Ring and Module Theory

Toma Albu
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2000 Mathematics Subject Classification 13, 16; 17W, 18E

Library of Congress Control Number: 2010927812

Bibliographic information published by Die Deutsche Bibliothek. Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at http://dnb.ddb.de

ISBN 978-3-0346-0006-4

P.O. Box 133, CH-4010 Basel, Switzerland
Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Cover Design: Alexander Faust, Basel, Switzerland
Printed in Germany

ISBN 978-3-0346-0006-4 e-ISBN 978-3-0346-0007-1

9 8 7 6 5 4 3 2 1 www.birkhauser.ch

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Preface

This volume is a collection of 13 peer reviewed papers consisting of expository/survey articles and research papers by 24 authors. Many of these papers were presented at the International Conference on Ring and Module Theory held at Hacettepe University in Ankara, Turkey, during August 18–22, 2008.

The selected papers and articles examine wide ranging and cutting edge developments in various areas of Algebra including Ring Theory, Module Theory, Hopf Algebras, and Commutative Algebra. The survey articles are by well-known experts in their respective areas and provide an overview which is useful for researchers in the area, as well as, for researchers looking for new or related fields to investigate. The research papers give a taste of current research. We feel the variety of topics will be of interest to both graduate students and researchers.

We wish to thank the large number of conference participants from over 20 countries, the contributors to this volume, and the referees. Encouragement and support from Hacettepe University, The Scientific and Technological Research Council of Turkey (TÜBİTAK) and Republic of Turkey Ministry of Culture and Tourism are greatly appreciated. We also appreciate Evrim Akalan, Sevil Barın, Canan Celep Yücel, Esra Demiryürek, Özlem Erdoğan, Fatih Karabacak, Didem Kavalcı, Mine Polat, Tuğçe Sivrikaya, Ayşe Sönmez, Figen Takıl, Muharrem Yavuz, Filiz Yıldız and Uğur Yücel for their assistance and efficient arrangement of the facilities which greatly contributed to the success of the conference.

Finally, we must thank Erkan Afacan of Gazi University for his excellent job of typing and uniformizing manuscripts.

December, 2009

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Ring and Module Theory
Trends in Mathematics, 1–26

A Seventy Years Jubilee: The Hopkins-Levitzki Theorem

Toma Albu

Dedicated to the memory of Mark L. Teply (1942–2006)

Abstract. The aim of this expository paper is to discuss various aspects of the Hopkins-Levitzki Theorem (H-LT), including the Relative H-LT, the Absolute or Categorical H-LT, the Latticial H-LT, as well as the Krull dimension-like H-LT.

Mathematics Subject Classification (2000). Primary 16-06, 16P20, 16P40; Secondary 16P70, 16S90, 18E15, 18E40.

Keywords. Hopkins-Levitzki Theorem, Noetherian module, Artinian module, hereditary torsion theory, τ-Noetherian module, τ-Artinian module, quotient category, localization, Grothendieck category, modular lattice, upper continuous lattice, Krull dimension, dual Krull dimension.

1. Introduction

In this expository paper we present a survey of the work done in the last forty years on various extensions of the *Classical Hopkins-Levitzki Theorem*: *Relative, Absolute* or *Categorical, Latticial,* and *Krull dimension-like*.

We shall also illustrate a *general strategy* which consists on putting a *module-theoretical* theorem in a *latticial frame*, in order to translate that theorem to Grothendieck categories and module categories equipped with hereditary torsion theories.

The author gratefully acknowledges partial financial support from the Grant ID-PCE 1190/2008 awarded by the Consiliul Naţional al Cercetării Ştiinţifice în Învăţământul Superior (CNCSIS), România.

The (Molien-)Wedderburn-Artin Theorem

One can say that the Modern Ring Theory begun in 1908, when *Joseph Henry Maclagan Wedderburn* (1882–1948) proved his celebrated Classification Theorem for finitely dimensional semi-simple algebras over a field F (see [49]). Before that, in 1893, *Theodor Molien* or *Fedor Eduardovich Molin* (1861–1941) proved the theorem for $F = \mathbb{C}$ (see [36]).

In 1921, *Emmy Noether* (1882–1935) considers in her famous paper [42], for the first time in the literature, the *Ascending Chain Condition* (ACC)

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

for ideals in a commutative ring R.

In 1927, *Emil Artin* (1898–1962) introduces in [17] the *Descending Chain Condition* (DCC)

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

for left/right ideals of a ring and extends the Wedderburn Theorem to rings satisfying both the DCC and ACC for left/right ideals, observing that both ACC and DCC are a good substitute for finite dimensionality of algebras over a field:

THE (MOLIEN-)WEDDERBURN-ARTIN THEOREM. *A ring R is semi-simple if and only if R is isomorphic to a finite direct product of full matrix rings over skew-fields*

$$R \simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

Recall that by a *semi-simple ring* one understands a ring R which is left (or right) Artinian and has Jacobson radical or prime radical zero. Since 1927, the (Molien-)Wedderburn-Artin Theorem became a cornerstone of the Noncommutative Ring Theory.

In 1929, Emmy Noether observes (see [43, p. 643]) that the ACC in Artin's extension of the Wedderburn Theorem can be omitted: *Im II. Kapitel werden die Wedderburnschen Resultate neu gewonnen und weitergefürt, ... Und zwar zeigt es sich das der "Vielfachenkettensatz" für Rechtsideale oder die damit identische "Minimalbedingung"* (*in jeder Menge von Rechtsidealen gibt es mindestens ein – in der Menge – minimales*) *als Endlichkeitsbedingung ausreicht* (*Die Wedderburnschen Schlußweissen lassen sich übertragen wenn "Doppelkettensatz" vorausgesezt wird. Vgl. E. Artin* [17]).

It took, however, ten years until it has been proved that always the DCC in a unital ring implies the ACC.

The Classical Hopkins-Levitzki Theorem (H-LT)

One of the most lovely result in Ring Theory is the *Hopkins-Levitzki Theorem*, abbreviated H-LT. This theorem, saying that any right Artinian ring with identity is right Noetherian, has been proved independently in 1939 by *Charles Hopkins*

[27][1] (1902–1939) for left ideals and by *Jacob Levitzki* [31][2] (1904–1956) for right ideals. Almost surely, the fact that the DCC implies the ACC for one-sided ideals in a unital ring was unknown to both E. Noether and E. Artin when they wrote their pioneering papers on chain conditions in the 1920's.

An equivalent form of the H-LT, referred in the sequel also as the *Classical H-LT*, is the following one:

Classical H-LT. *Let R be a right Artinian ring with identity, and let M_R be a right module. Then M_R is an Artinian module if and only if M_R is a Noetherian module.*

Proof. The standard proof of this theorem, as well as the original one of Hopkins [27, Theorem 6.4] for $M = R$, uses the Jacobson radical J of R. Since R is right Artinian, J is nilpotent and the quotient ring R/J is a semi-simple ring. Let n be a positive integer such that $J^n = 0$, and consider the descending chain of submodules of M_R

$$M \supseteq MJ \supseteq MJ^2 \supseteq \cdots \supseteq MJ^{n-1} \supseteq MJ^n = 0.$$

Since the quotients MJ^k/MJ^{k+1} are killed by J, $k = 0, 1, \ldots, n-1$, each MJ^k/MJ^{k+1} becomes a right module over the semi-simple ring R/J, so each MJ^k/MJ^{k+1} is a semi-simple (R/J)-module.

Now, observe that M_R is Artinian (resp. Noetherian) $\Longleftrightarrow$ all MJ^k/MJ^{k+1} are Artinian (resp. Noetherian) R (or R/J)-modules. Since a semi-simple module is Artinian if and only if it is Noetherian, it follows that M_R is Artinian if and only if it is Noetherian, which finishes the proof. □

Extensions of the H-LT

In the last fifty years, especially in the 1970's, 1980's, and 1990's the (Classical) H-LT has been generalized and dualized as follows:

1957 *Fuchs* [21] shows that a left Artinian ring A, not necessarily unital, is Noetherian if and only if the additive group of A contains no subgroup isomorphic to the Prüfer quasi-cyclic p-group $\mathbb{Z}_{p^\infty}$.

---

[1] In fact, he proved that any left Artinian ring (called by him *MLI* ring) with left or right identity is left Noetherian (see Hopkins [27, Theorems 6.4 and 6.7]).

[2] The result is however, surprisingly, neither stated nor proved in his paper, though in the literature, including our papers, the Hopkins' Theorem is also wrongly attributed to Levitzki. Actually, what Levitzki proved was that the ACC is superfluous in most of the main results of the original paper of Artin [17] assuming both the ACC and DCC for right ideals of a ring. This is also very clearly stated in the Introduction of his paper: "*In the present note it is shown that the maximum condition can be omitted without affecting the results achieved by Artin.*" Note that Levitzki considers rings which are not necessarily unital, so anyway it seems that he was even not aware about DCC implies ACC in unital rings; this implication does not hold in general in non unital rings, as the example of the ring with zero multiplication associated with any Prüfer quasi-cyclic p-group $\mathbb{Z}_{p^\infty}$ shows. Note also that though all sources in the literature, including Mathematical Reviews, indicate 1939 as the year of appearance of Levitzki's paper in *Compositia Mathematica*, the free reprint of the paper available at http://`www.numdam.org` indicates 1940 as the year when the paper has been published.

1972 *Shock* [46] provides necessary and sufficient conditions for a non unital Artinian ring and an Artinian module to be Noetherian; his proofs avoid the Jacobson radical of the ring and depend primarily upon the length of a composition series.

1976 *Albu* and *Năstăsescu* [9] prove the *Relative H-LT*, i.e., the H-LT relative to a hereditary torsion theory, but only for commutative unital rings, and conjecture it for arbitrary unital rings.

1978–1979 *Murase* [37] and *Tominaga* and *Murase* [48] show, among others, that a left Artinian ring A, not necessarily unital, is Noetherian if and only J/AJ is finite (where J is the Jacobson radical of R) if and only if the largest divisible torsion subgroup of the additive group of A is 0.

1979 *Miller* and *Teply* [35] prove the *Relative H-LT* for arbitrary unital rings.

1979–1980 *Năstăsescu* [38], [39] proves the *Absolute* or *Categorical H-LT*, i.e., the H-LT for an arbitrary Grothendieck category.

1980 *Albu* [3] proves the *Absolute Dual H-LT* for commutative Grothendieck categories.

1982 *Faith* [20] provides another module-theoretical proof of the Relative H-LT, and gives two interesting versions of it: Δ-Σ and *counter*.

1984 *Albu* [4] establishes the *Latticial H-LT* for upper continuous modular lattices.

1996 *Albu* and *Smith* [12] prove the *Latticial H-LT* for arbitrary modular lattices.

1996 *Albu, Lenagan*, and *Smith* [7] establish a *Krull dimension-like* extension of the Classical H-LT and Absolute H-LT.

1997 *Albu* and *Smith* [13] extend the result of Albu, Lenagan, and Smith [7] from Grothendieck categories to upper continuous modular lattices, using the technique of localization of modular lattices they developed in [12].

In the sequel we shall be discussing in full detail all the extensions of the HL-T for unital rings listed above.

2. The Relative H-LT

The next result is due to Albu and Năstăsescu [9, Théorème 4.7] for commutative rings, conjectured for noncommutative rings by Albu and Năstăsescu [9, Problème 4.8], and proved for arbitrary unital rings by Miller and Teply [35, Theorem 1.4].

Theorem 2.1. (Relative H-LT). *Let R be a ring with identity, and let τ be a hereditary torsion theory on* Mod-R*. If R is a right τ-Artinian ring, then every τ-Artinian right R-module is τ-Noetherian.*

Let us mention that the module-theoretical proofs available in the literature of the Relative H-LT, namely the original one in 1979 due to Miller and Teply [35, Theorem 1.4], and another one in 1982 due to Faith [20, Theorem 7.1 and Corollary 7.2], are very long and complicated.

