}_{n-1}^B X$ and

$$X \cong \varprojlim_n \operatorname{cosk}_n^B X.$$

Note that if $B = *$, the terminal object, then $\operatorname{cosk}_n^B X = \operatorname{cosk}_n X$.

For $X \in s\mathcal{C}$, define the n^{th} *matching object* M_nX by the formula

$$M_nX = (\operatorname{cosk}_{n-1} X)_n = \varprojlim_{\mathbf{k}\to\mathbf{n}} X_k \tag{1.18}$$

where $\mathbf{k} \to \mathbf{n}$ runs over all morphisms (or all monomorphisms) in $\mathbf{\Delta}$ with $k < n$.

REMARK 1.19. The matching object M_nX has a more explicit description:

(1) By convention, $M_0X = *$, where $*$ denotes the terminal object of the category $\mathcal{C}$.

(2) There is an isomorphism $M_1X \cong X_0 \times X_0$, and the canonical map $X_1 \to M_1X$ can be identified with the product $d = (d_0, d_1) : X_1 \to X_0 \times X_0$ of the face maps $d_0, d_1 : X_1 \to X_0$.

(3) For $n > 1$, the matching object M_nX is defined by an equalizer diagram

$$M_nX \to \prod_{i=0}^{n} X_{n-1} \rightrightarrows \prod_{0 \le i < j \le n} X_{n-2}.$$

Here, the parallel arrows are determined by the simplicial identities $d_id_j = d_{j-1}d_i$ for $i < j$; more explicitly, the images of these maps on the factor corresponding to $i < j$ are the maps

$$\prod_{i=0}^{n} X_{n-1} \xrightarrow{pr_j} X_{n-1} \xrightarrow{d_i} X_{n-2}$$

and

$$\prod_{i=0}^{n} X_{n-1} \xrightarrow{pr_i} X_{n-1} \xrightarrow{d_{j-1}} X_{n-2}.$$

The canonical map $d : X_n \to M_nX$ is induced by the face maps $d_i : X_n \to X_{n-1}$.

Claims (2) and (3) follow from the description of $(i^!_{n-1}X)_n = M_nX$ as an inverse limit indexed over ordinal number monomorphisms. This is an exercise for the reader.

To fit the map $\operatorname{cosk}^B_n X \to \operatorname{cosk}^B_{n-1} X$ into a pullback diagram, we make some definitions. Let $\rho_n : s\mathcal{C} \to \mathcal{C}$ be the functor $\rho_nX = X_n$. This is a restriction functor between diagram categories and has a right adjoint $\rho^!_n$. The functor $L_n : s\mathcal{C} \to \mathcal{C}$ assigning each simplicial object the latching object L_nX also has a right adjoint, which we will call $\mu^!_n$. To see this, see Lemma 1.25 below, or note that we may write

$$L_nX = (\operatorname{sk}_{n-1} X)_n = \rho_n i^*_{n-1} i_{(n-1)*} X$$

where $i_{(n-1)*} : s\mathcal{C} \to s_{n-1}\mathcal{C}$ is the restriction functor. Hence

$$\mu^!_n Z = i^!_{n-1} i_{(n-1)*} \rho^!_n Z = \operatorname{cosk}_{n-1}(\rho^!_n Z).$$

The natural map $s : L_nX \to X$ gives a natural transformation $\rho^!_n \to \mu^!_n$. The reader is invited to prove the following result.

PROPOSITION 1.20. *Let $f : X \to B$ be a morphism in $s\mathcal{C}$. Then for all $n \geq 0$ there is a pullback diagram*

$$\begin{tikzcd}
\operatorname{cosk}_n^B X \arrow[r] \arrow[d] & \rho_n^! X_n \arrow[d] \\
\operatorname{cosk}_{n-1}^B X \arrow[r] & \rho_n^! M_n(f) \times_{\mu_n^! M_n(f)} \mu_n^! X_n
\end{tikzcd}$$

where $M_n(f) = B_n \times_{M_n B} M_n X$.

We have never encountered the analog of degeneracy-free morphisms, and don't include an exposition here.

The latching and matching object functors L_n and M_n are examples of a much more general sort of functor, which we now introduce and analyze. We begin with generalized matching objects.

PROPOSITION 1.21. *Let $K \in \mathbf{S}$ be a simplicial set. Then the functor*

$$- \otimes K : \mathcal{C} \to s\mathcal{C}$$

has right adjoint M_K. For fixed $X \in s\mathcal{C}$, the assignment $K \mapsto M_K X$ induces a functor $\mathbf{S}^{\mathrm{op}} \to \mathcal{C}$ which has a left adjoint.

PROOF: For objects Z of $\mathcal{C}$ and all $n \geq 0$, there are isomorphisms

$$\hom_{s\mathcal{C}}(Z \otimes \Delta^n, X) \cong \hom_{\mathcal{C}}(Z, X_n).$$

It follows that there are isomorphisms

$$\begin{aligned}
\hom_{s\mathcal{C}}(Z \otimes K, X) &\cong \varprojlim_{\Delta^n \to K} \hom_{s\mathcal{C}}(Z \otimes \Delta^n, X) \\
&\cong \varprojlim_{\Delta^n \to K} \hom_{\mathcal{C}}(Z, X_n) \\
&\cong \hom_{\mathcal{C}}(Z, \varprojlim_{\Delta^n \to K} X_n) \\
&= \hom_{\mathcal{C}}(Z, M_K X).
\end{aligned}$$

Furthermore,

$$\hom_{\mathcal{C}}(Z, M_K X) \cong \hom_{s\mathcal{C}}(Z \otimes K, X) \cong \hom_{\mathbf{S}}(K, \mathbf{Hom}_{s\mathcal{C}}(Z, X)). \qquad \square$$

Notice the explicit description of $M_K X$ that arose in the proof of Proposition 1.21:

$$M_K X \cong \varprojlim_{(\mathbf{\Delta} \downarrow K)^{\mathrm{op}}} X \tag{1.22}$$

is the limit of a contravariant functor on the simplex category $\mathbf{\Delta} \downarrow K$ which sends an object $\Delta^n \to K$ of $\mathbf{\Delta} \downarrow K$ to the object X_n of $\mathcal{C}$. In particular, there is an isomorphism

$$M_{\Delta^n} X \cong X_n,$$

since the category $\Delta \downarrow \Delta^n$ has an initial object.

EXAMPLE 1.23. Let $\phi : \Delta^k \to \Delta^n$ be any morphism in $\mathbf{S}$. Then ϕ factors uniquely as a composition

$$\Delta^k \xrightarrow{\phi'} \Delta^m \xrightarrow{\psi} \Delta^n$$

where ϕ' is a surjection and ψ is an injection. Thus if $K \subseteq \Delta^n$ is any subcomplex, the full subcategory $\mathbf{\Delta} \downarrow K_0 \subseteq \mathbf{\Delta} \downarrow K$ with objects

$$\sigma : \Delta^m \to K$$

with σ an injection determines colimits on the larger category. Equivalently, restriction to $\mathbf{\Delta} \downarrow K_0^{\mathrm{op}}$ determines inverse limits for $\mathbf{\Delta} \downarrow K^{\mathrm{op}}$; it follows that there is an isomorphism

$$M_K X \cong \varprojlim_{\mathbf{\Delta}\downarrow K_0^{\mathrm{op}}} X.$$

In particular $M_{\partial\Delta^n} X \cong \varprojlim_{\phi:\mathbf{k}\to\mathbf{n}} X_k$ where $\phi \in \mathbf{\Delta}$ is an injection and $k < n$. Thus $M_{\partial\Delta^n} X \cong M_n X$. Note that the inclusion map $\partial\Delta^n \to \Delta^n$ induces the projection

$$X_n \to M_n X.$$

To generalize the latching objects we use the formulation of matching objects presented in (1.22). Let J be a small category and $F : J \to \mathbf{\Delta}^{\mathrm{op}}$. For $X \in s\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$, define the generalized latching object to be

$$\begin{aligned} L_J X &= \varinjlim_J (X \circ F) \\ &= \varinjlim_j X_{F(j)}. \end{aligned} \tag{1.24}$$

We are primarily interested in sub-categories J of $\mathbf{\Delta}^{\mathrm{op}}$. We write $X\big|_J$ for $X \circ F$.

