Chapter 3

Applications of tropical geometry to enumerative geometry

3.1 Introduction

The main purpose of this chapter is to present several applications of tropical geometry in enumerative geometry. The idea to use tropical curves in enumerative questions, and in particular in classical questions of enumeration of algebraic curves (satisfying some constraints) in algebraic varieties was suggested by M. Kontsevich. This idea was realized by G. Mikhalkin [40, 42] who established an appropriate correspondence theorem between the complex algebraic world and the tropical one. This correspondence allows one to calculate Gromov–Witten type invariants of toric surfaces, namely, to enumerate certain nodal complex curves of a given genus which pass through given points in a general position in a toric surface. Roughly speaking, Mikhalkin's theorem affirms that the number of complex curves in question is equal to the number of their tropical analogs passing through given points in a general position in $\mathbb{R}^2$ and counted with multiplicities. In addition, [40] suggests a combinatorial algorithm for an enumeration of the required tropical curves. An extension of Mikhalkin's correspondence theorem to the case of rational curves in toric varieties was proposed by T. Nishinou and B. Siebert in [48].

The tropical approach has important applications in enumerative real algebraic geometry as well. Enumerative geometric problems over the reals, such as counting real algebraic curves in real algebraic varieties, have a different character than in the complex case, since over the reals the answer typically depends on the configuration of the imposed constraints. Thus, the main question concerns the upper and lower bounds for the number of real solutions. For these problems of counting curves, the corresponding Gromov–Witten type invariant (i.e., the number of the corresponding complex curves) is an upper bound. No non-trivial lower bound was known until the recent discovery by J.-Y. Welschinger [73, 74, 75] of invariants which can be seen as real analogs of genus zero Gromov–Witten invariants. The theory of the Welschinger invariants is under an intensive development. The tropical approach allows one to calculate or estimate these invariants in some situations which leads to surprising results in enumerative real algebraic geometry (for example, the logarithmic equivalence of the Gromov–Witten and the Welschinger invariants of toric Del Pezzo surfaces [28]).

3.2 Tropical hypersurfaces in $\mathbb{R}^n$

We briefly recall here the definition and the basic properties of tropical hypersurfaces in $\mathbb{R}^n$ (in fact, $\mathbb{R}^n$ is the tropical analog of the complex torus $(\mathbb{C}^*)^n$ and should be viewed here as $(\mathbb{T}^*)^n = (\mathbb{T} \setminus \{-\infty\})^n$; cf. Chapter 1, Section 1.5).

Fix a positive integer n. A point in $\mathbb{R}^n$ is called *integer*, if all coordinates of this point are integer. Let A be a finite collection of integer points in $\mathbb{R}^n$, and $\varphi : A \to \mathbb{R}$ an arbitrary function. The pair (A, φ) gives rise to a **tropical hypersurface** in $\mathbb{R}^n$ in the following way.

Let $\widehat{\varphi} : \mathbb{R}^n \to \mathbb{R}$ be the *Legendre transform* of φ:

$$\widehat{\varphi}(x_1, \dots, x_n) = \max_{(i_1,\dots,i_n)\in A} \{i_1x_1 + \dots + i_nx_n - \varphi(i_1, \dots, i_n)\}.$$

The function $\widehat{\varphi}$ is a *tropical polynomial* defining the hypersurface under description (cf. Chapter 1, Section 1.5). Notice that $\widehat{\varphi}$ is convex piecewise-linear, and consider the corner locus $T(A, \varphi)$ of $\widehat{\varphi}$, i.e., the subset of $\mathbb{R}^n$ formed by the points where $\widehat{\varphi}$ is not locally affine-linear. The graph $\Gamma(A, \varphi)$ of $\widehat{\varphi}$ is naturally stratified. The set $T(A, \varphi)$ is stratified by the projections of the elements of the stratification of $\Gamma(A, \varphi)$, and defines a subdivision $\Theta(A, \varphi)$ of $\mathbb{R}^n$. The $(n-1)$-dimensional elements of the stratification of $T(A, \varphi)$ are called *facets*.

Each facet σ of $T(A, \varphi)$ can be equipped with a positive integer number. Namely, let σ be the projection of an $(n-1)$-dimensional polyhedron Σ in $\Gamma(A, \varphi)$. Denote by A_σ the subset of A formed by the points $(i_1, \dots, i_n)$ such that the graph of the affine-linear function $i_1x_1 + \dots + i_nx_n - \varphi(i_1, \dots, i_n)$ contains Σ. Notice that A_σ has at least two points, and all the points of A_σ belong to a straight line. Denote by I and J the two extremal points of A_σ, i.e., the points of A_σ such that the segment $[IJ]$ contains all the points of A_σ. Associate to the facet σ a *weight* $w(\sigma)$ equal to the integer length of $[IJ]$ (the *integer length* of a segment with integer endpoints is the number of its integer points diminished by 1).

Definition 3.1. The polyhedral complex $T(A, \varphi)$ whose facets are equipped with the corresponding weights is called the *tropical hypersurface* associated with the

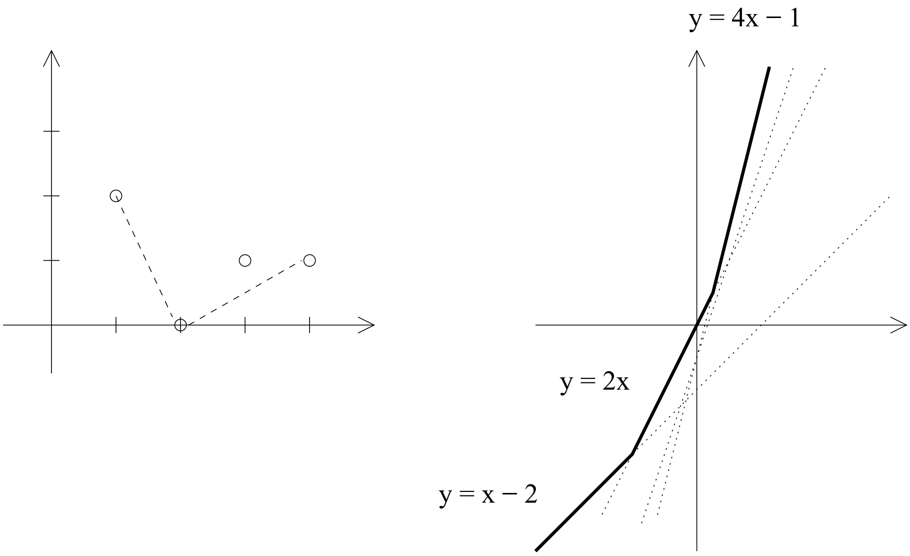


Figure 3.1: Legendre transform in dimension 1.

pair (A, φ). One says that $T(A, \varphi)$ is a tropical hypersurface with Newton polytope $\Delta(A)$, where $\Delta(A)$ is the convex hull of A. If $\Delta(A)$ is the simplex with vertices $(0, 0, \ldots, 0)$, $(m, 0, \ldots, 0)$, $(0, m, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, m)$, then the tropical hypersurface $T(A, f)$ is said to be *of degree* m.

Example 3.2. Let $A = \{(0, 0), (1, 0), (0, 1)\} \subset \mathbb{R}^2$, and $\varphi : A \to \mathbb{R}$ an arbitrary function. The tropical curve associated with the pair (A, φ) is the union of three rays in $\mathbb{R}^2$ which share a common extremal point; the directions of the rays are south, west and northeast (see Figure 3.2). In this case, a change of values of φ leads to a translation of the tropical curve. The common point of the three rays has the coordinates $(\varphi(1, 0) - \varphi(0, 0), \varphi(0, 1) - \varphi(0, 0))$.

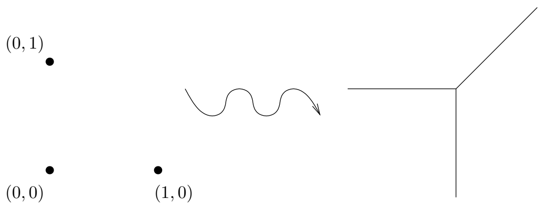


Figure 3.2: A tropical line.