The importance of the Relative H-LT in investigating the structure of some relevant classes of modules, including injectives as well as projectives, is revealed in Albu and Năstăsescu [10] and Faith [20], where the main body of both these monographs deals with this topic.

We are now going to explain all the terms occurring in the statement above.

Hereditary torsion theories

The concept of *torsion theory* for Abelian categories has been introduced by S.E. Dickson [19] in 1966. For our purposes, we present it only for module categories in one of the many equivalent ways that can be done. Basic torsion-theoretic concepts and results can be found in Golan [23] and Stenström [47].

All rings considered in this paper are associative with unit element $1 \neq 0$, and modules are unital right modules. If R is a ring, then Mod-R denotes the category of all right R-modules. We often write M_R to emphasize that M is a right R-module; $\mathcal{L}(M_R)$, or just $\mathcal{L}(M)$, stands for the lattice of all submodules of M. The notation $N \leqslant M$ means that N is a submodule of M.

A *hereditary torsion theory* on Mod-R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of nonempty subclasses $\mathcal{T}$ and $\mathcal{F}$ of Mod-R such that $\mathcal{T}$ is a *localizing subcategory* of Mod-R in the Gabriel's sense [22] (this means that $\mathcal{T}$ is a Serre class of Mod-R which is closed under direct sums) and $\mathcal{F} = \{ F_R \mid \mathrm{Hom}_R(T, F) = 0, \forall\, T \in \mathcal{T} \}$. Thus, any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is uniquely determined by its first component $\mathcal{T}$. Recall that a nonempty subclass $\mathcal{T}$ of Mod-R is a *Serre class* if for any short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ in Mod-R, one has $X \in \mathcal{T} \Longleftrightarrow X' \in \mathcal{T}$ & $X'' \in \mathcal{T}$, and $\mathcal{T}$ is *closed under direct sums* if for any family $(X_i)_{i\in I}$, I arbitrary set, with $X_i \in \mathcal{T}$, $\forall\, i \in I$, it follows that $\bigoplus_{i\in I} X_i \in \mathcal{T}$.

The prototype of a hereditary torsion theory is the pair $(\mathcal{A}, \mathcal{B})$ in Mod-$\mathbb{Z}$, where $\mathcal{A}$ is the class of all torsion Abelian groups, and $\mathcal{B}$ is the class of all torsion-free Abelian groups.

Throughout this paper $\tau = (\mathcal{T}, \mathcal{F})$ will be a fixed hereditary torsion theory on Mod-R. For any module M_R we denote

$$\tau(M) := \sum_{N \leqslant M,\, N \in \mathcal{T}} N.$$

Since $\mathcal{T}$ is a localizing subcategory of Mod-R, it follows that $\tau(M) \in \mathcal{T}$, and we call it the *τ-torsion submodule* of M. Note that, as for Abelian groups, we have

$$M \in \mathcal{T} \Longleftrightarrow \tau(M) = M \quad \text{and} \quad M \in \mathcal{F} \Longleftrightarrow \tau(M) = 0.$$

The members of $\mathcal{T}$ are called *τ-torsion modules*, while the members of $\mathcal{F}$ are called *τ-torsion-free modules*.

For any $N \leqslant M$ we denote by $\overline{N}$ the submodule of M such that $\overline{N}/N = \tau(M/N)$, called the *$\tau$-closure* or *$\tau$-saturation* of N (in M). One says that N is *τ-closed* or *τ-saturated* if $\overline{N} = N$, or equivalently, if $M/N \in \mathcal{F}$, and the set of all τ-closed submodules of M is denoted by $\mathrm{Sat}_\tau(M)$. It is well known that $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice. Note that though $\mathrm{Sat}_\tau(M)$ is a subset of the lattice $\mathcal{L}(M)$ of all submodules of M, it is not a sublattice, because the sum of two τ-closed submodules of M is not necessarily τ-closed.

Definition 2.2. A module M_R is said to be *τ-Noetherian* (resp. *τ-Artinian*) if $\mathrm{Sat}_\tau(M)$ is a Noetherian (resp. Artinian) poset. The ring R is said to be *τ-Noetherian* (resp. *τ-Artinian*) if the module R_R is *τ-Noetherian* (resp. *τ-Artinian*).

Recall that a partially ordered set, shortly poset, $(P, \leqslant)$ is called *Noetherian* (resp. *Artinian*) if it satisfies the ACC (resp. DCC), i.e., if there is no strictly ascending (resp. descending) chain $x_1 < x_2 < \cdots$ (resp. $x_1 > x_2 > \cdots$) in P.

Relativization

The Relative H-LT nicely illustrates a general direction in Module Theory, namely the so-called *Relativization*. Roughly speaking, this topic deals with the following matter:

Given a property $\mathbb{P}$ *in the lattice* $\mathcal{L}(M_R)$ *investigate*
the property $\mathbb{P}$ *in the lattice* $\mathrm{Sat}_\tau(M_R)$.

Since about forty years Module Theorists were dealing with the following problem:

Having a theorem $\mathbb{T}$ *on modules, is its relativization* τ-$\mathbb{T}$ *true?*

As we mentioned just after the statement of the Relative H-LT, its known module-theoretical proofs are very long and complicated; so, the relativization of a result on modules is not always a simple job, and as this will become clear with the next statement, sometimes it may be even impossible.

Theorem 2.3. (Metatheorem). *The relativization* $\mathbb{T} \rightsquigarrow \tau$-$\mathbb{T}$ *of a theorem* $\mathbb{T}$ *in Module Theory is not always true/possible.*

Proof. Consider the following lovely theorem (see Lenagan [30, Theorem 3.2]):

$\mathbb{T}$: *If R has right Krull dimension then the prime radical* $\mathrm{N}(R)$ *is nilpotent.*

The relativization of $\mathbb{T}$ is the following:

τ-$\mathbb{T}$: *If R has right τ-Krull dimension then the τ-prime radical* $\mathrm{N}_\tau(R)$ *is τ-nilpotent.*

Recall that $\mathrm{N}_\tau(R)$ is the intersection of all τ-closed two-sided prime ideals of R, and a right ideal I of R is said to be τ-nilpotent if $I^n \in \mathcal{T}$ for some integer $n > 0$.

The truth of the relativization τ-$\mathbb{T}$ of $\mathbb{T}$ has been asked by Albu and Smith [11, Problem 4.3]. Surprisingly, the answer is “no” in general, even if R is (left and right) Noetherian, by Albu, Krause, and Teply [6, Example 3.1]. This proves our Metatheorem.

However, τ-$\mathbb{T}$ is true for any ring R and any *ideal invariant* hereditary torsion theory τ, including any commutative ring R and any τ (see Albu, Krause, and Teply [6, Section 6]). □

3. The Absolute (or Categorical) H-LT

The next result is due to Năstăsescu, who actually gave two different short nice proofs: [38, Corollaire 1.3] in 1979, based on the Loewy length, and [39, Corollaire 2] in 1980, based on the length of a composition series.

Theorem 3.1. (Absolute H-LT). *Let $\mathcal{G}$ be a Grothendieck category having an Artinian generator. Then any Artinian object of $\mathcal{G}$ is Noetherian.*

Recall that a *Grothendieck category* is an Abelian category $\mathcal{G}$, with exact direct limits (or, equivalently, satisfying the axiom AB5 of Grothendieck), and having a generator G (this means that for every object X of $\mathcal{G}$ there exist a set I and an epimorphism $G^{(I)} \twoheadrightarrow X$). A family $(U_j)_{j\in J}$ of objects of $\mathcal{G}$ is said to be a *family of generators* of $\mathcal{G}$ if $\oplus_{j\in J} U_j$ is a generator of $\mathcal{G}$. The Grothendieck category $\mathcal{G}$ is called *locally Noetherian* (resp. *locally Artinian*) if it has a family of Noetherian (resp. Artinian) generators. Also, recall that an object $X \in \mathcal{G}$ is said to be *Noetherian* (resp. *Artinian*) if the lattice $\underline{\mathrm{Sub}}(X)$ of all subobjects of X is Noetherian (resp. Artinian).

Note that J.E. Roos [45] has produced in 1969 an example of a locally Artinian Grothendieck $\mathcal{C}$ category which is not locally Noetherian; thus, the so-called *Locally Absolute H-LT* fails. Even if a locally Artinian Grothendieck category $\mathcal{C}$ has a family of projective Artinian generators, then it is not necessarily locally Noetherian, as an example due to Menini [33] shows. However, the Locally Absolute H-LT is true if the family of Artinian generators of $\mathcal{C}$ is finite (because in this case $\mathcal{C}$ has an Artinian generator), as well as if the Grothendieck category $\mathcal{C}$ is *commutative*, by Albu and Năstăsescu [9, Corollaire 4.38] (see Section 6 for the definition of a commutative Grothendieck category).

Quotient categories and the Gabriel-Popescu Theorem

Clearly, for any ring R with identity element, the category Mod-R is a Grothendieck category. A procedure to construct new Grothendieck categories is by taking the *quotient category* Mod-$R/\mathcal{T}$ of Mod-R modulo any of its localizing subcategories $\mathcal{T}$. The construction of the quotient category of Mod-$R/\mathcal{T}$, or more generally, of the quotient category $\mathcal{A}/\mathcal{C}$ of any locally small Abelian category $\mathcal{A}$ modulo any of its Serre subcategories $\mathcal{C}$ is quite complicated and goes back to Serre's "langage modulo $\mathcal{C}$" (1953), Grothendieck (1957), and Gabriel (1962) [22].

Recall briefly this construction. The objects of the category $\mathcal{A}/\mathcal{C}$ are the same as those of $\mathcal{A}$, while the morphisms in this category are defined not so simple: for every objects X, Y of $\mathcal{A}$, one sets

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X,Y) := \varinjlim_{(X',Y')\in I_{X,Y}} \mathrm{Hom}_{\mathcal{A}}(X', Y/Y'),$$

where $I_{X,Y} := \{(X',Y') \mid X' \leqslant X,\ Y' \leqslant Y,\ X/X' \in \mathcal{C},\ Y' \in \mathcal{C}\}$ is considered as an ordered set in an obvious manner, and with this order it is actually a directed set (it is indeed a set because the given Abelian category $\mathcal{A}$ was supposed to be locally small, i.e., the class of all subobjects of every object of $\mathcal{A}$ is a set). Then $\mathcal{A}/\mathcal{C}$ is an Abelian category, and there exists a canonical covariant exact functor

$$T : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$$

defined as follows: for every objects X, Y of $\mathcal{A}$ and every $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ one sets $T(X) := X$ and $T(f) :=$ the image of f in the inductive limit. It turns out that the exact functor T annihilates $\mathcal{C}$ (i.e., "kills" each $X \in \mathcal{C}$), and, as for quotient modules, the pair $(\mathcal{A}/\mathcal{C}, T)$ is universal for exact functors, which annihilate $\mathcal{C}$, from $\mathcal{A}$ into Abelian categories. Moreover, the given Serre subcategory $\mathcal{C}$ of $\mathcal{A}$ is a localizing subcategory of $\mathcal{A}$ if and only if the functor T has a right adjoint, and in this case the quotient category $\mathcal{A}/\mathcal{C}$ is a Grothendieck category if $\mathcal{A}$ is so. In particular, for any unital ring R, the quotient category Mod-$R/\mathcal{T}$ of Mod-R modulo any of its localizing subcategories $\mathcal{T}$ is a Grothendieck category.