LEMMA 1.25. *For fixed $F : J \to \mathbf{\Delta}^{\mathrm{op}}$, the functor $L_J : s\mathcal{C} \to \mathcal{C}$ has a right adjoint and, hence, preserves colimits.*

PROOF: For $\hom_{\mathcal{C}}(L_J X, Z) \cong \hom_{\mathcal{C}^J}(X \circ F, Z)$ where Z is regarded as a constant diagram. But

$$\hom_{\mathcal{C}^J}(X\big|_J, Z) = \hom_{s\mathcal{C}}(X, F^! Z)$$

where $F^!$ is the right Kan extension functor. $\square$

An example of a generalized latching object is the following. Let $\mathcal{O}_n$ be the category with objects the morphisms $\phi : \mathbf{n} \to \mathbf{m}$ with ϕ surjective and $m < n$. The morphisms in $\mathcal{O}_n$ are commutative triangles in $\mathbf{\Delta}$ under $\mathbf{n}$. Define $F : \mathcal{O}_n^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ by $F(\phi : \mathbf{n} \to \mathbf{m}) = \mathbf{m}$. Then there is a natural isomorphism

$$L_n X \cong L_{\mathcal{O}_n^{\mathrm{op}}} X.$$

In the next section we will need a decomposition of $L_n X$. To accomplish this define sub-categories $\mathcal{M}_{n,k} \subseteq \mathcal{O}_n$, $0 \leq k \leq n$, to be the full sub-category of surjections $\phi : \mathbf{n} \to \mathbf{m}$, $m < n$, with $\phi(k) < k$. Define, for $X \in s\mathcal{C}$

$$L_{n,k}X = L_{\mathcal{M}_{n,k}^{\mathrm{op}}}X.$$

Then $L_{n,0}X = \phi$ is the initial object in $\mathcal{C}$ (since $\mathcal{M}_{n,0}$ is empty) and $L_{n,n}X = L_n X$, since $\mathcal{M}_{n,n} = \mathcal{O}_n$.

Also define $\mathcal{M}(k) \subseteq \mathcal{O}_n$ to be the full subcategory of surjections $\phi : \mathbf{n} \to \mathbf{m}$, $m < n$, with $\phi(k) = \phi(k+1)$. Notice that $s^k : \mathbf{n} \to \mathbf{n-1}$ is the initial object of $\mathcal{M}(k)$. Hence for $X \in s\mathcal{C}$

$$L_{\mathcal{M}(k)^{\mathrm{op}}}X \cong X_{n-1}.$$

The reader can verify the following statements about these subcategories.

LEMMA 1.26.
1) $\mathcal{M}_{n,k}$ and $\mathcal{M}(k)$ are subcategories of $\mathcal{M}_{n,k+1}$, and every object in $\mathcal{M}_{n,k+1}$ is in $\mathcal{M}_{n,k}$ or $\mathcal{M}(k)$ (or both).

2) There is an isomorphism of categories

$$- \circ s^k : \mathcal{M}_{n-1,k} \to \mathcal{M}(k) \cap \mathcal{M}_{n,k}$$

sending ϕ to $\phi \circ s^k$.

3) If ϕ is an object of $\mathcal{M}_{n,k}$ (or $\mathcal{M}(k)$) and $\phi \to \psi$ is a morphism in $\mathcal{M}_{n,k+1}$, then $\phi \to \psi$ is a morphism in $\mathcal{M}_{n,k}$ (or $\mathcal{M}(k)$).

The following example, illustrating the case $n = 3$, $k = 2$, might be helpful.

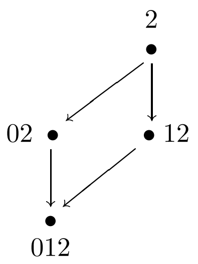
In the following diagram the symbol 012 near a dot ($\bullet$) indicates the object $s^0 s^1 s^2$ in the appropriate category. The unlabeled arrows indicated composition with s^i for some i; for example $0 \bullet \to \bullet\ 02$ means $s^0 \mapsto s^1 s^0 = s^0 s^2$.

$$\begin{tikzcd}
\overset{0}{\bullet} \arrow[d] & \overset{1}{\bullet} \arrow[dl] \\
\underset{01}{\bullet} &
\end{tikzcd}$$

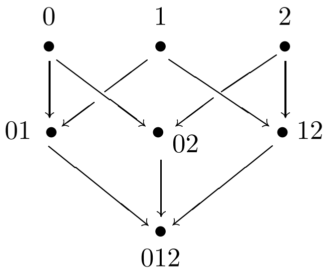
$\mathcal{M}_{2,2} = \mathcal{O}_2$

$$\begin{tikzcd}
\overset{0}{\bullet} \arrow[d] \arrow[dr] & \overset{1}{\bullet} \arrow[dl] \arrow[dr] & \\
01\ \bullet \arrow[dr] & \bullet\ 02 \arrow[d] & \bullet\ 12 \arrow[dl] \\
& \underset{012}{\bullet} &
\end{tikzcd}$$

$\mathcal{M}_{3,2}$



$\mathcal{M}(2)$



$\mathcal{M}_{3,3} = \mathcal{O}_3$

Proposition 1.27. *Let $X \in s\mathcal{C}$. Then there is a pushout diagram in $\mathcal{C}$*

$$\begin{tikzcd} L_{n-1,k}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ L_{n,k}X \arrow[r] & L_{n,k+1}X. \end{tikzcd}$$

Proof: Lemma 1.26 implies that

$$\begin{tikzcd} \mathcal{M}_{n-1,k}X \arrow[r] \arrow[d] & \mathcal{M}(k) \arrow[d] \\ \mathcal{M}_{n,k}X \arrow[r] & \mathcal{M}_{n,k+1} \end{tikzcd}$$

is a pushout in the category of small categories. The same is then true of the opposite categories. The result follows. □

2. Reedy model category structures.

Let $\mathcal{C}$ be a closed model category. Using the considerations of the previous section, this structure can be promoted to a closed model category structure on the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ that is particularly useful for dealing with geometric realization. The results of this section are a recapitulation of the highly influential, but unpublished paper of C.L. Reedy [81].

In the following definition, let L_0X and M_0X denote the initial and final object of $\mathcal{C}$ respectively.

Definition 2.1. A morphism $f : X \to Y$ in $s\mathcal{C}$ is a

1) Reedy weak equivalence if $f : X_n \to Y_n$ is a weak equivalence for all $n \geq 0$;
2) a Reedy fibration if

$$X_n \to Y_n \times_{M_n Y} M_n X$$

is a fibration for all $n \geq 0$;
3) a Reedy cofibration if

$$X_n \cup_{L_n X} L_n Y \to Y_n$$

is a cofibration for all $n \geq 0$.

The main result is that this defines a model category structure on $s\mathcal{C}$. Before proving this we give the following lemma.

Lemma 2.2. *A morphism $f : X \to Y$ in $s\mathcal{C}$ is a*

1) *Reedy trivial fibration if and only if*

$$X_n \to Y_n \times_{M_n Y} M_n X$$

is a trivial fibration for $n \geq 0$;
2) *a Reedy trivial cofibration if and only if*

$$X_n \cup_{L_n X} L_n Y \to Y_n$$

is a trivial cofibration for all $n \geq 0$.

Proof: Let $\Delta^{n,k} = d^0\Delta^{n-1} \cup \cdots \cup d^k\Delta^{n-1} \subseteq \partial\Delta^n$, $-1 \leq k \leq n$. Then $\Delta^{n,-1} = \phi$ and $\Delta^{n,n} = \partial\Delta^n$. There are pushout diagrams $-1 \leq k \leq n-1$

$$\begin{tikzcd} \Delta^{n-1,k} \arrow[r] \arrow[d] & \Delta^{n-1} \arrow[d] \\ \Delta^{n,k} \arrow[r] & \Delta^{n,k+1}. \end{tikzcd}$$

Taking matching objects yields a natural pullback square

$$\begin{tikzcd} M_{n,k+1}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ M_{n,k}X \arrow[r] & M_{n-1,k}X \end{tikzcd}$$

where we have written $M_{n,k}X$ for $M_{\Delta^{n,k}}X$. It follows that there is a pullback square

$$\begin{tikzcd} Y_n \times_{M_{n,k+1}Y} M_{n,k+1}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ Y_n \times_{M_{n,k}Y} M_{n,k}X \arrow[r] & Y_{n-1} \times_{M_{n-1,k}Y} M_{n-1,k}X. \end{tikzcd} \tag{2.3}$$

Assume $f : X \to Y$ is a Reedy fibration. Then $X_0 \to Y_0$ is a fibration and this begins an induction where the induction hypothesis is that

$$X_{n-1} \to Y_{n-1} \times_{M_{n-1,k}Y} M_{n-1,k}X \tag{2.4}$$

is a fibration. To complete the inductive step, one uses (2.3) to show

$$Y_n \times_{M_{n,k+1}Y} M_{n,k+1}X \to Y_n \times_{M_{n,k}Y} M_{n,k}X \tag{2.5}$$

is a fibration for all k, $-1 \leq k \leq n-1$. Since composites of fibrations are fibrations and $X_n \to Y_n \times_{M_nY} M_nX$ is a fibration, we close the loop.

Now suppose $f : X \to Y$ is a Reedy trivial fibration. Then, inductively, each of the maps of (2.4) is a trivial fibration. For the inductive step, use (2.3) to show each of the maps (2.5) is a trivial fibration. Then the axiom **CM2** and the fact that $X_n \to Y_n$ a trivial fibration finishes the argument. In particular $X_n \to Y_n \times_{M_nY} M_nX$ is a trivial fibration for all n.

