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# Simplicial Homotopy Theory

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# Simplicial Homotopy Theory

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# PREFACE

The origin of simplicial homotopy theory coincides with the beginning of algebraic topology almost a century ago. The thread of ideas started with the work of Poincaré and continued to the middle part of the 20th century in the form of combinatorial topology. The modern period began with the introduction of the notion of complete semi-simplicial complex, or simplicial set, by Eilenberg-Zilber in 1950, and evolved into a full blown homotopy theory in the work of Kan, beginning in the 1950s, and later Quillen in the 1960s.

The theory has always been one of simplices and their incidence relations, along with methods for constructing maps and homotopies of maps within these constraints. As such, the methods and ideas are algebraic and combinatorial and, despite the deep connection with the homotopy theory of topological spaces, exist completely outside any topological context. This point of view was effectively introduced by Kan, and later encoded by Quillen in the notion of a closed model category. Simplicial homotopy theory, and more generally the homotopy theories associated to closed model categories, can then be interpreted as a purely algebraic enterprise, which has had substantial applications throughout homological algebra, algebraic geometry, number theory and algebraic K-theory. The point is that homotopy is more than the standard variational principle from topology and analysis: homotopy theories are everywhere, along with functorial methods of relating them.

This book is, however, not quite so cosmological in scope. The theory has broad applications in many areas, but it has always been quite a sharp tool within ordinary homotopy theory — it is one of the fundamental sources of positive, qualitative and structural theorems in algebraic topology. We have concentrated on giving a modern account of the basic theory here, in a form that could serve as a model for corresponding results in other areas.

This book is intended to fill an obvious and expanding gap in the literature. The last major expository pieces in this area, namely [33], [67], [61] and [18], are all more than twenty-five years old. Furthermore, none of them take into account Quillen's ideas about closed model structures, which are now part of the foundations of the subject.

We have attempted to present an account that is as linear as possible and inclusive within reason. We begin in Chapter I with elementary definitions and examples of simplicial sets and the simplicial set category **S**, classifying objects, Kan complexes and fibrations, and then proceed quickly through much of the classical theory to proofs of the fundamental organizing theorems of the subject which appear in Section 11. These theorems assert that the category of simplicial sets satisfies Quillen's axioms for a closed model category, and that the associated homotopy category is equivalent to that arising from topological spaces. They are delicate but central results, and are the basis for all that follows.

Chapter I contains the definition of a closed model category. The foundations of abstract homotopy theory, as given by Quillen, start to appear in the first section of Chapter II. The "simplicial model structure" that most of the closed model structures appearing in nature exhibit is discussed in Sections 2–7. A simplicial model structure is an enrichment of the underlying category to simplicial sets which interacts with the closed model structure, like function spaces do for simplicial sets; the category of simplicial sets with function spaces is a standard example. Simplicial model categories have a singular technical advantage which is used repeatedly, in that weak equivalences can be detected in the associated homotopy category (Section 4). There is a detection calculus for simplicial model structures which leads to homotopy theories for various algebraic and diagram theoretic settings: this is given in Sections 5–7, and includes a discussion of cofibrantly generated closed model categories in Section 6 — it may be heavy going for the novice, but homotopy theories of diagrams almost characterize work in this area over the past ten years, and are deeply implicated in much current research. The chapter closes on a much more elementary note with a description of Quillen's non-abelian derived functor theory in Section 8, and a description of proper closed model categories, homotopy cartesian diagrams and gluing and cogluing lemmas in Section 9. All subsequent chapters depend on Chapters I and II.

Chapter III is a further repository of things that are used later, although perhaps not quite so pervasively. The fundamental groupoid is defined in Chapter I and then revisited here in Section III.1. Various equivalent formulations are presented, and the resulting theory is powerful enough to show, for example, that the fundamental groupoid of the classifying space of a small category is equivalent to the free groupoid on the category, and give a quick proof of the Van Kampen theorem. The closed model structure for simplicial abelian groups and the Dold-Kan correspondence relating simplicial abelian groups to chain complexes (ie. they're effectively the same thing) are the subject of Section 2. These ideas are the basis of most applications of simplicial homotopy theory and of closed model categories in homological algebra. Section 3 contains a proof of the Hurewicz theorem: Moore-Postnikov towers are introduced here in a self-contained way, and then treated more formally in Chapter VII. Kan's $Ex^\infty$-functor is a natural, combinatorial way of replacing a simplicial set up to weak equivalence by a Kan complex: we give updated proofs of its main properties in Section 4, involving some of the ideas from Section 1. The last section presents the Kan suspension, which appears later in Chapter V in connection with the loop group construction.

Chapter IV discusses the homotopy theory, or more properly homotopy theories, for bisimplicial sets and bisimplicial abelian groups, with major applications. Basic examples and constructions, including homotopy colimits and the diagonal complex, appear in the first section. Bisimplicial abelian groups, the subject of Section 2, are effectively bicomplexes, and hence have canonical associated spectral sequences. One of the central technical results is the generalized Eilenberg-Zilber theorem, which asserts that the diagonal and total complexes of a bisimplicial abelian group are chain homotopy equivalent. Three different closed model structures for bisimplicial sets, all of which talk about the same homotopy theory, are discussed in Section 3. They are all important, and in fact used simultaneously in the proof of the Bousfield-Friedlander theorem in Section 4, which gives a much used technical criterion for detecting fibre sequences arising from maps of bisimplicial sets. There is a small technical innovation in this proof, in that the so-called $\pi_*$-Kan condition is formulated in terms of certain fibred group objects being Kan fibrations. The chapter closes in Section 4 with proofs of Quillen's "Theorem B" and the group completion theorem. These results are detection principles for fibre sequences and homology fibre sequences arising from homotopy colimits, and are fundamental for algebraic $K$-theory and stable homotopy theory.

From the beginning, we take the point of view that simplicial sets are usually best viewed as set-valued contravariant functors defined on a category $\mathbf{\Delta}$ of ordinal numbers. This immediately leads, for example, to an easily manipulated notion of simplicial objects in a category $\mathcal{C}$: they're just functors $\mathbf{\Delta}^{op} \to \mathcal{C}$, so that morphisms between them become natural transformations, and so on. Chapter II contains a detailed treatment of the question of when the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ has a simplicial model structure.

Simplicial groups is one such category, and is the subject of Chapter V. We establish, in Sections 5 and 6, the classical equivalence of homotopy theories between simplicial groups and simplicial sets having one vertex, from a modern perspective. The method can the be souped up to give the Dwyer-Kan equivalence between the homotopy theories of simplicial groupoids and simplicial sets in Section 7. The techniques involve a new description of principal $G$-fibrations, for simplicial groups $G$, as cofibrant objects in a closed model structure on the category of $G$-spaces, or simplicial sets with $G$-action (Section 2). Then the classifying space for $G$ is the quotient by the $G$-action of any cofibrant model of a point in the category of $G$-spaces (Section 3); the classical $\overline{W}G$ construction is an example, but the proof is a bit interesting. We give a new treatment of $\overline{W}G$ as a simplicial object of universal cocycles in Section 4; one advantage of this method is that there is a completely analogous construction for simplicial groupoids, which is used for the results of Section 7. Our approach also depends on a specific closed model structure for simplicial sets with one vertex, which is given in Section 6. That same section contains a definition and proof of the main properties of the Milnor $FK$-construction, which is a functor taking values in simplicial groups that gives a model for loops suspension $\Omega\Sigma X$ of a given space $X$.

The first section of Chapter V contains a discussion of skeleta in the category of simplicial groups which is later used to show the technical (and necessary) result that the Kan loop group functor outputs cofibrant simplicial groups. Skeleta for simplicial sets first appear in a rather quick and dirty way in Section I.2. Skeleta for more general categories appear in various places: we have skeleta for simplicial groups in Chapter V, skeleta for bisimplicial sets in Section IV.3, and then skeleta for simplicial objects in more general categories later, in Section VII.1. In all cases, skeleta and coskeleta are left and right adjoints of truncation functors.

Chapter VI collects together material on towers of fibrations, nilpotent spaces, and the homotopy spectral sequence for a tower of fibrations. The first section describes a simplicial model structure for towers, which is used in Section 3 as a context for a formal discussion of Postnikov towers. The Moore-Postnikov tower, in particular, is a tower of fibrations that is functorially associated to a space $X$; we show, in Sections 4 and 5, that the fibrations appearing in the tower are homotopy pullbacks along maps, or $k$-invariants, taking values in homotopy colimits of diagrams of Eilenberg-Mac Lane spaces, which diagrams are functors defined on the fundamental groupoid of $X$. The homotopy pullbacks can be easily refined if the space is nilpotent, as is done in Section 6. The development includes an introduction of the notion of covering system of a connected space $X$, which is a functor defined on the fundamental groupoid and takes values in spaces homotopy equivalent to the covering space of $X$. The general homotopy spectral sequence for a tower of fibrations is introduced, warts and all, in Section 2 — it is the basis for the construction of the homotopy spectral sequence for a cosimplicial space that appears later in Chapter VIII.

Chapter VII contains a detailed treatment of the Reedy model structure for the category of simplicial objects in a closed model category. This theory simultaneously generalizes one of the standard model structures for bisimplicial sets that is discussed in Chapter IV, and specializes to the Bousfield-Kan model structures for the category of cosimplicial objects in simplicial sets, aka. cosimplicial spaces. The method of the application to cosimplicial spaces is to show that the category of simplicial objects in the category $\mathbf{S}^{op}$ has a Reedy model structure, along with an adequate notion of skeleta and an appropriate analogue of realization, and then reverse all arrows. There is one tiny wrinkle in this approach, in that one has to show that a cofibration in Reedy's sense coincides with the original definition of cofibration of Bousfield and Kan, but this argument is made, from two points of view, at the end of the chapter.

The standard total complex of a cosimplicial space is dual to the realization in the Reedy theory for simplicial objects in $\mathbf{S}^{op}$, and the standard tower of fibrations tower of fibrations from [14] associated to the total complex is dual to a skeletal filtration. We begin Chapter VIII with these observations, and then give the standard calculation of the $E_2$ term of the resulting spectral sequence. Homotopy inverse limits and $p$-completions, with associated spectral sequences, are the basic examples of this theory and its applications, and are the subjects of Sections 2 and 3, respectively. We also show that the homotopy inverse limit is a homotopy derived functor of inverse limit in a very precise sense, by introducing a "pointwise cofibration" closed model structure for small diagrams of spaces having a fixed index category.

The homotopy spectral sequence of a cosimplicial space is well known to be "fringed" in the sense that the objects that appear along the diagonal in total degree 0 are sets rather than groups. Standard homological techniques therefore fail, and there can be substantial difficulty in analyzing the path components of the total space. Bousfield has created an obstruction theory to attack this problem. We give here, in the last section of Chapter VII, a special case of this theory, which deals with the question of when elements in bidegree $(0,0)$ in the $E_2$-term lift to path components of the total space. This particular result can be used to give a criterion for maps between mod $p$ cohomology objects in the category of unstable algebras over the Steenrod algebra to lift to maps of $p$-completions.

Simplicial model structures return with a vengeance in Chapter IX, in the context of homotopy coherence. The point of view that we take is that a homotopy coherent diagram on a category $I$ in simplicial sets is a functor $X : \mathcal{A} \to \mathbf{S}$ which is defined on a category enriched in simplicial sets and preserves the enriched structure, subject to the object $\mathcal{A}$ being a resolution of $I$ in a suitable sense. The main results are due to Dwyer and Kan: there is a simplicial model structure on the category of simplicial functors $\mathbf{S}^{\mathcal{A}}$ (Section 1), and a large class of simplicial functors $f : \mathcal{A} \to \mathcal{B}$ which are weak equivalences induce equivalences of the homotopy categories associated to $\mathbf{S}^{\mathcal{A}}$ and $\mathbf{S}^{\mathcal{B}}$ (Section 2). Among such weak equivalences are resolutions $\mathcal{A} \to I$ — in practice, $I$ is the category of path components of $\mathcal{A}$ and each component of $\mathcal{A}$ is contractible. A realization of a homotopy coherent diagram $X : \mathcal{A} \to \mathbf{S}$ is then nothing but a diagram $Y : I \to \mathbf{S}$ which represents $X$ under the equivalence of homotopy categories. This approach subsumes the standard homotopy coherence phenomena, which are discussed in Section 3. We show how to promote some of these ideas to notions of homotopy coherent diagrams and realizations of same in more general simplicial model categories, including chain complexes and spectra, in the last section.

Frequently, one wants to take a given space and produce a member of a class of spaces for which homology isomorphisms are homotopy equivalences, without perturbing the homology. If the homology theory is mod $p$ homology, the $p$-completion works in many but not all examples. Bousfield's mod $p$ homology localization technique just works, for all spaces. The original approach to homology localization [8] appeared in the mid 1970's, and has since been incorporated into a more general theory of $f$-localization. The latter means that one constructs a minimal closed model structure in which a given map $f$ becomes invertible in the homotopy category — in the case of homology localization the map $f$ would be a disjoint union of maps of finite complexes which are homology isomorphisms. The theory of $f$-localization and the ideas underlying it are broadly applicable, and are still undergoing frequent revision in the literature. We present one of the recent versions of the theory here, in Sections 1–3 of Chapter X. The methods of proof involve little more than aggressive cardinal counts (the cogniscenti will note that there is no mention of regular cardinals): this is where the wide applicability of these ideas comes from — morally, if cardinality counts are available in a model category, then it admits a theory of localization. We describe Bousfield's approach to localization at a functor in Section 4, and then show that it leads to the Bousfield-Friedlander model for the stable category.

There are ten chapters in all; we use Roman numerals to distinguish them. Each chapter is divided into sections, plus an introduction. Results and equations are numbered consecutively within each section. The overall referencing system for the monograph is perhaps best illustrated with an example: Lemma 8.8 lives in Section 8 of Chapter II — it is referred to precisely this way from within Chapter II, and as Lemma II.8.8 from outside. Similarly, the corresponding section is called Section 8 inside Chapter II and Section II.8 from without.

Despite the length of this tome, much important material has been left out: there is not a word about traditional simplicial complexes and the vast modern literature related to them (trees, Tits buildings, Quillen's work on posets); the Waldhausen subdivision is not mentioned; we don't discuss the Hausmann-Husemoller theory of acyclic spaces or Quillen's plus construction; we have avoided all of the subtle aspects of categorical coherence theory, and there is very little about simplicial sheaves and presheaves. All of these topics, however, are readily available in the literature, and we have tried to include a useful bibliography.

This book should be accessible to mathematicians in the second year of graduate school or beyond, and is intended to be of interest to the research worker who wants to apply simplicial techniques, for whatever reason. We believe that it will be a useful introduction both to the theory and the current literature.

That said, this monograph does not have the structure of a traditional text book. We have, for example, declined to assign homework in the form of exercises, preferring instead to liberally sprinkle the text with examples and remarks that are designed to provoke further thought. Everything here depends on the first two chapters; the remaining material often reflects the original nature of the project, which amounted to separately written self contained tracts on defined topics. The book achieved its current more unified state thanks to a drive to achieve consistent notation and referencing, but it remains true that a more experienced reader should be able to read each of the later chapters in isolation, and find an essentially complete story in most cases.

This book had a lengthy and productive gestation period as an object on the Internet. There were many downloads, and many comments from interested readers, and we would like to thank them all. Particular thanks go to Frans Clauwens, who read the entire manuscript very carefully and made numerous technical, typographical, and stylistic comments and suggestions. The printed book differs substantially from the online version, and this is due in no small measure to his efforts.

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# CONTENTS

# Chapter I Simplicial sets

This chapter introduces the basic elements of the homotopy theory of simplicial sets. Technically, the purpose is twofold: to prove that the category of simplicial sets has a homotopical structure in the sense that it admits the structure of a closed model category (Theorem 11.3), and to show that the resulting homotopy theory is equivalent in a strong sense to the ordinary homotopy theory of topological spaces (Theorem 11.4). Insofar as simplicial sets are algebraically defined, and the corresponding closed model structure is combinatorial in nature, we obtain an algebraic, combinatorial model for standard homotopy theory.

The substance of Theorem 11.3 is that we can find three classes of morphisms within the simplicial set category **S**, called cofibrations, fibrations and weak equivalences, and then demonstrate that the following list of properties is satisfied:

**CM1:** **S** is closed under all finite limits and colimits.

**CM2:** Suppose that the following diagram commutes in **S**:

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\
& Z. &
\end{tikzcd}$$

If any two of $f$, $g$ and $h$ are weak equivalences, then so is the third.

**CM3:** If $f$ is a retract of $g$ in the category of maps of **S**, and $g$ is a weak equivalence, fibration or cofibration, then so is $f$.

**CM4:** Suppose that we are given a commutative solid arrow diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
V \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where $i$ is a cofibration and $p$ is a fibration. Then the dotted arrow exists, making the diagram commute, if either $i$ or $p$ is also a weak equivalence.

**CM5:** Any map $f : X \to Y$ may be factored:

(a) $f = p \cdot i$ where $p$ is a fibration and $i$ is both a cofibration and a weak equivalence, and

(b) $f = q \cdot j$ where $q$ is a fibration and a weak equivalence, and $j$ is a cofibration.

The fibrations in the simplicial set category are the Kan fibrations, which are defined by a lifting property that is analogous to the notion of Serre fibration. The cofibrations are the monomorphisms, and the weak equivalences are morphisms which induce homotopy equivalences of CW-complexes after passage to topological spaces. We shall begin to investigate the consequences of this list of axioms in subsequent chapters — they are the basis of a great deal of modern homotopy theory.

Theorem 11.3 and Theorem 11.4 are due to Quillen [76], but the development given here is different: the results are proved simultaneously, and their proofs jointly depend fundamentally on Quillen's later result that the realization of a Kan fibration is a Serre fibration [77]. The category of simplicial sets is historically the first full algebraic model for homotopy theory to have been found, but the verification of its closed model structure is still one of the more difficult proofs of abstract homotopy theory. These theorems and their proofs effectively summarize all of the classical homotopy theory of simplicial sets, as developed mostly by Kan in the 1950's. Kan's work was a natural outgrowth of the work of Eilenberg and Mac Lane on singular homology theory, and is part of a thread of ideas that used to be called "combinatorial homotopy theory" and which can be traced back to the work of Poincaré at the beginning of the twentieth century.

We give here, in the proof of the main results and the development leading to them, a comprehensive introduction to the homotopy theory of simplicial sets. Simplicial sets are defined, with examples, in Section 1, the functorial relationship with topological spaces via realization and the singular functor is described in Section 2, and we start to describe the combinatorial homotopical structure (Kan fibrations and Kan complexes) in Section 3. We introduce the Gabriel-Zisman theory of anodyne extensions in Section 4: this is the obstruction-theoretic machine that trivializes many potential difficulties related to the function complexes of Section 5, the notion of simplicial homotopy in Section 6, and the discussion of simplicial homotopy groups for Kan complexes in Section 7. The fundamental groupoid for a Kan complex is introduced in Section 8, by way of proving a major result about composition of simplicial sets maps which induce isomorphisms in homotopy groups (Theorem 8.2). This theorem, along with a lifting property result for maps which are simultaneously Kan fibrations and homotopy groups isomorphisms (Theorem 7.10 — later strengthened in Theorem 11.2), is used to demonstrate in Section 9 (Theorem 9.1) that the collection of Kan complexes and maps between them satisfies the axioms for a category of fibrant objects in the sense of Brown [15]. This is a first axiomatic approximation to the desired closed model structure, and is the platform on which the relation with standard homotopy theory is constructed with the introduction of minimal fibrations in Section 10. The basic ideas there are that every Kan fibration has a "minimal model" (Proposition 10.3 and Lemma 10.4), and the Gabriel-Zisman result that minimal fibrations induce Serre fibrations after realization (Theorem 10.9). It is then a relatively simple matter to show that the realization of a Kan fibration is a Serre fibration (Theorem 10.10).

The main theorems are proved in the final section, but Section 10 is the heart of the matter from a technical point of view once all the definitions and elementary properties have been established. We have not heard of a proof of Theorem 11.3 or Theorem 11.4 that avoids minimal fibrations. The minimality concept is very powerful wherever it appears, but not much has yet been made of it from a formal point of view.

## I.1. Basic definitions.

Let $\mathbf{\Delta}$ be the category of finite ordinal numbers, with order-preserving maps between them. More precisely, the objects for $\mathbf{\Delta}$ consist of elements $\mathbf{n}$, $n \geq 0$, where $\mathbf{n}$ is a string of relations

$$0 \to 1 \to 2 \to \cdots \to n$$

(in other words $\mathbf{n}$ is a totally ordered set with $n+1$ elements). A morphism $\theta : \mathbf{m} \to \mathbf{n}$ is an order-preserving set function, or alternatively a functor. We usually commit the abuse of saying that $\mathbf{\Delta}$ is the *ordinal number category.*

A *simplicial set* is a contravariant functor $X : \mathbf{\Delta}^{op} \to \mathbf{Sets}$, where $\mathbf{Sets}$ is the category of sets.

EXAMPLE 1.1. There is a standard covariant functor

$$\underset{\mathbf{n} \mapsto |\Delta^n|}{\mathbf{\Delta} \to \mathbf{Top}}.$$

The topological *standard n-simplex* $|\Delta^n| \subset \mathbb{R}^{n+1}$ is the space

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0\},$$

with the subspace topology. The map $\theta_* : |\Delta^n| \to |\Delta^m|$ induced by $\theta : \mathbf{n} \to \mathbf{m}$ is defined by

$$\theta_*(t_0, \dots, t_m) = (s_0, \dots, s_n),$$

where

$$s_i = \begin{cases} 0 & \theta^{-1}(i) = \emptyset \\ \sum_{j \in \theta^{-1}(i)} t_j & \theta^{-1}(i) \neq \emptyset \end{cases}$$

One checks that $\theta \mapsto \theta_*$ is indeed a functor (exercise). Let $T$ be a topological space. The *singular set* $S(T)$ is the simplicial set given by

$$\mathbf{n} \mapsto \hom(|\Delta^n|, T).$$

This is the object that gives the singular homology of the space $T$.

Among all of the functors $\mathbf{m} \to \mathbf{n}$ appearing in $\mathbf{\Delta}$ there are special ones, namely

$$\begin{aligned} &d^i : \mathbf{n-1} \to \mathbf{n} \qquad 0 \leq i \leq n \qquad \text{(cofaces)} \\ &s^j : \mathbf{n+1} \to \mathbf{n} \qquad 0 \leq j \leq n \qquad \text{(codegeneracies)} \end{aligned}$$

where, by definition,

$$d^i(0 \to 1 \to \cdots \to n-1) = (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$$

(ie. compose $i-1 \to i \to i+1$, giving a string of arrows of length $n-1$ in $\mathbf{n}$), and

$$s^j(0 \to 1 \to \cdots \to n+1) = (0 \to 1 \to \cdots \to j \xrightarrow{1} j \to \cdots \to n)$$

(insert the identity $1_j$ in the $j^{th}$ place, giving a string of length $n+1$ in $\mathbf{n}$). It is an exercise to show that these functors satisfy a list of identities as follows, called the *cosimplicial identities*:

$$\begin{cases} d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j \end{cases} \tag{1.2}$$

The maps $d^j$, $s^i$ and these relations can be viewed as a set of generators and relations for $\mathbf{\Delta}$ (see [66]). Thus, in order to define a simplicial set $Y$, it suffices to write down sets $Y_n$, $n \geq 0$ (sets of $n$-simplices) together with maps

$$\begin{aligned} &d_i : Y_n \to Y_{n-1}, \qquad 0 \leq i \leq n \quad \text{(faces)} \\ &s_j : Y_n \to Y_{n+1}, \qquad 0 \leq j \leq n \quad \text{(degeneracies)} \end{aligned}$$

satisfying the *simplicial identities*:

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j \end{cases} \tag{1.3}$$

This is the classical way to write down the data for a simplicial set $Y$.

From a simplicial set $Y$, one may construct a simplicial abelian group $\mathbb{Z}Y$ (ie. a contravariant functor $\mathbf{\Delta}^{op} \to \mathbf{Ab}$), with $\mathbb{Z}Y_n$ set equal to the free abelian group on $Y_n$. The simplicial abelian group $\mathbb{Z}Y$ has associated to it a chain complex, called its *Moore complex* and also written $\mathbb{Z}Y$, with

$$\mathbb{Z}Y_0 \xleftarrow{\partial} \mathbb{Z}Y_1 \xleftarrow{\partial} \mathbb{Z}Y_2 \leftarrow \dots \qquad \text{and}$$

$$\partial = \sum_{i=0}^{n} (-1)^i d_i$$

in degree $n$. Recall that the integral singular homology groups $H_*(X;\mathbb{Z})$ of the space $X$ are defined to be the homology groups of the chain complex $\mathbb{Z}SX$. The homology groups $H_n(Y, A)$ of a simplicial set $Y$ with coefficients in an abelian group $A$ are defined to be the homology groups $H_n(\mathbb{Z}Y \otimes A)$ of the chain complex $\mathbb{Z}Y \otimes A$.

EXAMPLE 1.4. Suppose that $\mathcal{C}$ is a (small) category. The *classifying space* (or nerve ) $B\mathcal{C}$ of $\mathcal{C}$ is the simplicial set with

$$B\mathcal{C}_n = \hom_{\mathbf{cat}}(\mathbf{n}, \mathcal{C}),$$

where $\hom_{\mathbf{cat}}(\mathbf{n}, \mathcal{C})$ denotes the set of functors from $\mathbf{n}$ to $\mathcal{C}$. In other words an $n$-simplex is a string

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

of composeable arrows of length $n$ in $\mathcal{C}$.

We shall see later that there is a topological space $|Y|$ functorially associated to every simplicial set $Y$, called the realization of $Y$. The term "classifying space" for the simplicial set $B\mathcal{C}$ is therefore something of an abuse – one really means that $|B\mathcal{C}|$ is the classifying space of $\mathcal{C}$. Ultimately, however, it does not matter; the two constructions are indistinguishable from a homotopy theoretic point of view.

EXAMPLE 1.5. If $G$ is a group, then $G$ can be identified with a category (or groupoid) with one object $*$ and one morphism $g : * \to *$ for each element $g$ of $G$, and so the classifying space $BG$ of $G$ is defined. Moreover $|BG|$ is an Eilenberg-Mac Lane space of the form $K(G, 1)$, as the notation suggests; this is now the standard construction.

EXAMPLE 1.6. Suppose that $\mathcal{A}$ is an exact category, like the category $\mathcal{P}(R)$ of finitely generated projective modules on a ring $R$ (see [79]). Then $\mathcal{A}$ has associated to it a category $Q\mathcal{A}$. The objects of $Q\mathcal{A}$ are those of $\mathcal{A}$. The arrows of $Q\mathcal{A}$ are equivalence classes of diagrams

$$\bullet \twoheadleftarrow \bullet \rightarrowtail \bullet$$

where both arrows are parts of exact sequences of $\mathcal{A}$, and composition is represented by pullback. Then $K_{i-1}(\mathcal{A}) := \pi_i|BQ\mathcal{A}|$ defines the $K$-groups of $\mathcal{A}$ for $i \geq 1$; in particular $\pi_i|BQ\mathcal{P}(R)| = K_{i-1}(R)$, the $i^{th}$ *algebraic $K$-group* of the ring $R$.

EXAMPLE 1.7. The *standard $n$-simplex, simplicial* $\Delta^n$ in the simplicial set category $\mathbf{S}$ is defined by

$$\Delta^n = \hom_{\mathbf{\Delta}}(\ ,\mathbf{n}).$$

In other words, $\Delta^n$ is the contravariant functor on $\mathbf{\Delta}$ which is represented by $\mathbf{n}$.

A *map $f : X \to Y$ of simplicial sets* (or, more simply, a *simplicial map*) is a natural transformation of contravariant set-valued functors defined on $\mathbf{\Delta}$. We shall use $\mathbf{S}$ to denote the resulting category of simplicial sets and simplicial maps.

The Yoneda Lemma implies that simplicial maps $\Delta^n \to Y$ classify $n$-simplices of $Y$ in the sense that there is a natural bijection

$$\hom_{\mathbf{S}}(\Delta^n, Y) \cong Y_n$$

between the set $Y_n$ of $n$-simplices of $Y$ and the set $\hom_{\mathbf{S}}(\Delta^n, Y)$ of simplicial maps from $\Delta^n$ to $Y$ (see [66], or better yet, prove the assertion as an exercise). More precisely, write $\iota_n = 1_{\mathbf{n}} \in \hom_{\mathbf{\Delta}}(\mathbf{n},\mathbf{n})$. Then the bijection is given by associating the simplex $\varphi(\iota_n) \in Y_n$ to each simplicial map $\varphi : \Delta^n \to Y$. This means that each simplex $x \in Y_n$ has associated to it a unique simplicial map $\iota_x : \Delta^n \to Y$ such that $\iota_x(\iota_n) = x$. One often writes $x = \iota_x$, since it's usually convenient to confuse the two.

$\Delta^n$ contains subcomplexes $\partial\Delta^n$ (*boundary* of $\Delta^n$) and $\Lambda^n_k$, $0 \le k \le n$ ($k^{th}$ *horn*, really the cone centred on the $k^{th}$ vertex). The simplicial set $\partial\Delta^n$ is the smallest subcomplex of $\Delta^n$ containing the faces $d_j(\iota_n)$, $0 \le j \le n$ of the standard simplex $\iota_n$. One finds that $\partial\Delta^n$ is specified in $j$-simplices by

$$\partial\Delta^n_j = \begin{cases} \Delta^n_j & \text{if } 0 \le j \le n-1, \\ \text{iterated degeneracies of elements of } \Delta^n_k, \\ \qquad 0 \le k \le n-1, \quad \text{if } j \ge n. \end{cases}$$

It is a standard convention to write $\partial\Delta^0 = \emptyset$, where $\emptyset$ is the "unique" simplicial set which consists of the empty set in each degree. The object $\emptyset$ is initial for the simplicial set category $\mathbf{S}$.

The $k^{th}$ horn $\Lambda^n_k \subset \Delta^n$ ($n \ge 1$) is the subcomplex of $\Delta^n$ which is generated by all faces $d_j(\iota_n)$ *except* the $k^{th}$ face $d_k(\iota_n)$. One could represent $\Lambda^2_0$, for example, by the picture

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & & & & 0 \arrow[dl] \arrow[dr] & & & \\ 1 & & 2 & \subset & 1 \arrow[rr] & & 2 & = & \Delta^2. \end{tikzcd}$$

## I.2. Realization.

Let **Top** denote the category of topological spaces. To go further, we have to get serious about the realization functor $| \ | : \mathbf{S} \to \mathbf{Top}$. There is a quick way to construct it which uses the *simplex category* $\mathbf{\Delta} \downarrow X$ of a simplicial set $X$. The objects of $\mathbf{\Delta} \downarrow X$ are the maps $\sigma : \Delta^n \to X$, or simplices of $X$. An arrow of $\mathbf{\Delta} \downarrow X$ is a commutative diagram of simplicial maps

$$\begin{tikzcd}
\Delta^n \arrow[dr, "\sigma"] \arrow[dd, "\theta"'] & \\
& X \\
\Delta^m \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Observe that $\theta$ is induced by a unique ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$.

Lemma 2.1. *There is an isomorphism*

$$X \cong \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta} \downarrow X}} \Delta^n.$$

Proof: The proof is the observation that any functor $\mathcal{C} \to \mathbf{Sets}$, which is defined on a small category $\mathcal{C}$, is a colimit of representable functors. $\square$

The *realization* $|X|$ of a simplicial set $X$ is defined by the colimit

$$|X| = \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta} \downarrow X}} |\Delta^n|.$$

in the category of topological spaces. The construction $X \mapsto |X|$ is seen to be functorial in simplicial sets $X$, by using the fact that any simplicial map $f : X \to Y$ induces a functor $f_* : \mathbf{\Delta} \downarrow X \to \mathbf{\Delta} \downarrow Y$ by composition with $f$.

Proposition 2.2. *The realization functor is left adjoint to the singular functor in the sense that there is an isomorphism*

$$\hom_{\mathbf{Top}}(|X|, Y) \cong \hom_{\mathbf{S}}(X, SY)$$

*which is natural in simplicial sets $X$ and topological spaces $Y$.*

Proof: There are isomorphisms

$$\begin{aligned}
\hom_{\mathbf{Top}}(|X|, Y) &\cong \varprojlim_{\Delta^n \to X} \hom_{\mathbf{Top}}(|\Delta^n|, Y) \\
&\cong \varprojlim_{\Delta^n \to X} \hom_{\mathbf{S}}(\Delta^n, S(Y)) \\
&\cong \hom_{\mathbf{S}}(X, SY)).
\end{aligned}$$

$\square$

Note that $\mathbf{S}$ has all colimits and the realization functor $| \ |$ preserves them, since it has a right adjoint.

PROPOSITION 2.3. *$|X|$ is a CW-complex for each simplicial set $X$.*

PROOF: Define the $n^{th}$ *skeleton* $\operatorname{sk}_n X$ of $X$ be the subcomplex of $X$ which is generated by the simplices of $X$ of degree $\leq n$. Then $X$ is a union

$$X = \bigcup_{n \geq 0} \operatorname{sk}_n X$$

of its skeleta, and there are pushout diagrams

$$\begin{tikzcd}
\displaystyle\bigsqcup_{x \in NX_n} \partial\Delta^n \arrow[r] \arrow[d, hook] & \operatorname{sk}_{n-1} X \arrow[d, hook] \\
\displaystyle\bigsqcup_{x \in NX_n} \Delta^n \arrow[r] & \operatorname{sk}_n X
\end{tikzcd}$$

of simplicial sets, where $NX_n \subset X_n$ is the set of *non-degenerate simplices* of degree $n$. In other words,

$$NX_n = \{x \in X_n | x \text{ not of the form } s_i y \text{ for any } 0 \leq i \leq n-1 \text{ and } y \in X_{n-1}\}.$$

The realization of $\Delta^n$ is the space $|\Delta^n|$, since $\mathbf{\Delta} \downarrow \Delta^n$ has terminal object $1 : \Delta^n \to \Delta^n$. Furthermore, one can show that there is a coequalizer

$$\bigsqcup_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

given by the relations $d^j d^i = d^i d^{j-1}$ if $i < j$ (exercise), and so there is a coequalizer diagram of spaces

$$\bigsqcup_{0 \leq i < j \leq n} |\Delta^{n-2}| \rightrightarrows \bigsqcup_{i=0}^{n} |\Delta^{n-1}| \to |\partial\Delta^n|$$

Thus, the induced map $|\partial\Delta^n| \to |\Delta^n|$ maps $|\partial\Delta^n|$ onto the $(n-1)$-sphere bounding $|\Delta^n|$. It follows that $|X|$ is a filtered colimit of spaces $|\operatorname{sk}_n X|$ where $|\operatorname{sk}_n X|$ is obtained from $|\operatorname{sk}_{n-1} X|$ by attaching n-cells according to the pushout diagram

$$\begin{tikzcd}
\displaystyle\bigsqcup_{x \in NX_n} |\partial\Delta^n| \arrow[r] \arrow[d, hook] & |\operatorname{sk}_{n-1} X| \arrow[d, hook] \\
\displaystyle\bigsqcup_{x \in NX_n} |\Delta^n| \arrow[r] & |\operatorname{sk}_n X|.
\end{tikzcd}$$

□

In particular $|X|$ is a *compactly generated Hausdorff space*, and so the realization functor takes values in the category **CGHaus** of all such. We shall interpret $|\ |$ as such a functor. Here is the reason:

PROPOSITION 2.4. *The functor* $|\ | : \mathbf{S} \to \mathbf{CGHaus}$ *preserves finite limits.*

We won't get into the general topology involved in proving this result; a demonstration is given in [33]. Proposition 2.4 avoids the problem that $|X \times Y|$ may not be homeomorphic to $|X| \times |Y|$ in general in the ordinary category of topological spaces, in that it implies that

$$|X \times Y| \cong |X| \times_{Ke} |Y|$$

(Kelley space product = product in **CGHaus**). We lose no homotopical information by working **CGHaus** since, for example, the definition of homotopy groups of a CW-complex does not see the difference between **Top** and **CGHaus**.

### I.3. Kan complexes.

Recall the "presentation"

$$\bigsqcup_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

of $\partial\Delta^n$ that was mentioned in the last section. There is a similar presentation for $\Lambda^n_k$.

LEMMA 3.1. *The "fork" defined by the commutative diagram*

$$\begin{tikzcd}
\Delta^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "in_{i<j}"'] & \Delta^{n-1} \arrow[d, "in_i"] \arrow[dr, "d^i"] & \\
\bigsqcup_{0\leq i<j\leq n} \Delta^{n-2} \arrow[r, shift left] \arrow[r, shift right] & \bigsqcup_{i\neq k} \Delta^{n-1} \arrow[r] & \Lambda^n_k \\
\Delta^{n-2} \arrow[r, "d^i"'] \arrow[u, "in_{i<j}"] & \Delta^{n-1} \arrow[u, "in_j"'] \arrow[ur, "d^j"'] &
\end{tikzcd}$$

*is a coequalizer in* $\mathbf{S}$.

PROOF: There is a coequalizer

$$\bigsqcup_{i<j} \Delta^{n-1} \times_{\Lambda^n_k} \Delta^{n-1} \rightrightarrows \bigsqcup_{\substack{i \neq k \\ 0 \leq i \leq n}} \Delta^{n-1} \to \Lambda^n_k.$$

But the fibre product $\Delta^{n-1} \times_{\Lambda^n_k} \Delta^{n-1}$ is isomorphic to

$$\Delta^{n-1} \times_{\Delta^n} \Delta^{n-1} \cong \Delta^{n-2}$$

since the diagram

$$\begin{tikzcd}
\mathbf{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & \mathbf{n-1} \arrow[d, "d^i"] \\
\mathbf{n-1} \arrow[r, "d^j"'] & \mathbf{n}
\end{tikzcd}$$

is a pullback in $\mathbf{\Delta}$. In effect, the totally ordered set $\{0 \dots \hat{\imath} \dots \hat{\jmath} \dots n\}$ is the intersection of the subsets $\{0 \dots \hat{\imath} \dots n\}$ and $\{0 \dots \hat{\jmath} \dots n\}$ of $\{0 \dots n\}$, and this poset is isomorphic to $\mathbf{n-2}$. □

The notation $\{0 \dots \hat{\imath} \dots n\}$ means that $i$ isn't there.

COROLLARY 3.2. *The set* $\hom_{\mathbf{S}}(\Lambda^n_k, X)$ *of simplicial set maps from* $\Lambda^n_k$ *to* $X$ *is in bijective correspondence with the set of* $n$*-tuples* $(y_0, \dots, \hat{y}_k, \dots, y_n)$ *of* $(n-1)$*-simplices* $y_i$ *of* $X$ *such that* $d_i y_j = d_{j-1} y_i$ *if* $i < j$*, and* $i, j \neq k$.

We can now start to describe the internal homotopy theory carried by $\mathbf{S}$. The central definition is that of a fibration of simplicial sets. A map $p : X \to Y$ of simplicial sets is said to be a *fibration* if for every commutative diagram of simplicial set homomorphisms

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

there is a map $\theta : \Delta^n \to X$ (the dotted arrow) making the diagram commute. The map $i$ is the inclusion of the subcomplex $\Lambda^n_k$ in $\Delta^n$.

This requirement was called the *extension condition* at one time (see [58], [67], for example), and fibrations were (and still are) called *Kan fibrations.* The condition amounts to saying that if $(x_0 \dots \hat{x}_k \dots x_n)$ is an $n$-tuple of simplices of $X$ such that $d_i x_j = d_{j-1} x_i$ if $i < j$, $i, j \neq k$, and there is an $n$-simplex $y$ of $Y$ such that $d_i y = p(x_i)$, then there is an $n$-simplex $x$ of $X$ such that $d_i x = x_i$, $i \neq k$, and such that $p(x) = y$. It is usually better to formulate it in terms of diagrams.

The same language may be used to describe Serre fibrations: a continuous map of spaces $f : T \to U$ is said to be a *Serre fibration* if the dotted arrow exists in each commutative diagram of continuous maps

$$\begin{tikzcd}
{|\Lambda^n_k|} \arrow[r] \arrow[d, hook] & T \arrow[d, "f"] \\
{|\Delta^n|} \arrow[r] \arrow[ur, dashed] & U
\end{tikzcd}$$

making it commute. By adjointness (Proposition 2.2), all such diagrams may be identified with diagrams

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & S(T) \arrow[d, "S(f)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S(U),
\end{tikzcd}$$

so that $f : T \to U$ is a Serre fibration if and only if $S(f) : S(T) \to S(U)$ is a (Kan) fibration. This is partial motivation for the definition of fibration of simplicial sets. The simplicial set $|\Lambda^n_k|$ is a strong deformation retract of $|\Delta^n|$, so that we've proved

LEMMA 3.3. *For each space $X$, the map $S(X) \to *$ is a fibration.*

The notation $*$ refers to the simplicial set $\Delta^0$, as is standard. It consists of a singleton set in each degree, and is therefore a terminal object in the category of simplicial sets.

A *fibrant* simplicial set (or *Kan complex*) is a simplicial set $Y$ such that the canonical map $Y \to *$ is a fibration. Alternatively, $Y$ is a Kan complex if and only if one of the following equivalent conditions is met:

**K1:** Every map $\alpha : \Lambda^n_k \to Y$ may be extended to a map defined on $\Delta^n$ in the sense that there is a commutative diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & Y \\
\Delta^n \arrow[ur] &
\end{tikzcd}$$

**K2:** For each $n$-tuple of $(n-1)$-simplices $(y_0 \dots \hat{y}_k \dots y_n)$ of $Y$ such that $d_i y_j = d_{j-1} y_i$ if $i < j$, $i, j \neq k$, there is an $n$-simplex $y$ such that $d_i y = y_i$.

The standard examples of fibrant simplicial sets are singular complexes, as we've seen, as well as classifying spaces $BG$ of groups $G$, and simplicial groups. A *simplicial group* $H$ is a simplicial object in the category of groups; this means that $H$ is a contravariant functor from $\mathbf{\Delta}$ to the category **Grp** of groups. We generally reserve the symbol $e$ for the identities of the groups $H_n$, for all $n \geq 0$.

LEMMA 3.4 (MOORE). *The underlying simplicial set of any simplicial group $H$ is fibrant.*

PROOF: Suppose that $(x_0, \dots, x_{k-1}, x_{\ell-1}, x_\ell, \dots, x_n)$, $\ell \geq k+2$, is a family of $(n-1)$-simplices of $H$ which is compatible in the sense that $d_i x_j = d_{j-1} x_i$ for $i < j$ whenever the two sides of the equation are defined. Suppose that there is an $n$-simplex $y$ of $H$ such that $d_i y = x_i$ for $i \leq k-1$ and $i \geq \ell$. Then the family

$$(e, \dots, e, \overset{\ell-1}{x_{\ell-1} d_{\ell-1}(y^{-1})}, e, \dots, e)$$

is compatible, and $d_i(s_{\ell-2}(x_{\ell-1} d_{\ell-1} y^{-1}) y) = x_i$ for $i \leq k-1$ and $i \geq \ell - 1$. This is the inductive step in the proof of the lemma. ☐

Recall that a *groupoid* is a category in which every morphism is invertible. Categories associated to groups as above are obvious examples, so that the following result specializes to the assertion that classifying spaces of groups are Kan complexes.

LEMMA 3.5. *Suppose that $G$ is a groupoid. Then $BG$ is fibrant.*

PROOF: If $\mathcal{C}$ is a small category, then its nerve $B\mathcal{C}$ is a 2-coskeleton in the sense that the set of simplicial maps $f : X \to B\mathcal{C}$ is in bijective correspondence with commutative (truncated) diagrams

$$\begin{tikzcd}
X_2 \arrow[r, "f_2"] \arrow[d, shift right] & B\mathcal{C}_2 \arrow[d, shift right] \\
X_1 \arrow[r, "f_1"] \arrow[u, shift right] \arrow[d, shift right] & B\mathcal{C}_1 \arrow[u, shift right] \arrow[d, shift right] \\
X_0 \arrow[r, "f_0"] \arrow[u, shift right] & B\mathcal{C}_0 \arrow[u, shift right]
\end{tikzcd}$$

in which the vertical maps are the relevant simplicial structure maps. It suffices to prove this for $X = \Delta^n$ since $X$ is a colimit of simplices. But any simplicial map $f : \Delta^n \to B\mathcal{C}$ can be identified with a functor $f : \mathbf{n} \to \mathcal{C}$, and this functor is completely specified by its action on vertices ($f_0$), and morphisms ($f_1$), and the requirement that $f$ respects composition ($f_2$, and $d_i f_2 = f_1 d_i$). Another way of saying this is that a simplicial map $X \to B\mathcal{C}$ is completely determined by its restriction to $\mathrm{sk}_2\, X$.

The inclusion $\Lambda^n_k \subset \Delta^n$ induces an isomorphism

$$\mathrm{sk}_{n-2}\, \Lambda^n_k \cong \mathrm{sk}_{n-2}\, \Delta^n.$$

To see this, observe that every simplex of the form $d_i d_j \iota_n$, $i < j$, is a face of some $d_r \iota_n$ with $r \neq k$: if $k \neq i, j$ use $d_i(d_j \iota_n)$, if $k = i$ use $d_k(d_j \iota_n)$, and if $k = j$ use $d_i(d_k \iota_n) = d_{k-1}(d_i \iota_n)$. It immediately follows that the extension problem

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & BG \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

is solved if $n \geq 4$, for in that case $\mathrm{sk}_2\, \Lambda^n_k = \mathrm{sk}_2\, \Delta^n$.

Suppose that $n = 3$, and consider the extension problem

$$\begin{tikzcd}
\Lambda^3_0 \arrow[r, "\alpha"] \arrow[d, hook] & BG \\
\Delta^3 \arrow[ur, dashed] &
\end{tikzcd}$$

Then $\mathrm{sk}_1\, \Lambda^3_0 = \mathrm{sk}_1\, \Delta^3$ and so we are entitled to write $\alpha_1 : a_0 \to a_1$, $\alpha_2 : a_1 \to a_2$ and $\alpha_3 : a_2 \to a_3$ for the images under the simplicial map $\alpha$ of the 1-simplices $0 \to 1$, $1 \to 2$ and $2 \to 3$, respectively. Write $x : a_1 \to a_3$ for the image of $1 \to 3$ under $\alpha$. Then the boundary of $d_0 \iota_3$ maps to the graph

$$\begin{tikzcd}
a_1 \arrow[rr, "\alpha_2"] \arrow[dr, "x"'] & & a_2 \arrow[dl, "\alpha_3"] \\
& a_3 &
\end{tikzcd}$$

in the groupoid $G$ under $\alpha$, and this graph bounds a 2-simplex of $BG$ if and only if $x = \alpha_3\alpha_2$ in $G$. But the images of the 2-simplices $d_2\iota_3$ and $d_1\iota_3$ under $\alpha$ together determine a commutative diagram

$$\begin{tikzcd}
& a_0 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2\alpha_1"] \arrow[dd, "\alpha_3(\alpha_2\alpha_1)"] & \\
a_1 \arrow[dr, "x"'] & & a_2 \arrow[dl, "\alpha_3"] \\
& a_3 &
\end{tikzcd}$$

in $G$, so that

$$x\alpha_1 = \alpha_3(\alpha_2\alpha_1),$$

and $x = \alpha_3\alpha_2$, by right cancellation. It follows that the simplicial map $\alpha : \Lambda^3_0 \to BG$ extends to $\partial\Delta^3 = \mathrm{sk}_2\,\Delta^3$, and the extension problem is solved.

The other cases corresponding to the inclusions $\Lambda^3_i \subset \Delta^3$ are similar.

If $n = 2$, then, for example, a simplicial map $\alpha : \Lambda^2_0 \to BG$ can be identified with a diagram

$$\begin{tikzcd}
& a_0 \arrow[dl, "\alpha_1"'] \arrow[dr, "x"] & \\
a_1 & & a_2
\end{tikzcd}$$

and $\alpha$ can be extended to a 2-simplex of $BG$ if and only if there is an arrow $\alpha_2 : a_1 \to a_2$ of $G$ such that $\alpha_2\alpha_1 = x$. But $\alpha_2 = x\alpha_1^{-1}$ does the trick. The other cases in dimension 2 are similar. $\square$

The standard $n$-simplex $\Delta^n = B\mathbf{n}$ fails to be fibrant for $n \geq 2$, precisely because the last step in the proof of Lemma 3.5 fails in that case.

## I.4. Anodyne extensions.

The homotopy theory of simplicial sets is based on the definition of fibration given in the last section. Originally, all statements involving fibrations were expressed in terms of the extension condition, and this often led to some rather difficult combinatorial manipulations based on the standard subdivision of a prism.

The algorithms involved in these manipulations are actually quite formal, and can be encoded in the Gabriel-Zisman theory of anodyne extensions [33].

This theory suppresses or engulfs most of the old combinatorial arguments, and is a basic element of the modern theory. We describe the Gabriel-Zisman theory in this section.

A class $M$ of (pointwise) monomorphisms of **S** is said to be *saturated* if the following conditions are satisfied:

**A:** All isomorphisms are in $M$.

**B:** $M$ is closed under pushout in the sense that, in a pushout square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\
B \arrow[r] & B \cup_A C,
\end{tikzcd}$$

if $i \in M$ then so is $i_*$ (Exercise: Show that $i_*$ is monic).

**C:** Each retract of an element of $M$ is in $M$. This means that, given a commutative diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"] \arrow[rr, bend left, "1"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\
B' \arrow[r] \arrow[rr, bend right, "1"'] & B \arrow[r] & B',
\end{tikzcd}$$

of simplicial set maps, if $i$ is in $M$ then so is $i'$.

**D:** $M$ is closed under countable compositions and arbitrary direct sums, meaning respectively that:

**D1:** Given

$$A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} \dots$$

with $i_j \in M$, the canonical map $A_1 \to \varinjlim A_i$ is in $M$.

**D2:** Given $i_j : A_j \to B_j$ in $M$, $j \in I$, the map

$$\sqcup\, i_j : \bigsqcup_{j\in I} A_j \to \bigsqcup_{j\in I} B_j$$

is in $M$.

A map $p : X \to Y$ is said to have the *right lifting property* (RLP is the standard acronym) with respect to a class of monomorphisms $M$ if in every solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with $i \in M$ the dotted arrow exists making the diagram commute.

Lemma 4.1. *The class $M_p$ of all monomorphisms which have the left lifting property (LLP) with respect to a fixed simplicial map $p : X \to Y$ is saturated.*

Proof: (trivial) For example, we prove the axiom **B**. Suppose given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[r] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B \arrow[r] & B \cup_A C \arrow[r] & Y, \end{tikzcd}$$

where the square on the left is a pushout. Then there is a map $\theta : B \to X$ such that the "composite" diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, "\theta"] & Y \end{tikzcd}$$

commutes. But then $\theta$ induces the required lifting $\theta_* : B \cup_A C \to X$ by the universal property of the pushout. $\square$

The *saturated class $M_B$ generated by a class of monomorphisms $B$* is the intersection of all saturated classes containing $B$. One also says that $M_B$ is the *saturation* of $B$.

Consider the following three classes of monomorphisms:

$\mathbf{B_1}$ := the set of all inclusions $\Lambda^n_k \subset \Delta^n$, $0 \leq k \leq n$, $n > 0$

$\mathbf{B_2}$ := the set of all inclusions

$$(\Delta^1 \times \partial\Delta^n) \cup (\{e\} \times \Delta^n) \subset (\Delta^1 \times \Delta^n), \qquad e = 0, 1$$

$\mathbf{B_3}$ := the set of all inclusions

$$(\Delta^1 \times Y) \cup (\{e\} \times X) \subset (\Delta^1 \times X),$$

arising from inclusions $Y \subset X$ of simplicial sets, where $e = 0, 1$.

PROPOSITION 4.2. *The classes* $\mathbf{B_1}$, $\mathbf{B_2}$ *and* $\mathbf{B_3}$ *have the same saturation.*

$M_{\mathbf{B_1}}$ is called the class of *anodyne extensions.*

COROLLARY 4.3. *A fibration is a map which has the right lifting property with respect to all anodyne extensions.*

PROOF OF PROPOSITION 4.2: We shall show only that $M_{\mathbf{B_2}} = M_{\mathbf{B_1}}$; it is relatively easy to see that $M_{\mathbf{B_2}} = M_{\mathbf{B_3}}$, since a simplicial set $X$ can be built up from a subcomplex $Y$ by attaching cells. To show that $M_{\mathbf{B_2}} \subset M_{\mathbf{B_1}}$, observe that any saturated set is closed under finite composition. The simplicial set $\Delta^n \times \Delta^1$ has non-degenerate simplices

$$h_j : \Delta^{n+1} \to \Delta^n \times \Delta^1, \qquad 0 \leq j \leq n,$$

where the $h_j$ may be identified with the strings of morphisms

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[r] & \dots \arrow[r] & (0,j) \arrow[d] & & \\
& & & (1,j) \arrow[r] & \dots \arrow[r] & (1,n)
\end{tikzcd}$$

of length $n + 1$ in $\mathbf{n} \times \mathbf{1}$ (anything longer must have a repeat). One can show that there are commutative diagrams

$$\begin{tikzcd}
\Delta^n \arrow[r, "d^0"] & \Delta^{n+1} \arrow[d, "h_0"] & \Delta^n \arrow[r, "d^{n+1}"] & \Delta^{n+1} \arrow[d, "h_n"] \\
\Delta^n \times \{1\} \arrow[u, "\cong"] \arrow[r] & \Delta^n \times \Delta^1 & \Delta^n \times \{0\} \arrow[u, "\cong"] \arrow[r] & \Delta^n \times \Delta^1
\end{tikzcd} \tag{4.4}$$

$$\left\{\begin{array}{ll}
\begin{tikzcd}[ampersand replacement=\&] \Delta^n \arrow[r, "d^i"] \arrow[d, "h_{j-1}"'] \& \Delta^{n+1} \arrow[d, "h_j"] \\ \Delta^{n-1}\times\Delta^1 \arrow[r, "d^i\times 1"'] \& \Delta^n\times\Delta^1 \end{tikzcd} & \text{if } i<j \\
\begin{tikzcd}[ampersand replacement=\&] \Delta^n \arrow[r, "d^{j+1}"] \arrow[d, "d^{j+1}"'] \& \Delta^{n+1} \arrow[d, "h_{j+1}"] \\ \Delta^{n+1} \arrow[r, "h_j"'] \& \Delta^n\times\Delta^1 \end{tikzcd} & \\
\begin{tikzcd}[ampersand replacement=\&] \Delta^n \arrow[r, "d^i"] \arrow[d, "h_j"'] \& \Delta^{n+1} \arrow[d, "h_j"] \\ \Delta^{n-1}\times\Delta^1 \arrow[r, "d^{i-1}\times 1"'] \& \Delta^n\times\Delta^1 \end{tikzcd} & \text{if } i>j+1.
\end{array}\right. \tag{4.5}$$

Moreover $d_{j+1}h_j \notin \partial\Delta^n\times\Delta^1$ for $j\geq 0$ since it projects to $\iota_n$ under the projection map $\Delta^n\times\Delta^1\to\Delta^n$. Finally, $d_{j+1}h_j$ is not a face of $h_i$ for $j\geq i+1$ since it has vertex $(0,j)$.

Let $(\Delta^n\times\Delta^1)^{(i)}$, $i\geq 1$ be the smallest subcomplex of $\Delta^n\times\Delta^1$ containing $\partial\Delta^n\times\Delta^1$ and the simplices $h_0,\dots,h_i$. Then $(\Delta^n\times\Delta^1)^{(n)}=\Delta^n\times\Delta^1$ and there is a sequence of pushouts, each having the form

$$\begin{tikzcd}
\Lambda^{n+1}_{i+2} \arrow[r, "{(d_0h_{i+1},\dots,\widehat{d_{i+2}h_{i+1}},\dots,d_{n+1}h_{i+1})}"] \arrow[d, hook] & (\Delta^n\times\Delta^1)^{(i)} \arrow[d] \\
\Delta^{n+1} \arrow[r, "h_{i+1}"'] & (\Delta^n\times\Delta^1)^{(i+1)}
\end{tikzcd}$$

for $n-1\geq i\geq -1$, by the observation above.

To see that $M_{\mathbf{B_1}}\subset M_{\mathbf{B_2}}$, suppose that $k<n$, and construct the functors

$$\mathbf{n}\xrightarrow{i}\mathbf{n}\times\mathbf{1}\xrightarrow{r_k}\mathbf{n},$$

where $i(j)=(j,1)$ and $r_k$ is defined by the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \dots \arrow[r] & k-1 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & \dots \arrow[r] & k \arrow[d] \\
0 \arrow[r] & 1 \arrow[r] & \dots \arrow[r] & k-1 \arrow[r] & k \arrow[r] & k+1 \arrow[r] & \dots \arrow[r] & n
\end{tikzcd}$$

in $\mathbf{n}$. Then $r \cdot i = 1_{\mathbf{n}}$, and $r$ and $i$ induce a retraction diagram of simplicial set maps

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & (\Lambda^n_k \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r] \arrow[d] & \Lambda^n_k \arrow[d, hook] \\
\Delta^n \arrow[r] & \Delta^n \times \Delta^1 \arrow[r] & \Delta^n
\end{tikzcd}$$

(apply the classifying space functor $B$). It follows that the inclusion $\Lambda^n_k \subset \Delta^n$ is in $M_{\mathbf{B_2}}$ if $k < n$.

Similarly, if $k > 0$, then the functor $v_k : \mathbf{n} \times \mathbf{1} \to \mathbf{n}$ defined by the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \dots \arrow[r] & k \arrow[r] \arrow[d] & k+1 \arrow[r] \arrow[d] & \dots \arrow[r] & n \arrow[d] \\
k \arrow[r] & k \arrow[r] & \dots \arrow[r] & k \arrow[r] & k+1 \arrow[r] & \dots \arrow[r] & n
\end{tikzcd}$$

may be used to show that the inclusion $\Lambda^n_k \subset \Delta^n$ is a retract of

$$(\Lambda^n_k \times \Delta^1) \cup (\Delta^n \times \{1\}).$$

Thus, $\Lambda^n_k \subset \Delta^n$ is in the class $M_{\mathbf{B_2}}$ for all $n$ and $k$. □

COROLLARY 4.6. *Suppose that $i : K \hookrightarrow L$ is an anodyne extension and that $Y \hookrightarrow X$ is an arbitrary inclusion. Then the induced map*

$$(K \times X) \cup (L \times Y) \to (L \times X)$$

*is an anodyne extension.*

PROOF: The set of morphisms $K' \to L'$ such that

$$(K' \times X) \cup (L' \times Y) \to (L' \times X)$$

is anodyne is a saturated set. Consider the inclusion

$$(\Delta^1 \times Y') \cup (\{e\} \times X') \subset (\Delta^1 \times X') \qquad (Y' \subset X')$$

and the induced inclusion

$$\begin{tikzcd}
(((\Delta^1 \times Y') \cup (\{e\} \times X')) \times X) \cup ((\Delta^1 \times X') \times Y) \arrow[r] \arrow[d, "\cong"'] & ((\Delta^1 \times X') \times X) \arrow[d, "\cong"] \\
(((\Delta^1 \times ((Y' \times X) \cup (X' \times Y))) \cup (\{e\} \times (X' \times X)) & \Delta^1 \times (X' \times X)
\end{tikzcd}$$

This inclusion is anodyne, and so the saturated set in question contains all anodyne morphisms. □

### I.5. Function complexes.

Let $X$ and $Y$ be simplicial sets. The *function complex* $\mathbf{Hom}(X,Y)$ is the simplicial set defined by

$$\mathbf{Hom}(X,Y)_n = \hom_{\mathbf{S}}(X \times \Delta^n, Y).$$

If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, then the induced function

$$\theta^* : \mathbf{Hom}(X,Y)_n \to \mathbf{Hom}(X,Y)_m$$

is defined by sending the map $f : X \times \Delta^n \to Y$ to the composite

$$X \times \Delta^m \xrightarrow{1\times\theta} X \times \Delta^n \xrightarrow{f} Y.$$

In other words, one thinks of $X \times \Delta^n$ as a cosimplicial "space" in the obvious way.

There is an evaluation map

$$ev : X \times \mathbf{Hom}(X,Y) \to Y$$

defined by $(x,f) \mapsto f(x,\iota_n)$. To show, for example, that $ev$ commutes with face maps $d_j$, one has to check that

$$f \cdot (1 \times d^j)(d_j x, \iota_{n-1}) = d_j f(x,\iota_n).$$

But

$$f \cdot (1 \times d^j)(d_j x, \iota_{n-1}) = f(d_j x, d_j \iota_n) = d_j f(x,\iota_n).$$

More generally, $ev$ commutes with all simplicial structure maps and is thus a simplicial set map which is natural in $X$ and $Y$.

Proposition 5.1 (Exponential Law). *The function*

$$ev_* : \hom_{\mathbf{S}}(K, \mathbf{Hom}(X,Y)) \to \hom_{\mathbf{S}}(X \times K, Y),$$

*which is defined by sending the simplicial map $g : K \to \mathbf{Hom}(X,Y)$ to the composite*

$$X \times K \xrightarrow{1\times g} X \times \mathbf{Hom}(X,Y) \xrightarrow{ev} Y,$$

*is a bijection which is natural in $K$, $X$ and $Y$.*

Proof: The inverse of $ev_*$ is the map

$$\hom_{\mathbf{S}}(X \times K, Y) \to \hom_{\mathbf{S}}(K, \mathbf{Hom}(X,Y))$$

defined by sending $g : X \times K \to Y$ to the map $g_* : K \to \mathbf{Hom}(X.Y)$, where, for $x \in K_n$, $g_*(x)$ is the composite

$$X \times \Delta^n \xrightarrow{1\times\iota_x} X \times K \xrightarrow{g} Y. \qquad \square$$

The relation between function complexes and the homotopy theory of simplicial sets is given by

PROPOSITION 5.2. *Suppose that $i : K \hookrightarrow L$ is an inclusion of simplicial sets and $p : X \to Y$ is a fibration. Then the map*

$$\mathbf{Hom}(L,X) \xrightarrow{(i^*,p_*)} \mathbf{Hom}(K,X) \times_{\mathbf{Hom}(K,Y)} \mathbf{Hom}(L,Y),$$

*which is induced by the diagram*

$$\begin{tikzcd}
\mathbf{Hom}(L,X) \arrow[r, "p_*"] \arrow[d, "i^*"'] & \mathbf{Hom}(L,Y) \arrow[d, "i^*"] \\
\mathbf{Hom}(K,X) \arrow[r, "p_*"'] & \mathbf{Hom}(K,Y),
\end{tikzcd}$$

*is a fibration.*

PROOF: Diagrams of the form

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & \mathbf{Hom}(L,X) \arrow[d, "(i^*,p_*)"] \\
\Delta^n \arrow[r] & \mathbf{Hom}(K,X) \times_{\mathbf{Hom}(K,Y)} \mathbf{Hom}(L,Y)
\end{tikzcd}$$

may be identified with diagrams

$$\begin{tikzcd}
(\Lambda^n_k \times L) \cup_{(\Lambda^n_k \times K)} (\Delta^n \times K) \arrow[r] \arrow[d, hook, "j"'] & X \arrow[d, "p"] \\
\Delta^n \times L \arrow[r] & Y
\end{tikzcd}$$

by the exponential law (Proposition 5.1). But $j$ is an anodyne extension by Corollary 4.6, so the required lifting exists. □

COROLLARY 5.3.

1. *If $p : X \to Y$ is a fibration, then so is $p_* : \mathbf{Hom}(K,X) \to \mathbf{Hom}(K,Y)$*
2. *If $X$ is fibrant then the induced map $i^* : \mathbf{Hom}(L,X) \to \mathbf{Hom}(K,X)$ is a fibration.*

Proof:

(1) The diagram

$$\begin{tikzcd}
\mathbf{Hom}(K,Y) \arrow[r, "1"] \arrow[d] & \mathbf{Hom}(K,Y) \arrow[d] \\
* \arrow[r] & *
\end{tikzcd}$$

is a pullback, and the following commutes:

$$\begin{tikzcd}
\mathbf{Hom}(K,X) \arrow[dr] \arrow[drr, "p_*"] \arrow[ddr] & & \\
& \mathbf{Hom}(K,Y) \arrow[r, "1"'] \arrow[d] & \mathbf{Hom}(K,Y) \arrow[d] \\
& \mathbf{Hom}(\emptyset,X) \arrow[r, "p_*"'] \arrow[d, equal] & \mathbf{Hom}(\emptyset,Y) \arrow[d, equal] \\
& * & *
\end{tikzcd}$$

for a uniquely determined choice of map $\mathbf{Hom}(K,Y) \to \mathbf{Hom}(\emptyset,X)$.

(2) There is a commutative diagram

$$\begin{tikzcd}
\mathbf{Hom}(L,X) \arrow[dr] \arrow[drr] \arrow[ddr, "i^*"'] & & \\
& \mathbf{Hom}(K,X) \arrow[r] \arrow[d, "1"'] & \mathbf{Hom}(L,*) = * \arrow[d] \\
& \mathbf{Hom}(K,X) \arrow[r] & \mathbf{Hom}(K,*) = *
\end{tikzcd}$$

where the inner square is a pullback. □

## I.6. Simplicial homotopy.

Let $f, g : K \to X$ be simplicial maps. We say that there is a *simplicial homotopy* $f \xrightarrow{\simeq} g$ from $f$ to $g$ if there is a commutative diagram

$$\begin{tikzcd}
K \times \Delta^0 = K \arrow[d, "1 \times d^1"'] \arrow[dr, "f"] & \\
K \times \Delta^1 \arrow[r, "h"] & X \\
K \times \Delta^0 = K \arrow[u, "1 \times d^0"] \arrow[ur, "g"'] &
\end{tikzcd}$$

The map $h$ is called a *homotopy.*

It's rather important to note that the commutativity of the diagram defining the homotopy $h$ implies that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all simplices $x \in K$. We have given a definition of homotopy which is intuitively correct elementwise — it is essentially the reverse of the definition that one is usually tempted to write down in terms of face (or coface) maps.

Suppose $i : L \subset K$ denotes an inclusion and that the restrictions of $f$ and $g$ to $L$ coincide. We say that there is a simplicial homotopy from $f$ to $g$, (rel $L$) and write $f \xrightarrow{\simeq} g$ (rel $L$), if the diagram exists above, and the following commutes as well:

$$\begin{tikzcd}
K \times \Delta^1 \arrow[r, "h"] & X \\
L \times \Delta^1 \arrow[u, hook, "i \times 1"] \arrow[r, "pr_L"'] & L \arrow[u, "\alpha"']
\end{tikzcd}$$

where $\alpha = f|_L = g|_L$, and $pr_L$ is projection onto the left factor ($pr_R$ will denote projection on the right). A homotopy of the form

$$L \times \Delta^1 \xrightarrow{pr_L} L \xrightarrow{\alpha} X$$

is called a *constant homotopy* (at $\alpha$).

The homotopy relation may fail to be an equivalence relation in general. Consider the maps $\iota_0, \iota_1 : \Delta^0 \rightrightarrows \Delta^n$, $(n \geq 1)$, which classify the vertices 0 and 1, respectively, of $\Delta^n$. There is a simplex $[0, 1] : \Delta^1 \to \Delta^n$ determined by these vertices, and so $\iota_0 \xrightarrow{\simeq} \iota_1$ (alternatively, $0 \xrightarrow{\simeq} 1$). But there is no 1-simplex which could give a homotopy $\iota_1 \xrightarrow{\simeq} \iota_0$, since $0 \leq 1$. This observation provides a second means (see Lemma 3.5) of seeing that $\Delta^n$ not fibrant, since we can prove

LEMMA 6.1. *Suppose that $X$ is a fibrant simplicial set. Then simplicial homotopy of vertices $x : \Delta^0 \to X$ of $X$ is an equivalence relation.*

PROOF: There is a homotopy $x \xrightarrow{\simeq} y$ if and only if there is a 1-simplex $v$ of $X$ such that $d_1 v = x$ and $d_0 v = y$ (alternatively $\partial v = (y, x)$; in general the *boundary* $\partial\sigma$ of an $n$-simplex $\sigma$ is denoted by $\partial\sigma = (d_0\sigma, \dots, d_n\sigma)$). But then the equation $\partial(s_0 x) = (x, x)$ gives the reflexivity of the homotopy relation.

Suppose that $\partial v_2 = (y, x)$ and $\partial v_0 = (z, y)$. Then $d_0 v_2 = d_1 v_0$, and so $v_0$ and $v_2$ determine a map $(v_0, \ , v_2) : \Lambda^2_1 \to X$. Choose a lifting

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "{(v_0,\ ,v_2)}"] \arrow[d, hook] & X. \\
\Delta^2 \arrow[ur, "\theta"']
\end{tikzcd}$$

Then

$$\begin{aligned}
\partial(d_1\theta) &= (d_0 d_1 \theta, d_1 d_1 \theta) \\
&= (d_0 d_0 \theta, d_1 d_2 \theta) \\
&= (z, x),
\end{aligned}$$

and so the relation is transitive. Finally, given $\partial v_2 = (y, x)$, set $v_1 = s_0 x$. Then $d_1 v_1 = d_1 v_2$ and so $v_1$ and $v_2$ define a map $( \ , v_1, v_2) : \Lambda^2_0 \to X$. Choose an extension

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "{(\ ,v_1,v_2)}"] \arrow[d, hook] & X. \\
\Delta^2 \arrow[ur, "\theta'"']
\end{tikzcd}$$

Then

$$\begin{aligned}
\partial(d_0\theta') &= (d_0 d_0 \theta', d_1 d_0 \theta') \\
&= (d_0 d_1 \theta', d_0 d_2 \theta') \\
&= (x, y),
\end{aligned}$$

and the relation is symmetric. □

COROLLARY 6.2. *Suppose $X$ is fibrant and that $L \subset K$ is an inclusion of simplicial sets. Then*

(a) *homotopy of maps $K \to X$ is an equivalence relation, and*

(b) *homotopy of maps $K \to X$ (rel $L$) is an equivalence relation.*

PROOF: (a) is a special case of (b), with $L = \emptyset$. But homotopy of maps $K \to X$ (rel $L$) corresponds to homotopy of vertices in the fibres of the Kan fibration

$$i^* : \mathbf{Hom}(K, X) \to \mathbf{Hom}(L, X)$$

via the exponential law (Proposition 5.1). The map $i^*$ is a fibration by Proposition 5.2. □

## I.7. Simplicial homotopy groups.

Let $X$ be a fibrant simplicial set and let $v \in X_0$ be a vertex of $X$. Define $\pi_n(X, v)$, $n \geq 1$, to be the set of homotopy classes of maps $\alpha : \Delta^n \to X$ (rel $\partial\Delta^n$) for maps $\alpha$ which fit into diagrams

$$\begin{tikzcd}
\Delta^n \arrow[r, "\alpha"] & X \\
\partial\Delta^n \arrow[u, hook] \arrow[r] & \Delta^0. \arrow[u, "v"']
\end{tikzcd}$$

One often writes $v : \partial\Delta^n \to X$ for the composition

$$\partial\Delta^n \to \Delta^0 \xrightarrow{v} X.$$

Define $\pi_0(X)$ to be the set of homotopy classes of vertices of $X$; this is the set of *path components* of $X$. The simplicial set $X$ is said to be *connected* if $\pi_0(X)$ is trivial (ie. a one-element set). We shall write $[\alpha]$ for the homotopy class of $\alpha$, in all contexts.

Suppose that $\alpha, \beta : \Delta^n \to X$ represent elements of $\pi_n(X, v)$. Then the simplices

$$\begin{cases} v_i = v, & 0 \leq i \leq n-2, \\ v_{n-1} = \alpha, & \text{and} \\ v_{n+1} = \beta & \end{cases}$$

satisfy $d_i v_j = d_{j-1} v_i$ if $i < j$ and $i, j \neq n$, since all faces of all simplices $v_i$ map through the vertex $v$. Thus, the $v_i$ determine a simplicial set map $(v_0, \dots, v_{n-1}, \ , v_{n+1}) : \Lambda^{n+1}_n \to X$, and there is an extension

$$\begin{tikzcd}
\Lambda^{n+1}_n \arrow[r, "{(v_0, \dots, v_{n-1}, \ , v_{n+1})}"] \arrow[d, hook] & X. \\
\Delta^{n+1} \arrow[ur, "\omega"'] &
\end{tikzcd}$$

Observe that

$$\begin{aligned}\partial(d_n\omega) &= (d_0d_n\omega, \dots, d_{n-1}d_n\omega, d_nd_n\omega)\\ &= (d_{n-1}d_0\omega, \dots, d_{n-1}d_{n-1}\omega, d_nd_{n+1}\omega)\\ &= (v, \dots, v),\end{aligned}$$

and so $d_n\omega$ represents an element of $\pi_n(X, v)$.

LEMMA 7.1. *The homotopy class of $d_n\omega$ (rel $\partial\Delta^n$) is independent of the choices of representatives of $[\alpha]$ and $[\beta]$ and of the choice of $\omega$.*

PROOF: Suppose that $h_{n-1}$ is a homotopy $\alpha \xrightarrow{\simeq} \alpha'$ (rel $\partial\Delta^n$) and $h_{n+1}$ is a homotopy $\beta \xrightarrow{\simeq} \beta'$ (rel $\partial\Delta^n$). Suppose further that

$$\partial\omega = (v, \dots, v, \alpha, d_n\omega, \beta)$$

and

$$\partial\omega' = (v, \dots, v, \alpha', d_n\omega', \beta').$$

Then there is a map

$$(\Delta^{n+1} \times \partial\Delta^1) \cup (\Lambda^{n+1}_n \times \Delta^1) \xrightarrow{(\omega', \omega, (v, \dots, h_{n-1},\ , h_{n+1}))} X$$

which is determined by the data. Choose an extension

$$\begin{tikzcd}
(\Delta^{n+1} \times \partial\Delta^1) \cup (\Lambda^{n+1}_n \times \Delta^1) \arrow[r, "{(\omega', \omega, (v, \dots, h_{n-1},\ , h_{n+1}))}"] \arrow[d, hook] & X. \\
\Delta^{n+1} \times \Delta^1 \arrow[ur, "w"'] &
\end{tikzcd}$$

Then the composite

$$\Delta^n \times \Delta^1 \xrightarrow{d^n \times 1} \Delta^{n+1} \times \Delta^1 \xrightarrow{w} X$$

is a homotopy $d_n\omega \xrightarrow{\simeq} d_n\omega'$ (rel $\partial\Delta^n$). □

It follows from Lemma 7.1 that the assignment

$$([\alpha], [\beta]) \mapsto [d_n\omega], \quad \text{where} \quad \partial\omega = (v, \dots, v, \alpha, d_n\omega, \beta),$$

gives a well-defined pairing

$$m : \pi_n(X, v) \times \pi_n(X, v) \to \pi_n(X, v).$$

Let $e \in \pi_n(X, v)$ be the homotopy class which is represented by the constant map

$$\Delta^n \to \Delta^0 \xrightarrow{v} X.$$

THEOREM 7.2. *With these definitions, $\pi_n(X, v)$ is a group for $n \geq 1$, which is abelian if $n \geq 2$.*

Proof: We shall demonstrate here that the $\pi_n(X, v)$ are groups; the abelian property for the higher homotopy groups will be proved later (Lemma 7.6).

It is easily seen (exercise) that $\alpha \cdot e = e \cdot \alpha = \alpha$ for any $\alpha \in \pi_n(X, v)$, and that the map $\pi_n(X, v) \to \pi_n(X, v)$ induced by left multiplication by $\alpha$ is bijective. The result follows, then, if we can show that the multiplication in $\pi_n(X, v)$ is associative.

To see that the multiplication is associative, let $x, y, z : \Delta^n \to X$ represent elements of $\pi_n(X, v)$. Choose $(n+1)$-simplices $\omega_{n-1}, \omega_{n+1}, \omega_{n+2}$ such that

$$\begin{aligned}
\partial\omega_{n-1} &= (v, \dots, v, x, d_n\omega_{n-1}, y), \\
\partial\omega_{n+1} &= (v, \dots, v, d_n\omega_{n-1}, d_n\omega_{n+1}, z), \quad \text{and} \\
\partial\omega_{n+2} &= (v, \dots, v, y, d_n\omega_{n+2}, z).
\end{aligned}$$

Then there is a map

$$\Lambda^{n+2}_n \xrightarrow{(v, \dots, v, \omega_{n-1},\ , \omega_{n+1}, \omega_{n+2})} X$$

which extends to a map $u : \Delta^{n+2} \to X$. But then

$$\partial(d_n u) = (v, \dots, v, x, d_n\omega_{n+1}, d_n\omega_{n+2}),$$

and so

$$\begin{aligned}
([x][y])[z] &= [d_n\omega_{n-1}][z] \\
&= [d_n\omega_{n+1}] \\
&= [d_n d_n u] \\
&= [x][d_n\omega_{n+2}] \\
&= [x]([y][z]).
\end{aligned}$$

□

In order to prove that $\pi_n(X, v)$ abelian for $n \geq 2$, it is most instructive to show that there is a loop-space $\Omega X$ such that $\pi_n(X, v) \cong \pi_{n-1}(\Omega X, v)$ and then to show that $\pi_i(\Omega X, v)$ is abelian for $i \geq 1$. This is accomplished with a series of definitions and lemmas, all of which we'll need in any case. The first step is to construct the long exact sequence of a fibration.

Suppose that $p : X \to Y$ is a Kan fibration and that $F$ is the fibre over a vertex $* \in Y$ in the sense that the square

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^0 \arrow[r, "*"'] & Y
\end{tikzcd}$$

is cartesian. Suppose that $v$ is a vertex of $F$ and that $\alpha : \Delta^n \to Y$ represents an element of $\pi_n(Y, *)$. Then in the diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "{(\ , v, \dots, v)}"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\alpha"'] \arrow[ur, "\theta"] & Y,
\end{tikzcd}$$

the element $[d_0\theta] \in \pi_{n-1}(F, v)$ is independent of the choice of $\theta$ and representative of $[\alpha]$. The resulting function

$$\partial : \pi_n(Y, *) \to \pi_{n-1}(F, v)$$

is called the *boundary map.*

LEMMA 7.3.

(a) *The boundary map* $\partial : \pi_n(Y, *) \to \pi_{n-1}(F, v)$ *is a group homomorphism if* $n > 1$.

(b) *The sequence*

$$\cdots \to \pi_n(F, v) \xrightarrow{i_*} \pi_n(X, v) \xrightarrow{p_*} \pi_n(Y, *) \xrightarrow{\partial} \pi_{n-1}(F, v) \to \dots$$
$$\dots \xrightarrow{p_*} \pi_1(Y, *) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y)$$

*is exact in the sense that kernel equals image everywhere. Moreover, there is an action of* $\pi_1(Y, *)$ *on* $\pi_0(F)$ *such that two elements of* $\pi_0(F)$ *have the same image under* $i_*$ *in* $\pi_0(X)$ *if and only if they are in the same orbit for the* $\pi_1(Y, *)$*-action.*

Most of the proof of Lemma 7.3 is easy, once you know

LEMMA 7.4. *Let* $\alpha : \Delta^n \to X$ *represent an element of* $\pi_n(X, v)$. *Then* $[\alpha] = e$ *if and only if there is an* $(n+1)$*-simplex* $\omega$ *of* $X$ *such that* $\partial\omega = (v, \dots, v, \alpha)$.

The proof of Lemma 7.4 is an exercise.

PROOF OF LEMMA 7.3: (a) To see that $\partial : \pi_n(Y, *) \to \pi_{n-1}(F, v)$ is a homomorphism if $n \geq 2$, suppose that we are given diagrams

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "v"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\alpha_i"'] \arrow[ur, "\theta_i"] & Y
\end{tikzcd} \qquad i = n-1, n, n+1,$$

where the $\alpha_i$ represent elements of $\pi_n(Y,*)$. Suppose that there is an $(n+1)$-simplex $\omega$ such that

$$\partial\omega = (*,\dots,*,\alpha_{n-1},\alpha_n,\alpha_{n+1}).$$

Then there is a commutative diagram

$$\begin{tikzcd}
\Lambda^{n+1}_0 \arrow[r, "{(\ ,v,\dots,v,\theta_{n-1},\theta_n,\theta_{n+1})}"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^{n+1} \arrow[r, "\omega"'] \arrow[ur, "\gamma"] & Y,
\end{tikzcd}$$

and

$$\begin{aligned}
\partial(d_0\gamma) &= (d_0d_0\gamma, d_1d_0\gamma,\dots,d_nd_0\gamma) \\
&= (d_0d_1\gamma, d_0d_2\gamma,\dots,d_0d_{n-1}\gamma, d_0d_n\gamma, d_0d_{n+1}\gamma) \\
&= (v,\dots,v,d_0\theta_{n-1},d_0\theta_n,d_0\theta_{n+1})
\end{aligned}$$

Thus $[d_0\theta_n] = [d_0\theta_{n-1}][d_0\theta_{n+1}]$, and so $\partial([\alpha_{n-1}][\alpha_{n+1}]) = \partial[\alpha_{n-1}]\partial[\alpha_{n+1}]$ in $\pi_{n-1}(F,v)$.

(b) We shall show that the sequence

$$\pi_n(X,v) \xrightarrow{p*} \pi_n(Y,*) \xrightarrow{\partial} \pi_{n-1}(F,v)$$

is exact; the rest of the proof is an exercise. The composite is trivial, since in the diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "v"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "p\alpha"'] \arrow[ur, "\alpha"] & Y
\end{tikzcd}$$

with $[\alpha] \in \pi_n(X,v)$, we find that $d_0\alpha = v$. Suppose that $\gamma : \Delta^n \to Y$ represents a class $[\gamma]$ such that $\partial[\gamma] = e$. Choose a diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "v"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\gamma"'] \arrow[ur, "\theta"] & Y
\end{tikzcd}$$

so that $[d_0\theta] = \partial[\gamma]$. Then there is a simplex homotopy

$$\Delta^{n-1} \times \Delta^1 \xrightarrow{h_0} F$$

giving $d_0\theta \simeq v$ (rel $\partial\Delta^n$). Thus, there is a diagram

$$\begin{tikzcd}
(\Delta^n \times 1) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "{(\theta, (h_0, v, \dots, v))}"] \arrow[d, hook] & X. \\
\Delta^n \times \Delta^1 \arrow[ur, "h"'] &
\end{tikzcd}$$

Moreover $p \cdot h$ is a homotopy $\gamma \simeq p \cdot (h \cdot d^1)$ (rel $\partial\Delta^n$). □

Now for some definitions. For a Kan complex $X$ and a vertex $*$ of $X$, the *path space* $PX$ is defined by requiring that the following square is a pullback.

$$\begin{tikzcd}
PX \arrow[r, "i_X"] \arrow[d] & \mathbf{Hom}(\Delta^1, X) \arrow[d, "(d^0)^*"] \\
\Delta^0 \arrow[r, "*"'] & \mathbf{Hom}(\Delta^0, X) \cong X.
\end{tikzcd}$$

Furthermore, the map $\pi : PX \to X$ is defined to be the composite

$$PX \xrightarrow{i_X} \mathbf{Hom}(\Delta^1, X) \xrightarrow{(d^1)^*} \mathbf{Hom}(\Delta^0, X) \cong X.$$

The maps $(d^\epsilon)^*$ are fibrations for $\epsilon = 0, 1$, by Corollary 5.3. In particular, $PX$ is fibrant.

LEMMA 7.5. *$\pi_i(PX, v)$ is trivial for $i \geq 0$ and all vertices $v$, and $\pi$ is a fibration.*

PROOF: $d^\epsilon : \Delta^0 \to \Delta^1$ is an anodyne extension, and so $(d^0)^*$ has the right lifting property with respect to all maps $\partial\Delta^n \subset \Delta^n$, $n \geq 0$, (see the argument for Proposition 5.2). Thus, the map $PX \to \Delta^0 = *$ has the right lifting property with respect to all such maps. Any two vertices of $PX$ are homotopic, by finding extensions

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r] \arrow[d, hook] & PX. \\
\Delta^1 \arrow[ur, dashed] &
\end{tikzcd}$$

If $\alpha : \Delta^n \to PX$ represents an element of $\pi_n(PX, v)$, then there is a commutative diagram

$$\begin{tikzcd}
\partial\Delta^{n+1} \arrow[r, "{(v,\ldots,v,\alpha)}"] \arrow[d, hook] & PX, \\
\Delta^{n+1} \arrow[ur, dashed] &
\end{tikzcd}$$

and so $[\alpha] = e$ in $\pi_n(PX, v)$. Finally, the map $\pi$ sits inside the pullback diagram

$$\begin{tikzcd}
PX \arrow[r] \arrow[dd, "\pi"'] & \mathbf{Hom}(\Delta^1, X) \arrow[d, "i^*"] \\
 & \mathbf{Hom}(\partial\Delta^1, X) \arrow[d, "\cong"] \\
X \arrow[r, "{(*, 1_X)}"'] & X \times X
\end{tikzcd}$$

and so $\pi$ is a fibration since $i^*$ is, by Corollary 5.3. □

Define the *loop space* $\Omega X$ to be the fibre of $\pi : PX \to X$ over the base point $*$. A simplex of $\Omega X$ is a simplicial map $f : \Delta^n \times \Delta^1 \to X$ such that the restriction of $f$ to $\Delta^n \times \partial\Delta^1$ maps into $*$. Now we can prove

LEMMA 7.6. *$\pi_i(\Omega X, *)$ is abelian for $i \geq 1$.*

PROOF: $\pi_n(\Omega X, *)$, as a set, consists of homotopy classes of maps of the form

$$\begin{tikzcd}
\Delta^n \times \Delta^1 \arrow[r, "\alpha"] & X, \\
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1) \arrow[u, hook] \arrow[ur, "*"'] &
\end{tikzcd}$$

rel the boundary $(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \partial\Delta^1)$. The group $\pi_n(\Omega X, *)$ has a second multiplication $[\alpha] \star [\beta]$ (in the 1-simplex direction) such that $[*]$ is an identity for this multiplication and such that $\star$ and the original multiplication together satisfy the *interchange law*

$$([\alpha_1] \star [\beta_1])([\alpha_2] \star [\beta_2]) = ([\alpha_1][\alpha_2]) \star ([\beta_1][\beta_2]).$$

It follows that $[\alpha][\beta] = [\alpha] \star [\beta]$, and that both multiplications are abelian. □

Corollary 7.7. *Suppose that $X$ is fibrant. Then $\pi_i(X,*)$ is abelian if $i \geq 2$.*

The proof of Theorem 7.2 is now complete.

Let $G$ be a group, and recall that the classifying space $BG$ is fibrant, by Lemma 3.5. The simplicial set $BG$ has exactly one vertex $*$. We can now show that $BG$ is an Eilenberg-Mac Lane space.

Proposition 7.8. *There are natural isomorphisms*

$$\pi_i(BG,*) \cong \begin{cases} G & \text{if } i=1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

Proof: $BG$ is a 2-coskeleton (see the proof of Lemma 3.5), and so $\pi_i(BG,*) = 0$ for $i \geq 2$, by Lemma 7.4. It is an elementary exercise to check that the identification $BG_1 = G$ induces an isomorphism of groups $\pi_1(BG,*) \xrightarrow{\cong} G$. The set $\pi_0(BG)$ of path components is trivial, since $BG$ has only one vertex. □

Suppose that $f : X \to Y$ is a map between fibrant simplicial sets. $f$ is said to be a *weak equivalence* if

$$\begin{cases} \text{for each vertex } x \text{ of } X \text{ the induced map } f_* : \pi_i(X,x) \to \pi_i(Y,f(x)) \\ \qquad \text{is an isomorphism for } i \geq 1, \text{ and} \\ \text{the map } f_* : \pi_0(X) \to \pi_0(Y) \text{ is a bijection.} \end{cases} \tag{7.9}$$

Theorem 7.10. *A map $f : X \to Y$ between fibrant simplicial sets is a fibration and a weak equivalence if and only if $f$ has the right lifting property with respect to all maps $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

Proof: ($\Rightarrow$) The simplicial homotopy $\Delta^n \times \Delta^1 \to \Delta^n$, given by the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \arrow[d] \\
0 \arrow[r] & 1 \arrow[r] & \dots \arrow[r] & n
\end{tikzcd}$$

in $\mathbf{n}$, contracts $\Delta^n$ onto the vertex 0. This homotopy restricts to a homotopy $\Lambda^n_0 \times \Delta^1 \to \Lambda^n_0$ which contracts $\Lambda^n_0$ onto 0.

Now suppose that we're given a diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\beta"'] & Y.
\end{tikzcd} \tag{D}$$

If there is a diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[ddd, "i"'] \arrow[dr, "d^0"'] \arrow[drr, "\alpha"] & & \\
& \partial\Delta^n \times \Delta^1 \arrow[r, "h"'] \arrow[d, hook, "i \times 1"'] & X \arrow[d, "p"] \\
& \Delta^n \times \Delta^1 \arrow[r, "g"] & Y \\
\Delta^n \arrow[ur, "d^0"] \arrow[urr, "\beta"'] & &
\end{tikzcd}$$

such that the lifting exists in the diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "h \cdot d^1"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "g \cdot d^1"'] \arrow[ur, "\theta"] & Y,
\end{tikzcd}$$

then the lifting exists in the original diagram **D**. This is a consequence of the existence of the commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "d^0"] \arrow[d, "i"'] \arrow[rr, bend left, "\alpha"] & (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{1\}) \arrow[r, "{(h,\theta)}"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "d^0"'] \arrow[rr, bend right, "\beta"'] & \Delta^n \times \Delta^1 \arrow[r, "g"'] \arrow[ur, "\gamma"] & Y
\end{tikzcd}$$

Now, the contracting homotopy $H : \Lambda^n_0 \times \Delta^1 \to \Lambda^n_0$ determines a diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "j"] \arrow[d, "d^0"'] & \partial\Delta^n \arrow[d, "\alpha"] \\
\Lambda^n_0 \times \Delta^1 \arrow[r, "h_1"] & X \\
\Lambda^n_0 \arrow[r] \arrow[u, "d^1"] & \Delta^0, \arrow[u, "\alpha(0)"']
\end{tikzcd}$$

where $h_1 = \alpha \cdot j \cdot H$. There is a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \{1\}) \cup (\Lambda^n_0 \times \Delta^1) \arrow[r, "{(\alpha, h_1)}"] \arrow[d, hook] & X \\
\partial\Delta^n \times \Delta^1 \arrow[ur, "h"'] &
\end{tikzcd}$$

since $X$ is fibrant. Moreover, there is a diagram

$$\begin{tikzcd}
(\Delta^n \times \{1\}) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "{(\beta, ph)}"] \arrow[d, hook] & Y \\
\Delta^n \times \Delta^1 \arrow[ur, "g"'] &
\end{tikzcd}$$

since $Y$ is fibrant. It therefore suffices to solve the problem for diagrams

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "{(x_0, *, \dots, *)}"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\omega"'] & Y
\end{tikzcd} \qquad \mathbf{D_1}$$

for some vertex $*$ $(= \alpha(0))$ of $X$, since the composite diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "d^1"] \arrow[d, hook, "i"'] & \partial\Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[d, hook, "i \times 1"] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "d^1"'] & \Delta^n \times \Delta^1 \arrow[r, "g"'] & Y
\end{tikzcd}$$

has this form. Then $x_0$ represents an element $[x_0]$ of $\pi_{n-1}(X,*)$ such that $p_*[x_0] = e$ in $\pi_{n-1}(Y,p*)$. Thus, $[x_0] = e$ in $\pi_{n-1}(X,*)$, and so the trivializing homotopy $h_0 : \Delta^{n-1} \times \Delta^1 \to X$ for $x_0$ determines a homotopy

$$h' = (h_0, *, \dots, *) : \partial\Delta^n \times \Delta^1 \to X.$$

But again there is a diagram

$$\begin{tikzcd}
(\Delta^n \times \{1\}) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "{(\omega, ph)}"] \arrow[d, hook] & Y, \\
\Delta^n \times \Delta^1 \arrow[ur, "g'"'] &
\end{tikzcd}$$

so it suffices to solve the lifting problem for diagrams

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "*"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\beta"'] & Y.
\end{tikzcd} \qquad \mathbf{D_2}$$

The map $p_*$ is onto, so $\beta \simeq p\alpha$ (rel $\partial\Delta^n$) via some homotopy $h'' : \Delta^n \times \Delta^1 \to Y$, and so there is a commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "d^0"] \arrow[d, hook] & (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r, "{(*, \alpha)}"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "d^0"'] & \Delta^n \times \Delta^1 \arrow[r, "h''"'] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

$\mathbf{D_2}$ is the composite of these two squares, and the lifting problem is solved.

($\Leftarrow$) Suppose that $p : X \to Y$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. Then $p$ has the right lifting property with respect to all inclusions $L \subset K$, and is a Kan fibration in particular. It follows that $p_* : \pi_0 X \to \pi_0 Y$ is a bijection. Also, if $x \in X$ is any vertex of $X$ and $F_x$ is the fibre over $p(x)$, then $F_x \to *$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. Then $F_x$ is fibrant, and $\pi_0(F_x) = *$ and $\pi_i(F_x, x) = 0$, $i \geq 1$, by the argument of Lemma 7.5. Thus, $p_* : \pi_i(X,x) \to \pi_i(Y,px)$ is an isomorphism for all $i \geq 1$. □

### I.8. Fundamental groupoid.

Let $X$ be a fibrant simplicial set. Provisionally, the fundamental groupoid $\pi_f X$ of $X$ is a category having as objects all vertices of $X$. An arrow $x \to y$ in $\pi_f X$ is a homotopy class of 1-simplices $\omega : \Delta^1 \to X$ (rel $\partial\Delta^1$) where the diagram

$$\begin{tikzcd}
\Delta^0 \arrow[d, "d^1"'] \arrow[dr, "x"] & \\
\Delta^1 \arrow[r, "\omega"] & X \\
\Delta^0 \arrow[u, "d^0"] \arrow[ur, "y"'] &
\end{tikzcd}$$

commutes. If $v_2$ represents an arrow $x \to y$ of $\pi_f X$ and $v_0$ represents an arrow $y \to z$; then the composite $[v_0][v_2]$ is represented by $d_1\omega$, where $\omega$ is a 2-simplex such that the following diagram commutes

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "{(v_0,\ \ ,v_2)}"] \arrow[d, hook] & X. \\
\Delta^2 \arrow[ur, "\omega"'] &
\end{tikzcd}$$

The fact that this is well-defined should be clear. The identity at $x$ is represented by $s_0 x$. This makes sense because, if $v_2 : x \to y$ and $v_0 : y \to z$ then $\partial s_0 v_0 = (v_0, v_0, s_0 y)$, and $\partial(s_1 v_2) = (s_0 y, v_2, v_2)$. The associativity is proved as it was for $\pi_1$. In fact, $\pi_f X(x, x) = \pi_1(X, x)$ specifies the group of homomorphisms $\pi_f X(x, x)$ from $x$ to itself in $\pi_f X$, by definition. By solving the lifting problem

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "{(\ \ ,s_0 x, v_2)}"] \arrow[d, hook] & X \\
\Delta^2 \arrow[ur, dashed, "\omega"'] &
\end{tikzcd}$$

for $v_2 : x \to y$, one finds a $v_0 : y \to x$ (namely $d_0\omega$) such that $[v_0][v_2] = 1_x$. But then $[v_2]$ is also epi since it has a right inverse by a similar argument. Thus $[v_2][v_0][v_2] = [v_2]$ implies $[v_2][v_0] = 1_y$, and so $\pi_f X$ really is a groupoid.

Now, let $\alpha : \Delta^n \to X$ represent an element of $\pi_n(X, x)$ and let $\omega : \Delta^1 \to X$ represent an element of $\pi_f X(x, y)$. Then there is a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[d, hook] \arrow[dr, "(\omega \cdot pr_R, \alpha)"] & \\
\Delta^n \times \Delta^1 \arrow[r, "h_{(\omega,\alpha)}"] & X, \\
\Delta^n \arrow[u, "d^0"] \arrow[ur, "\omega_*\alpha"'] &
\end{tikzcd}$$

and $\omega_*\alpha$ represents an element of $\pi_n(X, y)$.

PROPOSITION 8.1. *The class* $[\omega_*\alpha]$ *is independent of choices of representatives. Moreover,* $[\alpha] \mapsto [\omega_*\alpha]$ *is a group homomorphism which is functorial in* $[\omega]$, *and so the assignment* $x \mapsto \pi_n(X, x)$ *determines a functor on* $\pi_f X$.

PROOF: We shall begin by establishing independence of the choice of representative for the class $[\omega]$. Suppose that $G : \omega \xrightarrow{\simeq} \eta$ (rel $\partial\Delta^1$) is a homotopy of paths from $x$ to $y$. Then there is a 2-simplex $\sigma$ of $X$ such that

$$\partial\sigma = (s_0 y, \eta, \omega).$$

Find simplices of the form $h_{(\eta,\alpha)}$ and $h_{(\omega,\alpha)}$ according to the recipe given above, and let $h_\sigma$ be the composite

$$\partial\Delta^n \times \Delta^2 \xrightarrow{pr} \Delta^2 \xrightarrow{\sigma} X.$$

There is a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_0) \arrow[r, "(h_\sigma, (\ , h(\eta,\alpha), h(\omega,\alpha)))"] \arrow[d, hook, "i"'] & X, \\
\Delta^n \times \Delta^2 \arrow[ur, "\theta_1"'] &
\end{tikzcd}$$

since the inclusion $i$ is anodyne. Then the composite

$$\Delta^n \times \Delta^1 \xrightarrow{1 \times d^0} \Delta^n \times \Delta^2 \xrightarrow{\theta_1} X$$

is a homotopy from $\omega_*\alpha$ to $\eta_*\alpha$ (rel $\partial\Delta^n$), and so $[\omega_*\alpha] = [\eta_*\alpha]$ in $\pi_n(X, y)$.

Suppose that $H : \Delta^n \times \Delta^1 \to X$ gives $\alpha \xrightarrow{\simeq} \beta$ (rel $\partial\Delta^n$), and choose a homotopy $h(\omega, \beta) : \Delta^n \times \Delta^1 \to X$ as above. Let $h_{s_0\omega}$ denote the composite

$$\partial\Delta^n \times \Delta^2 \xrightarrow{pr} \Delta^2 \xrightarrow{s_0\omega} X.$$

Then there is a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_1) \arrow[r, "{(h_{s_0\omega}, (h(\omega,\beta),\ ,H))}"] \arrow[d, hook, "j"'] & X, \\
\Delta^n \times \Delta^2 \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

for some map $\gamma$, since the inclusion $j$ is anodyne. But then the simplex given by the composite

$$\Delta^n \xrightarrow{d^0} \Delta^n \times \Delta^1 \xrightarrow{1 \times d^1} \Delta^n \times \Delta^2 \xrightarrow{\gamma} X$$

is a construction for both $\omega_*\alpha$ and $\omega_*\beta$, so that $[\omega_*\alpha] = [\omega_*\beta]$ in $\pi_n(X, y)$.

For the functoriality, suppose that $\omega : \Delta^1 \to X$ and $\eta : \Delta^1 \to X$ represent elements of $\pi_f X(x, y)$ and $\pi_f X(y, z)$ respectively, and choose a 2-simplex $\gamma$ such that $\partial\gamma = (\eta, d_1\gamma, \omega)$. Then $[d_1\gamma] = [\eta] \cdot [\omega]$ in $\pi_f X$. Choose $h_{(\omega,\alpha)}$ and $h_{(\eta,\omega_*\alpha)}$ according to the recipe above. Then there is a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_1) \arrow[d, hook] \arrow[dr, "{(\gamma \cdot pr, (h(\eta,\omega_*\alpha),\ , h(\omega,\alpha))}"] & \\
\Delta^n \times \Delta^2 & X, \\
\Delta^n \times \Delta^1 \arrow[u, "1 \times d^1"] \arrow[ur, "\xi"'] &
\end{tikzcd}$$

and hence a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[d, hook] \arrow[dr, "{(h_{d_1\gamma}, \alpha)}"] & \\
\Delta^n \times \Delta^1 \arrow[r, "\xi"] & X, \\
\Delta^n \arrow[u, "d^1"] \arrow[ur, "\eta_*\omega_*\alpha = (d_1\gamma)_*\alpha"'] &
\end{tikzcd}$$

where $h_{d_1\gamma}$ is the composite

$$\partial\Delta^n \times \Delta^1 \xrightarrow{pr} \Delta^1 \xrightarrow{d_1\gamma} X.$$

The statement that $\omega_*$ is a group homomorphism is easily checked. □

THEOREM 8.2. *Suppose that the following is a commutative triangle of simplicial set maps:*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \\ & Z \arrow[ur, "f"'] & \end{tikzcd}$$

*with $X, Y$, and $Z$ fibrant. If any two of $f, g$, or $h$ are weak equivalences, then so is the third.*

PROOF: There is one non-trivial case, namely to show that $f$ is a weak equivalence if $g$ and $h$ are. This is no problem at all for $\pi_0$. Suppose $y \in Y$ is a vertex. We must show that $f_* : \pi_n(Y, y) \to \pi_n(Z, fy)$ is an isomorphism. The vertex $y$ may not be in the image of $g$, but there is an $x \in X$ and a path $\omega : y \to gx$ since $\pi_0(g)$ is epi. But then there is a diagram

$$\begin{tikzcd} \pi_n(Y,y) \arrow[r, "{[\omega]_*}"] \arrow[d, "f_*"'] & \pi_n(Y,gx) \arrow[d, "f_*"] & \pi_n(X,x). \arrow[l, "g_*"'] \arrow[dl, "h_*"] \\ \pi_n(Z,fy) \arrow[r, "{[f\omega]_*}"'] & \pi_n(Z,fgx) & \end{tikzcd}$$

The maps $g_*$, $h_*$, $[\omega]_*$, and $[f\omega]_*$ are isomorphisms, and so both of the maps labelled $f_*$ are isomorphisms. □

There are three competing definitions for the fundamental groupoid of an arbitrary simplicial set $X$. The most obvious choice is the classical fundamental groupoid $\pi|X|$ of the realization of $X$; in the notation above, this is $\pi_f S|X|$. Its objects are the elements of $|X|$, and its morphisms are homotopy classes of paths in $|X|$. The second choice is the model $GP_*X$ of Gabriel and Zisman. The groupoid $GP_*X$ is the free groupoid associated to the *path category* $P_*X$ of $X$. The path category has, as objects, all the vertices (elements of $X_0$) of $X$. It is generated, as a category, by the 1-simplices of $X$, subject to the relation that, for each 2-simplex $\sigma$ of $X$, the diagram

$$\begin{tikzcd} v_0 \arrow[rr, "d_2\sigma"] \arrow[dr, "d_1\sigma"'] & & v_1 \arrow[dl, "d_0\sigma"] \\ & v_2 & \end{tikzcd}$$

commutes. The free groupoid $G(\mathbf{\Delta} \downarrow X)$ associated to the simplex category $\mathbf{\Delta} \downarrow X$ is also a good model. We shall see later on, in Section III.1, that $\pi|X|$, $GP_*X$ and $G(\mathbf{\Delta} \downarrow X)$ are all naturally equivalent.

### I.9. Categories of fibrant objects.

Let $\mathbf{S}_f$ be the full subcategory of the simplicial set category whose objects are the Kan complexes. The category $\mathbf{S}_f$ has all finite products. We have two distinguished classes of maps in $\mathbf{S}_f$, namely the fibrations (defined by the lifting property) and the weak equivalences (defined via simplicial homotopy groups). A *trivial fibration* $p : X \to Y$ in $\mathbf{S}_f$ is defined to be a map which is both a fibration and a weak equivalence. A *path object* for $X \in \mathbf{S}_f$ is a commutative diagram

$$\begin{tikzcd}
& X^I \arrow[d, "{(d_0, d_1)}"] \\
X \arrow[ur, "s"] \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

where $s$ is a weak equivalence and $(d_0, d_1)$ is a fibration. The maps $d_0$ and $d_1$ are necessarily trivial fibrations. Any Kan complex $X$ has a natural choice of path object, namely the diagram

$$\begin{tikzcd}
& \mathbf{Hom}(\Delta^1, X) \arrow[d] \arrow[dd, bend left=90, "{(d_0, d_1)}"] \\
& \mathbf{Hom}(\partial\Delta^1, X) \arrow[d, "\cong"] \\
X \arrow[uur, "s"] \arrow[r] & X \times X
\end{tikzcd}$$

where $s$ is the map

$$X \cong \mathbf{Hom}(\Delta^0, X) \xrightarrow{(s^0)^*} \mathbf{Hom}(\Delta^1, X).$$

$(s^0)^* = s$ is a weak equivalence: it is a right inverse for the map

$$\mathbf{Hom}(\Delta^1, X) \xrightarrow{(d^0)^*} \mathbf{Hom}(\Delta^0, X),$$

and $(d^0)^*$ is a trivial fibration, by Theorem 7.10, since it has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. The map $(d^0)^*$ is isomorphic to one of the components of the map $\mathbf{Hom}(\Delta^1, X) \to X \times X$.

The following list of properties of $\mathbf{S}_f$ is essentially a recapitulation of things that we've seen:

**(A)** Suppose given a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\
& Z. &
\end{tikzcd}$$

If any two of $f$, $g$ and $h$ are weak equivalences, then so is the third.

**(B)** The composite of two fibrations is a fibration. Any isomorphism is a fibration.

**(C)** Pullback diagrams of the form

$$\begin{tikzcd}
Z \times_Y X \arrow[r] \arrow[d, "pr"'] & X \arrow[d, "p"] \\
Z \arrow[r] & Y
\end{tikzcd}$$

exist in the case where $p$ is a fibration. Furthermore, $pr$ is a fibration which is trivial if $p$ is trivial.

**(D)** For any object $X$ there is at least one path space $X^I$.

**(E)** For any object $X$, the map $X \to *$ is a fibration.

Statement **(A)** is Theorem 8.2. **(B)** is an exercise. **(C)** holds because fibrations and trivial fibrations are defined by lifting properties, by Theorem 7.10. **(D)** was discussed above, and **(E)** isn't really worth mentioning.

Following K. Brown's thesis [15] (where the notion was introduced), a category $\mathcal{C}$ which has all finite products and has distinguished classes of maps called fibrations and weak equivalences which together satisfy axioms **(A)** – **(E)** is called a *category of fibrant objects* (for a homotopy theory). We've proved:

THEOREM 9.1. $\mathbf{S}_f$ *is a category of fibrant objects for a homotopy theory.*

Other basic examples are the category **CGHaus** of compactly generated Hausdorff spaces, and the category **Top** of topological spaces. In fact, more is true. The fibrations of **CGHaus** are the Serre fibrations, and the weak equivalences are the weak homotopy equivalences. A map $i : U \to V$ in **CGHaus** is said to be a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

PROPOSITION 9.2. *The category* **CGHaus** *and these three classes of maps satisfy the following list of axioms:*

**CM1:** **CGHaus** *is closed under all finite limits and colimits.*

**CM2:** *Suppose that the following diagram commutes in* **CGHaus***:*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z. & \end{tikzcd}$$

*If any two of $f$, $g$ and $h$ are weak equivalences, then so is the third.*

**CM3:** *If $f$ is a retract of $g$ and $g$ is a weak equivalence, fibration or cofibration, then so is $f$.*

**CM4:** *Suppose that we are given a commutative solid arrow diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

*where $i$ is a cofibration and $p$ is a fibration. Then the dotted arrow exists, making the diagram commute, if either $i$ or $p$ is also a weak equivalence.*

**CM5:** *Any map $f : X \to Y$ may be factored:*

(a) *$f = p \cdot i$ where $p$ is a fibration and $i$ is a trivial cofibration, and*

(b) *$f = q \cdot j$ where $q$ is a trivial fibration and $j$ is a cofibration.*

Recall that a *trivial fibration* is a map which is a fibration and a weak equivalence. Similarly, a *trivial cofibration* is a map which is both a cofibration and a weak equivalence.

PROOF: The category **CGHaus** has all small limits and colimits, giving **CM1** (see [66, p.182]). This fact is also used to prove the factorization axioms **CM5**; this is the next step.

The map $p : X \to Y$ is a Serre fibration if and only if it has the right lifting property with respect to all inclusions $j : |\Lambda^n_k| \to |\Delta^n|$. Each such $j$ is necessarily a cofibration. Now consider all diagrams

$$\begin{tikzcd} {|\Lambda^{n_D}_{k_D}|} \arrow[r, "\alpha_D"] \arrow[d, hook] & X \arrow[d, "f"] \\ {|\Delta^{n_D}|} \arrow[r, "\beta_D"'] & Y \end{tikzcd}$$

**D**

and form the pushout

$$\begin{tikzcd}
\bigsqcup_D |\Lambda^{n_D}_{k_D}| \arrow[r, "(\alpha_D)"] \arrow[d, hook] & X \arrow[d, "i_1"'] \arrow[r, equal] & X_0 \arrow[dr, "f = f_0"] & \\
\bigsqcup_D |\Delta^{n_D}| \arrow[r] & X_1 \arrow[rr, "f_1"'] & & Y.
\end{tikzcd}$$

Then we obtain a factorization

$$f = f_0 = f_1 \cdot i_1,$$

where $i_1$ is a cofibration since it's a pushout of such, and also a weak equivalence since it is a pushout of a map which has a strong deformation retraction. We repeat the process by considering all diagrams.

$$\begin{tikzcd}
|\Lambda^{n_D}_{k_D}| \arrow[r, "\alpha_D"] \arrow[d, hook] & X_1 \arrow[d, "f_1"] \\
|\Delta^{n_D}| \arrow[r, "\beta_D"'] & Y
\end{tikzcd}$$

and so on. Thus, we obtain a commutative diagram

$$\begin{tikzcd}
X = X_0 \arrow[r, "i_1"] \arrow[dr, "f_0"'] & X_1 \arrow[r, "i_2"] \arrow[d, "f_1"'] & X_2 \arrow[r] \arrow[dl, "f_2"] & \ldots, \\
& Y & &
\end{tikzcd}$$

which induces a diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "\tau_0"] \arrow[dr, "f = f_0"'] & & \varinjlim X_i \arrow[dl, "f_\infty"] \\
& Y &
\end{tikzcd}$$

But now $\tau_0$ has the left lifting property with respect to all trivial fibrations, so it's a cofibration. Moreover, $\tau_0$ is a weak equivalence since any compact subset of $\varinjlim X_i$ lies in some finite stage $X_i$, and all the $X_i \to X_{i+1}$ are weak equivalences. Finally, $f_\infty$ is a fibration: the space $|\Lambda^n_k|$ is a finite CW-complex, so that for each diagram

$$\begin{tikzcd}
|\Lambda^n_k| \arrow[r, "\alpha"] \arrow[d, hook] & \varinjlim X_i \arrow[d, "f_\infty"] \\
|\Delta^n| \arrow[r, "\beta"'] & Y,
\end{tikzcd}$$

there is an index $i$ and a map $\alpha_i$ making the following diagram commute:

$$\begin{tikzcd}
|\Lambda^n_k| \arrow[rr, "\alpha"] \arrow[dr, "\alpha_i"] \arrow[dd, hook] & & \varinjlim X_i \arrow[dd, "f_\infty"] \\
& X_i \arrow[ur] \arrow[dr, "f_i"] & \\
|\Delta^n| \arrow[rr, "\beta"'] & & Y.
\end{tikzcd}$$

But then

$$\begin{tikzcd}
|\Lambda^n_k| \arrow[r, "\alpha_i"] \arrow[d, hook] & X_i \arrow[d, "f_i"] \\
|\Delta^n| \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

is one of the diagrams defining $f_{i+1}$ and there is a diagram

$$\begin{tikzcd}
|\Lambda^n_k| \arrow[rr, "\alpha_i"] \arrow[dd, hook] & & X_i \arrow[dl] \arrow[dd, "f_i"] \\
& X_{i+1} \arrow[dr] & \\
|\Delta^n| \arrow[ur, "\theta_i"] \arrow[rr, "\beta"'] & & Y
\end{tikzcd}$$

which defines the lifting.

The other lifting property is similar, using

LEMMA 9.4. *The map $p : X \to Y$ is a trivial fibration if and only if $p$ has the right lifting property with respect to all inclusions $|\partial\Delta^n| \subset |\Delta^n|$.*

The proof is an exercise.

Quillen calls this proof a *small object argument* [76]. **CM4** is really a consequence of this argument as well. The bulk of the proof displayed for Proposition 9.2 consists of showing that any map $f : X \to Y$ has a factorization $f = p \cdot i$ such that $p$ is a fibration and $i$ is a weak equivalence which has the left lifting property with respect to all fibrations.

Suppose now that we have a diagram

$$\begin{tikzcd}
U \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
V \arrow[r, "\beta"'] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

where $p$ is a fibration and $i$ is a trivial cofibration. We want to construct the dotted arrow (giving the interesting part of **CM4**). There is a diagram

$$\begin{tikzcd}
U \arrow[r, "j"] \arrow[d, "i"'] & W \arrow[d, "\pi"] \\
V \arrow[r, "1_V"'] \arrow[ur, dashed, "r"] & V,
\end{tikzcd}$$

where $j$ is a weak equivalence which has the left lifting property with respect to all fibrations, and $\pi$ is a (necessarily trivial) fibration. Thus, the dotted arrow $r$ exists. But then $i$ is a retract of $j$, and so $i$ has the same lifting property. All of the other axioms are straightforward to verify, and so the proof of Proposition 9.2 is complete.

A *closed model category* is a category $\mathcal{C}$, together with three classes of maps called cofibrations, fibrations and weak equivalences, such that the axioms **CM1**–**CM5** are satisfied. Proposition 9.2 is the statement that **CGHaus** has the structure of a closed model category. The category **CGHaus** is also a category of fibrant objects for a homotopy theory, by the following

PROPOSITION 9.5. *The subcategory of fibrant objects in any closed model category $\mathcal{C}$ is a category of fibrant objects for a homotopy theory.*

Proof: The statement **(E)** is part of the definition. For **(D)**, the map $\Delta : X \to X \times X$ may be factored

$$\begin{tikzcd}
 & X^I \arrow[d, "{(d_0, d_1)}"] \\
X \arrow[ur, "s"] \arrow[r, "\Delta"'] & X \times X,
\end{tikzcd}$$

where $s$ is a trivial cofibration and $(d_0, d_1)$ is a fibration. For **(B)** and **(C)**, we prove:

Lemma 9.6.

(a) *A map $f : X \to Y$ in $\mathcal{C}$ has the right lifting property with respect to all cofibrations (respectively trivial cofibrations) if and only if $f$ is a trivial fibration (respectively fibration).*

(b) *$U \to V$ in $\mathcal{C}$ has the left lifting property with respect to all fibrations (respectively trivial fibrations) if and only if $i$ is a trivial cofibration (respectively cofibration).*

Proof: We'll show that $f : X \to Y$ has the right lifting property with respect to all cofibrations if and only if $f$ is a trivial fibration. The rest of the proof is an exercise.

Suppose that $f$ has the advertised lifting property, and form the diagram

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
V \arrow[ur, "r"] \arrow[r, "p"'] & Y,
\end{tikzcd}$$

where $i$ is a cofibration, $p$ is a trivial fibration, and $r$ exists by the lifting property. Then $f$ is a retract of $p$ and is therefore a trivial fibration. The reverse implication is **CM4**. □

Finally, since fibrations (respectively trivial fibrations) are those maps having the right lifting property with respect to all trivial cofibrations (respectively all cofibrations), they are stable under composition and pullback and include all isomorphisms, yielding **(B)** and **(C)**. **(A)** is **CM2**. This completes the proof of Proposition 9.5.

We shall see that the category of fibrant objects structure that we have displayed for $\mathbf{S}_f$ is the restriction of a closed model structure on the entire simplicial set category; this will be proved in Section 11.

## I.10. Minimal fibrations.

Minimal Kan complexes play roughly the same role in the homotopy theory of simplicial sets as minimal models play in rational homotopy theory. Minimal Kan complexes appear as fibres of minimal fibrations; it turns out that minimal fibrations are exactly the right vehicle for relating the homotopy theories of **S** and **CGHaus**.

A simplicial set map $q : X \to Y$ is said to be a *minimal fibration* if $q$ is a fibration, and for every diagram

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \arrow[r, "pr"] \arrow[d, hook] & \partial\Delta^n \arrow[d] \\
\Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[d, "pr"'] & X \arrow[d, "q"] \\
\Delta^n \arrow[r] & Y
\end{tikzcd} \tag{10.1}$$

the composites

$$\begin{tikzcd}
\Delta^n \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \Delta^n \times \Delta^1 \arrow[r, "h"] & X
\end{tikzcd}$$

are equal. This means that, if two simplices $x$ and $y$ in $X_n$ are fibrewise homotopic (rel $\partial\Delta^n$), then $x = y$.

Note that the class of minimal fibrations is preserved by pullback.

More generally, write $x \simeq_p y$ if there is a diagram of the form (10.1) such that $h(\Delta^n \times 0) = x$, and $h(\Delta^n \times 1) = y$. The relation $\simeq_p$ is an equivalence relation (exercise).

LEMMA 10.2. *Suppose that $x$ and $y$ are degenerate $r$-simplices of a simplicial set $X$ such that $\partial x = \partial y$. Then $x = y$.*

PROOF: (See also [67], p.36.) Suppose that $x = s_m z$ and $y = s_n w$. If $m = n$, then

$$z = d_m x = d_m y = w,$$

and so $x = y$. Suppose that $m < n$. Then

$$z = d_m x = d_m s_n w = s_{n-1} d_m w,$$

and so

$$x = s_m s_{n-1} d_m w = s_n s_m d_m w.$$

Thus

$$s_m d_m w = d_n x = d_n y = w.$$

Therefore $x = s_n w = y$. □

Now we can prove:

PROPOSITION 10.3. *Let $p : X \to Y$ be a Kan fibration. Then $p$ has a strong fibre-wise deformation retract $q : Z \to Y$ which is a minimal fibration.*

PROOF: Let $Z^{(0)}$ be the subcomplex of $X$ which is generated by a choice of vertex in each p-class, and let $i^{(0)} : Z^{(0)} \subset X$ be the canonical inclusion. There is a map $r^{(0)} : \mathrm{sk}_0 X \to Z^{(0)}$ which is determined by choices of representatives. Moreover $pi^{(0)}r^{(0)} = p|_{\mathrm{sk}_0 X}$, and $j_0 \simeq i^{(0)}r^{(0)}$, where $j_0 : \mathrm{sk}_0 X \subset X$ is the inclusion of the subcomplex, via a homotopy $h_0 : \mathrm{sk}_0 X \times \Delta^1 \to X$ such that $h_0(x, 0) = x$ and $h_0(x, 1) = r^{(0)}(x)$, and $h_0$ is constant on simplices of $Z^{(0)}$. The map $h_0$ can be constructed fibrewise in the sense that $p \cdot h_0$ is constant, by using the homotopies implicit in the definition of $\simeq_p$ . The subcomplex $Z^{(0)}$ has a unique simplex in each p-equivalence class that it intersects, by Lemma 10.2.

Let $Z^{(1)}$ be the subcomplex of $X$ which is obtained by adjoining to $Z^{(0)}$ a representative for each homotopy class of 1-simplices $x$ such that $\partial x \subset Z^{(0)}$ and $x$ is not p-related to a 1-simplex of $Z^{(0)}$. Again, $Z^{(1)}$ has a unique simplex in each p-equivalence class that it intersects, by construction in degrees $\leq 1$ and Lemma 10.2 in degrees $> 1$.

Let $x$ be a non-degenerate 1-simplex of $X$. Then there is a commutative diagram

$$\begin{tikzcd}
(\Delta^1 \times \{0\}) \cup (\partial\Delta^1 \times \Delta^1) \arrow[rr, "{(x, h_0|_{\partial x})}"] \arrow[d, hook] & & X \arrow[d, "p"] \\
\Delta^1 \times \Delta^1 \arrow[rr, phantom] \arrow[urr, "h_x"] \arrow[r, "pr_L"'] & \Delta^1 \arrow[r, "px"'] & Y,
\end{tikzcd}$$

by the homotopy lifting property, where the constant homotopy is chosen for $h_x$ if $x \in Z^{(1)}$. But then $\partial(h_x(\Delta^1 \times \{1\})) \subset Z^{(0)}$ and so $h_x(\Delta^1 \times \{1\},)$ is p-related to a unique 1-simplex $r^{(1)}(x)$ of $Z^{(1)}$ via some diagram

$$\begin{tikzcd}
\partial\Delta^1 \times \Delta^1 \arrow[r, "pr_L"] \arrow[d, hook] & \partial\Delta^1 \arrow[d, "\partial h_x(\Delta^1 \times \{1\})"] \\
\Delta^1 \times \Delta^1 \arrow[r, "g_x"] \arrow[d, "pr"'] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r, "px"'] & Y,
\end{tikzcd}$$

where $g_x$ is constant if $x \in Z^{(1)}$, $r^{(1)}(x) = g_x(\Delta^1 \times 1)$, and

$$g_x(\Delta^1 \times \{0\}) = h_x(\Delta^1 \times \{1\}).$$

This defines $r^{(1)} : \mathrm{sk}_1 X \to Z^{(1)}$.

We require a homotopy $h_1 : j_1 \simeq_p i^{(1)}r^{(1)}$, such that $i^{(1)} : Z^{(1)} \subset X$ and $j_1 : \mathrm{sk}_1 X \subset X$ are the subcomplex inclusions, and such that $h_1$ is consistent with $h_0$. We also require that the restriction of $h_1$ to $Z^{(1)}$ be constant. This is done for the simplex $x$ by constructing a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^1 \times \Delta^2) \cup (\Delta^1 \times \Lambda^2_1) \arrow[rr, "{(s_1h_0,(g_x,\ ,h_x))}"] \arrow[d, hook] & & X \arrow[d, "p"] \\
\Delta^1 \times \Delta^2 \arrow[r, "pr"'] \arrow[urr, "\theta_x"] & \Delta^1 \arrow[r, "px"'] & Y,
\end{tikzcd}$$

where the lifting $\theta_x$ is chosen to be the composite

$$\Delta^1 \times \Delta^2 \xrightarrow{pr_L} \Delta^1 \xrightarrow{x} X$$

if $x \in Z^{(1)}$. Then $h_0$ can be extended to the required homotopy $h_1 : j_1 \simeq_p i^{(1)}r^{(1)}$ by requiring that $h_1|_x = \theta_x \cdot (1 \times d^1)$.

Proceeding inductively gives $i : Z = \varinjlim Z^{(n)} \subset X$ and $r : X \to Z$ such that $1_X \simeq ir$ fibrewise, and such that $q : Z \to Y$ has the minimality property. Finally, $q$ is a Kan fibration, since it is a retract of a Kan fibration. $\square$

LEMMA 10.4. *Suppose that*

$$\begin{tikzcd}
Z \arrow[rr, "f"] \arrow[dr, "q"'] & & Z' \arrow[dl, "q'"] \\
& Y &
\end{tikzcd}$$

*is a fibrewise homotopy equivalence of minimal fibrations $q$ and $q'$. Then $f$ is an isomorphism of simplicial sets.*

To prove Lemma 10.4, one uses:

SUBLEMMA 10.5. *Suppose that two maps*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\
& Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\
& Y &
\end{tikzcd}$$

*are fibrewise homotopic, where $g$ is an isomorphism and $q$ is minimal. Then $f$ is an isomorphism.*

PROOF OF SUBLEMMA: Let the diagram

$$\begin{tikzcd}
X \arrow[d, "d^0"'] \arrow[dr, "f"] & \\
X \times \Delta^1 \arrow[r, "h"] & Z \\
X \arrow[u, "d^1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

represent the homotopy. Suppose that $f(x) = f(y)$ for $n$-simplices $x$ and $y$ of $X$. Then inductively $d_i x = d_i y$, $0 \leq i \leq n$, and so the composites

$$\partial\Delta^n \times \Delta^1 \xrightarrow{i \times 1} \Delta^n \times \Delta^1 \xrightarrow{x \times 1} X \times \Delta^1 \xrightarrow{h} Z,$$

and

$$\partial\Delta^n \times \Delta^1 \xrightarrow{i \times 1} \Delta^n \times \Delta^1 \xrightarrow{y \times 1} X \times \Delta^1 \xrightarrow{h} Z,$$

are both equal to a map $h_* : \partial\Delta^n \times \Delta^1 \to Y$. Write $h_x$ for the composite homotopy

$$\Delta^n \times \Delta^1 \xrightarrow{x \times 1} X \times \Delta^1 \xrightarrow{h} Z.$$

Then there is a commutative diagram

$$\begin{tikzcd}
(\Delta^n \times \Lambda^2_2) \cup (\partial\Delta^n \times \Delta^2) \arrow[rr, "{((h_x, h_y, \ ), s_0 h_*)}"] \arrow[d, hook] & & Z \arrow[d, "q"] \\
\Delta^n \times \Delta^2 \arrow[r, "pr_L"'] \arrow[urr, "G"] & \Delta^n \arrow[r, "px = py"'] & Y,
\end{tikzcd}$$

and the homotopy $G \cdot (1 \times d^2)$ shows that $x = y$. Thus, $f$ is monic.

To see that $f$ is epi, suppose inductively that $f : X_i \to Z_i$ is an isomorphism for $0 \leq i \leq n-1$, and let $x : \Delta^n \to Z$ be an $n$-simplex of $Z$. Then there is a commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "{(x_0, \dots, x_n)}"] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r, "x"'] & Z
\end{tikzcd}$$

by the inductive assumption, and so one can find a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{1\}) \arrow[rr, "{(h|_{(x_0,\dots,x_n)}, x)}"] \arrow[d, hook] & & Z \arrow[d, "q"] \\
\Delta^n \times \Delta^1 \arrow[r, "pr_L"'] \arrow[urr, "h_1"'] & \Delta^n \arrow[r, "qx"'] & Y.
\end{tikzcd}$$

Then there is a diagram

$$\begin{tikzcd}
 & & X \arrow[d, "g"] \\
\Delta^n \arrow[urr, "z"] \arrow[r, "d^1"'] & \Delta^n \times \Delta^1 \arrow[r, "h_1"'] & Z,
\end{tikzcd}$$

since $g$ is epi. The restriction of $g(z)$ to $\partial\Delta^n$ is the composite $g \cdot (x_0, \dots, x_n)$, so that $\partial z = (x_0, \dots, x_n)$ since $g$ is monic. Thus, there is a diagram

$$\begin{tikzcd}
(\Delta^n \times \Lambda^2_0) \cup (\partial\Delta^n \times \Delta^2) \arrow[rr, "{((\ , h_1, h_z), s_1 h|_{(x_0,\dots,x_n)})}"] \arrow[d, hook] & & Z \arrow[d, "q"] \\
\Delta^n \times \Delta^2 \arrow[r, "pr_L"'] \arrow[urr, "G'"] & \Delta^n \arrow[r, "qx"'] & Y.
\end{tikzcd}$$

Finally, the composite

$$\Delta^n \times \Delta^1 \xrightarrow{1 \times d^0} \Delta^n \times \Delta^2 \xrightarrow{G'} Z$$

is a fibrewise homotopy from $f(z)$ to $x$, and so $x = f(z)$. □

LEMMA 10.6. *Suppose given a Kan fibration $p$ and pullback diagrams*

$$\begin{tikzcd}
f_i^{-1}p \arrow[r] \arrow[d, "p_i"'] & X \arrow[d, "p"] \\
A \arrow[r, "f_i"'] & Y
\end{tikzcd} \qquad i = 0, 1.$$

*Suppose further that there is a homotopy* $h : f_0 \xrightarrow{\simeq} f_1$. *Then there is a fibrewise homotopy equivalence*

$$\begin{tikzcd}
f_1^{-1}p \arrow[rr, "\simeq"] \arrow[dr, "p_1"'] & & f_2^{-1}p. \arrow[dl, "p_2"] \\
& A &
\end{tikzcd}$$

Proof: Consider the diagrams of pullbacks

$$\begin{tikzcd}
f_\epsilon^{-1}p \arrow[r, "x_\epsilon"] \arrow[d, "p_\epsilon"'] & h^{-1}p \arrow[r] \arrow[d, "p_h"] & X \arrow[d, "p"] \\
A \arrow[r, "d^\epsilon"'] & A \times \Delta^1 \arrow[r, "h"'] & Y
\end{tikzcd} \qquad \epsilon = 0, 1.$$

Then there is a commutative diagram

$$\begin{tikzcd}
f_0^{-1}p \arrow[r, "x_0"] \arrow[d, "d^0"'] & h^{-1}p \arrow[d, "p_h"] \\
f_0^{-1}p \times \Delta^1 \arrow[r, "p_0 \times 1"'] \arrow[ur, "\theta"] & A \times \Delta^1,
\end{tikzcd}$$

by the homotopy lifting property. It follows that there is a diagram

$$\begin{tikzcd}
f_0^{-1}p \arrow[r, "d^1"] \arrow[dr, "\theta_*"] \arrow[ddr, "p_0"'] & f_0^{-1}p \times \Delta^1 \arrow[dr, "\theta"] & \\
& f_1^{-1}p \arrow[r, "x_1"] \arrow[d, "p_1"] & h^{-1}p \arrow[d, "p_h"] \\
& A \arrow[r, "d^1"'] & A \times \Delta^1
\end{tikzcd}$$

and hence an induced map $\theta_*$ as indicated. Similarly there are diagrams

$$\begin{tikzcd}
f_1^{-1}p \arrow[r, "x_1"] \arrow[d, "d^1"'] & h^{-1}p \arrow[d, "p_h"] \\
f_1^{-1}p \times \Delta^1 \arrow[r, "p_1 \times 1"'] \arrow[ur, "\omega"] & A \times \Delta^1,
\end{tikzcd}$$

and

$$\begin{tikzcd}
f_1^{-1}p \arrow[r, "d^0"] \arrow[dr, "\omega_*"] \arrow[ddr, "p_1" swap] & f_1^{-1}p \times \Delta^1 \arrow[dr, "\omega"] & \\
& f_0^{-1}p \arrow[r, "x_0"] \arrow[d, "p_0"] & h^{-1}p \arrow[d, "p_h"] \\
& A \arrow[r, "d^0" swap] & A \times \Delta^1.
\end{tikzcd}$$

Form the diagram

$$\begin{tikzcd}
f_0^{-1}p \times \Lambda^2_0 \arrow[r, "( \ ,\theta,\omega(\theta_* \times 1))"] \arrow[d, hook] & h^{-1}p \arrow[d, "p_h"] \\
f_0^{-1}p \times \Delta^2 \arrow[r, "p_0 \times s^1" swap] \arrow[ur, "\gamma"] & A \times \Delta^1,
\end{tikzcd}$$

by using the homotopy lifting property and the relations

$$d_1\theta = x_1\theta_* = d_1(\omega(\theta_* \times 1)).$$

Then there is a commutative diagram

$$\begin{tikzcd}
f_0^{-1}p \times \Delta^1 \arrow[r, "1 \times d^0"] \arrow[dr, "\gamma_*" swap] \arrow[dd, "pr_L" swap] & f_0^{-1}p \times \Delta^2 \arrow[dr, "\gamma"] & \\
& f_0^{-1}p \arrow[r, "x_0"] \arrow[d, "p_0" swap] & h^{-1}p \arrow[d, "p_h"] \\
f_0^{-1}p \arrow[r, "p_0" swap] & A \arrow[r, "d^0" swap] & A \times \Delta^1,
\end{tikzcd}$$

by the simplicial identities, so that $\gamma_* : \omega_*\theta_* \simeq 1$ is a fibrewise homotopy. There is a similar fibrewise homotopy $\theta_*\omega_* \simeq 1$. $\square$

Corollary 10.7. *Suppose that $q : Z \to Y$ is a minimal fibration, and that $f_i : X \to Y$, $i = 0, 1$ are homotopic simplicial maps. Then there is a commutative diagram*

$$\begin{tikzcd}
f_0^{-1}q \arrow[rr, "\cong"] \arrow[dr, "q_0"'] & & f_1^{-1}q. \arrow[dl, "q_1"] \\
& X &
\end{tikzcd}$$

*In particular, the pullbacks $f_0^{-1}q$ and $f_1^{-1}q$ are isomorphic.*

Corollary 10.8. *Suppose that $q : Z \to Y$ is a minimal fibration with $Y$ connected. Suppose that $F$ is the fibre of $q$ over a base point $*$ of $Y$. Then, for any simplex $\sigma : \Delta^n \to Y$ there is a commutative diagram*

$$\begin{tikzcd}
F \times \Delta^n \arrow[rr, "\cong"] \arrow[dr, "pr_R"'] & & \sigma^{-1}q. \arrow[dl] \\
& \Delta^n &
\end{tikzcd}$$

Proof: Suppose that $v$ and $w$ are vertices of $Y$ such that there is a 1-simplex $z$ of $Y$ with $\partial z = (v, w)$. Then the classifying maps $v : \Delta^0 \to Y$ and $w : \Delta^0 \to Y$ are homotopic, and so there is an isomorphism $F_v \cong F_w$ of fibres induced by the homotopy. In particular, there is an isomorphism $F_v \cong F$ for any vertex $v$ of $Y$. Now let $i_0 : \Delta^0 \to \Delta^n$ be the map that picks out the vertex 0 of $\Delta^n$. Finally, recall (see the proof of Theorem 7.10) that the composite

$$\Delta^n \to \Delta^0 \xrightarrow{i_0} \Delta^n$$

is homotopic to the identity on $\Delta^n$. □

Theorem 10.9 (Gabriel-Zisman). *Suppose that $q : X \to Y$ is a minimal fibration. Then its realization $|q| : |X| \to |Y|$ is a Serre fibration.*

Proof: It is enough to suppose that $Y$ has only finitely many non-degenerate simplices, since the image of any continuous map $|\Delta^n| \to |Y|$ is contained in some finite subcomplex of $|Y|$. We may also suppose that $Y$ is connected. The idea of the proof is to show that $|q| : |X| \to |Y|$ is locally trivial with fibre $|F|$, where $F$ is the fibre over some base point $*$ of $Y$.

Now suppose that there is a pushout diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook] & Z \arrow[d] \\
\Delta^n \arrow[r, "\beta"'] & Y,
\end{tikzcd}$$

where $Z$ is subcomplex of $Y$ with fewer non-degenerate simplices, and suppose that $U$ is an open subset of $|Z|$ such that there is a fibrewise homeomorphism

$$\begin{tikzcd}
U \times_{|Y|} |X| \arrow[rr, "\omega", "\cong"'] \arrow[dr] & & U \times |F|. \arrow[dl, "pr_L"] \\
& U &
\end{tikzcd}$$

Let $U^1 = |\alpha|^{-1}(U) \subset |\partial\Delta^n|$. Then there is an induced fibrewise homeomorphism

$$\begin{tikzcd}
U^1 \times_{|Y|} |X| \arrow[rr, "\omega", "\cong"'] \arrow[dr] & & U^1 \times |F|. \arrow[dl, "pr_L"] \\
& U^1 &
\end{tikzcd}$$

The fibrewise isomorphism

$$\begin{tikzcd}
\Delta^n \times F \arrow[rr, "\delta", "\cong"'] \arrow[dr, "pr_L"'] & & \Delta^n \times_Y X \arrow[dl] \\
& \Delta^n &
\end{tikzcd}$$

induces a homeomorphism

$$\begin{tikzcd}
V^1 \times F \arrow[rr, "\delta", "\cong"'] \arrow[dr, "pr_L"'] & & V^1 \times_{|Y|} |X| \arrow[dl] \\
& V^1 &
\end{tikzcd}$$

over some open subset $V^1$ of $|\Delta^n|$ such that $V^1 \cap |\partial\Delta^n| = U^1$ and there is a retraction $r : V^1 \to U^1$. The map $\delta$ restricts to a homeomorphism

$$\begin{tikzcd}
U^1 \times F \arrow[rr, "\delta", "\cong"'] \arrow[dr, "pr_L"'] & & U^1 \times_{|Y|} |X| \arrow[dl] \\
& U^1 &
\end{tikzcd}$$

over $U^1$. Now consider the fibrewise homeomorphism

$$\begin{tikzcd}
U^1 \times |F| \arrow[rr, "\delta^{-1}\omega", "\cong"'] \arrow[dr, "pr_L"'] & & U^1 \times |F|. \arrow[dl, "pr_L"] \\
& U^1 &
\end{tikzcd}$$

There is a homeomorphism

$$\begin{tikzcd}
V^1 \times |F| \arrow[rr, "r^*(\delta^{-1}\omega)", "\cong"'] \arrow[dr, "pr_L"'] & & V^1 \times |F|. \arrow[dl, "pr_L"] \\
& V^1 &
\end{tikzcd}$$

which restricts to $\delta^{-1}\omega$ over $U^1$. In effect $r^*(\delta^{-1}\omega)(v', f) = (v', \varphi(rv', f))$, where $\delta^{-1}\omega(w, f) = (w, \varphi(w, f))$, and this definition is "functorial". Thus, the fibrewise isomorphism

$$\begin{tikzcd}
V^1 \times F \arrow[rr, "\delta r^*(\delta^{-1}\omega)", "\cong"'] \arrow[dr, "pr_L"'] & & V^1 \times_{|Y|} |X| \arrow[dl] \\
& V^1 &
\end{tikzcd}$$

restricts to $\omega$ over $U^1$. It follows that there is a fibrewise homeomorphism

$$\begin{tikzcd}
(V^1 \cup_{U^1} U) \times F \arrow[rr, "\cong"'] \arrow[dr, "pr_L"'] & & (V^1 \cup_{U^1} U) \times_{|Y|} |X| \arrow[dl] \\
& (V^1 \cup_{U^1} U) &
\end{tikzcd}$$

over the open set. $V^1 \cup_{U^1} U$ of $|Y|$. □

The following result of Quillen [77] is the key to both the closed model structure of the simplicial set category, and the relation between simplicial homotopy theory and ordinary homotopy theory. These results will appear in the next section.

THEOREM 10.10 (QUILLEN). *The realization of a Kan fibration is a Serre fibration.*

PROOF: Let $p : X \to Y$ be a Kan fibration. According to Proposition 10.3, one can choose a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "j"] \arrow[dr, "q"'] & X \arrow[r, "g"] \arrow[d, "p"] & Z \arrow[dl, "q"] \\
& Y &
\end{tikzcd}$$

where $q$ is a minimal fibrations, $gj = 1_Z$ and $jg$ is fibrewise homotopic to $1_X$. In view of Theorem 10.9, it clearly suffices to prove the following two results:

LEMMA 10.11. *$g : X \to Z$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

LEMMA 10.12. *Suppose that $g : X \to Z$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. Then $|g| : |X| \to |Z|$ is a Serre fibration.*

PROOF OF LEMMA 10.11: Suppose that the diagrams

$$\begin{tikzcd}
X \times \Delta^1 \arrow[r, "h"] \arrow[d, "pr"'] & X \arrow[d, "p"] \\
X \arrow[r, "p"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "d^0"'] \arrow[dr, "1_X"] & \\
X \times \Delta^1 \arrow[r, "h"'] & X \\
X \arrow[u, "d^1"] \arrow[r, "g"'] & Z \arrow[u, "j"']
\end{tikzcd}$$

represent the fibrewise homotopy, and suppose that the diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "u"] \arrow[d, hook, "i"'] & X \arrow[d, "g"] \\
\Delta^n \arrow[r, "v"'] & Z
\end{tikzcd}$$

commutes. Then there are commutative diagrams

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \arrow[r, "u \times 1"] \arrow[d, hook, "i \times 1"'] & X \times \Delta^1 \arrow[r, "h"] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "pr_L"'] & \Delta^n \arrow[r, "qv"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "u"] \arrow[d, hook, "i"'] & X \arrow[r, "g"] & Z \arrow[r, "j"] & X \arrow[d, "p"] \\
\Delta^n \arrow[rr, "v"'] & & Z \arrow[r, "q"'] \arrow[ur, "j"] & Y,
\end{tikzcd}$$

and hence a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[rr, "{(h(u \times 1), jv)}"] \arrow[d, hook] & & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "pr_L"'] \arrow[urr, "h_1"] & \Delta^n \arrow[r, "qv"'] & Y.
\end{tikzcd}$$

Let $v_1$ be the simplex classified by the composite

$$\Delta^n \xrightarrow{d^0} \Delta^n \times \Delta^1 \xrightarrow{h_1} X.$$

The idea of the proof is now to show that $gv_1 = v$. The diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "u"] \arrow[d, hook, "i"'] & X \\
\Delta^n \arrow[ur, "v_1"'] &
\end{tikzcd}$$

commutes, and there is a composite

$$\Delta^n \times \Delta^1 \xrightarrow{v_1 \times 1} X \times \Delta^1 \xrightarrow{h} X \xrightarrow{g} Z.$$

The map $gh(v_1 \times 1)$ is a homotopy $gjgv_1 = gv_1 \xrightarrow{\simeq} gv_1$. Moreover, the homotopy on the boundary is $gh(u \times 1)$. It follows that there is a commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_2) \arrow[rr, "{(s_0(gh(u \times 1)), (gh_1, gh(v_1 \times 1),\ ))}"] \arrow[d, hook] & & Z \arrow[d, "q"] \\
\Delta^n \times \Delta^2 \arrow[r, "pr_L"'] \arrow[urr, "\xi"] & \Delta^n \arrow[r, "qv"'] & Y.
\end{tikzcd}$$

Then the diagram

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \arrow[r, "pr_L"] \arrow[d, hook, "i \times 1"'] & \partial\Delta^n \arrow[d, "gu"] \\
\Delta^n \times \Delta^1 \arrow[r, "\xi(1 \times d^2)"] \arrow[d, "pr_L"'] & Z \arrow[d, "q"] \\
\Delta^n \arrow[r, "qv"'] & Y
\end{tikzcd}$$

commutes, and so $gv_1 = gjv = v$ by the minimality of $q$. □

Proof of Lemma 10.12: Suppose that $f : X \to Y$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$, and hence with respect to all inclusions of simplicial sets. Then there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "{(1_X, f)}"'] & X \arrow[d, "f"] \\
X \times Y \arrow[r, "pr_R"'] \arrow[ur, "r"] & Y,
\end{tikzcd}$$

and so $f$ is a retract of the projection $pr : X \times Y \to Y$. But then $|f|$ is a Serre fibration. □

This also completes the proof of Theorem 10.10.

## I.11. The closed model structure.

The results stated and proved in this section are the culmination of the work that we have done up to this point. We shall prove here that the simplicial set category **S** has a closed model structure, and that the resulting homotopy theory is equivalent to the ordinary homotopy theory of topological spaces. These are the central organizational theorems of simplicial homotopy theory.

Proposition 11.1. *Suppose that $X$ is a Kan complex. Then the canonical map $\eta_X : X \to S|X|$ is a weak equivalence in the sense that it induces an isomorphism in all possible simplicial homotopy groups.*

Proof: Recall that $S|X|$ is also a Kan complex.

$\eta_X$ induces an isomorphism in $\pi_0$: every map $v : |\Delta^0| \to |X|$ factors through the realization of a simplex $|\sigma| : |\Delta^n| \to |X|$ and so $S|X|$ is connected if $\pi_0 X = *$. The simplicial set $X$ is a disjoint union of its path components and $S|\ |$ preserves disjoint unions, so that $\pi_0 X \to \pi_0 S|X|$ is monic.

Suppose that we have shown that $\eta_X$ induces an isomorphism

$$(\eta_X)_* : \pi_i(X, x) \xrightarrow{\cong} \pi_i(S|X|, \eta x)$$

for all choice of base points $x \in X$ and $i \leq n$. Then, using Theorem 10.10 for the path-loop fibration $\Omega X \to PX \to X$ determined by $x$ (see the discussion following the proof of Lemma 7.3), one finds a commutative diagram

$$\begin{tikzcd}
\pi_{n+1}(X, x) \arrow[r, "\eta_X"] \arrow[d, "\partial"', "\cong"] & \pi_{n+1}(S|X|, \eta x) \arrow[d, "\partial"] \\
\pi_n(\Omega X, x) \arrow[r, "\cong", "\eta_{\Omega X}"'] & \pi_n(S|\Omega X|, \eta x),
\end{tikzcd}$$

and so we're done if we can show that $PX$ and hence $S|PX|$ contracts onto its base point. But there is a diagram

$$\begin{tikzcd}
(\Delta^0 \times \Delta^1) \cup (PX \times \partial\Delta^1) \arrow[r, "{(x, (1_{PX}, x))}"] \arrow[d, hook] & PX \arrow[d] \\
PX \times \Delta^1 \arrow[r] \arrow[ur, "h"] & \Delta^0,
\end{tikzcd}$$

and $h$ exists because the unique map $PX \to \Delta^0$ has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. □

If $X$ is a Kan complex and $x$ is any vertex of $X$, then it follows from Proposition 11.1 and adjointness that $\eta_X$ induces a canonical isomorphism

$$\pi_n(X, x) \cong \pi_n(|X|, x), \quad n \geq 1,$$

where the group on the right is the ordinary homotopy group of the space $|X|$. It follows that a map $f : X \to Y$ of Kan complexes is a (simplicial) weak equivalence if and only if the induced map $|f| : |X| \to |Y|$ is a topological weak equivalence. Thus, we are entitled to define a map $f : X \to Y$ of arbitrary simplicial sets to be a *weak equivalence* if the induced map $|f| : |X| \to |Y|$ is a weak equivalence of spaces. Our last major technical result leading to the closed model structure of **S** is

THEOREM 11.2. *Suppose that $g : X \to Y$ is a map between arbitrary simplicial sets. Then $g$ is a Kan fibration and a weak equivalence if and only if $g$ has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.*

PROOF: Suppose that $g : X \to Y$ is a Kan fibration with the advertised lifting property. We have to show that $S|g| : S|X| \to S|Y|$ is a weak equivalence. The simplicial set $Y$ is arbitrary, so we must define $\pi_0 Y$ to be the set of equivalence classes of vertices of $Y$ for the relation generated by the vertex homotopy relation. In other words, $y \simeq z$ if and only if there is a string of vertices

$$y = y_0, y_1, \dots, y_n = z$$

and a string of 1-simplices

$$v_1, \dots, v_n$$

of $X$ such that $\partial v_i = (y_{i-1}, y_i)$ or $\partial v_i = (y_i, y_{i-1})$ for $i = 1, \dots, n$. If $Y$ is a Kan complex, then this definition of $\pi_0 Y$ coincides with the old definition. Moreover, the canonical map $\eta_Y : Y \to S|Y|$ induces an isomorphism $\pi_0 Y \xrightarrow{\cong} \pi_0 S|Y|$ for all simplicial sets $Y$. The lifting property implies that $g_* : \pi_0 X \to \pi_0 Y$ is an isomorphism, so that the induced map $\pi_0 S|X| \to \pi_0 S|Y|$ is an isomorphism as well. Finally, it suffices to show that the induced maps $\pi_i(S|X|, x) \to \pi_i(S|Y|, gx)$ of simplicial homotopy groups are isomorphisms for all vertices $x$ of $X$ and all $i \geq 1$. But Theorem 10.9 implies that the the map $S|g| : S|X| \to S|Y|$ is a Kan fibration with fibre $S|F_x|$ over $g(x)$, where

$$\begin{tikzcd}
F_x \arrow[r] \arrow[d] & X \arrow[d, "g"] \\
\Delta^0 \arrow[r, "gx"'] & Y
\end{tikzcd}$$

is a pullback in the simplicial set category. The fibre $F_x$ is a contractible Kan complex (see the corresponding argument for $PX$ in the proof of Proposition 11.1), and so $S|F_x|$ is contractible as well. The result then follows from a long exact sequence argument.

For the reverse implication, it suffices (see the proof of Theorem 10.10) to assume that $g : X \to Y$ is a minimal fibration and a weak equivalence and then prove that it has the lifting property. We may also assume that $Y$ is connected. Consider a diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "g"] \\
\Delta^n \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

and the induced diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha_*"] \arrow[d, hook] & \Delta^n \times_Y X \arrow[d] \\
\Delta^n \arrow[r, "1"'] & \Delta^n.
\end{tikzcd}$$

It suffices to find a lifting for this last case. But there is a fibrewise isomorphism

$$\begin{tikzcd}
\Delta^n \times_Y X \arrow[rr, "\cong"] \arrow[dr] & & \Delta^n \times F_y \arrow[dl, "pr_L"] \\
& \Delta^n &
\end{tikzcd}$$

by Corollary 10.8, where $F_y$ is the fibre over some vertex $y$ of $Y$. Thus, it suffices to find a lifting of the following sort:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & F_y. \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

This can be done, by Theorem 7.10, since $F_y$ is a Kan complex such that $\pi_0(S|F_y|)$ is trivial, and $\pi_i(S|F_y|, *) = 0$, $i \geq 1$ for any base point $*$, and $\eta : F_y \to S|F_y|$ is a weak equivalence by Proposition 11.1. ☐

A *cofibration* of simplicial sets is an inclusion map.

THEOREM 11.3. *The simplicial set category* **S**, *together with the specified classes of Kan fibrations, cofibrations and weak equivalences, is a closed model category.*

PROOF: **CM1** is satisfied, since **S** is complete and cocomplete. **CM2** follows from **CM2** for **CGHaus**. The retract axiom **CM3** is easy.

To prove the factorization axiom **CM5**, observe that a small object argument and the previous theorem together imply that any simplicial set map $f : X \to Y$ may be factored as:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& Z \arrow[ur, "p"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y, \\
& W \arrow[ur, "q"'] &
\end{tikzcd}$$

where $i$ is anodyne, $p$ is a fibration, $j$ is an inclusion, and $q$ is a trivial fibration. The class of inclusions $i : U \to V$ of simplicial sets such that $|i| : |U| \to |V|$ is a trivial cofibration is saturated, by adjointness, and includes all $\Lambda^n_k \subset \Delta^n$. Thus all anodyne extensions are trivial cofibrations of **S**. To prove **CM4** we must show that the lifting (dotted arrow) exists in any commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

where $p$ is a fibration and $i$ is a cofibration, and either $i$ or $p$ is trivial. The case where $p$ is trivial is Theorem 11.2. If $i$ is a weak equivalence, then there is a diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ V \arrow[r, "1_V"'] \arrow[ur, "s"] & V \end{tikzcd}$$

where $j$ is anodyne and $p$ is a (necessarily) trivial fibration, so that $s$ exists. But then $i$ is a retract of an anodyne extension, so $i$ has the left lifting property with respect to all fibrations (compare the proof of Lemma 9.4). □

The homotopy category Ho(**S**) is obtained from **S** by formally inverting the weak equivalences. There are several ways to do this [76], [33], [15] — see also Section II.1. One may also form the category Ho(**Top**) by formally inverting the weak homotopy equivalences; this category is equivalent to the category of CW-complexes and ordinary homotopy classes of maps. For the same reason (see [76]), Ho(**S**) is equivalent to the category of Kan complexes and simplicial homotopy classes of maps. The realization functor preserve weak equivalences, by definition. One may use Theorem 10.10 (see the argument for Proposition 11.1) to show that the canonical map $\epsilon : |S(Y)| \to Y$ is a weak equivalence, for any topological space $Y$, and so the singular functor preserves weak equivalences as well. It follows that the realization and singular functors induce functors

$$\begin{tikzcd} \mathrm{Ho}(\mathbf{S}) \arrow[r, shift left, "|\ |_*"] & \mathrm{Ho}(\mathbf{Top}) \arrow[l, shift left, "S_*"] \end{tikzcd}$$

of the associated homotopy categories.

THEOREM 11.4. *The realization and singular functors induce an equivalence of categories of* Ho(**S**) *with* Ho(**Top**).

PROOF: We have just seen that $\epsilon : |S(Y)| \to Y$ is a weak equivalence for all topological spaces $Y$. It remains to show that $\eta : X \to S|X|$ is a weak equivalence for all simplicial sets $X$. But $\eta$ is a weak equivalence if $X$ is a Kan complex, by Proposition 11.1, and every simplicial set is weakly equivalent to a Kan complex by **CM5**. The composite functor $S|\ |$ preserves weak equivalences. □

The original proof of Theorem 11.3 appears in [76], modulo some fiddling with axioms (see [78]). Theorem 11.4 has been known in some form since the late 1950's (see [67], [33], [59]).

Although it may now seem like a moot point, the function complex trick of Proposition 5.2 was a key step in the proof of Theorem 11.3. We can now amplify the statement of Proposition 5.2 as follows:

PROPOSITION 11.5. *The category* **S** *of simplicial sets satisfies the simplicial model axiom*

**SM7**: *Suppose that $i : U \to V$ is a cofibration and $p : X \to Y$ is a fibration. Then the induced map*

$$\mathbf{Hom}(V, X) \xrightarrow{(i^*, p_*)} \mathbf{Hom}(U, X) \times_{\mathbf{Hom}(U,Y)} \mathbf{Hom}(V, X)$$

*is a fibration, which is trivial if either $i$ or $p$ is trivial.*

PROOF: Use Proposition 5.2 and Theorem 11.2. □

# Chapter II Model Categories

The closed model axioms have a list of basic abstract consequences, including an expanded notion of homotopy and a Whitehead theorem. The associated homotopy category is defined to be the result of formally inverting the weak equivalences within the ambient closed model category, but can be constructed in the CW-complex style by taking homotopy classes of maps between objects which are fibrant and cofibrant. These topics are presented in the first section of this chapter.

The simplicial set category has rather more structure than just that of a closed model category: the set $\hom(X, Y)$ of maps between simplicial sets $X$ and $Y$ is the set of vertices of the function complex $\mathbf{Hom}(X, Y)$, and the collection of all such function complexes determines a simplicial category. We've already seen that the function complexes satisfy an exponential law and respect cofibrations and fibrations in a suitable sense. The existence of the function complexes and the interaction with the closed model structure can be abstracted to a definition of a simplicial model category, which is given in Sections 2 and 3 along with various examples. Basic homotopical consequences of the additional simplicial structure are presented in Section 3.

Sections 4, 5 and 6 are concerned with detection principles for simplicial model category structures. Generally speaking, such a structure for the category $s\mathcal{C}$ of simplicial objects in a category $\mathcal{C}$ is induced from the simplicial model category structure on simplicial sets in the presence of an adjoint pair of functors

$$F : \mathbf{S} \rightleftarrows s\mathcal{C} : G,$$

(or a collection of adjoint pairs) if $G$ satisfies extra conditions, such as preservation of filtered colimits, in addition to being a right adjoint — this is Theorem 4.1. In one major stream of examples, the category $\mathcal{C}$ is some algebraic species, such as groups or abelian groups, and $G$ is a forgetful functor. There is, however, an extra technical requirement for Theorem 4.1, namely that every cofibration of $s\mathcal{C}$ having the left lifting property with respect to all fibrations should be a weak equivalence. This condition can often be verified by brute force, as can be done in the presence of a small object argument for the factorization axioms (eg. simplicial abelian groups), but there is a deeper criterion, namely the existence of a natural fibrant model (Lemma 5.1). The other major source of examples has to do with $G$ being a representable functor of the form $G = \hom(Z,\ )$, where $Z$ is either small in the sense that $\hom(Z,\ )$ respects filtered colimits, or is a disjoint union of small objects. In this setting, Kan's $\mathrm{Ex}^\infty$-construction (see Section III.4) is used to construct the natural fibrant models required by Lemma 5.1. This line of argument is generalized significantly in Section 6, at the cost of the introduction of cofibrantly generated closed model categories and transfinite small object arguments.

Section 7 is an apparent return to basics. We develop a criterion for a pair of adjoint functors between closed model categories to induce adjoint functors on the homotopy category level, known as Quillen's total derived functor theorem. Quillen's result is, at the same time, a non-abelian version of the calculus of higher direct images, and a generalization of the standard result that cohomology is homotopy classes of maps taking values in Eilenberg-Mac Lane spaces.

The category of simplicial sets, finally, has even more structure: it is a *proper* simplicial model category, which means that, in addition to everything else, weak equivalences are preserved by pullback over fibrations and by pushout along cofibrations. This property is discussed in Section 8. Properness is the basis of the standard results about homotopy cartesian diagrams, as well as being of fundamental importance in stable homotopy theory. We discuss homotopy cartesian diagrams in the context of Gunnarsson's axiomatic approach to the gluing and cogluing lemmas [40].

## 1. Homotopical algebra.

Recall that a *closed model category* $\mathcal{C}$ is a category which is equipped with three classes of morphisms, called cofibrations, fibrations and weak equivalences which together satisfy the following axioms:

**CM1:** The category $\mathcal{C}$ is closed under all finite limits and colimits.

**CM2:** Suppose that the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\
& Z. &
\end{tikzcd}$$

If any two of $f$, $g$ and $h$ are weak equivalences, then so is the third.

**CM3:** If $f$ is a retract of $g$ and $g$ is a weak equivalence, fibration or cofibration, then so is $f$.

**CM4:** Suppose that we are given a commutative solid arrow diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
V \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where $i$ is a cofibration and $p$ is a fibration. Then the dotted arrow exists, making the diagram commute, if either $i$ or $p$ is also a weak equivalence.

**CM5:** Any map $f : X \to Y$ may be factored:

(a) $f = p \cdot i$ where $p$ is a fibration and $i$ is a trivial cofibration, and
(b) $f = q \cdot j$ where $q$ is a trivial fibration and $j$ is a cofibration.

Recall that a map is said to be a *trivial fibration* (aka. *acyclic fibration*) if it is both a fibration and a weak equivalence. Dually, a *trivial cofibration* is a map which is simultaneously a cofibration and a weak equivalence.

According to **CM1**, a closed model category $\mathcal{C}$ has an initial object $\emptyset$ and a terminal object $*$. Say that an object $A$ of $\mathcal{C}$ is *cofibrant* if the map $\emptyset \to A$ is a cofibration. Dually, an object $X$ is *fibrant* if the map $X \to *$ is a fibration of $\mathcal{C}$.

This set of axioms has a list of standard consequences which amplifies the interplay between cofibrations, fibrations and weak equivalences, giving rise to collection of abstract techniques that has been known as homotopical algebra since Quillen introduced the term in [76]. This theory is is really an older friend in modern dress, namely obstruction theory made axiomatic. The basic results, along with their proofs, are sketched in this section.

