Chapter 1

Introduction to tropical geometry

In this section the notion of an amoeba of a variety will be introduced and several examples of such amoebas are given. Then we consider a degenerations process where an amoeba becomes a piecewise-linear object.

1.1 Images under the logarithm

We start with an algebraic variety over $(\mathbb{C}^*)^n$. Namely, let I be an ideal in the ring of polynomials in n variables over $\mathbb{C}$. Then the variety is given by

$$V = \{x \in (\mathbb{C}^*)^n \mid f(x) = 0 \text{ for all } f \in I\}.$$

We define the map $\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$,

$$(z_1, z_2, \ldots, z_n) \longmapsto (\log|z_1|, \log|z_2|, \ldots, \log|z_n|).$$

Definition 1.1. Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety. Then we define its *amoeba* as $A(V) = \mathrm{Log}(V)$. This is a subset of $\mathbb{R}^n$:

$$\mathcal{A}(V) = \mathrm{Log}\, V \subset \mathbb{R}^n \ .$$

0-dimensional amoebas

If V is 0-dimensional, then it is just a collection of points and so is $\mathrm{Log}(V)$.

Amoeba of a line in $\mathbb{P}^2$

For our first example of an amoeba of a 1-dimensional variety, consider the case when $V \subset (\mathbb{C}^*)^2 \subset \mathbb{CP}^2$ is a line given by equation

$$z + w + 1 = 0. \tag{1.1}$$

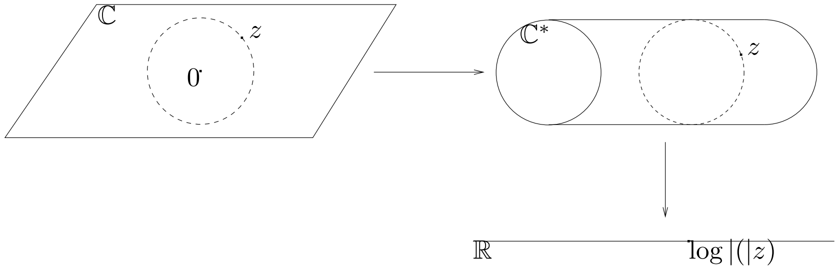

Figure 1.1: Going from $\mathbb{C}$ to $\mathbb{R}$ with Log.

Solving it for w we get $w = -z - 1$.

Set $x = \log|z|$, $y = \log|w|$. The value of x does not determine y as it also depends on the argument of z, but we get the following inequalities. If $x \geq 0$, then

$$\log(e^x - 1) \leq y \leq (1 + e^x);$$

if $x \leq 0$, then

$$\log(1 - e^x) \leq y \leq (1 + e^x).$$

Assume now that $V \subset (\mathbb{C}^*)^2$ is given by $az + bw + c = 0$ with $a, b, c \in (\mathbb{C}^*)$. In coordinates $z' = \frac{a}{c}z$, $w' = \frac{b}{c}w$, we get the equation (1.1) again. Thus the amoeba $\mathcal{A}(V)$ in this case is just a translation of the one pictured in Figure 1.2 by

$$x \mapsto x + \log|c| - \log|a|, \;\; y \mapsto y + \log|c| - \log|b|.$$

If $a, b, c \in R$, then the variety V is defined over the reals and thus we may consider its real locus $\mathbb{R}V$. Note that in this case the amoeba $\mathrm{Log}(\mathbb{R}V)$ is the boundary of the amoeba $\mathcal{A}(V)$.

In the case of a general hypersurface $V \subset (\mathbb{C}^*)^n$ defined over the reals, $\mathrm{Log}(\mathbb{R}V)$ is a subset of the discriminant locus of $\mathrm{Log}|_V : V \to \mathbb{R}^n$, i.e., the locus of the critical values of $\mathrm{Log}|_V$. There is a class of real varieties $\mathbb{R}V$ such that $\mathrm{Log}(\mathbb{R}V)$ *coincides* with the corresponding discriminant locus (see [38] for the case of curves). These varieties have some extremal topological and geometric properties. The lines (and hyperplanes in $(\mathbb{C}^*)^n$) are examples of such extremal hypersurfaces.

Geometric properties of the amoeba

Amoebas of hypersurfaces have the following properties, see [16], [11], [72], [38], [51], [44]. Let $V \subset (\mathbb{C}^*)^n$ be the zero locus of a polynomial

$$f(z) = \sum_{j \in A} a_j z^j,$$

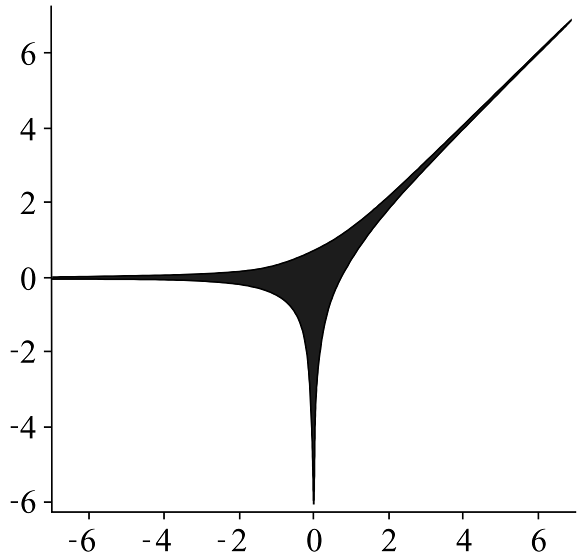

Figure 1.2: The amoeba of $z + w + 1 = 0$.