Conversely, suppose $X_n \to X_n \times_{M_nY} M_nX$ is a trivial fibration. Then one runs a similar argument to conclude $X_n \to Y_n$ is a trivial fibration.

The argument for part 2) is similar, using the pushout diagram

$$\begin{tikzcd} L_{n-1,k}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ L_{n,k}X \arrow[r] & L_{n,k+1}X \end{tikzcd}$$

of Proposition 1.27. □

As a corollary of the proof of Lemma 2.2 we have the following:

COROLLARY 2.6.

(1) *Every Reedy fibration $p : X \to Y$ of $s\mathcal{C}$ is a level fibration in the sense that all component maps $p : X_n \to Y_n$ are fibrations of $\mathcal{C}$.*

(2) *Every Reedy cofibration $i : A \to B$ of $s\mathcal{C}$ is a level cofibration in the sense that all component maps $i : A_n \to B_n$ are cofibrations of $\mathcal{C}$.*

We break the proof of the verification of the Reedy model category structure into several steps.

Lemma 2.7. *The Reedy structure on $s\mathcal{C}$ satisfies the lifting axiom* **CM4**.

Proof: Suppose we are given a lifting problem in $s\mathcal{C}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\ B \arrow[r] & Y \end{tikzcd}$$

where j is cofibration, q is a fibration and either j or q is a weak equivalence. We inductively solve the lifting problems

$$\begin{tikzcd} \mathrm{sk}^A_{n-1} B \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{sk}^A_n B \arrow[r] & Y \end{tikzcd}$$

for $n \geq 0$. Because of the pushout diagram of Proposition 1.9 it is sufficient to solve the lifting problems, with $L_n(j) = A_n \cup_{L_n A} L_n B$

$$\begin{tikzcd} L_n(j) \otimes \Delta^n \cup_{L_n(j) \otimes \partial\Delta^n} B_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ B_n \otimes \Delta^n \arrow[r] & Y. \end{tikzcd}$$

By an adjunction argument this is equivalent to the lifting problem

$$\begin{tikzcd} A_n \cup_{L_n A} L_n B \arrow[r] \arrow[d] & X_n \arrow[d] \\ B_n \arrow[r] & Y_n \times_{M_n Y} M_n X. \end{tikzcd}$$

This is solvable by Lemma 2.2. □

For the proof of the factorization axiom we need to know how much data we need to build a simplicial object.

Lemma 2.8. *Let $X \in s\mathcal{C}$. Then $\mathrm{sk}_n X$ is determined up to natural isomorphism by the following natural data: $\mathrm{sk}_{n-1} X$, X_n, and maps $L_n X \to X_n \to M_n X$ so that the composite $L_n X \to M_n X$ is the canonical map.*

Proof: The given map $X_n \to M_n X$ is adjoint to a map $X_n \otimes \partial\Delta^n \to X$ which factors uniquely through $\mathrm{sk}_{n-1} X$. The data listed thus yields the pushout square

$$\begin{tikzcd} L_n X \otimes \Delta^n \cup_{L_n X \otimes \partial\Delta^n} X_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} X \arrow[d] \\ X_n \otimes \Delta^n \arrow[r] & \mathrm{sk}_n X \end{tikzcd}$$

of Proposition 1.7. □

This can be restricted: let $s_n\mathcal{C}$ be the functor category $\mathcal{C}^{\mathbf{\Delta}_n^{\mathrm{op}}}$ and $X \in s_n\mathcal{C}$. Let $r_* X \in s_{n-1}\mathcal{C}$ be the restriction and $r^* : s_{n-1}\mathcal{C} \to s_n\mathcal{C}$ the left Kan extension functor. Hence $(r^* r_* X)_n = L_n X$.

Lemma 2.8 immediately implies

Lemma 2.9. *Let $X \in s_n\mathcal{C}$. Then X is determined up to natural isomorphism by the following natural data: $r_* X$, X_n, and maps $L_n X \to X_n \to M_n X$ so that the composite $L_n X \to M_n X$ is the canonical map.*

Proof: Let $i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$ be the restriction and $i_n^* : s_n\mathcal{C} \to s\mathcal{C}$ the left adjoint. Then $i_{n*} i_n^* \cong 1$ and $i_n^* i_{n*} = \mathrm{sk}_n$. Thus

$$i_n^* X \cong \mathrm{sk}_n i_n^* X$$

is determined up to natural isomorphism by $\mathrm{sk}_{n-1} i_n^* X \cong i_{n-1}^* r_* X$, the object X_n and maps $L_n i_n^* X \to X_n \to M_n i_n^* X$ that compose to the canonical map $L_n i_n^* X \to M_n i_n^* X$. But $L_n X \cong L_n i_n^* X$ and $M_n X \cong M_n i_n^* X$. □

Lemma 2.10. *The Reedy structure on $s\mathcal{C}$ satisfies the factorization axiom* **CM5**.

Proof: Let us do the trivial cofibration-fibration factorization (compare the proof of Lemma IV.3.6).

Let $X \to Y$ be a morphism in $s\mathcal{C}$ and let $i_{n*} X \to i_{n*} Y$ be the induced morphism in $s_n\mathcal{C}$. For each $n \geq 0$ we construct a factorization

$$i_{n*} X \to Z(n) \to i_{n*} Y$$

in $s_n\mathcal{C}$ with the property that restricted to $s_{n-1}\mathcal{C}$ we get the factorization $i_{(n-1)*} X \to Z(n-1) \to i_{(n-1)*} Y$.

For $n = 0$, simply factor $X_0 \to Y_0$ as

$$X_0 \xrightarrow{j} Z(0) \xrightarrow{q} Y_0$$

where j is a trivial cofibration in $\mathcal{C}$ and q is a fibration in $\mathcal{C}$. Suppose the factorization in $s_{n-1}\mathcal{C}$ has been constructed. Then there is a commutative diagram

$$\begin{tikzcd}
L_nX \arrow[r] \arrow[d] & X_n \arrow[r] & M_nX \arrow[d] \\
L_nZ \arrow[d] & & M_nZ \arrow[d] \\
L_nY \arrow[r] & Y_n \arrow[r] & M_nY
\end{tikzcd}$$

and hence a map

$$X_n \cup_{L_nX} L_nZ \to Y_n \times_{M_nY} M_nZ.$$

Factor this map as

$$X_n \cup_{L_nX} L_nZ \xrightarrow{j} Z_n \xrightarrow{q} Y_n \times_{M_nY} M_nZ \tag{2.11}$$

where j is a trivial cofibration and q is a fibration. The morphisms j and q yield diagrams

$$\begin{tikzcd}
L_nX \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & M_nX \arrow[d] \\
L_nZ \arrow[r] \arrow[d] & Z_n \arrow[r] \arrow[d] & M_nZ \arrow[d] \\
L_nY \arrow[r] & Y_n \arrow[r] & M_nY
\end{tikzcd}$$

and so Lemma 2.9 produces a factorization

$$i_{n*}X \to Z(n) \to i_{n*}Y.$$

Finally, define $Z \in s\mathcal{C}$ by $Z_k = Z(n)_k$, $k \leq n$. There is a factoring

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

and using (2.11) j is a trivial cofibration by Lemma 2.2 and q is a fibration by definition.

The other factorization is similar. $\square$

We now state

THEOREM 2.12. *With the definitions of Reedy weak equivalence, cofibration, and fibration given in Definition 2.1, the category $s\mathcal{C}$ is a closed model category.*

Proof: The axioms **CM1–CM3** are easy and Lemmas 2.7 and 2.10 prove **CM4** and **CM5** respectively. □

Next suppose $\mathcal{C}$ is in fact a simplicial model category. For $K \in \mathbf{S}$ and $Y, Z \in \mathcal{C}$ write $Z \square K$, $\mathbf{hom}_{\mathcal{C}}(K, Z)$, and $\mathbf{Hom}_{\mathcal{C}}(Y, Z)$ for the tensor object, the exponential object, and the mapping space. We use this notation to distinguish the internal object $Z \square K$ from the construction $Z \otimes K$ defined by the simplicial structure.

The category $s\mathcal{C}$ inherits a simplicial structure. If $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, then $X \square K$ and $\mathbf{hom}_{\mathcal{C}}(K, X)$ are defined level-wise

$$(X \square K)_n = X_n \square K \text{ and } \mathbf{hom}_{\mathcal{C}}(K, X)_n = \mathbf{hom}_{\mathcal{C}}(K, X_n).$$

The mapping space is defined by the usual formula

$$\mathbf{Hom}_{s\mathcal{C}}(X, Y)_n \cong \hom_{s\mathcal{C}}(X \square \Delta^n, Y).$$

We call this the internal structure on $s\mathcal{C}$.