Notice that different polytopes can play the role of Newton polytope for the same tropical hypersurface in $\mathbb{R}^n$. For example, given a finite collection A of integer points in $\mathbb{R}^n$, a function $\varphi : A \to \mathbb{R}$, and any integer point c in $\mathbb{R}^n$, consider the pair (A', φ'), where $A' = A + c$ and $\varphi : A' \to \mathbb{R}$ is the function such that $\varphi'(x) = \varphi(x - c)$ for any point $x \in A'$. Then, the tropical hypersurfaces $T(A, \varphi)$ and $T(A', \varphi')$ coincide. This example is a typical one: any two Newton polytopes of a given tropical hypersurface in $\mathbb{R}^n$ differ by a translation by an integer vector.

Consider a tropical hypersurface $T(A, \varphi) \subset \mathbb{R}^n$ associated with a pair (A, φ). The function φ gives rise to a subdivision of the convex hull $\Delta(A)$ of A. Namely, let $F : \Delta(A) \to \mathbb{R}$ be the convex piecewise-linear function whose graph is the lower part of the convex hull of the graph of φ. The linearity domains of F are n-dimensional polytopes with integer vertices. These polytopes produce a subdivision $S(A, \varphi)$ of $\Delta(A)$.

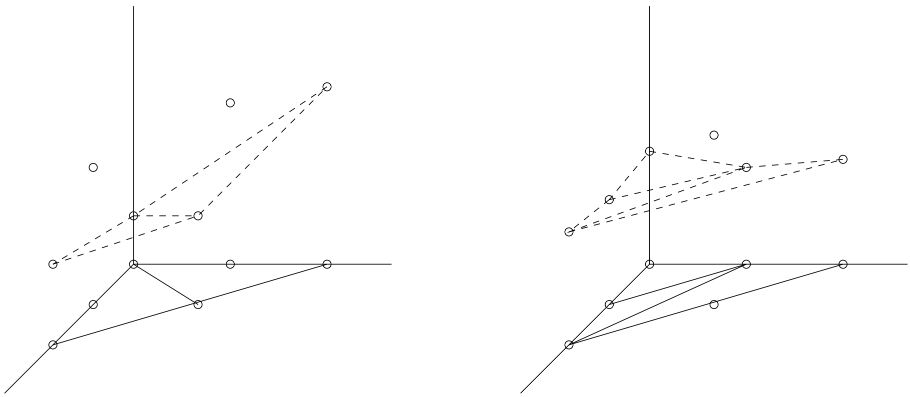


Figure 3.3: Examples of subdivisions of the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.

The subdivision $S(A, \varphi)$ is dual to the subdivision $\Theta(A, \varphi)$ in the following sense.

Theorem 3.3 (Duality theorem). *There exists a one-to-one correspondence $\mathcal{B}$ between the elements of $S(A, \varphi)$ on one side and the elements of $\Theta(A, \varphi)$ on the other side such that*

- *if e is an element of $S(A, \varphi)$ having dimension i, then the element $\mathcal{B}(e)$ of $\Theta(A, \varphi)$ has dimension $n - i$, and the linear spans of e and $\mathcal{B}(e)$ are orthogonal,*
- *the correspondence $\mathcal{B}$ reverses the incidence relation.*

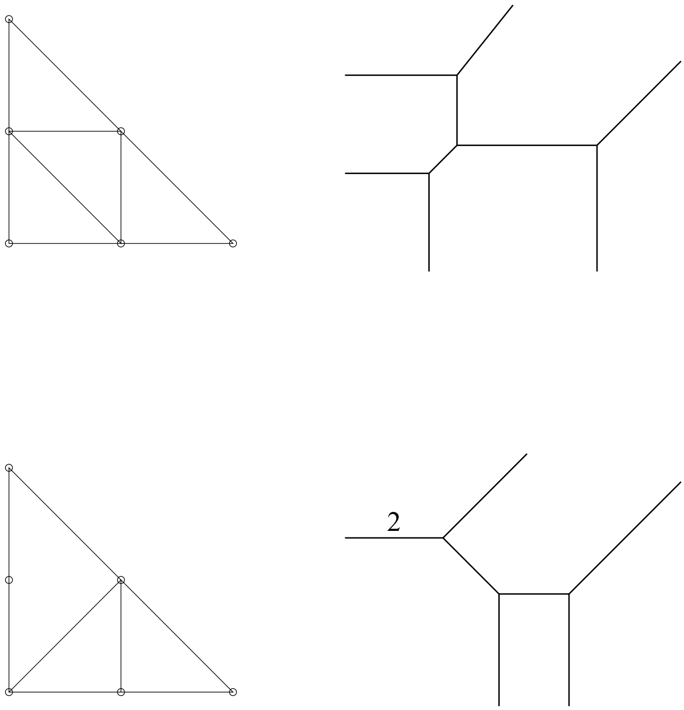


Figure 3.4: Examples of tropical conics in $\mathbb{R}^2$ and corresponding dual subdivisions (the weight of an edge is indicated only if this weight is different from 1).

3.3 Geometric description of plane tropical curves

We will now restrict ourselves to the study of tropical curves in $\mathbb{R}^2$. These curves can be described in the following geometric way.

Let V be a finite collection of distinct points in $\mathbb{R}^2$, E_b a collection of segments whose endpoints belong to V, and E_u a finite collection of half-infinite rays whose endpoints belong to V. Assume that the intersection of any two elements in $E_b \cup E_u$ is either a point in V or empty. Let $w : E_b \cup E_u \to \mathbb{N}$ be a function (for an element $e \in E_b \cup E_u$, the number $w(e)$ is called the *weight* of e). Such a quadruple (V, E_b, E_u, w) is called a *weighted rectilinear graph.* A weighted rectilinear graph (V, E_b, E_u, w) is *balanced* if

- each element in $E_b \cup E_u$ has a rational slope,
- no element in V is adjacent to exactly two elements in $E_b \cup E_u$,

- for any element v in V, one has $\sum_{e_k \in E(v)} w(e_k) \cdot \overrightarrow{e_k} = 0$, where $E(v) \subset E_b \cup E_u$ is the set formed by the elements of $E_b \cup E_u$ which are adjacent to v, and $\overrightarrow{e_i}$ is the smallest vector with integer coordinates based at v and pointed along e_i.

The last property in the definition above is called the *balancing condition.*

Theorem 3.4. *Any tropical curve in $\mathbb{R}^2$ represents a balanced weighted rectilinear graph. Conversely, any balanced weighted rectilinear graph represents a tropical curve.*

Proof. Let $T(A, \varphi)$ be a tropical curve associated with a pair (A, φ). Consider the weighted rectilinear graph Γ whose set V (respectively, E_b, E_u) is formed by the vertices (respectively, bounded edges, unbounded edges) of $T(A, \varphi)$, and whose weights coincide with the corresponding weights of $T(A, \varphi)$. Since

- any edge in $S(A, \varphi)$ has a rational slope,
- any polygon in $S(A, \varphi)$ has at least three sides,
- for any polygon in $S(A, \varphi)$ with vertices $p_1, p_2, \ldots, p_n$, one has $\overrightarrow{p_1 p_2} + \cdots + \overrightarrow{p_{n-1} p_n} + \overrightarrow{p_n p_1} = 0$,

the Duality Theorem 3.3 implies that the graph Γ verifies all three properties appearing in the definition of balanced graphs, and thus, is balanced.