We begin with the original meaning of the word "closed":

LEMMA 1.1. *Suppose that $\mathcal{C}$ is a closed model category. Then we have the following:*

(1) *A map $i : U \to V$ of $\mathcal{C}$ is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations.*
(2) *The map $i$ is a trivial cofibration if and only if it has the left lifting property with respect to all fibrations.*
(3) *A map $p : X \to Y$ of $\mathcal{C}$ is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations.*
(4) *The map $p$ is a trivial fibration if and only if it has the right lifting property with respect to all cofibrations.*

The point of Lemma 1.1 is that the various species of cofibrations and fibrations determine each other via lifting properties.

PROOF: We shall only prove the first statement; the other proofs are similar.

Suppose that $i$ is a cofibration, $p$ is a trivial fibration, and that there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r, "\beta"'] & Y \end{tikzcd} \tag{1.2}$$

Then there is a map $\theta : V \to X$ such that $p\theta = \beta$ and $\theta i = \alpha$, by **CM4**. Conversely suppose that $i : U \to V$ is a map which has the left lifting property with respect to all trivial fibrations. By **CM5**, $i$ has a factorization

$$\begin{tikzcd}
U \arrow[r, "j"] \arrow[dr, "i"'] & W \arrow[d, "q"] \\
& V
\end{tikzcd}$$

where $j$ is a cofibration and $q$ is a trivial fibration. But then there is a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "j"] \arrow[d, "i"'] & W \arrow[d, "q"] \\
V \arrow[ur] \arrow[r, equal] & V
\end{tikzcd}$$

and so $i$ is a retract of $j$. **CM3** then implies that $i$ is a cofibration. □

The proof of the Lemma 1.1 contains one of the standard tricks that is used to prove that the axiom **CM4** holds in a variety of settings, subject to having an adequate proof of the factorization axiom **CM5**. Lemma 1.1 also immediately implies the following:

Corollary 1.3.

(1) *The classes of cofibrations and trivial cofibrations are closed under composition and pushout. Any isomorphism is a cofibration.*

(2) *The classes of fibrations and trivial fibrations are closed under composition and pullback. Any isomorphism is a fibration.*

The statements in Corollary 1.3 are part of Quillen's original definition of a model category [76].

Quillen defines a *cylinder object* for an object $A$ in a closed model category $\mathcal{C}$ to be a commutative triangle

$$\begin{tikzcd}
A \sqcup A \arrow[d, "i"'] \arrow[dr, "\nabla"] & \\
\tilde{A} \arrow[r, "\sigma"'] & A
\end{tikzcd} \tag{1.4}$$

where $\nabla : A \sqcup A \to A$ is the canonical fold map which is defined to be the identity on $A$ on each summand, $i$ is a cofibration, and $\sigma$ is a weak equivalence.

Then a *left homotopy* of maps $f, g : A \to B$ is a commutative diagram

$$\begin{tikzcd}
A \sqcup A \arrow[d, "i"'] \arrow[dr, "{(f,g)}"] & \\
\tilde{A} \arrow[r, "h"'] & B
\end{tikzcd}$$

where $(f, g)$ is the map on the disjoint union which is defined by $f$ on one summand and $g$ on the other, and the data consisting of

$$i = (i_0, i_1) : A \sqcup A \to \tilde{A}$$

comes from *some choice* of cylinder object for $A$.

There are many choices of cylinder object for a given object $A$ of a closed model category $\mathcal{C}$: any factorization of $\nabla : A \sqcup A \to A$ into a cofibration followed by a trivial fibration that one might get out of **CM5** gives a cylinder object for $A$. In general, however, the object $A$ needs to be cofibrant for its cylinder objects to be homotopically interesting:

LEMMA 1.5.

(1) *Suppose that $A$ is a cofibrant object of a closed model category $\mathcal{C}$, and that the diagram (1.4) is a cylinder object for $A$. Then the maps $i_0, i_1 : A \to \tilde{A}$ are trivial cofibrations.*

(2) *Left homotopy of maps $A \to B$ in a closed model category $\mathcal{C}$ is an equivalence relation if $A$ is cofibrant.*

PROOF: Denote the initial object of $\mathcal{C}$ by $\emptyset$.

For the first part, observe that the diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d, "in_R"] \\
A \arrow[r, "in_L"'] & A \sqcup A
\end{tikzcd}$$

is a pushout since cofibrations are closed under pushout by Lemma 1.1, and the unique map $\emptyset \to A$ is a cofibration by assumption. It follows that the inclusions $in_L$ and $in_R$ are cofibrations, so that the compositions $i_0 = (i_0, i_1) \cdot in_L$ and $i_1 = (i_0, i_1) \cdot in_R$ are cofibrations as well. Finally, the maps $i_0$ and $i_1$ are weak equivalences by **CM2**, since the map $\sigma$ is a weak equivalence.

To prove the second statement, first observe that if $\tau : A \sqcup A \to A \sqcup A$ is the automorphism which flips summands, then the diagram

$$\begin{tikzcd}
A \sqcup A \arrow[d, "i \cdot \tau"'] \arrow[dr, "\nabla"] & \\
\tilde{A} \arrow[r, "\sigma"'] & A,
\end{tikzcd}$$

which is constructed from (1.4) by twisting by $\tau$, is a cylinder object for $A$. This implies that the left homotopy relation is symmetric.

Subject to the same definitions, the map $f\sigma : A \to B$ is clearly a left homotopy from $f : A \to B$ to itself, giving reflexivity.

Suppose given cylinder objects

$$\begin{tikzcd}
A \sqcup A \arrow[d, "(i_0^\epsilon, i_1^\epsilon)"'] \arrow[dr, "\nabla"] & \\
A_\epsilon \arrow[r, "\sigma^\epsilon"'] & A,
\end{tikzcd}$$

where $\epsilon = 0, 1$, and form the pushout

$$\begin{tikzcd}
A \arrow[r, "i_0^1"] \arrow[d, "i_1^0"'] & A_1 \arrow[d, "i_{1*}"] \\
A_0 \arrow[r, "i_{0*}"'] & \tilde{A}
\end{tikzcd}$$

Then the map

$$A \sqcup A \xrightarrow{(i_{0*}i_0^0, i_{1*}i_1^1)} \tilde{A}$$

is a composite

$$A \sqcup A \xrightarrow{i_0^0 \sqcup 1} A_0 \sqcup A \xrightarrow{(i_{0*}, i_{1*}i_1^1)} \tilde{A}.$$

The map $i_0^0 \sqcup 1$ is a cofibration by the first statement of the lemma, and there is a pushout diagram

$$\begin{tikzcd}
A \sqcup A \arrow[r, "i_1^0 \sqcup 1"] \arrow[d, "(i_0^1, i_1^1)"'] & A_0 \sqcup A \arrow[d, "(i_{0*}, i_{1*}i_1^1)"] \\
A_1 \arrow[r, "i_{1*}"'] & \tilde{A}.
\end{tikzcd}$$

In particular, there is a cylinder object for $A$

$$\begin{tikzcd}
A \sqcup A \arrow[d, "{(i_{0*}i_0^0, i_{1*}i_1^1)}"'] \arrow[dr, "\nabla"] & \\
A_\epsilon \arrow[r, "\sigma_*"'] & A.
\end{tikzcd}$$

It follows that if there are left homotopies $h_0 : A_0 \to B$ from $f_0$ to $f_1$ and $h_1 : A_1 \to B$ from $f_1$ to $f_2$, then there is an induced left homotopy $h_* : \tilde{A} \to B$ from $f_0$ to $f_2$. $\square$

A *path object* for an object $B$ in a closed model category $\mathcal{C}$ is a commutative triangle

$$\begin{tikzcd}
 & \hat{B} \arrow[d, "{p = (p_0, p_1)}"] \\
B \arrow[ur, "s"] \arrow[r, "\Delta"'] & B \times B
\end{tikzcd} \tag{1.6}$$

where $\Delta$ is the diagonal map, $s$ is a weak equivalence, and $p$ (which is given by $p_0$ on one factor and by $p_1$ on the other) is a fibration.

Once again, the factorization axiom **CM5** dictates that there is an ample supply of path objects for each object of an arbitrary closed model category. If a simplicial set $X$ is a Kan complex, then the function complex $\mathbf{hom}(\Delta^1, X)$ is a path object for $X$, and the function space $Y^I$ is a path object for each compactly generated Hausdorff space $Y$.

There is a notion of right homotopy which corresponds to path objects: two maps $f, g : A \to B$ are said to be *right homotopic* if there is a diagram

$$\begin{tikzcd}
 & \hat{B} \arrow[d, "{(p_0, p_1)}"] \\
A \arrow[ur, "h"] \arrow[r, "{(f,g)}"'] & B \times B
\end{tikzcd}$$

where the map $(p_0, p_1)$ arises from some path object (1.6), and $(f, g)$ is the map which projects to $f$ on the left hand factor and $g$ on the right hand factor.

LEMMA 1.7.

(1) *Suppose that $B$ is a fibrant object of a closed model category $\mathcal{C}$, and that $\hat{B}$ is a path object for $B$ as in (1.6). Then the maps $p_0$ and $p_1$ are trivial fibrations.*

(2) *Right homotopy of maps $A \to B$ in $\mathcal{C}$ is an equivalence relation if $B$ is fibrant.*

Lemma 1.7 is *dual* to Lemma 1.5 in a precise sense. If $\mathcal{C}$ is a closed model category, then its opposite $\mathcal{C}^{op}$ is a closed model category whose cofibrations (respectively fibrations) are the opposites of the fibrations (respectively cofibrations) in $\mathcal{C}$. A map in $\mathcal{C}^{op}$ is a weak equivalence for this structure if and only if its opposite is a weak equivalence in $\mathcal{C}$. Then Lemma 1.7 is an immediate consequence of the instance of Lemma 1.5 which occurs in $\mathcal{C}^{op}$. This sort of duality is ubiquitous in the theory: observe, for example, that the two statements of Corollary 1.3 are dual to each other.

Left and right homotopies are linked by the following result:

PROPOSITION 1.8. *Suppose that $A$ is cofibrant. Suppose further that*

$$\begin{tikzcd}
A \sqcup A \arrow[d, "(i_0, i_1)"'] \arrow[dr, "(f,g)"] & \\
\tilde{A} \arrow[r, "h"'] & B
\end{tikzcd}$$

*is a left homotopy between maps $f, g : A \to B$, and that*

$$\begin{tikzcd}
 & \hat{B} \arrow[d, "p = (p_0, p_1)"] \\
B \arrow[ur, "s"] \arrow[r, "\Delta"'] & B \times B
\end{tikzcd}$$

*is a fixed choice of path object for $B$. Then there is a right homotopy*

$$\begin{tikzcd}
 & \hat{B} \arrow[d, "(p_0, p_1)"] \\
A \arrow[ur, "H"] \arrow[r, "(f,g)"'] & B \times B.
\end{tikzcd}$$

This result has a dual, which the reader should be able to formulate independently. Proposition 1.8 and its dual together imply

COROLLARY 1.9. *Suppose given maps $f, g : A \to B$, where $A$ is cofibrant and $B$ is fibrant. Then the following are equivalent:*

(1) *$f$ and $g$ are left homotopic.*

(2) *$f$ and $g$ are right homotopic with respect to a fixed choice of path object.*

(3) *$f$ and $g$ are right homotopic.*

(4) *$f$ and $g$ are left homotopic with respect to a fixed choice of cylinder object.*

In other words, all possible definitions of homotopy of maps $A \to B$ are the same if $A$ is cofibrant and $B$ is fibrant.

PROOF OF PROPOSITION 1.8: The map $i_0$ is a trivial cofibration since $A$ is cofibrant, and $(p_0, p_1)$ is a fibration, so that there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "sf"] \arrow[d, "i_0"'] & \hat{B} \arrow[d, "{(p_0, p_1)}"] \\
\tilde{A} \arrow[r, "{(f\sigma, h)}"'] \arrow[ur, "K"] & B \times B
\end{tikzcd}$$

for some choice of lifting $K$. Then the composite $K \cdot i_1$ is the desired right homotopy. □

We can now, unambiguously, speak of homotopy classes of maps between objects $X$ and $Y$ of a closed model category $\mathcal{C}$ which are both fibrant and cofibrant. We can also discuss homotopy equivalences between such objects. The classical Whitehead Theorem asserts that any weak equivalence $f : X \to Y$ of CW-complexes is a homotopy equivalence. CW-complexes are spaces which are both cofibrant and cofibrant. The analogue of this statement in an arbitrary closed model category is the following:

THEOREM 1.10 (WHITEHEAD). *Suppose that $f : X \to Y$ is a morphism of a closed model category $\mathcal{C}$ such that the objects $X$ and $Y$ are both fibrant and cofibrant. Suppose also that $f$ is a weak equivalence. Then the map $f$ is a homotopy equivalence.*

PROOF: Suppose, first of all, that $f$ is a trivial fibration, and that

$$\begin{tikzcd}
X \sqcup X \arrow[r, "\nabla"] \arrow[d, "{(i_0, i_1)}"'] & X \\
\tilde{X} \arrow[ur, "\sigma"] &
\end{tikzcd}$$

is a cylinder object for $X$. Then one proves that $f$ is a homotopy equivalence by finding, in succession, maps $\theta$ and $h$ making the following diagrams commute:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r, equal] \arrow[ur, "\theta"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \sqcup X \arrow[r, "(\theta \cdot f, 1)"] \arrow[d, "(i_0, i_1)"'] & X \arrow[d, "f"] \\
\tilde{X} \arrow[r, "f \cdot \sigma"'] \arrow[ur, "h"] & Y.
\end{tikzcd}$$

Dually, if $f$ is a trivial cofibration, then $f$ is a homotopy equivalence.

Every weak equivalence $f : X \to Y$ between cofibrant and fibrant objects has a factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& Z \arrow[ur, "p"'] &
\end{tikzcd}$$

in which $i$ is a trivial cofibration and $p$ is a trivial fibration. The object $Z$ is both cofibrant and fibrant, so $i$ and $p$ are homotopy equivalences. □

Suppose that $X$ and $Y$ are objects of a closed model category $\mathcal{C}$ which are both cofibrant and fibrant. Quillen denotes the set of homotopy classes of maps between such objects $X$ and$Y$ by $\pi(X, Y)$. There is a category $\pi\mathcal{C}_{cf}$ associated to any closed model $\mathcal{C}$: the objects are the cofibrant and fibrant objects of $\mathcal{C}$, and the morphisms from $X$ to $Y$ in $\pi\mathcal{C}_{cf}$ are the elements of the set $\pi(X, Y)$.

For each object $X$ of $\mathcal{C}$, use **CM5** to choose, in succession, maps

$$* \xrightarrow{i_X} QX \xrightarrow{p_X} X$$

and

$$QX \xrightarrow{j_X} RQX \xrightarrow{q_X} *,$$

where $i_X$ is a cofibration, $p_X$ is a trivial fibration, $j_X$ is a trivial cofibration, and $q_X$ is a fibration. We can and will presume that $p_X$ is the identity map if $X$ is cofibrant, and that $j_X$ is the identity map if $QX$ is fibrant. Then $RQX$ is an object which is both fibrant and cofibrant, and $RQX$ is weakly equivalent to $X$, via the maps $p_X$ and $j_X$.

Any map $f : X \to Y$ lifts to a map $Qf : QX \to QY$, and then $Qf$ extends to a map $RQf : RQX \to RQY$. The map $Qf$ is not canonically defined: it is any morphism which makes the following diagram commute:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & QY \arrow[d, "\pi_Y"] \\
QX \arrow[r, "f \cdot \pi_X"'] \arrow[ur, "Qf"] & Y
\end{tikzcd}$$

Note, however, that any two liftings $f_1, f_2 : QX \to QY$ of the morphism $f \cdot \pi_X$ are left homotopic since $\pi_Y$ is a trivial fibration.

The argument for the existence of the morphism $RQf : RQX \to RQY$ is dual to the argument for the existence of $Qf$. If the maps $f_1, f_2 : QX \to QY$ are liftings of $f \cdot \pi_X$ and $g_i : RQX \to RQY$ is an extension of the map $j_Y \cdot f_i$ for $i = 1, 2$, then $f_1$ is left homotopic to $f_2$ by what we've already seen, and so the composites $j_Y \cdot f_1$ and $j_Y \cdot f_2$ are right homotopic, by Lemma 1.8. Observe finally that any right homotopy between the maps $j_Y f_1, j_Y f_2 : QX \to RQY$ can be extended to a right homotopy between the maps $g_1, g_2 : RQY \to RQY$. It follows that the assignment $f \mapsto RQf$ is well defined up to homotopy.

The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ associated to a closed model category $\mathcal{C}$ can be defined to have the same objects as $\mathcal{C}$, and with morphism sets defined by

$$\hom_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \pi(RQX, RQY).$$

There is a functor

$$\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$$

which is the identity on objects, and sends a morphism $f : X \to Y$ to the homotopy class $[RQf]$ which is represented by any choice of map $RQf : RQX \to RQY$ defined as above. If $f : X \to Y$ is a weak equivalence of $\mathcal{C}$, then $RQf : RQX \to RQY$ is a homotopy equivalence by the Whitehead Theorem, and so $\gamma(f)$ is an isomorphism of $\mathrm{Ho}(\mathcal{C})$.

This functor $\gamma$ is universal with respect to all functors $F : \mathcal{C} \to \mathcal{D}$ which invert weak equivalences:

THEOREM 1.11. *Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(f)$ is an isomorphism of $\mathcal{D}$ for all weak equivalences $f : X \to Y$ of $\mathcal{C}$. Then there is a unique functor $F_* : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $F_* \cdot \gamma = F$.*

PROOF: The functor $F : \mathcal{C} \to \mathcal{D}$ takes (left or right) homotopic maps of $\mathcal{C}$ to the same map of $\mathcal{D}$, since it inverts weak equivalences. It follows that, if $g : RQX \to RQY$ represents a morphism from $X$ to $Y$ in $\mathrm{Ho}(\mathcal{C})$, one can specify a well-defined morphism $F_*([g])$ of $\mathcal{D}$ by the assignment

$$F_*([g]) = F(\pi_Y)F(j_Y)^{-1}F(g)F(j_X)F(\pi_X)^{-1}. \tag{1.12}$$

This assignment plainly defines a functor $F_* : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $F_*\gamma = F$.

Also, the morphisms $\gamma(\pi_X)$ and $\gamma(j_X)$ are both represented by the identity map on $RQX$, and so the composite

$$\gamma(\pi_Y)\gamma(j_Y)^{-1}\gamma(g)\gamma(j_X)\gamma(j_X)^{-1}$$

coincides with the morphism $[g] : X \to Y$ of $\mathrm{Ho}(\mathcal{C})$. The morphism $F_*([g])$ must therefore have the form indicated in (1.12) if the composite functor $F_*\gamma$ is to coincide with $F$. □

REMARK 1.13. One can always formally invert a class $\Sigma$ of morphisms of a category $\mathcal{C}$ to get a functor $\gamma : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ which is initial among functors $F : \mathcal{C} \to \mathcal{D}$ which invert all members of the class of morphisms $\Sigma$ (see Schubert's book [83]), provided that one is willing to construct $\mathcal{C}[\Sigma^{-1}]$ in some higher set theoretic universe. This means that the morphism "things" $\hom_{\mathcal{C}[\Sigma^{-1}]}(X, Y)$ of $\mathcal{C}[\Sigma^{-1}]$ may no longer be sets. In Theorem 1.11, we have found an explicit way to formally invert the class $WE$ of weak equivalences of a closed model category $\mathcal{C}$ to obtain the category $\mathrm{Ho}(\mathcal{C})$ without invoking a higher universe. After the fact, all models of $\mathcal{C}[WE^{-1}]$ must be isomorphic as categories to $\mathrm{Ho}(\mathcal{C})$ on account of the universal property of the functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, so that all possible constructions have small hom sets.

Let $\pi\mathcal{C}_{cf}$ denote the category whose objects are the cofibrant fibrant objects of the closed model category $\mathcal{C}$, and whose sets of morphisms have the form

$$\hom_{\pi\mathcal{C}_{cf}}(X, Y) = \pi(X, Y).$$

The functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ induces a fully faithful embedding

$$\gamma_* : \pi\mathcal{C}_{cf} \to \mathrm{Ho}(\mathcal{C}),$$

and that every object of $\mathrm{Ho}(\mathcal{C})$ is isomorphic to an object which is in the image of the functor $\gamma_*$. In other words the category $\pi\mathcal{C}_{cf}$ of homotopy classes of maps between cofibrant fibrant objects of $\mathcal{C}$ is equivalent to the homotopy category $\mathrm{Ho}(\mathcal{C})$.

This observation specializes to several well-known phenomena. In particular, the category of homotopy classes of maps between CW-complexes is equivalent to the full homotopy category of topological spaces, and the homotopy category of simplicial sets is equivalent to the category of simplicial homotopy classes of maps between Kan complexes.

We close this section by showing that the weak equivalences in a closed model category $\mathcal{C}$ are exactly those maps which induce isomorphisms in the homotopy category $\mathrm{Ho}(\mathcal{C})$.

PROPOSITION 1.14. *Suppose that $f : X \to Y$ is a morphism of a closed model category $\mathcal{C}$ which induces an isomorphism in the homotopy category* $\mathrm{Ho}(\mathcal{C})$. *Then $f$ is a weak equivalence.*

PROOF: Suppose that the objects $X$ and $Y$ are both fibrant and cofibrant. In view of the construction of the functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, the idea is to show that any map $f : X \to Y$ which has a homotopy inverse must be a weak equivalence. Any such map $f$ has a factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& Z \arrow[ur, "p"'] &
\end{tikzcd}$$

where $p$ is a fibration and $i$ is a trivial cofibration, by the factorization axiom **CM5**. The trivial cofibration $i$ is a homotopy equivalence, by the Whitehead Theorem (Theorem 1.10), so it suffices to assume that the map $f$ is a fibration. We show that such a fibration $f$ must have the right lifting property with respect to all cofibrations, so that Lemma 1.1 may be invoked to conclude that $f$ is a weak equivalence.

Subject to proving Lemma 1.15 below, we can assume that the homotopy inverse $\theta : Y \to X$ is a section of $f$, and that there is a homotopy $h : \tilde{X} \to X$ from $\theta \cdot f$ to $1_X$ which is fibrewise in the sense that $f \cdot h = f \cdot \sigma_X$. One constructs path objects $\hat{X}$ and $\hat{Y}$ for $X$ and $Y$ which are compatible with $f$ by factorizing the map

$$X \xrightarrow{(\Delta, s_Y f)} (X \times X) \times_{(Y \times Y)} \hat{Y}$$

as a trivial cofibration $s_X : X \to \hat{X}$ followed by a fibration

$$\pi : \hat{X} \to (X \times X) \times_{(Y \times Y)} \hat{Y}.$$

Write $\hat{f}$ for the composite

$$\hat{X} \xrightarrow{\pi} (X \times X) \times_{(Y \times Y)} \hat{Y} \to \hat{Y}.$$

The composite

$$\hat{X} \xrightarrow{\pi} (X \times X) \times_{(Y \times Y)} \hat{Y} \to X \times X$$

is the fibration $(p_0, p_1) : \hat{X} \to X \times X$ for a path object $\hat{X}$ for $X$.

The dotted arrow $Q$ exists in the diagram

$$\begin{tikzcd}
X \arrow[r, "s_X"] \arrow[d, "i_1"] & \hat{X} \arrow[d, "\pi"] \\
\tilde{X} \arrow[r, "{((h, \sigma_X), s_Y f \sigma_X)}"'] \arrow[ur, dashed, "Q"'] & (X \times X) \times_{(Y \times Y)} \hat{Y},
\end{tikzcd}$$

making it commute, since $i_1$ is a trivial cofibration and $\pi$ is a fibration. The composite $k = Q \cdot i_0 : X \to \hat{X}$ is therefore a right homotopy from $\theta p$ to $1_X$ such that $\hat{f} k = s_Y f$

There is a pullback diagram

$$\begin{tikzcd}
(X \times X) \times_{(Y \times Y)} \hat{Y} \arrow[r] \arrow[d, "pr"'] & X \arrow[d, "f"] \\
X \times_Y \hat{Y} \arrow[r] & Y
\end{tikzcd}$$

so that the projection map $pr$ defined by $(x_0, x_1, \omega) \mapsto (x_0, \omega)$ is a fibration. It follows that the map $(p_0, \hat{f}) : \hat{X} \to X \times_Y \hat{Y}$ is a fibration.

Finally, given any commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
B \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

with $i$ a cofibration, the lifting $H$ exists in the diagram

$$\begin{tikzcd}
A \arrow[r, "k\alpha"] \arrow[d, "i"'] & \hat{X} \arrow[rr, "p_1"] \arrow[d, "{(p_0, \hat{f})}"] & & X \arrow[d, "f"] \\
B \arrow[r, "{(\theta\beta, s_Y\beta)}"'] \arrow[ur, dashed] & X \times_Y \hat{Y} \arrow[r] & \hat{Y} \arrow[r, "p_1"'] & Y
\end{tikzcd}$$

making it commute, and the composite square solves the lifting problem. □

Lemma 1.15. *Suppose that $X$ and $Y$ are cofibrant and fibrant objects of a closed model category $\mathcal{C}$, and that the map $f : X \to Y$ is a fibration and a homotopy equivalence. Then $f$ has a section $\theta : Y \to X$ with a left homotopy $h : \tilde{X} \to X$ from $\theta f$ to $1_X$ which is fibred over $f$ in the sense that the composite $fh : \tilde{X} \to Y$ is the constant homotopy $f\sigma_X$ at $f$.*

Proof: The map $f$ has a homotopy inverse $g : Y \to X$; in particular, there is a left homotopy $H : \tilde{Y} \to Y$ from $fg$ to $1_Y$. The homotopy lifting property for the fibration $f$ can be used to construct a left homotopy from $g$ to a section $\theta$ of $f$.

Suppose that $k : \tilde{X} \to X$ is a choice of left homotopy from $1_X$ to $\theta f$. Write $k^{-1} = k : \tilde{X} \to X$ for the homotopy from $\theta f$ to $1_X$ defined on the twisted cylinder object

$$\begin{tikzcd}
X \sqcup X \arrow[r, "\tau"] \arrow[d, "\nabla"'] & X \sqcup X \arrow[r, "{(i_0, i_1)}"] & \tilde{X} \arrow[dll, "\sigma_X"] \\
X & &
\end{tikzcd}$$

Here, $\tau$ is the isomorphism which flips direct summands. Now write $\theta f k^{-1} * k : \overline{X} \to X$ for the composite homotopy from $1_X$ to $\theta f$, where $\overline{X}$ is defined by the pushout

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "i_1"'] & \tilde{X} \arrow[d] \\
\tilde{X} \arrow[r] & \overline{X}
\end{tikzcd}$$

according to the recipe for composing homotopies given in the proof of Lemma 1.5. Then there is a diagram

$$\begin{tikzcd}
 & X \arrow[d, "f"] \\
\overline{X} \arrow[ur, "\theta f k^{-1} * k"] \arrow[r, "f k^{-1} * f k"'] & Y
\end{tikzcd}$$

The game is now to show that the homotopy $fk^{-1} * fk$ is homotopic to to the constant homotopy $f\sigma_X : \overline{X} \to Y$ in the sense that there is a commutative diagram

$$\begin{tikzcd}
\overline{X} \cup_{(X \sqcup X)} \overline{X} \arrow[r, "(fk^{-1} * k, f\sigma_X)"] \arrow[d, "j"'] & Y \\
X' \arrow[ur, "H"'] &
\end{tikzcd}$$

where $j$ is a cofibration appearing in a factorization

$$\begin{tikzcd}
\overline{X} \cup_{(X \sqcup X)} \overline{X} \arrow[r, "(\sigma_X, \sigma_X)"] \arrow[d, "j"] & X \\
X' \arrow[ur, "\pi"'] &
\end{tikzcd}$$

such that $\pi$ is a trivial fibration. Write $in_L : \overline{X} \to \overline{X} \cup_{(X \sqcup X)} \overline{X}$. Then if we have such a map $H$, there is a commutative diagram

$$\begin{tikzcd}
\overline{X} \arrow[r, "\theta f k^{-1} * k"] \arrow[d, "j \cdot in_L"'] & X \arrow[d, "f"] \\
X' \arrow[ur, dashed, "K"] \arrow[r, "H"'] & Y
\end{tikzcd}$$

Then the dotted arrow $K$ exists since the composite $j \cdot in_L$ is a trivial cofibration, and the composite $Kj \cdot in_R$ is the desired fibrewise homotopy from $1_X$ to $\theta f$.

In general, we claim that if $h : \tilde{X} \to Y$ is a homotopy $\alpha \to \beta$ of maps $X \to Y$, then there is a commutative diagram

$$\begin{tikzcd}
\overline{X} \cup_{(X \sqcup X)} \overline{X} \arrow[r, "(h^{-1} * h, \alpha\sigma_X)"] \arrow[d, "j"'] & Y \\
X' \arrow[ur, "H"'] &
\end{tikzcd}$$

subject to the choices made above. To see this, take a fixed path object $\hat{Y}$ for $Y$, and construct a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "s_Y\alpha"] \arrow[d, "i_0"'] & \hat{Y} \arrow[d, "(p_0, p_1)"] \\
\tilde{X} \arrow[r, "(\alpha\sigma_X, h)"'] \arrow[ur, "\gamma"] & Y \times Y
\end{tikzcd}$$

where the lifting $\gamma$ exists since $i_0$ is a trivial cofibration. Then two instances of the map $\gamma$ define a map $\overline{\gamma} : \overline{X} \to Y$ which fits into a commutative diagram

$$\begin{tikzcd}
X \sqcup X \arrow[r, "(s_Y\alpha, s_Y\alpha)"] \arrow[d, "i_X"'] & \hat{Y} \arrow[d, "(p_0, p_1)"] \\
\overline{X} \arrow[r, "(\alpha\sigma_X, h^{-1} * h)"'] \arrow[ur, "\overline{\gamma}"] & Y \times Y
\end{tikzcd}$$

It follows that there is a commutative diagram

$$\begin{tikzcd}
\overline{X} \cup_{(X \sqcup X)} \overline{X} \arrow[r, "(\overline{\gamma}, s_Y\alpha\sigma_X)"] \arrow[d, "j"'] & \hat{Y} \arrow[d, "p_0"] \\
X' \arrow[r, "\alpha\pi"'] \arrow[ur, "K"] & Y
\end{tikzcd}$$

and the desired map $H$ is the composite $p_1 K$. □

REMARK 1.16. Proposition 1.14 and its proof are to Quillen [76,p.5.2]; the proof of Lemma 1.15 that is displayed here is a special case of his method of correspondences [76,p.2.2].

## 2. Simplicial categories.

A simplicial model category is, roughly speaking, a closed model category equipped with a notion of a mapping space between any two objects. This has to be done in such a way that it makes homotopy theoretic sense. Thus, besides the new structure, there is an additional axiom, which is called Axiom **SM7** (See 3.1 below).

The initial property one wants is the following: let $\mathbf{S}$ be the category of simplicial sets and let $\mathcal{C}$ be a model category, and suppose $A \in \mathcal{C}$ is cofibrant and $X \in \mathcal{C}$ is fibrant. Then, the space of maps in $\mathcal{C}$ should be a functor to simplicial sets

$$\mathbf{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{S}$$

with the property that

$$\pi_0 \mathbf{Hom}_{\mathcal{C}}(A, X) \cong [A, X]_{\mathcal{C}} \ .$$

In addition, one would want to interpret $\pi_n \mathbf{Hom}_{\mathcal{C}}(A, X)$ in $\mathcal{C}$.

There are other desirable properties; for example, if $A$ is cofibrant and $X \to Y$ a fibration in $\mathcal{C}$, one would want

$$\mathbf{Hom}_{\mathcal{C}}(A, X) \to \mathbf{Hom}_{\mathcal{C}}(A, Y)$$

to be a fibration of spaces — that is, of simplicial sets.

Before imposing the closed model category structure on $\mathcal{C}$, let us make the following definition:

DEFINITION 2.1. *A category $\mathcal{C}$ is a simplicial category if there is a mapping space functor*

$$\mathbf{Hom}_{\mathcal{C}}(\cdot, \cdot) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{S}$$

*with the properties that for $A$ and $B$ objects in $\mathcal{C}$*

(1) $\mathbf{Hom}_{\mathcal{C}}(A, B)_0 = \hom_{\mathcal{C}}(A, B)$*;*

(2) *the functor $\mathbf{Hom}_{\mathcal{C}}(A, \cdot) : \mathcal{C} \to \mathbf{S}$ has a left adjoint*

$$A \otimes \cdot : \mathbf{S} \to \mathcal{C}$$

*which is associative in the sense that there is a isomorphism*

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L,$$

*natural in $A \in \mathcal{C}$ and $K, L \in \mathbf{S}$;*

(3) *The functor $\mathbf{Hom}_{\mathcal{C}}(\cdot, B) : \mathcal{C}^{op} \to \mathbf{S}$ has left adjoint*

$$\mathbf{hom}_{\mathcal{C}}(\cdot, B) : \mathbf{S} \to \mathcal{C}^{op}.$$

Of course, the adjoint relationship in (3) is phrased

$$\hom_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \hom_{\mathcal{C}}(A, \mathbf{hom}_{\mathcal{C}}(K, B)) \ .$$

*Warning:* The tensor product notation goes back to Quillen, and remains for lack of a better operator. But be aware that in this context we do not usually have a tensor product in the sense of algebra; that is, we don't have a pairing arising out of bilinear maps. Instead, we have an adjoint to an internal hom functor, and this is the sole justification for the notation — Lemma 2.2 says that there is a right adjoint $B \mapsto \mathbf{hom}_{\mathcal{C}}(K, B)$ to the functor $A \mapsto A \otimes K$ for a fixed simplicial set $K$.

Note the plethora of distinct mapping objects. As usual, $\hom_{\mathcal{C}}(A, B)$ is the set of morphisms from $A$ to $B$ in the category $\mathcal{C}$, whereas the simplicial set $\mathbf{Hom}_{\mathcal{C}}(A, B)$ is the function complex, and $\mathbf{hom}_{\mathcal{C}}(K, B)$ is an object of $\mathcal{C}$ which is defined for simplicial sets $K$ and objects $B \in \mathcal{C}$. The functor $\mathbf{hom}_{\mathcal{C}}(K, A)$ is often denoted by $A^K$ in the literature.

Observe finally that the objects $\mathbf{hom}_{\mathcal{C}}(K, A)$ and $\mathbf{Hom}_{\mathcal{C}}(K, A)$ coincide when $\mathcal{C}$ is the category of simplicial sets, but they are necessarily quite different elsewhere.

LEMMA 2.2.

(1) *For fixed* $K \in \mathbf{S}$, *the functor*

$$\cdot \otimes K : \mathcal{C} \to \mathcal{C}$$

*is left adjoint to the functor*

$$\mathbf{hom}_{\mathcal{C}}(K, \cdot) : \mathcal{C} \to \mathcal{C} \ .$$

(2) *For all* $K$ *and* $L$ *in* $\mathbf{S}$ *and* $B$ *in* $\mathcal{C}$ *there is a natural isomorphism*

$$\mathbf{hom}_{\mathcal{C}}(K \times L, B) \cong \mathbf{hom}_{\mathbf{S}}(K, \mathbf{hom}_{\mathcal{C}}(L, B)).$$

(3) *For all* $n \geq 0$, $\mathbf{Hom}_{\mathcal{C}}(A, B)_n \cong \hom_{\mathcal{C}}(A \otimes \Delta^n, B)$.

PROOF: Part 1 is a consequence of the string of natural isomorphisms

$$\hom(A \otimes K, B) \cong \hom(K, \mathbf{Hom}(A, B)) \cong \hom(A, \mathbf{hom}(K, B)).$$

Part 2 then follows from the associativity built into 2.1.2. Part 3 follows from 2.1.2 and the fact that $\hom_{\mathbf{S}}(\Delta^n, X) \cong X_n$. □

REMARK: A consequence of Lemma 2.2.1 is that there is a composition pairing of simplicial sets

$$\mathbf{Hom}_{\mathcal{C}}(A, B) \times \mathbf{Hom}_{\mathcal{C}}(B, C) \to \mathbf{Hom}_{\mathcal{C}}(A, C)$$

defined as follows. If $f : A \otimes \Delta^n \to B$ is an $n$-simplex of $\mathbf{Hom}_{\mathcal{C}}(A, B)$ and $g : B \otimes \Delta^n \to C$ is an $n$-simplex of $\mathbf{Hom}_{\mathcal{C}}(B, C)$ then their pairing in $\mathbf{Hom}_{\mathcal{C}}(A, C)$ is the composition

$$A \otimes \Delta^n \xrightarrow{1 \otimes \Delta} A \otimes (\Delta^n \times \Delta^n) \cong A \otimes \Delta^n \otimes \Delta^n \xrightarrow{f \otimes 1} B \otimes \Delta^n \xrightarrow{g} C.$$

Here $\Delta : \Delta^n \to \Delta^n \times \Delta^n$ is the diagonal. This pairing is associative, and reduces to the composition pairing in $\mathcal{C}$ in simplicial degree zero. It is also unital in the sense that if $* \to \mathbf{Hom}_{\mathcal{C}}(A, A)$ is the vertex corresponding to the identity, then the following diagram commutes

$$\begin{tikzcd}
* \times \mathbf{Hom}_{\mathcal{C}}(A, B) \arrow[d] \arrow[dr, "="] & \\
\mathbf{Hom}_{\mathcal{C}}(A, A) \times \mathbf{Hom}_{\mathcal{C}}(A, B) \arrow[r] & \mathbf{Hom}_{\mathcal{C}}(A, B).
\end{tikzcd}$$

There is also a diagram using the identity of $B$. A shorthand way of encoding all this structure is to say that $\mathcal{C}$ is enriched over simplicial sets. □

Another immediate consequence of the definition is the following result.

LEMMA 2.3. *For a simplicial category $\mathcal{C}$ then the following extended adjointness isomorphisms hold:*

(1) $\mathbf{Hom}_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \mathbf{Hom}_{\mathcal{C}}(A \otimes K, B)$.
(2) $\mathbf{Hom}_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \mathbf{Hom}_{\mathcal{C}}(A, \mathbf{hom}(K, B))$.

PROOF: This is an easy exercise using Lemma 2.2. □

Note that, in fact, Definition 2.1 implies that there are functors

$$\cdot \otimes \cdot : \mathcal{C} \times \mathbf{S} \to \mathcal{C}$$

and

$$\mathbf{hom}_{\mathcal{C}}(\cdot, \cdot) : \mathbf{S}^{op} \times \mathcal{C} \to \mathcal{C}$$

satisfying 2.1.1, 2.1.2 and 2.2.1. In order to produce examples of simplicial categories, we note the following:

LEMMA 2.4. *Let $\mathcal{C}$ be a category equipped with a functor*

$$\cdot \otimes \cdot : \mathcal{C} \times \mathbf{S} \to \mathcal{C} \ .$$

*Suppose the following three conditions hold:*

(1) *For fixed $K \in \mathbf{S}$, $\cdot \otimes K : \mathcal{C} \to \mathcal{C}$ has a right adjoint $\mathbf{hom}_{\mathcal{C}}(K, \cdot)$.*
(2) *For fixed $A$, the functor $A \otimes \cdot : \mathbf{S} \to \mathcal{C}$ commutes with arbitrary colimits and $A \otimes * \cong A$.*
(3) *There is an isomorphism $A \otimes (K \times L) \cong (A \otimes K) \otimes L$ natural is $A \in \mathcal{C}$ and $K, L \in \mathbf{S}$.*

*Then $\mathcal{C}$ is a simplicial category with* $\mathbf{Hom}_{\mathcal{C}}(A, B)$ *defined by:*

$$\mathbf{Hom}_{\mathcal{C}}(A, B)_n = \hom_{\mathcal{C}}(A \otimes \Delta^n, B)$$

PROOF: We first prove 2.1.2 holds. If $K \in \mathbf{S}$, write $K$ as the coequalizer in a diagram

$$\bigsqcup_q \Delta^{n_q} \rightrightarrows \bigsqcup_p \Delta^{n_p} \to K \ .$$

Then there is a coequalizer diagram

$$\bigsqcup_q A \otimes \Delta^{n_q} \rightrightarrows \bigsqcup_p A \otimes \Delta^{n_p} \to A \otimes K$$

Hence there is an equalizer diagram

$$\hom_{\mathcal{C}}(A \otimes K, B) \to \hom_{\mathcal{C}}(A \otimes (\bigsqcup_p \Delta^{n_p}), B) \rightrightarrows \hom_{\mathcal{C}}(A \otimes (\bigsqcup_q \Delta^{n_q}), B) \ .$$

This, in turn, is equivalent to the assertion that the equalizer of the maps

$$\hom_{\mathbf{S}}(\bigsqcup_p \Delta^{n_p}, \mathbf{hom}_{\mathcal{C}}(A, B)) \rightrightarrows \hom_{\mathbf{S}}(\bigsqcup_q \Delta^{n_q}, \mathbf{hom}_{\mathcal{C}}(A, B))$$

is the induced map

$$\hom_{\mathbf{S}}(K, \mathbf{hom}_{\mathcal{C}}(A, B)) \to \hom_{\mathbf{S}}(\bigsqcup_p \Delta^{n_p}, \mathbf{hom}_{\mathcal{C}}(A, B))$$

so 2.1.2 holds. If we let $\hom_{\mathcal{C}}(K, \cdot)$ be adjoint to $\cdot \otimes K$, as guaranteed by the hypotheses, 2.1.3 holds. Then finally, 2.1.1 is a consequence of the fact that $A \otimes * = A$. $\square$

We now give some examples. Needless to say $\mathbf{S}$ itself is a simplicial category with, for $A, B, K \in \mathbf{S}$

$$A \otimes K = A \times K$$

and (a tautology)

$$\mathbf{Hom}_{\mathbf{S}}(A, B) = \mathbf{Hom}_{\mathbf{S}}(A, B)$$

and

$$\mathbf{hom}_{\mathbf{S}}(K, B) = \mathbf{Hom}_{\mathbf{S}}(K, B) \ .$$

Only slightly less obvious is the following: let $\mathbf{S}_*$ denote the category of pointed (i.e., based) simplicial sets. Then $\mathbf{S}_*$ is a simplicial category with

$$A \otimes K = A \wedge K_+ = A \times K / * \times K$$

where ( $)_+$ denote adding a disjoint basepoint

$$\mathbf{Hom}_{\mathbf{S}_*}(A,B)_n = \hom_{\mathbf{S}_*}(A \wedge \Delta^n_+, B)$$

and

$$\mathbf{hom}_{\mathbf{S}_*}(K,B) = \mathbf{Hom}_{\mathbf{S}}(K,B)$$

with basepoint given by the constant map

$$K \to * \to B.$$

Note that $\mathbf{Hom}_{\mathbf{S}_*}(A,B) \in \mathbf{S}$, but $\mathbf{hom}_{\mathbf{S}_*}(K,B) \in \mathbf{S}_*$.

This example can be radically generalized. Suppose $\mathcal{C}$ is a category that is co-complete; that is, $\mathcal{C}$ has all colimits. Let $s\mathcal{C}$ denote the simplicial objects in $\mathcal{C}$. Then if $K \in \mathbf{S}$, we may define, for $A \in s\mathcal{C}$, an object $A \otimes K \in s\mathcal{C}$ by

$$(A \otimes K)_n = \bigsqcup_{k \in K_n} A_n$$

where $\bigsqcup$ denotes the coproduct in $\mathcal{C}$, and if $\phi : \mathbf{n} \to \mathbf{m}$ is an ordinal number map $\phi^* : (A \otimes K)_m \to (A \otimes K)_n$ is given by

$$\bigsqcup_{k \in K_m} A_m \xrightarrow{\bigsqcup \phi^*} \bigsqcup_{k \in K_m} A_n \to \bigsqcup_{k \in K_n} A_n \ .$$

The first map is induced by $\phi^* : A_m \to A_n$, the second by $\phi^* : K_m \to K_n$.

THEOREM 2.5. *Suppose that $\mathcal{C}$ is complete and complete. Then with this functor $\cdot \otimes \cdot : s\mathcal{C} \times \mathbf{S} \to s\mathcal{C}$, the category $s\mathcal{C}$ becomes a simplicial category with*

$$\mathbf{Hom}_{s\mathcal{C}}(A,B)_n = \hom_{s\mathcal{C}}(A \otimes \Delta^n, B) \ .$$

PROOF: This is an application of Lemma 2.4. First note that it follows from the construction that there is a natural isomorphism

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L.$$

And one has $A \otimes * \cong A$. Thus, we need only show that, for fixed $K \in \mathbf{S}$, the functor $\cdot \otimes K : s\mathcal{C} \to s\mathcal{C}$ has a right adjoint. To show this, one changes focus slightly. For $Y \in s\mathcal{C}$, define a functor

$$F_Y : \mathcal{C}^{op} \to \mathbf{S}$$

by

$$F_Y(A) = \hom_{\mathcal{C}}(A, Y) \ .$$

Then the functor $\mathcal{C}^{op} \to$ Sets given by

$$A \mapsto \mathbf{Hom}_{\mathbf{S}}(K, F_Y(A))_n = \hom_{\mathbf{S}}(K \times \Delta^n, \hom_{\mathcal{C}}(A, Y))$$

is representable. To see this, write $K \times \Delta^n$ as a coequalizer

$$\bigsqcup_q \Delta^{n_q} \rightrightarrows \bigsqcup_p \Delta^{n_p} \to K \times \Delta^n$$

then the representing object is defined by the equalizer diagram

$$\prod_q Y_{n_q} \leftleftarrows \prod_q Y_{n_p} \leftarrow \mathbf{hom}_{s\mathcal{C}}(K, Y)_n \ .$$

Letting the ordinal number vary yields an object $\mathbf{hom}_{s\mathcal{C}}(K, Y)$ and a natural isomorphism of simplicial sets

$$\hom_{\mathcal{C}}(A, \mathbf{hom}_{s\mathcal{C}}(K, Y)) \cong \mathbf{Hom}_{\mathbf{S}}(K, \hom_{\mathcal{C}}(A, Y)) \ , \tag{2.6}$$

or a natural equivalence of functors

$$F_{\mathbf{hom}_{s\mathcal{C}}(K,Y)}(\cdot) \cong \mathbf{Hom}_{\mathbf{S}}(K, F_Y(\cdot)) \ .$$

Now the morphisms $X \to Y$ in $s\mathcal{C}$ are in one-to-one correspondence with the natural transformations $F_X \to F_Y$, by the Yoneda lemma. In formulas, this reads

$$\hom_{s\mathcal{C}}(X, Y) \cong \mathrm{Nat}(F_X, F_Y) \ .$$

Now if $K \in \mathbf{S}$ and $X \in s\mathcal{C}$ we can define a new functor

$$F_X \otimes K : \mathcal{C}^{op} \to \mathbf{S}$$

by

$$(F_X \otimes K)(A) = F_X(A) \times K$$

we will argue below that

$$\mathrm{Nat}(F_{X \otimes K}, F_Y) \cong \mathrm{Nat}(F_X \otimes K, F_Y) \ .$$

Assuming this one has:

$$\begin{aligned} \hom_{s\mathcal{C}}(X, \mathbf{hom}_{s\mathcal{C}}(K, Y)) &\cong \mathrm{Nat}(F_X, F_{\mathbf{hom}_{s\mathcal{C}}(K,Y)}) \\ &\cong \mathrm{Nat}(F_X, \mathbf{Hom}_{\mathbf{S}}(K, F_Y)) \end{aligned}$$

by (2.6). Continuing, one has that this is isomorphic to

$$\mathrm{Nat}(F_X \otimes K, F_Y) \cong \mathrm{Nat}(F_{X \otimes K}, F_Y) \cong \hom_{s\mathcal{C}}(X \otimes K, Y)$$

so that

$$\hom_{s\mathcal{C}}(X, \mathbf{hom}_{s\mathcal{C}}(K, Y)) \cong \hom_{s\mathcal{C}}(X \otimes K, Y) \ .$$

as required. Thus we are left with

LEMMA 2.7. *There is an isomorphism*

$$\mathrm{Nat}(F_{X\otimes K}, F_Y) \cong \mathrm{Nat}(F_X \otimes K, F_Y)\ .$$

PROOF: It is easiest to show

$$\mathrm{Nat}(F_X \otimes K, F) \cong \hom_{s\mathcal{C}}(X \otimes K, Y)\ .$$

Given a natural transformation

$$\Phi : F_X \otimes K \to F_Y$$

note that

$$(F_X \otimes K)(X_n)_n = \prod_{k\in K_n} \hom_{\mathcal{C}}(X_n, X_n)\ .$$

Thus, for each $k \in K_n$, there is a map

$$\Phi(1)_k : X_n \to Y_n$$

corresponding to the identity in the factor corresponding to $k$. These assemble into a map

$$f_n : (X \otimes K)_n = \coprod_{k\in K_n} X_n \to Y_n\ .$$

We leave it to the reader to verify that yields a morphism

$$f : X \otimes K \to Y$$

of simplicial objects, and that the assignment $\Phi \to f$ yields the desired isomorphism. □

EXAMPLES 2.8. One can now assemble a long list of simplicial categories: We note in particular

1) Let $\mathcal{C}$ be one of the following "algebraic" categories: groups, abelian groups, rings, commutative rings, modules over a ring $R$, algebras or commutative algebras over a commutative ring $R$, or Lie algebras. Then $s\mathcal{C}$ is a simplicial category.

2) Let $\mathcal{C}$ be the graded analog of one of the categories in the previous example. Then $s\mathcal{C}$ is a simplicial category.

3) Let $\mathcal{C} = \mathcal{CA}$ be the category of coalgebras over a field $\mathbb{F}$. Then $s\mathcal{CA}$ is a simplicial category.

4) Note that the hypotheses of $\mathcal{C}$ used Theorem 2.5 apply equally to $\mathcal{C}^{op}$. Thus $s(\mathcal{C}^{op})$ is also a simplicial category. But if $s(\mathcal{C}^{op})$ is a simplicial category, so is $(s(\mathcal{C}^{op}))^{op}$. But this is the category $c\mathcal{C}$ of cosimplicial objects in $\mathcal{C}$. One must interpret the functors $\cdot \otimes \cdot$, $\hom_{c\mathcal{C}}(\cdot,\cdot)$, etc. in light of Theorem 2.5. Thus if $K \in \mathbf{S}$,

$$\mathbf{hom}_{c\mathcal{C}}(K, A)^n = \prod_{k\in K_n} A^n$$

and

$$\mathbf{Hom}_{c\mathcal{C}}(A, B)_n = \hom_{c\mathcal{C}}(A, \mathbf{hom}_{c\mathcal{C}}(\Delta^n, B))$$

and $A \otimes K$ is defined via Theorem 2.5.

To conclude this section, we turn to the following question: suppose given simplicial categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $G : \mathcal{D} \to \mathcal{C}$ with left adjoint $F$. We want a criterion under which the simplicial structure is preserved.

Lemma 2.9. *Suppose that for all $K \in \mathbf{S}$ and $A \in \mathcal{C}$ there is a natural isomorphism $F(A \otimes K) \cong F(A) \otimes K$. Then*

(1) *the adjunction extends to a natural isomorphism*

$$\mathbf{Hom}_{\mathcal{D}}(FA, B) \cong \mathbf{Hom}_{\mathcal{C}}(A, GB);$$

(2) *for all $K \in \mathbf{S}$ and $B \in \mathcal{D}$, there is a natural isomorphism*

$$G\mathbf{hom}_{\mathcal{D}}(K, B) \cong \mathbf{hom}_{\mathcal{C}}(K, GB) \ .$$

Proof: Part (1) uses that $\mathbf{Hom}_{\mathcal{D}}(FA, B)_n \cong \hom_{\mathcal{D}}(FA \otimes \Delta^n, B)$. Part (2) is an exercise in adjunctions. □

We give some examples.

Examples 2.10.

1) Let $G : \mathcal{D} \to \mathcal{C}$ have a left adjoint $F$. Extend this to a pair of adjoint functors by prolongation:

$$G : s\mathcal{D} \to s\mathcal{C}$$

with adjoint $F$. Thus $G(X)_n = G(X)_n$, and so on. Then, in the simplicial structure of Theorem 2.5, $F(X \otimes K) \cong F(X) \otimes K$, since $F$ commutes with colimits.

2) Let $\mathcal{C}$ be an arbitrary simplicial category and $A \in \mathcal{C}$. Define

$$G : \mathcal{C} \to \mathbf{S}$$

by $G(B) = \mathbf{Hom}_{\mathcal{C}}(A, B)$. Then $F(X) = A \otimes X$ and the requirement on 2.9 is simply the formula

$$A \otimes (X \times K) \cong (A \otimes X) \otimes K \ .$$

Remark 2.11. A functor $F : \mathcal{C} \to \mathcal{D}$ between simplicial model categories which has an associated natural isomorphism

$$F(A \otimes K) \xrightarrow[\cong]{\omega_{A,K}} F(A) \otimes K$$

as in the statement of Lemma 2.9 is said to be *continuous*, provided that it also satisfies the requirements that the diagrams

$$\begin{tikzcd}
F(A \otimes K \otimes L) \arrow[r, "\omega_{A\otimes K,L}"] \arrow[d, "\cong"'] & F(A \otimes K) \otimes L \arrow[r, "\omega_{A,K} \otimes L"] & F(A) \otimes K \otimes L \arrow[d, "\cong"] \\
F(A \otimes (K \times L)) \arrow[rr, "\omega_{A,K\times L}"'] & & F(A) \otimes (K \times L)
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(A \otimes \Delta^0) \arrow[rr, "\omega_{A,\Delta^0}"] \arrow[dr, "\cong"'] & & F(A) \otimes \Delta^0 \arrow[dl, "\cong"] \\
& F(A) &
\end{tikzcd}$$

commute, where the unnamed isomorphisms are induced by the simplicial structure on $\mathcal{C}$ and $\mathcal{D}$. Such a functor $F : \mathcal{C} \to \mathcal{D}$ induces simplicial set maps

$$F : \mathbf{Hom}_{\mathcal{C}}(A, B) \to \mathbf{Hom}_{\mathcal{D}}(F(A), F(B))$$

of function spaces which respects composition.

## 3. Simplicial model categories.

If a category $\mathcal{C}$ is at once a simplicial category and a closed model category, we would like the mapping space functor to have homotopy theoretic content. This is accomplished by imposing the following axiom.

3.1 AXIOM **SM7**. *Let $\mathcal{C}$ be a closed model category and a simplicial category. Suppose $j : A \to B$ is a cofibration and $q : X \to Y$ is a fibration. Then*

$$\mathbf{Hom}_{\mathcal{C}}(B, X) \xrightarrow{(j^*, q_*)} \mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

*is a fibration of simplicial sets, which is trivial if $j$ or $q$ is trivial.*

A category satisfying this axiom will be called a *simplicial model category*. The next few sections will be devoted to producing a variety of examples, but in this section we will explore the consequences of this axiom.

PROPOSITION 3.2. *Let $\mathcal{C}$ be a simplicial model category and $q : X \to Y$ a fibration. Then if $B$ is cofibrant*

$$q_* : \mathbf{Hom}_{\mathcal{C}}(B, X) \to \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

*is a fibration in* **S**. *Similarly, if $j : A \to B$ is a cofibration and $X$ is fibrant, then*

$$j^* : \mathbf{Hom}_{\mathcal{C}}(B, X) \to \mathbf{Hom}_{\mathcal{C}}(A, X)$$

*is a fibration.*

PROOF: One sets $A$ to be the initial object and $Y$ to be the final object, respectively, in Axiom **SM7**. □

In other words, $\mathbf{Hom}_{\mathcal{C}}(\cdot, \cdot)$ has entirely familiar homotopical behavior. This is one way to regard this axiom. Another is that **SM7** is a considerable strengthening of the lifting axiom **CM4** of a closed model category.

PROPOSITION 3.3. *Axiom* **SM7** *implies axiom* **CM4**; *that is, given a lifting problem in simplicial category $\mathcal{C}$ satisfying* **SM7**

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

*with $j$ a cofibration and $q$ a fibration, then the dotted arrow exists if either $j$ or $q$ is trivial.*

PROOF: Such a square is a zero-simplex in

$$\mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

and a lifting is a pre-image in the zero simplices of $\mathbf{Hom}_{\mathcal{C}}(B, X)$. Since trivial fibrations are surjective, the result follows. □

But more is true: Axiom **SM7** implies that the lifting built in **CM4** is unique up to homotopy. To explain that, however, requires a few words about homotopy. First we record

PROPOSITION 3.4. *Let $\mathcal{C}$ be a simplicial model category and $j : K \to L$ a cofibration of simplicial sets. If $A \in \mathcal{C}$ is cofibrant, then*

$$1 \otimes j : A \otimes K \to A \otimes L$$

*is a cofibration in $\mathcal{C}$. If $X \in \mathcal{C}$ is fibrant*

$$j^* : \mathbf{hom}_{\mathcal{C}}(L, X) \to \mathbf{hom}_{\mathcal{C}}(K, X)$$

*is a fibration. If $j$ is trivial, then so are $1 \otimes j$ and $j^*$.*

PROOF: For example, one needs to show $1 \otimes j$ has the left lifting property with respect to all trivial fibrations $q : X \to Y$ in $\mathcal{C}$. This is equivalent, by adjointness, to show $j$ has the left lifting property with respect to

$$q_* : \mathbf{Hom}_{\mathcal{C}}(A, X) \to \mathbf{Hom}_{\mathcal{C}}(A, Y)$$

for all trivial fibrations $q$. But $q_*$ is a trivial fibration of simplicial sets by **SM7**. The other three claims are proved similarly. □

Recall the definitions of left and right homotopy from Section 1. The following implies that if $A$ is cofibrant, then $A \otimes \Delta^1$ is a model for the cylinder on $A$.

LEMMA 3.5. *Let $\mathcal{C}$ be a simplicial model category and let $A \in \mathcal{C}$ be cofibrant. Then if $q : \Delta^1 \to *$ is the unique map*

$$1 \otimes q : A \otimes \Delta^1 \to A \otimes * \cong A$$

*is a weak equivalence. Furthermore,*

$$d_1 \sqcup d_0 : A \sqcup A \to A \otimes \Delta^1$$

*is a cofibration and the composite*

$$A \sqcup A \xrightarrow{d_0 \sqcup d_1} A \otimes \Delta^1 \xrightarrow{1 \otimes q} A$$

*is the fold map.*

PROOF: The first claim follows from Proposition 3.4, since

$$d_1 : A \cong A \otimes \Delta^0 \to A \otimes \Delta^1$$

is a weak equivalence. The second claim follows from 3.4 also since $d_1 \bigsqcup d_0$ is equivalent to

$$1 \otimes j : A \otimes \partial\Delta^1 \to A \otimes \Delta^1$$

where $j : \partial\Delta \to \Delta^1$ is inclusion of the boundary. For the third claim one checks that $(1 \otimes q) \cdot d_1 = (1 \otimes q) \cdot d_0 = 1$. □

Thus, if $\mathcal{C}$ is a simplicial model category and $A \in \mathcal{C}$ is cofibrant and $X$ is fibrant, then two morphisms $f, g : A \to X$ are homotopic if and only if there is a factoring

$$\begin{tikzcd}
A \sqcup A \arrow[r, "d_1 \sqcup d_0"] \arrow[d, "f \sqcup g"'] & A \otimes \Delta^1 \arrow[dl, "H"] \\
X &
\end{tikzcd} \tag{3.6}$$

This, too, is no surprise. As a further exercise, note that if one prefers right homotopy for a particular application, one could require a factoring

$$\begin{tikzcd}
& \mathbf{hom}_{\mathcal{C}}(\Delta^1, X) \arrow[d, "j^*"] \\
A \arrow[ur, "G"] \arrow[r, "f \times g"'] & X \times X \cong \mathbf{hom}_{\mathcal{C}}(\partial\Delta^1, X)
\end{tikzcd} \tag{3.7}$$

In using this formulation, one wants $X$ to be fibrant so that $j^*$ is a fibration.

To formulate the next notion, let $A$ be cofibrant and $j : A \to B$ a cofibration. Given two maps $f, g : B \to X$ so that $j \cdot f = j \cdot g$, we say $f$ and $g$ are *homotopic under* $A$ if there is a homotopy

$$H : B \otimes \Delta^1 \to X$$

so that $H \cdot (j \otimes 1) : A \otimes \Delta^1 \to X$ is the constant homotopy on $j \cdot f$. That is, $h \otimes (j \otimes 1)$ is the composite

$$A \otimes \Delta^1 \xrightarrow{1 \otimes q} A \otimes * \cong A \xrightarrow{j \cdot f} X$$

where $q : \Delta^1 \to *$ is the unique map. There is a dual notion of homotopic over $Y$.

The following result says that in a simplicial model category, the liftings required by axiom **CM4** are unique in a strong way.

Proposition 3.8. *Let $\mathcal{C}$ be a simplicial model category and $A$ a cofibrant object. Consider a commutative square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\
B \arrow[r] & Y
\end{tikzcd}$$

*where $j$ is a cofibration, $q$ is a fibration and one of $j$ or $q$ is trivial. Then any two solutions $f, g : B \to X$ of the lifting problem are homotopic under $A$ and over $Y$.*

Proof: The commutative square is a zero-simplex $\alpha$ in

$$\mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

Let $s_0\alpha$ be the corresponding degenerate 1-simplex. Then $s_0\alpha$ is the commutative square

$$\begin{tikzcd}
A \otimes \Delta^1 \arrow[r] \arrow[d, "j \otimes 1"'] & X \arrow[d, "q"] \\
B \otimes \Delta^1 \arrow[r] & Y
\end{tikzcd}$$

where the horizontal maps are the constant homotopies. Let

$$f, g \in \mathbf{Hom}_{\mathcal{C}}(B, X)_0 = \hom_{\mathcal{C}}(B, X)$$

be two solutions to the lifting problem. Thus $(i^*, q_*)f = (i^*, q_*)g = \alpha$. Then by **SM7**, there is a 1-simplex

$$\beta \in \mathbf{Hom}_{\mathcal{C}}(B, X)_1 \cong \hom_{\mathcal{C}}(B \otimes \Delta^1, X)$$

so that $d_1\beta = f,\ d_0\beta = g$, and $(i^*, q_*)\beta = s_0\alpha$. Then

$$\beta : B \otimes \Delta^1 \to X$$

is the required homotopy. □

We now restate a concept from Section 1. For a simplicial model category $\mathcal{C}$, we define the homotopy category $\mathrm{Ho}(\mathcal{C})$ as follows: the objects are the objects of $\mathcal{C}$ and the morphisms are defined by

$$[A, X]_{\mathcal{C}} = \hom_{\mathcal{C}}(B, Y)/\sim \tag{3.9}$$

where $q : B \to A$ is a trivial fibration with $B$ cofibrant, $i : X \to Y$ is a trivial cofibration with $Y$ fibrant and $f \sim g$ if and only if $f$ is homotopic to $g$. It is a consequence of Lemma 1.5 that $\sim$ is an equivalence relation, and it is a consequence of the proof of Theorem 1.11 $[A, X]_{\mathcal{C}}$ coincides with morphisms from $A$ to $X$ in the homotopy category $\mathrm{Ho}(\mathcal{C})$.

There is some ambiguity in the notation: $[A, X]_{\mathcal{C}}$ depends not only on $\mathcal{C}$, but on the particular closed model category structure. In the sequel, $[\ ,\ ]$ means $[\ ,\ ]_{\mathbf{S}}$.

The following result gives homotopy theoretic content to the functors $\cdot \otimes K$ and $\mathbf{hom}_{\mathcal{C}}(K, \cdot)$.

PROPOSITION 3.10. *Let $\mathcal{C}$ be a simplicial model category and $A$ and $B$ a cofibrant and a fibrant object of $\mathcal{C}$, respectively. Then*

$$[K, \mathbf{Hom}_{\mathcal{C}}(A, B)] \cong [A \otimes K, B]_{\mathcal{C}}$$

*and*

$$[K, \mathbf{Hom}_{\mathcal{C}}(A, B)] \cong [A, \mathbf{hom}_{\mathcal{C}}(K, B)]_{\mathcal{C}}\ .$$

PROOF: Note that $\mathbf{Hom}_{\mathcal{C}}(A, B)$ is fibrant, by Proposition 3.2. Hence

$$[K, \mathbf{Hom}_{\mathcal{C}}(A, B)] = \hom_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B))/\sim$$

where $\sim$ means "homotopy" as above. But, since

$$A \otimes (K \times \Delta^1) = (A \otimes K) \otimes \Delta^1$$

we have that

$$\begin{aligned} \hom_{\mathbf{S}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B))/\sim\ &\cong \hom_{\mathcal{C}}(A \otimes K, B)/\sim \\ &\cong [A \otimes K, B]_{\mathcal{C}} \end{aligned}$$

where we use Proposition 3.4 to assert that $A \otimes K$ is cofibrant. □

We now concern ourselves with developing a way of recognizing when **SM7** holds.

Proposition 3.11. *Let $\mathcal{C}$ be a closed model category and a simplicial category. Then the axiom* **SM7** *holds if and only if for all cofibrations $i : K \to L$ in* **S** *and cofibrations $j : A \to B$ in $\mathcal{C}$, the map*

$$(j \otimes 1) \cup (1 \otimes i) : (A \otimes L) \cup_{(A \otimes K)} (B \otimes K) \to B \otimes L\ .$$

*is a cofibration which is trivial if either $j$ or $i$ is.*

Proof: A diagram of the form

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \mathbf{Hom}_{\mathcal{C}}(B,X) \arrow[d] \\
L \arrow[r] & \mathbf{Hom}_{\mathcal{C}}(A,X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B,Y)
\end{tikzcd}$$

is equivalent, by adjointness, to a diagram

$$\begin{tikzcd}
(A \otimes L) \cup_{(A \otimes K)} (B \otimes K) \arrow[r] \arrow[d] & X \arrow[d] \\
B \otimes L \arrow[r] & Y\ .
\end{tikzcd}$$

The result follows by using the fact that fibrations and cofibrations are determined by various lifting properties. □

From this we deduce

Corollary 3.12. (Axiom **SM7b**) *Let $\mathcal{C}$ be a closed model category and a simplicial category. The axiom* **SM7** *is equivalent to the requirement that for all cofibrations $j : A \to B$ in $\mathcal{C}$*

$$(A \otimes \Delta^n) \cup_{(A \otimes \partial\Delta^n)} (B \otimes \partial\Delta^n) \to B \otimes \Delta^n$$

*is a cofibration (for $n \geq 0$) that is trivial if $j$ is, and that*

$$(A \otimes \Delta^1) \cup_{(A \otimes \{e\})} (B \otimes \{e\}) \to B \otimes \Delta^1$$

*is the trivial cofibration for $e = 0$ or $1$.*

Proof: Let $i : K \to L$ be a cofibration of simplicial sets. Then, since $i$ can be built by attaching cells to $K$, the first condition implies

$$(A \otimes L) \cup_{(A \otimes K)} (B \otimes K) \to (B \otimes L)$$

is a cofibration which is trivial if $j$ is. The second condition and proposition I.4.2 (applied to $B_2$) yields that $(j \otimes 1) \cup (1 \otimes i)$ is trivial if $i$ is. □

In the usual duality that arises in these situations, we also have

PROPOSITION 3.13. *Let $\mathcal{C}$ be a simplicial category and a model category and suppose $i : K \to L$ is a cofibration in $S$ and $q : X \to Y$ a fibration in $\mathcal{C}$. Then the following are equivalent:*

1. **SM7**,
2. $\mathbf{hom}_{\mathcal{C}}(L, X) \to \mathbf{hom}_{\mathcal{C}}(K, X) \times_{\mathbf{hom}_{\mathcal{C}}(K,Y)} \mathbf{hom}_{\mathcal{C}}(L, Y)$ *is a fibration which is trivial if $q$ or $j$ is;*
3. *(***SM7a***)* $\mathbf{hom}_{\mathcal{C}}(\Delta^n, X) \to \mathbf{hom}_{\mathcal{C}}(\partial\Delta^n, X) \times_{\mathbf{hom}_{\mathcal{C}}(\partial\Delta^n,Y)} \mathbf{hom}_{\mathcal{C}}(\Delta^n, Y)$ *is a fibration which is trivial if $q$ is, and*
$$\mathbf{hom}_{\mathcal{C}}(\Delta^1, X) \to \mathbf{hom}_{\mathcal{C}}(e, X) \times_{\mathbf{hom}_{\mathcal{C}}(e,Y)} \mathbf{hom}_{\mathcal{C}}(\Delta^1, Y)$$
*is a trivial fibration for $e = 0, 1$.*

EXAMPLE 3.14. *A simplicial model category structure on* **CGHaus**

We can now show that the category **CGHaus** of compactly generated Hausdorff spaces is a simplicial model category. To supply the simplicial structure let $X \in \mathbf{CGHaus}$ and $K \in \mathbf{S}$. Define

$$X \otimes K = X \times_{Ke} |K|$$

where $|\cdot|$ denotes the geometric realization and $\times_{Ke}$ the Kelley product, which is the product internal to the category **CGHaus**. Then if $X$ and $Y$ are in **CGHaus**, the simplicial set of maps between $X$ and $Y$ is given by

$$\mathbf{Hom}_{\mathbf{CGHaus}}(X, Y)_n = \hom_{\mathbf{CGHaus}}(X \times \Delta^n, Y)$$

regarded as a set. And the right adjoint to $\cdot \otimes K$ is given by

$$\mathbf{hom}_{\mathbf{CGHaus}}(K, X) = \mathbf{F}(|K|, X)$$

where $\mathbf{F}$ denotes the internal function space to **CGHaus**.

We have seen that **CGHaus** is a closed model category with the usual weak equivalences and Serre fibrations. In addition, Proposition 3.13 immediately implies that **CGHaus** is a simplicial model category.

It is worth pointing out that the realization functor $|\cdot|$ and its adjoint $S(\cdot)$ the singular set functor pass to the level of simplicial categories. Indeed, we've seen (Proposition I.2.4) that if $X \in \mathbf{S}$ and $K \in \mathbf{S}$, then

$$|X \times K| = |X| \times_{Ke} |K|.$$

This immediately implies that if $Y \in \mathbf{CGHaus}$, then

$$S\mathbf{F}(|K|, Y) = \mathbf{Hom}_{\mathbf{S}}(K, SY)$$

and

$$\mathbf{Hom}_{\mathbf{CGHaus}}(|X|, Y) = \mathbf{Hom}_{\mathbf{S}}(X, SY).$$

We close this section with the following lemma, which gives a standard method for detecting weak equivalences in a closed simplicial model category:

Lemma 3.15. *Suppose that $f : A \to B$ is a map between cofibrant objects in a simplicial model category $\mathcal{C}$. Then $f$ is a weak equivalence if and only if the induced map*

$$f^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(A, Z)$$

*is a weak equivalence of simplicial sets for each fibrant object $Z$ of $\mathcal{C}$.*

Proof: We use the fact, which appears as Lemma 8.4 below, that a map $f : A \to B$ between cofibrant objects in a closed model category has a factorization

$$\begin{tikzcd}
& X \arrow[d, "q"] \\
A \arrow[ur, "j"] \arrow[r, "f"'] & B
\end{tikzcd}$$

such that $j$ is a cofibration and the map $q$ is left inverse to a trivial cofibration $i : B \to X$.

If $f : A \to B$ is a weak equivalence, then the map $j : A \to X$ is a trivial cofibration, and hence induces a trivial fibration $j^* : \mathbf{Hom}(X, Z) \to \mathbf{Hom}(A, Z)$ for all fibrant objects $Z$. Similarly, the trivial cofibration $i$ induces a trivial fibration $i^*$, so that the map $q^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(X, Z)$ is a weak equivalence.

Suppose that the map $f^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(A, Z)$ is a weak equivalence for all fibrant $Z$. To show that $f$ is a weak equivalence, we can presume that the objects $A$ and $B$ are fibrant as well as cofibrant. In effect, there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

in which the objects $A'$ and $B'$ are fibrant, and the vertical maps are trivial cofibrations, and then one applies the functor $\mathbf{Hom}(\ , Z)$ for $Z$ fibrant and invokes the previous paragraph.

Finally, suppose that $A$ and $B$ are fibrant as well as cofibrant, and presume that $f^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(A, Z)$ is a weak equivalence for all fibrant $Z$. We can assume further that $f$ is a cofibration, by taking a suitable factorization. The map $f^* : \mathbf{Hom}(B, A) \to \mathbf{Hom}(A, A)$ is therefore a trivial Kan fibration, and hence surjective in all degrees, so that there is a map $g : B \to A$ such that $g \cdot f = 1_A$. The maps $f \cdot g$ and $1_B$ are both pre-image of the vertex $f$ under the trivial fibration $f^* : \mathbf{Hom}(B, B) \to \mathbf{Hom}(A, B)$, so that there is a homotopy $f \cdot g \simeq 1_B$. In particular, $f$ is a homotopy equivalence and therefore a weak equivalence, by Lemma 1.14. □

## 4. The existence of simplicial model category structures.

Here we concern ourselves with the following problem: Let $\mathcal{C}$ be a category and $s\mathcal{C}$ the category of simplicial objects over $\mathcal{C}$. Then, does $s\mathcal{C}$ have the structure of a simplicial model category? We will assume that there is a functor $G : s\mathcal{C} \to \mathbf{S}$ with a left adjoint

$$F : \mathbf{S} \to s\mathcal{C}\ .$$

Examples include algebraic categories such as the categories of groups, abelian groups, algebras over some ring $R$, commutative algebras, Lie algebras, and so on. In these cases, $G$ is a forgetful functor. See the examples in 2.10.

Define a morphism $f : A \to B$ is $s\mathcal{C}$ to be

a) a weak equivalence if $Gf$ is a weak equivalence in $\mathbf{S}$;

b) a fibration if $Gf$ is a fibration in $\mathbf{S}$;

c) a cofibration if it has the left lifting property with respect to all trivial fibrations in $s\mathcal{C}$.

A final definition is necessary before stating the result. Let $\{X_\alpha\}_{\alpha\in I}$ be a diagram in $\mathcal{C}$. Then, assuming the category $\mathcal{C}$ has enough colimits, there is a natural map

$$\varinjlim_I G(X_\alpha) \longrightarrow G(\varinjlim_I X_\alpha).$$

This is not, in general, an isomorphism. We say that $G$ commutes with filtered colimits if this is an isomorphism whenever the index category $I$ is filtered.

THEOREM 4.1. *Suppose $\mathcal{C}$ has all limits and colimits and that $G$ commutes with filtered colimits. Then with the notions of weak equivalence, fibration, and cofibration defined above, $s\mathcal{C}$ is a closed model category provided the following assumption on cofibrations holds: every cofibration with the left lifting property with respect to fibrations is a weak equivalence.*

We will see that, in fact, $s\mathcal{C}$ is a simplicial model category with the simplicial structure of Theorem 2.5.

The proof of Theorem 4.1 turns on the following observation. As we have seen, a morphism $f : X \to Y$ is a fibration of simplicial sets if and only if it has the right lifting property with respect to the inclusions for all $n, k$

$$\Lambda^n_k \hookrightarrow \Delta^n$$

and $f$ is a trivial fibration if and only if it has the right lifting property with respect to the inclusions $\partial\Delta^n \to \Delta^n$ of the boundary for all $n$. The objects $\Lambda^n_k, \partial\Delta^n$, and $\Delta^n$ are *small* in the following sense: the natural map

$$\varinjlim_I \hom_{\mathbf{S}}(\Lambda^n_k, X_\alpha) \longrightarrow \hom_{\mathbf{S}}(\Lambda^n_k, \varinjlim_I X_\alpha)$$

is an isomorphism for all filtered colimits in $\mathbf{S}$. This is because $\Lambda^n_k$ has only finitely many non-degenerate simplices. Similar remarks hold for $\partial\Delta^n$ and $\Delta^n$.

Lemma 4.2. *Any morphism $f : A \to B$ in $s\mathcal{C}$ can be factored*

$$A \xrightarrow{j} X \xrightarrow{q} B$$

*where the morphism $j$ is a cofibration and $q$ is a trivial fibration.*

Proof: Coproducts of cofibrations are cofibrations, and given a pushout diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "i"'] & A_2 \arrow[d, "j"] \\ A_1 \arrow[r] & B \end{tikzcd}$$

in $s\mathcal{C}$, then $i$ a cofibration implies $j$ is a cofibration, and that if $X \to Y$ is a cofibration in $\mathbf{S}$, then $FX \to FY$ is a cofibration in $s\mathcal{C}$. Inductively construct objects $X_n \in s\mathcal{C}$ with the following properties:

a) One has $A = X_0$ and there is a cofibration $j_n : X_n \to X_{n+1}$.

b) There are maps $q_n : X_n \to B$ so that $q_n = q_{n+1} \cdot j_n$ and the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & X_n \arrow[d, "q_n"] \\ & B \end{tikzcd}$$

commutes, where $A \to X_n$ is the composite $j_{n-1} \cdot \cdots \cdot j_0$.

c) Any diagram

$$\begin{tikzcd} F\partial\Delta^m \arrow[r, "\varphi"] \arrow[d] & X_n \arrow[d, "q_n"] \\ F\Delta^m \arrow[r, "\psi"] & B \end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd} F\partial\Delta^m \arrow[r, "\varphi"] \arrow[d] & X_n \arrow[d, "j_n"] \arrow[dr, "q_n"] & \\ F\Delta^m \arrow[r] & X_{n+1} \arrow[r, "q_{n+1}"'] & B \end{tikzcd}$$

where the bottom morphism is $\psi$.

Condition c) indicates how to construct $X_{n+1}$ given $X_n$. Define $j_n : X_n \to X_{n+1}$ by the pushout diagrams

$$\begin{tikzcd}
\bigsqcup F\partial\Delta^m \arrow[r] \arrow[d] & X_n \arrow[d, "j_n"] \\
\bigsqcup F\Delta^m \arrow[r] & X_{n+1}
\end{tikzcd}$$

where the coproduct is over all diagrams of the type presented in c).

Then condition c) automatically holds. Further, $q_{n+1} : X_{n+1} \to B$ is defined and satisfies condition b) by the universal property of pushouts. Lastly, condition a) holds by the remarks at the beginning of the proof.

Now define $X = \varinjlim X_n$ and notice that we have a factoring

$$A \xrightarrow{j} X \xrightarrow{q} B$$

of the original morphism. The morphism $j$ is a cofibration since directed colimits of cofibrations are cofibrations. We need only show $q : X \to B$ is a trivial fibration. This amounts to showing that any diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & GX \arrow[d, "Gq"] \\
\Delta^n \arrow[r] & GB
\end{tikzcd}$$

can be completed. But $GX \cong \varinjlim_n GX_n$ by hypothesis on $G$, and the result follows by the small object argument. □

The same argument, but using the trivial cofibrations $\Lambda^m_k \hookrightarrow \Delta^m$ in **S**, proves the following lemma.

LEMMA 4.3. *Any morphism $f : A \to B$ in $s\mathcal{C}$ can be factored*

$$A \xrightarrow{j} X \xrightarrow{q} B$$

*where $q$ is a fibration and $j$ is a cofibration which has the left lifting property with respect to all fibrations.*

PROOF OF THEOREM 4.1: The axioms **CM1**–**CM3** are easily checked. The axiom **CM5b** is Lemma 4.2; the axiom **CM5a** follows from Lemma 4.3 and the assumption on cofibrations. For axiom **CM4**, one half is the definition of cofibration. For the other half, one proceeds as follows. Let

$$i : A \to B$$

be a trivial cofibration. Then by Lemma 4.3 we can factor the morphism $i$ as

$$A \xrightarrow{j} X \xrightarrow{q} B$$

where $j$ is a cofibration with the left lifting with respect to all fibrations, and $q$ is a fibration. By the hypothesis on cofibrations, $j$ is a weak equivalence. Since $i$ is a weak equivalence, so is $q$. Hence, one can complete the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "q"] \\ B \arrow[r, "="] & B \end{tikzcd}$$

and finds that $i$ is a retract of $j$. Hence $i$ has the left lifting property with respect to fibrations, because $j$ does. This completes the proof. □

We next remark that, in fact, $s\mathcal{C}$ is a simplicial model category. For this, we impose the simplicial structure guaranteed by Theorem 2.5. Thus if $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, we have that

$$(A \otimes K)_n = \bigsqcup_{k \in K_n} A_n.$$

From this, one sees that if $X \in \mathbf{S}$

$$F(X \times K) \cong F(X) \otimes K.$$

This is because $F$, as a left adjoint, preserves coproducts. Thus Lemma 2.9 applies and

$$G\mathbf{hom}_{s\mathcal{C}}(K, B) \cong \mathbf{hom}_{\mathbf{S}}(K, GB).$$

THEOREM 4.4. *With this simplicial structure, $s\mathcal{C}$ becomes a simplicial model category.*

PROOF: Apply Proposition 3.13.1. If $j : K \to L$ is a cofibration in $\mathbf{S}$ and $q : X \to Y$ is a fibration in $s\mathcal{C}$, the map

$$G\mathbf{hom}_{s\mathcal{C}}(L, X) \longrightarrow G(\mathbf{hom}_{s\mathcal{C}}(K, X) \times_{\mathbf{hom}_{s\mathcal{C}}(K,Y)} \mathbf{hom}_{s\mathcal{C}}(L, Y))$$

is isomorphic to

$$\mathbf{hom}_{\mathbf{S}}(L, GX) \longrightarrow \mathbf{hom}_{\mathbf{S}}(K, GX) \times_{\mathbf{hom}_{\mathbf{S}}(K,GY)} \mathbf{hom}_{\mathbf{S}}(L, GY)$$

by the remarks above and the fact that $G$, as a right adjoint, commutes with pullbacks. Since the simplicial set category $\mathbf{S}$ has a simplicial model structure, the result holds. □

4.5 A REMARK ON THE HYPOTHESES. Theorem 4.1 and, by extension, Theorem 4.4 require the hypothesis that every cofibration with the left lifting property with respect to all fibrations is, in fact, a weak equivalence. This is so Lemma 4.3 produces the factoring of a morphism as a trivial cofibration followed by a fibration. In the next section we will give some general results about when this hypothesis holds; however, in a particular situation, one might be able to prove directly that the factoring produced in Lemma 4.3 actually yields a trivial cofibration. Then the hypothesis on cofibrations required by these theorems holds because any cofibration with the left lifting property with respect to all fibrations will be a retract of a trivial cofibration. Then one need say no more.

For example, in examining the proof of Lemma 4.3 (see Lemma 4.2), one sees that we would have a factorization of $f : A \to B$ as a trivial cofibration followed by a fibration provided one knows that 1.) $F(\Lambda^n_k) \to F(\Delta^n)$ is a weak equivalence or, more generally, that $F$ preserves trivial cofibrations, and 2.) trivial cofibrations in $s\mathcal{C}$ are closed under coproducts, pushouts, and colimits over the natural numbers.

## 5. Examples of simplicial model categories.

As promised, we prove that a variety of simplicial categories satisfy the hypotheses necessary for Theorem 4.4 of the previous section to apply.

We begin with a crucial lemma.

LEMMA 5.1. *Assume that for every $A \in s\mathcal{C}$ there is a natural weak equivalence*

$$\varepsilon_A : A \to QA$$

*where $QA$ is fibrant. Then every cofibration with the left lifting property with respect to all fibrations is a weak equivalence.*

Proof: This is the argument given by Quillen, on page II.4.9 of [76]. Let $j : A \to B$ be the given cofibration. Then by hypothesis, we may factor

$$\begin{tikzcd}
A \arrow[r, "\varepsilon_A"] \arrow[d, "j"'] & QA \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}$$

to get a map $u : B \to QA$ so that $uj = \varepsilon_A$. Then we contemplate the lifting problem

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & \mathbf{hom}_{s\mathcal{C}}(\Delta^1, QB) \arrow[d, "q"] \\
B \arrow[r, "g"] \arrow[ur, dashed] & \mathbf{hom}_{s\mathcal{C}}(\partial\Delta^1, QB) \cong QB \times QB
\end{tikzcd}$$

where $q$ is induced by $\partial\Delta^1 \subseteq \Delta^1$, $f$ is the composite

$$A \xrightarrow{j} B \xrightarrow{\varepsilon_B} QB = \mathbf{hom}_{s\mathcal{C}}(*, QB) \to \mathbf{hom}_{s\mathcal{C}}(\Delta^1, QB)$$

and

$$g = (\varepsilon_B, Qj \cdot u) \ .$$

Note that $f$ is adjoint to the constant homotopy on

$$\varepsilon_B \cdot j = Qj \cdot \varepsilon_A : A \to QB \ .$$

Then $q$ is a fibration since

$$G\mathbf{hom}_{s\mathcal{C}}(K, X) = \mathbf{hom}_{\mathbf{S}}(K, GX) \ ,$$

and $\mathbf{S}$ is a simplicial model category. Hence, since $j$ is a cofibration having the left lifting property with respect to all fibrations, there exists

$$H : B \to \mathbf{hom}_{s\mathcal{C}}(\Delta^1, QB)$$

making both triangles commute. Then $H$ is a right homotopy from

$$\varepsilon_B : B \to QB$$

to $Qj \cdot u$, and this homotopy restricts to the constant homotopy on $\varepsilon_B \cdot j = Qj \cdot \varepsilon_A : A \to QB$. In other words, we have a diagram

$$\begin{tikzcd}
A \arrow[r, "\epsilon_A"] \arrow[d, "j"'] & QA \arrow[d, "Qj"] \\
B \arrow[r, "\epsilon_B"'] \arrow[ur, "u"] & QB
\end{tikzcd}$$

such that the upper triangle commutes and the lower triangle commutes up to homotopy. Apply the functor $G$ to this diagram. Then $G$ preserves right homotopies, and one checks directly on the level of homotopy groups that $Gj$ is a weak equivalence, which, by definition, implies $j$ is a weak equivalence. □

EXAMPLE 5.2. Suppose that every object of $s\mathcal{C}$ is fibrant. Then we may take $\varepsilon_A : A \to QA$ to be the identity. This happens, for example, if the functor $G : s\mathcal{C} \to \mathbf{S}$ factors through the sub-category of simplicial groups and simplicial group homomorphisms. Thus, Theorem 4.4 applies to

(1) simplicial groups, simplicial abelian groups and simplicial $R$-modules, where $G$ is the forgetful functor;
(2) more generally to simplicial modules over a simplicial ring $R$, where $G$ is the forgetful functor,
(3) for a fixed commutative ring $R$; simplicial $R$-algebras, simplicial commutative $R$-algebras and simplicial Lie algebras over $R$. Again $G$ is the forgetful functor.

Another powerful set of examples arises by making a careful choice of the form the functor $G$ can take.

Recall that an object $A \in \mathcal{C}$ is small if $\hom_{\mathcal{C}}(A, \cdot)$ commutes with filtered colimits. Fix a small $Z \in \mathcal{C}$ and define

$$G : s\mathcal{C} \to \mathbf{S} \tag{5.3}$$

by

$$G(X) = \mathbf{hom}_{s\mathcal{C}}(Z, X) \ .$$

Then $G$ has left adjoint

$$FK = Z \otimes K$$

and $G(\cdot)$ commutes with filtered colimits. Thus, to apply Theorem 4.4, we need to prove the existence of the natural transformation

$$\varepsilon : A \to QA$$

as in Lemma 5.1. Let

$$\mathrm{Ex} : \mathbf{S} \to \mathbf{S}$$

be Kan's Extension functor[1]. Then for all $K \in \mathbf{S}$ there is a natural map

$$\varepsilon_K : K \to \mathrm{Ex}\, K$$

which is a weak equivalence. Furthermore, most crucially for the application here, $\mathrm{Ex}(\cdot)$ commutes with all limits. This is because it's a right adjoint. Finally, if $\mathrm{Ex}^n K$ is this functor applied $n$ times and

$$\mathrm{Ex}^n \varepsilon_K : \mathrm{Ex}^n K \longrightarrow \mathrm{Ex}^{n+1} K$$

the induced morphism, then $\mathrm{Ex}^\infty K = \varinjlim \mathrm{Ex}^n K$ is fibrant in $\mathbf{S}$ and the induced map

$$K \longrightarrow \mathrm{Ex}^\infty K$$

is a trivial cofibration.

LEMMA 5.4. *Suppose the category $\mathcal{C}$ is complete and cocomplete. Fix $n \geq 0$. Then there is a functor*

$$Q_0(\cdot)_n : s\mathcal{C} \longrightarrow \mathcal{C}$$

*so that, for all $Z \in \mathcal{C}$, there is a natural isomorphism of sets*

$$\hom_{\mathcal{C}}(Z, (Q_0 A)_n) \cong \mathrm{Ex}(\mathbf{Hom}_{s\mathcal{C}}(Z, A))_n \ .$$

PROOF: Recall that that the functor Ex on $\mathbf{S}$ is right adjoint to the the subdivision functor sd. Then one has a sequence of natural isomorphisms

$$\begin{aligned}
\mathrm{Ex}\,\mathbf{Hom}_{s\mathcal{C}}(Z, A)_n &\cong \hom_{\mathbf{S}}(\Delta^n, \mathrm{Ex}\,\mathbf{Hom}_{s\mathcal{C}}(Z, A)) \\
&\cong \hom_{\mathbf{S}}(\mathrm{sd}\,\Delta^n, \mathbf{Hom}_{s\mathcal{C}}(Z, A)) \\
&\cong \hom_{s\mathcal{C}}(Z \otimes \mathrm{sd}\,\Delta^n, A) \\
&\cong \hom_{s\mathcal{C}}(Z, \mathbf{hom}_{s\mathcal{C}}(\mathrm{sd}\,\Delta^n, A)) \\
&\cong \hom_{\mathcal{C}}(Z, \mathbf{hom}_{s\mathcal{C}}(\mathrm{sd}\,\Delta^n, A)_0).
\end{aligned}$$

The last isomorphism is due to the fact that $Z$ is a constant simplicial object and maps out of a constant simplicial object are completely determined by what happens on zero simplices. Thus we can set

$$(Q_0 A)_n = \mathbf{hom}_{s\mathcal{C}}(\mathrm{sd}\,\Delta^n, A)_0. \qquad \square$$

---

[1] This construction is discussed in Section III.4 below.

The simplicial object $Q_0A$ defined by

$$\mathbf{n} \mapsto (Q_0A)_n = \mathbf{hom}_{s\mathcal{C}}(\mathrm{sd}\,\Delta^n, A)_0$$

is natural in A; that is, we obtain a functor $Q_0 : s\mathcal{C} \to s\mathcal{C}$. Since we regard $Z \in \mathcal{C}$ as a constant simplicial object in $s\mathcal{C}$

$$\begin{aligned}\mathbf{Hom}_{s\mathcal{C}}(Z, Y)_n &\cong \hom_{s\mathcal{C}}(Z \otimes \Delta^n, Y)\\ &\cong \hom_{\mathcal{C}}(Z, Y_n)\end{aligned}$$

one immediately has that

$$\mathbf{Hom}_{s\mathcal{C}}(Z, Q_0A) \cong \mathrm{Ex}\,\mathbf{Hom}_{s\mathcal{C}}(Z, A).$$

Finally the natural transformation $\varepsilon_K : K \to \mathrm{Ex}\,K$ yields a natural map

$$\varepsilon_A : A \longrightarrow Q_0A$$

and, by iteration, maps

$$Q_0^n\varepsilon_A : Q_0^nA \longrightarrow Q_0^{n+1}A.$$

Define $QA = \varinjlim Q_0^nA$. The reader will have noticed that $Q_0A$ and $QA$ are independent of $Z$.

Now fix a small object $Z \in \mathcal{C}$ and regard $Z$ as a constant simplicial object in $s\mathcal{C}$. Then we define a morphism $A \to B$ in $s\mathcal{C}$ to be a weak equivalence (or fibration) if and only if the induced map

$$\mathbf{Hom}_{s\mathcal{C}}(Z, A) \longrightarrow \mathbf{Hom}_{s\mathcal{C}}(Z, B)$$

is a weak equivalence (or fibration) of simplicial sets.

PROPOSITION 5.5. *If $Z$ is small, the morphism $\varepsilon_A : A \to QA$ is a weak equivalence and $QA$ is fibrant.*

PROOF: Since $Z$ is small, we have that

$$\mathbf{Hom}_{s\mathcal{C}}(Z, QA) \cong \mathrm{Ex}^\infty\,\mathbf{Hom}_{s\mathcal{C}}(Z, A).$$

The morphism $\varepsilon_A$ is a weak equivalence if and only if

$$\mathbf{Hom}_{s\mathcal{C}}(Z, A) \to \mathbf{Hom}_{s\mathcal{C}}(Z, QA) \cong \mathrm{Ex}^\infty\,\mathbf{Hom}_{s\mathcal{C}}(Z, A)$$

is a weak equivalence and $QA \to *$ is a fibration if and only if

$$\mathbf{Hom}_{s\mathcal{C}}(Z, QA) \cong \mathrm{Ex}^\infty\,\mathbf{Hom}_{s\mathcal{C}}(Z, A) \to \mathbf{Hom}_{s\mathcal{C}}(Z, *) = *$$

is a fibration. Both of these facts follow from the properties of the functor $\mathrm{Ex}^\infty(\cdot)$. $\square$

Corollary 5.6. *Let $\mathcal{C}$ be a complete and cocomplete category and $Z \in \mathcal{C}$ a small object. Then $s\mathcal{C}$ is a simplicial model category with $A \to B$ a weak equivalence (or fibration) if and only if*

$$\mathbf{Hom}_{s\mathcal{C}}(Z, A) \longrightarrow \mathbf{Hom}_{s\mathcal{C}}(Z, B)$$

*is a weak equivalence (or fibration) of simplicial sets.*

In practice one wants an intrinsic definition of weak equivalence and fibration, in the manner of the following example.

Example 5.7. All the examples of 5.2 can be recovered from Corollary 5.6. For example, $\mathcal{C}$ be the category of algebras over a commutative ring. Then $\mathcal{C}$ has a single projective generator; namely $A[x]$, the algebra on one generator. Then one sets $Z = A[x]$, which is evidently small, and one gets a closed model category structure from the previous result. However, if $B \in s\mathcal{C}$, then

$$\mathbf{Hom}_{s\mathcal{C}}(A[x], B) \cong B$$

in the category of simplicial sets, so one recovers the same closed model category structure as in Example 5.2.

If $\mathcal{C}$ is a category satisfying 4.1 with a single small projective generator, then $\mathcal{C}$ is a category of universal algebras. Setting $Z$ to be the generator, one immediately gets a closed model category structure on $s\mathcal{C}$ from Corollary 5.6. This is the case for all the examples of 5.2.

To go further, we generalize the conditions of Theorem 4.1 a little, to require the existence of a collection of functors $G_i : s\mathcal{C} \to \mathbf{S}$, $i \in I$, each of which has a left adjoint $F_i : \mathbf{S} \to s\mathcal{C}$. We now say that a morphism $f : A \to B$ of $s\mathcal{C}$ is

a) a weak equivalence if $G_i f$ is a weak equivalence of simplicial sets for all $i \in I$;

b) a fibration if all induced maps $G_i f$ are fibrations of $\mathbf{S}$;

c) a cofibration if it has the left lifting property with respect to all trivial cofibrations of $s\mathcal{C}$.

Then Theorem 4.1 and Theorem 4.4 together have the following analogue:

Theorem 5.8. *Suppose that $\mathcal{C}$ has all small limits and colimits and that all of the functors $G_i : \mathcal{C} \to \mathbf{S}$ preserve filtered colimits. Then with the notions of weak equivalence, fibration and cofibration defined above, and if every cofibration with the left lifting property with respect to all fibrations is a weak equivalence, then $s\mathcal{C}$ is a simplicial model category.*

PROOF: The proof is the same as that of Theorem 4.1, except that the small object arguments for the factorization axiom are constructed from all diagrams of the form

$$\begin{tikzcd} F_i\partial\Delta^m \arrow[r] \arrow[d] & A \arrow[d] \\ F_i\Delta^m \arrow[r] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_i\Lambda^n_k \arrow[r] \arrow[d] & A \arrow[d] \\ F_i\Delta^n \arrow[r] & B \end{tikzcd}$$

□

Theorem 5.8 will be generalized significantly in the next section — it is a special case of Theorem 6.8.

Now fix a set of small objects $Z_i \in \mathcal{C}$, $i \in I$, and regard each $Z_i$ as a constant simplicial object in $s\mathcal{C}$. Then we define a morphism $A \to B$ in $s\mathcal{C}$ to be a weak equivalence (or fibration) if and only if the induced map

$$\mathbf{Hom}_{s\mathcal{C}}(Z_i, A) \longrightarrow \mathbf{Hom}_{s\mathcal{C}}(Z_i, B)$$

is a weak equivalence (or fibration) of simplicial sets. In the case where $\mathcal{C}$ is complete and cocomplete, we are still entitled to the construction of the natural map $\epsilon_A : A \to QA$ in $s\mathcal{C}$. Furthermore, each of the objects $Z_i$ is small, so that Proposition 5.5 holds with $Z$ replaced by $Z_i$, implying that the map $\epsilon_A$ is a weak equivalence and that $QA$ is fibrant. Then an analogue of Lemma 5.1 holds for the setup of Theorem 5.8 (with $G$ replaced by $G_i$ in the proof), and we obtain the following result:

THEOREM 5.9. *Suppose that $\mathcal{C}$ is a small complete and cocomplete category, and let $Z_i \in \mathcal{C}$, $i \in I$, be a set of small objects. Then $s\mathcal{C}$ is a simplicial model category with $A \to B$ a weak equivalence (respectively fibration) if and only if the induced map*

$$\mathbf{Hom}_{s\mathcal{C}}(Z_i, A) \to \mathbf{Hom}_{s\mathcal{C}}(Z_i, B)$$

*is a weak equivalence (respectively fibration) for all $i \in I$*

EXAMPLE 5.10. Suppose that $\mathcal{C}$ is small complete and cocomplete, and has a set $\{P_\alpha\}$ of small projective generators. Theorem 5.9 implies that $\mathcal{C}$ has a simplicial model category structure, where $A \to B$ is a weak equivalence (or fibration) if

$$\mathbf{Hom}_{s\mathcal{C}}(P_\alpha, A) \to \mathbf{Hom}_{s\mathcal{C}}(P_\alpha, B)$$

is a weak equivalence (or fibration) for all $\alpha$.

Note that the requirement that the objects $P_\alpha$ are projective generators is not necessary for the existence of the closed model structure. However, if we also assume that the category $\mathcal{C}$ has sufficiently many projectives in the sense that there is an effective epimorphism $P \to C$ with $P$ projective for all objects

$C \in \mathcal{C}$, then it can be shown that a morphism $f : A \to B$ of $s\mathcal{C}$ is a weak equivalence (respectively fibration) if every induced map

$$\mathbf{Hom}_{s\mathcal{C}}(P, A) \to \mathbf{Hom}_{s\mathcal{C}}(P, B)$$

arising from a projective object $P \in \mathcal{C}$ is a weak equivalence (respectively fibration) of simplicial sets. This is a result of Quillen [76, II.4], and its proof is the origin of the stream of ideas leading to Theorem 5.9. We shall go further in this direction in the next section.

To be more specific now, let $\mathcal{C}$ be the category of graded $A$-algebras for some commutative ring $A$ and let, for $n \geq 0$,

$$P_n = A[x_n]$$

be the free graded algebra or an element of degree $n$. Then $\{P_n\}_{n\geq 0}$ form a set of projective generators for $\mathcal{C}$. Thus $s\mathcal{C}$ gets a closed model category structure and $B \to C$ in $s\mathcal{C}$ is a weak equivalence if and only if

$$(B)_n \to (C)_n$$

is a weak equivalence of simplicial sets for all $n$. Here $(\cdot)_n$ denotes the elements of degree $n$. This is equivalent to the following: if $M$ is a simplicial graded $A$-module, define

$$\pi_* M = H_*(M, \partial)$$

where $\partial$ is the alternating of the face operators. Then $B \to C$ in $s\mathcal{C}$ is a weak equivalence if and only if

$$\pi_* B \to \pi_* C$$

is an isomorphism of bigraded $A$-modules.

This formalism works for graded groups, graded abelian groups, graded $A$-modules, graded commutative algebras, graded Lie algebras, and so on.

EXAMPLE 5.11. Let $\mathbb{F}$ be a field and let $\mathcal{C} = \mathcal{CA}$ be the category of coalgebras over $\mathbb{F}$. Then, by [88, p.46] every coalgebra $C \in \mathcal{CA}$ is the filtered colimit of its finite dimensional sub-coalgebras. Thus $\mathcal{CA}$ has a set of generators $\{C_\alpha\}$ where $C_\alpha$ runs over a set of representatives for the finite dimensional coalgebras. These are evidently small. Hence, $s\mathcal{CA}$ has a closed model category structure where $A \to B$ is a weak equivalence if and only if

$$\mathbf{Hom}_{s\mathcal{CA}}(C_\alpha, A) \to \mathbf{Hom}_{s\mathcal{CA}}(C_\alpha, B)$$

is a weak equivalence for all $C_\alpha$. The significance of this example is that the $C_\alpha$ are not necessarily projective.

## 6. A generalization of Theorem 4.1.

The techniques of the previous sections are very general and accessible to vast generalization. We embark some ways on this journey here. First we expand on what it means for an object in a category to be small. Assume for simplicity that we are considering a category $\mathcal{C}$ which has all limits and colimits. We shall use the convention that a *cardinal number* is the smallest ordinal number in a given bijection class.

Fix an infinite cardinal number $\gamma$, and let $\mathbf{Seq}(\gamma)$ denote the well-ordered set of ordinals less than $\gamma$. Then $\mathbf{Seq}(\gamma)$ is a category with $\hom(s,t)$ one element if $s \leq t$ and empty otherwise. A $\gamma$-diagram in $\mathcal{C}$ is a functor $X : \mathbf{Seq}(\gamma) \to \mathcal{C}$. We will write $\varinjlim_{\gamma} X_s$ for the colimit. We shall say that $X$ is a $\gamma$-diagram of cofibrations if each of the transition morphisms $X_s \to X_t$ is a cofibration of $\mathcal{C}$.

DEFINITION 6.1. *Suppose that $\beta$ is an infinite cardinal. An object $A \in \mathcal{C}$ is $\beta$-small if for all $\gamma$-diagrams of cofibrations $X$ in $\mathcal{C}$ with $\gamma \geq \beta$, the natural map*

$$\varinjlim_{\gamma} \hom_{\mathcal{C}}(A, X_s) \to \hom_{\mathcal{C}}(A, \varinjlim_{\gamma} X_s)$$

*is an isomorphism. A morphism $A \to B$ of $\mathcal{C}$ is said to be $\beta$-small if the objects $A$ and $B$ are both $\beta$-small.*

EXAMPLE 6.2. The small objects of the previous sections were $\omega$-small, where $\omega$ is the first infinite cardinal. Compact topological spaces are also $\omega$-small, but this assertion requires proof.

Suppose that $X : \mathbf{Seq}(\gamma) \to \mathbf{Top}$ is a $\gamma$-diagram of cofibrations. Then $X$ is a retract of a $\gamma$-diagram of cofibrations $\overline{X}$, where each of the transition morphisms $\overline{X}_s \hookrightarrow \overline{X}_t$ is a relative CW-complex. In effect, set $\overline{X}_0 = X_0$, and set $\overline{X}_\alpha = \varinjlim_{s<\alpha} \overline{X}_s$ for limit ordinals $\alpha < \gamma$. Suppose given maps

$$X_s \xrightarrow{r_s} \overline{X}_s \xrightarrow{\pi_s} X_s$$

with $\pi_s r_s = 1$. Then $\overline{X}_{s+1}$ is defined by choosing a trivial fibration $\pi_{s+1}$ and a relative CW-complex map $j_{s+1} : \overline{X}_s \hookrightarrow \overline{X}_{s+1}$ (ie. $\overline{X}_{s+1}$ is obtained from $\overline{X}_s$ by attaching cells, and $j_{s+1}$ is the corresponding inclusion), such that the following diagram commutes:

$$\begin{tikzcd}
\overline{X}_s \arrow[r, "j_{s+1}"] \arrow[d, "\pi_s"] & \overline{X}_{s+1} \arrow[d, "\pi_{s+1}"] \\
X_s \arrow[r, "i_{s+1}"'] & X_{s+1},
\end{tikzcd}$$

where the map $i_{s+1} : X_s \to X_{s+1}$ is the cofibration associated to the relation $s \leq s+1$ by the functor $X$. Then there is a lifting in the diagram

$$\begin{tikzcd}
X_s \arrow[r, "j_{s+1}r_s"] \arrow[d, "i_{s+1}"'] & \overline{X}_{s+1} \arrow[d, "\pi_{s+1}"] \\
X_{s+1} \arrow[r, "1"'] \arrow[ur, "r_{s+1}"] & X_{s+1}
\end{tikzcd}$$

so that the section $r_s$ extends to a section $r_{s+1}$ of the trivial fibration $\pi_{s+1}$. The inclusion

$$X_0 = \overline{X}_0 \hookrightarrow \varinjlim_{\gamma} \overline{X}_s$$

is a relative CW-complex map, and every compact subset of the colimit only meets finitely many cells outside of $X_0$. Every compact subset of $\varinjlim_{\gamma} \overline{X}_s$ is therefore contained in some subspace $\overline{X}_s$. It follows that every compact subset of $\varinjlim_{\gamma} X_s$ is contained in some $X_s$.

We next produce an appropriate generalization of saturation.

Definition 6.3. *Suppose that $\beta$ is an infinite cardinal. A class $\mathcal{M}$ of morphisms in $\mathcal{C}$ is $\beta$-saturated if it is closed under*

1) retracts*: Suppose there is a commutative diagram in $\mathcal{C}$*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i"'] & X' \arrow[r] \arrow[d, "i'"] & X \arrow[d, "i"] \\
Y \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

   *with the horizontal composition the identity. Then if $i' \in \mathcal{M}$ , then $i \in \mathcal{M}$.*

2) coproducts*: if each $j_\alpha : X_\alpha \to Y_\alpha$ is in $\mathcal{M}$, then $\bigsqcup_\alpha j_\alpha : \bigsqcup_\alpha X_\alpha \to \bigsqcup_\alpha Y_\alpha$ is in $\mathcal{M}$;*

3) pushouts*: given a pushout diagram in $\mathcal{C}$*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "j"] \\
B \arrow[r] & Y,
\end{tikzcd}$$

   *if $i$ is in $\mathcal{M}$, then so is $j$.*

4) colimits of $\beta$-sequences*: Suppose we are given a $\beta$-sequence*

$$X : \mathbf{Seq}(\beta) \to \mathcal{C}$$

*with the following properties: a) for each successor ordinal $s+1 \in \mathbf{Seq}(\beta)$, the map $X_s \to X_{s+1}$ is in $\mathcal{M}$, and b) for each limit ordinal $s \in \mathbf{Seq}(\beta)$, the map $\varinjlim_{t<s} X_t \to X_s$ is in $\mathcal{M}$. Then*

$$X_s \to \varinjlim_{\beta} X_s$$

*is in $\mathcal{M}$ for all $s \in \mathbf{Seq}(\beta)$.*

Up until now we have considered only saturated classes of morphisms with $\beta = \omega$, the cardinality of a countable ordinal. In this case, one doesn't need the extra care required in making the definition of what it means to be closed under colimits.

LEMMA 6.4. *Let $\mathcal{C}$ be a closed model category. Then the class of cofibrations and the class of trivial cofibrations are both $\beta$-saturated for all $\beta$.*

PROOF: This is an exercise using the fact that cofibrations (or trivial cofibrations) are characterized by the fact that they have the left lifting property with respect to trivial fibrations (or fibrations). □

The next step is to turn these concepts around.

DEFINITION 6.5. *Let $\mathcal{M}_0$ be a class of morphisms in $\mathcal{C}$. Then the $\beta$-saturation of $\mathcal{M}_0$ is the smallest $\beta$-saturated class of morphisms in $\mathcal{C}$ containing $\mathcal{M}_0$.*

We now come to the crucial axiom.

DEFINITION 6.6. *A closed model category is* cofibrantly generated *with respect to a cardinal $\beta$ if the class of cofibrations and the class of trivial cofibrations are the $\beta$-saturations of* sets *of $\beta$-small morphisms $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively.*

REMARKS 6.7.

1) Suppose that $\beta$ and $\gamma$ are cardinals such that $\beta \leq \gamma$. Then every $\gamma$-saturated class is $\beta$-saturated, because every sequence $X : \mathbf{Seq}(\beta) \to \mathcal{C}$ can be extended to a sequence $X_* : \mathbf{Seq}(\gamma) \to \mathcal{C}$ having the same colimit. It follows that the $\beta$-saturation of any set of morphisms is contained in its $\gamma$-saturation. Observe also that every $\beta$-small object is $\gamma$-small, directly from Definition 6.1. The size of the cardinal $\beta$ in Definition 6.6 therefore doesn't matter, so long as it exists. One says that the closed model category $\mathcal{C}$ is *cofibrantly generated* in cases where the cardinal $\beta$ can be ignored.

2) Until now, we've taken $\beta$ to $\omega$. Then the category of simplicial sets is cofibrantly generated, for example, by the usual small object argument. Similarly, modulo the care required for the assertion that finite CW-complexes are $\omega$-small

(Example 6.2), the category of topological spaces is cofibrantly generated with respect to $\omega$. We will see larger cases later.

3) One could require one cardinal $\beta_0$ for cofibrations and $\beta_1$ for trivial cofibrations. However, $\beta = \max\{\beta_0, \beta_1\}$ would certainly work in either case, by 1).

To give the generalization of Theorem 4.1 we establish a situation. We fix a simplicial model category $\mathcal{C}$ and a simplicial category $\mathcal{D}$. Suppose we have a set of functors $G_i : \mathcal{D} \to \mathcal{C}$, indexed by the elements $i$ in some set $I$, and suppose each $G_i$ has a left adjoint $F_i$ which preserves the simplicial structure in the sense that there is a natural isomorphism

$$F_i(X \otimes K) \cong F(X_i) \otimes K$$

for all $X \in \mathcal{C}$ and $K \in \mathbf{S}$. Define a morphism $f : A \to B$ in $\mathcal{D}$ to be a weak equivalence (or fibration) is

$$G_i f : G_i A \to G_i B$$

is a weak equivalence (or fibration). A cofibration of $\mathcal{D}$ is a map which has the left lifting property with respect to all trivial fibrations.

THEOREM 6.8. *Suppose the simplicial model category $\mathcal{C}$ is cofibrantly generated with respect to a cardinal $\beta$, and that*

(1) *all of the functors $G_i$ commute with colimits over* $\mathbf{Seq}(\beta)$, *and*
(2) *the functors $G_i$ take the $\beta$-saturation of the collection of all maps $F_j A \to F_j B$ arising from maps $A \to B$ in the generating family for the cofibrations of $\mathcal{C}$ and elements $j$ of $I$ to cofibrations of $\mathcal{C}$.*

*Then if every cofibration in $\mathcal{D}$ with the left lifting property with respect to all fibrations is a weak equivalence, $\mathcal{D}$ is a simplicial model category.*

PROOF (OUTLINE): There are no new ideas — only minor changes from the arguments of Section 5. The major difference is in how the factorizations are constructed. For example, to factor $X \to Y$ as $X \xrightarrow{j} Z \xrightarrow{q} Y$ where $j$ is a cofibration which has the left lifting property with respect to all fibrations and $q$ is a fibration, one forms a $\beta$-diagram $\{Z_s\}$ in $\mathcal{D}$ where

i) $Z_0 = X$;
ii) if $s \in \mathbf{Seq}(\beta)$ is a limit ordinal, $Z_s = \varinjlim_{t<s} Z_t$ and
iii) if $s + 1$ is a successor ordinal, there is a pushout diagram

$$\begin{tikzcd}
\bigsqcup_i \bigsqcup_f F_i(A) \arrow[r] \arrow[d] & Z_s \arrow[d] \\
\bigsqcup_i \bigsqcup_f F_i(B) \arrow[r] & Z_{s+1}
\end{tikzcd}$$

where $f$ runs over all diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & G_i(Z_s) \arrow[d] \\
B \arrow[r] & G_i(Y)
\end{tikzcd}$$

where $A \to B$ is in the set $\mathcal{M}_1$ of $\beta$-small cofibrations in $\mathcal{C}$ whose $\beta$-saturation is all trivial cofibrations. □

EXAMPLE 6.9. Suppose that $\mathcal{C}$ is a cofibrantly generated simplicial model category and $I$ is a fixed small category. Write $\mathcal{C}^I$ for the category of functors $X : I \to \mathcal{C}$ and natural transformations between them. There are $i$-section functors $G_i : \mathcal{C}^I \to \mathcal{C}$ defined by $G_iX = X(i)$, $i \in I$, and each such $G_i$ has a left adjoint $F_i : \mathcal{C} \to \mathcal{C}^I$ defined by

$$F_iD(j) = \bigsqcup_{i \to j \text{ in } I} D.$$

Say that a map $X \to Y$ of $\mathcal{C}^I$ is a *pointwise cofibration* if each of the maps $G_iX \to G_iY$ is a respectively cofibration of $\mathcal{C}$. If $A \to B$ is a generating cofibration for $\mathcal{C}$, the induced maps $F_iA(j) \to F_iB(j)$ are coproducts of cofibrations and hence are cofibrations of $\mathcal{C}$. The induced maps maps $F_iA \to F_iB$ are therefore pointwise cofibrations of $\mathcal{C}^I$. The functors $G_j$ preserve all colimits, and so the collection of pointwise cofibrations of $\mathcal{C}^I$ is saturated (meaning $\beta$-saturated for some infinite cardinal $\beta$ — similar abuses follow). The saturation of the collection of maps $F_iA \to F_iB$ therefore consists of pointwise cofibrations of $\mathcal{C}^I$.

A small object argument for $\mathcal{C}^I$ produces a factorization

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "q"] \\
& Y
\end{tikzcd}$$

for an arbitrary map $f : X \to Y$ of $\mathcal{C}^I$, with $q$ a fibration, and for which $j$ is in the saturation of the collection of maps $F_iC \to F_iD$ arising from the generating set $C \to D$ for the class of trivial cofibrations of $\mathcal{C}$. But again, each induced map $F_iC(j) \to F_iD(j)$ is a trivial cofibration of $\mathcal{C}$, and the $j$-section functors preserve all colimits. The collection of maps of $\mathcal{C}^I$ which are trivial cofibrations in sections is therefore saturated, and hence contains the saturation of the maps $F_iC \to F_iD$. It follows that the map $j$ is a weak equivalence as well as a cofibration. In particular, by a standard argument, every map of $\mathcal{C}^I$ which has the left lifting property with respect to all fibrations is a trivial cofibration.

It therefore follows from Theorem 6.8 that every diagram category $\mathcal{C}^I$ taking values in a cofibrantly generated simplicial model category has a simplicial model structure for which the fibrations and weak equivalences are defined pointwise. This result applies in particular to diagram categories $\mathbf{Top}^I$ taking values in topological spaces.

Here's the analog of Lemma 5.1:

Proposition 6.10. *Suppose there is a functor $Q : \mathcal{D} \to \mathcal{D}$ so that $QX$ is fibrant for all $X$ and there is a natural weak equivalence $\epsilon_X : X \to QX$. Then every cofibration with the left lifting property with respect to all fibrations is a weak equivalence.*

Proof: The argument is similar to that of Lemma 5.1; in particular, it begins the same way.

The map $j$ has the advertised lifting property, so we may form the diagrams

$$\begin{tikzcd}
A \arrow[r, "\epsilon_A"] \arrow[d, "j"'] & QA \arrow[d] \\
B \arrow[r] \arrow[ur, "u"'] & *
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "s\epsilon_B j"] \arrow[d, "j"'] & \mathbf{hom}(\Delta^1, QB) \arrow[d] \\
B \arrow[r, "{(\epsilon_B, Qj \cdot u)}"'] \arrow[ur] & QB \times QB
\end{tikzcd}$$

where $s\epsilon_B j$ is the constant (right) homotopy on the composite

$$A \xrightarrow{j} B \xrightarrow{\epsilon_B} QB.$$

The functors $G_i$ preserve right homotopies, so the diagram

$$\begin{tikzcd}
A \arrow[r, "\epsilon_A"] \arrow[d, "j"'] & QA \arrow[d, "Qj"] \\
B \arrow[r, "\epsilon_B"'] \arrow[ur, "u"] & QB
\end{tikzcd}$$

remains homotopy commutative after applying each of the functors $G_i$. It follows that the map $G_i(j)$ is a retract of the map $G_i(\epsilon_A)$ in the homotopy category $\mathrm{Ho}(\mathcal{C})$, and is therefore an isomorphism in $\mathrm{Ho}(\mathcal{C})$. But then a map in a simplicial model category which induces an isomorphism in the associated homotopy category must itself be a weak equivalence: this is Lemma 1.14. ☐

EXAMPLE 6.11. The factorization axioms for a cofibrantly generated simplicial model category $\mathcal{C}$ can always be proved with a possibly transfinite small object argument (see, for example, the proofs of Proposition V.6.2, Lemma VIII.2.10 and Lemma X.1.8). Such arguments necessarily produce factorizations which are natural in morphisms in $\mathcal{C}$, so that there is a natural fibrant model $X \hookrightarrow \tilde{X}$ for all objects $X$ of $\mathcal{C}$. It follows that there are natural fibrant models for the objects of any diagram category $\mathcal{C}^I$ taking values in $\mathcal{C}$. We therefore obtain a variation of the proof of the existence of the closed model structure for $\mathcal{C}^I$ of Example 6.9 which uses Proposition 6.10. This means that the requirement in Theorem 6.8 that every cofibration which has the left lifting property with respect to all fibrations should be a weak equivalence is not particularly severe.

EXAMPLE 6.12. As an instance where the cofibrant generators are not $\omega$-small, we point out that in Section IX.3 we will take the category of simplicial sets, with its usual simplicial structure and impose a new closed model category structure. Let $E_*$ be any homology theory and we demand that a morphism $f : X \to Y$ in $\mathbf{S}$ be a

1) $E_*$ equivalence if $E_* f$ is an isomorphism
2) $E_*$ cofibration if $f$ is a cofibration as simplicial sets
3) $E_*$ fibration if $f$ has the right lifting property with respect to all $E_*$ trivial cofibrations.

The $E_*$ fibrant objects are the Bousfield local spaces. In this case the $E_*$-trivial cofibrations are the saturation of a set of $E_*$ trivial cofibrations $f : A \to B$ where $B$ is $\beta$-small with $\beta$ some infinite cardinal greater than the cardinality of $E_*(pt)$. One has functorial factorizations, so Example 6.11 can be repeated to show that $\mathbf{S}^I$ has a simplicial model category structure with $f : X \to Y$ a weak equivalence (or fibration) if and only if $X(i) \to Y(i)$ is an $E_*$ equivalence (or $E_*$ fibration) for all $i$.

## 7. Quillen's total derived functor theorem.

Given two closed model categories $\mathcal{C}$ and $\mathcal{D}$ and adjoint functors between them, we wish to know when these induce adjoint functors on the homotopy categories. This is Quillen's Total Derived Functor Theorem. We also give criteria under which the induced adjoint functors give an equivalence of the homotopy categories.

The main result of this section is a generalization to non-abelian settings of an old idea of Grothendieck which can be explained by the following example. If $R$ is a commutative ring and $M, N$ are two $R$-modules, one might want to compute $\operatorname{Tor}_p^R(M, N)$, $p \geq 0$. However, there is a finer invariant, namely, the chain homotopy type of $M \otimes_R P_*$ where $P_*$ is a projective resolution of $N$. One calls the chain homotopy equivalence class of $M \otimes_R P_*$ by the name $\operatorname{Tor}^R(M, N)$. This is the total derived functor. The individual Tor terms can be recovered by taking homology groups.

For simplicity we assume we are working with simplicial model categories, although many of the results are true without this assumption.

DEFINITION 7.1. *Let $\mathcal{C}$ be a simplicial model category and $\mathcal{A}$ any category. Suppose $F : \mathcal{C} \to \mathcal{A}$ is a functor that sends weak equivalences between cofibrant objects to isomorphisms. Define the total left derived functor*

$$\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathcal{A}.$$

*by $\mathbf{L}F(X) = F(Y)$ where $Y \to X$ is a trivial fibration with $Y$ cofibrant.*

It is not immediately clear that $\mathbf{L}F$ is defined on morphisms or a functor. If $f : X \to X'$ is a morphism in $\mathcal{C}$ and $Y \to X$ and $Y' \to X'$ are trivial cofibrations with $Y$ and $Y'$ cofibrant, then there is a morphism $g$ making the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{7.2}$$

and we set $\mathbf{L}F(f) = F(g)$.

LEMMA 7.3. *The objects $\mathbf{L}F(X)$ and morphisms $\mathbf{L}F(f)$ are independent of the choices and $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathcal{A}$ is a functor.*

PROOF: Note that $\mathbf{L}F(f)$ is independent of the choice of $g$ in diagram (7.2) up to isomorphism. This is because any two lifts $g$ and $g'$ are homotopic and one has

$$\begin{tikzcd} F(Y) \sqcup F(Y) \arrow[r] \arrow[dr, "F(1) \sqcup F(1)"'] & F(Y \otimes \Delta^1) \arrow[r, "FH"] \arrow[d, "\cong"] & F(Y') \\ & F(Y) & \end{tikzcd}$$

where $H$ is the homotopy. Next, if we let $f$ be the identity in (7.2), the same argument implies $\mathbf{L}F(X)$ is independent of the choice of $Y$. Finally, letting $f = f_1 \cdot f_2$ be a composite in diagram (7.2) the same argument shows $\mathbf{L}F(f_1 \cdot f_2) = \mathbf{L}F(f_1) \cdot \mathbf{L}F(f_2)$. □

REMARK 7.4. For those readers attuned to category theory we note that $\mathbf{L}F$ is in fact a Kan extension in the following sense. Let $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ be the localization functor and

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, "F"'] & \mathrm{Ho}(\mathcal{C}) \arrow[dl, dashed] \\ \mathcal{A} & \end{tikzcd}$$

the diagram of categories. There may or may not be a functor $\mathrm{Ho}(\mathcal{C}) \to \mathcal{A}$ completing the diagram; however, one can consider functors $T : \mathrm{Ho}(\mathcal{C}) \to \mathcal{A}$ equipped with a natural transformation

$$\varepsilon_T : T\gamma \to F.$$

The Kan extension is the final such functor $T$, if it exists. If $R$ denotes this Kan extension, final means that given any such $T$, there is a natural transformation $\sigma : T \to R$ so that $\varepsilon_T = \varepsilon_R \sigma\gamma$. The Kan extension is unique if it exists. To see that it exists one applies Theorem 1, p. 233 of Mac Lane's book [66]. This result, in this context reads as follows: one forms a category $X \downarrow \gamma$ consisting of pairs $(Z, f)$ where $Z \in \mathcal{C}$ and $f : X \to Z$ is a morphism in $\mathrm{Ho}(\mathcal{C})$. Then if

$$R(X) = \varprojlim_{X\downarrow\gamma} F(Z)$$

exists for all $X$, then $R$ exists. However, the argument of Lemma 6.3 says that the diagram $F : (X \downarrow \gamma) \to \mathcal{A}$ has a terminal object. In fact, $X \downarrow \gamma$ has a terminal object, namely $X \to Y$ where $Y \to X$ is a trivial fibration (which has an inverse in $\mathrm{Ho}(\mathcal{C})$) with $Y$ cofibrant. This shows that $R = \mathbf{L}F$. □

COROLLARY 7.5. *Let $X \in \mathcal{C}$. If $X$ is cofibrant, then $\mathbf{L}F(X) \cong F(X)$. If $Y \to X$ is any weak equivalence, with $Y$ cofibrant, then $\mathbf{L}FX \cong FY$.*

PROOF: The first statement is obvious, and the second follows from

$$FY \cong \mathbf{L}F(Y) \xrightarrow{\cong} \mathbf{L}F(X)$$

since $Y \to X$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. □

EXAMPLE 7.6. Let $\mathcal{C} = \mathcal{C}_*R$ be chain complexes of left modules over a ring $R$, and let $\mathcal{A} = n\mathcal{A}b$ be graded abelian groups. Define

$$F(C) = H_*(M \otimes_R C)$$

for some right module $M$. Then

$$\mathbf{L}F(C) = H_*(M \otimes_R D)$$

where $D \to C$ is a projective resolution of $\mathcal{C}$. There is a spectral sequence

$$\mathrm{Tor}_p^R(M, H_qC) \Rightarrow (\mathbf{L}F(C))_{p+q}.$$

In particular, if $H_*C = N$ concentrated in degree 0,

$$\mathbf{L}F(C) \cong \mathrm{Tor}_*^R(M, N),$$

bringing us back to what we normally mean by derived functors.

If $G : \mathcal{C} \to \mathcal{A}$ sends weak equivalences between fibrant objects to isomorphisms, one also gets a total right derived functor

$$\mathbf{R}G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{A}.$$

It is also a Kan extension, suitably interpreted: it is initial among all functors $S : \mathrm{Ho}(\mathcal{C}) \to \mathcal{A}$ equipped with a natural transformation $\eta_S : F \to S\gamma$.

Now suppose we are given two simplicial model categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $F : \mathcal{C} \to \mathcal{D}$ with a right adjoint $G$. The following is one version of the total derived functor theorem:

THEOREM 7.7. *Suppose $F$ preserves weak equivalences between cofibrant objects and $G$ preserves weak equivalences between fibrant objects. Then $\mathbf{L}F$ : $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ and $\mathbf{R}G : \mathrm{Ho}(\mathcal{D}) \to \mathrm{Ho}(\mathcal{C})$ exist, and $\mathbf{R}G$ is right adjoint to $\mathbf{L}F$.*

**Note:** This result is stronger than the original statement of Quillen [76, p.I.4.5]: there is no assumption that $F$ preserves cofibrations and $G$ preserves fibrations.

PROOF: That $\mathbf{L}F$ and $\mathbf{R}G$ exist is a consequence of Lemma 7.3 and its analog for total right derived functors. We need only prove adjointness.

If $X \in \mathcal{C}$, choose $Y \to X$ a trivial fibration with $Y$ cofibrant. Hence $\mathbf{L}F(X) \cong F(Y)$. Now choose $F(Y) \to Z$, a trivial cofibration with $Z$ fibrant. Then $\mathbf{R}G \cdot \mathbf{L}F(X) \cong G(Z)$ and one gets a unit

$$\eta : X \to \mathbf{R}G \cdot \mathbf{L}F(X)$$

by $X \leftarrow Y \to GF(Y) \to G(Z)$.

Similarly, let $A \in \mathcal{D}$. Choose $A \to B$ a trivial cofibration with $B$ fibrant. Then $\mathbf{R}G(A) \cong G(B)$. Next choose $C \to G(B)$ a trivial fibration with $C$ cofibrant. Then $\mathbf{L}F \cdot \mathbf{R}G(A) \cong F(C)$ and one gets a counit $\varepsilon : \mathbf{L}F \cdot \mathbf{R}G(A) \to A$ by

$$F(C) \to FGB \to B \leftarrow A.$$

We now wish to show

$$\mathbf{L}F(X) \xrightarrow{\mathbf{L}F\eta} \mathbf{L}F \cdot \mathbf{R}G \cdot \mathbf{L}F(X) \xrightarrow{\varepsilon_{\mathbf{R}G}} \mathbf{L}F(X)$$

is the identity. In evaluating $\varepsilon_{\mathbf{R}G}$ we set $A = \mathbf{L}F(X) = F(Y)$, so that $B = Z$. Factor the composite $Y \to GF(Y) \to G(Z)$ by

$$Y \xrightarrow{j} C \xrightarrow{q} G(Z)$$

where $j$ is a cofibration (so $C$ is cofibrant) and $q$ is a trivial fibration. Then $\varepsilon_{\mathbf{R}G}$ is given by

$$F(C) \xrightarrow{Fq} FG(Z) \to Z \leftarrow F(Y).$$

Furthermore there is a commutative square

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "j"'] & GF(Y) \arrow[d] \\
\mathcal{C} \arrow[r, "q"'] & G(Z)
\end{tikzcd}$$

and $Fj \cong \mathbf{L}F\eta_X$. Expanding the diagram gives:

$$\begin{tikzcd}
F(Y) \arrow[r] \arrow[d, "Fj"'] & FGFY \arrow[r] \arrow[d] & FY \arrow[d, "\cong"] \\
F(C) \arrow[r, "Fq"'] & FGZ \arrow[r] & Z
\end{tikzcd}$$

The line across the top is the identity and represents the composite

$$\mathbf{L}F(X) \xrightarrow{\mathbf{L}F\eta} \mathbf{L}F \cdot \mathbf{R}G \cdot \mathbf{L}F(X) \xrightarrow{\varepsilon_{\mathbf{R}G}} \mathbf{L}F(X).$$

Hence we have proved the assertion.

The other assertion — that

$$\mathbf{R}G(A) \xrightarrow{\eta_\varepsilon} \mathbf{R}G \cdot \mathbf{L}F \cdot \mathbf{R}G(A) \xrightarrow{\mathbf{R}G_\varepsilon} \mathbf{R}G(A)$$

is an isomorphism — is proved similarly. The result now holds by standard arguments; e.g., [66: Thm. 2v), p.81]. □

An immediate corollary used several times in the sequel is:

COROLLARY 7.8. *Under the hypotheses of Theorem 7.7 assume further that for $X \in \mathcal{C}$ cofibrant and $A \in \mathcal{D}$ fibrant*

$$X \to GA$$

*is a weak equivalence if and only if its adjoint $FX \to A$ is a weak equivalence. Then $\mathbf{L}F$ and $\mathbf{R}G$ induce an adjoint equivalence of categories:*

$$\mathrm{Ho}(\mathcal{C}) \cong \mathrm{Ho}(\mathcal{D}).$$

PROOF: We need to check that $\eta : X \to \mathbf{R}G \cdot \mathbf{L}F(X)$ is an isomorphism and $\varepsilon : \mathbf{L}F \cdot \mathbf{R}G(A) \to A$ is an isomorphism. Using the notation established in the previous argument, we have a sequence of arrows that define $\eta$:

$$X \leftarrow Y \to GZ.$$

Now $Y$ is cofibrant and $Z$ is fibrant, and $FY \to Z$ is a weak equivalence; so $Y \to GZ$ is a weak equivalence and this shows $\eta$ is an isomorphism. The other argument is identical. □

In practice one may not know *a priori* that $F$ and $G$ satisfy the hypotheses of Theorem 7.7. The following result is often useful. We shall assume that the model categories at hand are, in fact, simplicial model categories; however, it is possible to prove the result more generally.

LEMMA 7.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between simplicial model categories, and suppose $F$ has a right adjoint $G$. If $G$ preserves fibrations and trivial fibrations, then $F$ preserves cofibrations, trivial cofibrations and weak equivalences between cofibrant objects.*

PROOF: It follows from an adjointness argument that $F$ preserves trivial cofibrations and cofibrations; for example, suppose $j : X \to Y$ is a cofibration in $\mathcal{C}$. To show $Fj$ is a cofibration, one need only solve the lifting problem

$$\begin{tikzcd}
FX \arrow[r] \arrow[d, "Fj"'] & A \arrow[d, "q"] \\
FY \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

for every trivial fibration $q$ in $\mathcal{D}$. This problem is adjoint to

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & GA \arrow[d, "Gq"] \\
Y \arrow[r] \arrow[ur, dashed] & GB
\end{tikzcd}$$

which has a solution by hypothesis.

Now suppose $f : X \to Y$ is a weak equivalence between cofibrant objects. Factor $f$ as $X \xrightarrow{j} Z \xrightarrow{q} Y$ where $j$ is a trivial cofibration and $q$ a trivial fibration. We have just shown $Fj$ is a weak equivalence. Also $q$ is actually a homotopy equivalence: there is a map $s : Y \to Z$ so that $qs = 1_Y$ and $sq \simeq 1_Z$. Here $X$ and $Y$ are cofibrant. We claim that $Fq$ is a homotopy equivalence, so that it is a weak equivalence by Lemma 1.14.

To see that $Fq$ is a homotopy equivalence, note that $F(q)F(s) = 1_{FY}$. Next note that since $Z$ is cofibrant, $Z \otimes \Delta^1$ is a cylinder object for $Z$ and, since $F$ preserves trivial cofibrations $F(Z \otimes \Delta^1)$ is a cylinder object for $F(Z)$. Hence $F(s)F(q) \simeq 1_{FZ}$. □

REMARK 7.10. As usual, the previous result has an analog that reverses the roles of $F$ and $G$; namely, if $F$ preserves cofibrations and trivial cofibrations, then $G$ preserves fibrations, trivial fibrations, and weak equivalences between fibrant objects. The proof is the same, *mutatis mutandis.*

EXAMPLE 7.11. Let $I$ be a small category and $\mathbf{S}^I$ the category of $I$ diagrams. Then $\mathbf{S}^I$ becomes a simplicial model category, where a morphism of diagrams $X \to Y$ is a weak equivalence or fibration of $I$ diagrams if and only if each $X(i) \to Y(i)$ is a weak equivalence or fibration of simplicial sets. The constant functor $\mathbf{S} \to \mathbf{S}^I$ preserves fibrations and weak equivalences, so (by Lemma 7.9), the left adjoint

$$F = \varinjlim_I : \mathbf{S}^I \to \mathbf{S}$$

preserves weak equivalences among cofibrant diagrams. Hence the total left derived functor

$$\mathbf{L}\varinjlim_{I} : \mathrm{Ho}(\mathbf{S}^I) \to \mathrm{Ho}(\mathbf{S})$$

exists. This functor is the homotopy colimit and we write $\mathbf{L}\varinjlim_I = \underrightarrow{\mathrm{holim}}_I$. In a certain sense, made precise by the notion of Kan extensions in Remark 7.4, this is the closest approximation to colimit that passes to the homotopy category. In any application it is useful to have an explicit formula for $\underrightarrow{\mathrm{holim}}_I X$ in terms of the original diagram $X$; this is given by the coend formula

$$\underrightarrow{\mathrm{holim}}_{I} X = \int^{i} B(i \downarrow I)^{\mathrm{op}} \otimes X(i).$$

These are studied in detail elsewhere — see Chapter IV.

This example can be greatly generalized. If $\mathcal{C}$ is any cofibrantly generated simplicial model category, $\mathcal{C}^I$ becomes a simplicial model category and one gets

$$\underrightarrow{\mathrm{holim}}_{I} : \mathrm{Ho}(\mathcal{C}^I) \to \mathrm{Ho}(\mathcal{C})$$

in an analogous manner. □

## 8. Homotopy cartesian diagrams.

We return, in this last section, to concepts which are particular to the category of simplicial sets and its close relatives. The theory of homotopy cartesian diagrams of simplicial sets is, at the same time, quite deep and essentially axiomatic. The axiomatic part of the theory is valid in arbitrary categories of fibrant objects such as the category of Kan complexes, while the depth is implicit in the passage from the statements about Kan complexes to the category of simplicial sets as a whole. This passage is non-trivial, even though it is completely standard, because it involves (interchangeably) either Quillen's theorem that the realization of a Kan fibration is a Serre fibration (Theorem I.10.10) or Kan's $\mathrm{Ex}^\infty$ construction (see III.4).

A *proper closed model category* $\mathcal{C}$ is a closed model category such that

**P1** the class of weak equivalences is closed under base change by fibrations, and

**P2** the class of weak equivalences is closed under cobase change by cofibrations.

In plain English, axiom **P1** says that, given a pullback diagram

$$\begin{tikzcd} X \arrow[r, "g_*"] \arrow[d] & Y \arrow[d, "p"] \\ Z \arrow[r, "g"'] & W \end{tikzcd}$$

of $\mathcal{C}$ with $p$ a fibration, if $g$ is a weak equivalence then so is $g_*$. Dually, axiom **P2** says that, given a pushout diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d] \\
C \arrow[r, "f_*"'] & D
\end{tikzcd}$$

with $i$ a cofibration, if $f$ is a weak equivalence then so is $f_*$.

The category of simplicial sets is a canonical example of a proper closed model category (in fact, a proper simplicial model category) — see Corollary 8.6. Furthermore, this is the generic example: most useful examples of proper closed model categories inherit their structure from simplicial sets. The assertion that the category of simplicial sets satisfies the two axioms above requires proof, but this proof is in part a formal consequence of the fact that every simplicial set is cofibrant and every topological space is fibrant. The formalism itself enjoys wide applicability, and will be summarized here, now.

A *category of cofibrant objects* is a category $\mathcal{D}$ with all finite coproducts (including an initial object $\varphi$ ), with two classes of maps, called weak equivalences and cofibrations, such that the following axioms are satisfied:

**(A)** Suppose given a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\
& Z &
\end{tikzcd}$$

in $\mathcal{D}$. If any two of $f$, $g$ and $h$ are weak equivalences, then so is the third.

**(B)** The composite of two cofibrations is a cofibration. Any isomorphism is a cofibration.

**(C)** Pushout diagrams of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "i_*"] \\
C \arrow[r] & D
\end{tikzcd}$$

exist in the case where $i$ is a cofibration. Furthermore, $i_*$ is a cofibration which is trivial if $i$ is trivial.

**(D)** For any object $X$ there is at least one cylinder object $X \otimes I$.

**(E)** For any object $X$, the unique map $\emptyset \to X$ is a cofibration.

To explain, a *trivial cofibration* is a morphism of $\mathcal{D}$ which is both a cofibration and a weak equivalence. A *cylinder object* $X \otimes I$ for $X$ is a commutative diagram

$$\begin{tikzcd}
X \sqcup X \arrow[r, "\nabla"] \arrow[d, "{i=(i_0,i_1)}"'] & X \\
X \otimes I \arrow[ur, "\sigma"']
\end{tikzcd}$$

in which $i$ is a cofibration and $\sigma$ is a weak equivalence, just like in the context of a closed model category (see Section 1 above). Each of the components $i_\epsilon$ of $i$ must therefore be a trivial cofibration.

The definition of category of cofibrant objects is dual to the definition of category of fibrant objects given in Section I.9. All results about categories of fibrant objects therefore imply dual results for categories of cofibrant objects, and conversely. In particular, we immediately have the dual of one of the assertions of Proposition I.9.5:

PROPOSITION 8.1. *The full subcategory of cofibrant objects $\mathcal{C}_c$ in a closed model category $\mathcal{C}$, together with the weak equivalences and cofibrations between them, satisfies the axioms* **(A)**–**(E)** *for a category of cofibrant objects.*

REMARK 8.2. One likes to think that a category of cofibrant objects structure (respectively a category of fibrant objects structure) is half of a closed model structure. This intuition fails, however, because it neglects the power of the axiom **CM4**.

COROLLARY 8.3.

(1) *The category of simplicial sets is a category of cofibrant objects.*
(2) *The category of compactly generated Hausdorff spaces is a category of fibrant objects.*

LEMMA 8.4. *Suppose that $f : A \to B$ is an arbitrary map in a category of cofibrant objects $\mathcal{D}$. Then $f$ has a factorization $f = q \cdot j$, where $j$ is a cofibration and $q$ is left inverse to a trivial cofibration. In particular, $q$ is a weak equivalence.*

PROOF: The proof of this result is the mapping cylinder construction. It's also dual of the classical procedure for replacing a map by a fibration.

Choose a cylinder object

$$\begin{tikzcd}
A \sqcup A \arrow[r, "\nabla"] \arrow[d, "{(i_0,i_1)}"'] & A \\
A \otimes I \arrow[ur, "\sigma"']
\end{tikzcd}$$

for $A$, and form the pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & B \arrow[d, "i_{0*}"] \\ A \otimes I \arrow[r, "f_*"'] & B_*. \end{tikzcd}$$

Then $(f\sigma) \cdot i_0 = f$, and so there is a unique map $q : B_* \to B$ such that $q \cdot f_* = f\sigma$ and $q \cdot i_{0*} = 1_B$. Then $f = q \cdot (f_* i_1)$.

The composite map $f_* i_1$ is a cofibration, since the diagram

$$\begin{tikzcd} A \sqcup A \arrow[r, "f \sqcup 1_A"] \arrow[d, "{(i_0, i_1)}"'] & B \sqcup A \arrow[d, "{(i_{0*}, f_* i_1)}"] \\ A \otimes I \arrow[r, "f_*"'] & B_* \end{tikzcd}$$

is a pushout. $\square$

Lemma 8.5. *Suppose that*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & B \arrow[d] \\ C \arrow[r, "u_*"'] & D \end{tikzcd}$$

*is a pushout in a category of cofibrant objects $\mathcal{D}$, such that $i$ is a cofibration and $u$ is a weak equivalence. Then the map $u_*$ is a weak equivalence.*

Proof: Trivial cofibrations are stable under pushout, so Lemma 8.4 implies that it suffices to assume that there is a trivial cofibration $j : B \to A$ such that $u \cdot j = 1_B$.

Form the pushout diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "j"'] & A \arrow[dd, "j_*"] \\ A \arrow[d, "i"'] & \\ C \arrow[r, "\tilde{j}"'] & \tilde{D} \end{tikzcd}$$

Then $\tilde{j}$ is a trivial cofibration.

Let $f : \tilde{D} \to C$ be the unique map which is determined by the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "j"] \arrow[d, "j"'] & A \arrow[dd, "i"] \\
A \arrow[d, "i"'] & \\
C \arrow[r, "1_C"'] & C.
\end{tikzcd}$$

Form the prism

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[d, "j_*"'] \arrow[ddr, "i"] & & B \arrow[d] \arrow[ddr] & \\
\tilde{D} \arrow[rr, "\tilde{u}"] \arrow[dr, "f"'] & & B_* \arrow[dr, "f_*"] & \\
& C \arrow[rr, "u_*"'] & & D
\end{tikzcd}$$

such that the front and back faces are pushouts (ie. push out the triangle on the left along $u$). Then $\tilde{u}$ is a weak equivalence, since $\tilde{j}$ is a weak equivalence and $u \cdot j = 1_B$. It therefore suffices to show that the map $f_*$ is a weak equivalence.

The bottom face

$$\begin{tikzcd}
\tilde{D} \arrow[r, "\tilde{u}"] \arrow[d, "f"'] & B_* \arrow[d, "f_*"] \\
C \arrow[r, "u_*"'] & D
\end{tikzcd}$$

is a pushout, and the map $f$ is a weak equivalence. The morphism

$$\begin{tikzcd}
& A \arrow[dl, "j_*"'] \arrow[dr, "i"] & \\
\tilde{D} \arrow[rr, "f"'] & & C
\end{tikzcd}$$

is therefore a weak equivalence in the category $A \downarrow \mathcal{D}$, and the argument of Lemma 8.4 says that this map has a factorization in $A \downarrow \mathcal{D}$ of the form $f = q \cdot j$, where $j$ is a trivial cofibration and $q$ is left inverse to a trivial cofibration. It follows that pushing out along $u$ preserves weak equivalences of $A \downarrow \mathcal{D}$, so that $f_*$ is a weak equivalence of $\mathcal{D}$. $\square$

Corollary 8.6. *The category* **S** *of simplicial sets is a proper simplicial model category.*

Proof: Axiom **P2** is a consequence of Lemma 8.5 and Corollary 8.3. The category **CGHaus** of compactly generated Hausdorff spaces is a category of fibrant objects, so the dual of Lemma 8.5 implies Axiom **P1** for that category. One infers **P1** for the simplicial set category from the exactness of the realization functor (Proposition I.2.4), the fact that the realization functor preserves fibrations (Theorem I.10.9), and the assertion that the canonical map $\eta : X \to S|X|$ is a weak equivalence for all $X$ (see the proof of Theorem I.11.4). □

Remark 8.7. Axiom **P1** for the category of simplicial sets can alternatively be seen by observing that Kan's $\mathrm{Ex}^\infty$ preserves fibrations and pullbacks (Lemma III.4.5), and preserves weak equivalences as well (Theorem III.4.6). Thus, given a pullback diagram

$$\begin{tikzcd}
X \arrow[r, "g_*"] \arrow[d] & Y \arrow[d, "p"] \\
Z \arrow[r, "g"'] & W
\end{tikzcd}$$

with $p$ a fibration and $g$ a weak equivalence, if we want to show that $g_*$ is a weak equivalence, it suffices to show that the induced map $\mathrm{Ex}^\infty g_*$ in the pullback diagram

$$\begin{tikzcd}
\mathrm{Ex}^\infty X \arrow[r, "\mathrm{Ex}^\infty g_*"] \arrow[d] & \mathrm{Ex}^\infty Y \arrow[d, "\mathrm{Ex}^\infty p"] \\
\mathrm{Ex}^\infty Z \arrow[r, "\mathrm{Ex}^\infty g"'] & \mathrm{Ex}^\infty W
\end{tikzcd}$$

is a weak equivalence. But all of the objects in this last diagram are fibrant and the map $\mathrm{Ex}^\infty g$ is a weak equivalence, so the desired result follows from the dual of Lemma 8.5.

The following result is commonly called the *gluing lemma*. The axiomatic argument for it that is given here is due to Thomas Gunnarsson [40].

LEMMA 8.8. *Suppose given a commutative cube*

$$\begin{tikzcd}
A_1 \arrow[rr, "j_1"] \arrow[dr, "i_1"] \arrow[dd, "f_A"'] & & B_1 \arrow[dr] \arrow[dd, "f_B"] & \\
& C_1 \arrow[rr] \arrow[dd, "f_C"] & & D_1 \arrow[dd, "f_D"] \\
A_2 \arrow[rr, "j_2"] \arrow[dr, "i_2"'] & & B_2 \arrow[dr] & \\
& C_2 \arrow[rr] & & D_2
\end{tikzcd} \tag{8.9}$$

*in a category of cofibrant objects $\mathcal{D}$. Suppose further that the top and bottom faces are pushouts, that $i_1$ and $i_2$ are cofibrations, and that the maps $f_A$, $f_B$ and $f_C$ are weak equivalences. Then $f_D$ is a weak equivalence.*

PROOF: It suffices to assume that the maps $j_1$ and $j_2$ are cofibrations. To see this, use Lemma 8.4 to factorize $j_1$ and $j_2$ as cofibrations followed by weak equivalences, and then use Lemma 8.5 to analyze the resulting map of cubes.

Form the diagram

$$\begin{tikzcd}
A_1 \arrow[rr, "j_1"] \arrow[dr, "i_1"] \arrow[dddd, "f_A"'] & & B_1 \arrow[dr] \arrow[dd, "f_{A*}"] & \\
& C_1 \arrow[rr, "j_{1*}"] \arrow[dddd, "f_C"] & & D_1 \arrow[dd, "f_{C*}"] \\
& & B' \arrow[dr, "\theta"] \arrow[dd, "n_B"'] & \\
& & & D' \arrow[dd, "n_D"] \\
A_2 \arrow[uurr] \arrow[rr, "j_2"] \arrow[dr, "i_2"'] & & B_2 \arrow[dr, "i_{2*}"'] & \\
& C_2 \arrow[uurr] \arrow[rr] & & D_2.
\end{tikzcd}$$

by pushing out the top face along the left face of the cube (8.9). The square

$$\begin{tikzcd} A_2 \arrow[r] \arrow[d, "i_2"'] & B' \arrow[d, "\theta"] \\ C_2 \arrow[r] & D' \end{tikzcd}$$

is a pushout, so the square

$$\begin{tikzcd} B' \arrow[r, "n_B"] \arrow[d, "\theta"'] & B_2 \arrow[d, "i_{2*}"] \\ D' \arrow[r, "n_D"'] & D_2 \end{tikzcd}$$

is a pushout, and $\theta$ is a cofibration. The map $f_{A*}$ is a weak equivalence, since $j_1$ is a cofibration and $f_A$ is a weak equivalence. Similarly, $f_{C*}$ is a weak equivalence, since $j_{1*}$ is a cofibration and $f_C$ is a weak equivalence. The map $f_B = n_B \cdot f_{A*}$ is assumed to be a weak equivalence, so it follows that $n_B$ is a weak equivalence. Then $n_D$ is a weak equivalence, so $f_D = n_D \cdot f_{C*}$ is a weak equivalence. $\square$

The dual of Lemma 8.8 is the *cogluing lemma* for categories of fibrant objects:

Lemma 8.10. *Suppose given a commutative cube*

$$\begin{tikzcd} A_1 \arrow[rr] \arrow[dr] \arrow[dd, "f_A"'] & & B_1 \arrow[dr, "p_1"] \arrow[dd, "f_B" near start] & \\ & C_1 \arrow[rr] \arrow[dd, "f_C" near start] & & D_1 \arrow[dd, "f_D"] \\ A_2 \arrow[rr] \arrow[dr] & & B_2 \arrow[dr, "p_2"] & \\ & C_2 \arrow[rr] & & D_2 \end{tikzcd} \tag{8.11}$$

*in a category of fibrant objects $\mathcal{E}$. Suppose further that the top and bottom squares are pullbacks, that the maps $p_1$ and $p_2$ are fibrations, and that the maps $f_B$, $f_C$ and $f_D$ are weak equivalences. Then the map $f_A$ is a weak equivalence.*

The gluing lemma also holds in an arbitrary proper closed model category $\mathcal{C}$; the proof is exactly that of Lemma 8.8.

Lemma 8.12. *Let $\mathcal{C}$ be a proper closed model category. Suppose given a commutative diagram*

$$\begin{tikzcd} D_1 \arrow[d] & C_1 \arrow[l, "j_1"'] \arrow[r] \arrow[d] & X_1 \arrow[d] \\ D_2 & C_2 \arrow[l, "j_2"] \arrow[r] & X_2 \end{tikzcd}$$

*where $j_1$ and $j_2$ are cofibrations and the three vertical maps are weak equivalences. Then the map*

$$D_1 \cup_{C_1} X_1 \to D_2 \cup_{C_2} X_2$$

*is a weak equivalence.*

The dual statement is the cogluing lemma for proper closed model categories:

COROLLARY 8.13. *Suppose that $\mathcal{C}$ is a proper closed model category. Consider a diagram*

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] & Z_1 \arrow[l, "p_1"'] \arrow[d] \\
X_2 \arrow[r] & Y_2 & Z_2 \arrow[l, "p_2"]
\end{tikzcd}$$

*where the maps $p_1$ and $p_2$ are fibrations and the three vertical maps are weak equivalences. Then the induced map*

$$X_1 \times_{Y_1} Z_1 \to X_2 \times_{Y_2} Z_2$$

*is a weak equivalence.*

Corollary 8.13 is the basis for the theory of homotopy cartesian diagrams in a proper closed model category $\mathcal{C}$. We say that a commutative square of morphisms

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
W \arrow[r] & Z
\end{tikzcd} \tag{8.14}$$

is *homotopy cartesian* if for any factorization

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "i"'] & & Z \\
& \tilde{Y} \arrow[ur, "p"'] &
\end{tikzcd} \tag{8.15}$$

of $f$ into a trivial cofibration $i$ followed by a fibration $p$ the induced map

$$X \xrightarrow{i_*} W \times_Z \tilde{Y}$$

is a weak equivalence.

In fact (and this is the central point), for the diagram (8.14) to be homotopy cartesian, it suffices to find only one such factorization $f = p \cdot i$ such that the map $i_*$ is a weak equivalence. This is a consequence of the following:

Lemma 8.16. *Suppose given a commutative diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
W \arrow[r] & Z
\end{tikzcd}$$

*of morphisms in a proper closed model category $\mathcal{C}$, and factorizations*

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "i_j"'] & & Z \\
& Y_j \arrow[ur, "p_j"'] &
\end{tikzcd}$$

*of $f$ as trivial cofibration $i_j$ followed by a fibration $p_j$ for $j = 1, 2$. Then the induced map $i_{1*} : X \to W \times_Z Y_1$ is a weak equivalence if and only if the map $i_{2*} : X \to W \times_Z Y_2$ is a weak equivalence.*

Proof: There is a lifting $\theta$ in the diagram

$$\begin{tikzcd}
Y \arrow[r, "i_2"] \arrow[d, "i_1"'] & Y_2 \arrow[d, "p_2"] \\
Y_1 \arrow[r, "p_1"'] \arrow[ur, "\theta"] & Z,
\end{tikzcd}$$

by the closed model axioms. Form the commutative cube

$$\begin{tikzcd}
W \times_Z Y_1 \arrow[rr] \arrow[dr] \arrow[dd, "\theta_*"'] & & Y_1 \arrow[dr, "p_1"] \arrow[dd, "\theta"] & \\
& W \arrow[rr] \arrow[dd, "1_W"'] & & Z \arrow[dd, "1_Z"] \\
W \times_Z Y_2 \arrow[rr] \arrow[dr] & & Y_2 \arrow[dr, "p_2"] & \\
& W \arrow[rr] & & Z.
\end{tikzcd}$$

Then the map $\theta_*$ is a weak equivalence by Corollary 8.13. There is a commutative diagram

$$\begin{tikzcd}
 & W \times_Z Y_1 \arrow[dd, "\theta_*"] \\
X \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & \\
 & W \times_Z Y_2,
\end{tikzcd}$$

and the desired result follows. □

REMARK 8.17. The argument of Lemma 8.16 implies that the definition of homotopy cartesian diagrams can be relaxed further: the diagram (8.14) is homotopy cartesian if and only if there is a factorization (8.15) with $p$ a fibration and $i$ a weak equivalence, and such that the induced map $i_* : X \to W \times_Z \tilde{Y}$ is a weak equivalence.

The way that the definition of homotopy cartesian diagrams has been phrased so far says that the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
W \arrow[r, "g"'] & Z
\end{tikzcd}$$

is homotopy cartesian if a map induced by a factorization of the map $f$ into a fibration following a trivial cofibration is a weak equivalence. In fact, it doesn't matter if we factor $f$ or $g$:

LEMMA 8.18. *Suppose given a commutative diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
W \arrow[r, "g"'] & Z
\end{tikzcd}$$

*in a proper closed model category $\mathcal{C}$. Suppose also that we are given factorizations*

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "i"'] & & Z & W \arrow[rr, "g"] \arrow[dr, "j"'] & & Z \\
 & \tilde{Y} \arrow[ur, "p"'] & & & \tilde{W} \arrow[ur, "q"'] &
\end{tikzcd}$$

*of $f$ and $g$ respectively such that $i$ and $j$ are trivial cofibrations and $p$ and $q$ are fibrations. Then the induced map $i_* : X \to W \times_Z \tilde{Y}$ is a weak equivalence if and only if the map $j_* : X \to \tilde{W} \times_Z Y$ is a weak equivalence.*

PROOF: There is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_*"] \arrow[d, "j_*"'] & W \times_Z \tilde{Y} \arrow[d, "j \times 1"] \\
\tilde{W} \times_Z Y \arrow[r, "1 \times i"'] & \tilde{W} \times_Z \tilde{Y}.
\end{tikzcd}$$

The map $p$ is a fibration, so the map $j \times 1$ is a weak equivalence, and $q$ is a fibration, so $1 \times i$ is a weak equivalence, all by Corollary 8.13. ☐

The cogluing lemma also has the following general consequence for homotopy cartesian diagrams:

COROLLARY 8.19. *Suppose given a commutative cube*

$$\begin{tikzcd}
X_1 \arrow[rr] \arrow[dr] \arrow[dd, "f_X"'] & & Y_1 \arrow[dr] \arrow[dd, "f_Y"] & \\
& W_1 \arrow[rr] \arrow[dd, "f_W"'] & & Z_1 \arrow[dd, "f_Z"] \\
X_2 \arrow[rr] \arrow[dr] & & Y_2 \arrow[dr] & \\
& W_2 \arrow[rr] & & Z_2
\end{tikzcd}$$

*of morphisms in a proper closed model category $\mathcal{C}$. Suppose further that the top and bottom squares are homotopy cartesian diagrams, and that the maps $f_Y$, $f_W$ and $f_Z$ are weak equivalences. Then the map $f_X$ is a weak equivalence.*

EXAMPLE 8.20. A *homotopy fibre sequence* of simplicial sets is a homotopy cartesian diagram in **S**

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d] & Y \arrow[d, "f"] \\
* \arrow[r, "x"'] & Z.
\end{tikzcd}$$

In effect, one requires that the composite $f \cdot j$ factor through the base point $x$ of $Z$, and that if $f = p \cdot i$ is a factorization of $f$ into a trivial cofibration followed by a fibration, then the canonical map $X \to F$ is a weak equivalence, where $F$ is the fibre of $p$ over $x$. More colloquially (see also Remark 8.17), this means that $X$ has the homotopy type of the fibre $F$ of any replacement of the map $f$ by a fibration up to weak equivalence. It is common practice to abuse notation and say that

$$X \xrightarrow{j} Y \xrightarrow{f} Z$$

is a homotopy fibre sequence, and mean that these maps are a piece of a homotopy cartesian diagram as above. Every fibration sequence

$$F \to E \to B$$

is plainly a homotopy fibre sequence.

EXAMPLE 8.21. Suppose that

$$X \xrightarrow{j} Y \xrightarrow{f} Z$$

is a homotopy fibre sequence, relative to a base point $x$ of $Z$, and that there is a vertex $y \in Y$ such that $f(y) = x$. Suppose that the canonical map $Y \to *$ is a weak equivalence. Then $X$ is weakly equivalent to the loop space $\Omega\tilde{Z}$ for some (and hence any) fibrant model $\tilde{Z}$ for $Z$. To see this, choose a trivial cofibration $j : Z \to \tilde{Z}$, where $\tilde{Z}$ is a Kan complex, and use the factorization axioms to form the commutative square

$$\begin{tikzcd}
Y \arrow[r, "j"] \arrow[d, "f"'] & \tilde{Y} \arrow[d, "p"] \\
Z \arrow[r, "j"'] & \tilde{Z},
\end{tikzcd}$$

where both maps labelled $j$ are trivial cofibrations and $p$ is a fibration. Let $F$ denote the fibre of the fibration $p$ over the image of the base point $x$ in $\tilde{Z}$. Then Corollary 8.19 implies that the induced map $X \to F$ is a weak equivalence. Now consider the diagram

$$\begin{tikzcd}
\tilde{Y} \times_{\tilde{Z}} P\tilde{Z} \arrow[r, "pr_R"] \arrow[d, "pr_L"'] & P\tilde{Z} \arrow[d, "\pi"] \\
\tilde{Y} \arrow[r, "p"'] & \tilde{Z},
\end{tikzcd}$$

where $P\tilde{Z}$ is the standard path space for the Kan complex $\tilde{Z}$ and the base point $x$, and $\pi$ is the canonical fibration. Then the map $y : * \to \tilde{Y}$ is a weak equivalence, so that the inclusion $\Omega\tilde{Z} \to \tilde{Y} \times_{\tilde{Z}} P\tilde{Z}$ of the fibre of the fibration $pr_L$ is a weak equivalence, by properness, as is the inclusion $F \to \tilde{Y} \times_{\tilde{Z}} P\tilde{Z}$ of the fibre of $pr_R$. In summary, we have constructed weak equivalences

$$X \xrightarrow{\simeq} F \xrightarrow{\simeq} \tilde{Y} \times_{\tilde{Z}} P\tilde{Z} \xleftarrow{\simeq} \Omega\tilde{Z}.$$

This collection of ideas indicates that it makes sense to define the loop space of a connected simplicial set $X$ to be the loops $\Omega\tilde{X}$ of a fibrant model $\tilde{X}$ for $X$ — the loop space of $X$ is therefore an example of a total right derived functor, in the sense of Section II.7.