Roughly speaking, the renowned *Gabriel-Popescu Theorem*, discovered exactly forty five years ago, states that in this way we obtain, up to an equivalence of categories, *all* the Grothendieck categories. More precisely,

Theorem 3.2. (The Gabriel-Popescu Theorem). *For any Grothendieck category $\mathcal{G}$ there exist a unital ring R and a localizing subcategory $\mathcal{T}$ of* Mod-R *such that $\mathcal{G} \simeq$ Mod-$R/\mathcal{T}$.*

Notice that the ring R and the localizing subcategory $\mathcal{T}$ of Mod-R can be obtained in the following (noncanonical) way: Let U be any generator of the Grothendieck category $\mathcal{G}$, and let R_U be the ring $\mathrm{End}_{\mathcal{G}}(U)$ of endomorphims of U. If $S_U : \mathcal{G} \longrightarrow$ Mod-R_U is the functor $\mathrm{Hom}_{\mathcal{G}}(U, -)$, then S_U has a left adjoint T_U, $T_U \circ S_U \simeq 1_{\mathcal{G}}$, and $\mathrm{Ker}(T_U) := \{ M \in \text{Mod-}R_U \mid T_U(M) = 0 \}$ is a localizing subcategory of Mod-R_U. Take now as R any such R_U and as $\mathcal{T}$ such a $\mathrm{Ker}(T_U)$.

The reader is referred to Albu and Năstăsescu [10], Gabriel [22], and Stenström [47] for the concepts, constructions, and facts presented in this subsection.

Absolutization

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Then, because $\mathcal{T}$ is a localizing subcategory of Mod-R one can form the quotient category Mod-$R/\mathcal{T}$. Denote by

$$T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

the canonical functor from the category Mod-R to its quotient category Mod-$R/\mathcal{T}$.

Proposition 3.3. (Albu and Năstăsescu [10, Proposition 7.10]). *With the notation above, for every module M_R there exists a lattice isomorphism*

$$\mathrm{Sat}_\tau(M) \simeq \underline{\mathrm{Sub}}(T_\tau(M)).$$

In particular, M is a τ-Noetherian (resp. τ-Artinian) module if and only if $T_\tau(M)$ is a Noetherian (resp. Artinian) object of Mod-$R/\mathcal{T}$.

Absolutization is a technique to pass from τ-relative results in Mod-R to absolute properties in the quotient category Mod-$R/\mathcal{T}$ via the canonical functor T_τ : Mod-$R \longrightarrow$ Mod-$R/\mathcal{T}$. This technique is, in a certain sense, opposite to *relativization*, meaning that absolute results in a Grothendieck category $\mathcal{G}$ can be translated, via the Gabriel-Popescu Theorem, into τ-relative results in Mod-R as follows:

Let U be any generator of the Grothendieck category $\mathcal{G}$, let R_U be the ring $\mathrm{End}_{\mathcal{G}}(U)$ of endomorphims of U. As we have already mentioned above, if $S_U : \mathcal{G} \longrightarrow$ Mod-R_U is the functor $\mathrm{Hom}_{\mathcal{G}}(U, -)$, then S_U has a left adjoint T_U, $T_U \circ S_U \simeq 1_{\mathcal{G}}$, and $\mathrm{Ker}(T_U) := \{ M \in \text{Mod-}R_U \mid T_U(M) = 0 \}$ is a localizing subcategory of Mod-R_U. Let now τ_U be the hereditary torsion theory (uniquely) determined by the localizing subcategory $\mathrm{Ker}(T_U)$ of Mod-R_U. Many properties of an object $X \in \mathcal{G}$ can now be translated as τ_U-relative properties of the right R_U-module $S_U(X)$; e.g., $X \in \mathcal{G}$ is an Artinian (resp. Noetherian) object if and only if $S_U(X)$ is a τ_U-Artinian (resp. τ_U-Noetherian) right R_U-module. Observe that this relativization strongly depends on the choice of the generator U of $\mathcal{G}$.

As mentioned before, the two module-theoretical proofs available in the literature of the Relative H-LT due to Miller and Teply [35] and Faith [20], are very long and complicated. On the contrary, the two categorical proofs of the Absolute H-LT due to Năstăsescu [38], [39] are very short and simple.

Using the interaction *relativization* $\longleftrightarrow$ *absolutization*, we shall prove in Section 5 that *Relative H-LT* $\Longleftrightarrow$ *Absolute H-LT*; this means exactly that any of this theorems can be deduced from the other one. In this way we can obtain two short categorical proofs of the Relative H-LT.

However, some module theorists are not so comfortable with categorical proofs of module-theoretical theorems: they cannot touch the elements of an object because categories work only with objects and morphisms and not with elements of an object.

Good news for those people: There exists an alternative, namely the *latticial setting*. Why? If τ is a hereditary torsion theory on Mod-R and M_R is any module then $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice, and if $\mathcal{G}$ is a Grothendieck category then the lattice $\underline{\mathrm{Sub}}(X)$ of all subobjects of any object $X \in \mathcal{G}$ is also an upper continuous modular lattice. Therefore, a strong reason to study such kinds of lattices exists.

A latticial strategy

Let $\mathbb{P}$ be a problem, involving subobjects or submodules, to be investigated in Grothendieck categories or in module categories with respect to hereditary torsion theories. Our *main strategy* in this direction since more than twenty five years consists of the following three steps:

I. *Translate/formulate*, if possible, the problem $\mathbb{P}$ to be investigated in a Grothendieck category or in a module category equipped with a hereditary torsion theory into a *latticial setting*.

II. *Investigate* the obtained problem $\mathbb{P}$ in this *latticial frame*.

III. *Back to basics*, i.e., to Grothedieck categories and module categories equipped with hereditary torsion theories.

The advantage to deal in such a way, is, in our opinion, that this is the most *natural* and the most *simple* as well, because we ignore the specific context of Grothendieck categories and module categories equipped with hereditary torsion theories, focussing only on those latticial properties which are relevant in our given specific categorical or relative module-theoretical problem $\mathbb{P}$. The best illustration of this approach is, as we will see later, that both the *Relative H-LT* and the *Absolute H-LT* are immediate consequences of the so-called *Latticial H-LT*, which will be amply discussed in Sections 4 and 5.

4. The latticial H-LT and latticial dual H-LT

The Classical/Relative/Absolute H-LT deals with the question when a particular Artinian lattice $\mathcal{L}(M_R)/\mathrm{Sat}_\tau(M_R)/\underline{\mathrm{Sub}}(X)$ is Noetherian. Our contention is that the natural setting for the H-LT and its various extensions is *Lattice Theory*, being concerned as it is with descending and ascending chains in certain lattices. Therefore we shall present in this section the Latticial H-LT which gives an exhaustive answer to the following more general question:

When an arbitrary Artinian modular lattice is Noetherian?

The answer, given in an "if and only" form, is due to Albu and Smith [11, Theorem 1.9], and will be discussed in the next subsections.

Lattice background

All lattices considered in this paper are assumed to have a least element denoted by 0 and a last element denoted by 1, and $(L, \leqslant, \wedge, \vee, 0, 1)$, or more simply, just L, will always denote such a lattice. We denote by $\mathcal{M}$ the class of all modular lattices with 0 and 1. The opposite lattice of L will be denoted by L^0. We shall use $\mathbb{N}$ to denote the set $\{0, 1, \ldots\}$ of all natural numbers.

Recall that a lattice L is called *modular* if

$$a \wedge (b \vee c) = b \vee (a \wedge c), \forall\, a, b, c \in L \text{ with } b \leqslant a.$$

A lattice L is said to be *upper continuous* if L is complete and

$$a \wedge \Big(\bigvee_{c \in C} c\Big) = \bigvee_{c \in C} (a \wedge c)$$

for every $a \in L$ and every chain (or, equivalently, directed subset) $C \subseteq L$.

If x, y are elements in L with $x \leqslant y$, then y/x will denote the interval $[x, y]$, i.e.,

$$y/x = \{\, a \in L \mid x \leqslant a \leqslant y \,\}.$$

An element e of L is called *essential* if $e \wedge a \neq 0$ for all $0 \neq a \in L$. Dually, an element s of L is called *superfluous* or *small* if $s \vee b \neq 1$ for all $1 \neq b \in L$, i.e., if s is an essential element of L^0. A *composition series* of a lattice L is a chain $0 = a_0 < a_1 < \cdots < a_n = 1$ in L which has no refinement, except by introducing repetitions of the given elements a_i, and the integer n is called the *length* of the chain. If L is a modular lattice having a composition series, then we say that L is a lattice of *finite length*, and in this case any two composition series of L have the same length, called the *length* of L and denoted by $l(L)$. A modular lattice is of finite length if and only if L is both Noetherian and Artinian.

For all undefined notation and terminology on lattices, the reader is referred to Crawley and Dilworth [18], Grätzer [26], and Stenström [47].

The H-LT and Dual H-LT for arbitrary modular lattices

In this subsection we present a very general form of the H-LT for an arbitrary modular lattice, saying that an Artinian lattice L is Noetherian if and only if it satisfies two conditions, one of which guaranteeing that L has a good supply of essential elements and the second ensuring that there is a bound for the composition lengths of certain intervals of L.

More precisely, consider the following two properties that a lattice L may have ("$\mathcal{E}$" for Essential and "$\mathcal{BL}$" for Bounded Length):

($\mathcal{E}$) *for all* $a \leqslant b$ *in* L *there exists* $c \in L$ *such that* $b \wedge c = a$ *and* $b \vee c$ *is an essential element of* $1/a$ *.*

($\mathcal{BL}$) *there exists a positive integer* n *such that for all* $x < y$ *in* L *with* $y/0$ *having a composition series there exists* $c_{xy} \in L$ *with* $c_{xy} \leqslant y$, $c_{xy} \nleqslant x$, *and* $l(c_{xy}/0) \leqslant n$ *.*

Any pseudo-complemented modular lattice, in particular any upper continuous modular lattice satisfies ($\mathcal{E}$). Also, any Noetherian lattice satisfies ($\mathcal{E}$).

The dual properties of ($\mathcal{E}$) and ($\mathcal{BL}$) are respectively:

($\mathcal{E}^0$) *for all* $a \leqslant b$ *in* L *there exists* $c \in L$ *such that* $a \vee c = b$ *and* $a \wedge c$ *is a superfluous element of* $b/0$ *.*

($\mathcal{BL}^0$) *there exists a positive integer* n *such that for all* $x < y$ *in* L *with* $1/x$ *having a composition series there exists* c_{xy} *in* L *with* $x \leqslant c_{xy}$, $y \nleqslant c_{xy}$, *and* $l(1/c_{xy}) \leqslant n$ *.*

The next result, due to Albu and Smith [12, Theorem 1.9] is the *Latticial H-LT* for an arbitrary modular lattice, which, on one hand, is interesting in its own right, being the most general form of the H-LT we know, and, on the other hand is crucial in proving other versions of the H-LT.

Theorem 4.1. (Latticial H-LT). *Let L be an Artinian modular lattice. Then L is Noetherian if and only if L satisfies both conditions ($\mathcal{E}$) and ($\mathcal{BL}$).*

Since the opposite of a modular lattice is again a modular lattice, it follows that the above result can be dualized as follows (see Albu and Smith [12, Theorem 1.11]):

Theorem 4.2. (Latticial Dual H-LT). *Let L be a Noetherian modular lattice. Then L is Artinian if and only if L satisfies both conditions ($\mathcal{E}^0$) and ($\mathcal{BL}^0$).*

The condition (l^*) and lattice generation

The following condition for a lattice L has been considered in Albu [4]:

(l^*) *there exists a positive integer* n *such that for all* $x < y$ *in* L *there exists* $c_{xy} \in L$ *with* $c_{xy} \leqslant y$, $c_{xy} \nleqslant x$, $c_{xy}/0$ *Artinian, and* $l^*(c_{xy}/0) \leqslant n$.