where $A \subset \mathbb{Z}^n$ is finite, $a_j \in \mathbb{C}$ and $z^j = z_1^{j_1} \dots z_n^{j_n}$ if $z = (z_1, \dots, z_n)$ and $j = (j_1, \dots, j_n)$. Let Δ_f be the *Newton polyhedron* of f, i.e.,

$$\Delta_f = \text{Convex Hull}\{j \in A \mid a_j \neq 0\}.$$

Then

- Every connected component of $\mathbb{R}^n \setminus \mathcal{A}$ is convex.
- The number of components of $\mathbb{R}^n \setminus \mathcal{A}$ is not greater than $\#(\Delta_f \cap \mathbb{Z}^n)$ and not less than the number of vertices of Δ_f.
- There is a naturally defined injection from the set of components of $\mathbb{R}^n \setminus \mathcal{A}$ to $\Delta_f \cap \mathbb{Z}^n$. The vertices of Δ_f are always in the image of this injection. A component of $\mathbb{R}^n \setminus \mathcal{A}$ is bounded if and only if its image is in the interior of Δ_f.
- If $V \subset (\mathbb{C}^*)^2$, then the area of $\mathcal{A}(V)$ is not greater than $\pi^2 \operatorname{Area}(\Delta_f)$.
- If $V \subset (\mathbb{C}^*)^2$ is such that $\operatorname{Area}(\mathcal{A}(V)) = \pi^2 \operatorname{Area}(\Delta_f)$, then the result of coordinatewise multiplication of V by $(c_1, c_2) \in (\mathbb{C}^*)^2$ is the zero set of a polynomial with real coefficients. Furthermore, the real zero set of this polynomial is a real curve with a particular topology in $(\mathbb{R}^*)^2$. If it is non-singular, then its isotopy class in $(\mathbb{R}^*)^2$ depends only on Δ_f. Such curves are called *simple Harnack curves*; historically these were the first examples of curves of degree d in $\mathbb{RP}^2$ with the maximal number of ovals.

- If the coordinates in $(\mathbb{C}^*)^n$ are changed by

$$z_j \mapsto \alpha_j \prod_{k=1}^{n} z_k^{p_{jk}},$$

with $(p_{jk}) \in \mathrm{GL}(n, \mathbb{Z})$, then the amoeba of V is changed by the affine-linear map of R^n, namely by the linear map corresponding to the matrix (p_{jk}) composed with the translation by $(\log|\alpha_1|, \log|\alpha_2|, \ldots, \log|\alpha_n|)$.

1.2 Families of amoebas

Now, when we are acquainted with amoebas (at least in the case of hypersurfaces), let us consider their families and their limits under an appropriate renormalization.

Let the coefficients of f depend on a parameter t. Let

$$V_t = \{z \in (\mathbb{C}^*)^n \mid \sum_{j \in A \subset \mathbb{Z}^n} a_j(t) \cdot z^j = 0\}$$

be the corresponding variety. To begin let us assume that $a_j(t)$ are polynomial functions in $t > 0$.

Let $\mathcal{A}_t(V_t)$ be the amoeba of V_t under the coordinatewise logarithm with base t, i.e.,

$$\begin{aligned} \mathrm{Log}_t : (\mathbb{C}^*)^n &\longrightarrow \mathbb{R}^n, \\ (z_1, z_2, \ldots, z_n) &\longmapsto (\log_t|z_1|, \log_t|z_2|, \ldots, \log_t|z_n|). \end{aligned}$$

We shall see that the limit of $\mathcal{A}_t(V_t)$, $t \to \infty$, exists in the Hausdorff metric on compacts of $\mathbb{R}^n$.

Recall the Hausdorff metric

Let A and B be subsets of some metric space (X, d). Then the Hausdorff distance between A and B is given by

$$d_H(A, B) = \max\{d_{\text{asym}}(A, B), d_{\text{asym}}(B, A)\}$$

where d_{asym} denotes the asymmetric Hausdorff distance

$$d_{\text{asym}}(A, B) = \sup_{a \in A} d(a, B),$$

as usual, $d(a, B) = \inf_{b \in B} d(a, b)$. Note that d_H is indeed a metric on the collection of all closed subsets of X. If $d(A, B) = 0$ and both A and B are closed, then $A = B$.

We say that a family $\{A_t\}$, $t \to \infty$, of subsets of X converges in the *Hausdorff metric on compacts* to a set $A \subset X$, if for every compact set $D \subset X$ there exists a neighborhood $U \supset D$ such that $\lim_{t \to \infty} d_H(A_t \cap U, A \cap U) = 0$.

Proposition 1.2. *The family of subsets $\mathcal{A}_t \subset \mathbb{R}^n$ has a limit in the Hausdorff metric on compacts when $t \to \infty$.*

To get an idea of the proof we suggest the following exercise. Let

$$V_t = \{(z, w) \in (\mathbb{C}^*)^2 \mid a(t)z + b(t)w + c(t) = 0\},$$

where $a(t), b(t), c(t)$ are polynomials.