Here is a clutch of results that illustrates the formal similarities between homotopy cartesian diagrams and pullbacks:

LEMMA 8.22. *Suppose that $\mathcal{C}$ is a proper closed model category.*

(1) *Suppose that*

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d] \\
Z \arrow[r, "\beta"'] & W
\end{tikzcd}$$

*is a commutative diagram in $\mathcal{C}$ such that the maps $\alpha$ and $\beta$ are weak equivalences. Then the diagram is homotopy cartesian.*

(2) *Suppose given a commutative diagram*

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{II}"] & X_3 \arrow[d] \\
Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3
\end{tikzcd}$$

*in $\mathcal{C}$. Then*

(a) *if the diagrams* **I** *and* **II** *are homotopy cartesian then so is the composite diagram* $\mathbf{I}+\mathbf{II}$

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
Y_1 \arrow[r] & Y_3,
\end{tikzcd}$$

(b) *if the diagrams* $\mathbf{I}+\mathbf{II}$ *and* **II** *are homotopy cartesian, then* **I** *is homotopy cartesian.*

The proof of this lemma is left to the reader as an exercise.

We close with a further application of categories of cofibrant objects structures. Let $\mathcal{C}$ be a fixed choice of simplicial model category having an adequate supply of colimits. Suppose that $\beta$ is a limit ordinal, and say that a *cofibrant $\beta$-sequence* in $\mathcal{C}$ is a functor $X : \mathbf{Seq}(\beta) \to \mathcal{C}$, such that all objects $X_i$ are cofibrant, each map $X_i \to X_{i+1}$ is a cofibration, and $X_t = \varinjlim_{i<t} X_i$ for all limit ordinals $t < \beta$. The cofibrant $\beta$-sequences, with ordinary natural transformations between them, form a category which will be denoted by $\mathcal{C}_\beta$. Say that a map $f : X \to Y$ in $\mathcal{C}_\beta$ is a weak equivalence if all of its components $f : X_i \to Y_i$ are weak equivalences of $\mathcal{C}$, and say that $g : A \to B$ is a cofibration of $\mathcal{C}_\beta$ if the maps $g : A_i \to B_i$ are cofibrations of $\mathcal{C}$, as are all induced maps $B_i \cup_{A_i} A_{i+1} \to B_{i+1}$.

LEMMA 8.23. *Let $\mathcal{C}$ be a simplicial model category having all filtered colimits. With these definitions, the category $\mathcal{C}_\beta$ of cofibrant $\beta$-sequences in $\mathcal{C}$ satisfies the axioms for a category of cofibrant objects.*

PROOF: Suppose that $A \to B \to C$ are cofibrations of $\mathcal{C}_\beta$. To show that the composite $A \to C$ is a cofibration, observe that the canonical map $C_i \cup_{A_i} A_{i+1} \to C_{i+1}$ has a factorization

$$\begin{tikzcd}
C_i \cup_{A_i} A_{i+1} \arrow[rr] \arrow[dr] & & C_{i+1} \\
& C_i \cup_{B_i} B_{i+1} \arrow[ur] &
\end{tikzcd}$$

and there is a pushout diagram

$$\begin{tikzcd}
B_i \cup_{A_i} A_{i+1} \arrow[r] \arrow[d] & B_{i+1} \arrow[d] \\
C_i \cup_{A_i} A_{i+1} \arrow[r] & C_i \cup_{B_i} B_{i+1}.
\end{tikzcd}$$

Suppose that

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "i_*"] \\
C \arrow[r] & D
\end{tikzcd}$$

is a pushout diagram of $\mathbf{Seq}(\beta)$-diagrams in $\mathcal{C}$, where $A$, $B$ and $C$ are cofibrant $\beta$-sequences and the map $i$ is a cofibration of same. We show that $D$ is a cofibrant $\beta$-sequence and that $i_*$ is a cofibration by observing that there are pushouts

$$\begin{tikzcd}
C_i \cup_{A_i} A_{i+1} \arrow[r] \arrow[d] & D_i \cup_{B_i} B_{i+1} \arrow[d] \\
C_{i+1} \arrow[r] & D_{i+1},
\end{tikzcd}$$

and that the maps $D_i \to D_i \cup_{B_i} B_{i+1}$ are cofibrations since $B$ is a cofibrant $\beta$-sequence.

Suppose that $A$ is a cofibrant $\beta$-sequence, and let $K$ be a simplicial set. Then the functor $A \otimes K : \mathbf{Seq}(\beta) \to \mathcal{C}$ is defined by $(A \otimes K)_i = A_i \otimes K$. The functor $X \mapsto X \otimes K$ preserves cofibrations and filtered colimits of $\mathcal{C}$, so that $A \otimes K$ is a cofibrant $\beta$-sequence. Furthermore, if $K \to L$ is a cofibration of $\mathbf{S}$ then the induced map $A \otimes K \to A \otimes L$ is a cofibration of $\mathcal{C}_\beta$: the proof is an instance of **SM7**. It follows that the diagram

$$\begin{tikzcd}
A \otimes \partial\Delta^1 \arrow[r] \arrow[d] & A \otimes \Delta^0 \\
A \otimes \Delta^1 \arrow[ur]
\end{tikzcd}$$

is a candidate for the cylinder object required by the category of cofibrant objects structure for the category $\mathcal{C}_\beta$. □

Lemma 8.24. *Suppose that $\mathcal{C}$ is a simplicial model category having all filtered colimits. Suppose that $f : A \to B$ is a cofibration and a weak equivalence of $\mathcal{C}_\beta$. Then the induced map*

$$f_* : \varinjlim_{i<\beta} A_i \to \varinjlim_{i<\beta} B_i$$

*is a trivial cofibration of $\mathcal{C}$.*

Proof: Suppose given a diagram

$$\begin{tikzcd}
\varinjlim_{i<\beta} A_i \arrow[r] \arrow[d, "f_*"'] & X \arrow[d, "p"] \\
\varinjlim_{i<\beta} B_i \arrow[r] & Y
\end{tikzcd}$$

where $p$ is a fibration of $\mathcal{C}$. We construct a compatible family of lifts

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "f_i"'] & X \arrow[d, "p"] \\ B_i \arrow[r] \arrow[ur, "\theta_i"] & Y \end{tikzcd} \tag{8.25}$$

as follows:

1) Let $\theta_s$ be the map induced by all $\theta_i$ for $i < s$ at limit ordinals $s < \beta$.
2) Given a lifting $\theta_i$ as in diagram (8.25), form the induced diagram

$$\begin{tikzcd} B_i \cup_{A_i} A_{i+1} \arrow[r, "\theta_*"] \arrow[d, "f_*"'] & X \arrow[d, "p"] \\ B_{i+1} \arrow[r] \arrow[ur, dashed, "\theta_{i+1}"] & Y. \end{tikzcd}$$

The map $f_*$ is a trivial cofibration of $\mathcal{C}$, since $f$ is a cofibration and a weak equivalence of $\mathcal{C}_\beta$, so the indicated lift $\theta_{i+1}$ exists. □

COROLLARY 8.26. *Suppose that $\mathcal{C}$ is a simplicial model category having all filtered colimits, and that $f : X \to Y$ is a weak equivalence of cofibrant $\beta$-sequences in $\mathcal{C}$. Then the induced map*

$$f_* : \varinjlim_{i<\beta} X_i \to \varinjlim_{i<\beta} Y_i$$

*is a weak equivalence of $\mathcal{C}$.*

PROOF: We have it from Lemma 8.23 that $\mathcal{C}_\beta$ is a category of cofibrant objects, and Lemma 8.4 says that $f : X \to Y$ has a factorization $f = q \cdot j$, where $j$ is a cofibration and $q$ is left inverse to a trivial cofibration. Then $j$ is a trivial cofibration since $f$ is a weak equivalence, and so Lemma 8.24 implies that both $j$ and $p$ induce weak equivalences after taking filtered colimits. □

The dual assertion for Corollary 8.26 is entertaining. Suppose again that $\beta$ is a limit ordinal and that $\mathcal{C}$ is a simplicial model category having enough filtered inverse limits. Define a *fibrant $\beta$-tower* in $\mathcal{C}$ to be (contravariant) functor $X : \mathbf{Seq}(\beta)^{op} \to \mathcal{C}$ such that each $X_i$ is a fibrant object of $\mathcal{C}$, each map $X_{i+1} \to X_i$ is a fibration of $\mathcal{C}$, and $X_t = \varprojlim_{i<t} X_i$ for all limit ordinals $t < \beta$. Then the dual of Lemma 8.23 asserts that, for pointwise weak equivalences and a suitable definition of fibration, the category of fibrant $\beta$-towers in $\mathcal{C}$ has a category of fibrant objects structure. The dual of Lemma 8.24 asserts that the inverse limit functor takes trivial fibrations of fibrant $\beta$-towers to trivial fibrations of $\mathcal{C}$, and then we have

LEMMA 8.27. *Suppose that $\mathcal{C}$ is a simplicial model category having all filtered inverse limits, and that $f : X \to Y$ is a weak equivalence of fibrant $\beta$-towers in $\mathcal{C}$. Then the induced map*

$$f_* : \varprojlim_{i<\beta} X_i \to \varprojlim_{i<\beta} Y_i$$

*is a weak equivalence of $\mathcal{C}$.*

For fibrant $\beta$-towers $X : \mathbf{Seq}(\beta)^{op} \to \mathbf{S}$ taking values in simplicial sets, one can take a different point of view, in a different language. In that case, fibrant $\beta$-towers are globally fibrant $\mathbf{Seq}(\beta)^{op}$-diagrams, and inverse limits and homotopy inverse limits coincide up to weak equivalence for globally fibrant diagrams, for all $\beta$. Homotopy inverse limits preserve weak equivalences, so inverse limits preserve weak equivalences of fibrant $\beta$-towers. Homotopy inverse limits and homotopy theories for categories of diagrams will be discussed in Chapters 6 and 7.

# Chapter III Classical results and constructions

This chapter is a rather disparate collection of stories, most of them old and well known, but told from a modern point of view.

The first section contains several equivalent descriptions of the fundamental groupoid, one of which (ie. left adjoint to a classifying space functor) is powerful enough to show that the fundamental groupoid of a classifying space of a small category is the free groupoid on that category (Corollary 1.2), as well as prove the Van Kampen theorem (Theorem 1.4).

The second section, on simplicial abelian groups, contains a complete development of the Dold-Kan correspondence. This correspondence is an equivalence of categories between chain complexes and simplicial abelian groups; the result appears as Corollary 2.3. We also give an elementary description of the proper simplicial model structure for the category of simplicial abelian groups (Theorem 2.8, Proposition 2.13, and Remark 2.14), and then use this structure to derive the standard isomorphism

$$H^n(X, A) \cong [X, K(A, n)]$$

relating cohomology to homotopy classes of maps which take values in an Eilenberg-Mac Lane space (Theorem 2.19). We close Section 2 by showing that every simplicial abelian group is non-canonically a product of Eilenberg-Mac Lane spaces up to homotopy equivalence (Proposition 2.20).

We have included Section 3, on the Hurewicz theorem, as further evidence for the assertion that many results in the Algebraic Topology canon have very clean simplicial homotopy proofs. We use Postnikov towers and the Serre spectral sequence, both of which appear here for the first time in the book and are described more fully in later chapters. The Hurewicz homomorphism itself has a very satisfying functorial description in this context: it is the adjunction map $X \to \mathbb{Z}X$ from a simplicial set $X$ to the corresponding free abelian simplicial group $\mathbb{Z}X$.

Section 4 contains a modernized treatment of Kan's $\mathrm{Ex}^\infty$ functor. This functor gives a combinatorial (even intuitionistic), natural way of mapping a simplicial set into a Kan complex, via a weak equivalence. The construction is therefore preserved by left exact functors which have right adjoints — these appear throughout topos theory [38]. We have seen similar applications already in Chapter II in connection with detecting simplicial model structures on categories of simplicial objects. The main theorem here is Theorem 4.8. The proof is an updated version of the original: the fundamental groupoid trick in Lemma 4.2 may be new, but the heart of the matter is Lemma 4.7.

There are two different suspension functors for a pointed simplicial set $X$, namely the smash product $S^1 \wedge X$ (where $S^1 = \Delta^1/\partial\Delta^1$ is the simplicial circle), and the Kan suspension $\Sigma X$. These are homotopy equivalent but not isomorphic constructions which have naturally homeomorphic realizations.

They further represent two of the standard subdivisions for the suspension of a pointed simplicial complex. Both have their uses; in particular, the Kan suspension is more easily related to the classifying space of a simplicial group which appears in Chapter V, and hence to Eilenberg-Mac Lane spectra [52]. We give a full treatment of the Kan suspension in Section 5, essentially to have it "in the bank" for later. Along the way, we say formally what it means for a simplicial set to have an extra degeneracy. This last idea has been in the folklore for a long time — it means, most succinctly, that the identity map on a simplicial set factors through a cone.

## 1. The fundamental groupoid, revisited.

Recall from Section I.8 that the classical fundamental groupoid $\pi|X|$ of the realization of a simplicial set $X$ coincides with the groupoid $\pi_f S|X|$ associated to the singular complex $S|X|$. In that section, there is a remark to the effect that this groupoid is equivalent to the free groupoids $GP_*X$ and $G(\mathbf{\Delta} \downarrow X)$ which are associated, respectively, to the path category $P_*X$ and the simplex category $\mathbf{\Delta} \downarrow X$ for $X$. This claim has the following precise form:

THEOREM 1.1. *The groupoids $G(\mathbf{\Delta} \downarrow X)$, $GP_*X$ and $\pi|X|$ are naturally equivalent as categories.*

PROOF: A functor $f : G \to H$ between groupoids is an equivalence if and only if

(1) the induced function $f : \hom_G(a, b) \to \hom_H(f(a), f(b))$ is a bijection for every pair of objects $a, b$ of $G$, and
(2) for every object $c$ of $H$ there is a morphism $c \to f(a)$ in $H$.

The groupoids $GP_*S|X|$ and $\pi|X|$ are naturally isomorphic. The 1-simplices of $S|X|$ are paths of $|X|$, and $[d_1\sigma] = [d_0\sigma] \cdot [d_2\sigma]$ for every 2-simplex $\sigma : |\Delta^2| \to |X|$ of $S|X|$. It follows that there is a functor $\gamma_X : GP_*S|X| \to \pi|X|$ which is defined by sending a path to its associated homotopy class. The inverse of $\gamma_X$ is constructed by observing that homotopic paths in $|X|$ represent the same element of $GP_*S|X|$.

The next step is to show that the functor $GP_*$ takes weak equivalences of simplicial sets to equivalences of groupoids. If $f$ is a weak equivalence of $\mathbf{S}$, then $f$ has a factorization $f = q \cdot j$ where $q$ is a trivial fibration and $j$ is a trivial cofibration. The map $q$ is left inverse to a trivial cofibration, and every trivial cofibration is a retract of a map which is a filtered colimit of pushouts of maps of the form $\Lambda^n_k \subset \Delta^n$. It suffices, therefore to show that $GP_*$ takes pushouts of maps $\Lambda^n_k \subset \Delta^n$ to equivalences of categories.

The induced map $GP_*(\Lambda^n_k) \to GP_*(\Delta^n)$ is an isomorphism of groupoids if $n \geq 2$. The groupoid $GP_*(\Lambda^1_i)$ is a strong deformation retract of $GP_*(\Delta^1)$ in the category of groupoids for $i = 0, 1$. Isomorphisms and strong deformation retractions of groupoids are closed under pushout, and so $GP_*$ takes weak equivalences of simplicial sets to equivalences of groupoids as claimed. It follows in particular that the groupoids $GP_*X$ and $\pi|X|$ are naturally equivalent.

We can assign, to each 1-simplex $x : d_1x \to d_0x$, the morphism $(d^0)^{-1}(d^1)$ of $G(\mathbf{\Delta} \downarrow X)$ arising from the diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, "d^1"] \arrow[dr, "d_1x"'] & \Delta^1 \arrow[d, "x"'] & \Delta^0 \arrow[l, "d^0"'] \arrow[dl, "d_0x"] \\
 & X. &
\end{tikzcd}$$

This assignment defines a functor $\omega_X : GP_*X \to G(\mathbf{\Delta} \downarrow X)$, since the following diagram of simplicial maps commutes:

$$\begin{tikzcd}
\Delta^0 \arrow[rr, "d^0"] \arrow[ddr, "d^1"'] & & \Delta^1 \arrow[d, "d^2"] & & \Delta^0 \arrow[ll, "d^1"'] \arrow[ddl, "d^1"] \\
 & & \Delta^2 & & \\
 & \Delta^1 \arrow[ur, "d^0"] & & \Delta^1 \arrow[ul, "d^1"'] & \\
 & & \Delta^0 \arrow[ul, "d^0"] \arrow[ur, "d^0"'] & &
\end{tikzcd}$$

Let $v_n : \Delta^0 \to \Delta^n$ denote the simplicial map which picks out the last vertex $n$ of the ordinal number $\mathbf{n}$. Then the assignment

$$\Delta^n \xrightarrow{\sigma} X \mapsto \Delta^0 \xrightarrow{v_n} \Delta^n \xrightarrow{\sigma} X$$

is the object function of a functor $\mathbf{\Delta} \downarrow X \to P_*X$. Write $\nu_X : G(\mathbf{\Delta} \downarrow X) \to GP_*X$ for the induced functor on associated groupoids. Then the composite functor $\nu_X \cdot \omega_X$ is the identity on $GP_*X$, and the composite $\omega_X \cdot \nu_X$ is naturally isomorphic to the identity on $G(\mathbf{\Delta} \downarrow X)$. The natural isomorphism is determined by the maps

$$\begin{tikzcd}
\Delta^0 \arrow[r, "\sigma \cdot v_n"] \arrow[d, "v_n"'] & X \\
\Delta^n \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

of the simplex category $\mathbf{\Delta} \downarrow X$. $\square$

From now on, the fundamental groupoid of a simplicial set $X$, in any of its forms, will be denoted by $\pi X$.

A simplicial set map $f : X \to BC$ associates to each $n$-simplex $x$ of $X$ a functor $f(x) : \mathbf{n} \to C$ which is completely determined by the 1-skeleton of $x$ and the composition relations arising from 2-simplices of the form

$$\Delta^2 \to \Delta^n \xrightarrow{x} X.$$

It follows that $f$ can be identified with the graph morphism

$$\begin{tikzcd}
X_1 \arrow[r, "f"] \arrow[d, shift right=2] \arrow[d, shift left=2] & \mathrm{Mor}(C) \arrow[d, shift right=2] \arrow[d, shift left=2] \\
X_0 \arrow[r, "f"'] \arrow[u] & \mathrm{Ob}(C) \arrow[u]
\end{tikzcd}$$

subject to the relations $f(d_1\sigma) = f(d_0\sigma) \cdot f(d_2\sigma)$ arising from all 2-simplices $\sigma$ of $X$. The path category $P_*X$ is the category which is freely associated to the graph

$$\begin{tikzcd}
X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & X_0 \arrow[l]
\end{tikzcd}$$

and the 2-simplex relations, so that there is an adjunction isomorphism

$$\hom_{\mathbf{cat}}(P_*X, C) \cong \hom_{\mathbf{S}}(X, BC).$$

A small category $C$ is completely determined by its set of arrows and composition relations. It follows that the path category $P_*BC$ of the nerve $BC$ is isomorphic to $C$. We therefore immediately obtain the following assertion:

Corollary 1.2. *Let $C$ be a small category. Then the fundamental groupoid $\pi BC$ of the nerve of $C$ is equivalent, as a category, to the free groupoid $GC$ on the category $C$.*

Corollary 1.2 may be used to give a direct proof of the fact that the fundamental group $\pi_1 BQ\mathcal{A}$ of the nerve of the Q-construction $Q\mathcal{A}$ on an exact category $\mathcal{A}$ is isomorphic to the $0^{th}$ $K$-group $K_0\mathcal{A}$ of $\mathcal{A}$ (see [48]).

Remark 1.3. In the proof of Theorem 1.1, observe that the composite

$$S|X| \to BP_*S|X| \to BGP_*S|X| \xrightarrow{\gamma_{X*}} B\pi|X|$$

sends a path $\alpha : \Delta^1 \to S|X|$ of $|X|$ to its homotopy class $[\alpha]$ (rel $\partial\Delta^1$). It follows that the composite

$$S|X| \to BP_*S|X| \to BGP_*S|X|$$

induces isomomorphisms $\pi_1(S|X|,x) \cong \pi_1(BGP_*S|X|,x)$ for all choices of vertex $x \in S|X|$. We have also seen that the functor $GP_*$ takes weak equivalences of simplicial sets to equivalences of groupoids, so that the commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\eta"'] & BP_*X \arrow[r] \arrow[d, "\eta_*"'] & BGP_*X \arrow[d, "\eta_*"] \\
S|X| \arrow[r] & BP_*S|X| \arrow[r] & BGP_*S|X|
\end{tikzcd}$$

implies that the composite

$$X \to BP_*X \to BGP_*X$$

induces isomorphisms

$$\pi_1(X,x) \cong \pi_1(BGP_*X,x) \cong \hom_{GP_*X}(x,x)$$

for any choice of vertex $x \in X$, if $X$ is a Kan complex.

Note, in particular, that if $X$ is connected, then there is a deformation retraction map of groupoids $r : GP_*X \to \hom_{GP_*X}(x,x)$ for any choice of vertex of $X$. This map $r$ determines a composite map

$$X \to BP_*X \to BGP_*X \xrightarrow{r_*} B(\hom_{GP_*X}(x,x)),$$

which induces isomorphisms

$$\pi_1(X,y) \cong \pi_1(B\hom_{GP_*X}(x,x)) \cong \hom_{GP_*X}(x,x)$$

for all vertices $y$ of $X$. It follows that there is a map

$$X \to B(\pi_1(X,x))$$

which induces isomorphisms on all fundamental groups if $X$ is a connected Kan complex with base point $x$.

Again, the fundamental groupoid construction takes weak equivalences of simplicial sets to equivalences of groupoids. It follows, in particular, that there is a purely categorical definition of the fundamental group $\pi_1(X,x)$ of an arbitrary simplicial set $X$ at a vertex $x$, given by

$$\pi_1(X,x) = \hom_{\pi X}(x,x).$$

This definition can be used, along with the observation that the fundamental groupoid construction $\pi = GP_*$ has a right adjoint and therefore preserves colimits, to give a rather short proof of the Van Kampen theorem:

Theorem 1.4 (Van Kampen). *Suppose that*

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "j"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

*is a pushout diagram of simplicial sets, where the map $j$ is a cofibration, and $A$, $B$ and $X$ are connected. Then, for any vertex $x$ of $A$, the induced diagram*

$$\begin{tikzcd} \pi_1(A,x) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(X,x) \arrow[d] \\ \pi_1(B,x) \arrow[r] & \pi_1(Y,x) \end{tikzcd}$$

*is a pushout in the category of groups.*

Proof: The glueing lemma II.8.8 implies that we can presume that the map $i$ is also a cofibration.

The induced maps $i_* : \pi A \to \pi X$ and $j_* : \pi A \to \pi B$ of fundamental groupoids are monomorphisms on objects, and so the strong deformation $r : \pi A \to \pi_1(A,x)$, by suitable choice of paths in $X$ and $B$, can be extended to a strong deformation

$$\begin{tikzcd} \pi B \arrow[d, "r"'] & \pi A \arrow[l, "i_*"'] \arrow[r, "j_*"] \arrow[d, "r"'] & \pi X \arrow[d, "r"] \\ \pi_1(B,x) & \pi_1(A,x) \arrow[l, "i_*"] \arrow[r, "j_*"'] & \pi_1(X,x) \end{tikzcd}$$

in a suitable diagram category in groupoids, meaning that the homotopy $h : \pi A \to \pi A^{\mathbf{1}}$ giving the deformation extends to a homotopy

$$\begin{tikzcd} \pi B \arrow[d, "h"'] & \pi A \arrow[l, "i_*"'] \arrow[r, "j_*"] \arrow[d, "h"'] & \pi X \arrow[d, "h"] \\ \pi B^{\mathbf{1}} & \pi A^{\mathbf{1}} \arrow[l, "i_*"] \arrow[r, "j_*"'] & \pi X^{\mathbf{1}} \end{tikzcd}$$

But then, if

$$\begin{tikzcd}
\pi_1(A,x) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(X,x) \arrow[d] \\
\pi_1(B,x) \arrow[r] & G
\end{tikzcd}$$

is a pushout in the category of groups, it is also a pushout in the category of groupoids, and so the group $G$ is a strong deformation retraction of the groupoid $\pi Y$, since the diagram

$$\begin{tikzcd}
\pi A \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi X \arrow[d] \\
\pi B \arrow[r] & \pi Y
\end{tikzcd}$$

is a pushout in groupoids. The group $G$ is therefore isomorphic to $\pi_1(Y,x)$. □

## 2. Simplicial abelian groups.

Suppose that $A$ is a simplicial abelian group, and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n.$$

The maps

$$NA_n \xrightarrow{(-1)^n d_n} NA_{n-1}$$

form a chain complex, on account of the simplicial identity

$$d_{n-1}d_n = d_{n-1}d_{n-1}.$$

Denote the corresponding chain complex by $NA$; this is the *normalized chain complex* associated to the simplicial abelian group $A$. The assignment $A \mapsto NA$ is plainly a functor from the category $s\mathbf{Ab}$ of simplicial abelian groups to the category $\mathbf{Ch}_+$ of chain complexes.

The *Moore complex* of a simplicial abelian group $A$ has the group $A_n$ of $n$-simplices of $A$ as $n$-chains, and has boundary $\partial : A_n \to A_{n-1}$ defined by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$

Of course, one has to verify that $\partial^2 = 0$, but this is a consequence of the simplicial identities. The notation $A$ will be used for the second purpose of denoting the Moore complex of the simplicial abelian group $A$ — this could be confusing, but it almost never is.

Let $DA_n$ denote the subgroup of $A_n$ which is generated by the degenerate simplices. The boundary map $\partial$ of the Moore complex associated to $A$ induces a homomorphism

$$\partial : A_n/DA_n \to A_{n-1}/DA_{n-1}.$$

The resulting chain complex will be denoted by $A/D(A)$, meaning "$A$ modulo degeneracies". One sees directly from the definitions that there are chain maps

$$NA \xrightarrow{i} A \xrightarrow{p} A/D(A),$$

where $i$ is the obvious inclusion and $p$ is the canonical projection.

THEOREM 2.1. *The composite*

$$NA \xrightarrow{p \cdot i} A/D(A)$$

*is an isomorphism of chain complexes.*

PROOF: Write

$$N_j A_n = \bigcap_{i=0}^{j} \ker(d_i) \subset A_n,$$

and let $D_j(A_n)$ be the subgroup of $A_n$ which is generated by the images of the degeneracies $s_i$ for $i \leq j$. One shows that the composite

$$N_j A_n \hookrightarrow A_n \xrightarrow{p} A_n/D_j(A_n)$$

is an isomorphism for all $n$ and $j < n$. Let $\phi$ denote this composite.

The claim is proved by induction on $j$. Here is the case $j = 0$: any class $[x] \in A_n/s_0(A_{n-1})$ is represented by $x - s_0 d_0 x$, and $d_0(x - s_0 d_0 x) = 0$, so $\phi$ is onto; if $d_0 x = 0$ and $x = s_0 y$, then

$$0 = d_0 x = d_0 s_0 y = y,$$

so $x = 0$.

Suppose that the map

$$\phi : N_k A_m \to A_m/D_k(A_m)$$

is known to be an isomorphism if $k < j$, and consider the map

$$\phi : N_j A_n \to A_n/D_j(A_n).$$

Form the commutative diagram

$$\begin{tikzcd}
N_{j-1}A_n \arrow[r, "\phi", "\cong"'] & A_n/D_{j-1}(A_n) \arrow[d, two heads] \\
N_jA_n \arrow[u, hook] \arrow[r, "\phi"'] & A_n/D_j(A_n).
\end{tikzcd}$$

On account of the displayed isomorphism, any class $[x] \in A_n/D_j(A_n)$ can be represented by an element $x \in N_{j-1}A_n$. But then $x - s_jd_jx$ is in $N_jA_n$ and represents $[x]$, so the bottom map $\phi$ in the diagram is onto. The simplicial identities imply that the degeneracy $s_j : A_{n-1} \to A_n$ maps $N_{j-1}A_{n-1}$ into $N_{j-1}A_n$, and takes $D_{j-1}(A_{n-1})$ to $D_{j-1}(A_n)$, and so there is a commutative diagram

$$\begin{tikzcd}
N_{j-1}A_{n-1} \arrow[r, "\phi", "\cong"'] \arrow[d, "s_j"'] & A_{n-1}/D_{j-1}(A_{n-1}) \arrow[d, "s_j"] \\
N_{j-1}A_n \arrow[r, "\cong", "\phi"'] & A_n/D_{j-1}(A_n).
\end{tikzcd}$$

Furthermore, the sequence

$$0 \to A_{n-1}/D_{j-1}(A_{n-1}) \xrightarrow{s_j} A_n/D_{j-1}(A_n) \to A_n/D_j(A_n) \to 0$$

is exact. Thus, if $\phi(x) = 0$ for some $x \in N_jA_n$, then $x = s_jy$ for some $y \in N_{j-1}A_n$. But (again), $d_jx = 0$, so that

$$0 = d_jx = d_js_jy = y,$$

so that $x = 0$, and our map is injective. □

Every simplicial structure map $d^* : A_n \to A_m$ corresponding to a monomorphism $d : \mathbf{m} \hookrightarrow \mathbf{n}$ of ordinal numbers takes $NA_n$ into $NA_m$. In fact, such maps are 0 unless $d$ is of the form $d = d^n : \mathbf{n-1} \to \mathbf{n}$. Put a different way, suppose given a collection of abelian group homomorphisms

$$\partial : C_n \to C_{n-1}, \;\; n \geq 0.$$

Associate to each ordinal number $\mathbf{n}$ the group $C_n$, and map each ordinal number monomorphism to an abelian group homomorphism by the rule

$$d \mapsto \begin{cases} 0 & \text{if } d \text{ is not some } d^n\text{, and} \\ C_n \xrightarrow{(-1)^n\partial} C_{n-1} & \text{if } d = d^n. \end{cases}$$

Then we get a contravariant functor on the category of ordinal number morphisms from such an assignment if and only if we started with a chain complex.

There is a simplicial abelian group whose $n$-simplices have the form

$$\bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} NA_k.$$

The map

$$\theta^* : \bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} NA_k \to \bigoplus_{\mathbf{m} \twoheadrightarrow \mathbf{r}} NA_r$$

associated to the ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ is given on the summand corresponding to $\sigma : \mathbf{n} \to \mathbf{k}$ by the composite

$$NA_k \xrightarrow{d^*} NA_s \xrightarrow{in_t} \bigoplus_{\mathbf{m} \twoheadrightarrow \mathbf{r}} NA_r,$$

where

$$\mathbf{m} \overset{t}{\twoheadrightarrow} \mathbf{s} \overset{d}{\hookrightarrow} \mathbf{k}$$

is the epi-monic factorization of the composite

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \overset{\sigma}{\twoheadrightarrow} \mathbf{k}.$$

Note as well that there is a morphism of simplicial abelian groups which is given in degree $n$ by the map

$$\Psi : \bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} NA_k \to A_n,$$

which is given at the summand corresponding to $\sigma : \mathbf{n} \to \mathbf{k}$ by the composite

$$NA_k \hookrightarrow A_k \xrightarrow{\sigma^*} A_n.$$

PROPOSITION 2.2. *The map $\Psi$ is a natural isomorphism of simplicial abelian groups.*

PROOF: An induction on the degree $n$ starts with the observation that $NA_0 = A_0$, and that there's only one map from the ordinal number $\mathbf{0}$ to itself. Suppose that $\Psi$ is known to be an isomorphism in degrees less than $n$. Then any degeneracy $s_j x \in A_n$ is in the image of $\Psi$, because $x$ is in the image of $\Psi$ in degree $n-1$. On the other hand, $\Psi$ induces an isomorphism of normalized complexes, so $\Psi$ is epi in degree $n$ by Theorem 2.1.

Suppose that $(x_\sigma)$ maps to 0 under $\Psi$, where $x_\sigma$ is the component of $(x_\sigma)$ which corresponds to $\sigma : \mathbf{n} \twoheadrightarrow \mathbf{k}$. If $k < n$, then $\sigma$ has a section $d : \mathbf{k} \to \mathbf{n}$, and the component of $d^*(x_\sigma)$ which corresponds to the identity on $\mathbf{k}$ is $x_\sigma$. But

$\Psi d^*(x_\sigma) = 0$, so $d^*(x_\sigma) = 0$ by the inductive hypothesis, and so $x_\sigma = 0$ in $NA_k$. Thus, $x_\sigma = 0$ for $\sigma : \mathbf{n} \twoheadrightarrow \mathbf{k}$ with $k < n$. The remaining component is $x_{1_\mathbf{n}} \in NA_n$, but the restriction of $\Psi$ to $NA_n$ is the inclusion $NA_n \hookrightarrow A_n$, so that $x_{1_\mathbf{n}} = 0$ as well. □

We have implicitly defined a functor

$$\Gamma : \mathbf{Ch}_+ \to s\mathbf{Ab}$$

from chain complexes to simplicial abelian groups, with

$$\Gamma(C)_n = \bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} C_k$$

for a chain complex $C$, and with simplicial structure maps given by the recipe above. The following result is now clear from the work that we have done:

COROLLARY 2.3 (DOLD-KAN CORRESPONDENCE). *The functors*

$$N : s\mathbf{Ab} \to \mathbf{Ch}_+ \quad \textit{and} \quad \Gamma : \mathbf{Ch}_+ \to s\mathbf{Ab}$$

*form an equivalence of categories.*

PROOF: The natural isomorphism $\Gamma N(A) \cong A$ is Proposition 2.2, and the natural isomorphism $N\Gamma(C) \cong C$ can be derived from Theorem 2.1 by collapsing $\Gamma(C)$ by degeneracies. □

There is a subcomplex $N_jA$ of the Moore complex $A$, which is defined for $j \geq 0$ by

$$N_jA_n = \begin{cases} \bigcap_{i=0}^{j} \ker(d_i) & \text{for } n \geq j+2, \\ NA_n & \text{for } n \leq j+1. \end{cases}$$

To see that these groups form a subcomplex, one has to verify that given $x \in N_jA_n$ with $n \geq j+2$, then

$$d_k(\sum_{i=j+1}^{n} (-1)^i d_i(x)) = 0$$

if $k \leq j$. This is a consequence of the simplicial identities $d_k d_i = d_{i-1} d_k$ that hold for $i > k$.

Set $N_{-1}A = A$. Observe that $N_{j+1}A \subset N_jA$, and that $NA = \cap_{j \geq 0} N_jA$. Let $i_j$ denote the inclusion of $N_{j+1}A$ into $N_jA$.

Now define abelian group homomorphisms $f_j : N_jA_n \to N_{j+1}A_n$ by specifying that

$$f_j(x) = \begin{cases} x - s_{j+1}d_{j+1}(x) & \text{if } n \geq j+2, \text{ and} \\ x & \text{if } n \leq j+1. \end{cases}$$

One has to check that $f_j$ takes values in $N_{j+1}A_n$, but this is a simplicial identity argument. The simplicial identities also imply that the collection of maps $f_j$ defines a chain map $f_j : N_jA \to N_{j+1}A$. The composite $f_j \cdot i_j$ is the identity on the chain complex $N_{j+1}A$.

Now define an abelian group homomorphisms $t_j : N_jA_n \to N_jA_{n+1}$ by

$$t_j(x) = \begin{cases} (-1)^j s_{j+1} & \text{if } n \geq j+1 \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

Then a little more calculating shows that

$$1 - i_j f_j = \partial t_j + t_j \partial : N_jA_n \to N_jA_n$$

in all degrees $n$, and that both sides of the equation are 0 in degrees $n \leq j+1$.

It follows that the composite abelian group homomorphisms

$$A_n = N_0A_n \xrightarrow{f_0} N_1A_n \xrightarrow{f_1} N_2A_n \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} N_{n-1}A_n = NA_n$$

define a chain map $f : A \to NA$ such that $f \cdot i : NA \to NA$ is the identity. The collection of homomorphisms $T : A_n \to A_{n+1}$ defined by

$$T = i_0 \cdots i_{n-2} t_{n-1} f_{n-2} \cdots f_0 + i_0 \cdots i_{n-3} t_{n-2} f_{n-3} \cdots f_0 + \cdots + i_0 t_1 f_0 + t_0$$

specifies a chain homotopy $i \cdot f \simeq 1_A$.

The chain maps $i$, $f$ and the chain homotopy $T$ are natural with respect to morphisms of simplicial abelian groups $A$. We have proved

THEOREM 2.4. *The inclusion $i : NA \to A$ of the normalized chain complex in the Moore complex of a simplicial abelian group $A$ is a chain homotopy equivalence. This equivalence is natural with respect to simplicial abelian groups $A$.*

The development starting at Theorem 2.1 and finishing at Theorem 2.4 can be generalized to the category $s\mathcal{A}$ of simplicial objects in an abelian category $\mathcal{A}$. Let $\mathbf{Ch}_+(\mathcal{A})$ denote th category of chain complexes in $\mathcal{A}$. There is a normalization functor $N$ and a Moore complex functor $s\mathcal{A} \to \mathbf{Ch}_+(A)$ defined by analogy with the construction for simplicial abelian groups, as well as a functor $\Gamma : \mathbf{Ch}_+(\mathcal{A}) \to s\mathcal{A}$. The degeneracy subobject $DA_n$ is defined for a simplicial object $A$ to be the image of the map

$$\bigoplus_{i=0}^{n-1} A_{n-1} \xrightarrow{s} A_n$$

defined by adding up the degeneracy maps taking values in $A_n$.

THEOREM 2.5. *Suppose that $\mathcal{A}$ is an abelian category, and let $A$ be a simplicial object in $\mathcal{A}$. Then we have the following:*

(1) *The objects $DA_n$, $n \geq 0$, define a subcomplex of the Moore complex for $A$, and the composition*

$$NA \to A \to A/DA$$

*is an isomorphism.*

(2) *The functors $N : s\mathcal{A} \to \mathbf{Ch}_+(\mathcal{A})$ and $\Gamma : \mathbf{Ch}_+(\mathcal{A}) \to s\mathcal{A}$ define an equivalence of categories.*

(3) *The inclusion $NA \to A$ is a natural chain homotopy equivalence.*

PROOF: This can be checked directly, by replacing the element chases in the proofs of Theorem 2.1, Corollary 2.3 and Theorem 2.4 by "class" chases in the sense of Mac Lane [66, p.200]. □

There is a further detail that is used and requires independent proof, namely that the degenerate objects $DA_n$ for a simplicial object $A$ in an abelian category should be expressible in a form that one expects from simplicial sets.

LEMMA 2.6. *Suppose that $A$ is a simplicial object in an abelian category $\mathcal{A}$, and let $DA_n \subset A_n$ be the degenerate part of the object of $n$-simplices $A_n$. Then there is a coequalizer*

$$\bigoplus_{0\leq i<j\leq n-1} A_{n-2} \rightrightarrows \bigoplus_{i=0}^{n-1} A_{n-1} \to D_nA$$

*where for $i < j$ the restrictions of the two displayed maps $f, g$ to $A_{n-2}$ are given, respectively, by the composites*

$$A_{n-2} \xrightarrow{s_i} A_{n-1} \xrightarrow{in_j} \bigoplus_{i=0}^{n-1} A_{n-1}$$

*and*

$$A_{n-2} \xrightarrow{s_{j-1}} A_{n-1} \xrightarrow{in_i} \bigoplus_{i=0}^{n-1} A_{n-1}.$$

PROOF: The proof ultimately devolves onto the fact that, given subobjects $C$ and $D$ of an object $E$ in an abelian category $\mathcal{A}$, the diagram

$$\begin{tikzcd} C \cap D \arrow[r] \arrow[d] & D \arrow[d] \\ C \arrow[r] & C + D \end{tikzcd}$$

is a pushout, where $C + D$ is the image of the map $C \oplus D \to E$. This is an elementary exercise in manipulating exact sequences.

The statement of the lemma amounts to the requirement that any collection of morphisms $f_i : A_{n-1} \to B$, $0 \leq i \leq n-1$, with $f_j s_i = f_i s_{j-1}$ for $i < j$ together determine a map

$$f : \bigoplus_{0 \leq i \leq n-1} A_{n-1} \to B$$

which factors through a morphism $f_* : DA_n \to B$.

Write $D^{(r)} A_n$ for the image of the map

$$s : \bigoplus_{0 \leq i \leq r} A_{n-1} \to A_n$$

which is determined by the degeneracies $s_i$ with $0 \leq i \leq r$. Suppose that every collection of maps $f_i : A_{n-1} \to B$ with $0 \leq i \leq r$ satisfying $f_j s_i = f_i s_{j-1}$ for $i < j \leq r$ determines a unique map $f_* : D^{(r)} A_n \to B$. Then there is a commutative diagram

$$\begin{tikzcd}
\bigoplus_{0 \leq i \leq r} A_{n-2} \arrow[r, "\oplus s_r"] \arrow[d] & \bigoplus_{0 \leq i \leq r} A_{n-1} \arrow[d] \\
D^{(r)} A_{n-1} \arrow[r] \arrow[d] & D^{(r)} A_n \arrow[d] \\
A_{n-1} \arrow[r, "s_{r+1}"'] & A_n
\end{tikzcd}$$

in which both squares are pullbacks by the simplicial identities. Given a compatible collection of maps $f_i : A_{n-1} \to B$, $0 \leq i \leq r+1$, there is a unique induced map $f_* : D^{(r)} A_n \to B$ which restricts to all $f_i$, $0 \leq i \leq r$. The morphisms $f_*$ and $f_{r+1}$ restrict to the same map on $D^{(r)} A_{n-1}$, and hence determine a unique map

$$D^{(r+1)} A_n \cong A_{n-1} + D^{(r)} A_{n-1} \to B. \qquad \square$$

Suppose that $A$ is a simplicial abelian group. Then there is an induced abelian group structure on the set

$$\pi_n(A, 0) = [(\Delta^n, \partial \Delta^n), (A, 0)]$$

of homotopy classes of pairs of maps, which structure satisfies an interchange law with respect to the standard group structure for $\pi_n(A,0)$. It follows that the homotopy group structure and the induced abelian group structure coincide. In particular, there is a natural isomorphism

$$\pi_n(A,0) \cong H_n(NA)$$

for $n \geq 0$. Theorem 2.4 therefore immediately implies the following:

COROLLARY 2.7. *Suppose that $A$ is a simplicial abelian group. Then there are isomorphisms*

$$\pi_n(A,0) \cong H_n(NA) \cong H_n(A),$$

*where $H_n(A)$ is the $n^{th}$ homology group of the Moore complex associated to $A$. These isomorphisms are natural in simplicial abelian groups $A$.*

The group $A_0$ of vertices of $A$ acts on the simplicial set underlying $A$, via the composite

$$A_0 \times A \xrightarrow{c \times 1} A \times A \xrightarrow{+} A,$$

where $c : A_0 \to A$ is the simplicial abelian group homomorphism given by inclusion of vertices, and $+$ is the abelian group structure on $A$. It follows that multiplication by a vertex $a$ induces an isomorphism of homotopy groups

$$\pi_n(A,0) \xrightarrow[\cong]{+a_*} \pi_n(A,a)$$

for $n \geq 0$. It also follows that a homomorphism $f : A \to B$ of simplicial abelian groups is a weak equivalence of the underlying Kan complexes if and only if $f$ induces a homology isomorphism (or quasi-isomorphism) $f_* : NA \to NB$ of the associated normalized chain complexes. Equivalently, $f$ is a weak equivalence if and only if the induced map $f : A \to B$ of Moore complexes is a homology isomorphism.

Say that a simplicial abelian group homomorphism $f : A \to B$ is a *weak equivalence* if $f$ is a weak equivalence of the underlying Kan complexes. We say that $f$ is a *fibration* if the underlying simplicial set map is a Kan fibration. Finally, *cofibrations* in the category $s\mathbf{Ab}$ of simplicial abelian groups are morphisms which have the left lifting property with respect to all maps which are fibrations and weak equivalences.

THEOREM 2.8. *With these definitions, the category $s\mathbf{Ab}$ of simplicial abelian groups satisfies the axioms for a closed model category.*

This result is a consequence of Theorem II.4.1 — see also Remark II.5.2. One can also argue directly as follows:

PROOF: The limit axiom **CM1**, the weak equivalence axiom **CM2** and the retraction axiom **CM3** are easy to verify.

A map $p : A \to B$ of simplicial abelian groups is a fibration if and only if it has the right lifting property with respect to all morphisms $i : \mathbb{Z}\Lambda^n_k \to \mathbb{Z}\Delta^n$ induced by the inclusions $\Lambda^n_k \subset \Delta^n$. The simplicial abelian group $\mathbb{Z}\Lambda^n_k$ is a degreewise direct summand of $\mathbb{Z}\Delta^n$. Note as well that each of these maps $i$ is a weak equivalence, because the underlying inclusions are weak equivalences and therefore integral homology isomorphisms. A small object argument therefore implies that any map $f : C \to D$ of simplicial abelian groups has a factorization

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "j"'] & & D \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

such that $p$ is a fibration and $j$ is a morphism which has the left lifting property with respect to all fibrations, is a weak equivalence, and is a monomorphism in each degree. In particular, $j$ is a cofibration and a weak equivalence, and so the corresponding factorization axiom is verified.

Similarly, a map $p : A \to B$ is a fibration and a weak equivalence if and only if it has the right lifting property with respect to all morphisms $\mathbb{Z}\partial\Delta^n \to \mathbb{Z}\Delta^n$ induced by the inclusions $\partial\Delta^n \subset \Delta^n$. It follows again by a small object argument that any map $f : C \to D$ has a factorization

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "i"'] & & D \\ & \overline{E} \arrow[ur, "q"'] & \end{tikzcd}$$

such that $q$ is a fibration and a weak equivalence, and such that $i$ is a cofibration and a levelwise monomorphism. We have therefore completely verified the factorization axiom **CM5**.

By standard nonsense, any map $\alpha$ which is a cofibration and a weak equivalence is a retract of a map of the form $j$ in the proof of the factorization axiom, so that $\alpha$ has the left lifting property with respect to all fibrations. This implies the lifting axiom **CM4**. $\square$

REMARK 2.9. It is a corollary of the proof of Theorem 2.8 that all cofibrations of $s\mathbf{Ab}$ are levelwise monomorphisms.

LEMMA 2.10. *Suppose that $f : A \to B$ is a homomorphism of simplicial abelian groups which is surjective in all degrees. Then $f$ is a fibration of simplicial abelian groups.*

PROOF: Suppose given a commutative diagram of simplicial set maps

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & A \arrow[d, "f"] \\
\Delta^n \arrow[r, "\beta"'] & B.
\end{tikzcd}$$

Then, by the assumptions, there is a simplex $\theta \in A_n$ such that $f(\theta) = \beta$. But then

$$\theta|_{\Lambda^n_k} - \alpha : \Lambda^n_k \to A$$

factors through the kernel $K$ of $f$, and so $\theta|_{\Lambda^n_k} - \alpha$ extends to an $n$-simplex $x$ of $K$ by Lemma I.3.4, in the sense that there is a commutative diagram of simplicial set maps

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\theta|_{\Lambda^n_k} - \alpha"] \arrow[d, hook] & K \\
\Delta^n \arrow[ur, "x"'] &
\end{tikzcd}$$

Then $(\theta - x)|_{\Lambda^n_k} = \alpha$ and $f(\theta - x) = \beta$. □

LEMMA 2.11.

(1) *A homomorphism $f : A \to B$ of simplicial abelian groups is surjective (in all degrees) if and only if the associated chain complex map $f : NA \to NB$ is surjective in all degrees.*

(2) *The homomorphism $f : A \to B$ of simplicial abelian groups is a fibration if and only if the induced abelian group maps $f : NA_n \to NB_n$ are surjective for $n \geq 1$.*

PROOF: For (1), note that the map $f : NA \to NB$ of normalized chain complexes is a retract of the map $f : A \to B$ of Moore complexes by Theorem 2.1. Thus, if the simplicial abelian group homomorphism $f : A \to B$ is surjective in all degrees, then so is the associated map of normalized chain complexes.

Conversely, one shows that if $f : N_{j+1}A \to N_{j+1}B$ is surjective in all degrees, then the abelian group homomorphisms $f : N_jA_n \to N_jB_n$ are surjective for all $n \geq 0$. This is proved by induction on $n$. Take $x \in N_jB_n$. Then $x - s_{j+1}d_{j+1}x$ is in $N_{j+1}B_n$ and so is in the image of $f : N_{j+1}A_n \to N_{j+1}B_n$. Also, $d_{j+1}x \in N_jB_{n-1}$ and is therefore in the image of $f : N_jA_{n-1} \to N_jB_{n-1}$ by the inductive assumption. It follows that $x$ is in the image of $f : N_jA_n \to N_jB_n$.

For (2), suppose that $f : A \to B$ is a fibration of simplicial abelian groups. Then the existence of solutions to the lifting problems of the form

$$\begin{tikzcd}
\Lambda^n_n \arrow[r, "0"] \arrow[d, hook] & A \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

implies that $f : NA_n \to NB_n$ is surjective for $n \geq 1$.

Conversely, suppose that $f : NA_n \to NB_n$ is surjective in non-zero degrees. Form the diagram

$$\begin{tikzcd}
A \arrow[dr, "\theta"] \arrow[drr, bend left, "f"] \arrow[ddr] & & \\
& K(\pi_0 A, 0) \times_{K(\pi_0 B, 0)} B \arrow[r, "pr"'] \arrow[d] & B \arrow[d] \\
& K(\pi_0 A, 0) \arrow[r, "f_*"'] & K(\pi_0 B, 0),
\end{tikzcd}$$

where (for example) $K(\pi_0 A, 0)$ is the constant simplicial abelian group on the abelian group $\pi_0 A$. The hypotheses imply that applying the normalization functor to the map $\theta$ gives a surjective chain map

$$\theta : NA \to NK(\pi_0 A, 0) \times_{NK(\pi_0 B, 0)} NB \cong \pi_0 A[0] \times_{\pi_0 B[0]} NB,$$

and so (1) implies that the simplicial abelian group map

$$\theta : A \to K(\pi_0 A, 0) \times_{K(\pi_0 B, 0)} B$$

is surjective in all degrees and is therefore a fibration by Lemma 2.10. The map $f_* : K(\pi_0 A, 0) \to K(\pi_0 B, 0)$ is a fibration, so that $pr$ is a fibration. It follows that $f = pr \cdot \theta$ is a fibration. $\square$

COROLLARY 2.12. *The homomorphism $f : A \to B$ is a trivial fibration of simplicial abelian groups if and only if the induced morphism $f : NA \to NB$ of normalized chain complexes is surjective in all degrees, with acyclic kernel.*

It follows that the category $\mathbf{Ch}_+$ of chain complexes of abelian groups inherits a closed model structure from the simplicial abelian group category, in which the fibrations are the chain maps $f : C \to D$ such that $f$ is surjective in degree $n$ for $n \geq 1$, and where the weak equivalences are the *quasi-isomorphisms*, or rather the maps which induce isomorphisms in all homology groups. The cofibrations of $\mathbf{Ch}_+$ are those maps which have the left lifting property with respect to all trivial fibrations. After the fact, it turns out that the cofibrations are those monomorphisms of chain complexes having degreewise projective cokernels.

One can, alternatively, give a direct proof of the existence of this closed model structure on the chain complex category $\mathbf{Ch}_+$. The proof of the factorization axioms is a small object argument which is based on some rather elementary constructions. Explicitly, let $\mathbb{Z}[n]$ be the chain complex consisting of a copy of the integers $\mathbb{Z}$ in degree $n$ and 0 elsewhere, and let $\mathbb{Z}\langle n+1\rangle$ be the chain complex

$$\cdots \to 0 \to \overset{n+1}{\mathbb{Z}} \xrightarrow{\partial=1} \overset{n}{\mathbb{Z}} \to 0 \to \cdots .$$

Then maps $\mathbb{Z}[n] \to C$ classify $n$-cycles of $C$, and $\mathbb{Z}\langle n+1\rangle$ is the free chain complex on an element of degree $n+1$. Write $x$ for the generator of $\mathbb{Z}[n]$ in degree $n$ and write $y$ for the generator of $\mathbb{Z}\langle n+1\rangle$ in degree $n+1$. There is a canonical map $j : \mathbb{Z}[n] \to \mathbb{Z}\langle n+1\rangle$ which is defined by $j(x) = \partial(y)$. Then $f : C \to D$ is a fibration if and only if $f$ has the right lifting property with respect to all chain maps $0 \to \mathbb{Z}\langle n+1\rangle$ for $n \geq 0$. Further, one can show that $f$ is a trivial fibration if and only if $f : C_0 \to D_0$ is surjective and $f$ has the right lifting property with respect to all maps $j : \mathbb{Z}[n] \to \mathbb{Z}\langle n+1\rangle$, $n \geq 0$.

The chain complex category $\mathbf{Ch}_+$ has a natural cylinder construction. For any chain complex $C$, there is a chain complex $C^I$ with

$$C^I_n = \begin{cases} C_n \oplus C_n \oplus C_{n+1} & \text{if } n \geq 1, \text{ and} \\ \{(x,y,z) \in C_0 \oplus C_0 \oplus C_1 | (x-y) + \partial z = 0\} & \text{if } n = 0. \end{cases}$$

and with

$$\partial(x,y,z) = (\partial x, \partial y, (-1)^n(x-y) + \partial z).$$

Then there is a commutative diagram of chain maps

$$\begin{tikzcd} & C^I \arrow[d, "p"] \\ C \arrow[ur, "s"] \arrow[r, "\Delta"'] & C \oplus C, \end{tikzcd}$$

where $p$ is the fibration defined by $p(x,y,z) = (x,y)$ and $s$ is a weak equivalence which is defined by $s(x) = (x,x,0)$. It is an exercise to show that there is a homotopy $h : D \to C^I$ from $f$ to $g$ if and only if the maps $f, g : C \to D$ are chain homotopic.

PROPOSITION 2.13. *The category of simplicial abelian groups admits a simplicial model structure.*

PROOF: If $K$ is a simplicial set and $A$ is a simplicial abelian group, then there is a simplicial abelian group $A \otimes K$, which is defined by

$$A \otimes K = A \otimes \mathbb{Z}K,$$

where (in this case) $\mathbb{Z}K$ denotes the free simplicial abelian group associated to $K$. Equivalently, on the level of $n$-simplices, there is a canonical isomorphism

$$A \otimes K = \bigoplus_{\sigma \in K_n} A_n,$$

with simplicial structure maps induced by the corresponding maps for $A$ and the simplicial set $K$. Dually, the simplicial abelian group structure on $A$ induces a simplicial abelian group structure on the simplicial function space $\mathbf{Hom}(K, A)$. Finally, for simplicial abelian groups $A$ and $B$, one defines the simplicial set $\mathbf{Hom}_{s\mathbf{Ab}}(A, B)$ to have $n$-simplices given by the set (actually abelian group) of simplicial abelian group homomorphisms $A \otimes \Delta^n \to B$. Then there are natural isomorphisms

$$\begin{aligned} \mathbf{Hom}_{s\mathbf{Ab}}(A \otimes K, B) &\cong \mathbf{Hom}(K, \mathbf{Hom}_{s\mathbf{Ab}}(A, B) \\ &\cong \mathbf{Hom}_{s\mathbf{Ab}}(A, \mathbf{Hom}(K, B)). \end{aligned}$$

The first of these isomorphisms follows from the exponential law

$$\hom_{s\mathbf{Ab}}(A \otimes K, B) \cong \hom_{\mathbf{S}}(K, \mathbf{Hom}_{s\mathbf{Ab}}(A, B)),$$

which itself is a specialization of the simplicial set exponential law. The second follows from the definition of $\mathbf{Hom}_{s\mathbf{Ab}}(A, B)$, together with the observation that the simplicial set $K$ is a colimit of its simplices. If $f : A \to B$ is a fibration of simplicial abelian groups and $i : K \hookrightarrow L$ is a cofibration of simplicial sets, then the induced map

$$\mathbf{Hom}(L, A) \xrightarrow{(i^*, f_*)} \mathbf{Hom}(K, A) \times_{\mathbf{Hom}(K,B)} \mathbf{Hom}(L, B)$$

is a fibration of simplicial abelian groups which is trivial if either $i$ or $f$ is trivial, by remembering that the underlying simplicial set map has the same properties. This is Quillen's axiom **SM7(a)**, whence the simplicial model structure on $s\mathbf{Ab}$. □

Proposition 2.13 can also be proved by appealing to Theorem II.5.4.

Remark 2.14. The category of simplicial abelian groups is also a *proper* simplicial model category (see Section II.8). In particular,

(1) weak equivalences are stable under pullback along fibrations, and

(2) the pushout of a weak equivalence along a cofibration is a weak equivalence.

The first requirement means that, in the pullback diagram

$$\begin{tikzcd}
C \times_B A \arrow[r, "g_*"] \arrow[d] & A \arrow[d, "p"] \\
C \arrow[r, "g"'] & B,
\end{tikzcd}$$

if $p$ is a fibration and $g$ is a weak equivalence, then $g_*$ is a weak equivalence. This follows from the corresponding property for the category of simplicial sets.

The second requirement says that, given a pushout diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\
B \arrow[r, "f_*"'] & B \cup_A C,
\end{tikzcd}$$

if $i$ is a cofibration and $f$ is a weak equivalence, then $f_*$ is a weak equivalence. But of course $i$ and $i_*$ are monomorphisms which have the same cokernel, and so a comparison of long exact sequences in homology shows that $f_*$ is a homology isomorphism.

The simplicial model structure on the simplicial abelian group category gives rise to a canonical choice of cylinder for cofibrant objects $A$. In effect, tensoring such an $A$ with the commutative simplicial set diagram

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r] \arrow[d, hook, "i"'] & \Delta^0 \\
\Delta^1 \arrow[ur] &
\end{tikzcd}$$

gives a natural diagram of simplicial abelian group homomorphisms of the form

$$\begin{tikzcd} A \oplus A \arrow[r, "\nabla"] \arrow[d, "i_* = (d^0, d^1)"'] & A \\ A \otimes \Delta^1 \arrow[ur, "s"'] & \end{tikzcd}$$

The map $i_*$ coincides up to isomorphism with the map

$$1 \otimes i : A \otimes \partial\Delta^1 \to A \otimes \Delta^1,$$

while the map $d^0$ coincides with

$$1 \otimes d^0 : A \otimes \Delta^0 \to A \otimes \Delta^1.$$

Since $A$ is cofibrant, the simplicial model structure implies that $1 \otimes i$ is a cofibration and $1 \otimes d^0$ is a trivial cofibration. It follows that the map $s$ is a weak equivalence.

The free simplicial abelian group $\mathbb{Z}X$ on a simplicial set $X$ is a cofibrant simplicial abelian group, and there is a natural isomorphism $\mathbb{Z}X \otimes K \cong \mathbb{Z}(X \times K)$. It follows that an ordinary simplicial homotopy $X \times \Delta^1 \to B$ taking values in a simplicial abelian group $B$ induces a homotopy $\mathbb{Z}X \otimes \Delta^1 \to B$ for the cylinder $\mathbb{Z}X \otimes \Delta^1$. Since $\mathbb{Z}X$ is cofibrant and $B$ is fibrant, the existence of a simplicial homotopy $X \times \Delta^1 \to B$ between maps $f$ and $g$ is equivalent to the existence of a chain homotopy between the induced maps $f_*, g_* : \mathbb{Z}X \to B$ of Moore complexes: the induced maps $Nf_*, Ng_* : N\mathbb{Z}X \to NB$ must be chain homotopic by formal nonsense, and so the induced maps of Moore complexes must be chain homotopic by Theorem 2.4. There is a classical alternative method of seeing this point, based on the following

Lemma 2.15. *Any "homotopy" $h : A \otimes \Delta^1 \to B$ from $f : A \to B$ to $g : A \to B$ gives rise to a chain homotopy between the associated maps of Moore complexes.*

Proof: Let $\theta_j : \mathbf{n} \to \mathbf{1}$ be the ordinal number morphism such that $\theta_j(i) = 0$ if and only if $i \leq j$. Now, given the map $h$, define abelian group homomorphisms $h_j : A_n \to B_{n+1}$, $0 \leq j \leq n$, by specifying that

$$h_j(a) = h(s_j(a) \otimes \theta_j).$$

Then, as a consequence of the simplicial identities and the relations

$$d_i(\theta_j) = \begin{cases} \theta_{j-1} & \text{if } i \leq j, \\ \theta_j & \text{if } i > j, \end{cases}$$

and

$$s_i(\theta_j) = \begin{cases} \theta_{j+1} & \text{if } i \leq j, \\ \theta_j & \text{if } i > j \end{cases}$$

in the simplicial set $\Delta^1$, one finds the relations

$$\begin{aligned} &d_0 h_0 = f, \\ &d_{n+1} h_n = g, \\ &d_i h_j = h_{j-1} d_i && \text{if } i < j, \\ &d_{j+1} h_j = d_{j+1} h_{j+1}, \\ &d_i h_j = h_j d_{i-1} && \text{if } i > j+1, \\ &s_i h_j = h_{j+1} s_i && \text{if } i \leq j, \text{ and} \\ &s_i h_j = h_j s_{i-1} && \text{if } i > j. \end{aligned}$$

It's now straightforward to verify that the collection of alternating sums

$$s = \sum_{i=0}^{n} (-1)^i h_i : A_n \to B_{n+1}$$

forms an explicit chain homotopy between the Moore complex maps $f$ and $g$. □

It follows that every simplicial abelian group homomorphism $A \otimes \Delta^1 \to B$ gives rise to a chain homotopy of maps $NA \to NB$ between the associated normalized complexes. The converse is far from clear, unless $A$ is cofibrant.

The following result establishes the relation between weak equivalence and homology isomorphism:

PROPOSITION 2.16. *The free abelian simplicial group functor $X \mapsto \mathbb{Z}X$ preserves weak equivalences.*

PROOF: Any weak equivalence $f : X \to Y$ can be factored $f = q \cdot j$, where $j$ is a trivial cofibration and $q$ has a section by a trivial cofibration. It therefore suffices to show that the free abelian group functor takes trivial cofibrations to weak equivalences of simplicial abelian groups. But every fibration of simplicial abelian groups is a fibration of simplicial sets, so the free abelian group functor takes trivial cofibrations of simplicial sets to trivial cofibrations of simplicial abelian groups. □

Write $\pi(\mathbb{Z}X, A)$ to denote homotopy classes of maps between the named objects in the simplicial abelian group category, computed with respect to the cylinder object $\mathbb{Z}X \otimes \Delta^1$.

The free abelian simplicial group functor $X \mapsto \mathbb{Z}X$ and the inclusion functor $i : s\mathbf{Ab} \subset \mathbf{S}$ both preserve weak equivalences. These functors are also adjoint. It follows that they induce corresponding functors $i : \mathrm{Ho}(s\mathbf{Ab}) \to \mathrm{Ho}(\mathbf{S})$ and $\mathbb{Z} : \mathrm{Ho}(\mathbf{S}) \to \mathrm{Ho}(s\mathbf{Ab})$, and that these functors are adjoint.

One way of seeing this (see also Section II.7, or Brown's "adjoint functor lemma" [15, p. 426]) begins with the observation that there is a composite

$$[X, A] \to [\mathbb{Z}X, \mathbb{Z}A]_{s\mathbf{Ab}} \xrightarrow{\epsilon_*} [\mathbb{Z}X, A]_{s\mathbf{Ab}},$$

where $[\ ,\ ]_{s\mathbf{Ab}}$ denotes morphisms in the homotopy category $\mathrm{Ho}(s\mathbf{Ab})$. The first function is induced by the free simplicial abelian group functor, and $\epsilon : \mathbb{Z}A \to A$ is one of the adjunction maps. The corresponding composite

$$\pi(X, A) \to \pi(\mathbb{Z}X, \mathbb{Z}A) \xrightarrow{\epsilon_*} \pi(\mathbb{Z}X, A)$$

is an isomorphism, since $\mathbb{Z}(X \times \Delta^1) \cong \mathbb{Z}X \otimes \Delta^1$. All sources are cofibrant and all targets are fibrant, so there is a commutative diagram

$$\begin{tikzcd}
\pi(X, A) \arrow[r] \arrow[d, "\cong"'] & \pi(\mathbb{Z}X, \mathbb{Z}A) \arrow[r, "\epsilon_*"] \arrow[d, "\cong"'] & \pi(\mathbb{Z}X, A) \arrow[d, "\cong"] \\
{[X, A]} \arrow[r] & {[\mathbb{Z}X, \mathbb{Z}A]} \arrow[r, "\epsilon_*"'] & {[\mathbb{Z}X, A]}
\end{tikzcd}$$

in which all vertical maps are canonical bijections. It follows that the bottom horizontal composite is a bijection.

Now observe that there is a commutative diagram of isomorphisms

$$\begin{tikzcd}
\pi(\mathbb{Z}X, A) \arrow[d, "N"', "\cong"] \arrow[rrd, dashed] & & \\
\pi_{\mathbf{Ch}_+}(N\mathbb{Z}X, NA) & \pi_{\mathbf{Ch}_+}(\mathbb{Z}X, NA) \arrow[l, "\cong", "i^*"'] \arrow[r, "\cong", "i_*"'] & \pi_{\mathbf{Ch}_+}(\mathbb{Z}X, A)
\end{tikzcd} \tag{2.17}$$

in which the map labelled by $N$ is induced by the normalization functor, and is well defined because $\mathbb{Z}X$ is cofibrant and $A$ is fibrant, and the other labelled maps are induced by the chain homotopy equivalences $i : NA \subset A$ and $i : N\mathbb{Z}X \subset \mathbb{Z}X$. The dotted arrow takes the homotopy class of a simplicial abelian group map $f : \mathbb{Z}X \to A$ to the chain homotopy class which is represented by the map of Moore complexes induced by $f$. In particular, we have proved

PROPOSITION 2.18. *Suppose that $X$ is a simplicial set and $A$ is a simplicial abelian group. Then the group of simplicial homotopy classes $\pi(\mathbb{Z}X, A)$ can be canonically identified up to isomorphism with the group of chain homotopy classes $\pi_{\mathbf{Ch}_+}(\mathbb{Z}X, A)$ between the associated Moore complexes.*

We also have the following well known result, which represents cohomology as homotopy classes of maps. It is implicit in the proof of this theorem that the Eilenberg-Mac Lane object $K(B,n) = \Gamma B[n]$ represents normalized $n$-cocycles with coefficients in $B$ on the simplicial abelian group category, by adjointness.

THEOREM 2.19. *Suppose that $X$ is a simplicial set and $B$ is an abelian group. Then there are canonical isomorphisms*

$$[X, K(B,n)] \cong H^n(X,B),$$

*for $n \geq 0$.*

PROOF: The simplicial abelian group $K(B,n)$ is $\Gamma B[n]$, where $B[n]$ is the chain complex which consists of the abelian group $B$ concentrated in degree $n$. We know that the set $[X, K(B,n)]$ of morphisms in $\operatorname{Ho}(\mathbf{S})$ is canonically isomorphic to the set $\pi(X, K(B,n))$ of simplicial homotopy classes, which in turn is isomorphic to the set (really group) of homotopy classes

$$\pi(\mathbb{Z}X, K(B,n)) = \pi(\mathbb{Z}X, \Gamma B[n])$$

in the simplicial abelian group category. But from the diagram (2.17), there are isomorphisms

$$\begin{aligned}\pi(\mathbb{Z}X, \Gamma B[n]) &\cong \pi_{\mathbf{Ch}_+}(N\mathbb{Z}X, N\Gamma B[n]) \\ &\cong \pi_{\mathbf{Ch}_+}(\mathbb{Z}X, B[n]).\end{aligned}$$

The group $\pi_{\mathbf{Ch}_+}(\mathbb{Z}X, B[n])$ is isomorphic to $H^n(X,B)$. ☐

Suppose that $C$ is a chain complex, let $Z_n$ denote the subgroup of $n$-cycles and let $B_n = \partial(C_{n+1})$ be the subgroup of boundaries in $C_n$. Pick an epimorphism $p : F_n \twoheadrightarrow Z_n$, where $F_n$ is a free abelian group. Then the kernel $K_n$ of the composite

$$F_n \twoheadrightarrow Z_n \twoheadrightarrow H_nC$$

is free abelian, and the composite

$$K_n \subset F_n \xrightarrow{p} Z_n$$

factors through a map $p' : K_n \to B_n$. Since $K_n$ is free abelian and the map $C_{n+1} \to B_n$ is surjective, the map $p'$ lifts to a map $p : K_n \to C_{n+1}$. Write $F_nC$ for the chain complex

$$\cdots \to 0 \to \overset{n+1}{K_n} \hookrightarrow \overset{n}{F_n} \to 0 \to \cdots.$$

Then the epimorphism $F_n \to H_nC$ defines a quasi-isomorphism

$$q_n : F_nC \to H_nC[n],$$

while the maps labelled by $p$ define a chain map

$$p_n : F_nC \to C$$

which induces an isomorphism $H_n(F_nC) \cong H_nC$. It follows that there are quasi-isomorphisms

$$\bigoplus_{n\geq 0} H_nC[n] \xleftarrow{\oplus q_n} \bigoplus_{n\geq 0} F_nC \xrightarrow{\oplus p_n} C.$$

Note that the canonical map

$$\bigoplus_{n\geq 0} H_nC[n] \to \prod_{n\geq 0} H_nC[n]$$

is an isomorphism of chain complexes.

If the chain complex $C$ happens to be the normalized complex $NA$ of a simplicial abelian group $A$, then this construction translates through the functor $\Gamma$ into weak equivalences of simplicial abelian groups

$$\prod_{n\geq 0} K(\pi_nA, n) \leftarrow \bigoplus_{n\geq 0} \Gamma F_nNA \to A.$$

These objects are fibrant in the category of simplicial sets, and so the weak equivalences induce homotopy equivalences of simplicial sets, proving

PROPOSITION 2.20. *Suppose that $A$ is a simplicial abelian group. Then, as a simplicial set, $A$ is non-canonically homotopy equivalent to the product of Eilenberg-Mac Lane spaces*

$$\prod_{n\geq 0} K(\pi_nA, n).$$

The key point in the argument for Proposition 2.20 is that a subgroup of a free abelian group is free, or at least projective. An analogous statement holds for modules over a principal ideal domain $R$ so the simplicial abelian group $A$ in the statement of the proposition can be replaced (at least) by a simplicial module over such a ring.

This is implicit above, but there is a natural short exact sequence

$$0 \to K(A,n) \to WK(A,n) \xrightarrow{p} K(A,n+1) \to 0$$

which are constructed by applying the functor $\Gamma$ to the short exact sequence

$$0 \to A[n] \to A\langle n+1\rangle \to A[n+1] \to 0$$

of chain complexes. The simplicial abelian group $WK(A,n)$ is contractible, and the sequence

$$K(A,n) \to WK(A,n) \xrightarrow{p} K(A,n+1)$$

is one of the standard fibre sequences which is used to construct an equivalence $K(A,n) \simeq \Omega K(A,n+1)$ in the literature. There is one final observation about the Eilenberg-Mac Lane spaces $K(A,n) = \Gamma A[n]$ and the fibration $p$ which is very commonly used:

LEMMA 2.21. *The map $p : WK(A,n) \to K(A,n+1)$ is a minimal fibration, and $K(A,n)$ is a minimal Kan complex, for all $n \geq 0$.*

PROOF: There is a relation $x \simeq_p y$ (in the sense of Section I.10) if and only if $(x-y) \simeq_p 0$, so it suffices to show that $z \simeq_p 0$ implies that $z = 0$ for any simplex $z$ of $WK(A,n)$. But $z \simeq_p 0$ forces $z$ to be in the fibre $K(A,n)$, and so to show that $p$ is a minimal fibration we need only prove that $K(A,n)$ is a minimal Kan complex.

Suppose that $z$ is an $r$-simplex of $K(A,n)$. If $z \simeq 0$ in $K(A,n)$ (rel $\partial\Delta^r$), then $d_i z = 0$ for all $i$, so that $z$ is a normalized $r$-chain of $K(A,n)$. By the Dold-Kan correspondence, $NK(A,n)_r \cong A[n]_r$, which group is 0 if $r \neq n$, so it suffices to concentrate on the case $r = n$. There are identifications $NK(A,n)_n = K(A,n)_n = A$, and the resulting map $K(A,n)_n \to \pi_n(K(A,n),0)$ is an isomorphism. Thus $z \simeq 0$ (rel $\partial\Delta^n$) means that $z \mapsto 0$ in $\pi_n(K(A,n),0)$. But then $z = 0$. □

## 3. The Hurewicz map.

Suppose that $X$ is a simplicial set. The *Hurewicz map* $h : X \to \mathbb{Z}X$ is alternate notation for the adjunction homomorphism which associates to an $n$-simplex $x$ of $X$ the corresponding generator of the free abelian group $\mathbb{Z}X_n$ on $X_n$. If $X$ happens to be a connected Kan complex with base point $*$, then $h$ induces a composite homomorphism of groups

$$\pi_n X \to \pi_n(\mathbb{Z}X, *) \to \pi_n(\mathbb{Z}X/\mathbb{Z}*, 0) \cong \tilde{H}_n(X,\mathbb{Z}),$$

where $\tilde{H}_*(X,\mathbb{Z})$ is reduced homology of $X$. Write

$$h_* : \pi_n X \to \tilde{H}_n(X,\mathbb{Z})$$

for this composite — the map $h_*$ is the *Hurewicz homomorphism.* The simplicial set $X$ will be a connected Kan complex with base point $*$ throughout this section, and all homology groups will be integral, so we shall write $\tilde{H}_*X$ to mean $\tilde{H}_*(X,\mathbb{Z})$. For ease of notation, write

$$\tilde{\mathbb{Z}}X = \mathbb{Z}X/\mathbb{Z}*\,.$$

The Hurewicz homomorphism $h_* : \pi_n X \to \tilde{H}_n X$ is the traditional map. To see this, let $S^n = \Delta^n/\partial\Delta^n$, with the obvious choice of base point. Then $\pi_n X$ is pointed simplicial homotopy classes of maps $S^n \to X$, and the homotopy group

$$\pi_n(\tilde{\mathbb{Z}}S^n) \cong \tilde{H}_n S^n$$

is a copy of the integers, canonically generated by the homotopy element $\kappa_n$ which is represented by the composite map

$$S^n \xrightarrow{h} \mathbb{Z}S^n \to \tilde{\mathbb{Z}}S^n.$$

One sees, by drawing an appropriate commutative diagram, that if $\alpha : S^n \to X$ represents an element $[\alpha] \in \pi_n X$, then

$$h_*([\alpha]) = \alpha_*(\kappa_n) \in \tilde{H}_n X,$$

as in the standard definition.

Suppose that $p : E \to B$ is a Kan fibration, where $B$ is a simply connected Kan complex with base point $*$, and let $F$ be the fibre over $*$. There is a complete description of the Serre spectral sequence

$$E_2^{p,q} = H_p(B, H_q F) \Rightarrow H_{p+q} B$$

given below in IV.5.1, but it is somewhat non-standard. To recover the more usual form, filter the base $B$ by skeleta $\mathrm{sk}_n\, B$, and form the pullback diagrams

$$\begin{tikzcd}
F_n E \arrow[r, hook] \arrow[d, "p_*"'] & E \arrow[d, "p"] \\
\mathrm{sk}_n\, B \arrow[r, hook] & B
\end{tikzcd}$$

to obtain a filtration $F_n E$ of the total space $E$ and fibrations $p_* : F_n E \to \mathrm{sk}_n\, B$ for which $\pi_1\, \mathrm{sk}_n\, B$ acts trivially on the homology $H_* F$ of $F$.

Recall that $N_n B$ denotes the set of non-degenerate $n$-simplices of $B$. The pushout diagram

$$\begin{tikzcd}
\bigsqcup_{x \in N_n B} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sk}_{n-1}\, B \arrow[d, hook] \\
\bigsqcup_{x \in N_n B} \Delta^n \arrow[r] & \mathrm{sk}_n\, B
\end{tikzcd}$$

can be pulled back along $p$ to obtain an identification of the filtration quotient $F_n E / F_{n-1} E$ with the wedge

$$\bigvee_{x \in N_n B} p^{-1}(\Delta^n)/p^{-1}(\partial\Delta^n).$$

The spectral sequence

$$H_r(\partial\Delta^n, H_s F) \Rightarrow H_{r+s} p^{-1}(\partial\Delta^n)$$

of IV.5 is used, along with the fact that $H_*F$ splits off $H_*p^{-1}(\partial\Delta^n)$ to show that there is an isomorphism

$$H_*p^{-1}(\partial\Delta^n) \cong H_*F \oplus H_*F[n-1],$$

and that the map induced in homology by the inclusion $p^{-1}(\partial\Delta^n) \subset p^{-1}(\Delta^n)$ can be identified up to isomorphism with the projection

$$H_*F \oplus H_*F[n-1] \to H_*F$$

In particular, there is an isomorphism

$$\tilde{H}_*(p^{-1}(\Delta^n)/p^{-1}(\partial\Delta^n)) \cong H_*F[n],$$

and hence isomorphisms

$$\tilde{H}_*(F_nE/F_{n-1}E) \cong \bigoplus_{x \in N_nB} H_*F[n].$$

Thus,

$$\tilde{H}_{p+q}(F_pE/F_{p-1}E) \cong \bigoplus_{x \in N_pB} H_qF,$$

and it's a matter of chasing simplices through the boundary map of the normalized complex $N\mathbb{Z}B \otimes H_qF$ to see that the $E_2$-term of the spectral sequence for $H_*E$ arising from the skeletal filtration for $B$ has the form

$$E_2^{p,q} = H_p(B, H_qF).$$

All appeals to the Serre spectral sequence in the rest of this section will be specifically to this form.

Now suppose that $Y$ is a pointed Kan complex, and observe that the canonical path object $PY$ is that pointed function complex

$$PY = \mathbf{Hom}_*(\Delta^1_*, Y),$$

where $\Delta^1_*$ is a copy of the standard 1-simplex $\Delta^1$, pointed by the vertex 1. The loop space $\Omega Y$ can be identified with the complex $\mathbf{Hom}_*(S^1, Y)$, and the path loop fibration for $Y$ is the fibre sequence

$$\mathbf{Hom}_*(S^1, Y) \xrightarrow{\pi^*} \mathbf{Hom}_*(\Delta^1_*, Y) \xrightarrow{d_1} Y,$$

where $\pi : \Delta^1 \to S^1$ is the canonical map.

There is a canonical contracting pointed homotopy

$$h : \Delta^1_* \wedge \Delta^1_* \to \Delta^1_*,$$

which is defined by the picture

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr] & 1 \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}$$

in the poset **1**. This homotopy $h$ induces a contracting homotopy

$$\mathbf{Hom}_*(\Delta^1_*, Y) \wedge \Delta^1_* \xrightarrow{h_*} \mathbf{Hom}_*(\Delta^1_*, Y)$$

for the path space on $Y$, by adjointness.

Suppose that $f : X \to \mathbf{Hom}_*(S^1, Y)$ is a pointed map, and denote the composite

$$X \wedge \Delta^1_* \xrightarrow{\pi^* f \wedge 1} \mathbf{Hom}_*(\Delta^1_*, Y) \wedge \Delta^1_* \xrightarrow{h} \mathbf{Hom}_*(\Delta^1_*, Y)$$

by $f_*$. Then there is a commutative diagram of pointed simplicial set maps

$$\begin{tikzcd}
X \arrow[r, "d^1"] \arrow[d, "f"'] & X \wedge \Delta^1_* \arrow[r, "1 \wedge \pi"] \arrow[d, "f_*"'] & X \wedge S^1 \arrow[d, "\tilde{f}"] \\
\mathbf{Hom}_*(S^1, Y) \arrow[r, "\pi^*"'] & \mathbf{Hom}_*(\Delta^1_*, Y) \arrow[r, "d_1"'] & Y,
\end{tikzcd} \tag{3.1}$$

where the indicated map $\tilde{f}$ is the adjoint of $f$, or rather the composite

$$X \wedge S^1 \xrightarrow{f \wedge 1} \mathbf{Hom}_*(S^1, Y) \wedge S^1 \xrightarrow{ev} Y.$$

Note as well that $X \wedge \Delta^1_*$ is a model for the pointed cone on $X$ (a different model for the cone is given in Section III.5 below).

Suppose that

$$F \to E \xrightarrow{p} B$$

is a fibre sequence of pointed Kan complexes, and that $B$ is simply connected. The edge maps

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "p"'] & B \arrow[d, "1"] \\ * \arrow[r] & B \arrow[r] & B \end{tikzcd}$$

and the calculation of the $E_2$-terms for the Serre spectral sequences for the corresponding fibrations together imply that $E_n^{n,0}$ in the Serre spectral sequence for $H_*E$ is the subgroup of $H_nB$ consisting of elements $x$ which can be "lifted along the staircase"

$$\begin{tikzcd} & & H_{n-1}F \arrow[d] \\ & H_n(F_nE/F_{n-1}E) \arrow[r, "\partial"] \arrow[d, "p_*"'] & H_{n-1}F_{n-1}E \\ H_n(\mathrm{sk}_n\, B) \arrow[r] \arrow[d] & H_n(\mathrm{sk}_n\, B/\mathrm{sk}_{n-1}\, B) & \\ H_nB & & \end{tikzcd}$$

to an element $z \in H_{n-1}F$, in the sense that there are elements $x_1 \in H_n(\mathrm{sk}_n\, B)$ and $x_2 \in H_n(F_nE/F_{n-1}E)$ such that $x_1 \mapsto x$, $x_1 \mapsto p_*(x_2)$, and $z \mapsto \partial(x_2)$. Furthermore, the image of such an $x$ under the differential $d_n : E_n^{n,0} \to E_n^{0,n-1}$ is represented by the element $z$. But then comparing long exact sequences shows that the element $x_2$ is in the image of the map $H_n(F_nE/F) \to H_n(F_nE/F_{n-1}E)$. As a consequence, the elements $x$ of $E_n^{n,0}$ can be identified with elements of $H_n(B/*)$ which are in the image of the map $p_* : H_n(E/F) \to H_n(B/*)$, and for such $x$, if there is an element $x_1 \in H_n(E/F)$ such that $p_*(x_1) = x$, then $d_n(x)$ is represented by $\partial(x_1) \in H_{n-1}F$. This is a classical description of the *transgression*.

Suppose that $Y$ is an $n$-connected pointed Kan complex, where $n \geq 1$, and consider the Serre spectral sequence for the path loop fibre sequence

$$\Omega Y \to PY \xrightarrow{d_1} Y.$$

Then $\tilde{H}_iY = 0$ for $i \leq n$, while $\tilde{H}_j\Omega Y = 0$ for $j \leq n-1$. The assumption on the connectivity of Kan complex $Y$ implies that there is a strong deformation retract $Z$ of $Y$ such that the $n$-skeleton $\mathrm{sk}_n Z$ is a point[2]. It follows that $E_2^{i,0} = E_i^{i,0}$ and $E_2^{0,i-1} = E_i^{0,i-1}$ for $i \leq 2n$, and the only possible non-trivial differential into or out of either group is the transgression, so that there is an exact sequence

$$0 \to E_\infty^{i,0} \to E_i^{i,0} \xrightarrow{d_i} E_i^{0,i-1} \to E_\infty^{0,i-1} \to 0.$$

The space $PY$ is acyclic, so all $E_\infty$-terms vanish in non-zero total degree, so we have shown that the transgression

$$H_iY \xrightarrow{d_i} H_{i-1}\Omega Y$$

is an isomorphism for $i \leq 2n$ under the assumption that $Y$ is $n$-connected.

The transgression $d_i : H_iY \to H_{i-1}\Omega Y$ is, at the same time, related to the boundary map

$$\partial : \tilde{H}_i(X \wedge S^1) \xrightarrow{\cong} \tilde{H}_{i-1}X,$$

in the sense of the following result:

Lemma 3.2. *Suppose that $f : X \to \Omega Y$ is a map of pointed simplicial sets, where $Y$ is an $n$-connected Kan complex with $n \geq 1$. Then, for $i \leq 2n$, there is a commutative diagram of the form*

$$\begin{tikzcd}
\tilde{H}_i(X \wedge S^1) \arrow[r, "\partial", "\cong"'] \arrow[d, "\tilde{f}_*"'] & \tilde{H}_{i-1}X \arrow[d, "f_*"] \\
\tilde{H}_iY \arrow[r, "\cong", "d_i"'] & \tilde{H}_{i-1}\Omega Y,
\end{tikzcd}$$

*where $\tilde{f} : X \wedge S^1 \to Y$ is the adjoint of the map $f$.*

Proof: Write $\Sigma X = X \wedge S^1$ and $CX = X \wedge \Delta^1_*$, as usual. Then the diagram (3.1) induces a commutative diagram

$$\begin{tikzcd}
\tilde{H}_i\Sigma X \cong \tilde{H}_i(\Sigma X/*) \arrow[d, "\tilde{f}_*"'] & \tilde{H}_i(CX/X) \arrow[l, "\cong"'] \arrow[r, "\partial"] \arrow[d, "f_*"'] & \tilde{H}_{i-1}X \arrow[d, "f_*"] \\
\tilde{H}_iY \cong \tilde{H}_i(Y/*) & \tilde{H}_i(PY/\Omega Y) \arrow[l, "d_{1*}"] \arrow[r, "\partial"'] & \tilde{H}_{i-1}\Omega Y
\end{tikzcd}$$

The top composite in the diagram is the boundary map $\partial : \tilde{H}_i\Sigma X \to \tilde{H}_{i-1}X$. The bottom "composite" is the transgression, according to the discussion preceding the statement of the lemma. □

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[2] This is an old idea. People sometimes say that complexes of the form $Z$ are *n-reduced.* The construction of the deformation retraction (via an iterated homotopy extension property argument) is one of the early applications of the Kan complex concept, and should be done as an exercise.

COROLLARY 3.3. *Suppose that $Y$ is an $n$-connected pointed Kan complex, with $n \geq 1$. Then, for $i \leq 2n$, the transgression $d_i : \tilde{H}_i Y \to \tilde{H}_{i-1}\Omega Y$ can be identified with the composite*

$$\tilde{H}_i(Y) \xleftarrow[\cong]{\epsilon_*} \tilde{H}_i((\Omega Y) \wedge S^1) \xrightarrow[\cong]{\partial} \tilde{H}_{i-1}(\Omega Y).$$

PROOF: The adjunction map $\epsilon : (\Omega Y) \wedge S^1 \to Y$ is the adjoint of the identity map on the loop space $\Omega Y$. □

The link between the Hurewicz homomorphism and the transgression is the following:

PROPOSITION 3.4. *Suppose that $Y$ is an $n$-connected pointed Kan complex, with $n \geq 1$. Then, for $i \leq 2n$, there is a commutative diagram*

$$\begin{tikzcd}
\pi_i Y \arrow[r, "\partial", "\cong"'] \arrow[d, "h_*"'] & \pi_{i-1}(\Omega Y) \arrow[d, "h_*"] \\
\tilde{H}_i Y \arrow[r, "\cong", "d_i"'] & \tilde{H}_{i-1}(\Omega Y)
\end{tikzcd}$$

PROOF: The instance of the diagram (3.1) corresponding to the identity map on $\Omega Y$ and the naturality of the Hurewicz map together give rise to a commutative diagram

$$\begin{tikzcd}
\tilde{\mathbb{Z}}\Omega Y \arrow[r, equal] \arrow[d] & \tilde{\mathbb{Z}}\Omega Y \arrow[d] & \Omega Y \arrow[l, "h"'] \arrow[d] \\
\tilde{\mathbb{Z}}C\Omega Y \arrow[r] \arrow[d] & \tilde{\mathbb{Z}}PY \arrow[d, "d_{1*}"'] & PY \arrow[l, "h"'] \arrow[d] \\
\tilde{\mathbb{Z}}((\Omega Y) \wedge S^1) \arrow[r, "\epsilon_*"'] & \tilde{\mathbb{Z}}Y & Y \arrow[l, "h"]
\end{tikzcd}$$

The homomorphism $d_{1*} : \tilde{\mathbb{Z}}PY \to \tilde{\mathbb{Z}}Y$ has a factorization

$$\tilde{\mathbb{Z}}PY \xrightarrow{j} B \xrightarrow{q} \tilde{\mathbb{Z}}Y$$

in the category of simplicial abelian groups (or of simplicial sets — your choice), where $j$ is a trivial cofibration and $q$ is a fibration. Then, by comparing boundary maps for the resulting fibrations of simplicial sets, one finds a commutative diagram

$$\begin{tikzcd}
\pi_i Y \arrow[rr, "\partial", "\cong"'] \arrow[d, "h_*"'] & & \pi_{i-1}(\Omega Y) \arrow[d, "h_*"] \\
\tilde{H}_i Y & \tilde{H}_i((\Omega Y) \wedge S^1) \arrow[l, "\cong", "\epsilon_*"'] \arrow[r, "\cong", "\partial"'] & \tilde{H}_{i-1}(\Omega Y).
\end{tikzcd}$$

Now use Corollary 3.3. □

Write $x \overset{r}{\sim} y$ for $n$-simplices $x$ and $y$ of a simplicial set $Z$ if $x|_{\mathrm{sk}_r\,\Delta^n} = y|_{\mathrm{sk}_r\,\Delta^n}$, or equivalently if the composite simplicial set maps

$$\mathrm{sk}_r\,\Delta^n \subset \Delta^n \xrightarrow{\iota_x} Z \qquad \text{and} \qquad \mathrm{sk}_r\,\Delta^n \subset \Delta^n \xrightarrow{\iota_y} Z$$

coincide. The relation $\overset{r}{\sim}$ is clearly an equivalence relation on the simplices of $Z$, and we may form the quotient simplicial set $Z(r) = Z/\overset{r}{\sim}$ and write $p_r : Z \to Z(r)$. The simplicial set $Z(r)$ is called either the $r^{th}$ *Moore-Postnikov section* of $Z$, or the $r^{th}$ *Postnikov* section of $Z$. This construction is natural in $Z$: one can show quickly (see also Exercise V.3.8) that if $Z$ is a Kan complex then so is $Z(r)$, and furthermore $p_r$ induces isomorphisms $\pi_j(Z,x) \cong \pi_j(Z(r),x)$ for $j \leq r$ and all vertices $x$ of $Z$, and $\pi_j(Z(r),x) = 0$ for $j > r$.

Postnikov sections will be discussed more thoroughly in Chapter VI.

If $X$ is a pointed connected Kan complex, then the object $X(1)$ is a pointed connected Kan complex of type $K(\pi_1 X, 1)$, meaning that its homotopy groups consist of $\pi_1 X$ in dimension one and 0 elsewhere. There is, however, a more geometrically satisfying way to construct a space naturally having the homotopy type of $X(1)$ which uses the fundamental groupoid construction: we have seen in Section III.1 that there is a map

$$X \to B(\pi_1 X)$$

of Kan complexes which induces an isomorphism on fundamental groups. It follows, for example, that a connected Kan complex of type $K(\pi, 1)$ is weakly equivalent to the space $B\pi$.

The Hurewicz homomorphism $h_* : \pi_1(B\pi) \to \tilde{H}_1(B\pi)$ is isomorphic to the canonical group homomorphism $\pi \to \pi/[\pi,\pi]$ from $\pi$ to its abelianization — this may be seen directly, or by invoking the following result:

Lemma 3.5. *Suppose that $Z$ is a Kan complex, such that the set $Z_0$ of vertices of $Z$ consists of a single point. The the Hurewicz map*

$$h_* : \pi_1 Z \to \tilde{H}_1(Z)$$

*can be identified up to isomorphism with the canonical homomorphism*

$$\pi_1 Z \to \pi_1 Z/[\pi_1 Z, \pi_1 Z].$$

Proof: Since $Z$ is reduced, the integral homology group $H_1(Z)$ is the quotient

$$H_1(Z) = \bigoplus_{\omega \in Z_1} \mathbb{Z}/\langle d_0\sigma - d_1\sigma + d_2\sigma | \sigma \in Z_2\rangle.$$

Up to this identification, the Hurewicz map

$$h : \pi_1 Z \to \bigoplus_{\omega \in Z_1} \mathbb{Z}/\langle d_0\sigma - d_1\sigma + d_2\sigma | \sigma \in Z_2\rangle$$

is defined by $[\omega] \mapsto [\omega]$, for 1-simplices $\omega$ of $Z$. One then shows, by chasing elements, that the map $h$ is initial among all group homomorphisms $f : \pi_1 Z \to A$ which take values in abelian groups $A$. □

COROLLARY 3.6. *Suppose that $X$ is a connected pointed Kan complex. Then the Hurewicz homomorphism $h_* : \pi_1 X \to \tilde{H}_1 X$ is isomorphic to the canonical homomorphism*

$$\pi_1 X \to \pi_1 X / [\pi_1 X, \pi_1 X]$$

*from the fundamental group of $X$ to its abelianization.*

PROOF: The space $X$ has a strong deformation retract $Z$ which is reduced. Now apply Lemma 3.5. □

THEOREM 3.7 (HUREWICZ). *Suppose that $X$ is an $n$-connected Kan complex, where $n \geq 1$. Then the Hurewicz homomorphism $h_* : \pi_i X \to \tilde{H}_i X$ is an isomorphism if $i = n+1$ and an epimorphism if $i = n+2$.*

PROOF: Suppose that $F$ is the homotopy fibre of the map $p_{n+1} : X \to X(n+1)$. Then there are commutative diagrams

$$\begin{tikzcd}
\pi_{n+1} X \arrow[r, "\cong"] \arrow[d, "h_*"'] & \pi_{n+1} X(n+1) \arrow[d, "h_*"] & \pi_{n+2} F \arrow[r, "\cong"] \arrow[d, "h_*"'] & \pi_{n+2} X \arrow[d, "h_*"] \\
\tilde{H}_{n+1} X \arrow[r] & \tilde{H}_{n+1} X(n+1) & \tilde{H}_{n+2} F \arrow[r] & \tilde{H}_{n+2} X
\end{tikzcd}$$

A Serre spectral sequence argument for the fibration $p_{n+1}$ shows that the map $\tilde{H}_{n+1} X \to \tilde{H}_{n+1} X(n+1)$ is an isomorphism, since the space $F$ is $(n+1)$-connected, and that it suffices to show

(1) that the Hurewicz homomorphism $h_* : \pi_{n+1} X(n+1) \to \tilde{H}_{n+1} X(n+1)$ is an isomorphism, and

(2) $\tilde{H}_{n+2} X(n+1) = 0$.

If these two statements are demonstrated (for all $n$), then the general statement that $h_* : \pi_{n+1} X \to \tilde{H}_{n+1} X$ is an isomorphism would be true, so that the map $h_* : \pi_{n+2} F \to \tilde{H}_{n+2} F$ would be an isomorphism as well. Furthermore, the assertion that $\tilde{H}_{n+2} X(n+1) = 0$ implies, via the Serre spectral sequence for $p_{n+1}$, that the map $\tilde{H}_{n+2} F \to \tilde{H}_{n+2} X$ is an epimorphism.

But the claim that $h_* : \pi_{n+1} X(n+1) \to \tilde{H}_{n+1} X(n+1)$ is an isomorphism reduces, by an inductive transgression argument involving Proposition 3.4, to Corollary 3.6, so statement (1) is proved. Similarly, statement (2) is reduced to showing that $\tilde{H}_3 Y = 0$ for any connected Kan complex $Y$ of type $K(A, 2)$.

Let $Y$ be such a Kan complex, and pick a strong deformation retraction map $r : Y \to W$ onto a 2-reduced subcomplex. Then there is a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "h"] \arrow[d, "r"'] & \tilde{\mathbb{Z}}Y \arrow[d, "r_*"] \\
W \arrow[r, "h"'] & \tilde{\mathbb{Z}}W
\end{tikzcd}$$

The map $r_*$ is a weak equivalence of simplicial abelian groups, hence of simplicial sets, and the complex $\tilde{\mathbb{Z}}W$ is 0 in degrees less than 2, as is its associated normalized chain complex $N\tilde{\mathbb{Z}}W$. There is a map of chain complexes $N\tilde{\mathbb{Z}}W \to \tilde{H}_2Y[2]$ which induces an isomorphism in $H_2$. It follows that the induced composite

$$Y \xrightarrow{h} \tilde{\mathbb{Z}}Y \xrightarrow{r_*} \tilde{\mathbb{Z}}W \to \Gamma\tilde{H}_2Y[2] = K(\tilde{H}_2Y, 2)$$

is a weak equivalence.

We are therefore required only to show that $H_3K(A, 2) = 0$ for any abelian group $A$. The functor $A \mapsto H_3K(A, 2)$ preserves filtered colimits, so it suffices to presume that $A$ is finitely generated. The functor $A \mapsto K(A, 2)$ takes finite direct sums to products of simplicial sets, and a Künneth exact sequence argument shows that $H_3K(A \oplus B, 2) = 0$ if $H_3K(A, 2) = H_3K(B, 2) = 0$. Finally, a few more Serre spectral sequence arguments, combined with knowing that the circle has type $K(\mathbb{Z}, 1)$ imply (successively) that $H_3K(\mathbb{Z}, 2) = 0$ and $H_3K(\mathbb{Z}/n, 2) = 0$ for any $n$. □

One can form what could be called the $n^{th}$ Postnikov section $C(n)$ of a chain complex $C$ as follows: define $C(n)$ to be the chain complex

$$\dots \to 0 \to C_n/Im\ \partial \to C_{n-1} \to \dots \to C_0.$$

Here, $Im\ \partial$ is the image of the boundary map $\partial : C_{n+1} \to C_n$. There is a chain map $p_n : C \to C(n)$ which induces an isomorphism in $H_i$ for $i \leq n$, and $H_iC(n)$ is plainly trivial for $i > n$. There is also a natural chain map $i_n : H_nC[n] \to C(n)$, and this map is a homology isomorphism in degree $n$. We could have used this construction in place of the retraction onto the 2-reduced complex $W$ in the proof of the Hurewicz Theorem. This construction is also used to prove the following:

COROLLARY 3.8. *Let $Y$ be a connected Kan complex with $\pi_nY \cong A$, and $\pi_jY = 0$ for $j \neq n$, where $n \geq 2$. Then $Y$ is naturally weakly equivalent to the space $K(A, n) = \Gamma A[n]$.*

The naturality in the statement of this result is with respect to maps $Y \to Z$ of connected Kan complexes having only one non-trivial homotopy group, in degree $n$.

PROOF: Let $p_{n*} : \tilde{\mathbb{Z}}Y \to \Gamma(N\tilde{\mathbb{Z}}Y(n))$ be the simplicial abelian group map induced by the $n^{th}$ Postnikov section map $p_n : N\tilde{\mathbb{Z}}Y \to N\tilde{\mathbb{Z}}(n)$ of the associated normalized chain complex. Then the composite

$$Y \xrightarrow{h} \tilde{\mathbb{Z}}Y \xrightarrow{p_{n*}} \Gamma(N\tilde{\mathbb{Z}}Y(n))$$

is a natural weak equivalence, by the Hurewicz Theorem and the construction of the chain map $p_n$. But then the following maps are both natural weak equivalences

$$Y \xrightarrow{p_{n*}h} \Gamma(N\tilde{\mathbb{Z}}Y(n)) \xleftarrow{i_{n*}} \Gamma A[n] = K(A,n),$$

and the result is proved. □

REMARK 3.9. The construction in this last proof is the only known way of showing that a diagram of spaces having only one non-trivial presheaf of homotopy groups is weakly equivalent to a diagram of spaces $K(A,n) = \Gamma A[n]$, for some presheaf (aka. diagram) of abelian groups $A$.

THEOREM 3.10 (FREUDENTHAL). *Suppose that $X$ is an $n$-connected pointed space, where $n \geq 0$. Then the canonical map $\eta : X \to \Omega(X \wedge S^1)$ induces a map $\pi_i X \to \pi_i \Omega(X \wedge S^1)$ which is an isomorphism if $i \leq 2n$ and an epimorphism if $i = 2n+1$.*

PROOF: We shall suppose that $n \geq 1$, and leave the case $n = 0$ for the reader.

From the characterization of the transgression for the path-loop fibre sequence of Corollary 3.3, the map

$$\epsilon_* : \tilde{H}_i(\Omega X \wedge S^1) \to \tilde{H}_i X$$

is an isomorphism for $i \leq 2n$. One then uses the triangle identity

$$\begin{tikzcd}
X \wedge S^1 \arrow[r, "\eta \wedge S^1"] \arrow[dr, "1"'] & \Omega(X \wedge S^1) \wedge S^1 \arrow[d, "\epsilon"] \\
 & X \wedge S^1
\end{tikzcd}$$

to infer that $\eta_* : \tilde{H}_i X \to \tilde{H}_i \Omega(X \wedge S^1)$ is an isomorphism if $i \leq 2n+1$. The space $\Omega(X \wedge S^1)$ is simply connected by assumption, so a Serre spectral sequence argument says that the homotopy fibre $F$ of the map $\eta : X \to \Omega(X \wedge S^1)$ has homology groups $\tilde{H}_i F$ which vanish for $i \leq 2n$. But $F$ is a simply connected space, by the Hurewicz theorem together with the fact that $\eta$ is a homology isomorphism in degree 2, so that $F$ is $2n$-connected (by Hurewicz again), giving the result. □

Theorem 3.10 is the classical Freudenthal suspension theorem, since the homomorphism $\eta_* : \pi_i X \to \pi_i \Omega(X \wedge S^1)$ is isomorphic to the suspension homomorphism $\pi_i X \to \pi_{i+1}(X \wedge S^1)$. We shall finish this section with the relative Hurewicz theorem:

THEOREM 3.11. *Suppose that $f : X \to Y$ is a map with homotopy fibre $F$ and homotopy cofibre $Y/X$. Suppose that $F$ is $n$-connected for some $n \geq 0$, the total space $X$ is simply connected, and the base $Y$ is connected. Then the homotopy fibre of the induced map $f_* : F \wedge S^1 \to Y/X$ is $(n+2)$-connected.*

PROOF: The diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & * \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

induces the map $f_* : F \wedge S^1 \to Y/X$ of associated homotopy cofibres.

The spaces $X$ and $Y$ are both simply connected, by the assumptions on the map $f$, and so the Serre exact sequence for the map $f$

$$H_{n+2}X \to H_{n+2}Y \xrightarrow{d_{n+2}} H_{n+1}F \to H_{n+1}X \to \dots$$

extends to a sequence of the form

$$H_{n+3}Y \xrightarrow{d_{n+3}} E^{0,n+2}_{n+3} \to H_{n+2}X \to H_{n+2}Y \to \dots.$$

The classical description of the transgression says that there is a comparison of exact sequences

$$\begin{tikzcd}
H_{n+3}(X/F) \arrow[r, "\partial"] \arrow[d] & H_{n+2}F \arrow[r] \arrow[d] & H_{n+2}X \arrow[r] \arrow[d, "="] & H_{n+2}(X/F) \arrow[r, "\partial"] \arrow[d] & \dots \\
H_{n+3}Y \arrow[r, "d_{n+3}"'] & E^{0,n+2}_{n+3} \arrow[r] & H_{n+2}X \arrow[r] & H_{n+2}Y \arrow[r, "d_{n+2}"'] & \dots
\end{tikzcd}$$

Chasing elements shows that the map $H_i(X/F) \to H_iY$ is an isomorphism for $i \leq n+2$; it is an epimorphism for $i = n+3$ because every element of $H_{n+3}Y$ is transgressive. Comparing Puppe sequences gives a diagram

$$\begin{tikzcd}
H_{n+3}X \arrow[r] \arrow[d, "="'] & H_{n+3}(X/F) \arrow[r] \arrow[d] & H_{n+3}(F \wedge S^1) \arrow[r] \arrow[d, "f_*"] & H_{n+3}(X \wedge S^1) \arrow[r] \arrow[d, "="] & \dots \\
H_{n+3}X \arrow[r] & H_{n+3}Y \arrow[r] & H_{n+3}(Y/X) \arrow[r] & H_{n+3}(X \wedge S^1) \arrow[r] & \dots
\end{tikzcd}$$

Then one chases some more elements to show that $f_* : H_i(F \wedge S^1) \to H_i(Y/X)$ is an epimorphism if $i = n+3$ and is an isomorphism in all lower degrees. The space $Y/X$ is simply connected, so the homotopy fibre of $f_* : F \wedge S^1 \to Y/X$ must be $(n+2)$-connected. □

One has a right to ask why Theorem 3.11 should be called a relative Hurewicz theorem. Here's the usual statement:

COROLLARY 3.12. *Suppose that $A$ is a simply connected subcomplex of $X$, and that the pair $(X, A)$ is $n$-connected for some $n \geq 1$. Then the Hurewicz map*

$$\pi_i(X, A) \xrightarrow{h} \tilde{H}_i(X, A)$$

*is an isomorphism if $i = n + 1$ and is surjective if $i = n + 2$.*

PROOF: The relative homotopy group $\pi_i(X, A)$ is the homotopy group $\pi_{i-1}F$ of the homotopy fibre $F$ of the inclusion $j : A \hookrightarrow X$, and the relative Hurewicz map $h$ is the composite

$$\pi_i(X, A) = \pi_{i-1}F \xrightarrow{\Sigma} \pi_i(F \wedge S^1) \xrightarrow{j_*} \pi_i(X/A) \xrightarrow{h} \tilde{H}_i(X/A) = \tilde{H}_i(X, A)$$

It is standard to say that $(X, A)$ is $n$-connected (for $n \geq 0$), and mean both that the homotopy fibre $F$ is $(n-1)$-connected and the function $\pi_0 A \to \pi_0 X$ is surjective. The Freudenthal suspension theorem says that the suspension map $\Sigma : \pi_{i-1}F \to \pi_i(F \wedge S^1)$ is an isomorphism if $i = n + 1$ and an epimorphism if $i = n+2$ (for all $n \geq 2$). Theorem 3.11 implies that $j_* : \pi_i(F \wedge S^1) \to \pi_i(X/A)$ is an isomorphism if $i = n + 1$ and an epi if $i = n + 2$. Theorem 3.11 also implies that the space $X/A$ is $n$-connected, so that the ordinary Hurewicz map $h : \pi_i(X/A) \to \tilde{H}_i(X/A)$ is an isomorphism if $i = n+1$ and an epimorphism if $i = n + 2$. □

THEOREM 3.13 (HOMOTOPY ADDITION). *Suppose that $X$ is a pointed Kan complex and that $\alpha_i : \Delta^n/\partial\Delta^n \to X$, $0 \leq i \leq n + 1$, are pointed maps. Then there is an $(n + 1)$-simplex $w$ of $X$ such that $d_i w = \alpha_i$ for $0 \leq i \leq n + 1$ if and only if*

$$\sum_{i=0}^{n+1} (-1)^i [\alpha_i] = 0$$

*in $\pi_n X$.*

PROOF: Suppose that the simplex $w$ exists, with $d_i w = \alpha_i$.

The statement in degree 1 is that

$$[\alpha_0][\alpha_1]^{-1}[\alpha_2] = e$$

in $\pi_1 X$ if the $\alpha_i$ bound a 2-simplex, and it's a trivial consequence of the definition of the multiplication in the fundamental group, so we'll assume that $n \geq 2$.

Consider the fibre sequence

$$F \xrightarrow{j} X \xrightarrow{p} X(n-1),$$

where $X(n - 1)$ is the $(n - 1)^{st}$ Postnikov section of $X$. Then $F$ is $(n - 1)$-connected, and we can assume, by a homotopy extension argument, that the maps $\alpha_i$ factor through $F$.

We can also assume that the simplex $w$ lives in $F$. The simplex $p(w)$ is homotopic rel boundary to $*$ in $X(n-1)$ since $\pi_{n+1}X(n-1) = 0$. Then there is a diagram

$$\begin{tikzcd}
\Delta^{n+1} \times \{0\} \cup \partial\Delta^{n+1} \times \Delta^1 \arrow[r, "{(w, c_{(\alpha_0,\dots,\alpha_{n+1})})}"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^{n+1} \times \Delta^1 \arrow[r, "h"'] \arrow[ur, dashed] & X(n-1)
\end{tikzcd}$$

where $h$ be a specific choice of homotopy $p(w) \simeq *$ rel boundary, and the map $(w, c_{(\alpha_0,\dots,\alpha_{n+1})})$ is defined by the simplex $w$ and the constant homotopy on its boundary. Then the $(n+1)$-simplex at the other end of the indicated lifted homotopy is in the fibre $F$.

We can therefore assume that our space $X$ is $(n-1)$-connected, where $n \geq 2$. Consider the Hurewicz map

$$X \to \mathbb{Z}X \to \mathbb{Z}X/\mathbb{Z}*\,.$$

The Hurewicz theorem implies that this composite induces an isomorphism

$$\pi_n X \cong \tilde{H}_n(X, \mathbb{Z}),$$

and of course

$$\partial w = \sum_{i=0}^{n+1} (-1)^i \alpha_i$$

in the reduced chain complex $\mathbb{Z}X/\mathbb{Z}*$. Each $\alpha_i$ is a cycle in the reduced chain complex, and

$$\sum_{i=0}^{n+1} (-1)^i [\alpha_i] = 0$$

in $\tilde{H}_n(X, \mathbb{Z})$.

For the converse, suppose that the simplices $\alpha_i$ satisfy the formula

$$\sum_{i=0}^{n+1} (-1)^i [\alpha_i] = 0$$

where $n \geq 2$ (the case $n = 1$ is similar).

In the diagram

$$\begin{tikzcd}
\Lambda^{n+1}_0 \arrow[r, "{(\alpha_1, \dots, \alpha_{n+1})}"] \arrow[d, hook] & X \\
\Delta^{n+1} \arrow[ur, "\theta"'] &
\end{tikzcd}$$

the boundary simplices of the simplex $\theta$ satisfy the formula

$$[d_0\theta] - [\alpha_1] + \cdots + (-1)^{n+1}[\alpha_{n+1}] = 0,$$

so that $[d_0\theta] = [\alpha_0]$. It follows that the map $(\alpha_0, \dots, \alpha_n) : \partial\Delta^{n+1} \to X$ is homotopic to a map extends to $\Delta^{n+1}$, namely $(d_0\theta, \alpha_1, \dots, \alpha_{n+1})$, so a homotopy extension property argument shows that $(\alpha_0, \dots, \alpha_{n+1})$ has the required extension. □

The standard proof for the homotopy addition theorem [18] is a basic combinatorial manipulation, but the arguments involved become quite complicated when one tries to replace the boundary of a simplex by a slightly more exotic shape, such as the boundary of a prism.

Write

$$\partial(\Delta^n \times \Delta^1) = (\Delta^n \times \partial\Delta^1) \cup (\partial\Delta^n \times \Delta^1).$$

Let $X$ be a pointed Kan complex and consider the set of pointed homotopy classes of maps

$$\alpha : \Delta^n \times \Delta^1/\partial(\Delta^n \times \Delta^1) \to X.$$

This set is in bijective correspondence with $\pi_{n+1}X$ by a standard homotopy extension argument: the map $\alpha$ determines a map

$$((\alpha, *, \dots, *), *) : (\partial\Delta^{n+1} \times \Delta^1) \cup (\Delta^{n+1} \times \{0\}) \to X$$

which is $\alpha$ on the $0^{th}$ face of $\partial\Delta^{n+1} \times \Delta^1$ and constant at $*$ on all other faces. Then there is an extension

$$\begin{tikzcd}
(\partial\Delta^{n+1} \times \Delta^1) \cup (\Delta^{n+1} \times \{0\}) \arrow[r, "{((\alpha, *, \dots, *), *)}"] \arrow[d, hook] & X \\
\Delta^{n+1} \times \Delta^1 \arrow[ur, dashed, "h_\alpha"'] &
\end{tikzcd}$$

and the restriction $\gamma_\alpha$ of the extension $h_\alpha$ to $\Delta^{n+1} \times \{1\}$ determines a well defined element $[\gamma_\alpha]$ in $\pi_{n+1}X$. One can reverse the construction, and show that $[\alpha] \mapsto [\gamma_\alpha]$ is a bijection.

In the fibre sequence

$$F \xrightarrow{j} X \xrightarrow{p} X(n),$$

$p_*(\pi_j X) = 0$ for $j \geq n+1$, so that by arguing successively as in the proof of Theorem 3.13 that $\alpha$, $\gamma_\alpha$ and the prism $h_\alpha$ relating them are in the image of $j : F \to X$ up to pointed homotopy. Keep the same notation for the corresponding maps taking values in the fibre $F$. We are then entitled to study the image of the prism $h_\alpha : \Delta^{n+1} \times \Delta^1 \to F$ under the Hurewicz map

$$F \to \mathbb{Z}F \to \mathbb{Z}F/\mathbb{Z}*.$$

Given a map $\beta : \Delta^n \times \Delta^1 \to X$, we have seen in the proof of Lemma 2.16 that the non-degenerate simplex $h_j = (s_j \iota_n, \theta_j)$ determines an $(n+1)$-simplex $h_j(\beta) = \beta(h_j)$, and the chain

$$s(\beta) = \sum_{i=0}^{n} (-1)^i h_j(\beta) \in \mathbb{Z}X_{n+1}$$

satisfies an equation

$$\partial s(\beta) + s(\partial \beta) = d_0 h_0(\beta) - d_{n+1} h_n(\beta).$$

Here,

$$\partial \beta = \sum_{i=0}^{n} (-1)^i \beta \cdot (d^i \times \Delta^1).$$

It follows that if $\beta|_{\partial(\Delta^n \times \Delta^1)} = *$, then the chain $s(\beta)$ is an $(n+1)$-cycle of $\mathbb{Z}X/\mathbb{Z}*$.

In the notation above, we see that

$$\partial s(h_\alpha) + s(\alpha) = \gamma_\alpha$$

in the reduced chain complex $\mathbb{Z}F/\mathbb{Z}*$, and so it follows that the image of the simplex $\gamma_\alpha$ under the Hurewicz map represents the cohomology class $[s(\alpha)] \in \tilde{H}_{n+1}(F, \mathbb{Z})$.

We are now ready to prove the following:

THEOREM 3.14. *Suppose that $X$ is a pointed Kan complex, and that $n \geq 2$. Suppose that $\beta : \Delta^n \times \Delta^1 \to X$ is a prism which maps the boundary of each end face*

$$\omega_\epsilon(\beta) : \Delta^n \cong \Delta^n \times \{\epsilon\} \subset \Delta^n \times \Delta^1 \xrightarrow{\beta} X,$$

*$\epsilon = 0, 1$, and the boundary of each side*

$$\beta_j : \Delta^{n-1} \times \Delta^1 \xrightarrow{d^j \times 1} \Delta^n \times \Delta^1 \xrightarrow{\beta} X,$$

*$0 \leq j \leq n$, into the base point. Associate elements $[\gamma_{\beta_j}] \in \pi_n X$ to each map $\beta_j$ according to the recipe above. Then there is a relation*

$$\sum_{i=0}^{n} (-1)^i [\gamma_{\beta_i}] = [\omega_1(\beta)] - [\omega_0(\beta)]$$

*in $\pi_n X$. Conversely, if the formula holds, then the map $\partial(\Delta^n \times \Delta^1) \to X$ determined by the simplices $\omega_\epsilon(\beta)$ and the prisms $\beta_i$ extends to a map $\beta : \Delta^n \times \Delta^1 \to X$.*

PROOF: Form the fibre sequence

$$F \xrightarrow{j} X \xrightarrow{p} X(n)$$

as before, and argue as previously to push the simplices $\omega_\epsilon(\beta)$, the prisms $\beta_i$ and $\beta$ itself into $F$ up to pointed homotopy. Consider the image of $\beta$ under the Hurewicz map

$$F \to \mathbb{Z}F \to \mathbb{Z}F/\mathbb{Z}* .$$

Then on the chain level

$$\partial s(\beta) + \sum_{i=0}^{n} (-1)^i s(\beta_i) = \omega_1(\beta) - \omega_0(\beta),$$

so that

$$\sum_{i=0}^{n} (-1)^i [s(\beta_i)] = [\omega_1(\beta)] - [\omega_0(\beta)]$$

in $\tilde{H}_n(F, \mathbb{Z})$. But $[s(\beta_i)]$ is the image of the homotopy class $[\gamma_{\beta_i}]$, so that

$$\sum_{i=0}^{n} (-1)^i [\gamma_{\beta_i}] = [\omega_1(\beta)] - [\omega_0(\beta)]$$

in $\pi_n F$, hence in $\pi_n X$.

The existence of the extension $\Delta^n \times \Delta^1 \to X$ extending the map $\partial(\Delta^n \times \Delta^1) \to X$ determined by $\omega_\epsilon(\beta)$ and the $\beta_i$ satisfying the boundary formula

$$\sum_{i=0}^{n} (-1)^i [\gamma_{\beta_i}] = [\omega_1(\beta)] - [\omega_0(\beta)]$$

follows from the reverse implication and a homotopy extension argument, as in the proof of Theorem 3.13. □

REMARK 3.15. The statement corresponding to Theorem 3.14 for $n = 1$ asserts that under the stated conditions there is a relation

$$[\omega_1(\beta)][\beta_1] = [\beta_0][\omega_0(\beta)]$$

in $\pi_1 X$. The proof of this assertion is an easy exercise.

Rather a lot of standard homotopy theory is amenable to proof by simplicial techniques. The reader may find it of particular interest to recast the Hausmann-Husemoller treatment of acyclic spaces and the Quillen plus construction [41] in this setting. In order to achieve this, it's helpful to know at the outset that the *universal cover* $\tilde{X}$ of a connected pointed Kan complex $X$ is, generally, the homotopy fibre of the map $X \to B(\pi_1 X)$ — see as well the definition of *covering system* in Section VI.3. This means, in particular, that the homotopy type of $\tilde{X}$ can be recovered from the pullback diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d, "\pi"'] & E(\pi_1 X) \arrow[d] \\
X \arrow[r] & B(\pi_1 X),
\end{tikzcd}$$

so that the map $\pi : \tilde{X} \to X$ is a principal $\pi_1 X$-fibration.

## 4. The $\mathrm{Ex}^\infty$ functor.

Kan's $\mathrm{Ex}^\infty$ functor is a combinatorial construction which associates a Kan complex $\mathrm{Ex}^\infty X$ to an arbitrary simplicial set $X$, up to natural weak equivalence. It is constructed as an inductive limit of spaces $\mathrm{Ex}^n X$, in such a way that the $m$-simplices of $\mathrm{Ex}^{n+1} X$ are a finite inverse limit of sets of simplices of $\mathrm{Ex}^n X$. This means in particular that this construction has very useful analogues in categories other than simplicial sets. It remains interesting in its own right in the simplicial set context, since it involves subdivision in a fundamental way.

We give the details of this construction and establish its basic properties in this section. It is one of the few remaining areas of simplicial homotopy theory in which the original combinatorial flavour of the subject (see the proof of Lemma 4.7) has not been engulfed by the calculus of anodyne extensions.

Recall that the non-degenerate simplices of the standard $n$-simplex

$$\Delta^n = \hom_{\mathbf{\Delta}}(\ ,\mathbf{n})$$

are the monic ordinal number maps $\mathbf{m} \hookrightarrow \mathbf{n}$. There is exactly one such monomorphism for each subset of $\mathbf{n}$ of cardinality $m+1$. It follows that the non-degenerate simplices of $\Delta^n$ form a poset $P\Delta^n$, ordered by the face relation, and this poset is isomorphic to the non-empty subsets of the ordinal number $\mathbf{n}$, ordered by inclusion.

The poset has a nerve $BP\Delta^n$. We shall write

$$\mathrm{sd}\,\Delta^n = BP\Delta^n,$$

and call it the *subdivision* of $\Delta^n$.

Lemma 4.1. *There is a homeomorphism*

$$h : |\,\mathrm{sd}\,\Delta^n| \xrightarrow{\cong} |\Delta^n|,$$

*where $h$ is the affine map which takes a vertex $\sigma = \{v_0, \dots, v_k\}$ of $\mathrm{sd}\,\Delta^n$ to the barycentre $\frac{1}{k+1}(v_0 + \dots + v_k)$ of the corresponding vertices.*

In other words, $|\,\mathrm{sd}\,\Delta^n|$ is the barycentric subdivision of $|\Delta^n|$.

Proof: To see the co-ordinate transformation, take

$$\alpha_0 v_0 + \alpha_1 v_1 + \dots + \alpha_n v_n \in |\Delta^n|,$$

and rewrite it as

$$t_1 X_1 + t_2 X_2 + \dots t_r X_r,$$

where $0 < t_1 < t_2 < \dots < t_r$ and $X_i = v_{j_0} + v_{j_1} + \dots + v_{j_{n_i}}$. More precisely, the numbers $t_i$ are the distinct values of the $\alpha_j$, arranged in order, and $X_i$ is the sum of the vertices having coefficient $t_i$.

Write

$$N_j = \sum_{k=j}^{r} (n_k + 1).$$

Then

$$\begin{aligned} & t_1 X_1 + \dots + t_r X_r \\ & = t_1(X_1 + \dots + X_r) + (t_2 - t_1)(X_2 + \dots + X_r) + \dots + (t_r - t_{r-1})X_r \\ & = t_1 N_1 (\frac{1}{N_1})(X_1 + \dots + X_r) + (t_2 - t_1) N_2 (\frac{1}{N_2})(X_2 + \dots + X_r) + \dots \\ & \qquad + (t_r - t_{r-1}) N_r (\frac{1}{N_r}) X_r. \end{aligned}$$

Note that

$$t_1 N_1 + (t_2 - t_1) N_2 + \dots + (t_r - t_{r-1}) N_r = 1,$$

so that we've rewritten $\alpha_0 v_0 + \dots + \alpha_n v_n$ as an affine sum of uniquely determined barycentres. □

Any *function* $f : \mathbf{n} \to \mathbf{m}$ determines a map of posets $f_* : P\Delta^n \to P\Delta^m$ via $f_*(X) = f(X) =$ image of $X$ under $f$. It follows that any poset morphism $\theta : \mathbf{n} \to \mathbf{m}$ determines a poset map $\tilde{\theta} : P\Delta^n \to P\Delta^m$, and hence induces a simplicial set map $\tilde{\theta} : \mathrm{sd}\,\Delta^n \to \mathrm{sd}\,\Delta^m$. This assignment is functorial, so we obtain a cosimplicial object $\mathbf{n} \mapsto \mathrm{sd}\,\Delta^n$ in the category of simplicial sets.

The *subdivision* $\mathrm{sd}\,X$ of a simplicial set $X$ is defined by

$$\mathrm{sd}\,X = \varinjlim_{\sigma : \Delta^n \to X} \mathrm{sd}\,\Delta^n,$$

where the colimit is indexed over the simplex category $\mathbf{\Delta} \downarrow X$ for $X$.

The functor $X \mapsto \mathrm{sd}\,X$ is left adjoint to a functor $Y \mapsto \mathrm{Ex}\,Y$, where the simplicial set $\mathrm{Ex}\,Y$ is defined to have $n$-simplices given by the set of all simplicial set maps $\mathrm{sd}\,\Delta^n \to Y$.

There is a natural map $h : \mathrm{sd}\,\Delta^n \to \Delta^n$, called the *last vertex map*. It is specified as a map of posets $P\Delta^n \to \mathbf{n}$ by the assignment

$$[v_0, v_1, \dots, v_k] \mapsto v_k,$$

where $[v_0,\dots,v_k] : \mathbf{k} \to \mathbf{n}$ is a non-degenerate simplex of $\Delta^n$ specified by $i \mapsto v_i$. There is also a map of posets $g : \mathbf{n} \to P\Delta^n$ defined by $g(i) = [0,1,\dots,i]$. Clearly $hg = 1$ and there is a relation

$$[v_0, v_1, \dots, v_k] \leq [0, 1, \dots, v_k]$$

in $P\Delta^n$. It follows that the last vertex map $h : \operatorname{sd}\Delta^n \to \Delta^n$ is a simplicial homotopy equivalence.

The natural maps $h : \operatorname{sd}\Delta^n \to \Delta^n$ induce a natural simplicial map

$$\eta = h^* : Y \to \operatorname{Ex} Y,$$

which is given on $n$-simplices by precomposition by the indicated map $h$.

LEMMA 4.2. *The map $\eta : Y \to \operatorname{Ex} Y$ is a $\pi_0$ isomorphism, and induces a surjection on fundamental groupoids.*

PROOF: The map $\eta$ is an isomorphism on the vertex level. A 1-simplex of $\operatorname{Ex} Y$ is a diagram

$$x \xrightarrow{\alpha} y \xleftarrow{\beta} z$$

of 1-simplices of $Y$, and $\eta(\alpha)$ is the diagram

$$x \xrightarrow{\alpha} y \xleftarrow{s_0 y} y$$

for any 1-simplex $x \xrightarrow{\alpha} y$ of $Y$ (incidentally, the notation means that $d_0(\alpha) = y$ and $d_1(\alpha) = x$). Thus, two vertices of $\operatorname{Ex} Y$ are related by a string of 1-simplices of $\operatorname{Ex} Y$ if and only if the corresponding vertices are related by a string on 1-simplices of $Y$, so that $\eta$ is a $\pi_0$-isomorphism as claimed.

For each 1-simplex

$$x \xrightarrow{\alpha} y \xleftarrow{\beta} z$$

of $\operatorname{Ex} Y$, there is a 2-simplex $\sigma$ of $\operatorname{Ex} Y$

$$\begin{tikzcd}
 & & y \arrow[ddl, "y"'] \arrow[ddr, "y"] \arrow[ddd, "y"] & & \\
 & & & & \\
 & y \arrow[dr, "y"] & & y \arrow[dl, "y"'] & \\
 & & y & & \\
x \arrow[uur, "\alpha"] \arrow[urr, "\alpha"] \arrow[rr, "\alpha"'] & & y \arrow[u, "y"] & & z \arrow[uul, "\beta"'] \arrow[ull, "\beta"'] \arrow[ll, "\beta"]
\end{tikzcd}$$

Here, $x$ is the $0^{th}$ vertex $v_0$ of $\sigma$, $v_1 = z$ and $v_2$ is the top copy of $y$. The two lower left 2-simplices of $y$ are copies of $s_1\alpha$, the two lower right 2-simplices are copies of $s_1\beta$, and the two upper 2-simplices are constant simplices of $Y$ associated to the vertex $y$. It follows that there is a relation

$$[\eta(\alpha)] = [x \xrightarrow{\alpha} y \xleftarrow{\beta} z][\eta(\beta)]$$

in the path category associated to $\mathrm{Ex}\, Y$. But then

$$[x \xrightarrow{\alpha} y \xleftarrow{\beta} z] = [\eta(\alpha)][\eta(\beta)]^{-1}$$

in the fundamental groupoid $\pi(\mathrm{Ex}\, Y)$. Every generator of $\pi(\mathrm{Ex}\, Y)$ is therefore in the image of the induced functor $\eta_* : \pi(Y) \to \pi(\mathrm{Ex}\, Y)$. □

LEMMA 4.3. *The map $\eta : Y \to \mathrm{Ex}\, Y$ induces an isomorphism*

$$\eta_* : H_*(Y, \mathbb{Z}) \xrightarrow{\cong} H_*(\mathrm{Ex}\, Y, \mathbb{Z})$$

*in integral homology.*

PROOF: The natural maps $\eta : \Delta^k \to \mathrm{Ex}\, \Delta^k$ can be used to show that the Ex functor preserves homotopies. It follows that $\mathrm{Ex}\, \mathrm{sd}\, \Delta^n$ is contractible; in effect, the poset $P\Delta^n$ contracts onto the top non-degenerate simplex of $\Delta^n$. It follows that $\mathrm{Ex}\, \mathrm{sd}\, \Delta^n$ has the homology of a point.

The natural map $\eta : Y \to \mathrm{Ex}\, Y$ induces an isomorphism $\pi_0 Y \cong \pi_0\, \mathrm{Ex}\, Y$, by Lemma 4.2. It follows that $\eta$ induces a natural isomorphism

$$\eta_* : H_0(Y, \mathbb{Z}) \xrightarrow{\cong} H_0(\mathrm{Ex}\, Y, \mathbb{Z})$$

in the $0^{th}$ integral homology group. The simplices $\sigma : \Delta^n \to \mathrm{Ex}\, Y$ factor through maps $\mathrm{Ex}\, \sigma_* : \mathrm{Ex}\, \mathrm{sd}\, \Delta^n \to \mathrm{Ex}\, Y$. A standard acyclic models argument therefore implies that there is a natural chain map $\gamma : \mathbb{Z}\mathrm{Ex}\, Y \to \mathbb{Z}Y$ between the associated Moore complexes which induces the map $\eta_*^{-1} : H_0(\mathbb{Z}\mathrm{Ex}\, Y) \to H_0(\mathbb{Z}Y)$, and any two natural chain maps which induce $\eta_*^{-1}$ are naturally chain homotopic. Similarly, the composite natural chain map

$$\mathbb{Z}\mathrm{Ex}\, Y \xrightarrow{\gamma} \mathbb{Z}Y \xrightarrow{\eta_*} \mathbb{Z}\mathrm{Ex}\, Y$$

is naturally chain homotopic to the identity map, and the models $\mathbb{Z}\Delta^n$ are used to show that the composite

$$\mathbb{Z}Y \xrightarrow{\eta_*} \mathbb{Z}\mathrm{Ex}\, Y \xrightarrow{\gamma} \mathbb{Z}Y$$

is naturally chain homotopic to the identity. The map $\eta_* : \mathbb{Z}Y \to \mathbb{Z}\mathrm{Ex}\, Y$ is therefore a natural chain homotopy equivalence, and so the map $\eta : Y \to \mathrm{Ex}\, Y$ is a homology isomorphism. □

COROLLARY 4.4. *If the canonical map $Y \to *$ is a weak equivalence, then so is the map* $\operatorname{Ex} Y \to *$.

PROOF: The fundamental groupoid $\pi Y$ for $Y$ is trivial, and $\eta$ induces a surjection $\eta_* : \pi Y \to \pi \operatorname{Ex} Y$ by Lemma 4.2, so that $\operatorname{Ex} Y$ has a trivial fundamental groupoid as well. But $\operatorname{Ex} Y$ is acyclic by Lemma 4.3, so that $\operatorname{Ex} Y$ is weakly equivalent to a point, by the Hurewicz Theorem (Theorem 3.7). □

LEMMA 4.5. *The functor* Ex *preserves Kan fibrations.*

PROOF: We show that the induced map $\operatorname{sd} \Lambda^n_k \hookrightarrow \operatorname{sd} \Delta^n$ is a weak equivalence.

The simplicial set $\operatorname{sd} \Lambda^n_k$ can be identified with the nerve of the poset of non-degenerate simplices of $\Lambda^n_k$, and the homeomorphism

$$h : |\operatorname{sd} \Delta^n| \xrightarrow{\cong} |\Delta^n|$$

restricts to a homeomorphism

$$|\operatorname{sd} \Lambda^n_k| \xrightarrow{\cong} |\Lambda^n_k|.$$

It follows that $\operatorname{sd} \Lambda^n_k \hookrightarrow \operatorname{sd} \Delta^n$ is a weak equivalence. □

THEOREM 4.6. *The natural map* $\eta : Y \to \operatorname{Ex} Y$ *is a weak equivalence.*

PROOF: Let $Y_f$ be a fibrant model for $Y$ in the sense that there is a weak equivalence $\alpha : Y \to Y_f$. Pick a base point $y$ for $Y$, and let $PY_f$ be path space for $Y_f$ corresponding to the base point $\alpha(y)$. Form the pullback diagram

$$\begin{tikzcd}
\alpha^{-1} PY_f \arrow[r, "\alpha_*"] \arrow[d, "\pi_*"'] & PY_f \arrow[d, "\pi"] \\
Y \arrow[r, "\alpha"'] & Y_f,
\end{tikzcd}$$

where $\pi : PY_f \to Y_f$ is the canonical fibration. Then $\alpha_*$ is a weak equivalence (Corollary II.8.6), so that $\alpha^{-1} PY_f$ is weakly equivalent to a point, and the fibre sequence

$$\Omega Y_f \to \alpha^{-1} PY_f \xrightarrow{\pi_*} Y.$$

gives rise to a comparison of fibre sequences

$$\begin{tikzcd}
\Omega Y_f \arrow[r] \arrow[d, "\eta"'] & \alpha^{-1} PY_f \arrow[r, "\pi_*"] \arrow[d, "\eta"'] & Y \arrow[d, "\eta"] \\
\operatorname{Ex} \Omega Y_f \arrow[r] & \operatorname{Ex} \alpha^{-1} PY_f \arrow[r, "\operatorname{Ex} \pi_*"'] & \operatorname{Ex} Y,
\end{tikzcd}$$

by Lemma 4.5. Computing in homotopy groups of realizations (and thereby implicitly using the fact that the realization of a Kan fibration is a Serre fibration — Theorem I.10.10), we find a commutative diagram

$$\begin{tikzcd}
\pi_1(Y, y) \arrow[r, "\cong"] \arrow[d, "\eta_*"'] & \pi_0(\Omega Y_f) \arrow[d, "\eta_*", "\cong"'] \\
\pi_1(\mathrm{Ex}\, Y, y) \arrow[r, "\cong"'] & \pi_0(\mathrm{Ex}\, \Omega Y_f).
\end{tikzcd}$$

One uses Corollary 4.4 to see that the indicated boundary maps are isomorphisms. It follows that $\eta$ induces an isomorphism

$$\eta_* : \pi_1(Y, y) \xrightarrow{\cong} \pi_1(\mathrm{Ex}\, Y, y)$$

for all choices of base point $y$ of $Y$. Inductively then, one shows that all maps

$$\eta_* : \pi_i(Y, y) \xrightarrow{\cong} \pi_i(\mathrm{Ex}\, Y, y)$$

are isomorphisms for all choices of base point $y$. □

LEMMA 4.7. *For any map $\lambda : \Lambda^n_k \to \mathrm{Ex}\, Y$, the dotted arrow exists in the diagram*

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\lambda"] \arrow[d, hook] & \mathrm{Ex}\, Y \arrow[d, "\eta"] \\
\Delta^n \arrow[r, dashed] & \mathrm{Ex}^2\ Y.
\end{tikzcd}$$

PROOF: The adjoint of the composite map

$$\Lambda^n_k \xrightarrow{\lambda} \mathrm{Ex}\, Y \xrightarrow{\eta} \mathrm{Ex}^2\ Y$$

is the composite

$$\mathrm{sd}\, \Lambda^n_k \xrightarrow{h} \Lambda^n_k \xrightarrow{\lambda} \mathrm{Ex}\, Y.$$

It therefore suffices to show that the dotted arrow $h_*$ exists in the diagram

$$\begin{tikzcd}
\mathrm{sd}\, \Lambda^n_k \arrow[r, "h"] \arrow[d, hook] & \Lambda^n_k \arrow[d, "\tilde{\eta}"] \\
\mathrm{sd}\, \Delta^n \arrow[r, dashed] & \mathrm{Ex}\, \mathrm{sd}\, \Lambda^n_k,
\end{tikzcd}$$

making it commute, where $\tilde{\eta} : K \to \operatorname{Ex}\operatorname{sd} K$ is the counit of the adjunction, and is defined by sending a simplex $\sigma : \Delta^n \to K$ to the simplex of $\operatorname{Ex}\operatorname{sd} K$ given by the induced map $\operatorname{sd}\sigma : \operatorname{sd}\Delta^n \to \operatorname{sd} K$.

Let $\sigma = (\sigma_0, \dots, \sigma_q)$ be a $q$-simplex of $\operatorname{sd}\Delta^n$, where the maps $\sigma_i : \mathbf{n}_i \hookrightarrow \mathbf{n}$ are simplices of $\Delta^n$. Define a function $f_\sigma : \mathbf{q} \to \mathbf{n}$ by the assignment

$$f_\sigma(i) = \begin{cases} \sigma_i(n_i) & \text{if } \sigma_i \neq d_k\iota_n \text{ or } \iota_n \text{, and} \\ k & \text{if } \sigma_i = d_k\iota_n \text{ or } \sigma_i = \iota_n. \end{cases}$$

The assignment $\sigma \mapsto f_\sigma$ is natural with respect to morphisms of ordinal numbers $\theta : \mathbf{q}' \to \mathbf{q}$ in the sense that $f_\sigma\theta = f_{\sigma\tilde{\theta}}$. There is a unique pair $(X_\sigma, \pi_\sigma)$ consisting of a poset monic $X_\sigma : \mathbf{r} \hookrightarrow \mathbf{n}$ and a surjective function $\pi_\sigma : \mathbf{q} \twoheadrightarrow \mathbf{r}$ such that the diagram of functions

$$\begin{tikzcd}
\mathbf{q} \arrow[rr, "f_\sigma"] \arrow[dr, "\pi_\sigma"', two heads] & & \mathbf{n} \\
& \mathbf{r} \arrow[ur, hook, "X_\sigma"'] &
\end{tikzcd}$$

commutes. Then a simplicial map $h_* : \operatorname{sd}\Delta^n \to \operatorname{Ex}\operatorname{sd}\Lambda^n_k$ is defined by the assignment $\sigma \mapsto \pi^*_\sigma(\tilde{\eta}(X_\sigma))$. Note that, whereas $\pi_\sigma : \mathbf{q} \to \mathbf{r}$ is only a function, it induces a poset map $\pi_{\sigma *} : P\Delta^q \to P\Delta^r$ and hence a simplicial map $\pi_{\sigma *} : \operatorname{sd}\Delta^q \to \operatorname{sd}\Delta^r$, so that the definition of the map $h_*$ makes sense. $\square$

For any simplicial set $X$, define $\operatorname{Ex}^\infty X$ to be the colimit in the simplicial set category of the string of maps

$$X \xrightarrow{\eta_X} \operatorname{Ex} X \xrightarrow{\eta_{\operatorname{Ex} X}} \operatorname{Ex}^2 X \xrightarrow{\eta_{\operatorname{Ex}^2 X}} \operatorname{Ex}^3 X \to \dots$$

Then the assignment $X \mapsto \operatorname{Ex}^\infty X$ defines a functor from the simplicial set category to itself, which is commonly called the $\operatorname{Ex}^\infty$ *functor* . Write $\nu : X \to \operatorname{Ex}^\infty X$ for the canonical natural map which arises from definition of $\operatorname{Ex}^\infty X$. The results of this section imply the following:

Theorem 4.8.

1. *The canonical map* $\nu : X \to \operatorname{Ex}^\infty X$ *is a weak equivalence, for any simplicial set* $X$.
2. *For any* $X$*, the simplicial set* $\operatorname{Ex}^\infty X$ *is a Kan complex.*
3. *The* $\operatorname{Ex}^\infty$ *functor preserves Kan fibrations.*

Proof: The first statement is a consequence of Theorem 4.6. Statement (2) is implied by Lemma 4.7. The third statement follows from Lemma 4.5. $\square$

## 5. The Kan suspension.

The ordinal number map $d^0 : \mathbf{n} \to \mathbf{n+1}$ induces an inclusion $d^0 : \Delta^n \to \Delta^{n+1}$. Let the vertex 0 be a base point for $\Delta^{n+1}$, and observe that any simplicial map $\theta : \Delta^n \to \Delta^m$ uniquely extends to a simplicial map $\tilde{\theta} : \Delta^{n+1} \to \Delta^{m+1}$, which is pointed in the sense that $\tilde{\theta}(0) = 0$, and such that $\tilde{\theta} d^0 = d^0 \theta$. Note that $\tilde{d^i} = d^{i+1}$ and $\tilde{s^j} = s^{j+1}$ for all $i$ and $j$.

The *cone* $CY$ of a simplicial set $Y$ is the pointed simplicial set

$$CY = \varinjlim_{\Delta^n \to Y} \Delta^{n+1},$$

where the colimit is indexed in the *simplex category* of all simplices $\Delta^n \to Y$, and is formed in the pointed simplicial set category. The maps $d^0 : \Delta^n \to \Delta^{n+1}$ induce a natural map $j : Y \to CY$.

The cone $CY$ on a simplicial set $Y$ is contractible: the contracting homotopies $h : \Delta^{n+1} \times \Delta^1 \to \Delta^{n+1}$ given by the transformations $0 \to i$ in the ordinal numbers $\mathbf{n+1}$ glue together along the simplices of $Y$ to give a contracting homotopy $h : CY \times \Delta^1 \to CY$ onto the base point of $Y$, since there are commutative diagrams

$$\begin{tikzcd}
\Delta^{n+1} \times \Delta^1 \arrow[r, "h"] \arrow[d, "\tilde{\theta} \times 1"'] & \Delta^{n+1} \arrow[d, "\tilde{\theta}"] \\
\Delta^{m+1} \times \Delta^1 \arrow[r, "h"'] & \Delta^{m+1}
\end{tikzcd}$$

for any ordinal number map $\theta : \mathbf{n} \to \mathbf{m}$.

We can see now that a simplicial set map $f : Y \to X$ can be extended to a map $g : CY \to X$ in the sense that there is a commutative diagram

$$\begin{tikzcd}
CY \arrow[dr, "g"] & \\
Y \arrow[u, "j"] \arrow[r, "f"'] & X
\end{tikzcd}$$

if and only if for each $n$-simplex $x$ of $Y$ there is an $(n+1)$-simplex $g(x)$ of $X$ such that

(1) $d_0(g(x)) = f(x)$,

(2) $d_1 d_2 \cdots d_{n+1}(g(x))$ is some fixed vertex $v$ of $X$ for all simplices $x$ of $Y$, and

(3) for all $i, j \geq 0$ and all simplices $x$ of $Y$, we have

$$d_{i+1}(g(x)) = g(d_i x) \quad \text{and} \quad s_{j+1}(g(x)) = g(s_j x).$$

For the moment, given a simplicial set $X$, let $X_{-1} = \pi_0 X$ and write $d_0 : X_0 \to X_{-1}$ for the canonical map $X_0 \to \pi_0 X$.

A simplicial set $X$ is said to have an *extra degeneracy* if there are functions $s_{-1} : X_n \to X_{n+1}$ for all $n \geq -1$, such that, in all degrees,

(1) $d_0 s_{-1}$ is the identity on $X_n$,

(2) for all $i, j \geq 0$, we have the identities

$$d_{i+1}s_{-1}(x) = s_{-1}d_i(x) \quad \text{and} \quad s_{j+1}s_{-1}(x) = s_{-1}s_j(x).$$

LEMMA 5.1. *Suppose that a simplicial set $X$ has an extra degeneracy. Then the canonical map $X \to K(\pi_0 X, 0)$ is a homotopy equivalence.*

PROOF: It suffices to assume that $X$ is connected. Then the association $x \mapsto s_{-1}(x)$ determines an extension

$$\begin{tikzcd} CX \arrow[dr, "s_{-1}"] & \\ X \arrow[u, "j"] \arrow[r, "1"'] & X \end{tikzcd}$$

of the identity map on $X$, according to the criteria given above. Also, $CX$ contracts onto its base point, which point maps to $s_{-1}(*)$ in $X$, where $*$ denotes the unique element of $\pi_0 X$. □

EXAMPLE 5.2. Suppose that $G$ is a group. The *translation category $EG$* associated to the $G$-action $G \times G \to G$ has the elements of $G$ for objects, and has morphisms of the form $h : g \to hg$. The nerve of this category is commonly also denoted by $EG$. Note in particular that an $n$-simplex of the resulting simplicial set $EG$ has the form

$$g_0 \xrightarrow{g_1} g_1 g_0 \xrightarrow{g_2} \dots \xrightarrow{g_n} g_n \cdot \dots \cdot g_0,$$

and may therefore be identified with a string $(g_0, g_1, \dots, g_n)$ of elements of the group $G$. The simplicial set $EG$ is plainly connected. By thinking in terms of strings of arrows, it is seen that the assignment

$$(g_0, g_1, \dots, g_n) \mapsto (e, g_0, g_1, \dots, g_n)$$

defines an extra degeneracy $s_{-1} : EG_n \to EG_{n+1}$ for $EG$, so that $EG$ is contractible.

If $K$ is a pointed simplicial set, then the *pointed cone* $C_*K$ is defined by the pushout

$$\begin{tikzcd}
C\Delta^0 \arrow[r, "C*"] \arrow[d] & CK \arrow[d] \\
* \arrow[r] & C_*K.
\end{tikzcd}$$

Here, the map $C*$ is induced by the inclusion of the base point $* : \Delta^0 \to K$ in $K$.

The maps $d^0 : \Delta^n \to \Delta^{n+1}$ induce a natural pointed map $i : K \to C_*K$. The *Kan suspension* $\Sigma K$ of $K$ is defined to be the quotient

$$\Sigma K = C_*K/K.$$

The Kan suspension $\Sigma K$ is a reduced simplicial set, and is a concrete model for the suspension of the associated pointed space $|K|$, in the sense that the realization $|\Sigma K|$ is naturally homeomorphic to the topological suspension of $|K|$. The existence of this homeomorphism is one of the reasons that the Kan suspension functor $\Sigma : \mathbf{S}_* \to \mathbf{S}_*$ preserves weak equivalences — one could also argue directly from the definitions by using the cofibre sequence

$$K \subset C_*K \to \Sigma K$$

and the contracting homotopy on the cone $C_*K$.

A pointed simplicial set map $\phi : \Sigma K \to Y$ consists of pointed functions

$$\phi_n : K_n \to Y_{n+1} \tag{5.3}$$

such that

(1) $d_1 \dots d_{n+1}\phi_n(x) = *$, and $d_0\phi_n(x) = *$ for each $x \in K_n$, and

(2) for each ordinal number map $\theta : \mathbf{n} \to \mathbf{m}$, the diagram of pointed functions

$$\begin{tikzcd}
K_n \arrow[r, "\phi_n"] \arrow[d, "\theta^*"'] & Y_{n+1} \arrow[d, "(\tilde{\theta})^*"] \\
K_m \arrow[r, "\phi_m"'] & Y_{m+1}
\end{tikzcd}$$

commutes.

Pointed simplicial set maps of the form $\psi : C_*K \to Y$ have a very similar characterization; one simply deletes the requirement that $d_0\psi_n(x) = *$. It follows that the pointed cone and Kan suspension functors preserve colimits of pointed simplicial sets.

An equivalent description of $C_*K$ starts from the observation that the pointed simplicial set $K$ is a member of a coequalizer diagram of the form

$$\bigvee_{\theta:\mathbf{n}\to\mathbf{m}} K_m \wedge \Delta^n_+ \rightrightarrows \bigvee_{n\geq 0} K_n \wedge \Delta^n_+ \to K$$

where, for example, $K_m \wedge \Delta^n_+$ is the wedge of the pointed set of $m$-simplices $K_m$, thought of as a discrete pointed simplicial set, with $\Delta^n_+$, and $\Delta^n_+$ is notation for the simplicial set $\Delta^n \sqcup \{*\}$, pointed by the disjoint vertex $*$. Then $C_*K$ is defined by the coequalizer diagram

$$\bigvee_{\theta:\mathbf{n}\to\mathbf{m}} K_m \wedge \Delta^{n+1} \rightrightarrows \bigvee_{n\geq 0} K_n \wedge \Delta^{n+1} \to C_*K.$$

The set of $m$-simplices of $\Delta^{n+1}$ is the set of ordinal number maps of the form $\gamma : \mathbf{m} \to \mathbf{n+1}$. Each such $\gamma$ fits into a pullback diagram

$$\begin{tikzcd}
\mathbf{j} \arrow[r, "\gamma_*"] \arrow[d, "(d^0)^{m-j}"'] & \mathbf{n} \arrow[d, "d^0"] \\
\mathbf{m} \arrow[r, "\gamma"'] & \mathbf{n+1}
\end{tikzcd}$$

in the ordinal number category, for some uniquely determined map $\gamma_* : \mathbf{j} \to \mathbf{n}$ if $\gamma^{-1}(\mathbf{n}) \neq \emptyset$. It follows that, as a pointed set, $\Delta^{n+1}_m$ has the form

$$\begin{aligned}
\Delta^{n+1}_m &= \Delta^n_m \sqcup \Delta^n_{m-1} \sqcup \cdots \sqcup \Delta^n_0 \sqcup \{*\} \\
&= (\Delta^n_+)_m \vee (\Delta^n_+)_{m-1} \vee \cdots \vee (\Delta^n_+)_0,
\end{aligned}$$

where the base point corresponds to the case $\gamma(\mathbf{m}) = 0$. Now take another map $\zeta : \mathbf{k} \to \mathbf{m}$; there is a pullback diagram of ordinal number maps

$$\begin{tikzcd}
\zeta^{-1}(\mathbf{j}) \arrow[r, "\hat{\zeta}"] \arrow[d] & \mathbf{j} \arrow[d, "(d^0)^{m-j}"] \\
\mathbf{k} \arrow[r, "\zeta"'] & \mathbf{m}
\end{tikzcd} \tag{5.4}$$

in the case where $\mathbf{r} = \zeta^{-1}(\mathbf{j}) \neq \emptyset$. It follows that the restriction of $\zeta^* : \Delta^{n+1}_m \to \Delta^{n+1}_k$ to the summand $(\Delta^n_+)_j$ is the map

$$\hat{\zeta}^* : (\Delta^n_+)_j \to (\Delta^n_+)_r$$

if $\zeta^{-1}(\mathbf{j}) \neq \emptyset$, and is the map to the base point otherwise.

Suppose that $K$ is a pointed simplicial set. There is a a pointed simplicial set $\tilde{C}_*(K)$ whose set of $n$-simplices is given by

$$\tilde{C}_*(K)_n = K_n \vee K_{n-1} \vee \cdots \vee K_0.$$

The map $\zeta^* : \tilde{C}_*(K)_m \to \tilde{C}_*(K)_k$ associated to $\zeta : \mathbf{k} \to \mathbf{m}$ is given on the summand $K_j$ by the composite

$$K_j \xrightarrow{\hat{\zeta}^*} K_r \hookrightarrow \tilde{C}_*(K_k)$$

in the case where $\zeta^{-1}(\mathbf{j}) \neq \emptyset$, $\mathbf{r} = \zeta^{-1}(\mathbf{j})$, and the map $\hat{\zeta}$ is defined by the diagram (5.4). If $\zeta^{-1}(\mathbf{j}) = \emptyset$, then the restriction of $\zeta^*$ to $K_j$ maps to the base point.

One checks that $\tilde{C}_*(K)$ is indeed a pointed simplicial set, and that the construction is functorial in $K$. Furthermore, the functor preserves colimits, so that the diagram

$$\bigvee_{\theta:\mathbf{n}\to\mathbf{m}} \tilde{C}_*(K_m \wedge \Delta^n_+) \rightrightarrows \bigvee_{n\geq 0} \tilde{C}_*(K_n \wedge \Delta^n_+) \to \tilde{C}_*K.$$

is a coequalizer. On the other hand, the definitions imply that there are isomorphisms

$$\begin{aligned} \tilde{C}_*(Y \wedge \Delta^n_+) &\cong Y \wedge \tilde{C}_*(\Delta^n_+) \\ &\cong Y \wedge \Delta^{n+1}, \end{aligned}$$

which are natural in the pointed sets $Y$ and simplices $\Delta^n$. This is enough to prove

LEMMA 5.5. *There is a pointed simplicial set isomorphism*

$$\tilde{C}_*(K) \cong C_*(K),$$

*which is natural in $K$.*

The composite

$$K \xrightarrow{i} C_*(K) \cong \tilde{C}_*(K),$$

in degree $n$ is the inclusion of the wedge summand $K_n$. The canonical map $i : K \to C_*(K)$ is therefore an inclusion. Collapsing by $K$ in each degree also gives a nice description of the Kan suspension $\Sigma K$: the set of $(n+1)$-simplices of $\Sigma K$ is given by the wedge sum

$$\Sigma K_{n+1} = K_n \vee K_{n-1} \vee \cdots \vee K_0.$$

It's also a worthy exercise to show that the maps $\eta_n : K_n \to \Sigma K_{n+1}$ corresponding to the identity map $\Sigma K \to \Sigma K$ under the association (5.3) are inclusions $K_n \to K_n \vee K_{n-1} \vee \cdots \vee K_0$ of wedge summands.

REMARK 5.6. The abelian version of Kan suspension is the Eilenberg-Mac Lane $\overline{W}$ construction $A \mapsto \overline{W}A$, for simplicial abelian groups $A$. This construction is a also a special case of a functor $G \mapsto \overline{W}G$ which is defined for simplicial groups $G$ — the functor for simplicial groups is discussed at some length in Section V.4.

The group of $(n+1)$-simplices of the simplicial abelian group $\overline{W}A$ is given by the direct sum

$$\overline{W}A_{n+1} = A_n \oplus A_{n-1} \oplus \cdots \oplus A_0,$$

and the simplicial structure maps are defined on direct summands by analogy with the structure maps for the Kan suspension functor. Collapsing by degeneracies according to Theorem 2.1 yields natural isomorphisms

$$N\overline{W}A_n \cong NA_{n-1},$$

and these isomorphisms fit into a commutative diagram

$$\begin{tikzcd}
N\overline{W}A_n \arrow[r, "\cong"] \arrow[d, "\partial"'] & NA_{n-1} \arrow[d, "(-1)^n d_{n-1}"] \\
N\overline{W}A_{n-1} \arrow[r, "\cong"'] & NA_{n-2}
\end{tikzcd}$$

The $\overline{W}$ construction for simplicial abelian groups thus corresponds to a shift on the chain complex level, and it is therefore the "good" suspension for simplicial abelian groups. This phenomenon and its consequences are discussed more fully in Section 4.6 of [52].

# Chapter IV Bisimplicial sets

This chapter is a basic exposition of the homotopy theory of bisimplicial sets and bisimplicial abelian groups.

A bisimplicial set can be viewed either as a simplicial object in the category of simplicial sets or a contravariant functor on the product $\mathbf{\Delta} \times \mathbf{\Delta}$ of two copies of ordinal number category $\mathbf{\Delta}$: both points of view are constantly exploited. Similar considerations apply to bisimplicial objects in any category, and to bisimplicial abelian groups in particular.

Categories of bisimplicial objects are ubiquitous sources of spectral sequence constructions. In many contexts, bisimplicial sets and bisimplicial abelian groups function as analogs of projective resolutions for homotopy theoretic objects. The Serre spectral sequence is one of the original examples: pullbacks over simplices of the base of a map $p : E \to B$ form a bisimplicial resolution of the total space. Then every bisimplicial set has canonically associated bisimplicial abelian groups and hence bicomplexes, and so a spectral sequence (5.3) drops out. If the map $p$ happens to be a fibration, the resulting spectral sequence is the Serre spectral sequence (5.5). It is not much of a conceptual leap from this construction to the notion of a homology fibre sequence, which is the basis for the group completion theorem (Theorem 5.15). These ideas are essentially non-abelian, so the theory can be pushed to give the basic detection principle for homotopy cartesian diagrams (Lemma 5.7) that is the basis of proof for Quillen's Theorem B (Theorem 5.6). Group completion and Theorem B are fundamental tools for algebraic $K$-theory. This collection of results appears in Section 5.

We begin in Section 1 with some definitions and examples of bisimplicial sets and abelian groups, which examples include the bisimplicial sets underlying homotopy colimits. Section 2 contains a discussion of the basic features of bisimplicial abelian groups, including homotopy colimit objects and the generalized Eilenberg-Zilber theorem (Theorem 2.4). This theorem asserts that the two standard ways of extracting a chain complex from a bisimplicial abelian group, namely the chain complex associated to the diagonal simplicial abelian group and the total complex, are naturally chain homotopy equivalent. We show (Lemma 2.11) that the homotopy groups of homotopy colimit simplicial abelian groups are the left derived functors of the colimit functor.

A description of the formal homotopy theory of bisimplicial sets is given in Section 3. This homotopy theory is a little complicated, because there are closed model structures associated to multiple definitions of weak equivalence for these objects. The diagonal functor creates an external notion — one says that a map of bisimplicial sets is a diagonal weak equivalence if it induces a weak equivalence of associated diagonal simplicial sets. There are also two internal descriptions of weak equivalence, corresponding to viewing a bisimplicial set as a diagram in its vertical or horizontal simplicial sets. In particular, we say that a bisimplicial set map is a pointwise (or vertical) weak equivalence if each of the induced maps of vertical simplicial sets is a weak equivalence. We discuss closed model structures associated to all of these definitions. Diagonal weak equivalences are the objects of study in the Moerdijk structure, whereas pointwise weak equivalences figure into two different structures, namely the Bousfield-Kan structure in which the fibrations are defined pointwise, and the Reedy structure where the cofibrations are defined pointwise. All of these theories are useful, and they are used jointly in the applications that follow, but this is by no means the end of the story: there is a further notion of $E_2$-weak equivalence due to Dwyer, Kan and Stover [28], [29] and a corresponding closed model structure that is not discussed here.

We confine ourselves here to applications of the homotopy theory of bisimplicial sets that involve detection of cartesian squares of bisimplicial set morphisms that become homotopy cartesian after applying the diagonal functor. There are two extant non-trivial techniques. One of these is the circle of ideas related to the Serre spectral sequence in Section 5, which has already been discussed. The other is the theorem of Bousfield and Friedlander (Theorem 4.9) which is dealt with in Section 4, and arises from the Reedy closed model structure in the presence of the so-called $\pi_*$-Kan condition. The $\pi_*$-Kan condition is satisfied widely in nature, in particular for all pointwise connected bisimplicial sets; it is best expressed by saying that the canonical maps from the homotopy group objects fibred over the simplicial set of vertical vertices of a bisimplicial set to the vertices are Kan fibrations. The Bousfield-Friedlander theorem leads to a spectral sequence (see (4.14)) for the homotopy groups of the diagonal of a pointwise connected bisimplicial set. This spectral sequence is the origin of the definition of $E_2$-weak equivalence that is referred to above.