If A is an Artinian lattice, then $l^*(A)$ denotes the so-called *reduced length* of A, that is $l(1/a^*)$, where a^* is the least element of the set $\{ a \in A \mid 1/a \text{ is Noetherian}\}$, see Albu [4, Lemma 0.3]. It is clear that for an Artinian lattice L, the condition (l^*) implies the condition $(\mathcal{BL})$.

Recall that if M_R and U_R are two modules, then the module M is said to be *U-generated* if there exists a set I and an epimorphism $U^{(I)} \twoheadrightarrow M$. The fact that M is U-generated can also be expressed as follows: for any proper submodule N of M there exists a submodule P of M which is not contained in N, such that P is isomorphic to a quotient of the module U. Further, M is said to be *completely U-generated* in case every submodule of M is U-generated. These concepts have been naturally extended in Albu [5] to posets as follows:

We say that a poset L is *generated* by a poset G, or is *G-generated*, if for every $a \neq 1$ in L there exist $c \in L$ and $g \in G$ such that $c \nleqslant a$ and $c/0 \simeq 1/g$. The poset L is called *completely generated* by G or *completely G-generated* if for every $b \in L$, the interval $b/0$ is G-generated, that is, for every $a < b$ in L, there exist $c \in L$ and $g \in G$ such that $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$.

Clearly, if the module M is (completely) U-generated, then the lattice $\mathcal{L}(M_R)$ is (completely) $\mathcal{L}(U_R)$-generated, but not conversely.

Note that if L and G are two Artinian lattices, and if L is completely G-generated, then the lattice L satisfies the condition (l^*), and so, also the condition $(\mathcal{BL})$. This immediately implies the following version of the Latticial H-LT (Theorem 4.1) in terms of lattice complete generation:

Theorem 4.3. *If* L *is a modular Artinian lattice which is completely generated by a modular Artinian lattice* G, *then* L *is Noetherian if and only if* L *satisfies* $(\mathcal{E})$.

The H-LT for upper continuous modular lattices

We present below a version in terms of condition (l^*), due to Albu [4, Corollary 1.8], of the Latticial H-LT for modular lattices which additionally are upper continuous:

Theorem 4.4. (Latticial H-LT for upper continuous lattices). *Let* L *be an Artinian upper continuous modular lattice. Then* L *is Noetherian if and only if* L *satisfies the condition* (l^*).

Observe that Theorem 4.1 is an extension of Theorem 4.4 from upper continuous modular lattices to arbitrary modular lattices. More precisely, the upper continuity from Theorem 4.4 is replaced by the less restrictive condition $(\mathcal{E})$, while the condition (l^*) by the condition $(\mathcal{BL})$.

5. Connections between various forms of the H-LT

In this section we are going to discuss the connections between the *Classical H-LT, Relative H-LT, Absolute H-LT*, and *Latticial H-LT*, and to present the *Faith's* Δ-Σ and *counter* versions of the Relative H-LT.

Latticial H-LT $\Longrightarrow$ Relative H-LT

As mentioned above, the module-theoretical proofs available in the literature of the Relative H-LT (namely, the original one in 1979 due to Miller and Teply [35], and another one in 1982 due to Faith [20]) are very long and complicated. We present below a very short proof based on the Latticial H-LT in terms of complete generation (Theorem 4.3).

So, let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Assume that R is τ-Artinian, and let M_R be a τ-Artinian module. The Relative H-LT states that M_R is a τ-Noetherian module.

Set $G := \mathrm{Sat}_\tau(R_R)$ and $L := \mathrm{Sat}_\tau(M_R)$. Then G and L are Artinian upper continuous modular lattices. We have to prove that M_R is a τ-Noetherian module, i.e., L is a Noetherian lattice. By Theorem 4.3, it is sufficient to check that L is completely G-generated, i.e., for every $a < b$ in L, there exist $c \in L$ and $g \in G$ such that $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$.

Since $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(M/\tau(M))$ we may assume, without loss of generality, that $M \in \mathcal{F}$. Let $a = A < B = b$ in $L = \mathrm{Sat}_\tau(M_R)$. Then, there exists $x \in B \setminus A$. Set $C := \overline{xR}$ and $I = \mathrm{Ann}_R(x)$. We have $R/I \simeq xR \leqslant M \in \mathcal{F}$, so $R/I \in \mathcal{F}$, i.e., $I \in \mathrm{Sat}_\tau(R_R) = G$. Using known properties of lattices of type $\mathrm{Sat}_\tau(N)$, we deduce that

$$[I, R] \simeq \mathrm{Sat}_\tau(R/I) \simeq \mathrm{Sat}_\tau(xR) \simeq \mathrm{Sat}_\tau(\overline{xR}) = \mathrm{Sat}_\tau(C) = [0, C],$$

where the intervals $[I, R]$ and $[0, C]$ are considered in the lattices G and L, respectively. Then, if we denote $c = C$ and $g = I$, we have $c \in L$, $g \in G$, $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$, which shows that L is completely G-generated, as desired.

Absolute H-LT $\Longrightarrow$ Relative H-LT

We are going to show how the Relative H-LT can be deduced from the Absolute H-LT. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Assume that R is τ-Artinian ring, and let M_R be a τ-Artinian module. We pass from Mod-R to the Grothendieck category Mod-$R/\mathcal{T}$ with the use of the canonical functor $T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$. Since R_R is a generator of Mod-R and T_τ is an exact functor we deduce that $T_\tau(R)$ is a generator of Mod-$R/\mathcal{T}$, which is Artinian by Proposition 3.3. Now, again by Proposition 3.3, $T_\tau(M)$ is an Artinian object of Mod-$R/\mathcal{T}$, so, it is also Noetherian by the Absolute H-LT, i.e., M is τ-Noetherian, and we are done.

Relative H-LT $\Longrightarrow$ Absolute H-LT

We prove that the Absolute H-LT is a consequence of the Relative H-LT. Let $\mathcal{G}$ be a Grothendieck category having an Artinian generator U. Set $R_U := \mathrm{End}_{\mathcal{G}}(U)$, and let $S_U = \mathrm{Hom}_{\mathcal{G}}(U, -) : \mathcal{G} \longrightarrow \text{Mod-}R_U$ and $T_U : \text{Mod-}R_U \longrightarrow \mathcal{G}$ be the pair of functors from the Gabriel-Popescu Theorem setting, described in Section 3, after Proposition 3.3. Then $T_U \circ S_U \simeq 1_{\mathcal{G}}$, and $\mathrm{Ker}(T_U) := \{ M \in \text{Mod-}R_U \mid T_U(M) = 0\}$ is a localizing subcategory of Mod-R_U. Let now τ_U be the hereditary torsion theory (uniquely) determined by the localizing subcategory $\mathcal{T}_U := \mathrm{Ker}(T_U)$ of Mod-R_U. Then, the Gabriel-Popescu Theorem says that $\mathcal{G} \simeq \text{Mod-}R_U/\mathcal{T}_U$ and

$$U \simeq (T_U \circ S_U)(U) = T_U(S_U(U)) = T_U(R_U).$$

Since U is an Artinian object of $\mathcal{G}$, so is also $T_U(R_U)$, which implies, by Proposition 3.3, that R_U is a τ_U-Artinian ring.

Now, let $X \in \mathcal{G}$ be an Artinian object of $\mathcal{G}$. Then, there exists a right R_U-module M such that $X \simeq T_U(M)$, so $T_U(M)$ is an Artinian object of $\mathcal{G}$, i.e., M is a τ_U-Artinian module. By the Relative H-LT, M is τ_U-Noetherian, so, again by Proposition 3.3, $X \simeq T_U(M)$ is a Noetherian object of $\mathcal{G}$, as desired.

The Faith's Δ-Σ version of the Relative H-LT

Recall that an injective module Q_R is said to be Σ-*injective* if any direct sum of copies of Q is injective. This concept is related with the concept of τ-Noetherian module as follows:

Let Q_R be an injective module, and denote $\mathcal{T}_Q := \{ M_R \mid \mathrm{Hom}_R(M, Q) = 0\}$. Then $\mathcal{T}_Q$ is a localizing category of Mod-R, and let τ_Q be the hereditary torsion theory on Mod-R (uniquely) determined by $\mathcal{T}_Q$. Note that for any hereditary torsion theory τ on Mod-R there exists an injective module Q_R such that $\tau = \tau_Q$.

A renowned theorem of Faith (1966) says that an injective module Q_R is Σ-injective if and only if R_R is τ_Q-Noetherian, or equivalently, if R satisfies the ACC on annihilators of subsets of Q. In order to uniformize the notation, Faith [20] introduced the concept of a Δ-*injective module* as being an injective module Q such that R_R is τ_Q-Artinian, or equivalently, R satisfies the DCC on annihilators of subsets of Q. Thus, the Relative H-LT is equivalent with the following:

Theorem 5.1. (Faith [20, p. 3]). *Any Δ-injective module is Σ-injective.*

Faith also proved a converse of Theorem 5.1: *An injective module Q_R is Δ-injective if and only Q_R is Σ-injective and the ring* $\mathrm{Biend}_R(Q)$ *of biendomorphisms of Q_R is semiprimary* (see Faith [20, Theorem 8.9]).

The Faith's counter version of the Relative H-LT

Let M_R be a module, and let $S := \mathrm{End}_R(M)$. Then M becomes a left S-module, and the module ${}_SM$ is called the *counter-module* of M_R. We say that M_R is *counter-Noetherian* (resp. *counter-Artinian*) if ${}_SM$ is a Noetherian (resp. Artinian) module.

The next result is an equivalent version, in terms of counter-modules, of the Relative H-LT.

Theorem 5.2. (Faith [20, Theorem 7.1]). *If Q_R is an injective module which is counter-Noetherian, then Q_R is counter-Artinian.*

In fact, Faith proved a sharper form of Theorem 5.2, namely for quasi-injective modules.

Absolute H-LT $\Longleftrightarrow$ Classical H-LT

The Grothendieck categories having an Artinian generator are very special in view of the following surprising result.

Theorem 5.3. (Năstăsescu [41, Théorème 3.3]). *A Grothendieck category $\mathcal{G}$ has an Artinian generator if and only if $\mathcal{G} \simeq$ Mod-A, with A a right Artinian ring with identity.*

Note that a heavy artillery has been used in the original proof of Theorem 5.3, namely: the Gabriel-Popescu Theorem, the Relative H-LT, as well as structure theorems for Δ-injective and Δ^*-projective modules. The Σ^*-*projective* and Δ^*-*projective* modules, introduced and investigated by Năstăsescu [40], [41], are in a certain sense dual to the notions of Σ-injective and Δ-injective modules.

A more general result, whose original proof is direct, without involving the many facts listed above, is the following one:

Theorem 5.4. (Albu and Wisbauer [16, Theorem 2.2]). *Let $\mathcal{G}$ be a Grothedieck category having a (finitely generated) generator U with $S = \mathrm{End}_{\mathcal{G}}(U)$ a right perfect ring. Then $\mathcal{G}$ has a (finitely generated) projective generator.*

Observe now that if $\mathcal{G}$ has an Artinian generator U, then, by the Absolute H-LT, U is also Noetherian, so, an object of finite length. Then $S = \mathrm{End}_{\mathcal{G}}(U)$ is a semi-primary ring, in particular it is right perfect. Now, by Theorem 5.4, $\mathcal{G}$ has a finitely generated projective generator, say P. If $A = \mathrm{End}_{\mathcal{G}}(P)$ then A is a right Artinian ring, and $\mathcal{G} \simeq$ Mod-A, which shows how Theorem 5.3 is an immediate consequence of Theorem 5.4.