Exercise.

1. If the polynomials a b and c are non-zero constants, then

$$\lim_{t\to\infty} \mathcal{A}_t = Y \subset \mathbb{R}^2,$$

where

$$Y = \{(s, 0) \in \mathbb{R}^2 \mid s \le 0\} \cup \{(0, s) \in \mathbb{R}^2 \mid s \le 0\} \cup \{(s, s) \in \mathbb{R}^2 \mid s \ge 0\}. \quad (1.2)$$

2. If a is a polynomial of degree k, b is a polynomial of degree l and c is a non-zero constant, then

$$\lim_{t\to\infty} \mathcal{A}_t = \tau_{k,l}(Y) \subset \mathbb{R}^2,$$

where

$$\tau_{k,l} : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x - k, y - l)$$

and Y is defined by (1.2).

1.3 Non-Archimedean amoebas

Alternatively we can view a family of algebraic varieties depending on t as a single variety over a field whose elements are functions of t.

The simplest algebraic functions are just polynomials in t. All such polynomials (with complex coefficients) form the ring $\mathbb{C}[t]$. To introduce division we have to pass to a larger ring $\mathbb{C}((t))$ formed by the Laurent power series in t, i.e., functions

$$\phi(t) = \sum_{j=k}^{+\infty} a_j t^j$$

with $a_j \in \mathbb{C}$. Here we may restrict our attention only to the Laurent series $\phi(t)$ that converge in a neighborhood $U \ni 0$. The ring $\mathbb{C}((t))$ is a field, but this field is not algebraically closed. E.g., the equation $z^2 = t$ does not have solutions in $\mathbb{C}((t))$. To make it algebraically closed one has to consider fractional powers of t and the Puiseux series formed by them.

For our purposes it is convenient to allow not only rational but any real powers of t. We define the field $K\{t\}$ of *Puiseux series with real powers locally converging at zero* by

$$K\{t\} = \{\phi : U \to \mathbb{R} \mid \phi(t) = \sum_{j \in I} a_j t^j,\ a_j \in (\mathbb{C}^*),\ t \in U\},$$

where $0 \in U \in \mathbb{R}$ is some open neighborhood of 0 and $I \subset \mathbb{R}$ is a well ordered set (i.e., any subset of I has a minimum). It can be checked that $K\{t\} \supset \mathbb{C}((t))$ is an algebraically closed field.

Furthermore, there exists a non-Archimedean valuation

$$\text{val} : K\{t\} \to \mathbb{R} \cup \{-\infty\},$$

i.e., a map which satisfies the following properties:

1. $\text{val}(f) = -\infty$ if and only if $f = 0$,
2. $\text{val}(fg) = \text{val}(f) + \text{val}(g)$,
3. $\text{val}(f + g) \leq \max\{\text{val}(f), \text{val}(g)\}$.

We define this valuation by

$$\text{val}(\sum_{j \in I} a_j t^j) = -\min(I).$$

As in the degeneration that we considered earlier we have $t \to \infty$, we set $K = K\{\frac{1}{t}\}$ for our function field.

Every valuation defines a norm by

$$|f|_{\text{val}} = e^{\text{val}(f)}.$$

This norm satisfies the stronger, non-Archimedean, version of the triangle inequality

$$|\phi + \psi|_{\text{val}} \leq \max\{|\phi|_{\text{val}}, |\psi|_{\text{val}}\}. \tag{1.3}$$

Remark 1.3. Recall that the Archimedes axiom states that for any numbers $a, b \in \mathbb{R}$ we will have

$$|na| = |a + \cdots + a| > |b|$$

for some number $n \in \mathbb{N}$, where $|a|$ stands for the standard absolute value in $\mathbb{R}$.

If $\phi, \psi \in K$ and $|\phi| < |\psi|$, then

$$|\psi + \cdots + \psi| = |\psi| < |\psi|,$$

so the Archimedes axiom is not satisfied. However, in the modern terminology, being non-Archimedean refers not simply to the absence of the Archimedes axiom, but specifically to the inequality (1.3) that guarantees its absence.

Since $|\cdot|_{\text{val}}$ is the only norm on the Puiseux series in this section, the subscript val to $|\cdot|$ will be omitted from now on.

As in the case of complex varieties, we may use the norm on K to define amoebas of algebraic varieties over K. Let $V_K \subset (K^*)^n$ be an algebraic variety. Again we define the componentwise logarithm of the absolute values

$$\operatorname{Log}_K = \operatorname{Val} : (K^*)^n \to \mathbb{R}^n$$

by

$$\operatorname{Log}_K(\phi_1, \dots, \phi_n) = (\log|\phi_1|, \dots, \log|\phi_n|) = (\operatorname{val}(\phi_1), \dots, \operatorname{val}(\phi_n)).$$

Then $\mathcal{A}(V_K) = \operatorname{Val}(V_K)$ is the (non-Archimedean) amoeba of V_K.