Corollary 2.13. *With this internal simplicial structure on $s\mathcal{C}$, the Reedy model category structure is a simplicial model category.*

Proof: We claim $M_n(\mathbf{hom}_{\mathcal{C}}(K, Y)) \cong \mathbf{hom}_{\mathcal{C}}(K, M_n Y)$. For there is a sequence of natural isomorphisms, $Z \in \mathcal{C}$,

$$\begin{aligned}
\hom_{\mathcal{C}}(Z, M_n(\mathbf{hom}_{\mathcal{C}}(K, Y))) &\cong \hom_{s\mathcal{C}}(Z \otimes \partial\Delta^n, \mathbf{hom}_{\mathcal{C}}(K, Y)) \\
&\cong \hom_{s\mathcal{C}}((Z \otimes \partial\Delta^n) \square K, Y) \\
&\cong \hom_{s\mathcal{C}}((Z \square K) \otimes \partial\Delta^n, Y) \\
&\cong \hom_{s\mathcal{C}}(Z \square K, M_n Y) \\
&\cong \hom_{s\mathcal{C}}(Z, \mathbf{hom}_{\mathcal{C}}(K, M_n Y)).
\end{aligned}$$

The isomorphism $(Z \otimes \partial\Delta^n) \square K \cong (Z \square K) \otimes \partial\Delta^n$ follows by a level-wise calculation. The result follows from the claim using Proposition II.3.13. □

One can ask if $s\mathcal{C}$ in the Reedy model category is a simplicial model category in the standard simplicial structure obtained by Quillen's method (as in the previous section). The answer is no; for if $Z \in \mathcal{C}$ is cofibrant in $\mathcal{C}$, then

$$1 \otimes d^0 : Z \otimes \Delta^0 \to Z \otimes \Delta^1$$

is a Reedy cofibration, but not, in general, a Reedy weak equivalence (see Remark IV.3.13).

As a corollary of the proof of Theorem 2.12 (more specifically Lemma 2.7), we have the following:

Corollary 2.14. *Suppose that $j : A \to B$ is a Reedy cofibration. Then all maps*

$$L_n(j) \otimes \Delta^n \cup_{L_n(j)\otimes\partial\Delta^n} B_n \otimes \partial\Delta^n \to B_n \otimes \Delta^n$$

are Reedy cofibrations. In particular, if X is a Reedy cofibrant object, then all maps

$$L_nX \otimes \Delta^n \cup_{L_nX\otimes\partial\Delta^n} X_n \otimes \partial\Delta^n \to X_n \otimes \Delta^n$$

are Reedy cofibrations.

It is true (Proposition 2.15 below), however, that if $f : X \to Y$ is a Reedy cofibration in $s\mathcal{C}$ and $j : K \to L$ is a cofibration of simplicial sets, then the induced map

$$X \otimes L \cup_{X\otimes K} Y \otimes K \to Y \otimes L$$

is a Reedy cofibration which is a Reedy weak equivalence if f is a Reedy weak equivalence. The proof of this statement requires that we first recast the definition of Reedy fibration, and properly describe function complex objects for the external structure.

Recall the definition: a map $p : Z \to W$ is a Reedy fibration of $s\mathcal{C}$ if the induced map

$$Z_n \to W_n \times_{M_nW} M_nZ$$

is a fibration of the closed model category $\mathcal{C}$ for every $n \geq 0$. This means that all such maps should have the right lifting property with respect to all trivial cofibrations of $\mathcal{C}$, so an adjointness argument says that $p : Z \to W$ is a Reedy fibration if it has the right lifting property in $s\mathcal{C}$ with respect to all maps

$$B \otimes \partial\Delta^n \cup_{A\otimes\partial\Delta^n} A \otimes \Delta^n \to B \otimes \Delta^n$$

associated to trivial cofibrations $A \to B$ of $\mathcal{C}$ and simplicial set inclusions $\partial\Delta^n \hookrightarrow \Delta^n$. The underlying category $\mathcal{C}$ has all colimits, so p is a Reedy fibration if and only if it has the right lifting property with respect to all maps

$$B \otimes K \cup_{A\otimes K} A \otimes L \to B \otimes L$$

induced by trivial cofibrations $A \to B$ of $\mathcal{C}$ and simplicial set inclusions $K \hookrightarrow L$.

We have seen in Proposition 1.21 that the functor $M_K : s\mathcal{C} \to \mathcal{C}$ is right adjoint to $Z \mapsto Z \otimes K$, and this adjunction is defined and natural for all simplicial sets K. For such K, define a functor $\mathbf{M}_K : s\mathcal{C} \to s\mathcal{C}$ by specifying $\mathbf{M}_KY_n = M_{\Delta^n\times K}Y$. Then the adjunction isomorphisms

$$\hom(X_n, M_{\Delta^n\times K}Y) \cong \hom(X_n \otimes \Delta^n \otimes K, Y)$$

jointly induce an adjunction isomorphism

$$\hom(X, \mathbf{M}_KY) \cong \hom(X \otimes K, Y).$$

To see this, it helps to know that there is a natural coequalizer

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} X_n \otimes \Delta^m \rightrightarrows \bigsqcup_n X_n \otimes \Delta^n \to X$$

in the simplicial object category $s\mathcal{C}$, and that tensoring with K preserves colimits.

Proposition 2.15.

(1) *Let* $f : X \to Y$ *be a Reedy cofibration in* $s\mathcal{C}$ *and* $j : K \to L$ *a cofibration in* $\mathbf{S}$. *Then*

$$X \otimes L \cup_{X \otimes K} Y \otimes K \to Y \otimes L$$

is a Reedy cofibration which is a Reedy weak equivalence if f *is a Reedy weak equivalence.*

(2) *Suppose* $f : X \to Y$ *is a Reedy fibration in* $s\mathcal{C}$ *and* $j : K \to L$ *is a cofibration in* $\mathbf{S}$. *Then*

$$\mathbf{M}_L X \to \mathbf{M}_K X \times_{\mathbf{M}_K Y} \mathbf{M}_L Y$$

is a Reedy fibration which is a Reedy weak equivalence if f *is a Reedy weak equivalence.*

Proof: These two statements are equivalent by an adjunction argument, and we shall prove the second.

The map

$$\mathbf{M}_L X \to \mathbf{M}_K X \times_{\mathbf{M}_K Y} \mathbf{M}_L Y$$

has the right lifting property with respect to a class of maps $C \to D$ if and only if $f : X \to Y$ has the right lifting property with respect to all induced maps

$$D \otimes K \cup_{C \otimes K} C \otimes L \to D \otimes L$$

The corresponding map induced by the morphism

$$B \otimes \partial\Delta^n \cup_{A \otimes \partial\Delta^n} A \otimes \Delta^n \to B \otimes \Delta^n$$

arising from a trivial cofibration $A \to B$ of $\mathcal{C}$ has the form

$$B \otimes K' \cup_{A \otimes K'} A \otimes L' \to B \otimes L'$$

where the simplicial set inclusion $K' \hookrightarrow L'$ is the morphism

$$L \times \partial\Delta^n \cup_{K \times \partial\Delta^n} K \times \Delta^n \hookrightarrow L \times \Delta^n.$$

Any Reedy fibration $f : X \to Y$ has the right lifting property with respect to all such morphisms.

The second part of claim (2) follows in a similar way from Lemma 2.2. $\square$

3. Geometric realization.

Suppose $\mathcal{C}$ is a simplicial category and $X \in s\mathcal{C}$. Then the geometric realization $|X| \in \mathcal{C}$ is defined by the coequalizer diagram

$$\bigsqcup_{\phi:\mathbf{n}\to\mathbf{m}} X_m \Box \Delta^n \underset{d_1}{\overset{d_0}{\rightrightarrows}} \bigsqcup_{n\geq 0} X_n \Box \Delta^n \to |X| \tag{3.1}$$

where ϕ runs over the morphisms of $\mathbf{\Delta}$, and d_0 and d_1 on the factor associated to $\phi : \mathbf{n} \to \mathbf{m}$ are respectively

$$X_m \Box \Delta^n \xrightarrow{\phi^*\Box 1} X_n \Box \Delta^n \longrightarrow \bigsqcup_{n\geq 0} X_n \Box \Delta^n$$

$$X_m \Box \Delta^n \xrightarrow{1\Box\phi} X_m \Box \Delta^m \longrightarrow \bigsqcup_{n\geq 0} X_n \Box \Delta^n.$$

This is the obvious generalization of the geometric realization of Chapters I and III. Note that $|X|$ is a coend:

$$|X| = \int^{\mathbf{\Delta}} X \Box \Delta,$$

where Δ denotes the covariant functor $\mathbf{n} \mapsto \Delta^n$ on $\mathbf{\Delta}$. We discuss the homotopical properties of $|X|$.

First note that $|\cdot| : s\mathcal{C} \to \mathcal{C}$ has a right adjoint

$$Y \mapsto Y^{\Delta} = \{\mathbf{hom}_{\mathcal{C}}(\Delta^n, Y)\}. \tag{3.2}$$

If we give $s\mathcal{C}$ the internal (or level-wise) simplicial structure induced from $\mathcal{C}$, it follows immediately that if $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, then

$$|X \Box K| \cong |X| \Box K. \tag{3.3}$$

Indeed, $\hom_{s\mathcal{C}}(X \Box K, Y^{\Delta}) \cong \hom_{s\mathcal{C}}(X, \mathbf{hom}_{\mathcal{C}}(K, Y)^{\Delta})$.