## 1. Bisimplicial sets: first properties.

A *bisimplicial set* $X$ is a simplicial object in the category of simplicial sets, or equivalently a functor $X : \mathbf{\Delta}^{op} \to \boldsymbol{S}$ where $\mathbf{\Delta}$ is the ordinal number category and $\boldsymbol{S}$ denotes the category of simplicial sets as before. Write $\boldsymbol{S}^2$ for the category of bisimplicial sets.

A bisimplicial set $X$ can also be viewed as a functor

$$X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \boldsymbol{S},$$

or as a contravariant functor on the category $\mathbf{\Delta} \times \mathbf{\Delta}$, by the categorical exponential law. From this point of view, the data for $X$ consists of sets $X(m,n)$ with appropriately defined functions between them. The set $X(m,n)$ will often be called the set of *bisimplices of* $X$ *of bidegree* $(m,n)$, or the $(m,n)$-*bisimplices* of $X$. We shall also say that a bisimplex $x \in X(m,n)$ has horizontal degree $m$ and vertical degree $n$.

EXAMPLE 1.1. The bisimplicial set $\Delta^{k,l}$ is the simplicial object in $\boldsymbol{S}$ which is composed of the simplicial sets

$$\bigsqcup_{\mathbf{n} \xrightarrow{\theta} k} \Delta^l,$$

where $\Delta^l$ is the standard $l$-simplex in $\mathbf{S}$ and the disjoint union is indexed over morphisms $\theta : \mathbf{n} \to \mathbf{k}$ in the ordinal number category $\boldsymbol{\Delta}$. The bisimplicial set $\Delta^{k,l}$ classifies bisimplices of bidegree $(k,l)$: there is a simplex

$$\iota_{k,l} \in \bigsqcup_{\mathbf{k} \xrightarrow{\tau} \mathbf{k}} \Delta^l,$$

which is a *classifying* $(k,l)$*-simplex* in the sense that the bisimplices $x \in X(k,l)$ in a bisimplicial set $X$ are in one to one correspondence with maps $\iota_x : \Delta^{k,l} \to X$ such that $\iota(\iota_{k,l}) = x$. Specifically, the *classifying bisimplex* $\iota_{k,l}$ is the copy of the classifying $l$-simplex $\iota_l \in \Delta^l_l$ in the summand corresponding to the identity map $1 : \mathbf{k} \to \mathbf{k}$.

It follows that $\Delta^{k,l}$ is the contravariant functor on $\boldsymbol{\Delta} \times \boldsymbol{\Delta}$ which is represented by the object $(\mathbf{k}, \mathbf{l})$.

EXAMPLE 1.2. Suppose that $K$ and $L$ are simplicial sets. Then there is a bisimplicial set $K \tilde{\times} L$ with $(m,n)$-bisimplices specified by

$$K \tilde{\times} L(m,n) = K_m \times L_n.$$

The bisimplicial set $K \tilde{\times} L$ will be called the *external product* of $K$ with $L$. Note that the bisimplicial set $\Delta^{k,l}$ may be alternatively described as the external product $\Delta^k \tilde{\times} \Delta^l$.

The *diagonal simplicial set* $d(X)$ of a bisimplicial set $X$ has $n$-simplices given by

$$d(X)_n = X(n,n).$$

It can also be viewed as the composite functor

$$\boldsymbol{\Delta}^{op} \xrightarrow{\Delta} \boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \xrightarrow{X} \mathbf{S},$$

where $\Delta$ is the diagonal functor.

Think of the bisimplicial set $X$ as a simplicial object in the simplicial set category by defining the *vertical simplicial sets* $X_n = X(n,*)$. Any morphism $\theta : \mathbf{m} \to \mathbf{n}$ gives rise to a diagram

$$\begin{tikzcd}
X_n \times \Delta^m \arrow[r, "1 \times \theta_*"] \arrow[d, "\theta^* \times 1"'] & X_n \times \Delta^n \\
X_m \times \Delta^m.
\end{tikzcd} \tag{1.3}$$

The collection of all such maps $\theta$ therefore determines a pair of maps

$$\bigsqcup_{\substack{\theta \\ \mathbf{m} \longrightarrow \mathbf{n}}} X_n \times \Delta^m \rightrightarrows \bigsqcup_n X_n \times \Delta^n,$$

by letting the restriction of the displayed maps to the summand corresponding to $\theta$ be $1 \times \theta_*$ and $\theta^* \times 1$, respectively. There are also simplicial set maps

$$\gamma_n : X_n \times \Delta^n \to d(X)$$

defined on $r$-simplices by

$$\gamma_n(x, \mathbf{r} \xrightarrow{\tau} \mathbf{n}) = \tau^*(x).$$

Here, $x$ is an $r$-simplex of $X_n$, and $\tau^* : X_n \to X_r$ is a simplicial structure map of $X$, so that $\tau^*(x)$ is an $r$-simplex of $X(r, *)$ and is therefore an $r$-simplex of $d(X)$. The collection of the maps $\gamma_n$ together determine a simplicial set map

$$\gamma : \bigsqcup_n X_n \times \Delta^n \to d(X).$$

EXERCISE 1.4. *Show that the resulting diagram*

$$\bigsqcup_{\substack{\theta \\ \mathbf{m} \longrightarrow \mathbf{n}}} X_n \times \Delta^m \rightrightarrows \bigsqcup_n X_n \times \Delta^n \xrightarrow{\gamma} d(X)$$

*is a coequalizer in the category of simplicial sets.*

The exercise implies that the diagonal simplicial set $d(X)$ is a coend in the category of simplicial sets for the data given by all diagrams of the form (1.3).

The diagonal simplicial set $d(X)$ has a natural filtration by subobjects $d(X)^{(p)}$, $p \geq 0$, where

$$d(X)^{(p)} = \text{image of } \Big( \bigsqcup_{0 \leq n \leq p} X_n \times \Delta^n \Big) \text{ in } d(X).$$

The degenerate part (with respect to the horizontal simplicial structure) of $X_{p+1}$ is filtered by subobjects

$$s_{[r]} X_p = \bigcup_{0 \leq i \leq r} s_i(X_p) \subset X_{p+1}.$$

It follows from the simplicial identities that there are pushout diagrams

$$\begin{tikzcd}
s_{[r]}X_{p-1} \arrow[r, "s_{r+1}"] \arrow[d, hook] & s_{[r]}X_p \arrow[d, hook] \\
X_p \arrow[r, "s_{r+1}"'] & s_{[r+1]}X_p,
\end{tikzcd} \tag{1.5}$$

and

$$\begin{tikzcd}
(s_{[p]}X_p \times \Delta^{p+1}) \cup (X_{p+1} \times \partial\Delta^{p+1}) \arrow[r] \arrow[d, hook] & d(X)^{(p)} \arrow[d, hook] \\
X_{p+1} \times \Delta^{p+1} \arrow[r] & d(X)^{(p+1)}.
\end{tikzcd} \tag{1.6}$$

Diagrams (1.5) and (1.6) and the gluing lemma (Lemma II.8.8) are the basis of an inductive argument leading to the proof of

PROPOSITION 1.7. *Suppose that $f : X \to Y$ is a map of bisimplicial sets which is a pointwise weak equivalence in the sense that all of the maps $f : X_n \to Y_n$ are weak equivalences of simplicial sets. Then the induced map $f_* : d(X) \to d(Y)$ of associated diagonal simplicial sets is a weak equivalence.*

EXAMPLE 1.8. Any simplicial set-valued functor $Z : I \to \boldsymbol{S}$ gives rise to a bisimplicial set $BEZ = BE_IZ$, with $(m,n)$-bisimplices

$$BE_IZ_{m,n} = \bigsqcup_{i_0 \to i_1 \to \cdots \to i_m} Z(i_0)_n. \tag{1.9}$$

Note that the indexing is over simplices of degree $m$ in the nerve $BI$ of the category $I$, or equivalently over strings of arrows of length $m$ in $I$.

The *homotopy colimit* of the functor $Z$ is the diagonal $d(BE_IZ)$; one usually writes $\underrightarrow{\operatorname{holim}}_I Z = d(BE_IZ)$.

As the notation suggests, each of the $n$-simplex functors $Z_n$ gives rise to a *translation category* $E_IZ_n$ having objects $(i,x)$ with $i$ an object of $I$ and $x \in Z_n(i)$, and with morphisms $\alpha : (i,x) \to (j,y)$ where $\alpha : i \to j$ is a morphism of $I$ such that $Z_n(\alpha)(x) = y$. Then the set of $m$-simplices of the nerve $BE_IZ_n$ is the set displayed in (1.9). Furthermore, the data is simplicial in $n$, so the simplicial object $BE_IZ$ is indeed a bisimplicial set.

This definition of homotopy colimit is standard and we have given one of the standard notations for it, but there is some tendency in the literature (eg. [48]) to confuse homotopy colimits with their underlying bisimplicial sets. The diagonal simplicial set functor associates the "correct" homotopy type to a bisimplicial set (see Sections 3 and 4), so the distinction between the internal homotopy type of a bisimplicial set and that of its associated diagonal is rather slight in practice.

EXAMPLE 1.10. Suppose that $G$ is a group and that $X$ is a simplicial set admitting a left $G$-action. The homotopy colimit $\underrightarrow{\text{holim}}_G\, X$ for the corresponding functor $X : G \to \mathbf{S}$ has $n$-simplices is isomorphic to the traditional Borel construction $EG \times_G X$ for the action of $G$ on $X$. To see this, recall that $EG$ is the nerve of the translation category arising from the left action of $G$ on itself given by the multiplication map $G \times G \to G$. Then the set of $n$-simplices of the Borel construction $EG \times_G X$ is defined to be the quotient of the product $EG_n \times X_n$ for the $G$-action given in terms of the evident right $G$-action on $EG$ by

$$g \cdot (\omega, x) = (\omega g^{-1}, gx).$$

It follows that there is a bijection

$$(EG \times_G X)_n \cong \underset{G}{\underrightarrow{\text{holim}}}\, X_n$$

which is defined on the $n$-simplex level by sending the class

$$[(e \xrightarrow{g_1} g_1 \xrightarrow{g_2} g_2 g_1 \to \cdots, x)]$$

to the $n$-simplex

$$x \xrightarrow{g_1} g_1 x \xrightarrow{g_2} g_2 g_1 x \to \cdots$$

of the homotopy colimit. One shows that the simplex level bijections determine an isomorphism

$$EG \times_G X \cong \underset{G}{\underrightarrow{\text{holim}}}\, X$$

by checking that simplicial structure maps are respected.

## 2. Bisimplicial abelian groups.

We collect here the basic theory of bisimplicial abelian groups, in two subsections.

The first of these is effectively about homotopy colimit constructions in the category of simplicial abelian groups. Any functor $A : I \to s\mathbf{Ab}$ taking values in simplicial abelian groups has an associated bisimplicial abelian group which we call translation object, and is formed by analogy with the homotopy colimit of a diagram of simplicial sets. We derive the basic technical result that translation object for the diagram $A$ is weakly equivalent to the simplicial abelian group $A(t)$ if the index category $I$ has a terminal object $t$.

When we say that a bisimplicial abelian group $B$ is weakly equivalent to a simplicial abelian group $C$, we mean that the diagonal simplicial abelian group $d(B)$ is weakly equivalent to $C$ within the simplicial abelian group category. One could alternatively interpret $B$ as a bicomplex and $C$ as a chain complex, and then ask for a weak equivalence between the chain complexes $\text{Tot}(B)$ and $C$. The generalized Eilenberg-Zilber theorem says that $\text{Tot}(B)$ and the chain complex $d(B)$ are naturally chain homotopy equivalent, so in fact there is no difference between the two approaches to defining such weak equivalences. This is the subject of the second subsection. We further show that the standard spectral sequence

$$E_2^{p,q} = H_p(H_q A_*) \Rightarrow H_{p+q}(d(A))$$

for a bisimplicial abelian group $A$ can be derived by methods which are completely internal to the simplicial abelian group category. The section closes with the example of this spectral sequence which calculates the homology of a homotopy colimit; the $E_2$ term is identified by showing that the homotopy groups of the translation object for a functor $A : I \to \mathbf{Ab}$ are the higher derived functors of the colimit functor.

### 2.1. The translation object.

This section contains technical results concerning a simplicial abelian group-valued analogue of the homotopy colimit construction for diagrams of abelian groups, called the translation object. This construction is of fundamental importance for the description of the Serre spectral sequence that appears in a subsequent section of this chapter. More generally, it appears canonically in any homology spectral sequence arising from a homotopy colimit of a diagram of simplicial sets.

Suppose that $A : I \to \mathbf{Ab}$ is an abelian group valued functor, where $I$ is a small category. There is a simplicial abelian group $EA$, with

$$EA_n = \bigoplus_{\gamma:\mathbf{n}\to I} A_{\gamma(0)},$$

where the direct sum is indexed by the $n$-simplices of the nerve $BI$ of the index category $I$. The abelian group homomorphism $\theta^* : EA_n \to EA_m$ induced by an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ is specified by requiring that all diagrams

$$\begin{tikzcd}
A_{\gamma(0)} \arrow[r] \arrow[d, "in_{\gamma}"'] & A_{\gamma\theta(0)} \arrow[d, "in_{\gamma\theta}"] \\
\bigoplus_{\gamma:\mathbf{n}\to I} A_{\gamma(0)} \arrow[r, "\theta^*"'] & \bigoplus_{\zeta:\mathbf{m}\to I} A_{\zeta(0)}
\end{tikzcd}$$

commute, where the homomorphism $A_{\gamma(0)} \to A_{\gamma\theta(0)}$ is induced by the relation $0 \leq \theta(0)$ in the ordinal number $\mathbf{n}$. The simplicial abelian group $EA$ is called the *translation object* associated to the functor $A$.

Note that $EA$ is not the nerve of a category, even though its definition is analogous to that of the nerve of a translation category for a set-valued functor.

Suppose given functors

$$J \xrightarrow{F} I \xrightarrow{A} \mathbf{Ab},$$

where $J$ and $I$ are small categories. The functor $F$ induces a simplicial abelian group homomorphism $F_* : E(AF) \to EA$, which is defined on $n$-simplices in such a way that the diagram

$$\begin{tikzcd}
AF_{\theta(0)} \arrow[r, "1"] \arrow[d, "in_\theta"'] & A_{F\theta(0)} \arrow[d, "in_{F\cdot\theta}"] \\
\bigoplus\limits_{\theta:\mathbf{n}\to J} AF_{\theta(0)} \arrow[r, "F_*"'] & \bigoplus\limits_{\gamma:\mathbf{n}\to I} A_{\gamma(0)}
\end{tikzcd}$$

commutes.

Any natural transformation $\omega : A \to B$ of functors $I \to \mathbf{Ab}$ determines a morphism $\omega_* : EA \to EB$ of simplicial abelian groups. This morphism $\omega_*$ is defined on $n$-simplices by the requirement that the following diagram commutes:

$$\begin{tikzcd}
A_{\theta(0)} \arrow[r, "\omega"] \arrow[d, "in_\theta"'] & B_{\theta(0)} \arrow[d, "in_\theta"] \\
\bigoplus\limits_{\theta:\mathbf{n}\to I} A_{\theta(0)} \arrow[r, "\omega_*"'] & \bigoplus\limits_{\theta:\mathbf{n}\to I} B_{\theta(0)}.
\end{tikzcd}$$

Now consider a functor $B : I \times \mathbf{1} \to \mathbf{Ab}$, and let $d^1 : I \to I \times \mathbf{1}$ and $d^0 : I \to I \times \mathbf{1}$ be defined, respectively, by $d^1(i) = (i, 0)$ and $d^0(i) = (i, 1)$. The maps $(i, 0) \to (i, 1)$ in $I \times \mathbf{1}$ induce a natural transformation $\eta : Bd^1 \to Bd^0$ of functors $I \to \mathbf{Ab}$, and hence induce a simplicial abelian group homomorphism $\eta_* : EBd^1 \to EBd^0$. In general, the group of $n$-simplices of the simplicial abelian group $EA \otimes \Delta^1$ can be identified with the direct sum

$$\bigoplus_{(\theta,\gamma):\mathbf{n}\to I\times\mathbf{1}} A_{\theta(0)}.$$

In the case at hand, one finds a canonical map $h : EBd^1 \otimes \Delta^1 \to EB$, which is defined on $n$-simplices by the requirement that the diagram

$$\begin{tikzcd}
B_{(\theta(0),0)} \arrow[r] \arrow[d, "in_{(\theta,\gamma)}"'] & B_{(\theta(0),\gamma(0))} \arrow[d, "in_{(\theta,\gamma)}"] \\
\bigoplus\limits_{(\theta,\gamma):\mathbf{n}\to I\times\mathbf{1}} B_{(\theta(0),0)} \arrow[r, "h"'] & \bigoplus\limits_{(\theta,\gamma):\mathbf{n}\to I\times\mathbf{1}} B_{(\theta(0),\gamma(0))}
\end{tikzcd}$$

commutes, where the top horizontal map is induced by the morphism

$$(\theta(0), 0) \to (\theta(0), \gamma(0))$$

in $I \times \mathbf{1}$. One can now check that there is a commutative diagram of simplicial abelian group homomorphisms

$$\begin{tikzcd}
EBd^1 \arrow[d, "d^1"'] \arrow[drr, "d^1_*"] & & \\
EBd^1 \otimes \Delta^1 \arrow[rr, "h"'] & & EB. \\
EBd^1 \arrow[u, "d^0"] \arrow[r, "\eta_*"'] & EBd^0 \arrow[ur, "d^0_*"'] &
\end{tikzcd} \tag{2.1}$$

LEMMA 2.2. *Let $A : I \to \mathbf{Ab}$ be an abelian group valued functor on a small category $I$, and suppose that $I$ has a terminal object $t$. Then there is a canonical weak equivalence $EA \to K(A_t, 0)$, which is specified on $n$-simplices by the homomorphism*

$$\bigoplus_{\theta:\mathbf{n}\to I} A_{\theta(0)} \to A_t$$

*given on the summand corresponding to $\theta : \mathbf{n} \to I$ by the map $A_{\theta(0)} \to A_t$ induced by the unique morphism $\theta(0) \to t$ of the index category $I$.*

PROOF: Let $t$ also denote the composite functor

$$I \to \{t\} \subset I.$$

Then the discrete category $\{t\}$ is a strong deformation retract of the category $I$ in the sense that there is a commutative diagram of functors

$$\begin{tikzcd}
I \arrow[d, "d^1"'] \arrow[dr, "1"] & \\
I \times \mathbf{1} \arrow[r, "H"] & I. \\
I \arrow[u, "d^0"] \arrow[ur, "t"'] &
\end{tikzcd}$$

Now, $EAHd^1 = EA$, so that (2.1) can be used to show there is a commutative diagram of simplicial abelian group homomorphisms

$$\begin{tikzcd}
EA \arrow[d, "d^1"'] \arrow[drr, "1"] & & \\
EA \otimes \Delta^1 \arrow[r, "h"'] & EAH \arrow[r, "H_*"'] & EA. \\
EA \arrow[u, "d^0"] \arrow[r, "\eta_*"'] & EAt \arrow[ur, "t_*"'] &
\end{tikzcd}$$

Note that $\eta_* t_*$ is the identity map on the simplicial abelian group $EAt$, so that the Moore complex of $EA$ is chain homotopy equivalent to the Moore complex of $EAt$. The map $EA \to K(A_t, 0)$ factors as the weak equivalence $\eta_* : EA \to EA_t$, followed by the map $EAt \to K(A_t, 0)$, which map is defined on $n$-simplices by the codiagonal map

$$\bigoplus_{\theta:\mathbf{n}\to I} A_t \overset{\nabla}{\longrightarrow} A_t.$$

This last homomorphism is a weak equivalence, because the space $BI$ is contractible. ☐

### 2.2. The generalized Eilenberg-Zilber theorem.

A *bisimplicial abelian group* $A$ is a simplicial object in the category of simplicial abelian groups, or equivalently a functor of the form

$$A : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{Ab},$$

where **Ab** denotes the category of abelian groups, as before. Subject to the latter description, the simplicial abelian groups $A(n, *)$ will be referred to as the vertical simplicial abelian groups, while the objects $A(*, m)$ are the horizontal simplicial abelian groups. The category of bisimplicial abelian groups and natural transformations between them will be denoted by $s^2\mathbf{Ab}$.

It is often convenient to write $A_n = A(n, *)$ for the vertical simplicial abelian group in horizontal degree $n$. The simplicial abelian group morphism $A_n \to A_m$ associated to an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ will sometimes be denoted by $\theta^{*h}$: this morphism is given on $k$-simplices by the abelian group homomorphism $(\theta, 1)^* : A(n, k) \to A(m, k)$. We shall also occasionally write $\tau^{*v} = (1, \tau)^* : A(n, k) \to A(n, p)$ for the vertical structure maps associated to $\tau : \mathbf{p} \to \mathbf{k}$.

The diagonal simplicial abelian group $d(A)$ associated to the bisimplicial abelian group $A$ has $n$-simplices given by $d(A)_n = A(n,n)$, and the association $A \mapsto d(A)$ defines a functor $d : s^2\mathbf{Ab} \to s\mathbf{Ab}$. The diagonal functor is plainly exact. Furthermore, if $B$ is a simplicial abelian group and $K(B,0)$ is the associated horizontally (or vertically) constant bisimplicial abelian group, then there is a natural isomorphism $d(K(B,0)) \cong B$.

The *Moore bicomplex* for a bisimplicial abelian group $A$ is the bicomplex having $(p,q)$-chains $A(p,q)$, horizontal boundary

$$\partial_h = \sum_{i=0}^{p} (-1)^i d_i : A(p,q) \to A(p-1,q),$$

and vertical boundary

$$\partial_v = \sum_{j=0}^{q} (-1)^{p+j} d_j : A(p,q) \to A(p,q-1).$$

We shall also write $A$ for the Moore bicomplex of a bisimplicial abelian group $A$, and $\operatorname{Tot} A$ will denote the associated total complex. Write also $A_n = A(n,*)$ for the Moore complex in horizontal degree $n$. Then filtering the bicomplex $A$ in the horizontal direction gives a spectral sequence

$$E_2^{p,q} = H_p(H_q A_*) \Rightarrow H_{p+q}(\operatorname{Tot} A). \tag{2.3}$$

It follows that, if the bisimplicial abelian group map $f : A \to B$ is a *pointwise weak equivalence* in the sense that all of the maps $f : A_n \to B_n$, $n \geq 0$, of vertical simplicial abelian group are weak equivalences, then $f$ induces an homology isomorphism $f_* : \operatorname{Tot} A \to \operatorname{Tot} B$ of the associated total complexes. Of course, the meaning of "vertical" and "horizontal" are in the eyes of the beholder, so it follows immediately that any map of bisimplicial abelian groups which consists of weak equivalences on the horizontal simplicial abelian group level must again induce a homology isomorphism of total complexes. One can, alternatively, make an argument with the second spectral sequence for the homology of $\operatorname{Tot} A$ (constructed by filtering in the Moore bicomplex in the vertical direction).

The generalized Eilenberg-Zilber theorem of Dold and Puppe [20] asserts the following:

THEOREM 2.4. *The chain complexes $d(A)$ and $\operatorname{Tot} A$ are chain homotopy equivalent. This equivalence is natural with respect to morphisms of bisimplicial abelian groups $A$.*

PROOF: Suppose that $K$ and $L$ are simplicial sets. The usual Eilenberg-Zilber theorem asserts that there are natural chain maps

$$f : \mathbb{Z}(K \times L) \to \operatorname{Tot}(\mathbb{Z}K \otimes \mathbb{Z}L),$$

and

$$g : \operatorname{Tot}(\mathbb{Z}K \otimes \mathbb{Z}L) \to \mathbb{Z}(K \times L),$$

and that there are natural chain homotopies $fg \simeq 1$ and $gf \simeq 1$. Specializing $K$ and $L$ to the standard simplices gives bicosimplicial chain maps

$$f : \mathbb{Z}(\Delta^p \times \Delta^q) \to \mathrm{Tot}(\mathbb{Z}\Delta^p \otimes \mathbb{Z}\Delta^q),$$

and

$$g : \mathrm{Tot}(\mathbb{Z}\Delta^p \otimes \mathbb{Z}\Delta^q) \to \mathbb{Z}(\Delta^p \times \Delta^q),$$

as well as bicosimplicial chain homotopies $fg \simeq 1$ and $gf \simeq 1$.

Observe that $\Delta^p \times \Delta^q = d(\Delta^{p,q})$, where $\Delta^{p,q}$ is the bisimplicial set represented by the pair of ordinal numbers $(\mathbf{p}, \mathbf{q})$. It follows that the chain complex $\mathbb{Z}(\Delta^p \times \Delta^q)$ can be identified up to natural isomorphism with $d(\mathbb{Z}\Delta^{p,q})$. Note as well that, up to natural isomorphism, $\mathrm{Tot}(\mathbb{Z}\Delta^p \otimes \mathbb{Z}\Delta^q)$ is the total complex of the bisimplicial abelian group $\mathbb{Z}\Delta^{p,q}$. Every bisimplicial abelian group $A$ sits in a functorial exact sequence

$$\bigoplus_{\tau \to \sigma} \mathbb{Z}\Delta^{r,s} \to \bigoplus_{\sigma : \Delta^{p,q} \to A} \mathbb{Z}\Delta^{p,q} \to A \to 0.$$

The functors Tot and $d$ are both right exact and preserve direct sums, so the chain maps

$$f : d(\mathbb{Z}\Delta^{p,q}) \to \mathrm{Tot}\, \mathbb{Z}\Delta^{p,q}$$

uniquely extend to a natural chain map

$$f : d(A) \to \mathrm{Tot}\, A.$$

which is natural in bisimplicial abelian groups $A$. Similarly, the chain maps

$$g : \mathrm{Tot}\, \mathbb{Z}\Delta^{p,q} \to d(\mathbb{Z}\Delta^{p,q})$$

induce a natural chain map

$$g : \mathrm{Tot}\, A \to d(A).$$

The same argument implies that the bicosimplicial chain homotopies $fg \simeq 1$ and $gf \simeq 1$ extend uniquely to chain homotopies which are natural in bisimplicial abelian groups. □

REMARK 2.5. The maps $f$ and $g$ in the proof of Theorem 2.4 can be precisely specified as the unique extensions of the classical Alexander-Whitney and shuffle maps, respectively. The definitions will not be written down here (see [20], [64, pp. 241–243]).

The underlying acyclic models argument for the Eilenberg-Zilber theorem is somewhat less than conceptual, so that the usual approach of using the spectral sequence (2.3) and the generalized Eilenberg-Zilber theorem to construct the standard spectral sequence

$$E_2^{p,q} = \pi_p(\pi_q A_*) \Rightarrow \pi_{p+q} d(A)$$

is rather indirect. We can now give an alternative construction — the method is to arrange for some independent means of showing the following:

LEMMA 2.6. *A pointwise weak equivalence $f : A \to B$ of bisimplicial abelian groups induces a weak equivalence $f_* : d(A) \to d(B)$ of the associated diagonal complexes.*

PROOF: There is a bisimplicial abelian group given in vertical degree $m$ by the simplicial abelian group

$$\bigoplus_{\mathbf{m}\to\mathbf{n}_0\to\cdots\to\mathbf{n}_k} A(n_k, m).$$

This simplicial abelian group is the translation object associated to the functor $A(*, m) : (\mathbf{m} \downarrow \mathbf{\Delta})^{op} \to \mathbf{Ab}$ which is defined by associating to the object $\mathbf{m} \to \mathbf{n}$ the abelian group $A(n, m)$. The category $(\mathbf{m} \downarrow \mathbf{\Delta})^{op}$ has a terminal object, namely the identity map $1 : \mathbf{m} \to \mathbf{m}$, so Lemma 2.2 implies that the canonical simplicial abelian group map $EA(*, m) \to K(A(m, m), 0)$ is a weak equivalence. It follows that the Moore complex for $EA(*, m)$ is canonically weakly equivalent to the chain complex $A(m, m)[0]$ consisting of the group $A(m, m)$ concentrated in degree 0. The morphism of bicomplexes which is achieved by letting $m$ vary therefore induces a natural weak equivalence of chain complexes $\operatorname{Tot} EA(*, *) \to d(A)$. On the other hand, the vertical simplicial abelian group of $EA(*, *)$ in horizontal degree $k$ has the form

$$\bigoplus_{\mathbf{n}_0\to\cdots\to\mathbf{n}_k} \Delta^{n_0} \otimes A(n_k, *).$$

It follows that any pointwise equivalence $f : A \to B$ induces a homology isomorphism $f_* : \operatorname{Tot} EA(*, *) \to \operatorname{Tot} EB(*, *)$, and hence a weak equivalence $f_* : d(A) \to d(B)$. □

Define the *horizontal normalization* $N_hA$ of a bisimplicial abelian group $A$ to be the simplicial chain complex whose "$n$-simplices" are given by the chain complex $N_hA_n = NA(*, n)$. The bisimplicial abelian group $A$ can be recovered from the simplicial chain complex $N_hA$ by applying the functor $\Gamma$ in all vertical degrees. The simplicial chain complex $N_hA$ can be filtered: one defines a simplicial chain complex $F_pN_hA$ as a chain complex object by specifying

$$F_pN_hA_i = \begin{cases} N_hA_i & \text{if } i \leq p, \text{ and} \\ 0 & \text{if } i > p. \end{cases}$$

Now, $F_{p-1}N_hA \subset F_pN_hA$, with quotient $N_hA_p[p]$, which can be thought of as the simplicial chain complex which is the simplicial abelian group $N_hA_p$ in chain degree $p$ and is 0 in other chain degrees. Applying the functor $\Gamma$ to the corresponding short exact sequence gives a short exact sequence of bisimplicial abelian groups

$$0 \to \Gamma F_{p-1} N_h A \to \Gamma F_p N_h A \to \Gamma N_h A_p[p] \to 0.$$

It follows that the bisimplicial abelian groups $\Gamma F_p N_h A$ filter $A$. In vertical degree $n$, we have an identification

$$\Gamma N_h A_p[p](*, n) = K(NA(*, n)_p, p),$$

since, in general, the Eilenberg-Mac Lane space $K(B, n)$ can be defined to be the simplicial abelian group $\Gamma B[n]$ which arises by applying the functor $\Gamma$ to the chain complex $B[n]$ which consists of $B$ concentrated in degree $n$ (see Corollary III.3.8).

LEMMA 2.7. *There is an isomorphism*

$$\pi_n N_h A_p \cong N(\pi_n A_*)_p.$$

PROOF: Write

$$N_h^j A_p = \bigcap_{i=0}^{j} \ker(d_i^h) \subset A_p$$

for $0 \leq j \leq p-1$. We show that the canonical map

$$\pi_n N_h^j A_p \to N^j(\pi_n A_*)_p \tag{2.8}$$

is an isomorphism for all $j$.

The degree 0 case is shown by observing that there is a short exact sequence

$$0 \to N_h^0 A_p \to A_p \xrightarrow{d_0^h} A_{p-1} \to 0$$

which is split by $s_0^h : A_{p-1} \to A_p$, so that the map $\pi_n N_h^0 A_p \to N^0(\pi_n A_*)_p$ is an isomorphism. Furthermore, the induced map $\pi_n N_h^0 A_p \to \pi_n A_p$ is a monomorphism.

Assume that the map (2.8) is an isomorphism, and that the induced map

$$\pi_n N_j^h A_p \to \pi_n A_p$$

is monic. Consider the pullback diagram of simplicial abelian groups

$$\begin{tikzcd}
N_h^{j+1} A_p \arrow[r] \arrow[d, hook] & \ker(d_{j+1}^h) \arrow[d] \\
N_h^j A_p \arrow[r] & A_p,
\end{tikzcd}$$

and form the pushout

$$\begin{tikzcd}
N_h^{j+1}A_p \arrow[r] \arrow[d, hook] & \ker(d_{j+1}^h) \arrow[d] \\
N_h^jA_p \arrow[r] & N_h^jA_p + \ker(d_{j+1}^h).
\end{tikzcd}$$

The inclusion map $N_h^{j+1}A_p \subset N_h^jA_p$ is split by the map $x \mapsto x - s_{j+1}^h d_{j+1}^h(x)$, and so a comparison of long exact sequences shows that the induced diagram of abelian group homomorphisms

$$\begin{tikzcd}
\pi_n N_h^{j+1}A_p \arrow[r] \arrow[d, hook] & \pi_n \ker(d_{j+1}^h) \arrow[d] \\
\pi_n N_h^jA_p \arrow[r] & \pi_n(N_h^jA_p + \ker(d_{j+1}^h))
\end{tikzcd}$$

is a pushout. But the inclusion $\ker(d_{j+1}^h) \subset A_p$ is a split monomorphism, so that the induced map $\pi_n \ker(d_{j+1}^h) \to \pi_n A_p$ is monic, and the map $\pi_n N_h^jA_p \to \pi_n A_p$ is monic by the inductive assumption. It follows that the induced map

$$\pi_n(N_h^jA_p + \ker(d_{j+1}^h)) \to \pi_n A_p$$

is a monomorphism, and that the canonical sequence

$$0 \to \pi_n N_h^{j+1}A_p \to \pi_n N_h^jA_p \oplus \pi_n \ker(d_{j+1}^h) \to \pi_n A_p$$

is exact. □

The alternative method for constructing a spectral sequence

$$E_2^{p,q} = \pi_p(\pi_q A_*) \Rightarrow \pi_{p+q} d(A) \tag{2.9}$$

for a bisimplicial abelian group $A$ is now clear. The filtration $\Gamma F_p N_h A$ for the bisimplicial abelian group $A$ gives rise to short exact sequences

$$0 \to \Gamma F_{p-1} N_h A \to \Gamma F_p N_h A \to \Gamma N_h A_p[p] \to 0$$

Let $N_hA_p\langle p\rangle$ be simplicial chain complex

$$\cdots \to 0 \to \overset{p}{N_hA_p[p]} \xrightarrow{1} \overset{p-1}{N_hA_p[p]} \to 0 \to \cdots$$

which is non-trivial in horizontal degrees $p$ and $p-1$ only. Then $N_hA_p\langle p\rangle$ is horizontally acyclic, and there is a short exact sequence of simplicial chain complexes

$$0 \to N_hA_p[p-1] \to N_hA_p\langle p\rangle \to N_hA_p[p] \to 0.$$

It therefore follows from Lemma 2.7 that there are natural isomorphisms

$$\pi_{p+q}\Gamma N_hA_p[p] \cong \pi_q N_hA_p \cong N(\pi_qA_*)_p. \tag{2.10}$$

Furthermore, if we define a spectral sequence by letting

$$E_1^{p,q} = \pi_{p+q}\Gamma N_hA_p[p],$$

then the differential $d_1 : E_1^{p,q} \to E_1^{p-1,q}$ is identified by the isomorphisms (2.10) with the standard differential

$$\partial : N(\pi_qA_*)_p \to N(\pi_qA_*)_{p-1}$$

in the normalized complex for the simplicial abelian group $\pi_qA_*$. It follows that there are canonical isomorphisms

$$E_2^{p,q} \cong \pi_p(\pi_qA_*),$$

and the construction of the spectral sequence (2.9) is complete.

Suppose again that $A : I \to \mathbf{Ab}$ is an abelian group valued functor on a small index category $I$. The homotopy groups of the simplicial abelian group $EA$ are the higher derived functors of the colimit functor:

LEMMA 2.11. *Suppose that $A : I \to \mathbf{Ab}$ is a functor which is defined on a small index category $I$. Then there is a natural isomorphism*

$$\pi_n(EA, 0) \cong L_n(\varinjlim)A$$

*for each $n \geq 0$.*

PROOF: The groups $\pi_n(EA, 0)$ coincide up to isomorphism with the homology groups $H_nEA$ of the associated Moore complex, by Theorem III.2.4. The group $\pi_0EA \cong H_0EA$ sits in an exact sequence

$$\bigoplus_{i\to j} A_i \xrightarrow{d_0-d_1} \bigoplus_{i\in I} A_i \to H_0EA \to 0,$$

and therefore coincides up to natural isomorphism with the group $\varinjlim A$.

The functor $F_i\mathbb{Z}$ is defined at $j \in I$ by

$$F_i\mathbb{Z}(j) = \bigoplus_{i\to j} \mathbb{Z}.$$

The functors $F_i\mathbb{Z}$ are projective, and give an adequate supply of projectives to construct a projective resolution $F_* \to A$. The derived functor $L_n(\varinjlim)A$ is defined to be the homology group $H_n(\varinjlim F_*)$, as is standard.

In particular, the simplicial abelian group $EF_i\mathbb{Z}$ has $n$-simplices

$$(EF_i\mathbb{Z})_n \cong \bigoplus_{i \to j_0 \to \dots \to j_n} \mathbb{Z},$$

so that there is an isomorphism of simplicial abelian groups

$$EF_i\mathbb{Z} \cong \mathbb{Z}B(i \downarrow I).$$

The space $B(i \downarrow I)$ is contractible, so there are isomorphisms

$$H_nEF_i\mathbb{Z} \cong 0$$

for $n > 0$. All members $F_m$ of the projective resolution $F_*$ are direct sums of objects $F_i\mathbb{Z}$, so that $H_nEF_m \cong 0$ for $n > 0$ and all $m$. Both spectral sequences for the bicomplex $EF_*$ therefore collapse at the $E_2$ level, yielding isomorphisms $H_nEA \cong L_n(\varinjlim)A$. □

COROLLARY 2.12. *Suppose that $X : I \to \mathbf{S}$ is a simplicial set valued functor which is defined on a small index category $I$, and let $A$ be an abelian group. Then there is a convergent spectral sequence*

$$E_2^{p,q} \cong L_p(\varinjlim)H_q(X, A) \Rightarrow H_{p+q}(\underrightarrow{\operatorname{holim}} X, A).$$

PROOF: This is the spectral sequence (2.9) for the bisimplicial abelian group $\mathbb{Z}BE_IX \otimes A$. The $E_2$ term is identified by using Lemma 2.11. □

With a little more care (see Section 4.3 of [52]), one can derive a similar convergent spectral sequence

$$E_2^{p,q} = L_p(\varinjlim)h_qX \Rightarrow h_{p+q}(\underrightarrow{\operatorname{holim}} X)$$

for the homology of the homotopy colimit $\underrightarrow{\operatorname{holim}} X$ for any decently behaved homology theory $h_*$.

## 3. Closed model structures for bisimplicial sets.

There are three closed model structures on the category $\mathbf{S}^2$ of bisimplicial sets that will be discussed in this section, namely

(1) the *Bousfield-Kan structure*, in which a fibration is a pointwise fibration and a weak equivalence is a pointwise weak equivalence,

(2) the *Reedy structure*, in which a cofibration is a pointwise cofibration (aka. inclusion) of bisimplicial sets, and a weak equivalence is a pointwise weak equivalence, and

(3) the *Moerdijk structure*, in which a fibration (respectively weak equivalence) is a map $f$ which induces a fibration (respectively weak equivalence) $d(f)$ of associated diagonal simplicial sets.

In all of the above, a map $f : X \to Y$ is said to be a *pointwise* weak equivalence (respectively fibration, cofibration) if each of the simplicial set maps $f : X_n \to Y_n$ is a weak equivalence (respectively fibration, cofibration). This idea was partially introduced at the end of Section 1.

### 3.1. The Bousfield-Kan structure.

The Bousfield-Kan closed model structure on the category $\mathbf{S}^2$ is a special case of a closed model structure introduced by Bousfield and Kan for the category $\mathbf{S}^I$ of $I$-diagrams of simplicial sets arising from a small category $I$ and natural transformations between such — see Example II.7.11. The special case in question corresponds to letting $I$ be the opposite category $\mathbf{\Delta}^{op}$ of the ordinal number category.

A map $f : X \to Y$ is defined to be a weak equivalence (respectively fibration) in the Bousfield-Kan structure on $\mathbf{S}^I$ if each induced simplicial set map $f : X(i) \to Y(i)$ ("in sections") is a weak equivalence (respectively fibration). One says that such maps are *pointwise weak equivalences* (respectively *pointwise fibrations*). Cofibrations are defined by the left lifting property with respect to pointwise trivial fibrations, suitably defined.

The closed model axioms for the Bousfield-Kan structure on $\mathbf{S}^I$ are verified in Example II.5.9. They can also be seen directly by using a small object argument based on the observation that a map $f : X \to Y$ of $\mathbf{S}^I$ is a pointwise fibration if it has the right lifting property with respect to all maps

$$F_i\Lambda^n_k \to F_i\Delta^n, \quad i \in I,$$

and $f$ is a pointwise trivial fibration if it has the right lifting property with respect to all induced maps

$$F_i\partial\Delta^n \to F_i\Delta^n, \quad i \in I.$$

Here, we need to know that $K \mapsto F_iK$ is the left adjoint of the $i$-sections functor $X \mapsto X(i)$, and explicitly

$$F_iK(j) = \bigsqcup_{i \to j} K$$

defines the functor $F_iK$ at $j \in I$.

### 3.2. The Reedy structure.

First, a word or two about skeleta and coskeleta of simplicial sets — the skeleton construction for more general simplicial objects will be discussed in Section V.1.

Write $\mathbf{\Delta}_n$ for the full subcategory of the ordinal numbers on the objects $\mathbf{0}, \dots, \mathbf{n}$. An *n-truncated simplicial set* $Y$ is a functor $Y : \mathbf{\Delta}^{op}_n \to \mathbf{Sets}$. Let $\mathbf{S}_n$ denote the category of $n$-truncated simplicial sets.

Every simplicial set $X : \mathbf{\Delta}^{op} \to \mathbf{Sets}$ gives rise to an $n$-truncated simplicial set $i_{n*}X$ by composition with the inclusion functor $i_n : \mathbf{\Delta}_n \subset \mathbf{\Delta}$. The $n$-truncation functor $X \mapsto i_{n*}X$ has a left adjoint $Y \mapsto i^*_nY$ and a right adjoint $Z \mapsto i^!_nZ$, and these adjoints are defined by left and right Kan extension respectively. Explicitly, the set $i^*_nY_m$ of $m$-simplices of $i^*_nY$ is defined by

$$i^*_nY_m = \varinjlim_{\mathbf{m} \to \mathbf{k} \leq \mathbf{n}} Y_k, \qquad \text{while} \qquad i^!_nZ_m = \varprojlim_{\mathbf{n} \geq \mathbf{k} \to \mathbf{m}} Z_k,$$

where the indicated morphisms in both cases are in the ordinal number category $\mathbf{\Delta}$.

EXERCISE 3.1. *Show that the canonical maps*

$$\eta : Y \to i_{n*}i_n^*Y \qquad \text{and} \qquad \epsilon : i_{n*}i_n^!Z \to Z$$

*are isomorphisms.*

There is a canonical map of simplicial sets $\phi : i_n^*Z \to i_n^!Z$ and a commutative diagram

$$\begin{tikzcd}
Z_m \arrow[r, "(\theta\gamma)^*"] \arrow[d, "\theta^*"'] & Z_k \\
i_n^*Z_{n+1} \arrow[r, "\phi"'] & i_n^!Z_{n+1} \arrow[u, "\gamma^*"']
\end{tikzcd}$$

for each composite

$$\mathbf{k} \xrightarrow{\gamma} \mathbf{n+1} \xrightarrow{\theta} \mathbf{m}$$

in the ordinal number category with $k, m \leq n$. It follows that an extension of the $n$-truncated simplicial set $Z$ to an $(n+1)$-truncated simplicial set consists precisely of a factorization

$$\begin{tikzcd}
i_n^*Z_{n+1} \arrow[r, hook] \arrow[dr, "\phi"'] & Z_{n+1} \arrow[d] \\
& i_n^!Z_{n+1}
\end{tikzcd} \tag{3.2}$$

of the function $\phi : i_n^*Z_{n+1} \to i_n^!Z_{n+1}$. The indicated map is a monomorphism, because $i_n^*Z_{n+1}$ must be the degenerate part of $Z_{n+1}$ on account of the universal property implicit in diagram (1.5).

All of the foregoing is completely natural, and gives corresponding results for diagrams of simplicial sets. In particular, an extension of an $n$-truncated bisimplicial set $Z : \mathbf{\Delta}_n^{op} \to \mathbf{S}$ to an $(n+1)$-truncated bisimplicial set consists of a factorization of the canonical simplicial set map $\phi : i_n^*Z_{n+1} \to i_n^!Z_{n+1}$ of the form (3.2). Of course, the extended object consists of the simplicial sets $Z_0, \dots, Z_n, Z_{n+1}$.

In the literature, the simplicial set $i_{n*}i_{n-1}^!i_{(n-1)*}X$ arising from a bisimplicial set $X$ is denoted by $M_nX$, and is called the $n^{th}$ *matching object* for $X$ [28], [29].

A map $p : X \to Y$ of bisimplicial sets is said to be *Reedy fibration* if

(1) the map $p : X_0 \to Y_0$ is a Kan fibration of simplicial sets, and

(2) each of the induced maps $p_* : X_n \to Y_n \times_{M_nY} M_nX$ is a Kan fibration for $n > 0$.

It is common to write

$$\operatorname{cosk}_n X = i_n^! i_{n*} X,$$

and

$$\operatorname{sk}_n X = i_n^* i_{n*} X,$$

and we shall have occasion to think about the sets of simplices

$$\operatorname{cosk}_n X_m = (\operatorname{cosk}_n X)_m.$$

From this point of view, the canonical maps in condition (2) in the definition of a Reedy fibration are induced by the commutative diagram of simplicial set maps

$$\begin{tikzcd}
X_n \arrow[r, "\eta"] \arrow[d, "p"'] & \operatorname{cosk}_{n-1} X_n \arrow[d, "p_*"] \\
Y_n \arrow[r, "\eta"'] & \operatorname{cosk}_{n-1} Y_n
\end{tikzcd}$$

which arises from the naturality of the adjunction maps. In particular, a bisimplicial set $X$ is Reedy fibrant if the simplicial set $X_0$ is a Kan complex and each of the maps $X_n \to \operatorname{cosk}_{n-1} X_n$, $n > 0$, is a Kan fibration.

Lemma 3.3.

(1) *Suppose that a map $p : X \to Y$ is a Reedy fibration. Then $p$ has the right lifting property with respect to all maps of bisimplicial sets which are pointwise cofibrations and pointwise weak equivalences.*

(2) *Suppose that $p : X \to Y$ is a Reedy fibration such that each of the fibrations*

$$X_n \to Y_n \times_{\operatorname{cosk}_{n-1} Y_n} \operatorname{cosk}_{n-1} X_n$$

*is also a weak equivalence. Then $p$ has the right lifting property with respect to all maps which are pointwise cofibrations.*

Proof: We' ll prove the first assertion. The second is similar.

Suppose given a commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
V \arrow[r] & Y
\end{tikzcd}$$

of bisimplicial set maps in which the map $i$ is a pointwise cofibration and a pointwise weak equivalence. Suppose inductively that there is a commutative diagram of $n$-truncated bisimplicial set maps of the form

$$\begin{tikzcd}
i_{n*}U \arrow[r] \arrow[d, "i"'] & i_{n*}X \arrow[d, "p"] \\
i_{n*}V \arrow[r] \arrow[ur, "\theta"] & i_{n*}Y
\end{tikzcd}$$

Then there is an induced solid arrow diagram of simplicial set maps

$$\begin{tikzcd}
\mathrm{sk}_n V_{n+1} \cup_{\mathrm{sk}_n U_{n+1}} U_{n+1} \arrow[r] \arrow[d, "i_*"'] & X_{n+1} \arrow[d, "p_*"] \\
V_{n+1} \arrow[r] \arrow[ur, dashed] & Y_{n+1} \times_{\mathrm{cosk}_n Y_{n+1}} \mathrm{cosk}_n X_{n+1}
\end{tikzcd}$$

The map $i_*$ is a trivial cofibration, since the functor $U \mapsto \mathrm{sk}_n U_{n+1}$ takes maps which are pointwise weak equivalences to weak equivalences of simplicial sets via diagram (1.5). The map $p_*$ is a Kan fibration by assumption, so the indicated dotted arrow exists. □

COROLLARY 3.4. *Suppose that $p : X \to Y$ is a Reedy fibration. Then all of the fibrations*

$$X_n \to Y_n \times_{\mathrm{cosk}_{n-1} Y_n} \mathrm{cosk}_{n-1} X_n \tag{3.5}$$

*are weak equivalences if and only if $p$ is a pointwise weak equivalence.*

PROOF: The left adjoint $F_{\mathbf{n}}$ of the "level $n$" functor $X \mapsto X_n$ can be defined in terms of the external product by $F_{\mathbf{n}}(K) = \Delta^n \tilde{\times} K$. The functor $F_{\mathbf{n}}$ takes cofibrations to pointwise cofibrations. If all of the maps (3.5) are trivial fibrations, then the map $p$ is a has the right lifting property with respect to the cofibrations $F_{\mathbf{n}}\partial\Delta^n \to F_{\mathbf{n}}\Delta^n$ by Lemma 3.3, so that $p$ is a pointwise trivial fibration. The converse is left to the reader. □

LEMMA 3.6. *Suppose that $f : X \to Y$ is a map of bisimplicial sets. Then $f$ has factorizations*

$$\begin{tikzcd}
 & Z \arrow[dr, "p"] & \\
X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\
 & W, \arrow[ur, "q"'] &
\end{tikzcd}$$

*where*

(1) *the map $i$ is a pointwise cofibration and a pointwise weak equivalence and $p$ is a Reedy fibration, and*
(2) *the map $j$ is a pointwise cofibration and $q$ is a Reedy fibration and a pointwise weak equivalence.*

PROOF: We'll prove the second claim. The first has a similar argument.

It suffices to find a factorization $f = q \cdot j$, where $j$ is a pointwise cofibration and $q$ meets the conditions of Corollary 3.4.

Suppose, inductively, that we've found a factorization

$$\begin{tikzcd}
i_{n*}X \arrow[r, "j"] \arrow[dr, "f"'] & W \arrow[d, "q"] \\
& i_{n*}Y
\end{tikzcd}$$

in the category of $n$-truncated bisimplicial sets, such that $j$ is a pointwise cofibration, and such that the maps $q : W_0 \to Y_0$ and

$$q_* : W_m \to Y_m \times_{i^!_{m-1}Y_m} i^!_{m-1}W_m, \qquad 0 < m \leq n$$

are trivial fibrations of simplicial sets. The commutative diagram

$$\begin{tikzcd}
i_n^*X_{n+1} \arrow[rr] \arrow[d, "j_*"'] \arrow[ddr, "f_*"] & & X_{n+1} \arrow[rr] \arrow[ddr, "f_*"] & & i_n^!X_{n+1} \arrow[d, "j_*"'] \arrow[ddr, "f_*"] & \\
i_n^*W_{n+1} \arrow[rrrr] \arrow[dr, "q_*"'] & & & & i_n^!W_{n+1} \arrow[dr, "q_*"'] & \\
& i_n^*Y_{n+1} \arrow[rr] & & Y_{n+1} \arrow[rr] & & i_n^!Y_{n+1}
\end{tikzcd}$$

induces a diagram

$$\begin{tikzcd}
i_n^*W_{n+1} \arrow[r] \arrow[d] & i_n^*W_{n+1} \cup_{i_n^*X_{n+1}} X_{n+1} \arrow[d] \arrow[dl, "\gamma"'] \\
Y_{n+1} \times_{i_n^!Y_{n+1}} i_n^!W_{n+1} \arrow[r] & i_n^!W_{n+1}.
\end{tikzcd}$$

Choose a factorization

$$\begin{tikzcd}
i_n^* W_{n+1} \cup_{i_n^* X_{n+1}} X_{n+1} \arrow[r, "j'"] \arrow[dr, "\gamma"'] & W_{n+1} \arrow[d, "q'"] \\
& Y_{n+1} \times_{i_n^! Y_{n+1}} i_n^! W_{n+1}
\end{tikzcd}$$

such that $j'$ is a cofibration and $q'$ is a trivial fibration of simplicial sets. Then the desired factorization of $f$ at level $n+1$ is given by the maps

$$X_{n+1} \xrightarrow{j' \cdot j_*} W_{n+1} \xrightarrow{q_* \cdot q'} Y_{n+1}.$$

Note that the map

$$j_* : X_{n+1} \to i_n^* W_{n+1} \cup_{i_n^* X_{n+1}} X_{n+1}$$

is a cofibration. □

The matching space $M_n X = \operatorname{cosk}_{n-1} X_n$ is a special case of a construction which associates a simplicial set $M_K X$ to each pair consisting of a simplicial set $K$ and a bisimplicial set $X$. Explicitly, the $p$-simplices of $M_K X$ are defined to be a set of simplicial set morphisms by setting

$$M_K X_p = \hom_{\mathbf{S}}(K, X(*, p)).$$

The simplicial set $M_K X$ is a *matching space* for $K$ in the bisimplicial set $X$.

Subject to the tacit indentification

$$X_n = X(n, *),$$

the bisimplicial set $\operatorname{cosk}_{n-1} X$ has $(m,p)$-bisimplices specified by

$$\operatorname{cosk}_{n-1} X(m,p) = (\operatorname{cosk}_{n-1} X(*, p))_m.$$

It follows that the $p$-simplices of the simplicial set $\operatorname{cosk}_{n-1} X_n$ have the form

$$\begin{aligned}
\operatorname{cosk}_{n-1} X(n,p) &= (\operatorname{cosk}_{n-1} X(*, p))_n \\
&= \hom_{\mathbf{S}}(\operatorname{sk}_{n-1} \Delta^n, X(*, p)) \\
&= \hom_{\mathbf{S}}(\partial\Delta^n, X(*, p)) \\
&= M_{\partial\Delta^n} X_p,
\end{aligned}$$

so that $M_{\partial\Delta^n}X$ is naturally isomorphic to $M_nX = \operatorname{cosk}_{n-1} X_n$. The functor $K \mapsto M_K X$ is right adjoint to the functor $\mathbf{S} \to \mathbf{S}^2$ which is defined by $L \mapsto K\tilde{\times}L$.

Suppose given integers $s_0, \dots, s_r$ such that $0 \leq s_0 < s_1 < \cdots < s_r \leq n$, and let $\Delta^n\langle s_0, \dots, s_r\rangle$ be the subcomplex of $\partial\Delta^n$ which is generated by the simplices $d_{s_j}\iota_n$, $j = 0, \dots, r$. Then the simplicial identities $d_id_j = d_{j-1}d_i$ for $i < j$ imply that the complex $\Delta^n\langle s_0, \dots, s_r\rangle$ can be inductively constructed by pushout diagrams or the form

$$\begin{tikzcd}
\Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle \arrow[r, "d^{s_r-1}"] \arrow[d, hook] & \Delta^n\langle s_0, \dots, s_{r-1}\rangle \arrow[d, hook] \\
\Delta^{n-1} \arrow[r, "d^{s_r}"'] & \Delta^n\langle s_0, \dots, s_r\rangle .
\end{tikzcd} \tag{3.7}$$

If $k \neq s_j$ for $j = 0, \dots, r$, so that $\Delta^n\langle s_0, \dots, s_r\rangle$ is a subcomplex of $\Lambda^n_k$, then $k$ is a vertex of all of the generating simplices of the object $\Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle$ in the diagram (3.7). It follows that this copy of $\Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle$ is a subcomplex of $\Lambda^{n-1}_q \subset \Delta^{n-1}$, where $q = k$ if $k < s_r$ and $q = k - 1$ if $k > s_r$.

Following Bousfield and Friedlander, we shall write

$$M_n^{(s_0,\dots,s_r)}X = M_{\Delta^n\langle s_0,\dots,s_r\rangle}X.$$

Then, in particular, $M_n^{(0,1,\dots,n)}X$ is yet another notation for $\operatorname{cosk}_{n-1} X_n$.

LEMMA 3.8. *Suppose that the map $p : X \to Y$ is a Reedy fibration and a pointwise weak equivalence. Then $p$ has the right lifting property with respect to all pointwise cofibrations.*

PROOF: We show that each of the fibrations

$$X_m \to Y_m \times_{\operatorname{cosk}_{m-1} Y_m} \operatorname{cosk}_{m-1} X_m$$

is a weak equivalence, and then apply Lemma 3.3.

There are canonical simplicial set morphisms

$$X_{n+1} \to Y_{n+1} \times_{M_{n+1}^{(0,\dots,k)}Y} M_{n+1}^{(0,\dots,k)}X$$

which generalize the map

$$X_{n+1} \to Y_{n+1} \times_{\operatorname{cosk}_n Y_{n+1}} \operatorname{cosk}_n X_{n+1}.$$

The idea of proof is to show inductively that each of the maps

$$X_{n+1} \to Y_{n+1} \times_{M_{n+1}^{(0,\dots,k)}Y} M_{n+1}^{(0,\dots,k)}X$$

is a trivial fibration of simplicial sets, by induction on $n$.

There is a pullback diagram

$$\begin{tikzcd}
Y_{n+1} \times_{M^{(0,\dots,k+1)}_{n+1}Y} M^{(0,\dots,k+1)}_{n+1}X \arrow[r] \arrow[d] & X_n \arrow[d] \\
Y_{n+1} \times_{M^{(0,\dots,k)}_{n+1}Y} M^{(0,\dots,k)}_{n+1}X \arrow[r] & Y_n \times_{M^{(0,\dots,k)}_{n}Y} M^{(0,\dots,k)}_{n}X,
\end{tikzcd}$$

and so each of the vertical maps in the diagram

$$\begin{tikzcd}
X_{n+1} \arrow[r] \arrow[ddddr, "p"'] & Y_{n+1} \times_{M^{(0,\dots,n+1)}_{n+1}Y} M^{(0,\dots,n+1)}_{n+1}X \arrow[d] \\
& Y_{n+1} \times_{M^{(0,\dots,n)}_{n+1}Y} M^{(0,\dots,n)}_{n+1}X \arrow[d] \\
& \vdots \arrow[d] \\
& Y_{n+1} \times_{M^{(0)}_{n+1}Y} M^{(0)}_{n+1}X \arrow[d] \\
& Y_{n+1}
\end{tikzcd}$$

is a trivial fibration by the inductive hypothesis. The map $p$ is a weak equivalence, so that all of the intermediate maps

$$X_{n+1} \rightarrow Y_{n+1} \times_{M^{(0,\dots,k)}_{n+1}Y} M^{(0,\dots,k)}_{n+1}X$$

are weak equivalences as well. □

We've now done all the work that goes into the proof of

THEOREM 3.9. *The category $\mathbf{S}^2$ of bisimplicial sets, together with the classes of pointwise weak equivalences, pointwise cofibrations and Reedy fibrations, satisfies the axioms for a proper closed simplicial model category.*

PROOF: The factorization axiom **CM5** is Lemma 3.6, and the lifting axiom **CM4** is Lemma 3.3 together with Lemma 3.8. The simplicial model structure is the internal one, for which $X\square K$ is specified in horizontal degree $n$ by $(X\square K)_n = X_n \times K$, for a bisimplicial set $X$ and a simplicial set $K$ (see Section VII.4). Properness is a consequence of properness for the category of simplicial sets, since all Reedy fibrations are pointwise fibrations by Lemma 3.3.1. □

REMARK 3.10. There is a completely different approach to proving Theorem 3.9, which involves a closed model structure for a certain category of presheaves of simplicial sets [46], [51] (ie. for the "chaotic topology" on the ordinal number category $\mathbf{\Delta}$). The global fibrations for that theory coincide with the Reedy fibrations: seeing this requires having both the Reedy structure and the closed model structure for the simplicial presheaf category in hand. The applications of the Reedy structure given in the next section depend on the matching space description of Reedy fibrations.

REMARK 3.11. The *external* simplicial model structure for bisimplicial sets is specified by

$$(X\otimes K)_n = \bigsqcup_{\sigma\in K_n} X_n = X_n \times K_n.$$

In other words, $X \times K$ is defined in vertical degree $m$ by

$$(X \otimes K)_{*,m} = X_{*,m} \times K,$$

and is therefore the "vertical" analogue of the "horizontal" construction $X\square K$. Quillen's simplicial model axiom **SM7** fails for the external structure: a cofibration $i : K \to L$ of simplicial sets and a pointwise (aka. Reedy) cofibration $j : X \to Y$ together induce a pointwise cofibration

$$(i, j)_* : X \otimes L \cup_{X\otimes K} Y \otimes K \to Y \otimes L$$

which is a pointwise weak equivalence if $j$ is trivial, but can fail to be a pointwise weak equivalence if $i$ is trivial. In particular, it is easily seen that the map $Z \otimes \Delta^0 \to Z \otimes \Delta^1$ induced by $d^0 : \Delta^0 \to \Delta^1$ is almost never a pointwise weak equivalence. This is a symptom of a general phenomenon, which is discussed further in Section VII.2.

### 3.3. The Moerdijk structure.

A map $f : X \to Y$ is said to be a *diagonal fibration* (respectively *diagonal weak equivalence*) if the induced map $f_* : d(X) \to d(Y)$ of associated diagonal simplicial sets is a Kan fibration (respectively weak equivalence). A *Moerdijk cofibration* is a map which has the left lifting property with respect to all maps which are diagonal fibrations and diagonal weak equivalences.

The diagonal functor $d : \mathbf{S}^2 \to \mathbf{S}$ has a left adjoint $d^* : \mathbf{S} \to \mathbf{S}^2$, which is completely determined by the requirements

(1) $d^*(\Delta^n) = \Delta^{n,n}$, and

(2) $d^*$ preserves colimits.

This is a consequence of the fact that every simplicial set is a colimit of its simplices. It follows that the bisimplicial set $d^*(\Lambda^n_k)$ is the subcomplex of $\Delta^{n,n}$

which is generated by the bisimplices $(d_i\iota_n, d_i\iota_n)$ for $i \neq k$, whereas $d^*(\partial\Delta^n)$ is the subcomplex of $\Delta^{n,n}$ which is generated by all $(d_i\iota_n, d_i\iota_n)$.

Alternatively, the set of bisimplices $d^*(\Lambda^n_k)(r,s)$ can be characterized as the set of all pairs of ordinal number maps $(\alpha : \mathbf{r} \to \mathbf{n}, \beta : \mathbf{s} \to \mathbf{n})$ such that the images of the functions $\alpha$ and $\beta$ miss some common element $i$, where $i \neq k$. Put a different way, the simplicial set of bisimplices $d^*(\Lambda^n_k)(*,\beta)$ can be identified with the subcomplex $C_\beta$ of $\Lambda^n_k$ which is generated by the faces $d_i\iota_n$ which contain the $s$-simplex $\beta$. This observation is the heart of the proof of

LEMMA 3.12. *The inclusion map $d^*(\Lambda^n_k) \subset d^*(\Delta^n) = \Delta^{n,n}$ is a diagonal weak equivalence of bisimplicial sets.*

PROOF: The projection map

$$(\alpha : \mathbf{r} \to \mathbf{n}, \beta : \mathbf{s} \to \mathbf{n}) \mapsto \beta$$

defines a map of bisimplicial sets

$$\bigsqcup_{\beta \in \Lambda^n_k} C_\beta \to \bigsqcup_{\beta \in \Lambda^n_k} *,$$

which in turn induces a map of simplicial sets

$$pr : d(d^*(\Lambda^n_k)) \to \Lambda^n_k$$

after applying the diagonal functor. The complex $C_\beta$ of $\Delta^n$ is covered by subcomplexes isomorphic to $\Delta^{n-1}$, each of which contains the vertex $k$. It follows that the contracting homotopy

$$(x,t) \mapsto tx + (1-t)v_k$$

of the affine simplex $|\Delta^n|$ onto the vertex $v_k$ corresponding to the element $k \in \mathbf{n}$ restricts to a contracting homotopy

$$|C_\beta| \times I \to |C_\beta|.$$

The map *pr* is therefore a weak equivalence of simplicial sets, by Proposition 1.7, so $d(d^*(\Lambda^n_k))$ is contractible. □

REMARK 3.13. In the proof of Lemma 3.12, the complex $C_\beta$ is a subcomplex of $\Lambda^n_k$ of the form

$$C_\beta = \Delta^n\langle s_0, \dots, s_r\rangle.$$

One can alternatively give a combinatorial argument for the "contractibility" of this complex by making an inductive argument based on the existence of the pushout diagram (3.7).

The diagonal functor $d$ also has a right adjoint $d_* : \mathbf{S} \to \mathbf{S}^2$: the bisimplicial set $d_*K$ which is associated to the simplicial set $K$ by this functor is defined by

$$d_*K_{p,q} = \hom_{\mathbf{S}}(\Delta^p \times \Delta^q, K).$$

LEMMA 3.14. *The functor $d_* : \mathbf{S} \to \mathbf{S}^2$ takes fibrations to diagonal fibrations.*

PROOF: Suppose that $p : X \to Y$ is a Kan fibration, and suppose that there is a commutative diagram of simplicial set maps

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, "j"'] & d(d_*X) \arrow[d, "d(d_*p)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & d(d_*Y)
\end{tikzcd}$$

Then the indicated lifting exists if and only if the lifting exists in the diagram

$$\begin{tikzcd}
d(d^*\Lambda^n_k) \arrow[r] \arrow[d, "d(d^*j)"'] & X \arrow[d, "p"] \\
d(d^*\Delta^n) \arrow[r] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

and Lemma 3.12 says that the inclusion $d(d^*j)$ is a trivial cofibration. □

Now suppose that the map $i : U \to V$ of bisimplicial sets has the left lifting property with respect to all diagonal fibrations. Then the lifting exists in all diagrams

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & d_*X \arrow[d, "d_*p"] \\
V \arrow[r] \arrow[ur, dashed] & d_*Y
\end{tikzcd}$$

where $p : X \to Y$ is a Kan fibration, so that the induced simplicial set map $i_* : d(U) \to d(V)$ has the left lifting property with respect to all Kan fibrations. In particular, the map $i$ is a diagonal weak equivalence. In view of Theorem II.4.1, this implies the following:

THEOREM 3.15. *The category $\mathbf{S}^2$ of bisimplicial sets, together with the classes of Moerdijk cofibrations, diagonal fibrations and diagonal weak equivalences, satisfies the axioms for a closed model category.*

An alternative proof of this result can be given by using a small object argument (as was done originally in [73]), based on the observation that a map $p : X \to Y$ of bisimplicial sets is a diagonal fibration (respectively a diagonal fibration and a diagonal weak equivalence) if and only if $f$ has the right lifting property with respect to all maps $d^*(\Lambda^n_k) \subset \Delta^{n,n}$ (respectively with respect to all maps $d^*(\partial\Delta^n) \subset \Delta^{n,n}$). One of the outcomes of the small object argument is the assertion that every Moerdijk cofibration is a monomorphism of bisimplicial sets.

## 4. The Bousfield-Friedlander theorem.

Suppose that $X$ is a pointwise fibrant bisimplicial set. The simplicial set $\pi_n X$ having $m$-simplices

$$\bigsqcup_{x \in X(m,0)} \pi_n(X_m, x)$$

and the simplicial map

$$\pi_n X_m = \bigsqcup_{x \in X(m,0)} \pi_n(X_m, x) \to \bigsqcup_{x \in X(m,0)} * = X(m,0)$$

together form a group object in the category $\mathbf{S} \downarrow X(*,0)$ of simplicial sets over the vertex simplicial set $X(*,0)$. This group object is abelian if $m > 1$.

Recall that a vertex $v \in M_K X$ is a simplicial set morphism $v : K \to X(*,0)$. The set of morphisms

$$M_K(\pi_m X, v) = \hom_{\mathbf{S}\downarrow X(*,0)}(v, \pi_m X)$$

therefore has a group structure for $m \geq 1$, which is abelian for $m \geq 2$.

Suppose that $i \mapsto K(i)$ defines an $I$-diagram $K : I \to \mathbf{S}$ in the category of simplicial sets, and let

$$v : \varinjlim_{i \in I} K(i) \to X(*,0)$$

be a map of simplicial sets. Let $v(i) : K(i) \to X(*,0)$ be the composite

$$K(i) \xrightarrow{in_i} \varinjlim_{i \in I} K(i) \xrightarrow{v} X(*,0)$$

of $v$ with the canonical map $in_i$ of the colimiting cone. Then

$$v = \varinjlim_{i \in I} v_i$$

in the category $\mathbf{S} \downarrow X(*,0)$ of simplicial sets over $X(*,0)$, and so there is an isomorphism

$$\hom_{\mathbf{S}\downarrow X(*,0)}(v, \pi_m X) \cong \varprojlim_{i \in I} \hom_{\mathbf{S}\downarrow X(*,0)}(v_i, \pi_m X).$$

It follows that there is an isomorphism

$$M_{\varinjlim_i K(i)}(\pi_m X, v) \cong \varprojlim_{i \in I} M_{K(i)}(\pi_m X, v_i).$$

In particular, the group $M_{\Delta^n}(\pi_m X, v)$ is canonically isomorphic to the group $\pi_m(X_n, v)$, where the map $v : \Delta^n \to X(*, 0)$ is identified with a vertex $v \in X(n, 0)$. Any such vertex $v : \Delta^n \to X(*, 0)$ restricts to a composite map

$$\Lambda^n_k \subset \Delta^n \xrightarrow{v} X(*, 0),$$

which will also be denoted by $dv$. It follows in particular that the corresponding group $M_{\Lambda^n_k}(\pi_m X, dv)$ fits into an equalizer diagram

$$M_{\Lambda^n_k}(\pi_m X, dv) \to \prod_{i \neq k} \pi_m(X_{n-1}, d_i v) \rightrightarrows \prod_{i<j;\, i,j \neq k} \pi_m(X_{n-2}, d_i d_j v), \qquad (4.1)$$

where the parallel pair of arrows is defined by the simplicial identities $d_i d_j = d_{j-1} d_i$.

A pointwise fibrant bisimplicial set $X$ is said to satisfy the *$\pi_*$-Kan condition* if the map

$$d : \pi_m(X_n, v) \to M_{\Lambda^n_k}(\pi_m X, dv)$$

induced by restriction along the inclusion $\Lambda^n_k \subset \Delta^n$ is a surjective group homomorphism for all $m \geq 1$ and all $n, k$ that make sense. The $\pi_*$-Kan condition for $X$ is equivalent to the requirement that all of the structure maps

$$\pi_m X \to X(*, 0)$$

for the homotopy group objects $\pi_m X$, $m \geq 1$, are Kan fibrations, on account of the description of $M_{\Lambda^n_k}(\pi_m X, dv)$ given in (4.1).

Suppose that $Y$ is an arbitrary bisimplicial set, and write $\pi_0 Y$ for the simplicial set having $n$-simplices $\pi_0 Y_n = \pi_0 Y(n, *)$. This is the simplicial set of *vertical path components* of the bisimplicial set $Y$. There is a canonical simplicial set map $Y(*, 0) \to \pi_0 Y$.

LEMMA 4.2.

(1) *A pointwise fibrant bisimplicial set $X$ satisfies the $\pi_*$-Kan condition if all of the vertical simplicial sets $X_n = X(n, *)$ are path connected.*

(2) *Suppose that $f : X \to Y$ is a pointwise weak equivalence of pointwise fibrant bisimplicial sets. Then $X$ satisfies the $\pi_*$-Kan condition if and only if $Y$ satisfies the $\pi_*$-Kan condition.*

Proof: In the case of statement (1), there is a path $\omega$ from a given vertex $x \in X(n,0)$ to a horizontally degenerate vertex $s(y)$, where $y \in X(0,0)$ and so the action the corresponding morphism $[\omega]$ of the fundamental groupoid for $X_n$ induces an isomorphism of maps

$$\begin{tikzcd}
\pi_m(X_n, x) \arrow[r, "d"] \arrow[d, "{[\omega]_*}"', "\cong"] & M_{\Lambda^n_k}(\pi_m X, x) \arrow[d, "\cong"', "{[\omega]_*}"] \\
\pi_m(X_n, s(y)) \arrow[r, "d"'] & M_{\Lambda^n_k}(\pi_m X, s(y)),
\end{tikzcd}$$

and the simplicial group $n \mapsto \pi_m(X_n, s(y))$ is a Kan complex.

To prove statement (2), note first of all that the map $f : X \to Y$ is a pointwise weak equivalence of pointwise fibrant bisimplicial sets if and only if

(a) the induced map $f_* : \pi_0 X \to \pi_0 Y$ is an isomorphism of simplicial sets, and

(b) the induced simplicial set diagrams

$$\begin{tikzcd}
\pi_m X \arrow[r, "f_*"] \arrow[d] & \pi_m Y \arrow[d] \\
X(*,0) \arrow[r, "f_*"'] & Y(*,0)
\end{tikzcd} \tag{4.3}$$

are pullbacks for $m \geq 1$.

Kan fibrations are stable under pullback, so if $f$ is a pointwise weak equivalence and $Y$ satisfies the $\pi_*$-Kan condition, then $X$ satisfies the $\pi_*$-Kan condition.

Suppose that $X$ satisfies the $\pi_*$-Kan condition, and that there is a diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & \pi_m Y \arrow[d] \\
\Delta^n \arrow[r, "v"'] & Y(*,0)
\end{tikzcd}$$

The simplicial set map $f_* : \pi_0 X \to \pi_0 Y$ is an isomorphism, so that the vertex $v \in Y(n,*)$ is homotopic to a vertex $f(w)$ for some vertex $w \in X(n,*)$. The argument for statement (1) implies that the $\pi_*$-Kan condition for the vertex $v$ is equivalent to the $\pi_*$-Kan condition for the vertex $f(w)$, so we can replace $v$ by $f(w)$. But now the diagram (4.3) is a pullback, by assumption, and so there is a commutative diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\alpha_*"] \arrow[d, hook] & \pi_m X \arrow[r, "f_*"] \arrow[d] & \pi_m Y \arrow[d] \\
\Delta^n \arrow[r, "w"'] \arrow[ur, dashed] & X(*,0) \arrow[r, "f_*"'] & Y(*,0)
\end{tikzcd}$$

where $f_* \cdot \alpha_* = \alpha$. Then $X$ satisfies the $\pi_*$-Kan condition, by assumption, so that the indicated lifting exists, and the $\pi_*$-Kan condition for the vertex $f(w)$ is verified. ☐

A *pointwise fibrant model* of a bisimplicial set $X$ consists of a pointwise weak equivalence $j : X \to Z$, where $Z$ is a pointwise fibrant bisimplicial set.

We shall say that an arbitrary bisimplicial set $X$ satisfies the *$\pi_*$-Kan condition* if for any pointwise fibrant model $j : X \to Z$ of $X$, the pointwise fibrant object $Z$ satisfies the $\pi_*$-Kan condition in the sense described above. Lemma 4.2 says that the $\pi_*$-Kan condition for pointwise fibrant bisimplicial sets is an invariant of pointwise weak equivalence, so (by appropriate manipulation of the Bousfield-Kan closed model structure for bisimplicial sets) it suffices to find only one pointwise fibrant model $j : X \to Z$ which satisfies the $\pi_*$-Kan condition. The $\pi_*$-Kan condition for arbitrary bisimplicial sets is also an invariant of weak equivalence.

Suppose that $f : X \to Y$ is a Reedy fibration. Let $i : K \subset L$ be an inclusion of simplicial sets, and observe that all induced bisimplicial set maps

$$(K \tilde{\times} \Delta^n) \cup_{(K \tilde{\times} \Lambda^n_k)} (L \tilde{\times} \Lambda^n_k) \hookrightarrow L \tilde{\times} \Delta^n$$

are pointwise trivial cofibrations of bisimplicial sets. The functor $X \mapsto M_K X$ is right adjoint to the functor $Y \mapsto K \tilde{\times} Y$, so it follows that the simplicial set map

$$M_L X \xrightarrow{(f_*, i^*)} M_L Y \times_{M_K Y} M_K X \tag{4.4}$$

which is jointly induced by the Reedy fibration $f$ and the inclusion $i$ is a Kan fibration.

LEMMA 4.5. *Suppose $X$ is a Reedy fibrant bisimplicial set $X$ that satisfies the $\pi_*$-Kan condition. Take a vertex $x \in X(n, *)$. Then there is a canonical isomorphism*

$$\pi_m(M_{\Lambda^n_k} X, dx) \cong M_{\Lambda^n_k}(\pi_m X, dx).$$

*There is also an isomorphism*

$$\pi_0(M_{\Lambda^n_k}X) \cong M_{\Lambda^n_k}(\pi_0 X).$$

PROOF: Choose integers $0 \leq s_0 < s_1 < \cdots < s_r \leq n$ with $s_i \neq k$, and recall that

$$M_n^{(s_0,\dots,s_r)}X = M_{\Delta^n\langle s_0,\dots,s_r\rangle}X$$

is a subcomplex of $M_{\Lambda^n_k}X$. Write

$$M_n^{(s_0,\dots,s_r)}(\pi_m X, dx) = M_{\Delta^n\langle s_0,\dots,s_r\rangle}(\pi_m X, dx),$$

and let $M_n^{(s_0,\dots,s_r)}\pi_0 X$ denote the set

$$M_{\Delta^n\langle s_0,\dots,s_r\rangle}(\pi_0 X) = \hom_{\mathbf{S}}(\Delta^n\langle s_0,\dots,s_r\rangle, \pi_0 X).$$

The pushout diagram (3.7) induces a pullback diagram

$$\begin{tikzcd}
M_n^{(s_0,\dots,s_r)}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d, "d"] \\
M_n^{(s_0,\dots,s_{r-1})}X \arrow[r] & M_{n-1}^{(s_0,\dots,s_{r-1})}X
\end{tikzcd} \tag{4.6}$$

Then the map $d$ is an instance of the fibration (4.4), and inductively the canonical map

$$\pi_m(M_n^{(s_0,\dots,s_{r-1})}X, dx) \to M_n^{(s_0,\dots,s_{r-1})}(\pi_m X, dx)$$

is an isomorphism. The map

$$d : \pi_m(X_{n-1}, d_{s_r}x) \to M_{n-1}^{(s_0,\dots,s_{r-1})}(\pi_m X, dx)$$

is surjective for all $m \geq 1$, since $X$ satisfies the $\pi_*$-Kan condition. It follows that the induced map

$$d_* : \pi_m(X_{n-1}, x) \to \pi_m(M_{n-1}^{(s_0,\dots,s_{r-1})}X, dx)$$

is surjective for all $m \geq 1$, and so the commutative square

$$\begin{tikzcd}
\pi_m(M_n^{(s_0,\dots,s_r)}X, dx) \arrow[r] \arrow[d] & \pi_m(X_{n-1}, d_{s_r}x) \arrow[d, "d_*"] \\
\pi_m(M_n^{(s_0,\dots,s_{r-1})}X, dx) \arrow[r] & \pi_m(M_{n-1}^{(s_0,\dots,s_{r-1})}X, dd_{s_r}x)
\end{tikzcd}$$

of group homomorphisms is a pullback. The group $\pi_m(M_n^{(s_0,\dots,s_r)}X, dx)$ therefore has the required form.

The map

$$d_* : \pi_1(X_{n-1}, x) \to \pi_1(M_{n-1}^{(s_0,\dots,s_{r-1})}X, dx)$$

is surjective for all choices of base point $x \in X_{n-1}$, and so all inclusions $F_x \hookrightarrow X_{n-1}$ of fibres over $dx$ induce injections $\pi_0 F_x \to \pi_0 X_{n-1}$. It follows that applying the path component functor $\pi_0$ to all diagrams of the from (4.6) gives pullback diagrams of sets. This is what is required to show inductively that the canonical maps

$$\pi_0(M_n^{(s_0,\dots s_r)}X) \to M_n^{(s_0,\dots,s_r)}\pi_0 X$$

are bijections. □

LEMMA 4.7. *Suppose that $X$ and $Y$ are Reedy fibrant bisimplicial sets which satisfy the $\pi_*$-Kan condition, and that the bisimplicial set map $f : X \to Y$ is a Reedy fibration. Suppose further that the induced simplicial set map $f_* : \pi_0 X \to \pi_0 Y$ of vertical path components is a Kan fibration. Then the map $f$ is a horizontal pointwise Kan fibration.*

To understand the meaning of the word "horizontal" in the statement of Lemma 4.7, note that every Reedy fibration is a pointwise Kan fibration, since the maps $F_{\mathbf{n}}\Lambda^m_k \to F_{\mathbf{n}}\Delta^m$ are trivial cofibrations (see the proof of Corollary 3.4). The assumptions of the lemma therefore imply that the simplicial sets $X(n,*)$ and $Y(n,*)$ are Kan complexes, and that the simplicial set maps $f : X(n,*) \to Y(n,*)$ are Kan fibrations. The lemma asserts that under the stated conditions the simplicial set maps $f : X(*,m) \to Y(*,m)$ are Kan fibrations as well.

PROOF: We shall prove the lemma by showing that each canonical map

$$d : X_n \to Y_n \times_{M_{\Lambda^n_k}Y} M_{\Lambda^n_k}X$$

is a surjective simplicial set map. The map $d$ is an instance of (4.4), hence a Kan fibration, so it suffices to show that $d$ induces a surjective function

$$d_* : \pi_0 X_n \to \pi_0(Y_n \times_{M_{\Lambda^n_k}Y} M_{\Lambda^n_k}X)$$

in path components.

From the previous result, the induced map

$$\pi_0 X_n \to \pi_0(Y_n \times_{M_{\Lambda^n_k}Y} M_{\Lambda^n_k}X)$$

can be identified up to isomorphism with the map

$$\pi_0 X_n \to \pi_0 Y_n \times_{M_{\Lambda^n_k}\pi_0 Y} M_{\Lambda^n_k}\pi_0 X,$$

and the latter is surjective on account of the assumption that the induced map $\pi_0 X \to \pi_0 Y$ in vertical path components is a Kan fibration. □

Lemma 4.8. *Suppose that $f : X \to Y$ is a Reedy fibration and a horizontal pointwise Kan fibration in the sense that each of the maps $f : X(*, m) \to Y(*, m)$ is a Kan fibration. Then $f$ is a diagonal fibration.*

Proof: The cofibration

$$d^* \Lambda^n_k \to d^* \Delta^n = \Delta^{n,n}$$

factors as a composite of two maps

$$d^* \Lambda^n_k \subset \Lambda^n_k \tilde{\times} \Delta^n \subset \Delta^n \tilde{\times} \Delta^n.$$

The first map is a pointwise trivial cofibration, by the proof of Lemma 3.12. The second has the left lifting property with respect to all horizontal pointwise Kan fibrations. □

A commutative square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
Z \arrow[r] & W
\end{tikzcd}$$

of bisimplicial set maps is said to be *pointwise homotopy cartesian* if each of the induced squares of simplicial set maps

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\
Z_n \arrow[r] & W_n
\end{tikzcd}$$

is homotopy cartesian, for $n \geq 0$.

Theorem 4.9 (Bousfield-Friedlander). *Suppose given a pointwise homotopy cartesian square*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\
Z \arrow[r] & W
\end{tikzcd}$$

*in the category of bisimplicial sets such that $Y$ and $W$ satisfy the $\pi_*$-Kan condition. Suppose further that the induced map $p : \pi_0 Y \to \pi_0 W$ of vertical path components is a Kan fibration. Then the associated commutative square*

$$\begin{tikzcd}
d(X) \arrow[r] \arrow[d] & d(Y) \arrow[d, "d(p)"] \\
d(Z) \arrow[r] & d(W)
\end{tikzcd}$$

*of diagonal simplicial sets is homotopy cartesian.*

PROOF: Construct a diagram

$$\begin{tikzcd}
Y \arrow[r, "j_Y"] \arrow[d, "p"'] & Y' \arrow[d, "p'"] \\
W \arrow[r, "j_W"'] & W'
\end{tikzcd}$$

in which $j_Y$ and $j_W$ are pointwise trivial cofibrations, $W'$ is Reedy fibrant, and $p'$ is a Reedy fibration. Then the composite square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & Y \arrow[r, "j_Y"] \arrow[d, "p"] \arrow[dr, phantom, "\mathbf{II}"] & Y' \arrow[d, "p'"] \\
Z \arrow[r] & W \arrow[r, "j_W"'] & W'
\end{tikzcd}$$

is pointwise homotopy cartesian by Lemma II.8.22, and the bisimplicial sets $Y'$ and $W'$ satisfy the $\pi_*$-Kan condition by Lemma 4.2. The Reedy fibrant objects $Y'$ and $W'$ are both pointwise fibrant.

Applying the diagonal functor gives a composite diagram in the simplicial set category

$$\begin{tikzcd}
d(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & d(Y) \arrow[r, "j_{Y*}"] \arrow[d, "p_*"] \arrow[dr, phantom, "\mathbf{II}"] & d(Y') \arrow[d, "p'_*"] \\
d(Z) \arrow[r] & d(W) \arrow[r, "j_{W*}"'] & d(W'),
\end{tikzcd} \tag{4.10}$$

in which the induced maps $j_{Y*}$ and $j_{W*}$ are weak equivalences by Proposition 1.7. Thus, in order to demonstrate that the square

$$\begin{tikzcd}
d(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & d(Y) \arrow[d, "p_*"] \\
d(Z) \arrow[r] & d(W)
\end{tikzcd}$$

is homotopy cartesian, it suffices, by Lemma II.8.22, to show that the composite square (4.10) is homotopy cartesian.

The map $p'$ is a diagonal fibration, by Lemma 4.7 and Lemma 4.8, and the induced bisimplicial set map

$$X \to Z \times_{W'} Y'$$

is a pointwise weak equivalence, since the Reedy fibration $p'$ is a pointwise fibration. It follows that the induced map

$$d(X) \to d(Z) \times_{d(W')} d(Y')$$

of diagonal simplicial sets is a weak equivalence. □

COROLLARY 4.11. *Suppose that a pointed bisimplicial set $X$ is pointwise fibrant and pointwise connected. Then there is a weak equivalence*

$$d(\Omega X) \simeq \Omega d(X).$$

PROOF: We have tacitly chosen a base point $x \in X(0,0)$ for all of the vertical simplicial sets $X(n,*)$ in our assumption that $X$ is pointed. The loop object $\Omega X$ can then be characterized as the pointed bisimplicial set having vertical simplicial sets $\Omega X(n,*)$. There is a corresponding path space $PX$, which is the pointed bisimplicial set with vertical simplicial sets $PX(n,*)$, and a pointwise fibre sequence

$$\Omega X \to PX \xrightarrow{p} X.$$

The map $p$ induces an isomorphism $\pi_0 PX \cong \pi_0 X$ of vertical path component simplicial sets. The bisimplicial sets $PX$ and $X$ both satisfy the $\pi_*$-Kan condition, and so applying the diagonal functor gives a homotopy fibre sequence

$$d(\Omega X) \to d(PX) \xrightarrow{d(p)} d(X)$$

in the category of pointed simplicial sets, by Theorem 4.9. The simplicial set $d(PX)$ is contractible, by Proposition 1.7. □

A bisimplicial set $X$ is said to be *pointwise $k$-connected* if each of the associated simplicial sets $X_n$ is $k$-connected, for $n \geq 0$. A pointwise connected (or 0-connected) bisimplicial set can also be characterized as a bisimplicial set $X$ such that the associated vertical path component simplicial set $\pi_0 X$ is a copy of the terminal object $*$.

LEMMA 4.12. *Suppose that $X$ is a bisimplicial set which is pointwise connected. Then the diagonal simplicial set $d(X)$ is connected.*

PROOF: There is a coequalizer diagram

$$\begin{tikzcd} \pi_0 X(1,*) \arrow[r, shift left, "d_{0*}"] \arrow[r, shift right, "d_{1*}"'] & \pi_0 X(0,*) \arrow[r] & \pi_0 d(X), \end{tikzcd}$$

where $d_{0*}$ and $d_{1*}$ are induced by the horizontal face maps $d_0, d_1 : X(1,*) \to X(0,*)$. ☐

PROPOSITION 4.13. *Suppose that $X$ is a pointed bisimplicial set which is pointwise $k$-connected. Then the diagonal $d(X)$ is a $k$-connected simplicial set.*

PROOF: Choose a pointwise weak equivalence $i : X \to \tilde{X}$, where $\tilde{X}$ is pointwise fibrant. Then $d(i) : d(X) \to d(\tilde{X})$ is a weak equivalence. We may therefore presume that $X$ is pointwise fibrant.

The space $d(X)$ is connected, by Lemma 4.12. Also, the bisimplicial set $\Omega X$ is pointwise $(k-1)$-connected, and there are isomorphisms

$$\pi_j d(X) \cong \pi_{j-1} d(\Omega X),$$

for $j \geq 1$, by Corollary 4.11. This does it, by induction on $k$. ☐

Proposition 4.13 is also a consequence of a very general spectral sequence calculation. The existence of the spectral sequence in question is a consequence of Theorem 4.9, modulo a few technical observations.

First of all, if $G$ is a simplicial group, then there is a bisimplicial set $BG$ whose vertical simplicial in horizontal degree $n$ is the classifying space $BG_n$ of the group $G_n$ of $n$-simplices of $G$. Similarly, the translation categories of the various groups $G_n$ can be collected together to form a bisimplicial set $EG$ and a canonical map $\pi : EG \to BG$. These bisimplicial sets are pointwise fibrant and connected in each horizontal degree, and the map $\pi$ is a pointwise fibration. It therefore follows from Theorem 4.9 that there is an induced fibre sequence

$$G \to d(EG) \xrightarrow{\pi_*} d(BG).$$

Furthermore, the bisimplicial set $EG$ consists of contractible simplicial sets $EG_n$, so that the associated diagonal $d(EG)$ is contractible, by Proposition 1.7. It follows that there are natural isomorphisms

$$\pi_n d(BG) \cong \pi_{n-1} G$$

for $n \geq 1$. The space $d(BG)$ is connected, by Lemma 4.12.

Suppose that $X$ is a pointwise fibrant and pointed bisimplicial set such that each of the vertical simplicial sets $X_n$ is an Eilenberg-Mac Lane space of the form $K(\pi_m, m)$, for some fixed number $m \geq 2$. Then Corollary III.3.8 implies that there is a pointwise weak equivalence of bisimplicial sets of the form

$$X \to \Gamma(\pi_m X[m]),$$

where $\pi_m X[m]$ denotes the chain complex of simplicial groups, concentrated in chain degree $m$. Then, by Theorem 4.9 or via chain complex arguments, one sees that there are natural isomorphisms

$$\pi_j d(X) \cong \pi_{j-m}(\pi_m X)$$

for $j \geq 0$.

Suppose finally that $X$ is a pointwise fibrant bisimplicial set which is pointed and pointwise connected. Then the Postnikov tower construction applied to each of the vertical simplicial sets $X_n$ induces a tower of pointwise fibrations

$$\dots \to X(n) \to X(n-1) \to \dots \to X(1) \to X(0),$$

such that the the fibre $F_n$ of the map $X(n) \to X(n-1)$ is a diagram of Eilenberg-Mac Lane spaces of the form $K(\pi_n X, n)$. Each of the pointwise fibre sequences

$$F_n \to X(n) \to X(n-1)$$

induces a fibre sequence of associated diagonal simplicial sets, by Theorem 4.9. The resulting long exact sequences

$$\dots \to \pi_{j+1} d(X(n-1)) \xrightarrow{\partial} \pi_j d(F_n) \to \pi_j d(X(n)) \to \pi_j d(X(n-1)) \to \dots$$

determine an exact couple which gives rise to a convergent spectral sequence with

$$E_2^{s,t} = \pi_{s+t} F_t \cong \pi_s(\pi_t X) \Rightarrow \pi_{s+t} d(X), \quad t+s \geq 0. \tag{4.14}$$

This spectral sequence is due to Bousfield and Friedlander [12, p.122]. It is a reindexed example of the spectral sequence for a tower of fibrations, which will be discussed at more length in Section VI.2. Convergence for such spectral sequences is usually an issue, but it follows in this case from Proposition 4.13, which implies that the map $X \to X(n)$ induces isomorphisms

$$\pi_j d(X) \cong \pi_j d(X(n))$$

for $j \leq n$.

## 5. Theorem B and group completion.

The stream of ideas leading to Quillen's Theorem B begins with the most general formulation of the Serre spectral sequence.

### 5.1. The Serre spectral sequence.

Suppose that $f : E \to B$ is an arbitrary map of simplicial sets, and recall that $\mathbf{\Delta} \downarrow B$ denotes the simplex category for $B$. There is a functor $f^{-1} : \mathbf{\Delta} \downarrow B \to \mathbf{S}$ taking values in the simplicial set category which is defined by associating to the simplex $\sigma : \Delta^n \to B$ the simplicial set $f^{-1}(\sigma)$, where $f^{-1}(\sigma)$ is defined by the pullback diagram

$$\begin{tikzcd}
f^{-1}(\sigma) \arrow[r, "\sigma_*"] \arrow[d] & E \arrow[d, "f"] \\
\Delta^n \arrow[r, "\sigma"'] & B.
\end{tikzcd}$$

The homotopy colimit $\underrightarrow{\mathrm{holim}}\, f^{-1}$ arising from the functor $f^{-1} : \mathbf{\Delta} \downarrow B \to \mathbf{S}$ is the diagonal of a bisimplicial set $BEf^{-1}$ which has vertical simplicial set in horizontal degree $n$ given by

$$\bigsqcup_{\sigma_0 \to \dots \to \sigma_n} f^{-1}(\sigma_0).$$

Note that this disjoint union is indexed by strings of arrows of length $n$ in the simplex category $\mathbf{\Delta} \downarrow B$, and that these strings form the set of $n$-simplices of the nerve $B(\mathbf{\Delta} \downarrow B)$.

The simplicial set $B$ is a colimit of its simplices in the simplicial set category, and pulling back along $f : E \to B$ is right exact, so that the maps $\sigma_* : f^{-1}(\sigma) \to E$ induce an isomorphism of simplicial sets

$$\varinjlim_{\sigma \in \mathbf{\Delta} \downarrow B} f^{-1}(\sigma) \cong E.$$

In particular, the set $E_m$ of $m$-simplices of $E$ may be identified with the set of path components of the translation category $Ef^{-1}_m$ arising from the functor

$$\sigma \mapsto f^{-1}(\sigma)_m.$$

The objects of this category are pairs $(\sigma, x)$, where $\sigma : \Delta^n \to B$ is a simplex of $B$ and $x \in f^{-1}(\sigma)_m$, and a morphism $\theta : (\sigma, x) \to (\tau, y)$ consists of a morphism $\theta : \sigma \to \tau$ in $\mathbf{\Delta} \downarrow B$ such that $\tau_*(x) = y$. The nerve $BEf^{-1}_m$ for this translation category coincides with the horizontal simplicial set for $BEf^{-1}$ which appears in vertical degree $m$.

Write $Ef^{-1}_{m,x}$ for the path component of the translation category $Ef^{-1}_m$ corresponding to a simplex $x$ of $E_m$. This component $Ef^{-1}_{m,x}$ is the full subcategory of $Ef^{-1}_m$ on objects of the form $(\sigma, y)$, where $\sigma_*(y) = x$. In particular, $y$ must have the form $y = (\gamma, x)$ in $f^{-1}(\sigma) = \Delta^n \times_B E$. The object $(f(x), (\iota_m, x))$

is initial in the category $Ef^{-1}_{m,x}$, so that $BEf^{-1}_{m,x}$ is contractible, and the simplicial set map

$$BEf^{-1}_m \to E_m \tag{5.1}$$

is a weak equivalence. We have taken the liberty of identifying the set $E_m$ with the corresponding constant simplicial set $K(E_m, 0)$; by further abuse, the maps (5.1) are the horizontal components of a bisimplicial set map

$$BEf^{-1} \to E,$$

which is a weak equivalence in each vertical degree. Proposition 1.7 therefore implies the following:

LEMMA 5.2. *Suppose that $f : E \to B$ is a map of simplicial sets. Then the canonical simplicial set map $\underrightarrow{\text{holim}}\, f^{-1} \to E$ is a weak equivalence.*

Now let $A$ be an abelian group, and consider the bisimplicial abelian group $\mathbb{Z}(BEf^{-1}) \otimes A$. The diagonal of this object is $\mathbb{Z}(\underrightarrow{\text{holim}}\, f^{-1}) \otimes A$, and the weak equivalence $\underrightarrow{\text{holim}}\, f^{-1} \to E$ of Lemma 5.2 induces a weak equivalence of simplicial abelian groups

$$\mathbb{Z}(\underrightarrow{\text{holim}}\, f^{-1}) \otimes A \to \mathbb{Z}(E) \otimes A,$$

by universal coefficients and the fact that the free abelian group functor preserves weak equivalences (see Lemma III.2.16). Note as well that the bisimplicial abelian group $\mathbb{Z}(BEf^{-1}) \otimes A$ can be identified with the translation object associated to the functor defined on the simplex category $\mathbf{\Delta} \downarrow B$ by $\sigma \mapsto \mathbb{Z}(f^{-1}(\sigma)) \otimes A$ (Section II.4). It follows, by the results of Section 2, that there is a spectral sequence

$$E_2^{p,q} = \pi_p EH_q(\mathbb{Z}(f^{-1}) \otimes A) \Rightarrow H_{p+q}(E, A). \tag{5.3}$$

In other words, $E_2^{p,q}$ is the $p^{th}$ homotopy group of the translation object for the abelian group valued functor $H_q(\mathbb{Z}(f^{-1}) \otimes A) : \mathbf{\Delta} \downarrow B \to \mathbf{Ab}$ defined by

$$\sigma \mapsto H_q(\mathbb{Z}(f^{-1}(\sigma)) \otimes A) = H_q(f^{-1}(\sigma), A). \tag{5.4}$$

The spectral sequence (5.3) is sometimes called the *Grothendieck spectral sequence*, and is defined for any simplicial set map $f : E \to B$. This spectral sequence specializes to the *Serre spectral sequence* in the case where the map $f : E \to B$ is a fibration. When $f$ is a fibration, any map $\theta : \sigma \to \tau$ in the simplex category $\mathbf{\Delta} \downarrow B$ induces a weak equivalence $\theta_* : f^{-1}(\sigma) \to f^{-1}(\tau)$, and hence induces isomorphisms

$$H_q(f^{-1}(\sigma), A) \xrightarrow[\cong]{\theta_*} H_q(f^{-1}(\tau), A)$$

for all $q \geq 0$. It follows that the functors (5.4) factor through functors

$$H_q(\mathbb{Z}(f^{-1}) \otimes A) : \pi(B) = G(\mathbf{\Delta} \downarrow B) \to \mathbf{Ab},$$

which are defined on the fundamental groupoid $\pi(B)$ of the space $B$ (see Theorem III.1.1).

If in addition $B$ is simply connected, and $F$ is the fibre over a choice of base point for $B$, then the functor $H_q(\mathbb{Z}(f^{-1}) \otimes A)$ is naturally isomorphic to the constant functor $\sigma \mapsto H_q(F, A)$ on the simplex category for $B$, and so there is a natural isomorphism

$$E_2^{p,q} = \pi_p E H_q(\mathbb{Z}(f^{-1}) \otimes A) \cong H_p(B(\mathbf{\Delta} \downarrow B), H_q(F, A)).$$

The assertion that there is a natural weak equivalence $\underrightarrow{\text{holim}}\, f^{-1} \to E$ can be specialized to the case of the identity map $B \to B$, implying that the bisimplicial set map

$$\bigsqcup_{\sigma_0 \to \cdots \to \sigma_n} \Delta^{m_0} \to B$$

(where $\sigma_j : \Delta^{m_j} \to B$ are simplices of $B$) induces a weak equivalence of diagonal simplicial sets. One also knows that the canonical bisimplicial set map

$$\bigsqcup_{\sigma_0 \to \cdots \to \sigma_n} \Delta^{m_0} \to \bigsqcup_{\sigma_0 \to \cdots \to \sigma_n} *$$

is a pointwise weak equivalence, and hence induces a weak equivalence

$$d(\bigsqcup_{\sigma_0 \to \cdots \to \sigma_n} \Delta^{m_0}) \to B(\mathbf{\Delta} \downarrow B).$$

It follows that there are isomorphisms

$$H_p(B(\mathbf{\Delta} \downarrow B), H_q(F, A)) \cong H_p(B, H_q(F, A)),$$

$p \geq 0$, and we obtain the standard form of the Serre spectral sequence

$$E_2^{p,q} = H_p(B, H_q(F, A)) \Rightarrow H_{p+q}(E, A). \tag{5.5}$$

### 5.2. Theorem B.

Quillen's Theorem B is the following:

THEOREM 5.6. *Suppose that $F : C \to D$ is a functor between small categories such that for every morphism $\alpha : y \to y'$ of $D$ the induced simplicial set map $\alpha^* : B(y' \downarrow F) \to B(y \downarrow F)$ is a weak equivalence. Then, for every object $y$ of $D$, the commutative diagram*

$$\begin{tikzcd}
B(y \downarrow F) \arrow[r] \arrow[d] & BC \arrow[d, "F_*"] \\
B(y \downarrow D) \arrow[r] & BD
\end{tikzcd}$$

*of simplicial set maps is homotopy cartesian.*

Here, one should recall that the objects of the comma category $y \downarrow F$ are pairs $(\tau, x)$, where $x$ is an object of $C$ and $\tau : y \to F(x)$ is a morphism of $D$. A morphism $\alpha : (\tau_0, x_0) \to (\tau_1, x_1)$ of $y \downarrow F$ is a morphism $\alpha : x_0 \to x_1$ of $C$ such that the diagram

$$\begin{tikzcd}
& F(x_0) \arrow[dd, "F(\alpha)"] \\
y \arrow[ur, "\tau_0"] \arrow[dr, "\tau_1"'] & \\
& F(x_1)
\end{tikzcd}$$

commutes in the category $D$.

Theorem B has important applications in algebraic $K$-theory. In some sense, however, one of the steps in its proof is even more important, this being the following result:

LEMMA 5.7. *Suppose that $X : I \to \mathbf{S}$ is a simplicial set valued functor which is defined on a small category $I$. Suppose further that the induced simplicial set map $X(\alpha) : X(i) \to X(j)$ is a weak equivalence for each morphism $\alpha : i \to j$ of the index category $I$. Then, for each object $j$ of $I$ the pullback diagram of simplicial sets*

$$\begin{tikzcd}
X(j) \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}} X \arrow[d, "\pi"] \\
* \arrow[r, "j"'] & BI
\end{tikzcd} \tag{5.8}$$

*is homotopy cartesian.*

Proof: The diagram (5.8) is obtained by applying the diagonal functor to the following pullback diagram of bisimplicial sets:

$$\begin{tikzcd}
X(j) \arrow[r] \arrow[d] & \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0) \arrow[d] \\
* \arrow[r] & \bigsqcup_{i_0 \to \cdots \to i_n} *
\end{tikzcd}$$

The strategy of proof is to find a factorization

$$\begin{tikzcd}
* \arrow[rr, "j"] \arrow[dr, "i"'] & & BI \\
& U \arrow[ur, "p"'] &
\end{tikzcd}$$

of the inclusion of the vertex $j$ in $BI$ such that $i$ is a trivial cofibration, $p$ is a fibration, and such that the induced map $X(j) \to U \times_{BI} \underrightarrow{\operatorname{holim}}\, X$ is a weak equivalence.

Pulling back along the map $\pi : \underrightarrow{\operatorname{holim}}\, X \to BI$ preserves colimits in the category $\mathbf{S} \downarrow BI$ of simplicial set maps $K \to BI$. The small object argument therefore implies that it suffices to show that any diagram

$$\begin{tikzcd}
& & \underrightarrow{\operatorname{holim}}\, X \arrow[d, "\pi"] \\
\Lambda^n_k \arrow[r, "i"'] & \Delta^n \arrow[r, "\sigma"'] & BI
\end{tikzcd} \tag{5.9}$$

induces a weak equivalence

$$i_* : \Lambda^n_k \times_{BI} \underrightarrow{\operatorname{holim}}\, X \to \Delta^n \times_{BI} \underrightarrow{\operatorname{holim}}\, X. \tag{5.10}$$

The simplex $\sigma$ in the diagram (5.9) is a functor $\sigma : \mathbf{n} \to I$, and the space $\Delta^n \times_{BI} \underrightarrow{\operatorname{holim}}\, X$ is isomorphic to the homotopy colimit $\underrightarrow{\operatorname{holim}}\, X\sigma$ associated to the composite functor

$$\mathbf{n} \xrightarrow{\sigma} I \xrightarrow{X} \mathbf{S}.$$

Furthermore, the map $i_*$ in (5.10) can be identified with the diagonal of the map $i_*$ in the following pullback diagram of bisimplicial sets:

$$\begin{tikzcd}
\displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Lambda^n_k}} X\sigma(k_0) \arrow[r, "i_*"] \arrow[d] & \displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Delta^n}} X\sigma(k_0) \arrow[d] \\
\displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Lambda^n_k}} * \arrow[r, "i"'] & \displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Delta^n}} *.
\end{tikzcd}$$

The initial object $0 \in \mathbf{n}$ determines a natural transformation $\theta : X\sigma(0) \to X\sigma$, where $X\sigma(0)$ denotes the constant functor at the object of the same name, and there is an induced diagram of bisimplicial set maps

$$\begin{tikzcd}
\displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Lambda^n_k}} X\sigma(0) \arrow[r] \arrow[d, "\theta_*"'] & \displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Delta^n}} X\sigma(0) \arrow[d, "\theta_*"] \\
\displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Lambda^n_k}} X\sigma(k_0) \arrow[r, "i_*"'] & \displaystyle\bigsqcup_{\substack{k_0\to\cdots\to k_r\\ \in\Delta^n}} X\sigma(k_0).
\end{tikzcd}$$

The vertical maps $\theta_*$ induce weak equivalences of associated diagonal simplicial sets, by Proposition 1.7, and the diagonal of the top horizontal map is the weak equivalence

$$i \times 1 : \Lambda^n_k \times X\sigma(0) \to \Delta^n \times X\sigma(0).$$

It follows that the map $i_*$ of (5.10) is a weak equivalence. □

There is a homology version of Lemma 5.7 for every homology theory $h_*$ which satisfies the wedge axiom. Here is a specimen statement:

Lemma 5.11. *Suppose that $X : I \to \mathbf{S}$ is a simplicial set valued functor which is defined on a small category $I$, and that $A$ is an abelian group. Suppose further that the induced simplicial set map $X(\alpha) : X(i) \to X(j)$ induces an isomorphism $H_*(X(i), A) \cong H_*(X(j), A)$ for each morphism $\alpha : i \to j$ of the index category $I$. Then, for each object $j$ of $I$ the pullback diagram of simplicial* *sets*

$$\begin{tikzcd}
X(j) \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}} X \arrow[d, "\pi"] \\
* \arrow[r, "j"'] & BI
\end{tikzcd}$$

*is homology cartesian in the sense that the corresponding map $X(j) \to F_j$ from $X(j)$ to the homotopy fibre $F_j$ over $j$ induces an isomorphism $H_*(X(j), A) \cong H_*(F_j, A)$.*

The proof of Lemma 5.11 is a spectral sequence argument which follows the basic outline of the proof of Lemma 5.7.

Proof of Theorem 5.6: The functor $y \mapsto B(y \downarrow F)$ determines a contravariant simplicial set valued functor $D \to \mathbf{S}$, and hence a bisimplicial set having $(n, m)$-bisimplices

$$\bigsqcup_{y_n \to \cdots \to y_0} B(y_0 \downarrow F)_m$$

This set of bisimplices can also be identified with the set of all strings of arrows in $D$ of the form

$$y_n \to \cdots \to y_0 \to F(x_0) \to \cdots \to F(x_m).$$

The degenerate simplices

$$y \stackrel{1}{\to} y \stackrel{1}{\to} \dots \stackrel{1}{\to} y$$

determine a commutative diagram of bisimplicial set maps

$$\begin{tikzcd}
B(y \downarrow F)_m \arrow[r] \arrow[d] & \displaystyle\bigsqcup_{y_n \to \cdots \to y_0} B(y_0 \downarrow F)_m \arrow[r, "Q_*"] \arrow[d] & BC_m \arrow[d, "F_*"] \\
B(y \downarrow D)_m \arrow[r] \arrow[d, "\simeq"'] & \displaystyle\bigsqcup_{y_n \to \cdots \to y_0} B(y_0 \downarrow D)_m \arrow[r, "Q_*"'] \arrow[d, "\simeq"] & BD_m \\
* \arrow[r] & \displaystyle\bigsqcup_{y_n \to \cdots \to y_0} *. &
\end{tikzcd} \tag{5.12}$$

The bisimplicial set map

$$\bigsqcup_{y_n \to \cdots \to y_0} B(y_0 \downarrow F)_m \xrightarrow{Q_*} BC_m \tag{5.13}$$

is an alternate way of representing the forgetful map

$$\bigsqcup_{y_n \to \dots y_0 \to F(x_0) \to \dots \to F(x_m)} * \to \bigsqcup_{x_0 \to \dots \to x_m} *$$

corresponding to the functor $F$, and can also be identified with the map

$$\bigsqcup_{x_0 \to \dots \to x_m} B(F(x_0) \downarrow D)^{op} \to \bigsqcup_{x_0 \to \dots \to x_m} *.$$

The category $(F(x_0) \downarrow D)^{op}$ has a terminal object, so the map $Q_*$ in (5.13) induces a weak equivalence of associated diagonals, by Proposition 1.7. The bisimplicial set map

$$\bigsqcup_{y_n \to \dots \to y_0} B(y_0 \downarrow D)_m \xrightarrow{Q_*} BD_m$$

is an instance of the map in (5.13), corresponding to the case where $F$ is the identity functor on the category $D$, so it induces a weak equivalence of associated diagonal simplicial sets as well. The categories $y \downarrow D$ and $y_0 \downarrow D$ have initial objects, so Proposition 1.7 implies that the indicated maps in the diagram (5.12) induce weak equivalences of associated diagonals.

Thus (see Lemma II.8.22), to show that the simplicial set diagram

$$\begin{tikzcd}
B(y \downarrow F) \arrow[r] \arrow[d] & BC \arrow[d, "F_*"] \\
B(y \downarrow D) \arrow[r] & BD
\end{tikzcd}$$

is homotopy cartesian, it is enough to see that the bisimplicial set diagram

$$\begin{tikzcd}
B(y \downarrow F)_m \arrow[r] \arrow[d] & \bigsqcup_{y_n \to \dots \to y_0} B(y \downarrow F)_m \arrow[d] \\
* \arrow[r, "y"'] & \bigsqcup_{y_n \to \dots \to y_0} *
\end{tikzcd}$$

induces a homotopy cartesian diagram of the associated diagonal simplicial sets. This is a consequence of Lemma 5.7. □

### 5.3. The group completion theorem.

Suppose that $M$ is a simplicial monoid, and that $X$ is a simplicial set with an $M$-action $M \times X \to X$. There is a Borel construction for this action, namely a bisimplicial set $EM \times_M X$ having vertical simplicial set in horizontal degree $m$ given by $M^{\times m} \times X$. The fastest way to convince yourself that this thing actually exists is to observe that the action $M \times X \to X$ is composed of actions $M_n \times X_n \to X_n$ of monoids of $n$-simplices on the corresponding sets $X_n$. These actions admit Borel constructions $EM_n \times_{M_n} X_n$ (or nerves of translation categories), and this construction is natural in $n$.

The canonical maps $\pi : EM_n \times_{M_n} X_n \to BM_n$ are also natural in $n$, and therefore define a map

$$\pi : EM \times_M X \to BM$$

of bisimplicial sets, which is given in horizontal degree $m$ by the projection

$$M^{\times m} \times X \to M^{\times m}.$$

There is a pullback diagram of bisimplicial set maps

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & EM \times_M X \arrow[d, "\pi"] \\
* \arrow[r] & BM
\end{tikzcd} \tag{5.14}$$

The group completion theorem gives a criterion for this diagram to be homology cartesian.

THEOREM 5.15 (GROUP COMPLETION). *Suppose that $M \times X \to X$ is an action of a simplicial monoid $M$ on a simplicial set $X$, and let $A$ be an abelian group. Suppose further that the action of each vertex $v$ of $M$ induces an isomorphism $v_* : H_*(X, A) \cong H_*(X, A)$. Then the diagram (5.14) is homology cartesian in the sense that the map $X \to F$ to the homotopy fibre of the simplicial set map $d(\pi)$ induces an isomorphism in homology with coefficients in $A$.*

Theorem 5.15 is used, in the main, to analyze the output of infinite loop space machines. It implies, for example, that each connected component of the $0^{th}$ space of the $\Omega$-spectrum corresponding to the sphere spectrum is a copy of the space $B\Sigma_\infty^+$ obtained by applying Quillen's plus construction to the classifying space of the infinite symmetric group [6], [84]. Here is another typical calculation:

EXAMPLE 5.16. Suppose that $R$ is a ring with identity. Then matrix addition induces a simplicial monoid structure on the simplicial set

$$M(R) = \bigsqcup_{n \geq 0} BGl_n(R).$$

In particular, right multiplication by the vertex $e = * \in BGl_1(R)_0$ (and all of its degeneracies) induces a simplicial set map

$$? \oplus e : M(R) \to M(R),$$

which restricts, on the $n^{th}$ summand, to the map $BGl_n(R) \to BGl_{n+1}(R)$ which is induced by the canonical inclusion $Gl_n(R) \hookrightarrow Gl_{n+1}(R)$ defined by $A \mapsto A \oplus I$, where $I$ denotes the identity element of $Gl_1(R)$. The filtered colimit of the system

$$M(R) \xrightarrow{? \oplus e} M(R) \xrightarrow{? \oplus e} \dots$$

can be identified up to isomorphism with the simplicial set

$$X(R) = \bigsqcup_{\mathbb{Z}} BGl(R).$$

The simplicial set $X(R)$ has a left $M(R)$ action, and Theorem 5.15 implies that the diagram

$$\begin{tikzcd}
X(R) \arrow[r] \arrow[d] & EM(R) \times_{M(R)} X(R) \arrow[d] \\
* \arrow[r] & BM(R)
\end{tikzcd}$$

is homology cartesian. In effect, left multiplication by a vertex $n \in M(R)$ shifts the vertices of $X(R)$: the summand corresponding to $r \in \mathbb{Z}$ is taken to the summand corresponding to $n + r$. The map induced on the $r^{th}$ summand itself is the simplicial set map $BGl(R) \to BGl(R)$ which is induced by the group homomorphism $I_n \oplus ? : Gl(R) \to Gl(R)$ defined by $A \mapsto I_n \oplus A$, where $I_n$ is the $n \times n$ identity matrix. As such, the group homomorphism $I_n \oplus ?$ is a filtered colimit of group homomorphisms $Gl_m(R) \to Gl_{m+n}(R)$. The key point is that $I_n \oplus A$ is conjugate, via a suitable choice of permutation matrix, to $A \oplus I_n$ in $Gl_{m+n}(R)$. It follows that (vertical) components of the comparison map

$$\begin{tikzcd}
H_*(BGl_m(R), \mathbb{Z}) \arrow[r, "can_*"] \arrow[d, "(I_n \oplus ?)_*"'] & H_*(BGl_{m+1}(R), \mathbb{Z}) \arrow[r, "can_*"] \arrow[d, "(I_n \oplus ?)_*"'] & \dots \\
H_*(BGl_{m+n}(R), \mathbb{Z}) \arrow[r, "can_*"'] & H_*(BGl_{m+n+1}(R), \mathbb{Z}) \arrow[r, "can_*"'] & \dots
\end{tikzcd}$$

coincide with morphisms induced by canonical inclusions, and so the group homomorphism $I_n \oplus ?$ induces the identity map on $H_*(BGl(R), \mathbb{Z})$.

The space arising from the bisimplicial set $EM(R) \times_{M(R)} X(R)$ is contractible, since it is a filtered colimit of objects of the form $EM(R) \times_{M(R)} M(R)$. It follows that $X(R)$ has the homology of the loop space $\Omega d(BM(R))$, and that the component $\Omega d(BM(R))_0$ of $0 \in \mathbb{Z}$ is an H-space having the homology of $BGl(R)$. This component $\Omega d(BM(R))_0$ must therefore be a copy of Quillen's space $BGl(R)^+$.

Finally (without going into a lot of details), the monoidal structure on $M(R)$ is abelian up to coherent isomorphism, so effectively one is entitled to form a collection of connected objects

$$BM(R),\ BBM(R),\ B^3M(R), \dots$$

such that $B^{n+1}M(R)$ is a delooping of $B^nM(R)$ for all $n$, just like one could do if $M(R)$ happened to be a simplicial abelian group. The list of spaces corresponding to

$$\Omega BM(R),\ BM(R),\ BBM(R),\ B^3M(R), \dots$$

is a model for the algebraic $K$-theory spectrum of the ring $R$; we have used the group completion theorem to identify its $0^{th}$ term.

There are several proofs of the group completion theorem in the literature: [48], [49], [73], [70]. The most elementary of these, and the one that will be given here, involves an analogue of the construction leading to the Serre spectral sequence, for maps of bisimplicial sets.

Suppose that $f : X \to Y$ is a map of bisimplicial sets, and consider all bisimplices $\sigma : \Delta^{r,s} \to Y$ of $Y$. Form the pullback diagram

$$\begin{tikzcd}
f^{-1}(\sigma) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^{r,s} \arrow[r] & Y
\end{tikzcd}$$

in the category of bisimplicial sets. The bisimplices $\Delta^{m,n} \to Y$ of $Y$ are the objects of the category $\mathbf{\Delta}^{\times 2} \downarrow Y$, called the *category of bisimplices* of $Y$. A morphism $\sigma \to \tau$ of this category is a commutative diagram of bisimplicial set maps

$$\begin{tikzcd}
\Delta^{r,s} \arrow[dr, "\sigma"] \arrow[dd] & \\
 & Y. \\
\Delta^{m,n} \arrow[ur, "\tau"'] &
\end{tikzcd}$$

One sees immediately that the assignment $\sigma \mapsto f^{-1}(\sigma)$ defines a functor

$$f^{-1} : \mathbf{\Delta}^{\times 2} \downarrow Y \to \mathbf{S}^2.$$

LEMMA 5.17. *Suppose that $f : X \to Y$ is a map of bisimplicial sets. Then the corresponding map*

$$\bigsqcup_{\substack{\sigma_0 \to \dots \to \sigma_r \\ \in B(\mathbf{\Delta}^{\times 2} \downarrow Y)}} f^{-1}(\sigma_0) \to X$$

*of trisimplicial sets induces a weak equivalence of associated diagonal simplicial sets.*

PROOF: The bisimplicial set $Y$ is a colimit of its bisimplices, and so $X$ is a colimit of their pullbacks, giving rise to a coequalizer

$$\bigsqcup_{\sigma_0 \to \sigma_1} f^{-1}(\sigma_0) \rightrightarrows \bigsqcup_{\sigma} f^{-1}(\sigma) \to Y.$$

Also, the component of the simplicial set

$$\bigsqcup_{\sigma_0 \to \dots \to \sigma_r} f^{-1}(\sigma_0)(m,n)$$

corresponding to each bisimplex $x \in X(m,n)$ is contractible, since it's the nerve of a category having an initial object. It follows that partially diagonalizing the trisimplicial set map

$$\bigsqcup_{\substack{\sigma_0 \to \dots \to \sigma_r \\ \in B(\mathbf{\Delta}^{\times 2} \downarrow Y)}} f^{-1}(\sigma_0)(m,n) \to X(m,n)$$

with respect to the variables $r$ and $m$ gives a bisimplicial set map

$$\bigsqcup_{\substack{\sigma_0 \to \dots \to \sigma_r \\ \in B(\mathbf{\Delta}^{\times 2} \downarrow Y)}} f^{-1}(\sigma_0)(r,n) \to X(r,n)$$

which is a weak equivalence of simplicial sets in each vertical degree $n$, by Proposition 1.7. This same result then implies that the simplicial set map

$$\bigsqcup_{\substack{\sigma_0 \to \dots \to \sigma_r \\ \in B(\mathbf{\Delta}^{\times 2} \downarrow Y)}} f^{-1}(\sigma_0)(r,r) \to X(r,r)$$

is a weak equivalence. □

REMARK 5.18. There is a paradigm in the proof of Lemma 5.17 for the manipulation of trisimplicial sets. The diagonal of a trisimplicial set $X$ is the simplicial set whose set of $n$-simplices is the set $X(n,n,n)$. This simplicial set defines the homotopy type arising from $X$, and it can be formed by iterating the diagonal construction for bisimplicial sets in three different ways. One picks the most convenient iteration for the problem at hand. Similar considerations apply, more generally, to $n$-fold simplicial sets

Consider the diagram of trisimplicial set maps

$$\begin{tikzcd}
f^{-1}(v) \arrow[r] \arrow[d] & \displaystyle\bigsqcup_{\sigma_0 \to \cdots \to \sigma_r} f^{-1}(\sigma_0) \arrow[r, "\simeq"] \arrow[d] & X \arrow[d] \\
\Delta^{0,0} \arrow[r] \arrow[d, "\simeq"'] & \displaystyle\bigsqcup_{\sigma_0 \to \cdots \to \sigma_r} \Delta^{m_0,n_0} \arrow[r, "\simeq"'] \arrow[d, "\simeq"] & Y \\
* \arrow[r, "v"'] & \displaystyle\bigsqcup_{\sigma_0 \to \cdots \to \sigma_r} *, &
\end{tikzcd}$$

where $v : \Delta^{0,0} \to Y$ is a vertex of $Y$. Observe that $* = \Delta^{0,0}$ is the terminal object in the bisimplicial set category. The labelled horizontal maps induce weak equivalences of the corresponding diagonal simplicial sets by Lemma 5.17, while the corresponding vertical maps induce weak equivalences of diagonals by iterated application of Proposition 1.7. It follows from Lemma 5.11 that the pullback diagram

$$\begin{tikzcd}
f^{-1}(v) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
* \arrow[r] & Y
\end{tikzcd}$$

of bisimplicial sets induces a homology cartesian diagram of associated diagonals if each morphism of bisimplices

$$\begin{tikzcd}
\Delta^{r_0,s_0} \arrow[dr, "\sigma_0"] \arrow[dd] & \\
& Y. \\
\Delta^{r_1,s_1} \arrow[ur, "\sigma_1"'] &
\end{tikzcd}$$

induces an isomorphism

$$H_*(f^{-1}(\sigma_0)) \xrightarrow{\cong} H_*(f^{-1}(\sigma_1)).$$

By this, it is meant that there should be an induced isomorphism

$$H_*(d(f^{-1}(\sigma_0)), A) \xrightarrow{\cong} H_*(d(f^{-1}(\sigma_1)), A),$$

relative to some choice of coefficient group $A$. Let's agree to suppress mention of the diagonals and the coefficient groups in the rest of this section.

PROOF OF THEOREM 5.15: We will show that each morphism

$$\begin{tikzcd} \Delta^{r,s} \arrow[d, "(\zeta_1,\zeta_2)"'] \arrow[dr, "\tau"] & \\ \Delta^{k,\ell} \arrow[r, "\sigma"'] & BM. \end{tikzcd} \tag{5.19}$$

induces an isomorphism

$$(\zeta_1, \zeta_2)_* : H_*(\pi^{-1}(\tau)) \xrightarrow{\cong} H_*(\pi^{-1}(\sigma)),$$

where $\pi : EM \times_M X \to BM$ is the canonical map.

Recall that, in horizontal degree $m$, $\Delta^{k,\ell}_m$ may be identified with the simplicial set

$$\bigsqcup_{\mathbf{m}\to\mathbf{k}} \Delta^\ell.$$

There is a pullback diagram of simplicial sets

$$\begin{tikzcd} \bigsqcup_{\mathbf{m}\to\mathbf{k}} (\Delta^\ell \times X) \arrow[r] \arrow[d] & M^{\times m} \times X \arrow[d, "pr"] \\ \bigsqcup_{\mathbf{m}\to\mathbf{k}} \Delta^\ell \arrow[r] & M^{\times m}, \end{tikzcd}$$

where the map on the left is a disjoint union of projections. Each ordinal number map $\theta : \mathbf{n} \to \mathbf{m}$ induces a simplicial map

$$\bigsqcup_{\mathbf{m}\to\mathbf{k}} (\Delta^\ell \times X) \xrightarrow{\theta^*} \bigsqcup_{\mathbf{n}\to\mathbf{k}} (\Delta^\ell \times X),$$

according to the horizontal structure of the bisimplicial set $\pi^{-1}(\sigma)$. There is a commutative diagram

$$\begin{tikzcd}
\Delta^{\ell} \times X \arrow[r, "\theta_{\gamma}^{*}"] \arrow[d, "in_{\gamma}"'] & \Delta^{\ell} \times X \arrow[d, "in_{\gamma\theta}"] \\
\bigsqcup_{\mathbf{m} \to \mathbf{k}} (\Delta^{\ell} \times X) \arrow[r, "\theta^{*}"'] & \bigsqcup_{\mathbf{n} \to \mathbf{k}} (\Delta^{\ell} \times X),
\end{tikzcd}$$

where $\theta_{\gamma}^{*}$ is defined on the simplex level by

$$(\zeta, x) \mapsto (\zeta, m_{*}(\zeta, x)).$$

Here, $m_{*}$ is the composite simplicial set map

$$\Delta^{\ell} \times X \xrightarrow{\nu \times 1} M^{\times m} \times X \to X,$$

where the map $M^{\times m} \times X \to X$ is an iterate of the action of $M$ on $X$, and $\nu$ is some $\ell$-simplex of $M$. The assumptions on the action $m$ therefore imply that $\theta_{\gamma}^{*}$ is a homology isomorphism.

One of the spectral sequences for the homology of the diagonal of the bisimplicial set

$$\bigsqcup_{\mathbf{m} \to \mathbf{k}} (\Delta^{\ell} \times X)$$

has $E_2$-term

$$H_{*}(\bigoplus_{\omega : \mathbf{m} \to k} A(\omega)),$$

where $A$ is a contravariant functor on the simplices of $\Delta^{k}$: the group $A(\omega)$ is a copy of $H_q(\Delta^{\ell} \times X)$. We have just seen that every morphism

$$\begin{tikzcd}
\mathbf{m} \arrow[dr, "\gamma\theta"] \arrow[dd, "\theta"'] & \\
& \mathbf{k} \\
\mathbf{n} \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

induces an isomorphism $\theta^{*} : A(\gamma) \cong A(\gamma\theta)$. The morphism $1_{\mathbf{k}}$ is terminal in the simplex category for $\Delta^{k}$, so there is a natural isomorphism which is defined by diagrams

$$\begin{tikzcd}
A(1_{\mathbf{k}}) \arrow[r, "\cong"] \arrow[d, "1"'] & A(\gamma) \arrow[d, "\theta^*"] \\
A(1_{\mathbf{k}}) \arrow[r, "\cong"'] & A(\gamma\theta).
\end{tikzcd}$$

Lemma 2.2 says that there are isomorphisms

$$H_i(\bigoplus_{\omega:\mathbf{m}\to k} A(\omega)) \cong \begin{cases} A(1_{\mathbf{k}}) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

In particular, the inclusion

$$X \times \Delta^{\ell} \xrightarrow{in_v} \bigsqcup_{\mathbf{0}\to\mathbf{k}} (X \times \Delta^{\ell})$$

corresponding to a vertex $v$ of $\Delta^k$ induces an isomorphism

$$H_q(X \times \Delta^{\ell}) \cong H_q(\pi^{-1}(\sigma))$$

Finally, observe that every bisimplex map of the form (5.19) induces a commutative diagram of simplicial set maps

$$\begin{tikzcd}
X \times \Delta^s \arrow[r, "1 \times \zeta_2"] \arrow[d, "in_w"'] & X \times \Delta^{\ell} \arrow[d, "in_{\zeta_1 w}"] \\
\bigsqcup_{\mathbf{0}\to\mathbf{r}} (X \times \Delta^s) \arrow[r, "(\zeta_1, \zeta_2)_*"'] & \bigsqcup_{\mathbf{0}\to\mathbf{k}} (X \times \Delta^{\ell}),
\end{tikzcd}$$

where $w$ is a vertex of $\Delta^r$. It follows that the map $(\zeta_1, \zeta_2)_*$ is a homology isomorphism. □

# Chapter V Simplicial groups

This is a somewhat complex chapter on the homotopy theory of simplicial groups and groupoids, divided into seven sections. Many ideas are involved. Here is a thumbnail outline:

Section 1, *Skeleta*: Skeleta for simplicial sets were introduced briefly in Chapter I, and then discussed more fully in the context of the Reedy closed model structure for bisimplicial sets in Section IV.3.2. Skeleta are most precisely described as Kan extensions of truncated simplicial sets. The current section gives a general description of such Kan extensions in a more general category $\mathcal{C}$, followed by a particular application to a description of the skeleta of almost free morphisms of simplicial groups. The presentation of this theory is loosely based on the Artin-Mazur treatment of hypercovers of simplicial schemes [3], but the main result for applications that appear in later sections is Proposition 1.9. This result is used to show in Section 5 that the loop group construction outputs cofibrant simplicial groups.

Section 2, *Principal Fibrations I: Simplicial $G$-spaces*: The main result of this section asserts that the category $\mathbf{S}_G$ of simplicial sets admitting an action by a fixed simplicial group $G$ admits a closed model structure: this is Theorem 2.3. Principal $G$-fibrations in the classical sense may then be identified with cofibrant objects of $\mathbf{S}_G$, by Corollary 2.10, and an equivariant map between two such objects is an isomorphism if and only if it induces an isomorphism of coinvariants (Lemma 2.11).

Section 3, *Principal Fibrations II: Classifications*: This section contains a proof of the well known result (Theorem 3.9) that isomorphism classes of principal $G$-fibrations $p : E \to B$ can be classified by homotopy classes of maps $B \to BG$, where $BG = EG/G$, and $EG$ is an arbitrary cofibrant object of $\mathbf{S}_G$ admitting a trivial fibration $EG \to *$, all with respect to the the closed model structure for $\mathbf{S}_G$ of Section 2.

Section 4, *Universal cocycles and $\overline{W}G$*: It is shown here that the classical model $\overline{W}G$ for the classifying object $BG$ of Section 3 can be constructed as a simplicial set of cocycles taking values in the simplicial group $G$. This leads to "global" descriptions of the simplicial structure maps for $\overline{W}G$, as well as for the $G$-bundles associated to simplicial set maps $X \to \overline{W}G$. The total space $WG$ for the canonical bundle associated to the identity map on $\overline{W}G$ is contractible (Lemma 4.6).

Section 5, *The loop group construction*: The functor $G \mapsto \overline{W}G$ has a left adjoint $X \mapsto GX$, defined on reduced simplicial sets $X$ (Lemma 5.3). The simplicial group $GX$ is the loop group of the reduced simplicial set $X$, in the sense that the total space of the bundle associated to the adjunction map $X \to \overline{W}GX$ is contractible: this is Theorem 5.10. The proof of this theorem is a modernized version of the Kan's original geometric proof, in that it involves a reinterpretation of the loop group $GX$ as an object constructed from equivalence classes of loops.

Section 6, *Reduced simplicial sets, Milnor's $FK$-construction*: This section gives a closed model structure for the category $\mathbf{S}_0$ of reduced simplicial sets. This structure is used to show (in conjunction with the results of Section 1) that the loop group functor preserves cofibrations and weak equivalences, and that $\overline{W}$ preserves fibrations and weak equivalences (Proposition 6.3). In particular, the loop group functor and the functor $\overline{W}$ together induce an equivalence between the homotopy categories associated to the categories of reduced simplicial sets and simplicial groups (Corollary 6.4). Furthermore, any space of the form $\overline{W}G$ is a Kan complex (Corollary 6.8); this is the last piece of the proof of the assertion that $\overline{W}G$ is a classifying space for the simplicial group $G$, as defined in Section 3. Milnor's $FK$-construction is a simplicial group which gives a fibrant model for the space $\Omega\Sigma K$: Theorem 6.15 asserts that $FK$ is a copy of $G(\Sigma K)$, by which is meant the loop group of the Kan suspension of $K$. The Kan suspension was introduced in Section III.5.

Section 7, *Simplicial groupoids*: The main result of Section 5, which leads to the equivalence of homotopy theories between reduced simplicial sets and simplicial groups of Section 6, fails badly for non-reduced simplicial sets. We can nevertheless recover an analogous statement for the full category of simplicial sets if we replace simplicial groups by simplicial groupoids, by a series of results of Dwyer and Kan. This theory is presented in this section. There is a closed model structure on the category $s\mathbf{Gd}$ of simplicial *groupoids* (Theorem 7.6) whose associated homotopy category is equivalent to that of the full simplicial set category (Corollary 7.11). The classifying object and loop group functors extend, respectively, to functors $\overline{W} : s\mathbf{Gd} \to \mathbf{S}$ and $G : \mathbf{S} \to s\mathbf{Gd}$; the object $\overline{W}A$ associated to a simplicial groupoid $A$ is a simplicial set of cocycles in a way that engulfs the corresponding object for simplicial groups, and the extended functor $G$ is its left adjoint.

## 1. Skeleta.

Suppose that $\mathcal{C}$ is a category having all finite colimits, and let $s\mathcal{C}$ denote the category of simplicial objects in $\mathcal{C}$. Recall that simplicial objects in $\mathcal{C}$ are contravariant functors of the form $\mathbf{\Delta}^{op} \to \mathcal{C}$, defined on the ordinal number category $\mathbf{\Delta}$.

The ordinal number category contains a full subcategory $\mathbf{\Delta}_n$, defined on the objects $\mathbf{m}$ with $0 \leq m \leq n$. Any simplicial object $X : \mathbf{\Delta}^{op} \to \mathcal{C}$ restricts to a contravariant functor $i_{n*}X : \mathbf{\Delta}_n^{op} \to \mathcal{C}$, called the *$n$-truncation of $X$*. More generally, an *$n$-truncated simplicial object* in $\mathcal{C}$ is a contravariant functor $Y : \mathbf{\Delta}_n^{op} \to \mathcal{C}$, and the category of such objects (functors and natural transformations between them) will be denoted by $s_n\mathcal{C}$.

The $n$-truncation functor $s\mathcal{C} \to s_n\mathcal{C}$ defined by $X \mapsto i_{n*}X$ has a left adjoint $i_n^* : s_n\mathcal{C} \to s\mathcal{C}$, on account of the completeness assumption on the category $\mathcal{C}$. Explicitly, the theory of left Kan extensions dictates that, for an $n$-truncated object $Y$, $i_n^*Y_m$ should be defined by

$$i_n^*Y_m = \varinjlim_{\mathbf{m} \xrightarrow{\theta} \mathbf{i},\ i \leq n} Y_i.$$

As the notation indicates, the colimit is defined on the finite category whose objects are ordinal number morphisms $\theta : \mathbf{m} \to \mathbf{i}$ with $i \leq n$, and whose morphisms $\gamma : \theta \to \tau$ are commutative diagrams

$$\begin{tikzcd}
& \mathbf{i} \arrow[dd, "\gamma"] \\
\mathbf{m} \arrow[ur, "\theta"] \arrow[dr, "\tau"'] & \\
& \mathbf{j}
\end{tikzcd}$$

in the ordinal number category. The simplicial structure map $\omega^* : i_n^*Y_m \to i_n^*Y_k$ is defined on the index category level by precomposition with the morphism $\omega : \mathbf{k} \to \mathbf{m}$.

The functor $Y \mapsto i_n^*Y$ is left adjoint to the $n$-truncation functor: this can be seen by invoking the theory of Kan extensions, or directly.

If $m \leq n$, then the index category of arrows $\mathbf{m} \to \mathbf{i}$, $i \leq n$, has an initial object, namely $1_{\mathbf{m}} : \mathbf{m} \to \mathbf{m}$, so that the canonical map

$$Y_m \xrightarrow{in_{1_{\mathbf{m}}}} \varinjlim_{\mathbf{m} \xrightarrow{\theta} \mathbf{i},\ i \leq n} Y_i$$

is an isomorphism by formal nonsense. Furthermore, maps of this form in $\mathcal{C}$ are the components of the adjunction map

$$Y \xrightarrow{\eta} i_{n*}i_n^*Y,$$

so that this map is an isomorphism of $s_n\mathcal{C}$.

The objects $i_n^*Y_m$, $m > n$, require further analysis. The general statement that is of the most use is the following:

Lemma 1.1. *There is a coequalizer diagram*

$$\bigsqcup_{i<j} Y_{n-1} \rightrightarrows \bigsqcup_{i=0}^{n} Y_n \xrightarrow{s} i_n^*Y_{n+1},$$

*where the maps in the coequalizer are defined by the commutativity of the following diagram:*

$$\begin{tikzcd}
Y_{n-1} \arrow[r, "s_i"] \arrow[d, "in_{i<j}"'] & Y_n \arrow[d, "in_j"'] \arrow[dr, "in_{s^j}"] & \\
\bigsqcup\limits_{i<j} Y_{n-1} \arrow[r, shift left] \arrow[r, shift right] & \bigsqcup\limits_{i=0}^{n} Y_n \arrow[r, "s"'] & i_n^* Y_{n+1}. \\
Y_{n-1} \arrow[r, "s_{j-1}"'] \arrow[u, "in_{i<j}"] & Y_n \arrow[u, "in_i"] &
\end{tikzcd}$$

Proof: Write $\mathbf{D}$ for the category of ordinal number morphisms $\theta : \mathbf{n+1} \to \mathbf{j}$, $j \leq n$. Suppose that $t : \mathbf{n+1} \to \mathbf{i}$ is an ordinal number epimorphism, where $i \leq n$, and write $\mathbf{D}_t$ for the category of ordinal number morphisms $\theta : \mathbf{n+1} \to \mathbf{j}$, $j \leq n$, which factor through $t$. Then $\mathbf{D}_t$ has an initial object, namely $t$, so that the canonical map $in_t$ induces an isomorphism

$$Y_i \xrightarrow[\cong]{in_t} \varinjlim_{\mathbf{n+1} \xrightarrow{\theta} \mathbf{j} \ \in \mathbf{D}_t} Y_j$$

Furthermore, if $t$ has a factorization

$$\begin{tikzcd}
\mathbf{n+1} \arrow[rr, "t"] \arrow[dr, "r"'] & & \mathbf{i} \\
& \mathbf{m} \arrow[ur, "s"'] &
\end{tikzcd}$$

where $r$ and $s$ are ordinal number epimorphisms, the inclusion $\mathbf{D}_t \subset \mathbf{D}_r$ induces a morphism $s^*$ of colimits which fits into a commutative diagram

$$\begin{tikzcd}
Y_i \arrow[r, "s^*"] \arrow[d, "in_t"', "\cong"] & Y_m \arrow[d, "in_r", "\cong"'] \\
\varinjlim\limits_{\mathbf{n+1} \to \mathbf{j} \ \in \mathbf{D}_t} Y_j \arrow[r, "s^*"'] & \varinjlim\limits_{\mathbf{n+1} \to \mathbf{j} \ \in \mathbf{D}_r} Y_j.
\end{tikzcd}$$

Write $\mathbf{D}_j$ for the category $\mathbf{D}_{s^j}$, $0 \leq j \leq n$.

For $i < j$, the diagram

$$\begin{tikzcd}
\mathbf{n+1} \arrow[r, "s^j"] \arrow[d, "s^i"'] & \mathbf{n} \arrow[d, "s^i"] \\
\mathbf{n} \arrow[r, "s^{j-1}"'] & \mathbf{n-1}
\end{tikzcd} \tag{1.2}$$

is a pushout in the ordinal number category: this is checked by fiddling with simplicial identities. Now, suppose given a collection of maps

$$f_j : \varinjlim_{\mathbf{n+1}\xrightarrow{\theta}\mathbf{i}\ \in\mathbf{D}_j} Y_i \to X,$$

$0 \leq j \leq n$, such that the diagrams

$$\begin{tikzcd}
\varinjlim\limits_{\mathbf{n+1}\to\mathbf{i}\ \in\mathbf{D}_t} Y_i \arrow[r, "s_{j-1}"] \arrow[d, "s_i"'] & \varinjlim\limits_{\mathbf{n+1}\to\mathbf{i}\ \in\mathbf{D}_{s^i}} Y_i \arrow[d, "f_i"] \\
\varinjlim\limits_{\mathbf{n+1}\to\mathbf{i}\ \in\mathbf{D}_{s^j}} Y_i \arrow[r, "f_j"'] & X
\end{tikzcd} \tag{1.3}$$

commute, where $t = s^i s^j = s^{j-1}s^i$. Let $\theta : \mathbf{n+1} \to \mathbf{k}$ be an object of $\mathbf{D}$. Then $\theta \in \mathbf{D}_i$ for some $i$, and we define a morphism $f_\theta : Y_k \to X$ to be the composite

$$Y_k \xrightarrow{in_\theta} \varinjlim_{\mathbf{n+1}\to\mathbf{k}\ \in\mathbf{D}_i} Y_k \xrightarrow{f_i} X.$$

The pushout diagram (1.2) and the commutativity conditions (1.3) together imply that the definition of $f_\theta$ is independent of $i$. The collection of maps $f_\theta$, $\theta \in \mathbf{D}$, determine a unique map

$$f_* : \varinjlim_{\mathbf{n+1}\to\mathbf{k}\ \in\mathbf{D}} Y_k$$

which restricts to the maps $f_i$, for $0 \leq i \leq n$, and the lemma is proved. $\square$

Write $\mathrm{sk}_n\, Y = i_n^* i_{n*} Y$, and write $\epsilon : \mathrm{sk}_n\, Y \to Y$ for the counit of the adjunction. The simplicial set $sk_nY$ is the *n-skeleton* of $Y$.

LEMMA 1.4. *Let $Y$ be a simplicial object in the category $\mathcal{C}$, and suppose that there is a morphism $f : N \to Y_{n+1}$ such that the canonical map $\epsilon : \operatorname{sk}_n Y \to Y$ and $f$ together induce an isomorphism*

$$\operatorname{sk}_n Y_{n+1} \sqcup N \xrightarrow[\cong]{(\epsilon,f)} Y_{n+1}.$$

*Then an extension of a map $g : \operatorname{sk}_n Y \to Z$ to a map $g' : \operatorname{sk}_{n+1} Y \to Z$ corresponds to a map $\tilde{g} : N \to Z_{n+1}$ such that $d_i\tilde{g} = gd_if$ for $0 \leq i \leq n+1$.*

PROOF: Given such a map $\tilde{g}$, define a map

$$g'' : Y_{n+1} \cong \operatorname{sk}_n Y_{n+1} \sqcup N \to Z_{n+1}$$

by $g' = (g, \tilde{g})$. In effect, we are looking to extend a map $g : i_{n*}Y \to i_{n*}Z$ to a map $g' : i_{(n+1)*}Y \to i_{(n+1)*}Z$. The truncated map $g'$ will be the map $g''$ in degree $n+1$ and will coincide with the map $g$ in degrees below $n+1$, once we show that $g'$ respects simplicial identities in the sense that the following diagram commutes:

$$\begin{tikzcd}
Y_{n+1} \arrow[r, "g'"] \arrow[d, shift right, "\gamma^*"'] & Z_{n+1} \arrow[d, shift right, "\gamma^*"'] \\
Y_m \arrow[r, "g"'] \arrow[u, shift right, "\theta^*"'] & Z_m \arrow[u, shift right, "\theta^*"']
\end{tikzcd}$$

for all ordinal number maps $\gamma : \mathbf{m} \to \mathbf{n+1}$ and $\theta : \mathbf{n+1} \to \mathbf{m}$, where $m < n+1$. The canonical map $\epsilon : \operatorname{sk}_n Y \to Y$ consists of isomorphisms

$$\operatorname{sk}_n Y_i \xrightarrow[\cong]{\epsilon} Y_i$$

in degrees $i \leq n$, so that $\theta^* : Y_m \to Y_{n+1}$ factors through the map $\epsilon : \operatorname{sk}_n Y_{n+1} \to Y_{n+1}$; the restriction of $g'$ to $\operatorname{sk}_n Y_{n+1}$ is a piece of a simplicial map, so that $g'$ respects $\theta^*$. The map $\gamma^*$ factors through some face map $d_i$, so it's enough to show that $g'$ respects the face maps, but this is automatic on $\operatorname{sk}_n Y_{n+1}$ and is an assumption on $\tilde{g}$.

The converse is obvious. □

LEMMA 1.5. *Suppose that $i : A \to B$ is a morphism of $s_{n+1}\mathcal{C}$ which is an isomorphism in degrees $j \leq n$. Suppose further that there is a morphism $f : N \to B_{n+1}$ such that the maps $i$ and $f$ together determine an isomorphism*

$$A_{n+1} \sqcup N \xrightarrow[\cong]{(i,f)} B_{n+1}.$$

*Suppose that $g : A \to Z$ is a morphism of $s_{n+1}\mathcal{C}$. Then extensions*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i"'] & Z \\
B \arrow[ur, "g'"'] &
\end{tikzcd}$$

*of the morphism $g$ to morphisms $g' : B \to Z$ are in one to one correspondence with morphisms $\tilde{g} : N \to Z_{n+1}$ of $\mathcal{C}$ such that $d_i\tilde{g} = gd_if$ for $0 \leq i \leq n$.*

PROOF: This lemma is an abstraction of the previous result. The proof is the same. □

A morphism $j : G \to H$ of simplicial groups is said to be *almost free* if there is a contravariant set-valued functor $X$ defined on the epimorphisms of the ordinal number category $\mathbf{\Delta}$ such that there are isomorphisms

$$G_n * F(X_n) \xrightarrow[\cong]{\theta_n} H_n$$

which

(1) are compatible with the map $j$ in the sense that $\theta_n \cdot in_{G_n} = j_n$ for all $n$, and
(2) respect the functorial structure of $X$ in the sense that the diagram

$$\begin{tikzcd}
G_n * F(X_n) \arrow[r, "\theta_n"] \arrow[d, "t^* * F(t^*)"'] & H_n \arrow[d, "t^*"] \\
G_m * F(X_m) \arrow[r, "\theta_m"'] & H_m
\end{tikzcd}$$

commutes for every ordinal number epimorphism $t : \mathbf{m} \to \mathbf{n}$.

The *n-skeleton* $\mathrm{sk}_n(j)$ of the simplicial group homomorphism $j : G \to H$ is defined by the pushout diagram

$$\begin{tikzcd}
\mathrm{sk}_n G \arrow[r, "j_*"] \arrow[d] & \mathrm{sk}_n H \arrow[d] \\
G \arrow[r] & \mathrm{sk}_n(j)
\end{tikzcd}$$

in the category of groups. There are maps $\mathrm{sk}_n(j) \to \mathrm{sk}_{n+1}(j)$ and morphisms $\mathrm{sk}_n(j) \to H$ such that the diagrams

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ \mathrm{sk}_n(j) \arrow[rr] \arrow[dr] & & \mathrm{sk}_{n+1}(j) \arrow[dl] \\ & H & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r] \arrow[dr, "j"'] & \mathrm{sk}_n(j) \arrow[d] \\ & H \end{tikzcd}$$

commute, and such that $j : G \to H$ is a filtered colimit of the maps $G \to \mathrm{sk}_n(j)$ in the category of simplicial groups under $G$. The maps $\mathrm{sk}_n(j)_i \to H_i$ are group isomorphisms for $i \leq n$, so the map $\mathrm{sk}_n(j) \to \mathrm{sk}_{n+1}(j)$ consists of group isomorphisms in degrees up to $n$.

Write $DX_n$ for the degenerate part of $X_{n+1}$. This subset can be described (as usual) as the union of the images of the functions $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$. For $i < j$ the diagram of group homomorphisms

$$\begin{tikzcd} H_{n-1} \arrow[r, "s_i"] \arrow[d, "s_{j-1}"'] & H_n \arrow[d, "s_j"] \\ H_n \arrow[r, "s_i"'] & H_{n+1} \end{tikzcd} \tag{1.6}$$

is a pullback, by manipulating the simplicial identities. Pullback diagrams are closed under retraction, so the diagram of group homomorphisms

$$\begin{tikzcd} F(X_{n-1}) \arrow[r, "s_i"] \arrow[d, "s_{j-1}"'] & F(X_n) \arrow[d, "s_j"] \\ F(X_n) \arrow[r, "s_i"'] & F(X_{n+1}) \end{tikzcd} \tag{1.7}$$

is also a pullback. All the homomorphisms in (1.7) are monomorphisms (since they are retracts of such), so an argument on reduced words shows that (1.7) restricts on generators to a pullback

$$\begin{tikzcd} X_{n-1} \arrow[r, "s_i"] \arrow[d, "s_{j-1}"'] & X_n \arrow[d, "s_j"] \\ X_n \arrow[r, "s_i"'] & X_{n+1} \end{tikzcd} \tag{1.8}$$

in the set category. It follows that the degenerate part $DX_n$ of the set $X_{n+1}$ can be defined by a coequalizer

$$\bigsqcup_{i<j} X_{n-1} \rightrightarrows \bigsqcup_{i=0}^{n} X_n \xrightarrow{s} DX_n$$

such as one would expect if $X$ were part of the data for a simplicial set, in which case $DX_n$ would be a copy of $\mathrm{sk}_n\, X_{n+1}$.

Lemma 1.1 implies that the diagram of group homomorphisms

$$\begin{tikzcd}
\mathrm{sk}_n\, G_{n+1} \arrow[r] \arrow[d] & \mathrm{sk}_n\, H_{n+1} \arrow[d] \\
G_{n+1} \arrow[r] & \mathrm{sk}_n(j)_{n+1}
\end{tikzcd}$$

can be identified up to canonical isomorphism with the diagram

$$\begin{tikzcd}
\mathrm{sk}_n\, G_{n+1} \arrow[r] \arrow[d] & \mathrm{sk}_n\, G_{n+1} * F(DX_n) \arrow[d] \\
G_{n+1} \arrow[r] & G_{n+1} * F(DX_n).
\end{tikzcd}$$

The map $\mathrm{sk}_n(i)_{n+1} \to \mathrm{sk}_{n+1}(i)_{n+1}$ can therefore be identified up to isomorphism with the monomorphism

$$G_{n+1} * F(DX_n) \to G_{n+1} * F(X_{n+1})$$

which is induced by the inclusion $DX_n \subset X_{n+1}$.

Let $NX_{n+1} = X_{n+1} - DX_n$ be the non-degenerate part of $X_{n+1}$. The truncation at level $n+1$ of the map $\mathrm{sk}_n(j) \to \mathrm{sk}_{n+1}(j)$ is an isomorphism in degrees up to $n$, and is one of the components of an isomorphism

$$\mathrm{sk}_n(j)_{n+1} * F(NX_{n+1}) \cong \mathrm{sk}_{n+1}(j)_{n+1}.$$

in degree $n+1$.

Proposition 1.9. *Suppose that $j : G \to H$ is an almost free simplicial group homomorphism, with $H$ generated over $G$ by the functor $X$ as described above. Let $NX_{n+1}$ be the non-degenerate part of $X_{n+1}$. Then there is pushout diagram of simplicial groups of the form*

$$\begin{tikzcd}
\underset{x \in NX_{n+1}}{*} F(\partial\Delta^{n+1}) \arrow[r] \arrow[d] & \mathrm{sk}_n(j) \arrow[d] \\
\underset{x \in NX_{n+1}}{*} F(\Delta^{n+1}) \arrow[r] & \mathrm{sk}_{n+1}(j)
\end{tikzcd} \tag{1.10}$$

*for each $n \geq -1$.*

Proof: From the discussion above, truncating the diagram (1.10) at level $n+1$ gives a pushout of $(n+1)$-truncated simplicial groups. All objects in (1.10) diagram are isomorphic to their $(n+1)$-skeleta, so (1.10) is a pushout. □

Corollary 1.11. *Any almost free simplicial group homomorphism $j : G \to H$ is a cofibration of simplicial groups.*

## 2. Principal fibrations I: simplicial $G$-spaces.

A principal fibration is one in which the fibre is a simplicial group acting in a particular way on the total space. They will be defined completely below and we will classify them, but it simplifies the discussion considerably if we discuss more general actions first.

Definition 2.1. *Let $G$ be a simplicial group and $X$ a simplicial set. Then $G$ acts on $X$ if there is a morphism of simplicial sets*

$$\mu : G \times X \to X$$

*so that the following diagrams commute:*

$$\begin{tikzcd}
G \times G \times X \arrow[r, "1 \times \mu"] \arrow[d, "m \times 1"'] & G \times X \arrow[d, "\mu"] \\
G \times X \arrow[r, "\mu"'] & X
\end{tikzcd}$$

*and*

$$\begin{tikzcd}
X \arrow[d, "i"'] \arrow[dr, "1_X"] & \\
G \times X \arrow[r, "\mu"'] & X
\end{tikzcd}$$

*where $m$ is the multiplication in $G$ and $i(X) = (e, X)$.*

In other words, at each level, $X_n$ is a $G_n$-set and the actions are compatible with the simplicial structure maps.

Let $\mathbf{S}_G$ be the category of simplicial sets with $G$-action, hereinafter known as $G$-spaces. Note that $\mathbf{S}_G$ is a simplicial category. Indeed, if $K \in \mathbf{S}$, then $K$ can be given the trivial $G$-action. Then for $X \in \mathbf{S}_G$ set

$$X \otimes K = X \times K \tag{2.2.1}$$

with diagonal action,

$$\mathbf{hom}_{\mathbf{S}_G}(K, X) = \mathbf{Hom}_{\mathbf{S}}(K, X) \tag{2.2.2}$$

with action in the target, and for $X$ and $Y$ in $\mathbf{S}_G$,

$$\mathbf{Hom}_{\mathbf{S}_G}(X, Y)_n = \hom_{\mathbf{S}_G}(X \otimes \Delta^n, Y). \tag{2.2.3}$$

Then the preliminary result is:

THEOREM 2.3. *There is a simplicial model category structure on* $\mathbf{S}_G$ *such that* $f : X \to Y$ *is*

1) *a weak equivalence if and only if* $f$ *is a weak equivalence in* $\mathbf{S}$*;*
2) *a fibration if and only if* $f$ *is a fibration in* $\mathbf{S}$*; and*
3) *a cofibration if and only if* $f$ *has the left lifting property with respect to all trivial fibrations.*

PROOF: The forgetful functor $\mathbf{S}_G \to S$ has a left adjoint given by

$$X \mapsto G \times X.$$

Thus we can apply Theorem II.6.8 once we show that every cofibration having the left lifting property with respect to all fibrations is a weak equivalence. Every morphism $X \to Y$ can be factored as $X \xrightarrow{j} Z \xrightarrow{q} X$ where $q$ is a fibration and $j$ is obtained by setting $Z = \varinjlim_n Z_n$ with $Z_0 = X$ and $X_n$ defined by a pushout diagram

$$\begin{tikzcd}
\bigsqcup\limits_{\alpha} G \times \Lambda^n_k \arrow[r] \arrow[d, "i_n"'] & Z_{n-1} \arrow[d, "j_n"] \\
\bigsqcup\limits_{\alpha} G \times \Delta^n \arrow[r] & Z_n
\end{tikzcd}$$

where $\alpha$ runs over all diagrams in $\mathbf{S}_G$

$$\begin{tikzcd}
G \times \Lambda^n_k \arrow[r] \arrow[d, "i_n"'] & Z_{n-1} \arrow[d] \\
G \times \Delta^n \arrow[r] & Y.
\end{tikzcd}$$

Since $i_n$ is a trivial cofibration in $\mathbf{S}$, we have that $j_n$ a trivial cofibration in $\mathbf{S}$ (and also in $\mathbf{S}_G$). So $j : X \to Z$ is a trivial cofibration in $\mathbf{S}$ (and $\mathbf{S}_G$).

If $i : X \to Y$ is a cofibration having the left lifting property with respect to all fibrations, then $i$ has a factorization $i = q \cdot j$ as above, so that $i$ is a retract of the cofibration $j$ by the standard argument. □

A crucial structural fact about $\mathbf{S}_G$ is the following:

Lemma 2.4. *Let $f : X \to Y$ be a cofibration in $\mathbf{S}_G$. Then $f$ is an inclusion and at each level $Y_k - f(X_k)$ is a free $G_k$-set.*

Proof: Every cofibration is a retract of a cofibration $j : X \to Z$ where $Z = \varinjlim Z_n$ and $Z_n$ is defined recursively by setting $Z_0 = X$ and defining $Z_n$ by a pushout diagram

$$\begin{tikzcd}
\bigsqcup_\alpha G \times \partial\Delta^n \arrow[r] \arrow[d] & Z_{n-1} \arrow[d, "j_n"] \\
\bigsqcup_\alpha G \times \Delta^n \arrow[r] & Z_n.
\end{tikzcd}$$

So it is sufficient to prove the result for these more specialized cofibrations. Now each $j_n$ is an inclusion, so $j : X \to Z$ is an inclusion. Also, at each level, we have a formula for $k$ simplices

$$(Z_n)_k - (Z_{n-1})_k = (\amalg_\alpha G \times \Delta^n)_k - (\amalg_\alpha G \times \partial\Delta^n)_k$$

is free. Hence

$$(Z)_k - (X)_k = \bigcup_n (Z_n)_k - (Z_{n-1})_k$$

is free. □

For $X \in \mathbf{S}_G$, let $X/G$ be the quotient space by the $G$-action. Let $q : X \to X/G$ be the quotient map. If $X \in \mathbf{S}_G$ is cofibrant this map has special properties.

Lemma 2.5. *Let $X \in \mathbf{S}_G$ have the property that $X_n$ is a free $G_n$ set for all $n$. Let $x \in (X/G)_n$ be an $n$-simplex. If $f_x : \Delta^n \to X$ represents $x$, define $F_x$ by the pullback diagram*

$$\begin{tikzcd}
F_x \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
\Delta^n \arrow[r, "f_x"'] & X/G.
\end{tikzcd}$$

*Then for every $z \in X$ so that $q(z) = x$, there is an isomorphism in $\mathbf{S}_G$*

$$\varphi_z : G \times \Delta^n \to F_x$$

*so that the following diagram commutes*

$$\begin{tikzcd}
G \times \Delta^n \arrow[r, "\varphi_z"] \arrow[d, "\pi_2"'] & F_x \arrow[d] \\
\Delta^n \arrow[r, "="'] & \Delta^n.
\end{tikzcd} \tag{2.6}$$

PROOF: First note that there is a natural $G$-action on $F_x$ so that $F_x \to X$ is a morphism of $G$-spaces. Fix $z \in X_n$ so that $q(z) = x$. Now every element of $\Delta^n$ can be written uniquely as $\theta^*\iota_n$ where $\iota_n \in \Delta^n_n$ is the canonical $n$-simplex and $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map. Define $\varphi_z$ by the formula, for $g \in G_m$:

$$\varphi_z(g, \theta^*\iota_n) = (\theta^*\iota_n, g\theta^*z).$$

One must check this is a simplicial $G$-map. Having done so, the diagram (2.6) commutes, so we need only check $\varphi_z$ is a bijection.