Theorem 1.4. *[a version of Viro patchworking] Let V_t be an algebraic variety with parameter t as defined above and V_K be the corresponding variety in $(K^*)^n$. Then the non-Archimedean amoeba $\mathcal{A}(V_K)$ is the limit of the amoebas $\mathcal{A}(V_t)$ as t goes to infinity with respect to the Hausdorff metric on compacts. In particular, $\lim_{t\to\infty} \mathcal{A}(V_t)$ exists in this sense.*

Theorem 1.5 (Kapranov). *If $V_K \subset (K^*)^n$ is a hypersurface, then the non-Archimedean amoeba $\mathcal{A}(V_K) = \operatorname{Val}(V_K)$ depends only on the valuation of the coefficients of the defining equation for V_K. In other words, if $V_K = \{\sum \alpha_j(t) z^j = 0\}$, $\alpha_j \in K$, then $A(V_K)$ is determined by the values* $\operatorname{val}(\alpha_j(t))$.

As it was noticed by Kapranov [8] the choice of a particular algebraically closed non-Archimedean field K does not affect the geometry of non-Archimedean amoebas as long as the non-Archimedean valuation $K^* \to \mathbb{R}$ is surjective. Another useful choice for such K is provided by the non-standard analysis and considered in the following subsection. Although the following content does not depend on this subsection and uses $K = K\{\frac{1}{t}\}$ as the ground non-Archimedean field, some people might find this other example more intuitive.

1.4 Non-standard complex numbers

Here we present another construction for a non-Archimedean field K with a surjective valuation to $\mathbb{R}$.[1]

We start by recalling of one of the constructions for a generalized limit in analysis. By an *ultrafilter* on the set of natural numbers $\mathbb{N}$, we mean a finitely additive measure υ with the following three properties.

- For any set $S \subset \mathbb{N}$, either $\upsilon(S) = 0$ or $\upsilon(S) = 1$.
- We have $\upsilon(S) = 0$, if $S \subset \mathbb{N}$ is finite.
- $\upsilon(\mathbb{N}) = 1$.

[1] The author is indebted to M. Kapovich for an illuminating explanation of the asymptotic cone construction in geometry and the relevant point of view on non-standard analysis.

Existence of such υ can be deduced from the axiom of choice.

Definition 1.6. A sequence $a_k \in \mathbb{C}$, $k \in \mathbb{N}$ is called converging to $L \in \mathbb{C}$ with respect to the ultrafilter υ if for any $\epsilon > 0$ we have

$$\upsilon\{k \in \mathbb{N} \mid |a_k - L| \geq \epsilon\} = 0.$$

We say that $a_k \in \mathbb{C}$, $k \in \mathbb{N}$ converges to $L = \infty$ if for any $\epsilon > 0$ we have $\upsilon\{k \in \mathbb{N} \mid |a_k| \leq \epsilon\} = 0$.

We write $\lim\limits_{k \to +\infty}{}^{\upsilon} a_k = L$.

It is easy to see that every sequence of complex number has a limit with respect to υ. E.g., the sequence $0, 1, 0, 1, \dots$ has the limit 0 or 1 depending on whether the measure of all odd numbers is 1 or the measure of all even numbers is 1 (we should get exactly one of these cases for our υ).

Let $\mathbb{C}^\infty = \{\{a_k\}_{k=1}^{+\infty}\}$ be the set of all sequences $\{a_k\}$ with $a_k \in \mathbb{C}$. Define the equivalence relation by setting $\{a_k\} \sim_\upsilon \{b_k\}$ if $\upsilon\{k \in \mathbb{N} \mid a_k \neq b_k\} = 0$. Let $\mathbb{C}^\infty_\upsilon$ be the set of the corresponding equivalence classes.

We may operate with elements of $\mathbb{C}^\infty_\upsilon$ as with usual complex numbers. We can add them, subtract, multiply and divide coordinatewise. Furthermore, the functions on $\mathbb{C}$ extend to $\mathbb{C}^\infty_\upsilon$ by coordinatewise application. Clearly, $\mathbb{C}^\infty_\upsilon$ is an algebraically closed field.

We say that $\rho = \{\rho_k\} \in \mathbb{C}^\infty_\upsilon$ is positive if $\upsilon\{k \in \mathbb{N} \,|\rho_k > 0\} = 1$ (as usual, $\rho_k > 0$ implies, in particular, that $\rho_k \in \mathbb{R}$). We say that ρ is infinitely large if $\lim\limits_{k \to +\infty}{}^{\upsilon} \rho_k = \infty$. Let us fix once and for all a positive infinitely large ρ.

The inequality $|a| < \rho^N$ for $a = \{a_k\} \in \mathbb{C}^\infty_\upsilon$ means that $\upsilon\{k \in \mathbb{N} \mid |a_k| \leq \rho_k^N\} = 1$. Define

$$A = \{a \in \mathbb{C}^\infty_\upsilon \mid \exists N \in \mathbb{N} \ : \ |a| < \rho^N\}.$$

Similarly, we define

$$B = \{a \in \mathbb{C}^\infty_\upsilon \mid |a| < \rho^N \ \forall N \in \mathbb{Z}\}.$$

Define $\mathbb{C}^\rho_\upsilon = A/B$. Clearly, it is an algebraically closed field. Furthermore,

$$\mathrm{Log}_\rho : \mathbb{C}^\rho_\upsilon \to \mathbb{R} \cup \{-\infty\}, \ a \mapsto \log_\rho(|a|) = \{\log_{\rho_k} |a_k|\}$$

is a surjective valuation.