Now assume $\mathcal{C}$ is a simplicial model category. Endow $s\mathcal{C}$ with the Reedy model category structure. By Corollary 2.13, this is a simplicial model category in the internal simplicial structure.

Lemma 3.4. *The functor $(\cdot)^{\Delta} : \mathcal{C} \to s\mathcal{C}$ preserves fibrations and trivial fibrations.*

PROOF: The first point to be proved is this: if $K \in \mathbf{S}$ and $Y \in \mathcal{C}$, then

$$M_K(Y^\Delta) \cong \mathbf{hom}_{\mathcal{C}}(K, Y). \tag{3.5}$$

There are isomorphisms

$$\begin{aligned} M_K(Y^\Delta) &\cong \varprojlim_{\Delta^n \to K} (Y^\Delta)_n \\ &\cong \varprojlim_{\Delta^n \to K} \mathbf{hom}_{\mathcal{C}}(\Delta^n, Y) \\ &\cong \mathbf{hom}_{\mathcal{C}}(\varinjlim_{\Delta^n \to K} \Delta^n, Y) \\ &\cong \mathbf{hom}_{\mathcal{C}}(K, Y), \end{aligned}$$

where the limits and colimits are indexed over objects $\Delta^n \to K$ of the simplex category $\mathbf{\Delta} \downarrow K$ of K (cf. Example 1.23).

Now let $X \to Y$ be a fibration in $\mathcal{C}$. Then

$$(X^\Delta)_n \to (Y^\Delta)_n \times_{M_n(Y^\Delta)} M_n(X^\Delta)$$

is isomorphic to

$$\mathbf{hom}_{\mathcal{C}}(\Delta^n, X) \to \mathbf{hom}_{\mathcal{C}}(\Delta^n, Y) \times_{\mathbf{hom}_{\mathcal{C}}(\partial\Delta^n, Y)} \mathbf{hom}_{\mathcal{C}}(\partial\Delta^n, X)$$

and the result follows from Lemma 2.2 and **SM7** for $\mathcal{C}$.

The claim about preservation of trivial fibrations has a similar proof. □

PROPOSITION 3.6. *The geometric realization functor* $|\cdot| : s\mathcal{C} \to \mathcal{C}$ *preserves cofibrations, trivial cofibrations and weak equivalences between Reedy cofibrant objects.*

PROOF: Use Lemma 3.4 and Lemma II.7.9. □

The proof of Lemma 3.4 implicitly involves the assertion that if $Z \in s\mathcal{C}$ is constant and $K \in \mathbf{S}$ then there is a natural isomorphism

$$|Z \otimes K| \cong Z \square K. \tag{3.7}$$

Indeed, using Proposition 1.10 the isomorphism (3.5), we have

$$\hom_{\mathcal{C}}(|Z \otimes K|, Y) \cong \hom_{\mathcal{C}}(Z, \mathbf{hom}_{\mathcal{C}}(K, Y)),$$

and the assertion follows. Therefore, for $X \in s\mathcal{C}$ Reedy cofibrant, the realization comes with a natural skeletal filtration. Define

$$\mathrm{sk}_n |X| = |\,\mathrm{sk}_n X|.$$

Then Proposition 1.7 and the natural isomorphism of (3.7) together show that there are natural pushout squares

$$\begin{array}{ccc} X_n\Box\partial\Delta^n \cup_{L_n X\Box\partial\Delta^n} L_n\Box\Delta^n & \longrightarrow & \mathrm{sk}_{n-1}|X| \\ \downarrow & & \downarrow \\ X_n\Box\Delta^n & \longrightarrow & \mathrm{sk}_n|X|. \end{array} \tag{3.8}$$

This is because the realization functor $|\cdot|$ is a left adjoint and hence commutes with all colimits. If X is cofibrant, then Proposition 3.6 and Corollary 2.14 together imply that each of the maps $\mathrm{sk}_{n-1}|X| \to \mathrm{sk}_n|X|$ is a cofibration. Furthermore, again since the functor $|\cdot|$ commutes with colimits

$$\varinjlim_n \mathrm{sk}_n|X| \cong |X|. \tag{3.9}$$

Finally if X happens to be degeneracy free on some set $\{Z_n\}$ of objects in $\mathcal{C}$, then (3.8) can be refined (as in Corollary 1.14) to a pushout diagram

$$\begin{array}{ccc} Z_n\Box\partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1}|X| \\ \downarrow & & \downarrow \\ Z_n\Box\Delta^n & \longrightarrow & \mathrm{sk}_n|X|. \end{array} \tag{3.10}$$

The object $\mathrm{sk}_n|X|$ can also be described as a coend. Let $\mathrm{sk}_n\Delta$ be the functor from $\mathbf{\Delta}$ to $\mathbf{S}$ with

$$\mathbf{m} \mapsto \mathrm{sk}_n\Delta^m.$$

PROPOSITION 3.11. *Let $X \in s\mathcal{C}$. Then there is a natural isomorphism*

$$\mathrm{sk}_n|X| \cong \int^{\mathbf{\Delta}} X\Box\,\mathrm{sk}_n\Delta$$

and this isomorphism is compatible with the skeletal filtrations of source and target.

PROOF: There is a sequence of natural isomorphisms, where

$$i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$$

is the restriction functor

$$\begin{aligned} \hom_{\mathcal{C}}(\mathrm{sk}_n|X|, Y) &\cong \hom_{s\mathcal{C}}(\mathrm{sk}_n X, Y^{\Delta}) \\ &\cong \hom_{s\mathcal{C}}(i_n^* i_{n*} X, Y^{\Delta}) \\ &\cong \hom_{s\mathcal{C}}(X, i_n^! i_{n*} Y^{\Delta}) \end{aligned}$$

Now $i_n^! i_{n*} Y^\Delta \cong \mathrm{cosk}_n(Y^\Delta) \cong Y^{\mathrm{sk}_n \Delta}$, where $Y^{\mathrm{sk}_n \Delta}$ is defined on the simplex level by

$$Y_r^{\mathrm{sk}_n \Delta} = \mathbf{hom}_{\mathcal{C}}(\mathrm{sk}_n \Delta^r, Y).$$

It follows that

$$\begin{aligned} \hom_{\mathcal{C}}(\mathrm{sk}_n |X|, Y) &\cong \hom_{s\mathcal{C}}(X, Y^{\mathrm{sk}_n \Delta}) \\ &\cong \hom_{\mathcal{C}}(\int^{\Delta} X \square \mathrm{sk}_n \Delta, Y). \end{aligned} \qquad \square$$

4. Cosimplicial spaces.

The language and technology of the previous three sections can be used to give a discussion of the homotopy theory of cosimplicial spaces; that is, of the category $c\mathbf{S} = \mathbf{S}^{\mathbf{\Delta}}$ of functors from the ordinal number category to simplicial sets. We go through some of the details and give some examples. It turns out that cofibrations in $c\mathbf{S}$ have a very simple characterization; we close the section with a proof of this fact.

We begin with two important examples.

Example 4.1. Let $R = \mathbb{F}_p$, the prime field with $p > 0$, or let R be a subring of the rationals. The forgetful functor from R-modules to sets has left adjoint $X \mapsto RX$, where RX is the free R-module on X. These functors prolong to an adjoint pair between simplicial R-modules and simplicial sets. By abuse of notation we write

$$R : \mathbf{S} \to \mathbf{S}$$

for the composite of these two functors. Then R is the functor underlying a triple on $\mathbf{S}$ and, if $X \in \mathbf{S}$,

$$\pi_* RX \cong H_*(X; R).$$

Let $T : \mathbf{S} \to \mathbf{S}$ be any triple (or monad) on $\mathbf{S}$ with natural structure maps $\eta : X \to TX$ and $\epsilon : T^2 X \to TX$. If $X \in \mathbf{S}$ is any object, there is a natural augmented cosimplicial space

$$X \to T^\bullet X$$

with $(T^\bullet X)^n = T^{n+1} X$ and

$$\begin{aligned} d^i &= T^i \eta T^{n+1-i} : (T^\bullet X)^n \to (T^\bullet X)^{n+1} \\ s^i &= T^i \epsilon T^{n-i} : (T^\bullet X)^{n+1} \to (T^\bullet X)^n. \end{aligned}$$

The augmentation is given by $\eta : X \to TX = (T^\bullet X)^0$; note that

$$d^0 \eta = d^1 \eta : X \to (T^\bullet X)^1.$$

In particular, if we let $T = R : \mathbf{S} \to \mathbf{S}$ we get an augmented cosimplicial space

$$X \to R^\bullet X$$

with the property that d^i, $i \geq 1$, and s^i, $i \geq 0$, are all morphisms of simplicial R-modules. Furthermore, if we apply R one more time, the augmented cosimplicial R-module

$$RX \to R(R^\bullet X)$$

has a cosimplicial contraction; hence

$$H^s(H_*(R^\bullet X; R)) \cong \begin{cases} H_*(X; R), & s = 0 \\ 0, & s > 0. \end{cases}$$

The object $X \to R^\bullet X$ is a variation on the Bousfield-Kan R-resolution of X.