To prove the converse statement, consider a balanced weighted graph Γ, and choose a region R of the complement of Γ in $\mathbb{R}^2$. Associate to R an arbitrary affine-linear function $\widehat{\varphi}_R : \mathbb{R}^2 \to \mathbb{R}$, $\widehat{\varphi}_R(x, y) = i_R x + j_R y - \varphi_R$. Let R' be a region neighboring to R, i.e., such that the intersection e of the closures of R and R' is an element of $E_b \cup E_u$. Associate to R' the affine-linear function $\widehat{\varphi}_{R'} : \mathbb{R}^2 \to \mathbb{R}$, $\widehat{\varphi}_{R'}(x, y) = i_{R'} x + j_{R'} y - \varphi_{R'}$ such that $((i_{R'} - i_R)/w(e), (j_{R'} - j_R)/w(e))$ is the smallest integer vector normal to e and pointed inside R', and the restrictions of $\widehat{\varphi}_R$ and $\widehat{\varphi}_{R'}$ on e coincide. Continuing in the same manner, we associate to any region P of the complement of Γ in $\mathbb{R}^2$ an affine-linear function $\widehat{\varphi}_P : \mathbb{R}^2 \to \mathbb{R}$, $\widehat{\varphi}(x, y) = i_P x + j_P y - \varphi_P$. The balancing condition insures that the function $\widehat{\varphi}_P$ does not depend on the sequence of regions used in the definition of $\widehat{\varphi}_P$. We obtain a finite collection A of integer points (i_P, j_P) and a function $\varphi : A \to \mathbb{R}$ defined by $(i_P, j_P) \mapsto \varphi_P$ such that Γ represents the tropical curve associated with (A, φ). $\square$

The *sum* $T(A_1, \varphi_1) + \cdots + T(A_n, \varphi_n)$ of plane tropical curves $T(A_1, \varphi_1)$, ..., $T(A_n, \varphi_n)$ is the plane tropical curve defined by the tropical polynomial $\widehat{\varphi}_1 + \cdots + \widehat{\varphi}_n$. The underlying set of the tropical curve $T(A_1, \varphi_1) + \cdots + T(A_n, \varphi_n)$ is the union of underlying sets of $T(A_1, \varphi_1)$, ..., $T(A_n, \varphi_n)$, and the weight of any edge of $T(A_1, \varphi_1) + \cdots + T(A_n, \varphi_n)$ is equal to the sum of the weights of the corresponding edges of summands. A tropical curve in $\mathbb{R}^2$ is *reducible* if it is the sum of two proper tropical subcurves. A non-reducible tropical curve in $\mathbb{R}^2$ is called *irreducible.*

Tropical curves have many properties in common with algebraic curves. For example, one can prove the following analog of the Bézout theorem (see, e.g., [64]).

Theorem 3.5 (Tropical Bézout theorem). *Let T_1 and T_2 be two tropical curves of degrees m_1 and m_2, respectively, such that T_1 and T_2 are in general position with respect to each other (the latter condition means that T_1 and T_2 intersect each other only in inner points of edges); then the number of intersection points (counted with certain multiplicities) of T_1 and T_2 is equal to m_1m_2. The multiplicities of intersection points are defined as follows. Consider an intersection point of an edge e_1 of T_1 and an edge e_2 of T_2. Let (a_1, b_1) and (a_2, b_2) be smallest integer vectors along e_1 and e_2, respectively. Then, the multiplicity of the intersection point is equal to $w(e_1)w(e_2)|a_1b_2 - a_2b_1|$.*

Notice also that for any two points in general position in $\mathbb{R}^2$ there exists exactly one tropical line passing through these points. This observation has an important generalization which is the core of the remaining part of this chapter.

3.4 Count of complex nodal curves

In this section, we formulate certain enumerative problems concerning nodal curves in the complex projective plane $\mathbb{C}P^2$.

Fix a positive integer m, and choose $\frac{m(m+3)}{2}$ points in general position in $\mathbb{C}P^2$. There exists exactly one curve of degree m in $\mathbb{C}P^2$ which passes through the chosen points. Indeed, the space $\mathbb{C}C_m$ of all the curves of degree m in $\mathbb{C}P^2$ can be identified with a complex projective space $\mathbb{C}P^N$ of dimension $N = \frac{m\cdot(m+3)}{2}$: the coefficients of a polynomial defining a given curve can be taken for homogeneous coordinates of the corresponding point in $\mathbb{C}C_m$. The condition to pass through a given point in $\mathbb{C}P^2$ produces a linear equation in coefficients of a polynomial defining the curve, and thus determines a hyperplane in $\mathbb{C}C_m$. If the configuration of $\frac{m(m+3)}{2}$ chosen points is sufficiently generic, the corresponding $\frac{m(m+3)}{2}$ hyperplanes in $\mathbb{C}C_m$ have exactly one common point (and in addition this point corresponds to a nonsingular curve).

Choose now $\frac{m\cdot(m+3)}{2} - 1$ points in general position in $\mathbb{C}P^2$. How many curves of degree m with one non-degenerate double point each pass through the chosen points? Consider the hypersurface $D \subset \mathbb{C}C_m$ formed by the points corresponding to singular curves. The hypersurface D is called the *discriminant* of $\mathbb{C}C_m$. The smooth part of D is formed by the points corresponding to curves whose only singular point is non-degenerate double. If the configuration of $\frac{m(m+3)}{2} - 1$ chosen points is sufficiently generic, the intersection of the hyperplanes corresponding to these points is a line in $\mathbb{C}C_m$, and moreover, this line intersects the discriminant only in the smooth part and transversally. Thus, the number we are interested in is the degree of D.

Consider the following generalization of the above questions. Pick an integer δ verifying the inequalities $0 \le \delta \le \frac{(m-1)(m-2)}{2}$, and choose a collection U of $\frac{m(m+3)}{2} - \delta$ points in $\mathbb{C}P^2$. Consider the curves of degree m in $\mathbb{C}P^2$ which pass through all the points of U and have δ non-degenerate double points each. If U is sufficiently generic, then the number of these curves is finite and does not depend on U. Denote by $N_m(\delta)$ (respectively, $N_m^{\mathrm{irr}}(\delta)$) the number of curves (respectively, of irreducible curves) of degree m in $\mathbb{C}P^2$ which pass through the points of a generic configuration of $\frac{m(m+3)}{2} - \delta$ points in $\mathbb{C}P^2$ and have δ non-degenerate double points each. The expression "sufficiently generic" in the description of the numbers $N_m(\delta)$ can be made precise in the following way. Denote by $S_m(\delta)$ the subset of $\mathbb{C}C_m$ formed by the points corresponding to curves of degree m having δ non-degenerate double points each and no other singularities. The *Severy variety* $\overline{S}_m(\delta)$ is the closure of $S_m(\delta)$ in $\mathbb{C}C_m$. It is an algebraic variety of codimension δ in $\mathbb{C}C_m$. Its smooth part is $S_m(\delta)$. We say that a collection U of $\frac{m(m+3)}{2} - \delta$ points is *generic*, if the dimension of the projective subspace $\Pi(U) \subset \mathbb{C}C_m$ defined by the points of U is equal to δ, the intersection $\Pi(U) \cap \overline{S}_m(\delta)$ is contained in $S_m(\delta)$, and this intersection is transverse. Generic collections form an open dense subset in the space of all collections of $\frac{m(m+3)}{2} - \delta$ points in $\mathbb{C}P^2$. If U is generic, the number of curves of degree m in $\mathbb{C}P^2$ which pass through the points of U and have δ non-degenerate double points each is equal to the number of elements in the finite intersection $\Pi(U) \cap \overline{S}_m(\delta)$. Thus, the number $N_m(\delta)$ is the degree of the Severi variety $\overline{S}_m(\delta)$.

The numbers $N_m(\delta)$ can be calculated starting with the numbers $N_m^{\mathrm{irr}}(\delta)$ and *vice versa* (see, for example, [4]). The numbers $N_m^{\mathrm{irr}}(\delta)$ are *Gromov–Witten invariants* of $\mathbb{C}P^2$. The number $N_m^{\mathrm{irr}}(\delta)$, where $\delta = \frac{(m-1)(m-2)}{2}$, is the number of rational curves of degree m which pass through a generic collection of $\frac{m(m+3)}{2} - \frac{(m-1)(m-2)}{2} = 3m - 1$ points in $\mathbb{C}P^2$. A recursive formula for the numbers $N_m^{\mathrm{irr}}(\delta)$, with $\delta = \frac{(m-1)(m-2)}{2}$, was found by M. Kontsevich (see [33]). A recursive formula that allows one to calculate the numbers $N_m(\delta)$ with an arbitrary δ was obtained by L. Caporaso and J. Harris [4].