To see $\varphi_z$ is onto, for fixed $(a, b) \in F_x$ one has $f_x a = q(b)$. We can write $a = \theta^*\iota_n$ for some $\theta$, so

$$f_x a = \theta^* f_x \iota_n = \theta^* x = q\theta^* z$$

so $b$ is in the same orbit as $\theta^*z$, as required.

To see $\varphi_z$ is one-to-one, suppose

$$(\theta^*\iota_n, g\theta^*z) = (\psi^*\iota_n, h\psi^*z).$$

Then $\theta = \psi$ and, hence, $g\theta^*z = h\theta^*z$. The action is level-wise free by assumption, so $g = h$. $\square$

COROLLARY 2.7. *Let $X \in \mathbf{S}_G$ have the property that each $X_n$ is a free $G_n$ set. The quotient map $q : X \to X/G$ is a fibration in $\mathbf{S}$. It is a minimal fibration if $G$ is minimal as a Kan complex.*

PROOF: Consider a lifting problem

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "q"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & X/G.
\end{tikzcd}$$

This is equivalent to a lifting problem

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & F_x \arrow[d] \\
\Delta^n \arrow[r, "="'] \arrow[ur, dashed] & \Delta^n.
\end{tikzcd}$$

By Lemma 2.5, this is equivalent to a lifting problem

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & G \times \Delta^n \arrow[d, "\pi_j"] \\
\Delta^n \arrow[r, "="'] \arrow[ur, dashed] & \Delta^n.
\end{tikzcd}$$

Because $G$ is fibrant in $\mathbf{S}$ (Lemma I.3.4), $\pi_j$ is a fibration, so the problem has a solution. If $G$ is minimal, the lifting has the requisite uniqueness property to make $q$ a minimal fibration (see Section I.10). □

LEMMA 2.8. *Let $X \in \mathbf{S}_G$ have the property that each $X_n$ is a free $G_n$ set. Then $X = \varinjlim X^{(n)}$ where $X^{(-1)} = \emptyset$ and for each $n \geq 0$ there is a pushout diagram*

$$\begin{tikzcd}
\bigsqcup_\alpha \partial\Delta^n \times G \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\
\bigsqcup_\alpha \Delta^n \times G \arrow[r] & X^{(n)}
\end{tikzcd}$$

*where $\alpha$ runs over the non-degenerate $n$-simplices of $X/G$.*

PROOF: Define $X^{(n)}n$ by the pullback diagram

$$\begin{tikzcd}
X^{(n)} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
\mathrm{sk}_n(X/G) \arrow[r] & X/G.
\end{tikzcd}$$

Then $X^{(-1)} = \emptyset$ and $\varinjlim X^{(n)} = X$. Also, the pushout diagram

$$\begin{tikzcd}
\bigsqcup_\alpha \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(X/G) \arrow[d] \\
\bigsqcup_\alpha \Delta^n \arrow[r] & \mathrm{sk}_n(X/G)
\end{tikzcd}$$

pulls back along the canonical map $X \to X/G$ to a diagram

$$\begin{tikzcd}
\bigsqcup_\alpha F(\alpha)|_{\partial\Delta^\eta} \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\
\bigsqcup_\alpha F(\alpha) \arrow[r] & X^{(n)},
\end{tikzcd} \tag{2.9}$$

where $F(\alpha)$ is defined to be the pullback along $\alpha$. But Lemma 2.5 rewrites $F(\alpha) \cong \Delta^n \times G$ and, hence, $F(\alpha)|_{\partial\Delta^n} \cong \partial\Delta^n \times G$. Finally, pulling back along $\pi$ preserves pushouts, so the diagram (2.9) is a pushout. □

COROLLARY 2.10. *An object $X \in \mathbf{S}_G$ is cofibrant if and only if $X_n$ is a free $G_n$ set for all $n$.*

PROOF: One implication is Lemma 2.4. The other is a consequence of Lemma 2.8. □

LEMMA 2.11. *Suppose given a morphism in $\mathbf{S}_G$ $f : Y \to X$ so that*

1) *$X_n$ is a free $G_n$ set for all $n$*
2) *the induced map $Y/G \to X/G$ is an isomorphism*

*then $f$ is an isomorphism.*

PROOF: This is a variation on the proof of the 5-lemma. To show $f$ is onto, choose $z \in X$. Let $q_X : X \to X/G$ and $q_Y : Y \to Y/G$ be the quotient maps. Then there is a $w \in Y/G$ so that $(f/G)(w) = q_X(z)$. Let $y \in Y$ be so that $q_Y(y) = w$. Then there is a $g \in G$ so that $gf(y) = f(gy) = z$. To show $f$ is one-to-one suppose $f(y_1) = f(y_2)$. Then $q_X f(y_1) = q_X f(y_2)$ so $q_Y(y_1) = q_Y(y_2)$ or there is a $g \in G$ so that $gy_1 = y_2$. Then

$$gf(y_1) = f(y_2) = f(y_1)$$

Since $X$ is free at each level, $g = e$, so $y_1 = y_2$. □

## 3. Principal fibrations II: classifications.

In this section we will define and classify principal fibrations. Let $G$ be a fixed simplicial group.

DEFINITION 3.1. *A principal fibration (or principal $G$-fibration) $f : E \to B$ is a fibration in $\mathbf{S}_G$ so that*

1) *$B$ has trivial $G$-action;*
2) *$E$ is a cofibrant $G$-space; and*
3) *the induced map $E/G \to B$ is an isomorphism.*

Put another way, $f : E \to B$ is isomorphic to a quotient map

$$q : X \to X/G$$

where $X \in \mathbf{S}_G$ is cofibrant. Such a map $q$ is automatically a fibration by Corollary 2.7. Cofibrant objects can be recognized by Corollary 2.10, and Lemma 2.5 should be regarded as a local triviality condition. Finally, there is a diagram

$$\begin{tikzcd}
G \times E \arrow[r, "\mu"] \arrow[d, "* \times f"'] & E \arrow[d, "f"] \\
* \times B \arrow[r, "\cong"'] & B
\end{tikzcd}$$

where $\mu$ is the action; such diagrams figure in the topological definition of principal fibration.

In the same vein, it is quite common to say that a principal $G$-fibration is a *$G$-bundle.*

DEFINITION 3.2. *Two principal fibrations $f_1 : E_1 \to B$ and $f_2 : E_2 \to B$ will be called isomorphic if there is an isomorphism $g : E_1 \to E_2$ of $G$-spaces making the diagram commute*

$$\begin{tikzcd}
E_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & E_2 \arrow[dl, "f_2"] \\
& B &
\end{tikzcd}$$

REMARK 3.3. By Lemma 2.11 it is sufficient to construct a $G$-equivariant map $g : E_1 \to E_2$ making the diagram commute. Then $g$ is automatically an isomorphism.

Let $PF_G(B)$ be the set of isomorphism class of principal fibrations over $B$. The purpose of this section is to classify this set.

To begin with, note that $PF_G(\cdot)$ is a contravariant functor. If $q : E \to B$ is a principal fibration and $f : B' \to B$ is any map of spaces, and if $q' : E(f) \to B'$ is defined by the pullback diagram

$$\begin{tikzcd}
E(f) \arrow[r] \arrow[d, "q'"'] & E \arrow[d, "q"] \\
B' \arrow[r, "f"'] & B,
\end{tikzcd}$$

then $f'$ is a principal fibration. Indeed

$$E(g) = \{(b, e) \in B' \times E \mid f(b) = q(e)\}$$

has $G$ action given by $g(b, e) = (b, ge)$. Then parts 1) and 3) of Definition 3.1 are obvious and part 2) follows from Corollary 2.10.

But, in fact, $PF_G(\cdot)$ is a homotopy functor. Recall that two maps $f_0, f_1 : B' \to B$ are simplicially homotopic if there is a diagram

$$\begin{tikzcd}
B' \sqcup B' \arrow[rr, "d^0 \sqcup d^1"] \arrow[dr, "f_0 \sqcup f_1"'] & & B' \times \Delta^1 \arrow[dl] \\
& B &
\end{tikzcd}$$

LEMMA 3.4. *If $f_0$ and $f_1$ are simplicially homotopic, $PF_G(f_0) = PF_G(f_1)$.*

PROOF: It is sufficient to show that given $q : E \to B$ a principal fibration, the pullbacks $E(f_0) \to B'$ and $E(f_1) \to B'$ are isomorphic. For this it is sufficient to consider the universal example: given a principal fibration $E \to B \times \Delta^1$, the pullbacks $E(d^0) \to B$ and $E(d^1) \to B$ are isomorphic. For this consider the lifting problem in $\mathbf{S}_G$

$$\begin{tikzcd}
E(d^0) \arrow[r] \arrow[d, "d^0"'] & E \arrow[d] \\
E(d^0) \times \Delta^1 \arrow[r] \arrow[ur, dashed] & B \times \Delta^1.
\end{tikzcd}$$

Since $E(d^0)$ is cofibrant in $\mathbf{S}_G$, $d^0$ is a trivial cofibration, so the lifting exists and by Lemma 1.8 defines an isomorphism of principal fibrations

$$\begin{tikzcd}
E(d^0) \times \Delta^1 \arrow[rr, "\cong"] \arrow[dr] & & E \arrow[dl] \\
& B \times \Delta^1 &
\end{tikzcd}$$

Pulling back this diagram along $d^1$ gives the desired isomorphism. □

A similar sort of argument proves the following lemma:

LEMMA 3.5. *Let $B \in \mathbf{S}$ be contractible. Then any principal fibration over $B$ is isomorphic to $\pi_2 : G \times B \to B$.*

PROOF: The isomorphism is given by lifting in the diagram (in $\mathbf{S}_G$).

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "j"'] & E \arrow[d] \\
G \times B \arrow[r, "\pi_2"'] \arrow[ur, dashed] & B
\end{tikzcd}$$

Here $j$ is induced by any basepoint $* \to B$; since $G$ is cofibrant in $\mathbf{S}_G$, $j$ is a trivial cofibration in $\mathbf{S}_G$. □

We can now define the classifying object for principal fibrations.

DEFINITION 3.6. *Let $EG \in \mathbf{S}_G$ be any cofibrant object so that the unique map $EG \to *$ is a fibration and a weak equivalence. Let $BG = EG/G$ and $q : EG \to BG$ the resulting principal fibration.*

Note that $EG$ is unique up to equivariant homotopy equivalence, so $q : EG \to BG$ is unique up to homotopy equivalence.

In other words we require more than that $EG$ be a free contractible $G$-space; $EG$ must also be fibrant. The extra condition is important for the proof of Theorem 3.9 below. It also makes the following result true.

LEMMA 3.7. *The space $BG$ is fibrant as a simplicial set.*

PROOF: By Corollary 2.7, the map $q : EG \to BG$ is a Kan fibration. It is also surjective, so that any map $\Lambda^n_k \to BG$ lifts to a map $\Lambda^n_k \to EG$. But then $EG$ is fibrant, so that the map $\Lambda^n_k \to EG$ extends to an $n$-simplex $\Delta^n \to EG$ in $EG$, hence in $BG$. □

EXERCISE 3.8. There is a general principle at work in the proof of Lemma 3.7. Suppose given a diagram of simplicial set maps

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "q \cdot p"'] & Y \arrow[d, "q"] \\ & Z \end{tikzcd}$$

such that $p$ and the composite $q \cdot p$ are Kan fibrations, and that $p$ is surjective. Show that $q$ is a Kan fibration.

Note that the same argument proves that if $E \in \mathbf{S}_G$ is cofibrant and fibrant, the resulting principal fibration $E \to E/G$ has fibrant base.

We now come to the main result.

THEOREM 3.9. *For all spaces $B \in \mathbf{S}$, the map*

$$\theta : [B, BG] \to PF_G(B)$$

*sending the class $[f] \in [B, BG]$ to the pullback of $EG \to BG$ along $f$ is a bijection.*

Here, $[B, BG]$ denotes morphisms in the homotopy category $\mathrm{Ho}(\mathbf{S})$ from $B$ to $BG$. The space $BG$ is fibrant, so this morphism set cat be identified with the set of simplicial homotopy classes of maps from $B$ to $BG$.

PROOF: Note that $\theta$ is well-defined by Lemma 3.4. To prove the result we construct an inverse. If $q : E \to B$ is a principal fibration, there is a lifting in the diagram in $\mathbf{S}_G$

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & EG \arrow[d] \\ E \arrow[r] \arrow[ur] & * \end{tikzcd} \tag{3.10}$$

since $E$ is cofibrant and $EG$ is fibrant, and this lifting is unique up to equivariant homotopy. Let $f : B \to BG$ be the quotient map. Define $\Psi : PF_G(B) \to [B, BG]$, by sending $q : E \to B$ to the class of $f$.

Note that if $E(f)$ is the pullback of $f$, there is a diagram

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr] & & E(f) \arrow[dl] \\
& B &
\end{tikzcd}$$

so Lemma 2.11 implies $\theta\Psi = 1$. On the other hand, given a representative $g : B \to BG$ of a homotopy class in $[B, BG]$, the map $g'$ in the diagram

$$\begin{tikzcd}
E(g) \arrow[r, "g'"] \arrow[d] & EG \arrow[d] \\
B \arrow[r, "g"'] & BG
\end{tikzcd}$$

makes the diagram (3.10) commute, so by the homotopy uniqueness of liftings $\Psi\theta = 1$. □

## 4. Universal cocycles and $\overline{W}G$.

In the previous sections, we took a simplicial group $G$ and assigned to it a homotopy type $BG$; that is, the space $BG$ depended on a choice $EG$ of a fibrant, cofibrant contractible $G$-space.

In this section we give a natural, canonical choice for $EG$ and $BG$ called, respectively, $WG$ and $\overline{W}G$. The spaces $WG$ and $\overline{W}G$ are classically defined by letting $WG$ be the simplicial set with

$$WG_n = G_n \times G_{n-1} \times \cdots \times G_0$$

and

$$d_i\,(g_n, g_{n-1}, \dots g_0)$$
$$= \begin{cases} (d_i g_n, d_{i-1}g_{n-1}, \dots , (d_0 g_{n-i})g_{n-i-1}, g_{n-i-2}, \dots g_0) & i < n, \\ (d_n g_n, d_{n-1}g_{n-1}, \dots d_1 g_1) & i = n. \end{cases}$$

$$s_i(g_n, g_{n-1} \dots , g_0) = (s_i g_n, s_{i-1}g_{n-1} \dots s_0 g_{n-1}, e, g_{n-i-1}, \dots g_0)$$

where $e$ is always the unit. Note that $WG$ becomes a $G$-space if we define $G \times WG \to WG$ by:

$$(h, (g_n, g_{n-1}, \dots , g_0) \longrightarrow (hg, g_{n-1}, \dots g_0).$$

Then $\overline{W}G$ is the quotient of $WG$ by the left $G$-action; write $q = q_G : WG \to \overline{W}G$ for the quotient map. We establish the most of the basic properties of this construction in this section; $\overline{W}G$ will be shown to be fibrant in Corollary 6.8.

LEMMA 4.1. *The map $q : WG \to \overline{W}G$ is a fibration.*

PROOF: This follows from Corollary 2.7 since $(WG)_n$ is a free $G_n$ set. □

The functor $G \mapsto \overline{W}G$ takes values in the category $\mathbf{S}_0$ of reduced simplicial sets, where a *reduced simplicial set* is a simplicial set having only one vertex. The salient deeper feature of the functor $\overline{W} : s\mathbf{Gr} \to \mathbf{S}_0$ is that it has a left adjoint $G : \mathbf{S}_0 \to s\mathbf{Gr}$, called the loop group functor, such that the canonical maps $G(\overline{W}G) \to G$ and $X \to \overline{W}(GX)$ are weak equivalences for all simplicial groups $G$ and reduced simplicial sets $X$. A demonstration of these assertions will occupy this section and the following two. These results are originally due to Kan, and have been known since the late 1950's. The original proofs were calculational — we recast them in modern terms here. Kan's original geometric insights survive and are perhaps sharpened, in the presence of the introduction of a closed model structure for reduced simplicial sets and a theory of simplicial cocycles.

A *segment* of an ordinal number $\mathbf{n}$ is an ordinal number monomorphism $\mathbf{n} - \mathbf{j} \hookrightarrow \mathbf{n}$ which is defined by $i \mapsto i + j$. This map can also be variously characterized as the unique monomorphism $\mathbf{n} - \mathbf{j} \hookrightarrow \mathbf{n}$ which takes 0 to $j$, or as the map $(d^0)^j$. This map will also be denoted by $[j, n]$, as a means of identifying its image. There is a commutative diagram of ordinal number maps

$$\begin{tikzcd}
\mathbf{n} - \mathbf{k} \arrow[dr, "{[k,n]}"] \arrow[dd, "\tau"'] & \\
& \mathbf{n} \\
\mathbf{n} - \mathbf{j} \arrow[ur, "{[j,n]}"'] &
\end{tikzcd}$$

if and only if $j \leq k$. The map $\tau$ is uniquely determined and must be a segment map if it exists: it's the map $(d^0)^{k-j}$. Thus, we obtain a poset $\mathrm{Seg}(\mathbf{n})$ of segments of the ordinal number $n$. This poset is plainly isomorphic to the poset opposite to the ordinal $\mathbf{n}$.

Suppose that $G$ is a simplicial group. An *$n$-cocycle* $f : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ associates to each relation $\tau : [k, n] \leq [j, n]$ in $\mathrm{Seg}(\mathbf{n})$ an element $f(\tau) \in G_{n-k}$, such that the following conditions hold:

(1) $f(1_j) = e \in G_{n-j}$, where $1_j$ is the identity relation $[j, n] \leq [j, n]$,

(2) for any composeable pair of relations $[l, n] \xrightarrow{\zeta} [k, n] \xrightarrow{\tau} [j, n]$, there is an equation
$$\zeta^*(f(\tau))f(\zeta) = f(\tau\zeta).$$

Any ordinal number map $\gamma : \mathbf{r} \to \mathbf{s}$ has a unique factorization

$$\begin{tikzcd}
\mathbf{r} \arrow[rr, "\gamma"] \arrow[dr, "\gamma_*"'] & & \mathbf{s} \\
& \mathbf{s} - \gamma(\mathbf{0}) \arrow[ur, "{[\gamma(0), s] = (d^0)^{\gamma(0)}}"'] &
\end{tikzcd}$$

where $\gamma_*$ is an ordinal number map such that $\gamma_*(0) = 0$. It follows that any relation $\tau : [k, m] \leq [j, m]$ in $\mathrm{Seg}(\mathbf{m})$ induces a commutative diagram of ordinal number maps

$$\begin{tikzcd}
\mathbf{m} - \mathbf{k} \arrow[rr, "\theta_k"] \arrow[dr, "\tau"] \arrow[dd, "{[k,m]}"'] & & \mathbf{n} - \theta(\mathbf{k}) \arrow[dr, "\tau_*"] \arrow[dd, "{[\theta(k),n]}"' near end] & \\
& \mathbf{m} - \mathbf{j} \arrow[rr, "\theta_j" near start] \arrow[dl, "{[j,m]}"] & & \mathbf{n} - \theta(\mathbf{j}) \arrow[dl, "{[\theta(j),n]}"] \\
\mathbf{m} \arrow[rr, "\theta"'] & & \mathbf{n} &
\end{tikzcd} \tag{4.2}$$

where the maps $\theta_j$ and $\theta_k$ take 0 to 0. Given an $n$-cocyle $f : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, define, for each relation $\tau : [k, m] \leq [j, m]$ in $\mathrm{Seg}(\mathbf{m})$, an element $\theta^*(f)(\tau) \in G_{m-k}$ by

$$\theta^*(f)(\tau) = \theta_k^*(f(\tau_*)).$$

It's not hard to see now that the collection of all such elements $\theta^*(f)(\tau)$ defines an $m$-cocycle $\theta^*(f) : \mathrm{Seg}(\mathbf{m}) \rightsquigarrow G$, and that the assignment $\theta \mapsto \theta^*$ is contravariantly functorial in ordinal maps $\theta$. We have therefore constructed a simplicial set whose $n$-simplices are the $n$-cocycles $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, and whose simplicial structure maps are the induced maps $\theta^*$.

This simplicial set of $G$-cocycles is $\overline{W}G$ . This claim is checked by chasing the definition through faces and degeneracies, while keeping in mind the observation that an $n$-cocycle $f : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ is completely determined by the string of relations

$$[n, n] \xrightarrow{\tau_0} [n-1, n] \xrightarrow{\tau_1} \dots \xrightarrow{\tau_{n-2}} [1, n] \xrightarrow{\tau_{n-1}} [0, n], \tag{4.3}$$

and the corresponding element

$$(f(\tau_{n-1}), f(\tau_{n-2}), \dots, f(\tau_0)) \in G_{n-1} \times G_{n-2} \times \dots \times G_0.$$

Of course, each $\tau_i$ is an instance of the map $d^0$.

The identification of the simplicial set of $G$-cocycles with $\overline{W}G$ leads to a "global" description of the simplicial structure of $\overline{W}G$. Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and let

$$\overline{g} = (g_{n-1}, g_{n-2}, \dots, g_0)$$

be an element of $G_{n-1} \times G_{n-2} \times \cdots \times G_0$. Let $F_{\overline{g}}$ be the cocycle $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ associated to the n-tuple $\overline{g}$. Then, subject to the notation appearing in diagram (4.2), we have the relation

$$\theta^*(g_{n-1}, g_{n-2}, \dots, g_0) = (\theta_1^* F_{\overline{g}}(\tau_{m-1*}), \theta_2^* F_{\overline{g}}(\tau_{m-2*}), \dots, \theta_m^* F_{\overline{g}}(\tau_{0*})),$$

where $\tau_{m-i*} = (d^0)^{\theta(i)-\theta(i-1)}$ is the induced relation $[\theta(i), n] \leq [\theta(i-1), n]$ of $\mathrm{Seg}(\mathbf{n})$.

A simplicial map $f : X \to \overline{W}G$, from this point of view, assigns to each $n$-simplex $x$ a cocycle $f(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, such that for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ and each map $\tau : [k,m] \to [j,m]$ in $\mathrm{Seg}(\mathbf{m})$ there is a relation

$$\theta_k^* f(x)(\tau_*) = f(\theta^*(x))(\tau).$$

Any element $j \in \mathbf{n}$ determines a unique diagram

$$\begin{tikzcd}
\mathbf{n} \arrow[dr, "{[0,n]=1}"] & \\
 & \mathbf{n} \\
\mathbf{n}-\mathbf{j} \arrow[uu, "{[j,n]}"] \arrow[ur, "{[j,n]}"'] &
\end{tikzcd}$$

and hence unambiguously gives rise to elements

$$f(x)([j,n]) \in G_{n-j}.$$

Observe further that if $j \leq k$ and $\tau : [k,n] \leq [j,n]$ denotes the corresponding relation in $\mathrm{Seg}(\mathbf{n})$, then the cocyle condition for the composite

$$[k,n] \xrightarrow{\tau} [j,n] \xrightarrow{[j,n]} [0,n]$$

can be rephrased as the relation

$$\tau^*(f(x)([j,n])) = f(x)([k,n]) f(x)(\tau)^{-1}.$$

Now, given a map (cocycle) $f : X \to \overline{W}G$, and an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$, there is an induced function

$$\theta^* : G_n \times X_n \to G_m \times X_m,$$

which is defined by

$$(g, x) \mapsto (\theta^*(g)\theta_0^*(f(x)([\theta(0), n])), \theta^*(x)), \tag{4.4}$$

where $\theta_0 : \mathbf{m} \to \mathbf{n} - \theta(0)$ is the unique ordinal number map such that

$$[\theta(0), n] \cdot \theta_0 = \theta.$$

LEMMA 4.5. *The maps $\theta^*$ defined in (4.4) are functorial in ordinal number maps $\theta$.*

PROOF: Suppose given ordinal number maps

$$\mathbf{k} \xrightarrow{\gamma} \mathbf{m} \xrightarrow{\theta} \mathbf{n},$$

and form the diagram

$$\begin{tikzcd}
\mathbf{k} \arrow[d, "\gamma_0"'] \arrow[dr, "\gamma"] & & \\
\mathbf{m}-\gamma(0) \arrow[r, "{[\gamma(0),m]}"'] \arrow[d, "\theta_{\gamma(0)}"'] & \mathbf{m} \arrow[d, "\theta_0"'] \arrow[dr, "\theta"] & \\
\mathbf{n}-\theta(\gamma(0)) \arrow[r, "{[\gamma(0),m]_*}"'] & \mathbf{n}-\theta(0) \arrow[r, "{[\theta(0),n]}"'] & \mathbf{n}
\end{tikzcd}$$

in the ordinal number category. In order to show that $\gamma^*\theta^*(g,x) = (\theta\gamma)^*(g,x)$ in $G_k \times X_k$, we must show that

$$\gamma^*\theta_0^*(f(x)([\theta(0),n]))\gamma_0^*(f(\theta^*(x))([\gamma(0),m])) = \gamma_0^*\theta_{\gamma(0)}^*(f(x)([\theta(\gamma(0)),n]))$$

in $G_k$. But

$$\gamma^*\theta_0^* = \gamma_0^*\theta_{\gamma(0)}^*[\gamma(0),m]_*^*,$$

and

$$[\gamma(0),m]_*^*(f(x)[\theta(0),n]) = f(x)([\theta\gamma(0),n])(f(x)([\gamma(0),m]_*))^{-1}$$

by the cocycle condition. Finally,

$$\theta_{\gamma(0)}^*(f(x)([\gamma(0),m]_*)) = f(\theta^*(x))([\gamma(0),m]),$$

since $f$ is a simplicial map. The desired result follows. □

The simplicial set constructed in Lemma 4.5 from the map $f : X \to \overline{W}G$ will be denoted by $X_f$. The projection maps $G_n \times X_n \to X_n$ define a simplicial map $\pi : X_f \to X$, and this map $\pi$ has the structure of a $G$-bundle, or principal fibration. This is a natural construction: if $h : Y \to X$ is a simplicial set map, then the maps $G_n \times Y_n \to G_n \times X_n$ defined by $(g,y) \mapsto (g,h(y))$ define a $G$-equivariant simplicial set map $h_* : Y_{fh} \to X_f$ such that the diagram

$$\begin{tikzcd}
Y_{fh} \arrow[r, "h_*"] \arrow[d, "\pi"'] & X_f \arrow[d, "\pi"] \\
Y \arrow[r, "h"'] & X
\end{tikzcd}$$

commutes. Furthermore, this diagram is a pullback.

The simplicial set $\overline{W}G_1$ associated to the identity map $1 : \overline{W}G \to \overline{W}G$ is $WG$, and the $G$-bundle $\pi : WG \to \overline{W}G$ is called the *canonical $G$-bundle.*

LEMMA 4.6. *$WG$ is contractible.*

PROOF: Suppose given an element $(g_n, (g_{n-1}, \dots, g_0)) \in WG_n$. Then the $(n+2)$-tuple $(e, (g_n, g_{n-1}, \dots g_0))$ defines an element of $WG_{n+1}$, in such a way that the following diagram of simplicial set maps commutes:

$$\begin{tikzcd}
\Delta^{n+1} \arrow[drr, "{(e,(g_n,g_{n-1},\dots,g_0))}"] & & \\
\Delta^n \arrow[u, "d^0"] \arrow[rr, "{(g_n,(g_{n-1},\dots,g_0))}"'] & & WG
\end{tikzcd}$$

commutes. Furthermore, if $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and $\theta_* : \mathbf{m+1} \to \mathbf{n+1}$ is the unique map such that $\theta_*(0) = 0$ and $\theta_* d^0 = d^0\theta$, then

$$\theta_*^*(e, (g_n, g_{n-1}, \dots, g_0)) = (e, \theta^*(g_n, (g_{n-1}, \dots, g_0)).$$

It follows that the simplices $(e, g_n, \dots, g_0)$ define an extra degeneracy on $WG$ in the sense of Section III.5, and so Lemma III.5.1 implies that $WG$ is contractible. □

REMARK 4.7. Every principal $G$-fibration $p : Y \to X$ is isomorphic to a principal fibration $X_f \to X$ for some map $f : X \to \overline{W}G$. In effect, let $\mathbf{\Delta}_*$ denote the subcategory of the category $\mathbf{\Delta}$ consisting of all ordinal number morphisms $\gamma : \mathbf{m} \to \mathbf{n}$ such that $\gamma(0) = 0$. Then the map $p$ restricts to a natural transformation $p_* : Y|_{\mathbf{\Delta}_*} \to X|_{\mathbf{\Delta}_*}$, and this transformation has a section $\sigma : X|_{\mathbf{\Delta}_*} \to X|_{\mathbf{\Delta}_*}$ in the category of contravariant functors on $\mathbf{\Delta}_*$, essentially since the simplicial map $p$ is a surjective Kan fibration. Classically, the map $\sigma$ is called a *pseudo cross-section* for the bundle $p$. The pseudo cross-section $\sigma$ defines $G_n$-equivariant isomorphisms

$$\phi_n : G_n \times X_n \cong Y_n$$

given by $(g, x) \mapsto g \cdot \sigma(x)$. If $\tau : \mathbf{n-k} \to \mathbf{n-j}$ is a morphism of Seg($\mathbf{n}$) then

$$\tau^*(\sigma(d_0^j x)) = f_x(\tau)\sigma(\tau^* d_0^j x)$$

for some unique element $f_x(\tau) \in G_{n-k}$. The elements $f_x(\tau)$ define a cocycle $f_x : \text{Seg}(\mathbf{n}) \rightsquigarrow G$ for each simplex $x$ of $X$, and the collection of cocycles $f_x$, $x \in X$, defines a simplicial map $f : X \to \overline{W}G$ such that $Y$ is $G$-equivariantly isomorphic to $X_f$ over $X$ via the maps $\phi_n$. The classical approach to the classification of principal $G$-bundles is based on this construction, albeit not in these terms.

## 5. The loop group construction.

Suppose that $f : X \to \overline{W}G$ is a simplicial set map, and let $x \in X_n$ be an $n$-simplex of $X$. Recall that the associated cocycle $f(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ is completely determined by the group elements

$$f(x)(d^0 : (d^0)^{k+1} \to (d^0)^k).$$

On the other hand,

$$f(x)(d^0 : (d^0)^{k+1} \to (d^0)^k) = f(d_0^k(x))(d^0 : d^0 \to 1_{\mathbf{n-k}}).$$

It follows that the simplicial map $f : X \to \overline{W}G$ is determined by the elements

$$f(x)(d^0) = f(x)(d^0 \to 1_{\mathbf{n}}) \in G_{n-1},$$

for $x \in X_n$, $n \geq 1$. Note in particular that $f(s_0 x)(d^0) = e \in G_{n-1}$.

Turning this around, suppose that $x \in X_{n+1}$, and the ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ has the factorization

$$\begin{tikzcd} & \mathbf{n} - \theta(\mathbf{0}) \arrow[d, "\tau"] \\ \mathbf{m} \arrow[ur, "\theta_0"] \arrow[r, "\theta"'] & \mathbf{n}, \end{tikzcd}$$

where $\theta_0(0) = 0$ and $\tau$ is a segment map, and suppose that $d^0 : d^0 \to 1_{\mathbf{n+1}}$ is the inclusion in $\mathrm{Seg}(\mathbf{n+1})$. Then

$$\tau^*(f(x)(d^0)) = f(x)(d^0\tau)(f(d_0(x))(\tau))^{-1}.$$

by the cocycle condition for $f(x)$, and so

$$\begin{aligned} \theta^*(f(x)(d^0)) &= \theta_0^*\tau^*(f(x)(d^0)) \\ &= \theta_0^*(f(x)(d^0\tau))\theta_0^*(f(d_0(x))(\tau))^{-1} \\ &= f(\tilde{\theta}^*(x))(d^0)(f((c\theta)^*(d_0(x)))(d^0))^{-1}, \end{aligned}$$

where $\tilde{\theta} : \mathbf{m+1} \to \mathbf{n+1}$ is defined by

$$\tilde{\theta}(i) = \begin{cases} 0 & \text{if } i = 0, \text{ and} \\ \theta(i-1)+1 & \text{if } i \geq 1, \end{cases}$$

and $c\theta : \mathbf{m+1} \to \mathbf{n}$ is the ordinal number map defined by $(c\theta)(0) = 0$ and $(c\theta)(i) = \theta(i-1)$ for $i \geq 1$. Observe that $c\theta = s^0\tilde{\theta}$.

Define a group $GX_n = F(X_{n+1})/s_0F(X_n)$ for $n \geq 0$, where $F(Y)$ denotes the free group on a set $Y$. Note that $GX_n$ may also be described as the free group on the set $X_{n+1} - s_0X_n$.

Given an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$, define a group homomorphism $\theta^* : GX_n \to GX_m$ on generators $[x]$, $x \in X_{n+1}$ by specifying

$$\theta^*([x]) = [\tilde{\theta}^*(x)][(c\theta)^*(d_0(x))]^{-1}. \tag{5.1}$$

If $\gamma : \mathbf{k} \to \mathbf{m}$ is an ordinal number map which is composeable with $\theta$, then the relations

$$\begin{aligned}(c\gamma)^* d_0\tilde{\theta}^*(x) &= (c\gamma)^*\theta^* d_0(x)\\ &= (c\gamma)^* d_0(c\theta)^* d_0(x)\end{aligned}$$

and

$$\tilde{\gamma}^*(c\theta)^* d_0(x) = (c(\theta\gamma))^* d_0(x)$$

together imply that $\gamma^*\theta^*([x]) = (\theta\gamma)^*([x])$ for all $x \in X_{n+1}$, so that we have a simplicial group, called the *loop group* of $X$, which will be denoted $GX$. This construction is plainly functorial in simplicial sets $X$.

Each $n$-simplex $x \in X$ gives rise to a string of elements

$$([x], [d_0x], [d_0^2x], \dots, [d_0^{n-1}x]) \in GX_{n-1} \times GX_{n-2} \times \cdots \times GX_0,$$

which together determine a cocycle $F_x : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow GX$. Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map such that $\theta(0) = 0$. The game is now to obtain a recognizable formula for $[\theta^*x]$, in terms of the simplicial structure of $GX$.

Obviously, if $\theta(1) = \theta(0)$, then $[\theta^*x] = e \in GX_{m-1}$. Suppose that $\theta(1) > 0$. Then there is a commutative diagram of ordinal number maps

$$\begin{tikzcd}
\mathbf{m}-\mathbf{1} \arrow[r, "\theta_1"] \arrow[dd, "d^0"'] & \mathbf{n}-\theta(\mathbf{1}) \arrow[dr, "(d^0)^{\theta(1)-1}"] \arrow[dd, "\tau"'] & \\
& & \mathbf{n}-\mathbf{1} \arrow[dl, "d^0"] \\
\mathbf{m} \arrow[r, "\theta"'] & \mathbf{n} &
\end{tikzcd}$$

If $\gamma = (d^0)^{\theta(1)-1}\theta_1$, then $\theta = \tilde{\gamma}$, and so

$$[\theta^*(x)] = (\theta_1^* d_0^{\theta(1)-1}[x])[f(\theta)^*(d_0x)],$$

where $f(\theta)$ is defined by $f(\theta) = c\gamma$. We have $f(\theta)(0) = 0$ by construction, and there is a commutative diagram

$$\begin{tikzcd}
\mathbf{m}-\mathbf{1} \arrow[r, "\theta_1"] \arrow[d, "d^0"'] & \mathbf{n}-\theta(\mathbf{1}) \arrow[d, "(d^0)^{\theta(1)-1}"] \\
\mathbf{m} \arrow[r, "f(\theta)"'] & \mathbf{n}-\mathbf{1},
\end{tikzcd}$$

so an inductive argument on the exponent $\theta(1) - 1$ implies that there is a relation

$$[f(\theta)^*(d_0x)] = (\theta_1^* d_0^{\theta(1)-2}[d_0x]) \dots (\theta_1^*[d_0^{\theta(1)-1}(x)]).$$

It follows that

$$[\theta^*(x)] = (\theta_1^* d_0^{\theta(1)-1}[x])(\theta_1^* d_0^{\theta(1)-2}[d_0x]) \dots (\theta_1^*[d_0^{\theta(1)-1}(x)]) = \theta_1^*(F_x(\tau)). \quad (5.2)$$

LEMMA 5.3.

(a) *The assignment*

$$x \mapsto ([x], [d_0x], [d_0^2x], \dots, [d_0^{n-1}x])$$

*defines a natural simplicial map* $\eta : X \to \overline{W}GX$.

(b) *The map* $\eta$ *is one of the canonical homomorphisms for an adjunction*

$$\hom_{s\mathbf{Gr}}(GX, H) \cong \hom_{\mathbf{S}}(X, \overline{W}H),$$

*where* $s\mathbf{Gr}$ *denotes the category of simplicial groups.*

PROOF:

(a) Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and recall the decomposition of (4.2). It will suit us to observe once again that the map $[j, m]$ is the composite $(d^0)^j$, and that $\tau_* = (d^0)^{\theta(k)-\theta(j)}$. Note in particular that $\theta = (d^0)^{\theta(0)}\theta_0$, and recall that $\theta_0(0) = 0$. It is also clear that there is a commutative diagram

$$\begin{tikzcd}
X_n \arrow[r, "\eta"] \arrow[d, "d_0^{\theta(0)}"'] & \overline{W}GX_n \arrow[d, "d_0^{\theta(0)}"] \\
X_{n-\theta(0)} \arrow[r, "\eta"'] & \overline{W}GX_{n-\theta(0)}
\end{tikzcd}$$

Let $F_x$ be the cocycle $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow GX$ associated to the element

$$([x], [d_0x], [d_0^2x], \dots, [d_0^{n-1}x]).$$

Then, for $x \in X_n$,

$$\begin{aligned}\theta_0^*([d_0^{\theta(0)}x],[d_0^{\theta(0)+1}x],\dots,[d_0^{n-1}x]) &= (\theta_1^*F_x(\tau_{m-1*}),\dots,\theta_m^*F_x(\tau_{0*}))\\ &= ([\theta_0^*d_0^{\theta(0)}x],[\theta_1^*d_0^{\theta(1)}x],\dots,[\theta_{m-1}^*d_0^{\theta(m-1)}x])\\ &= ([\theta^*x],[d_0\theta^*x],\dots,[d_0^{m-1}\theta^*x]),\end{aligned}$$

where $\tau_{m-i*} = (d^0)^{\theta(i)-\theta(i-1)}$ as before, and this by repeated application of the formula (5.2). In particular, $\eta$ is a simplicial set map. The naturality is obvious.

(b) Suppose that $f : X \to \overline{W}H$ is a simplicial set map, where $H$ is a simplicial group. Recall that the cocycle $f(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow H$ can be identified with the element

$$(f(x)(d^0), f(d_0x)(d^0),\dots,f(d_0^{n-1}x)(d^0)) \in H_{n-1}\times H_{n-2}\times\cdots\times H_0.$$

The simplicial structure for $GX$ given by the formula (5.1) implies that $f : X \to \overline{W}H$ induces a simplicial group map $f_* : GX \to H$ which is specified on generators by $f_*([x]) = f(x)(d^0)$. It follows that the function

$$\hom_{s\mathbf{Gr}}(GX,H) \to \hom_{\mathbf{S}}(X,\overline{W}H)$$

defined by $g \mapsto (\overline{W}g)\cdot\eta$ is surjective. Furthermore, any map $f : X \to \overline{W}H$ is uniquely specified by the elements $f(x)(d^0)$, and hence by the simplicial group homomorphism $f_*$. □

REMARK 5.4. Any simplicial group homomorphism $f : G \to H$ induces a $f$-equivariant morphism of associated principal fibrations of the form

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & H \arrow[d] \\ WG \arrow[r, "Wf"] \arrow[d] & WH \arrow[d] \\ \overline{W}G \arrow[r, "\overline{W}f"'] & \overline{W}H, \end{tikzcd}$$

as can be seen directly from the definitions. The canonical map $\eta : X \to \overline{W}GX$ induces a morphism

$$\begin{tikzcd} GX \arrow[r, equal] \arrow[d] & GX \arrow[d] \\ X_\eta \arrow[r] \arrow[d] & WGX \arrow[d] \\ X \arrow[r, "\eta"'] & \overline{W}GX \end{tikzcd}$$

of $GX$-bundles. It follows that, for any simplicial group homomorphism $f : GX \to H$, the map $f$ and its adjoint $f_* = \overline{W}f \cdot \eta$ fit into a morphism of bundles

$$\begin{tikzcd}
GX \arrow[r, "f"] \arrow[d] & H \arrow[d] \\
X_\eta \arrow[r] \arrow[d] & WH \arrow[d] \\
X \arrow[r, "f_*"'] & \overline{W}H.
\end{tikzcd}$$

Suppose now that the simplicial set $X$ is *reduced* in the sense that it has only one vertex. A *closed $n$-loop of length $2k$* in $X$ is defined to be a string

$$(x_{2k}, x_{2k-1}, \dots, x_2, x_1)$$

of $(n+1)$-simplices $x_j$ of $X$ such that $d_0x_{2i-1} = d_0x_{2i}$ for $1 \leq i \leq k$. Define an equivalence relation on loops by requiring that

$$(x_{2k}, \dots, x_1) \sim (x_{2k}, \dots, x_{i+2}, x_{i-1}, \dots, x_1)$$

if $x_i = x_{i+1}$. Let

$$\langle x_{2k}, \dots, x_1\rangle$$

denote the equivalence class of the loop $(x_{2k}, \dots, x_1)$. Write $G'X_n$ for the set of equivalence classes of $n$-loops under the relation $\sim$. Loops may be concatenated, giving $G'X_n$ the structure of a group having identity represented by the empty $n$-loop. Any ordinal number morphism $\theta : \mathbf{m} \to \mathbf{n}$ induces a group homomorphism

$$\theta^* : G'X_n \to G'X_m,$$

which is defined by the assignment

$$\langle x_{2k}, \dots, x_2, x_1\rangle \mapsto \langle\tilde{\theta}^* x_{2k}, \dots \tilde{\theta}^* x_2, \tilde{\theta}^* x_1\rangle.$$

The corresponding simplicial group will be denote by $G'X$. This construction is clearly functorial with respect to morphisms of reduced simplicial sets.

There is a homomorphism

$$\phi_n : G'X_n \to GX_n$$

which is defined by

$$\phi_n\langle x_{2k}, x_{2k-1}, \dots, x_2, x_1\rangle = [x_{2k}][x_{2k-1}]^{-1} \dots [x_2][x_1]^{-1}.$$

Observe that

$$\begin{aligned}\theta^*([x_{2i}][x_{2i-1}]^{-1}) &= [\tilde{\theta}^*(x_{2i})][(c\theta)^*d_0(x_{2i})]^{-1}[(c\theta)^*d_0(x_{2i-1})][\tilde{\theta}^*(x_{2i-1}]^{-1}\\ &= [\tilde{\theta}^*(x_{2i})][\tilde{\theta}^*(x_{2i-1}]^{-1},\end{aligned}$$

so that

$$\theta^*([x_{2k}][x_{2k-1}]^{-1}\dots[x_2][x_1]^{-1}) = [\tilde{\theta}^*x_{2k}][\tilde{\theta}^*x_{2k-1}]^{-1}\dots[\tilde{\theta}^*x_2][\tilde{\theta}^*x_1]^{-1}.$$

The homomorphisms $\phi_n : G'X_n \to GX_n$, taken together, therefore define a simplicial group homomorphism $\phi : G'X \to GX$.

LEMMA 5.5. *The homomorphism $\phi : G'X \to GX$ is an isomorphism of simplicial groups which is natural with respect to morphisms of reduced simplicial sets $X$.*

PROOF: The homomorphism $\phi_n : G'X_n \to GX_n$ has a section, which is defined on generators by

$$[x] \mapsto \langle x, s_0d_0x\rangle,$$

and elements of the form $\langle x, s_0d_0x\rangle$ generate $G'X_n$. □

Again, let $X$ be a reduced simplicial set. The set $E'X_n$ consists of equivalence classes of strings of $(n+1)$-simplices

$$(x_{2k}, \dots, x_1, x_0)$$

with $d_0x_{2i} = d_0x_{2i-1}$, $i \geq 1$, subject to an equivalence relation generated by relations if the form

$$(x_{2k}, \dots, x_0) \sim (x_{2k}, \dots, x_{i+2}, x_{i-1}, \dots, x_0)$$

if $x_i = x_{i+1}$. We shall write $\langle x_{2k}, \dots, x_0\rangle$ for the equivalence class containing the element $(x_{2k}, \dots, x_0)$. Any ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ determines a function $\theta^* : E'X_n \to E'X_m$, which is defined by

$$\theta^*\langle x_{2k}, \dots, x_0\rangle = \langle\tilde{\theta}^*(x_{2k}), \dots, \tilde{\theta}^*(x_0)\rangle,$$

and so we obtain a simplicial set $E'X$. Concatenation induces a left action $G'X \times E'X \to E'X$ of the simplicial group $G'X$ on $E'X$.

There is a function

$$\phi'_n : E'X_n \to GX_n \times X_n$$

which is defined by

$$\phi'_n\langle x_{2k}, \dots, x_1, x_0\rangle = ([x_{2k}][x_{2k-1}]^{-1}\dots[x_2][x_1]^{-1}[x_0], d_0x_0).$$

The function $\phi'_n$ is $\phi_n$-equivariant, and so

$$\phi'_n(\phi_n^{-1}(g)\langle s_0 x\rangle) = (g, x)$$

for any $(g, x) \in GX_n \times X_n$, and $\phi'_n$ is surjective. There is an equation

$$\langle x_{2k}, \dots, x_0\rangle = \langle x_{2k}, \dots, x_0, s_0d_0(x_0)\rangle\langle s_0d_0(x_0)\rangle$$

for every element of $E'X_n$, so that $E'X_n$ consists of $G'X_n$-orbits of elements $\langle s_0x\rangle$. The function $\phi'_n$ preserves orbits and $\phi_n$ is a bijection, so that $\phi'_n$ is injective as well.

The set $GX_n \times X_n$ is the set of $n$-simplices of the $GX$-bundle $X_\eta$ which is associated to the natural map $\eta : X \to \overline{W}GX$. If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, then the associated simplicial structure map $\theta^*$ in $X_\eta$ has the form

$$\begin{aligned}&\theta^*([x_{2k}]\dots[x_1]^{-1}[x_0], d_0x_0)\\ &\quad = ([\tilde{\theta}^*(x_{2k})]\dots[\tilde{\theta}^*(x_1)]^{-1}[\tilde{\theta}^*(x_0)][(c\theta)^*(d_0x_0)]^{-1}\theta_0^*(\eta(x)([\theta(0), n])), d_0\tilde{\theta}^*(x_0))\end{aligned}$$

since $d_0\tilde{\theta}^*(x_0) = \theta^*(d_0x_0)$. But

$$[(c\theta)^*(d_0x_0))] = \theta_0^*(\eta(x)([\theta(0), n]),$$

by equation (5.2). The bijections $\phi'_n$ therefore define a $\phi$-equivariant simplicial map, and so we have proved

LEMMA 5.6. *There is a $\phi$-equivariant isomorphism*

$$\phi' : E'X \to X_\eta.$$

*This isomorphism is natural with respect to maps of reduced simplicial sets.*

There is a simplicial set $E''X$ whose $n$-simplices consist of the strings of $(n+1)$-simplices $(x_{2k}, \dots, x_0)$ of $X$ as above, and with simplicial structure maps defined by

$$\theta^*(x_{2k}, \dots, x_0) = (\tilde{\theta}^*x_{2k}, \dots, \tilde{\theta}^*x_0)$$

for $\theta : \mathbf{m} \to \mathbf{n}$. Observe that $E'X = E''X/\sim$.

Given this description of the simplicial structure maps in $E''X$, the best way to think of the members of an $n$-simplex is as a string $(x_{2k}, \dots, x_0)$ of cones on their $0^{th}$ faces, with the obvious incidence relations. A homotopy $\Delta^n \times \Delta^1 \to E''X$ can therefore be identified with a string

$$(h_{2k}, \dots, h_1, h_0),$$

where

(1) $h_i : C(\Delta^n \times \Delta^1) \to X$ is a map defined on the cone $C(\Delta^n \times \Delta^1)$ for the simplicial set $\Delta^n \times \Delta^1$, and

(2)
$$h_{2i}|_{\Delta^n\times\Delta^1} = h_{2i-1}|_{\Delta^n\times\Delta^1}$$

for $1 \leq i \leq k$.

We shall say that maps of the form $C(\Delta^n \times \Delta^1) \to Y$ are *cone homotopies.* Examples of such include the following:

(1) The canonical contracting homotopy

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[d] \\
0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & n+1
\end{tikzcd}$$

of $\Delta^{n+1}$ onto the vertex 0 induces a map $C(\Delta^n \times \Delta^1) \to \Delta^{n+1}$ which is jointly specified by the vertex 0 and the restricted homotopy

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[d] \\
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n+1.
\end{tikzcd}$$

This map is a "contracting" cone homotopy.

(2) The vertex 0 and the constant homotopy

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & 2 \arrow[r] \arrow[d] & \cdots \arrow[r] & n+1 \arrow[d] \\
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n+1.
\end{tikzcd}$$

jointly specify a "constant" cone homotopy $C(\Delta^n \times \Delta^1) \to \Delta^{n+1}$.

In both of these cases, it's helpful to know that the cone $CBP$ on the nerve $BP$ of a poset $P$ can be identified with the nerve of the cone poset $CP$ which is obtained from $P$ by adjoining a disjoint initial object. Furthermore, a poset map $\gamma : P \to Q$ can be extended to a map $CP \to Q$ by mapping the initial object of $CP$ to some common lower bound of the objects in the image of $\gamma$, if such a lower bound exists.

LEMMA 5.7. *$E'X$ is acyclic in the sense that $\tilde{H}_*(E'X, \mathbb{Z}) = 0$.*

PROOF: Both the contracting and constant cone homotopies defined above are natural in $\Delta^n$ in the sense that the diagram

$$\begin{tikzcd}
C(\Delta^m \times \Delta^1) \arrow[r, "h"] \arrow[d, "C(\theta \times 1)"'] & \Delta^{m+1} \arrow[d, "\tilde{\theta}_*"] \\
C(\Delta^n \times \Delta^1) \arrow[r, "h"'] & \Delta^{n+1}
\end{tikzcd}$$

commutes for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$, where $h$ denotes one of the two types. It follows that there is a homotopy from the identity map on $E''X$ to the map $E''X \to E''X$ defined by

$$(x_{2k}, \dots, x_1, x_0) \mapsto (x_{2k}, \dots, x_1, *),$$

and that this homotopy can be defined on the level of simplices by strings of cone homotopies

$$(h(x_{2k}), \dots, h(x_1), h(x_0)),$$

where $h(x_0)$ is contracting on $d_0x_0$, and all other $h(x_i)$ are constant. This homotopy, when composed with the canonical map $E''X \to E'X$, determines a chain homotopy $\mathbf{S}$ from the induced map $\mathbb{Z}E''X \to \mathbb{Z}E'X$ to the map $\mathbb{Z}E''X \to \mathbb{Z}E'X$ which is induced by the simplicial set map defined by

$$(x_{2k}, \dots, x_1, x_0) \mapsto \langle x_{2k}, \dots, x_1, * \rangle.$$

For each element $(x_{2k}, \dots, x_1, x_0)$, the chain $S(x_{2k}, \dots, x_1, x_0)$ is an alternating sum of the simplices comprising the homotopy $(h(x_{2k}), \dots, h(x_1), h(x_0))$. It follows in particular, that if $x_i = x_{i+1}$ for some $i \geq 1$, then the corresponding adjacent simplices of the components of $S(x_{2k}, \dots, x_1, x_0)$ are also equal.

It also follows that there is a chain homotopy defined by

$$(x_{2k}, \dots, x_1, x_0) \mapsto S(x_{2k}, \dots, x_1, x_0) - S(x_{2k}, \dots, x_1, x_1),$$

and that this is a chain homotopy from the chain map induced by the canonical map $E''X \to E'X$ to the chain map induced by the simplicial set map

$$(x_{2k}, \dots, x_1, x_0) \mapsto \langle x_{2k}, \dots, x_3, x_2 \rangle$$

This construction can be iterated, to produce a chain homotopy $H$ defined by

$$(x_{2k}, \dots, x_0) \mapsto (\sum_{i=0}^{k-1} (S(x_{2k}, \dots, x_{2i+1}, x_{2i}) - S(x_{2k}, \dots, x_{2i+1}, x_{2i+1}))) + S(x_{2k})$$

from the chain map $\mathbb{Z}E''X \to \mathbb{Z}E'X$ to the chain map induced by the simplicial set map $E''X \to E'X$ which takes all simplices to the base point $*$. One can show that

$$H(x_{2k}, \dots, x_0) = H(x_{2k}, \dots x_{i+2}, x_{i-1}, \dots, x_0)$$

if $x_i = x_{i+1}$. It follows that $H$ induces a contracting chain homotopy on the complex $\mathbb{Z}E'X$. □

LEMMA 5.8. *$E'X$ is simply connected.*

PROOF: Following Lemma 5.6, we shall do a fundamental groupoid calculation in $X_\eta \cong E'X$.

The boundary of the 1-simplex $(s_0g, x)$ in $X_\eta$ has the form

$$\partial(s_0g, x) = ((g[x], *), (g, *)).$$

There is an oriented graph $T(X)$ (hence a simplicial set) having vertices coinciding with the elements of $GX_0$ and with edges $x : g \to gx$ for $x \in X_1 - \{*\}$. There is plainly a simplicial set map $T(X) \to X_\eta$ which is the identity on vertices and sends each edge $x : g \to gx$ to the 1-simplex $(s_0g, x)$. This map induces a map of fundamental groupoids

$$\pi T(X) \to \pi X_\eta$$

which is bijective on objects. A reduced word argument shows that $T(X)$ is contractible, hence has trivial fundamental groupoid, so we conclude that $X_\eta$ is simply connected if we can show that the 1-simplices $(s_0g, x)$ generate the fundamental groupoid $\pi X_\eta$.

There are boundary relations

$$\begin{aligned}\partial(s_1g, s_0x) &= (d_0(s_1g, s_0x), d_1(s_1g, s_0x), d_2(s_1g, s_0x))\\ &= (s_0d_0g, x), (g, x), (g, *))\end{aligned}$$

and in the same notation,

$$\partial(s_0g, y) = ((g[y], d_0y), (g, d_1y), (s_0d_1g, d_2y)).$$

The upshot is that there are commuting diagrams in $\pi X_\eta$ of the form

$$\begin{tikzcd}
* \arrow[rr, "(s_0d_1g, d_2y)"] \arrow[dr, "(g, d_1y)"'] & & * \arrow[rr, "(g[y], x)"] \arrow[dl, "(g[y], d_0y)"'] \arrow[dr, "(g[y], *)"] & & * \\
& * & & * \arrow[ll, "(s_0d_0(g[y]), d_0y)"] \arrow[ur, "(s_0d_0(g[y]), x)"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
* \arrow[dr, "(g, d_0y)"'] & & * \arrow[ll, "(s_0d_1(g[y]^{-1}), d_2y)"'] \arrow[rr, "(g[y]^{-1}, x)"] \arrow[dl, "(g[y]^{-1}, d_1y)"'] \arrow[dr, "(g[y]^{-1}, *)"] & & * \\
& * & & * \arrow[ll, "(s_0d_0(g[y]^{-1}), d_1y)"] \arrow[ur, "(s_0d_0(g[y]^{-1}), x)"'] &
\end{tikzcd}$$

It follows that any generator $(g[y], x)$ (respectively $(g[y]^{-1}, x)$ of $\pi X_\eta$ can be replaced by a generator $(g, d_1 y)$ (respectively $(g, d_0 y)$) of $\pi X_\eta$ up to multiplication by elements of $\pi T(X)$. In particular, any generator $(h, x)$ of $\pi X_\eta$ can be replaced up to multiplication by elements of $\pi T(X)$ by a generator $(h', x')$ such that $h'$ has strictly smaller word length as an element of the free group $GX_1$. An induction on word length therefore shows that the groupoid $\pi X_\eta$ is generated by the image of $T(X)$. □

REMARK 5.9. The object $T(X)$ in the proof of Lemma 5.8 is the Serre tree associated to the generating set $X_1 - \{*\}$ of the free group $GX_0$. See p.16 and p.26 of [85].

An acyclic space which has a trivial fundamental group is contractible in the sense that it is weakly equivalent to a point, by a standard Hurewicz argument, so Lemmas 5.6 through 5.8 together imply the following:

THEOREM 5.10. *Suppose that $X$ is a reduced simplicial set. Then the total space $X_\eta$ of the principal $GX$-fibration $X_\eta \to X$ is weakly equivalent to a point.*

COROLLARY 5.11. *There are weak equivalences*

$$GX \xrightarrow{\simeq} X_\eta \times_X PX \xleftarrow{\simeq} \Omega X,$$

*which are natural with respect to morphisms of reduced Kan complexes $X$.*

## 6. Reduced simplicial sets, Milnor's $FK$-construction.

The proof of Theorem 5.10 depends on an explicit geometric model for the space $X_\eta$, and the construction of this model uses the assumption that the simplicial set $X$ is reduced. There is no such restriction on the loop group functor: $GY$ is defined for all simplicial sets $Y$. The geometric model for $X_\eta$ can be expanded to more general simplicial sets (see Kan's paper), but Theorem 5.10 fails badly in the non-reduced case: the loop group $G(\Delta^1)$ on the simplex $\Delta^1$ is the constant simplicial group on the free group $\mathbb{Z}$ on one letter, which is manifestly not contractible. This sort of example forces us (for the time being — see Section 8) to restrict our attention to spaces with one vertex.

We now turn to the model category aspects of the loop group and $\overline{W}$ functors.

LEMMA 6.1. *Let $f : X \to Y$ be a cofibration of simplicial sets. Then $Gf : GX \to GY$ is a cofibration of simplicial groups. In particular, for all simplicial sets $X$, $GX$ is a cofibrant simplicial group.*

PROOF: This result is a consequence of Corollary 1.11.

Note that since $s_0 X_n \subseteq X_{n+1}$ there is an isomorphism of groups

$$GX_n \cong F(X_{n+1} - s_0 X_n).$$

Furthermore, for all $i \geq 0$, the map $s_{i+1} : X_n \to X_{n+1}$ restricts to a map

$$s_{i+1} : X_n - s_0 X_{n-1} \to X_{n+1} - s_0 X_n$$

since $s_{i+1} s_0 X = s_0 s_i X$. Hence there is a diagram

$$\begin{tikzcd}
GX_{n-1} \arrow[r, "\cong"] \arrow[d, "s_i"'] & F(X_n - s_0 X_{n-1}) \arrow[d, "Fs_{i+1}"] \\
GX_n \arrow[r, "\cong"] & F(X_{n+1} - s_0 X_n)
\end{tikzcd}$$

and $GX$ is almost free, hence cofibrant. For the general case, if $X \to Y$ is a level-wise inclusion

$$Y_{n+1} - s_0 Y_n = (X_{n+1} - s_0 X_n) \cup Z_{n+1}$$

where $Z_{n+1} = Y_{n+1} - (X_{n+1} \cup s_0 Y_n)$. Thus

$$GY_n \cong GX_n * FZ_{n+1}$$

where the $*$ denotes the free product. Now $s_{i+1} : Y_n \to Y_{n+1}$ restricts to a map $s_{i+1} : Z_n \to Z_{n+1}$ and, hence, the inclusion $GX \to GY$ is almost-free and a cofibration. □

As a result of Theorem 5.10, Lemma 6.1 and a properness argument one sees that $G$ preserves cofibrations and weak equivalences between spaces with one vertex. This suggests that the proper domain category for $G$ — at least from a model category point of view — is the category $\mathbf{S}_0$ of simplicial sets with one vertex. Our next project then is to give that category a closed model structure.

PROPOSITION 6.2. *The category $\mathbf{S}_0$ has a closed model category structure where a morphism $f : X \to Y$ is a*

1) *a weak equivalence if it is a weak equivalence as simplicial sets;*
2) *a cofibration if it is a cofibration as simplicial sets; and*
3) *a fibration if it has the right lifting property with respect to all trivial cofibrations.*

The proof is at the end of the section, after we explore some consequences.

PROPOSITION 6.3.

1) *The functor $G : \mathbf{S}_0 \to s\mathbf{Gr}$ preserves cofibrations and weak equivalences.*
2) *The functor $\overline{W} : s\mathbf{Gr} \to \mathbf{S}_0$ preserves fibrations and weak equivalences.*
3) *Let $X \in \mathbf{S}_0$ and $G \in s\mathbf{Gr}$. Then a morphism $f : GX \to G$ is a weak equivalence in $s\mathbf{Gr}$ if only if the adjoint $f_* : X \to \overline{W}G$ is a weak equivalence in $\mathbf{S}_0$.*

PROOF: Part 1) follows from Lemma 6.1 and Theorem 5.10. For part 2) notice that since $\overline{W}$ is right adjoint to a functor which preserves trivial cofibrations, it preserves fibrations. The clause about weak equivalences follows from Lemma 4.6 Finally, part 3), follows from Remark 5.4, Lemma 4.6, Theorem 5.10 and properness for simplicial sets.

COROLLARY 6.4. *Let* $\operatorname{Ho}(\mathbf{S}_0)$ *and* $\operatorname{Ho}(s\mathbf{Gr})$ *denote the homotopy categories. Then the functors* $G$ *and* $\overline{W}$ *induce an equivalence of categories*

$$\operatorname{Ho}(\mathbf{S}_0) \cong \operatorname{Ho}(s\mathbf{Gr}).$$

PROOF: Proposition 6.3 implies that the natural maps $\epsilon : G\overline{W}H \to H$ and $\eta : X \to \overline{W}GX$ are weak equivalences for all simplicial groups $H$ and reduced simplicial sets $X$. □

REMARK 6.5. If $\operatorname{Ho}(\mathbf{S})_c \subseteq \operatorname{Ho}(\mathbf{S})$ is the full sub-category of the usual homotopy category with objects the connected spaces, then the inclusion $\operatorname{Ho}(\mathbf{S}_0) \to \operatorname{Ho}(\mathbf{S})_c$ is an equivalence of categories. To see this, it is sufficient to prove if $X$ is connected there is a $Y$ weakly equivalent to $X$ with a single vertex. One way is to choose a weak equivalence $X \to Z$ with $Z$ fibrant and then let $Y \subseteq Z$ be a minimal subcomplex weakly equivalent to $Z$.

We next relate the fibrations in $\mathbf{S}_0$ to the fibrations in $\mathbf{S}$.

LEMMA 6.6. *Let* $f : X \to Y$ *be a fibration in* $\mathbf{S}_0$. *Then* $f$ *is a fibration in* $\mathbf{S}$ *if and only if* $f$ *has the right lifting property with respect to*

$$* \to S^1 = \Delta^1/\partial\Delta^1.$$

PROOF: First suppose $f$ is a fibration in $\mathbf{S}$. Consider a lifting problem

$$\begin{tikzcd}
\Delta^0 \arrow[r] \arrow[d, "d^i"'] & * \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^1 \arrow[r] & S^1 \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd} \tag{6.7}$$

Since $f$ is a fibration in $\mathbf{S}$, there is a map $g : \Delta^1 \to X$ solving the lifting problem for the outer rectangle. Since $X$ has one vertex $g$ factors through the quotient map,

$$\Delta^1 \to \Delta^1/\operatorname{sk}_0 \Delta^1 = S^1 \xrightarrow{\overline{g}} X$$

and $\overline{g}$ solves the original lifting problem.

Now suppose $f$ has the stipulated lifting property. Then one must solve all lifting problems

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

If $n > 1$, this diagram can be expanded to

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & \Lambda^n_k / \operatorname{sk}_0 \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] & \Delta^n / \operatorname{sk}_0 \Delta^n \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

The map

$$\Lambda^n_k / \operatorname{sk}_0 \Lambda^n_k \to \Delta^n / \operatorname{sk}_0 \Delta^n$$

is still a trivial cofibration, now in $\mathbf{S}_0$. So the lift exists. If $n = 1$, the expanded diagram is an instance of diagram (6.7), and the lift exists by hypothesis. □

COROLLARY 6.8. *Let $X \in \mathbf{S}_0$ be fibrant in $\mathbf{S}_0$, then $X$ is fibrant in $\mathbf{S}$. In particular, if $G \in s\mathbf{Gr}$, then $\overline{W}G$ is fibrant in $\mathbf{S}$.*

PROOF: The first clause follows from the previous lemma. For the second, note that every object of $s\mathbf{Gr}$ is fibrant. Since $\overline{W} : s\mathbf{Gr} \to \mathbf{S}_0$ preserves fibrations, $\overline{W}G$ is fibrant in $\mathbf{S}_0$. □

COROLLARY 6.9. *Let $f : X \to Y$ be a fibration in $\mathbf{S}_0$ between fibrant spaces. Then $f$ is a fibration in $\mathbf{S}$ if and only if*

$$f_* : \pi_1 X \to \pi_1 Y$$

*is onto. In particular, if $G \to H$ is a fibration of simplicial groups, $\overline{W}G \to \overline{W}H$ is a fibration of simplicial sets if and only if $\pi_0 G \to \pi_0 H$ is onto.*

PROOF: Consider a lifting problem

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & X \arrow[d] \\
S^1 \arrow[r, "\alpha"'] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

This can be solved up to homotopy; that is there is a diagram

$$\begin{tikzcd}
 & S^1 \arrow[r] \arrow[d, "d^1"] & X \arrow[d] \\
S^1 \arrow[r, "d^0"] & S^1 \wedge \Delta^1_+ \arrow[r, "h"] & Y.
\end{tikzcd}$$

where $h \cdot d^0 = \alpha$. But $d^1 : S^1 \to S^1 \wedge \Delta^1_+$ is a trivial cofibration in $\mathbf{S}_0$ so the homotopy $h$ can be lifted to $\widetilde{h} : S^1 \wedge \Delta^1_+ \to X$ and $\widetilde{h} \cdot d^0$ solves the original lifting problem.

For the second part of the corollary, note that Corollary 2.7 and Lemma 4.6 together imply that $\pi_1 \overline{W} G \cong \pi_0 G$. □

We now produce the model category structure promised for $\mathbf{S}_0$. The following lemma sets the stage. If $X$ is a simplicial set, let $\#X$ denote the cardinality of the non-degenerate simplices in $X$. Let $\omega$ be the first infinite cardinal.

LEMMA 6.10.

1) *Let $A \to B$ be a cofibration in* $\mathbf{S}$ *and $x \in B_k$ a $k$-simplex. Then there is a subspace $C \subseteq B$ so that $\#C < \omega$ and $x \in C$.*
2) *Let $A \to B$ be a trivial cofibration in* $\mathbf{S}$ *and $x \in B_k$ a $k$-simplex. Then there is a subspace $D \subseteq B$ so that $\#D \leq \omega$, $x \in D$ and $A \cap D \to D$ is a trivial cofibration.*

PROOF: Part 1) is a reformulation of the statement that every simplicial set is the filtered colimit of its finite subspaces. For part 2) we will construct an expanding sequence of subspaces

$$D_1 \subseteq D_2 \subseteq \cdots \subseteq B$$

so that $x \in D_1$, $\#D_n \leq \omega$ and

$$\pi_p(|D_n|, |D_n \cap A|) \to \pi_p(|D_{n+1}|, |D_{n+1} \cap A|)$$

is the zero map. Then we can set $D = \bigcup_n D_n$.

To get $D_1$, simply choose a finite subspace $D_1 \subseteq B$ with $x \in D_1$. Now suppose $D_q, q \leq n$, have been constructed and satisfy the above properties. Let

$$\alpha \in \pi_*(|D_n|, |D_n \cap A|).$$

Since $\alpha$ maps to zero under

$$\pi_*(|D_n|, |D_n \cap A|) \to \pi_*(|B|, |A|)$$

there must be a subspace $D_\alpha \subseteq B$, such that $\#D < \omega$ and so that $\alpha$ maps to zero under

$$\pi_*(|D_n|, |D_n \cap A|) \to \pi_*(|D_n \cup D_\alpha|, (D_n \cup D_\alpha) \cap A|).$$

Set $D_{n+1} = D_n \cup (\bigcup_\alpha D_\alpha)$. □

Remark 6.11. The relative homotopy groups $\pi_*(|B|,|A|)$ for a cofibration $i : A \to B$ of simplicial sets are defined to be the homotopy groups of the homotopy fibre of the realized map $i_* : |A| \hookrightarrow |B|$, up to a dimension shift. The realization of a Kan fibration is a Serre fibration (Theorem I.10.10), so it follows that these groups coincide up to isomorphism with the simplicial homotopy groups $\pi_* F_i$ of any choice of homotopy fibre $F_i$ in the simplicial set category. One can use Kan's $Ex^\infty$ functor along with an analog of the classical method of replacing a continuous map by a fibration to give a rigid construction of the Kan complex $F_i$ which satisfies the property that the assignment $i \mapsto F_i$ preserves filtered colimits in the maps $i$. The argument for part 2) of Lemma 6.10 can therefore be made completely combinatorial. This observation becomes quite important in contexts where functoriality is vital — see [38].

Lemma 6.12. *A morphism $f : X \to Y$ in $\mathbf{S}_0$ is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations $C \to D$ in $\mathbf{S}_0$ with $\#D \leq \omega$.*

Proof: Consider a lifting problem

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "j"'] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where $j$ is a trivial cofibration. We solve this by a Zorn's Lemma argument. Consider the set $\Lambda$ of pairs $(Z,g)$ where $A \subseteq Z \subseteq B$, $A \to Z$ is a weak equivalence and $g$ is a solution to the restricted lifting problem

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d] & X \arrow[d, "f"] \\
Z \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

Partially order $\Lambda$ by setting $(Z,g) < (Z',g')$ if $Z \subseteq Z'$ and $g'$ extends $g$. Since $(A,a) \in \Lambda$, $\Lambda$ is not empty and any chain

$$\cdots < (Z_i, g_i) < (Z_{i+1}, g_{i+1}) < \cdots$$

in $\Lambda$ has an upper bound, namely $(\cup Z_i, \cup g_i)$. Thus $\Lambda$ satisfies the hypotheses of Zorn's lemma and has a maximal element $(B_0, g_0)$. Suppose $B_0 \neq B$. Consider the diagram

$$\begin{tikzcd}
B_0 \arrow[r, "g_0"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
B \arrow[r] & Y.
\end{tikzcd}$$

Then $i$ is a trivial cofibration. Choose $x \in B$ with $x \notin B_0$. By Lemma 6.10.2 there is a subspace $D \subseteq B$ with $x \in D$, $\# D \leq \omega$ and $B_0 \cap D \to D$ a trivial cofibration. The restricted lifting problem

$$\begin{tikzcd}
B_0 \cap D \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
D \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

has a solution, by hypotheses. Thus $g_0$ can be extended over $B_0 \cup D$. This contradicts the maximality of $(B_0, g_0)$. Hence $B_0 = B$. □

REMARK 6.13. The proofs of Lemma 6.10 and Lemma 6.12 are actually standard moves. The same circle of ideas appears in the arguments for the closed model structures underlying both the Bousfield homology localization theories [8], [9] and the homotopy theory of simplicial presheaves [46], [51], [38]. We shall return to this topic in Chapter IX.

THE PROOF OF PROPOSITION 6.2: Axioms **CM1**–**CM3** for a closed model category are easy in this case. Also, the "trivial cofibration-fibration" part of **CM4** is the definition of fibration. We next prove the factorization axiom **CM5** holds, then return to finish **CM4**.

Let $f : X \to Y$ be a morphism in $\mathbf{S}_0$. To factor $f$ as a cofibration followed by a trivial fibration, use the usual small object argument with pushout along cofibrations $A \to B$ in $\mathbf{S}_0$ with $\# B < \omega$ to factor $f$ as $X \xrightarrow{j} Z \xrightarrow{q} Y$ where $j$ is cofibration and $q$ is a map with the right lifting property with respect to all cofibrations $A \to B$ with $\# B < \omega$. The evident variant on the Zorn's lemma argument given in the proof of Lemma 6.12 using 6.10.1 implies that $q$ has the right lifting property with respect to all cofibrations in $\mathbf{S}_0$. Hence $q$ is a fibration. We claim it is a weak equivalence and, in fact, a trivial fibration in $\mathbf{S}$. To see this consider a lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & Z \arrow[d, "q"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

If $n = 0$ this has a solution, since $Z_0 \cong Y_0$. If $n > 0$, this extends to a diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & \partial\Delta^n/\operatorname{sk}_0(\partial\Delta^n) \arrow[r] \arrow[d, "j"] & Z \arrow[d, "q"] \\
\Delta^n \arrow[r] & \Delta^n/\operatorname{sk}_0\Delta^n \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

Since $n \geq 1$, $sk_0(\partial\Delta^n) = \operatorname{sk}_0(\Delta^n)$, so $j$ is a cofibration between finite complexes in $\mathbf{S}_0$ and the lift exists.

Return to $f : X \to Y$ in $\mathbf{S}_0$. To factor $f$ as a trivial cofibration followed by a fibration, we use a transfinite small object argument.

We follow the convention that a cardinal number is the smallest ordinal number within a given bijection class; we further interpret a cardinal number $\beta$ as a poset consisting of strictly smaller ordinal numbers, and hence as a category. Choose a cardinal number $\beta$ such that $\beta > 2^\omega$.

Take the map $f : X \to Y$, and define a functor $X : \beta \to \mathbf{S}_0$ and a natural transformation $f_s : X(s) \to Y$ such that

(1) $X(0) = X$,

(2) $X(t) = \varinjlim_{s<t} X(s)$ for all limit ordinals $t < \beta$, and

(3) the map $X(s) \to X(s+1)$ is defined by the pushout diagram

$$\begin{tikzcd}
\bigvee_D A_D \arrow[r, "(\alpha_D)"] \arrow[d, "\vee i_D"'] & X(s) \arrow[d] \\
\bigvee_D B_D \arrow[r] & X(s+1)
\end{tikzcd}$$

where the index $D$ refers to a set of representatives for all diagrams

$$\begin{tikzcd}
A_D \arrow[r, "\alpha_D"] \arrow[d, "i_D"'] & X(s) \arrow[d, "f_s"] \\
B_D \arrow[r] & Y
\end{tikzcd}$$

such that $i_D : A_D \to B_D$ is a trivial cofibration in $\mathbf{S}_0$ with $\#B_D \leq \omega$.

Then there is a factorization

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[dr, "f"'] & X_\beta \arrow[d, "f_\beta"] \\
& Y
\end{tikzcd}$$

for the map $f$, where $X_\beta = \varinjlim_s X(s)$, and $i_0 : X = X(0) \to X_\beta$ is the canonical map into the colimit. A pushout of a trivial cofibration in $\mathbf{S}_0$ is a trivial cofibration in $\mathbf{S}_0$ because the same is true in $\mathbf{S}$, so $i_0$ is a trivial cofibration. Also, any map $A \to X_\beta$ must factor through one of the canonical maps $i_s : X(s) \to X_\beta$ if $\#A \leq \omega$, for otherwise $A$ would have too many subobjects on account of the size of $\beta$. It follows that the map $f_\beta : X_\beta \to Y$ is a fibration of $\mathbf{S}_0$. This finishes **CM5**.

To prove **CM4** we must show any trivial fibration $f : X \to Y$ in $\mathbf{S}_0$ has the right lifting property with respect to all cofibrations. However, we factored $f$ as a composite

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where $j$ is a cofibration and $q$ is a trivial fibration with the right lifting property with respect to all cofibrations. Now $j$ is a trivial cofibration, since $f$ is a weak equivalence. Thus there is a lifting in

$$\begin{tikzcd}
X \arrow[r, "="] \arrow[d, "j"'] & X \arrow[d, "f"] \\
Z \arrow[r, "q"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

since $f$ is a fibration. This shows $f$ is a retract of $q$ and has the requisite lifting property, since $q$ does. □

As an artifact of the proof we have:

LEMMA 6.14. *A morphism $f : X \to Y$ in $\mathbf{S}_0$ is a trivial fibration in $\mathbf{S}_0$ if and only if it is a trivial fibration in* $\mathbf{S}$.

The *Milnor $FK$ construction* associates to a pointed simplicial set $K$ the simplicial group $FK$, which is given in degree $n$ by

$$FK_n = F(K_n - \{*\}),$$

so that $FK_n$ is the free group on the set $K_n - \{*\}$. This construction gives a functor from pointed simplicial sets to simplicial groups. The group $FK$ is also a loop group:

THEOREM 6.15. *There is a natural isomorphism*

$$G(\Sigma K) \cong FK,$$

*for pointed simplicial sets $K$.*

PROOF: Recall that $\Sigma K$ denotes the Kan suspension of $K$. The group of $n$-simplices of $G(\Sigma K)$ is defined to be the quotient

$$G(\Sigma K)_n = F(\Sigma K_{n+1})/F(s_0 \Sigma K_n).$$

The map $s_0 : \Sigma K_n \to \Sigma K_{n+1}$ can be identified with the wedge summand inclusion

$$K_{n-1} \vee \cdots \vee K_0 \hookrightarrow K_n \vee K_{n-1} \vee \cdots \vee K_0,$$

so that the composite group homomorphism

$$F(K_n) \xrightarrow{\eta_{n*}} F(\Sigma K_{n+1}) \to F(\Sigma K_{n+1})/F(s_0 \Sigma K_n)$$

can be identified via an isomorphism

$$F(\Sigma K_{n+1})/F(s_0 \Sigma K_n) \cong FK_n \tag{6.16}$$

with the quotient map

$$F(K_n) \to F(K_n)/F(*) \cong FK_n.$$

Recall that for $\theta : \mathbf{m} \to \mathbf{n}$, the map $\theta^* : G(\Sigma K)_n \to G(\Sigma K)_m$ is specified on generators $[x]$ by

$$\theta^*([x]) = [\tilde{\theta}^*(x)][(c\theta)^*(d_0(x))]^{-1}.$$

But then

$$\begin{aligned} \theta^*([\eta_n(x)]) &= [\tilde{\theta}^*(\eta_n(x))][(c\theta)^*(d_0(\eta_n(x)))]^{-1} \\ &= [\tilde{\theta}^*(\eta_n(x))] \\ &= [\eta_m(\theta^*(x))], \end{aligned}$$

since $d_0(\eta_n(x)) = *$. It follows that the isomorphisms (6.16) respect the simplicial structure maps. □

The proof of Theorem 6.15 is easy enough, but this result has important consequences:

COROLLARY 6.17.

(1) *The Milnor $FK$ construction takes weak equivalences of pointed simplicial sets to weak equivalences of simplicial groups.*

(2) *The simplicial group $FK$ is a natural fibrant model for $\Omega\Sigma K$, in the category of pointed simplicial sets.*

PROOF: The first assertion is proved by observing that the Kan suspension functor preserves weak equivalences; the loop group construction has the same property by Theorem 5.10 (see Section III.5).

Let $\Sigma K \to Y$ be a fibrant model for $\Sigma K$ in the category of reduced simplicial sets. Then $Y$ is a Kan complex which is weakly equivalent to $\Sigma K$, so that $\Omega Y$ is a model for $\Omega\Sigma K$. The loop group functor preserves weak equivalences, so that the induced map $G(\Sigma K) \to GY$ is a weak equivalence of simplicial groups. Finally, we know that $GY$ is weakly equivalent to $\Omega Y$, so that $G(\Sigma K)$ and hence $FK$ is a model for $\Omega\Sigma K$. $\square$

## 7. Simplicial groupoids.

A *simplicial groupoid* $G$, for our purposes, is a simplicial object in the category of groupoids whose simplicial set of objects is discrete. In other words, $G$ consists of small groupoids $G_n$, $n \geq 0$ with a functor $\theta^* : G_m \to G_n$ for each ordinal number map $\theta : \mathbf{n} \to \mathbf{m}$, such that all sets of objects $\mathrm{Ob}(G_n)$ coincide with a fixed set $\mathrm{Ob}(G)$, and all functors $\theta^*$ induce the identity function on $\mathrm{Ob}(G)$. Of course, $\theta \mapsto \theta^*$ is also contravariantly functorial in ordinal number maps $\theta$. The set of morphisms from $x$ to $y$ in $G_n$ will be denoted by $G_n(x,y)$, and there is a simplicial set $G(x,y)$ whose $n$-simplices are the morphism set $G_n(x,y)$ in the groupoid $G_n$. We shall denote the category of simplicial groupoids by $s\mathbf{Gd}$.

The free groupoid $G(X)$ on a graph $X$ has the same set of objects as $X$, and has morphisms consisting of reduced words in arrows of $X$ and their inverses. There is a canonical graph morphism $\eta : X \to G(X)$ which is the identity on objects, and takes an arrow $\alpha$ to the reduced word represented by the string consisting of $\alpha$ alone. Any graph morphism $f : X \to H$ taking values in a groupoid $H$ extends uniquely to a functor $f_* : G(X) \to H$, in the sense that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta"'] & H \\
G(X) \arrow[ur, "f_*"'] &
\end{tikzcd}$$

There is a similar construction of a free groupoid $G\mathcal{C}$ on a category $\mathcal{C}$, which has been used without comment until now. The groupoid $G\mathcal{C}$ is obtained by the free groupoid on the graph underlying the category $\mathcal{C}$ by killing the normal subgroupoid generated by the composition relations of $\mathcal{C}$ and the strings associated to the identity morphisms of $\mathcal{C}$ (see also Sections I.8 and III.1). The category of groupoids has all small coproducts, given by disjoint unions. This category also has pushouts, which are actually pushouts in the category of small categories, so the category of groupoids is cocomplete. Note that filtered colimits are formed in the category of groupoids as filtered colimits of sets on the object and morphism levels. The initial object in the category of groupoids has an empty set of morphisms and an empty set of objects and is denoted by $\emptyset$.

It is also completely straightforward to show that the category of simplicial groupoids has all small inverse limits.

Dwyer and Kan define [25], for every simplicial set $X$, a groupoid $F'X$ having object set $\{0,1\}$, such that the set of $n$-simplices $X_n$ is identified with a set of arrows from 0 to 1, and such that $F'X_n$ is the free groupoid on the resulting graph.

The groupoid $F'K$ is morally the same thing as the Milnor construction, for pointed simplicial sets $K$. If $x$ denotes the base point of $K$, then there is a homomorphism of simplicial groups

$$g : FK \to F'K(0,0)$$

which is defined on generators $y \in K_n - \{x\}$ by $y \mapsto x^{-1}y$. Also, regarding $FK$ as a simplicial groupoid with one object, we see that there is a map of simplicial groupoids

$$f : F'K \to FK$$

defined by sending $x$ to $e$ in all degrees and such that $y \in K_n - \{x\}$ maps to the arrow $y$. The collection of all products $y^{-1}z$, $y, z \in K_n$, generates $F'K(0,0)$ in degree $n$, and so it follows that the composite simplicial group homomorphism

$$F'K(0,0) \xrightarrow{f} FK \xrightarrow{g} F'K(0,0)$$

is the identity. The composite

$$FK \xrightarrow{g} F'K(0,0) \xrightarrow{f} FK$$

sends $y \in K_n$ to $x^{-1}y = y \in FK_n$, so the homomorphism $g$ is an isomorphism.

LEMMA 7.1. *Suppose that $K$ is a pointed simplicial set. Then the simplicial sets $F'K(a,b)$, $a, b \in \{0,1\}$, are all isomorphic to the Milnor $FK$ construction.*

PROOF: The base point $x$ of $K$ determines an isomorphism $x : 0 \to 1$ in the groupoid $F'K_n$ for all $n \geq 0$. Composition and precomposition with $x$ therefore determines a commutative diagram of simplicial set isomorphisms

$$\begin{tikzcd}
F'K(0,0) \arrow[r, "x_*", "\cong"'] & F'K(0,1) \\
F'K(1,0) \arrow[u, "x^*", "\cong"'] \arrow[r, "\cong", "x_*"'] & F'K(1,1), \arrow[u, "\cong", "x^*"']
\end{tikzcd} \tag{7.2}$$

and of course we've seen that $F'K(0,0) \cong FK$. □

COROLLARY 7.3. *A weak equivalence $f : X \to Y$ of simplicial sets induces weak equivalences $f_* : F'X(a,b) \to F'Y(a,b)$ for all objects $a, b \in \{0,1\}$.*

PROOF: We can suppose that $X$ is non-empty. Pick a base point $x$ in $X$, and observe that the diagram (7.2) is natural in pointed simplicial set maps, as is the isomorphism $F'X(0,0) \cong FX$. We've seen that the Milnor $FX$ construction preserves weak equivalences in Corollary 6.17. □

For an ordinary groupoid $H$, it's standard to write $\pi_0 H$ for the set of *path components* of $H$. By this one means that

$$\pi_0 H = \mathrm{Ob}(H)/\sim,$$

where there is a relations $x \sim y$ between two objects of $H$ if and only if there is a morphism $x \to y$ in $H$. This is plainly an equivalence relation since $H$ is a groupoid, but more generally $\pi_0 H$ is the specialization of a notion of the set of path components $\pi_0 \mathcal{C}$ for a small category $\mathcal{C}$.

If now $G$ is a simplicial groupoid, all of the simplicial structure functors $\theta^* : G_n \to G_m$ induce isomorphisms $\pi_0 G_n \cong \pi_0 G_m$. We shall therefore refer to $\pi_0 G_0$ as the set of *path components of the simplicial groupoid* $G$, and denote it by $\pi_0 G$.

A map $f : G \to H$ of simplicial groupoids is said to be a *weak equivalence of* $s\mathbf{Gd}$ if

(1) the morphism $f$ induces an isomorphism $\pi_0 G \cong \pi_0 H$, and
(2) each induced map $f : G(x,x) \to H(f(x), f(x))$, $x \in \mathrm{Ob}(G)$ is a weak equivalence of simplicial groups (or of simplicial sets).

Corollary 7.3 says that the functor $F' : \mathbf{S} \to s\mathbf{Gd}$ takes weak equivalences of simplicial sets to weak equivalences of simplicial groupoids.

A map $g : H \to K$ of simplicial groupoids is said to be a *fibration* if

(1) the morphism $g$ has the *path lifting property* in the sense for every object $x$ of $H$ and morphism $\omega : g(x) \to y$ of the groupoid $K_0$, there is a morphism $\hat{\omega} : x \to z$ of $H_0$ such that $g(\hat{\omega}) = \omega$, and
(2) each induced map $g : H(x,x) \to K(g(x), g(x))$, $x \in \mathrm{Ob}(H)$, is a fibration of simplicial groups (or of simplicial sets).

According to this definition, every simplicial groupoid $G$ is fibrant, since the map $G \to *$ which takes values in the terminal simplicial groupoid $*$ is a fibration. A *cofibration of simplicial groupoids* is defined to be a map which has the left lifting property with respect to all morphisms of $s\mathbf{Gd}$ which are both fibrations and weak equivalences.

Picking a representative $x \in [x]$ for each $[x] \in \pi_0 G$ determines a map of simplicial groupoids

$$i : \bigsqcup_{[x] \in \pi_0 G} G(x,x) \to G$$

which is plainly a weak equivalence. But more is true, in that the simplicial groupoid $\bigsqcup_{[x]\in\pi_0 G} G(x,x)$ is a deformation retract of $G$ in the usual groupoid-theoretic sense. To see this, pick morphisms $\omega_y : y \to x$ in $G_0$ for each $y \in [x]$ and for each $[x] \in \pi_0 G$, such that $\omega_x = 1_x$ for all the fixed choices of representatives $x$ of the various path components $x$. Then there is a simplicial groupoid morphism

$$r : G \to \bigsqcup_{[x]\in\pi_0 G} G(x,x),$$

which is defined by conjugation by the paths $\omega_y$, in that $r(y) = x$ if and only if $y \in [x]$ for all objects $y$ of $G$, and $r : G(y,z) \to G(x,x)$ is the map sending $\alpha : y \to z$ to the composite $\omega_z \alpha \omega_y^{-1} \in G(x,x)$ for all $y, z \in [x]$, and for each $[x] \in \pi_0 G$. The morphisms $\omega_y$ also determine a groupoid homotopy

$$h : G \times I \to G$$

where $I$ denotes the free groupoid on the ordinal number (category) $\mathbf{1}$. This homotopy is from the identity on $G$ to the composite $ir$, and is given by the obvious conjugation picture. It follows that the maps $r$ and $i$ are weak equivalences of simplicial groupoids.

The choices of the paths which define the retraction map $r$ are non-canonical and fail to be natural with respect to morphisms of simplicial groupoids, except in certain useful isolated cases.

LEMMA 7.4. *Suppose that $A$ is a connected simplicial groupoid, and that the morphism $j : A \to B$ of simplicial groupoids is a bijection on the object level. Pick an object $x$ of $A$. Then all squares in the diagram*

$$\begin{tikzcd}
A(x,x) \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[r, "r"] \arrow[d, "j"] & A(x,x) \arrow[d, "j"] \\
B(jx,jx) \arrow[r, "i"'] & B \arrow[r, "r"'] & B(jx,jx)
\end{tikzcd}$$

*are pushouts of simplicial groupoids.*

PROOF: The paths $\omega_y : y \to x$ in $A$ (with $\omega_x = 1_x$) are used to define both retraction maps $r$ in the diagram (so it makes sense), and the top and bottom horizontal compositions are the identity.

It suffices to show that the diagram

$$\begin{tikzcd}
A(x,x) \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "j"] \\
B(jx,jx) \arrow[r, "i"'] & B
\end{tikzcd}$$

is a pushout. But any commutative diagram

$$\begin{tikzcd}
A(x,x) \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "f"] \\
B(jx,jx) \arrow[r, "g"'] & D
\end{tikzcd}$$

completely determines a function $h : \mathrm{Ob}(B) \to \mathrm{Ob}(D)$, since $i$ is a bijection on the object level, and then a simplicial groupoid map $h : B \to D$ is specified by the observation that every morphism $\alpha : v \to w$ of $B$ has a representation $\alpha = i(\omega_w)^{-1}\theta i(\omega_v)$, where $\theta$ is a uniquely determined morphism $jx \to jx$. □

Write $F'\partial\Delta^0$ to denote the discrete simplicial groupoid on the object set $\{0,1\}$, and write $F'\Lambda^0_0$ to denote the terminal groupoid $*$. The statement of Lemma 7.4 fails for the map $j : F'\Lambda^0_0 \to F'\Delta^0$, precisely because the object sets do not agree.

LEMMA 7.5. *Suppose that*

$$\begin{tikzcd}
F'\Lambda^n_k \arrow[r] \arrow[d, "j"'] & C \arrow[d, "j_*"] \\
F'\Delta^n \arrow[r] & D
\end{tikzcd}$$

*is a pushout in the category of simplicial groupoids. Then the map $j_*$ is a weak equivalence.*

PROOF: The simplicial groupoid $F'\Lambda^0_0$ is a strong deformation retraction of $F'\Delta^0$ on the groupoid level, and such strong deformation retractions are closed under pushout in the category of simplicial groupoids. The maps involved in a strong deformation retraction are weak equivalences of simplicial groupoids.

If $n \geq 1$, it is harmless to suppose that $C$ is connected. The maps $j$ and $j_*$ are bijective on the object level, so that Lemma 7.4 applies, to give a composite pushout diagram

$$\begin{tikzcd}
F'\Lambda^n_k(0,0) \arrow[r, "i"] \arrow[d, "j"'] & F'\Lambda^n_k \arrow[r] \arrow[d, "j"'] & C \arrow[r, "r"] \arrow[d, "j_*"] & C(x,x) \arrow[d, "j_*"] \\
F'\Delta^n(0,0) \arrow[r, "i"'] & F'\Delta^n \arrow[r] & D \arrow[r, "r"'] & D(j_*x, j_*x)
\end{tikzcd}$$

The composite square is a pushout in the category of simplicial groups, so that the map $j_* : C(x,x) \to D(j_*x, j_*x)$ is a weak equivalence. □

THEOREM 7.6. *With the definitions of weak equivalence, fibration and cofibration given above, the category* $s\mathbf{Gd}$ *of simplicial groupoids satisfies the axioms for a closed model category.*

PROOF: Only the factorization axiom has an interesting proof. A map of simplicial groupoids $f : G \to H$ is a fibration if and only if it has the right lifting property with respect to all morphisms $F'\Lambda^n_k \to F'\Delta^n$, $0 \leq k \leq n$, and $f$ is a trivial fibration (aka. fibration and weak equivalence) if and only if it has the right lifting property with respect to all morphisms $F'\partial\Delta^n \to F'\Delta^n$, $n \geq 0$, and the morphism $\emptyset \to *$ (compare [25]). We can therefore use a small object argument to show that every simplicial groupoid morphism $g : K \to L$ has factorizations

$$\begin{tikzcd} & N \arrow[dr, "p"] & \\ K \arrow[ur, "i"] \arrow[rr, "g"] \arrow[dr, "j"'] & & L \\ & M \arrow[ur, "q"'] & \end{tikzcd}$$

where $p$ is a fibration and $i$ has the left lifting property with respect to all fibrations, and $q$ is a trivial fibration and $j$ is a cofibration. Lemma 7.5 further implies that $i$ is a weak equivalence.

The proof of the lifting axiom **CM4** is a standard consequence of the proof of the factorization axiom: any map which is both a cofibration and a weak equivalence (ie. a trivial cofibration) is a retract of a map which has the left lifting property with respect to all fibrations, and therefore has that same lifting property. □

There is a simplicial set $\overline{W}G$ for a simplicial groupoid $G$ that is defined by analogy with and extends the corresponding object for a simplicial group. Explicitly, suppose that $G$ is a simplicial groupoid. An *n-cocycle* $X : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ associates to each object $[k,n]$ an object $X_k$ of $G$, and assigns to each relation $\tau : [j,n] \leq [k,n]$ in $\mathrm{Seg}(\mathbf{n})$ a morphism $X(\tau) : X_j \to X_k$ in $G_{n-j}$, such that the following conditions hold:

(1) $X(1_j) = 1_{X_j} \in G_{n-j}$, where $1_j$ is the identity relation $[j,n] \leq [j,n]$,

(2) for any composeable pair of relations $[l,n] \xrightarrow{\zeta} [k,n] \xrightarrow{\tau} [j,n]$, there is a commutative diagram

$$\begin{tikzcd} X_l \arrow[r, "X(\zeta)"] \arrow[dr, "X(\tau\zeta)"'] & X_k \arrow[d, "\zeta^* X(\tau)"] \\ & X_j \end{tikzcd}$$

in the groupoid $G_{n-l}$.

Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map. As before, $\theta$ induces a functor $\theta_* : \mathrm{Seg}(\mathbf{m}) \to \mathrm{Seg}(\mathbf{n})$, which is defined by sending the morphism $\tau : [k, m] \to [j, m]$ to the morphism $\tau_* : [\theta(k), n] \to [\theta(j), n]$. "Composing" the $n$-cocycle $X : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ with $\theta_*$ gives a cocycle $\theta^* X : \mathrm{Seg}(\mathbf{m}) \rightsquigarrow G$, defined for each relation $\tau : [k, m] \leq [j, m]$ in $\mathrm{Seg}(\mathbf{m})$, (and in the notation of (4.2)) by the morphism

$$\theta^* X(\tau) = \theta_k^*(X(\tau_*)) : X_{\theta(k)} \to X_{\theta(j)}.$$

of $G_{m-k}$ The assignment $\theta \mapsto \theta^*$ is contravariantly functorial in ordinal maps $\theta$.

We have therefore constructed a simplicial set whose $n$-simplices are the $n$-cocycles $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, and whose simplicial structure maps are the induced maps $\theta^*$. This simplicial set of $G$-cocycles is $\overline{W}G$. In particular, an $n$-cocycle $X : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ is completely determined by the string of relations

$$[n, n] \xrightarrow{\tau_0} [n-1, n] \xrightarrow{\tau_1} \dots \xrightarrow{\tau_{n-2}} [1, n] \xrightarrow{\tau_{n-1}} [0, n],$$

and the corresponding maps

$$X_n \xrightarrow{X(\tau_0)} X_{n-1} \xrightarrow{X(\tau_1)} X_{n-2} \to \dots \to X_1 \xrightarrow{X(\tau_{n-1})} X_0.$$

Each $\tau_i$ is an instance of the map $d^0$, and $X(\tau_i)$ is a morphism of the groupoid $G_i$. Note, in particular, that the $i^{th}$ vertex of the cocycle $X : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ is the object $X_i$ of $G$: this means that $X_i$ can be identified with the "cocycle" $i^* X$, where $i : \mathbf{0} \to \mathbf{n}$.

Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and let $\overline{g}$ denote the string of morphisms

$$X_n \xrightarrow{g_0} X_{n-1} \xrightarrow{g_1} X_{n-2} \to \dots \to X_1 \xrightarrow{g_{n-1}} X_0$$

in $G$, with $g_i$ a morphism of $G_i$. Let $X_{\overline{g}}$ be the cocycle $\mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$ associated to the n-tuple $\overline{g}$. Then, subject to the notation appearing in diagram (4.2), $\theta^* X_{\overline{g}}$ is the string

$$X_{\theta(m)} \xrightarrow{\theta_m^* X_{\overline{g}}(\tau_{0*})} X_{\theta(m-1)} \to \dots \to X_{\theta(1)} \xrightarrow{\theta_1^* X_{\overline{g}}(\tau_{m-1*})} X_{\theta(0)}.$$

This definition specializes to the cocycle definition of $\overline{W}G$ in the case where $G$ is a simplicial group.

A simplicial map $f : X \to \overline{W}G$ assigns to each $n$-simplex $x$ a cocycle $f(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow G$, such that for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ and each map $\tau : [k, m] \to [j, m]$ in $\mathrm{Seg}(\mathbf{m})$ there is a relation

$$\theta_k^* f(x)(\tau_*) = f(\theta^*(x))(\tau).$$

Furthermore, $f(x)$ is determined by the string of maps

$$f(x_n) \xrightarrow{f(x)(\tau_0)} f(x_{n-1}) \xrightarrow{f(x)(\tau_1)} f(x_{n-2}) \to \dots \to f(x_1) \xrightarrow{f(x)(\tau_{n-1})} f(x_0),$$

in $G$, where $x_i$ is the $i^{th}$ vertex of $x$, and $\tau_{n-i}$ is the map $\tau_{n-i} = d^0 : [i,n] \to [i-1,n]$ of $\mathrm{Seg}(\mathbf{n})$. By the simplicial relations, $f(x)(\tau_{n-i}) = f(d_0^{i-1}(x))(\tau_{n-i})$, so that the simplicial map $f : X \to \overline{W}G$ is completely determined by the morphisms

$$f(x)(d^0 = \tau_{n-1} : [1,n] \to \mathbf{n}) : f(x_1) \to f(x_0)$$

in $G_{n-1}$, $x \in X$. In alternate notation then, the cocycle $f(x)$ is given by the string of morphisms

$$f(x_n) \xrightarrow{f(d_0^{n-1}x)(d^0)} f(x_{n-1}) \to \dots \xrightarrow{f(d_0x)(d^0)} f(x_1) \xrightarrow{f(x)(d^0)} f(x_0)$$

in $G$.

The morphism $f(s_0x)(d^0)$ is the identity on $f(x_0)$. We now can define a groupoid $GX_n$ to be the free groupoid on generators $x : x_1 \to x_0$, where $x \in X_{n+1}$, subject to the relations $s_0x = 1_{x_0}$, $x \in X_n$. The objects of this groupoid are the vertices of $X$. Following the description of the loop group from a previous section, we can define a functor $\theta^* : GX_n \to GX_m$ for each ordinal number morphism $\theta : \mathbf{m} \to \mathbf{n}$ by specifying that $\theta^*$ is the identity on objects, and is defined on generators $[x]$, $x \in X_{n+1}$, by requiring that the following diagram commutes:

$$\begin{tikzcd}
x_{\theta(0)+1} \arrow[r, "{[(c\theta)^* d_0 x]}"] \arrow[dr, "{[\tilde{\theta}^*(x)]}"'] & x_1 \arrow[d, "{\theta^*[x]}"] \\
& x_0,
\end{tikzcd}$$

or rather that

$$\theta^*[x] = [\tilde{\theta}^*(x)][(c\theta)^* d_0 x]^{-1}.$$

One checks, as before, that this assignment is functorial in ordinal number morphisms $\theta$, so that the groupoids $GX_n$, $n \geq 0$, and the functors $\theta^*$ form a simplicial groupoid $GX$, which we call the *loop groupoid* for $X$.

Any $n$-simplex $x$ of the simplicial set $X$ determines a string of morphisms

$$x_n \xrightarrow{[d_0^{n-1}x]} x_{n-1} \to \dots \xrightarrow{[d_0x]} x_1 \xrightarrow{[x]} x_0$$

in $GX$, which together determine a cocycle $\eta(x) : \mathrm{Seg}(\mathbf{n}) \rightsquigarrow GX$ in the simplicial groupoid $GX$. The calculations leading to Lemma 5.3 also imply the following:

LEMMA 7.7.

(a) *The assignment* $x \mapsto \eta(x)$ *defines a natural simplicial map* $\eta : X \to \overline{W}GX$.

(b) *The map* $\eta$ *is one of the canonical homomorphisms for an adjunction*

$$\hom_{s\mathbf{Gd}}(GX, H) \cong \hom_{\mathbf{S}}(X, \overline{W}H),$$

*where* $s\mathbf{Gd}$ *denotes the category of simplicial groupoids.*

Here's the homotopy theoretic content of these functors:

THEOREM 7.8.

(1) *The functor* $G : \mathbf{S} \to s\mathbf{Gd}$ *preserves cofibrations and weak equivalences.*

(2) *The functor* $\overline{W} : s\mathbf{Gd} \to \mathbf{S}$ *preserves fibrations and weak equivalences.*

(3) *A map* $K \to \overline{W}X \in \mathbf{S}$ *is a weak equivalence if and only if its adjoint* $GK \to X \in s\mathbf{Gd}$ *is a weak equivalence.*

PROOF: The heart of the matter for this proof is statement (2). We begin by showing that $\overline{W}$ preserves weak equivalences.

Suppose that $A$ is a simplicial groupoid, and choose a representative $x$ for each $[x] \in \pi_0 A$. Recall that the inclusion

$$i : \bigsqcup_{[x] \in \pi_0 A} A(x, x) \to A$$

is a homotopy equivalence of simplicial groupoids in the sense that there is a groupoid map

$$r : A \to \bigsqcup_{[x] \in \pi_0 A} A(x, x)$$

which is determined by paths, such that $ri$ is the identity and such that the paths defining $r$ determine a groupoid homotopy

$$h : A \times I \to A$$

from the identity on $A$ to the composite morphism $ir$. The object $I$ is the constant simplicial groupoid associated to the groupoid having two objects $0, 1$ and exactly one morphism $a \to b$ for any $a, b \in \{0, 1\}$. One sees that $\overline{W}I = BI$ and that $\overline{W}$ preserves products. It follows that the groupoid homotopy $h$ induces a homotopy of simplicial sets from the identity on $\overline{W}A$ to the composite map $\overline{W}i \cdot \overline{W}r$, and so $\overline{W}i$ is a weak equivalence. If $f : A \to B$ is a weak equivalence of simplicial groupoids, then $f$ induces an isomorphism $\pi_0 A \cong \pi_0 B$, and there is a commutative diagram of simplicial groupoid maps

$$\begin{tikzcd}
\bigsqcup_{x \in \pi_0 A} A(x,x) \arrow[r] \arrow[d, "i"'] & \bigsqcup_{x \in \pi_0 A} B(f(x),f(x)) \arrow[d, "i"] \\
A \arrow[r] & B
\end{tikzcd}$$

in which the vertical maps are homotopy equivalences. To see that $\overline{W}f$ is a weak equivalence, it therefore suffices to show that $\overline{W}$ takes the top horizontal map to a weak equivalence. But $\overline{W}$ preserves disjoint unions, and then one uses the corresponding result for simplicial groups (ie. Proposition 6.3).

To show that $\overline{W}$ preserves fibrations, we have to show that a lifting exists for all diagrams

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & \overline{W}A \arrow[d, "\overline{W}f"] \\
\Delta^n \arrow[r, "\beta"'] & \overline{W}B,
\end{tikzcd} \tag{7.9}$$

given that $f : A \to B$ is a fibration of $s\mathbf{Gd}$. We can assume that $A$ and $B$ are connected. The lifting problem is solved by the path lifting property for $f$ if $n = 1$.

Otherwise, take a fixed $x \in A_0$ and choose paths $\eta_i : y_i \to x$ in $A_0$, where $y_i$ is the image of the $i^{th}$ vertex in $\Lambda^n_k$. Note that the vertices of $\Lambda^n_k$ coincide with those of $\Delta^n$, since $n \geq 2$. These paths, along with their images in the groupoid $B_0$ determine "cocycle homotopies" from the diagram (7.9) to a diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\alpha'"] \arrow[d, hook] & \overline{W}A(x,x) \arrow[d, "\overline{W}f"] \\
\Delta^n \arrow[r, "\beta'"'] \arrow[ur, dashed, "\gamma"] & \overline{W}B(f(x),f(x)).
\end{tikzcd} \tag{7.10}$$

More explicitly, if the simplicial set map $\beta$ is determined by the string of morphisms

$$f(y_n) \xrightarrow{g_{n-1}} f(y_{n-1}) \xrightarrow{g_{n-2}} f(y_{n-2}) \to \cdots \to f(y_1) \xrightarrow{g_0} f(y_0)$$

in $B$, then the cocycle homotopy from $\beta$ to $\beta'$ is the diagram

$$\begin{tikzcd}
f(y_n) \arrow[r, "g_{n-1}"] \arrow[d, "f(\eta_n)"'] & f(y_{n-1}) \arrow[r, "g_{n-2}"] \arrow[d, "f(\eta_{n-1})"'] & \dots \arrow[r, "g_1"] & f(y_1) \arrow[r, "g_0"] \arrow[d, "f(\eta_1)"'] & f(y_0) \arrow[d, "f(\eta_0)"] \\
f(x) \arrow[r, "h_{n-1}"'] & f(x) \arrow[r, "h_{n-2}"'] & \dots \arrow[r, "h_1"'] & f(x) \arrow[r, "h_0"'] & f(x)
\end{tikzcd}$$

where $h_i = f(\eta_i)g_i f(\eta_{i+1})^{-1}$, and $\beta'$ is defined by the string of morphisms $h_i$. The cocycle $\beta'$ is a cocycle conjugate of $\beta$, in an obvious sense.

The indicated lift exists in the diagram (7.10), because the simplicial set map $\overline{W}f : A(x,x) \to B(f(x), f(x))$ satisfies the lifting property for $n \geq 2$ (see the proof of Lemma 6.6). The required lift for the diagram (7.9) is cocycle conjugate to $\gamma$.

We have therefore proved statement (2) of the theorem. An adjointness argument now implies that the functor $G$ preserves cofibrations and trivial cofibrations. Every weak equivalence $K \to L$ of simplicial sets can be factored as a trivial cofibration, followed by a trivial fibration, and every trivial fibration in $\mathbf{S}$ is left inverse to a trivial cofibration. It follows that $G$ preserves weak equivalences, giving statement (1).

Statement (3) is proved by showing that the unit and counit of the adjunction are both weak equivalences. Let $A$ be a simplicial groupoid. To show that the counit $\epsilon : G\overline{W}A \to A$ is a weak equivalence, we form the diagram

$$\begin{tikzcd}
G\overline{W}(\bigsqcup_{x\in\pi_0 A} A(x,x)) \arrow[r, "\epsilon"] \arrow[d, "G\overline{W}i"'] & \bigsqcup_{x\in\pi_0 A} A(x,x) \arrow[d, "i", "\simeq"'] \\
G\overline{W}A \arrow[r, "\epsilon"'] & A,
\end{tikzcd}$$

where we note that $G\overline{W}i$ is a weak equivalence by statements (1) and (2). The functors $G$ and $\overline{W}$ both preserve disjoint unions, so it's enough to show that the simplicial group map $\epsilon : G\overline{W}A(x,x) \to A(x,x)$ is a weak equivalence, but this is the traditional result for simplicial groups (Proposition 6.3; see also Corollary 6.4).

Let $K$ be a simplicial set. To show that the unit $\eta : K \to \overline{W}GK$ is a weak equivalence, it suffices to assume that $K$ is a reduced Kan complex, by statements (1) and (2). Now apply Proposition 6.3. □

COROLLARY 7.11. *The functors $G$ and $\overline{W}$ induce an equivalence of homotopy categories*

$$\mathrm{Ho}(s\mathbf{Gd}) \simeq \mathrm{Ho}(\mathbf{S}).$$

# Chapter VI The homotopy theory of towers

Towers of fibrations are everywhere. The homotopy spectral sequence for a tower of fibrations is a fundamental device in modern homotopy theory and its applications in other fields such as algebraic K-theory, while Postnikov towers and the analysis of $k$-invariants are basic classical themes. These are the major objects of study of this chapter.

We begin in Section 1 by describing an approach to constructing homotopy inverse limits of towers: a closed model structure of towers of spaces is introduced, for which towers of fibrations between Kan complexes are the fibrant objects. The homotopy inverse limit of a tower can then be defined as the inverse limit of an associated fibrant model — this is a special case of a general concept which is more fully described in Chapters VIII and X.

Section 2 contains a description of the homotopy spectral sequence of a tower of fibrations of Kan complexes

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

The controlling idea is that the long exact sequences in homotopy for each of the fibrations in the tower can be pieced together in a canonical way to give a spectral sequence, which converges in well behaved cases to the homotopy groups of the inverse limit of the tower.

That said, this spectral sequence involves non-abelian groups and pointed sets, so its homological behaviour can be difficult to analyze in general. It is also "fringed"as a result of the fact that the induced maps $\pi_0 X_s \to \pi_0 X_{s-1}$ are not necessarily surjective functions. Further, the general question of convergence of such spectral sequences is famously difficult: we give the usual partial answer here (the "complete convergence lemma", Lemma 2.20), and then consider the problem further in a discussion of Bousfield's obstruction theory for the homotopy spectral sequence for a cosimplicial space in Chapter VIII. This spectral sequence for a cosimplicial space occurs quite naturally in descriptions of homotopy theoretic resolutions, and is one of the most widely used tools in modern homotopy theory: it is the subject of Chapter VIII, and its construction depends in part on the material presented here.

The Postnikov tower construction gives a method of breaking up a space $X$ into a collection of spaces $X(n)$, $n \geq 0$, such that $X(n)$ carries the homotopy groups of $X$ up to level n, along with a tower of fibrations $X(n) \to X(n-1)$ each of which gives a calculus of adding the $n^{th}$ homotopy group of $X$ to $X(n-1)$ to create the space $X(n)$. We've already seen an application, namely the proof of the Hurewicz theorem in Section III.3. Section 3 of this chapter contains a formal introduction to Postnikov towers, both for spaces and maps. The construction is the standard one for simplicial sets, which is due to Moore.

In many applications, say in rational homotopy theory or more generally in the theory of $R$-completions and localizations, one is presented with a suite of results which says

1) Eilenberg-Mac Lane spaces have property **P**,
2) If $X \to Y$ is a principal $K(A,n)$-fibration and $Y$ has property **P**, then so does $X$, and
3) If $\cdots \to X_1 \to X_0 \to *$ is a tower of fibrations such that all $X_i$ have property **P**, then so does $\varprojlim X_i$.

This gives a method of inferring that a certain class of spaces $X$ has property **P**, which involves "crawling up a Postnikov tower" or some refinement thereof *provided* that all of the fibrations in the tower are principal fibrations, at least up to homotopy equivalence. There is a good class of spaces for which this yoga works, namely nilpotent spaces, and the method for establishing it involves a careful analysis of $k$-invariants.

For us, a $k$-invariant is a map

$$k_n : X(n-1) \to \underset{\overrightarrow{\Gamma}}{\mathrm{holim}}\, K(\pi_n X, n+1)$$

for a connected space $X$, taking values in a homotopy colimit arising from the action of the fundamental groupoid $\Gamma$ on the homotopy group $\pi_n X$. This map is implicitly fibred over the classifying space $B\Gamma$ of the fundamental groupoid $\Gamma$, and can be interpreted as a representative of a class in a suitably defined equivariant cohomology group $H^{n+1}_{\Gamma}(X(n-1), \pi_n X)$. The main point overall (Proposition 5.1) is that the fibration $X(n) \to X(n-1)$ in the Postnikov tower for $X$ sits in a homotopy cartesian diagram

$$\begin{tikzcd} X(n) \arrow[r] \arrow[d] & \underset{\overrightarrow{\Gamma}}{\mathrm{holim}}\, L(\pi_n X, n+1) \arrow[d, "p_*"] \\ X(n-1) \arrow[r, "k_n"'] & \underset{\overrightarrow{\Gamma}}{\mathrm{holim}}\, K(\pi_n X, n+1) \end{tikzcd}$$

where the map $p_*$ is induced by a natural (hence equivariant) contractible covering $L(\pi_n X, n+1) \to K(\pi_n X, n+1)$ of the space $K(\pi_n X, n+1)$. From this, it's pretty much immediate (Corollary 5.3) that the fibration $X(n) \to X(n-1)$ is homotopy equivalent to a principal $K(\pi_n, n)$-fibration if the fundamental group acts trivially on $\pi_n X$. Similarly, if $X$ is nilpotent then the covering $L(\pi_n X, n+1) \to K(\pi_n, n+1)$ has a finite refinement by fibrations that induce principal fibrations after taking homotopy colimit (Proposition 6.1), giving the refined Postnikov tower for a nilpotent space.

This collection of results is the subject of the last three sections of this chapter. The main techniques involve relating homotopy classes of maps

$$[X, \underset{\overrightarrow{\Gamma}}{\mathrm{holim}}\, K(A,n)]$$

fibred over the classifying space $B\Gamma$ of a groupoid $\Gamma$ to equivariant cohomology in various forms (Theorems 4.10, 4.11) — this is the subject of Section 4, along with some formalities about equivariant homotopy theory. The introduction of $k$-invariants and the proofs of their main properties occupy Section 5, and we construct the refined Postnikov tower for a nilpotent space in Section 6.

## 1. A model category structure for towers of spaces.

The purpose of this section is to introduce the structure of a simplicial model category on the category $\mathbf{tow}(\mathbf{S})$ of towers of simplicial sets. This will have implications for Postnikov systems as well as allowing us to define homotopy inverse limits for towers of spaces.

Let $\mathcal{C}$ be a category having an adequate supply of inverse limits, and let $\mathbf{tow}(\mathcal{C})$ denote the category of towers in $\mathcal{C}$. An object in $\mathbf{tow}(\mathcal{C})$ is a diagram

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$$

in $\mathcal{C}$, and a morphism in $\mathbf{tow}(\mathcal{C})$ is a morphism of diagrams; that is, a "commutative ladder". It will be convenient to write $\mathbf{X}$ for a tower $\{X_n\}$

First notice that if $\mathcal{C}$ is a simplicial category, then $\mathbf{tow}(\mathcal{C})$ is a simplicial category. Indeed if $\mathbf{X} \in \mathbf{tow}(\mathcal{C})$ and $K \in \mathbf{S}$, let

$$\mathbf{X} \otimes K = \{X_n \otimes K\}$$

and

$$\mathbf{hom}(K, \mathbf{X}) = \{\mathbf{hom}_{\mathcal{C}}(K, X_n)\}.$$

Then $\mathbf{tow}(\mathcal{C})$ is a simplicial category with

$$\mathbf{Hom}(\mathbf{X}, \mathbf{Y})_n = \hom(\mathbf{X} \otimes \Delta^n, \mathbf{Y}).$$

The subscript in the last equation is the simplicial degree.

Definition 1.1. *Define a morphism $f : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{tow}(\mathcal{C})$ to be*

1) *a weak equivalence if $f_n : X_n \to Y_n$ is a weak equivalence for all $n \geq 0$;*
2) *a cofibration if $f_n : X_n \to Y_n$ is a cofibration for all $n \geq 0$; and*
3) *a fibration if $f_0 : X_0 \to Y_0$ is a fibration and for all $n \geq 1$, the induced map*

$$X_n \to Y_n \times_{Y_{n-1}} X_{n-1}$$

*is a fibration.*

A useful preliminary lemma is:

Lemma 1.2. *Let $q : \mathbf{X} \to \mathbf{Y}$ be a fibration in $\mathbf{tow}(\mathcal{C})$. Then for all $n \geq 0$, $q_n : X_n \to Y_n$ is a fibration.*

PROOF: This is true if $n = 0$. If it is true for $n - 1$, contemplate the induced diagram

$$\begin{tikzcd}
X_n \arrow[r, "p_n"] \arrow[dr, "q_n"'] & Y_n \times_{Y_{n-1}} X_{n-1} \arrow[r] \arrow[d, "\overline{q}_n"] & X_{n-1} \arrow[d, "q_{n-1}"] \\
& Y_n \arrow[r] & Y_{n-1}
\end{tikzcd}$$

The morphism $p_n$ is a fibration by hypothesis; the morphism $\overline{q}_n$ is a fibration because the pullback of a fibration is a fibration. So $q_n$ is a fibration. □

PROPOSITION 1.3. *With these definitions,* $\mathbf{tow}(\mathcal{C})$ *becomes a simplicial model category.*

PROOF: Axioms **CM1–CM3** are obvious. Suppose given a lifting problem in $\mathbf{tow}(\mathcal{C})$

$$\begin{tikzcd}
\mathbf{A} \arrow[r] \arrow[d, "j"'] & \mathbf{X} \arrow[d, "q"] \\
\mathbf{B} \arrow[r] & \mathbf{Y}
\end{tikzcd}$$

with $j$ a cofibration and $q$ a fibration. If $j$ is also a weak equivalence, one can recursively solve the lifting problem

$$\begin{tikzcd}
A_n \arrow[r] \arrow[d, "j"'] & X_n \arrow[d] \\
B_n \arrow[r] \arrow[ur, dashed] & Y_n \times_{Y_{n-1}} X_{n-1}
\end{tikzcd} \tag{1.4}$$

to solve the lifting problem in $\mathbf{tow}(\mathcal{C})$. If $q$ is a weak equivalence, the pullback diagram

$$\begin{tikzcd}
Y_n \times_{Y_{n-1}} X_{n-1} \arrow[r] \arrow[d, "q'_n"'] & X_{n-1} \arrow[d, "q_{n-1}"] \\
Y_n \arrow[r] & Y_{n-1}
\end{tikzcd}$$

shows $q'_n$ is a trivial fibration, since $q_{n-1}$ is a fibration and a weak equivalence. Hence $X_n \to Y_n \times_{Y_{n-1}} X_{n-1}$ is a trivial fibration and we can again recursively solve the lifting problem of (1.4). This proves **CM4**.

To prove **CM5** fix a morphism $f : \mathbf{X} \to \mathbf{Y}$. To factor $f$ as a cofibration followed by a trivial fibration, proceed inductively as follows. First, factor $f_0 : X_0 \to Y_0$ as a cofibration followed by a trivial fibration

$$X_0 \xrightarrow{j_0} Z_0 \xrightarrow{q_0} Y_0.$$

Then, having factored through level $n-1$, consider the induced maps

$$\begin{tikzcd}
X_n \arrow[r, "p_n"] & Z_{n-1} \times_{Y_{n-1}} Y_n \arrow[r, "\overline{q}_{n-1}"] \arrow[d] & Y_n \arrow[d] \\
& Z_{n-1} \arrow[r, "q_{n-1}"'] & Y_{n-1}.
\end{tikzcd}$$

Since $q_{n-1}$ is a trivial fibration so is $\overline{q}_{n-1}$. Factor $p_n$ as

$$X_n \xrightarrow{j_n} Z_n \xrightarrow{q'_n} Z_{n-1} \times_{Y_{n-1}} Y_n$$

where $j_n$ is a cofibration and $q'_n$ is a trivial fibration, so that the composite map $q_n = \overline{q}_{n-1} q'_n$ is a trivial fibration. The other factoring is similar, but easier.

Finally, **SM7** is equivalent to **SM7b** (Corollary II.3.12), which is obvious in this case. □

REMARK 1.5.

1) If every object of $\mathcal{C}$ is cofibrant, then every object of $\mathbf{tow}(\mathcal{C})$ is cofibrant. This applies, for example, if $\mathcal{C}$ is the category $\mathbf{S}$ of simplicial sets or the pointed simplicial set category $\mathbf{S}_*$.
2) The fibrant objects of $\mathbf{tow}(\mathcal{C})$ are the ones where $X_0$ is fibrant in $\mathcal{C}$ and every $q_n : X_n \to X_{n-1}$ is a fibration.

REMARK 1.6. The model structure of Proposition 1.3 extends easily to bigger towers. Suppose that $\beta$ is a limit ordinal, and identify it with a poset. A $\beta$-tower is a simplicial set-valued functor $X : \beta^{op} \to \mathbf{S}$ which is contravariant on $\beta$. Say that a map $f : X \to Y$ between $\beta$-towers is a cofibration (respectively weak equivalence) if the maps $f : X_s \to Y_s$ are cofibrations (respectively weak equivalences) for $s < \beta$. A map $g : Z \to W$ is a fibration if the following conditions hold:

1) the map $g : Z_0 \to W_0$ is a fibration,
2) for all ordinals $s < \beta$ the induced map $Z_{s+1} \to W_{s+1} \times_{W_s} Z_s$ is a fibration, and
3) for all limit ordinals $\alpha < \beta$ the map

$$Z_\alpha \to W_\alpha \times_{(\varprojlim_{s<\alpha} W_s)} (\varprojlim_{s<\alpha} Z_s)$$

is a fibration.

With these definitions, the category of $\beta$-towers satisfies the conditions for a simplicial model category. This model structure could be arrived at in a different way, by using the model structure for $\beta^{op}$-diagrams of simplicial sets which appears in Section IX.5 below, but that method does not produce the completely explicit description of fibration that you see here.

The notion of homotopy inverse limit is extremely flexible. Here is one of its equivalent formulations for towers:

DEFINITION 1.7. *Let* $\mathbf{X} \in \mathbf{tow}(\mathcal{C})$. *Choose a weak equivalence* $\mathbf{X} \to \mathbf{Y}$ *where* $\mathbf{Y}$ *is fibrant in* $\mathbf{tow}(\mathcal{C})$. *The homotopy inverse limit* $\underleftarrow{\text{holim}}\ \mathbf{X}$ *of the tower* $\mathbf{X}$ *is defined to be the inverse limit* $\underleftarrow{\lim}\ \mathbf{Y}$ *of the tower* $\mathbf{Y}$.

As usual, $\underleftarrow{\text{holim}}\ \mathbf{X}$ is well-defined and functorial up to homotopy; indeed, $\underleftarrow{\text{holim}}(\cdot)$ is the total right derived functor of $\underleftarrow{\lim}(\cdot)$. Notice that $\underleftarrow{\lim}(\cdot) : \mathbf{tow}(\mathcal{C}) \to \mathcal{C}$ is right adjoint to the constant tower functor, which preserves cofibrations and weak equivalences, so $\underleftarrow{\lim}(\cdot)$ preserves fibrations and trivial fibrations and hence preserves weak equivalences between fibrant objects. Furthermore if $\mathbf{X} \to \mathbf{X}'$ is a weak equivalence, $\underleftarrow{\text{holim}}\ \mathbf{X} \to \underleftarrow{\text{holim}}\ \mathbf{X}'$ is a weak equivalence. It is this invariance property that justifies the name homotopy inverse limit.

EXERCISE 1.8 (HOMOTOPY PULLBACKS). Let $I$ be the category with objects 1, 2, and 12 and non-identity morphisms as follows:

$$1 \to 12 \leftarrow 2.$$

If $\mathcal{C}$ is a category, let $\mathcal{C}^I$ denote the resulting diagram category. An object in $\mathcal{C}^I$ is a diagram

$$\begin{tikzcd} & Y_2 \arrow[d] \\ Y_1 \arrow[r] & Y_{12} \end{tikzcd}$$

in $\mathcal{C}$. Suppose $\mathcal{C}$ is a simplicial model category. Then $\mathcal{C}^I$ becomes a simplicial model category with, for $K \in S$,

$$\left(\begin{tikzcd} & Y_2 \arrow[d] \\ Y_1 \arrow[r] & Y_{12} \end{tikzcd}\right) \otimes K = \begin{tikzcd} & Y_2 \otimes K \arrow[d] \\ Y_1 \otimes K \arrow[r] & Y_{12} \otimes K. \end{tikzcd}$$

The techniques of this section can be adapted to prove the following:

THEOREM 1.9. *The category* $\mathcal{C}^I$ *is a simplicial model category with a morphism*

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "f_1"'] & X_{12} \arrow[d, "f_{12}"'] & X_2 \arrow[l] \arrow[d, "f_2"] \\ Y_1 \arrow[r] & Y_{12} & Y_2 \arrow[l] \end{tikzcd}$$

1) *a weak equivalence if* $f_1, f_2$*, and* $f_{12}$ *are;*
2) *a cofibration if* $f_1, f_2$*, and* $f_{12}$ *are; and*
3) *a fibration if* $f_{12} : X_{12} \to Y_{12}$ *is a fibration and for* $i = 1, 2$ *the induced maps*

$$X_i \to X_{12} \times_{Y_{12}} Y_i$$

*are fibrations.*

A crucial lemma is:

LEMMA 1.10. *If a morphism* $f : X_\bullet \to Y_\bullet$ *in* $\mathcal{C}^I$ *is a trivial fibration, then the induced maps*

$$X_i \to X_{12} \times_{Y_{12}} Y_i$$

*are trivial fibrations.*

Note the fibrant objects in $\mathcal{C}^I$ are those diagrams

$$Y_1 \to Y_{12} \leftarrow Y_2$$

with $Y_{12}$ fibrant and $Y_i \to Y_{12}$ a fibration.

If $X_\bullet \in \mathcal{C}^I$ is a diagram, define the homotopy pullback by taking the actual pullback of a fibrant replacement. This has the usual homotopy invariance properties, much as after Definition 1.7. Prove that if $\mathcal{C}$ is a proper model category and

$$Y_1 \to Y_{12} \leftarrow Y_2$$

is a diagram with $Y_{12}$ fibrant and one of $Y_1 \to Y_{12}$ or $Y_2 \to Y_{12}$ a fibration, then $Y_1 \times_{Y_{12}} Y_2$ is weakly equivalent to the homotopy pullback.

Here is another description of the homotopy inverse limit of a tower that is often useful for computations. Fix a simplicial model category $\mathcal{C}$ and write, for $X \in \mathcal{C}$,

$$D : \mathbf{hom}_{\mathcal{C}}(\Delta^1, X) \to \mathbf{hom}_{\mathcal{C}}(\partial\Delta^1, X) \cong X \times X.$$

Then define, for a tower $\mathbf{X} = \{X_n\} \in \mathbf{tow}(\mathcal{C})$, the object $T(\mathbf{X}) \in \mathcal{C}$ by the pullback diagram

$$\begin{tikzcd} T(\mathbf{X}) \arrow[r] \arrow[d] & \prod_n \mathbf{hom}(\Delta^1, X_n) \arrow[d, "\prod D_n"] \\ \prod_n X_n \arrow[r, "(1,q)"'] & \prod_n X_n \times X_n \end{tikzcd} \tag{1.11}$$

where $(1, q)$ is the product of the maps

$$\prod_n X_n \to X_n \times X_{n+1} \xrightarrow{1 \times q_n} X_n \times X_n.$$

LEMMA 1.12. *Let $\mathbf{X} = \{X_n\} \in \mathbf{tow}(\mathcal{C})$ be a tower so that each $X_n$ is fibrant in $\mathcal{C}$. Then there is a weak equivalence*

$$T(\mathbf{X}) \simeq \underleftarrow{\operatorname{holim}} \mathbf{X}.$$

PROOF: The functor $T : \mathbf{tow}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint defined as follows. Note that specifying a map $f : Y \to T(\mathbf{X})$ is equivalent to specifying a sequence of maps $g_n : Y \to X_n$ and right homotopies

$$Y \to \mathbf{hom}(\Delta^1, X_n)$$

between $g_n$ and $q_{n+1} \cdot g_{n+1}$. Define $[0, \infty) \in \mathbf{S}$ to be the simplicial "half-line"; thus $[0, \infty)$ has non-degenerate 1-simplices $[n, n+1]$, $0 \leq n < \infty$, and $d_0[n-1, n] = d_1[n, n+1]$. Let $[n, \infty) \subseteq [0, \infty)$ be the evident sub-complex. Then the left adjoint to $T$ is given by the functor $F(\cdot)$:

$$F(Y) = \{Y \otimes [n, \infty)\}.$$

Since this left adjoint preserves cofibrations and weak equivalences among cofibrant objects, $F(\cdot)$ has a total left derived functor $\mathbf{L}F$. But the unique maps $[n, \infty) \to *$ induces a weak equivalence $F(Y) \to \{Y\}_{n \geq 0}$ from $F(Y)$ to the constant tower for $Y$ cofibrant, so $\mathbf{L}F : \operatorname{Ho}(\mathcal{C}) \to \operatorname{Ho}(\mathbf{tow}(\mathcal{C}))$ is the total left derived functor of the constant tower functor.

The functor $T$ preserves fibrations, since it is right adjoint to a functor that preserves trivial cofibrations; $T$ also preserves weak equivalences between objects $\mathbf{X} = \{X_n\}$ where $X_n$ is fibrant for each $n$. This is because

$$\prod D_n : \prod_n \mathbf{hom}(\Delta^1, X_n) \to \prod_n X_n \times X_n$$

is a fibration, so we may apply the definition and homotopy invariance property of homotopy pullback given in Exercise 1.8. Thus $T$ has a total right derived functor $\mathbf{R}T : \operatorname{Ho}(\mathbf{tow}(\mathcal{C})) \to \operatorname{Ho}(\mathcal{C})$, right adjoint to $\mathbf{L}F$. However, $\mathbf{L}F$ is the total left derived functor of the constant diagram functor; hence, by uniqueness of adjoints

$$\mathbf{R}T \cong \underleftarrow{\operatorname{holim}} : \operatorname{Ho}(\mathbf{tow}(\mathcal{C})) \to \operatorname{Ho}(\mathcal{C}).$$

Since $T$ preserves weak equivalences among objects $\mathbf{X} = \{X_n\}$ with all $X_n$ fibrant one has for such $X$, $\underleftarrow{\operatorname{holim}} \mathbf{X} \simeq \mathbf{R}T(\mathbf{X}) \simeq T(\mathbf{X})$. □

## 2. The spectral sequence of a tower of fibrations.

Given a tower of pointed fibrations, the exact sequences in homotopy give rise to a spectral sequence, which is introduced in this section.

There are a number of problems with this spectral sequence. First, there is the problem that $\pi_0$ and $\pi_1$ are not generally abelian groups; second, the spectral sequence need not converge to anything homotopical; third, the spectral sequence is "fringed" in the sense that $E_\infty$ can contain extra elements; and, fourth, the $E_2$ term may not have a sensible description in terms of homological algebra.

We analyze these difficulties in turn. The first two difficulties are addressed in this section, and a homological description of the $E_2$ term and the fringing effect are both considered in the case of cosimplicial spaces in Chapter VIII.

Let

$$\begin{tikzcd}
\cdots \arrow[r] & X_s \arrow[r, "p_s"] & X_{s-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r, "p_1"] & X_0 \arrow[r] & * \\
 & F_s \arrow[u, "i_s"'] & F_{s-1} \arrow[u, "i_{s-1}"'] & & F_1 \arrow[u, "i_1"'] & F_0 \arrow[u, equal] &
\end{tikzcd}$$

be a tower of pointed fibrations, with $F_s$ the fibre of $p_s$. Write

$$X = \varprojlim_s X_s.$$

Applying homotopy groups to each of the fibre sequences yields

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_* X_s \arrow[r, "p_{s*}"] & \pi_* X_{s-1} \arrow[r] \arrow[dl, dashed, "\partial"] & \cdots \arrow[r] \arrow[dl, dashed] & \pi_* X_1 \arrow[r, "p_{1*}"] & \pi_* X_0 \arrow[dl, dashed, "\partial"] \\
 & \pi_* F_s \arrow[u, "i_{s*}"] & \pi_* F_{s-1} \arrow[u] & & \pi_* F_1 \arrow[u, "i_{1*}"] & \pi_* F_0 \arrow[u, equal]
\end{tikzcd}$$

where each of the dotted maps is of degree $-1$. This gives an exact couple and hence a spectral sequence. We will be explicit because $\pi_1$ is not abelian and because

$$\pi_1 X_{s-1} \to \pi_0 F_s \to \pi_0 X_s \to \pi_0 X_{s-1}$$

is exact only in the sense of pointed sets, with the additional proviso that the action of $\pi_1 X_{s-1}$ on $\pi_0 F_s$ extends to an injection of sets

$$* \to \pi_0 F_s / \pi_1 X_{s-1} \to \pi_0 X_s.$$

Note that $\pi_0 X_s \to \pi_0 X_{s-1}$ need not be onto, although the pre-image of the basepoint is $\pi_0 F_s/\pi_1 X_{s-1}$.

The spectral sequence is now defined as follows. Let

$$\pi_i X_s^{(r)} = \mathrm{im}(\pi_i X_{s+r} \to \pi_i X_s) \subseteq \pi_i X_s, \tag{2.1}$$

and for $t-s \geq 0$, let

$$Z_r^{s,t} = \mathrm{Ker}(\pi_{t-s} F_s \to \pi_{t-s} X_s/\pi_{t-s} X_s^{(r-1)}). \tag{2.2}$$

Then let

$$B_r^{s,t} = \mathrm{Ker}(\pi_{t-s+1} X_{s-1} \to \pi_{t-s+1} X_{s-r}). \tag{2.3}$$

Then we set

$$E_r^{s,t} = Z_r^{s,t}/B_r^{s,t} \tag{2.4}$$

where this formula must be interpreted: if $t-s \geq 1$ and $\partial : \pi_{t-s+1} X_{s-1} \to \pi_{t-s} F_s$ is the connecting homomorphism, $\partial B_r^{s,t} \subseteq Z_r^{s,t}$ is a normal subgroup and $E_r^{s,t}$ is the quotient group, and if $t-s=0$, $B_r^{s,t}$ acts on $Z_r^{s,t}$ and $E_r^{s,t}$ is the set of orbits. Define differentials for $t-s \geq 1$

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1} \tag{2.5}$$

by the composite map

$$E_r^{s,t} \to \pi_{t-s} X_s^{(r-1)} \to E_r^{s+r,t+r-1} \tag{2.6}$$

where the first map is induced by noticing that if $x \in Z_r^{s,t} \subseteq \pi_{t-s} F_s$, then $i_{s*} x \in \pi_{t-s} X_s^{(r-1)}$ and the second by noticing that if $y$ is in $\pi_{t-s} X_s^{(r-1)}$ we may choose $z \in \pi_{t-s} X_{s+r-1}$ mapping to $y$ and

$$j_* z \in E_r^{s+r,t+r-1}$$

is independent of that choice. Notice that no differential is defined on $E_r^{s,s}$ and, indeed, that $E_r^{s,t}$ is not defined for $t-s<0$.

The following result is left as an exercise:

LEMMA 2.7.
*1) $E_r^{s,t}$ is a pointed set if $t-s \geq 0$, a group if $t-s \geq 1$, and an abelian group if $t-s \geq 2$;*

*2) The function $d_r : E_r^{s,t} \to E_s^{s+r,t+r-1}$ is a homomorphism if $t-s \geq 2$ and a map of pointed sets if $t-s \geq 1$.*

*3) The image of $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is in the center of $E_r^{s+r,t+r-1}$ if $t-s \geq 2$ and if $t-s \geq 1$, $E_{r+1}^{s,t} = \mathrm{Ker}(d_r)/\mathrm{im}(d_r)$.*

*4) The map $d_r : E_r^{s,s+1} \to E_r^{s+r,s+r}$ extends to an action of the source on the target and $E_{r+1}^{s,s} \subseteq E_r^{s,s}/E_r^{s-r,s-r+1}$.*

PROOF: All these facts follow from the properties of the long exact sequence in homotopy of a fibration. □

The long exact sequences for the fibrations $X_s \to X_{s-1}$ can be "derived" to give the following:

LEMMA 2.8. *Write $X_i = *$ for $i < 0$. Then there are long exact sequences*

$$\begin{aligned}\cdots \to \pi_{t-s+1} X^{(r-1)}_{s-r+1} &\to \pi_{t-s+1} X^{(r-1)}_{s-r} \\ &\to E^{s,t}_r \to \pi_{t-s} X^{(r-1)}_s \pi_{t-s} X^{(r-1)}_{s-1} \to \cdots\end{aligned}$$

*In addition, there is an action*

$$\pi_1 X^{(r-1)}_{s-r} \times E^{s,s}_r \to E^{s,s}_r$$

*such that two elements of $E^{s,s}_r$ have the same image under the map $E^{s,s}_r \to \pi_0 X^{(r-1)}_s$ if and only if they are related by the action of $\pi_1 X^{(r-1)}_{s-r}$.*

REMARK 2.9. The fact that $E^{s,s}_{r+1} \subseteq E^{s,s}_r / E^{s-r,s-r+1}_r$ and is not necessarily equal to what is meant by saying the spectral sequence is *fringed.* The failure of equality arises from the fact that $\pi_0 X_s \to \pi_0 X_{s-1}$ need not be onto.

The collection $\{E^{s,t}_r ; d_r\}_{r \geq 1}$ is the homotopy spectral sequence of the tower of fibrations. It is natural in the tower and we may write $\{E^{s,t}_r X\}$. We now begin to address what it might converge to.

Notice that if $r > s$, $E^{s,t}_{r+1} \subset E^{s,t}_r$. Define

$$E^{s,t}_\infty = \varprojlim_{r>s} E^{s,t}_r = \bigcap_{r>s} E^{s,t}_r. \tag{2.10}$$

One hopes that $E^{s,t}_\infty$ may have something to do with the homotopy groups of $\varprojlim X_s$. For this, we must first decide how to compute $\pi_*(\varprojlim X_s)$ in terms of the tower of groups $\{\pi_* X_s\}$. This is done by a $\varprojlim^1$ exact sequence, once we have defined $\varprojlim^1$ for non-abelian groups.

First recall that $\varprojlim^1$ for a tower of abelian groups $A = \{A_n\}_{n \geq 0}$ is defined by the exact sequence

$$0 \to \varprojlim A_n \to \prod_n A_n \xrightarrow{\partial} \prod_n A_n \to \varprojlim^1 A_n \to 0$$

where $\partial(a_n) = (a_n - f(a_{n+1}))$, with $f : A_{n+1} \to A_n$ the maps in the tower. The following is an easy exercise in homological algebra.

LEMMA 2.11. *The functor $\varprojlim^1 : \mathbf{tow}(\mathbf{Ab}) \to \mathbf{Ab}$ is characterized up to natural isomorphism by:*

*1) If $0 \to A \to B \to C \to 0$ is a short exact of towers of abelian groups, then there is a six-term exact sequence*

$$0 \to \varprojlim A \to \varprojlim B \to \varprojlim C \to \varprojlim^1 A \to \varprojlim^1 B \to \varprojlim^1 C \to 0.$$

*2) If $A$ is any tower so that $A_{n+1} \to A_n$ is surjective for all $n$, then $\varprojlim^1 A = 0$.*

More generally, let $G = \{G_n\}$ be a tower of groups. Then the group $\prod_n G_n$ acts on the set $Z = \prod_n G_n$ by $(g_n) \circ (x_n) = (g_n x_n f(g_{n+1})^{-1})$. Then the stabilizer subgroup of $e \in Z$ is $\varprojlim G_n$, and we define $\varprojlim^1 G_n$ to be the pointed set of orbits. If $G$ is a tower of abelian groups these definitions agree with those above.

LEMMA 2.12. *The functor* $\varprojlim^1 : \mathbf{tow}(\mathbf{Gr}) \to$ pointed sets *has the following properties:*

*1) If* $* \to G_1 \to G_2 \to G_2 \to *$ *is a short exact sequence of towers of groups then there is a six term sequence*

$$* \to \varprojlim G_1 \to \varprojlim G_2 \to \varprojlim G_3 \to \varprojlim^1 G_1 \to \varprojlim^1 G_2 \to \varprojlim^1 G_3 \to *$$

*which is exact as groups at the first three terms, as pointed sets at the last three terms, and* $\varprojlim G_3$ *acts on* $\varprojlim^1 G_1$ *and there is an induced injection*

$$\varprojlim^1 G_1 / \varprojlim G_3 \to \varprojlim^1 G_2.$$

*2) If* $G$ *is any tower so that* $G_{n+1} \to G_n$ *is surjective for all* $n$*, then* $\varprojlim^1 G_n = *$.

PROOF: The proof is the same as in the abelian case. □

REMARK 2.13. It is good to have examples where $\varprojlim^1$ is non-zero. Let $B$ be any group and let $B$ also be the constant tower on that group. let $A_n \subseteq B$ be a descending (ie. $A_{n+1} \subset A_n$) sequence of normal subgroups so that $\bigcap_n A_n = e$. Let $A = \{A_n\}$ be the resulting tower. Then there is a short exact sequence

$$* \to B \to \varprojlim_n (B/A_n) \to \varprojlim^1 A \to * \tag{2.14}$$

so that $\varprojlim^1 A = *$ if and only if $B$ is complete in the topology defined by the $A_n$.

For example, if $B = \mathbb{Z}$ and $A_n = p^n\mathbb{Z}$ for some prime $p$, (2.14) becomes

$$0 \to \mathbb{Z} \to \mathbb{Z}_p \to \mathbb{Z}_p/\mathbb{Z} \to 0$$

where $\mathbb{Z}_p$ is the group of $p$-adic integers, which is uncountable.

A second example comes from setting $B = \bigoplus_{s\geq 1} \mathbb{Z}$ and $A_n = \bigoplus_{s\geq n} \mathbb{Z} \subseteq B$. Then (2.14) becomes

$$0 \to \bigoplus_{s\geq 1} \mathbb{Z} \to \prod_{s\geq 1} \mathbb{Z} \to \prod_{s\geq 1} \mathbb{Z} / \bigoplus_{s\geq 1} \mathbb{Z} \to 0.$$

Now let $\{X_s\}$ be a tower of pointed fibrations.

PROPOSITION 2.15 (MILNOR EXACT SEQUENCE). *For every $i \geq 0$ there is a sequence*

$$* \to \varprojlim{}^1 \pi_{i+1} X_s \to \pi_i(\varprojlim X_s) \to \varprojlim \pi_i X_s \to *$$

*which is exact as groups if $i \geq 1$ and as pointed sets if $i = 0$.*

PROOF: Consider the description of the homotopy inverse limit given in Section VI.1. Then one has a homotopy fibration sequence

$$\prod_s \Omega X_s \to \varprojlim X_s \to \prod X_s$$

by Lemma 1.12. A simple calculation with the homotopy exact sequence of this fibration gives the result except possibly for the claim that $\pi_0 \varprojlim X_s \to \varprojlim \pi_0 X_s$ is onto. But it is an easy exercise to show that for a tower of fibrations $\pi_0 \varprojlim X_s \to \varprojlim \pi_0 X_s$ is onto. $\square$

We now return to computing $\pi_* \varprojlim X_s$ by the homotopy spectral sequence. Recall the notation $X = \varprojlim X_s$.

Because of the definition of $E_r^{s,t}$ given in (2.1)–(2.4) and $E_\infty^{s,t}$ in (2.10), we can think of an element $z = [x] \in E_\infty^{s,t}$ as an equivalence class of elements $x \in \pi_{t-s} F_s$ so that for each $r \geq 0$ there is an element $y_r \in \pi_{t-s} X_{s+r}$ mapping to $i_{s*}x \in \pi_{t-s} X_s$. (Notice we do not say $y_r$ maps to $y_{r-1}$; see Example 2.18.2 below.) If an element $u \in \pi_{t-s} X$ maps to zero in $\pi_{t-s} X_{s-1}$, then one gets such an equivalence class. To make this precise, let

$$G_s \pi_i X = \operatorname{im}(\pi_i X \to \pi_i X_s)$$

and

$$e_\infty^{s,t} = \ker\{G_s \pi_{t-s} X \to G_{s-1} \pi_{t-s} X\}. \tag{2.16}$$

If $y \in e_\infty^{s,t}$, let $x \in \pi_{t-s} F_s$ be any element so that $i_* x = y$.

LEMMA 2.17.

*1) The assignment $y \mapsto x$ induces a well-defined monomorphism $e_\infty^{s,t} \to E_\infty^{s,t}$.*

*2) The inclusion maps $G_s \pi_* X \to \pi_* X_s$ induce an isomorphism*

$$\varprojlim_s G_s \pi_* X \xrightarrow{\cong} \varprojlim_s \pi_* X_s.$$

PROOF: This is an exercise in unraveling the definitions. $\square$

Notice that any element of $\varprojlim{}^1 \pi_* X_s$ maps to zero in all $G_s \pi_* X$, so will not be detected in any $e_\infty^{s,t}$. It also turns out that $e_\infty^{s,t} \to E_\infty^{s,t}$ need not be an isomorphism.

Here are two examples of the potential difficulties.

EXAMPLE 2.18.

1) Consider the tower of abelian groups

$$\cdots \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z}$$

where $p \in \mathbb{Z}$ is a prime number. The resulting tower of Eilenberg-Mac Lane spaces

$$\to K(\mathbb{Z}, n) \xrightarrow{\times p} K(\mathbb{Z}, n) \xrightarrow{\times p} \cdots \xrightarrow{\times p} K(\mathbb{Z}, n)$$

with $n \geq 1$ can be transformed into a tower of fibrations. One easily calculates that $E_1^{0,n} = \mathbb{Z}$ and $E_1^{s,s+n-1} = \mathbb{Z}/p\mathbb{Z}$ and $E_1^{s,t} = 0$ otherwise. Next $E_r^{0,n} = p^{r-1}\mathbb{Z} \subseteq \mathbb{Z} = E_1^{0,n}$ and $d_r : E_r^{0,n} \to E_r^{r,r+n-1} = \mathbb{Z}/p\mathbb{Z}$ is onto.. Thus $E_\infty^{s,t} = 0$ for all $t - s \geq 0$. However

$$\pi_{n-1} \varprojlim X_s = \mathbb{Z}_p/\mathbb{Z} \cong \varprojlim_r{}^1 (p^r\mathbb{Z}) \neq 0.$$

Note that in this case $\varprojlim^1 E_r^{0,n} \neq 0$.

2) Consider the tower of abelian groups, where $a$ is the addition map,

$$\to \bigoplus_{s\geq 3} \mathbb{Z} \to \bigoplus_{s\geq 2} \mathbb{Z} \to \bigoplus_{s\geq 1} \mathbb{Z} \xrightarrow{a} \mathbb{Z}$$

and let $\{X_m\}$ be the tower obtained by applying $K(\cdot, n)$ with $n > 1$, and then convert to a tower of fibrations. Now

$$\pi_{n-1} \varprojlim X_m \cong \varprojlim{}^1 \pi_n X_m \cong \prod_{s\geq 1} \mathbb{Z} / \bigoplus_{s\geq 1} \mathbb{Z}$$

and $\pi_k \varprojlim X_m = 0$ otherwise. Then

$$E_1^{s,t} = \begin{cases} \mathbb{Z} & s = 0, t = n \\ \bigoplus_{i\geq 1} \mathbb{Z} & s = 1, t = n+1 \\ \mathbb{Z} & s \geq 2, t = s+n-1 \end{cases}$$

and $d_r : E_r^{1,n+1} \to E_r^{r+1,n+r}$ is onto. Thus $E_\infty^{0,n} = \mathbb{Z}$ and $E_\infty^{s,t} = 0$ otherwise. Thus $e_\infty^{0,n} \neq E_\infty^{0,n}$. Note also $\varprojlim^1 E_r^{1,n+1} \neq 0$. □

The following definition now seems appropriate.

DEFINITION 2.19. *The spectral sequence of the pointed tower of fibrations* $\{X_s\}$ *converges completely if*

1) $\varprojlim^1 \pi_i X_s = *$ *for* $i \geq 1$, *and*
2) $e_\infty^{s,t} \cong E_\infty^{s,t}$ *for all* $(s, t)$ *with* $t - s \geq 1$.

When 2.19.2 holds, the spectral sequence effectively computes $\varprojlim \pi_* X_s$, subject to the fringing effect when $t-s=0$, and when 2.19.1 holds, we have $\pi_*(\varprojlim X_s) \cong \varprojlim \pi_* X_s$. One would like to be able to decide when one has complete convergence from knowledge of the spectral sequence, rather than using knowledge of $\pi_* \varprojlim X_s$, which Definition 2.19 demands — after all one is trying to compute $\pi_* \varprojlim X_s$. The next lemma is the best possible result. Compare the examples of 2.18.

LEMMA 2.20 (COMPLETE CONVERGENCE LEMMA). *The spectral sequence of the pointed tower of fibrations $\{X_s\}$ converges completely if and only if*

$$\varprojlim_r{}^1 E_r^{s,t} = *, \qquad t-s \geq 1.$$

This will be proved below, after some examples and preliminaries.

COROLLARY 2.21. *Suppose that for each integer $i \geq 0$ there are only finitely many $s$, so that $E_2^{s,s+i} \neq *$. Then the spectral sequence converges completely.*

More generally we have

COROLLARY 2.22. *Suppose that for each integer $i \geq 0$ and each integer $s$, there is an integer $N$ so that*

$$E_\infty^{s,s+i} \cong E_N^{s,s+i}.$$

*Then the spectral sequence converges completely.*

PROOF: In this case $\varprojlim^1 E_r^{s,t} = *$. See 2.23.1 below. □

The phenomenon encountered in Corollary 2.22 is called *Mittag-Leffler convergence.* The wider implications are explored in [14, p.264].

As a preliminary to proving the complete convergence lemma, we discuss derived towers of group. Suppose that $\{G_n\}$ is a tower of groups. Define $G_n^{(r)} = \operatorname{im}\{G_{n+r} \to G_n\}$. We already encountered $\pi_* X_s^{(r)} = (\pi_* X_s)^{(r)}$ in (2.1)–(2.4). There are maps

$$G_n^{(r)} \to G_{n-1}^{(r+1)} \subseteq G_{n-1}^{(r)}$$

and one has two new towers of groups

$$\{G_n^{(r)}\}_{n\geq 0} \qquad\qquad \{G_n^{(r)}\}_{r\geq 0}.$$

LEMMA 2.23.
*1) The inclusions $G_n^{(r)} \subseteq G_n$ induce isomorphisms*

$$\varprojlim_n G_n^{(r)} \cong \varprojlim_n G_n \quad \textit{and} \quad \varprojlim_n{}^1 G_n^{(r)} \cong \varprojlim_n{}^1 G_n.$$

*2) There is a natural isomorphism*

$$\varprojlim_n \varprojlim_r G_n^{(r)} \cong \varprojlim_n G_n$$

*and a natural short exact sequence*

$$* \to \varprojlim_n{}^1 \varprojlim_r G_n^{(r)} \to \varprojlim{}^1 G_n \to \varprojlim_n \varprojlim_r{}^1 G_n^{(r)} \to *.$$

PROOF: For 1) note that the functor $\{G_n\}_{n\geq 0} \mapsto \{G_n^{(r)}\}_{n\geq 0}$ is exact and takes tower of surjections to surjections. By inspection, one has isomorphisms

$$\varprojlim_n G_n^{(1)} \cong \varprojlim_n G_n \quad \text{and} \quad \varprojlim_n{}^1 G_n^{(1)} \cong \varprojlim_n{}^1 G_n.$$

One shows that $G_n^{(1)(r)} \cong G_n^{(r+1)}$, so there are isomorphisms

$$\varprojlim_n G_n^{(r)} \cong \varprojlim_n G_n \quad \text{and} \quad \varprojlim_n{}^1 G_n^{(r)} \cong \varprojlim_n{}^1 G_n$$

for all $r \geq 1$.

For 2), note that if the groups were abelian, what we would have is a degenerate form of a composite functor spectral sequence. In the non-abelian case, it is better to proceed topologically. For fixed $r$ define a tower $Y_r = \{Y_{r,n}\}$ with $Y_{r,n} = BG_n^{(r)}$. There are maps of towers $Y_{r+1} \to Y_r$; convert the resulting tower of towers into a tower of fibrations in $\mathbf{tow}(\mathbf{S}_*)$:

$$\cdots \to Z_2 \to Z_1 \to Z_0.$$

Thus $Y_0 \to Z_0$ is a weak equivalence in $\mathbf{tow}(\mathbf{S}_*)$ with $Z_0$ fibrant (i.e., a tower of fibrations) and $Y_r \to Z_r \to Z_{r-1}$ factors $Y_r \to Y_{r-1} \to Z_{r-1}$ as a weak equivalence followed by a fibration. Then

$$\cdots \to \varprojlim_n Z_2 \to \varprojlim_n Z_1 \to \varprojlim_n Z_0$$

is a tower of fibrations and for all $r$, $\pi_1 \varprojlim_n Z_r \cong \varprojlim_n G_n$ and $\pi_0 \varprojlim_n Z_r \cong \varprojlim_n{}^1 G_n$, by part 1). Hence

$$\pi_1(\varprojlim_r \varprojlim_n Z_{r,n}) \cong \varprojlim_n G_n$$

$$\pi_0(\varprojlim_r \varprojlim_n Z_{r,n}) \cong \varprojlim_n{}^1 G_n.$$

Now one takes limits in the other direction: for fixed $n$, $\{Z_{r,n}\}_{r\geq 0}$ is a tower of fibrations, so

$$\pi_1 \varprojlim_r Z_{r,n} \cong \varprojlim_r G_n^{(r)}$$

$$\pi_0 \varprojlim_r Z_{r,n} \cong \varprojlim_r{}^1 G_n^{(r)}$$

Now we calculate $\pi_* \varprojlim_n \varprojlim_r Z_{r,n}$ and use the fact that inverse limits commute to finish the proof. □

PROOF OF THE COMPLETE CONVERGENCE LEMMA 2.20: There are exact sequences

$$* \to E_r^{s,s+i} \to \pi_i X_s^{(r-1)} \to \pi_i X_{s-1}^{(r)} \to * \tag{2.24}$$

for $r > s$ and $i \geq 1$, so there is a comparison of exact sequences

$$\begin{tikzcd}
* \arrow[r] & e_\infty^{s,s+i} \arrow[r] \arrow[d, "\cong"'] & G_s\pi_i X \arrow[r] \arrow[d] & G_{s-1}\pi_i X \arrow[r] \arrow[d] & * \\
* \arrow[r] & E_\infty^{s,s+i} \arrow[r] & \varprojlim_r \pi_i X_s^{(r)} \arrow[r] & \varprojlim_r \pi_i X_{s-1}^{(r)} &
\end{tikzcd} \tag{2.25}$$

in which the bottom sequence arises by taking an inverse limit of the sequences (2.24). It follows from the definitions that there are isomorphisms

$$E_{r+1}^{0,i} \cong \pi_i X_0^{(r)}$$

for $r \geq 0$ and $i \geq 1$.

Suppose that $\varprojlim_r{}^1 E_r^{s,s+i} = *$ for $i \geq 1$. Then the map

$$\varprojlim_r \pi_i X_s^{(r)} \to \varprojlim_r \pi_i X_{s-1}^{(r)} \tag{2.26}$$

is surjective for $i, s \geq 1$. It follows that the composite map

$$\pi_i X \to \varprojlim_s \pi_i X_s \to \varprojlim_r \pi_i X_s^{(r)}$$

is surjective, so that the canonical inclusion

$$G_s\pi_i X \to \varprojlim_r \pi_i X_s^{(r)}$$

is a bijection, as is the canonical map

$$e_\infty^{s,s+i} \to E_\infty^{s,s+i}$$

on account of the comparison diagram (2.25).

The surjectivity of the maps (2.26) also implies that

$$\varprojlim_s{}^1 \varprojlim_r \pi_i X_s^{(r)} = *.$$

The isomorphism

$$\pi_i X_0^{(r)} \cong E_{r+1}^{0,i},$$

and the sequences (2.24) together inductively imply that

$$\varprojlim_r{}^1 \pi_i X_s^{(r)} = *.$$

It follows from Lemma 2.23 that

$$\varprojlim_s{}^1 \pi_i X_s = *$$

for $i \geq 1$.

For the converse, suppose that $i \geq 1$, and presume that

$$e_\infty^{s,s+i} \cong E_\infty^{s,s+i} \quad \text{and} \quad \varprojlim_s{}^1 \pi_i X_s = *$$

for $s \geq 0$ and $i \geq 1$. In particular, the map

$$G_0 \pi_i X \to \varprojlim_r \pi_i X_0^{(r)}$$

is an isomorphism, so inductively all canonical maps $G_s \pi_i X \to \varprojlim_r \pi_i X_s^{(r)}$ are isomorphisms, and the maps

$$\varprojlim_r \pi_i X_s^{(r)} \to \varprojlim_r \pi_i X_{s-1}^{(r)}$$

are surjective.

The exact sequence

$$\varprojlim_s{}^1 \pi_i X_s \to \varprojlim_s \varprojlim_r{}^1 \pi_i X_s^{(r)} \to *$$

from Lemma 2.23 and the assumption $\varprojlim_s{}^1 \pi_i X_s = *$ together force

$$\varprojlim_s \varprojlim_r{}^1 \pi_i X_s^{(r)} = *.$$

At the same time, the maps

$$\varprojlim_r{}^1 \pi_i X_s^{(r)} \to \varprojlim_r{}^1 \pi_i X_{s-1}^{(r)}$$

are surjective, so that $\varprojlim_r{}^1 \pi_i X_s^{(r)} = *$. It follows that $\varprojlim_r{}^1 E_r^{s,s+i} = *$. □

It is worth pointing out that even without convergence one has a comparison result.