Remark. If $\mathcal{G}$ is a Grothendieck category having an Artinian generator U, then the right Artinian ring A in Theorem 5.3 for which $\mathcal{G} \simeq$ Mod-A is far from being the endomorphism ring of U, and does not seem to be canonically associated with $\mathcal{G}$. The existence of a right Artinian ring B, canonically associated with $\mathcal{G}$, such that $\mathcal{G} \simeq$ Mod-B, is an easy consequence of a more general and more sophisticated construction of the *basic ring* of an arbitrary locally Artinian Grothendieck category, due to Menini and Orsatti [34, Section 3].

Clearly Relative H-LT $\Longrightarrow$ Classical H-LT by taking as τ the hereditary torsion theory $(0, \text{Mod-}R)$ on Mod-R, and Absolute H-LT $\Longrightarrow$ Classical H-LT by taking as $\mathcal{G}$ the category Mod-R.

We conclude that the following implications between the various aspects of the H-LT discussed so far hold:

| **Latticial H-LT $\Longrightarrow$ Relative H-LT $\Longleftrightarrow$ Absolute H-LT $\Longleftrightarrow$ Classical H-LT** |

| **Faith's Δ-Σ Theorem $\Longleftrightarrow$ Relative H-LT $\Longleftrightarrow$ Faith's counter Theorem** |

6. The Absolute and Relative Dual H-LT

Remember that the Absolute H-LT states that if $\mathcal{G}$ is a Grothendieck category with an Artinian generator, then any Artinian object of $\mathcal{G}$ is necessarily Noetherian, so it is natural to ask whether its dual holds, that is:

Problem 6.1. (Absolute Dual H-LT). *If $\mathcal{G}$ is a Grothendieck category with a Noetherian cogenerator, then does it follow that any Noetherian object of $\mathcal{G}$ is Artinian?*

Bad News: The Absolute Dual H-LT fails even for a module category Mod-R. To see this, let k be a universal differential field of characteristic zero with derivation D; then, the *Cozzens domain* $R = k[y, D]$ of differential polynomials over k in the derivation D is a principal right ideal domain which has a simple injective cogenerator S. So, $C = R \oplus S$ is both a Noetherian generator and cogenerator of Mod-R, which is clearly not Artinian (see Albu [3, Section 4]).

Good News: The Absolute Dual H-LT holds for large classes of Grothendieck categories, namely for the so-called *commutative Grothendieck categories*, introduced by Albu and Năstăsescu [8]. A Grothendieck category $\mathcal{C}$ is said to be *commutative* if there exists a commutative ring A with identity such that $\mathcal{G} \simeq$ Mod-$A/\mathcal{T}$ for some localizing subcategory $\mathcal{T}$ of Mod-A. By Albu [2, Proposition], these are exactly those Grothendieck categories $\mathcal{G}$ having at least a generator U with a commutative ring of endomorphisms.

Recall that an object G of a Grothendieck category $\mathcal{G}$ is a *generator* of $\mathcal{G}$ if every object X of $\mathcal{G}$ is an epimorphic image $G^{(I)} \twoheadrightarrow X$ of a direct sum of copies of G for some set I. Dually, an object $C \in \mathcal{G}$ is said to be a *cogenerator* of $\mathcal{G}$ if every object X of $\mathcal{G}$ can be embedded $X \rightarrowtail C^I$ into a direct product of copies of C for some set I.

Theorem 6.2. (Albu [3, Theorem 3.2]). *The following assertions are equivalent for a commutative Grothendieck category $\mathcal{G}$.*

(1) *$\mathcal{G}$ has a Noetherian cogenerator.*
(2) *$\mathcal{G}$ has an Artinian generator.*
(1) *$\mathcal{G} \simeq$ Mod-A for some commutative Artinian ring with identity.*

Corollary 6.3. (Absolute Dual HL-T). *If $\mathcal{G}$ is any commutative Grothendieck category having a Noetherian cogenerator, then every Noetherian object of $\mathcal{G}$ is Artinian.*

We are now going to present the relative version of the Absolute Dual H-LT. If $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on Mod-R. then a module C_R is said to be a *τ-cogenerator* of Mod-R if $C \in \mathcal{F}$ and every module in $\mathcal{F}$ is cogenerated by C.

Theorem 6.4. (Relative Dual HL-T). *Let R be a commutative ring with identity, and let τ be a hereditary torsion theory on* Mod-R *such that* Mod-R *has a τ-Noetherian τ-cogenerator. Then every τ-Noetherian R-module is τ-Artinian.*

7. The Krull dimension-like H-LT

In this section we shall discuss extensions of the Absolute H-LT and Latticial H-LT in terms of Krull dimension, due to Albu, Lenagan, and Smith [7] and Albu and Smith [12], [13], respectively. For the last one we present an outline of a localization technique of modular lattices, which has been invented exactly in order to prove such an extension.

Krull dimension: a brief history

1923 *E. Noether* explores the relationship between chains of prime ideals and dimensions of algebraic varieties.

1928 *W. Krull* develops Noether's idea into a powerful tool for arbitrary commutative Noetherian rings. Later, writers gave the name (*classical*) *Krull dimension* to the supremum of lengths of finite chains of prime ideals in a ring.

1962 *P. Gabriel* [22] introduces an ordinal-valued dimension which he named "Krull dimension" for objects in an Abelian category using a transfinite sequence of localizing subcategories.

1967 *R. Rentschler and P. Gabriel* introduce the *deviation* of an arbitrary poset, but only for finite ordinals.

1970 *G. Krause* introduces the ordinal-valued version of the Rentschler-Gabriel definition, but only for modules over an arbitrary unital ring.

1972 *B. Lemonnier* [28] introduces the general ordinal-valued notion of *deviation* of an arbitrary poset, called in the sequel *Krull dimension*.

1973 *R. Gordon and J.C. Robson* [24] give the name *Gabriel dimension* to Gabriel's original definition after shifting the finite values by 1, and provide an incisive investigation of the Krull dimension of modules and rings.

1985 *M. Pouzet and N. Zaguia* [44] introduce the more general concept of Γ-*deviation* of an arbitrary poset, where Γ is any set of ordinals.

The definition of the Krull dimension of a poset

Denote by $\mathcal{P}$ the class of all partially ordered sets, shortly *posets*. Without loss of generality we can suppose that all the posets which occur have a least element 0 and a greatest element 1.

If $(P, \leqslant)$ is a poset, shortly denoted by P, and x, y are elements in P with $x \leqslant y$, then recall that y/x denotes the interval $[x, y]$, i.e.,

$$y/x = \{\, a \in P \mid x \leqslant a \leqslant y \,\}.$$

The *Krull dimension* of a poset P (also called *deviation* of P and denoted by $\mathrm{dev}(P)$) is an ordinal number denoted by $k(P)$, which may or may not exist, and is defined recursively as follows:

- $k(P) = -1 \iff P = \{0\}$, where -1 is assumed to be the predecessor of 0.
- $k(P) = 0 \iff P \neq \{0\}$ and P is Artinian.

- Let $\alpha \geqslant 1$ be an ordinal number, and assume that we have already defined which posets have Krull dimension β for any ordinal $\beta < \alpha$. Then we define what it means for a poset P to have Krull dimension α: $k(P) = \alpha$ if and only if we have not defined $k(P) = \beta$ for some $\beta < \alpha$, and for any descending chain
$$x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant x_{n+1} \geqslant \cdots$$
of elements of P, $\exists\, n_0 \in \mathbb{N}$ such that $\forall\, n \geqslant n_0$, $k(x_n/x_{n+1}) < \alpha$, i.e., $k(x_n/x_{n+1})$ has previously been defined and it is an ordinal $< \alpha$.
- If no ordinal α exists such that $k(P) = \alpha$, we say that P does not have Krull dimension.

As in Albu and Smith [13], an alternative more compact equivalent definition is that involving the concept of an *Artinian poset relative* to a class of posets. If $\mathcal{X}$ is an arbitrary nonempty subclass of $\mathcal{P}$, a poset P is said to be $\mathcal{X}$-*Artinian* if for every descending chain $x_1 \geqslant x_2 \geqslant \cdots$ in P, $\exists\, k \in \mathbb{N}$ such that $x_i/x_{i+1} \in \mathcal{X}$, $\forall\, i \geqslant k$. The notion of an $\mathcal{X}$-*Noetherian* poset is defined similarly.

For every ordinal $\alpha \geqslant 0$, we denote by $\mathcal{P}_\alpha$ the class of all posets having Krull dimension $< \alpha$. Then, by Albu and Smith [13, Proposition 3.3], a poset P has Krull dimension an ordinal $\alpha \geqslant 0$ if and only if $P \notin \mathcal{P}_\alpha$ and P is $\mathcal{P}_\alpha$-Artinian. So, roughly speaking, the Krull dimension of a poset P measures how close P is to being Artinian.

The definition of the dual Krull dimension of a poset

The *dual Krull dimension* of a poset P (also called *codeviation* of P and denoted by $\mathrm{codev}(P)$), denoted by $k^0(P)$, is defined as being (if it exists!) the Krull dimension $k(P^0)$ of the opposite poset P^0 of P.

If α is an ordinal, then the notation $k(P) \leqslant \alpha$ (resp. $k^0(P) \leqslant \alpha$) will be used to indicate that P has Krull dimension (resp. dual Krull dimension) and it is less than or equal to α.

The existence of the dual Krull dimension $k^0(P)$ of a poset P is equivalent with the existence of the Krull dimension $k(P)$ of P in view of the following nice result of Lemonnier:

Theorem 7.1. (Lemonnier [28, Théorème 5, Corollaire 6]). *An arbitrary poset P does not have Krull dimension if and only if P contains a copy of the* (*usually*) *ordered set $\mathbb{D} = \{\, m/2^n \mid m \in \mathbb{Z},\ n \in \mathbb{N} \,\}$ of diadic real numbers. Consequently, P has Krull dimension if and only if P has dual Krull dimension.*

Remember that
$$P \text{ is Artinian (resp. Noetherian)} \Longleftrightarrow k(P) \leqslant 0 \ \text{(resp. } k^0(P) \leqslant 0).$$
So, we immediately deduce from Theorem 7.1 the following fact: *Any Noetherian poset has Krull dimension*, which usually is proved in a more complicated way.

The following problem naturally arises:

Problem. *Let P be a poset with Krull dimension. Then P also has dual Krull dimension. How are the ordinals $k(P)$ and $k^0(P)$ related?*

For other basic facts on the Krull dimension and dual Krull dimension of an arbitrary poset the reader is referred to Lemonnier [28] and McConnell and Robson [32].

Krull dimension of modules and rings

Recall that for a module M_R one denotes by $\mathcal{L}(M)$ the lattice of all submodules of M. The following ordinals (if they exist) are defined in terms of the lattice $\mathcal{L}(M)$:

- *Krull dimension* of M: $k(M) := k(\mathcal{L}(M))$
- *Dual Krull dimension* of M: $k^0(M) := k^0(\mathcal{L}(M))$
- *Right Krull dimension* of R: $k(R) := k(R_R))$
- *Right dual Krull dimension* of R: $k^0(R) := k^0(R_R))$

The problem we presented above for arbitrary posets can be specialized to modules and rings as follows (see also Section 8):

Problem. *Compare the ordinals $k(M)$ and $k^0(M)$ of a given module M_R with Krull dimension. In particular, compare the ordinals $k(R)$ and $k^0(R)$ of a ring R with right Krull dimension.*

A crucial concept in *Commutative Algebra*, which actually originated the general concept of Krull dimension of an arbitrary poset, is that of *classical Krull dimension* cl.K.dim(R) of a commutative ring R, introduced by W. Krull in 1928: this is the supremum of length of chains of prime ideals of R, which is either a natural number or ∞. The following result relates the Krull dimension of a ring with its classical Krull dimension.