EXAMPLE 4.2. Let J be a small category and $\mathbf{S}^J$ the category of J-diagrams in $\mathbf{S}$. Let J^δ be the category with the same objects as J but no non-identity morphisms—J^δ is J made discrete. There is an inclusion functor $J^\delta \to J$, hence a restriction functor

$$r_* : \mathbf{S}^J \to \mathbf{S}^{J^\delta}.$$

The functor r_* has a right adjoint $r^!$ given by right Kan extension; in formulas

$$r^! X(j) = \prod_{j \to i} X(i)$$

where the product is over morphisms in J with source j. Let

$$T = r^! r_* : \mathbf{S}^J \to \mathbf{S}^J.$$

Then T is the functor of a triple on $\mathbf{S}^J$, and if $Y \in \mathbf{S}^J$, there is a natural cosimplicial object in $\mathbf{S}^J$

$$Y \to T^\bullet Y. \tag{4.3}$$

This cosimplicial object has the property that the underlying J^δ diagram has a cosimplicial contraction. Put another way, for each $j \in J$, the augmented cosimplicial space

$$Y(j) \to T^\bullet Y(j)$$

has a cosimplicial contraction. We can apply the functor $\varprojlim_J(\cdot)$ to (4.3) to obtain an augmented cosimplicial space

$$\varprojlim_J Y \to \varprojlim_J T^\bullet Y. \tag{4.4}$$

Note that $\varprojlim_J (T^\bullet Y)^n$ can be easily computed because

$$\varprojlim_J r^! X \cong \prod_j X(j)$$

where j runs over the objects on J. This last assertion follows from the isomorphisms

$$\begin{aligned}\hom_{\mathbf{S}}(Z, \varprojlim_J r^! X) &\cong \hom_{\mathbf{S}^J}(Z, r^! X) \\ &\cong \hom_{\mathbf{S}^{J^\delta}}(Z, X) \cong \prod_j \hom_{\mathbf{S}}(Z, X(j))\end{aligned}$$

where $Z \in \mathbf{S}$ is regarded as a constant diagram in $\mathbf{S}^J$ or $\mathbf{S}^{J^s}$.

It follows that the functor $T^n Y$ is defined for objects j of J by

$$T^n Y(j) = \prod_{j \to j_0 \to \cdots \to j_n} Y(j_n),$$

and that

$$\varprojlim_j T^n Y(j) \cong \prod_{j_0 \to \cdots \to j_n} Y(j_n).$$

The canonical map $\varprojlim_j T^n(j) \to T^n(j)$ can therefore be identified with the simplicial set map

$$\prod_{j_0 \to \cdots \to j_n} Y(j_n) \to \prod_{j \to j_0 \to \cdots \to j_n} Y(j_n)$$

whose projection onto the factor $Y(j_n)$ corresponding to the string

$$j \to j_0 \xrightarrow{\alpha_1} j_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} j_n$$

is the projection

$$pr_\alpha : \prod_{j_0 \to \cdots \to j_n} Y(j_n) \to Y(j_n)$$

corresponding to the string

$$\alpha : j_0 \xrightarrow{\alpha_1} j_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} j_n.$$

One also finds that the cosimplicial structure map associated to $\theta : \mathbf{m} \to \mathbf{n}$ for the object $\varprojlim_j T^\bullet Y(j)$ can be identified with the unique simplicial set map

$$\prod_{i_0 \to \cdots \to i_m} Y(i_m) \xrightarrow{\theta_*} \prod_{j_0 \to \cdots \to j_n} Y(j_n)$$

which makes the diagrams

$$\begin{tikzcd}
\prod_{i_0 \to \cdots \to i_m} Y(i_m) \arrow[r, "\theta^*"] \arrow[d, "pr_{\alpha \cdot \theta}"'] & \prod_{j_0 \to \cdots \to j_n} Y(j_n) \arrow[d, "pr_\alpha"] \\
Y(j_{\theta(m)}) \arrow[r, "(\alpha_n \cdots \alpha_{\theta(m)+1})_*"'] & Y(j_n)
\end{tikzcd}$$

commute.

This is the standard description of the cosimplicial object which underlies the homotopy inverse limit of $Y \in \mathbf{S}^J$ — see Section VIII.1.

With these examples in hand, we begin to analyze the homotopy theory of $c\mathbf{S}$. Since $\mathbf{S}$ is a simplicial model category, so is the opposite category $\mathbf{S}^{\mathrm{op}}$. Thus the category $(s(\mathbf{S}^{\mathrm{op}}))^{\mathrm{op}} = c\mathbf{S}$ acquires a Reedy model category structure as in Section 2. We take some care with the definitions as the opposite category device can be confusing.

To begin with, $c\mathbf{S}$ is a simplicial category: if $K \in \mathbf{S}$ and $X \in c\mathbf{S}$ define $X \square K$ and $\mathbf{hom}_{c\mathbf{S}}(K, X)$ in $c\mathbf{S}$ by

$$(X\square K)^n = X^n \times K \tag{4.5}$$

and

$$\mathbf{hom}_{c\mathbf{S}}(K, X)^n = \mathbf{hom}(K, X^n). \tag{4.6}$$

One often writes $X \otimes K$ for $X\square K$; however, we are—in this chapter—reserving the tensor product notation for the external operation on $s\mathcal{C}$. The mapping spaces functor is then

$$\mathbf{Hom}_{c\mathbf{S}}(X, Y)_n \cong \hom_{c\mathbf{S}}(X\square\Delta^n, Y). \tag{4.7}$$

There are also latching and matching objects, but at this point the literature goes to pieces. The matching objects in $c\mathbf{S}$ defined in [BK] are the latching objects in $s(\mathbf{S}^{\mathrm{op}})$ as defined in Section 1. Since we would hope the reader will turn to the work of Bousfield and Kan as needed, we will be explicit.

Let $X \in c\mathbf{S}$. The n^{th} *matching space* $M^n X \in \mathbf{S}$ is the $(n+1)^{\text{st}}$ *latching* object $L_{n+1}X$ of $X \in s(\mathbf{S}^{\mathrm{op}})$. Specifically,

$$M^n X \cong \varprojlim_{\phi:\mathbf{n+1}\to\mathbf{k}} X^k \tag{4.8}$$

where $\phi : \mathbf{n+1} \to \mathbf{k}$ runs over the surjections in $\mathbf{\Delta}$ with $k \le n$. Thus $n \ge -1$, and Remark 1.8 implies the following:

LEMMA 4.9.

(1) *The simplicial set $M^{-1}X$ is isomorphic to the terminal object $*$.*

(2) *There is an isomorphism $M^0 X \cong X^0$, and the canonical map $X^1 \to M^0 X$ can be identified with the codegeneracy map $s^0 : X^1 \to X^0$.*

(3) *For $n > 0$, the object $M^n X$ is defined by the equalizer*

$$M^n X \to \prod_{i=0}^{n} X^n \rightrightarrows \prod_{0\le i<j\le n} X^{n-1}$$

where the images of the two displayed maps on the factor corresponding to the relation $i < j$ on X_{n-1} are given by the composites

$$\prod_{i=0}^{n} X^n \xrightarrow{pr_j} X^n \xrightarrow{s^i} X^{n-1} \quad \text{and} \quad \prod_{i=0}^{n} X^n \xrightarrow{pr_i} X^n \xrightarrow{s^{j-1}} X^{n-1}.$$

The canonical map $s : X^{n+1} \to M^n X$ is induced by the codegeneracies $s^i : X^{n+1} \to X^n$.

The reader should be aware of 1) the shift in indices $M^n X = L_{n+1} X$ and 2) the superscript versus the subscript: $M^n X \neq M_n X$.

A map $f : X \to Y$ in $c\mathbf{S}$ is a *fibration* if and only if

$$X^{n+1} \to Y^{n+1} \times_{M^n Y} M^n X \tag{4.10}$$

is a fibration for $n \geq -1$.

Similarly, there are latching objects. If $X \in c\mathbf{S}$, $L^n X = M_{n+1} X$, the matching spaces of $X \in s(\mathbf{S}^{\mathrm{op}})$, where $n \geq -1$; thus

$$L^n X = \varinjlim_{\phi : \mathbf{k} \to \mathbf{n+1}} X^k$$

where ϕ runs over the injections in $\mathbf{\Delta}$ with $k \leq n$. The following is a consequence of Remark 1.19:

LEMMA 4.11.