PROPOSITION 2.27. *Suppose* $f : X = \{X_s\} \to Y = \{Y_s\}$ *is a map of towers of pointed fibrations and suppose there is a (finite)* $N \geq 1$ *so that*

$$f_* : E_N^{s,t} X \to E_N^{s,t} Y$$

*is an isomorphism for all* $(s,t)$*. If* $E_r^{s,s} X = *$ *for all* $s$*, then the map* $\varprojlim X_s \to \varprojlim Y_s$ *is a weak equivalence of connected spaces.*

PROOF: Note that $f_* : E_r^{s,t} X \to E_r^{s,t} Y$ is an isomorphism for all $r \geq N$ by Lemma 2.7. Then induction on $s$ and the exact sequence

$$* \to E_r^{s+1,t+1} \to \pi_{t-s} X_{s+1}^{(r-1)} \to \pi_{t-s} X_s^{(r)} \to *$$

(see (2.24)) implies that $\pi_k X_s^{(r)} \to \pi_k Y_s^{(r)}$ is an isomorphism for $r > s, N$ and $k \geq 1$. It follows that the induced maps

$$f_* : \varprojlim_r \pi_k X_s^{(r)} \to \varprojlim_r \pi_k Y_s^{(r)}$$

and

$$f_* : \varprojlim_r{}^1 \pi_k X_s^{(r)} \to \varprojlim_r{}^1 \pi_k Y_s^{(r)}$$

are isomorphisms for $k \geq 1$ and all $s$. Lemma 2.23 then implies that the induced maps

$$f_* : \varprojlim_s \pi_k X_s \to \varprojlim_s \pi_k Y_s$$

and

$$f_* : \varprojlim_s{}^1 \pi_k X_s \to \varprojlim{}^1 \pi_k Y_s$$

are isomorphisms for $k \geq 1$. It follows from the Milnor exact sequence (Proposition 2.15) that the induced map

$$f_* : \pi_k(\varprojlim_s X_s) \to \pi_k(\varprojlim_s Y_s)$$

is an isomorphism for $k \geq 1$.

The hypotheses on $E_r^{s,s}$ and the low end

$$\pi_1 X_{s-r}^{(r)} \to \pi_1 X_{s-(r+1)}^{(r)} \to E_{r+1}^{s,s} \to \pi_0 X_s^{(r)} \to \pi_0 X_{s-1}^{(r)}$$

of the exact sequence in Lemma 2.8 together imply $\pi_0 X_n^{(r)} = * = \pi_0 Y_n^{(r)}$ and $\pi_1 X_n^{(r)} \to \pi_1 X_{n-1}^{(r)}$ is onto. It follows from Proposition 2.15 that the spaces $\varprojlim_s X_s$ and $\varprojlim Y_s$ are path connected. □

## 3. Postnikov towers.

This section presents basic facts about Postnikov towers, including the construction of Moore. The material on $k$-invariants, which uses cohomology with twisted coefficients is presented in Section 4.

DEFINITION 3.1. *Let $X$ be a space. A Postnikov tower $\{X_n\}$ for $X$ is a tower of spaces*

$$\cdots \to X_2 \xrightarrow{q_1} X_1 \xrightarrow{q_0} X_0$$

*equipped with maps $i_n : X \to X_n$ so that $q_n i_n = i_{n-1} : X \to X_{n-1}$ and so that for all vertices $v \in X$, $\pi_i X_n = 0$ for $i > n$ and*

$$(i_n)_* : \pi_i X \xrightarrow{\cong} \pi_i X_n$$

*for $i \leq n$.*

The purpose of the next few sections is to construct such and to prove a uniqueness theorem.

REMARK 3.2. If $X$ is not connected we may write $X = \bigsqcup_\alpha X_\alpha$ as the disjoint union of its components. If $\{(X_\alpha)_n\}$ is a Postnikov tower for $X_\alpha$, then, by setting $X_n = \bigsqcup_\alpha (X_\alpha)_n$ we create a Postnikov tower for $X$. Conversely, if $\{X_n\}$ is a Postnikov tower for $X$ and $v \in X_\alpha \subset X$ is a vertex of $X_\alpha$, then if $(X_\alpha)_n \subseteq X_n$ is the component of $i_n(v)$, $\{(X_\alpha)_n\}$ is a Postnikov tower for $X_\alpha$. Hence without loss of generality, we may assume $X$ is connected.

REMARK 3.3. We also may assume that $X$ is fibrant. For if $X \to Y$ is a weak equivalence with $Y$ fibrant, then a Postnikov tower for $Y$ is a Postnikov tower for $X$.

We now give a specific model for the Postnikov tower, due to Moore. For this reason it is called the Moore-Postnikov tower. This construction is functorial in fibrant $X$.

DEFINITION 3.4. *Let $X$ be a fibrant simplicial set. Define, for each integer $n \geq 0$, an equivalence relation $\sim_n$ on the simplices of $X$ as follows: two $q$-simplices*

$$\alpha, \beta : \Delta^q \to X$$

*are equivalent if*

$$\alpha = \beta : \operatorname{sk}_n \Delta^q \to X;$$

*that is, the classifying maps $\alpha$ and $\beta$ agree on the $n$-skeleton. Define $X(n) = X/\sim_n$.*

Then there are evident maps $q_n : X(n) \to X(n-1)$ and $i_n : X \to X(n)$ yielding a map of towers $\iota : \{X\} \to \{X(n)\}$. The principal result of this section is

THEOREM 3.5. *The tower $\{X(n)\}$ is a Postnikov tower for $X$, and it is a tower of fibrations. Furthermore, the evident map*

$$X \to \varprojlim X(n)$$

*is an isomorphism.*

Each map $X \to X(n)$ is a surjective fibration, so $\{X(n)\}$ is a tower of fibrations (see Exercise V.3.8). Thus it suffices to prove

LEMMA 3.6. *For any choice of base point in $X$, $\pi_k X(n) = 0$ for $k > n$ and $(i_n)_* : \pi_k X \to \pi_k X(n)$ is an isomorphism for $k \leq n$.*

Recall from Section I.7 that an element in $\pi_k X$ can be represented by a pointed map

$$f : S^k = \Delta^k / \partial\Delta^k \to X$$

Two such maps yield the same element in $\pi_k X$ if they are related by a pointed homotopy

$$H : S^k \wedge \Delta^1_+ \to X.$$

PROOF OF LEMMA 3.6: First notice that $\pi_k X(n) = 0$ for $k > n$. This is because any representative of an element in $\pi_k X(n)$

$$f : S^k \to X(n)$$

has the property that the composite

$$\Delta^k \to S^k \xrightarrow{f} X(n)$$

is the constant map, by the definition of $\sim_n$. Next, let $E(n) \subseteq X$ be the fibre of the projection $X \to X(n)$ at some vertex $v$. Then $E(n)$ consists of those simplices $f : \Delta^q \to X$ so that

$$f : \mathrm{sk}_n \Delta^q \to X$$

is constant. In particular $E(n)$ is fibrant and $E(n)_k = \{v\}$ for $k \leq n$, where $v$ is the chosen base point. Hence $\pi_k E(n) = 0$ for $k \leq n$. The result now follows from the long exact sequence of the fibration

$$E(n) \to X \to X(n). \qquad \square$$

The complex $E(n) \subseteq X$ of this proof is called, by Moore, the $n$-th Eilenberg subcomplex of $X$. It depends on the choice of base point. It is worth pointing out that $X(0)$ has contractible components; however, it is not necessarily true that these components have a single vertex. Indeed $X(0)_0 = X_0$, so that, for example, $X(0)$ is a one-point space if and only if $X_0 = \{v\}$.

COROLLARY 3.7. *Let $X$ be fibrant and connected, and $v \in X$ a vertex. Let $v \in X(n)$ be the image of $v$ under $X \to X(n)$ and $K(n)$ the fibre at $v$ of the projection $q_n : X(n) \to X(n-1)$. Then there is a weak equivalence*

$$K(n) \to K(\pi_n X, n).$$

PROOF: See Corollary III.3.8. This follows from the long exact sequence of the fibration sequence

$$K(n) \to X(n) \to X(n-1). \qquad \square$$

In particular, since $X$ is assumed to be connected,

$$K(1) \to X(1)$$

is a weak equivalence, so there is a weak equivalence

$$X(1) \simeq K(\pi_1 X, 1) = B\pi_1 X.$$

The following is also worth noting:

PROPOSITION 3.8. *Suppose $X$ is connected, fibrant, and minimal. Then $q_n : X(n) \to X(n-1)$ is a minimal fibration and $X(0) = \{v\}$ where $v \in X$ is the unique vertex. Furthermore $K(n)$ is a minimal complex and there are isomorphisms, $n \geq 1$,*

$$K(n) \cong K(\pi_n X, n)$$

*and*

$$K(1) \cong B\pi_1 X.$$

Next, let us remark that there is a relative version of the Moore construction.

DEFINITION 3.9. *Let $f : X \to B$ be a morphism of simplicial sets. Then a Postnikov tower for $f$ is a tower of space $\{X_n\}$ equipped with a map*

$$i : \{X\} \to \{X_n\}$$

*from the constant tower and a map*

$$p : \{X_n\} \to \{B\}$$

*to the constant tower so that*

1) *for all $n$, the composite $X \xrightarrow{i_n} X_n \xrightarrow{p_n} B$ is $f$.*
2) *for any choice of vertex of $X$, the map $(i_n)_* : \pi_k X \to \pi_k X_n$ is an isomorphism for $k \leq n$;*
3) *for any choice of vertex of $X$, the map $(p_n)_* : \pi_k X_n \to \pi_k B$ is an isomorphism for $k > n+1$.*
4) *for any choice of vertex $v$ of $X$, there is an exact sequence*

$$0 \to \pi_{n+1} X_n \to \pi_{n+1} B \xrightarrow{\partial} \pi_n F_v$$

*where $F_v$ is the homotopy fibre of $f$ at $v$ and $\partial$ is the connecting homomorphism in the long exact sequence of the homotopy fibration $F_v \to X \to B$.*

DEFINITION 3.10. *Suppose $f : X \to B$ is a fibration and $B$ is fibrant. Define an equivalence relation $\sim_n$ as simplices*

$$\alpha, \beta : \Delta^q \to X$$

*by saying $\alpha \sim_n \beta$ if and only if*

1) $f\alpha = f\beta$ *and*
2) $\alpha = \beta : \mathrm{sk}_n \Delta^q \to X$.

Then there are maps $i_n : X \to X(n)$ and $p_n : X(n) \to B$ and the appropriate generalization of Theorem 3.5 is:

THEOREM 3.11. *Suppose that $X$ is a fibrant simplicial set. Then the tower $\{X(n)\}$ is a Postnikov tower for $f$, and it is a tower of fibrations. Furthermore, the evident map*

$$X \to \varprojlim X(n)$$

*is an isomorphism.*

PROOF: Again we need only check the statements about homotopy groups. Choose a vertex in $X$ and let $F(n)$ be the fibre of the fibration $X(n) \to B$. Then a moment's thought shows the notation $F(n)$ is unambiguous — $F(n)$ really is the $n$-th stage in the Moore-Postnikov tower of the fibre $F$ of $f : X \to B$. The result thus follows from Lemma 3.6 and the long-exact sequence of the fibration $F(n) \to X(n) \to B$. □

Note that we have used

LEMMA 3.12. *Let $f : X \to B$ be a fibration with $B$ fibrant and fibre $F$, for some vertex of $B$. Then the fibre of $p_n : X(n) \to B$ is the $n$-th stage $F(n)$ in the Moore-Postnikov tower for $F$.*

COROLLARY 3.13. *If $F$ is connected $p_0 : X(0) \to B$ is a weak equivalence.*

## 4. Local coefficients and equivariant cohomology.

The purpose of this section is to demonstrate that cohomology with local coefficients is representable in an appropriate homotopy category, and to identify the homotopy type of the representing object. We then develop a calculation scheme in terms of equivariant cohomology on the universal cover.

Let $X$ be a fibrant space (meaning fibrant simplicial set) and $\Gamma = \pi_f X$ its fundamental groupoid (see Sections I.8, III.1). There is a canonical simplicial map $\phi : X \to B\Gamma$ which associates to the simplex $\tau : \Delta^n \to X$ the string

$$\tau(0) \to \tau(1) \to \cdots \to \tau(n)$$

in $\Gamma$ arising from the image of the string

$$0 \to 1 \to \cdots \to n$$

in the 1-skeleton of $\Delta^n$.

We shall, more generally, consider the category of spaces $\mathbf{S} \downarrow B\Gamma$ over $B\Gamma$. The objects of this category are the simplicial set maps $Y \to B\Gamma$, and the morphisms are commutative diagrams of simplicial set maps

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & B\Gamma & \end{tikzcd}$$

The following is an exercise in formal homotopical algebra:

Lemma 4.1. *Suppose that $X$ is an object of a closed model category $\mathcal{C}$. Then the category $\mathcal{C} \downarrow X$ has a closed model structure, in which a morphism*

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

*is a weak equivalence, fibration, or cofibration if the same is true for the map $f : Z \to Y$ of $\mathcal{C}$.*

In particular, the category $\mathbf{S} \downarrow B\Gamma$ inherits a closed model structure from the category of simplicial sets. Furthermore, the fibrant objects in $\mathbf{S} \downarrow B\Gamma$ are fibrations $Y \to B\Gamma$. At this level of generality, the following result gives us an good selection of fibrations and fibrant objects (compare with Lemma IV.5.7):

Lemma 4.2. *Suppose that $p : Z \to Y$ is a natural transformation of functors $\Gamma \to \mathbf{S}$ such that each map $p : Z_v \to Y_v$ is a fibration. Then the induced map*

$$p_* : \underset{\overrightarrow{\Gamma}}{\text{holim}}\, Z \to \underset{\overrightarrow{\Gamma}}{\text{holim}}\, Y$$

*is a fibration. If all fibrations $p_v$ are minimal then $p_*$ is minimal.*

Proof: Consider a commutative solid arrow diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d] & \underset{\overrightarrow{\Gamma}}{\text{holim}}\, Z \arrow[d, "p_*"] \\ \Delta^n \arrow[r, "\beta"'] \arrow[ur, dashed] & \underset{\overrightarrow{\Gamma}}{\text{holim}}\, Y. \end{tikzcd} \tag{4.3}$$

We need to show that the dotted arrow exists, making the diagram commute.

The pullback of the canonical map $\pi : \underrightarrow{\mathrm{holim}}_{\Gamma}\, Y \to B\Gamma$ along the composite

$$\Delta^n \xrightarrow{\beta} \underrightarrow{\mathrm{holim}}_{\Gamma}\, Y \xrightarrow{\pi} B\Gamma$$

is the homotopy colimit of the functor $Y\pi(\beta) : \mathbf{n} \to \mathbf{S}$. The category $\Gamma$ is a groupoid, so that there is a natural isomorphism of functors

$$\begin{tikzcd}
Y(\pi(\beta)(0)) \arrow[r, "1"] \arrow[d, "\cong"'] & Y(\pi(\beta)(0)) \arrow[r, "1"] \arrow[d, "\cong"'] & \dots \arrow[r, "1"] & Y(\pi(\beta)(0)) \arrow[d, "\cong"] \\
Y(\pi(\beta)(0)) \arrow[r] & Y(\pi(\beta)(1)) \arrow[r] & \dots \arrow[r] & Y(\pi(\beta)(n))
\end{tikzcd}$$

on the category $\mathbf{n}$. It follows that the induced map $\Delta^n \times_{B\Gamma} \underrightarrow{\mathrm{holim}}_{\Gamma}\, Y \to \Delta^n$ is isomorphic over $\Delta^n$ to $\Delta^n \times Y(\pi(\beta)(0)) \to \Delta^n$. Similarly, the map

$$\Delta^n \times_{B\Gamma} \underrightarrow{\mathrm{holim}}_{\Gamma}\, Z \to \Delta^n$$

is isomorphic over $\Delta^n$ to the projection $\Delta^n \times Z(\pi(\beta)(0)) \to \Delta^n$. The original diagram (4.3) therefore factors (uniquely) through a diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\alpha_*"] \arrow[d] & \Delta^n \times Z(\pi(\beta)(0)) \arrow[d, "1 \times p"] \\
\Delta^n \arrow[r, "\beta_*"'] \arrow[ur, dashed] & \Delta^n \times Y(\pi(\beta)(0))
\end{tikzcd}$$

The map $1 \times p$ is a fibration since $p$ is a fibration by assumption, so the dotted arrow exists. Observe further that if all fibrations $p : Z_v \to Y_v$ are minimal, then $1 \times p$ is minimal, and so $p_*$ is minimal. □

COROLLARY 4.4. *Suppose that $K : \Gamma \to \mathbf{S}$ is a functor such that $K_v$ is fibrant for each $v$, then the canonical map $\pi : \underrightarrow{\mathrm{holim}}_{\Gamma}\, K \to B\Gamma$ is a fibration.*

A *local coefficient system* is a functor $A : \Gamma \to \mathbf{Ab}$. We shall write $A_x$ for $A(x)$. Note there is nothing special about abelian groups. If $\mathcal{C}$ is a category, a local system of objects in $\mathcal{C}$ is simply a functor $\Gamma \to \mathcal{C}$. In particular, the functors of Lemma 4.2 are local systems of simplicial sets.

The cohomology of $X$ with local coefficients in $A$ can be defined via cochains. The group $C^n_\Gamma(X, A)$ of $n$-cochains with coefficients in $A$ is defined to be the collection of commutative diagrams

$$\begin{tikzcd}
X_n \arrow[rr, "\alpha"] \arrow[dr, "\phi"'] & & \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, A_n \arrow[dl] \\
& B\Gamma_n &
\end{tikzcd}$$

or rather $C^n_\Gamma(X, A)$ is the collection of all assignments $x \mapsto \alpha(x) \in A_{\phi(x)(0)}$, $x \in X_n$.

All calculations in such groups of cochains depend (initially) on the following observation:

LEMMA 4.5. *Suppose that $\psi : Y \to B\Gamma$ is a second space over $B\Gamma$, $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map, and that $f : Y_n \to X_m$ is a function such that the diagram*

$$\begin{tikzcd}
Y_n \arrow[r, "f"] \arrow[d, "\psi"'] & X_m \arrow[d, "\phi"] \\
B\Gamma_n \arrow[r, "\theta^*"'] & B\Gamma_m
\end{tikzcd}$$

*commutes. Then, given any cochain $\alpha : X_m \to \underrightarrow{\operatorname{holim}}_{\Gamma} A_m$, there is a unique cochain $f^*(\alpha) : Y_n \to \underrightarrow{\operatorname{holim}}_{\Gamma} A_n$ such that the diagram*

$$\begin{tikzcd}
Y_n \arrow[r, "f^*(\alpha)"] \arrow[d, "f"'] & \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, A_n \arrow[d, "\theta^*"] \\
X_m \arrow[r, "\alpha"'] & \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, A_m
\end{tikzcd}$$

*commutes.*

PROOF: Recall that the simplicial structure map $\theta^* : \underrightarrow{\operatorname{holim}}_{\Gamma} A_n \to \underrightarrow{\operatorname{holim}}_{\Gamma} A_m$ is defined on the component $A_{\sigma(0)}$ of

$$\underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, A_n = \bigsqcup_{\sigma(0) \to \sigma(1) \to \cdots \to \sigma(n)} A_{\sigma(0)}$$

corresponding to the simplex $\sigma : \mathbf{n} \to \Gamma$ by the composite

$$A_{\sigma(0)} \xrightarrow{\sigma(0,\theta(0))_*} A_{\sigma(\theta(0))} \xrightarrow{in_{\sigma\theta}} \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, A_m,$$

where $\sigma(0,\theta(0))$ denotes the composite of the morphisms

$$\sigma(0) \to \sigma(1) \to \cdots \to \sigma(\theta(0))$$

in $\Gamma$. For $y \in Y_n$, $f^*(\alpha)(y)$ is therefore the unique element in $A_{\psi(y)(0)}$ which maps to $\alpha(f(y)) \in A_{\phi(f(y))(0)} = A_{\theta^*\psi(y)(0)} = A_{\psi(y)\theta(0)}$ under the isomorphism

$$A_{\psi(y)(0)} \xrightarrow[\cong]{\psi(y)(0,\theta(0))} A_{\psi(y)\theta(0)}. \qquad \square$$

Lemma 4.5 gives a cosimplicial object $C^*_\Gamma(X,A)$ — write $H^*_\Gamma(X,A)$ for the corresponding cohomology groups.

Suppose that $y$ is an object of the groupoid $\Gamma$. Recall that the groupoid $\Gamma \downarrow y$ has for objects all morphisms $\tau : x \to y$, and has morphisms given by commutative diagrams

$$\begin{tikzcd}
x \arrow[rr, "\alpha"] \arrow[dr, "\tau"'] & & x' \arrow[dl, "\tau'"] \\
& y &
\end{tikzcd}$$

in $\Gamma$. As usual, write $B(\Gamma \downarrow y)$ for the nerve of the category $\Gamma \downarrow y$. The functor $\Gamma \downarrow y \to \Gamma$ which forgets $y$ induces a simplicial set map $\pi_y : B(\Gamma \downarrow y) \to B\Gamma$, whereas composition with $\beta : y \to z$ induces a functor $\beta_* : \Gamma \downarrow y \to \Gamma \downarrow z$ which commutes with forgetful functors. It follows that there is a functor $\Gamma \to \mathbf{S}$ defined by $y \mapsto B(\Gamma \downarrow y)$ which is fibred over $B\Gamma$. Note that each of the categories $\Gamma \downarrow y$ has a terminal object, namely the identity $1_y : y \to y$, so that all spaces $B(\Gamma \downarrow y)$ are contractible. Observe finally that the maps $\pi_y : B(\Gamma \downarrow y) \to B\Gamma$ are Kan fibrations.

Suppose that $\phi : X \to B\Gamma$ is a simplicial set map, and define a collection of spaces $\tilde{X}_y$ by forming pullbacks

$$\begin{tikzcd}
\tilde{X}_y \arrow[r] \arrow[d] & B(\Gamma \downarrow y) \arrow[d, "\pi_y"] \\
X \arrow[r, "\phi"'] & B\Gamma
\end{tikzcd}$$

In this way, we define a functor $\tilde{X} : \Gamma \to \mathbf{S}$. We shall call $\tilde{X}$ the *covering system* for $\phi$; the construction specializes to a $\Gamma$-diagram of covering spaces for $X$ in the case where $\Gamma$ is the fundamental groupoid of $X$ and $\phi : X \to B\Gamma$ is the canonical map.

Write $\mathbf{S}_\Gamma$ for the category of diagrams $X : \Gamma \to \mathbf{S}$ taking values in simplicial sets. There is plainly a functor $\underrightarrow{\operatorname{holim}}_\Gamma : \mathbf{S}_\Gamma \to \mathbf{S} \downarrow B\Gamma$ defined by taking homotopy colimits. The covering system construction $X \mapsto \tilde{X}$ defines a functor $\mathbf{S} \downarrow B\Gamma \to \mathbf{S}_\Gamma$. These two functors are adjoint:

LEMMA 4.6. *There is a natural bijection*

$$\hom_{\mathbf{S}_\Gamma}(\tilde{X}, Y) \cong \hom_{\mathbf{S}\downarrow B\Gamma}(X, \underrightarrow{\operatorname{holim}}_{\Gamma} Y).$$

PROOF: For each object $y \in \Gamma$, the simplicial set $\tilde{X}_y$ has $n$-simplices $(\tilde{X}_y)_n$ consisting of all pairs $(x, \alpha)$, where $x \in X_n$ and $\alpha : \phi(x)(0) \to y$ is a morphism of $\Gamma$. The simplicial structure map $\theta^* : (\tilde{X}_y)_n \to (\tilde{X}_y)_m$ associated to $\theta : \mathbf{m} \to \mathbf{n}$ is defined by

$$\theta^*(x, \alpha) = (\theta^*(x), \alpha\phi(x)(0, \theta(0))^{-1}).$$

There is a natural bijection

$$\hom_{\mathbf{Sets}_\Gamma}(\tilde{X}_n, A) \cong \hom_{\mathbf{Sets}\downarrow B\Gamma_n}(X_n, \underrightarrow{\operatorname{holim}}_{\Gamma} A_n) \tag{4.7}$$

in each degree $n$. In effect, if $A : \Gamma \to \mathbf{Sets}$ is an arbitrary set valued functor, then a natural transformation

$$g_y : (\tilde{X}_y)_n \to A_y \ \ y \in \Gamma,$$

is completely determined by its effect on the elements $(x, 1_{\phi(x)(0)})$, $x \in X_n$, and $g(x, 1_{\phi(x)(0)}) \in A_{\phi(x)(0)}$, so the assignment $x \mapsto g(x, 1_{\phi(x)(0)})$ determines an element of $\hom_{\mathbf{Sets}\downarrow B\Gamma_n}(X_n, \underrightarrow{\operatorname{holim}}_{\Gamma} A_n)$.

One checks to see that the isomorphisms (4.7) assemble to give the desired adjunction on the simplicial set level. □

If $Y : \Gamma \to \mathbf{S}$ is a $\Gamma$-diagram of simplicial sets and $A : \Gamma \to \mathbf{Ab}$ is a local coefficient system, there is an obvious way to form a cochain complex (aka. cosimplicial abelian group) $\hom_\Gamma(Y, A)$, having $n$-cochains given by the group $\hom_\Gamma(Y_n, A)$ of all natural transformations from $Y_n$ to $A$.

COROLLARY 4.8. *There is a natural isomorphism of cosimplicial abelian groups*

$$C^*_\Gamma(X, A) \cong \hom_\Gamma(\tilde{X}, A)$$

*for all spaces $X$ over $B\Gamma$ and all local coefficient systems $A$.*

PROOF: Use the calculation in the proof of Lemma 4.5 to show that the natural bijections (4.7) from the proof of Lemma 4.6 preserve cosimplicial structure. □

The category $\mathbf{S}_\Gamma$ has a simplicial model structure for which a map $f : Z \to W$ is a weak equivalence (respectively fibration) if and only if each of the components $f : Z_v \to W_v$, $v \in \Gamma$, is a weak equivalence (respectively fibration) of simplicial sets. This is the Bousfield-Kan structure for $\Gamma$-diagrams — see Example II.6.9. The homotopy colimit construction takes weak equivalences of $\mathbf{S}_\Gamma$ to weak equivalences of $\mathbf{S} \downarrow B\Gamma$ (Proposition IV.1.7), whereas the covering system functor $\mathbf{S} \downarrow B\Gamma \to \mathbf{S}_\Gamma$ preserves weak equivalences by the coglueing lemma (Lemma II.8.13). There are therefore induced functors

$$\mathrm{Ho}(\mathbf{S}_\Gamma) \leftrightarrows \mathrm{Ho}(\mathbf{S} \downarrow B\Gamma)$$

which are adjoint, and so the natural isomorphism of Lemma 4.6 extends to a natural bijection

$$[\tilde{X}, Y]_{\mathbf{S}_\Gamma} \cong [X, \underset{\Gamma}{\underrightarrow{\mathrm{holim}}}\, Y]_{\mathbf{S}\downarrow B\Gamma}$$

on the homotopy category level.

To go further, we need to know the following:

LEMMA 4.9. *For any object $\phi : X \to B\Gamma$, the corresponding covering system $\tilde{X}$ is a cofibrant object of* $\mathbf{S}_\Gamma$.

PROOF: It is enough to suppose that the groupoid $\Gamma$ is connected, and pick an object $v \in \Gamma$. Write $\pi = \hom_\Gamma(v, v)$, and pick a morphism $\eta_x : x \to v$ in $\Gamma$ for each object $x$, such that $\eta_v = 1_v$.

There is an inclusion functor $i : \pi \to \Gamma$ which identifies the group $\pi$ with the group of automorphisms of $v$ in $\Gamma$. There is also a functor $r : \Gamma \to \pi$ which is defined by sending $\omega : x \to y$ to the composite $\eta_y \omega \eta_x^{-1} : v \to v$, and the functors $r$ and $i$ are the component functors of an equivalence of the (connected) groupoid $\Gamma$ with the group $\pi$: $r \cdot i = 1_\pi$, and the isomorphisms $\eta_x : x \to v$ determine a natural isomorphism $1_\Gamma \cong i \cdot r$.

The functors $i$ and $r$ induce functors $i^* : \mathbf{S}_\Gamma \to \mathbf{S}_\pi$ and $r^* : \mathbf{S}_\pi \to \mathbf{S}_\Gamma$, by precomposition. The composite $i^* \cdot r^*$ is the identity functor on $\mathbf{S}_\pi$, while the isomorphisms $\eta_{x*} : X_v \to X_x$ define a natural isomorphism $r^* \cdot i^*(X) \cong X$, so that $\mathbf{S}_\Gamma$ and $\mathbf{S}_\pi$ are equivalent categories.

The functor $i^*$ reflects cofibrations, so it suffices to show that $\tilde{X}_v$ is a cofibrant $\pi$-space. To see this, observe that the functor $r$ induces a commutative diagram

$$\begin{tikzcd}
E\Gamma_v \arrow[r] \arrow[d] & E\pi \arrow[d] \\
B\Gamma \arrow[r] & B\pi,
\end{tikzcd}$$

and this diagram is a pullback. But then there is a $\pi$-equivariant isomorphism $(E\Gamma_v)_n \cong \pi \times B\Gamma_n$; in other words, $E\Gamma_v$ is a free $\pi$-set in each degree, so it must be a cofibrant $\pi$-space, by Corollary V.2.11. □

We can now relate elements in the cohomology groups $H^n_\Gamma(X, A)$ arising from fibred spaces $X \to B\Gamma$ and local coefficient systems $A : \Gamma \to \mathbf{Ab}$ to morphisms in some homotopy category. The method is to use a $\Gamma$-equivariant version of the Dold-Kan correspondence (Section III.2), in conjunction with Lemma 4.9.

First of all, recall from Corollary 4.8 that $H^n_\Gamma(X, A)$ is the $n^{th}$ cohomology group of the cochain complex $\hom_\Gamma(\tilde{X}, A)$ which is defined in degree $n$ by the set $\hom_\Gamma(\tilde{X}_n, A)$ of natural transformations $\tilde{X}_n \to A$. It follows that $H^n_\Gamma(X, A)$ is canonically isomorphic to the group $\pi(\mathbb{Z}(\tilde{X}), A[n])$ of chain homotopy classes of morphisms of $\Gamma$-diagrams of chain complexes from the $\Gamma$-diagram of Moore chains $\mathbb{Z}(\tilde{X})$ to $A[n]$, where $A[n]$ is the $\Gamma$-diagram of chain complexes which consists of a copy of $A$ concentrated in degree $n$. The normalized chain complex $N\mathbb{Z}(\tilde{X})$ is naturally chain homotopy equivalent to $\mathbb{Z}(\tilde{X})$, so that there is an isomorphism

$$\pi(\mathbb{Z}(\tilde{X}), A[n]) \xrightarrow[\cong]{i^*} \pi(N\mathbb{Z}(\tilde{X}), A[n])$$

which is induced by the inclusion $i : N\mathbb{Z}(\tilde{X}) \hookrightarrow \mathbb{Z}(\tilde{X})$. But now there are isomorphisms

$$\begin{aligned} \pi(N\mathbb{Z}(\tilde{X}), A[n]) &\cong \pi(\mathbb{Z}(\tilde{X}), K(A, n)) \\ &\cong \pi(\tilde{X}, K(A, n)) \\ &\cong [\tilde{X}, K(A, n)]_\Gamma, \end{aligned}$$

where the first isomorphism is induced by the Dold-Kan correspondence, the second isomorphism relates naive homotopy classes in $\Gamma$-diagrams of simplicial abelian groups to naive homotopy classes in $\Gamma$-diagrams of simplicial sets, and the last isomorphism relating naive homotopy classes to morphisms in the homotopy category of $\Gamma$-spaces results from the fact that $\tilde{X}$ is cofibrant and the Eilenberg-Mac Lane object $K(A, n)$ is fibrant. Subject to chasing an explicit cochain through these identifications (which is left to the reader), we have proved:

THEOREM 4.10. *Suppose that $X \to B\Gamma$ is a space over $B\Gamma$, and let $A : \Gamma \to$ **Ab** be a local coefficient system. Then there is an isomorphism*

$$[\tilde{X}, K(A, n)]_\Gamma \xrightarrow{\cong} H^n_\Gamma(X, A),$$

*which is defined by sending a class represented by a map $f : \tilde{X} \to K(A, n)$ to the class represented by the cocycle $f_n : X_n \to \underrightarrow{\operatorname{holim}}_\Gamma A_n$.*

In the connected case, the equivalence of categories $i^* : \mathbf{S}_\Gamma \leftrightarrow \mathbf{S}_\pi : r^*$ induces an equivalence of associated homotopy categories, giving an isomorphism

$$[\tilde{X}, K(A, n)]_\Gamma \xrightarrow[\cong]{i^*} [\tilde{X}_v, K(A_v, n)]_\pi,$$

while the identifications leading to Theorem 4.10 are preserved by $r^*$ as well as being valid in the category of $\pi$-diagrams. This leads to an identification of $H^*_\Gamma(X, A)$ with ordinary $\pi$-equivariant cohomology:

THEOREM 4.11. *Suppose that $X$ and $A$ are as in the statement of Theorem 4.10, suppose that $\Gamma$ is a connected groupoid and let $\pi$ be the group* $\hom_{\Gamma}(v,v)$ *of an object $v \in \Gamma$. Then restriction to $\pi$-spaces determines a commutative diagram of isomorphisms*

$$\begin{tikzcd}
{[\tilde{X}, K(A,n)]_{\Gamma}} \arrow[r, "\cong"] \arrow[d, "i^*"', "\cong"] & H^n_{\Gamma}(X,A) \arrow[d, "\cong"', "i^*"] \\
{[\tilde{X}_v, K(A_v,n)]_{\pi}} \arrow[r, "\cong"'] & H^n_{\pi}(\tilde{X}_v, A_v).
\end{tikzcd}$$

Observe, from the proof of Lemma 4.9, that $\tilde{X}_v$ is a copy of the universal cover of $X$ if the underlying map $\phi : X \to B\Gamma$ is the canonical map arising from the fundamental groupoid $\Gamma$ of $X$.

REMARK 4.12. Suppose that $\Gamma$ is a connected groupoid, and let $F : \Gamma \to \mathbf{S}$ be a functor taking values in simplicial sets. Let $v$ be an object of $\Gamma$, and let $\pi = \hom_{\Gamma}(v,v)$ denote the group of automorphisms of $v$ in $\Gamma$. Let $F_v$ denote the composite functor

$$\pi \subset \Gamma \xrightarrow{F} \mathbf{S}.$$

Then the induced map

$$\underset{\overrightarrow{\pi}}{\text{holim}}\, F_v \to \underset{\overrightarrow{\Gamma}}{\text{holim}}\, F$$

is a weak equivalence. This follows from the observation that the translation categories $E_{\pi}(F_v)_n$ and $E_{\Gamma}F_n$ for the set-valued functors on the $n$-simplex level are homotopy equivalent for each $n$ (see Example IV.1.8 and Proposition IV.1.7). Finally, the homotopy colimit $\underset{\overrightarrow{\pi}}{\text{holim}}\, F_v$ is a copy of the Borel construction $E\pi \times_{\pi} F_v$ for the $\pi$-space $F_v$ (Example IV.1.10).

There is a notion of reduced cohomology with local coefficients. Suppose $\Gamma = \pi_f X$ is the fundamental groupoid and the map $X \to B\Gamma$ has a section $s : B\Gamma \to X$. Such a section is a map from the terminal object to $X$ in $\mathbf{S} \downarrow B\Gamma$, which is the natural notion of base point. For example, we could take

$$X = \underset{\overrightarrow{\Gamma}}{\text{holim}}\, K(A,n)$$

and let $s(\sigma) = (0,\sigma)$. A *pointed map* in $\mathbf{S} \downarrow \Gamma$

$$X \to \underset{\overrightarrow{\Gamma}}{\text{holim}}\, K(A,n)$$

is a map commuting with sections.

In gross generality, if $\mathcal{C}$ is a simplicial model category, then the category $\mathcal{C}_*$ of pointed objects in $\mathcal{C}$ (meaning morphisms $t \to X$, where $t$ is terminal) satisfies the axioms for a closed model category, where a map of $\mathcal{C}_*$ is a fibration, cofibration or weak equivalence if and only if it is so as a map of $\mathcal{C}$ (compare Lemma 4.1).

A reduced cochain $\varphi \in C^n_\Gamma(X,A)$ is a cochain so that $\varphi(s(x)) = 0$ for all $x \in B\Gamma$; if $\widetilde{C}^n_\Gamma(X,A)$ is the group of such there is a split short exact sequence of cosimplicial abelian groups

$$0 \to \widetilde{C}^*_\Gamma(X,A) \to C^*_\Gamma(X,A) \xrightarrow{\sigma^*} C^*_\Gamma(B\Gamma,A) \to 0$$

and a split short exact sequence of graded groups

$$0 \to \widetilde{H}^*_\Gamma(X,A) \to H^*_\Gamma(X,A) \to H^*_\Gamma(B\Gamma,A) \to 0.$$

Of course, if $X$ is connected, then for any vertex $v \in X$

$$H^*_\Gamma(B\Gamma,A) \cong H^*(\pi_1(X,v),A_v),$$

by Theorem 4.11, so that the groups $H^*_\Gamma(B\Gamma,A)$ for the groupoid $\Gamma$ coincide up to isomorphism with traditional group cohomology groups in the connected case.

The following is left as an exercise for anyone who has read the proof of Theorem 4.10. Let $\mathbf{S} \downarrow B\Gamma_*$ denote the category of pointed objects over $B\Gamma$.

LEMMA 4.13. *For $X \in \mathbf{S} \downarrow B\Gamma_*$ there are natural isomorphisms*

$$[X, \underset{\overrightarrow{\Gamma}}{\operatorname{holim}}\, K(A,n)]_{\mathbf{S}\downarrow B\Gamma_*} \cong \widetilde{H}^n_\Gamma(X;A).$$

Now suppose $X \subseteq Y$ — that is, $X$ is a subspace of $Y$ — and $\Gamma = \pi_f X \cong \pi_f Y$: This would happen for example, if $X$ and $Y$ have the same 2-skeleton. Define $B\Gamma \cup_X Y$ by the push-out diagram in $\mathbf{S} \downarrow B\Gamma$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ B\Gamma \arrow[r] & B\Gamma \cup_X Y. \end{tikzcd}$$

Then $B\Gamma \cup_X Y$ is an object of $\mathbf{S} \downarrow B\Gamma_*$ and we define

$$H^*_\Gamma(Y,X;A) = \widetilde{H}^*_\Gamma(B\Gamma \cup_X Y;A).$$

For a different point of view, observe that the covering system construction preserves cofibrations and pushouts, and the covering system $\widetilde{B\Gamma}$ for $B\Gamma$

consists of contractible spaces, so that the covering system for $B\Gamma \cup_X Y$ is equivalent to the homotopy cofibre $\tilde{Y}/\tilde{X}$ of the inclusion $\tilde{X} \hookrightarrow \tilde{Y}$ in the category of $\Gamma$-diagrams. It follows that there is an isomorphism

$$H^n_\Gamma(Y, X; A) \cong [\tilde{Y}/\tilde{X}, K(A,n)]_{\Gamma *}.$$

PROPOSITION 4.14. *There is a long exact sequence*

$$\cdots \to H^n_\Gamma(Y, X; A) \to H^n_\Gamma(Y; A) \to H^n_\Gamma(X; A) \to H^{n+1}_\Gamma(Y, X; A) \to \cdots$$

PROOF: One easily checks

$$0 \to \widetilde{C}^*_\Gamma(B\Gamma \cup_X Y; A) \to C^*_\Gamma(Y; A) \to C^*_\Gamma(X; A) \to 0$$

is exact.

Alternatively, one could use Theorem 4.10, Corollary 4.13, and the Puppe sequence. For this, let $\Omega_{\mathcal{C}}$ denote the "loop space" functor in any simplicial model category $\mathcal{C}$. Then Lemma 4.2 implies that there is a natural weak equivalence

$$\Omega_{\mathbf{S}\downarrow B\Gamma}(\underset{\overrightarrow{\Gamma}}{\operatorname{holim}}\, K(A,n)) \simeq \underset{\overrightarrow{\Gamma}}{\operatorname{holim}}\, K(A, n-1). \qquad \square$$

We shall now write down some variants of a spectral sequence (Proposition 4.17) which computes equivariant cohomology.

Fix a discrete group $\pi$, and observe that, if $A$ is a $\mathbb{Z}\pi$ module then for all $n \geq 0$, $K(A,n)$ is a fibrant object in the model structure for $\mathbf{S}_\pi$ of Section V.1.

Let $X \in \mathbf{S}_\pi$ be cofibrant, and suppose that $n \geq 0$. We have already seen that there is a natural isomorphism

$$H^n_\pi(X, A) \cong [X, K(A,n)]_\pi$$

in the proof of Theorem 4.11. If $X$ is not cofibrant we make the following:

DEFINITION 4.15. *For arbitrary $\pi$-spaces $X$, define $H^n_\pi(X, A)$ by setting*

$$H^n_\pi(X, A) = [X, K(A,n)]_\pi.$$

The moral is that if $X$ is cofibrant, then $H^n_\pi(X, A)$ has a cochain description; otherwise, take a weak equivalence $Y \to X$ and $Y$ cofibrant in $\mathbf{S}_\pi$, so that

$$[X, K(A,n)]_\pi \cong [Y, K(A,n)]_\pi \cong H^n_\pi(Y, A).$$

While less important for our applications, this also tells one how to define homology.

DEFINITION 4.16. *Let $A$ be a $\mathbb{Z}\pi$ module, $X \in \mathbf{S}_\pi$ and $Y \to X$ a weak equivalence of $\pi$-spaces with $Y$ cofibrant. Then define*

$$H_n^\pi(X, A) = \pi_n(\mathbb{Z}Y \otimes_{\mathbb{Z}\pi} A).$$

The standard model category arguments (cf. Section II.1) show that $H_*^\pi(X, A)$ is independent of the choice of $Y$.

For calculational purposes one has the following result. If $X \in \mathbf{S}_\pi$, then $H_*X = H_*(X, \mathbb{Z})$ is a graded $\mathbb{Z}\pi$ module.

PROPOSITION 4.17. *There is a first quadrant cohomology spectral sequence*

$$\mathrm{Ext}^p_{\mathbb{Z}\pi}(H_qX, A) \Rightarrow H_\pi^{p+q}(X, A).$$

PROOF: We may assume $X$ is cofibrant and, indeed, that $X_n$ is a free $\pi$-set for all $n$, by Lemma V.2.4. Let $P_\bullet(\cdot) = \{\cdots \to P_1(\cdot) \to P_0(\cdot) \to (\cdot)\}$ denote a functorial projective resolution — for example the bar resolution — of $\mathbb{Z}\pi$ modules. We require each $P_s(\cdot)$ to be exact. Form the double complex

$$\hom_{\mathbb{Z}\pi}(P_\bullet(\mathbb{Z}X), A) = \{\hom_{\mathbb{Z}\pi}(P_p(\mathbb{Z}X_q), A\}.$$

Filtering by degree in $p$ we get a spectral sequence with

$$E_1^{p,q} = \hom_{\mathbb{Z}\pi}(P_p(H_qX), A)$$

so that $E_2$ is as required. To determine what the spectral sequence abuts to, filter by degree in $p$, whence

$$E_1^{p,q} = \mathrm{Ext}^p_{\mathbb{Z}\pi}(\mathbb{Z}X_q, A).$$

But $\mathbb{Z}X_q$ is a free $\mathbb{Z}\pi$ module, so $E_1^{p,q} = 0$ if $p > 0$ and

$$E_1^{0,q} = \hom_{\mathbb{Z}\pi}(\mathbb{Z}X_q, A).$$

Hence the spectral sequence abuts to $E_2^{0,*} = H_\pi^*(X, A)$, as required. □

Proposition 4.17 is a special case of a result that holds in great generality, in the context of homotopy theories of simplicial sheaves and presheaves [45], [46]. We shall confine ourselves here to displaying a few of its close relatives.

An argument similar to the one given in Proposition 4.17 yields a first quadrant homology spectral sequence

$$\mathrm{Tor}_p^{\mathbb{Z}\pi}(H_qX, A) \Rightarrow H^\pi_{p+q}(X, A). \tag{4.18}$$

There are further refinements. For example, if $A$ is an $\mathbb{F}\pi$ module where $\mathbb{F}$ is a field one has

$$\mathrm{Ext}^p_{\mathbb{F}\pi}(H_q(X; \mathbb{F}), A) \Rightarrow H_\pi^{p+q}(X; A). \tag{4.19}$$

EXAMPLE 4.20. Let $X = *$ be a point. Then $H_*X = \mathbb{Z}$ concentrated in degree 0 and

$$H^p_\pi(*; A) \cong \mathrm{Ext}^p_{\mathbb{Z}\pi}(\mathbb{Z}, A) = H^q(\pi, A).$$

Note that unlike ordinary homology, $H^*_\pi(*; A)$ need not split off of $H^*_\pi(X; A)$. The reader is invited to calculate $H^*_{\mathbb{Z}/2\mathbb{Z}}(S^n, \mathbb{Z})$ where $S^n$ has the "antipodal point" action and $\mathbb{Z}$ the trivial action. Use the next observation.

EXAMPLE 4.21. If $A$ is a trivial $\pi$ module and $X$ is a cofibrant $\pi$-space, there is an isomorphism

$$H^n_\pi(X, A) \cong H^n(X/\pi, A).$$

This follows from the definition.

There is a relative version of this cohomology. In fact, let $\mathbf{S}_{\pi *}$ denote pointed $\pi$-spaces. For all $\mathbb{Z}\pi$ modules $A$, $K(A, n)$ has an evident base point.

DEFINITION 4.22. *For $X \in \mathbf{S}_{\pi *}$, let*

$$\widetilde{H}^n_\pi(X; A) = [X, K(A, n)]_{\mathbf{S}_{\pi *}}.$$

Hence one factors $* \to X$ as $* \xrightarrow{i} Y \to X$ where $i$ is a cofibration in $\mathbf{S}_\pi$ and

$$\widetilde{H}^n_\pi(X; A) = \pi_0 \, \mathbf{Hom}_{\mathbf{S}_{\pi *}}(Y, K(A, n)).$$

PROPOSITION 4.23.

1) *For $X \in \mathbf{S}_{\pi *}$ there is a split natural short exact sequence*

$$0 \to \widetilde{H}^*_\pi(X; A) \to H^*_\pi(X; A) \to H^*(\pi; A) \to 0.$$

2) *There is a first quadrant cohomology spectral sequence*

$$\operatorname{Ext}^p_{\mathbb{Z}\pi}(\widetilde{H}_q X, A) \Rightarrow \widetilde{H}^{p+q}_\pi(X; A).$$

We leave the proof of this, and of the homology analogs, as an exercise.

DEFINITION 4.24. *Let $X \to Y$ be an inclusion in $\mathbf{S}_\pi$. Define relative cohomology by*

$$H^n_\pi(Y, X; A) = \widetilde{H}^n_\pi(Y/X; A).$$

PROPOSITION 4.25. *There is a natural long exact sequence*

$$\cdots \to H^n_\pi(Y, X; A) \to H^n_\pi(Y; A) \to H^n_\pi(X; A) \to H^{n+1}_\pi(Y, X; A) \to \cdots.$$

PROOF: Consider the diagram in $\mathbf{S}_\pi$

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "\simeq"'] & V \arrow[d, "\simeq"] \\ X \arrow[r] & Y \end{tikzcd}$$

where $U \to X$ is a weak equivalence with $U$ cofibrant, $U \xrightarrow{i} V \to Y$ factors $U \to X \to Y$ as a cofibration followed by a weak equivalence. Then $* \to V/U$ is a cofibration and $\widetilde{H}_*(Y/X) \cong H_*(V/U)$, so

$$\widetilde{H}^n_\pi(Y/X; A) \cong \widetilde{H}^n_\pi(V/U, A).$$

Now use the chain description of the various cohomology groups. □

## 5. On $k$-invariants.

This section is devoted to $k$-invariants — their definition and properties. We will end with a discussion of the uniqueness of Postnikov towers.

Let $X$ be a connected fibrant space and $X = \{X_n\}_{n\geq 0}$ a Postnikov tower for $X$. Let $\pi_f X_{n-1}$ be the fundamental groupoid for $X_{n-1}$ and $\pi_n$ the local coefficient system on $X_{n-1}$ obtained as follows: factor $X_n \to X_{n-1}$ as

$$X_n \xrightarrow{i} Y \xrightarrow{q} X_{n-1}$$

with $i$ a weak equivalence and $q$ a fibration. Then if $v \in X_{n-1}$ a vertex, let $F_v = q^{-1}(v)$ and

$$\pi_n(v) = \pi_n(F_v).$$

If $\pi_f X_n = \pi_f X_{n-1}$, as in the case of the Moore-Postnikov tower, there is obviously no ambiguity in $\pi_n(v)$, since the fibre has a single vertex. In general, however, the fibre is simply connected, and for any two choice of vertices $x$ and $y$ in $F_v$ there is a canonical isomorphism $\pi_n(F_v, x) \cong \pi_n(F_v, y)$, so we identify the two groups via this isomorphism and obtain $\pi_n(F_v)$.

Let $p : L(\pi, n+1) \to K(\pi, n+1)$ be a fixed functorial fibration with $L(\pi, n+1)$ contractible. Setting $L(\pi, n+1) = WK(\pi, n)$ would certainly suffice — this has the advantage that the corresponding fibration $p : L(\pi, n+1) \to K(\pi, n+1)$ is minimal (Lemma III.2.21).

The cohomology class $k_n$ in the statement of the following result is known as the $n^{th}$ $k$-invariant.

PROPOSITION 5.1. *Suppose that $X$ is a connected fibrant simplicial set. There is a cohomology class $k_n \in H^{n+1}_{\Gamma}(X_{n-1}, \pi_n)$ and a homotopy pullback diagram over $B\Gamma$*

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\Gamma} L(\pi_n, n+1) \arrow[d] \\
X_{n-1} \arrow[r, "k_n"'] & \underrightarrow{\operatorname{holim}}_{\Gamma} K(\pi_n, n+1).
\end{tikzcd}$$

This is a consequence of the more general result Theorem 5.9 below. For now, we state some consequences.

COROLLARY 5.2. *Suppose $X$ is connected and pointed, with base point $v$. Let $\pi_1 = \pi_1(X, v)$. Then there is a homotopy pullback diagram of over $B\pi_1$*

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\pi_1} L(\pi_n(v), n+1) \arrow[d] \\
X_{n-1} \arrow[r, "k_n"'] & \underrightarrow{\operatorname{holim}}_{\pi_1} K(\pi_n(v), n+1).
\end{tikzcd}$$

PROOF: This result is equivalent to Proposition 5.1, since there is a homotopy cartesian square

$$\begin{tikzcd}
\underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, L(\pi_n, n+1) \arrow[r, "\simeq"] \arrow[d] & \underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, L(\pi_n(v), n+1) \arrow[d] \\
\underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n, n+1) \arrow[r, "\simeq"'] & \underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n(v), n+1).
\end{tikzcd}$$ □

COROLLARY 5.3. *Suppose that $X$ is simple and pointed, with base point $v$. Then there is a homotopy pullback diagram*

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & L(\pi_n(v), n+1) \arrow[d] \\
X_{n-1} \arrow[r] & K(\pi_n(v), n+1).
\end{tikzcd}$$

PROOF: Again, let $\pi_1 = \pi_1(X, v)$ and consider the homotopy pullback diagram that appears in Corollary 5.2. Then

$$\underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n(v), n+1) = K(\pi_n(v), n+1) \times B\pi_1$$

as spaces over $B\pi_1$ and similarly for $\underrightarrow{\operatorname{holim}}_{\pi_1} L(\pi_n(v))$, and the diagram

$$\begin{tikzcd}
L(\pi_n(v), n+1) \times B\pi_1 \arrow[r, "pr_L"] \arrow[d, "p \times 1"'] & L(\pi_n(v), n+1) \arrow[d, "p"] \\
K(\pi_n(v), n+1) \times B\pi_1 \arrow[r, "pr_L"'] & K(\pi_n(v), n+1)
\end{tikzcd}$$

is homotopy cartesian. □

To prove Proposition 5.1, we return to essentials. Let $f : Y \to X$ be a map of spaces over $B\Gamma$, where $\Gamma$ is the fundamental groupoid of $Y$ and $Y$ is connected. If we factor $f$ as

$$Y \xrightarrow{j} Z \xrightarrow{q} X$$

where $j$ is a trivial cofibration and $q$ is a fibration, then for each vertex $v$ of $Y$ we get a relative homotopy group

$$\pi_n(f, v) = \pi_{n-1}(F_v, v),$$

where $F_v$ is the fibre of $q$ over $v$, or in other words the homotopy fibre of $Y \to X$. This gives a local system $\pi_n(f)$ over $B\Gamma$ and, if $\Gamma$ is equivalent to the fundamental groupoid of $X$, there is a long exact sequence of local systems

$$\cdots \to \pi_{n+1}(f) \to \pi_n(Y) \xrightarrow{f_*} \pi_n(X) \to \pi_n(f) \to \cdots .$$

LEMMA 5.4. *Suppose $f : Y \to X$ is a morphism of spaces over $B\Gamma$, where $\Gamma$ is the fundamental groupoid of $Y$ and $Y$ is connected. Suppose $n \geq 2$ and for all choices of base point of $Y$, the map $f_* : \pi_k(Y, v) \to \pi_k(X, fv)$ is an isomorphism for $k < n$ and a surjection for $k = n$. Then for any local system of abelian groups $A$ over $B\Gamma$, there is an isomorphism*

$$f^* : H^k_\Gamma(X, A) \xrightarrow{\cong} H^k_\Gamma(Y, A), \qquad k < n$$

*and an exact sequence*

$$\begin{aligned} 0 \to H^n_\Gamma(X, A) \xrightarrow{f^*} H^n_\Gamma(Y, A) &\to \hom_\Gamma(\pi_{n+1}(f), A) \\ &\xrightarrow{d} H^{n+1}_\Gamma(X, A) \xrightarrow{f^*} H^{n+1}_\Gamma(Y, A). \end{aligned}$$

*The sequence is natural in maps $f$ satisfying the hypotheses.*

PROOF: The hypotheses on homotopy groups in degree 0 and degree 1 imply that $\Gamma$ is equivalent to the fundamental groupoid on $X$. Hence, the long exact sequence of local systems above exists, and the groups $H^*_\Gamma(X, A)$ are defined.

We may assume that $f : Y \to X$ is a cofibration over $B\Gamma$ and use the long exact sequence of 4.14:

$$\cdots \to H^k_\Gamma(X, A) \to H^k_\Gamma(Y, A) \to H^{k+1}_\Gamma(X, Y, A) \to H^{k+1}_\Gamma(X, A) \to \cdots .$$

Note that the results of Section 4 imply

$$H^k_\Gamma(Y, X, A) \cong \tilde{H}^k_\Gamma(B\Gamma \cup_Y X, A) \cong \tilde{H}^k_\pi(Z, A_v)$$

where $Z$ is the universal cover of $B\Gamma \cup_Y X$. The latter group may be computed using Proposition 4.23.2, and one gets $\tilde{H}^k_\pi(Z, A_v) = 0$ for $k \leq n$ and

$$\begin{aligned} \tilde{H}^{n+1}_\pi(Z, A_v) &\cong \hom_{\mathbb{Z}\pi}(\tilde{H}_{n+1} Z, A_v) \\ &\cong \hom_{\mathbb{Z}\pi}(\pi_{n+1}(f, v)), A_v) \\ &\cong \hom_\Gamma(\pi_{n+1}(f), A), \end{aligned}$$

by appropriate use of the relative Hurewicz theorem (Theorem III.3.12). □

In particular, in Lemma 5.4, one can set $A = \pi_{n+1}(f)$ and define the *$k$-invariant* of $f$ to be

$$k(f) = d(1_{\pi_{n+1}(f)}) \in H^{n+1}_\Gamma(X, \pi_{n+1}(f)). \tag{5.5}$$

REMARK 5.6. Given the identifications that have been made, the $k$-invariant $k(f)$ is also represented by the composite

$$\tilde{X} \to \tilde{X}/\tilde{Y} \to \tilde{X}/\tilde{Y}(n+1) \simeq K(\pi_{n+1}(\tilde{X}/\tilde{Y}), n+1) \cong K(\pi_{n+1}(f), n+1)$$

in the homotopy category of $\Gamma$-diagrams. The object $\tilde{X}/\tilde{Y}(n+1)$ is the $(n+1)^{st}$ Postnikov section of a fibrant model of $\tilde{X}/\tilde{Y}$ — this is sensible, because the Postnikov section construction is functorial. The relative Hurewicz isomorphism is also functorial, so that there is an isomorphism $\pi_{n+1}(\tilde{X}/\tilde{Y}) \cong \pi_{n+1}(f)$ of local coefficient systems.

One possible way to build maps $f : Y \to X$ satisfying the hypotheses of Lemma 5.4 is to kill a cohomology class by the following method. Let $X$ be a connected space with fundamental groupoid $\Gamma$ and let $A$ be a local coefficient system. Fix $x \in H^{n+1}_{\Gamma}(X, A)$ with $n \geq 2$ and form the pullback diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"'] & \underrightarrow{\mathrm{holim}}_{\Gamma} L(A, n+1) \arrow[d] \\ X \arrow[r, "\theta"'] & \underrightarrow{\mathrm{holim}}_{\Gamma} K(A, n+1) \end{tikzcd} \tag{5.7}$$

where $\theta$ represents $x$. By construction, $\pi_{n+1}(p) \cong A$ and we have the following result.

LEMMA 5.8. *There is an equality* $k(p) = x \in H^{n+1}_{\Gamma}(X, A)$.

PROOF: By naturality, we may take $\theta$ to the identity, so that $p$ is the projection

$$q : \underrightarrow{\mathrm{holim}}_{\Gamma} L(A, n+1) \to \underrightarrow{\mathrm{holim}}_{\Gamma} K(A, n+1)$$

and $x$ is the universal class. The zero section $s : B\Gamma \to \underrightarrow{\mathrm{holim}}_{\Gamma} L(A, n+1)$ is a weak equivalence, and we may calculate using the composite

$$q \cdot s : B\Gamma \to \underrightarrow{\mathrm{holim}}_{\Gamma} K(A, n+1).$$

Then $B\Gamma \cup_{B\Gamma} \underrightarrow{\mathrm{holim}}_{\Gamma} K(A, n+1)$ has as universal cover

$$Z = \underrightarrow{\mathrm{holim}}_{\Gamma} K(A, n+1) \simeq K(A_v, n+1).$$

Thus

$$\tilde{H}^{n+1}_{\pi}(Z, A_v) \cong \hom(A_v, A_v) \cong \hom_{\Gamma}(A, A)$$

and

$$d : \hom_{\Gamma}(A, A) \to H^{n+1}_{\Gamma}(\underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n+1), A)$$

is the standard isomorphism taking $1_A$ to the universal class. □

We point out that in the diagram (5.7) the object $\underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n)$ is a group object over $B\Gamma$ and it acts on the space $Y$ over $B\Gamma$; that is, there is a map

$$\mu : \underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n) \times_{B\Gamma} Y \to Y$$

over $B\Gamma$ satisfying the usual associativity and unital conditions. Notice further that this action descends to the trivial action on $X$ in the sense that the following diagram commutes

$$\begin{tikzcd}
\underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n) \times_{B\Gamma} Y \arrow[r, "\mu"] \arrow[d, "p_2"'] & Y \arrow[d, "p"] \\
X \arrow[r, "="'] & X
\end{tikzcd}$$

where $p_2(a, b) = p(b)$.

Now suppose we are given $f : Y \to X$ satisfying the hypotheses of Lemma 5.4. Then we may form the $k$-invariant $k(f)$ as in (5.5), and the pullback diagram with $A = \pi_{n+1}(f)$,

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "p"'] & \underrightarrow{\operatorname{holim}}_{\Gamma} L(A, n+1) \arrow[d] \\
X \arrow[r, "k(f)"'] & \underrightarrow{\operatorname{holim}}_{\Gamma} K(A, n+1).
\end{tikzcd}$$

Notice that we have confused the cohomology class $k(f)$ with a map representing it. This abuse of notation pervades the literature, and is suggestive: the homotopy type of Z does not depend on the choice of a representative for $k(f)$.

THEOREM 5.9. *The map $f : Y \to X$ lifts to a map $g : Y \to Z$ inducing an isomorphism*

$$g_* : \pi_{n+1}(f) \to \pi_{n+1}(p).$$

PROOF: Since $f^* k(f)$ is in the image of the composite

$$\hom_{\Gamma}(A, A) \xrightarrow{d} H^{n+1}_{\Gamma}(X, A) \xrightarrow{f^*} H^{n+1}_{\Gamma}(Y, A)$$

the composite $k(f) \cdot f$ is null-homotopic, and hence lifts to $\underrightarrow{\operatorname{holim}}_{\Gamma} L(A, n+1)$. Choose a lifting, and let $h : Y \to Z$ be the induced map. We wish to modify $h$, if necessary, to a homotopy equivalence.

By the naturality clause of Lemma 5.4, there is a diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^n_\Gamma(X,A) \arrow[r] \arrow[d, "="'] & H^n_\Gamma(Z,A) \arrow[r] \arrow[d, "h^*"'] & \hom_\Gamma(A,A) \arrow[r, "d_Z"] \arrow[d, "h^\sharp"'] & H^{n+1}_\Gamma(X,A) \arrow[d, "="] \\
0 \arrow[r] & H^n_\Gamma(X,A) \arrow[r] & H^n_\Gamma(Y,A) \arrow[r] & \hom_\Gamma(A,A) \arrow[r, "d_Y"'] & H^{n+1}_\Gamma(X,A).
\end{tikzcd}$$

Note that $d_Y(1_A) = k(f)$ by definition and $d_Z(1_A) = k(f)$ by Lemma 4.5. Furthermore $h^\sharp(1_A) = h_*$ where we write $h_*$ for the composite

$$A = \pi_{n+1}(f) \to \pi_{n+1}(p) \cong A.$$

Since $d_Y(1_A - h_*) = k(f) - k(f) = 0$, there is a class $x \in H^n_\Gamma(Y, A)$ mapping to $1_A - h_*$. Let $g : Y \to Z$ be the composite

$$Y \xrightarrow{x \times h} \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, K(A,n) \times_{B\Gamma} Z \xrightarrow{\mu} Z.$$

Then the map $g^\sharp : \hom_\Gamma(A,A) \to \hom_\Gamma(A,A)$ takes $1_A$ to $1_A$, so the induced map $\pi_{n+1}(f) \to \pi_{n+1}(p)$ must be an isomorphism. □

REMARKS 5.10.

1) Note that the map $g : Y \to Z$ of Theorem 5.9 is not unique, but may be modified by any element of $H^n(X, A)$.

2) The proof of Theorem 5.9 can also be carried out entirely within the Γ-diagram category. Writing the argument that way is a good exercise.

Now let $X$ be space and let $\{X_n\}$ be any Postnikov tower for $X$. Let $q_n : X_n \to X_{n-1}$ be the projection. Then $\pi_k(q_n) = 0$ for $k \neq n+1$ and $\pi_{n+1}(q_n) = \pi_n X$. Thus Proposition 5.1 follows immediately from Theorem 5.9.

More generally, let $f : X \to B$ be any map and $\{X_n\}$ a Postnikov tower for $f$ (see Definition 3.9). Again, let $q_n : X_n \to X_{n-1}$ be the projection. Then $\pi_k(q_n) = 0$ for $k \neq n+1$ and

$$\pi_{n+1}(q_n) = \pi_n(F)$$

by the relative Hurewicz theorem, and one obtains a homotopy pullback square

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, L(\pi_n F, n+1) \arrow[d] \\
X_{n-1} \arrow[r, "k_n"'] & \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n F, n+1).
\end{tikzcd}$$

We now examine the consequences for Postnikov towers. Let $X$ be a fibrant connected space and $\{X_n\}$ a Postnikov tower for $X$. Let $\pi_n = \pi_n X$ regarded as a local coefficient system as $X_{n-1}$. Then there is a homotopy cartesian square, with $\Gamma = \pi_f X_{n-1}$

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, "q_n"'] & \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, L(\pi_n, n+1) \arrow[d, "p_*"] \\
X_{n-1} \arrow[r, "k_n"'] & \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n, n+1)
\end{tikzcd} \tag{5.11}$$

and so $X_n$ is weakly equivalent to the pullback. More is true:

Corollary 5.12. *If the tower map $q_n : X_n \to X_{n-1}$ is a minimal fibration then the diagram (5.11) is a pullback.*

Proof: The map

$$p_* : \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, L(\pi_n, n+1) \to \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n, n+1)$$

is a minimal fibration (Lemma 4.2), and so the induced map

$$\begin{tikzcd}
X_n \arrow[r, "k_{n*}"] \arrow[dr, "q_n"'] & X_{n-1} \times_{\underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n, n+1)} \underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, L(\pi_n, n+1) \arrow[d] \\
& X_{n-1}
\end{tikzcd}$$

is a weak equivalence (and hence a homotopy equivalence) of minimal fibrations over $X_{n-1}$. Lemma I.10.4 implies that the map $k_{n*}$ is an isomorphism. □

This applies, for example, to the Moore-Postnikov tower $\{X(n)\}$. In that case $\Gamma = \pi_f X = \pi_f X_n$ for all $n$. If $X$ is minimal, then $q_n : X(n) \to X(n-1)$ is a minimal fibration.

Corollary 5.13. *Let $X$ be a connected fibrant minimal space. Then for all $n \geq 2$ there is a pullback diagram*

$$\begin{tikzcd}
X(n) \arrow[r] \arrow[d, "q_n"'] & \underset{\pi}{\underrightarrow{\operatorname{holim}}}\, L(\pi_n, n+1) \arrow[d] \\
X(n-1) \arrow[r, "k_n"'] & \underset{\pi}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n, n+1)
\end{tikzcd}$$

*where $\pi = \pi_1 X$. If the fundamental group $\pi_1(X)$ acts trivially on $\pi_n(X)$, then the map $q_n : X(n) \to X(n-1)$ is a principal $K(\pi_n, n)$-fibration.*

PROOF: If $X$ is minimal and connected it has a single vertex, so

$$\underset{\Gamma}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n, n+1) = \underset{\pi}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n, n+1),$$

etc. If $\pi_1$ acts trivially on $\pi_n$, then

$$p_* : \underset{\pi}{\underrightarrow{\operatorname{holim}}}\, L(\pi_n, n+1) \to \underset{\pi}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n, n+1)$$

can be identified up to isomorphism with the map

$$p \times 1 : L(\pi_n, n+1) \times B\pi \to K(\pi_n, n+1)$$

as in the proof of Corollary 5.3, and $p \times 1$ is a principal $K(\pi_n, n)$-fibration. □

THEOREM 5.14. *Let $X$ be a connected fibrant space. Then any two Postnikov towers for $X$ are weakly equivalent as towers under $X$.*

PROOF: Choose a minimal subcomplex $X_0 \subseteq X$ which is a weak equivalence and let $v \in X_0$ be the vertex. Choose a retraction $g : X \to X_0$. Then the Moore-Postnikov tower $\{X_0(n)\}$ is a Postnikov tower under $X$. We will show that any Postnikov tower $\{X_n\}$ under $X$ is weakly equivalent (under $X$) to $\{X_0(n)\}$.

Inductively define tower maps $f_n : X_n \to X_0(n)$ as follows. Let $\pi = \pi_1(X, v)$. Then for $n = 1$ choose a weak equivalence $X_1 \to X_0(1) = B\pi$. Suppose $X_k \to X_0(k)$ have been defined, and are compatible and weak equivalences for $k < n$. Then Theorem 5.9 implies that there is a diagram

$$\begin{tikzcd}
 & X_0(n) \arrow[r] \arrow[d] & \underset{\pi}{\underrightarrow{\operatorname{holim}}}\, L(\pi_n, n+1) \arrow[d] \\
X_n \arrow[ur, "f_n"] \arrow[r] & X_0(n-1) \arrow[r] & \underset{\pi}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n, n+1)
\end{tikzcd}$$

where $\pi_n = \pi_n(X, v)$, and $f_n$ is a weak equivalence. Thus we have a weak equivalence of towers $f : \{X_n\} \to \{X_0(n)\}$. To modify this is into a weak equivalence of towers under $X$, consider the induced map

$$f : \underleftarrow{\operatorname{holim}}\, X_n \to \underleftarrow{\lim}\, X_0(n) = X_0.$$

The canonical map $X \to \underleftarrow{\operatorname{holim}}\, X_n$ is a weak equivalence, and the induced map $\overline{f} : X \to X_0$ is a weak equivalence. There is an isomorphism $\theta : X_0 \to X_0$ so that

$$X \xrightarrow{\overline{f}} X_0 \xrightarrow{\theta} X_0$$

is $g$. Since the Moore-Postnikov tower is natural we get an induced isomorphism of towers $\theta : \{X_0(n)\} \to \{X_0(n)\}$ and the composite

$$\theta \cdot f : \{X_n\} \to \{X_0(n)\}$$

is a weak equivalence of towers under $X$. □

## 6. Nilpotent spaces.

We now describe how to refine a Postnikov tower for a nilpotent space. A group $G$ is nilpotent if the lower central series eventually stabilizes at the trivial group. Thus, if we define $F_nG \subseteq G$ by $F_0G = G$ and $F_nG = [F_{n-1}G, G]$, we are asking that there be an integer $k$ so that $F_kG = \{e\}$. If $G$ is a group, a $G$-module $M$ is nilpotent if there is finite filtration of $M$ by $G$-modules

$$0 = F_kM \subseteq F_{k-1}M \subseteq \cdots F_1M \subseteq M$$

so that $G$ acts trivially on the successive quotients. A simplicial set $X$ is *nilpotent* if $X$ is connected, $\pi_1X$ is a nilpotent group and $\pi_nX$ is nilpotent $\pi_1X$ module for $n \geq 2$.

PROPOSITION 6.1. *Let $X$ be a nilpotent space and $\{X_n\}$ a Postnikov tower for $X$. Each of the maps $q_n : X_n \to X_{n-1}$ can be refined to a finite composition*

$$X_n = Y_k \to Y_{k-1} \to \cdots Y_1 \to Y_0 = X_{n-1}$$

*so that each of the maps $Y_i \to Y_{i-1}$ fits into a homotopy pullback square*

$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d] & L(A_i, n+1) \arrow[d] \\
Y_{i-1} \arrow[r] & K(A_i, n+1)
\end{tikzcd}$$

*for some abelian group $A_i$*

PROOF: We begin with the case $n = 1$. Then $X_1 \to X_0$ is weakly equivalent to $B\pi_1 \to *$ where $\pi_1 = \pi_1X$. Let $\{F_i\}$ be the lower central series of $\pi_1X$ and let $G_i = \pi_1/F_i$. Let $k$ be an integer so that $F_k = \{e\}$. Then there is a tower

$$B\pi_1 = BG_k \to BG_{k-1} \to \cdots BG_1 \to *.$$

Since

$$\{e\} \to F_{i-1}/F_i \to G_i \to G_{i-1} \to \{e\}$$

is a central extension, there is homotopy pullback diagram, with $A_i = F_{i-1}/F_i$,

$$\begin{tikzcd}
BG_i \arrow[r] \arrow[d] & L(A_i, 2) \arrow[d] \\
BG_{i-1} \arrow[r] & K(A_i, 2).
\end{tikzcd}$$

In effect, the map $BG_{i-1} \to K(A_i, 2)$ classifies the principal $BA_i$-fibration $BG_i \to BG_{i-1}$.

Next we assume that $n \geq 2$. Let $\{F_i\}$ be a filtration of $\pi_nX$ by $\pi_1$ modules so that each of the successive quotients is a trivial $\pi_1$ module. We define $Y_i$ by the homotopy pullback diagram

$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d] & \underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, K(F_i, n+1) \arrow[d] \\
X_{n-1} \arrow[r] & \underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, K(\pi_n X, n+1)
\end{tikzcd}$$

where the bottom map is the $k$-invariant. Since there is a homotopy pullback diagram

$$\begin{tikzcd}
\underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, K(F_i, n+1) \arrow[r, "0_*"] \arrow[d] & \underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, L(F_{i-1}/F_i, n+1) \arrow[d] \\
\underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, K(F_{i-1}, n+1) \arrow[r] & \underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, K(F_{i-1}/F_i, n+1).
\end{tikzcd}$$

there is a homotopy pullback diagram

$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d] & \underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, L(F_{i-1}/F_i, n+1) \arrow[d] \\
Y_{i-1} \arrow[r] & \underset{\pi_1}{\underrightarrow{\operatorname{holim}}}\, K(F_{i-1}/F_i, n+1).
\end{tikzcd}$$

Since $\pi_1$ acts trivially on $F_{i-1}/F_i$, one can finish the proof by arguing as in Proposition 5.3. □

The previous result has the following consequence.

COROLLARY 6.2. *Let $X$ be a nilpotent space. Then there is a tower of fibrations $\{Z_j\}$ so that $X \simeq \varprojlim Z_j$ and each of the maps $Z_j \to Z_{j-1}$ fits into a homotopy pullback square*

$$\begin{tikzcd}
Z_j \arrow[r] \arrow[d] & L(A_j, n_j+1) \arrow[d] \\
Z_{j-1} \arrow[r] & K(A_j, n_j+1)
\end{tikzcd}$$

*where each $A_j$ is an abelian group and $\{n_j\}$ is a non-decreasing sequence of positive integers so that $\varinjlim n_j = \infty$*

Further variations on this result are possible. For example, one can remove the hypothesis that $X$ be connected, or consider nilpotent fibrations. It is also possible to construct this refined Postnikov tower by a method similar to the construction of the Moore-Postnikov tower. See Bousfield and Kan [14].

# Chapter VII Reedy model categories

This chapter contains an exposition of the Bousfield-Kan model structure on the category $c\mathbf{S}$ of cosimplicial objects in simplicial sets, also known as cosimplicial spaces. It appears here as the dual of a Reedy model category structure on the category of simplicial objects $s\mathcal{C}$ in a suitable closed model category $\mathcal{C}$. Another standard example of a Reedy structure on a simplicial object category is the Reedy structure on the category of bisimplicial sets, or simplicial objects in simplicial sets — see Section IV.3.

Much of the chapter concerns the general Reedy theory. We preface this development in Section 1 with a discussion of skeleta in a very general context. The main results are Proposition 1.9 and Corollary 1.14; they discuss to what extent a simplicial object in a category $\mathcal{C}$ with enough colimits can be built by "attaching cells". One application is a characterization of cofibrations in the kind of model category considered in Section II.4. See Example 1.15.

In general, if a category $\mathcal{C}$ is a closed model category, then the Reedy structure of the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ has as weak equivalences those morphisms $X \to Y$ in $s\mathcal{C}$ for which each $X_n \to Y_n$ is a weak equivalence for all $n \geq 0$. One of the main auxiliary results is that there is a geometric realization functor

$$|\cdot| : s\mathcal{C} \longrightarrow \mathcal{C}$$

which preserves weak equivalences between Reedy cofibrant objects — see Proposition 3.6. The Reedy model category structure is discussed in detail in Section 2, and the geometric realization functor is the subject of Section 3.

This theory is specialized in Section 4 to the case of cosimplicial spaces $c\mathbf{S}$; that is, to the case of the opposite category to the category of simplicial objects in $\mathbf{S}^{op}$. The resulting model category structure on $c\mathbf{S}$ is the standard one discussed by Bousfield and Kan. In particular, we show in Proposition 4.18 that the cofibrations $A \to B$ defined by the Reedy structure are exactly those maps which are monomorphisms in all levels and induce isomorphisms $H^0 A \cong H^0 B$ on maximal augmentations.

The material in Section 4, along with that appearing in Section VI.2, is the basis for the construction of the homotopy spectral sequence for a cosimplicial space which is given in Chapter VIII.

## 1. Decomposition of simplicial objects.

Let $\mathcal{C}$ be a category with all limits and colimits. The purpose of this section is to analyze how simplicial objects are constructed out of smaller components. We will use this inductive argument in later sections.

We begin with skeleta. The category $s\mathcal{C}$ is the functor category $\mathcal{C}^{\mathbf{\Delta}^{\text{op}}}$. Let $i_n : \mathbf{\Delta}_n \subseteq \mathbf{\Delta}$ be the inclusion of the full subcategory with objects $\mathbf{k}$, $k \leq n$, and let $s_n\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}_n^{\text{op}}}$. There is a restriction function $i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$ which simply forgets the $k$-simplices, $k > n$. This restriction functor has a left adjoint given by

$$(i_n^*X)_m = \varinjlim_{\mathbf{m}\to\mathbf{k}} X_k = \varinjlim_{\mathbf{m}\downarrow\mathbf{\Delta}_n} X_k \tag{1.1}$$

and the colimit is over morphisms $\mathbf{m} \to \mathbf{k}$ in $\mathbf{\Delta}$ with $k \leq n$. This is an example of a left Kan extension. Every morphism $\mathbf{m} \to \mathbf{k}$ in $\mathbf{\Delta}$ can be factored uniquely as

$$\mathbf{m} \xrightarrow{\phi} \mathbf{k}' \xrightarrow{\psi} \mathbf{k}$$

where $\phi$ is a surjection and $\psi$ is one-to-one, so the surjections $\mathbf{m} \to \mathbf{k}$, $k \leq n$, can be used to define the colimit of (1.1) and we get

$$(i_n^*X)_m \cong \varinjlim_{\mathbf{m}\downarrow\mathbf{\Delta}_n^+} X_k \tag{1.2}$$

where $\mathbf{\Delta}_+ \subseteq \mathbf{\Delta}$ is the subcategory with the same objects but only surjections as morphisms, and $\mathbf{\Delta}_n^+ = \mathbf{\Delta}_n \cap \mathbf{\Delta}_+$.

If $X \in s\mathcal{C}$ we define the $n^{\text{th}}$ *skeleton* of $X$ by the formula

$$\operatorname{sk}_n X = i_n^* i_{n*} X. \tag{1.3}$$

There are natural maps $\operatorname{sk}_m X \to \operatorname{sk}_n X$, $m \leq n$, and $\operatorname{sk}_n X \to X$. The morphism $1_{\mathbf{m}}$ is an initial object in the category $\mathbf{m} \downarrow \mathbf{\Delta}_n$ if $m \leq n$, so there is an isomorphism $(\operatorname{sk}_n X)_m \cong X_m$ in that range. It follows that there is a natural isomorphism

$$\varinjlim_n \operatorname{sk}_n X \xrightarrow{\cong} X.$$

Here is an example. Because $\mathcal{C}$ has limits and colimits, $s\mathcal{C}$ has a canonical structure as a simplicial category. (See Section II.2). In particular if $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, then

$$(X \otimes K)_n = \bigsqcup_{K_n} X_n.$$

It is now a straightforward exercise to prove

PROPOSITION 1.4. *If $X \in s\mathcal{C}$ is constant and $K \in \mathbf{S}$, then there are natural isomorphisms*

$$\operatorname{sk}_n X \cong X \text{ and } X \otimes \operatorname{sk}_n K \cong \operatorname{sk}_n(X \otimes K).$$

To explain how $\operatorname{sk}_n X$ is built from $\operatorname{sk}_{n-1} X$ we define the $n^{\text{th}}$ *latching object* $L_nX$ of $X$ by the formula

$$\begin{aligned} L_nX &= (\operatorname{sk}_{n-1} X)_n \\ &\cong \varinjlim_{\mathbf{n}\downarrow\mathbf{\Delta}_{n-1}^+} X_k. \end{aligned} \tag{1.5}$$

If $Z \in \mathcal{C}$, we may regard $Z$ as a constant object in $s\mathcal{C}$ and there is an adjoint isomorphism

$$\hom_{\mathcal{C}}(Z, X_n) \cong \hom_{s\mathcal{C}}(Z \otimes \Delta^n, X)$$

for all $n \geq 0$. This immediately supplies maps in $s\mathcal{C}$

$$L_n X \otimes \Delta^n \to \operatorname{sk}_{n-1} X$$

and

$$X_n \otimes \Delta^n \to \operatorname{sk}_n X.$$

Furthermore, by Proposition 1.4, $\operatorname{sk}_{n-1}(X_n \otimes \Delta^n) = X_n \otimes \operatorname{sk}_{n-1} \Delta^n = X_n \otimes \partial\Delta^n$ and we obtain a diagram

$$\begin{tikzcd}
L_n X \otimes \partial\Delta^n \arrow[r] \arrow[d] & L_n X \otimes \Delta^n \arrow[d] \\
X_n \otimes \partial\Delta^n \arrow[r] & \operatorname{sk}_{n-1} X.
\end{tikzcd} \tag{1.6}$$

PROPOSITION 1.7. *For all $X \in s\mathcal{C}$ there is a natural pushout diagram, $n \geq 0$,*

$$\begin{tikzcd}
X_n \otimes \partial\Delta^n \cup_{L_n X \otimes \partial\Delta^n} L_n X \otimes \Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} X \arrow[d] \\
X_n \otimes \Delta^n \arrow[r] & \operatorname{sk}_n X
\end{tikzcd}$$

PROOF: In light of Proposition 1.4, if we apply $i_n^* i_{n*} = \operatorname{sk}_n(\cdot)$ to this diagram we obtain an isomorphic diagram. Since $i_n^* : s_n\mathcal{C} \to s\mathcal{C}$ is a left adjoint, we need only show this is a pushout diagram in degrees less than or equal to $n$.

In degrees $m < n$, the map $(L_n X \otimes \partial\Delta^n)_m \to (L_n X \otimes \Delta^n)_m$ is an isomorphism, so

$$(X_n \otimes \partial\Delta^n \cup_{L_n X \otimes \partial\Delta^n} L_n X \otimes \Delta^n)_m \to (X_n \otimes \Delta^m)_m$$

is an isomorphism, and the assertion is that $(\operatorname{sk}_{n-1} X)_m \cong (\operatorname{sk}_n X)_m$, which is true since both are isomorphic to $X_m$.

In degree $n$, the left vertical map is isomorphic to

$$\Big(\bigsqcup_{(\partial\Delta^n)_n} X_n\Big) \sqcup L_n X \to \Big(\bigsqcup_{(\partial\Delta^n)_n} X_n\Big) \sqcup X_n$$

and the right vertical map is isomorphic to the natural map $L_n X \to X_n$, by definition of $L_n X$. This is enough to show that the diagram is a pushout in degree $n$. □

REMARK 1.8. There is another, more explicit, description of the latching objects $L_nX$, which can be summarized as follows:

1. By convention, $L_0X = \emptyset$, where $\emptyset$ denotes the initial object of the category $\mathcal{C}$.
2. There is an isomorphism $L_1X \cong X_0$, and the canonical map $L_1X \to X_1$ can be identified with the degeneracy map $s_0 : X_0 \to X_1$.
3. For $n > 1$, the object $L_nX$ is defined by the coequalizer

$$\bigsqcup_{0 \leq i < j \leq n-1} X_{n-2} \rightrightarrows \bigsqcup_{i=0}^{n-1} X_{n-1} \to L_nX$$

where for $i < j$ the restrictions of the two displayed maps to $X_{n-2}$ are given by the composites

$$X_{n-2} \xrightarrow{s_i} X_{n-1} \xrightarrow{in_j} \bigsqcup_{i=0}^{n-1} X_{n-1}$$

and

$$X_{n-2} \xrightarrow{s_{j-1}} X_{n-1} \xrightarrow{in_i} \bigsqcup_{i=0}^{n-1} X_{n-1}$$

(this definition corresponds to the simplicial identity $s_js_i = s_is_{j-1}$). The canonical map $s : L_nX \to X_n$ is induced by the degeneracies $s_i : X_{n-1} \to X_n$.

Claims (2) and (3) follow from the description of $L_nX = \mathrm{sk}_{n-1} X_n$ given in (1.2) — this is an exercise for the reader.

Morphisms also have skeletal filtrations. If $f : A \to X$ is a morphism in $s\mathcal{C}$, define $\mathrm{sk}_n^A X$ by setting $\mathrm{sk}_{-1}^A = A$ and, for $n \geq 0$, defining $\mathrm{sk}_n^A$ by the pushout diagram

$$\begin{tikzcd}
\mathrm{sk}_n A \arrow[r] \arrow[d, "\mathrm{sk}_n f"'] & A \arrow[d] \\
\mathrm{sk}_n X \arrow[r] & \mathrm{sk}_n^A X.
\end{tikzcd}$$

The analog of Proposition 1.7 is the next result, which is proved in an identical manner. Let $L_n(f) = (\mathrm{sk}_{n-1}^A X)_n = A_n \cup_{L_nA} L_nX$.

PROPOSITION 1.9. *For all morphisms $A \to X$ in $s\mathcal{C}$ there is a pushout diagram*

$$\begin{tikzcd}
X_n \otimes \partial\Delta^n \cup_{L_n(f) \otimes \partial\Delta^n} L_n(f) \otimes \Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}^A X \arrow[d] \\
X_n \otimes \Delta^n \arrow[r] & \mathrm{sk}_n^A X.
\end{tikzcd}$$

□

There is a situation under which the pushout diagrams of Proposition 1.7 and 1.9 simplify considerably. This we now explain.

If $I$ is a small category, let $I^\delta$ be the discrete category with the same objects as $I$, but no non-identity morphism. The left adjoint $r^*$ to the restriction functor $r_* : \mathcal{C}^I \to \mathcal{C}^{I^\delta}$ has a very simple form

$$(r^* Z)_j = \bigsqcup_{i \to j} Z_i.$$

We call such a diagram *I-free*. More generally, a morphism $f : A \to X$ in $\mathcal{C}^I$ is *I-free* if there is an $I$-free object $X' \in \mathcal{C}^I$ and an isomorphism under $A$ of $f$ with the inclusion of the summand $A \to A \sqcup X'$. This implies that there is an object $\{Z_i\} \in \mathcal{C}^{I^\delta}$ so that

$$X_j \cong A_j \sqcup (\bigsqcup_{i \to j} Z_i).$$

Notice an object $X$ is $I$-free if and only if the morphism $\phi \to X$ from the initial object is $I$-free.

DEFINITION 1.10. An object $X \in s\mathcal{C}$ is *degeneracy free* if the underlying degeneracy diagram is free. That is, if $\mathbf{\Delta}_+ \subseteq \mathbf{\Delta}$ is the subcategory with same objects but only surjective morphisms, then $X$ regarded as an object in $\mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$ is $\mathbf{\Delta}_+^{\mathrm{op}}$-free. A morphism $A \to X$ is degeneracy free if, when regarded as a morphism in $\mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$, is $\mathbf{\Delta}_+^{\mathrm{op}}$ free.

If $X$ is degeneracy free, then there is a sequence $\{Z_n\}_{n \geq 0}$ of objects in $\mathcal{C}$ so that

$$X_n \cong \bigsqcup_{\phi : \mathbf{n} \to \mathbf{m}} Z_m$$

where $\phi$ runs over the epimorphisms in $\mathbf{\Delta}$. A degeneracy free map $A \to X$ yields a similar decomposition:

$$X_n \cong A_n \sqcup \bigsqcup_{\phi : \mathbf{n} \to \mathbf{m}} Z_m. \tag{1.11}$$

We say $A \to X$ is degeneracy free on $\{Z_m\}$.

The following result says that degeneracy free maps are closed under a variety of operations.

LEMMA 1.12.
*1) Let $f_\alpha : A_\alpha \to X_\alpha$ be a set of maps so that $f_\alpha$ is degeneracy free on $\{Z_n\}$. Then $\bigsqcup_\alpha f_\alpha$ is free on $\{\bigsqcup_\alpha Z_n^\alpha\}$.*

*2) Suppose $f : A \to X$ is degeneracy free on $\{Z_n\}$ and*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\
X \arrow[r] & Y
\end{tikzcd}$$

*is a pushout diagram. Then $g$ is degeneracy free on $\{Z_n\}$.*
*3) Let $A_0 \to A_1 \to A_2 \to \cdots$ be a sequence of morphisms so that $A_{j-1} \to A_j$ is degeneracy free on $\{Z^j_n\}$. Then $A_0 \to \varinjlim A_j$ is degeneracy free on $\{\bigsqcup_j Z^j_n\}$.*
*4) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that preserves coproducts. If $f : A \to X$ in $s\mathcal{C}$ is degeneracy free on $\{Z_n\}$, then $Ff : FA \to FX$ is degeneracy free on $\{FZ_n\}$.*

LEMMA 1.13. *A morphism $f : A \to X$ is degeneracy free if and only if there are objects $Z_n$ and maps $Z_n \to X_n$ so that the induced map*

$$(A_n \cup_{L_nA} L_nX) \sqcup Z_n \to X_n$$

*is an isomorphism.*

PROOF: If $f$ is degeneracy free on $\{Z_n\}$, the decomposition follows from the formulas (1.5) and (1.11). To prove the converse, fix the given isomorphisms. Then Proposition 1.9 implies there is a pushout diagram

$$\begin{tikzcd}
Z_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}^A_{n-1} X \arrow[d] \\
Z_n \otimes \Delta^n \arrow[r] & \operatorname{sk}^A_n X
\end{tikzcd}$$

in $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}_+}$.

However, the morphism of simplicial sets $\partial\Delta^n \to \Delta^n$ is degeneracy free on the canonical $n$-simplex in $\Delta^n_n$; hence, $\operatorname{sk}^A_{n-1} X \to \operatorname{sk}^A_n X$ is degeneracy free on $Z_n$ in degree $n$. The result now follows from Lemma 1.12.3; indeed $A \to X$ is degeneracy free on $\{Z_n\}$. □

Here is a consequence of the proof of Lemma 1.13:

COROLLARY 1.14. *Suppose $A \to X$ is degeneracy-free on $\{Z_n\}$ in $s\mathcal{C}$. Then for all $n \geq 0$ there is a pushout diagram,*

$$\begin{tikzcd}
Z_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}^A_{n-1} X \arrow[d] \\
Z_n \otimes \Delta^n \arrow[r] & \operatorname{sk}^A_n X.
\end{tikzcd}$$

Notice that one can interpret this result as saying $\operatorname{sk}^A_n X$ is obtained from $\operatorname{sk}^A_{n-1} X$ by attaching $n$-cells.

EXAMPLE 1.15. Lemmas 1.12 and 1.13 provide any number of examples of degeneracy free morphisms. For example, a cofibration in simplicial sets is degeneracy free, by Lemma 1.13. Also, consider a category $\mathcal{C}$ equipped with a functor $G : \mathcal{C} \to \mathbf{Sets}$ with a left adjoint $F$ and satisfying the hypotheses of Theorem II.4.1. Define a morphism $f : A \to X$ in $s\mathcal{C}$ to be *free* (this terminology is from Quillen) if there is a sequence of sets $\{Z_n\}$ so that $f$ is degeneracy free on $\{FZ_n\}$. Then a morphism in $s\mathcal{C}$ is a cofibration if and only if it is a retract of a free map. To see this, note that the small object argument of Lemma II.4.2, coupled with Lemma 1.12 factors any morphism $A \to B$ as

$$A \xrightarrow{j} X \xrightarrow{q} Y$$

where $j$ is a free map and a cofibration and $q$ is a trivial fibration. Thus any cofibration is a retract of a free map. Conversely, Corollary 1.14 and Proposition II.3.4 imply any free map is a cofibration. Similar remarks apply to the model categories supplied in Theorems II.5.8 and II.6.8. In the latter case one must generalize Lemma 1.12.3 to longer colimits.

The notion of coskeleta is dual to the notion of skeleta. The theory is analogous, and we give only an outline.

The restriction functor $i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$ has right adjoint $i_n^!$ with

$$(i_n^! X)_m = \varprojlim_{\mathbf{k} \to \mathbf{m}} X_k,$$

with the limit over all morphism $\mathbf{k} \to \mathbf{m}$ in $\mathbf{\Delta}$ with $k \leq n$. Equally, one can take the limit over morphisms $\mathbf{k} \to \mathbf{m}$ which are injections. The composite gives the $n^{\text{th}}$ coskeleton functor:

$$\operatorname{cosk}_n X = i_n^! i_{n*} X. \tag{1.16}$$

More generally, if $f : X \to B$ is a morphism in $s\mathcal{C}$, let $\operatorname{cosk}_{-1}^B X = B$ and let $\operatorname{cosk}_n^B X$ be defined by the pullback

$$\begin{tikzcd}
\operatorname{cosk}_n^B X \arrow[r] \arrow[d] & \operatorname{cosk}_n X \arrow[d] \\
B \arrow[r] & \operatorname{cosk}_n B.
\end{tikzcd} \tag{1.17}$$

Then there are maps $\operatorname{cosk}_n^B X \to \operatorname{cosk}_{n-1}^B X$ and

$$X \cong \varprojlim_n \operatorname{cosk}_n^B X.$$

Note that if $B = *$, the terminal object, then $\operatorname{cosk}_n^B X = \operatorname{cosk}_n X$.

For $X \in s\mathcal{C}$, define the $n^{\text{th}}$ *matching object* $M_nX$ by the formula

$$M_nX = (\operatorname{cosk}_{n-1} X)_n = \varprojlim_{\mathbf{k} \to \mathbf{n}} X_k \tag{1.18}$$

where $\mathbf{k} \to \mathbf{n}$ runs over all morphisms (or all monomorphisms) in $\mathbf{\Delta}$ with $k < n$.

REMARK 1.19. The matching object $M_nX$ has a more explicit description:

(1) By convention, $M_0X = *$, where $*$ denotes the terminal object of the category $\mathcal{C}$.

(2) There is an isomorphism $M_1X \cong X_0 \times X_0$, and the canonical map $X_1 \to M_1X$ can be identified with the product $d = (d_0, d_1) : X_1 \to X_0 \times X_0$ of the face maps $d_0, d_1 : X_1 \to X_0$.

(3) For $n > 1$, the matching object $M_nX$ is defined by an equalizer diagram

$$M_nX \to \prod_{i=0}^{n} X_{n-1} \rightrightarrows \prod_{0 \leq i < j \leq n} X_{n-2}.$$

Here, the parallel arrows are determined by the simplicial identities $d_i d_j = d_{j-1} d_i$ for $i < j$; more explicitly, the images of these maps on the factor corresponding to $i < j$ are the maps

$$\prod_{i=0}^{n} X_{n-1} \xrightarrow{pr_j} X_{n-1} \xrightarrow{d_i} X_{n-2}$$

and

$$\prod_{i=0}^{n} X_{n-1} \xrightarrow{pr_i} X_{n-1} \xrightarrow{d_{j-1}} X_{n-2}.$$

The canonical map $d : X_n \to M_nX$ is induced by the face maps $d_i : X_n \to X_{n-1}$.

Claims (2) and (3) follow from the description of $(i^!_{n-1}X)_n = M_nX$ as an inverse limit indexed over ordinal number monomorphisms. This is an exercise for the reader.

To fit the map $\operatorname{cosk}^B_n X \to \operatorname{cosk}^B_{n-1} X$ into a pullback diagram, we make some definitions. Let $\rho_n : s\mathcal{C} \to \mathcal{C}$ be the functor $\rho_n X = X_n$. This is a restriction functor between diagram categories and has a right adjoint $\rho^!_n$. The functor $L_n : s\mathcal{C} \to \mathcal{C}$ assigning each simplicial object the latching object $L_nX$ also has a right adjoint, which we will call $\mu^!_n$. To see this, see Lemma 1.25 below, or note that we may write

$$L_nX = (\operatorname{sk}_{n-1} X)_n = \rho_n i^*_{n-1} i_{(n-1)*} X$$

where $i_{(n-1)*} : s\mathcal{C} \to s_{n-1}\mathcal{C}$ is the restriction functor. Hence

$$\mu^!_n Z = i^!_{n-1} i_{(n-1)*} \rho^!_n Z = \operatorname{cosk}_{n-1}(\rho^!_n Z).$$

The natural map $s : L_nX \to X$ gives a natural transformation $\rho^!_n \to \mu^!_n$. The reader is invited to prove the following result.

PROPOSITION 1.20. *Let $f : X \to B$ be a morphism in $s\mathcal{C}$. Then for all $n \geq 0$ there is a pullback diagram*

$$\begin{tikzcd}
\operatorname{cosk}_n^B X \arrow[r] \arrow[d] & \rho_n^! X_n \arrow[d] \\
\operatorname{cosk}_{n-1}^B X \arrow[r] & \rho_n^! M_n(f) \times_{\mu_n^! M_n(f)} \mu_n^! X_n
\end{tikzcd}$$

*where $M_n(f) = B_n \times_{M_n B} M_n X$.*

We have never encountered the analog of degeneracy-free morphisms, and don't include an exposition here.

The latching and matching object functors $L_n$ and $M_n$ are examples of a much more general sort of functor, which we now introduce and analyze. We begin with generalized matching objects.

PROPOSITION 1.21. *Let $K \in \mathbf{S}$ be a simplicial set. Then the functor*

$$- \otimes K : \mathcal{C} \to s\mathcal{C}$$

*has right adjoint $M_K$. For fixed $X \in s\mathcal{C}$, the assignment $K \mapsto M_K X$ induces a functor $\mathbf{S}^{\mathrm{op}} \to \mathcal{C}$ which has a left adjoint.*

PROOF: For objects $Z$ of $\mathcal{C}$ and all $n \geq 0$, there are isomorphisms

$$\hom_{s\mathcal{C}}(Z \otimes \Delta^n, X) \cong \hom_{\mathcal{C}}(Z, X_n).$$

It follows that there are isomorphisms

$$\begin{aligned}
\hom_{s\mathcal{C}}(Z \otimes K, X) &\cong \varprojlim_{\Delta^n \to K} \hom_{s\mathcal{C}}(Z \otimes \Delta^n, X) \\
&\cong \varprojlim_{\Delta^n \to K} \hom_{\mathcal{C}}(Z, X_n) \\
&\cong \hom_{\mathcal{C}}(Z, \varprojlim_{\Delta^n \to K} X_n) \\
&= \hom_{\mathcal{C}}(Z, M_K X).
\end{aligned}$$

Furthermore,

$$\hom_{\mathcal{C}}(Z, M_K X) \cong \hom_{s\mathcal{C}}(Z \otimes K, X) \cong \hom_{\mathbf{S}}(K, \mathbf{Hom}_{s\mathcal{C}}(Z, X)). \qquad \square$$

Notice the explicit description of $M_K X$ that arose in the proof of Proposition 1.21:

$$M_K X \cong \varprojlim_{(\mathbf{\Delta} \downarrow K)^{\mathrm{op}}} X \tag{1.22}$$

is the limit of a contravariant functor on the simplex category $\mathbf{\Delta} \downarrow K$ which sends an object $\Delta^n \to K$ of $\mathbf{\Delta} \downarrow K$ to the object $X_n$ of $\mathcal{C}$. In particular, there is an isomorphism

$$M_{\Delta^n} X \cong X_n,$$

since the category $\Delta \downarrow \Delta^n$ has an initial object.

EXAMPLE 1.23. Let $\phi : \Delta^k \to \Delta^n$ be any morphism in $\mathbf{S}$. Then $\phi$ factors uniquely as a composition

$$\Delta^k \xrightarrow{\phi'} \Delta^m \xrightarrow{\psi} \Delta^n$$

where $\phi'$ is a surjection and $\psi$ is an injection. Thus if $K \subseteq \Delta^n$ is any subcomplex, the full subcategory $\mathbf{\Delta} \downarrow K_0 \subseteq \mathbf{\Delta} \downarrow K$ with objects

$$\sigma : \Delta^m \to K$$

with $\sigma$ an injection determines colimits on the larger category. Equivalently, restriction to $\mathbf{\Delta} \downarrow K_0^{\mathrm{op}}$ determines inverse limits for $\mathbf{\Delta} \downarrow K^{\mathrm{op}}$; it follows that there is an isomorphism

$$M_K X \cong \varprojlim_{\mathbf{\Delta} \downarrow K_0^{\mathrm{op}}} X.$$

In particular $M_{\partial\Delta^n} X \cong \varprojlim_{\phi:\mathbf{k}\to\mathbf{n}} X_k$ where $\phi \in \mathbf{\Delta}$ is an injection and $k < n$. Thus $M_{\partial\Delta^n} X \cong M_n X$. Note that the inclusion map $\partial\Delta^n \to \Delta^n$ induces the projection

$$X_n \to M_n X.$$

To generalize the latching objects we use the formulation of matching objects presented in (1.22). Let $J$ be a small category and $F : J \to \mathbf{\Delta}^{\mathrm{op}}$. For $X \in s\mathcal{C} = \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$, define the generalized latching object to be

$$\begin{aligned} L_J X &= \varinjlim_J (X \circ F) \\ &= \varinjlim_j X_{F(j)}. \end{aligned} \tag{1.24}$$

We are primarily interested in sub-categories $J$ of $\mathbf{\Delta}^{\mathrm{op}}$. We write $X\big|_J$ for $X \circ F$.

LEMMA 1.25. *For fixed* $F : J \to \mathbf{\Delta}^{\mathrm{op}}$, *the functor* $L_J : s\mathcal{C} \to \mathcal{C}$ *has a right adjoint and, hence, preserves colimits.*

PROOF: For $\hom_{\mathcal{C}}(L_J X, Z) \cong \hom_{\mathcal{C}^J}(X \circ F, Z)$ where $Z$ is regarded as a constant diagram. But

$$\hom_{\mathcal{C}^J}(X\big|_J, Z) = \hom_{s\mathcal{C}}(X, F^! Z)$$

where $F^!$ is the right Kan extension functor. □

An example of a generalized latching object is the following. Let $\mathcal{O}_n$ be the category with objects the morphisms $\phi : \mathbf{n} \to \mathbf{m}$ with $\phi$ surjective and $m < n$. The morphisms in $\mathcal{O}_n$ are commutative triangles in $\mathbf{\Delta}$ under $\mathbf{n}$. Define $F : \mathcal{O}_n^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ by $F(\phi : \mathbf{n} \to \mathbf{m}) = \mathbf{m}$. Then there is a natural isomorphism

$$L_n X \cong L_{\mathcal{O}_n^{\mathrm{op}}} X.$$

In the next section we will need a decomposition of $L_nX$. To accomplish this define sub-categories $\mathcal{M}_{n,k} \subseteq \mathcal{O}_n$, $0 \leq k \leq n$, to be the full sub-category of surjections $\phi : \mathbf{n} \to \mathbf{m}$, $m < n$, with $\phi(k) < k$. Define, for $X \in s\mathcal{C}$

$$L_{n,k}X = L_{\mathcal{M}^{\mathrm{op}}_{n,k}}X.$$

Then $L_{n,0}X = \phi$ is the initial object in $\mathcal{C}$ (since $\mathcal{M}_{n,0}$ is empty) and $L_{n,n}X = L_nX$, since $\mathcal{M}_{n,n} = \mathcal{O}_n$.

Also define $\mathcal{M}(k) \subseteq \mathcal{O}_n$ to be the full subcategory of surjections $\phi : \mathbf{n} \to \mathbf{m}$, $m < n$, with $\phi(k) = \phi(k+1)$. Notice that $s^k : \mathbf{n} \to \mathbf{n-1}$ is the initial object of $\mathcal{M}(k)$. Hence for $X \in s\mathcal{C}$

$$L_{\mathcal{M}(k)^{\mathrm{op}}}X \cong X_{n-1}.$$

The reader can verify the following statements about these subcategories.

LEMMA 1.26.
*1) $\mathcal{M}_{n,k}$ and $\mathcal{M}(k)$ are subcategories of $\mathcal{M}_{n,k+1}$, and every object in $\mathcal{M}_{n,k+1}$ is in $\mathcal{M}_{n,k}$ or $\mathcal{M}(k)$ (or both).*

*2) There is an isomorphism of categories*

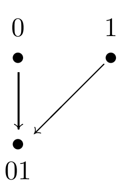
$$- \circ s^k : \mathcal{M}_{n-1,k} \to \mathcal{M}(k) \cap \mathcal{M}_{n,k}$$

*sending $\phi$ to $\phi \circ s^k$.*

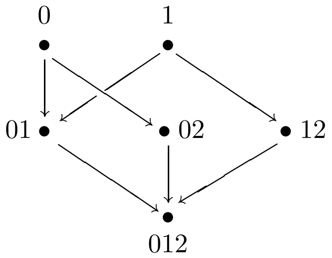
*3) If $\phi$ is an object of $\mathcal{M}_{n,k}$ (or $\mathcal{M}(k)$) and $\phi \to \psi$ is a morphism in $\mathcal{M}_{n,k+1}$, then $\phi \to \psi$ is a morphism in $\mathcal{M}_{n,k}$ (or $\mathcal{M}(k)$).*

The following example, illustrating the case $n = 3$, $k = 2$, might be helpful.

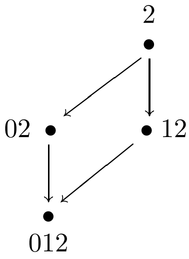
In the following diagram the symbol 012 near a dot (•) indicates the object $s^0s^1s^2$ in the appropriate category. The unlabeled arrows indicated composition with $s^i$ for some $i$; for example 0 • → • 02 means $s^0 \mapsto s^1s^0 = s^0s^2$.



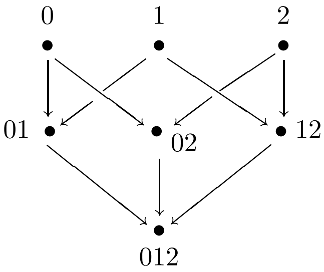
$\mathcal{M}_{2,2} = \mathcal{O}_2$



$\mathcal{M}_{3,2}$



$\mathcal{M}(2)$



$\mathcal{M}_{3,3} = \mathcal{O}_3$

Proposition 1.27. *Let $X \in s\mathcal{C}$. Then there is a pushout diagram in $\mathcal{C}$*

$$\begin{tikzcd}
L_{n-1,k}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
L_{n,k}X \arrow[r] & L_{n,k+1}X.
\end{tikzcd}$$

Proof: Lemma 1.26 implies that

$$\begin{tikzcd}
\mathcal{M}_{n-1,k}X \arrow[r] \arrow[d] & \mathcal{M}(k) \arrow[d] \\
\mathcal{M}_{n,k}X \arrow[r] & \mathcal{M}_{n,k+1}
\end{tikzcd}$$

is a pushout in the category of small categories. The same is then true of the opposite categories. The result follows. □

## 2. Reedy model category structures.

Let $\mathcal{C}$ be a closed model category. Using the considerations of the previous section, this structure can be promoted to a closed model category structure on the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ that is particularly useful for dealing with geometric realization. The results of this section are a recapitulation of the highly influential, but unpublished paper of C.L. Reedy [81].

In the following definition, let $L_0X$ and $M_0X$ denote the initial and final object of $\mathcal{C}$ respectively.

DEFINITION 2.1. A morphism $f : X \to Y$ in $s\mathcal{C}$ is a

1) Reedy weak equivalence if $f : X_n \to Y_n$ is a weak equivalence for all $n \geq 0$;
2) a Reedy fibration if
$$X_n \to Y_n \times_{M_nY} M_nX$$
is a fibration for all $n \geq 0$;
3) a Reedy cofibration if
$$X_n \cup_{L_nX} L_nY \to Y_n$$
is a cofibration for all $n \geq 0$.

The main result is that this defines a model category structure on $s\mathcal{C}$. Before proving this we give the following lemma.

LEMMA 2.2. *A morphism $f : X \to Y$ in $s\mathcal{C}$ is a*

1) *Reedy trivial fibration if and only if*
$$X_n \to Y_n \times_{M_nY} M_nX$$
*is a trivial fibration for $n \geq 0$;*
2) *a Reedy trivial cofibration if and only if*
$$X_n \cup_{L_nX} L_nY \to Y_n$$
*is a trivial cofibration for all $n \geq 0$.*

PROOF: Let $\Delta^{n,k} = d^0\Delta^{n-1} \cup \cdots \cup d^k\Delta^{n-1} \subseteq \partial\Delta^n$, $-1 \leq k \leq n$. Then $\Delta^{n,-1} = \phi$ and $\Delta^{n,n} = \partial\Delta^n$. There are pushout diagrams $-1 \leq k \leq n-1$

$$\begin{tikzcd}
\Delta^{n-1,k} \arrow[r] \arrow[d] & \Delta^{n-1} \arrow[d] \\
\Delta^{n,k} \arrow[r] & \Delta^{n,k+1}.
\end{tikzcd}$$

Taking matching objects yields a natural pullback square

$$\begin{tikzcd}
M_{n,k+1}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
M_{n,k}X \arrow[r] & M_{n-1,k}X
\end{tikzcd}$$

where we have written $M_{n,k}X$ for $M_{\Delta^{n,k}}X$. It follows that there is a pullback square

$$\begin{tikzcd}
Y_n \times_{M_{n,k+1}Y} M_{n,k+1}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
Y_n \times_{M_{n,k}Y} M_{n,k}X \arrow[r] & Y_{n-1} \times_{M_{n-1,k}Y} M_{n-1,k}X.
\end{tikzcd} \tag{2.3}$$

Assume $f : X \to Y$ is a Reedy fibration. Then $X_0 \to Y_0$ is a fibration and this begins an induction where the induction hypothesis is that

$$X_{n-1} \to Y_{n-1} \times_{M_{n-1,k}Y} M_{n-1,k}X \tag{2.4}$$

is a fibration. To complete the inductive step, one uses (2.3) to show

$$Y_n \times_{M_{n,k+1}Y} M_{n,k+1}X \to Y_n \times_{M_{n,k}Y} M_{n,k}X \tag{2.5}$$

is a fibration for all $k$, $-1 \leq k \leq n-1$. Since composites of fibrations are fibrations and $X_n \to Y_n \times_{M_nY} M_nX$ is a fibration, we close the loop.

Now suppose $f : X \to Y$ is a Reedy trivial fibration. Then, inductively, each of the maps of (2.4) is a trivial fibration. For the inductive step, use (2.3) to show each of the maps (2.5) is a trivial fibration. Then the axiom **CM2** and the fact that $X_n \to Y_n$ a trivial fibration finishes the argument. In particular $X_n \to Y_n \times_{M_nY} M_nX$ is a trivial fibration for all $n$.

Conversely, suppose $X_n \to X_n \times_{M_nY} M_nX$ is a trivial fibration. Then one runs a similar argument to conclude $X_n \to Y_n$ is a trivial fibration.

The argument for part 2) is similar, using the pushout diagram

$$\begin{tikzcd}
L_{n-1,k}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
L_{n,k}X \arrow[r] & L_{n,k+1}X
\end{tikzcd}$$

of Proposition 1.27. □

As a corollary of the proof of Lemma 2.2 we have the following:

Corollary 2.6.

(1) *Every Reedy fibration $p : X \to Y$ of $s\mathcal{C}$ is a level fibration in the sense that all component maps $p : X_n \to Y_n$ are fibrations of $\mathcal{C}$.*

(2) *Every Reedy cofibration $i : A \to B$ of $s\mathcal{C}$ is a level cofibration in the sense that all component maps $i : A_n \to B_n$ are cofibrations of $\mathcal{C}$.*

We break the proof of the verification of the Reedy model category structure into several steps.

LEMMA 2.7. *The Reedy structure on $s\mathcal{C}$ satisfies the lifting axiom* **CM4**.

PROOF: Suppose we are given a lifting problem in $s\mathcal{C}$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\
B \arrow[r] & Y
\end{tikzcd}$$

where $j$ is cofibration, $q$ is a fibration and either $j$ or $q$ is a weak equivalence. We inductively solve the lifting problems

$$\begin{tikzcd}
\operatorname{sk}^A_{n-1} B \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{sk}^A_n B \arrow[r] & Y
\end{tikzcd}$$

for $n \geq 0$. Because of the pushout diagram of Proposition 1.9 it is sufficient to solve the lifting problems, with $L_n(j) = A_n \cup_{L_nA} L_nB$

$$\begin{tikzcd}
L_n(j) \otimes \Delta^n \cup_{L_n(j)\otimes\partial\Delta^n} B_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\
B_n \otimes \Delta^n \arrow[r] & Y.
\end{tikzcd}$$

By an adjunction argument this is equivalent to the lifting problem

$$\begin{tikzcd}
A_n \cup_{L_nA} L_nB \arrow[r] \arrow[d] & X_n \arrow[d] \\
B_n \arrow[r] & Y_n \times_{M_nY} M_nX.
\end{tikzcd}$$

This is solvable by Lemma 2.2. □

For the proof of the factorization axiom we need to know how much data we need to build a simplicial object.

Lemma 2.8. *Let $X \in s\mathcal{C}$. Then $\mathrm{sk}_n X$ is determined up to natural isomorphism by the following natural data: $\mathrm{sk}_{n-1} X$, $X_n$, and maps $L_nX \to X_n \to M_nX$ so that the composite $L_nX \to M_nX$ is the canonical map.*

Proof: The given map $X_n \to M_nX$ is adjoint to a map $X_n \otimes \partial\Delta^n \to X$ which factors uniquely through $\mathrm{sk}_{n-1} X$. The data listed thus yields the pushout square

$$\begin{tikzcd}
L_nX \otimes \Delta^n \cup_{L_nX \otimes \partial\Delta^n} X_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} X \arrow[d] \\
X_n \otimes \Delta^n \arrow[r] & \mathrm{sk}_n X
\end{tikzcd}$$

of Proposition 1.7. □

This can be restricted: let $s_n\mathcal{C}$ be the functor category $\mathcal{C}^{\mathbf{\Delta}_n^{\mathrm{op}}}$ and $X \in s_n\mathcal{C}$. Let $r_*X \in s_{n-1}\mathcal{C}$ be the restriction and $r^* : s_{n-1}\mathcal{C} \to s_n\mathcal{C}$ the left Kan extension functor. Hence $(r^*r_*X)_n = L_nX$.

Lemma 2.8 immediately implies

Lemma 2.9. *Let $X \in s_n\mathcal{C}$. Then $X$ is determined up to natural isomorphism by the following natural data: $r_*X$, $X_n$, and maps $L_nX \to X_n \to M_nX$ so that the composite $L_nX \to M_nX$ is the canonical map.*

Proof: Let $i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$ be the restriction and $i_n^* : s_n\mathcal{C} \to s\mathcal{C}$ the left adjoint. Then $i_{n*}i_n^* \cong 1$ and $i_n^*i_{n*} = \mathrm{sk}_n$. Thus

$$i_n^*X \cong \mathrm{sk}_n\, i_n^*X$$

is determined up to natural isomorphism by $\mathrm{sk}_{n-1}\, i_n^*X \cong i_{n-1}^*r_*X$, the object $X_n$ and maps $L_ni_n^*X \to X_n \to M_ni_n^*X$ that compose to the canonical map $L_ni_n^*X \to M_ni_n^*X$. But $L_nX \cong L_ni_n^*X$ and $M_nX \cong M_ni_n^*X$. □

Lemma 2.10. *The Reedy structure on $s\mathcal{C}$ satisfies the factorization axiom* **CM5**.

Proof: Let us do the trivial cofibration-fibration factorization (compare the proof of Lemma IV.3.6).

Let $X \to Y$ be a morphism in $s\mathcal{C}$ and let $i_{n*}X \to i_{n*}Y$ be the induced morphism in $s_n\mathcal{C}$. For each $n \geq 0$ we construct a factorization

$$i_{n*}X \to Z(n) \to i_{n*}Y$$

in $s_n\mathcal{C}$ with the property that restricted to $s_{n-1}\mathcal{C}$ we get the factorization $i_{(n-1)*}X \to Z(n-1) \to i_{(n-1)*}Y$.

For $n = 0$, simply factor $X_0 \to Y_0$ as

$$X_0 \xrightarrow{j} Z(0) \xrightarrow{q} Y_0$$

where $j$ is a trivial cofibration in $\mathcal{C}$ and $q$ is a fibration in $\mathcal{C}$. Suppose the factorization in $s_{n-1}\mathcal{C}$ has been constructed. Then there is a commutative diagram

$$\begin{tikzcd}
L_nX \arrow[r] \arrow[d] & X_n \arrow[r] & M_nX \arrow[d] \\
L_nZ \arrow[d] & & M_nZ \arrow[d] \\
L_nY \arrow[r] & Y_n \arrow[r] & M_nY
\end{tikzcd}$$

and hence a map

$$X_n \cup_{L_nX} L_nZ \to Y_n \times_{M_nY} M_nZ.$$

Factor this map as

$$X_n \cup_{L_nX} L_nZ \xrightarrow{j} Z_n \xrightarrow{q} Y_n \times_{M_nY} M_nZ \tag{2.11}$$

where $j$ is a trivial cofibration and $q$ is a fibration. The morphisms $j$ and $q$ yield diagrams

$$\begin{tikzcd}
L_nX \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & M_nX \arrow[d] \\
L_nZ \arrow[r] \arrow[d] & Z_n \arrow[r] \arrow[d] & M_nZ \arrow[d] \\
L_nY \arrow[r] & Y_n \arrow[r] & M_nY
\end{tikzcd}$$

and so Lemma 2.9 produces a factorization

$$i_{n*}X \to Z(n) \to i_{n*}Y.$$

Finally, define $Z \in s\mathcal{C}$ by $Z_k = Z(n)_k$, $k \leq n$. There is a factoring

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

and using (2.11) $j$ is a trivial cofibration by Lemma 2.2 and $q$ is a fibration by definition.

The other factorization is similar. □

We now state

THEOREM 2.12. *With the definitions of Reedy weak equivalence, cofibration, and fibration given in Definition 2.1, the category $s\mathcal{C}$ is a closed model category.*

PROOF: The axioms **CM1–CM3** are easy and Lemmas 2.7 and 2.10 prove **CM4** and **CM5** respectively. $\square$

Next suppose $\mathcal{C}$ is in fact a simplicial model category. For $K \in \mathbf{S}$ and $Y, Z \in \mathcal{C}$ write $Z\square K$, $\mathbf{hom}_{\mathcal{C}}(K, Z)$, and $\mathbf{Hom}_{\mathcal{C}}(Y, Z)$ for the tensor object, the exponential object, and the mapping space. We use this notation to distinguish the internal object $Z\square K$ from the construction $Z \otimes K$ defined by the simplicial structure.

The category $s\mathcal{C}$ inherits a simplicial structure. If $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, then $X\square K$ and $\mathbf{hom}_{\mathcal{C}}(K, X)$ are defined level-wise

$$(X\square K)_n = X_n\square K \text{ and } \mathbf{hom}_{\mathcal{C}}(K, X)_n = \mathbf{hom}_{\mathcal{C}}(K, X_n).$$

The mapping space is defined by the usual formula

$$\mathbf{Hom}_{s\mathcal{C}}(X, Y)_n \cong \hom_{s\mathcal{C}}(X\square\Delta^n, Y).$$

We call this the internal structure on $s\mathcal{C}$.

COROLLARY 2.13. *With this internal simplicial structure on $s\mathcal{C}$, the Reedy model category structure is a simplicial model category.*

PROOF: We claim $M_n(\mathbf{hom}_{\mathcal{C}}(K, Y)) \cong \mathbf{hom}_{\mathcal{C}}(K, M_nY)$. For there is a sequence of natural isomorphisms, $Z \in \mathcal{C}$,

$$\begin{aligned} \hom_{\mathcal{C}}(Z, M_n(\mathbf{hom}_{\mathcal{C}}(K, Y))) &\cong \hom_{s\mathcal{C}}(Z \otimes \partial\Delta^n, \mathbf{hom}_{\mathcal{C}}(K, Y)) \\ &\cong \hom_{s\mathcal{C}}((Z \otimes \partial\Delta^n)\square K, Y) \\ &\cong \hom_{s\mathcal{C}}((Z\square K) \otimes \partial\Delta^n, Y) \\ &\cong \hom_{s\mathcal{C}}(Z\square K, M_nY) \\ &\cong \hom_{s\mathcal{C}}(Z, \mathbf{hom}_{\mathcal{C}}(K, M_nY)). \end{aligned}$$

The isomorphism $(Z \otimes \partial\Delta^n)\square K \cong (Z\square K) \otimes \partial\Delta^n$ follows by a level-wise calculation. The result follows from the claim using Proposition II.3.13. $\square$

One can ask if $s\mathcal{C}$ in the Reedy model category is a simplicial model category in the standard simplicial structure obtained by Quillen's method (as in the previous section). The answer is no; for if $Z \in \mathcal{C}$ is cofibrant in $\mathcal{C}$, then

$$1 \otimes d^0 : Z \otimes \Delta^0 \to Z \otimes \Delta^1$$

is a Reedy cofibration, but not, in general, a Reedy weak equivalence (see Remark IV.3.13).

As a corollary of the proof of Theorem 2.12 (more specifically Lemma 2.7), we have the following:

COROLLARY 2.14. *Suppose that $j : A \to B$ is a Reedy cofibration. Then all maps*

$$L_n(j) \otimes \Delta^n \cup_{L_n(j)\otimes\partial\Delta^n} B_n \otimes \partial\Delta^n \to B_n \otimes \Delta^n$$

*are Reedy cofibrations. In particular, if $X$ is a Reedy cofibrant object, then all maps*

$$L_nX \otimes \Delta^n \cup_{L_nX\otimes\partial\Delta^n} X_n \otimes \partial\Delta^n \to X_n \otimes \Delta^n$$

*are Reedy cofibrations.*

It is true (Proposition 2.15 below), however, that if $f : X \to Y$ is a Reedy cofibration in $s\mathcal{C}$ and $j : K \to L$ is a cofibration of simplicial sets, then the induced map

$$X \otimes L \cup_{X\otimes K} Y \otimes K \to Y \otimes L$$

is a Reedy cofibration which is a Reedy weak equivalence if $f$ is a Reedy weak equivalence. The proof of this statement requires that we first recast the definition of Reedy fibration, and properly describe function complex objects for the external structure.

Recall the definition: a map $p : Z \to W$ is a Reedy fibration of $s\mathcal{C}$ if the induced map

$$Z_n \to W_n \times_{M_nW} M_nZ$$

is a fibration of the closed model category $\mathcal{C}$ for every $n \geq 0$. This means that all such maps should have the right lifting property with respect to all trivial cofibrations of $\mathcal{C}$, so an adjointness argument says that $p : Z \to W$ is a Reedy fibration if it has the right lifting property in $s\mathcal{C}$ with respect to all maps

$$B \otimes \partial\Delta^n \cup_{A\otimes\partial\Delta^n} A \otimes \Delta^n \to B \otimes \Delta^n$$

associated to trivial cofibrations $A \to B$ of $\mathcal{C}$ and simplicial set inclusions $\partial\Delta^n \hookrightarrow \Delta^n$. The underlying category $\mathcal{C}$ has all colimits, so $p$ is a Reedy fibration if and only if it has the right lifting property with respect to all maps

$$B \otimes K \cup_{A\otimes K} A \otimes L \to B \otimes L$$

induced by trivial cofibrations $A \to B$ of $\mathcal{C}$ and simplicial set inclusions $K \hookrightarrow L$.

We have seen in Proposition 1.21 that the functor $M_K : s\mathcal{C} \to \mathcal{C}$ is right adjoint to $Z \mapsto Z \otimes K$, and this adjunction is defined and natural for all simplicial sets $K$. For such $K$, define a functor $\mathbf{M}_K : s\mathcal{C} \to s\mathcal{C}$ by specifying $\mathbf{M}_KY_n = M_{\Delta^n\times K}Y$. Then the adjunction isomorphisms

$$\hom(X_n, M_{\Delta^n\times K}Y) \cong \hom(X_n \otimes \Delta^n \otimes K, Y)$$

jointly induce an adjunction isomorphism

$$\hom(X, \mathbf{M}_KY) \cong \hom(X \otimes K, Y).$$

To see this, it helps to know that there is a natural coequalizer

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} X_n \otimes \Delta^m \rightrightarrows \bigsqcup_{n} X_n \otimes \Delta^n \to X$$

in the simplicial object category $s\mathcal{C}$, and that tensoring with $K$ preserves colimits.

PROPOSITION 2.15.

(1) *Let* $f : X \to Y$ *be a Reedy cofibration in* $s\mathcal{C}$ *and* $j : K \to L$ *a cofibration in* $\mathbf{S}$. *Then*

$$X \otimes L \cup_{X \otimes K} Y \otimes K \to Y \otimes L$$

*is a Reedy cofibration which is a Reedy weak equivalence if* $f$ *is a Reedy weak equivalence.*

(2) *Suppose* $f : X \to Y$ *is a Reedy fibration in* $s\mathcal{C}$ *and* $j : K \to L$ *is a cofibration in* $\mathbf{S}$. *Then*

$$\mathbf{M}_L X \to \mathbf{M}_K X \times_{\mathbf{M}_K Y} \mathbf{M}_L Y$$

*is a Reedy fibration which is a Reedy weak equivalence if* $f$ *is a Reedy weak equivalence.*

PROOF: These two statements are equivalent by an adjunction argument, and we shall prove the second.

The map

$$\mathbf{M}_L X \to \mathbf{M}_K X \times_{\mathbf{M}_K Y} \mathbf{M}_L Y$$

has the right lifting property with respect to a class of maps $C \to D$ if and only if $f : X \to Y$ has the right lifting property with respect to all induced maps

$$D \otimes K \cup_{C \otimes K} C \otimes L \to D \otimes L$$

The corresponding map induced by the morphism

$$B \otimes \partial\Delta^n \cup_{A \otimes \partial\Delta^n} A \otimes \Delta^n \to B \otimes \Delta^n$$

arising from a trivial cofibration $A \to B$ of $\mathcal{C}$ has the form

$$B \otimes K' \cup_{A \otimes K'} A \otimes L' \to B \otimes L'$$

where the simplicial set inclusion $K' \hookrightarrow L'$ is the morphism

$$L \times \partial\Delta^n \cup_{K \times \partial\Delta^n} K \times \Delta^n \hookrightarrow L \times \Delta^n.$$

Any Reedy fibration $f : X \to Y$ has the right lifting property with respect to all such morphisms.

The second part of claim (2) follows in a similar way from Lemma 2.2. □

## 3. Geometric realization.

Suppose $\mathcal{C}$ is a simplicial category and $X \in s\mathcal{C}$. Then the geometric realization $|X| \in \mathcal{C}$ is defined by the coequalizer diagram

$$\bigsqcup_{\phi:\mathbf{n}\to\mathbf{m}} X_m \square \Delta^n \underset{d_1}{\overset{d_0}{\rightrightarrows}} \bigsqcup_{n\geq 0} X_n \square \Delta^n \to |X| \tag{3.1}$$

where $\phi$ runs over the morphisms of $\mathbf{\Delta}$, and $d_0$ and $d_1$ on the factor associated to $\phi : \mathbf{n} \to \mathbf{m}$ are respectively

$$X_m \square \Delta^n \xrightarrow{\phi^* \square 1} X_n \square \Delta^n \longrightarrow \bigsqcup_{n\geq 0} X_n \square \Delta^n$$

$$X_m \square \Delta^n \xrightarrow{1 \square \phi} X_m \square \Delta^m \longrightarrow \bigsqcup_{n\geq 0} X_n \square \Delta^n.$$

This is the obvious generalization of the geometric realization of Chapters I and III. Note that $|X|$ is a coend:

$$|X| = \int^{\mathbf{\Delta}} X \square \Delta,$$

where $\Delta$ denotes the covariant functor $\mathbf{n} \mapsto \Delta^n$ on $\mathbf{\Delta}$. We discuss the homotopical properties of $|X|$.

First note that $|\cdot| : s\mathcal{C} \to \mathcal{C}$ has a right adjoint

$$Y \mapsto Y^{\Delta} = \{\mathbf{hom}_{\mathcal{C}}(\Delta^n, Y)\}. \tag{3.2}$$

If we give $s\mathcal{C}$ the internal (or level-wise) simplicial structure induced from $\mathcal{C}$, it follows immediately that if $X \in s\mathcal{C}$ and $K \in \mathbf{S}$, then

$$|X \square K| \cong |X| \square K. \tag{3.3}$$

Indeed, $\hom_{s\mathcal{C}}(X \square K, Y^{\Delta}) \cong \hom_{s\mathcal{C}}(X, \mathbf{hom}_{\mathcal{C}}(K, Y)^{\Delta})$.

Now assume $\mathcal{C}$ is a simplicial model category. Endow $s\mathcal{C}$ with the Reedy model category structure. By Corollary 2.13, this is a simplicial model category in the internal simplicial structure.

LEMMA 3.4. *The functor $(\cdot)^{\Delta} : \mathcal{C} \to s\mathcal{C}$ preserves fibrations and trivial fibrations.*

PROOF: The first point to be proved is this: if $K \in \mathbf{S}$ and $Y \in \mathcal{C}$, then

$$M_K(Y^{\Delta}) \cong \mathbf{hom}_{\mathcal{C}}(K, Y). \tag{3.5}$$

There are isomorphisms

$$\begin{aligned} M_K(Y^{\Delta}) &\cong \varprojlim_{\Delta^n \to K} (Y^{\Delta})_n \\ &\cong \varprojlim_{\Delta^n \to K} \mathbf{hom}_{\mathcal{C}}(\Delta^n, Y) \\ &\cong \mathbf{hom}_{\mathcal{C}}( \varinjlim_{\Delta^n \to K} \Delta^n, Y) \\ &\cong \mathbf{hom}_{\mathcal{C}}(K, Y), \end{aligned}$$

where the limits and colimits are indexed over objects $\Delta^n \to K$ of the simplex category $\mathbf{\Delta} \downarrow K$ of $K$ (cf. Example 1.23).

Now let $X \to Y$ be a fibration in $\mathcal{C}$. Then

$$(X^{\Delta})_n \to (Y^{\Delta})_n \times_{M_n(Y^{\Delta})} M_n(X^{\Delta})$$

is isomorphic to

$$\mathbf{hom}_{\mathcal{C}}(\Delta^n, X) \to \mathbf{hom}_{\mathcal{C}}(\Delta^n, Y) \times_{\mathbf{hom}_{\mathcal{C}}(\partial\Delta^n, Y)} \mathbf{hom}_{\mathcal{C}}(\partial\Delta^n, X)$$

and the result follows from Lemma 2.2 and **SM7** for $\mathcal{C}$.

The claim about preservation of trivial fibrations has a similar proof. □

PROPOSITION 3.6. *The geometric realization functor* $| \cdot | : s\mathcal{C} \to \mathcal{C}$ *preserves cofibrations, trivial cofibrations and weak equivalences between Reedy cofibrant objects.*

PROOF: Use Lemma 3.4 and Lemma II.7.9. □

The proof of Lemma 3.4 implicitly involves the assertion that if $Z \in s\mathcal{C}$ is constant and $K \in \mathbf{S}$ then there is a natural isomorphism

$$|Z \otimes K| \cong Z \square K. \tag{3.7}$$

Indeed, using Proposition 1.10 the isomorphism (3.5), we have

$$\hom_{\mathcal{C}}(|Z \otimes K|, Y) \cong \hom_{\mathcal{C}}(Z, \mathbf{hom}_{\mathcal{C}}(K, Y)),$$

and the assertion follows. Therefore, for $X \in s\mathcal{C}$ Reedy cofibrant, the realization comes with a natural skeletal filtration. Define

$$\operatorname{sk}_n |X| = |\operatorname{sk}_n X|.$$

Then Proposition 1.7 and the natural isomorphism of (3.7) together show that there are natural pushout squares

$$\begin{tikzcd}
X_n\Box\partial\Delta^n \cup_{L_nX\Box\partial\Delta^n} L_n\Box\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}|X| \arrow[d] \\
X_n\Box\Delta^n \arrow[r] & \mathrm{sk}_n|X|.
\end{tikzcd} \tag{3.8}$$

This is because the realization functor $|\cdot|$ is a left adjoint and hence commutes with all colimits. If $X$ is cofibrant, then Proposition 3.6 and Corollary 2.14 together imply that each of the maps $\mathrm{sk}_{n-1}|X| \to \mathrm{sk}_n|X|$ is a cofibration. Furthermore, again since the functor $|\cdot|$ commutes with colimits

$$\varinjlim_n \mathrm{sk}_n|X| \cong |X|. \tag{3.9}$$

Finally if $X$ happens to be degeneracy free on some set $\{Z_n\}$ of objects in $\mathcal{C}$, then (3.8) can be refined (as in Corollary 1.14) to a pushout diagram

$$\begin{tikzcd}
Z_n\Box\partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}|X| \arrow[d] \\
Z_n\Box\Delta^n \arrow[r] & \mathrm{sk}_n|X|.
\end{tikzcd} \tag{3.10}$$

The object $\mathrm{sk}_n|X|$ can also be described as a coend. Let $\mathrm{sk}_n\Delta$ be the functor from $\mathbf{\Delta}$ to $\mathbf{S}$ with

$$\mathbf{m} \mapsto \mathrm{sk}_n\Delta^m.$$

PROPOSITION 3.11. *Let $X \in s\mathcal{C}$. Then there is a natural isomorphism*

$$\mathrm{sk}_n|X| \cong \int^{\mathbf{\Delta}} X\Box\,\mathrm{sk}_n\Delta$$

*and this isomorphism is compatible with the skeletal filtrations of source and target.*

PROOF: There is a sequence of natural isomorphisms, where

$$i_{n*} : s\mathcal{C} \to s_n\mathcal{C}$$

is the restriction functor

$$\begin{aligned}
\hom_{\mathcal{C}}(\mathrm{sk}_n|X|, Y) &\cong \hom_{s\mathcal{C}}(\mathrm{sk}_n X, Y^{\Delta}) \\
&\cong \hom_{s\mathcal{C}}(i_n^* i_{n*} X, Y^{\Delta}) \\
&\cong \hom_{s\mathcal{C}}(X, i_n^! i_{n*} Y^{\Delta})
\end{aligned}$$

Now $i_n^! i_{n*} Y^{\Delta} \cong \operatorname{cosk}_n(Y^{\Delta}) \cong Y^{\operatorname{sk}_n \Delta}$, where $Y^{\operatorname{sk}_n \Delta}$ is defined on the simplex level by

$$Y_r^{\operatorname{sk}_n \Delta} = \mathbf{hom}_{\mathcal{C}}(\operatorname{sk}_n \Delta^r, Y).$$

It follows that

$$\begin{aligned} \hom_{\mathcal{C}}(\operatorname{sk}_n |X|, Y) &\cong \hom_{s\mathcal{C}}(X, Y^{\operatorname{sk}_n \Delta}) \\ &\cong \hom_{\mathcal{C}}(\int^{\Delta} X \square \operatorname{sk}_n \Delta, Y). \end{aligned} \qquad \square$$

## 4. Cosimplicial spaces.

The language and technology of the previous three sections can be used to give a discussion of the homotopy theory of cosimplicial spaces; that is, of the category $c\mathbf{S} = \mathbf{S}^{\boldsymbol{\Delta}}$ of functors from the ordinal number category to simplicial sets. We go through some of the details and give some examples. It turns out that cofibrations in $c\mathbf{S}$ have a very simple characterization; we close the section with a proof of this fact.

We begin with two important examples.

Example 4.1. Let $R = \mathbb{F}_p$, the prime field with $p > 0$, or let $R$ be a subring of the rationals. The forgetful functor from $R$-modules to sets has left adjoint $X \mapsto RX$, where $RX$ is the free $R$-module on $X$. These functors prolong to an adjoint pair between simplicial $R$-modules and simplicial sets. By abuse of notation we write

$$R : \mathbf{S} \to \mathbf{S}$$

for the composite of these two functors. Then $R$ is the functor underlying a triple on $\mathbf{S}$ and, if $X \in \mathbf{S}$,

$$\pi_* RX \cong H_*(X; R).$$

Let $T : \mathbf{S} \to \mathbf{S}$ be any triple (or monad) on $\mathbf{S}$ with natural structure maps $\eta : X \to TX$ and $\epsilon : T^2X \to TX$. If $X \in \mathbf{S}$ is any object, there is a natural augmented cosimplicial space

$$X \to T^{\bullet} X$$

with $(T^{\bullet}X)^n = T^{n+1}X$ and

$$\begin{aligned} d^i &= T^i \eta T^{n+1-i} : (T^{\bullet}X)^n \to (T^{\bullet}X)^{n+1} \\ s^i &= T^i \epsilon T^{n-i} : (T^{\bullet}X)^{n+1} \to (T^{\bullet}X)^n. \end{aligned}$$

The augmentation is given by $\eta : X \to TX = (T^{\bullet}X)^0$; note that

$$d^0 \eta = d^1 \eta : X \to (T^{\bullet}X)^1.$$

In particular, if we let $T = R : \mathbf{S} \to \mathbf{S}$ we get an augmented cosimplicial space

$$X \to R^\bullet X$$

with the property that $d^i$, $i \geq 1$, and $s^i$, $i \geq 0$, are all morphisms of simplicial $R$-modules. Furthermore, if we apply $R$ one more time, the augmented cosimplicial $R$-module

$$RX \to R(R^\bullet X)$$

has a cosimplicial contraction; hence

$$H^s(H_*(R^\bullet X; R)) \cong \begin{cases} H_*(X; R), & s = 0 \\ 0, & s > 0. \end{cases}$$

The object $X \to R^\bullet X$ is a variation on the Bousfield-Kan $R$-resolution of $X$.

EXAMPLE 4.2. Let $J$ be a small category and $\mathbf{S}^J$ the category of $J$-diagrams in $\mathbf{S}$. Let $J^\delta$ be the category with the same objects as $J$ but no non-identity morphisms—$J^\delta$ is $J$ made discrete. There is an inclusion functor $J^\delta \to J$, hence a restriction functor

$$r_* : \mathbf{S}^J \to \mathbf{S}^{J^\delta}.$$

The functor $r_*$ has a right adjoint $r^!$ given by right Kan extension; in formulas

$$r^! X(j) = \prod_{j \to i} X(i)$$

where the product is over morphisms in $J$ with source $j$. Let

$$T = r^! r_* : \mathbf{S}^J \to \mathbf{S}^J.$$

Then $T$ is the functor of a triple on $\mathbf{S}^J$, and if $Y \in \mathbf{S}^J$, there is a natural cosimplicial object in $\mathbf{S}^J$

$$Y \to T^\bullet Y. \tag{4.3}$$

This cosimplicial object has the property that the underlying $J^\delta$ diagram has a cosimplicial contraction. Put another way, for each $j \in J$, the augmented cosimplicial space

$$Y(j) \to T^\bullet Y(j)$$

has a cosimplicial contraction. We can apply the functor $\varprojlim_J(\cdot)$ to (4.3) to obtain an augmented cosimplicial space

$$\varprojlim_J Y \to \varprojlim_J T^\bullet Y. \tag{4.4}$$

Note that $\varprojlim_J (T^\bullet Y)^n$ can be easily computed because

$$\varprojlim_J r^! X \cong \prod_j X(j)$$

where $j$ runs over the objects on $J$. This last assertion follows from the isomorphisms

$$\begin{aligned}\hom_{\mathbf{S}}(Z, \varprojlim_J r^! X) &\cong \hom_{\mathbf{S}^J}(Z, r^! X)\\ &\cong \hom_{\mathbf{S}^{J^\delta}}(Z, X) \cong \prod_j \hom_{\mathbf{S}}(Z, X(j))\end{aligned}$$

where $Z \in \mathbf{S}$ is regarded as a constant diagram in $\mathbf{S}^J$ or $\mathbf{S}^{J^s}$.

It follows that the functor $T^nY$ is defined for objects $j$ of $J$ by

$$T^nY(j) = \prod_{j \to j_0 \to \dots \to j_n} Y(j_n),$$

and that

$$\varprojlim_j T^nY(j) \cong \prod_{j_0 \to \dots \to j_n} Y(j_n).$$

The canonical map $\varprojlim_j T^n(j) \to T^n(j)$ can therefore be identified with the simplicial set map

$$\prod_{j_0 \to \dots \to j_n} Y(j_n) \to \prod_{j \to j_0 \to \dots \to j_n} Y(j_n)$$

whose projection onto the factor $Y(j_n)$ corresponding to the string

$$j \to j_0 \xrightarrow{\alpha_1} j_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} j_n$$

is the projection

$$pr_\alpha : \prod_{j_0 \to \dots \to j_n} Y(j_n) \to Y(j_n)$$

corresponding to the string

$$\alpha : j_0 \xrightarrow{\alpha_1} j_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} j_n.$$

One also finds that the cosimplicial structure map associated to $\theta : \mathbf{m} \to \mathbf{n}$ for the object $\varprojlim_j T^\bullet Y(j)$ can be identified with the unique simplicial set map

$$\prod_{i_0 \to \dots \to i_m} Y(i_m) \xrightarrow{\theta_*} \prod_{j_0 \to \dots \to j_n} Y(j_n)$$

which makes the diagrams

$$\begin{tikzcd}
\prod_{i_0 \to \dots \to i_m} Y(i_m) \arrow[r, "\theta^*"] \arrow[d, "pr_{\alpha \cdot \theta}"'] & \prod_{j_0 \to \dots \to j_n} Y(j_n) \arrow[d, "pr_\alpha"] \\
Y(j_{\theta(m)}) \arrow[r, "(\alpha_n \cdots \alpha_{\theta(m)+1})_*"'] & Y(j_n)
\end{tikzcd}$$

commute.

This is the standard description of the cosimplicial object which underlies the homotopy inverse limit of $Y \in \mathbf{S}^J$ — see Section VIII.1.

With these examples in hand, we begin to analyze the homotopy theory of $c\mathbf{S}$. Since $\mathbf{S}$ is a simplicial model category, so is the opposite category $\mathbf{S}^{\mathrm{op}}$. Thus the category $(s(\mathbf{S}^{\mathrm{op}}))^{\mathrm{op}} = c\mathbf{S}$ acquires a Reedy model category structure as in Section 2. We take some care with the definitions as the opposite category device can be confusing.

To begin with, $c\mathbf{S}$ is a simplicial category: if $K \in \mathbf{S}$ and $X \in c\mathbf{S}$ define $X \square K$ and $\mathbf{hom}_{c\mathbf{S}}(K, X)$ in $c\mathbf{S}$ by

$$(X \square K)^n = X^n \times K \tag{4.5}$$

and

$$\mathbf{hom}_{c\mathbf{S}}(K, X)^n = \mathbf{hom}(K, X^n). \tag{4.6}$$

One often writes $X \otimes K$ for $X \square K$; however, we are—in this chapter—reserving the tensor product notation for the external operation on $s\mathcal{C}$. The mapping spaces functor is then

$$\mathbf{Hom}_{c\mathbf{S}}(X, Y)_n \cong \hom_{c\mathbf{S}}(X \square \Delta^n, Y). \tag{4.7}$$

There are also latching and matching objects, but at this point the literature goes to pieces. The matching objects in $c\mathbf{S}$ defined in [BK] are the latching objects in $s(\mathbf{S}^{\mathrm{op}})$ as defined in Section 1. Since we would hope the reader will turn to the work of Bousfield and Kan as needed, we will be explicit.

Let $X \in c\mathbf{S}$. The $n^{\mathrm{th}}$ *matching space* $M^n X \in \mathbf{S}$ is the $(n+1)^{\mathrm{st}}$ *latching* object $L_{n+1}X$ of $X \in s(\mathbf{S}^{\mathrm{op}})$. Specifically,

$$M^n X \cong \varprojlim_{\phi : \mathbf{n+1} \to \mathbf{k}} X^k \tag{4.8}$$

where $\phi : \mathbf{n+1} \to \mathbf{k}$ runs over the surjections in $\mathbf{\Delta}$ with $k \leq n$. Thus $n \geq -1$, and Remark 1.8 implies the following:

LEMMA 4.9.

1. *The simplicial set $M^{-1}X$ is isomorphic to the terminal object $*$.*
2. *There is an isomorphism $M^0 X \cong X^0$, and the canonical map $X^1 \to M^0 X$ can be identified with the codegeneracy map $s^0 : X^1 \to X^0$.*
3. *For $n > 0$, the object $M^n X$ is defined by the equalizer*

$$M^n X \to \prod_{i=0}^{n} X^n \rightrightarrows \prod_{0 \leq i < j \leq n} X^{n-1}$$

*where the images of the two displayed maps on the factor corresponding to the relation* $i < j$ *on* $X_{n-1}$ *are given by the composites*

$$\prod_{i=0}^{n} X^n \xrightarrow{pr_j} X^n \xrightarrow{s^i} X^{n-1} \quad \textit{and} \quad \prod_{i=0}^{n} X^n \xrightarrow{pr_i} X^n \xrightarrow{s^{j-1}} X^{n-1}.$$

*The canonical map* $s : X^{n+1} \to M^n X$ *is induced by the codegeneracies* $s^i : X^{n+1} \to X^n$.

The reader should be aware of 1) the shift in indices $M^n X = L_{n+1} X$ and 2) the superscript versus the subscript: $M^n X \neq M_n X$.

A map $f : X \to Y$ in $c\mathbf{S}$ is a *fibration* if and only if

$$X^{n+1} \to Y^{n+1} \times_{M^n Y} M^n X \tag{4.10}$$

is a fibration for $n \geq -1$.

Similarly, there are latching objects. If $X \in c\mathbf{S}$, $L^n X = M_{n+1} X$, the matching spaces of $X \in s(\mathbf{S}^{\mathrm{op}})$, where $n \geq -1$; thus

$$L^n X = \varinjlim_{\phi : \mathbf{k} \to \mathbf{n+1}} X^k$$

where $\phi$ runs over the injections in $\mathbf{\Delta}$ with $k \leq n$. The following is a consequence of Remark 1.19:

LEMMA 4.11.

(1) *The space* $L^{-1} X$ *is the initial object* $\emptyset$ *in the category of simplicial sets.*

(2) *There is an isomorphism* $L^0 X \cong X^0 \sqcup X^0$*, and the canonical map* $L^0 X \to X^1$ *can be identified with the coproduct* $d = (d^0, d^1) : X^0 \sqcup X^0 \to X^1$ *of the coface maps* $d^0, d^1 : X^0 \to X^1$.

(3) *For* $n > 1$*, the latching object* $L^n X$ *is defined by a coequalizer diagram*

$$\bigsqcup_{0 \leq i < j \leq n+1} X^{n-1} \rightrightarrows \bigsqcup_{i=0}^{n+1} X^n \to L^n X.$$

*Here, the restrictions of the displayed maps on the summand* $X^{n-1}$ *corresponding to the relation* $i < j$ *are the composites*

$$X^{n-1} \xrightarrow{d^i} X^n \xrightarrow{in_j} \bigsqcup_{i=0}^{n+1} X^n \quad \textit{and} \quad X^{n-1} \xrightarrow{d^{j-1}} X^n \xrightarrow{in_i} \bigsqcup_{i=0}^{n+1} X^n.$$

*The canonical map* $d : L^n X \to X^{n+1}$ *is induced by the coface maps* $d^i : X^n \to X^{n+1}$.

A morphism $X \to Y$ in $c\mathbf{S}$ is a *cofibration* if and only if

$$X^{n+1} \cup_{L^n X} L^n Y \to Y^{n+1} \tag{4.12}$$

is a cofibration (that is, inclusion) of simplicial sets.

Finally, we define a morphism $X \to Y$ in $c\mathbf{S}$ to be a weak equivalence if $X^n \to Y^n$ is a weak equivalence for all $n \geq 1$.

THEOREM 4.13. *With the definitions above, the category $c\mathbf{S}$ of cosimplicial spaces is a proper closed simplicial model category.*

PROOF: Applying Theorem 2.12 and Corollary 2.13 to the case of the category $s\mathbf{S}^{op}$ of simplicial objects in $\mathbf{S}^{op}$ gives the simplicial model structure. Properness is a consequence of Corollary 2.6 □

We can give a simple characterization of cofibrations in $c\mathbf{S}$, and along the way show that $c\mathbf{S}$ is cofibrantly generated. First, let us define a set of specific cofibrations.

The functors from $c\mathbf{S} \to \mathbf{S}$

$$\rho_n : X \mapsto X^n, n \geq 0$$

and

$$\mu_n : X \mapsto M^n X, n \geq -1$$

all have left adjoints, given by variations on left Kan extension. Indeed, the adjoint to $\rho_n$ is given by the formula

$$(\rho_n^* Y)^k = \bigsqcup_{\phi:\mathbf{n}\to\mathbf{k}} Y$$

where $\phi$ runs over all morphisms in $\mathbf{\Delta}$ with source $\mathbf{n}$. This is a left Kan extension.

The adjoint to $\mu_n$ is slightly more complicated: if $J$ is the category with objects surjections $\mathbf{n+1} \to \mathbf{k}$ in $\mathbf{\Delta}$, and $r : J \to \mathbf{\Delta}$ the functor sending $\mathbf{n+1} \to \mathbf{k}$ to $\mathbf{k}$, then the left adjoint $\mu_n^*$ to $\mu_n$ is characterized by

$$\begin{aligned} \hom_{c\mathbf{S}}(\mu_n^* Z, X) &\cong \hom_{\mathcal{S}^J}(Z, r_* X) \\ &\cong \hom_{\mathbf{S}}(Z, M^n X) \end{aligned}$$

where $Z \in \mathbf{S}^J$ is the constant diagram. Thus $\mu_n^* Z$ is a left Kan extension of a constant diagram. Alternatively, one can use the equalizer description of $M^n X$ given in Lemma 4.9 to show that there is a natural coequalizer

$$\bigsqcup_{0\leq i<j\leq n} \rho_{n-1} Z \rightrightarrows \bigsqcup_{i=o}^{n} \rho_n Z \to \mu_n Z$$

for $n > 0$, and that $\mu_0 Z \cong \rho_0 Z$.

Note that the natural transformation

$$s : X^n \to M^{n-1}X$$

induces a natural transformation

$$\mu_{n-1}^* Z \to \rho_n^* Z.$$

Define morphisms in $c\mathbf{S}$ as follows:

$$\rho_n^* \partial\Delta^m \cup_{\mu_{n-1}^* \partial\Delta^m} \mu_{n-1}^* \Delta^m = \partial\Delta \begin{bmatrix} m \\ n \end{bmatrix} \xrightarrow{i_n^m} \Delta \begin{bmatrix} m \\ n \end{bmatrix} = \rho_n^* \Delta^m, \tag{4.14}$$

for $n \geq 0$, $m \geq 0$, and

$$\rho_n^* \Lambda_k^m \cup_{\mu_{n-1}^* \Lambda_k^m} \mu_{n-1}^* \Delta^m = \Delta \begin{bmatrix} m \\ n,k \end{bmatrix} \xrightarrow{j_{n,k}^m} \Delta \begin{bmatrix} m \\ n \end{bmatrix} = \rho_n^* \Delta^m, \tag{4.15}$$

for $n \geq 0$, $0 \leq k \leq m$, $m \geq 1$.

LEMMA 4.16. *A morphism $f : X \to Y$ in $c\mathbf{S}$ is a fibration if and only if it has the right lifting property with respect to the morphisms $j_{n,k}^m$ of (4.15). A morphism in $c\mathbf{S}$ is a trivial fibration if and only if it has the right lifting property with respect to the morphisms $i_n^m$ of (4.14).*

PROOF: We prove the trivial fibration case; the other is similar. A lifting problem

$$\begin{tikzcd}
\partial\Delta \begin{bmatrix} m \\ n \end{bmatrix} \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta \begin{bmatrix} m \\ n \end{bmatrix} \arrow[r] & Y
\end{tikzcd}$$

is equivalent, by adjointness to a lifting problem

$$\begin{tikzcd}
\partial\Delta^m \arrow[r] \arrow[d] & X^n \arrow[d, "s"] \\
\Delta^m \arrow[r] & Y^n \times_{M^{n-1}Y} M^{n-1}X.
\end{tikzcd}$$

Lemma 2.2.2 implies that $f : X \to Y$ is a trivial fibration if and only if

$$(f, s) : X^n \to Y^n \times_{M^{n-1}Y} M^{n-1}X$$

is a trivial fibration for all $n$. The result follows. ☐

PROPOSITION 4.17. *The simplicial model category structure on $c\mathbf{S}$ is cofibrantly generated: the morphisms $i_n^m$ of (4.14) generate the cofibrations and the morphisms $j_{n,k}^m$ of (4.15) generate the trivial cofibrations.*

PROOF: In light of Lemma 4.16, the small object argument now applies. □

We can use this result to characterize cofibrations. If $X \in c\mathcal{C}$ is a cosimplicial object in any category $\mathcal{C}$ with enough limits, define the maximal augmentation $H^0X$ by the equalizer diagram

$$H^0X \to X^0 \overset{d^0}{\underset{d^1}{\rightrightarrows}} X^1.$$

Let $d^0 :^0 X \to X^0$ be the natural map.

PROPOSITION 4.18. *A morphism $f : X \to Y$ in $c\mathbf{S}$ is a cofibration if and only if $X^n \to Y^n$ is a cofibration in $\mathbf{S}$ for all $n \geq 0$ and the induced map $H^0X \to H^0Y$ of maximal augmentations is an isomorphism.*

We give the proof below, after some technical preliminaries.

Let $\mathbf{\Delta}_{-1}$ be the augmented ordinal number category; this has objects

$$\mathbf{n} = \{0, 1, \dots, n-1\}, n \geq -1,$$

$(-\mathbf{1} = \phi)$ and ordering preserving maps. An augmented cosimplicial object in $\mathcal{C}$ is a functor $X : \mathbf{\Delta}_{-1} \to \mathcal{C}$. We write $d^0 : X^{-1} \to X^0$ for the unique map. Note that the maximal augmentation extends any cosimplicial object to an augmented cosimplicial object.

LEMMA 4.19. *Let $X$ be an augmented cosimplicial set, and*

$$Z^n = X^n - \bigcup_{i=0}^{n} d^i X^{n-1}$$

*If $d^0 : X^{-1} \to X^0$ is isomorphic to the inclusion of the maximal augmentation, then the map*

$$\bigsqcup_{\phi} Z^k \to X^n$$

*with $\phi : \mathbf{k} \to \mathbf{n}$ running over all injections, $-1 \leq k \leq n$, is an isomorphism.*

PROOF: Out of any cosimplicial set $X$ we may construct a simplicial set $Y$ "without $d^0$" as follows: $Y_n = X^n$ and

$$d_i = s^{n-i} : Y_n \to Y_{n-1}, \ 1 \leq i \leq n$$
$$s_i = d^{n-i} : Y_n \to Y_{n+1}, \ 0 \leq i \leq n.$$

Notice that this construction does not use $d^n : X^{n-1} \to X^n$. Let $Z_n \subseteq Y_n$ be the non-degenerate simplices:

$$Z_n = Y_n - \bigcup_i s_i Y_{n-1}.$$

The standard argument for simplicial sets (see Example 1.15) shows that

$$Y_n \cong \bigsqcup_{\psi:\mathbf{n}\to\mathbf{k}} Z_k$$

where $\psi$ runs over the surjections in $\mathbf{\Delta}$. Unraveling the definitions shows that our claim will follow if we can show that $d^n : X^{n-1} \to X^n$ induces an isomorphism $Z^n \sqcup Z^{n-1} \xrightarrow{\cong} Z_n$ or, equivalently, an isomorphism

$$Z^{n-1} \xrightarrow{\cong} Z_n \cap d^n X^{n-1}.$$

First note that $d^n$ does induce an injection

$$d^n : Z^{n-1} \to Z_n \cap d^n X^{n-1}. \tag{4.20}$$

For this, it is sufficient to show that if $y \in Z^{n-1}$, then $d^n y \in Z_n$; that is, if $y \notin \cup_{i=0}^{n-1} d^i X^{n-2}$, then $d^n y \notin \cup_{i=0}^{n-1} d^i X^{n-1}$. The contrapositive of this statement reads: if $d^n y \in \cup_{i=0}^{n-1} d^i X^{n-1}$, then $y \in \cup_{i=0}^{n-1} d^i X_{n-2}$. So assume $d^n y = d^i z$ with $i < n$. If $i < n-1$, then

$$z = s^i d^n y = d^{n-1} s^i y$$

so

$$d^n y = d^i z = d^i d^{n-1} s^i y = d^n d^i s^i y$$

and $y = d^i s^i y$. If $i = n-1$ and $n > 1$, then

$$y = s^{n-1} d^n y = s^{n-1} d^{n-1} z = z$$

so $d^n y = d^{n-1} y$; hence

$$y = s^{n-2} d^{n-1} y = s^{n-2} d^n y = d^{n-1} s^{n-2} y.$$

If $n = 1$ and $i = n-1$, we have $d^1 y = d^0 z$, hence $y = z$; since $d^0 : X^{-1} \to X^0$ is the inclusion of the maximal augmentation, $y = d^0 w$ for some $w$ This is where the hypothesis is used.

We now must show that $d^n$, as in (4.20) is onto. If $n = 0$, $Z_0 = X^0$ and the result is clear. If $n \geq 1$, we need to show that if $x = d^n y$ and $x \notin \cup_{i=0}^{n-1} d^i X^{n-1}$, then $y \notin \cup_{i=0}^{n-1} d^i X^{n-2}$. The contrapositive of this statement is: if $x = d^n y$ and $y = d^i w$, $i \leq n-1$, then $x = d^j z$ for $j \leq n-1$. But

$$x = d^n y = d^n d^i w = d^i d^{n-1} w. \qquad \square$$

We can now prove Proposition 4.18:

PROOF OF 4.18: For the purposes of this argument, we say a morphism $f : X \to Y$ in $c\mathbf{S}$ has *Property* $\mathbf{C}$ if $X^n \to Y^n$ is a cofibration in $\mathbf{S}$ for all $n \geq 0$ and $H^0X \cong H^0Y$. We leave it as an exercise to show that the class of morphisms satisfying Property $\mathbf{C}$ is closed under isomorphisms, coproducts, retracts, cobase change, and colimits over ordinal numbers. Only the statement about cobase change is non-trivial. Furthermore, the generating cofibrations $\iota_n^m : \partial\Delta\left[{m \atop n}\right] \to \Delta\left[{m \atop n}\right]$ have Property $\mathbf{C}$. Hence Proposition 4.17 implies all cofibrations have Property $\mathbf{C}$.

For the converse, suppose $f : X \to Y$ has Property $\mathbf{C}$. Referring to Lemma 4.19, write $Z^n(X_m)$ for $Z^n$ obtained from the cosimplicial set of $m$ simplices $X_m$. Then, Lemma 4.19 implies

$$(L^0X)_m \cong Z^0(X_m) \sqcup Z^{-1}(X_m) \sqcup Z^0(X_m) \sqcup Z^{-1}(X_m),$$

$$X^1_m = Z^{-1}(X_m) \sqcup Z^0(X_m) \sqcup Z^0(X_m) \sqcup Z^1(X_m),$$

and if $n > 1$,

$$(L^{n-1}X)_m \cong \bigsqcup_{\phi:\mathbf{k}\to\mathbf{n}} Z^k(X_m)$$

with $\phi$ running over injections with $-1 \leq k < n$. Since $f : X^n_m \to Y^n_m$ is one-to-one, $Z^k(\cdot)$ is natural in $f$. Since $Z^{-1}(X_m) \cong Z^{-1}(Y_m)$,

$$[X^n \cup_{L^{n-1}X} L^{n-1}Y]_m \cong Z^n(X_m) \sqcup \bigsqcup_{\phi:\mathbf{k}\to\mathbf{n}} Z^k(Y_m)$$

for all $n \geq 0$. Again $\phi$ runs over injections $k$, $-1 \leq k < n$. The result follows. □

SECOND PROOF OF PROPOSITION 4.18: Note, first of all, that by manipulating cosimplicial identities, one can show that all of the diagrams

$$\begin{tikzcd}
X^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & X^{n-1} \arrow[d, "d^i"] \\
X^{n-1} \arrow[r, "d^j"'] & X^n
\end{tikzcd}$$

are pullbacks. It follows that the maps $d : L^{n-1}X \to X^n$ is a monomorphism if $n > 1$. Note further that Lemma 4.11 says that $L^0X = X^0 \sqcup X^0$.