Theorem. (Gordon and Robson [24, Proposition 7.8]). *If R is a commutative ring with finite Krull dimension $k(R)$, then $k(R) =$ cl.K.dim(R).*

A more accurate *ordinal-valued* variant of the original Krull's classical Krull dimension of a not necessarily commutative ring R is due to Gabriel (1962) [22] and Krause (1970). Note that this can be easily carried out from the poset Spec(R) of all two-sided prime ideals of R to an arbitrary poset P to define a so-called *classical Krull dimension* cl.K.dim(P) of P (see Albu [1, Section 1]). A different classical Krull dimension cl.k.dim(R) of a ring R, called *little classical Krull dimension* of R is also defined in the literature, and its relations with $k(R)$ and cl.K.dim(R) are established in Gordon and Robson [24, Proposition 7.9].

A Krull dimension-like extension of the Absolute H-LT

If $\mathcal{G}$ is a Grothendieck category and X is an object of $\mathcal{G}$, then the *Krull dimension* of X, denoted by $k(X)$, is defined as $k(X) := k(\underline{\mathrm{Sub}}(X))$.

The definition of the Krull dimension of an object in a Grothendieck category $\mathcal{G}$ can also be given using a transfinite sequence of Serre subcategories of $\mathcal{G}$ and suitable quotient categories of $\mathcal{G}$ (see Gordon and Robson [25, Proposition 1.5]). Using this approach, the following extension of the Absolute H-LT has been proved:

Theorem 7.2. (Albu, Lenagan, and Smith [7, Theorem 3.1]). *Let $\mathcal{G}$ be a Grothendieck category, and let U be a generator of $\mathcal{G}$ such that $k(U) = \alpha + 1$ for some* *ordinal* $\alpha \geqslant -1$. *Then, for every object* X *of* $\mathcal{G}$ *having Krull dimension and for every ascending chain*

$$X_1 \leqslant X_2 \leqslant \cdots \leqslant X_n \leqslant \cdots$$

of subobjects of X, $\exists\, m \in \mathbb{N}$ *such that* $k(X_{i+1}/X_i) \leqslant \alpha, \forall\, i \geqslant m$.

Note that for $\alpha = -1$ we obtain exactly the Absolute H-LT, because in this case, $X \in \mathcal{G}$ has Krull dimension if and only if $k(X) \leqslant 0$, i.e., if and only if X is Artinian.

It seems that the above result is really a *categorical property* of Grothendieck categories. As we already stressed before, the natural frame for the H-LT and its various extensions is *Lattice Theory*, being concerned as it is with descending and ascending chains in certain lattices, and therefore we shall present in the next subsection a very general version of Theorem 7.2 for upper continuous modular lattices.

A Krull dimension-like extension of the Latticial H-LT

In order to present an extension of Theorem 7.2 to lattices, which, on one hand, is interesting in his own right, and, on the other hand, provides another proof of Theorem 7.2, avoiding the use of quotient categories, we need first a *good substitute* for the notion of generator of a Grothendieck category, which has already been presented in Section 4.

Theorem 7.3. (Albu and Smith [13, Theorem 3.16]). *Let* L *and* G *be upper continuous modular lattices. Suppose that* $k(G) = \alpha + 1$ *for some ordinal* $\alpha \geqslant -1$, *and* L *is completely generated by* G. *If* L *has Krull dimension, then* $k(L) \leqslant \alpha+1$, *and for every ascending chain*

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots$$

of elements of L, $\exists\, m \in \mathbb{N}$ *such that* $k(x_{i+1}/x_i) \leqslant \alpha$, $\forall\, i \geqslant m$.

Two main ingredients are used in the proof of this result, namely:

- first, the *Latticial H-LT*, and
- second, a *localization* technique for modular lattices, developed by Albu and Smith [12], [13] in analogy with that for Grothendieck categories.

In the next subsection we shall briefly discuss this technique, and thereafter we shall provide a sketch of the proof of Theorem 7.3.

Localization of modular lattices

The terminology and notation below are taken from the localization theory in Grothendieck categories. First, in analogy with the notion of a Serre subcategory of an Abelian category, we present below, as in Albu and Smith [12], the notion of a *Serre class of lattices*:

Definition 7.4. By an *abstract class of lattices* we mean a nonempty subclass $\mathcal{X}$ of the class $\mathcal{M}$ of all modular lattices with 0 and 1, which is closed under lattice isomorphisms (this means that if $L, K \in \mathcal{M}$, $K \simeq L$ and $L \in \mathcal{X}$, then $K \in \mathcal{X}$).

We say that a subclass $\mathcal{X}$ of $\mathcal{M}$ is a *Serre class for* $L \in \mathcal{M}$ if $\mathcal{X}$ is an abstract class of lattices, and for all $a \leqslant b \leqslant c$ in L, $c/a \in \mathcal{X}$ if and only if $b/a \in \mathcal{X}$ and $c/b \in \mathcal{X}$. A *Serre class of lattices* is an abstract class of lattices which is a Serre class for all lattices $L \in \mathcal{M}$.

Let $\mathcal{X}$ be an arbitrary nonempty subclass of $\mathcal{M}$ and let $L \in \mathcal{M}$ be a lattice. Define a relation $\sim_{\mathcal{X}}$ on L by:

$$a \sim_{\mathcal{X}} b \iff (a \vee b)/(a \wedge b) \in \mathcal{X}.$$

Then $\sim_{\mathcal{X}}$ is a congruence on L if and only if $\mathcal{X}$ is a Serre class for L. Recall that a *congruence* on a lattice L is an equivalence relation $\sim$ on L such that for all $a, b, c \in L$, $a \sim b$ implies $a \vee c \sim b \vee c$ and $a \wedge c \sim b \wedge c$. It is well known that in this case the quotient set $L/\sim$ has a natural lattice structure, and the canonical mapping $L \longrightarrow L/\sim$ is a lattice morphism. If $\mathcal{X}$ is a Serre class for $L \in \mathcal{M}$, then the lattice $L/\sim_{\mathcal{X}}$ is called the *quotient lattice of* L *by* (or *modulo*) $\mathcal{X}$.

We define now for any nonempty subclass $\mathcal{X}$ of $\mathcal{M}$ and for any lattice L, a certain subset $\mathrm{Sat}_{\mathcal{X}}(L)$ of L, called the $\mathcal{X}$*-saturation* of L or the $\mathcal{X}$*-closure* of L:

$$\mathrm{Sat}_{\mathcal{X}}(L) = \{\, x \in L \mid x \leqslant y \in L,\ y/x \in \mathcal{X} \implies x = y \,\}.$$

This is the precise analogue of the subset $\mathrm{Sat}_{\tau}(M) = \{\, N \leqslant M_R \mid M/N \in \mathcal{F} \,\}$ of the lattice $\mathcal{L}(M_R)$ of all submodules of a given module M_R, where $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on Mod-R.

Definition 7.5. Let $\mathcal{X}$ be an arbitrary nonempty subclass of $\mathcal{M}$. We say that a lattice L has an $\mathcal{X}$*-closure* or an $\mathcal{X}$*-saturation* if there exists a mapping, called the $\mathcal{X}$*-closure* or $\mathcal{X}$*-saturation* of L

$$L \longrightarrow \mathrm{Sat}_{\mathcal{X}}(L)\,, \quad x \longmapsto \overline{x},$$

such that

(1) $x \leqslant \overline{x}$ and $\overline{x}/x \in \mathcal{X}$ for all $x \in L$.
(2) $x \leqslant y$ in $L \implies \overline{x} \leqslant \overline{y}$.

If $\mathcal{X}$ is a Serre class for $L \in \mathcal{M}$ such that L has an $\mathcal{X}$-closure $x \longmapsto \overline{x}$, and if we define $x \overline{\vee} y = \overline{x \vee y}$, $\forall\, x, y \in \mathrm{Sat}_{\mathcal{X}}(L)$, then it is easy to check that $\mathrm{Sat}_{\mathcal{X}}(L)$ becomes a modular lattice with respect to $\leqslant, \wedge, \overline{\vee}, \overline{0}, 1$.

By Proposition 3.3, for any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-R, and any module M_R, the lattice $\mathrm{Sat}_{\tau}(M)$ is isomorphic to the lattice $\underline{\mathrm{Sub}}(T_{\tau}(M))$ of all subobjects of the object $T_{\tau}(M)$ in the quotient category Mod-$R/\mathcal{T}$, where T_{τ} : Mod-$R \longrightarrow$ Mod-$R/\mathcal{T}$ is the canonical functor. The same happens also in our latticial frame: if $\mathcal{X}$ is a Serre class for $L \in \mathcal{M}$ such that L has an $\mathcal{X}$-closure, then

$$L/\sim_{\mathcal{X}} \simeq \mathrm{Sat}_{\mathcal{X}}(L).$$

This implies that the lattice L is $\mathcal{X}$-Noetherian (resp. $\mathcal{X}$-Artinian) $\Longleftrightarrow$ the lattice $\mathrm{Sat}_{\mathcal{X}}(L)$ is Noetherian (resp. Artinian) $\Longleftrightarrow$ the lattice $L/\sim_{\mathcal{X}}$ is Noetherian (resp. Artinian).

The Serre classes of lattices which are closed under taking arbitrary joins, which we next introduce, are called *localizing classes of lattices* and they play the same role as that of localizing subcategories in the setting of Grothendieck categories. More precisely, we have the following:

Definition 7.6. Let $\mathcal{X}$ be a nonempty subclass of $\mathcal{M}$ and let L be a complete modular lattice. We say that $\mathcal{X}$ is a *localizing class for* L if $\mathcal{X}$ is a Serre class for L, and for any $x \in L$ and for any family $(x_i)_{i\in I}$ of elements of $1/x$ such that $x_i/x \in \mathcal{X}$ for all $i \in I$, we have $(\bigvee_{i\in I} x_i)/x \in \mathcal{X}$. By a *localizing class of lattices* we mean a Serre class of lattices which is a localizing class for every complete modular lattice.

Note that if $\mathcal{X}$ is a localizing class for a complete modular lattice L then L has an $\mathcal{X}$-closure, which is uniquely determined.

Sketch of the proof of Theorem 7.3

We shall preserve the notation from the statement of Theorem 7.3. The full details of the proof can be found in Albu and Smith [13, Section 3].

Denote for every ordinal $\beta \geqslant 0$,

$$\mathcal{K}_\beta = \{ X \in \mathcal{M} \mid k(X) \leqslant \beta \}.$$

It is easy to check that $\mathcal{K}_\beta$ is a Serre class, but, in general, not a localizing class of lattices. For a nonempty subclass $\mathcal{A}$ of $\mathcal{M}$ we denote

$$\langle \mathcal{A} \rangle = \{ X \in \mathcal{M} \mid \forall\, a \in X,\ a \neq 1,\ \exists\, b \in X,\ a < b,\ b/a \in \mathcal{A} \}.$$

If $\mathcal{A}$ is a Serre class of lattices, then $\langle \mathcal{A} \rangle$ is a localizing class of lattices which contains $\mathcal{A}$.

Because L is completely generated by G we deduce that $L \in \langle \mathcal{K}_{\alpha+1} \rangle$. Now, if X is an upper continuous modular lattice with Krull dimension, and $\beta \geqslant 0$ is an arbitrary ordinal, then, using a latticial version of a nice result of Lemonnier [29, Lemme 1.1] originally proved for modules, we deduce that

$$X \in \mathcal{K}_\beta \iff X \in \langle \mathcal{K}_\beta \rangle.$$

Thus $L \in \mathcal{K}_{\alpha+1}$, i.e., $k(L) \leqslant \alpha + 1$, in other words, L is $\mathcal{K}_\alpha$-Artinian.