The field $\mathbb{C}^\rho_\upsilon$ can be considered as a field of non-standard complex numbers and we have just seen that it is a non-Archimedean field. The choice of $K = \mathbb{C}^\rho_\upsilon$ allows us to consider the map $\mathrm{Val} = \mathrm{Log}_\rho$. This map is a limiting map in the family Log_t and the limit can be obtained by substitution of the infinitely large value $t = \rho$.

1.5 The tropical semifield $\mathbb{T}$

Definition 1.7. We set the *tropical semifield* $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ to be the real numbers enhanced with $-\infty$ and equipped with the following arithmetic operations (we use quotation marks to distinguish them from the classical arithmetic operations with real numbers)

$$\text{“}x + y\text{”} = \max\{x, y\},$$
$$\text{“}xy\text{”} = x + y.$$

It is easy to check that $\mathbb{T}$ is a commutative semigroup with respect to addition (here $-\infty \in \mathbb{T}$ plays the rôle of the additive zero), a commutative group with respect to multiplication (here $0 \in \mathbb{T}$ plays the rôle of the multiplicative unit) and that we have the distribution law

$$\text{“}x(y + z) = xy + xz\text{”}.$$

In other words, $\mathbb{T}$ is a true semifield. This semifield drastically lacks subtraction: as tropical addition is an idempotent operation, we have “$x + x = x$”.

Remark 1.8. The term *tropical* is borrowed from Computer Science (where it was introduced to commemorate a Brazilian scientist Imre Simon). These arithmetic operations under different names appeared even earlier. E.g., Litvinov and Maslov [36] used the term *idempotent analysis* and related the process of passing from the classical arithmetics to the tropical arithmetics to the *quantization* process of Schrödinger, but in the opposite direction. This is the source for another name for passing to the tropical limit, “*dequantization*”.

We finish this remark with the corresponding deformation of arithmetic operations on $\mathbb{T}$ from “classical” to “tropical” (cf. e.g., [36], [72]). Let $\mathbb{R}_{\geq 0}$ be the semifield of non-negative real numbers equipped with classical arithmetic operations. The map

$$\log_t : \mathbb{R}_{\geq 0} \to \mathbb{T},$$

$t > 1$, induces some arithmetic operations on $\mathbb{T}$. Namely, we have

$$x \oplus_t y = \log_t(t^x + t^y), \ x \otimes_t y = x + y$$

for $x, y \in \mathbb{T}$. Clearly, for any finite $t > 0$ the set $\mathbb{T}$ equipped with these operations is a semifield isomorphic to $\mathbb{R}_{\geq 0}$ (the isomorphism is provided by $\log_t$ itself). In particular, the semifields $(\mathbb{T}, \oplus_t, \otimes_t)$ are mutually isomorphic. However, we have

$$\lim_{t \to +\infty} x \oplus y = \text{“}x + y\text{”}, \ x \otimes_t y = \text{“}xy\text{”},$$

thus the tropical semifield $\mathbb{T}$ (which not isomorphic to $\mathbb{R}_{\geq 0}$ as $\mathbb{T}$ is idempotent and $\mathbb{R}_{\geq 0}$ is not) is the limit of a family of semifields isomorphic to $\mathbb{R}_{\geq 0}$.

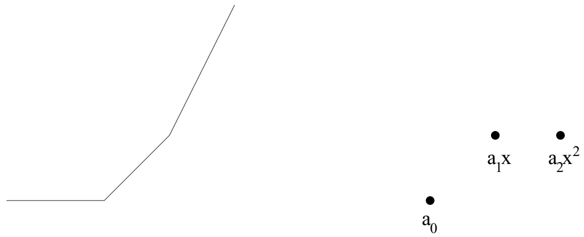

Figure 1.3: The graph of a tropical parabola $a_0 + a_1 x + a_2 x^2$ and the graph of the corresponding function $j \mapsto a_j$.

Tropical polynomials and corresponding tropical hypersurfaces

We do not have subtraction in $\mathbb{T}$, but we do not need it to define polynomials, as a polynomial is a sum of monomials. Denote $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$.

Definition 1.9. Let $A \subset (\mathbb{Z}_{\geq 0})^n$ be finite and $a_j \in \mathbb{T}$ for all $j \in A$. Then a tropical polynomial is given by

$$f(x) = \text{“}\sum_{j \in A} a_j x^j\text{”} = \max_{j \in A}(a_j + \langle j, x \rangle), \qquad (1.4)$$

$x \in \mathbb{T}^n$.