Now suppose that $f : X \to Y$ is a cofibration. Then $f : X^0 \to Y^0$ is monic, so that $f : L^0X \to L^0Y$ is monic, and the assumption that the map

$$L^0Y \cup_{L^0X} X^1 \to Y^1$$

is a monomorphism implies that $f : X^1 \to Y^1$ is a monomorphism. One uses cosimplicial identities (using Remark 1.19) to show that if $f : X^i \to Y^i$ is a monomorphism for $i \leq n$ then the induced map $L^n X \to L^n Y$ is a monomorphism. Then the assumption that $f$ is a cofibration implies that $f : X^{n+1} \to Y^{n+1}$ is a monomorphism in degree $n+1$. In particular, $f$ is a monomorphism in all degrees.

To see that the map $f : H^0 X \to H^0 Y$ on maximal augmentations is an isomorphism, observe that there is a natural coequalizer

$$H^0 X \rightrightarrows X^0 \sqcup X^0 \to \mathrm{im}(d),$$

where

$$\mathrm{im}(d) = \mathrm{im}(d^0) \cup \mathrm{im}(d^1) \subset X^1.$$

Write $PO = \mathrm{im}(f) \cup \mathrm{im}(d) \subset Y^1$ for the diagram

$$\begin{tikzcd}
X^0 \sqcup X^0 \arrow[r, "f \sqcup f"] \arrow[d, "d"'] & Y^0 \sqcup Y^0 \arrow[d, "d"] \\
X^1 \arrow[r, "f"'] & Y^1
\end{tikzcd}$$

This diagram is a pullback by cosimplicial identities, so the induced diagram

$$\begin{tikzcd}
X^0 \sqcup X^0 \arrow[r, "f \sqcup f"] \arrow[d, "d"'] & Y^0 \sqcup Y^0 \arrow[d, "d_*"] \\
X^1 \arrow[r, "f_*"'] & PO
\end{tikzcd} \tag{4.21}$$

is a pullback. This latter diagram (4.21) is also a pushout if and only if the induced diagram

$$\begin{tikzcd}
X^0 \sqcup X^0 \arrow[r, "f \sqcup f"] \arrow[d, "d"'] & Y^0 \sqcup Y^0 \arrow[d, "d_*"] \\
\mathrm{im}(d) \arrow[r] & \mathrm{im}(d_*)
\end{tikzcd}$$

is a pushout, since epi-monic factorizations are preserved by pushout. The diagram (4.21) is therefore a pushout if and only if $f$ induces an isomorphism $H^0 X \cong H^0 Y$. The map

$$L^0 Y \cup_{L^0 X} Y^1 \to X^1$$

is therefore a monomorphism if and only if the diagram (4.21) is a pushout.

It follows that the map $f : H^0X \to H^0Y$ on maximal augmentations is a bijection.

For the converse, one can show that any levelwise monomorphism $f : X \to Y$ induces monomorphisms $L^nX \to L^nY$, and that all induced diagrams

$$\begin{tikzcd}
L^nX \arrow[r, "d"] \arrow[d] & X^{n+1} \arrow[d] \\
L^nY \arrow[r, "d"'] & Y^{n+1}
\end{tikzcd}$$

are pullbacks. The maps $d$ are monomorphisms if $n > 0$, as are the vertical maps, so the induced maps

$$L^nY \cup_{L^nX} X^{n+1} \to Y^{n+1}$$

are monomorphisms for $n > 0$. The assertion that the map

$$L^0Y \cup_{L^0X} X^1 \to Y^1$$

is a monomorphism when $f$ is a levelwise monic that induces an isomorphism of maximal augmentations is proved in the previous paragraph. □

Proposition 4.18 makes it very easy to decide when an object of $c\mathbf{S}$ is cofibrant for the Reedy structure. For example, a constant object on a non-empty simplicial set is not cofibrant, but the standard simplices $\Delta^n$ form a cosimplicial space $\Delta$ which is cofibrant. Also, every subobject of a cofibrant simplicial space is cofibrant.

# Chapter VIII Cosimplicial spaces: applications

The homotopy spectral sequence of a cosimplicial space is one of the most commonly used tools in homotopy theory. It first appeared in the work of Bousfield and Kan [14] and has been further analyzed by Bousfield [10]. Two of the standard examples include the Bousfield-Kan spectral sequence — an unstable Adams spectral sequence that arose before the general example [7] — and the spectral sequence for computing the homotopy groups of the homotopy inverse limit of a diagram of pointed spaces. The main purpose of this chapter is to define and discuss this spectral sequence, and outline some of its applications.

We begin with the point of view that the total space $\mathrm{Tot}(X)$ of a cosimplicial space $X$ is dual to the geometric realization of $X$, when interpreted as an element of $s\mathbf{S}^{op}$; that is,

$$\mathrm{Tot}(\cdot) = |\cdot|^{op} : c\mathbf{S} = (s(\mathbf{S})^{op})^{op} \to (\mathbf{S}^{op})^{op},$$

where the realization functor on $s(\mathbf{S}^{op})$ is that of Chapter VII. Simplicial objects are filtered colimits of skeleta, so that $\mathrm{Tot}(X)$ is the inverse limit of a tower of fibrations

$$\mathrm{Tot}_0(X) \leftarrow \mathrm{Tot}_1(X) \leftarrow \mathrm{Tot}_2(X) \leftarrow \cdots$$

if $X$ is a fibrant cosimplicial space. It is then a matter of manipulating adjoints to show that $\mathrm{Tot}(X)$ can expressed in the standard way in terms of function complexes as

$$\mathrm{Tot}(X) = \mathbf{Hom}(\Delta, X),$$

and that

$$\mathrm{Tot}_n(X) = \mathbf{Hom}(\mathrm{sk}_n\,\Delta, X).$$

If $X$ is pointed, the tower is pointed, and there is spectral sequence in homotopy groups arising from the homotopy spectral sequence for a tower of fibrations, as constructed in Section VI.2. This is the homotopy spectral sequence of a fibrant pointed cosimplicial space, and its construction occupies Section 1. One of the main results is the standard cohomological identification of the $E_2$-term.

The discussion of the two fundamental examples VII.4.1 and VII.4.2 continues in the following two sections.

We have collected together the basic results on homotopy inverse limits in Section 2. We give various constructions of the homotopy inverse limit functor, including an identification with the total right derived functor for inverse limit which arises from a pointwise cofibration model structure for a diagram category of simplicial sets (Proposition 2.4). The section closes with an identification of the $E_2$-term of the homotopy spectral sequence for a homotopy inverse limit, expressed in terms of the higher derived functors of inverse limit for categories of groups — the resulting spectral sequence is (2.18).

Bousfield-Kan completions are the object of study of Section 3, and appear again at the end of Section 4. In particular, the $E_2$-term of the homotopy spectral sequence for the $p$-completion $X_p = \operatorname{Tot}\mathbb{F}_p^\bullet X$ of a space $X$ is identified with higher ext groups, computed in the category of unstable coalgebras over the Steenrod algebra, and yielding the well known unstable Adams spectral sequence for the homotopy groups of the $X_p$ (3.6) of Bousfield and Kan. We give a brief overview of the homotopy properties of the $p$-completion of a nilpotent space at the end of the section.

The spectral sequence of a cosimplicial space inherits several technical difficulties from the general construction of the homotopy spectral sequence for a tower of fibrations, including the definition of the spectral sequence in low degrees in homotopy, where one might not have groups, let alone abelian groups, and the question of whether the spectral sequence converges to $\pi_* \operatorname{Tot}(X)$ or not. There is also a possibility that $\operatorname{Tot}(X)$ might be empty — it is an inverse limit, and inverse limits can be empty. To analyze this question, and to address the calculation of $\pi_0 \operatorname{Tot}(X)$ in general, Bousfield has developed an extensive obstruction theory in [10]. We give a small, but very useful, example of this theory in Section 4: the main results are Lemma 4.6 and Theorem 4.9. The section closes with an application (Proposition 4.16) which asserts that a map $H^*Y \to H^*X$ in mod $p$ cohomology can be lifted to a map of spaces $X \to Y_p$ if certain obstructions vanish.

## 1. The homotopy spectral sequence of a cosimplicial space.

The results on geometric realization from Section VII.3, interpreted in the context of cosimplicial spaces, lead to the study of the total space of a cosimplicial space and an associated tower of fibrations. The purpose of this section is to work through these results.

If $X \in c\mathbf{S} \cong (s(\mathbf{S}^{\mathrm{op}}))^{\mathrm{op}}$, we can define the geometric realization $|X| \in (\mathbf{S}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{S}$; we will write $\operatorname{Tot}(X)$ for this object. Using the coequalizer VII.(3.1), suitably interpreted, we find that $\operatorname{Tot}(X)$ fits into an equalizer diagram

$$\operatorname{Tot}(X) \to \prod_{n\geq 0} \mathbf{Hom}(\Delta^n, X^n) \rightrightarrows \prod_{\phi:\mathbf{n}\to\mathbf{m}} \mathbf{Hom}(\Delta^m X^m). \tag{1.1}$$

If $\Delta = \{\Delta^n\} \in c\mathbf{S}$ is the cosimplicial space composed of the standard simplices, (1.1) implies there is a natural isomorphism in $\mathbf{S}$

$$\operatorname{Tot}(X) \cong \mathbf{Hom}_{s\mathbf{S}}(\Delta, X). \tag{1.2}$$

Since $\Delta \in c\mathbf{S}$ is cofibrant, **SM7** for $c\mathbf{S}$ implies that $\operatorname{Tot}(\cdot)$ preserves weak equivalences between fibrant objects; alternatively, this fact is a consequence of Proposition VII.3.6.

The skeletal filtration on the geometric realization (VII.(3.8), VII.(3.9)) dualizes to $\operatorname{Tot}(X)$ as the inverse limit of a tower. We define $\operatorname{Tot}_n(X) = \operatorname{sk}_n |X| \in (\mathbf{S}^{\mathrm{op}})^{\mathrm{op}}$; arguing as in (1.1) and (1.2) and using Proposition VII.3.11, we have

$$\operatorname{Tot}_n(X) \cong \mathbf{Hom}_{c\mathbf{S}}(\operatorname{sk}_n \Delta, X) \tag{1.3}$$

and the natural projection $\mathrm{Tot}_n(X) \to \mathrm{Tot}_{n-1}(X)$ is induced by the inclusion $\mathrm{sk}_{n-1}\Delta \to \mathrm{sk}_n \Delta$. Note that

$$\mathrm{Tot}(X) \cong \varprojlim \mathrm{Tot}_n(X). \tag{1.4}$$

By Proposition VII.4.18, $\mathrm{sk}_{n-1}\Delta \to \mathrm{sk}_n \Delta$ is a cofibration in $c\mathbf{S}$, hence, if $X \in c\mathbf{S}$ is fibrant, **SM7** for $c\mathbf{S}$ implies that

$$\{\mathrm{Tot}_n(X)\}_{n\geq 0} \tag{1.5}$$

is a tower of fibrations. This fact equally follows from Proposition VII.3.6 and Corollary VII.2.14. This is the *total tower* of $X$.

The diagrams VII.(3.8) and VII.(3.10) tell us how $\mathrm{Tot}_n(X)$ is built from $\mathrm{Tot}_{n-1}(X)$. Recalling that for $X \in c\mathbf{S}$, $M^{n-1}X \cong L_nX$, $X \in s(\mathbf{S}^{\mathrm{op}})$, VII.(3.8) implies that for all $X \in c\mathbf{S}$ there is a pullback diagram

$$\begin{tikzcd}
\mathrm{Tot}_n(X) \arrow[r] \arrow[d] & \mathbf{Hom}(\Delta^n, X^n) \arrow[d] \\
\mathrm{Tot}_{n-1}(X) \arrow[r] & \mathbf{Hom}(\partial\Delta^n, X^n) \times_{\mathbf{Hom}(\partial\Delta^n, M^{n-1}X)} \mathbf{Hom}(\Delta^n, M^{n-1}X).
\end{tikzcd} \tag{1.6}$$

Note $\mathrm{Tot}_0(X) \cong X^0$. If $X$ is fibrant, the vertical maps in (1.6) are fibrations. Finally, if the cosimplicial object $X \in c\mathbf{S}$ is codegeneracy-free in the sense that the underlying codegeneracy diagram is the right Kan extension of a discrete diagram $\{Z^n\}$, then VII.(3.10) implies that there is a pullback diagram in $\mathbf{S}$.

$$\begin{tikzcd}
\mathrm{Tot}_n X \arrow[r] \arrow[d] & \mathbf{Hom}(\Delta^n, Z^n) \arrow[d] \\
\mathrm{Tot}_{n-1} X \arrow[r] & \mathbf{Hom}(\partial\Delta^n, Z^n).
\end{tikzcd} \tag{1.7}$$

If $X$ is a pointed fibrant cosimplicial space, then the tower $\{\mathrm{Tot}_n X\}_{n\geq 0}$ has a homotopy spectral sequence which arises from the homotopy spectral sequence for a tower of fibrations of Section VI.2. The constant map is the basepoint of $\mathrm{Tot}_n X$ and by the diagram (1.6), the fibre of $\mathrm{Tot}_n X \to \mathrm{Tot}_{n-1} X$ is $\Omega^n N^n X$, where $N^n X$ is the fibre of $s : X^n \to M^{n-1}X$. Thus the $E_1$ term of the associated spectral sequence is, for $t - s \geq 0$,

$$E_1^{s,t} = \pi_{t-s}\Omega^s N^s X \cong \pi_t N^s X.$$

If $G$ is a cosimplicial (not necessarily abelian) group, define

$$N^s G = \bigcap_{i=0}^{s-1} \ker\{s^i : G^s \to G^{s-1}\}.$$

If $G$ is a cosimplicial abelian group, then $NG = \{N^s G, \Sigma(-1)^i d^i\}$ becomes a cochain complex. Define $N\pi_0 X = \pi_0 X^0$.

LEMMA 1.8. *For a fibrant pointed cosimplicial space $X$, there is a natural isomorphism*

$$E_1^{s,t} \cong \pi_t N^s X \cong N^s \pi_t X, \ t - s \geq 0$$

PROOF: First assume $t \geq 2$. The cases $t = 0, 1$ will be handled at the end. We use the pullback square

$$\begin{tikzcd}
M^{n-1,k+1}X \arrow[r] \arrow[d] & X^{n-1} \arrow[d] \\
M^{n-1,k}X \arrow[r] & M^{n-2,k}X
\end{tikzcd} \tag{1.9}$$

of Proposition VII.1.25, and a double induction on the following two statements. Notice that $M^{n,k}$ is a functor defined on $c\mathcal{C}$ where $\mathcal{C}$ is any category with finite limits.

> $H(n,k)$: The natural map $\pi_t M^{n-1,k}X \to M^{n-1,k}\pi_t X$ is an isomorphism.
>
> $I(n,k)$: Let $N^{n,k}X$ be the fibre of $s : X^n \to M^{n-1,k}X$. Then
>
> $$0 \to \pi_t N^{n,k}X \to \pi_t X^n \to \pi_t M^{n-1,k}X \to 0$$
>
> is short exact.

The result follows from setting $n = k = s$ in $I(n,k)$ and $H(n,k)$.

First notice that $I(0,0)$ is trivial since $M^{-1,0}X = *$. Also $H(n,0)$ and $I(n,0)$ are trivial since $M^{n-1,0}X = *$ and $M^{n-1,0}\pi_t X = 0$.

Next, $H(n,k), H(n-1,k)$, and $I(n-1,k)$ yield $H(n,k+1)$, since $\pi_t(\cdot)$ applied to (1.9) gives a pullback diagram

$$\begin{tikzcd}
\pi_t M^{n-1,k+1}X \arrow[r] \arrow[d] & \pi_t X_{n-1} \arrow[d] \\
M^{n-1,k}\pi_t X \arrow[r] & M^{n-2,k}\pi_t X.
\end{tikzcd}$$

Finally, $H(n,k) \Rightarrow I(n,k)$ because $\pi_t X^n \to M^{n-1,k}\pi_t X$ is surjective. Indeed,

$$M^{n-1,k}\pi_t X \cong \bigoplus_{\phi:\mathbf{n}\to\mathbf{m}} N^m \pi_t X$$

where $\phi$ runs over the objects of $\mathcal{M}_{n,k}$.

For the cases $t = 0$ and $1$ of Lemma 1.8, note that if $t = 0$ then $s = 0$, so one need only check $\pi_0 N^0 X = \pi_0 X^0 = N\pi^0 X$ which is true since $N^0 X = X^0$.

If $t = 1$ then $s = 0$ or 1. If $s = 0$, one has $\pi_1 N^0 X = \pi_1 X^0$ as required. If $s = 1$, the sequence

$$* \to \pi_1 N^1 X \to \pi_1 X^1 \xrightarrow{s^0} \pi_1 X^0 \to *$$

is exact, since $s^0 d^0 = 1$ and $X^0 = M^0 X$. □

We single out the following statement from the proof of Lemma 1.8:

COROLLARY 1.10. *Suppose that $X$ is a fibrant pointed cosimplicial space. Then there are natural short exact sequences*

$$* \to \pi_t N^n X \to \pi_t X^n \to \pi_t M^n X \to *$$

*for all $n, t \geq 1$.*

Lemma 1.8 is used as the input for the calculation of the $E_2$ term. In fact, we wish to claim that $E_2^{s,t}$ is a cohomology group $H^s(\pi_t X)$, but we need to take some care with the definitions, especially for $t = 1$.

If $A$ is a cosimplicial abelian group, we can make $A$ into a cochain complex

$$A^0 \xrightarrow{\partial} A^1 \xrightarrow{\partial} A^2 \xrightarrow{\partial} \cdots$$

with $\partial = \Sigma(-1)^i d^i$. Following Bousfield and Kan, we shall write $\pi^s A = H^s A$ for the $s^{th}$ cohomology group of the cochain complex associated to $A$, say that these cohomology groups are the *cohomotopy groups* of the cosimplicial abelian group $A$.

It follows from Theorem III.2.5 that computing cohomotopy groups for $A$ is equivalent to computing $H^s N^* A$, where $N^* A$ is the normalized cochain complex with

$$N^* A = \{N^s A, \Sigma(-1)^i d^i\}.$$

This means that one has natural isomorphisms

$$\pi^s A = H^s A \cong H^s N^* A. \tag{1.11}$$

If $G$ is a cosimplicial (not necessarily abelian) group, one still has cohomotopy objects $\pi^0 G$ and $\pi^1 G$. The group $\pi^0 G$ is the equalizer of $d^0$, $d^1$:

$$\pi^0 G \to G^0 \overset{d^0}{\underset{d^1}{\rightrightarrows}} G^1. \tag{1.12}$$

The pointed set $\pi^1 G$ is defined by the formula

$$\pi^1 G = Z^1 G / G^0 \tag{1.13}$$

where $Z^1 G$ is the pointed set of cocycles

$$Z^1 G = \{x \in G^1 | (d^2 x)(d^1 x)^{-1}(d^0 x) = e\}$$

and $G^0$ acts on $Z^1G$ by

$$g \circ x = d^0g \cdot x(d^1g)^{-1}.$$

Notice that if $G$ is a cosimplicial abelian group, the two definitions of $\pi^1G$ agree. One can also normalize in the non-abelian case; indeed, the situation is already normalized in the sense that if $x \in Z^1G$, then $x \in N^1G$ because

$$e = s^0[(d^2x)(d^1x)^{-1}(d^0x)] = d^1s^0x$$

so

$$e = s^0d^1s^0x = s^0x.$$

Thus $N^1G \cap Z^1G = Z^1G$ and

$$\pi^1G = (N^1G \cap Z^1G)/G^0. \tag{1.14}$$

If $X$ is a cosimplicial set we define $\pi^0X$ to be the equalizer of $d^0, d^1 : X^0 \to X^1$.

PROPOSITION 1.15. *Let $X$ be a pointed fibrant cosimplicial space. Then the $E_2$ term of the spectral sequence of the tower of fibrations $\{\operatorname{Tot}_s X\}$ can be calculated by natural isomorphisms*

$$E_2^{s,t} \cong \pi^s\pi_tX, \qquad t-s \geq 0.$$

Because of this result, one often writes

$$\pi^s\pi_tX \Rightarrow \pi_{t-s}\operatorname{Tot} X$$

for the homotopy spectral sequence arising from a pointed cosimplicial space. The double arrow is not meant to prejudice anyone for or against convergence of the spectral sequence; for this, one must appeal to the results of Section VI.2.

Before proving Proposition 1.15, we line up some notation and ideas. We are interested in the tower $\{\operatorname{Tot}_s X\}$, where the tower maps

$$\operatorname{Tot}_s X = \mathbf{Hom}_{c\mathbf{S}}(\operatorname{sk}_s \Delta, X) \to \mathbf{Hom}_{c\mathbf{S}}(\operatorname{sk}_{s-1} \Delta, X) = \operatorname{Tot}_{s-1} X$$

are induced by the inclusion $\operatorname{sk}_{s-1}\Delta \to \operatorname{sk}_s \Delta$. Thus the fibre at the constant map is the space of pointed maps

$$\mathbf{Hom}_{c\mathbf{S}_*}(\operatorname{sk}_s \Delta / \operatorname{sk}_{s-1}\Delta, X).$$

If $f : \mathrm{sk}_s \Delta / \mathrm{sk}_{s-1} \Delta \to X$ is a pointed map one gets a diagram

$$\begin{tikzcd}
\Delta^s/\partial\Delta^s \arrow[r, "f"] \arrow[d] & X^s \arrow[d] \\
* \arrow[r] & M_s X
\end{tikzcd}$$

and hence a map of spaces

$$\mathbf{Hom}_{c\mathbf{S}_*}(\mathrm{sk}_s \Delta / \mathrm{sk}_{s-1} \Delta, X) \to \mathbf{Hom}_*(\Delta^s/\partial\Delta^s, N_s X).$$

Diagram (1.6) implies that this map is an isomorphism. To calculate $E_2^{s,t} = Z_2^{s,t}/B_2^{s,t}$ (see VI.(2.1)–VI.(2.4)) we calculate $Z_2^{s,t}$ and the action of $B_2^{s,t}$ in Lemmas 1.17, 1.19, and 1.20 below. The proof of Proposition 1.15 will follow.

DEFINITION 1.16. Recall that for $t \geq 2$, we may define

$$\partial = \sum_{i=0}^{s+1} (-1)^i d^i : \pi_t N^s X \to \pi_t N^{s+1} X,$$

and this defines the sub-group $\ker(\partial) \subseteq \pi_t N^s X$. If $t = 1$, define a pointed set $\ker(\partial) \subseteq \pi_1 N^1 X$ by

$$\ker(\partial) = Z^1 \pi_1 N^* X$$

(see (1.12)–(1.13)).

If $t = 0$ or 1 define $\ker(\partial) \subseteq \pi_t N^0 X = \pi_t X^0$ to be the equalizer of

$$d^0, d^1 : \pi_t X^0 \to \pi_t X^1.$$

If $t = 1$, $\ker(\partial)$ is a group; if $t = 0$, $\ker(\partial)$ is a pointed set.

LEMMA 1.17. *Suppose that $t - s \geq 1$. If also $t \geq 2$ then the composition*

$$\pi_{t-s}\mathbf{Hom}_*(S^s, NX^s) \to \pi_{t-s} \mathrm{Tot}_s X \xrightarrow{\partial} \pi_{t-s-1}\mathbf{Hom}_*(S^{s+1}, NX^{s+1}) \quad (1.18)$$

*can be identified up to isomorphism with the restricted homomorphism*

$$\sum_{i=0}^{s+1} (-1)^{i+1} d^i : N\pi_t X^s \to N\pi_t X^{s+1}.$$

*If $t = 1$, the composite (1.18) can be identified with the function*

$$N\pi_1 X^0 \to N\pi_1 X^1$$

*which is defined by sending $\alpha$ to $d^0(\alpha)d^1(\alpha)^{-1}$.*

PROOF: First of all we note something elementary. Suppose that $p : X \to Y$ is a pointed Kan fibration with fibre $F$, with inclusion $i : F \to X$. Recall that there is an isomomorphism

$$[S^n, Y]_* \cong [S^{n-1} \wedge S^1, Y]_*$$

which can be given explicitly (see the preamble to Theorem III.3.14) as follows. Suppose that $\alpha : \Delta^{n-1} \times \Delta^1 \to Y$ takes boundary to base point, and hence determines a pointed map $S^{n-1} \wedge S^1 \to Y$. Choose a lifting

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r, "{((\alpha, *, \dots, *), *)}"] \arrow[d, hook] & Y \\
\Delta^n \times \Delta^1 \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Then the restriction of $\theta$ to $\Delta^n \times \{1\}$ represents the image of $[\alpha]$ in $\pi_n Y$. Using this, it's an exercise to show that there is a commutative diagram

$$\begin{tikzcd}
\pi_n Y \arrow[r, "\partial"] \arrow[d, "\cong"'] & \pi_{n-1} F \arrow[d, "\cong"] \\
\pi_1 \mathbf{Hom}_*(S^{n-1}, Y) \arrow[r, "\partial"'] & \pi_0 \mathbf{Hom}_*(S^{n-1} F),
\end{tikzcd}$$

where the vertical isomorphisms are canonical, and the bottom boundary map is for the induced fibre sequence

$$\mathbf{Hom}_*(S^{n-1}, F) \to \mathbf{Hom}_*(S^{n-1}, X) \to \mathbf{Hom}_*(S^{n-1}, Y),$$

and $S^{n-1} = \Delta^{n-1}/\partial\Delta^{n-1}$. By taking loop spaces of the pointed fibrant cosimplicial space $X$ within the closed simplicial model structure for cosimplicial spaces, one sees that it suffices to show that the composite map

$$\pi_1 \mathbf{Hom}_*(S^s, NX^s) \to \pi_1 \operatorname{Tot}_s X \xrightarrow{\partial} \pi_0 \mathbf{Hom}_*(S^{s+1}, NX^{s+1})$$

is isomorphic to the map

$$\sum_{i=0}^{s+1} (-1)^{i+1} d^i : N\pi_{s+1} X^s \to N\pi_{s+1} X^{s+1}.$$

Interpret representatives for elements of $\pi_1 F_s \cong \pi_1 \mathbf{Hom}_*(S^s, NX^s)$ as morphisms $\alpha_s : \Delta^s \times \Delta^1 \to NX^s$ such that $\alpha(\partial(\Delta^s \times \Delta^1)) = *$. Then the image of $[\alpha_s]$ in $\pi_1 \operatorname{Tot}(X)$ is represented by the map $\alpha : \operatorname{sk}_s \Delta \times \Delta^1 \to X$ which is defined to be the composite

$$\Delta^s \times \Delta^1 \xrightarrow{\alpha_s} NX^s \subset X^s$$

in degree $s$, and is the basepoint in lower degrees. Then the lifting $\theta$ in the diagram

$$\begin{tikzcd}
\Delta^0 \arrow[rr, "*"] \arrow[d, "d^1"'] & & \operatorname{Tot}_{s+1} X \arrow[d] \\
\Delta^1 \arrow[r, "\alpha_s"'] \arrow[urr, dashed, "\theta"] & F_s \arrow[r] & \operatorname{Tot}_s X
\end{tikzcd}$$

amounts to a choice of simplicial set map $\theta_{s+1} : \Delta^{s+1} \times \Delta^1 \to X^{s+1}$ such that the following diagram commutes

$$\begin{tikzcd}
(\partial\Delta^{s+1} \times \Delta^1) \cup (\Delta^{s+1} \times \{0\}) \arrow[r, "((d^i\alpha_s), *)"] \arrow[d] & X^{s+1} \arrow[d] \\
\Delta^{s+1} \times \Delta^1 \arrow[r, "\alpha_*"'] \arrow[ur, "\theta_{s+1}"] & M^s X
\end{tikzcd}$$

The restriction $\theta_{s+1}(1 \times d^0)$ of $\theta_{s+1}$ represents the image of $\partial[\alpha_s]$ in

$$[S^{s+1}, X^{s+1}] \cong \pi_{s+1} X^{s+1},$$

and Theorem III.3.14 says that

$$\sum_{i=0}^{s+1} (-1)^i d^i [\alpha_s] = -[\theta_{s+1}(1 \times d^0)]$$

in $\pi_1 \mathbf{Hom}_*(S^s, X^{s+1})$. The inclusion $NX^{s+1} \subset X^{s+1}$ induces an inclusion on homotopy groups (Lemma 1.8, Corollary 1.10), so there is a commutative diagram of group homomorphisms

$$\begin{tikzcd}
\pi_1 \mathbf{Hom}_*(S^s, NX^s) \arrow[r] \arrow[rrd, "\sum_{i=0}^{s+1}(-1)^{i+1} d^i"'] & \pi_1 \operatorname{Tot}_s X \arrow[r, "\partial"] & \pi_0 \mathbf{Hom}_*(S^{s+1}, NX^{s+1}) \arrow[d, "\cong"] \\
& & \pi_1 \mathbf{Hom}_*(S^s, NX^{s+1})
\end{tikzcd}$$

where the indicated isomorphism is the one discussed at the beginning of the proof. □

For the moment (see Section 4), the cases not covered by this last result are taken care of by the following:

LEMMA 1.19. *Consider the diagram*

$$\begin{tikzcd}
 & \pi_0 \operatorname{Tot}_{s+1} X \arrow[d] \\
\pi_0 \mathbf{Hom}_*(S^s, NX^s) \arrow[r, "i_*"'] & \pi_0 \operatorname{Tot}_s(X)
\end{tikzcd}$$

*and take an element* $x \in \pi_0 \mathbf{Hom}_*(S^s, NX^s)$.

(1) *If* $s \geq 2$, *then* $i_*(x)$ *lifts to* $\pi_0 \operatorname{Tot}_{s+1}(X)$ *if and only if*

$$\sum_{i=0}^{s+1} (-1)^i d^i(x) = 0$$

*in* $\pi_s X^{s+1}$.

(2) *If* $s = 1$, *then* $i_*(x)$ *lifts to* $\pi_0 \operatorname{Tot}_{s+1}(X)$ *if and only if*

$$d^1(x) = d^0(x) d^2(x)$$

*in* $\pi_1 X^{s+1}$.

(3) *If* $s = 0$, *then* $x \in \pi_0 \mathbf{Hom}_*(S^0, NX^0) = \pi_0 X^0$ *lifts to* $\pi_0 \operatorname{Tot}_1 X$ *if and only if* $d^0 x = d^1 x$ *in* $\pi_0 X^1$.

PROOF: We shall prove only part (1) — the other cases are similar.

Let $\alpha : \Delta^s / \partial \Delta^s \to NX^s$ represent an element of $\pi_0 \mathbf{Hom}_*(S^s, NX^s)$. Then its image in $\pi_0 \operatorname{Tot}_s(X)$ lifts to $\pi_0 \operatorname{Tot}_{s+1}(X)$ if and only if there is a simplex $\theta : \Delta^{s+1} \to X^{s+1}$ making the following diagram commute:

$$\begin{tikzcd}
\partial \Delta^{s+1} \arrow[r, "(d^i \alpha)"] \arrow[d, hook] & X^{s+1} \arrow[d] \\
\Delta^{s+1} \arrow[r, "\alpha_*"'] \arrow[ur, "\theta"'] & M^s X
\end{tikzcd}$$

In particular, if such a lift $\theta$ exists then $\sum_{i=0}^{s+1} (-1)^i d^i[\alpha] = 0$ by the homotopy addition theorem (Theorem III.3.13). The converse is a formal exercise which depends on knowing that the fibration $X^{s+1} \to M^s X$ is surjective in all $\pi_i$ — this is Corollary 1.10. □

LEMMA 1.20. *The group $\pi_1\mathbf{Hom}_*(S^s, NX^s)$ acts on $\pi_0\mathbf{Hom}_*(S^{s+1}, NX^{s+1})$ through the action of $\pi_1 \operatorname{Tot}_s(X)$ on the fibre of $\operatorname{Tot}_{s+1}(X) \to \operatorname{Tot}_s(X)$. Given $[\alpha] \in \pi_1\mathbf{Hom}_*(S^s, NX^s)$ and $[\beta_\epsilon] \in \pi_0\mathbf{Hom}_*(S^{s+1}, NX^{s+1})$, we have the following:*

(1) *If $s > 0$, then $[\alpha] \cdot [\beta_0] = [\beta_1]$ if and only if*

$$[\beta_{1*}] - [\beta_{0*}] = \sum_{i=0}^{s+1} (-1)^i [d^i\alpha]$$

*in $\pi_1\mathbf{Hom}_*(S^s, NX^{s+1})$.*

(2) *If $s = 0$, then $[\alpha] \cdot [\beta_0] = [\beta_1]$ if and only if*

$$[\beta_1][d^1\alpha] = [d^0\alpha][\beta_0]$$

*in $\pi_1 NX^1$.*

PROOF: We shall prove only part (1).

Suppose that the simplices $\beta_0, \beta_1 : \Delta^{s+1} \to NX^{s+1}$ represent the corresponding elements of $\pi_0\mathbf{Hom}_*(S^{s+1}, NX^{s+1})$, and that the map $\alpha : \Delta^s \times \Delta^1 \to NX^s$ represents the element $[\alpha]$. Then $[\alpha] \cdot [\beta_0] = [\beta_1]$ if and only if there is a map $\theta : \Delta^{s+1} \times \Delta^1 \to X^{s+1}$ making the following diagram commute:

$$\begin{tikzcd}
(\partial\Delta^{s+1} \times \Delta^1) \cup \Delta^{s+1} \times \partial\Delta^1 \arrow[r, "((d^i\alpha), (\beta_0, \beta_1))"] \arrow[d, hook] & X^{s+1} \arrow[d] \\
\Delta^{s+1} \times \Delta^1 \arrow[r, "\alpha_*"'] \arrow[ur, "\theta"] & M^s X
\end{tikzcd}$$

Let $[\beta_{\epsilon *}]$ denote the elements corresponding to $[\beta_\epsilon]$ in $\pi_1\mathbf{Hom}_*(S^s, NX^{s+1})$ according to the method described at the beginning of the proof of Lemma 1.17. Then Theorem III.3.14 and the existence of the lift $\theta$ together imply that

$$[\beta_{1*}] - [\beta_{0*}] = \sum_{i=0}^{s+1} (-1)^i [d^i\alpha]$$

in $\pi_1\mathbf{Hom}_*(S^s, NX^{s+1})$. The converse, as in the proof of Lemma 1.19, is a formal consequence of the surjectivity in homotopy groups of the fibration $X^{s+1} \to M^s X$. □

PROOF OF PROPOSITION 1.15: The identification

$$Z_2^{s,t} \cong \ker(\partial) \subseteq \pi_t N^s X$$

(see Definition 1.16) is a consequence of Lemma 1.17 for $t - s > 0$ and Lemma 1.19 for $t = s$. The identification

$$E_2^{s,t} \cong H^s N\pi_t X$$

then follows from Lemma 1.17 if $t - s \geq 1$, and from Lemma 1.20 if $t = s$. The isomorphisms

$$H^s N\pi_t X \cong H^s \pi_t X = \pi^s \pi_t X$$

appear in (1.12)–(1.14). □

## 2. Homotopy inverse limits.

Recall the cosimplicial object $T^\bullet Y \in c\mathbf{S}^J$ constructed in Example VII.4.2 for a functor $Y : J \to \mathbf{S}$ taking values in simplicial sets. Examining the definitions we have

$$(T^\bullet Y)^n(j) = T^{n+1}Y(j) = \prod_{j \to i_0 \to \cdots \to i_n} Y(i_n).$$

It follows that $T^\bullet Y$ is naturally codegeneracy free as an object in $c\mathbf{S}^J$ on $Z^n(Y)$ where

$$Z^n(Y) = \prod_{j \to i_0 \to i_1 \to \cdots \to i_n} Y_{i_n}$$

where $i_t \to i_{t+1}$ is not the identity for any $t$, $0 \leq t \leq n-1$. It follows that $\varprojlim_J T^\bullet Y$ at $c\mathbf{S}$ is also codegeneracy free, since the inverse limit functor preserves products.

Note that the matching space $M^n \varprojlim_J T^\bullet Y$ can be identified with the product

$$\prod_{\sigma \in DBJ_n} Y(\sigma(n))$$

indexed over all degenerate $n$-simplices

$$\sigma : \sigma(0) \to \cdots \to \sigma(n)$$

of the nerve $BJ$, and the canonical map $s : \varprojlim_J T^n Y \to M^{n-1} \varprojlim_J T^\bullet Y$ can be naturally identified with the projection

$$\prod_{\alpha \in BJ_n} Y(\alpha(n)) \to \prod_{\sigma \in DBJ_n} Y(\sigma(n)).$$

It is a simple exercise to use this observation to show the following central result:

LEMMA 2.1. *Suppose that $p : X \to Y$ is a natural transformation of functors $J \to \mathbf{S}$ such that each component map $p : X(j) \to Y(j)$ is a fibration of simplicial sets. Then the induced map*

$$p_* : \varprojlim_J T^\bullet X \to \varprojlim_J T^\bullet Y$$

*is a fibration of cosimplicial spaces.*

Suppose that $Y : J \to \mathbf{S}$ is a small diagram of simplicial sets which is pointwise fibrant in the sense that each $Y(j)$ is a Kan complex. Then Lemma 2.1 implies that the cosimplicial space $\varprojlim_J T^\bullet Y$ is a fibrant cosimplicial space, so that the simplicial set

$$\operatorname{Tot} \varprojlim_J T^\bullet Y = \mathbf{Hom}_{c\mathbf{S}}(\Delta, \varprojlim_J T^\bullet Y)$$

is a Kan complex. It is standard to write

$$\underset{\overleftarrow{\;J\;}}{\operatorname{holim}}\, Y = \operatorname{Tot} \varprojlim_J T^\bullet Y,$$

and say that $\underset{\overleftarrow{\;J\;}}{\operatorname{holim}}\, Y$ is the *homotopy inverse limit* of the diagram $Y$. If $X : J \to \mathbf{S}$ does not consist of Kan complexes, $\underset{\overleftarrow{\;J\;}}{\operatorname{holim}}\, X$ is defined by finding a pointwise fibrant replacement $i : X \to Y$ according to the pointwise fibration closed model structure given in Example II.6.9.

Recall that a natural transformation $p : X \to Y$ of functors $J \to \mathbf{S}$ is said to be a *pointwise fibration* if each component map $p : X(j) \to Y(j)$ is a fibration of simplicial sets. Lemma 2.1 can now be paraphrased by saying that the homotopy inverse limit functor takes pointwise fibrations to fibrations. The homotopy inverse limit functor also takes pointwise weak equivalences between pointwise fibrant diagrams to weak equivalences, on account of the following:

LEMMA 2.2. *Suppose that $f : X \to Y$ is a natural transformation of pointwise fibrant diagrams $J \to \mathbf{S}$ of simplicial sets such that each component $f : X(j) \to Y(j)$ is a weak equivalence of simplicial sets. Then the induced map*

$$f_* : \underset{\overleftarrow{\;J\;}}{\operatorname{holim}}\, X \to \underset{\overleftarrow{\;J\;}}{\operatorname{holim}}\, Y$$

*is a weak equivalence.*

PROOF: The pointwise equivalence $f$ induces weak equivalences

$$\prod_{j_0 \to \cdots \to j_n} X(j_n) \to \prod_{j_0 \to \cdots \to j_n} Y(j_n)$$

for all $n \geq 0$, and hence induces a weak equivalence of fibrant cosimplicial spaces

$$\varprojlim_J T^\bullet X \to \varprojlim_J T^\bullet Y,$$

by Lemma 2.1. It follows that the induced map

$$\mathbf{Hom}_{c\mathbf{S}}(\Delta, \varprojlim_J T^\bullet X) \to \mathbf{Hom}_{c\mathbf{S}}(\Delta, \varprojlim_J T^\bullet Y)$$

is a weak equivalence of simplicial sets. □

Suppose that the functor $Y : J \to \mathbf{S}$ is pointwise fibrant. The description of the Tot functor given in (1.1) implies that $\underleftarrow{\operatorname{holim}}_J X$ is the equalizer of a pair of morphisms

$$\prod_{n\geq 0} \mathbf{Hom}(\Delta^n, \prod_{j_0\to\dots\to j_n} Y(j_n)) \rightrightarrows \prod_{\theta:\mathbf{m}\to\mathbf{n}} \mathbf{Hom}(\Delta^m, \prod_{j_0\to\dots\to j_n} Y(j_n)).$$

At the same time, the standard function complex for $J$-diagrams of simplicial sets has the form

$$\mathbf{Hom}_{\mathbf{S}^J}(X,Y)_n = \hom_{\mathbf{S}^J}(X\square\Delta^n, Y),$$

where the functor $X\square\Delta^n : J \to \mathbf{S}$ is defined by

$$(X\square\Delta^m)(j) = X(j)\times\Delta^n$$

for all objects $j$ of $J$. The homotopy inverse limit $\underleftarrow{\operatorname{holim}}_J Y$ of a $J$-diagram has a function complex description with respect to the category of $J$-diagrams in the following sense:

LEMMA 2.3. *Suppose that $Y : J \to \mathbf{S}$ is a diagram of simplicial sets, and let $B(J\downarrow ?)$ be the diagram which is defined by $j \mapsto B(J\downarrow j)$. Then there is a natural isomorphism of simplicial sets*

$$\underleftarrow{\operatorname{holim}}_J Y \cong \mathbf{Hom}_{\mathbf{S}^J}(B(J\downarrow ?), Y).$$

PROOF: It suffices to prove the result in degree 0, since

$$\underleftarrow{\operatorname{holim}}_J Y_n \cong (\underleftarrow{\operatorname{holim}}_J \mathbf{Hom}(\Delta^n, Y))_0.$$

From the equalizer description, a 0-simplex of $\underleftarrow{\operatorname{holim}}_J Y$ consists of a collection of simplicial set maps

$$\Delta^n \xrightarrow{\alpha_\sigma} Y(j_n),$$

one for each simplex $\sigma : j_0 \to \dots \to j_n$, and these should patch properly in the sense that every ordinal number map $\theta : \mathbf{m}\to\mathbf{n}$ determines a commutative diagram

$$\begin{tikzcd}
\Delta^m \arrow[r, "\alpha_{\sigma\cdot\theta}"] \arrow[d, "\theta_*"'] & Y(j_{\theta(m)}) \arrow[d] \\
\Delta^n \arrow[r, "\alpha_\sigma"'] & Y(j_n).
\end{tikzcd}$$

Any natural transformation $f : B(J \downarrow j) \to Y(j)$ determines and is completely determined by the collection of simplices

$$\alpha_f(\sigma) = f(j_0 \to \cdots \to j_n \xrightarrow{1} j_n) \in Y(j_n)_n,$$

indexed over the simplices $\sigma : j_0 \to \cdots \to j_n$ of $BJ$. The collection of simplices $\{\alpha_f(\sigma)\}$ patches, and the assignment $f \mapsto \{\alpha_f(\sigma)\}$ gives the desired isomorphism. □

It has been shown (in various places: Section IV.3.1, Example II.6.11) that there is a simplicial model structure on the category $\mathbf{S}^J$ of $J$-diagrams in simplicial sets, for which a map $f : X \to Y$ is a fibration (respectively weak equivalence) if all the component maps $f : X(i) \to Y(i)$ are fibrations (respectively weak equivalences) of simplicial sets. This is the *pointwise fibration structure.*

There is, however, another model category structure on diagrams which is especially suited for discussing homotopy limits, and is often called the *pointwise cofibration structure.*

PROPOSITION 2.4. *With its standard simplicial structure, the category* $\mathbf{S}^J$ *has a simplicial model category structure where a morphism* $f : X \to Y$ *is*

1) *a weak equivalence if each component map* $f : X(i) \to Y(i)$ *is a weak equivalence;*
2) *a cofibration if each component map* $f : X(i) \to Y(i)$ *is a cofibration (that is, an injection), and*
3) *a fibration if it has the right lifting property with respect to all trivial cofibrations.*

Fibrations for the pointwise cofibration structure are called *global fibrations.*

It helps to know that the functor

$$G_i : \mathbf{S}^J \to \mathbf{S} \tag{2.5}$$

given by $G_i X = X(i)$ has both a left and right adjoint. The left adjoint is given by

$$(L_i Y)(j) = \bigsqcup_{i \to j} Y \tag{2.6}$$

and the right adjoint given by

$$(T_i Y)(j) = \prod_{j \to i} Y \tag{2.7}$$

PROOF OF PROPOSITION 2.4: Assume first that we can prove **CM5**. Then axioms **CM1**–**CM3** clearly hold, and so we need only prove the "cofibration-trivial fibration" half of **CM4** to get a closed model category. For this, consider a lifting problem in $\mathbf{S}^J$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where $i$ is a cofibration and $p$ is a trivial fibration. Form the diagram

$$\begin{tikzcd}
X \arrow[rr, "="] \arrow[d, "\bar{\imath}"'] & & X \arrow[d, "p"] \\
X \sqcup_A B \arrow[r, "j"] & Z \arrow[r, "q"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where we have used **CM5** to factor $X \sqcup_A B \to Y$ as a cofibration $j$ by a trivial fibration $q$. Then $q(j\bar{\imath}) = p$ is a weak equivalence and $q$ is a weak equivalence, so $ji$ is a trivial cofibration and the indicated lift exists. The composite $B \to X \sqcup_A B \to Z \to X$ solves the original lifting problem. Finally, Axiom **SM7** follows from the corresponding property for simplicial sets.

This leaves **CM5**. The argument is completed in Lemma 2.10, once we prove two preliminary lemmas. □

If $X \in \mathbf{S}^J$ let $\#X$ be the cardinality of the simplices of $\bigsqcup_i X(i)$. Let $\beta$ be a fixed infinite cardinal greater than the cardinality of the set

$$\bigsqcup_{(i,j)} \hom_J(i,j),$$

of all morphisms of $J$.

LEMMA 2.8. *Suppose that $A \to B$ is any trivial cofibration, and that $x \in B(i)$ is a simplex. Then there is a sub-object $C \subseteq B$ so that $x \in C(i)$, $\#C \leq \beta$, and $A \cap C \to C$ is a trivial cofibration.*

PROOF: One constructs objects $C_n \subseteq B$ so that $x \in C_0(i)$, $\#C_n \leq \beta$, $C_n \subseteq C_{n+1}$ and $\pi_k(|C_n|, |C_n \cap A|) \to \pi_k(|C_{n+1}|, |C_{n+1} \cap A|)$ is trivial for all $k \geq 0$ (Note that $\pi_0(P, Q)$ is the quotient set $\pi_0 P/\pi_0 Q$.) and all choices of basepoint. Then $C = \bigcup_n C_n$.

To construct the objects $C_n$, choose $C_0$ to be any sub-object with $x \in C_0(i)$ and $\#C_0 \leq \beta$. This is possible: choose $x \in K_0 \subseteq B(i)$ where $K_0$ has only finitely non-degenerate simplices, and let $C_0$ be the image of the induced map $L_iK_0 \to B$ where $L_i$ is as in (2.6). Having produced $C_n$, proceed as follows. For each $i$ and each $k$ and any choice of basepoint in $C_n(i) \cap A$, let $z \in \pi_k(|C_n(i)|, |C_n(i) \cap A|)$. There is a sub-complex $K_z \subseteq B(i)$ with finitely many non-degenerate simplices so that $z$ is trivial in $\pi_k(|C_n(i) \cup K_z|, |(C_n(i) \cup K_z) \cap A(i)|)$. This is because $A(i) \to B(i)$ is a trivial cofibration. Let $D_z$ be the image of $L_iK_z \to B$ and let $C_{n+1} = C_n \cup (\bigcup_z D_z)$. □

LEMMA 2.9. *A morphism $q : X \to Y$ is a global fibration if and only if it has the right lifting property with respect to all trivial cofibrations $j : A \to B$ with $\#B \leq \beta$.*

PROOF: Suppose that $q : X \to Y$ has the right lifting property with respect to all trivial cofibrations $j : A \to B$ with $\#B \leq \beta$, and consider a diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "i"'] & X \arrow[d, "q"] \\
D \arrow[r] & Y
\end{tikzcd}$$

where $i$ is a trivial cofibration. Consider also the collection of partial lifts

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "i'"'] & X \arrow[dd, "q"] \\
C' \arrow[d, "i''"'] \arrow[ur] & \\
D \arrow[r] & Y
\end{tikzcd}$$

where $i'$ and $i''$ are trivial cofibrations, $i = i'' \cdot i'$, and $C \neq C'$. This collection of partial lifts is non-empty, since every simplex $x \in D(i) - C(i)$ is contained in a subobject $B \subset D$ with $\#B \leq \beta$ and $B \cap C \to B$ a trivial cofibration by Lemma 2.8, and there is a diagram

$$\begin{tikzcd}
B \cap C \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "i'"'] & X \arrow[dd] \\
B \arrow[r] & B \cup C \arrow[d, "i''"'] \arrow[ur, dashed] & \\
& D \arrow[r] & Y
\end{tikzcd}$$

where the dotted arrow exists making the diagram commute, since $i'$ is a pushout of the trivial cofibration $B \cap C \to B$ where $B$ has appropriately bounded cardinality.

The proof is finished with a Zorn's lemma argument: the collection of partial lifts has maximal elements, and such maximal elements must solve the lifting problem by an argument similar to that for existence. □

LEMMA 2.10. *Axiom* **CM5** *holds for the level-wise cofibration structure on* $\mathbf{S}^J$.

PROOF: The "trivial cofibration-fibration" half of this axiom is a transfinite small object argument which is based on Lemma 2.9. Let $\gamma$ be a cardinal which is strictly larger than $2^\beta$, take a map $f : X \to Y$, and define a functor $\mathbf{X} : \gamma \to \mathbf{S}^J$ and factorizations

$$X \xrightarrow{i_s} X(s) \xrightarrow{p_s} Y$$

of $f$, which are jointly defined by specifying

(1) $\mathbf{X}(0) = X$,

(2) the diagram

$$\begin{tikzcd} \bigsqcup_D A_D \arrow[r] \arrow[d] & \mathbf{X}(s) \arrow[d] \\ \bigsqcup_D B_D \arrow[r] & \mathbf{X}(s+1) \end{tikzcd}$$

should be a pushout, where $D$ refers to the set of all diagrams

$$\begin{tikzcd} A_D \arrow[r] \arrow[d] & \mathbf{X}(s) \arrow[d] \\ B_D \arrow[r] & Y \end{tikzcd}$$

with $A_D \to B_D$ a trivial cofibration with $\# B_D \leq \beta$ and the map $p_{s+1} : X(s+1) \to Y$ is the obvious induced map,

(3) $X(t) = \varinjlim_{s<t} X(s)$ at limit ordinals $t$.

Write $\mathbf{X}(\gamma) = \varinjlim_{s<\gamma} \mathbf{X}(s)$. Then the induced factorization

$$X \xrightarrow{i_\gamma} \mathbf{X}(\gamma) \xrightarrow{p_\gamma} Y$$

is a factorization of $f$ as a trivial cofibration followed by a global fibration. Note in particular that if $\alpha : A \to \mathbf{X}(\gamma)$ is a map with $\#A \leq \beta$, then $\alpha$ must factor through some $\mathbf{X}(s)$ with $s < \gamma$, and so $p_\gamma$ is a global fibration by Lemma 2.9.

An argument which is similar to that for Lemma 2.9 shows that a map $p : Z \to W$ has the right lifting property with respect to all cofibrations if and only if it has the right lifting property with respect to the set of all cofibrations $A \subset L_i\Delta^n$. A transfinite small object then shows that every map $f : X \to Y$ has a factorization

$$X \xrightarrow{j} W \xrightarrow{p} Y$$

where $j$ is a cofibration and $p$ has the right lifting property with respect to all cofibrations. The functor $L_i$ preserves cofibrations, so that all component maps $p : W(i) \to Y(i)$ must be trivial fibrations. In particular, $p$ is a weak equivalence as well as a global fibration. □

The constant diagram functor $\mathbf{S} \to \mathbf{S}^J$ (sending $X \in \mathbf{S}$ to the diagram $X(i) = X$ with all the maps the identity) sends cofibrations to level-wise cofibrations and preserves weak equivalences. Thus the inverse limit functor

$$\varprojlim = \varprojlim_J : \mathbf{S}^J \to \mathbf{S}$$

preserves weak equivalences between globally fibrant objects (in the level-wise cofibration structure) and hence there is a total right derived functor

$$\mathbf{R}\varprojlim_J : \mathrm{Ho}(\mathbf{S}^J) \to \mathrm{Ho}(\mathbf{S}).$$

If $X \in \mathbf{S}^J$ one chooses a level-wise trivial cofibration $X \to Y$ to a fibrant object, and then $\varprojlim_J Y$ is a model for $\mathbf{R}\varprojlim_J X$. The total derived functor for inverse limit coincides in the homotopy category with the homotopy inverse limit on account of the following:

LEMMA 2.11. *Suppose that $Y : J \to \mathbf{S}$ is an $J$-diagram of simplicial sets such that every simplicial set $Y(i)$ is a Kan complex. Let $j : Y \to Z$ be a globally fibrant model in the sense for $Y$ in the sense that $Z$ is a globally fibrant diagram and $j$ is a pointwise weak equivalence. Then there is a weak equivalence*

$$\underset{\longleftarrow J}{\mathrm{holim}}\, Y \simeq \varprojlim_J Z.$$

PROOF: A simple adjunction trick shows that every globally fibrant diagram is pointwise fibrant, so that the map $j$ induces a weak equivalence

$$j_* : \underset{\longleftarrow J}{\mathrm{holim}}\, Y \to \underset{\longleftarrow J}{\mathrm{holim}}\, Z$$

by Lemma 2.2. At the same time, the canonical map $B(J \downarrow ?) \to *$ to the terminal diagram is a pointwise weak equivalence, so that the induced map of function complexes

$$\mathbf{Hom}_{\mathbf{S}^J}(*, Z) \to \mathbf{Hom}_{\mathbf{S}^J}(B(J \downarrow ?), Z)$$

is a weak equivalence, since every diagram is cofibrant for the pointwise cofibration structure. Finally, there is an isomorphism

$$\mathbf{Hom}_{\mathbf{S}^J}(*, Z) \cong \varprojlim_J Z. \qquad \square$$

Note that Lemma 2.11 specializes to the definition of homotopy inverse limit for a tower given in Lemma VI.1.12.

LEMMA 2.12. Suppose that $X$ is a fibrant cosimplicial space, and interpret $X$ as a diagram of spaces $X : \mathbf{\Delta} \to \mathbf{S}$ on the ordinal number category $\mathbf{\Delta}$. Then there is a natural weak equivalence

$$\mathrm{Tot}(X) \simeq \underset{\overleftarrow{\mathbf{\Delta}}}{\mathrm{holim}}\, X.$$

PROOF: According to Proposition 2.4, there is a globally fibrant replacement $j : X \to Z$ in the category of cosimplicial spaces. The map $j$ is a weak equivalence of cosimplicial spaces in particular, and every global fibration is a fibration of cosimplicial spaces on account of Proposition VII.4.18, so that $Z$ is a fibrant cosimplicial space. It follows that the map $j$ induces a weak equivalence

$$\mathrm{Tot}(X) = \mathbf{Hom}(\Delta, X) \xrightarrow{j_*} \mathbf{Hom}(\Delta, Z) = \mathrm{Tot}(Z).$$

The map $\Delta \to *$ is a weak equivalence of cofibrant objects in the $\mathbf{\Delta}$-diagram category, so that the induced map

$$\varprojlim_{\mathbf{\Delta}} Z \cong \mathbf{Hom}(*, Z) \to \mathbf{Hom}(\Delta, Z)$$

is a weak equivalence. Finally, there is a weak equivalence

$$\varprojlim_{\mathbf{\Delta}} Z \cong \mathbf{Hom}(*, Z) \simeq \underset{\overleftarrow{\mathbf{\Delta}}}{\mathrm{holim}}\, X$$

by Lemma 2.11. $\square$

We finish this section by using the results of Section 1 to construct a spectral sequence for computing the homotopy groups of a pointed homotopy inverse limit.

We begin with a discussion of the derived functors of limit on $J$-diagrams of abelian groups. Let $\mathbf{Ab}^J$ be the category of these diagrams. Then the functor $G_i : \mathbf{Ab}^J \to \mathbf{Ab}$ with $G_i A = A(i)$ has both a left and a right adjoint; in particular, the right adjoint is given by

$$T_i B(j) = \prod_{j \to i} B. \tag{2.13}$$

If $B \in \mathbf{Ab}$ is injective, then $T_i B$ is injective in $\mathbf{Ab}^J$ and this implies $\mathbf{Ab}^J$ has enough injectives; indeed, if we embed each $A(i)$ in an injective abelian group $B_i$, then $A \to \prod_i T_i B_i$ is an embedding of $A$ into an injective in $\mathbf{Ab}^J$. We define the functors $\varprojlim^s = \varprojlim^s_J$ to be the right derived functors of $\varprojlim_J$. Since limit is left exact, $\varprojlim = \varprojlim^0$. The standard observation is that one need not go all the way to injectives to compute the derived functors. As with sheaf cohomology one can resolve by "flabby" objects rather than injective ones. The crucial result is:

LEMMA 2.14. *Let $B$ be an abelian group. Then for all $i$ we have $\varprojlim T_i B \cong B$ and $\varprojlim^s T_i B = 0$ if $s > 0$.*

PROOF: To prove $\varprojlim T_i B \cong B$, let $A \in \mathbf{Ab}^J$ be a constant diagram on an abelian group $A$. Then

$$\hom_{\mathbf{Ab}}(A, \varprojlim T_i B) \cong \hom_{\mathbf{Ab}^J}(A, T_i B) \cong \hom_{\mathbf{Ab}}(A, B).$$

To obtain the vanishing of the higher derived functors, let $B \to I^*$ be an injective resolution of $B$. Then (see (2.13)), $T_i B \to T_i I^*$ is an injective resolution of $T_i B$. Applying $\varprojlim$ to this resolution, one recovers $B \to I^*$, which is acyclic by construction. □

We use this construction to build a cosimplicial resolution of $A \in \mathbf{Ab}^J$ suitable for computing $\varprojlim^s$. For our category $J$ let $J^\delta$ be the associated discrete category; i.e., $J^\delta$ has the same objects as $J$ but no non-identity morphisms. There is a functor $J^\delta \to J$, inducing a forgetful functor $\mathbf{Ab}^J \to \mathbf{Ab}^{J^\delta}$. The category $\mathbf{Ab}^{J^\delta}$ has for objects all $J$-indexed families of abelian groups $A = \{A_i\}_{i \in J}$. This forgetful functor has a right adjoint given by

$$TA = \prod_i T_i A_i.$$

We also write $T : \mathbf{Ab}^J \to \mathbf{Ab}^J$ for the composite functor. This functor is analogous to the functor $T$ of Example VII.4.2. Then $T$ is the functor of a triple on $\mathbf{Ab}^J$ and one, therefore, obtains a natural augmented cosimplicial object in $\mathbf{Ab}^J$

$$A \to T^\bullet A. \tag{2.15}$$

If one forgets to $\mathbf{Ab}^{J^\delta}$ this cosimplicial object has a retraction and so is acyclic. This implies it is acyclic in $\mathbf{Ab}^J$ and Lemma 2.14 implies:

LEMMA 2.16. *For all $A \in \mathbf{Ab}^J$ there is a natural isomorphism*

$$\varprojlim^s A \cong \pi^s \varprojlim T^\bullet A.$$

By analogy we use this construction to define $\varprojlim^1 G$ where $G$ is an $J$-diagram of non-abelian groups. The reader will have noticed that there is nothing special about abelian groups in the construction of the cosimplicial object $A \to T^\bullet A$ of (2.15). Indeed, if $\mathcal{C}$ is any category and $C \in \mathcal{C}^J$ is an $J$-diagram, we obtain a cosimplicial object $C \to T^\bullet C$. In particular if $G$ is an $J$-diagram of (not necessarily abelian) groups, one has a cosimplicial $J$-diagram of groups

$$G \to T^\bullet G$$

and we define

$$\varprojlim^1 G = \pi^1 \varprojlim T^\bullet G. \tag{2.17}$$

Thus $\varprojlim^1 G$ is a pointed set and is a group of cycles modulo a group of boundaries. See (1.13). At this point we have potentially two different definitions of $\varprojlim^1 G_n$ for a tower of groups, one from (2.17) and the other from Lemma VI.2.12; this difficulty will be resolved by Proposition 2.19 below.

Now let $X \in \mathbf{S}_*^J$ be an $J$-diagram of pointed fibrant spaces. We wish to calculate with $\operatorname{holim} X$. We use the cosimplicial diagram $T^\bullet X$ of Example VII.4.2, and note that by definition

$$\underset{J}{\operatorname{holim}}\, X = \operatorname{Tot} \varprojlim_J T^\bullet X.$$

The cosimplicial space $\varprojlim T^\bullet X$ can be identified up to isomorphism in cosimplicial degree $n$ with the space

$$\prod_{j_0 \to \cdots \to j_n} X(j_n),$$

where the product is indexed over $n$-simplices of the classifying space $BJ$. The isomorphism is formal, and there is a similar isomorphism

$$\varprojlim T^n A \cong \prod_{j_0 \to \cdots \to j_n} A(j_n)$$

for any functor $A$ on $J$ taking values in either groups or abelian groups. It follows that there is an isomorphism of cosimplicial groups

$$\pi_i \varprojlim T^\bullet X \cong \varprojlim T^\bullet \pi_i X,$$

and so there are isomorphisms

$$E_2^{s,t} \cong \pi^s \pi_t \varprojlim T^\bullet X \cong \varprojlim_J{}^s \pi_t X,$$

and hence a spectral sequence

$$\varprojlim_J{}^s \pi_t X \Rightarrow \pi_{t-s} \operatorname{holim}_{\overleftarrow{J}} X. \tag{2.18}$$

This follows from Lemma 2.14 and a simple calculation in the case $t = s = 0$.

Finally, we show that our two potentially different definitions of $\varprojlim^1 G_n$ agree up to natural isomorphism for a tower of groups $\{G_n\}$. Let $\varprojlim^1 G_n$ be defined as in Section VI.2, and suppose that $\underline{\lim}^1 G_n$ is a temporary name for $\varprojlim^1 G_n$ as defined in (2.17).

LEMMA 2.19. *There are natural isomorphisms*

$$\pi_0 \underleftarrow{\operatorname{holim}} BG_n \cong \underline{\lim}^1 G_n \cong \varprojlim{}^1 G_n.$$

PROOF: We will show $\pi_0 \underleftarrow{\operatorname{holim}} BG_n \cong \underline{\lim}^1 G_n$ and $\pi_0 \underleftarrow{\operatorname{holim}} BG_n \cong \varprojlim^1 G_n$. The second of these isomorphisms is a consequence of the Milnor exact sequence (Proposition VI.2.15) and Lemma 2.11.

For the first, we have that

$$\pi_0 \underleftarrow{\operatorname{holim}} BG_n \cong \pi_0 \operatorname{Tot} \varprojlim T^\bullet\{BG_n\}.$$

There is an isomorphism

$$\pi_0 \operatorname{Tot}_0 \varprojlim T^\bullet\{BG_n\} = \pi_0 \prod_n BG_n = *,$$

while the diagram (1.7) implies that the canonical maps induce isomorphisms

$$\pi_0 \operatorname{Tot}_{s+1} \varprojlim T^\bullet\{BG_n\} \cong \pi_0 \operatorname{Tot}_s \varprojlim T^\bullet\{BG_n\}$$

for $s \geq 2$ and a monomorphism

$$\pi_0 \operatorname{Tot}_2 \varprojlim T^\bullet\{BG_n\} \to \pi_0 \operatorname{Tot}_1 \varprojlim T^\bullet\{BG_n\}.$$

Also, for dimensional reasons, the image of this last monomorphism can be identified with the set $E_2^{1,1}$, so that there is an induced bijection

$$\varprojlim_s \pi_0 \operatorname{Tot}_s \varprojlim T^\bullet\{BG_n\} \cong \underline{\lim}^1 G_n.$$

The fibre of the map

$$\operatorname{Tot}_{s+1} \varprojlim T^\bullet\{BG_n\} \to \operatorname{Tot}_s \varprojlim T^\bullet\{BG_n\}$$

is contractible for $s \geq 2$, again on account of (1.7), so that the maps

$$\pi_1 \operatorname{Tot}_{s+1} \varprojlim T^\bullet\{BG_n\} \cong \pi_1 \operatorname{Tot}_s \varprojlim T^\bullet\{BG_n\}$$

are isomorphisms in that range. The Milnor exact sequence then implies that there is an isomorphism

$$\pi_0 \operatorname{Tot} \varprojlim T^\bullet\{BG_n\} \cong \varprojlim_s \pi_0 \operatorname{Tot}_s \varprojlim T^\bullet\{BG_n\},$$

and so we have the desired isomorphism. □

## 3. Completions.

The category of cosimplicial abelian groups is equivalent to $(s(\mathbf{Ab}^{\mathrm{op}}))^{\mathrm{op}}$. The category $\mathbf{Ab}^{\mathrm{op}}$ is abelian; hence the normalization functor $N : s(\mathbf{Ab}^{\mathrm{op}}) \to C_*\mathbf{Ab}^{\mathrm{op}}$ defines an equivalence of categories, by Theorem III.2.5. In particular every object $A \in s(\mathbf{Ab}^{\mathrm{op}})$ is naturally degeneracy free on $\{NA_s\}$. An examination of the proof shows that $d_0$ plays no role in the decomposition of $A \in s(\mathbf{Ab}^{\mathrm{op}})$; specifically if $n\mathbf{Ab}^{\mathrm{op}}$ denotes the category with objects $\{X_n\}_{n\geq 0}$, $X_n \in \mathbf{Ab}^{\mathrm{op}}$, then normalization and its adjoint define an equivalence of categories

$$N : s_0(\mathbf{Ab})^{\mathrm{op}} \to n\mathbf{Ab}^{\mathrm{op}}.$$

Here $s_0\mathcal{C}$ denotes the category of functors $\mathbf{\Delta}_0^{\mathrm{op}} \to \mathcal{C}$ where $\mathbf{\Delta}_0 \subseteq \mathbf{\Delta}$ is the subcategory with the same objects and morphisms satisfying $\phi(0) = 0$. These are simplicial objects without $d_0$.

If $c^+\mathbf{Ab}$ denotes the cosimplicial abelian groups without $d^0$, these remarks supply an equivalence categories

$$N^* : c^+\mathbf{Ab} \to n\mathbf{Ab}$$

sending $A$ to $N^*A$ with

$$(N^*A)^n = \bigcap_{s=0}^{n-1} \ker\{s^i : A^n \to A^{n-1}\}.$$

In particular any object $A \in c^+\mathbf{Ab}$ is naturally codegeneracy free on $N^*A$. Now let $X \in \mathbf{S}$ and $R^\bullet X$ the Bousfield-Kan $R$-resolution of Example VII.4.1. Then $R^\bullet X \in c\mathbf{S}$ is not a cosimplicial simplicial abelian group; however, when restricted, an object $R^\bullet X \in c^+\mathbf{S}$ has a natural structure as an object in $c^+(s\mathbf{Ab})$. Since products in $s\mathbf{Ab}$ are products in $\mathbf{S}$, we have that $R^\bullet X$ is naturally codegeneracy free on $N^*R^\bullet X$. Then (1.7) gives the existence of a pullback diagram

$$\begin{tikzcd}
\operatorname{Tot}_n R^\bullet X \arrow[r] \arrow[d] & \mathbf{Hom}(\Delta^n, N^*R^nX) \arrow[d] \\
\operatorname{Tot}_{n-1} R^\bullet X \arrow[r] & \mathbf{Hom}(\partial\Delta^n, N^*R^nX).
\end{tikzcd}$$

In particular, $\operatorname{Tot} R^\bullet X$ is a Kan complex. One can see, alternatively, that the Bousfield-Kan $R$-resolution is a fibrant cosimplicial space by observing that all maps $s : R^{n+1}X \to M^nR^\bullet X$ are surjective simplicial $R$-module maps. The surjectivity follows from the fact that the map is dual to the inclusion of the degenerate subobject in $R^{n+1}X$ in the category $s_0(\mathbf{Ab})^{op}$ — see the proof of Lemma III.2.6, and note that it does not involve $d_0$.

The total complex $\operatorname{Tot} R^{\bullet}X$ is called the Bousfield-Kan *R-completion* of the space $X$. One often writes

$$R^{\infty}X = \operatorname{Tot} R^{\bullet}X.$$

Let $X$ be a pointed space and let $X \to \mathbb{F}_p^{\bullet}X$ be the Bousfield-Kan resolution of $X$. Then the total space $\operatorname{Tot}(\mathbb{F}_p^{\bullet}X) = X_p$ is more usually called the Bousfield-Kan $p$-completion of $X$, and $\pi_*X_p$ may be calculated by the spectral sequence

$$\pi^s\pi_t\mathbb{F}_p^{\bullet}X \Rightarrow \pi_{t-s}X_p.$$

We will now show how the $E_2$ term can be rewritten in terms of homological algebra. We assume the reader is familiar with the category $\mathcal{CA}$ of unstable coalgebras over the Steenrod algebra. See [71, Sec. 1].

If $X$ is a space, $H_*X = H_*(X, \mathbb{F}_p)$ is a member of the category $\mathcal{CA}$. Let $\mathcal{CA}_*$ be the category of augmented (or pointed) coalgebras in $\mathcal{CA}$; thus an object in $\mathcal{CA}_*$ is an unstable coalgebra $C$ over the Steenrod algebra equipped with a morphism of such coalgebras $\mathbb{F}_p \to C$. The homology of a pointed space is in $\mathcal{CA}_*$.

First notice that if $V$ is any simplicial vector space, then, choosing $0 \in V$ as the basepoint,

$$\pi_tV \cong \hom_{\mathcal{CA}_*}(H_*S^t, H_*V). \tag{3.1}$$

One way to see this is to note that there is a (non-canonical) weak equivalence $V \simeq \bigoplus_\alpha K(\mathbb{Z}/p, n_\alpha)$ for some set of non-negative integers $\{n_\alpha\}$, and to use the calculations of Serre and Cartan on $H_*K(\mathbb{Z}/p, n_\alpha)$. Now the fact that $V$ is a simplicial abelian group implies that (3.1) holds for any choice of basepoint. Since a choice of basepoint for $X$ induces a basepoint for $\mathbb{F}_p^{\bullet}X$,

$$\pi^s\pi_t\mathbb{F}_p^{\bullet}X \cong \pi^s \hom_{\mathcal{CA}_*}(H_*S^t, H_*\mathbb{F}_p^{\bullet}X). \tag{3.2}$$

Now, the forgetful functor $\mathcal{CA} \to n\mathbb{F}_p$ to the category of graded vector spaces has a right adjoint $G$. The functor $G$ is characterized by the fact that it commutes with products and filtered colimits, and by the fact that if $W \in n\mathbb{F}_p$ is of dimension 1 concentrated in degree $n$, then

$$G(W) \cong H_*K(\mathbb{Z}/p, n). \tag{3.3}$$

(Note that $G$ is the functor of a triple on $n\mathbb{F}_p$ and $\mathcal{CA}$ is the category of coalgebras over $G$.)

Let $\overline{G} : \mathcal{CA} \to \mathcal{CA}$ be the composite of $G$ and the forgetful functor. Then $\overline{G}$ is a triple on $\mathcal{CA}$, so if $C \in \mathcal{CA}$, we have an augmented cosimplicial object (see Example VII.4.1).

$$C \to \overline{G}^{\bullet}C. \tag{3.4}$$

This is a resolution of augmented cosimplicial graded vector spaces, so $\pi^*\overline{G}^\bullet C \cong C$ via the augmentation. Thus we define, for $C \in \mathcal{CA}_*$,

$$\operatorname{Ext}^s_{\mathcal{CA}_*}(H_*S^t, C) = \pi^s \hom_{\mathcal{CA}_*}(H_*S^t, \overline{G}^\bullet C). \tag{3.5}$$

Now if $X$ is a space, $H_*\mathbb{F}_p^\bullet X \cong \overline{G}(H_*X)$. This follows from (3.1), (3.3) and the properties of $\overline{G}$. Thus combining (3.2) with (3.5) we have

$$\pi^s\pi_t\mathbb{F}_p^\bullet X \cong \operatorname{Ext}^s_{\mathcal{CA}_*}(H_*S^t, H_*X)$$

and the homotopy spectral sequence of $\mathbb{F}_p^\bullet X$ is the Bousfield-Kan spectral sequence

$$\operatorname{Ext}^s_{\mathcal{CA}_*}(H_*S^t, H_*X) \Rightarrow \pi_{t-s}X_p. \tag{3.6}$$

This is an unstable Adams spectral sequence.

The reader might object that, for pointed coalgebras, one should define the Ext groups differently. If $C \in \mathcal{CA}_*$, define the "coaugmentation ideal" by

$$JC = \operatorname{coker}\{\mathbb{F}_p \to C\}.$$

Then $J : \mathcal{CA}_* \to n\mathbb{F}_p$ has a right adjoint $G_*$ with $G_*V = GV$ augmented by $G(0) \cong \mathbb{F}_p \to G(V)$. Let $\overline{G}_* = G_* \circ J : \mathcal{CA}_* \to \mathcal{CA}_*$ and one might demand that the Ext object be

$$\pi^s \hom_{\mathcal{CA}_*}(H_*S^t, \overline{G}_*^\bullet C).$$

However, a bicomplex argument with

$$\hom_{\mathcal{CA}_*}(H_*S^t, \overline{G}^\bullet\overline{G}_*C)$$

shows that this definition agrees with the previous one (3.5); furthermore, (3.5) offers additional flexibility with basepoints.

We shall now give a brief sketch of an alternative construction of the Bousfield-Kan $p$-completion. The main source is [14], but many of the ideas here are also examined in [37] and [75].

Let **tow** be the category of towers $X = \{X_n\}$ of simplicial sets with morphisms the pro-maps

$$\hom(X, Y) = \varprojlim_n \varinjlim_k \hom_{\mathbf{S}}(X_k, Y_n).$$

Then a morphism $X \to Y$ is an equivalence class of tower maps, meaning a "commutative ladder" of the form

$$\begin{tikzcd}
\cdots \arrow[r] & X_{k_{s+1}} \arrow[r] \arrow[d] & X_{k_s} \arrow[r] \arrow[d] & \cdots X_{k_2} \arrow[r] \arrow[d] & X_{k_1} \arrow[d] \\
\cdots \arrow[r] & Y_{n_{s+1}} \arrow[r] & Y_{n_s} \arrow[r] & \cdots Y_{n_2} \arrow[r] & Y_{n_1}
\end{tikzcd}$$

where the horizontal maps are induced from the tower projections of $X$ and $Y$ and $\varinjlim k_s = \infty = \varinjlim n_s$.

There is a simplicial model category structure on **tow** where a pro-map $f : X \to Y$ is a weak equivalence if the induced map

$$f^* : \varinjlim H^*(Y_n, \mathbb{F}_p) \to \varinjlim H^*(X_n, \mathbb{F}_p)$$

is an isomorphism, and $f : X \to Y$ is a cofibration if $\varprojlim X_n \to \varprojlim Y_n$ is an injection of simplicial sets. If $X \in \mathbf{S}$ is a space, the *p-completion* of $X$ is defined as follows: regard $X \in \mathbf{tow}$ as a constant tower, choose a weak equivalence $X \to Y$ with $Y$ fibrant and set

$$X_p = \varprojlim Y_n.$$

Note that $Y$ is actually a tower of fibrations. There is a map $\eta : X \to X_p$ called the completion map, and $X$ is *p-complete* if this map is a weak equivalence. Note also that if $X \to Y$ is an $H_*(\cdot, \mathbb{F}_p)$ isomorphism, then $X_p \simeq Y_p$.

If $X \to \mathbb{F}_p^\bullet X$ is the Bousfield-Kan resolution of $X$ then the total tower of this cosimplicial space $\{\mathrm{Tot}_n \, \mathbb{F}_p^\bullet X\}$ (see Section VII.5) is a fibrant model for $X$ in **tow** by [21], hence

$$X_p \simeq \mathrm{Tot}(\mathbb{F}_p^\bullet X).$$

This is the original definition of the completion.

To analyze the homotopy type of $X_p$ we introduce and discuss the $p$-completion functor on abelian groups. If $A$ is an abelian group define

$$A_p = \varprojlim(\mathbb{Z}/p^n\mathbb{Z} \otimes A).$$

A group is $p$-complete if the map $A \to A_p$ is an isomorphism. We write $\mathbf{Ab}_p$ for the category of $p$-complete groups. Then completion is left adjoint to the inclusion functor $\mathbf{Ab}_p \to \mathbf{Ab}$. Note that the functor $A \mapsto A_p$ is neither left nor right exact, which implies that $\mathbf{Ab}_p$ is not an abelian sub-category. Nonetheless, completion has left derived functors $L_s(\cdot)_p$ and one has that $L_s(A)_p = 0$ for $s > 1$, a short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Tor}(\mathbb{Z}/p^n\mathbb{Z}, A) \to L_0(A)_p \to A_p \to 0,$$

and a natural isomorphism

$$L_1(A)_p \cong \varprojlim \mathrm{Tor}(\mathbb{Z}/p^n\mathbb{Z}, A).$$

See [39]. For example, if $\mathbb{Z}/p^\infty\mathbb{Z} = \varinjlim \mathbb{Z}/p^n\mathbb{Z}$, then $L_0(\mathbb{Z}/p^\infty\mathbb{Z})_p = 0$ and $L_1(\mathbb{Z}/p^\infty\mathbb{Z})_p \cong \mathbb{Z}_p$, the $p$-adic numbers. Since

$$\mathrm{Tor}(\mathbb{Z}/p^n\mathbb{Z}, A) \cong \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A)$$

one has

$$L_1(A)_p \cong \varprojlim \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A) \cong \mathrm{Hom}(\mathbb{Z}/p^\infty\mathbb{Z}, A)$$

from which it follows that

$$L_0(A)_p \cong \mathrm{Ext}(\mathbb{Z}/p^\infty\mathbb{Z}, A).$$

The smallest abelian sub-category of **Ab** containing the $p$-complete groups is the category of groups so that

$$\mathrm{Hom}(\mathbb{Z}[1/p], A) = 0 = \mathrm{Ext}(\mathbb{Z}[1/p], A).$$

These are known variously as Ext $-p$ *complete groups* , *p-cotorsion groups* , or *weakly p-complete groups.* See Section 4 of [44].

The functors of $p$-completion and its derived functors can be extended to the class of nilpotent groups, which are those groups for which the lower central series is eventually zero. See [14].

Now we consider *nilpotent spaces.* These are connected simplicial sets $X$ for which $\pi_1 X$ is a nilpotent group and for which $\pi_1 X$ acts nilpotently on $\pi_n$ for all $n$. See Section VI.6 for complete definitions. The main result is the following.

THEOREM 3.7. *Let $X$ be a nilpotent space. Then*

*1) $H_*(X, \mathbb{F}_p) \to H_*(X_p, \mathbb{F}_p)$ is an isomorphism and $X_p$ is the p-completion of $X$;*

*2) for all $n \geq 1$ there is short exact sequence*

$$0 \to L_0(\pi_n X)_p \to \pi_n X_p \to L_1(\pi_{n-1} X)_p \to 0.$$

*This sequence is split, but not naturally.*

REMARK 3.8. Note that the $p$-completion $X_p$ is $H_*(\cdot, \mathbb{F}_p)$-local in the sense of Chapter X, according to the construction above. It must therefore be a model for the $H_*(\cdot, \mathbb{F}_p)$-localization of $X$ if it has the right mod $p$ homology. More generally, can also show directly that the Bousfield-Kan $R$-completion $R^\infty X$ is weakly equivalent to an $H_*(\cdot, R)$-local space. See Remark 3.7 below.

Here is an outline of the proof of Theorem 3.7. First one shows the result for $K(A, n)$ where $A$ is an abelian group and $n \geq 1$. If

$$0 \to F_1 \to F_0 \to A \to 0$$

is a free resolution of $A$, let $K$ be the simplicial abelian group with normalized chain complex

$$\cdots \to 0 \to F_1 \to F_0 \to 0 \to \cdots \to 0$$

with $F_0$ in degree $n$. Then $K$ is weakly equivalent to $K(A,n)$ and the simplicial abelian group $K_p$ is a model for $K(A,n)_p$.

Next is $X$ is nilpotent, we have, by the results of Section VI.6, a refined Postnikov tower for $X$; in particular, $X$ may be written as $X \simeq \varprojlim_k X_k$ where the $X_k$ fit into a tower and each successive stage is built by a pullback diagram

$$\begin{tikzcd}
X_k \arrow[r] \arrow[d] & WK(A,n_k) \arrow[d] \\
X_{k-1} \arrow[r] & K(A,n_k+1),
\end{tikzcd}$$

and $\varinjlim_k n_k = \infty$. Thus one can proceed inductively using the "nilpotent fibre lemma" of Bousfield and Kan [14]. The necessary corollary of that result which is needed here is the following.

PROPOSITION 3.9. *Let $K(A,n) \to E \to B$ be a principal fibration, and suppose $n \geq 1$. Then*

$$K(A,n)_p \to E_p \to B_p$$

*is a fibration sequence up to homotopy.*

This can be proved by arguing directly that it holds for

$$K(A,n) \to WK(A,n) \to K(A,n+1)$$

and then using the Serre spectral sequence. Specifically, if we define $E'$ by the pullback diagram

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] & WK(A,n)_p \arrow[d] \\
B_p \arrow[r] & K(A,n)_p,
\end{tikzcd}$$

there is a map $E \to E'$ which is an $H_*(\cdot,\mathbb{F}_p)$ isomorphism by the Serre spectral sequence. Since $E'$ is $p$-complete — the class of $p$-complete spaces is closed under homotopy inverse limits — we have $E_p \simeq E'$ and the result follows.

## 4. Obstruction theory.

The purpose of this section is to develop a small amount of the theory of Bousfield's paper [10], and to discuss to some extent the meaning of the elements on the fringe of the spectral sequence of a cosimplicial space. We close the section with an extended example intended to make the theory concrete.

We wish to address the following question: when is an element $[\alpha] \in \pi^0\pi_0 X$ in the image of $\pi_0 \operatorname{Tot} X \to \pi_0 \operatorname{Tot}_0 X = \pi_0 X$? We will develop some elementary cohomological obstructions to lifting $[\alpha]$ in successive stages up the Tot tower. The enterprise is complicated by the fact that the term "cohomological" has to be interpreted with some care at the bottom of the tower, where the obstructions most naturally lie in pointed sets or non-abelian groups.

In order to make the arguments of the general theory work, it is common to assume [10] at the outset that $X$ is a fibrant cosimplicial space with the property that for all $n \geq 0$ and every choice of basepoint $v \in X^n$, Whitehead products in $\pi_*(X^n, v)$ vanish. Thus $\pi_1(X^n, v)$ is abelian and the action of $\pi_1(X^n, v)$ on $\pi_m(X^n, v)$, $m \geq 1$, is trivial. We do not assume $\pi_0(X^n, v) = *$, as one of our main examples will be $X = \mathbf{Hom}(Y, R^\bullet Z)$, where $Y \in \mathbf{S}$, and $R^\bullet Z$ is the $R$-resolution of $Z \in \mathbf{S}$.

The obstructions considered in this section are at a low level, and a weaker assertion suffices, namely that the fundamental groupoid $\pi X^n$ acts trivially on all fundamental groups $\pi_1(X^n, x)$. This assumption means explicitly that any two paths $\omega, \eta : x \to y$ should induce the same morphism $\omega_* = \eta_* : \pi_1(X^n, x) \to \pi_1(X^n, y)$ by conjugation at the level of fundamental groups of $X^n$ for all $n \geq 0$. In particular, all fundamental groups $\pi_1(X^n, x)$ of $X^n$ must be abelian. The reader will note that this assumption is only necessary for computing obstructions that take values in $\pi_1$, and in fact can be further weakened with a little care so that it's necessary for only low values of $n$. No such assumption on fundamental groupoid actions is necessary to define or compute the obstructions that take values in higher homotopy groups.

Let $\alpha : \operatorname{sk}_n \Delta \to X$ be a cosimplicial map representing $[\alpha] \in \pi_0 \operatorname{Tot}_n X$, where $n \geq 2$. Suppose that $\theta : \mathbf{r} \to \mathbf{s}$ is an ordinal number map, and let $\omega_\theta : 0 \to \omega(0)$ be the obvious path (aka. relation) in $\operatorname{sk}_n \Delta^s$. It $\tau : \mathbf{s} \to \mathbf{t}$ is a second ordinal number map, then by the assumption on $n$ there is a 2-simplex

$$\begin{tikzcd}
0 \arrow[r, "\omega_\tau"] \arrow[dr, "\omega_{\tau\theta}"'] & \tau(0) \arrow[d, "\tau(\omega_\theta)"] \\
& \tau\theta(0)
\end{tikzcd}$$

in $\operatorname{sk}_n \Delta^t$.

The ordinal number map $\theta : \mathbf{r} \to \mathbf{s}$ induces a homomorphism

$$\tilde{\theta} : \pi_i(X^r, \alpha(0)) \to \pi_i(X^s, \alpha(0)),$$

which is defined to be the composite

$$\pi_i(X^r, \alpha(0)) \xrightarrow{\theta_*} \pi_i(X^s, \theta\alpha(0)) \xleftarrow[\cong]{\alpha(\omega_\theta)_*} \pi_i(X^s, \alpha(0)),$$

where $\alpha(\omega_\theta)_*$ is the isomorphism induced by the fundamental groupoid element represented by $\alpha(\omega_\theta)$. There is a commutative diagram

$$\begin{tikzcd}
\pi_i(X^r,\alpha(0)) \arrow[r, "\theta_*"] \arrow[dr, "\tilde{\theta}"'] & \pi_i(X^s,\theta\alpha(0)) \arrow[r, "\tau_*"] & \pi_i(X^t,\tau\theta\alpha(0)) \\
 & \pi_i(X^s,\alpha(0)) \arrow[u, "\alpha(\omega_\theta)_*"'] \arrow[r, "\tau_*"] \arrow[dr, "\tilde{\tau}"'] & \pi_i(X^t,\tau\alpha(0)) \arrow[u, "\alpha(\tau(\omega_\theta))_*"'] \\
 & & \pi_i(X^t,\alpha(0)) \arrow[u, "\alpha(\omega_\tau)_*"']
\end{tikzcd}$$

But $\tau(\omega_\theta) = \omega_{\tau\theta}$, so that $\tilde{\tau}\tilde{\theta} = \widetilde{\tau\theta}$, and so the groups $\pi_i(X^r,\alpha(0))$ and the homomorphisms $\tilde{\theta}$ determine a cosimplicial group.

We shall need to know that the Hurewicz map respects this construction. Specifically, let $\alpha : \mathrm{sk}_n\,\Delta \to X$ be a cosimplicial space map as before, and form the cosimplicial space map $h : X \to \mathbb{Z}X$ by applying the Hurewicz map $h : X^r \to \mathbb{Z}X^r$ in each degree. There is a diagram

$$\begin{tikzcd}
\pi_i(X^r,\alpha(0)) \arrow[r, "\theta_*"] \arrow[d, "h"'] & \pi_i(X^s,\theta\alpha(0)) \arrow[d, "h"] & \pi_i(X^s,\alpha(0)) \arrow[l, "\alpha(\omega_\theta)_*"'] \arrow[d, "h"] \\
\pi_i(\mathbb{Z}X^r,h\alpha(0)) \arrow[r, "\theta_*"'] & \pi_i(\mathbb{Z}X^s,h\theta\alpha(0)) & \pi_i(\mathbb{Z}X^s,h\alpha(0)) \arrow[l, "h\alpha(\omega_\theta)_*"] \\
\pi_i(\mathbb{Z}X^r,0) \arrow[r, "\theta_*"'] \arrow[u, "\gamma_{h\alpha(0)}", "\cong"'] & \pi_i(\mathbb{Z}X^s,0) \arrow[r, equal] \arrow[u, "\cong", "\gamma_{h\theta\alpha(0)}"'] & \pi_i(\mathbb{Z}X^s,0) \arrow[u, "\cong", "\gamma_{h\alpha(0)}"']
\end{tikzcd} \tag{4.1}$$

Here, the map $\gamma_x : \pi_i(\mathbb{Z}Y,0) \to \pi_i(\mathbb{Z}Y,x)$ is defined by $[\beta] \mapsto [\beta + x]$. It is easily seen that the simplicial abelian group structure on $\mathbb{Z}X^s$ implies that $h\alpha(\omega_\theta)$ coincides with the map $[\beta] \mapsto [\beta - h\alpha(0) + h\theta\alpha(0)]$, so the diagram (4.1) commutes. It follows that there is a commutative diagram

$$\begin{tikzcd}
\pi_i(X^r,\alpha(0)) \arrow[r, "\tilde{\theta}"] \arrow[d, "h_*"'] & \pi_i(X^s,\alpha(0)) \arrow[d, "h_*"] \\
H_i(X^r,\mathbb{Z}) \arrow[r, "\theta_*"'] & H_i(X^s,\mathbb{Z})
\end{tikzcd} \tag{4.2}$$

for each ordinal number map $\theta : \mathbf{r} \to \mathbf{s}$.

Suppose that $n \geq 1$. The element $[\alpha] \in \pi_0 \operatorname{Tot}_n X$ lifts to $\pi_0 \operatorname{Tot}_{n+1} X$ if and only if the lifting $\beta$ exists in the diagram

$$\begin{tikzcd}
\partial\Delta^{n+1} \arrow[r, "\alpha"] \arrow[d] & X^{n+1} \arrow[d, "s"] \\
\Delta^{n+1} \arrow[r, "\alpha_*"'] \arrow[ur, dashed, "\beta"] & M^n X
\end{tikzcd} \tag{4.3}$$

The indicated map $\alpha$ is the piece of the cosimplicial map $\alpha : \operatorname{sk}_n \Delta \to X$ that lives in degree $n+1$. It represents an element of the collection of pointed homotopy classes of maps $[(\partial\Delta^{n+1}, 0), X^{n+1}, \alpha(0)]$. The isomorphism

$$[(\partial\Delta^{n+1}, 0), (X^{n+1}, \alpha(0))] \cong \pi_n(X^{n+1}, \alpha(0)) \tag{4.4}$$

is geometrically obvious, and it's an exercise to show that a pointed map $\gamma : (\partial\Delta^{n+1}, 0) \to (X^{n+1}, \alpha(0))$ is pointed homotopically trivial if and only $\gamma$ extends to a map $\Delta^{n+1} \to X^{n+1}$, since $X^{n+1}$ is a Kan complex.

We know, therefore, that if the lifting $\beta$ exists in the diagram (4.3), then $\alpha : \partial\Delta^{n+1} \to X^{n+1}$ represents the trivial element of $\pi_n(X^{n+1}, \alpha(0))$. Conversely, if $[\alpha] = 0$, then there is a pointed homotopy $h_\alpha : \partial\Delta^{n+1} \times \Delta^1 \to X^{n+1}$ from $\alpha$ to the constant map at the vertex $\alpha(0)$, and this homotopy extends to a homotopy of diagrams

$$\begin{tikzcd}
\partial\Delta^{n+1} \times \Delta^1 \arrow[r, "h_\alpha"] \arrow[d] & X^{n+1} \arrow[d, "s"] \\
\Delta^{n+1} \times \Delta^1 \arrow[r, "H_\alpha"'] & M^n X
\end{tikzcd}$$

from the diagram (4.3) to a diagram

$$\begin{tikzcd}
\partial\Delta^{n+1} \arrow[r, "\alpha(0)"] \arrow[d] & X^{n+1} \arrow[d, "s"] \\
\Delta^{n+1} \arrow[r, "\hat{\alpha}"'] \arrow[ur, dashed, "\theta"] & M^n X
\end{tikzcd} \tag{4.5}$$

and a lift $\beta$ exists in the diagram (4.3) if and only if the dotted arrow exists in (4.5), making it commute.

The same argument as for Lemma 1.8 shows that there is an isomorphism

$$\pi_i(M^n X, \alpha(0)) \cong M^n \pi_i(X, \alpha(0)),$$

and so all induced maps

$$s_* : \pi_i(X^{n+1}, \alpha(0)) \to \pi_i(M^n X, \alpha(0))$$

are surjective for $i \geq 2$. Note that $\omega_\theta$ is the identity relation if $\theta : \mathbf{r} \to \mathbf{s}$ is a surjective ordinal number map, so that

$$\tilde{\theta} = \theta_* : \pi_i(X^r, \alpha(0)) \to \pi_i(X^s, \alpha(0)),$$

and the last assertion makes sense. The surjectivity of $s_*$ in degree $n+1$ therefore implies that the dotted lift exists in the diagram (4.5). It follows that the pointed homotopy class of the map

$$\alpha : (\partial\Delta^{n+1}, 0) \to (X^{n+1}, \alpha(0))$$

is the obstruction to extending the cosimplicial space map $\alpha : \mathrm{sk}_n \Delta \to X$ to $\mathrm{sk}_{n+1} \Delta$.

We can be more precise about where the obstruction class $[\alpha]$ lies if $n \geq 2$. First of all, $s_*([\alpha]) \in \pi_n(M^n X, \alpha(0))$ is plainly trivial, so that $[\alpha]$ is in $N\pi_n(X, \alpha(0))^{n+1}$. Secondly, there are commutative diagrams

$$\begin{tikzcd}
H_n(|\partial\Delta^{n+1}|) \arrow[d, "d^j_*"'] & \pi_n(|\partial\Delta^{n+1}|, 0) \arrow[l, "h_*"', "\cong"] \arrow[r, "\alpha_*"] \arrow[d, "\tilde{d}^j"'] & \pi_n(|X^{n+1}|, \alpha(0)) \arrow[d, "\tilde{d}^j"] \\
H_n(|\mathrm{sk}_n \Delta^{n+2}|) & \pi_n(|\mathrm{sk}_n \Delta^{n+2}|, 0) \arrow[l, "\cong", "h_*"'] \arrow[r, "\alpha_*"'] & \pi_n(|X^{n+2}|, \alpha(0))
\end{tikzcd}$$

The cycle $z = \sum_{i=0}^{n+1} (-1)^i d_i(\iota_{n+1})$ represents a generator of the integral homology group $H_n(|\partial\Delta^{n+1}|) \cong \mathbb{Z}$, and

$$\sum_{j=0}^{n+2} (-1)^j d^j_*(z) = 0$$

in $H_n(|\mathrm{sk}_n \Delta^{n+2}|)$. Chasing the generator of $\pi_n(|\partial\Delta^{n+1}|, 0)$ under $\alpha_*$ into $\pi_n(|X^{n+1}|, \alpha(0))$ shows that

$$\sum_{j=0}^{n+2} (-1)^j d^j_*([\alpha]) = 0.$$

We have proved most of the following:

LEMMA 4.6. *Suppose that $X$ is a fibrant cosimplicial space, and that the map $\alpha : \mathrm{sk}_n \Delta \to X$ represents an element of $\pi_0 \operatorname{Tot}_n X$.*

(1) *If $n = 0$, then $[\alpha]$ lifts to $\pi_0 \operatorname{Tot}_1 X$ if and only if $[\alpha] \in \pi^0\pi_0 X$.*

(2) *If $n \geq 1$, then $[\alpha]$ lifts to $\pi_0 \operatorname{Tot}_{n+1} X$ if and only if the component $\alpha : \partial\Delta^{n+1} \to X^{n+1}$ in cosimplicial degree $n+1$ represents the trivial element of $ZN\pi_n(X^{n+1}, \alpha(0))$.*

PROOF: Part (2) is Lemma 1.19.3. For part (1), we have only to consider the case $n = 1$.

If $\alpha : \Delta^1 \to X^1$ is the component of the cosimplicial map $\alpha : \mathrm{sk}_1 \Delta \to X$ in cosimplicial degree 1, then the component in cosimplicial degree 2 is the map $\alpha^2 = (d^0\alpha, d^1\alpha, d^2\alpha) : \partial\Delta^2 \to X^2$, and the corresponding obstruction class in $\pi_1(X^2, \alpha(0))$ can be identified with the fundamental groupoid element $(d^1\alpha)^{-1}(d^0\alpha)(d^2\alpha)$. One can check directly in the fundamental groupoid for $\mathrm{sk}_1 \Delta^3$ that there is a relation

$$d^2(\alpha^2)^{-1} d^1(\alpha^2) d^3(\alpha^2) = \alpha(\omega_{d^0})^{-1} d^0(\alpha^2) \alpha(\omega_{d^0}),$$

so that

$$\tilde{d}^2(\alpha^2)^{-1} \tilde{d}^1(\alpha^2) \tilde{d}^3(\alpha^2) = \tilde{d}^0(\alpha^2)$$

in $\pi_1(X^3, \alpha(0))$, and $[\alpha^2]$ is a cocycle. □

Suppose that $n \geq 1$ and that the simplices $\beta, \beta' : \Delta^{n+1} \to X^{n+1}$ define maps $\beta, \beta' : \mathrm{sk}_{n+1} \Delta \to X$ which restrict to $\alpha : \mathrm{sk}_n \Delta \to X$. Then $\beta$ and $\beta'$ together determine a morphism

$$(\beta, \beta') : \mathrm{sk}_{n+1} \Delta \cup_{\mathrm{sk}_n \Delta} \mathrm{sk}_{n+1} \Delta \to X$$

of cosimplicial spaces which has component

$$(\beta, \beta') : \Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1} \to X^{n+1}$$

in cosimplicial degree $n+1$.

The map $(\beta, \beta')$ determines an element $[(\beta, \beta')] \in \pi_{n+1}(X^{n+1}, \alpha(0))$, and there are commutative diagrams

$$\begin{tikzcd}
H_{n+1}(\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}) \arrow[d, "d^j"'] & \pi_{n+1}(|\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}|, 0) \arrow[l, "h", "\cong"'] \arrow[d, "\tilde{d}^j"] \\
H_{n+1}(\partial\Delta^{n+2} \cup_{\mathrm{sk}_n \Delta^{n+2}} \partial\Delta^{n+2}) & \pi_{n+1}(|\partial\Delta^{n+2} \cup_{\mathrm{sk}_{n-1} \Delta^{n+2}} \partial\Delta^{n+2}|, 0) \arrow[l, "\cong", "h"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\pi_{n+1}(|\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}|, 0) \arrow[r, "(\beta,\beta')_*"] \arrow[d, "\tilde{d}^j"'] & \pi_{n+1}(|X^{n+1}|, \alpha(0)) \arrow[d, "\tilde{d}^j"] \\
\pi_{n+1}(|\partial\Delta^{n+2} \cup_{\mathrm{sk}_n \Delta^{n+2}} \partial\Delta^{n+2}|, 0) \arrow[r, "(\beta,\beta')_*"'] & \pi_{n+1}(|X^{n+2}|, \alpha(0))
\end{tikzcd}$$

The two inclusions $i_L, i_R : \Delta^{n+1} \to \Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}$ represent the top dimensional non-degenerate simplices of $\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}$, and the cycle $i_L - i_R$ represents the generator of $H_{n+1}(\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}) \cong \mathbb{Z}$. Recall that the cycle $\zeta = \sum_{j=0}^{n+2}(-1)^j d_j(\iota_{n+2})$ represents the generator of $H_{n+1}(\partial\Delta^{n+2})$, and write $\partial = \sum_{j=0}^{n+2}(-1)^j d^j$. Then there is a relation

$$\partial([i_L - i_R]) = i_{L*}([\zeta]) - i_{R*}([\zeta])$$

in $H_{n+1}(\partial\Delta^{n+2} \cup_{\mathrm{sk}_n \Delta^{n+2}} \partial\Delta^{n+2})$, and so

$$\partial[(\beta, \beta')] = [\beta] - [\beta']$$

in $\pi_{n+1}(X^{n+2}, \alpha(0))$.

Suppose that $n = 0$ and that the 1-simplices $\beta, \beta' : \Delta^1 \to X^1$ separately extend the map $\alpha : \mathrm{sk}_0 \Delta \to X$ to a map $\mathrm{sk}_1 \Delta \to X$. Under these circumstances, all vertices of $\mathrm{sk}_0 \Delta^n$ are mapped into the same path component of $X^n$, but there are no canonical choices of paths up to homotopy in $\mathrm{sk}_1 \Delta^n$ between these vertices. We circumvent this problem by (somewhat grossly) assuming that the fundamental groupoid $\pi X^m$ acts trivially on all groups $\pi_1(X^m, x)$ for all $m \geq 0$. This means precisely that any two 1-simplices $\omega, \eta : x \to y$ in $X^m$ induce the same group homomorphism

$$\omega_* = \eta_* : \pi_1(X^m, x) \to \pi_1(X^m, y)$$

by conjugation in the fundamental groupoid. It follows that all fundamental groups $\pi_1(X^m, x)$ are abelian. This assumption further implies that there is a cosimplicial group, given by assigning the group homomorphism $\tilde{\theta}_*$ to each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$, where $\tilde{\theta}_*$ is the composite

$$\pi_1(X^m, \alpha(0)) \xrightarrow{\theta_*} \pi_1(X^n, \alpha(\theta(0))) \xrightarrow{\omega_{\alpha(\theta(0))}} \pi_1(X^n, \alpha(0)),$$

where $\omega_{\alpha(\theta(0))} : \alpha(\theta(0)) \to \alpha(0)$ is a 1-simplex connecting the vertices $\alpha(\theta(0))$ and $\alpha(0)$ in $X^n$. We can and will assume that $\omega_{\alpha(\theta(0))}$ is the identity map if $\theta(0) = 0$.

The 1-simplices $\beta, \beta' : \alpha(0) \to \alpha(1)$ of $X^1$ determine an element

$$\beta^{-1}\beta' \in \pi X^1(\alpha(0), \alpha(0)) \cong \pi_1(X^1, \alpha(0)),$$

and there are relations in the fundamental groupoid $\pi X^2$ of the form

$$\begin{aligned}
&d^1(\beta^{-1}\beta')^{-1}\tilde{d}^0(\beta^{-1}\beta')d^2(\beta^{-1}\beta')\\
&= d^1(\beta^{-1}\beta')^{-1}d^2(\beta)^{-1}d^0(\beta^{-1}\beta')d^2(\beta)d^2(\beta^{-1}\beta')\\
&= d^1(\beta')^{-1}d^1(\beta)d^2(\beta)^{-1}d^0(\beta)^{-1}d^0(\beta')d^2(\beta')\\
&= d^1(\beta')^{-1}d^1(\beta)d^2(\beta)^{-1}d^0(\beta)^{-1}d^1(\beta)d^1(\beta)^{-1}d^1(\beta')d^1(\beta')^{-1}d^0(\beta')d^2(\beta')\\
&= d^1(\beta^{-1}\beta')^{-1}d^2(\beta)^{-1}d^0(\beta)^{-1}d^1(\beta)d^1(\beta^{-1}\beta')d^1(\beta')^{-1}d^0(\beta')d^2(\beta')\\
&= (d^1(\beta)^{-1}d^0(\beta)d^2(\beta))^{-1}(d^1(\beta')^{-1}d^0(\beta')d^2(\beta')).
\end{aligned}$$

We have proved the following:

LEMMA 4.7. *Suppose that $\beta, \beta' : \mathrm{sk}_{n+1} \to X$ both extend a map $\alpha : \mathrm{sk}_n \Delta \to X$.*

(1) *If $n \geq 1$, then the simplices $\beta, \beta' : \Delta^{n+1} \to X^{n+1}$ determine a canonical class $[(\beta, \beta')] \in \pi_{n+1}(X^{n+1}, \alpha(0))$ such that*

$$\partial[(\beta, \beta')] = [\beta] - [\beta'] \in \pi_{n+1}(X^{n+2}, \alpha(0)),$$

*where $[\beta]$ and $[\beta']$ are the obstructions to extending the corresponding maps to cosimplicial space maps $\mathrm{sk}_{n+2}\Delta \to X$.*

(2) *Suppose that $n = 0$ and that the fundamental groupoid $\pi X^m$ acts trivially on all groups $\pi_1(X^m, x)$ for $m \geq 0$ if $n = 0$. Then the class $[(\beta)^{-1}\beta'] \in \pi_1(X^1, \alpha(0))$ satisfies*

$$\partial[(\beta)^{-1}\beta'] = [\beta'] - [\beta] \in \pi_1(X^2, \alpha(0)),$$

*where $[\beta]$ and $[\beta']$ are the obstructions to extending the corresponding maps to cosimplicial space maps $\mathrm{sk}_2\Delta \to X$.*

It follows that the obstruction cocycle $\beta : \partial\Delta^{n+2} \to X^{n+2}$ associated to an extension $\beta$ of $\alpha : \mathrm{sk}_n \Delta \to X$ determines an element

$$[\beta] \in \pi^{n+2}\pi_{n+1}(X, \alpha(0))$$

which is independent of the choice of extension $\beta$ for $n \geq 1$, and that there is a corresponding statement for $n = 0$ subject to the assumption about fundamental groupoid actions in Lemma 4.7.2.

LEMMA 4.8. *Suppose that $\beta : \mathrm{sk}_{n+1}\Delta \to X$ is an extension of $\alpha : \mathrm{sk}_n \Delta \to X$ and that $[\omega]$ is an element of $N\pi_{n+1}(X, \alpha(0))^{n+1}$. Then there is an extension $\gamma : \mathrm{sk}_{n+1}\Delta \to X$ of $\alpha$ such that*

$$\partial([\omega]) = [\beta] - [\gamma] \in \pi_{n+1}(X^{n+2}, \alpha(0)).$$

The proof of Lemma 4.8 will appear after the following:

THEOREM 4.9. *Suppose that $\beta : \mathrm{sk}_{n+1}\Delta \to X$ extends $\alpha : \mathrm{sk}_n \Delta \to X$.*

(1) *Suppose that $n \geq 1$. Then the element $[\beta] \in \pi^{n+2}\pi_{n+1}(X, \alpha(0))$ associated to the obstruction cocycle represented by $\beta : \partial\Delta^{n+2} \to X^{n+2}$ is 0 if and only if $\alpha$ extends to a map $\mathrm{sk}_{n+2}\Delta \to X$.*

(2) *Suppose that $n = 0$ and that the fundamental groupoid $\pi X^m$ acts trivially on all $\pi_1(X^m, x)$ for $m \geq 0$. Then the element $[\beta] \in \pi^2\pi_1(X, \alpha(0))$ associated to the obstruction cocycle represented by $\beta : \partial\Delta^2 \to X^2$ is 0 if and only if $\alpha$ extends to a map $\mathrm{sk}_2 \Delta \to X$.*

Before proving Theorem 4.9, we mention an obvious corollary:

COROLLARY 4.10. *Suppose that $X$ is a fibrant cosimplicial space such that the fundamental groupoid $\pi X^m$ acts trivially on all fundamental groups $\pi_1(X^m, x)$ for $m \geq 0$. Suppose that $\alpha : \mathrm{sk}_0 \Delta \to X$ represents an element of $\pi^0\pi_0 X$, and that*

$$\pi^{n+1}\pi_n(X, \alpha(0)) = 0$$

*for all $n \geq 1$. Then $[\alpha]$ lifts to $\pi_0 \operatorname{Tot} X$.*

PROOF OF THEOREM 4.9: If $\alpha$ extends to a map $\gamma : \mathrm{sk}_{n+2}\Delta \to X$, then $[\gamma] = [\beta] \in \pi^{n+2}\pi_{n+1}(X, \alpha(0))$ and $[\gamma] = 0$, so that the map $\beta : \partial\Delta^{n+2} \to X$ represents a boundary.

If $\beta$ represents a boundary, say

$$[\beta] = \partial(\omega)$$

for some $\omega \in N\pi_{n+1}(X, \alpha(0))^{n+1}$, then there is an extension $\gamma : \mathrm{sk}_{n+1}\Delta \to X$ such that

$$\partial(\omega) = [\beta] - [\gamma].$$

But then $[\gamma] = 0 \in \pi_{n+1}(X^{n+2}, \alpha(0))$, so $\gamma$ extends to a map $\mathrm{sk}_{n+2}\Delta \to X$. □

PROOF OF LEMMA 4.8: Suppose that $\tau : (\Delta^{n+1}, \partial\Delta^{n+1}) \to (NX^{n+1}, \alpha(0))$ represents an element of $N\pi_{n+1}(X, \alpha(0))^{n+1}$, and let $h : \Delta^{n+1} \times \Delta^1 \to \Delta^{n+1}$ be the canonical contracting homotopy onto the vertex 0. Then there is an extension

$$\begin{tikzcd}
(\Delta^{n+1} \times \Delta^1) \cup ((\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}) \times \{0\}) \arrow[r, "{(\beta \cdot h, (\alpha(0), \tau))}"] \arrow[d] & X^{n+1} \\
(\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}) \times \Delta^1 \arrow[ur, "H"'] &
\end{tikzcd}$$

and the restriction of $H$ to $(\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}) \times \{1\}$ determines a commutative diagram

$$\begin{tikzcd}
\Delta^{n+1} \arrow[r, "\beta"] \arrow[d, "in_L"'] & X^{n+1} \\
\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1} \arrow[ur, "\tau_*"'] &
\end{tikzcd}$$

where the map $\tau_*$ represents an element of the pointed homotopy class

$$[(\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}, 0), (NX^{n+1}, \alpha(0))].$$

The association $[\tau] \mapsto [\tau_*]$ is well defined (so in particular $[\tau_*]$ is independent of the choice of lift $H$), and gives an explicit bijection

$$\pi_{n+1}(NX^{n+1}, \alpha(0)) \cong [(\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}, 0), (NX^{n+1}, \alpha(0))].$$

We can assume that the representative $\omega$ of $[\omega]$ is chosen such that the composite

$$\Delta^{n+1} \xrightarrow{in_L} \Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1} \xrightarrow{\omega} NX^{n+1}$$

is the simplex $\beta$, by a standard argument. One can therefore reverse the process of the previous paragraph to produce a pointed map $\omega_* : (\Delta^{n+1}, \partial\Delta^{n+1}) \to (NX^{n+1}, \alpha(0))$ where $[\omega_*] \mapsto [\omega]$.

The map $s : \Delta^{n+1} \to M^n\Delta$ and the map of cosimplicial spaces $\beta : \mathrm{sk}_{n+1}\Delta \to X$ together determine a pointed composite $s_*$

$$\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1} \xrightarrow{(s,s)} M^n\Delta \xrightarrow{\beta_*} M^nX.$$

The proof of the lemma will be complete if we can show that the lifting $\theta$ exists in the diagram

$$\begin{tikzcd}
\Delta^{n+1} \arrow[r, "\beta"] \arrow[d, "in_L"'] & X^{n+1} \arrow[d, "s"] \\
\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1} \arrow[r, "s_*"'] \arrow[ur, dashed, "\theta"] & M^nX
\end{tikzcd}$$

and $[\theta] = [\omega]$, for then $\theta \cdot in_R : \Delta^{n+1} \to X^{n+1}$ defines the required second extension $\gamma$ of $\alpha$, and $\partial[\omega] = [\beta] - [\gamma]$ according to the proof of the Lemma 4.7.

The map $s_*$ is pointed homotopy trivial, since

$$\pi_{n+1}(M^nX, \alpha(0)) \cong M^n\pi_{n+1}(X, \alpha(0))$$

and the composite with each codegeneracy $s^j : M^n X \to X^n$ factors through the space $\Delta^n$. It follows that there is an extension

$$\begin{tikzcd}
(\Delta^{n+1} \times \Delta^1) \cup ((\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}) \times \partial\Delta^1) \arrow[r, "{(s\beta \cdot h, (\alpha(0), s_*))}"] \arrow[d] & M^n X \\
(\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}) \times \Delta^1 \arrow[ur, "K"'] &
\end{tikzcd}$$

Now form the diagram

$$\begin{tikzcd}
(\Delta^{n+1} \times \Delta^1) \cup ((\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}) \times \{0\}) \arrow[r, "{(\beta \cdot h, (\alpha(0), \omega_*))}"] \arrow[d] & X^{n+1} \arrow[d, "s"] \\
(\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1}) \times \Delta^1 \arrow[r, "K"'] \arrow[ur, "L"] & M^n X
\end{tikzcd}$$

The restriction of the lift $L$ to $\Delta^{n+1} \cup_{\partial\Delta^{n+1}} \Delta^{n+1} \times \{1\}$ is the required lift $\theta$, and $[\theta] = [\omega]$ by independence of the choice of lift defining $[\omega]$ from $[\omega_*]$. $\square$

Now we come to the example. Let $Y \in \mathbf{S}$ be a space and let $\mathbb{F}_p^\bullet Y$ be the Bousfield-Kan resolution of $Y$. Given another space $X$ there a cosimplicial space $\mathbf{Hom}(X, \mathbb{F}_p^\bullet Y)$ with

$$\begin{aligned}
\operatorname{Tot} \mathbf{Hom}(X, \mathbb{F}_p^\bullet Y) &\cong \mathbf{Hom}(X, \operatorname{Tot}(\mathbb{F}_p^\bullet Y)) \\
&\cong \mathbf{Hom}(X, Y_p)
\end{aligned}$$

where $Y_p$ is the Bousfield-Kan completion of $Y$. We now describe the sets

$$\pi^s \pi_t \mathbf{Hom}(X, \mathbb{F}_p^\bullet Y).$$

Let $H^* = H^*(\cdot, \mathbb{F}_p)$ and let $\mathcal{K}$ be the category of unstable algebras over the Steenrod algebra. Then for any space $Z$ there is a Hurewicz homomorphism

$$\pi_0 \mathbf{Hom}(X, Z) \to \hom_{\mathcal{K}}(H^* Z, H^* Y) \tag{4.11}$$

sending $f$ to $f^*$. If $f \in \mathbf{Hom}(X, Z)$ is a chosen basepoint, and $\varphi : S^t \to \mathbf{Hom}(X, Z)$ represents a class in $\pi_t \mathbf{Hom}(X, Z)$ with this basepoint, then $\varphi$ is adjoint to a map $\psi : S^t \times X \to Z$ that appears in the diagram

$$\begin{tikzcd}
S^t \times X \arrow[r, "\psi"] & Z \\
* \times X. \arrow[u] \arrow[ur, "f"'] &
\end{tikzcd}$$

The morphism $\psi^* : H^*Z \to H^*(S^t \times X) \cong H^*S^t \otimes H^*X$ can be decomposed

$$\psi^*(x) = 1 \otimes f^*(x) + x_t \otimes \partial(x)$$

where $x_t \in H^tS^t$ is a chosen generator. Because $\psi^*$ is a morphism in $\mathcal{K}$, the induced map of graded vector spaces

$$\partial : H^*Z \to \Sigma^t H^*X$$

is a morphism of unstable modules over the Steenrod algebra and a derivation over $f^*$:

$$\partial(xy) = (-1)^{t|x|} f^*x\partial(y) + \partial(x)f^*y.$$

We write $\mathrm{Der}_{\mathcal{K}}(H^*Z, \Sigma^t H^*X; f^*)$ or simply $\mathrm{Der}_{\mathcal{K}}(H^*Z, \Sigma^t H^*X)$ for the vector space of such derivations. Hence we get a Hurewicz map

$$\pi_t \mathbf{Hom}(X, Z) \longrightarrow \mathrm{Der}_{\mathcal{K}}(H^*Z, \Sigma^t H^*X; f^*). \tag{4.12}$$

If $Z$ is a simplicial $\mathbb{F}_p$ vector space and $Z$ and $X$ are of finite type in the sense that $H^nZ$ and $H^nX$ are finite dimensional for all $n$, then the Hurewicz maps of (4.11) and (4.12) are isomorphisms. Therefore, if $X$ and $Y$ are of finite type then

$$\pi^0\pi_0 \mathbf{Hom}(X, \mathbb{F}_p^\bullet Y) \cong \hom_{\mathcal{K}}(H^*Y, H^*X) \tag{4.13}$$

and for any $\varphi \in \hom_{\mathcal{K}}(H^*Y, H^*X)$

$$\pi^s\pi_t \mathbf{Hom}(X, \mathbb{F}_p^\bullet Y) \cong \pi^s \, \mathrm{Der}_{\mathcal{K}}(H^*\mathbb{F}_p^\bullet Y, \Sigma^t X; \varphi). \tag{4.14}$$

In the last equation we have written $\varphi$ for any composition of face operators

$$H^*(\mathbb{F}_p^s Y) \to H^*Y \xrightarrow{\varphi} H^*X.$$

The category $s\mathcal{K}$ of simplicial objects in in $\mathcal{K}$ has a simplicial model category structure, as in Example II.5.2. Also the objects $H^*K(\mathbb{Z}/p, n)$ form a set of projective generators for $\mathcal{K}$. If we regard $H^*Y$ as a constant object in $s\mathcal{K}$, then the augmentation

$$H^*\mathbb{F}_p^\bullet Y \to H^*Y$$

is a weak equivalence in $s\mathcal{K}$ and $H^*\mathbb{F}_p^\bullet Y$ is cofibrant — indeed, degeneracy free as in Example VII.1.15. Thus

$$\mathrm{Der}_{\mathcal{K}}(H^*\mathbb{F}_p^\bullet Y, \Sigma^t X; \varphi)$$

is a model for the total derived functor of $\mathrm{Der}_{\mathcal{K}}(\cdot, \Sigma^t H^*X; \varphi)$ applied to $H^*Y$, and we may as well write

$$\pi^s \, \mathrm{Der}_{\mathcal{K}}(H^*\mathbb{F}_p^\bullet Y, \Sigma^t X; \varphi) = R^s \, \mathrm{Der}_{\mathcal{K}}(H^*Y, \Sigma^t X; \varphi). \tag{4.15}$$

This is an example of an André-Quillen cohomology, and a great deal of work has gone into learning how to compute this object in special cases. For a general viewpoint one might look at [35]. However, the most successful application of this and related techniques is given by Lannes in [62].

Combining (4.15) with (4.13) and (4.14) and appealing to Corollary 4.10 we have

PROPOSITION 4.16. *Suppose $X$ and $Y$ are spaces of finite type. A morphism $\varphi : H^*Y \to H^*X$ in $\mathcal{K}$ can be lifted to a map $X \to Y_p$ if*

$$R^{s+1}\operatorname{Der}_{\mathcal{K}}(H^*Y, \Sigma^s X; \varphi) = 0$$

*for $s \geq 1$.*

The techniques of Lannes's paper [62] show that if $X = B(\mathbb{Z}/p)^n$ is the classifying space of an elementary abelian $p$-group and $A \in \mathcal{K}$ is any unstable algebra, then

$$R^{s+1}\operatorname{Der}_{\mathcal{K}}(A, \Sigma^t X; \varphi) = 0$$

for all $\varphi \in \hom_{\mathcal{K}}(A, H^*B(\mathbb{Z}/p)^n)$ and all $s \geq 0, t \geq 0$.

# Chapter IX Simplicial functors and homotopy coherence

Suppose that $\mathcal{A}$ is a simplicial category. The main objects of study in this chapter are the functors $X : \mathcal{A} \to \mathbf{S}$ taking values in simplicial sets, and which respect the simplicial structure of $\mathcal{A}$. In applications, the simplicial category $\mathcal{A}$ is typically a resolution of a category $I$, and the simplicial functor $X$ describes a homotopy coherent diagram. The main result of this chapter (due to Dwyer and Kan, Theorem 2.13 below) is a generalization of the assertion that simplicial functors of the form $X : \mathcal{A} \to \mathbf{S}$ are equivalent to diagrams of the form $I \to \mathbf{S}$ in the case where $\mathcal{A}$ is a resolution of the category $I$. The proof of this theorem uses simplicial model structures for categories of simplicial functors, given in Section 1, and then the result itself is proved in Section 2.

The Dwyer-Kan theorem immediately leads to realization theorems for homotopy coherent diagrams in cases where the simplicial categories $\mathcal{A}$ model homotopy coherence phenomena. A realization of a homotopy coherent diagram $X : \mathcal{A} \to \mathbf{S}$ is a functor $Y : I \to \mathbf{S}$ which is weakly equivalent to $X$ in a strong sense. Insofar as the information arising from $X$ typically consists of simplicial set maps $X(\alpha) : X(i) \to X(j)$, one for each morphism $\alpha$ of $I$ which only respect the composition laws of $I$ up to some system of higher homotopies, a realization $Y$ is a replacement of $X$, up to weak equivalence, by a collection of maps $Y(\alpha) : Y(i) \to Y(j)$ which define a functor on the nose. Diagrams of spaces which are not quite functorial are really very common: the machines which produce the algebraic $K$-theory spaces, for example, are not functors (on scheme categories in particular), but they have homotopy coherent output for categorical reasons [49].

Approaches to homotopy coherence for simplicial set diagrams arising from some specific resolution constructions are discussed in Section 3; traditional homotopy coherence, as in [91], is one of the examples.

More generally, we take the point of view that a homotopy coherent diagram on a fixed index category $I$ is a simplicial functor $X : \mathcal{A} \to \mathbf{S}$ defined on *any* simplicial resolution $\mathcal{A}$ of $I$. We can further ask for realization results concerning homotopy coherent diagrams $\mathcal{A} \to \mathcal{M}$ taking values in more general simplicial model categories $\mathcal{M}$. This is the subject of Section 4. We derive, in particular, realization theorems for homotopy coherent diagrams taking values in pointed simplicial sets, spectra and simplicial abelian groups (aka. chain complexes).

## 1. Simplicial functors.

We shall take the point of view throughout this chapter that a *simplicial category* $\mathcal{A}$ is a simplicial object in the category of categories having a discrete simplicial class of objects. This is a weaker definition than that appearing in Section II.2. The full simplicial set category $\mathbf{S}$ with the function complexes $\mathbf{Hom}(X, Y)$ is a simplicial category in this sense, but of course it also, canonically, has the extra structure required by Definition II.2.1.

The simplicial set of morphisms from $A$ to $B$ in a simplicial category $\mathcal{A}$ is denoted by $\mathbf{Hom}(A,B)$; from this point of view, the corresponding set of $n$-simplices $\mathbf{Hom}(A,B)_n$ is the set of morphisms from $A$ to $B$ in the category at level $n$. Any morphism $\alpha : B \to C$ in $\mathbf{Hom}(B,C)_0$ induces a simplicial set map

$$\alpha_* : \mathbf{Hom}(A,B) \to \mathbf{Hom}(A,C),$$

which one understands to be composition with $\alpha$, and which is identified with the composite simplicial set map

$$\mathbf{Hom}(A,B) \times \Delta^0 \xrightarrow{1\times\iota_\alpha} \mathbf{Hom}(A,B) \times \mathbf{Hom}(B,C) \xrightarrow{\circ} \mathbf{Hom}(A,C).$$

Similarly, any morphism $\beta : A \to B$ induces a simplicial set map

$$\beta^* : \mathbf{Hom}(B,C) \to \mathbf{Hom}(A,C),$$

that one thinks of as precomposition with $\beta$. Composition and precomposition respect composition in $\alpha$ and $\beta$ in the traditional sense.

A *simplicial functor* $f : \mathcal{A} \to \mathcal{B}$ is a morphism of simplicial categories. This means that $f$ consists of a function $f : Ob(\mathcal{A}) \to Ob(\mathcal{B})$ and simplicial set maps $f : \mathbf{Hom}(A,B) \to \mathbf{Hom}(f(A),f(B))$ which respect identities and composition at all levels.

A *natural transformation* $\eta : f \to g$ of simplicial functors $f,g : \mathcal{A} \to \mathcal{B}$ consists of morphisms

$$\eta_A : f(A) \to g(A)$$

in $\hom(f(A),g(A)) = \mathbf{Hom}(f(A),g(A))_0$, one for each object $A$ of $\mathcal{A}$, such that the following diagram of simplicial set maps commutes

$$\begin{tikzcd}
\mathbf{Hom}(A,B) \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{Hom}(f(A),f(B)) \arrow[d, "\eta_{B*}"] \\
\mathbf{Hom}(g(A),g(B)) \arrow[r, "\eta_A^*"'] & \mathbf{Hom}(f(A),g(B))
\end{tikzcd}$$

for each pair of objects $A,B$ of $\mathcal{A}$. Notice that this is just another way of saying that the various degeneracies of the morphisms $\eta_A$ are natural transformations between the functors induced by $f$ and $g$ at all levels. The collection of all simplicial functors from $\mathcal{A}$ to $\mathcal{B}$ and all natural transformations between them form a category, which we shall denote by $\mathcal{B}^{\mathcal{A}}$. Write $\mathrm{Nat}(f,g)$ for the set of all natural transformations from $f$ to $g$.

The category $\mathbf{S}^{\mathcal{A}}$ of simplicial functors taking values in the simplicial set category $\mathbf{S}$ are of particular interest, and can be given a much more explicit description. Suppose that $\mathcal{A}_n$ denotes the small category at level $n$ within the simplicial category $\mathcal{A}$. Then a simplicial functor $X : \mathcal{A} \to \mathbf{S}$ consists of a function $X : Ob(\mathcal{A}) \to Ob(\mathbf{S})$ as before, together with a collection of simplicial set maps $X(\alpha) : X(A) \times \Delta^n \to X(B)$, one for each morphism $\alpha : A \to B$ in $\mathcal{A}_n$, $n \geq 0$, such that

(1) the simplicial set map $X(\beta\alpha)$ is the composite

$$X(A) \times \Delta^n \xrightarrow{1\times\Delta} X(A) \times \Delta^n \times \Delta^n \xrightarrow{X(\alpha)\times 1} X(B) \times \Delta^n \xrightarrow{X(\beta)} X(C)$$

for each composeable string of morphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

in $\mathcal{A}_n$,

(2) the simplicial set map $X(1_A)$ associated to the identity on $A \in \mathcal{A}_n$ is the projection $X(A) \times \Delta^n \to X(A)$, and

(3) for each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ and each morphism $\alpha : A \to B$ of $\mathcal{A}_n$, the following diagram commutes:

$$\begin{tikzcd}
X(A) \times \Delta^m \arrow[d, "1 \times \theta_*"'] \arrow[dr, "X(\theta^*(\alpha))"] & \\
X(A) \times \Delta^n \arrow[r, "X(\alpha)"'] & X(B).
\end{tikzcd}$$

Then, from this point of view, a natural transformation $\eta : X \to Y$ of simplicial functors taking values in simplicial sets consists of simplicial set maps $\eta_A : X(A) \to Y(A)$, one for each $A \in Ob(\mathcal{A})$, such that the diagram

$$\begin{tikzcd}
X(A) \times \Delta^n \arrow[r, "X(\alpha)"] \arrow[d, "\eta_A \times 1"'] & X(B) \arrow[d, "\eta_B"] \\
Y(A) \times \Delta^n \arrow[r, "Y(\alpha)"'] & Y(B)
\end{tikzcd}$$

commutes for each morphism $\alpha : A \to B$ of $\mathcal{A}_n$, and for all $n \geq 0$.

Now it's easy to see that the category $\mathbf{S}^{\mathcal{A}}$ is the category at level 0 of a simplicial category in the sense of Definition II.2.1. Given a simplicial functor $X : \mathcal{A} \to \mathbf{S}$ and a simplicial set $K$, there is a simplicial functor $K \times X$ which assigns to each $A \in Ob(\mathcal{A})$ the simplicial set $K \times X(A)$ and to each morphism $\alpha : A \to B$ in $\mathcal{A}_n$ the simplicial set map

$$K \times X(A) \times \Delta^n \xrightarrow{1 \times X(\alpha)} K \times X(B).$$

Dually, there is a simplicial functor $\mathbf{hom}(K, X)$ which associates to each $A \in Ob(\mathcal{A})$ the simplicial set $\mathbf{hom}(K, X(A))$, and to each morphism $\alpha : A \to B$ of $\mathcal{A}_n$ the simplicial set map $\mathbf{hom}(K, X(A)) \times \Delta^n \to \mathbf{hom}(K, X(B))$ which is defined to be the adjoint of the composite

$$K \times \mathbf{hom}(K, X(A)) \times \Delta^n \xrightarrow{ev \times 1} X(A) \times \Delta^n \xrightarrow{X(\alpha)} X(B),$$

where $ev : K \times \mathbf{hom}(K, X(A)) \to X(A)$ is the standard evaluation map. It follows immediately that the collection of all such evaluation maps induces a natural bijection

$$\mathrm{Nat}(K \times Y, X) \cong \mathrm{Nat}(Y, \mathbf{hom}(K, X)).$$

There is also, plainly, a simplicial set $\mathbf{Hom}(Y, X)$ whose set of $n$-simplices is the set $\mathrm{Nat}(\Delta^n \times Y, X)$. Finally, since the simplicial set $K$ is a colimit of its simplices, there is a natural bijection (exponential law)

$$\hom(K, \mathbf{Hom}(Y, X)) \cong \mathrm{Nat}(K \times Y, X) \tag{1.1}$$

relating morphisms in the simplicial set category to natural transformations of simplicial functors.

The *representable simplicial functor* $\mathbf{Hom}(A, \ )$ is the simplicial functor which associates to each $B \in Ob(\mathcal{A})$ the simplicial set $\mathbf{Hom}(A, B)$, and associates to each morphism $\beta : B \to C$ of $\mathcal{A}_n$ the composite simplicial set map

$$\mathbf{Hom}(A, B) \times \Delta^n \xrightarrow{1 \times \iota_\beta} \mathbf{Hom}(A, B) \times \mathbf{Hom}(B, C) \xrightarrow{\circ} \mathbf{Hom}(A, C).$$

It is central to observe that the representable simplicial functors in $\mathbf{S}^{\mathcal{A}}$ satisfy the Yoneda lemma:

LEMMA 1.2. *Suppose that $A$ is an object of a simplicial category $\mathcal{A}$, and that $X : \mathcal{A} \to \mathbf{S}$ is a simplicial functor. Then there is a natural isomorphism of simplicial sets*

$$\mathbf{Hom}(\mathbf{Hom}(A, \ ), X) \cong X(A).$$

PROOF: The natural transformations $\eta : \mathbf{Hom}(A, \ ) \to X$ are in one to one correspondence with the vertices of $X(A)$. In effect, given a morphism $\alpha : A \to B$ of $\mathcal{A}_m$, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{Hom}(A, A) \times \Delta^m \arrow[r, "\eta_A \times 1"] \arrow[d, "\alpha_*"'] & X(A) \times \Delta^m \arrow[d, "X(\alpha)"] \\
\mathbf{Hom}(A, B) \arrow[r, "\eta_B"'] & X(B),
\end{tikzcd}$$

so that

$$\eta_B(\alpha) = X(\alpha)(\eta_A(1_A), \iota_m),$$

where $\iota_m = 1_{\mathbf{m}} \in \Delta^m_m$ is the classifying simplex. But $1_A = s(1_{A,0})$ where $1_{A,0}$ denotes the identity on $A$ in $\mathcal{A}_0$. Thus, $\eta$ is completely determined by the vertex $\eta_A(1_{A,0})$ of $X(A)$.

It follows that there are natural bijections

$$\begin{aligned}\mathrm{Nat}(\Delta^n \times \mathbf{Hom}(A,\ ), X) &\cong \mathrm{Nat}(\mathbf{Hom}(A,\ ), \mathbf{hom}(\Delta^n, X)) \\ &\cong \mathbf{hom}(\Delta^n, X(A))_0 \\ &\cong X(A)_n\end{aligned}$$

and that these bijections respect the simplicial structure. □

A natural transformation $\eta : \Delta^n \times \mathbf{Hom}(A,\ ) \to X$ is completely determined by the $n$-simplex $\eta(\iota_n, 1_A)$ of the simplicial set $X(A)$: this is a corollary of the proof of Lemma 1.2.

The functor $\mathbf{S} \to \mathbf{S}$ defined by $K \mapsto K \times \Delta^n$ preserves all small colimits of simplicial sets and takes limits to limits fibred over $\Delta^n$. It follows that the category $\mathbf{S}^{\mathcal{A}}$ of simplicial functors from $\mathcal{A}$ to $\mathbf{S}$ is complete and co-complete, and that all limits and colimits are formed pointwise.

There is an analogue of the simplex category for each simplicial functor $X \in \mathbf{S}^{\mathcal{A}}$. Write $\mathcal{A} \downarrow X$ for the category whose objects are the transformations

$$\Delta^n \times \mathbf{Hom}(A,\ ) \to X,$$

and whose objects are all commutative triangles of transformations of the form

$$\begin{tikzcd}
\Delta^n \times \mathbf{Hom}(A,\ ) \arrow[d] \arrow[dr] & \\
& X \\
\Delta^m \times \mathbf{Hom}(B,\ ) \arrow[ur] &
\end{tikzcd}$$

The objects of this category are simplices of sections of $X$, and the morphisms form its structural data. It follows that $X$ is a colimit of its simplices in $\mathbf{S}^{\mathcal{A}}$ in the sense that there is a natural isomorphism

$$X \cong \varinjlim_{\Delta^n \times \mathbf{Hom}(A,\ ) \to X} \Delta^n \times \mathbf{Hom}(A,\ ). \tag{1.3}$$

Say that a map $f : X \to Y$ of $\mathbf{S}^{\mathcal{A}}$ is a *fibration* (respectively *weak equivalence*) if the component maps $f : X(A) \to Y(A)$ are fibrations (respectively weak equivalences) of simplicial sets, for all objects $A$ of $\mathcal{A}$. Such maps will be called *pointwise fibrations* and *pointwise weak equivalences*, respectively. A *cofibration* of $\mathbf{S}^{\mathcal{A}}$ is a map which has the left lifting property with respect to all trivial fibrations.

PROPOSITION 1.4. *With these definitions, the category* $\mathbf{S}^{\mathcal{A}}$ *of simplicial set-valued simplicial functors defined on a small simplicial category* $\mathcal{A}$ *satisfies the axioms for a simplicial model category.*

PROOF: **CM1** is a consequence of the completeness and cocompleteness of $\mathbf{S}^{\mathcal{A}}$, as described above. The weak equivalence axiom **CM2** and the retract axiom **CM3** are both trivial consequences of the corresponding axioms for simplicial sets.

Note that a map $f : X \to Y$ of $\mathbf{S}^{\mathcal{A}}$ is a fibration if and only if $f$ has the right lifting property with respect to all maps

$$\Lambda^n_k \times \mathbf{Hom}(A,\ ) \xrightarrow{i\times 1} \Delta^n \times \mathbf{Hom}(A,\ ),$$

and $f$ is a trivial fibration if and only if it has the right lifting property with respect to all maps

$$\partial\Delta^n \times \mathbf{Hom}(A,\ ) \xrightarrow{i'\times 1} \Delta^n \times \mathbf{Hom}(A,\ ),$$

where $i : \Lambda^n_k \hookrightarrow \Delta^n$ and $i' : \partial\Delta^n \hookrightarrow \Delta^n$ denote the respective canonical inclusions. Both statements follow from the exponential law (1.1) and Lemma 1.2. The map $i \times 1$ is a weak equivalence.

We may therefore apply standard small object arguments to prove the factorization axiom **CM5** for $\mathbf{S}^{\mathcal{A}}$. In particular, any map $f : X \to Y$ has a factorization

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\
& Y,
\end{tikzcd}$$

where $p$ is a fibration and $j$ is a filtered colimit of maps of the form $j_n : X_n \to X_{n+1}$, each of which is defined by a pushout diagram

$$\begin{tikzcd}
\bigsqcup_r \mathbf{Hom}(A_r,\ ) \times \Lambda^{n_r}_{k_r} \arrow[r] \arrow[d, "\bigsqcup_r (i\times 1)"'] & X_n \arrow[d, "j_n"] \\
\bigsqcup_r \mathbf{Hom}(A_r,\ ) \times \Delta^{n_r} \arrow[r] & X_{n+1}.
\end{tikzcd}$$

It follows that each of the maps $j_n$ is a pointwise weak equivalence, and has the left lifting property with respect to all fibrations, and so the map $j$ has the same properties, giving one of the factorizations required by **CM5** in particular. The other factorization has a similar construction.

The axiom **CM4** is a standard consequence of the method of proof of the factorization axiom **CM5**, in that any map $f : X \to Y$ which is a cofibration and a pointwise weak equivalence has a factorization $f = p \cdot j$, where $p$ is a fibration and where $j$ is a pointwise weak equivalence and has the right lifting property with respect to all fibrations. In particular, $p$ is a pointwise weak equivalence, and so the indicated lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "1_G"'] \arrow[ur, dashed] & Y. \end{tikzcd}$$

The map $f$ is therefore a retract of $j$, and therefore has the left lifting property with respect to all fibrations.

The axiom **SM7** is a consequence of the corresponding statement for simplicial sets. □

Remark 1.5. Proposition 1.4 is a special case of a result of Dwyer and Kan [24]. The proof given here is simpler.

Remark 1.6. The Bousfield-Kan closed model structure for small $I$-diagrams of simplicial sets given in Example II.6.11 and Section IV.3.1 is a special case of Proposition 1.4.

## 2. The Dwyer-Kan theorem.

Suppose, throughout this section, that $\mathcal{A}$ and $\mathcal{B}$ are small simplicial categories. A simplicial functor $X : \mathcal{A} \to \mathbf{S}$ can alternatively be described as a "rule" which associates to each object $A$ of $\mathcal{A}$ a simplicial set $X(A)$, and to each morphism $\alpha : A \to B$ of $\mathcal{A}_n$ a function $\alpha_* : X(A)_n \to X(B)_n$ on the $n$-simplex level such that

(1) the assignment $\alpha \mapsto \alpha_*$ is functorial in morphisms $\alpha$ of $\mathcal{A}_n$, for all $n \geq 0$, and

(2) for each morphism $\alpha : A \to B$ of $\mathcal{A}_n$ and each ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$, the following diagram commutes:

$$\begin{tikzcd} X(A)_n \arrow[r, "\alpha_*"] \arrow[d, "\theta^*"] & X(B)_n \arrow[d, "\theta^*"] \\ X(A)_m \arrow[r, "(\theta^*\alpha)_*"'] & X(B)_m. \end{tikzcd}$$

Given a simplicial functor $X$, with induced maps $\alpha_* : X(A) \times \Delta^n \to X(B)$ as defined above for $\alpha : A \to B$, the corresponding morphism $\alpha_* : X(A)_n \to X(B)_n$ is the assignment $x \mapsto \alpha_*(x, \iota_n)$. Conversely, given a system of maps $\alpha_* : X(A)_n \to X(B)_n$ having the properties described in (1) and (2) above, one defines a map $\alpha_* : X(A) \times \Delta^n \to X(B)$ on $m$-simplices $(x, \theta)$ by the assignment $(x, \theta) \mapsto (\theta^* \alpha)_*(x)$. These two assignments are inverse to each other.

EXAMPLE 2.1. Suppose that $C$ is an object of the simplicial category $\mathcal{A}$. Then, according to this new description, the representable simplicial functor

$$\mathbf{Hom}(C, \ ) : \mathcal{A} \to \mathbf{S}$$

associates to each object $A$ the simplicial set $\mathbf{Hom}(C, A)$ and to each morphism $\alpha : A \to B$ of $A_n$ the function

$$\mathbf{Hom}(C, A)_n \xrightarrow{\alpha_*} \mathbf{Hom}(C, B)_n$$

defined by composition with $\alpha$ in $A_n$.

From this new point of view, a natural transformation of simplicial functors $f : X \to Y$ is a collection of simplicial set maps $f : X(A) \to Y(A)$ such that, for each morphism $\alpha : A \to B$ of $\mathcal{A}_n$ the diagram of functions

$$\begin{tikzcd}
X(A)_n \arrow[r, "\alpha_*"] \arrow[d, "f"'] & X(B)_n \arrow[d, "f"] \\
Y(A)_n \arrow[r, "\alpha_*"'] & Y(B)_n
\end{tikzcd}$$

commutes.

Suppose that $f : \mathcal{A} \to \mathcal{B}$ and $X : \mathcal{A} \to \mathbf{S}$ are simplicial functors and that $B$ is an object of the simplicial category $\mathcal{B}$. Then the bisimplicial object $\tilde{f}^* X(B)$ is defined by setting $\tilde{f}^* X(B)_{n,m}$ to be the disjoint union

$$\bigsqcup X(A)_{0,m} \times \mathbf{Hom}(A_0, A_1)_m \times \cdots \times \mathbf{Hom}(A_{n-1}, A_n)_m \times \mathbf{Hom}(fA_n, B)_m.$$

The horizontal simplicial set $\tilde{f}^* X(B)_{*,m}$ is the nerve of the translation category associated to the composite functor

$$f \downarrow B \xrightarrow{Q} \mathcal{A}_m \xrightarrow{X_m} \mathbf{Sets},$$

and each morphism $\omega : B \to C$ of $\mathcal{B}_m$ determines a simplicial set map

$$\omega_* : \tilde{f}^* X(B)_{*,m} \to \tilde{f}^* X(C)_{*,m}.$$

On the other hand, if $\tau : \mathbf{k} \to \mathbf{m}$ is an ordinal number map, then there is a commutative diagram of simplicial set maps

$$\begin{tikzcd}
\tilde{f}^*X(B)_{*,m} \arrow[r, "\omega_*"] \arrow[d, "\tau^*"'] & \tilde{f}^*X(C)_{*,m} \arrow[d, "\tau^*"] \\
\tilde{f}^*X(B)_{*,k} \arrow[r, "(\tau^*\omega)_*"'] & \tilde{f}^*X(C)_{*,k}.
\end{tikzcd} \tag{2.2}$$

It follows that applying the diagonal simplicial set functor gives a simplicial functor $\mathcal{B} \to \mathbf{S}$ defined by $(\mathbf{n}, B) \mapsto \tilde{f}^*X(B)_{n,n}$, which will also be denoted by $\tilde{f}^*X$. The simplicial functor $\tilde{f}^*X$ is the *homotopy left Kan extension* of $X$ along the simplicial functor $f$.

We've already noted that the simplicial set $\tilde{f}^*X(B)_{*,m}$ is the nerve of a translation category. Furthermore, the simplicial set diagram (2.2) is induced by a commutative diagram of functors. Suppose, for example, that the simplicial functor $X$ is the representable functor $\mathbf{Hom}(A,\ )$. Then the corresponding simplicial set $\tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m}$ can be identified with the nerve of the translation category associated to the composite contravariant functor

$$A \downarrow \mathcal{A}_m \xrightarrow{Q} \mathcal{A}_m \xrightarrow{f} \mathcal{B}_m \xrightarrow{\mathbf{Hom}(\ ,B)_m} \mathbf{Sets}.$$

The identity morphism on $A$ is initial in $A \downarrow A_m$, so the *category*

$$\tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m}$$

contracts canonically onto the discrete subcategory on the set of objects

$$\mathbf{Hom}(f(A), B)_m.$$

In this way, we obtain maps

$$\begin{cases} \mathbf{Hom}(f(A), B)_m \xrightarrow{s} \tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m}, \\ \tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m} \xrightarrow{r} \mathbf{Hom}(f(A), B)_m \end{cases} \tag{2.3}$$

such that $rs = 1$, as well as a natural transformation

$$\tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m} \times \mathbf{1} \xrightarrow{H} \tilde{f}^*\mathbf{Hom}(A,\ )(B)_{*,m}$$

from the composite functor $sr$ to the identity. All of this data is natural in $B$, and simplicial, and so we have proved

LEMMA 2.4. *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a simplicial functor between small simplicial categories, and let $A$ be an object of $\mathcal{A}$. Then the simplicial functor $\mathbf{Hom}(f(A),\ )$ is a strong deformation retract of $\tilde{f}^*\mathbf{Hom}(A,\ )$.*

We shall also need

LEMMA 2.5. *The simplicial functors $\mathbf{Hom}(f(A),\ )$ and $\tilde{f}^*\mathbf{Hom}(A,\ )$ are cofibrant.*

PROOF: The assertion about $\mathbf{Hom}(f(A),\ )$ is a consequence of the Yoneda lemma and the observation that any trivial fibration $p : X \to Y$ of $\mathbf{S}^{\mathcal{B}}$ consists of maps $p : X(B) \to Y(B)$ which are trivial fibrations of simplicial sets and are therefore surjective on vertices.

The object $\tilde{f}^*\mathbf{Hom}(A,\ )$ may also be interpreted as a simplicial object in the category $\mathbf{S}^{\mathcal{B}}$ of simplicial functors defined on $\mathcal{B}$.

Suppose, for the moment, that $Z$ is an arbitrary simplicial object in $\mathbf{S}^{\mathcal{B}}$. Then $Z$ consists of simplicial functors $Z_n$, $n \geq 0$, and simplicial structure maps $\theta^* : Z_n \to Z_m$ in $\mathbf{S}^{\mathcal{B}}$ corresponding to ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$. The object $Z$ is, in other words, a type of diagram taking values in bisimplicial sets. The standard constructions on bisimplicial sets apply to $Z$. There is, in particular, an associated diagonal simplicial functor $d(Z)$ which is defined by $d(Z)_n = Z_{n,n}$, and $d(Z)$ is a co-end in the simplicial functor category for the diagrams of simplicial functor maps

$$\begin{tikzcd}
Z_n \times \Delta^m \arrow[r, "1 \times \theta_*"] \arrow[d, "\theta^* \times 1"'] & Z_n \times \Delta^n \\
Z_m \times \Delta^m &
\end{tikzcd}$$

arising from ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$. This means that there are simplicial functor maps $\gamma_n : Z_n \times \Delta^n \to d(Z)$ defined "in sections" by $(x, \tau) \mapsto \tau^*(x)$, and these maps assemble to give the universal arrow in a coequalizer diagram

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} Z_n \times \Delta^m \rightrightarrows \bigsqcup_{n} Z_n \times \Delta^n \to d(Z).$$

Finally, $d(Z)$ has a filtration $d(Z)^{(p)} \subset d(Z)^{(p+1)} \subset \dots$, where $d(Z)^{(p)}$ is generated by the images of the maps $\gamma_r$, $0 \leq r \leq p$. We are interested in proving that the simplicial functor $d(Z)$ is cofibrant, so the key, for us, is the existence of the pushout diagram

$$\begin{tikzcd}
(s_{[p]}Z_p \times \Delta^{p+1}) \cup (Z_{p+1} \times \partial\Delta^{p+1}) \arrow[r] \arrow[d, hook] & d(Z)^{(p)} \arrow[d, hook] \\
Z_{p+1} \times \Delta^{p+1} \arrow[r, hook] & d(Z)^{(p+1)}.
\end{tikzcd}$$

in the category $\mathbf{S}^{\mathcal{B}}$. Here, the inclusion on the left is induced by the inclusion $s_{[p]}Z_p \hookrightarrow Z_{p+1}$ of the "horizontally degenerate part" in $Z_{p+1}$. The central observation is that the simplicial model structure on $\mathbf{S}^{\mathcal{B}}$ implies that $d(Z)$ is cofibrant if each of the inclusions $s_{[p]}Z_p \hookrightarrow Z_{p+1}$ is a cofibration.

The object at level $n$ for the simplicial object $\tilde{f}^*\mathbf{Hom}(A,\ )$ is the disjoint union

$$\bigsqcup \mathbf{Hom}(A, A_0) \times \dots \mathbf{Hom}(A_{n-1}, A_n) \times \mathbf{Hom}(f(A_n),\ ),$$

which may in turn be written as a disjoint union

$$\bigsqcup_{C \in Ob(\mathcal{B})} B(A \downarrow \mathcal{A})_n^C \times \mathbf{Hom}(f(C),\ ),$$

where $B(A \downarrow \mathcal{A})_n^C$ indicates the simplicial set of strings

$$A \to A_0 \to \dots \to A_{n-1} \to A_n = C,$$

ending at $C$. The horizontal degeneracies preserve this decomposition, and so the inclusion of the degenerate part in

$$\bigsqcup_{C \in Ob(\mathcal{A})} B(A \downarrow \mathcal{A})_{n+1}^C \times \mathbf{Hom}(f(C),\ ),$$

can be identified with the map

$$\bigsqcup_C DB(A \downarrow \mathcal{A})_{n+1}^C \times \mathbf{Hom}(f(C),\ ) \hookrightarrow \bigsqcup_C B(A \downarrow \mathcal{A})_{n+1}^C \times \mathbf{Hom}(f(C),\ ), \quad (2.6)$$

which is induced by the simplicial set inclusions

$$DB(A \downarrow \mathcal{A})_{n+1}^C \subset B(A \downarrow \mathcal{A})_{n+1}^C.$$

Each of the simplicial functors $\mathbf{Hom}(f(C),\ )$ is cofibrant, so the map (2.6) is a cofibration of simplicial functors, and the simplicial functor $\tilde{f}^*\mathbf{Hom}(A,\ )$ is cofibrant, as claimed. □

Suppose now that $Y : \mathcal{B} \to \mathbf{S}$ is a simplicial functor. Then a simplicial functor $\tilde{f}_* Y : \mathcal{A} \to \mathbf{S}$ is specified at an object $A$ of $\mathcal{A}$ by

$$\tilde{f}_* Y(A) = \mathbf{Hom}(\tilde{f}^* \mathbf{Hom}(A,\ ), Y).$$

In other words, an $n$-simplex of $\tilde{f}_*(Y)(A)$, or rather a morphism

$$\Delta^n \times \mathbf{Hom}(A,\ ) \to \tilde{f}_* Y$$

is defined to be a morphism

$$\tilde{f}^* \mathbf{Hom}(A,\ ) \times \Delta^n \to Y.$$

There is an isomorphism

$$\tilde{f}^* \mathbf{Hom}(A,\ ) \times \Delta^n \cong \tilde{f}^*(\mathbf{Hom}(A,\ ) \times \Delta^n),$$

by formal nonsense. The resulting isomorphisms

$$\begin{aligned} \mathrm{Nat}(\mathbf{Hom}(A,\ ) \times \Delta^n, \tilde{f}_* Y) &\cong \mathrm{Nat}(\tilde{f}^* \mathbf{Hom}(A,\ ) \times \Delta^n, Y) \\ &\cong \mathrm{Nat}(\tilde{f}^*(\mathbf{Hom}(A,\ ) \times \Delta^n), Y) \end{aligned}$$

are natural with respect to maps

$$\mathbf{Hom}(B,\ ) \times \Delta^m \to \mathbf{Hom}(A,\ ) \times \Delta^n.$$

Furthermore, every simplicial functor $X : \mathcal{A} \to \mathbf{S}$ is a colimit of its simplices, as in (1.3), and the functor $\tilde{f}^*$ preserves colimits. It follows that there is an adjunction isomorphism

$$\mathrm{Nat}(\tilde{f}^* X, Y) \cong \mathrm{Nat}(X, \tilde{f}_* Y). \tag{2.7}$$

The simplicial functor $f : \mathcal{A} \to \mathcal{B}$ also induces a functor $f_* : \mathbf{S}^{\mathcal{B}} \to \mathbf{S}^{\mathcal{A}}$, which is defined by $f_* Y(A) = Y(f(A))$. The notation is supposed to remind one of a direct image functor.

Note in particular, that the adjoint of a $n$-simplex

$$\mathbf{Hom}(A,\ ) \times \Delta^n \xrightarrow{x} \tilde{f}_* Y$$

is the composite

$$\tilde{f}^*(\mathbf{Hom}(A,\ ) \times \Delta^n) \cong \tilde{f}^* \mathbf{Hom}(A,\ ) \times \Delta^n \xrightarrow{x} Y$$

It follows that the canonical map

$$\eta : \mathbf{Hom}(A,\ ) \times \Delta^n \to \tilde{f}_* \tilde{f}^*(\mathbf{Hom}(A,\ ) \times \Delta^n)$$

is the map which sends the classifying simplex $(1_A, \iota_n)$ to the canonical isomorphism

$$\tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^n \xrightarrow{\cong} \tilde{f}^*(\mathbf{Hom}(A,\ ) \times \Delta^n),$$

since $\eta$ must be adjoint to the identity on $\tilde{f}^*(\mathbf{Hom}(A,\ ) \times \Delta^n)$. Thus, if $x$ is an $n$-simplex of $X(A)$, then $\eta(x) \in \tilde{f}_*\tilde{f}^*X(A)$ is the simplex defined by the composite

$$\tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^n \xrightarrow{\cong} \tilde{f}^*(\mathbf{Hom}(A,\ ) \times \Delta^n) \xrightarrow{\tilde{f}^*(x)} \tilde{f}^*X. \tag{2.8}$$

The adjunction (2.7) is easily promoted to a natural isomorphism of simplicial sets

$$\mathbf{Hom}(\tilde{f}^*X, Y) \cong \mathbf{Hom}(X, \tilde{f}_*Y). \tag{2.9}$$

We have therefore proved the first part of the following result:

PROPOSITION 2.10. *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a simplicial functor between small simplicial categories. Then*

(1) *The functors*

$$\tilde{f}^* : \mathbf{S}^{\mathcal{A}} \leftrightarrow \mathbf{S}^{\mathcal{B}} : \tilde{f}_*$$

*are simplicially adjoint in the sense that there is a natural isomorphism of simplicial sets*

$$\mathbf{Hom}(\tilde{f}^*X, Y) \cong \mathbf{Hom}(X, \tilde{f}_*Y).$$

(2) *the functor $\tilde{f}_*$ preserves weak equivalences and fibrations.*

(3) *the functor $\tilde{f}^*$ preserves weak equivalences and cofibrations.*

PROOF: To prove Part (2), observe that $\tilde{f}^*\mathbf{Hom}(A,\ )$ is cofibrant for every object $A \in \mathcal{A}$, by Lemma 2.5, so that every map

$$\Lambda^n_k \times \tilde{f}^*\mathbf{Hom}(A,\ ) \to \Delta^n \times \tilde{f}^*\mathbf{Hom}(A,\ )$$

is a trivial cofibration on account of the simplicial model structure of the category $\mathbf{S}^{\mathcal{B}}$. It follows that the functor $\tilde{f}_*$ preserves fibrations.

The retraction map

$$r : \tilde{f}^*\mathbf{Hom}(A,\ ) \to \mathbf{Hom}(f(A),\ )$$

is natural in $A$, and therefore induces a map

$$r^* : f_*Y \to \tilde{f}_*Y$$

which is natural in $Y \in \mathbf{S}^{\mathcal{B}}$. It follows that $\tilde{f}_*$ preserves weak equivalences, if it can be shown that the map $r^*$ is a natural weak equivalence.

Recall that the map $r$ has a section

$$s : \mathbf{Hom}(f(A),\ ) \to \tilde{f}^*\mathbf{Hom}(A,\ ),$$

and that there is a homotopy

$$H : \tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^1 \to \tilde{f}^*\mathbf{Hom}(A,\ )$$

from the composite $sr$ to the identity. The induced map

$$H^* : \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ ), Y) \to \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^1, Y)$$

can be composed with the canonical map

$$\mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^1, Y) \to \mathbf{hom}(\Delta^1, \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ ), Y))$$

to give a map which is adjoint to a simplicial homotopy

$$\mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ ), Y) \times \Delta^1 \to \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ ), Y)$$

from the identity map to $s^*r^*$. This implies that $r^*$ is a pointwise weak equivalence, as required.

For Part (3), the functor $\tilde{f}^*$ preserves weak equivalences as a consequence of its definition as a type of homotopy colimit. Also, $\tilde{f}^*$ is left adjoint to the functor $\tilde{f}_*$, and it is a consequence of Part (2) that the latter preserves trivial fibrations. Standard closed model category tricks therefore imply that the functor $\tilde{f}^*$ preserves cofibrations. □

COROLLARY 2.11. *The functor $f_* : \mathbf{S}^{\mathcal{B}} \to \mathbf{S}^{\mathcal{A}}$ induces an equivalence*

$$f_* : \mathrm{Ho}(\mathbf{S}^{\mathcal{B}}) \to \mathrm{Ho}(\mathbf{S}^{\mathcal{A}})$$

*of homotopy categories if and only if the adjunction maps*

$$\eta : X \to \tilde{f}_*\tilde{f}^*X \qquad \text{and} \qquad \epsilon : \tilde{f}^*\tilde{f}_*Y \to Y$$

*are weak equivalences.*

PROOF:

The functors $\tilde{f}^*$ and $\tilde{f}_*$ induce an adjoint pair of functors

$$\tilde{f}^* : \mathrm{Ho}(\mathbf{S}^{\mathcal{A}}) \leftrightarrow \mathrm{Ho}(\mathbf{S}^{\mathcal{B}}) : \tilde{f}_*$$

between the associated homotopy categories, by Proposition 2.10. If the induced functor $\tilde{f}_*$ is an equivalence of categories, then it is part of an adjoint equivalence (see [66, p.91]). The adjoint to $\tilde{f}_*$ of that adjoint equivalence is therefore naturally isomorphic to $\tilde{f}^*$ in $\mathrm{Ho}(\mathbf{S}^{\mathcal{B}})$, and so the adjunction maps $\eta$ and $\epsilon$ induce isomorphisms on the homotopy category level. It follows that $\eta$ and $\epsilon$ are weak equivalences.

The converse is clear. □

There is a simplicial map $i : X(A) \to \tilde{f}^*X(f(A))$ that is defined on the bisimplicial set level to be the map from $X(A)$ to

$$\bigsqcup X(A_0) \times \mathbf{Hom}(A_0, A_1) \times \cdots \times \mathbf{Hom}(A_{n-1}, A_n) \times \mathbf{Hom}(f(A_n), f(A))$$

which is given by sending an $n$-simplex $x$ to the element $(x, 1_A, \dots, 1_A, 1_{f(A)})$ in the summand

$$X(A) \times \mathbf{Hom}(A, A) \times \cdots \times \mathbf{Hom}(A, A) \times \mathbf{Hom}(f(A), f(A))$$

corresponding to $A_i = A$ for $0 \leq i \leq n$. It follows from the description of the adjunction map $\eta$ given in (2.8) that $i$ is the composite of the map

$$X(A) \xrightarrow{\eta} \tilde{f}_*\tilde{f}^*X(A) = \mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ ), \tilde{f}^*X)$$

with the map

$$\mathbf{Hom}(\tilde{f}^*\mathbf{Hom}(A,\ ), \tilde{f}^*X) \xrightarrow{s^*} \mathbf{Hom}(\mathbf{Hom}(f(A),\ ), \tilde{f}^*X) \cong \tilde{f}^*X(f(A))$$

induced by precomposition with the section $s : \mathbf{Hom}(f(A),\ ) \to \tilde{f}^*\mathbf{Hom}(A,\ )$ given in (2.3). Neither $s$ nor $i$ is natural in $A$.