In order to pass to an Artinian lattice related to L, which is suitable for the application of the Relative H-LT (Theorem 4.1), we need a Serre class of lattices $\mathcal{X}$ for which L has an $\mathcal{X}$-closure. This cannot be $\mathcal{K}_\alpha$, so take as such an $\mathcal{X}$ the localizing class of lattices $\langle \mathcal{K}_\alpha \rangle$ "generated" by $\mathcal{K}_\alpha$. As we have seen above, with this $\mathcal{X}$, L has an $\mathcal{X}$-closure, and L is still $\mathcal{X}$-Artinian because $\mathcal{K}_\alpha \subseteq \mathcal{X}$. Also, G is $\mathcal{X}$-Artinian. For simplicity denote $\overline{L} = \mathrm{Sat}_{\mathcal{X}}(L)$, and $\overline{G} = \mathrm{Sat}_{\mathcal{X}}(G)$. Then $\overline{L}$ and $\overline{G}$ are both Artinian modular lattices, and since L is upper continuous, it satisfies the condition $(\mathcal{E})$, so too does $\overline{L}$, because this condition behaves well under localization. The same argument shows that $\overline{L}$ satisfies the condition $(\mathcal{BL})$

too, hence, by the Relative H-LT, we deduce that $\overline{L}$ is a Noetherian lattice, and so, L is $\mathcal{X}$-Noetherian.

Now let

$$x_1 \leqslant x_2 \leqslant \cdots$$

be an ascending chain of elements in L. Then, there exists $m \in \mathbb{N}$ such that $x_{i+1}/x_i \in \mathcal{X} = \langle \mathcal{K}_\alpha \rangle, \forall\, i \geqslant m$. But x_{i+1}/x_i has Krull dimension because L does. As above, it follows that $x_{i+1}/x_i \in \mathcal{K}_\alpha$, i.e., $k(x_{i+1}/x_i) \leqslant \alpha, \forall\, i \geqslant m$, which completes the proof. □

A Krull dimension-like extension of the Classical H-LT

If we specialize Theorem 7.2 to Mod-R, one obtains at once the following result, which can be also proved using only module-theoretical tools (see Albu, Lenagan, and Smith [7, Section 2]:

Corollary 7.7. *Let R is a ring having Krull dimension $k(R) = \alpha + 1$ for some ordinal $\alpha \geqslant -1$. Then, for any module M_R having Krull dimension and for any ascending chain*

$$N_1 \leqslant N_2 \leqslant \cdots \leqslant N_n \leqslant \cdots$$

of submodules of M, $\exists\, m \in \mathbb{N}$ such that $k(N_{i+1}/N_i) \leqslant \alpha, \forall\, i \geqslant m$.

Note that a relative version of Theorem 7.2, in terms of τ-*Krull dimension* also holds (see Albu and Smith [13, Theorem 4.1]).

8. Four open problems

We present below a list of four open problems related with the topics discussed in this paper.

1. *Compare the ordinals $k(M)$ and $k^0(M)$ of a given module M_R with Krull dimension*. In particular, *compare the ordinals $k(R)$ and $k^0(R)$ of a ring R with right Krull dimension*.
2. *If R is a ring with right Krull dimension, is it true that $k^0(R) \leqslant k(R)$*? This question has been raised by Albu and Smith in 1991, and also mentioned in Albu and Smith [14, Question 1].

   Observe that this is true for $k(R) = 0$; this is exactly the Classical H-LT. Other cases when the answer is *yes*, according to Albu and Smith [14], are when R is one of the following types of rings:
   - a commutative Noetherian ring, or
   - a commutative ring with Krull dimension 1, or
   - a commutative domain with Krull dimension 2, or
   - a valuation domain with Krull dimension, or
   - a right Noetherian right V-ring.
3. Similarly with the *right global homological dimension* of a ring R, two kinds of "global dimension" related to the Krull dimension and dual Krull dimension of a ring R have been defined in Albu and Smith [14]: the *right global Krull*

dimension r.gl.$k(R)$ and the *right global dual Krull dimension* r.gl.$k^0(R)$ of a ring, as being the supremum of $k(M_R)$ and $k^0(M_R)$, respectively, when M_R is running in the class of all modules having Krull dimension.

Similarly with Question 1, one may ask: *what is the order relation between* r.gl.$k(R)$ *and* r.gl.$k^0(R)$*?*

Note that, though according to Albu and Vámos [15, Corollary 1.3], $k^0(R) \leqslant k(R)$ for any *valuation ring* (this is a commutative ring with identity whose ideals are totally ordered by inclusion) having Krull dimension, unexpectedly one has the opposite order relation r.gl.$k(R) \leqslant$r.gl.$k^0(R)$ for any valuation ring, by Albu and Vámos [15, Theorem 2.4].

4. *Does the result of Corollary* 7.7 *fail when* $k(R)$ *is a limit ordinal?* We suspect that the answer is *yes*.

Acknowledgment

Many thanks are due to *Patrick F. Smith* for the careful reading of the manuscript and helpful suggestions, as well as to *Carl Faith* for sending me his book [20] and interesting recent phone conversations on the Hopkins-Levitzki Theorem.

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Ring and Module Theory
Trends in Mathematics, 27–71

A Theory of Hulls for Rings and Modules

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Abstract. In this expository paper, we survey results on the concept of a hull of a ring or a module with respect to a specific class of rings or modules. A hull is a ring or a module which is minimal among essential overrings or essential overmodules from a specific class of rings or modules, respectively. We begin with a brief history highlighting various types of hulls of rings and modules. The general theory of hulls is developed through the investigation of four problems with respect to various classes of rings including the (quasi-) Baer and (FI-) extending classes. In the final section, application to C^*-algebras are provided.

Mathematics Subject Classification (2000). Primary 16N60, 16S90; Secondary 16D50, 16S34, 46L05.

Keywords. FI-extending, ring hull, module hull, quasi-Baer rings, principally quasi-Baer ring, Baer ring, ring of quotients, essential overring, involution, C^*-algebra, local multiplier algebra, boundedly centrally closed.

1. Introduction

Throughout this paper all rings are associative with unity unless indicated otherwise and R denotes such a ring. Subrings and overrings preserve the unity of the base ring. Ideals without the adjective "right" or "left" mean two-sided ideals. All modules are unital and we use M_R (resp., ${}_RM$) to denote a right (resp., left) R-module.

If N_R is a submodule of M_R, then N_R is called *essential* (resp., *dense* also called *rational*) in M_R if for each $0 \neq x \in M$, there exists $r \in R$ such that $0 \neq xr \in N$ (resp., for $x, y \in M$ with $y \neq 0$, there exists $r \in R$ such that $xr \in N$ and $yr \neq 0$). We use $N_R \leq^{\text{ess}} M_R$ and $N_R \leq^{\text{den}} M_R$ to denote that N_R is an essential submodule of M_R and N_R is a dense submodule of M_R, respectively.

Recall that a *right ring of quotients* T of R is an overring of R such that R_R is dense in T_R. The maximal right (resp., left) ring of quotients of R is denoted by $Q(R)$ (resp., $Q^\ell(R)$). We say that T is a *right essential overring* of R if T is an overring of R such that R_R is essential in T_R. The right injective hull of R is denoted by $E(R_R)$ and we use $\mathcal{E}_R$ to denote $\text{End}(E(R_R))$. Unless noted otherwise, we work with right-sided concepts. However most of the results and concepts have left-sided analogues.

One of the major efforts in Ring Theory has been, for a given ring R, to find a "well-behaved" overring Q in the sense that it has better properties than R and such that a rich information transfer between R and Q can take place. Alternatively, given a "well-behaved" ring, to find conditions which describe those subrings for which there is some fruitful transfer of information.

The search for such overrings motivates the notion of a hull (i.e., an overring that is "close to" the base ring, in some sense, so as to facilitate the transfer of information). Since we want the overring to have some "desirable properties" the hull should come from a class of rings possessing these properties.

In 1999, the authors embarked on a research program to develop methods that enable one to select a specific class $\mathfrak{K}$ of rings and then to describe all right essential overrings or all right rings of quotients of a given ring R which lie in $\mathfrak{K}$. Moreover, the transfer of information between the base ring R and the essential overring in the class $\mathfrak{K}$ is also investigated.

We have tried to make our definitions flexible enough to encompass the existing theory, apply to many classes of rings, and shed new light on the relationship between a base ring and its essential overrings.

Much of the current theory of rings of quotients emphasizes investigating when a relatively small number of right rings of quotients of R (e.g., its classical right ring of quotients $Q^r_{c\ell}(R)$, the symmetric ring of quotients $Q^s(R)$, the Martindale right ring of quotients $Q^m(R)$, and $Q(R)$) are in a few standard classes of rings (e.g., semisimple Artinian, right Artinian, right Noetherian, right self-injective, or regular).

Some of the deficiencies of this approach are illustrated in the following examples. First take $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$, where $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers, respectively. The ring R is neither right nor left Noetherian and its prime radical is nonzero. However, $Q(R)$ is simple Artinian. Next take R to be a domain which does not satisfy the right Ore condition. Then $Q(R)$ is a simple right self-injective regular ring which has an infinite set of orthogonal idempotents and an unbounded nilpotent index. The sharp disparity between R and $Q(R)$ in the aforementioned examples limits the transfer of information between R and $Q(R)$. These examples illustrate a need to find overrings of a given ring that have some weaker versions of the properties traditionally associated with right rings of quotients such as mentioned above. Furthermore, this need is reinforced when one studies classes of rings for which $R = Q(R)$ (e.g., right Kasch rings). For these classes the theory of right rings of quotients is virtually useless.

Our theory makes no particular restriction on the classes that we consider for our essential overrings. Further, the properties of the classes determine the existence and characterizations of the hulls which may not coincide with $Q^r_{c\ell}(R)$,

$Q^s(R)$, $Q^m(R)$, or $Q(R)$. However those classes which are generalizations of the class of right self-injective rings, regular rings, or classes which are closed under dense or essential extensions work especially well with our methods.

We recall the definitions of some of the classes that generalize the class of right self-injective rings or the class of regular right self-injective rings. A ring R is: right (*FI*-) *extending* if every (ideal) right ideal of R is essential in a right ideal generated by an idempotent; right (*quasi*-) *continuous* if R is right extending and (if A_R and B_R are direct summands of R_R with $A \cap B = 0$, then $A_R \oplus B_R$ is a direct summand of R_R) R satisfies the (C_2) condition, that is, if X and Y are right ideals of R with $X_R \cong Y_R$ and X_R is a direct summand of R_R, then Y_R is a direct summand of R_R; (*quasi*-) *Baer* if the right annihilator of every (ideal) nonempty subset of R is an idempotent generated right ideal. The classes of Baer rings, quasi-Baer rings, right extending rings, right FI-extending rings, right continuous rings, and right quasi-continuous rings are denoted by $\mathbf{B}, \mathbf{qB}$, $\mathbf{E}$, $\mathbf{FI}$, $\mathbf{Con}$, and $\mathbf{qCon}$, respectively (See [11, 58, 63, 76] for $\mathbf{B}$, [13, 15, 16, 17, 20, 23, 39, 73, 76] for $\mathbf{qB}$, [37, 38, 43] for $\mathbf{E}$, [18, 22, 23, 28] for $\mathbf{FI}$, and [49, 64, 65, 84, 85] for $\mathbf{Con}$ and $\mathbf{qCon}$.)

Recall from [14] that a ring R is called *right principally quasi-Baer* (simply, *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated, as a right ideal, by an idempotent (equivalently, R modulo the right annihilator of each principal right ideal is projective). Left principally quasi-Baer (simply, left p.q.-Baer) rings are defined similarly. Rings which are both right and left principally quasi-Baer are called *principally quasi-Baer* (simply, *p.q.-Baer*) rings. We use $\mathbf{pqB}$ to denote the class of right p.q.-Baer rings (see [14] and [30] for more details on right p.q.-Baer rings). A ring R is called *right PP* (also called *right Rickart*) if the right annihilator of each element is generated, as a right ideal, by an idempotent. Left PP (also called left Rickart) rings are defined in a similar way. Rings which are both right and left PP are called *PP* ring (also called Rickart rings).