Remark 1.10. Equation (1.4) may recall the definition of the *Legendre transform.* Recall that for an arbitrary function $\varphi : \mathbb{R}^n \to \mathbb{R}$ its Legendre transform L_φ is defined by

$$L_\varphi(x) = \max_j(\langle j, x \rangle - \varphi(j)).$$

This definition makes sense even if φ is defined only on a subset $A \subset \mathbb{R}^n$ — we just take the maximum over $j \in A$ or, equivalently, extend φ to $\mathbb{R}^n$ by setting $\varphi(j) = +\infty$, $j \notin A$. The function L_φ is convex even if φ is not convex. However, if φ is not convex, then there exists a (non-strictly) convex function

$$\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$$

such that $L_{\tilde{\varphi}} = L_\varphi$. It is easy to see that the graph of such $\tilde{\varphi}$ can be obtained from the overgraph of ϕ by taking the convex hull.

We have the tropical polynomial f equal to the Legendre transform of the function $j \mapsto -a_j$ defined on the finite subset $A \subset \mathbb{R}^n$. Since A is finite the function f is a piecewise-linear convex function.

Our next step is to define hypersurfaces associated to tropical polynomials. The neutral element with respect to addition is $-\infty$, but tropical polymomials almost never take value $-\infty$. Because of the lack of subtraction we have to be very careful in phrasing the classical definition of hypersurface in order to make this definition work also for $\mathbb{T}$. For such definition one can take the definition of the zero locus of a polynomial as the locus where the multiplicative inverse of the polynomial is not regular.

Definition 1.11. Let f be a tropical polynomial. Then the corresponding tropical hypersurface V_f is given by

$$V_f = \{x \in \mathbb{R}^n \mid f \text{ is not smooth in } x\} \subset \mathbb{R}^n = (\mathbb{T}^*)^n.$$

It is also called the corner locus of f.

Indeed, near V_f we have "$\frac{0}{f}$" not locally convex and, therefore, not regular.

Example 1.12. The locus of the tropical parabola from Figure 1.3 is formed by the projections onto the x-axis of the two corners of the graph. One can compute that these points are $a_0 - a_1$ and $a_1 - a_2$ if $2a_1 > a_0 + a_1$. Otherwise this locus consists of one *double* point $\frac{a_0 - a_2}{2} \in \mathbb{T}$.

Remark 1.13. For every polynomial f in $K[x_1, \ldots, x_n]$ we can define its *tropicalization* by replacing the Puiseux series coefficients from K with their valuations from $\mathbb{T}$.

Kapranov's Theorem 1.5 may be now reformulated in the following way. *The hypersurface of the tropicalization of a polynomial over K coincides with its (non-Archimedean) amoeba.*

While the study of tropical hypersurfaces (and, as a matter of fact, also complete intersections) is relatively easy, even the definition of general tropical varieties takes time (see [45] and [46]). In the next section we give some basic treatment of general tropical varieties in dimension 1, i.e., *tropical curves*. We treat them as abstract tropical varieties, i.e., in a manner analogous to the Riemann surfaces). The geometric structure serving as a tropical counterpart of complex structure turns out to be integer affine structure.

1.6 Tropical curves and integer affine structure

We start with a preliminary definition that is well-known in classical geometry.

Definition 1.14. Let M be a manifold. An integer affine structure on M is given by an open covering $\{U_j\}$ of M with embedding charts

$$\varphi_j : U_j \longrightarrow \mathbb{R}^n,$$

such that every overlapping transition map

$$\varphi_k \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_k) \to \varphi_k(U_j \cap U_k)$$

can be extended to a map $\mathbb{R}^n \to \mathbb{R}^n$ obtained by the composition of a map

$$\Phi_{kj} : \mathbb{R}^n \to \mathbb{R}^n,$$

linear over $\mathbb{Z}$, and a translation by an arbitrary vector in $\mathbb{R}^n$, in other words, by an element of $\mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$.

A manifold M equipped with an integer affine structure is called a $\mathbb{Z}$-*affine manifold*. Let M and N be two $\mathbb{Z}$-affine manifolds (of dimensions $\dim M$ and $\dim N$), and let $f : M \to N$ be a map. We say that f is a $\mathbb{Z}$-*affine map* if for every $x \in M$ there exist charts $U_j \ni x$ and $U_k^{(N)} \ni f(x)$ such that $\phi_k^{(N)} \circ f \circ \phi^{-1}$ can be extended to a map $\mathbb{R}^{\dim M} \to \mathbb{R}^{\dim N}$ obtained by the composition of a map $\mathbb{R}^{\dim M} \to \mathbb{R}^{\dim N}$, linear over $\mathbb{Z}$, and a translation by an arbitrary vector in $\mathbb{R}^{\dim N}$.

Given an integer affine structure on M, one can make it a full affine structure by including in the set of charts all $\mathbb{Z}$-affine embeddings $M \supset U \to \mathbb{R}^n$. Recall that an (integer) affine structure is called *complete* if for every chart $\varphi_j : U_j \to \mathbb{R}^n$ and $y \in \mathbb{R}^n$ there exists a finite sequence of charts $U_j = U_{j_0}, \ldots, U_{j_l}$ such that $U_{j_{m-1}} \cap U_{j_m} \neq \emptyset$ and $\phi_{j_l}(U_l) \ni y$.

Note that since $U_{j_{m-1}} \cap U_{j_m} \neq \emptyset$, affine-linear maps $\Phi_{j_m j_{m-1}} : \mathbb{R}^n \to \mathbb{R}^n$ are defined. One can check that the composition

$$\tilde{\Phi}_j(y) = \phi_{j_l}^{-1} \circ \Phi_{j_m j_{m-1}} \circ \ldots \Phi_{j_1 j_0}(y) \in M$$

does not depend on the choice of the chart sequence and gives a well-defined covering map $\tilde{\Phi}_j : \mathbb{R}^n \to M$ called the *developing map*, cf. [66].