G. Mikhalkin proposed a new formula for the numbers $N_m(\delta)$ (see [40, 42]). This formula has an immediate generalization to the case of an arbitrary toric surface (see [40, 42]). Mikhalkin's approach is based on a reformulation of the enumerative problem presented above into an enumerative problem concerning tropical curves.

3.5 Correspondence theorem

To formulate Mikhalkin's correspondence theorem, introduce additional definitions.

Let m be a positive integer, $\Delta_m \subset \mathbb{R}^2$ the triangle having the vertices $(0,0)$, $(m,0)$, and $(0,m)$, and T a tropical curve of degree m. The curve T is called *simple* if the corresponding dual subdivision S_T of Δ_m satisfies the following properties:

- any polygon of S_T is either a triangle or a parallelogram,
- any integer point on the boundary of Δ_m is a vertex of S_T.

In this case, the subdivision S_T is also called simple. Notice that if T is simple, then it can be represented in a unique possible way as a sum of irreducible tropical curves.

Assume that T is simple. Then, the *rank* of T is the difference diminished by 1 between the number of vertices of S_T and the number of parallelograms in S_T. The *multiplicity* $\mu(S_T)$ of S_T (and the multiplicity $\mu(T)$ of T) is the product of areas of all the triangles in S_T (we normalize the area in such a way that the area of a triangle whose only integer points are its vertices is equal to 1).

Let r be a positive integer, and $\mathcal{U}$ a generic collection of r points in $\mathbb{R}^2$. (One can formalize the expression "generic" used here and introduce the notion of a *tropically generic* collection of points in $\mathbb{R}^2$; this can be done in a way similar to the one used in the complex situation.) Consider the collection $\mathcal{C}(\mathcal{U})$ of simple tropical curves of degree m and of rank r which pass through all the points of $\mathcal{U}$. Denote by $\mathcal{C}^{\mathrm{irr}}(\mathcal{U})$ the collection of irreducible curves belonging to $\mathcal{C}(\mathcal{U})$.

Theorem 3.6. (G. Mikhalkin, [42]). *Let $\mathcal{U}$ be a generic set of $r = \frac{m(m+3)}{2} - \delta$ points in $\mathbb{R}^2$, where an integer δ satisfies the inequalities $0 \le \delta \le \frac{(m-1)(m-2)}{2}$. Then,*

$$N_m(\delta) = \sum_{T \in \mathcal{C}(\mathcal{U})} \mu(T) \quad \text{and} \quad N_m^{\mathrm{irr}}(\delta) = \sum_{T \in \mathcal{C}^{\mathrm{irr}}(\mathcal{U})} \mu(T).$$

Theorem 3.6 is a particular case of Mikhalkin's theorem which is valid in the more general setting of projective toric surfaces. Mikhalkin's proof of Theorem 3.6 provides a bijection between the multi-set $\mathcal{C}(\mathcal{U})$ and the set of complex curves of degree m which pass through certain r generic points in $\mathbb{C}P^2$ and have δ non-degenerate double points each. A slightly different approach establishing such a bijection was proposed by E. Shustin [59].

In addition, Mikhalkin [40, 42] found a combinatorial algorithm which gives a possibility to calculate the number of tropical curves in question. We present this algorithm in the next section.

3.6 Mikhalkin's algorithm

Let again m be a positive integer, and Δ_m the triangle with vertices $(0,0)$, $(m,0)$, and $(0,m)$. Fix a linear function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ which is injective on the integer points of Δ_m, and denote by p (respectively, q) the vertex of Δ_m where λ takes its minimum (respectively, maximum). The points p and q divide the boundary of Δ_m in two parts. Denote one of these parts by $\partial\Delta_+$, and the other part by $\partial\Delta_-$.

Let l be a natural number. A path $\gamma : [0,l] \to \Delta_m$ is called λ-*admissible* if

- $\gamma(0) = p$ and $\gamma(l) = q$,
- the composition $\lambda \circ \gamma$ is injective,
- for any integer $0 \le i \le l-1$ the point $\gamma(i)$ is integer, and $\gamma([i, i+1])$ is a segment.

The number l is called the *length* of γ, and the integer points of the form $\gamma(i)$, where i is an integer satisfying the inequalities $0 \le i \le l$, are called *vertices* of γ. A λ-admissible path γ divides Δ_m in two parts: the part $\Delta_+(\gamma)$ bounded by γ and $\partial\Delta_+$ and the part $\Delta_-(\gamma)$ bounded by γ and $\partial\Delta_-$. Define an operation of *compression* of $\Delta_+(\gamma)$ in the following way. Let j be the smallest positive integer $1 \le j \le l-1$ such that $\gamma(j)$ is the vertex of $\Delta_+(\gamma)$ with the angle less than π (a compression of $\Delta_+(\gamma)$ is defined only if such an integer j does exist). A compression of $\Delta_+(\gamma)$ is $\Delta_+(\gamma')$, where γ' is either the path defined by $\gamma'(i) = \gamma(i)$ for $i < j$ and $\gamma'(i) = \gamma(i+1)$ for $i \ge j$, or the path defined by $\gamma'(i) = \gamma(i)$ for $i \ne j$ and $\gamma'(j) = \gamma(j-1) + \gamma(j+1) - \gamma(j)$ (the latter path can be considered only if $\gamma(j-1) + \gamma(j-1) + \gamma(j) \in \Delta_m$). Note that γ' is also a λ-admissible path. A sequence of compressions started with $\Delta_+(\gamma)$ and ended with a path whose image coincides with $\partial\Delta_+$ defines a subdivision of $\Delta_+(\gamma)$ which is called *compressing.* A compression and a compressing subdivision of $\Delta_-(\gamma)$ is defined in a completely similar way. A pair $(S_+(\gamma), S_-(\gamma))$, where $S_\pm(\gamma)$ is a compressing subdivision of $\Delta_\pm(\gamma)$, produces a subdivision of Δ_m. The latter subdivision is called γ-*consistent.* Denote by $\mathcal{N}_\lambda(\gamma)$ the collection of simple γ-consistent subdivisions of Δ_m.

Theorem 3.7. (G. Mikhalkin, see [40, 42]). *Let $0 \le \delta \le \frac{(m-1)(m-2)}{2}$ be an integer. There exists a generic set $\mathcal{U}$ of $r = \frac{m(m+3)}{2} - \delta$ points in $\mathbb{R}^2$ such that the map associating to a simple tropical curve T of degree m the dual subdivision S_T of Δ_m establishes a one-to-one correspondence between the set $\mathcal{C}(\mathcal{U})$ and the disjoint union $\amalg_\gamma \mathcal{N}_\lambda(\gamma)$, where γ runs over all the λ-admissible paths in Δ_m of length r. In particular, $N_m(\delta) = \sum_\gamma \sum_{S \in \mathcal{N}_\lambda(\gamma)} \mu(S)$, where $\mu(S)$ is the multiplicity of S.*

Example 3.8. Figure 3.5 illustrates the algorithm in the case of rational cubics. The function λ is given by $\lambda(i,j) = i - \varepsilon j$, where ε is a positive sufficiently small number. In this case, the number of integer points of the Newton triangle Δ_3 is greater by 1 than the number $r+1 = 9$ of vertices of λ-admissible paths to consider. Therefore, each λ-admissible path γ of length $r = 8$ is uniquely determined by the integer point which is not a vertex of γ. It is easy to see that the integer points marked by small squares on Figure 3.5 are vertices of any path γ such that the set of simple γ-consistent subdivisions is not empty.

3.7 Welschinger invariants

Mikhalkin's correspondence theorem also gives a possibility to enumerate real curves passing through specific configurations of real points in $\mathbb{R}P^2$ (as well as on other projective toric surfaces). Of course, in the real case the result depends on the chosen point configuration in $\mathbb{R}P^2$. Fortunately, another important discovery was made recently by J.-Y. Welschinger [73, 74]. He found a way of attributing weights ± 1 to real rational curves which makes the number of curves counted with the weights to be independent of the configuration of points in $\mathbb{R}P^2$ and produces lower bounds for the number of real curves in question.