(1) *The space $L^{-1} X$ is the initial object $\emptyset$ in the category of simplicial sets.*

(2) *There is an isomorphism $L^0 X \cong X^0 \sqcup X^0$, and the canonical map $L^0 X \to X^1$ can be identified with the coproduct $d = (d^0, d^1) : X^0 \sqcup X^0 \to X^1$ of the coface maps $d^0, d^1 : X^0 \to X^1$.*

(3) *For $n > 1$, the latching object $L^n X$ is defined by a coequalizer diagram*

$$\bigsqcup_{0 \leq i < j \leq n+1} X^{n-1} \rightrightarrows \bigsqcup_{i=0}^{n+1} X^n \to L^n X.$$

Here, the restrictions of the displayed maps on the summand X^{n-1} corresponding to the relation $i < j$ are the composites

$$X^{n-1} \xrightarrow{d^i} X^n \xrightarrow{in_j} \bigsqcup_{i=0}^{n+1} X^n \quad \text{and} \quad X^{n-1} \xrightarrow{d^{j-1}} X^n \xrightarrow{in_i} \bigsqcup_{i=0}^{n+1} X^n.$$

The canonical map $d : L^n X \to X^{n+1}$ is induced by the coface maps $d^i : X^n \to X^{n+1}$.

A morphism $X \to Y$ in $c\mathbf{S}$ is a *cofibration* if and only if

$$X^{n+1} \cup_{L^n X} L^n Y \to Y^{n+1} \tag{4.12}$$

is a cofibration (that is, inclusion) of simplicial sets.

Finally, we define a morphism $X \to Y$ in $c\mathbf{S}$ to be a weak equivalence if $X^n \to Y^n$ is a weak equivalence for all $n \geq 1$.

Theorem 4.13. *With the definitions above, the category $c\mathbf{S}$ of cosimplicial spaces is a proper closed simplicial model category.*

Proof: Applying Theorem 2.12 and Corollary 2.13 to the case of the category $s\mathbf{S}^{op}$ of simplicial objects in $\mathbf{S}^{op}$ gives the simplicial model structure. Properness is a consequence of Corollary 2.6 $\square$

We can give a simple characterization of cofibrations in $c\mathbf{S}$, and along the way show that $c\mathbf{S}$ is cofibrantly generated. First, let us define a set of specific cofibrations.

The functors from $c\mathbf{S} \to \mathbf{S}$

$$\rho_n : X \mapsto X^n, n \geq 0$$

and

$$\mu_n : X \mapsto M^n X, n \geq -1$$

all have left adjoints, given by variations on left Kan extension. Indeed, the adjoint to ρ_n is given by the formula

$$(\rho_n^* Y)^k = \bigsqcup_{\phi:\mathbf{n}\to\mathbf{k}} Y$$

where ϕ runs over all morphisms in $\mathbf{\Delta}$ with source $\mathbf{n}$. This is a left Kan extension.

The adjoint to μ_n is slightly more complicated: if J is the category with objects surjections $\mathbf{n+1} \to \mathbf{k}$ in $\mathbf{\Delta}$, and $r : J \to \mathbf{\Delta}$ the functor sending $\mathbf{n+1} \to \mathbf{k}$ to $\mathbf{k}$, then the left adjoint μ_n^* to μ_n is characterized by

$$\begin{aligned} \hom_{c\mathbf{S}}(\mu_n^* Z, X) &\cong \hom_{S^J}(Z, r_* X) \\ &\cong \hom_{\mathbf{S}}(Z, M^n X) \end{aligned}$$

where $Z \in \mathbf{S}^J$ is the constant diagram. Thus $\mu_n^* Z$ is a left Kan extension of a constant diagram. Alternatively, one can use the equalizer description of $M^n X$ given in Lemma 4.9 to show that there is a natural coequalizer

$$\bigsqcup_{0 \leq i < j \leq n} \rho_{n-1} Z \rightrightarrows \bigsqcup_{i=o}^{n} \rho_n Z \to \mu_n Z$$

for $n > 0$, and that $\mu_0 Z \cong \rho_0 Z$.

Note that the natural transformation

$$s : X^n \to M^{n-1}X$$

induces a natural transformation

$$\mu_{n-1}^* Z \to \rho_n^* Z.$$

Define morphisms in $c\mathbf{S}$ as follows:

$$\rho_n^* \partial\Delta^m \cup_{\mu_{n-1}^* \partial\Delta^m} \mu_{n-1}^* \Delta^m = \partial\Delta \begin{bmatrix} m \\ n \end{bmatrix} \xrightarrow{i_n^m} \Delta \begin{bmatrix} m \\ n \end{bmatrix} = \rho_n^* \Delta^m, \tag{4.14}$$

for $n \geq 0$, $m \geq 0$, and

$$\rho_n^* \Lambda_k^m \cup_{\mu_{n-1}^* \Lambda_k^m} \mu_{n-1}^* \Delta^m = \Delta \begin{bmatrix} m \\ n,k \end{bmatrix} \xrightarrow{j_{n,k}^m} \Delta \begin{bmatrix} m \\ n \end{bmatrix} = \rho_n^* \Delta^m, \tag{4.15}$$

for $n \geq 0$, $0 \leq k \leq m$, $m \geq 1$.

LEMMA 4.16. *A morphism $f : X \to Y$ in $c\mathbf{S}$ is a fibration if and only if it has the right lifting property with respect to the morphisms $j_{n,k}^m$ of (4.15). A morphism in $c\mathbf{S}$ is a trivial fibration if and only if it has the right lifting property with respect to the morphisms i_n^m of (4.14).*

PROOF: We prove the trivial fibration case; the other is similar. A lifting problem

$$\begin{tikzcd} \partial\Delta \begin{bmatrix} m \\ n \end{bmatrix} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta \begin{bmatrix} m \\ n \end{bmatrix} \arrow[r] & Y \end{tikzcd}$$

is equivalent, by adjointness to a lifting problem

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d] & X^n \arrow[d, "s"] \\ \Delta^m \arrow[r] & Y^n \times_{M^{n-1}Y} M^{n-1}X. \end{tikzcd}$$

Lemma 2.2.2 implies that $f : X \to Y$ is a trivial fibration if and only if

$$(f, s) : X^n \to Y^n \times_{M^{n-1}Y} M^{n-1}X$$

is a trivial fibration for all n. The result follows. $\square$

PROPOSITION 4.17. *The simplicial model category structure on $c\mathbf{S}$ is cofibrantly generated: the morphisms i_n^m of (4.14) generate the cofibrations and the morphisms $j_{n,k}^m$ of (4.15) generate the trivial cofibrations.*

PROOF: In light of Lemma 4.16, the small object argument now applies. □

We can use this result to characterize cofibrations. If $X \in c\mathcal{C}$ is a cosimplicial object in any category $\mathcal{C}$ with enough limits, define the maximal augmentation H^0X by the equalizer diagram

$$H^0X \to X^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X^1.$$

Let $d^0 :^0 X \to X^0$ be the natural map.

PROPOSITION 4.18. *A morphism $f : X \to Y$ in $c\mathbf{S}$ is a cofibration if and only if $X^n \to Y^n$ is a cofibration in $\mathbf{S}$ for all $n \geq 0$ and the induced map $H^0X \to H^0Y$ of maximal augmentations is an isomorphism.*

We give the proof below, after some technical preliminaries.

Let $\mathbf{\Delta}_{-1}$ be the augmented ordinal number category; this has objects

$$\mathbf{n} = \{0, 1, \dots, n-1\}, n \geq -1,$$

$(-\mathbf{1} = \phi)$ and ordering preserving maps. An augmented cosimplicial object in $\mathcal{C}$ is a functor $X : \mathbf{\Delta}_{-1} \to \mathcal{C}$. We write $d^0 : X^{-1} \to X^0$ for the unique map. Note that the maximal augmentation extends any cosimplicial object to an augmented cosimplicial object.

LEMMA 4.19. *Let X be an augmented cosimplicial set, and*

$$Z^n = X^n - \bigcup_{i=0}^{n} d^i X^{n-1}$$

If $d^0 : X^{-1} \to X^0$ is isomorphic to the inclusion of the maximal augmentation, then the map

$$\bigsqcup_{\phi} Z^k \to X^n$$

with $\phi : \mathbf{k} \to \mathbf{n}$ running over all injections, $-1 \leq k \leq n$, is an isomorphism.