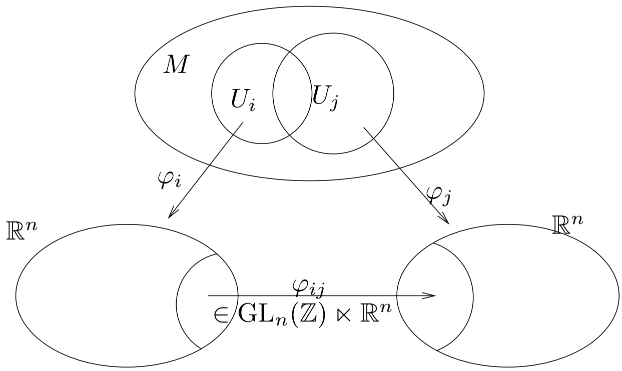

Figure 1.4: Illustrating the $\mathbb{Z}$-affine structure.

We have a well-defined notion of an *integer tangent vector* to a $\mathbb{Z}$-affine manifold M. These are the vectors corresponding to vectors in $\mathbb{Z}^n$ under the differential of charts ϕ_j. We say that an integer tangent vector is *primitive* if it is not a non-trivial (not ± 1) integer multiple of another integer tangent vector.

Note that specifying a $\mathbb{Z}$-affine structure on a 1-manifold is equivalent to specifying a metric. Indeed, we have

$$\mathrm{GL}_n(\mathbb{Z}) = O(1);$$

the metric is specified by setting the primitive tangent vector (which is unique up to the sign) to have a unit length.

The simplest class of examples of $\mathbb{Z}$-affine manifolds is provided by quotients $\mathbb{R}^n/\Lambda$ where Λ is a *lattice* in $\mathbb{R}^n$, i.e., a discrete subgroup of $\mathbb{R}^n$ isomorphic to $\mathbb{Z}^n$. These examples play the role of tropical tori; some of them admit a polarization and thus are tropical Abelian varieties (see [47]).

However, most tropical varieties of dimension n are not topologically manifolds. They are polyhedral complexes of dimension n which still come equipped with an integer affine structure. In this section we look only at the case $n = 1$. Topologically tropical curves are graphs. Compact tropical curves are finite graphs.

Let Γ be a finite graph. Instead of defining a $\mathbb{Z}$-affine structure in the non-manifold case for a general polyhedral complex (see [45]), we take advantage of dimension 1 where we can express a $\mathbb{Z}$-affine structure by using a metric. Denote

$$\Gamma^\circ = \Gamma \backslash \{1 - \text{valent vertices}\}. \tag{1.5}$$

Definition 1.15. A compact tropical curve is a connected finite graph Γ equipped with a complete inner metric on Γ°.

Thus the length of an edge of Γ is infinite if and only if this edge is adjacent to a 1-valent vertex. The integer affine structure near a k-valent vertex $x \in \Gamma$ with $k > 1$ can be thought as an isometric map

$$\phi : U \to Y \subset \mathbb{R}^{k-1} \tag{1.6}$$

of a neighborhood $U \ni x$ to the subspace $Y \subset \mathbb{R}^{k-1}$ obtained as the union of the negative part of the $k-1$ axes of $\mathbb{R}^{k-1}$ (considered with the metric induced by the Euclidean metric on $\mathbb{R}^n$) and the ray $R = \{(t, \dots, t) \in \mathbb{R}^{k-1} \mid t \geq 0\}$, where the metric is induced by the Euclidean metric on $\mathbb{R}^n$ scaled by $\sqrt{k-1}$ (so that the length of the primitive integer vector $(1, \dots, 1)$ is unity).

The restriction of tropical polynomials from $\mathbb{R}^{k-1}$ to Y define regular functions on open sets of Γ. Together they form *the structure sheaf* $\mathcal{O}$.

Note that if Γ is a 3-valent graph with n univalent vertices, then the number of finite edges of Γ is equal to $3(\dim H_1(\Gamma)) - 3 + n$ if $2g + n > 2$, where $g = \dim H_1(\Gamma)$. Furthermore, all tropical curves with n *marked* (i.e., numbered) 1-valent vertices and with $\dim H_1(\Gamma) = g$ form *the tropical moduli space* $\mathcal{M}_{g,n}^{\text{trop}}$ that can be naturally compactified. It can be shown that this compactification is a tropical orbifold (of dimension $3g - 3 + n$) as long as $2g + n > 2$. Furthermore, if $g = 0$, then it is a manifold.

Definition 1.16. The number $g = b_1(\Gamma)$ is called *the genus* of a tropical curve Γ.

As in the complex case the genus g can be interpreted as the dimension of regular 1-forms on Γ (see [47]). All regular forms can be used to form the *Jacobian variety* of Γ. As in classical geometry one has the tropical counterpart of the Abel–Jacobi theorem. The Riemann–Roch theorem holds in the form of an inequality and can be used to give a lower bound for the dimension of the space of deformations of a tropical curve.