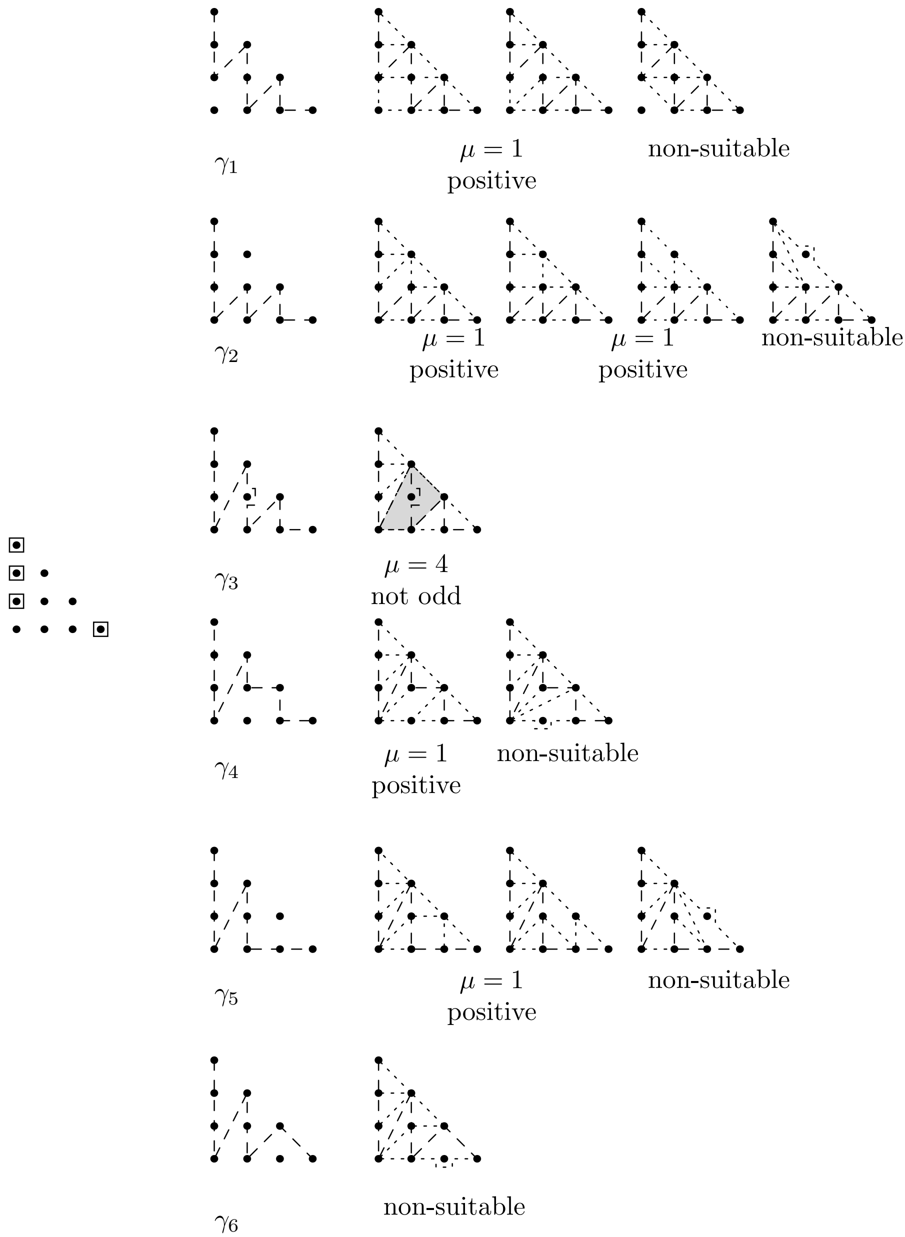


Figure 3.5: The algorithm for rational cubics.

For a given positive integer m and an integer δ satisfying $0 \le \delta \le \frac{(m-1)(m-2)}{2}$, choose a generic collection U of $\frac{m(m+3)}{2} - \delta$ points in $\mathbb{R}P^2$. Consider the set of real irreducible curves of degree m passing through all the points of U and having δ non-degenerate double points each. Denote by $R_m^{\mathrm{irr}}(\delta, U)$ the number of curves in the set considered, and by $R_m^{\mathrm{irr,\ even}}(\delta, U)$ (resp., $R_m^{\mathrm{irr,\ odd}}(\delta, U)$) the number of curves in this set which have even (resp., odd) number of solitary nodes (i.e., non-degenerate real double points locally given by the equation $x^2 + y^2 = 0$). The *Welschinger sign* of a nodal real curve is $(-1)^s$, where s is the number of solitary nodes of the curve. Define the *Welschinger number* as $W_m(\delta, U) = R_m^{\mathrm{irr,\ even}}(\delta, U) - R_m^{\mathrm{irr,\ odd}}(\delta, U)$.

Theorem 3.9. (J.-Y. Welschinger, see [73, 74]). *If $\delta = \frac{(m-1)(m-2)}{2}$ (i.e., if the considered curves are rational), then $W_m(\delta, U)$ does not depend on the choice of a (generic) configuration U.*

In fact, Theorem 3.9 is a particular case of Welschinger's theorem. The general statement in the case of real symplectic 4-manifolds and the proof can be found in [73, 74]. Higher dimensional generalizations are found in [75].

The number $W_m(\frac{(m-1)(m-2)}{2}, U)$ is called the *Welschinger invariant* and is denoted by W_m. Clearly, the Welschinger invariant W_m gives a lower bound for the number of real solutions to our interpolation problem: $R_m^{\mathrm{irr}}(\frac{(m-1)(m-2)}{2}, U) \ge |W_m|$.

Welschinger's theorem provides another type of applications of Mikhalkin's correspondence. The remaining part of this chapter is mostly devoted to these applications.

3.8 Welschinger invariants W_m for small m

Let us calculate the Welschinger invariants W_m for $m = 1$, 2, and 3.

If $m = 1$, then we should count straight lines passing through $3 \cdot 1 - 1 = 2$ points in general position in $\mathbb{R}P^2$. There is exactly one straight line passing through two points in general position in $\mathbb{R}P^2$. This is a nonsingular real rational curve, and its Welschinger sign is $+1$. Thus, $W_1 = 1$.

If $m = 2$, then we should count real conics passing through $3 \cdot 2 - 1 = 5$ points in general position in $\mathbb{R}P^2$. There is exactly one curve of degree 2 passing through five points in general position in $\mathbb{R}P^2$. Once again, this curve is real, rational and nonsingular, its Welschinger sign is $+1$. Thus, $W_2 = 1$.

The case $m = 3$ is more complicated. If $m = 3$, we should count real rational cubics passing through $3 \cdot 3 - 1 = 8$ points in general position in $\mathbb{R}P^2$. Let U be a generic configuration of 8 points in $\mathbb{R}P^2$. The configuration U defines a pencil $\mathcal{P}$ of real cubics passing through all the points of U. Any two cubics of $\mathcal{P}$ intersect in 8 points of U, and thus have one additional point of intersection in $\mathbb{C}P^2$. Denote this point by Q. Notice that Q is real, and all the cubics of $\mathcal{P}$ pass through Q. Let $\widetilde{\mathbb{R}P^2}$ be $\mathbb{R}P^2$ blown up at 8 points of U and at the point Q.

The Euler characteristic $\chi(\widetilde{\mathbb{R}P^2})$ of $\widetilde{\mathbb{R}P^2}$ is equal to $1 - 9 = -8$. On the other hand, the calculation of the Euler characteristic of $\widetilde{\mathbb{R}P^2}$ via the pencil $\mathcal{P}$ gives $\chi(\widetilde{\mathbb{R}P^2}) = R_3^{\text{irr, odd}}(1, U) - R_3^{\text{irr, even}}(1, U) = -W_3$. Thus, $W_3 = 8$.