PROOF: Out of any cosimplicial set X we may construct a simplicial set Y "without d^0" as follows: $Y_n = X^n$ and

$$d_i = s^{n-i} : Y_n \to Y_{n-1},\ 1 \leq i \leq n$$
$$s_i = d^{n-i} : Y_n \to Y_{n+1},\ 0 \leq i \leq n.$$

Notice that this construction does not use $d^n : X^{n-1} \to X^n$. Let $Z_n \subseteq Y_n$ be the non-degenerate simplices:

$$Z_n = Y_n - \bigcup_i s_i Y_{n-1}.$$

The standard argument for simplicial sets (see Example 1.15) shows that

$$Y_n \cong \bigsqcup_{\psi:\mathbf{n}\to\mathbf{k}} Z_k$$

where ψ runs over the surjections in $\mathbf{\Delta}$. Unraveling the definitions shows that our claim will follow if we can show that $d^n : X^{n-1} \to X^n$ induces an isomorphism $Z^n \sqcup Z^{n-1} \xrightarrow{\cong} Z_n$ or, equivalently, an isomorphism

$$Z^{n-1} \xrightarrow{\cong} Z_n \cap d^n X^{n-1}.$$

First note that d^n does induce an injection

$$d^n : Z^{n-1} \to Z_n \cap d^n X^{n-1}. \tag{4.20}$$

For this, it is sufficient to show that if $y \in Z^{n-1}$, then $d^n y \in Z_n$; that is, if $y \notin \cup_{i=0}^{n-1} d^i X^{n-2}$, then $d^n y \notin \cup_{i=0}^{n-1} d^i X^{n-1}$. The contrapositive of this statement reads: if $d^n y \in \cup_{i=0}^{n-1} d^i X^{n-1}$, then $y \in \cup_{i=0}^{n-1} d^i X_{n-2}$. So assume $d^n y = d^i z$ with $i < n$. If $i < n-1$, then

$$z = s^i d^n y = d^{n-1} s^i y$$

so

$$d^n y = d^i z = d^i d^{n-1} s^i y = d^n d^i s^i y$$

and $y = d^i s^i y$. If $i = n-1$ and $n > 1$, then

$$y = s^{n-1} d^n y = s^{n-1} d^{n-1} z = z$$

so $d^n y = d^{n-1} y$; hence

$$y = s^{n-2} d^{n-1} y = s^{n-2} d^n y = d^{n-1} s^{n-2} y.$$

If $n = 1$ and $i = n-1$, we have $d^1 y = d^0 z$, hence $y = z$; since $d^0 : X^{-1} \to X^0$ is the inclusion of the maximal augmentation, $y = d^0 w$ for some w This is where the hypothesis is used.

We now must show that d^n, as in (4.20) is onto. If $n = 0$, $Z_0 = X^0$ and the result is clear. If $n \geq 1$, we need to show that if $x = d^n y$ and $x \notin \cup_{i=0}^{n-1} d^i X^{n-1}$, then $y \notin \cup_{i=0}^{n-1} d^i X^{n-2}$. The contrapositive of this statement is: if $x = d^n y$ and $y = d^i w$, $i \leq n-1$, then $x = d^j z$ for $j \leq n-1$. But

$$x = d^n y = d^n d^i w = d^i d^{n-1} w. \qquad \square$$

We can now prove Proposition 4.18:

PROOF OF 4.18: For the purposes of this argument, we say a morphism $f : X \to Y$ in $c\mathbf{S}$ has *Property* $\mathbf{C}$ if $X^n \to Y^n$ is a cofibration in $\mathbf{S}$ for all $n \geq 0$ and $H^0 X \cong H^0 Y$. We leave it as an exercise to show that the class of morphisms satisfying Property $\mathbf{C}$ is closed under isomorphisms, coproducts, retracts, cobase change, and colimits over ordinal numbers. Only the statement about cobase change is non-trivial. Furthermore, the generating cofibrations $\iota_n^m : \partial\Delta\begin{bmatrix} m \\ n \end{bmatrix} \to \Delta\begin{bmatrix} m \\ n \end{bmatrix}$ have Property $\mathbf{C}$. Hence Proposition 4.17 implies all cofibrations have Property $\mathbf{C}$.

For the converse, suppose $f : X \to Y$ has Property $\mathbf{C}$. Referring to Lemma 4.19, write $Z^n(X_m)$ for Z^n obtained from the cosimplicial set of m simplices X_m. Then, Lemma 4.19 implies

$$(L^0 X)_m \cong Z^0(X_m) \sqcup Z^{-1}(X_m) \sqcup Z^0(X_m) \sqcup Z^{-1}(X_m),$$

$$X_m^1 = Z^{-1}(X_m) \sqcup Z^0(X_m) \sqcup Z^0(X_m) \sqcup Z^1(X_m),$$

and if $n > 1$,

$$(L^{n-1} X)_m \cong \bigsqcup_{\phi : \mathbf{k} \to \mathbf{n}} Z^k(X_m)$$

with ϕ running over injections with $-1 \leq k < n$. Since $f : X_m^n \to Y_m^n$ is one-to-one, $Z^k(\cdot)$ is natural in f. Since $Z^{-1}(X_m) \cong Z^{-1}(Y_m)$,

$$[X^n \cup_{L^{n-1}X} L^{n-1} Y]_m \cong Z^n(X_m) \sqcup \bigsqcup_{\phi : \mathbf{k} \to \mathbf{n}} Z^k(Y_m)$$

for all $n \geq 0$. Again ϕ runs over injections k, $-1 \leq k < n$. The result follows. $\square$

SECOND PROOF OF PROPOSITION 4.18: Note, first of all, that by manipulating cosimplicial identities, one can show that all of the diagrams

$$\begin{CD} X^{n-2} @>{d^{j-1}}>> X^{n-1} \\ @V{d^i}VV @VV{d^i}V \\ X^{n-1} @>>{d^j}> X^n \end{CD}$$

are pullbacks. It follows that the maps $d : L^{n-1} X \to X^n$ is a monomorphism if $n > 1$. Note further that Lemma 4.11 says that $L^0 X = X^0 \sqcup X^0$.

Now suppose that $f : X \to Y$ is a cofibration. Then $f : X^0 \to Y^0$ is monic, so that $f : L^0 X \to L^0 Y$ is monic, and the assumption that the map

$$L^0 Y \cup_{L^0 X} X^1 \to Y^1$$

is a monomorphism implies that $f : X^1 \to Y^1$ is a monomorphism. One uses cosimplicial identities (using Remark 1.19) to show that if $f : X^i \to Y^i$ is a monomorphism for $i \leq n$ then the induced map $L^n X \to L^n Y$ is a monomorphism. Then the assumption that f is a cofibration implies that $f : X^{n+1} \to Y^{n+1}$ is a monomorphism in degree $n+1$. In particular, f is a monomorphism in all degrees.

To see that the map $f : H^0 X \to H^0 Y$ on maximal augmentations is an isomorphism, observe that there is a natural coequalizer

$$H^0 X \rightrightarrows X^0 \sqcup X^0 \to \mathrm{im}(d),$$

where

$$\mathrm{im}(d) = \mathrm{im}(d^0) \cup \mathrm{im}(d^1) \subset X^1.$$

Write $PO = \mathrm{im}(f) \cup \mathrm{im}(d) \subset Y^1$ for the diagram

$$\begin{tikzcd} X^0 \sqcup X^0 \arrow[r, "f \sqcup f"] \arrow[d, "d"'] & Y^0 \sqcup Y^0 \arrow[d, "d"] \\ X^1 \arrow[r, "f"'] & Y^1 \end{tikzcd}$$

This diagram is a pullback by cosimplicial identities, so the induced diagram

$$\begin{tikzcd} X^0 \sqcup X^0 \arrow[r, "f \sqcup f"] \arrow[d, "d"'] & Y^0 \sqcup Y^0 \arrow[d, "d_*"] \\ X^1 \arrow[r, "f_*"'] & PO \end{tikzcd} \tag{4.21}$$

is a pullback. This latter diagram (4.21) is also a pushout if and only if the induced diagram

$$\begin{tikzcd} X^0 \sqcup X^0 \arrow[r, "f \sqcup f"] \arrow[d, "d"'] & Y^0 \sqcup Y^0 \arrow[d, "d_*"] \\ \mathrm{im}(d) \arrow[r] & \mathrm{im}(d_*) \end{tikzcd}$$

is a pushout, since epi-monic factorizations are preserved by pushout. The diagram (4.21) is therefore a pushout if and only if f induces an isomorphism $H^0 X \cong H^0 Y$. The map

$$L^0 Y \cup_{L^0 X} Y^1 \to X^1$$

is therefore a monomorphism if and only if the diagram (4.21) is a pushout.

It follows that the map $f : H^0X \to H^0Y$ on maximal augmentations is a bijection.

For the converse, one can show that any levelwise monomorphism $f : X \to Y$ induces monomorphisms $L^nX \to L^nY$, and that all induced diagrams

$$\begin{tikzcd} L^nX \arrow[r, "d"] \arrow[d] & X^{n+1} \arrow[d] \\ L^nY \arrow[r, "d"'] & Y^{n+1} \end{tikzcd}$$

are pullbacks. The maps d are monomorphisms if $n > 0$, as are the vertical maps, so the induced maps

$$L^nY \cup_{L^nX} X^{n+1} \to Y^{n+1}$$

are monomorphisms for $n > 0$. The assertion that the map

$$L^0Y \cup_{L^0X} X^1 \to Y^1$$

is a monomorphism when f is a levelwise monic that induces an isomorphism of maximal augmentations is proved in the previous paragraph. $\square$

Proposition 4.18 makes it very easy to decide when an object of $c\mathbf{S}$ is cofibrant for the Reedy structure. For example, a constant object on a non-empty simplicial set is not cofibrant, but the standard simplices Δ^n form a cosimplicial space Δ which is cofibrant. Also, every subobject of a cofibrant simplicial space is cofibrant.