COROLLARY 2.12. *The functor $f_* : \mathbf{S}^{\mathcal{B}} \to \mathbf{S}^{\mathcal{A}}$ induces an equivalence*

$$f_* : \mathrm{Ho}(\mathbf{S}^{\mathcal{B}}) \to \mathrm{Ho}(\mathbf{S}^{\mathcal{A}})$$

*of homotopy categories if and only if the following conditions are satisfied:*

(1) *for every simplicial functor $X \in \mathbf{S}^{\mathcal{A}}$ and every object $A \in \mathcal{A}$ the map*

$$i : X(A) \to \tilde{f}^*X(f(A)) = f_*\tilde{f}^*X(A)$$

*is a weak equivalence,*

(2) *a map $\alpha : Y_1 \to Y_2$ of $\mathbf{S}^{\mathcal{B}}$ is a weak equivalence if and only if the induced map $f_*\alpha : f_*Y_1 \to f_*Y_2$ is a weak equivalence of $\mathbf{S}^{\mathcal{A}}$.*

PROOF: If $f_*$ is an equivalence of homotopy categories, then (2) holds, by Proposition II.1.14. Furthermore, the canonical map map $\eta$ is a weak equivalence by Corollary 2.11. It follows from the proof of Proposition 2.10 that $s^*$ is a weak equivalence in general, and so the map $i$ is a weak equivalence as claimed.

Conversely, if conditions (1) and (2) are satisfied, then $\eta$ is a weak equivalence, and then $\tilde{f}_*(\epsilon)$ is a weak equivalence on account of a triangle identity. The conditions for Corollary 2.11 are therefore met, since $f_*(\epsilon)$ must also be a weak equivalence. □

Dwyer and Kan say that a simplicial functor $f : \mathcal{A} \to \mathcal{B}$ between small simplicial categories is a *weak $r$-equivalence* if two conditions hold:

(a) for every pair of objects $A_1, A_2$ of $\mathcal{A}$, the simplicial functor $f$ induces a weak equivalence of simplicial sets

$$\mathbf{Hom}(A_1, A_2) \to \mathbf{Hom}(f(A_1), f(A_2)),$$

(b) every object in the category of path components $\pi_0\mathcal{B}$ is a retract of an object in the image of $\pi_0 f$.

Here, the path component category $\pi_0\mathcal{B}$ is the category having the same objects as $\mathcal{B}$, and with morphisms specified by

$$\hom_{\pi_0\mathcal{B}}(A, B) = \pi_0\mathbf{Hom}(A, B).$$

Every simplicial functor $f : \mathcal{A} \to \mathcal{B}$ which satisfies condition (a) and is surjective on objects is a weak $r$-equivalence. Most examples of weak $r$-equivalences that are encountered in nature have this form.

The following is the main result of this chapter:

THEOREM 2.13 (DWYER-KAN). *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a simplicial functor between small simplicial categories. Then the induced functor*

$$f_* : \mathrm{Ho}(\mathbf{S}^{\mathcal{B}}) \to \mathrm{Ho}(\mathbf{S}^{\mathcal{A}})$$

*of homotopy categories is an equivalence of categories if and only if the functor $f$ is a weak $r$-equivalence.*

PROOF: Suppose that the functor $f$ is a weak $r$-equivalence. We shall verify the conditions of Corollary 2.12.

The instance of $i$ corresponding to the identity functor $1_{\mathcal{A}}$ on $\mathcal{A}$ has the form

$$i : X(A) \to \tilde{1}_{\mathcal{A}}^* X(A).$$

This map is a weak equivalence since $1_{\mathcal{A}}$ induces an equivalence $\mathrm{Ho}(\mathbf{S}^{\mathcal{A}}) \to \mathrm{Ho}(\mathbf{S}^{\mathcal{A}})$, and it fits into a commutative diagram

$$\begin{tikzcd}
 & \tilde{1}_{\mathcal{A}}^{*}X(A) \arrow[dd, "f_*"] \\
X(A) \arrow[ur, "i"] \arrow[dr, "i"'] & \\
 & \tilde{f}^{*}X(f(A)),
\end{tikzcd}$$

where the indicated map $f_*$ is induced by a bisimplicial set map given on summands by the simplicial set maps

$$X(A_0) \times \cdots \times \mathbf{Hom}(A_n, A) \to X(A_0) \times \cdots \times \mathbf{Hom}(f(A_n), f(A))$$

defined by $1 \times \cdots \times 1 \times f$. This map $f_*$ is a weak equivalence, by assumption, and so the map $i : X(A) \to \tilde{f}^{*}X(f(A))$ is a weak equivalence as well.

Suppose that $\alpha : Y_1 \to Y_2$ is a map of $\mathbf{S}^{\mathcal{B}}$ such that $f_*\alpha : f_*Y_1 \to f_*Y_2$ is a weak equivalence of $\mathbf{S}^{\mathcal{A}}$. Then the induced simplicial set maps $\alpha : Y_1(f(A)) \to Y_2(f(A))$ are weak equivalences for all objects $A \in \mathcal{A}$. The assumption that $f$ is a weak $r$-equivalence means, in part, that for each $B \in \mathcal{B}$ there is an object $A \in \mathcal{A}$ and maps $j : B \to f(A)$ and $q : f(A) \to B$ in $\mathcal{B}$ such that $qj$ maps to the identity in $\pi_0\mathcal{B}$. It follows that, for each $Y \in \mathbf{S}^{\mathcal{B}}$, there is a path of simplicial homotopies from the composite simplicial set map

$$Y(B) \xrightarrow{Y(j)} Y(f(A)) \xrightarrow{Y(q)} Y(B)$$

to the identity on $Y(B)$. It follows that this composite is a weak equivalence for any $Y$. In particular, in the diagram

$$\begin{tikzcd}
Y_1(B) \arrow[r, "Y_1(j)"] \arrow[d, "\alpha"'] & Y_1(f(A)) \arrow[r, "Y_1(q)"] \arrow[d, "\alpha"] & Y_1(B) \arrow[d, "\alpha"] \\
Y_2(B) \arrow[r, "Y_2(j)"'] & Y_2(f(A)) \arrow[r, "Y_2(q)"'] & Y_2(B)
\end{tikzcd}$$

both horizontal composites are weak equivalences, so that $\alpha : Y_1(B) \to Y_2(B)$ is a weak equivalence as well.

Conversely, suppose that $f : \mathcal{A} \to \mathcal{B}$ induces the indicated equivalence of homotopy categories. The following diagram commutes:

$$\begin{tikzcd}
\mathbf{Hom}(B,A) \arrow[r, "i"] \arrow[dr, "f"'] & \tilde{f}^*\mathbf{Hom}(B,\ )(A) \arrow[d, "r"] \\
& \mathbf{Hom}(f(B),f(A))
\end{tikzcd}$$

The map $i$ is a weak equivalence by assumption, and $r$ is the weak equivalence of Lemma 2.4, and so the condition (a) for $f$ to be weak $r$-equivalence is verified.

The simplicial functor $\tilde{f}^*\tilde{f}_*\mathbf{Hom}(B,\ )$ is the colimit

$$\varinjlim_{x:\tilde{f}^*\mathbf{Hom}(A,\ )\times\Delta^n\to\mathbf{Hom}(B,\ )} \tilde{f}^*\mathbf{Hom}(A,\ )\times\Delta,$$

and the canonical map

$$\epsilon : \tilde{f}^*\tilde{f}_*\mathbf{Hom}(B,\ ) \to \mathbf{Hom}(B,\ )$$

is the unique map having components given by the morphisms

$$x : \tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^n \to \mathbf{Hom}(B,\ )$$

in the definition of the colimit. The natural map $\epsilon$ is a weak equivalence by assumption, and the path component functor commutes with colimits, so there is a map

$$\tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^n \to \mathbf{Hom}(B,\ )$$

such that $1_B \in \mathbf{Hom}(B,B)_0$ lifts to a vertex of $\tilde{f}^*\mathbf{Hom}(A,B) \times \Delta^n$ up to (iterated) homotopy. The map

$$\mathbf{Hom}(f(A),B) \times \Delta^n \xrightarrow{s\times 1} \tilde{f}^*\mathbf{Hom}(A,B) \times \Delta^n$$

is a weak equivalence, so that $1_B$ is in the image of the composite

$$\mathbf{Hom}(f(A),B) \times \Delta^n \xrightarrow{s\times 1} \tilde{f}^*\mathbf{Hom}(A,B) \times \Delta^n \to \mathbf{Hom}(B,B)$$

up to homotopy. The composite

$$\mathbf{Hom}(f(A),\ ) \times \Delta^n \xrightarrow{s\times 1} \tilde{f}^*\mathbf{Hom}(A,\ ) \times \Delta^n \to \mathbf{Hom}(B,\ )$$

classifies an $n$-simplex $\alpha$ of $\mathbf{Hom}(B,f(A))$, and one verifies that this maps sends a 0-simplex $(\gamma,v)$ of $\mathbf{Hom}(f(A),B)\times\Delta^n$ maps to the composite $\gamma\circ v^*(\alpha)$.

□

## 3. Homotopy coherence.

The point of homotopy coherence theory is to determine when a diagram which commutes up to a system of higher homotopies can be replaced by a diagram which commutes on the nose. To this end, the game is either to recognize when systems of higher homotopies can be suitably defined to serve as input for the Dwyer-Kan theorem (Theorem 2.13), or to avoid that result altogether, as one does normally with lax functors. Various examples of these phenomena will be described here, in the context of homotopy coherent diagrams of simplicial sets, or spaces. A discussion of homotopy coherence phenomena for other selected simplicial model categories appears in Section 4.

### 3.1. Classical homotopy coherence.

Suppose that $I$ is a small category, and write $UI$ for the underlying directed graph of $I$. The graph $UI$ is *pointed* in the sense that there is a distinguished element, namely $1_a$, in the set $I(a,a)$ of arrows from $a$ to itself, for all objects $a \in I$. There is a free category $FX$ associated to each pointed directed graph $X$, which has the same objects as $X$, and all finite composeable strings of non-identity arrows in $X$ as morphisms. Composition in $FX$ is given by concatenation. The free category functor $F$ is left adjoint to the underlying graph functor $U$, with canonical maps $\eta : 1 \to UF$ and $\epsilon : FU \to 1$. The canonical maps can be used, along with various iterations of the composite functor $FU$ to define a simplicial category $F_*I$. The category $F_nI$ of $n$-simplices of $F_*I$ has the form $F_nI = (FU)^{n+1}I$, and the faces and degeneracies of $F_*I$ are defined by

$$\begin{cases} d_i = (FU)^{i-1}\epsilon(FU)^{n-i}, & \text{and} \\ s_j = (FU)^j F\eta U(FU)^{n-j}. \end{cases}$$

One can appeal to the dual of results of [66, p.134,171] to see that this definition works, or check the simplicial identities directly. The diagram

$$\begin{tikzcd} FUFU(I) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & FU(I) \arrow[r, "\epsilon"] & I \end{tikzcd}$$

is a coequalizer in the category of small categories. It follows that, for any pair of objects $a,b$ of $I$, the diagram of functions between morphism sets

$$\begin{tikzcd} FUFU(I)(a,b) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & FU(I)(a,b) \arrow[r, "\epsilon"] & I(a,b) \end{tikzcd}$$

is a coequalizer. It can be shown directly that the underlying simplicial graph $UF_*I$ has an extra degeneracy (see Section III.5), given by the functors

$$s_{-1} = \eta U(FU)^n(I) \to U(FU)^{n+1}(I).$$

Now, $UF_*(I)(a,b) = F_*(I)(a,b)$ as a simplicial set, so that $F_*(I)(a,b)$ has an extra degeneracy. The simplicial set map

$$F_*I(a,b) \to K(I(a,b),0)$$

induced by $\epsilon$ is therefore a weak equivalence by Lemma III.5.1, so we have constructed a weak $r$-equivalence

$$\epsilon : F_*I \to K(I,0)$$

of simplicial categories called *simplicial free resolution* of the category $I$.

The simplicial free resolution $F_*I$ of a category $I$ is the traditional basis of the definition of homotopy coherence. Explicitly, a *homotopy coherent* $I$-diagram is classically defined to be a simplicial functor $X : F_*I \to \mathbf{S}$.

Homotopy coherent diagrams are notoriously difficult to interpret, much less construct. Intuitively, $X : F_*I \to \mathbf{S}$ associates a simplicial map $\alpha_* : X(a) \to X(b)$ to each non-identity morphism $\alpha : a \to b$ of $I$ ($\alpha_*$ is the image of the string $(\alpha) \in FU(I)$). Given $\beta : b \to c$ in $I$, it's *not* the case that $(\beta\alpha)_* = \beta_*\alpha_*$, but rather that the two maps are homotopic in a precise way, via the homotopy $X((\beta,\alpha)) : X(a) \times \Delta^1 \to X(c)$ which is the image of the arrow $((\beta,\alpha)) \in FUFU(I)$. Given yet another $I$-morphism $\gamma : c \to d$, one sees that the associativity relationship $\gamma(\beta\alpha) = \gamma\beta\alpha$ in $I$ in the sense that the morphism $(((\gamma),(\beta,\alpha)))$ of $FUFUFU(I)$ determines a map $X(a) \times \Delta^2 \to X(d)$, which is a higher homotopy that has the homotopies

$$\begin{tikzcd}
(\gamma\beta\alpha)_* \arrow[r, "X((\gamma,\beta\alpha))"] \arrow[dr, "X((\gamma,\beta,\alpha))"'] & \gamma_*(\beta\alpha)_* \arrow[d, "X((\gamma),(\beta,\alpha))"] \\
& \gamma_*\beta_*\alpha_*
\end{tikzcd}$$

appearing in its 1-skeleton. The higher, or iterated, associativity relation

$$\omega(\gamma(\alpha\beta)) = \omega\gamma\alpha\beta$$

corresponds to the 3-simplex $((((\omega)),((\gamma),(\beta,\alpha))))$ of $F_*I$. The homotopy coherent diagram $X$ is determined by all higher associativities or iterated bracketing of strings $\alpha_n\alpha_{n-1}\cdots\alpha_1$ of morphisms in $I$.

Homotopy coherent diagrams are very rarely constructed from scratch, although there are obstruction theoretic techniques for doing so [27]. They nevertheless appear quite naturally, usually as the output of large categorical machines — more will be said about this below.

Most of the point of having a homotopy coherent diagram in hand is that one can immediately replace it up to pointwise weak equivalence by a diagram that commutes on the nose. This is a consequence of the following realization theorem:

Theorem 3.1 (Realization). *Suppose that $X : F_*I \to \mathbf{S}$ is a homotopy coherent diagram in the category of simplicial sets. Then $X$ is naturally pointwise weakly equivalent to a diagram $\epsilon_*Y$, for some ordinary $I$-diagram $Y : I \to \mathbf{S}$.*

Proof: This result follows easily from the Dwyer-Kan theorem 2.13 and its proof: the simplicial functor $\epsilon : F_*I \to K(I,0)$ is a weak $r$-equivalence, so there are natural pointwise weak equivalences

$$X \xrightarrow{i} \tilde{\epsilon}_*\tilde{\epsilon}^*X \xleftarrow{r^*} \epsilon_*\tilde{\epsilon}^*X.$$

Take $Y$ to be $\tilde{\epsilon}^*X$. □

### 3.2. Homotopy coherence: an expanded version.

The classical definition of homotopy coherence is interesting and complicated, but not flexible enough for all applications. It does not, for example, take into account diagrams where images of identity maps could wiggle away from actual identities up to controlled homotopy. The realization result, Theorem 3.1, is also obviously just a special case of a much broader statement.

Define a *resolution of a category $I$* to be a simplicial functor $\pi : \mathcal{A} \to K(I,0)$ such the $\pi$ is a weak $r$-equivalence. A *homotopy coherent diagram* on the category $I$ shall henceforth be defined to be a simplicial functor $X : \mathcal{A} \to \mathbf{S}$, where $\pi : \mathcal{A} \to K(I,0)$ is a resolution of $I$. We now have an expanded form of the realization theorem:

Theorem 3.2. *Suppose that $X : \mathcal{A} \to \mathbf{S}$ is a homotopy coherent diagram which is defined with respect to some resolution $\pi : \mathcal{A} \to K(I,0)$ of $I$. Then $X$ is naturally pointwise equivalent to a diagram of the form $\pi_*Y$, for some $I$-diagram $Y : I \to \mathbf{S}$.*

Proof: The proof is a copy of the proof of Theorem 3.1. □

The rest of this section will be taken up with a description of a natural resolution $BI_s \to K(I,0)$ of the category $I$ which is different from the simplicial free resolution of the previous section. Homotopy coherent diagrams $X : BI_s \to \mathbf{S}$ provide for variance of morphisms induced by the identities of $I$ up to homotopy.

Let $a$ and $b$ be objects of the small category $I$. There is a category, denoted $I_s(a,b)$, whose objects are the functors of the form $\theta : \mathbf{n} \to I$, with $\theta(0) = a$ and $\theta(n) = b$. The morphisms of $I_s(a,b)$ are commutative diagrams

$$\begin{tikzcd}
\mathbf{n}_0 \arrow[dd, "\gamma"'] \arrow[dr, "\theta_0"] & \\
& I \\
\mathbf{n}_1 \arrow[ur, "\theta_1"'] &
\end{tikzcd}$$

where $\theta_0$ and $\theta_1$ are objects of $I_s(a,b)$ and $\gamma : \mathbf{n}_0 \to \mathbf{n}_1$ is an ordinal number map which is *end point preserving* in the sense that $\gamma(0) = 0$ and $\gamma(n_0) = n_1$.

Suppose that $\theta : \mathbf{n} \to I$ is an object of $I_s(a,b)$ and $\omega : \mathbf{m} \to I$ is an object of $I_s(b,c)$. The poset join $\mathbf{n} * \mathbf{m}$ may be identified up to isomorphism with the ordinal number $\mathbf{n} + \mathbf{m}$, in such a way that the inclusion $\mathbf{n} \hookrightarrow \mathbf{n} * \mathbf{m}$ is identified with the ordinal number map $\mathbf{n} \to \mathbf{n} + \mathbf{m}$ defined by $j \mapsto j + n$ for $0 \leq j \leq m$. The objects $\theta$ and $\gamma$ together determine a functor $\theta * \gamma : \mathbf{n} + \mathbf{m} \to I$, defined by

$$\theta * \gamma(j) = \begin{cases} \theta(j), & \text{if } 0 \leq j \leq n, \\ \gamma(j-n), & \text{if } n \leq j \leq m+n. \end{cases}$$

This is plainly the object level description of a functor

$$I_s(a,b) \times I_s(b,c) \xrightarrow{*} I_s(a,c),$$

which is called the *join functor*. This operation is associative, and has a two-sided identity in each $I_s(a,a)$ given by the object $a : \mathbf{0} \to I$.

We have constructed a category $I_s$ which has the same objects as $I$, and is enriched in the category **cat** of small categories. Applying the nerve construction to each of the categories $I_s(a,b)$ gives simplicial sets $BI_s(a,b)$, which then form the morphism objects for a simplicial category $BI_s$, which has the same objects as $I$. These constructions are obviously natural: a functor $f : I \to J$ induces a functor $f : I_s \to J_s$ of categories enriched in **cat**, and hence determines a functor $f : BI_s \to BJ_s$ of simplicial categories.

The category $I_s(a,b)$ has the form

$$I_s(a,b) = \bigsqcup_{\substack{f : a \to b \\ \text{in } I}} I_s(a,b)_f,$$

where $I_s(a,b)_f$ is the subcategory of those strings $\alpha : \mathbf{n} \to I_s$ whose composite is $f : a \to b$. The category $I_s(a,b)_f$ has an initial object, namely the functor $f : \mathbf{1} \to I_s$ which is determined by $f$, so that, on the simplicial set level, we have a decomposition

$$BI_s(a,b) = \bigsqcup_{\substack{f : a \to b \\ \text{in } I}} BI_s(a,b)_f,$$

of $BI_s(a,b)$ into connected components, each of which is contractible. It follows that the path component functor determines a resolution $BI_s \to K(I,0)$ of the category $I$.

A homotopy coherent diagram $X : BI_s \to \mathbf{S}$ exhibits the standard homotopy coherence phenomena. Suppose that

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2$$

is a composeable pair of morphisms of $I$, and let these morphisms canonically determine a functor $\alpha : \mathbf{2} \to I$, which in turn is a 0-simplex of $BI_s(a_0, a_2)$. Then $X$ associates simplicial set morphisms $(\alpha_1)_* : X(a_0) \to X(a_1)$ and $(\alpha_2)_* : X(a_1) \to X(a_2)$ to the morphisms (aka. 0-simplices) $\alpha_1$ and $\alpha_2$ respectively, and associates the composite simplicial set map

$$(\alpha_2)_*(\alpha_1)_* : X(a_0) \to X(a_2)$$

to the simplex $\alpha$, since $X$ takes joins to composites. There is a 1-simplex $\sigma$ of $BI_s(a_0, a_2)$, defined by the picture

$$\begin{tikzcd}
\mathbf{2} \arrow[dr, "\alpha"] & \\
& I_s, \\
\mathbf{1} \arrow[uu, "d^1"] \arrow[ur, "\alpha_2\alpha_1"'] &
\end{tikzcd}$$

where $\mathbf{1} \to \mathbf{2}$ is the unique endpoint preserving ordinal number map, and this 1-simplex is mapped to a homotopy $X(\sigma) : X(a_0) \times \Delta^1 \to X(a_2)$ from $(\alpha_2\alpha_1)_*$ to the composite $(\alpha_2)_*(\alpha_1)_*$. In the same way, the associativity relation $\alpha_3(\alpha_2\alpha_1) = \alpha_3\alpha_2\alpha_1$ gives rise to a 2-simplex

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "d^1"] \arrow[dr, "\alpha_3\alpha_2\alpha_1"'] & \mathbf{2} \arrow[r, "d^2"] \arrow[d, "(\alpha_3\alpha_2{,}\alpha_1)"'] & \mathbf{3} \arrow[dl, "(\alpha_3{,}\alpha_2{,}\alpha_1)"] \\
& I &
\end{tikzcd}$$

and hence to a higher homotopy $X(a_0) \times \Delta^2 \to X(a_3)$.

If $a$ is an object of $I$, then the identity map $1_a : a \to a$ defines a 0-simplex $1_a : \mathbf{1} \to I$, and hence gets mapped to the simplicial set map $(1_a)_* : X(a) \to X(a)$, whereas the 0-simplex $a : \mathbf{0} \to I$ is an identity for $BI_s$, and is therefore sent to the identity $1_{X(a)}$ on $X(a)$. There is a 1-simplex $\eta$ of $BI_s(a, a)$, of the form

$$\begin{tikzcd}
\mathbf{0} \arrow[dr, "a"] & \\
& I \\
\mathbf{1} \arrow[uu] \arrow[ur, "1_a"'] &
\end{tikzcd}$$

which is then mapped to a homotopy $X(\eta) : X(a) \times \Delta^1 \to X(\alpha)$ from $(1_a)_*$ to $1_{X(a)}$.

### 3.3. Lax functors.

A *lax functor* $F : I \rightsquigarrow \mathbf{cat}$ associates a category $F(a)$ to each object $a$ of $I$, and associates a functor $\alpha_* : F(a) \to F(b)$ to each morphism $\alpha : a \to b$, in such a way that there are natural transformations

$$\theta(\beta, \alpha) : (\beta\alpha)_* \to \beta_*\alpha_* \quad \text{and} \quad \eta_a : (1_a)_* \to 1_{F(a)},$$

which together satisfy the cocycle conditions

$$\begin{tikzcd}
(\beta\alpha\gamma)_* \arrow[r, "\theta(\beta\alpha, \gamma)"] \arrow[d, "\theta(\beta, \alpha\gamma)"'] & (\beta\alpha)_*\gamma_* \arrow[d, "\theta(\beta, \alpha)\gamma_*"] \\
\beta_*(\alpha\gamma)_* \arrow[r, "\beta_*\theta(\alpha, \gamma)"'] & \beta_*\alpha_*\gamma_*
\end{tikzcd} \tag{3.3}$$

$$\begin{tikzcd}
\alpha_* 1_{F(a)} \arrow[d, equal] & \alpha_*(1_a)_* \arrow[l, "\alpha_*\eta_a"'] & (1_b)_*\alpha_* \arrow[r, "\eta_b\alpha_*"] & 1_{F(b)}\alpha_* \arrow[d, equal] \\
\alpha_* \arrow[r, equal] & (\alpha 1_a)_* \arrow[u, "\theta(\alpha, 1_a)"'] & (1_b\alpha)_* \arrow[u, "\theta(1_b, \alpha)"] \arrow[r, equal] & \alpha_*.
\end{tikzcd} \tag{3.4}$$

The category $\mathbf{cat}$ of (small) categories is enriched in categories. Its morphism categories $\mathbf{Hom}(A, B)$ have the functors from $A$ to $B$ as objects with natural transformations as morphisms. The composition functor

$$\mathbf{Hom}(A, B) \times \mathbf{Hom}(B, C) \to \mathbf{Hom}(A, C)$$

is defined on morphisms as follows: given natural transformations $\alpha_1$ and $\alpha_2$

$$\begin{tikzcd}
A \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\downarrow \alpha_1"] & B \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\downarrow \alpha_2"] & C
\end{tikzcd}$$

one defines the composition $\alpha_2\alpha_1$ of $\alpha_1$ and $\alpha_2$ to be the diagonal transformation appearing in the commutative diagram

$$\begin{tikzcd}
F_2F_1 \arrow[r, "F_2\alpha_1"] \arrow[d, "\alpha_2 F_1"'] \arrow[dr] & F_2G_1 \arrow[d, "\alpha_2 G_1"] \\
G_2F_1 \arrow[r, "G_2\alpha_1"'] & G_2G_1.
\end{tikzcd}$$

There is a one to one correspondence between functors

$$I_s \to \mathbf{cat}$$

and lax functors on $I$. In particular, given a lax functor $F$ as above, associate to each string $\mathbf{n} \to I$ in $I$ of the form

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

the composite functor

$$(\alpha_n)_* \dots (\alpha_1)_* : F(a_0) \to F(a_n).$$

The requisite functors

$$I_s(a_0, a_n) \to \mathbf{Hom}(F(a_0), F(a_n))$$

are defined on morphisms of $I_s(a_0, a_n)$ by first making a definition on cofaces and codegeneracies, which are completely determined up to join by the transformations $\theta$ and $\eta$, and then by showing that the relevant cosimplicial identities are satisfied. The non-trivial cosimplicial identities amount to the cocycle conditions which appear in the definition of the lax functor $F$.

The object **cat** has associated to it a simplicial category $B\mathbf{cat}$ in the obvious way: one uses the nerves $B\mathbf{Hom}(A, C)$ of the categories of morphisms $\mathbf{Hom}(A, C)$ appearing in **cat**. Note that $B\mathbf{Hom}(A, C)$ is canonically isomorphic to the function complex $\mathbf{Hom}(BA, BC)$. The collection of such canonical isomorphisms respects the composition laws of $B\mathbf{cat}$ and $\mathbf{S}$ and hence determines a functor $B\mathbf{cat} \to \mathbf{S}$. It follows in particular that any lax functor $F : I_s \to \mathbf{cat}$ determines a homotopy coherent diagram

$$BI_s \xrightarrow{BF} B\mathbf{cat} \to \mathbf{S}$$

on $I$.

### 3.4. The Grothendieck construction.

Homotopy coherent diagrams arising from lax functors are most often realized (or *rectified*) by using a category theoretic method that is known as the Grothendieck construction in place of a result like Theorem 3.2. Suppose that $F : I \rightsquigarrow \mathbf{cat}$ is a lax functor. The *Grothendieck construction* associated to $F$ is a category $LF$ whose set of objects consists of all pairs $(a, x)$, where $a$ is an object of $I$ and $x$ is an object of the category $F(a)$. A morphism $(\alpha, f) : (a, x) \to (b, y)$ is a pair consisting of a morphism $\alpha : a \to b$ of the base category $I$ and a morphism $f : \alpha_*(x) \to y$ of the category $F(b)$. The composite of $(\alpha, f)$ with the morphism $(\beta, g) : (b, y) \to (c, z)$ of $LF$ is defined to be the map $(\beta\alpha, g * f)$, where $g * f$ is the composite

$$(\beta\alpha)_*(x) \xrightarrow{\theta(\beta,\alpha)} \beta_*\alpha_*(x) \xrightarrow{\beta_*(f)} \beta_*(y) \xrightarrow{g} z$$

of $F(c)$. The identity morphism on an element $(a, x)$ of $LF$ is the morphism $(1_a, \eta_a)$. The associativity of the composition operation in $LF$ and the fact that the morphisms $(1_a, x)$ are two-sided identities are, respectively, consequences of the cocycle conditions (3.3) and (3.4).

Projection onto the first variable, for both objects and morphisms, defines a canonical functor $\pi_F : LF \to I$. The comma category $\pi_F \downarrow c$ associated to an object $c$ of $I$ has all morphisms $\alpha : \pi_F(a, x) \to c$ for objects. There is a functor $f_c : \pi_F \downarrow c \to F(c)$ which associates the object $\alpha_*(x) \in F(I)$ to the object $\alpha : \pi_F(a, x) \to c$. Also, there is a functor $g_c : F(c) \to \pi_F \downarrow c$ which associates the object $1_c : \pi_F(c, x) \to c$ to $x \in F(c)$. The functor $f_c$ is left adjoint to $g_c$, as can be seen by observing that the commutative diagram

$$\begin{tikzcd}
\pi_F(a,x) \arrow[dd, "{(\alpha,h)}"'] \arrow[dr, "\alpha"] & \\
& c \\
\pi_F(c,y) \arrow[ur, "1_c"'] &
\end{tikzcd}$$

is uniquely determined by the map $h : \alpha_*(x) \to y$ of $F(c)$.

Any morphism $\beta : c \to d$ induces a functor $\beta_* : \pi_F \downarrow c \to \pi_F \downarrow d$ in an obvious way, and this assignment determines a functor $\pi_F \downarrow ? : I \to \mathbf{cat}$. The diagram of functors

$$\begin{tikzcd}
F(c) \arrow[r, "g_c"] \arrow[d, "\beta_*"'] & \pi_F \downarrow c \arrow[d, "\beta_*"] \\
F(c) \arrow[r, "g_d"'] & \pi_F \downarrow d
\end{tikzcd}$$

commutes up to canonically determined natural transformation.

Generally, any functor $f : D \to E$ that has a left or right adjoint induces a homotopy equivalence $f_* : BD \to BE$ of the associated nerves. This is a result of the fact that any natural transformation of functors gives rise to a homotopy of the respective induced maps of simplicial sets. The Grothendieck construction therefore gives rise to homotopy equivalences

$$g_{c*} : BF(c) \xrightarrow{\simeq} B(\pi_F \downarrow c)$$

such that the diagrams

$$\begin{tikzcd}
BF(c) \arrow[r, "g_{c*}"] \arrow[d, "\beta_*"'] & B(\pi_F \downarrow c) \arrow[d, "\beta_*"] \\
BF(c) \arrow[r, "g_{d*}"'] & B(\pi_F \downarrow d)
\end{tikzcd}$$

commute up to homotopy. In other words, the $I$-diagram $c \mapsto B(\pi_F \downarrow c)$ is a realization of the homotopy coherent diagram $c \mapsto BF(c)$.

## 4. Realization theorems.

Suppose that $\pi : \mathcal{A} \to K(I,0)$ is a resolution of a small category $I$, and let $X : \mathcal{A} \to \mathbf{S}$ be a homotopy coherent diagram on $I$. A *realization* of $X$ is a simplicial set valued functor $Y : I \to \mathbf{S}$ such that there is a pointwise weak equivalence $X \simeq \pi_* Y$ in the simplicial functor category $\mathbf{S}^{\mathcal{A}}$. Theorem 3.2 says that any homotopy coherent diagram $X$ has a realization, and that the weak equivalence in $\mathbf{S}^{\mathcal{A}}$ can be chosen canonically.

Given an arbitrary simplicial model category $\mathcal{M}$ and a resolution $\pi : \mathcal{A} \to K(I,0)$, it is certainly sensible to say that a homotopy coherent diagram on $I$ is a simplicial functor $X : \mathcal{A} \to \mathcal{M}$. One analogously defines a realization of $X$ to be a functor $Y : I \to \mathcal{M}$ such that there is a pointwise weak equivalence $X \simeq \pi_* Y$. The purpose of this section is to show that all homotopy coherent diagrams in $\mathcal{M}$ admit realizations if the simplicial model category $\mathcal{M}$ has an adequate notion of homotopy colimit. This is done by giving a proof of Theorem 3.2 which does not depend on the Yoneda lemma 1.2. The proof is achieved by thinking about the functor $\tilde{\pi}^*$ in a different way; we lose the functor $\tilde{\pi}_*$ and along with it any notion of an equivalence of homotopy categories associated to the categories $\mathbf{S}^{\mathcal{A}}$ and $\mathbf{S}^I$. We begin with a weakened version of the Dwyer-Kan theorem.

Theorem 4.1. *Suppose that a simplicial functor $f : \mathcal{A} \to \mathcal{B}$ is homotopically full and faithful in the sense that all induced simplicial set maps*

$$f_* : \mathbf{Hom}(A,B) \to \mathbf{Hom}(f(A), f(B))$$

*are weak equivalences, and let $X : \mathcal{A} \to \mathbf{S}$ be a simplicial functor. Then $X$ is naturally pointwise weakly equivalent to the simplicial functor $f_* \tilde{f}^* X$.*

PROOF: The assumption on $f$ implies that the map of simplicial objects in $\mathbf{S}^{\mathcal{B}}$ defined by

$$\begin{tikzcd}
\bigsqcup_{(A_0,\dots,A_n)} X(A_0) \times \mathbf{Hom}(A_0,A_1) \times \dots \times \mathbf{Hom}(A_n,A) \arrow[d, "1 \times f_*"'] \\
\bigsqcup_{(A_0,\dots,A_n)} X(A_0) \times \mathbf{Hom}(A_0,A_1) \times \dots \times \mathbf{Hom}(f(A_n),f(A))
\end{tikzcd}$$

is a levelwise weak equivalence, which is natural in $A \in \mathcal{A}$. The simplicial object

$$\bigsqcup_{(A_0,\dots,A_n)} X(A_0) \times \mathbf{Hom}(A_0,A_1) \times \dots \times \mathbf{Hom}(A_n,A)$$

is, in $m$-simplices, the nerve of the translation category associated to the composite functor

$$\mathcal{A}_m \downarrow A \xrightarrow{Q} \mathcal{A}_m \xrightarrow{X_m} \mathbf{Sets},$$

and the identity element $1_A$ is terminal in $\mathcal{A}_m \downarrow A$. It follows (see the development around (2.3)) that there is a canonical weak equivalence induced by the bisimplicial set map

$$r : \bigsqcup_{(A_0,\dots,A_n)} X(A_0) \times \mathbf{Hom}(A_0,A_1) \times \dots \times \mathbf{Hom}(A_n,A) \to X(A)$$

which is induced on the translation category level by functors

$$(x, \alpha : A' \to A) \mapsto \alpha_*(x).$$

We therefore have natural pointwise weak equivalences

$$f_*\tilde{f}^*X \xleftarrow{\simeq} 1_*\tilde{1}^*X \xrightarrow[\simeq]{r} X. \qquad \square$$

Suppose now that $f : \mathcal{A} \to \mathcal{B}$ is a simplicial functor, and that $Y : \mathcal{A} \to \mathcal{M}$ is a simplicial functor taking values in a simplicial model category $\mathcal{M}$. We are entitled to an analogue of the functor $\tilde{f}^*$; in particular, we define $\tilde{f}^*Y(B)$ to be the simplicial object having $n$-simplices

$$\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0,A_1) \otimes \dots \otimes \mathbf{Hom}(f(A_n),B).$$

In this way, we define a functor

$$\tilde{f}^* : \mathcal{M}^{\mathcal{A}} \to \mathbf{S}(\mathcal{M}^{\mathcal{B}}),$$

where $\mathbf{S}(\mathcal{M}^{\mathcal{B}})$ denotes the category of simplicial objects in $\mathcal{M}^{\mathcal{B}}$. We also have the following analogues of the maps appearing in the proof of Theorem 4.1 in the category $\mathbf{S}(\mathcal{M}^{\mathcal{B}})$:

$$\begin{tikzcd}
\bigsqcup\limits_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(A_n, A) \arrow[r, "r"] \arrow[d, "f_*"'] & Y(A) \\
\bigsqcup\limits_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(f(A_n), f(A)). &
\end{tikzcd} \tag{4.2}$$

The trick, either for a given simplicial model category or for a particular class of objects of the simplicial functor category $\mathcal{M}^{\mathcal{A}}$, is to find a realization functor $\mathbf{S}(\mathcal{M}) \to \mathcal{M}$ which takes the maps in this diagram to weak equivalences of $\mathcal{M}^{\mathcal{B}}$.

The naive realization $d(Z)$ of a simplicial object $Z$ in $\mathcal{M}$ is the coend, given by the coequalizer

$$\bigsqcup_{\mathbf{m}\to\mathbf{n}} Z_n \otimes \Delta^m \rightrightarrows \bigsqcup_{n\geq 0} Z_n \otimes \Delta^n \to d(Z).$$

The object $d(Z)$ has a filtration

$$d(Z)^{(n)} \subset d(Z)^{(n+1)} \subset \dots,$$

where $d(Z)^{(n)}$ is the epimorphic image of the restricted map

$$\bigsqcup_{p\leq n} Z_p \otimes \Delta^p \to d(Z).$$

There are pushout diagrams

$$\begin{tikzcd}
(s_{[n]} Z_n \otimes \Delta^{n+1}) \cup (Z_{n+1} \otimes \partial\Delta^{n+1}) \arrow[r] \arrow[d, "j_*"'] & d(Z)^{(n)} \arrow[d] \\
Z_{n+1} \otimes \Delta^{n+1} \arrow[r] & d(Z)^{(n+1)},
\end{tikzcd}$$

in $\mathcal{M}$, where the map $j_*$ is canonically induced by the monomorphism $j : s_{[n]}Z_n \hookrightarrow Z_{n+1}$ given by taking the epimorphic image of the map

$$s : \bigsqcup_{0 \leq i \leq n} Z_n \to Z_{n+1}$$

which applies the $i^{th}$ degeneracy $s_i$ on the $i^{th}$ summand. The monomorphism $d(Z)^{(n)} \hookrightarrow d(Z)^{(n+1)}$ is a cofibration of $\mathcal{M}$ if $j : s_{[n]}Z_n \to Z_{n+1}$ is a cofibration.

Say that a simplicial object $Z$ in $\mathcal{M}$ is *diagonally cofibrant* if

(1) $Z_0$ is a cofibrant object of $\mathcal{M}$, and
(2) each morphism $j : s_{[n]}Z_n \to Z_{n+1}$ is a cofibration.

If $Z$ is a diagonally cofibrant simplicial object of $\mathcal{M}$ in this sense, then

$$d(Z)^{(0)} = Z_0$$

is cofibrant and all of the maps $d(Z)^{(n)} \hookrightarrow d(Z)^{(n+1)}$ are cofibrations, and so the realization $d(Z)$ is a cofibrant object of $\mathcal{M}$. It also follows that if $f : Z \to W$ is a map of diagonally cofibrant simplicial objects which is a levelwise weak equivalence in the sense that all of the maps $f : Z_n \to W_n$ are weak equivalences of $\mathcal{M}$, then the induced map $f_* : d(Z) \to d(W)$ is a weak equivalence as well.

THEOREM 4.3. *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a simplicial functor which is homotopically full and faithful. Suppose that $Y : \mathcal{A} \to \mathcal{M}$ is a simplicial functor taking values in simplicial model category $\mathcal{M}$, such that $Y(A)$ is a cofibrant object of $\mathcal{M}$ for all objects $A \in \mathcal{A}$. Then the morphisms $f_*$ and $r$ induce pointwise weak equivalences*

$$f_*(d\tilde{f}^*Y) \cong d(f_*\tilde{f}^*Y) \xleftarrow[\simeq]{f_*} d(1_*\tilde{1}^*Y) \xrightarrow[\simeq]{r} d(Y) = Y.$$

PROOF: The assumptions imply that the simplicial object $\tilde{f}^*Y(B)$ is diagonally cofibrant, for all $B \in \mathcal{B}$. This is seen by observing that the object

$$\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(f(A_n), B)$$

can be rewritten in the form

$$\bigsqcup_{A_0} Y(A_0) \otimes B(f \downarrow B)_n^{A_0},$$

where

$$B(f \downarrow B)_n^{A_0} = \mathbf{Hom}(A_0, A_1) \times \cdots \times \mathbf{Hom}(f(A_n), B)$$

means strings of length $n$ in the simplicial nerve $B(f \downarrow B)$ which begin at $A_0$. This decomposition is preserved by all degeneracies, and so we have

$$s_{[n]}\tilde{f}^*Y(B) = \bigsqcup_{A_0} Y(A_0) \otimes DB(f \downarrow B)^{A_0}_{n+1},$$

where $DB(f \downarrow B)^{A_0}_{n+1}$ denotes the degenerate strings of length $n+1$ which begin at $A_0$. Each map

$$Y(A_0) \otimes DB(f \downarrow B)^{A_0}_{n+1} \to Y(A_0) \otimes B(f \downarrow B)^{A_0}_{n+1}$$

induced by the simplicial set inclusion $DB(f \downarrow B)^{A_0}_{n+1} \subset B(f \downarrow B)^{A_0}_{n+1}$ is a cofibration, since $Y(A_0)$ is cofibrant, and the claim is verified.

The maps

$$\begin{tikzcd}
\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(A_n, A) \arrow[d, "1 \otimes f_*"'] \\
\bigsqcup_{(A_0,\dots,A_n)} Y(A_0) \otimes \mathbf{Hom}(A_0, A_1) \otimes \cdots \otimes \mathbf{Hom}(f(A_n), f(A))
\end{tikzcd}$$

are weak equivalences since all $Y(A_0)$ are cofibrant and $f_*$ is a weak equivalence by assumption. It follows that the map

$$f_* : d(1_*\tilde{1}^*Y(A)) \to d(f_*\tilde{f}^*(Y(A)))$$

is a weak equivalence, since the simplicial objects at issue are diagonally cofibrant.

The retraction map $r : 1_*\tilde{1}^*Y(A) \to Y(A)$ has a section $s : Y(A) \to 1_*\tilde{1}^*Y(A)$, along with an associated homotopy

$$h : 1_*\tilde{1}^*Y(A) \otimes \Delta^1 \to 1_*\tilde{1}^*Y(A)$$

from $sr$ to the identity on $1_*\tilde{1}^*Y(A)$. Applying the coend functor to $h$ gives a homotopy

$$d(h) : d(1_*\tilde{1}^*Y(A)) \otimes \Delta^1 \to d(1_*\tilde{1}^*Y(A)),$$

from $s_*r_*$ to the identity, again since $1_*\tilde{1}^*Y(A)$ is diagonally cofibrant. It follows from Lemma II.1.14 that $r_*$ is a weak equivalence of $\mathcal{M}$. □

All pointed simplicial sets are cofibrant, so Theorem 4.3 immediately implies

COROLLARY 4.4. *Suppose that a simplicial functor $f : \mathcal{A} \to \mathcal{B}$ is homotopically full and faithful, and let $X : \mathcal{A} \to \mathbf{S}_*$ be a simplicial functor taking values in the* *category* $\mathbf{S}_*$ *of pointed simplicial sets. Then* $X$ *is naturally pointwise weakly equivalent to the simplicial functor* $f_* d(\tilde{f}^* X)$.

Spectra, and the Bousfield-Friedlander model for the stable category, are formally discussed in Section X.6. Within that model, it is not true that all spectra are cofibrant, but homotopy colimits in the category of spectra are constructed levelwise within the category of pointed simplicial sets, giving

COROLLARY 4.5. *Suppose that a simplicial functor* $f : \mathcal{A} \to \mathcal{B}$ *is homotopically full and faithful, and let* $X : \mathcal{A} \to \mathbf{Spt}$ *be a simplicial functor taking values in the category* $\mathbf{Spt}$ *of spectra. Then* $X$ *is naturally pointwise strictly weakly equivalent to the simplicial functor* $f_* d(\tilde{f}^* X)$.

A simplicial functor $X : \mathcal{A} \to \mathbf{Spt}$ taking values in the category of spectra can be identified with a spectrum object in the category $\mathbf{S}_*^{\mathcal{A}}$ of pointed simplicial set valued simplicial functors. The object $X$ therefore consists of simplicial functors $X_n : \mathcal{A} \to \mathbf{S}_*$ and pointed transformations $S^1 \wedge X_n \to X_{n+1}$. A map $X \to Y$ in $\mathbf{Spt}^{\mathcal{A}}$ is a pointwise strict weak equivalence if all of the maps $X_n(A) \to Y_n(A)$ are weak equivalences of pointed simplicial sets.

Despite all the huffing and puffing above, Theorem 4.3 is a formal result which may not always be the best tool. To illustrate, suppose that $Y : \mathcal{A} \to s\mathbf{Ab}$ is a simplicial functor taking values in the category of simplicial abelian groups, and consider the maps of bisimplicial abelian groups corresponding to the diagram (4.2). One uses spectral sequence arguments and the generalized Eilenberg-Zilber theorem (Theorem IV.2.4) to see that, if $f : \mathcal{A} \to \mathcal{B}$ is homotopically faithful, then the maps $f_*$ and $r$ induce weak equivalences of the associated diagonal simplicial abelian groups, proving

LEMMA 4.6. *Suppose that a simplicial functor* $f : \mathcal{A} \to \mathcal{B}$ *is homotopically full and faithful, and let* $Y : \mathcal{A} \to s\mathbf{Ab}$ *be a simplicial functor taking values in the category* $s\mathbf{Ab}$ *of simplicial abelian groups. Then* $Y$ *is naturally pointwise weakly equivalent to the simplicial functor* $f_* d(\tilde{f}^* Y)$.

Here, $d$ is the diagonal functor. Note that the simplicial abelian groups $Y(A)$ do not have to be cofibrant.

The Dold-Kan equivalence (Corollary III.2.3) relating the categories $s\mathbf{Ab}$ of simplicial abelian groups and the category $\mathbf{Ch}_+$ of ordinary chain complexes, gives a simplicial model structure to the chain complex category. It also immediately implies the following result:

COROLLARY 4.7. *Suppose that a simplicial functor* $f : \mathcal{A} \to \mathcal{B}$ *is homotopically full and faithful, and let* $Y : \mathcal{A} \to \mathbf{Ch}_+$ *be a simplicial functor taking values in the category* $\mathbf{Ch}_+$ *of ordinary chain complexes. Then* $Y$ *is naturally pointwise weakly equivalent to the simplicial functor* $f_* d(\tilde{f}^* Y)$.

In this case, $d(\tilde{f}^* Y)$ is the normalized chain complex $Nd(\tilde{f}^*(\Gamma Y))$, where $\Gamma Y$ is the diagram of simplicial abelian groups associated to the diagram of chain complexes $Y$.

# Chapter X Localization

Localization is more a way of life than any specific collection of results. For example, under this rubric one can include Bousfield localization with respect to a homology theory, localization with respect to a map as pioneered by Bousfield, Dror-Farjoun and elaborated on by many others, and even the formation of the stable homotopy category. We will touch on all three of these subjects, but we also have another purpose. There is a body of extremely useful techniques that we will explore and expand on. These have come to be known as Bousfield factorization, which is a kind of "trivial cofibration-fibration" factorization necessary for producing localizations, and the Bousfield-Smith cardinality argument. This latter technique arises when one is confronted with a situation where a fibration is defined to be a map which has the right lifting property with respect to some class of maps. However, for certain arguments one needs to know it is sufficient to check that the map has the right lifting property with respect to a *set* of maps. We explain both Bousfield factorization and the cardinality argument and explore the implications in several contexts.

The concept of localization probably has its roots in the notion of a Serre class of abelian groups and the Whitehead Theorem mod a Serre class [87, §9.6]. This result is still useful and prevalent — so prevalent, in fact, that it is often used without reference. The idea of localizing a space with respect to a homology theory appeared in Sullivan's work on the Adams conjecture [86], where there is an explicit localization of a simply connected space with respect to ordinary homology with $\mathbb{Z}[1/p]$ coefficients. Bousfield and Kan [14] gave the first categorical definition of localization with respect to homology theory and provided a localization for nilpotent spaces with respect to $H_*(\cdot, R)$, where $R = \mathbb{F}_p$ for some prime $p$ or $R$ a subring of the rationals. Their original technique was the $R$-completion of space, which we have discussed in Section VIII.3.

Bousfield later introduced model category theoretic techniques to provide the localization of any space with respect to an arbitrary homology theory. His paper [8] has been enormously influential, as much for the methods as for the results, and it's hard to overestimate its impact. For example, the concept of localization with respect to a map and the proof of its existence, which appears in the work for Dror-Farjoun [22] and [23] is directly influenced by Bousfield's ideas. About the time Dror-Farjoun's papers were first circulating, a whole group of people began to explore these ideas, both in homotopy theory and in related algebraic subjects — the paper by Cascuberta [17] is a useful survey. One should also mention the important paper of Bousfield [11], which uses similar techniques for its basic constructions. The longest and most general work in this vein, a work that includes an exposition of the localization model category in a highly general setting, is that of Hirschhorn [42], available at this writing over the Internet.

We emphasize, however, that Bousfield's ideas had influence outside of the area of homotopy localization. For example, Jeff Smith realized very early on that one could use these constructions to put a model category structure on the category of small diagrams of simplicial sets, so that homotopy inverse limits can be computed as total right derived functors of inverse limit. This never made it into print under his name, but we have presented the arguments in Section VIII.2. Beyond this, there is the second author's work on the homotopy theory of simplicial presheaves [46], see also [38] as well as Joyal's result for simplicial sheaves [53]. In the context of the present discussion that work can be interpreted as follows: the category of presheaves on a Grothendieck site is a category of diagrams and there is a closed model category structure obtained by localizing with respect to a class of cofibrations determined by the topology of the underlying site.

## 1. Localization with respect to a map.

This section is an exposition of a technique due to Bousfield for defining localization with respect to a map in a simplicial model category. We explain the technique for the category **S** of simplicial sets.

In the category **S** of simplicial sets we fix a cofibration $f : A \to B$.

DEFINITION 1.1. *A space* $Z \in \mathbf{S}$ *is* $f$*-local if* $Z$ *is fibrant and*

$$f^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(A, Z)$$

*is a weak equivalence.*

REMARKS 1.2.

1) Because $f$ is a cofibration, $f^*$ is a fibration. Hence we could equally require that $f^*$ be a trivial fibration.

2) The hypothesis that $f$ be a cofibration is innocuous. Indeed, if we drop the hypothesis that $f$ be a cofibration, we have the following observation. Factor $f$ as

$$A \xrightarrow{f_0} B_0 \xrightarrow{q} B$$

where $f_0$ is a cofibration and $q$ is a trivial fibration. Then $f^*$ is a weak equivalence if and only if $f_0^*$ is a weak equivalence, because $q$ is left inverse to a trivial cofibration.

3) If $\mathcal{F}$ is a set of cofibrations $f_\alpha : A_\alpha \to B_\alpha$ we could define a space $Z$ to be $\mathcal{F}$-local if it is fibrant and $f_\alpha^*$ is a weak equivalence for all $f_\alpha \in \mathcal{F}$. However, $Z$ would be $\mathcal{F}$-local if and only if $Z$ were $f$-local where $f = \sqcup f_\alpha$. Hence we would achieve no greater generality.

We expand on the notion of $f$-local:

DEFINITION 1.3. *A map* $q : X \to Y$ *in* $\mathbf{S}$ *is an* $f$*-injective if* $q$ *is a fibration and*

$$(q_*, f^*) : \mathbf{Hom}(B, X) \to \mathbf{Hom}(B, Y) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(A, X)$$

*is a trivial fibration.*

Since this map is a fibration by **SM7**, we are only requiring it to be weak equivalence. In light of Remark 1.2.1, $Z$ is $f$-local if and only if the unique map $Z \to *$ is an $f$-injective.

It is convenient to have a recognition principle for $f$-injectives. First, if $j : C \to D$ and $q : X \to Y$ are maps in $\mathbf{S}$, let us write

$$D(j, q) = \mathbf{Hom}(D, Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C, X).$$

Note that $D(j, q)$ is a space of diagrams; indeed an $n$-simplex is a commutative diagram

$$\begin{tikzcd} C \times \Delta^n \arrow[r] \arrow[d, "j \times 1"'] & X \arrow[d, "q"] \\ D \times \Delta^n \arrow[r] & Y. \end{tikzcd}$$

If $L$ is a simplicial set, let $\#L$ denote the cardinality of the set of non-degenerate simplices in $L$.

LEMMA 1.4. *A morphism* $q : X \to Y$ *in* $\mathbf{S}$ *is an* $f$*-injective if and only if it is a fibration and has the right lifting property with respect to all maps*

$$A \times L \cup_{A \times K} B \times K \to B \times L$$

*where* $K \to L$ *is a cofibration in* $\mathbf{S}$ *with* $\#L$ *finite.*

PROOF: A map in $\mathbf{S}$ is a trivial fibration if and only if it has the right lifting property with respect to all cofibrations $K \to L$ with $\#L$ finite. Now use an adjunction argument. □

Note that one could specialize to the cases

$$K = \partial\Delta^n \hookrightarrow \Delta^n = L.$$

LEMMA 1.5. *If* $q : X \to Y$ *is an* $f$*-injective and* $K \in \mathbf{S}$*, then*

$$q_* : \mathbf{Hom}(K, X) \to \mathbf{Hom}(K, Y)$$

*is an* $f$*-injective.*

PROOF: Since $q$ is an $f$-injective

$$\mathbf{Hom}(B, X) \to D(f, q)$$

is a trivial fibration. Hence

$$\mathbf{Hom}(K, \mathbf{Hom}(B, X)) \to \mathbf{Hom}(K, D(f, q))$$

is a trivial fibration. But this is isomorphic to

$$\mathbf{Hom}(B, \mathbf{Hom}(K, X)) \to D(f, q_*). \qquad \square$$

From this it follows that the $f$-local spaces and $f$-injectives defined above, with the usual notion of weak equivalence, form a category of fibrant objects for a homotopy theory. This is direct from the definitions, using the previous two lemmas. In particular the mapping object is supplied by, for $f$-local $Z$, $\mathbf{Hom}(\Delta^1, Z)$. This uses Lemma 1.4.

We now let $\mathbf{Loc}_f$ be the resulting homotopy category of $f$-local spaces obtained as the full subcategory of $\mathrm{Ho}(\mathbf{S})$ with the $f$-local spaces. Since every object of $\mathbf{S}$ is cofibrant and every $f$-local space if fibrant, this is the equivalent to the category of $f$-local spaces and homotopy classes of maps. We will examine the inclusion functor

$$\mathbf{Loc}_f \to \mathrm{Ho}(\mathbf{S})$$

DEFINITION 1.6. *A localization with respect to $f$ is a functor*

$$L_f : \mathrm{Ho}(\mathbf{S}) \to \mathrm{Ho}(\mathbf{S})$$

*equipped with a natural transformation $\eta_X : X \to L_f X$ so that*

1) *$L_f X \in \mathbf{Loc}_f$ and the restricted functor $L_f : \mathrm{Ho}(\mathbf{S}) \to \mathbf{Loc}_f$ is left adjoint to the inclusion; and*
2) *for all $X$, the two morphisms $L_f \eta_X, \eta_{L_f X} : L_f X \to L_f L_f X$ are equal and isomorphisms.*

If it exists, such a localization will be unique up to isomorphism in the homotopy category $\mathrm{Ho}(\mathbf{S})$. The existence follows from Bousfield's factorization, embodied in Proposition 1.8 below. Before stating this we need a definition of a class of cofibrations which behave much like a class of trivial cofibrations.

DEFINITION 1.7. *A cofibration $j : C \to D$ is an $f$-*cofibration *if the map*

$$(q_*, j^*) : \mathbf{Hom}(D, X) \to \mathbf{Hom}(D, Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C, X)$$

*is a trivial fibration for all $f$-injectives $q : X \to Y$. Let $\mathcal{C}_f$ be the class of $f$-cofibrations.*

Proposition 1.8 (Bousfield factorization). *Every morphism $g : X \to Y$ may be factored*

$$X \xrightarrow{j} E_f \xrightarrow{q} Y$$

*where $j$ is an $f$-cofibration and $q$ is an $f$-injective. Furthermore, this factorization is natural in the morphism $g$.*

We prove this below after some further preliminaries and proving the existence of the localization demanded by Definition 1.6.

First some basic properties of $f$-cofibrations.

Lemma 1.9.

1) *The morphism $f : A \to B$ is an $f$-cofibration.*
2) *Any trivial cofibration is an $f$-cofibration.*
3) *Suppose $i : C \to D$ and $j : D \to E$ are cofibrations and $i$ is an $f$-cofibration. Then if either of $j$ or $ji$ is an $f$-cofibration so is the other.*

Proof: Only part 3) needs comment. We use the following fact: if one has a diagram

$$\begin{tikzcd}
X \arrow[r, "q"] \arrow[dr] & Y \arrow[d] \\
 & Z
\end{tikzcd}$$

and a morphism $W \to Z$, then there is a pullback square

$$\begin{tikzcd}
W \times_Z X \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
W \times_Z Y \arrow[r] & Y.
\end{tikzcd}$$

Hence if $q$ is a trivial fibration so is $W \times_Z X \to W \times_Z Y$. Applying this remark to the diagram

$$\begin{tikzcd}
\mathbf{Hom}(D, X) \arrow[r] \arrow[dr] & \mathbf{Hom}(D, Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C, X) \arrow[d] \\
 & \mathbf{Hom}(D, Y)
\end{tikzcd}$$

and the map $\mathbf{Hom}(E,Y) \to \mathbf{Hom}(D,Y)$ we have that if $i$ is an $f$-cofibration and $q : X \to Y$ is an $f$-injective, then

$$\mathbf{Hom}(E,Y) \times_{\mathbf{Hom}(D,Y)} \mathbf{Hom}(D,X) \to \mathbf{Hom}(E,Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C,X)$$

is a trivial fibration. Now consider the composite

$$\begin{tikzcd}
\mathbf{Hom}(E,X) \arrow[r] & \mathbf{Hom}(E,Y) \times_{\mathbf{Hom}(D,Y)} \mathbf{Hom}(D,X) \arrow[d] \\
 & \mathbf{Hom}(E,Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C,X).
\end{tikzcd}$$

If $j$ is an $f$-cofibration, the first map is a trivial fibration, so the composite is a trivial cofibration and $ji$ is an $f$-cofibration. The converse equally applies to show that if $ji$ is an $f$-cofibration so is $j$. □

More constructive properties of $f$-cofibrations are given in the next result.

LEMMA 1.10.

1) *Any retract of an $f$-cofibration is an $f$-cofibration. Any coproduct of $f$-cofibrations is an $f$-cofibration.*
2) *Given a pushout diagram with $j$ an $f$-cofibration*

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "j"'] & C' \arrow[d, "j'"] \\
D \arrow[r] & D',
\end{tikzcd}$$

*then $j'$ is an $f$-cofibration.*

3) *If $j : C \to D$ is an $f$-cofibration and $K \to L$ is any cofibration, then*

$$C \times L \cup_{C \times K} D \times K \to D \times L$$

*is an $f$-cofibration.*

PROOF: Again only part 3) needs comment. First note that if $K = \phi$, we are asserting $C \times L \to D \times L$ is an $f$-cofibration. This requires that for all $f$-injectives $q : X \to Y$

$$\mathbf{Hom}(D \times L, X) \to \mathbf{Hom}(D \times L, Y) \times_{\mathbf{Hom}(C \times L, Y)} \mathbf{Hom}(C \times L, X)$$

be a trivial fibration. But this is isomorphic to

$$\begin{tikzcd}
\mathbf{Hom}(D, \mathbf{Hom}(L, X)) \arrow[d] \\
\mathbf{Hom}(D, \mathbf{Hom}(L, Y)) \times_{\mathbf{Hom}(C, \mathbf{Hom}(L, Y))} \mathbf{Hom}(C, \mathbf{Hom}(L, X))
\end{tikzcd}$$

and $\mathbf{Hom}(L, X) \to \mathbf{Hom}(L, Y)$ is an $f$-injective by Lemma 1.5.

For the general case consider the diagram

$$\begin{tikzcd}
C \times K \arrow[rr] \arrow[d] & & C \times L \arrow[d, "i"] \\
D \times K \arrow[rr] & & C \times L \cup_{C \times K} D \times K \arrow[d, "j"] \\
& & D \times L.
\end{tikzcd}$$

Part 2) of this lemma says $i$ is a $f$-cofibration. Since $ji$ is a $f$-cofibration, the previous lemma says $j$ is. □

Next is the crucial lifting result.

PROPOSITION 1.11. *Given a lifting problem*

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\
D \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

*where $j$ is a $f$-cofibration and $q$ is an $f$-injective, the lift exists and is unique up to homotopy under $C$ and over $Y$.*

PROOF: Consider the trivial fibration

$$\mathbf{Hom}(D, X) \to \mathbf{Hom}(D, Y) \times_{\mathbf{Hom}(C, Y)} \mathbf{Hom}(C, X) = D(j, q).$$

A solid arrow diagram as above is a 0-simplex in the target. Since a trivial fibration is surjective such a lift exists. The required homotopy is adjoint to a lift in

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r] \arrow[d] & \mathbf{Hom}(D, X) \arrow[d] \\
\Delta^1 \arrow[r] \arrow[ur, dashed] & D(j, q).
\end{tikzcd}$$

□

Suppose that we have demonstrated the existence of Bousfield factorization, as in Proposition 1.8. Define a functor $\mathcal{L} : \mathbf{S} \to \mathbf{S}$ and a natural transformation $j : X \to \mathcal{L}X$, by taking the Bousfield factorization of $X \to *$ as

$$X \xrightarrow{j} \mathcal{L}X \xrightarrow{q} *$$

where $j$ is an $f$-cofibration and $q$ is a $f$-injective. This is a functor since the factorization is natural. Then $\mathcal{L}X$ is $f$-local. The previous result implies that $j$ has the following universal property: given a diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "j"'] & Z \\
\mathcal{L}X \arrow[ur, dashed, "\overline{g}"'] &
\end{tikzcd}$$

with $Z$ $f$-local, then the dotted arrow exists and is unique up to homotopy. If $g$ is a $f$-cofibration, $\overline{g}$ is a homotopy equivalence.

The first result is that $\mathcal{L}(\cdot)$ passes to a functor on the homotopy category.

LEMMA 1.12. *If $\varphi : X \to Y$ is a weak equivalence, then $\mathcal{L}\varphi : \mathcal{L}X \to \mathcal{L}Y$ is a weak equivalence.*

PROOF: First of all, $\mathcal{L}$ takes trivial cofibrations to homotopy equivalences. In effect, if $i : X \to Y$ is a trivial cofibration, then $i$ is an $f$-cofibration by Lemma 1.9, and so the composites

$$X \xrightarrow{j_X} \mathcal{L}X \to * \quad \text{and} \quad X \xrightarrow{j_Y i} \mathcal{L}Y \to *$$

are two factorizations of $X \to *$ as an $f$-injective following an $f$-cofibration. It follows from Proposition 1.11 that $\mathcal{L}i : \mathcal{L}X \to \mathcal{L}Y$ is a homotopy equivalence.

More generally, any weak equivalence $f : X \to Y$ has a factorization $f = p \cdot i$, where $i$ is a trivial cofibration and $p$ is left inverse to a trivial cofibration. Thus, $\mathcal{L}p$ is a weak equivalence, as is the map $\mathcal{L}f$. □

Let $L_f : \text{Ho}(\mathbf{S}) \to \text{Ho}(\mathbf{S})$ be the functor induced by $\mathcal{L}$, and $\eta : X \to L_f X$ be induced in $\text{Ho}(\mathbf{S})$ by the map $j_X : X \to \mathcal{L}X$.

PROPOSITION 1.13. *The functor* $L_f : \mathrm{Ho}(\mathbf{S}) \to \mathrm{Ho}(\mathbf{S})$ *is an* $f$*-localization.*

PROOF: Since $\mathcal{L}X$ is $f$-local, $L_f$ restricts to a functor

$$L_f : \mathrm{Ho}(\mathbf{S}) \to \mathbf{Loc}_f.$$

By the universal property of $\mathcal{L}X$, $L_f$ is left adjoint to the inclusion.

If $j : X \to Y$ is an $f$-cofibration, then any choice of extension $\mathcal{L}j : \mathcal{L}X \to \mathcal{L}Y$ is a homotopy equivalence, by an obvious extension of the argument giving Lemma 1.12. It follows that $\mathcal{L}j_X$ is a homotopy equivalence, for any $X$. Also, the map $j_X : X \to \mathcal{L}X$ and the composite

$$X \xrightarrow{j_X} \mathcal{L}X \xrightarrow{j_{\mathcal{L}X}} \mathcal{L}^2X$$

are $f$-cofibrations taking values in $f$-local spaces, and the diagram

$$\begin{tikzcd}
X \arrow[r, "j_X"] \arrow[d, "j_X"'] & \mathcal{L}X \arrow[r, "j_{\mathcal{L}X}"] & \mathcal{L}^2X \\
\mathcal{L}X \arrow[urr, "\theta"'] & &
\end{tikzcd}$$

commutes, where $\theta$ could be either $\mathcal{L}j_X$ or $j_{\mathcal{L}X}$. It follows that the maps $\mathcal{L}j_X$ and $j_{\mathcal{L}X}$ are homotopic. □

We now turn to the existence of Bousfield factorizations. To start, we must expand on Lemmas 1.9 and 1.10 and supply one more construction that preserves $f$-cofibrations, namely certain types of directed colimits.

Fix an infinite cardinal number $\beta$ and let $\mathbf{Seq}(\beta)$ denote the well-ordered set of ordinals less than $\beta$. Then $\mathbf{Seq}(\beta)$ is a category with $\hom(s,t)$ having one element of $s \leq t$ and empty otherwise.

LEMMA 1.14. *Let* $C_\bullet \in \mathbf{S}$ *be a diagram of spaces over* $\mathbf{Seq}(\beta)$. *If*

1) *for each successor ordinal* $s < \beta$, $C_{s-1} \to C_s$ *is an* $f$*-cofibration, and*
2) *for each limit ordinal* $t < \beta$, $\varinjlim C_s = C_t$,

*then the map*

$$C_0 \to \varinjlim_{s<\beta} C_s$$

*is an* $f$*-cofibration.*

PROOF: If $K$ is a simplicial set, denote by $\#K$ the cardinality of the member of non-degenerate simplices of $K$. Notice that $D \to E$ is an $f$-cofibration if and only if every lifting problem

$$\begin{tikzcd}
D \times L \cup_{D \times K} E \times K \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
E \times L \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

can be solved, where $q$ is an $f$-injective and $K \to L$ is a cofibration in $\mathbf{S}$ with $\#L$ finite. The strategy is this: we will use transfinite induction and solve the successive lifting problems

$$\begin{tikzcd}
C_0 \times L \cup_{C_0 \times K} C_s \times K \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
C_s \times L \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd} \tag{1.15}$$

in a compatible way, meaning that if $s < t$, the solution of the lifting problem for $t$ will restrict to the solution for $s$. Then taking the colimit over $s < \beta$ will solve the problem for $C_0 \to \varinjlim_{s<\beta} C_s$.

So regard the lifting problem for $s$ given in (1.15). If $s$ is a successor ordinal, we can complete the diagram

$$\begin{tikzcd}
C_0 \times L \cup_{C_0 \times K} C_{s-1} \times K \arrow[r] \arrow[d] & C_0 \times L \cup_{C_0 \times K} C_s \times K \arrow[r] \arrow[d] & X \arrow[d] \\
C_{s-1} \times L \arrow[r] & C_{s-1} \times L \cup_{C_{s-1} \times K} C_s \times K \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

since we have a solution for $s - 1$ and the left square is a pushout. Thus we need to solve a lifting problem

$$\begin{tikzcd}
C_{s-1} \times L \cup_{C_{s-1} \times K} C_s \times K \arrow[r] \arrow[d] & X \arrow[d] \\
C_s \times L \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

But this is possible since $C_{s-1} \to C_s$ is an $f$-cofibration. Note that the constructed lift satisfies the compatibility requirement spelled out above.

If $s$ is a limit ordinal, we have constructed compatible lifts, $t < s$,

$$\begin{tikzcd}
C_0 \times L \cup_{C_0 \times K} C_t \times K \arrow[r] \arrow[d] & X \arrow[d] \\
C_t \times L \arrow[r] \arrow[ur] & Y.
\end{tikzcd}$$

Taking the colimit over $t$ yields a solution to the lifting problem for $C_0 \to \varinjlim_{t<s} C_t = C_s$ compatible with all previous lifts. □

We now come to the factorization result Proposition 1.8. It turns on the following construction.

MAIN CONSTRUCTION 1.16.

Let $g : X \to Y$ be a map in $\mathbf{S}$, and let $I$ be the set of morphisms

$$A \times L \cup_{A \times K} B \times K \to B \times L,$$

where $K \subseteq L$ runs over representatives for isomorphism classes of cofibrations of simplicial sets so that $\# L$ is finite. Write $j_\alpha : C_\alpha \to D_\alpha$ for a typical element in $I$.

Define a factorization

$$X \xrightarrow{j_1} E_1 X \xrightarrow{g_1} Y$$

of $g$ by the pushout diagram

$$\begin{tikzcd}
\bigsqcup_I C_\alpha \times D(j_\alpha, g) \arrow[r, "\epsilon"] \arrow[d] & X \arrow[d, "j_1"] \\
\bigsqcup_I D_\alpha \times D(j_\alpha, g) \arrow[r] & E_1
\end{tikzcd}$$

where

$$D(j_\alpha, g) = \mathbf{Hom}(C_\alpha, X) \times_{\mathbf{Hom}(C_\alpha, Y)} \mathbf{Hom}(D_\alpha, Y)$$

is the space of commutative squares and $\epsilon$ is induced by evaluation. Evaluation also defines a map $D_\alpha \times D(j_\alpha, g) \to Y$ and $g_1 : E_1 \to Y$ is induced by the universal property of pushouts.

Note that since every morphism in $I$ is an $f$-cofibration, Lemma 1.10 implies $j_1$ is an $f$-cofibration. Equally important, note that if one has any diagram

$$\begin{tikzcd}
C_\alpha \arrow[r] \arrow[d, "j_\alpha"'] & X \arrow[d, "g"] \\
D_\alpha \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

where $j_\alpha \in I$, then $\varphi$ factors through $E_1$. This is because such a diagram is a 0-simplex of $D(j_\alpha, g)$. Finally, note this construction is natural in $g$.

PROOF OF PROPOSITION 1.8: This is a transfinite variation on the small object argument. Pick an infinite cardinal $\gamma > \#B$, and choose an infinite cardinal $\beta > 2^{\gamma}$. Note that $\#C_\alpha < \gamma$ for all $\alpha$. Fix a map $g : X \to Y$. Using recursion, we construct a diagram $\{E_s\}$ over $\mathbf{Seq}(\beta)$, with $E_0 = X$ and so that there is a map $\{g_s\} : \{E_s\} \to \{Y\}$ to the constant diagram on $Y$.

For $s = 0$, let $g_0 : X = E_0 \to Y$. If $s + 1$ is a successor ordinal, let $E_{s+1}$ be defined by applying the main construction 1.16 to $g_s : E_s \to Y$ to obtain a factorization

$$E_s \to E_1 E_s \xrightarrow{(g_s)_1} Y.$$

Then $E_{s+1} = E_1 E_s$ and $g_{s+1} = (g_s)_1$. If $s$ is a limit ordinal, let $E_s = \varinjlim_{t<s} E_t$ and $g_s$ the induced map. This defines $\{E_s\}$.

Define $E_f = \varinjlim_{s<\beta} E_s$. Then $g : X \to Y$ factors as

$$X \xrightarrow{j} E_f \xrightarrow{q} Y$$

where $j$ is an $f$-cofibration by Lemma 1.14. To see that $q$ is an $f$-injective, it is sufficient to show that any lifting problem

$$\begin{tikzcd}
C_\alpha \arrow[r, "\psi"] \arrow[d, "j_\alpha"'] & E_f \arrow[d, "g"] \\
D_\alpha \arrow[r, "\varphi"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

with $j_\alpha \in I$ can be solved. Since $2^{\#C_\alpha} < \beta$,

$$\hom(C_\alpha, E_f) \cong \varinjlim_{\beta} \hom(C_\alpha, E_s),$$

for otherwise $C_\alpha$ has too many subobjects. Thus, there is an $s < \beta$, and a factorization of $\psi$

$$C_\alpha \to E_s \to E_f.$$

Then the main construction implies that $\varphi$ factors through $E_{s+1}$ and the result follows. □

The constructive part of this argument allows us to identify the class of $f$-cofibrations in another way. Let $\beta$ be a cardinal, and recall from Definition II.6.3 that a class $\mathcal{M}$ of morphisms in a cocomplete category $\mathcal{C}$ is $\beta$-saturated if

1) coproducts and retracts of morphisms in $\mathcal{M}$ are in $\mathcal{M}$
2) given a pushout diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "j"'] & C' \arrow[d, "j'"] \\
D \arrow[r] & D'
\end{tikzcd}$$

with $j$ in $\mathcal{M}$, then $j'$ is in $\mathcal{M}$.

3) If $C : \mathbf{Seq}(\beta) \to \mathcal{C}$ is a diagram over $\beta$, and

   a) for each successor ordinal $s$, $C_{s-1} \to C_s$ is in $\mathcal{M}$ and

   b) for each limit ordinal $s$, $\varinjlim_{t<s} C_t \cong C_s$,

   then $C_0 \to \varinjlim_{t<\beta} C_t$ is in $\mathcal{M}$.

COROLLARY 1.17. *Suppose that $\beta$ is an infinite cardinal greater than $2^\gamma$, where $\gamma$ is an infinite cardinal larger than $\#B$. Then the class of $f$-cofibrations in* $\mathbf{S}$ *is the $\beta$-saturation of the morphisms*

$$A \times L \cup_{A\times K} B \times K \to B \times L$$

*with $K \to L$ a cofibration in* $\mathbf{S}$ *and $\#L < \infty$.*

PROOF: Let $\mathcal{C}_f$ be this saturation. Then every morphism in $\mathcal{C}_f$ is an $f$-cofibration since the class of $f$-cofibrations is $\beta$-saturated. Also if $g : X \to Y$ is factored

$$X \overset{j}{\to} Z \overset{q}{\to} Y$$

as in the proof of Proposition 1.8, then $j \in \mathcal{C}_f$. If $g$ is any $f$-cofibration, then there is a lifting

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "g"'] & Z \arrow[d, "q"] \\
Y \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

since $q$ is an $f$-injective. This lifting shows $g$ is a retract of $j$, hence in $\mathcal{C}_f$. $\square$

We end this section with a sequence of technical lemmas on the properties of the functor $\mathcal{L}(\cdot)$. The purpose here is to have in place the structure necessary to prove the existence of the model category structure in the next section.

Recall that $\mathcal{L}(X) = \varinjlim_{s<\beta} E_s(X)$ where $E_{s+1}(X)$ is obtained from $E_s = E_s(X)$ by a pushout diagram

$$\begin{tikzcd}
\bigsqcup_{j_\alpha \in J} C_\alpha \times \mathbf{Hom}(C_\alpha, E_s(X)) \arrow[r] \arrow[d] & E_s(X) \arrow[d] \\
\bigsqcup_{j_\alpha \in J} D_\alpha \times \mathbf{Hom}(C_\alpha, E_s(X)) \arrow[r] & E_{s+1}(X)
\end{tikzcd} \tag{1.18}$$

Of course $E_0(X) = X$, and if $s$ is a limit ordinal then $E_s(X) = \varinjlim_{t<s} E_t(X)$.

Lemma 1.19. *Let $\lambda$ be a cardinal larger than $2^\gamma$, where $\gamma$ is an infinite cardinal larger than $\#B$. Let $X : \mathbf{Seq}(\lambda) \to \mathbf{S}$ be $\lambda$-diagram. Suppose for each $s < \lambda$, $X_s \to X_{s+1}$ is a cofibration, and suppose for each limit ordinal $X_s = \varinjlim_{t<s} X_t$. Then the natural map*

$$\varinjlim_{s<\lambda} \mathcal{L}(X_s) \to \mathcal{L}(\varinjlim_{s<\lambda} X_s)$$

*is an isomorphism.*

Proof: It is sufficient, since colimits commute, to show this statement holds for each of the functors $E_s(\cdot)$. For the same reason, it is sufficient to show, that if this statement holds for $E_t(\cdot)$ it holds for $E_{t+1}(\cdot)$. Then it will automatically hold for limit ordinals. Now, since $2^{\#C_\alpha} < \lambda$ for all $\alpha$, the natural map

$$\begin{aligned}\varinjlim_{s<\lambda} \hom(C_\alpha \times \Delta^n, E_t(X_s)) \to &\hom(C_\alpha \times \Delta^n, \varinjlim_{s<\lambda} E_t X_s) \\ &\cong \hom(C_\alpha \times \Delta^n, E_t(\varinjlim_{s<\lambda} X_s))\end{aligned}$$

is an isomorphism in each degree $n$. It follows that the simplicial map

$$\varinjlim_{s<\lambda} \mathbf{Hom}(C_\alpha, E_t(X_s)) \to \mathbf{Hom}(C_\alpha, E_t(\varinjlim_{s<\lambda} X_s))$$

is an isomorphism. Now use the diagram (1.18) and the fact that colimits commute. □

For the next result we need a "Reedy Lemma". Suppose we have two pushout squares

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ C_1 \arrow[r] & D_1 \end{tikzcd} \qquad \begin{tikzcd} A_2 \arrow[r] \arrow[d] & B_2 \arrow[d] \\ C_2 \arrow[r] & D_2 \end{tikzcd}$$

and a map from the first square to the second.

Lemma 1.20. *If $B_1 \to B_2$ is a cofibration and the induced map $C_1 \cup_{A_1} A_2 \to C_2$ is a cofibration, then the map $D_1 \to D_2$ is a cofibration.*

Proof: The hypotheses are what is required to show $D_1 \to D_2$ has the left lifting property with respect to all trivial fibrations. □

Lemma 1.21. *The functor $\mathcal{L}(\cdot)$ preserves cofibrations.*

Proof: Suppose that $X \to Y$ is a cofibration, and presume that the induced map $E_s X \to E_s Y$ is a cofibration. We will show that the map $E_{s+1} X \to E_{s+1} Y$ is a cofibration.

The diagram (1.18) for $X$ maps to the corresponding diagram for $Y$. The diagram

$$\begin{tikzcd}
\bigsqcup_J C_\alpha \times \mathbf{Hom}(C_\alpha, E_s X) \arrow[r] \arrow[d] & \bigsqcup_J D_\alpha \times \mathbf{Hom}(C_\alpha, E_s X) \arrow[d] \\
\bigsqcup_J C_\alpha \times \mathbf{Hom}(C_\alpha, E_s Y) \arrow[r] & \bigsqcup_J D_\alpha \times \mathbf{Hom}(C_\alpha, E_s Y)
\end{tikzcd}$$

is a disjoint union of diagrams of the form

$$\begin{tikzcd}
C_\alpha \times K \arrow[r, "j_\alpha \times 1"] \arrow[d, "1 \times i"'] & D_\alpha \times K \arrow[d, "1 \times i"] \\
C_\alpha \times L \arrow[r, "j_\alpha \times 1"'] & D_\alpha \times L
\end{tikzcd}$$

which are induced by cofibrations $j_\alpha : C_\alpha \to D_\alpha$ and $i : K \to L$. Each induced map

$$C_\alpha \times L \cup_{C_\alpha \times K} D_\alpha \times K \to D_\alpha \times L$$

is plainly a cofibration. Now use Lemma 1.20. □

The next result is a variation on Lemma 1.19.

LEMMA 1.22. *Let $\lambda$ be a cardinal number with $\lambda > \gamma$. For $X \in \mathbf{S}$, let $\{X_j\}$ be the filtered system of sub-complexes $X_j \subseteq X$ with $\#X_j \leq \lambda$. Then the natural map*

$$\varinjlim_j \mathcal{L}(X_j) \to \mathcal{L}(X)$$

*is an isomorphism.*

PROOF: The proof is the same as that for Lemma 1.19, once one notices that

$$\varinjlim_j \mathbf{Hom}(C_\alpha, X_j) \to \mathbf{Hom}(C_\alpha, X)$$

is an isomorphism on account of the size of $C_\alpha$. □

We now have a counting argument:

LEMMA 1.23. *Let $\lambda$ be any cardinal such that $\lambda \geq \beta$. If $X \in S$ has $\#X \leq 2^\lambda$, then $\#\mathcal{L}(X) \leq 2^\lambda$.*

PROOF: Recall that $\beta$ was chosen such that $\beta > 2^\gamma$, where $\gamma$ is an infinite cardinal with $\gamma > \#B$.

We show that for each ordinal $s < \beta$, $\#E_s(X) \leq 2^\lambda$. This is done by induction: it is true for $E_0(X) = X$; if $s$ is a limit ordinal and it is true for $E_t(X)$, $t < s$, then it is true for $E_s(X)$. Finally, if it is true for $s$ it is true for $s+1$ by the defining diagram (1.18) and because

$$\#(D_\alpha \times \mathbf{Hom}(C_\alpha, E_s(X)) \leq \beta \cdot (2^\lambda)^\beta = \beta \cdot 2^{\lambda \cdot \beta} = 2^\lambda. \qquad \square$$

Because its definition involves a sequence of pushouts, one would not expect the functor $\mathcal{L}$ to preserve inverse limits; however, we do have the following result. Recall that $\mathcal{L}$ preserves cofibrations and the cofibrations are inclusions.

LEMMA 1.24. *Let $X$ be a simplicial set and $C, D \subseteq X$ two sub-simplicial sets. Then $\mathcal{L}(C \cap D) = \mathcal{L}(C) \cap \mathcal{L}(D)$.*

PROOF: First suppose that we can show that for every ordinal $s$ there is an equality $E_{s+1}(C \cap D) = E_{s+1}(C) \cap E_{s+1}(D)$. Then, because $E_s(\cdot) \subseteq E_{s+1}(\cdot)$, an inductive argument shows that we have the following equalities for any limit ordinal $s$

$$E_s(C \cap D) = \bigcup_{t<s}[E_t(C \cap D)] = \bigcup_{t<s}[E_t(C) \cap E_t(D)] = [\bigcup_{t<s} E_t(C)] \cap [\bigcup_{t<s} E_t(D)].$$

The last of these equalities uses the observation that we have a nested sequence of inclusions, and the last listed object is $E_s(C) \cap E_s(D)$.

Now suppose that we can show $E_1(C \cap D) = E_1(C) \cap E_1(D)$. Then another inductive argument implies

$$E_{s+1}(C \cap D) = E_1(E_s(C \cap D)) = E_1(E_s(C) \cap E_s(D)) = E_{s+1}(C) \cap E_{s+1}(D).$$

Finally, to see that $E_1(C \cap D) = E_1(C) \cap E_1(D)$, note that $E_1 X_n$ has the form

$$(E_1 X)_n = (\bigsqcup_\alpha (D_\alpha - C_\alpha)_n \ ) \ \times \mathbf{Hom}(C_\alpha, X)_n) \bigsqcup X_n$$

in each simplicial degree $n$. $\square$

## 2. The closed model category structure.

We have presented Bousfield factorization as if it were a "trivial cofibration-fibration" factorization for an appropriate closed model category structure on simplicial sets. This is not quite the case, but a minor variation makes this statement true and precise.

Fix a cofibration $f : A \to B$ and note that the previous section gives a functor $\mathcal{L} : \mathbf{S} \to \mathbf{S}$ equipped with a natural $f$-cofibration

$$j : X \to \mathcal{L}X.$$

The functor $\mathcal{L}$ preserves weak equivalences and induces the localization functor $L_f$ on the homotopy category. We now define a morphism $g : C \to D$ in $\mathbf{S}$ to be an *$f$-local equivalence* if the induced map

$$g^* : \mathbf{Hom}(D, X) \to \mathbf{Hom}(C, X)$$

is a weak equivalence for all $f$-local spaces $X$.

LEMMA 2.1.

1) *Any $f$-cofibration is an $f$-local equivalence.*
2) *A morphism $g : C \to D$ is an $f$-local equivalence if and only if $\mathcal{L}g : \mathcal{L}C \to \mathcal{L}D$ is a weak equivalence.*

PROOF: Part 1) is a consequence of Definition 1.7 with $q$ the unique morphism $X \to *$. For part 2) consider the induced diagram, with $X$ $f$-local

$$\begin{tikzcd}
\mathbf{Hom}(\mathcal{L}D, X) \arrow[r, "j^*"] \arrow[d, "\mathcal{L}g^*"'] & \mathbf{Hom}(D, X) \arrow[d, "g^*"] \\
\mathbf{Hom}(\mathcal{L}C, X) \arrow[r, "j^*"'] & \mathbf{Hom}(C, X).
\end{tikzcd}$$

By part 1) both maps labeled $j^*$ are weak equivalences. If $\mathcal{L}g$ is a weak equivalence, then $\mathcal{L}g^*$ and, hence, $g^*$ are weak equivalences, so $g$ is a $f$-local equivalence. Conversely, if $g^*$ is a weak equivalence, so is $\mathcal{L}g^*$ and

$$\mathcal{L}g^* : [\mathcal{L}D, X] \to [\mathcal{L}C, X]$$

is a bijection for all $f$-local $X$. This implies $\mathcal{L}g$ is a homotopy equivalence. To see this, set $X = \mathcal{L}C$, then there is a map $h : \mathcal{L}D \to \mathcal{L}C$ so that $\mathcal{L}g^* h = h \circ \mathcal{L}g \simeq 1_{\mathcal{L}C}$. Then set $X = \mathcal{L}D$ and compute

$$\mathcal{L}g^*(\mathcal{L}g \circ h) = \mathcal{L}g \circ h \circ \mathcal{L}g \simeq \mathcal{L}g = 1_{\mathcal{L}D} \circ \mathcal{L}g = \mathcal{L}g^*(1_{\mathcal{L}D})$$

so $\mathcal{L}g \circ h \simeq 1_{\mathcal{L}D}$. □

DEFINITION 2.2. *We define a morphism $g : C \to D$ to be*

1) *an $f$-local cofibration if it is a cofibration; and*
2) *an $f$-local fibration if it has the right lifting property with respect to all cofibrations which are also $f$-local equivalences.*

The main result of this section is:

THEOREM 2.3. *With its usual simplicial structure and the above definitions of $f$-local equivalence, cofibrations, and $f$-local fibration, the category* $\mathbf{S}$ *becomes a simplicial model category.*

REMARK 2.4. If every $f$-injective were an $f$-local fibration, this would be easy. However, this need not be the case as the following example shows. Let $f : \partial\Delta^{n+1} \to \Delta^{n+1}$ be the inclusion and consider the fibration $q : WK(\mathbb{Z}, n-1) \to K(\mathbb{Z}, n)$. Note that $q$ is an $f$-injective; that is,

$$\mathbf{Hom}(\Delta^{n+1}, WK(\mathbb{Z}, n-1)) \to D(f, q)$$

is a trivial fibration. This is equivalent to the assertion that $D(f,q)$ is contractible. For this, there is a pullback diagram

$$\begin{tikzcd}
D(f,q) \arrow[r] \arrow[d] & \mathbf{Hom}(\Delta^{n+1}, K(\mathbb{Z},n)) \arrow[d] \\
\mathbf{Hom}(\partial\Delta^{n+1}, WK(\mathbb{Z},n-1)) \arrow[r] & \mathbf{Hom}(\partial\Delta^{n+1}, K(\mathbb{Z},n))
\end{tikzcd}$$

so that the fibre of the induced fibration $D(f,q) \to \mathbf{Hom}(\partial\Delta^n, WK(\mathbb{Z},n-1))$ is the pointed function complex

$$\mathbf{hom}_*(\Delta^{n+1}/\partial\Delta^{n+1}, K(\mathbb{Z},n)) \simeq \Omega^{n+1}K(\mathbb{Z},n) \simeq *.$$

Now let $* \to \partial\Delta^{n+1}$ be any vertex. Then $* \to \Delta^{n+1}$ is a trivial cofibration and $\partial\Delta^{n+1} \to \Delta^{n+1}$ is an $f$-local equivalence (by Lemma 2.1.1 and Lemma 1.9.1) so $* \to \partial\Delta^{n+1}$ is an $f$-local equivalence. But if $i : \partial\Delta^{n+1} \to K(\mathbb{Z},n)$ is non-trivial in homotopy, there is no solution to the lifting problem

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & WK(\mathbb{Z},n-1) \arrow[d, "q"] \\
\partial\Delta^{n+1} \arrow[r, "i"'] \arrow[ur, dashed] & K(\mathbb{Z},n),
\end{tikzcd}$$

so $q$ is an $f$-injective which is not an $f$-local fibration. We will have more to say about $f$-local fibrations after the proof of Theorem 2.3.

We start the proof of Theorem 2.3 with the following lemmas.

LEMMA 2.5. *A morphism in* $\mathbf{S}$ *is at once an* $f$*-local fibration and an* $f$*-local equivalence if and only if it is a trivial fibration.*

PROOF: If $q : X \to Y$ is a trivial fibration it has the right lifting property with respect to all cofibrations. Furthermore $\mathcal{L}q$ is a weak equivalence by Lemma 1.12. So Lemma 2.1 implies $q$ is a $f$-local fibration and an $f$-local equivalence.

For the converse, fix $q : X \to Y$ which is an $f$-local fibration and an $f$-local equivalence. Factor $q$ as

$$X \xrightarrow{i} Z \xrightarrow{p} Y,$$

where $i$ is a cofibration and $p$ is a trivial fibration. Then Lemmas 1.12 and 2.1 imply $i$ is an $f$-local equivalence, so there is a solution to the lifting problem

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i"'] & X \arrow[d, "q"] \\
Z \arrow[r, "p"'] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

Thus $q$ is a retract of $p$ and is therefore a trivial fibration. □

LEMMA 2.6. *Any morphism $g : X \to Y$ in* **S** *can be factored as*

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

*where $q$ is an $f$-local fibration and $j$ is at once a cofibration and an $f$-local equivalence.*

This is the heart of the matter and will be proved below. We also record:

LEMMA 2.7.

1) *The class of morphisms which are at once cofibrations and $f$-local equivalences is closed under pushouts and colimits over ordinal numbers.*
2) *Let $j : C \to D$ be at once a cofibration and an $f$-local equivalence, and let $K \to L$ be any cofibration, then*

$$D \times K \cup_{C \times K} C \times L \to D \times L$$

*is a cofibration and an $f$-local equivalence.*

PROOF: Observe that $j : C \to D$ is a cofibration and an $f$-local equivalence if and only if the induced map

$$j^* : \mathbf{Hom}(D, X) \to \mathbf{Hom}(C, X)$$

is a trivial fibration for all $f$-local spaces $X$. Part 1) follows immediately.

For part 2), it suffices to show that the map $j \times 1 : C \times K \to D \times K$ is a cofibration and an $f$-local equivalence if the same is true for $j : C \to D$. But the map

$$\mathbf{Hom}(D \times K, X) \xrightarrow{(j\times 1)^*} \mathbf{Hom}(C \times K, X)$$

is isomorphic to

$$\mathbf{Hom}(K, \mathbf{Hom}(D, X)) \xrightarrow{j^*} \mathbf{Hom}(K, \mathbf{Hom}(C, X)),$$

which is a trivial fibration if $X$ is $f$-local and $j$ is a cofibration and $f$-local equivalence. □

PROOF OF THEOREM 2.3: The "trivial cofibration-fibration" factorization is Lemma 2.6. The other part of **CM5** follows from Lemma 2.5 and the axioms for **S** in its usual closed model category structure. The remaining closed model axioms are automatic. Finally, **SM7** follows from Lemma 2.7.2. and Corollary II.3.12. □

We now must prove Lemma 2.6. This argument that we shall give is a sequence of ideas originally due to Bousfield, and then later modified by by J. Smith and Hirschhorn. The argument that we shall give here is the iteration presented in [38]. The central device is the following:

LEMMA 2.8. *Suppose that* $\gamma$ *is an infinite cardinal, and suppose given a diagram of simplicial set maps*

$$\begin{tikzcd} & X \arrow[d, hook, "j"] \\ A \arrow[r, hook] & Y \end{tikzcd}$$

*where* $\#A \leq \gamma$ *and the inclusion* $j : X \hookrightarrow Y$ *is a weak equivalence. Then there is a subcomplex* $B \subset Y$ *containing* $A$*, such that* $\#B \leq \gamma$ *and the inclusion* $j_B : B \cap X \hookrightarrow B$ *is a weak equivalence.*

In the language of [38], Lemma 2.8 is called the "bounded cofibration property" for simplicial set cofibrations. To effect the proof, we need the following method of converting a map to a fibration:

LEMMA 2.9. Suppose that $f : X \to Y$ is a map of simplicial sets. Then there is a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\sigma_f"'] & & Y \\ & Z_f \arrow[ur, "\pi_f"'] & \end{tikzcd}$$

such that $\pi$ is a fibration and $\sigma$ is a weak equivalence. Furthermore, this factorization preserves filtered colimits in $f$.

PROOF: Let $f : X \to Y$ be any simplicial set map, and use Kan's $Ex^\infty$-construction (see III.4) to form the diagram

$$\begin{tikzcd} X \arrow[r, "\nu_X"] \arrow[d, "f"'] & Ex^\infty X \arrow[d, "f_*"] \\ Y \arrow[r, "\nu_Y"'] & Ex^\infty Y \end{tikzcd} \tag{2.10}$$

Note that the $Ex^\infty$ construction commutes with filtered colimits. The next step is to use the dual of Lemma II.8.4 to convert the map $f_* : Ex^\infty X \to Ex^\infty Y$ into a fibration according to the standard classical method — this works because $f_*$ is a map between Kan complexes. In the current context, one forms the diagram

$$\begin{tikzcd}
Ex^{\infty}X \times_{Ex^{\infty}Y} \mathbf{Hom}(\Delta^1, Ex^{\infty}Y) \arrow[r, "\tilde{f}"] \arrow[d, "pr_L"'] & \mathbf{Hom}(\Delta^1, Ex^{\infty}Y) \arrow[r, "d^{1*}"] \arrow[d, "d^{0*}"] & Ex^{\infty}Y \\
Ex^{\infty}X \arrow[r, "f_*"'] & Ex^{\infty}Y &
\end{tikzcd}$$

Write $\pi = d^{1*} \cdot \tilde{f}$, and observe that $pr_L$ has a section $\sigma$ which is induced by the map $\Delta^1 \to \Delta^0$. Then $\pi \cdot \sigma = f_*$, the map $\pi$ is a fibration, and $\sigma$ is a section of the trivial fibration $pr_L$ and is therefore a weak equivalence. Write $\tilde{Z}_f = Ex^{\infty}X \times_{Ex^{\infty}Y} \mathbf{Hom}(\Delta^1, Ex^{\infty}Y)$. Finally, one forms the pullback diagram

$$\begin{tikzcd}
Y \times_{Ex^{\infty}Y} \tilde{Z}_f \arrow[r, "\nu_*"] \arrow[d, "\pi_f"'] & \tilde{Z}_f \arrow[d, "\pi"] \\
Y \arrow[r, "\nu_Y"'] & Ex^{\infty}Y
\end{tikzcd} \tag{2.11}$$

Write $Z_f = Y \times_{Ex^{\infty}Y} \tilde{Z}_f$. The factorization $\pi \cdot \sigma = f_*$ determines a map from diagram (2.10) to diagram (2.11), and hence there is an induced map $\sigma_f : X \to Z_f$ such that $f = \pi_f \cdot \sigma_f$. Furthermore, $\pi$ is a fibration so that $\pi_f$ is a fibration and the map $\nu_*$ is a weak equivalence by properness of the closed model structure for the simplicial set category (Corollary II.8.6). It follows that the map $\sigma_f$ is a weak equivalence.

Note that all constructions here are natural in $f$ and either involve finite limits or filtered colimits, all of which commute with filtered colimits in $f$. □

PROOF OF LEMMA 2.8: Say that a subcomplex $B$ of $Y$ is $\gamma$-bounded if $\#B \leq \gamma$. In the language of Lemma 2.9, we must show that there is a $\gamma$-bounded subcomplex $B \subset Y$ containing $A$ such that the induced fibration $\pi_{j_B}$ is trivial. Note further that $Z_{j_B}$ is $\gamma$-bounded if $B$ is $\gamma$-bounded.

Write $A = B_0$, and consider all diagrams of the form

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & Z_{j_{B_0}} \arrow[r] & Z_j \arrow[d, "\pi_j"] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\theta"] & B_0 \arrow[r] & Y.
\end{tikzcd} \tag{2.12}$$

Observe that the lifting $\theta$ always exists, because $\pi_j$ is a trivial fibration by assumption. The simplicial set $Y$ is a filtered colimit of its $\gamma$-bounded subcomplexes $B$ so that $Z_j = \varinjlim_B Z_{j_B}$, and there are at most $\gamma$ such (solid arrow) diagrams. It follows that there is a $\gamma$-bounded subcomplex $B_1$ of $Y$ such that all such solid arrow lifting problems have solutions in $Z_{j_{B_1}}$. Repeat this construction countably many times to form a sequence of $\gamma$-bounded subcomplexes $A = B_0 \subset B_1 \subset B_2 \subset \dots$, so that all lifting problems of the form (2.12) over $B_i$ are solved over $B_{i+1}$. Then $B = \cup_{i\geq 0} B_i$ is $\gamma$-bounded, and each lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & Z_{j_B} \arrow[d, "\pi_{j_B}"] \\
\Delta^n \arrow[r] & B
\end{tikzcd}$$

factors through a corresponding lifting problem over some $B_i$ and is therefore solved over $B_{i+1}$. □

LEMMA 2.13. *Let $\lambda = 2^\beta$, where $\beta > 2^\gamma$ and $\gamma > \#B$ are the choices of cardinals appearing in the proof of Proposition 1.8 Let $j : X \to Y$ be at once a cofibration and an $f$-local equivalence, and suppose that $A$ is a $\lambda$-bounded subcomplex of $Y$. Then there is a $\lambda$-bounded sub-complex $B \subset Y$ so that $A \subset B$ and $B \cap X \hookrightarrow B$ is an $f$-local equivalence.*

PROOF: We inductively define a chain of $\lambda$-bounded sub-objects $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq Y$ over $\lambda$, and a chain of sub-objects

$$\mathcal{L}(A) = \mathcal{L}(A_0) \subseteq X_1 \subseteq \mathcal{L}(A_1) \subseteq X_2 \subseteq \mathcal{L}(A_2) \subseteq \cdots \mathcal{L}(Y),$$

also over $\lambda$, with the property that

$$\mathcal{L}(X) \cap X_s \to X_s$$

is a weak equivalence. Then we set $B = \varinjlim_{s<\lambda} A_s$ and, by Lemmas 1.19 and 1.24,

$$\begin{aligned} \mathcal{L}(X \cap B) = \mathcal{L}(X) \cap \mathcal{L}(B) &= \varinjlim_{s<\lambda} \mathcal{L}(X) \cap X_s \\ &\to \varinjlim_{s<\lambda} X_s \cong \mathcal{L}(B) \end{aligned}$$

is a weak equivalence as required.

The objects $A_s$ and $X_s$ are defined recursively. Suppose $s + 1$ is a successor ordinal and $A_s$ has been defined. Then, since $A_s$ is $\lambda$-bounded, $\mathcal{L}A_s$ is $\lambda$-bounded by Lemma 1.23. Then Lemma 2.8 implies that there is a $\lambda$-bounded sub-object $X_{s+1} \subseteq \mathcal{L}(Y)$ so that $\mathcal{L}(A_s) \subseteq X_{s+1}$ and $\mathcal{L}(X) \cap X_{s+1} \to X_{s+1}$

is a weak equivalence. Since $\mathcal{L}(Y) = \varinjlim_j \mathcal{L}(Y_j)$ where $Y_j \subseteq Y$ runs over the $\lambda$-bounded sub-objects of $Y$ (Lemma 1.22), there is a $\lambda$-bounded sub-object $A'_{s+1}$ so that $X_{s+1} \subseteq \mathcal{L}(A'_{s+1})$. Let $A_{s+1} = A_s \cup A'_{s+1}$. Finally, suppose $s$ is a limit ordinal. Then set $X_s = \varinjlim_{t<s} \mathcal{L}(A_t) \cong \varinjlim_{t<s} X_t$. The object $X_s$ is $\lambda$-bounded and $\mathcal{L}(X) \cap X_s \to X_s$ is a weak equivalence. Choose $A'_s \subseteq Y$ so that $A'_s$ is $\lambda$-bounded and $X_s \subseteq \mathcal{L}(A'_s)$ and set $A_s = \varinjlim_{t<s} A_t \cup A'_s$. □

LEMMA 2.14. *Suppose that $\lambda$ is the cardinal chosen in Lemma 2.13. A map $q : X \to Y$ in $\mathbf{S}$ is an $f$-local fibration if and only if it has the right lifting property with respect of all morphisms $j : C \to D$ which are at once cofibrations and $f$-local equivalences and so that $\#D \leq \lambda$.*

PROOF: The "only if" implication is clear. For the reverse implication we use a Zorn's Lemma argument. Consider an arbitrary lifting problem

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "j"'] & X \arrow[d, "q"] \\
D \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where $j$ is any cofibration which is also an $f$-local equivalence. We must complete the dotted arrow. Define $\Omega$ to be the set of pairs $(C', g)$ where $C \subseteq C' \subseteq D$ and $C \to C'$ is a $f$-local equivalence, and $g$ solves the lifting problem

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
C' \arrow[r] \arrow[ur, dashed, "g"] & Y.
\end{tikzcd}$$

Define $(C', g) < (C'', h)$ if $C' \subseteq C''$ and $h|_{C'} = g$. Then $\Omega$ satisfies the hypotheses of Zorn's Lemma and, thus, has a maximal element $(C_0, g_0)$. We show $C_0 = D$. Consider the new lifting problem

$$\begin{tikzcd}
C_0 \arrow[r, "g_0"] \arrow[d] & X \arrow[d, "q"] \\
D \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

If $C_0 \neq D$, choose $x \in D$ so that $x \notin C_0$. By the previous lemma, there is a $\lambda$-bounded subobject $D_0 \subseteq D$ so that $x \in D_0$ and $C_0 \cap D_0 \to D_0$ is an $f$-local equivalence. By hypothesis, the restricted lifting problem

$$\begin{tikzcd} C_0 \arrow[r, "g_0"] \arrow[d] & X \arrow[d, "q"] \\ C_0 \cup D_0 \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a solution. Lemma 2.7.1 implies $C_0 \to C_0 \cup D_0$ is an $f$-local equivalence, so we have a contradiction to the maximality of $(C_0, g_0)$. Thus $C_0 = D$ and the proof is complete. □

THE PROOF OF LEMMA 2.6:

We use a Bousfield factorization. Let $J$ be a set of maps containing one representative for each isomorphism class of cofibrations $j : C \to D$ which are $f$-local equivalences and so that $\#D \leq \lambda$. Define

$$\varphi = \bigsqcup_J j : \tilde{C} = \bigsqcup_J C \to \bigsqcup_J D = \tilde{D}$$

and factor $g : X \to Y$ as

$$X \xrightarrow{i} E_\varphi \xrightarrow{q} Y$$

where $i$ is the $\varphi$-cofibration of Proposition 1.8 and $q$ is a $\varphi$-injective.

The class of $\varphi$-cofibrations is the saturation of the class of all cofibrations

$$\tilde{C} \times L \cup_{\tilde{C} \times K} \tilde{D} \times K \to \tilde{D} \times L$$

induced by $\varphi$ and all inclusions $K \hookrightarrow L$ of finite simplicial sets, by Corollary 1.17. Each such map is an $f$-local equivalence, by Lemma 2.7.2, as well as a cofibration. The class of morphisms which are cofibrations and $f$-local equivalences is saturated, by Lemma 2.6.1. It follows that all $\varphi$-cofibrations are $f$-local equivalences. The definition of $\varphi$-injective (see 1.3) implies $q$ has the right lifting property with respect to $\varphi$ and, hence, with respect to all $j : C \to D$ of $J$. Hence the result follows from Lemma 2.14. □

One can also make an argument for Lemma 2.6 directly from Lemma 2.14, by means of a transfinite small object argument.

The reader who has come this far will have noticed that we used Bousfield factorization twice: once to produce the functor $\mathcal{L}(\cdot)$ and once to prove Lemma 2.6. Local objects are produced using $\mathcal{L}(X)$ and fibrant objects by using Lemma 2.6 to factor $X \to *$ as a trivial cofibration followed by a fibration; therefore, they might be different. However, we have:

PROPOSITION 2.15. *A space $X \in \mathbf{S}$ is $f$-local if and only if it is fibrant in the $f$-local model category structure.*

PROOF: Suppose that $j : C \to D$ is a cofibration and an $f$-local equivalence. in particular that the induced map

$$j^* : \mathbf{Hom}(D, X) \to \mathbf{Hom}(C, X)$$

is a trivial fibration for all $f$-local spaces $X$. The map $j^*$ is surjective in degree 0, so that the lifting problem

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "j"'] & X \arrow[d] \\
D \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}$$

can always be solved. Every $f$-local space is therefore fibrant in the $f$-local model structure.

Now suppose that $X$ is fibrant. The map $f : A \to B$ is an $f$-local cofibration by Lemma 1.9, and is therefore an $f$-local equivalence by Lemma 2.1 as well as a cofibration. Each induced map

$$A \times \Delta^n \cup_{A \times \partial\Delta^n} B \times \partial\Delta^n \to B \times \Delta^n$$

is an $f$-local equivalence and a cofibration, by Lemma 2.6, so that the map

$$f^* : \mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X)$$

is a trivial Kan fibration. In particular, the space $X$ is $f$-local. □

We close this section with a preliminary indication of the interaction of the $f$-local homotopy theory with the homotopy theory of cosimplicial spaces.

LEMMA 2.16. *Suppose that $p : X \to Y$ is a map of cosimplicial spaces such that*

(1) *the map $p : X^0 \to Y^0$ is an $f$-fibration, and*
(2) *the canonical maps $(p, s) : X^{n+1} \to Y^{n+1} \times_{M^nY} M^nX$ are $f$-fibrations.*

*Then the induced map $p_* : \mathrm{Tot}(X) \to \mathrm{Tot}(Y)$ is an $f$-fibration of simplicial sets.*

PROOF: The map $p_* : \mathrm{Tot}(X) \to \mathrm{Tot}(Y)$ is an $f$-fibration if the lifting $\theta$ exists in all diagrams of cosimplicial space maps

$$\begin{tikzcd}
\Delta \times C \arrow[r] \arrow[d, "1 \times g"'] & X \arrow[d, "p"] \\
\Delta \times D \arrow[r] \arrow[ur, dashed, "\theta"] & Y
\end{tikzcd}$$

where $g : C \to D$ is a cofibration and an $f$-local equivalence of simplicial sets.

The map $\theta$ is constructed by induction on cosimplicial degree, where $\theta^0 : \Delta^0 \times D \to X^0$ exists by assumption (1). Given compatible lifts $\theta^s : \Delta^s \times D \to X^s$ for $0 \leq s \leq n$, finding $\theta^{n+1}$ amounts to solving a lifting problem

$$\begin{tikzcd}
\partial\Delta^{n+1} \times D \cup_{\partial\Delta^{n+1}\times C} \Delta^{n+1} \times C \arrow[r] \arrow[d, "j"'] & X^{n+1} \arrow[d, "{(p,s)}"] \\
\Delta^{n+1} \times D \arrow[r] \arrow[ur, dashed] & Y^{n+1} \times_{M^nY} M^nX
\end{tikzcd}$$

in the category of simplicial sets. But the map $j$ is a cofibration and an $f$-local equivalence by the simplicial model category axiom **SM7** for the $f$-local structure, so that the lift exists by assumption (2). □

COROLLARY 2.17. *Suppose that $J$ is a small category and that $p : X \to Y$ is a map of $J$-diagrams of simplicial sets which is a pointwise $f$-fibration in the sense that the map $p : X(i) \to Y(i)$ is an $f$-fibration for each object $i$ of $J$. Then the induced map*

$$p_* : \underset{\overleftarrow{J}}{\text{holim}}\, X \to \underset{\overleftarrow{J}}{\text{holim}}\, Y$$

*is an $f$-fibration of simplicial sets.*

PROOF: For the purposes of this proof, we have to be careful to use the standard model for homotopy inverse limit. Write

$$\underset{\overleftarrow{J}}{\text{holim}}\, X = \text{Tot} \underset{\overleftarrow{J}}{\lim}\, T^\bullet X,$$

as in Section VIII.2, and recall that the cosimplicial space $\underset{\longleftarrow J}{\lim}\, T^\bullet X$ can be specified in cosimplicial degree $n+1$ by

$$(\underset{\overleftarrow{J}}{\lim}\, T^\bullet X)^{n+1} = \prod_{i_0 \to \cdots \to i_{n+1}} X(i_{n+1}),$$

where the product in indexed over the $n$-simplices of the nerve $BJ$. Recall further that there is an identification

$$M^n(\underset{\overleftarrow{J}}{\lim}\, T^\bullet X) = \prod_{\substack{i_0 \to \cdots \to i_{n+1} \\ \text{degenerate}}} X(i_{n+1})$$

where the product is indexed over degenerate $(n+1)$-simplices of $BJ$, and the map

$$s : (\underset{\overleftarrow{J}}{\lim}\, T^\bullet X)^{n+1} \to M^n(\underset{\overleftarrow{J}}{\lim}\, T^\bullet X)$$

is a projection.

The map $p_* : \prod_i X(i) \to \prod_i Y(i)$ in cosimplicial degree 0 is an $f$-fibration of simplicial sets, so that condition (1) of Lemma 2.16 is satisfied. The map $(p, s)$ of condition (2) can be identified up to isomorphism with the product of the map

$$p_* : \prod_{\substack{i_0 \to \cdots \to i_{n+1} \\ \text{non-degenerate}}} X(i_{n+1}) \to \prod_{\substack{i_0 \to \cdots \to i_{n+1} \\ \text{non-degenerate}}} Y(i_{n+1})$$

with the identity map on the product

$$\prod_{\substack{i_0 \to \cdots \to i_{n+1} \\ \text{degenerate}}} X(i_{n+1}).$$

Condition (2) of Lemma 2.16 is therefore satisfied. □

## 3. Bousfield localization.

Let **S** be the category of simplicial sets and let $E_*$ a generalized homology theory which satisfies the limit axiom in the sense that it preserves filtered colimits. A space $Z \in \mathbf{S}$ is $E_*$-local if and only if $Z$ is fibrant, and any diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] & Z \\ Y \arrow[ur, dashed] & \end{tikzcd}$$

with $E_*g$ an isomorphism can be completed uniquely up to homotopy. The $E_*$-localization of a space $X$ is a map $\eta : X \to Z$ with $Z$ $E_*$-local and $E_*\eta$ an isomorphism. One easily checks that if such a localization exists, it is unique up to homotopy. The existence hinges on the existence of an appropriate Bousfield factorization and, hence, is best described under the rubric of the previous two sections.

DEFINITION 3.1. *A morphism $g : X \to Y$ in* **S** *is said to be*

1) *an $E_*$-equivalence if $E_*g$ is an isomorphism,*
2) *an $E_*$-cofibration if it is a cofibration in* **S**, *and*
3) *an $E_*$-fibration if it has the right lifting property with respect to all $E_*$-trivial cofibrations.*

Then one has

THEOREM 3.2. *With these definitions and its usual simplicial structure,* **S** *becomes a simplicial model category.*

This will be proved below.

COROLLARY 3.3.

1) *A space $Z \in \mathbf{S}$ is $E_*$-local if and only if it is $E_*$-fibrant.*
2) *Every space $X \in \mathbf{S}$ has an $E_*$-localization.*

PROOF: For 1) first assume $Z$ is $E_*$-fibrant and factor any $E_*$-equivalence $X \to Y$ as

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where $q$ is a trivial fibration and $i$ is a cofibration. Then $i$ will be an $E_*$-trivial cofibration, so $\mathbf{Hom}(W, Z) \to \mathbf{Hom}(X, Z)$ is a weak equivalence. The trivial fibration $q$ has a section $\sigma : Y \to W$ which is a trivial cofibration, so $\mathbf{Hom}(Y, Z) \to \mathbf{Hom}(W, Z)$ is a weak equivalence. It follows that $Z$ is $E_*$-local. Conversely, if $Z$ is $E_*$-local and $X \to Y$ is an $E_*$-trivial cofibration, then we need to solve any lifting problem

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Z \\
Y \arrow[ur, dashed] &
\end{tikzcd}$$

Since $Z$ is $E_*$ local it can be solved up to homotopy. Since $Z$ is fibrant, the original lifting problem can be solved by appropriate use of the homotopy extension property.

For 2), simply factor the unique map $X \to *$ as

$$X \xrightarrow{j} Z \xrightarrow{q} *$$

where $j$ is a trivial $E_*$-cofibration and $q$ is an $E_*$-fibration. □

COROLLARY 3.4. *Let $j : X \to Z$ be an $E_*$-localization of $X$. Then the map $j$ is, up to homotopy, both the terminal $E_*$-equivalence out of $X$ and the initial map to an $E_*$-local space.*

PROOF: Both statements follow from the definition of localization. □

The difficulty in proving Theorem 3.3 arises in verifying the "trivial cofibration-fibration" factoring axiom. This is done by using Bousfield factorization for a particular map $f : A \to B$ in $\mathbf{S}$. The next three lemmas construct and identify the map $f$. Choose an infinite cardinal $\beta$ greater than the cardinality of $E_*$ — that is, greater than the cardinality of the set $\bigsqcup_n E_n(*)$.

LEMMA 3.5. *Let $g : C \to D$ be a cofibration and an $E_*$ equivalence. Let $x \in D$ be a simplex. Then there is a subcomplex $D_0 \subseteq D$ so that $x \in D$, $\#D_0 \leq \beta$, and*

$$C \cap D_0 \to D_0$$

*is an $E_*$ equivalence.*

PROOF: We use that for any simplicial set $Z$, the natural map

$$\varinjlim_{\alpha} E_* Z_\alpha \to E_* Z$$

is an isomorphism, where $Z_\alpha \subseteq Z$ runs over the sub-simplicial sets with finitely many non-degenerate simplices.

We recursively define a sequence of sub-simplicial sets

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq D$$

with the properties that $x \in K_0$, $\#K_s \leq \beta$, and

$$\widetilde{E}_*(K_n/K_n \cap C) \to \widetilde{E}_*(K_{n+1}/K_{n+1} \cap C)$$

is the zero map. Then we can set $D_0 = \bigcup K_n$.

For $K_0$, choose any sub-simplicial set with $\#K_0$ finite and $x \in K_0$. Having defined $K_n$, produce $K_{n+1}$ as follows. For each $y \in \widetilde{E}_*(K_n/K_n \cap C)$ there is a finite sub-complex $Z_y \subseteq D/C$ so that $y$ maps to zero in $\widetilde{E}_*((K_n/K_n \cap C) \cup Z_y)$. Choose a finite sub-complex $Y_y \subseteq D$ that maps onto $Z_y$ and then let $K_{n+1} = K_n \cup \bigcup_y Y_y$. The collection of all such elements $y$ has cardinality bounded above by $\beta$. □

LEMMA 3.6. *A morphism $q : Z \to W$ in **S** is an $E_*$ fibration if and only if it has the right lifting property with respect to all $E_*$-trivial cofibrations $X \to Y$ with $\#Y \leq \beta$.*

PROOF: Use the same Zorn's lemma argument as for Lemma 2.14. □

THE PROOF OF THEOREM 3.2: As in Lemma 2.5, a morphism is at once an $E_*$-fibration and an $E_*$-equivalence if and only if it is a trivial fibration. Then all but the "trivial cofibration-fibration" factorization axiom follow immediately. For the final factorizations, let $J$ be a set of $E_*$-trivial cofibrations $j_\alpha : C_\alpha \to D_\alpha$ containing one representative of each isomorphism class of $E_*$ trivial cofibrations $j : C \to D$ with $\#D \leq \beta$. Let

$$f = \bigsqcup j_\alpha : \bigsqcup C_\alpha \to \bigsqcup D_\alpha$$

with the coproduct over $J$. Fix a morphism $X \to Y$ in **S** and let

$$X \to E_f \to Y$$

be the Bousfield Factorization with respect to $f$. Then the Mayer-Vietoris sequence and the main construction 1.16 implies $X \to E_f$ is an $E_*$ trivial cofibration and, using Lemma 3.6 and arguing as in Proposition 1.8, one sees $E_f \to Y$ is an $E_*$-fibration.

To prove **SM7** we know it is sufficient to show that if $C \to D$ is an $E_*$ trivial cofibration and $K \to L$ is any cofibration, then

$$D \times K \cup_{C \times K} C \times L \to D \times L$$

is a trivial $E_*$-cofibration. Use the Mayer-Vietoris sequence. □

REMARK 3.7. Lemma 2.16 implies that the Bousfield-Kan $R$-completion $R^{\infty}X$ is $H_*(\cdot, R)$-local. To see this, note that the spaces $R^nX$ are simplicial $R$-modules and are therefore $H_*(\cdot, R)$-local, and all maps $s : R^{n+1}X \to M^nR^{\bullet}X$ are surjective simplicial $R$-module homomorphisms, and are therefore $H_*(\cdot, R)$-fibrations. It follows in particular that if a space $X$ is "$R$-good" (see [14]) in the sense that the canonical map $X \to R^{\infty}X$ induces an isomorphism in $H_*(\cdot, R)$, then $R^{\infty}X$ is a model for the $H_*(\cdot, R)$-localization of $X$. Warning: not all spaces are $R$-good, but the class of $R$-good spaces includes all nilpotent spaces [14, p.134]. See also Theorem VIII.3.7.

## 4. A model for the stable homotopy category.

We have so far concentrated on the concept of localization at a cofibration or set of cofibrations. This is not the only method for constructing localization theories: there are also theories which arise from classes of cofibrations satisfying certain axioms — this is the approach taken in [38], and it completely subsumes the material presented here in cases where the ambient closed model category admits cardinality arguments. There is another, older approach, which involves localizing at a sufficiently well-behaved functor $Q : \mathcal{C} \to \mathcal{C}$ from a proper simplicial model category to itself. This technique is due to Bousfield and Friedlander [12], and is the basis for their approach to constructing the stable homotopy category. We shall present this method here.

Suppose that $\mathcal{C}$ is a proper closed (simplicial) model category, and let $Q : \mathcal{C} \to \mathcal{C}$ be a functor. Suppose further that there is a natural transformation $\eta_X : X \to Q(X)$ from the identity functor on $\mathcal{C}$ to $Q$. Say that a map $f : X \to Y$ is a *$Q$-weak equivalence* if the induced map $Q(f) : Q(X) \to Q(Y)$ is a weak equivalence of $\mathcal{C}$, and say that a map is a *$Q$-fibration* if it has the right lifting property with respect to all cofibrations of $\mathcal{C}$ which are $Q$-weak equivalences.

We shall require that the functor $Q$ and the natural map $\eta$ together satisfy the following list of properties:

**A1** The functor $Q$ preserves weak equivalences.

**A2** The maps $\eta_{QX}, Q(\eta_X) : Q(X) \to Q^2(X)$ are weak equivalences of $\mathcal{C}$.

**A3** The class of $Q$-weak equivalences is closed under pullback along $Q$-fibrations: given maps

$$B \xrightarrow{g} Y \xleftarrow{p} X$$

with $p$ a $Q$-fibration and $g$ a $Q$-weak equivalence, the induced map $g_* : B \times_Y X \to X$ is a $Q$-weak equivalence. Dually, the class of $Q$-weak equivalences is closed under pushout along cofibrations of $\mathcal{C}$.

The first major result of this section is the following:

THEOREM 4.1. *Suppose that $\mathcal{C}$ is a proper closed model category, $Q : \mathcal{C} \to \mathcal{C}$ is a functor, and $\eta_X : X \to Q(X)$, $x \in \mathrm{Ob}(\mathcal{C})$, is a natural transformation, all* *satisfying the properties* **A1**–**A3** *above. Then there is a closed model structure on $\mathcal{C}$ for which the weak equivalences are the $Q$-weak equivalences, the cofibrations are the cofibrations of $\mathcal{C}$, and the fibrations are the $Q$-fibrations.*

We shall refer to the closed model structure of Theorem 4.1 as the *$Q$-structure* on the category $\mathcal{C}$. Note first that this result implies that there is an idempotent functor $G : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ in the sense of Adams, and hence an associated categorical localization theory (see [8, p.135]):

COROLLARY 4.2. *Suppose that the conditions for Theorem 4.1 hold. Then there is a functor $G : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ and a natural transformation $j : 1_{Ho(\mathcal{C})} \to G$ such that $j_{GX} = G(j_X)$, and $j_{GX}$ is an isomorphism of $Ho(\mathcal{C})$ for all $X$.*

PROOF: It suffices to work on the level of objects of $\mathcal{C}$ which are fibrant and cofibrant for the original closed model structure on $\mathcal{C}$, so that the morphisms of $\mathrm{Ho}(\mathcal{C})$ can be identified with homotopy classes of maps.

Choose a map $j_X : X \to GX$ such that $j_X$ is a cofibration and a $Q$-weak equivalence, and $GX$ is $Q$-fibrant. Every trivial cofibration of $\mathcal{C}$ is a $Q$-weak equivalence, by **A1**, so $GX$ is fibrant in the original structure on $\mathcal{C}$.

If $f : X \to Y$ is a map such that $Y$ is $Q$-fibrant, then there is an extension $f_* : GX \to Y$ such that $f_* \cdot j_X = f$. Furthermore, the choice of $f_*$, up to homotopy, depends only on the homotopy class of $f$. It follows that any map $f : X \to Y$ determines an extension $Gf : GX \to GY$ such that $Gf \cdot j_X = j_Y \cdot f$, so this construction induces a functor $G : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$, and a natural transformation $j : 1_{\mathrm{Ho}(\mathcal{C})} \to G$. The maps $j_{GX}$ and $Gj_X$ are two extensions of a common map $X \to G^2X$, so they coincide up to homotopy, and the map $Gj_X : GX \to G^2X$ is a $Q$-weak equivalence between objects which are $Q$-fibrant and cofibrant, so $Gj_X$ is a homotopy equivalence. □

The proof of Theorem 4.1 will be given in a series of lemmas.

LEMMA 4.3. *Assume the conditions of Theorem 4.1. Then a map $f : X \to Y$ is a fibration and a weak equivalence of $\mathcal{C}$ if and only if it is a $Q$-weak equivalence and a $Q$-fibration.*

PROOF: If $f$ is a fibration and a weak equivalence of $\mathcal{C}$, it is a $Q$-weak equivalence by **A1** and has the right lifting property with respect to all cofibrations, so it's a $Q$-fibration.

Suppose that $f$ is a $Q$-weak equivalence and a $Q$-fibration, and take a factorization $f = p \cdot i$, where where $i : X \to Z$ is a cofibration and $p : Z \to Y$ is a trivial fibration in $\mathcal{C}$. Then $i$ is a $Q$-weak equivalence by **A1**, so the lifting $\theta$ exists in the diagram

$$\begin{tikzcd}
X \arrow[r, "="] \arrow[d, "i"] & X \arrow[d, "f"] \\
Z \arrow[r, "p"'] \arrow[ur, "\theta"'] & Y
\end{tikzcd}$$

and so $f$ is a retract of $p$. □

Lemma 4.4. *Suppose that $f : X \to Y$ is a fibration of $\mathcal{C}$, and that $\eta_X : X \to QX$ and $\eta_Y : Y \to QY$ are weak equivalences of $\mathcal{C}$. Then $f$ is a $Q$-fibration.*

Proof: Start with a lifting problem

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
B \arrow[r, "\beta"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where $i$ is a cofibration and a $Q$-weak equivalence. We have to show that the dotted arrow exists.

There is a diagram

$$\begin{tikzcd}
QA \arrow[rr, "Q\alpha"] \arrow[dr, "j_\alpha"'] \arrow[dd, "Qi"'] & & QX \arrow[dd, "Qf"] \\
& Z \arrow[ur, "p_\alpha"'] \arrow[dd, "\pi"] & \\
QB \arrow[rr] \arrow[dr, "j_\beta"'] & & QY \\
& W \arrow[ur, "p_\beta"'] &
\end{tikzcd}$$

where $j_\alpha$ and $j_\beta$ are trivial cofibrations of $\mathcal{C}$ and $p_\alpha$ and $p_\beta$ are fibrations. To see this, find the factorization $Q\beta = p_\beta j_\beta$, and then factorize the induced map $QA \to W \times_{QY} QX$ as a trivial cofibration $j_\alpha$ followed by a fibration. There is an induced diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & Z \times_{QX} X \arrow[r] \arrow[d, "\pi_*"'] & X \arrow[d, "f"] \\
B \arrow[r] & W \times_{QY} Y \arrow[r] & Y
\end{tikzcd}$$

such that the top horizontal composite is $\alpha$ and the bottom composite is $\beta$.

We will show that the conditions on $f$ and $i$ imply that the map $\pi_*$ is a weak equivalence of $\mathcal{C}$. This suffices, for then $\pi_*$ has a factorization $\pi_* = q \cdot j$

where $q$ is a trivial fibration and $j$ is a trivial cofibration, and there is a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "i"'] & Z \times_{QX} X \arrow[r] \arrow[d, "j"'] & X \arrow[dd, "f"] \\
 & U \arrow[ur, "\omega"] \arrow[d, "q"] & \\
B \arrow[r] \arrow[ur, "\theta"] & W \times_{QY} Y \arrow[r] & Y
\end{tikzcd}$$

so that $\omega \cdot \theta$ is the desired lift.

There is finally a diagram

$$\begin{tikzcd}
QA \arrow[r, "j_\alpha"] \arrow[d, "Qi"'] & Z \arrow[d, "\pi"'] & Z \times_{QX} X \arrow[l, "pr"'] \arrow[d, "\pi_*"] \\
QB \arrow[r, "j_\beta"'] & W & W \times_{QY} Y \arrow[l, "pr"]
\end{tikzcd}$$

The map $Qi$ is a weak equivalence since $i$ is a $Q$-weak equivalence, so that $\pi$ is a weak equivalence. The maps $\eta_X$ and $\eta_Y$ are weak equivalences by assumption and $p_\alpha$ and $p_\beta$ are fibrations, so the maps labelled $pr$ are weak equivalences by properness for $\mathcal{C}$. It follows that $\pi_*$ is a weak equivalence of $\mathcal{C}$. □

Lemma 4.5. *Any map $f : QX \to QY$ has a factorization $f = p \cdot i$ where $p : Z \to QY$ is a $Q$-fibration and $i : QX \to Z$ is a cofibration and a $Q$-weak equivalence.*

Proof: Take a factorization $f = p \cdot i$ where $p : Z \to QY$ is a fibration and $i : QX \to Z$ is a cofibration and a weak equivalence. Applying the functor $Q$ to this diagram, one sees that $\eta_{QX}$ is a weak equivalence by **A2**, and $i$ is a weak equivalence so that $Qi$ is a weak equivalence by **A1**. It follows that $\eta_Z : Z \to QZ$ is a weak equivalence, as is $\eta_{QY}$, so that $p$ is a $Q$-fibration by Lemma 4.4. □

Lemma 4.6. *Any map $f : X \to Y$ of $\mathcal{C}$ has a factorization $f = q \cdot j$ where $q : Z \to Y$ is a $Q$-fibration and $j : X \to Z$ is a cofibration and a $Q$-weak equivalence.*

Proof: Take the factorization $f = p \cdot i$ of Lemma 4.5 and pull $\eta_Y : Y \to QY$ along $p$ to give a diagram

$$\begin{tikzcd}
X \arrow[r, "i_*"] \arrow[d, "\eta_X"'] & Z \times_{QY} Y \arrow[r, "p_*"] \arrow[d, "\eta_*"'] & Y \arrow[d, "\eta_Y"] \\
QX \arrow[r, "i"'] & Z \arrow[r, "p"'] & QY,
\end{tikzcd}$$

where $p_* \cdot i_* = f$. Then $\eta_*$ is a $Q$-weak equivalence by **A2** and **A3**, as are both $\eta_X$ and $i$. It follows that $i_*$ is a $Q$-weak equivalence.

Now factorize $i_*$ in $\mathcal{C}$ as $i_* = \pi \cdot j$, where $j$ is a cofibration and $\pi$ is a trivial fibration. Then $\pi$ is a $Q$-fibration by Lemma 4.3, so that $q = p_* \cdot \pi$ is a $Q$-fibration. □

PROOF OF THEOREM 4.1: The closed model axioms **CM1**–**CM3** are trivial to verify. One part of the factorization axiom is Lemma 4.6. The other is a consequence of Lemma 4.3, as is **CM4**. □

REMARK 4.7. Observe that if $\mathcal{C}$ has a simplicial model structure in addition to the conditions of Theorem 4.1, then the $Q$-structure on $\mathcal{C}$ is a simplicial model structure, essentially for free. Axiom **SM7** is a consequence of the fact that the two structures have the same cofibrations, and every trivial cofibration of $\mathcal{C}$ is a $Q$-weak equivalence.

Similarly, the full statement of **A3** (we've only used the fibration part of it up to now) implies that the $Q$-structure is proper.

Here's the second major result:

THEOREM 4.8. *Suppose that the proper closed model category $\mathcal{C}$ and the functor $Q$ together satisfy the conditions for Theorem 4.1. Then a map $f : X \to Y$ is a $Q$-fibration if and only if it is a fibration and the square*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & QX \arrow[d, "Qf"] \\ Y \arrow[r, "\eta_Y"'] & QY \end{tikzcd} \tag{4.9}$$

*is a homotopy cartesian diagram of $\mathcal{C}$.*

PROOF: Suppose that the diagram (4.9) is homotopy cartesian and that $f : X \to Y$ is a fibration of $\mathcal{C}$. We will show that the map $f$ is a retract of a $Q$-fibration.

Factorize $Qf$ as $Qf = p \cdot i$, where $p : Z \to QY$ is a fibration of $\mathcal{C}$ and $i : QX \to Z$ is a trivial cofibration. Then, as in the proof of Lemma 4.5, the map $p$ is a $Q$-fibration, so that the induced map $p_* : Y \times_{QY} Z \to Y$ is a $Q$-fibration. The induced map $i_* : X \to Y \times_{QY} Z$ is a weak equivalence of $\mathcal{C}$, since $\mathcal{C}$ is proper, and so it has a factorization $i_* = \pi \cdot j$, where $j : X \to W$ is a trivial cofibration and $\pi : W \to Y \times_{QY} Z$ of $\mathcal{C}$ is a trivial fibration of $\mathcal{C}$. Then the composite $p_*\pi : W \to Y$ is a $Q$-fibration by Lemma 4.3, and the lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, "j"'] & X \arrow[d, "f"] \\ W \arrow[r, "p_*\pi"'] \arrow[ur] & Y \end{tikzcd}$$

since $f$ is a fibration. It follows that $f$ is a retract of $p_*\pi$.

For the converse, suppose that $f : X \to Y$ is a $Q$-fibration. We show that $f$ is a retract of a fibration $g$ for which the diagram

$$\begin{tikzcd} Z \arrow[r, "\eta_Z"] \arrow[d, "g"'] & QZ \arrow[d, "Qg"] \\ Y \arrow[r, "\eta_Y"'] & QY \end{tikzcd} \tag{4.10}$$

is homotopy cartesian in $\mathcal{C}$.

Observe that the class of maps $g$ which are fibrations of $\mathcal{C}$ and for which the diagram (4.10) is homotopy cartesian is closed under composition, and includes all trivial fibrations of $\mathcal{C}$ by **A1**. It suffices therefore, with respect to the construction giving the first part of the proof, to show that the map $p_*$ is a candidate for one of these maps $g$.

The component square diagrams in

$$\begin{tikzcd} Y \times_{QY} Z \arrow[r, "\eta_*"] \arrow[d] & Z \arrow[r, "\eta_Z"] \arrow[d, "p"'] & QZ \arrow[d, "Qp"] \\ Y \arrow[r, "\eta_Y"'] & QY \arrow[r, "\eta_{QY}"'] & Q^2Y \end{tikzcd}$$

are homotopy cartesian since $\eta_Z$ and $\eta_{QY}$ are weak equivalences (see the proof of Lemma 4.5), so that the composite square is homotopy cartesian. This composite coincides with the composite of the squares

$$\begin{tikzcd} Y \times_{QY} Z \arrow[r, "\eta"] \arrow[d, "p_*"'] & Q(Y \times_{QY} Z) \arrow[r, "Q\eta_*"] \arrow[d, "Qp_*"'] & QZ \arrow[d, "Qp"] \\ Y \arrow[r, "\eta_Y"'] & QY \arrow[r, "Q\eta_Y"'] & QQY \end{tikzcd} \tag{4.11}$$

The map $\eta_*$ is a $Q$-weak equivalence by **A3**, so that $Q\eta_*$ is a weak equivalence, as is $Q\eta_Y$. It follows that the square on the left in (4.11) is homotopy cartesian, but this is what we had to prove. □

COROLLARY 4.12. *Suppose that the proper closed model category $\mathcal{C}$ and the functor $Q$ together satisfy the conditions for Theorem 4.1. Then an object $X$ of $\mathcal{C}$ is $Q$-fibrant if and only if it is fibrant in $\mathcal{C}$ and the map $\eta_X : X \to QX$ is a weak equivalence of $\mathcal{C}$.*

The main application of this theory is the Bousfield-Friedlander construction of the stable homotopy category.

A *spectrum* $X$ (or rather a spectrum object in simplicial sets) consists of pointed simplicial sets $X^n$, $n \geq 0$, together with pointed simplicial set maps $\sigma : S^1 \wedge X^n \to X^{n+1}$, which we call *bonding maps.* Here, $S^1 = \Delta^1/\partial\Delta^1$ is the simplicial circle. A *map of spectra* $f : X \to Y$ consists of pointed simplicial set maps $f : X^n \to Y^n$, $n \geq 0$, which respect structure in the sense that all diagrams

$$\begin{tikzcd}
S^1 \wedge X^n \arrow[r, "\sigma"] \arrow[d, "S^1 \wedge f"'] & X^{n+1} \arrow[d, "f"] \\
S^1 \wedge Y^n \arrow[r, "\sigma"'] & Y^{n+1}
\end{tikzcd}$$

commute. The resulting category of spectra will be denoted by **Spt**.

A map $f : X \to Y$ of spectra is said to be a *strict weak equivalence* (respectively *strict fibration*) if all of the component maps $f : X^n \to Y^n$ are weak equivalences (respectively fibrations) of simplicial sets. A map $j : A \to B$ of spectra is said to be a *cofibration* if the following two conditions are satisfied:

(1) the map $i : A^0 \to B^0$ is a cofibration (or monomorphism) of simplicial sets, and

(2) each induced map

$$S^1 \wedge B^n \cup_{S^1 \wedge A^n} A^{n+1} \to B^{n+1}$$

is a cofibration of simplicial sets.

PROPOSITION 4.13. *With the definitions of strict fibration, strict weak equivalence, and cofibration given above, the category* **Spt** *satisfies the axioms for a proper simplicial model category.*

It follows from the definition of cofibration that a spectrum $X$ is cofibrant if and only if the bonding maps $S^1 \wedge X^n \to X^{n+1}$ are all cofibrations. Notice further that if $i : A \to B$ is a cofibration of spectra, then all of the maps $i : A^n \to B^n$ are cofibrations of simplicial sets, but the converse may not be true. We usually say that a map $j : C \to D$ of spectra for which all the maps $j : C^n \to D^n$ are cofibrations is a *pointwise cofibration.* In the same terminology, strict fibrations are pointwise fibrations and strict weak equivalences are pointwise weak equivalences.

REMARK 4.14. To see the simplicial structure for **Spt** a little more clearly, note that if $X$ is a spectrum and $K$ is a pointed simplicial set, then there is a spectrum $X \wedge K$ with $(X \wedge K)^n = X^n \wedge K$, and having bonding maps of the form

$$S^1 \wedge X^n \wedge K \xrightarrow{\sigma \wedge K} X^{n+1} \wedge K.$$

Then, for an arbitrary simplicial set $L$ and a spectrum $X$, $X \otimes L = X \wedge L_+$, where $L_+ = L \sqcup *$ is $L$ with a disjoint base point attached. Dually, the pointed function complex spectrum $\mathbf{hom}_*(K, X)$ (denoted by $\mathbf{hom}_{\mathbf{S}_*}(K, X)$ in Section II.2) is the spectrum with

$$\mathbf{hom}_*(K, X)^n = \mathbf{hom}_*(K, X^n),$$

with bonding map $S^1 \wedge \mathbf{hom}_*(K, X^n) \to \mathbf{hom}_*(K, X^{n+1})$ adjoint to the composite

$$S^1 \wedge \mathbf{hom}_*(K, X^n) \wedge K \xrightarrow{S^1 \wedge ev} S^1 \wedge X^n \xrightarrow{\sigma} X^{n+1}$$

It follows that the evaluation maps $ev : \mathbf{hom}_*(K, X^n) \wedge K \to X^n$ determine a map of spectra $ev : \mathbf{hom}_*(K, X) \wedge K \to X$.

Given a spectrum $X$, there is a spectrum $\Sigma X$ having $(\Sigma X)^n = S^1 \wedge X^n$ and with bonding maps

$$S^1 \wedge S^1 \wedge X^n \xrightarrow{S^1 \wedge \sigma} S^1 \wedge X^{n+1}.$$

This spectrum $\Sigma X$ *is not* the suspension object $X \wedge S^1$ arising from the simplicial structure on **Spt**: the two differ by a twist of circle smash factors — this fact is an avatar of one of the standard dangerous bends in the foundations of stable homotopy theory. We say that $\Sigma X$ is the *fake suspension spectrum* of $X$.

Similarly, there is a *fake loop spectrum* $\Omega Y$, which does not coincide with the function complex object $\mathbf{hom}_*(S^1, Y)$. In effect, $(\Omega Y)^n = \mathbf{hom}_*(S^1, Y^n)$, but the bonding map is the composite

$$S^1 \wedge \mathbf{hom}_*(S^1, Y^n) \underset{\cong}{\xrightarrow{\tau}} \mathbf{hom}_*(S^1, Y^n) \wedge S^1 \xrightarrow{ev} Y^n \xrightarrow{\sigma_*} \mathbf{hom}_*(S^1, Y^{n+1}),$$

where $\sigma_* : Y^n \to \mathbf{hom}_*(S^1, Y^{n+1})$ is adjoint to the composite

$$Y^n \wedge S^1 \underset{\cong}{\xrightarrow{\tau}} S^1 \wedge Y^n \xrightarrow{\sigma} Y^{n+1}.$$

The fake loop construction is right adjoint to the fake suspension.

The closed model structure determined by Proposition 4.13 is usually called the *strict closed model structure* on the category of spectra. The associated homotopy category is not yet the stable category. We stabilize by localizing at a suitable functor $Q$.

The construction of the functor $Q : \mathbf{Spt} \to \mathbf{Spt}$ requires a natural strictly fibrant model. There are both geometric and combinatorial ways of producing such an object, corresponding to the two standard methods of functorially replacing simplicial set by a Kan complex. For the geometric method, observe that there is a natural map

$$S|K| \wedge S|L| \to S|K \wedge L|$$

for each pair of pointed simplicial sets $K$ and $L$. Given a spectrum $X$, the bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$ therefore induce composites

$$S^1 \wedge S|X^n| \xrightarrow{\eta\wedge 1} S|S^1| \wedge S|S^n| \to S|S^1 \wedge X^n| \xrightarrow{S|\sigma|} S|X^{n+1}|,$$

so that applying realization and singular functors levelwise to a spectrum $X$ determines a spectrum $S|X|$. It's not hard to see that the canonical weak equivalences $\eta : X^n \to S|X^n|$ collectively determine a natural strict equivalence of spectra $\eta : X \to S|X|$. Of course, $S|X|$ is strictly fibrant. Kan's $Ex^\infty$ construction can be goaded into performing a similar service: the natural composite

$$K \times Ex^\infty L \xrightarrow{\nu\wedge 1} Ex^\infty K \times Ex^\infty L \cong Ex^\infty(K \times L)$$

induces a natural map

$$\nu_* : K \wedge Ex^\infty L \to Ex^\infty(K \wedge L)$$

which, in the presence of a spectrum $X$ determines composites

$$S^1 \wedge Ex^\infty X^n \to Ex^\infty(S^1 \wedge X^n) \xrightarrow{Ex^\infty \sigma} Ex^\infty X^{n+1}.$$

Furthermore, the canonical maps $\nu : X^n \to Ex^\infty X^n$ together determine a natural strict equivalence $\nu : X \to Ex^\infty X$, taking values in a strictly fibrant spectrum. Observe that $Ex^\infty X$ and $S|X|$ are naturally strictly equivalent.

The indices in spectra can be shifted at will: if $X$ is a spectrum and $n$ is an integer, there is a spectrum $X[n]$ with

$$X[n]^k = \begin{cases} X^{n+k} & \text{if } n+k \geq 0, \text{ and} \\ * & \text{if } n+k < 0. \end{cases}$$

Shifting indices is functorial, and is cumulative in the sense that there are canonical natural isomorphisms $X[n][k] \cong X[n+k]$.

The bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$ of a fixed spectrum $X$ determine a map of spectra $\sigma : \Sigma X \to X[1]$. This map is natural in $X$, and has a natural adjoint $\sigma_* : X \to \Omega X[1]$, which is defined levelwise by the (twisted) adjoints $\sigma_* : X^n \to \mathbf{hom}_*(S^1, X^{n+1})$ of the maps $\sigma : S^1 \wedge X^n \to X^{n+1}$. This construction can be repeated, to form an inductive system of maps

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega\sigma_*[1]} \Omega^2 X[2] \xrightarrow{\Omega^2\sigma_*[2]} \dots .$$

Write $\Omega^\infty X = \varinjlim_n \Omega^n X[n]$

The functor $Q : \mathbf{Spt} \to \mathbf{Spt}$ is defined for spectra $X$ by $QX = \Omega^\infty \operatorname{Ex}^\infty X$, and there is a natural map $\eta_X : X \to QX$ given by the composite

$$X \xrightarrow{\nu} Ex^\infty X \xrightarrow{\tau} \Omega^\infty Ex^\infty X.$$

where $\tau : Y \to \Omega^\infty Y$ denotes the canonical map to the colimit.

Say that a map $f : X \to Y$ of spectra is a *stable equivalence* if it induces a strict equivalence $Qf : QX \to QY$. The map $p : Z \to W$ is said to be a *stable fibration* if it has the right lifting property with respect to all maps which are cofibrations and stable equivalences.

THEOREM 4.15. *With these definitions, the category* **Spt** *of spectra, together with cofibrations, stable equivalences and stable fibrations satisfies the axioms for a proper simplicial model category.*

PROOF: We need only verify that the functor $Q$ satisfies axioms **A1**–**A3**. Of these, **A1** is clear, and **A2** is a consequence of the observation that

$$\Omega^\infty \tau_Y = \tau_{\Omega^\infty Y} : \Omega^\infty Y \to \Omega^\infty \Omega^\infty Y$$

is an isomorphism by a cofinality argument. The functor $Q$ preserves strict fibrations and pullbacks, giving the pullback part of **A3**. Recall from the proof of Theorem 4.1 that this is enough to give the desired simplicial model structure.

Suppose given maps of spectra

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "j"'] & C \\ B & \end{tikzcd}$$

where $j$ is a cofibration and $g$ is a stable equivalence. To finish verifying **A3**, we have to show that the induced map $g_* : B \to B \cup_A C$ is a stable equivalence. Construct a diagram

$$\begin{tikzcd} & & \tilde{C} \arrow[d, "\pi_C"] \\ B & A \arrow[l, "j"] \arrow[r, "g"'] \arrow[ur, "\tilde{g}"] & C \end{tikzcd}$$

where $\tilde{g}$ is a cofibration, and the map $\pi_C$ is a strict fibration and a strict weak equivalence. The gluing lemma for simplicial sets implies that the induced map $\pi_* : B \cup_A \tilde{C} \to B \cup_A C$ is a strict weak equivalence. It is therefore enough to show that the induced map $\tilde{g}_* : B \to B \cup_A \tilde{C}$ is a stable equivalence. But $\tilde{g}$ is a stable equivalence as well as a cofibration, and such maps are closed under pushout. □

REMARKS 4.16.

1) In view of the coincidence of the homotopy groups of a space with the simplicial homotopy groups of its associated singular complex (Proposition I.11.1), the definition of $QX$ implies that the homotopy groups $\pi_k QX^n$ coincide up to isomorphism with the stable homotopy groups $\pi_{k-n}|X|$ of the (pre)spectrum $|X|$, so that $Q$-weak equivalence is stable equivalence in the traditional sense: a map $f : X \to Y$ of spectra is a $Q$-weak equivalence if and only if it induces an isomorphism in all stable homotopy groups.

2) Corollary 4.12 implies that a spectrum $X$ is stably fibrant if and only if $X$ is strictly fibrant and all associated adjoints $\sigma_* : X^n \to \mathbf{hom}_*(S^1, X^{n+1})$ of the bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$ are weak equivalences. In other words, a stably fibrant spectrum $X$ is a type of $\Omega$-*spectrum*, and one commonly lapses into describing it that way.

Write $\mathbf{Spt}^Q$ for the closed model structure on the category of spectra of Theorem 4.15. The associated homotopy category $\mathrm{Ho}(\mathbf{Spt}^Q)$ is a model for the stable category, in view of the above remarks. This is far from being the only (or even best) construction of the stable category, but it has the advantage of giving descriptions of stable homotopy theory in a variety of non-standard situations [52]. There is even a collection of ways to describe the closed model structure $\mathbf{Spt}^Q$: one could, for example, localize $\mathbf{Spt}$ at the family of stably trivial cofibrations, as is done in [38].

# REFERENCES

[1] J.F. Adams, *Stable homotopy and generalised cohomology*, University of Chicago Press, Chicago, 1974.

[2] J.F. Adams, *Infinite Loop Spaces*, Princeton University Press, Princeton, 1978.

[3] M. Artin and B. Mazur, *Étale homotopy*, Lecture Notes in Math., Vol. 100, Springer, Berlin-Heidelberg-New York, 1972.

[4] M. Barr and C. Wells, *Toposes, Triples and Theories*, Springer-Verlag, New York, 1985.

[5] M. Barratt, V.K.A.M. Guggenheim and J.C. Moore, *On semi-simplicial fibre bundles*, Amer. J. Math. **81** (1959), 639–657.

[6] M.G. Barratt and S. Priddy, *On the homology of non-connected monoids and their associated groups*, Comment. Math. Helv. **47** (1972), 1–14.

[7] A.K. Bousfield, *A second quadrant homotopy spectral sequence*, Trans. AMS **177** (1973), 305–318.

[8] A.K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150.

[9] A.K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281.

[10] A.K. Bousfield, *Homotopy spectral sequences and obstructions*, Isr. J. Math. **66** (1989), 54-104.

[11] A.K. Bousfield, *Localization and periodicity in unstable homotopy theory*, J. AMS **7** (1994), 831–873.

[12] A.K. Bousfield and E.M. Friedlander, *Homotopy theory of* $\Gamma$*-spaces, spectra, and bisimplicial sets*, Springer Lecture Notes in Math. **658** (1978), 80–150.

[13] A.K. Bousfield and V.K.A.M. Guggenheim, *On PL de Rham theory and rational homotopy type*, Mem. AMS **179** (1976).

[14] A.K. Bousfield and D.M. Kan, *Homotopy limits completions and localizations*, Springer Lecture Notes in Math., Vol. 304 ($2^{nd}$ corrected printing), Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[15] K.S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. AMS **186** (1973), 419–458.

[16] K.S. Brown, *Buildings*, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

[17] C. Cascuberta, *Recent advances in unstable localization*, The Hilton Symposium (1993), CRM Proc. Lecture Notes **6**, AMS, Providence, 1994, pp. 1–22.

[18] E.B. Curtis, *Simplicial homotopy theory*, Advances in Math. **6** (1971), 107–209.

[19] J. Dieudonné, *Les débuts de la topologie algébrique*, Expo. Math. **3** (1985), 347–357.

[20] A. Dold and D. Puppe, *Homologie nicht-additiver Funktoren; Anwendungen*, Ann. Inst. Fourier (Grenoble) **11** (1961), 201–312.

[21] E. Dror, *Pro-nilpotent representations of homotopy types*, Proc. AMS **38** (1973), 657–660.

[22] E. Dror-Farjoun, *Homotopy localization and* $v_1$*-periodic spaces*, Algebraic Topology (San Feliu de Guixols, 1990), Lecture Notes in Math. **1622**, Springer-Verlag, Berlin, 1992, pp. 104–113.

[23] E. Dror-Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Math. **1622**, Springer-Verlag, Berlin, 1996.

[24] W.G. Dwyer and D.M. Kan, *Singular functors and realization functors*, Ind. Math. **46** (1984), 147–153.

[25] W.G. Dwyer and D.M. Kan, *Homotopy theory and simplicial groupoids*, Indag. Math. **46** (1984), 379–385.

[26] W.G. Dwyer and D.M. Kan, *Equivalences between homotopy theories of diagrams*, Algebraic Topology and Algebraic $K$-theory, W. Browder, ed., Annals of Math. Studies, Vol. 113, pp. 180–205, Princeton Univ. Press, Princeton, 1987.

[27] W.G. Dwyer, D.M. Kan and J.H. Smith, *Homotopy commutative diagrams and their realizations*, J. Pure Applied Albegra **57** (1989), 5–24.

[28] W.G. Dwyer, D.M. Kan and C.R. Stover, *An $E^2$ model category structure for pointed simplicial spaces*, J. of Pure and Applied Algebra **90** (1993), 137–152.

[29] W.G. Dwyer, D.M. Kan and C.R. Stover, *The bigraded homotopy groups $\pi_{i,j}X$ of a pointed simplicial space $X$*, J. of Pure and Applied Algebra **103** (1995), 167–188.

[30] W.G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of Algebraic Topology, Elsevier, Amsterdam, 1995, pp. 73–126.

[31] S. Eilenberg, *Singular homology theory*, Ann. of Math. **45** (1944), 407–477.

[32] S. Eilenberg and J.A. Zilber, *On semi-simplicial complexes and singular homology*, Ann. of Math. **51** (1950), 409–513.

[33] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer-Verlag, New York, 1967.

[34] H. Gillet and D. Grayson, *The loop space of the $Q$-construction*, Ill. J. Math. **31(4)** (1987), 574–597.

[35] P. Goerss, *André-Quillen cohomology and the homotopy groups of mapping spaces: understanding the $E_2$ term of the Bousfield-Kan spectral sequence*, J. of Pure and Applied Algebra **63** (1990), 113–153.

[36] P.G. Goerss, *Homotopy fixed points for Galois groups*, Contemp. Math. **181** (1995), 187-224.

[37] P.G. Goerss, *Comparing completions of a space at a prime*, Contemp. Math. **220** (1998), 65-102.

[38] P.G. Goerss and J.F. Jardine, *Localization theories for simplicial presheaves*, Preprint (1996).

[39] J.P.C. Greenlees and J.P. May, *Derived functors of $I$-adic completion and local homology*, J. Algebra **149** (1992), 438–453.

[40] T. Gunnarsson, *Abstract Homotopy Theory and Related Topics*, Thesis, Chalmers Tekniska Högskola, Göteborg, 1978.

[41] J-C. Hausmann and D. Husemoller, *Acyclic maps*, Enseign. Math. **25** (1979), 53–75.

[42] P.S. Hirschhorn, *Localization of Model Categories*, http://www-math.mit.edu/∼psh.

[43] L. Illusie, *Complexe cotangent et déformations I*, Springer Lecture Notes in Math., Vol. 239, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[44] U. Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245.

[45] J.F. Jardine, *Simplicial objects in a Grothendieck topos*, Contemp. Math. **55** (1986), 193–239.

[46] J.F. Jardine, *Simplicial presheaves*, J. Pure Applied Algebra **47** (1987), 35–87.

[47] J.F. Jardine, *Stable homotopy theory of simplicial presheaves*, Can. J. Math. **39(3)** (1987), 733–747.

[48] J.F. Jardine, *The homotopical foundations of algebraic $K$-theory*, Contemp. Math. **83** (1989), 57–82.

[49] J.F. Jardine, *Supercoherence*, J. Pure Applied Algebra **75** (1991), 103–194.

[50] J.F. Jardine, *Homotopy and homotopical algebra*, Handbook of Algebra, Vol. 1, Elsevier, Amsterdam, 1996, pp. 639–670.

[51] J.F. Jardine, *Boolean localization, in practice*, Doc. Math. **1** (1996), 245–275.

[52] J.F. Jardine, *Generalized Etale Cohomology Theories*, Progress in Math., Vol. 146, Birkhäuser, Boston-Basel-Berlin, 1997.

[53] A. Joyal, *letter to A. Grothendieck*, (1984).

[54] D.M. Kan, *Semisimplicial spectra*, Ill. J. Math. **7** (1963), 463–478.

[55] D.M. Kan, *Abstract homotopy I*, Proc. of Nat. Acad. of Sci. **41** (1955), 1092–1096.

[56] D.M. Kan, *Abstract homotopy II*, Proc. of Nat. Acad. of Sci. **42** (1956), 225–228.

[57] D.M. Kan, *Abstract homotopy III*, Proc. of Nat. Acad. of Sci. **42** (1956), 419–421.

[58] D.M. Kan, *On c.s.s. complexes*, Amer. J. Math. **79** (1957), 449–476.

[59] D.M. Kan, *Adjoint functors*, Trans. AMS **87** (1958), 294–329.

[60] D.M. Kan, *A combinatorial definition of homotopy groups*, Ann. of Math. **67** (1958), 288–312.

[61] K. Lamotke, *Semisimpliziale algebraische Topologie*, Springer-Verlag, Berlin-Heidelberg-New York, 1968.

[62] J. Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, I.H.E.S. Publications Mathématiques **75** (1992), 135–244.

[63] L.G. Lewis, J.P. May and M. Steinberger, *Equivariant Stable Homotopy Theory*, Springer Lecture Notes in Math., Vol. 1213, Springer-Verlag, Berlin-Heidelberg-New York, 1986.

[64] S. Mac Lane, *Homology*, Third Corrected Printing, Springer-Verlag, New York Heidelberg Berlin, 1975.

[65] S. Mac Lane, *Natural associativity and commutativity*, Rice University Studies **49** (1963), 28–46.

[66] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Math., Vol. 5, Springer, Berlin-Heidelberg-New York, 1971.

[67] J.P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, Princeton, 1967.

[68] J.P. May (with contributions by F. Quinn, N. Ray, and J. Tornehave), *$E_\infty$ ring spaces and $E_\infty$ ring spectra*, Springer Lecture Notes in Math., Vol. 577, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[69] J.P. May, *Pairings of categories and spectra*, J. Pure Applied Algebra **19** (1980), 299–346.

[70] D. McDuff and G. Segal, *Homology fibrations and the group-completion theorem*, Invent. Math **31** (1976), 279–287.

[71] H. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2) **120** (1984), 39–87.

[72] J.S. Milne, *Étale Cohomology*, Princeton University Press, Princeton, 1980.

[73] I. Moerdijk, *Bisimplicial sets and the group-completion theorem*, Algebraic $K$-Theory: Connections with Geometry and Topology, NATO ASI Series C, Vol. 279, pp. 225–240, Kluwer, Dordrecht, 1989.

[74] J.C. Moore, *Semi-simplicial complexes and Postnikov systems*, Symposium Internationale de Topologica Algebraica, 1958, pp. 232–247.

[75] F. Morel, *Ensembles profinis simpliciaux et interprétation géométrique du foncteur $T$*, Bulletin de la SMF **124** (1996), 347–373.

[76] D. Quillen, *Homotopical Algebra*, Lecture Notes in Math., Vol. 43, Springer, Berlin-Heidelberg- New York, 1967.

[77] D. Quillen, *The geometric realization of a Kan fibration is a Serre fibration*, Proc. AMS **19** (1968), 1499–1500.

[78] D. Quillen, *Rational homotopy theory*, Ann. of Math. **90** (1969), 205–295.

[79] D. Quillen, *Higher algebraic $K$-theory I*, Springer Lecture Notes in Math. **341** (1973), 85–147.

[80] D. Quillen, *Homotopy properties of the poset of non-trivial p-subgroups of a group*, Advances in Math. **28** (1978), 101–128.

[81] C.L. Reedy, *Homotopy theory of model categories*, Preprint (1973).

[82] M. Ronan, *Lectures on Buildings*, Perspectives in Math., 7, Academic Press, Boston, 1989.

[83] H. Schubert, *Categories*, Springer, New York-Heidelberg-Berlin, 1972.

[84] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[85] J-P. Serre, *Trees*, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[86] D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. **100** (1974), 1–79.

[87] E.H. Spanier, *Algebraic Topology*, First corrected Springer edition, Springer-Verlag, New York, 1981.

[88] M. Sweedler, *Hopf Algebras*, W.A. Benjamin, New York, 1969.

[89] R.M. Switzer, *Algebraic Topology – Homotopy and Homology*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[90] D.H. Van Osdol, *Simplicial homotopy in an exact category*, Amer. J. Math. **99** (1977), 1193–1204.

[91] R.M. Vogt, *Homotopy limits and colimits*, Math. Z. **134** (1973), 11–52.

[92] F. Waldhausen, *Algebraic $K$-theory of spaces*, Springer Lecture Notes in Math. **1126** (1985), 318–419.

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