The lower bound 8 for the number of real rational cubics passing through 8 points in general position in $\mathbb{R}P^2$ is sharp and was proved by V. Kharlamov before the discovery of the Welschinger invariants (see, for example, [7]). It is not known whether the lower bounds provided by the Welschinger invariants W_m, $m \geq 4$ are sharp.

3.9 Tropical calculation of Welschinger invariants

A simple tropical curve T of degree m and the corresponding subdivision S_T of Δ_m are called *odd*, if each triangle in S_T has an odd (normalized) area. Such a curve T and the dual subdivision S_T are called *positive* (respectively, *negative*) if the sum of the numbers of interior integer points over all the triangles of S_T is even (respectively, odd). Associate to any simple tropical curve T of degree m and to the corresponding subdivision S_T of Δ_m the Welschinger multiplicity $\mathcal{W}(T)$ in the following way. If T is not odd, then put $\mathcal{W}(T) = 0$. If T is odd and positive (respectively, negative), then put $\mathcal{W}(T) = 1$ (respectively, $\mathcal{W}(T) = -1$).

Theorem 3.10. (cf. [40, 42], and [59]). *Let $\mathcal{U}$ be a generic collection of $r = 3m - 1$ points in $\mathbb{R}^2$. Then,*

$$W_m = \sum_{T \in \mathcal{C}^{\text{irr}}(\mathcal{U})} \mathcal{W}(T).$$

Take now a set $\mathcal{U}$ with the properties described in Theorem 3.7, and denote by $n_\lambda^+(\gamma)$ (respectively, $n_\lambda^-(\gamma)$) the number of odd positive (respectively, negative) subdivisions in $\mathcal{N}_\lambda(\gamma)$ which are dual to irreducible tropical curves. The following statement is an immediate corollary of Theorems 3.10 and 3.7.

Theorem 3.11. (see [40, 42]). *The Welschinger invariant W_m is equal to $\sum_\gamma (n_\lambda^+(\gamma) - n_\lambda^-(\gamma))$, where γ runs over all the λ-admissible paths in Δ_m of length $r = 3m - 1$.*

Figure 3.5 illustrates the tropical calculation of W_3. The subdivision with two grey triangles has multiplicity 4. This subdivision is not odd (the grey triangles are of area 2), and thus, it does not contribute to W_3.

Remark 3.12. As it was noticed by G. Mikhalkin, one can easily prove the following result comparing Theorems 3.7 and 3.11: *for any positive integer m, the Welschinger invariant W_m and the corresponding Gromov–Witten invariant $N_m = N_m^{\text{irr}}(\frac{(m-1)(m-2)}{2})$ are congruent modulo* 4.

Theorem 3.11 gives a possibility to calculate or to estimate the Welschinger invariants W_m. The following section is devoted to applications of Theorem 3.11.

3.10 Asymptotic enumeration of real rational curves

Consider the following question: *fix a positive integer m; whether for any generic collection of $3m-1$ points in the real projective plane there always exists a real rational curve of degree m which passes through the points of the collection* ? (The number $N_m = N_m^{\mathrm{irr}}(\frac{(m-1)(m-2)}{2})$ of complex rational curves (see [33]) is even for every $m \geq 3$, so the existence of a required real curve does not immediately follow from the computation in the complex case.)

The following statement is a corollary of Theorem 3.11.

Theorem 3.13. (I. Itenberg, V. Kharlamov, E. Shustin; see [27, 28]))

- *For any positive integer m, the Welschinger invariant W_m is positive.*
- *The sequences $\log W_m$ and $\log N_m$, $m \in \mathbb{N}$, are asymptotically equivalent, More precisely,*

$$\log W_m = \log N_m + O(m) \quad \textit{and}$$
$$\log N_m = 3m \log m + O(m).$$

As a corollary, the aforementioned question is answered in the affirmative. Moreover, Theorems 3.9 and 3.13 imply that asymptotically in the logarithmic scale all the complex solutions of our interpolation problem are real.

Let $\lambda^0 : \mathbb{R}^2 \to \mathbb{R}$ be a linear function defined by $\lambda^0(i,j) = i - \varepsilon j$, where ε is a sufficiently small positive number (so that λ^0 defines a kind of a lexicographical order on the integer points of the triangle Δ_m).

The following statement is a key point in the proof of Theorem 3.13.

Lemma 3.14. *For any λ^0-admissible path γ in Δ_m, the number $n^-_{\lambda^0}(\gamma)$ is equal to* 0.

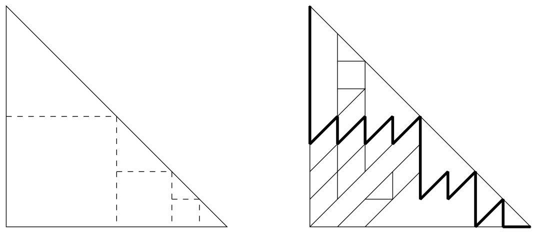
Proof. Let γ be a λ^0-admissible path in Δ_m, and S a subdivision in the collection $\mathcal{N}_{\lambda^0}(\gamma)$. The subdivision S does not have an edge with the endpoints (i_1, j_1) and (i_2, j_2) such that $|i_1 - i_2| > 1$; otherwise, at least one integer point on the boundary of Δ_m would not be a vertex of the corresponding compressing subdivision. This implies that no triangle in S has interior integer points. □

Lemma 3.14 implies that, for any λ_0-admissible path γ, the contribution of any γ-consistent subdivision of Δ_m to the Welschinger invariant W_m is non-negative. Thus, to prove Theorem 3.13, it is sufficient to present a λ_0-admissible path γ such that the contribution of certain γ-consistent subdivisions of Δ_m to the Welschinger invariant W_m is big enough.

Sketch of the proof of Theorem 3.13. Inscribe in Δ_m a sequence of maximal size squares as shown on Figure 3.6(a). Their right upper vertices have the coordinates

$$(x_i, y_i),\ i \geq 1,\ x_1 = y_1 = \left[\frac{m}{2}\right],\ y_{i+1} = \left[\frac{m - x_i}{2}\right],\ x_{i+1} = x_i + y_{i+1}.$$

Put $(x_0, y_0) = (0, m)$. Then pick a λ^0-admissible path γ consisting of segments of integer length 1 as shown on Figure 3.6(b). This path consists of sequences of vertical segments joining (x_i, y_i) with $(x_i, y_{i+1} - 1)$, zig-zag sequences joining $(x_i, y_{i+1} - 1)$ with (x_{i+1}, y_{i+1}) (in such a zig-zag sequence the segments of slope 1 alternate with vertical segments; it always starts and ends with segments of slope 1), and the segments $[(m-1, 1), (m-1, 0)]$ and $[(m-1, 0), (m, 0)]$. The length of this path is $3m - 1$.



(a) (b)

Figure 3.6: Path γ and γ-consistent subdivisions of Δ_m.

Now, we select some γ-consistent subdivisions of Δ_m. Subdivide the upper part $\Delta_+(\gamma)$ in vertical strips of integer width 1. Note that the rightmost strip consists of one primitive triangle (a triangle with integer vertices is called *primitive* if it is of (normalized) area 1). Pack into each strip but the rightmost one the maximal possible number of primitive parallelograms (a parallelogram with integer vertices is called primitive if it is of (normalized) area 2) and place in the remaining part of the strip two primitive triangles (see Figure 3.6(b)). Then subdivide $\Delta_-(\gamma)$ in slanted strips of slope 1 and horizontal width 1. Pack into each strip the maximal possible number of primitive parallelograms. This gives a subdivision of any slanted strip situated above the line $y = x$. For any strip situated below the line $y = x$ place in the remaining part of the strip one primitive triangle (see Figure 3.6(b)).

The total number of such γ-consistent subdivisions is

$$M_m \geq \prod_i \frac{y_i!(y_i + 1)!}{2^{y_i}} \cdot \prod_i y_i! \ , \tag{3.1}$$

where the first product corresponds to subdivisions of $\Delta_+(\gamma)$ and the second one to those of $\Delta_-(\gamma)$.