Many other (but not all, see e.g., [54]) classical theorems for complex curves also have their tropical counterpart. As we will not need them in the applications of tropical geometry considered in the next two chapters we just refer the reader to [45], [46], [47]. To relate abstract tropical curves to these applications we finish this section by looking at tropical maps $h : \Gamma^\circ \to \mathbb{R}^n$, where Γ° is defined by (1.5).

Such a map h is tropical if it is given by a $\mathbb{Z}$-affine map in every small chart (1.6). The following proposition translates this to the language of metric graphs.

Proposition 1.17. *Let Γ be a compact tropical curve and Γ° be its finite part. A map*

$$h : \Gamma^\circ \to \mathbb{R}^n$$

is tropical if and only if the following two conditions hold.

- *For every edge $E \subset \Gamma^\circ$, $h|_E : E \to \mathbb{R}^n$ is a smooth map such that dh maps every unit tangent vector to E to an integer vector in $\mathbb{R}^n$.*
- *For every k-valent vertex $x \in \Gamma^\circ$ denote with $v_1, \dots, v_k$ the outgoing unit tangent vectors to the edges $E_1, \dots, E_k$ adjacent to x. Then we have*

$$\sum_{j=1}^{k} (dh|_{E_j})_x v_j = 0.$$

Clearly for every edge $E \subset \Gamma^\circ$ the image $W \subset \mathbb{Z}^n$ of the unit tangent vector is the same for all points of E (up to multiplication by -1). The largest natural divisor $w \in \mathbb{N}$ of W (i.e., the GCD of its coordinates) is called *the weight* of the image $h(E)$. In the remaining part of our discussion we will be looking at the images $h(\Gamma^\circ) \subset \mathbb{R}^2$ and using them for the needs of classical real and complex geometry.

1.7 Exercises

Exercise 1.1. Let $V \subset (\mathbb{C}^*)^2$ be given by $z + w + 1 = 0$.

- Write down explicit inequalities defining the amoeba $\mathrm{Log}(V) \subset \mathbb{R}^2$.
- Write down explicit inequalities defining the image $\mu(V) \subset \mathbb{R}^2$, where $\mu : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ is the moment map given by

$$(z, w) \mapsto \left(\frac{|z|^2}{1 + |z|^2 + |w|^2}, \frac{|w|^2}{1 + |z|^2 + |w|^2} \right).$$

- Prove that $\mathrm{Log}\big|_{(\mathbb{R}_+)^2}$ and $\mu\big|_{(\mathbb{R}_+)^2}$ are both diffeomorphisms onto their images, and find these images.

- Prove that, for any convex lattice polytope $\Delta \subset \mathbb{R}^n$, the moment map

$$\mu_\Delta(x) = \frac{\sum_{\omega \in \Delta \cap \mathbb{Z}^n} x^\omega \cdot \omega}{\sum_{\omega \in \Delta \cap \mathbb{Z}^n} x^\omega}$$

is a real analytic diffeomorphism of the positive orthant $\mathbb{R}^n_+$ onto the interior $\mathrm{Int}(\Delta)$ of Δ.

Exercise 1.2. Find a family $\{A_t\}$, $t \to +\infty$ of subsets of $\mathbb{R}$, a set $A \subset \mathbb{R}$, and a set $U \subset \mathbb{R}$ such that

$$\lim_{t \to +\infty} d_H(A_t, A) = 0 \text{ and } \lim_{t \to +\infty} d_H(A_t \cap U, A \cap U) = +\infty.$$

Exercise 1.3. Denote by $\mathbb{K}$ the field of formal power series $\sum_{r \in J} a_r t^r$ with complex coefficients, where $J \subset \mathbb{R}$ is well-ordered. Let $V \subset (\mathbb{K}^*)^2$ be the non-Archimedean curve defined by the equation $1 + z + w + tzw = 0$, and let $V_f \subset \mathbb{R}^2$ be the tropical curve given by the tropical polynomial $f(x, y) =$ "$0 + x + y + 1 \cdot x \cdot y$". Prove that $\mathrm{Log}_{\mathbb{K}}(V) = V_f$.

Exercise 1.4. Let K be a field with a non-Archimedean valuation $\mathrm{val} : K^* \to \mathbb{R}$, $A \subset \mathbb{Z}^2$ a non-empty finite set, $F(z, w) = \sum_{(i,j) \in A} a_{ij} z^i w^j$ a Laurent polynomial over K. Show that the set

$$\mathcal{A}(F) = \{(\mathrm{val}(z), \mathrm{val}(w)) \ : \ F(z, w) = 0, \ z, w \in K^*\}$$

is contained in the tropical curve V_f defined by the tropical polynomial $f(x) = \max_{j \in A}(a_j + \langle j, x \rangle)$. Assuming in addition that K is algebraically closed of characteristic zero, and the valuation val is dominant (surjective), prove that $\mathcal{A}(F)$ is dense in (coincides with) V_f.

Exercise 1.5. (A research problem.) Can you find a tropical curve in $\mathbb{R}^3$ that is the limit of amoebas of real rational algebraic curves which are knotted (*e.g.*, realize a trefoil knot)?