All the constructed subdivisions of Δ_m are simple and odd, each of them is dual to an irreducible tropical curve and contributes 1 to the Welschinger invariant. The irreducibility of the dual tropical curve can easily proved by the following induction. Let us scan the subdivision by vertical lines from right to left. The rightmost fragment of the tropical curve is dual to the primitive triangle $\Delta_m \cap \{x \geq m-1\}$, so it is irreducible. At the i-th step, $i > 0$, we look at the irreducible components of the curve dual to the union of those elements of our subdivision which intersect the strip $m-i-1 < x < m-i$. Each of these irreducible components either connects the lines $x = m-i-1$ to $x = m-i$, or contains a pattern dual to a triangle with an edge on $x = m-i$, or contains a pattern dual to a slanted parallelogram. Therefore, each component joins the curve dual to the subdivision of $\Delta_m \cap \{x \geq m-i\}$.

We have $W_m \geq M_m$. This gives the first statement of Theorem 3.13, and since $\log N_m = 3m \log m + O(m)$, it remains to check that

$$\log M_m \geq 3m \log m + O(m). \qquad \square$$

Remark 3.15. The statements similar to Theorem 3.13 are proved for all unnodal (i.e., not containing any rational $(-n)$-curve, $n \geq 2$) toric Del Pezzo surfaces equipped with their standard real structure, see [28]. (A *real structure* on a complex variety X is an anti-holomorphic involution $conj : X \to X$. A subvariety $C \subset X$ is *real* with respect to $conj$ if $conj(C) = C$. The standard real structure on a toric variety is the one which is naturally compatible with the toric structure.) Recall that there are five unnodal toric Del Pezzo surfaces: $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, and P_k, $k = 1, 2, 3$, where P_k is the projective plane $\mathbb{C}P^2$ blown up at k points in general position. The same asymptotic statements are also proved for all unnodal toric Del Pezzo surfaces equipped with any non-standard real structure except the standard real $\mathbb{C}P^1 \times \mathbb{C}P^1$ blown up at two imaginary conjugated points; see [61, 30]. Recently, E. Brugallé and G. Mikhalkin [3] proved the statements similar to Theorem 3.13 and Remark 3.12 for Welschinger invariants of $\mathbb{C}P^3$.

3.11 Recurrence formula for Welschinger invariants

As it is shown in [29], the Welschinger invariants W_m can also be calculated using a recurrence formula. This formula can be seen as a real analog of the Caporaso–Harris formula [4] for relative Gromov–Witten invariants of $\mathbb{C}P^2$.

Denote by $\mathcal{G}$ the semigroup of sequences $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}^\infty$ with non-negative terms and finite norm $\|\alpha\| = \sum_i \alpha_i$. Each element of $\mathcal{G}$ contains only finitely many non-zero terms, so in the description of concrete sequences we omit zero terms after the last non-zero one. The only exception concerns the zero element of $\mathcal{G}$ (the sequence with all the terms equal to zero). This element is denoted by (0). For an element α in $\mathcal{G}$, put $J\alpha = \sum_{i=1}^\infty (2i-1)\alpha_i$. Define in $\mathcal{G}$ the following natural partial order: if each term of a sequence α is greater than or equal to the corresponding term of a sequence β, then we say that α is greater than or equal to β and write $\alpha \geq \beta$. For two elements $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ of $\mathcal{G}$ such that $\alpha \geq \beta$, the sequence $\alpha - \beta$, whose i-th term is equal to $\alpha_i - \beta_i$, is an element of $\mathcal{G}$. Denote by θ_k the element in $\mathcal{G}$ whose k-th term is equal to 1 and all the other terms are equal to 0.

For $\alpha, \alpha^{(1)}, \ldots, \alpha^{(s)} \in \mathcal{G}$ such that $\alpha \geq \alpha^{(1)} + \cdots + \alpha^{(s)}$ put

$$\binom{\alpha}{\alpha^{(1)}, \ldots, \alpha^{(s)}} = \prod_{i=1}^{\infty} \frac{\alpha_i!}{\alpha_i^{(1)}! \cdots \alpha_i^{(s)}!(\alpha_i - \sum_{k=1}^{s} \alpha_i^{(k)})!} .$$

Theorem 3.16. (I. Itenberg, V. Kharlamov, E. Shustin; cf. [29])) *Consider the family of numbers $W_m(\alpha, \beta)$ (indexed by a positive integer m and sequences α, β in $\mathcal{G}$ such that $J\alpha + J\beta = m$) defined by the initial conditions $W_1((1),(0)) = W_1((0),(1)) = 1$ and the recurrence relation (valid for any $m \geq 2$)*

$$W_m(\alpha, \beta) = \sum_{\substack{k \geq 1 \\ \beta_k > 0}} W_m(\alpha + \theta_k, \beta - \theta_k)$$

$$+ \sum \binom{\alpha}{\alpha^{(1)}, ..., \alpha^{(s)}} \frac{n!}{n_1! ... n_s!} \prod_{i=1}^{s} \left(\binom{\beta^{(i)}}{\widetilde{\beta}^{(i)}} W_{m^{(i)}}(\alpha^{(i)}, \beta^{(i)}) \right) , \qquad (3.2)$$

where

$$n = 2m + \|\beta\| - 2, \quad n_i = 2m^{(i)} + \|\beta^{(i)}\| - 1, \ i = 1, ..., s,$$

and the latter sum in formula (3.2) is taken over all collections $(m^{(1)}, \ldots, m^{(s)})$, $(\alpha^{(1)}, \ldots, \alpha^{(s)})$, $(\beta^{(1)}, \ldots, \beta^{(s)})$, and $(\widetilde{\beta}^{(1)}, \ldots, \widetilde{\beta}^{(s)})$ considered up to simultaneous permutations and satisfying the relations

$$m^{(i)} \in \mathbb{Z}, \quad m_i \geq 0, \quad \alpha^{(i)}, \beta^{(i)}, \widetilde{\beta}^{(i)} \in \mathcal{G}, \quad J\alpha^{(i)} + J\beta^{(i)} = m^{(i)}, \ i = 1, \ldots, s,$$

$$\sum_{i=1}^{s} m^{(i)} = m - 1, \quad \sum_{i=1}^{s} \alpha^{(i)} \leq \alpha, \quad \sum_{i=1}^{s} \beta^{(i)} = \beta + \sum_{i=1}^{s} \widetilde{\beta}^{(i)},$$

$$s = \| \sum_{i=1}^{s} \beta^{(i)} - \beta \|, \quad \|\widetilde{\beta}^{(i)}\| = 1, \quad \beta^{(i)} \geq \widetilde{\beta}^{(i)}, \ i = 1, \ldots, s .$$

Then, for any positive integer m, we have $W_m((0),(m)) = W_m$.

The numbers $W_m(\alpha, \beta)$ are *tropical relative Welschinger invariants*. They can be interpreted as numbers of tropical curves subject to certain constraints and counted with appropriate multiplicities (see [29]). Theorem 3.16 is a particular case of Theorem 4 in [29]. The latter theorem deals with tropical analogs of curves of arbitrary genus on any unnodal toric Del Pezzo surface. The proof follows ideas of A. Gathmann and H. Markwig [13, 14] who suggested a tropical version of the Caporaso–Harris formula.

3.12 Welschinger invariants $W_{m,i}$

We end these lectures with a definition and some properties of the Welschinger invariants $W_{m,i}$.

Let m be a positive integer. Consider a configuration U of $3m-1$ points in general position in $\mathbb{C}P^2$ such that U is *real*, that is invariant under the involution of complex conjugation c acting in $\mathbb{C}P^2$. If the configuration U contains a non-real point z, then U contains also the point $c(z)$. Denote by i the number of pairs of conjugated non-real points in U.

As in Section 3.7, consider the set of real rational curves of degree m passing through all the points of U. Denote by $R_m(U)$ the number of curves in the set considered, and by $R_m^{\mathrm{even}}(U)$ (resp., $R_m^{\mathrm{odd}}(U)$) the number of curves in this set which have an even (resp., odd) number of solitary nodes. Define the Welschinger number $W_{m,i}(U)$ as $R_m^{\mathrm{even}}(U) - R_m^{\mathrm{odd}}(U)$.

Theorem 3.17. (J.-Y. Welschinger [73, 74]). *The number $W_{m,i}(U)$ does not depend on the choice of a (generic) real configuration U provided that the number of pairs of conjugated non-real points in U is equal to i.*

The number $W_{m,i}(U)$ is also called Welschinger invariant and is denoted by $W_{m,i}$. Of course, $W_{m,0} = W_m$.

A calculation similar to that made in Section 3.8 shows that $W_{3,i} = 8 - 2i$ for any integer $0 \le i \le 4$. Notice that in this case the values $W_{3,i}$ are interpolated by a linear function. Whatever is the integer $0 \le i \le 4$, the number $R_{3,i}(U)$ of real rational cubics attains the value $W_{3,i}$ for a suitable generic configuration U.

To calculate the Welschinger invariants $W_{m,i}$ for quartics and quintics, one can use birational transformations and Welschinger's wall-crossing formula (see [74], Theorem 2.2) which expresses the first finite difference of the function $i \mapsto W_{m,i}$ as twice the Welschinger invariant of $\mathbb{C}P^2$ blown up at one real point. For quartics the answer is as follows (see [28] for details):

i	0	1	2	3	4	5
W	240	144	80	40	16	0

These values $W_{4,i}$ are interpolated by a polynomial of degree 3,

$$W_{4,i} = -\frac{4}{3}i(i-1)(i-2) + 16i(i-1) - 96i + 240.$$

For quintics the Welschinger invariants take the values

i	0	1	2	3	4	5	6	7
W	18264	9096	4272	1872	744	248	64	64

which are interpolated by a polynomial of degree 6,

$$\begin{aligned} W_{5,i} = {} & \tfrac{4}{45}i(i-1)(i-2)(i-3)(i-4)(i-5) \\ & -\tfrac{32}{15}i(i-1)(i-2)(i-3)(i-4) + 32i(i-1)(i-2)(i-3) \\ & -320i(i-1)(i-2) + 2172i(i-1) - 9168i + 18264. \end{aligned}$$

In the cases $m = 3$, 4, and 5, the degree of the interpolating polynomials happens to be smaller than for a generic interpolation data, that is, smaller than $[\frac{3d-1}{2}]$. It is no more the case for any $m \ge 6$.

One of the facts known about the Welschinger invariants $W_{m,i}$ is the following theorem.

Theorem 3.18. (I. Itenberg, V. Kharlamov, E. Shustin; see [28])) *Let $m \geq 3$ be an integer, and i a non-negative integer such that $i \leq 3$. Then, the Welschinger invariant $W_{m,i}$ is positive. Moreover,*

$$W_{m,0} > W_{m,1} > W_{m,2}.$$

Furthermore, for a family of Welschinger invariants $(W_{m,i})_{m\in\mathbb{N}}$, $m \geq 3$, with a given $i \leq 3$, one has

$$\log W_{m,i} = \log N_m + O(m).$$

The proof of Theorem 3.18 is based on the tropical formulas obtained by E. Shustin [60].

Remark 3.19. Statements of the same nature as Theorem 3.18 are proved for all unnodal toric Del Pezzo surfaces equipped with their standard real structure; see [28]. In the case of $\mathbb{C}P^1 \times \mathbb{C}P^1$, the Welschinger invariants depend on three integers: the bi-degree (m_1, m_2) of the real rational curves under consideration and the number i of conjugated non-real points in a given configuration of points. In this case, one can improve the result of Theorem 3.18 and show that all the Welschinger invariants $W_{(m_1,m_2),i}$ (these invariants are defined if m_1 and m_2 are positive integers, and i is a non-negative integer such that $i < m_1 + m_2$) of $\mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the standard real structure $(z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2})$ are positive; see [28].

3.13 Exercises

Exercise 3.1. Find a convex polygon $\Delta \subset \mathbb{R}^2$ with integer vertices and functions $\varphi_1, \varphi_2 : A \to \mathbb{R}$, where $A = \Delta \cap \mathbb{Z}^2$, such that the underlying sets of the tropical curves $T(A, \varphi_1)$ and $T(A, \varphi_2)$ coincide, but the subdivisions of Δ defined by φ_1 and φ_2 do not.

Exercise 3.2. Let $\Delta \subset \mathbb{R}^2$ be a convex polygon with integer vertices, and $\nu : \Delta \to \mathbb{R}$ a convex piecewise-linear function defining a primitive triangulation of Δ (i.e., a triangulation whose vertices are integer and whose triangles are primitive). Show that the tropical curve $T(A, \varphi)$, where $A = \Delta \cap \mathbb{Z}^2$ and $\varphi = \nu\big|_A$, is homotopy equivalent to a bouquet of n circles, n being the number of interior integer points of Δ.

Exercise 3.3. Let $A \subset \mathbb{Z}^2$ be a finite non-empty set, and $\varphi : A \to \mathbb{R}$ a function. For any $c \in \mathbb{Z}^2$, put $A' = A + c$ and consider the function $\varphi' : A' \to \mathbb{R}$ defined by $f'(x) = f(x - c)$. Prove that the tropical curves $T(A', f')$ and $T(A, f)$ coincide.

Exercise 3.4. Let $A = \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{Z}^2$. The tropical curves associated with the pairs of the form (A, φ), where $\varphi : A \to \mathbb{R}$ is a function, are called *tropical curves of bi-degree* $(1,1)$. Show that, for any three points in general position in $\mathbb{R}^2$, there exists precisely one tropical curve of bi-degree $(1,1)$ passing through these points.

Exercise 3.5. Let $A \subset \mathbb{Z}^2$ be a set of $n \geq 2$ points. Denote by $\mathcal{T}(A)$ the set of tropical curves associated with the pairs of the form (A, φ), where $\varphi : A \to \mathbb{R}$ is a function. Prove that, for any $n-1$ points in general position in $\mathbb{R}^2$, there exists precisely one tropical curve $T \in \mathcal{T}(A)$ passing through these points.

Exercise 3.6. Show that any tropical hypersurface T in $\mathbb{R}^n$ is *balanced*, that is, for any $(n-2)$-dimensional face σ of T,

$$\sum_{\delta \supset \sigma} w(\delta) \cdot e(\delta, \sigma) = 0 \ ,$$

where δ runs over all $(n-1)$-dimensional faces containing σ, $w(\delta)$ is the weight of δ, and $e(\delta, \sigma)$ is the smallest integer inner normal vector of $\sigma \subset \delta$. Formulate the converse statement and prove it.

Exercise 3.7. Let A_1 and A_2 be two finite nonempty sets of integer points in $\mathbb{R}^2$. Denote by Δ_i the convex hull of A_i, $i = 1, 2$. Consider functions $\varphi_1 : A_1 \to \mathbb{R}$ and $\varphi_2 : A_2 \to \mathbb{R}$ such that the tropical curves $T(A_1, \varphi_1)$ and $T(A_2, \varphi_2)$ intersect each other only at interior points of their edges. Prove the *tropical Bernstein theorem*: the number of intersection points of $T(A_1, \varphi_1)$ and $T(A_2, \varphi_2)$, counted with the same multiplicities as those defined in the tropical Bézout theorem, is equal to the mixed area of Δ_1 and Δ_2, that is, to the Euclidean area of the Minkowski sum $\Delta_1 + \Delta_2$ diminished by the Euclidean areas of Δ_1 and Δ_2.

Exercise 3.8. Compute the Welschinger invariants

- for rational curves of bi-degree $(2,2)$ on $\mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the real structure $(z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2})$,
- for rational curves of bi-degree $(2,2)$ on $\mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the real structure $(z_1, z_2) \mapsto (\overline{z_2}, \overline{z_1})$.

Exercise 3.9. Using Mikhalkin's algorithm, compute the number of uninodal curves (a curve is *uninodal* if its only singular point is non-degenerate double) of degree $m \geq 3$ which pass through given $(m^2 + 3m - 2)/2$ points in general position in $\mathbb{C}P^2$.