The right essential overrings, which are in some sense "minimal" with respect to belonging to a specific class of rings, are important tools in our investigations. Hence we define several types of ring hulls to accommodate the various notions of "minimality" among the class of right essential overrings of a given ring. Our search for such minimal overrings for a given ring R includes the seemingly unexplored region that lies between $Q(R)$ and $E(R_R)$ (e.g., when $R = Q(R)$). We consider two basic types (the others are their derivatives). Let S be a right essential overring of R and $\mathfrak{K}$ be a specific class of rings. We say that S is a $\mathfrak{K}$ *right ring hull* of R if S is minimal among the right essential overrings of R belonging to the class $\mathfrak{K}$ (i.e., whenever T is a subring of S where T is a right essential overring of R in the class $\mathfrak{K}$, then $T = S$). For the other basic type, we generate S with R and certain subsets of $E(R_R)$ so that S is in $\mathfrak{K}$ in some "minimal" fashion. This leads to our concepts of a $\mathfrak{C}$ pseudo and $\mathfrak{C}\,\rho$ pseudo right ring hull of R, where ρ is an equivalence relation on a certain set of idempotents from $\mathcal{E}_R$. These ring hull concepts are "tool" concepts in that they appear in the proofs of various results but do not appear in the statements of the results. Let $\mathbf{M}$ be a class of right

R-modules and let M_R be a right R-module. The smallest essential extension of M_R (if it exists) in a fixed injective hull of M_R, that belongs to $\mathbf{M}$ is called the *absolute* $\mathbf{M}$ *hull* of M_R (see Definition 8.1 for details).

The following four problems provide the driving force for our program.

Problem I. Assume that a ring R and a class $\mathfrak{K}$ of rings are given.

(i) Determine conditions to ensure the existence of right rings of quotients and that of right essential overrings of R which are, in some sense, "minimal" with respect to belonging to the class $\mathfrak{K}$.
(ii) Characterize the right rings of quotients and the right essential overrings of R which are in the class $\mathfrak{K}$, possibly by using the "minimal" ones obtained in part (i).

Problem II. Given a ring R and a class $\mathfrak{K}$ of rings, determine what information transfers between R and its right essential overrings in $\mathfrak{K}$ (especially the right essential overrings which are, in some sense, "minimal" with respect to belonging to $\mathfrak{K}$).

Problem III. Given classes of rings $\mathfrak{K}$ and $\mathcal{S}$, determine those $T \in \mathfrak{K}$ such that $Q(T) \in \mathcal{S}$.

Problem IV. Given a ring R and a class of rings $\mathfrak{K}$, let $X(R)$ denote some standard type of extension of R (e.g., $X(R) = R[x]$, or $X(R) = \mathrm{Mat}_n(R)$, the n-by-n matrix ring over R, etc.) and let $H(R)$ denote a right essential overring of R which is "minimal" with respect to belonging to the class $\mathfrak{K}$ (i.e., a hull). Determine when $H(X(R))$ is comparable to $X(H(R))$.

We recall from [13] that an idempotent e of a ring R is called *left* (resp., *right*) *semicentral* if $xe = exe$ (resp., $ex = exe$) for all $x \in R$. Observe that $e = e^2 \in R$ is left (resp., right) semicentral if and only if eR (resp., Re) is an ideal of R. We let $\mathbf{S}_\ell(R)$ (resp., $\mathbf{S}_r(R)$) denote the set of all left (resp., right) semicentral idempotents of R. Note that $\mathbf{S}_\ell(R) = \{0, 1\}$ if and only if $\mathbf{S}_r(R) = \{0, 1\}$. A ring R is said to be *semicentral reduced* if $\mathbf{S}_\ell(R) = \{0, 1\}$ or equivalently $\mathbf{S}_r(R) = \{0, 1\}$. We use $\mathcal{B}(R)$ to denote the set of all central idempotents of a ring R. It can be shown that $\mathcal{B}(R) = \mathbf{S}_\ell(R) \cap \mathbf{S}_r(R)$. If R is a semiprime ring, then $\mathcal{B}(R) = \mathbf{S}_\ell(R) = \mathbf{S}_r(R)$.

For a ring R, we use $\mathbf{I}(R), \mathbf{U}(R)$, $Z(R_R)$, $\mathrm{Cen}(R)$, $P(R)$, and $J(R)$ to denote the idempotents, units, right singular ideal, center, prime radical, and Jacobson radical of R, respectively. For ring extensions of R, we use $R\mathcal{B}(Q(R))$ and $T_n(R)$ to denote the idempotent closure (i.e., the subring of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$ [9]) and the n-by-n upper triangular matrix ring over R, respectively. For a nonempty subset X of a ring R, the symbols $r_R(X)$, $\ell_R(X)$, and $\langle X \rangle_R$ denote the right annihilator of X in R, the left annihilator of X in R, and the subring of R generated by X, respectively. Also $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{Z}_n$ denote the field of rational numbers, the ring of integers, and the ring of integers modulo n, respectively. We use $I \trianglelefteq R$ to denote that I is an ideal of a ring R. Finally recall that a ring R is called *reduced* if R has no nonzero nilpotent elements and *Abelian* if $\mathbf{I}(R) = \mathcal{B}(R)$. Throughout the paper, a regular ring means a von Neumann regular ring.

We let $\mathcal{Q}_R = \text{End}({}_{\mathcal{E}_R}E(R_R))$ (recall that $\mathcal{E}_R = \text{End}(E(R_R))$). Note that $Q(R) = 1 \cdot \mathcal{Q}_R$ (i.e., the canonical image of $\mathcal{Q}_R$ in $E(R_R)$) and that $\mathcal{B}(\mathcal{Q}_R) = \mathcal{B}(\mathcal{E}_R)$ [61, pp. 94–96]. Also, $\mathcal{B}(Q(R)) = \{b(1) \mid b \in \mathcal{B}(\mathcal{Q}_R)\}$ [60, p. 366]. Thus $R\mathcal{B}(\mathcal{E}_R) = R\mathcal{B}(Q(R))$. Recall that the *extended centroid* of R is $\text{Cen}(Q(R))$. If R is semiprime, then $\text{Cen}(Q(R)) = \text{Cen}(Q^m(R)) = \text{Cen}(Q^s(R))$ [60, pp. 389–390], where $Q^m(R)$ and $Q^s(R)$ denote the Martindale right ring of quotients of R and the symmetric ring of quotients of R, respectively. (See [2] for more details on $Q^m(R)$.)

2. Brief history of hulls

In this section, we summarize the definitions and results that provide the background for our definitions of various ring hulls. The story begins in 1940 with the famous paper of R. Baer [8]. In that paper, Baer introduced the concept of an injective module by calling a module M_R *complete* (*injective* in current terminology) if to every right ideal I of R and to every R-homomorphism h of I into M_R there is some $m \in M$ with $h(x) = mx$ for all $x \in I$. This definition incorporates the celebrated "Baer Criterion". Moreover he proved the following result.

Theorem 2.1. ([8, Baer])

(i) *A module M_R is injective if and only if whenever $M_R \leq N_R$ then M_R is a direct summand of N_R.*
(ii) *Every module is a submodule of an injective module.*

Further, Baer indicated that each module can be embedded in some "essentially smallest" injective module. In 1952, Shoda [79] and independently in 1953 Eckmann and Schopf [44] explicitly established the existence of a minimal (up to isomorphism) injective extension (hull) of a module. Eckmann and Schopf characterized the injective hull of a module as its maximal essential extension.

Johnson and Wong [56], in 1961, defined a module K_R to be *quasi-injective* if for every R-homomorphism $h : S \to K$, of a submodule S of K, there is an $f \in \text{End}(K_R)$ such that $f(s) = h(s)$ for all $s \in S$. They proved that every module M_R has a unique (up to isomorphism) quasi-injective hull in the following result.

Theorem 2.2 ([56, Johnson and Wong]) *Let $E(M_R)$ be an injective hull of a module M_R. Take $\mathcal{E}_M = \text{End}(E(M_R))$, and let $\mathcal{E}_M M_R$ denote the R-submodule of $E(M_R)$ generated by all the $h(M)$ where $h \in \mathcal{E}_M$. Then the following hold.*

(i) *$\mathcal{E}_M M_R$ is quasi-injective.*
(ii) *$\mathcal{E}_M M_R$ is the intersection of all quasi-injective submodules of $E(M_R)$ containing M_R.*
(iii) *M_R is quasi-injective if and only if $M_R = \mathcal{E}_M M_R$.*

In 1963, J. Kist [59] defined a commutative PP ring $\overline{R}$ to be a *Baer extension* of a commutative PP ring R if the following conditions hold.

(i) R is (isomorphic to) a subring of $\overline{R}$;

(ii) $\mathfrak{C}(R)$ is (isomorphic to) a dense semilattice of $\mathfrak{C}(\overline{R})$, and the Boolean subalgebra of $\mathfrak{C}(\overline{R})$ is generated by this dense subsemilattice is all of $\overline{R}$, where $\mathfrak{C}(-)$ consists of the open-and-closed sets in the hull-kernel topology on the set of minimal prime ideals of a ring; and

(iii) If $x \in \overline{R}$, then there exist finitely many idempotents $e_1, \ldots, e_n$ in $\overline{R}$ which are mutually orthogonal, and whose sum is 1; and elements $x_1 \ldots, x_n$ in R such that $x = e_1x_1 + \cdots + e_nx_n$.

We note that in [59], Kist uses the terminology "Baer ring" for what are more commonly called PP rings. Thus a Baer extension may not be a Baer ring in the sense of Kaplansky [58]. Kist proved the following result.

Theorem 2.3. ([59, Kist]) *If R is a commutative semiprime ring, then it has a Baer extension. Moreover, isomorphic rings have isomorphic Baer extensions.*

For a commutative semiprime ring R, in 1968 H. Storrer [82], called the intersection of all regular subrings of $Q(R)$ containing R the *epimorphic hull* of R. By showing this intersection was regular, he showed that every commutative semiprime ring has a smallest regular ring of quotients.

The Baer hull, namely the ring $\mathbf{B}(R)$ in the next theorem, for a commutative semiprime ring R, was defined by Mewborn [63] in 1971.

Theorem 2.4. ([63, Mewborn]) *Let R be a commutative semiprime ring. Let $\mathbf{B}(R)$ be the intersection of all Baer subrings of $Q(R)$ containing R. Then $\mathbf{B}(R)$ is a Baer ring and it is the subring of $Q(R)$ generated by R and $\mathbf{I}(Q(R))$.*

In [66], Oshiro used sheaf theoretic methods to construct the Baer hull of a commutative regular ring.

The absolute π-injective (equivalently, quasi-continuous) hull of a module was defined by Goel and Jain in 1978 [49]. The following theorem is an immediate consequence of their results.

Theorem 2.5. ([49, Goel and Jain]) *Let V be the subring of $\mathcal{E}_M = \mathrm{End}(E(M_R))$ generated by $\mathbf{I}(\mathcal{E}_M)$. Then VM_R is the unique (up to isomorphism) absolute quasi-continuous hull of M_R.*

For any submodule A of a quasi-continuous module M, there exists a direct summand $P = M \cap E(A)$ of M which contains A as an essential submodule. This P is called the *internal quasi-continuous hull* of A in M and was shown to be unique up to isomorphism by Müller and Rizvi [65].

Theorem 2.6. ([65, Müller and Rizvi]) *Let M be a quasi-continuous module, A_1, A_2 submodules of M, and P_1, P_2 internal hulls of A_1 and A_2, respectively. If $A_1 \cong A_2$, then $P_1 \cong P_2$.*

In 1982, Müller and Rizvi [64] defined three types of continuous hulls for modules as follows.

Definitions 2.7. ([64, Müller and Rizvi]) Let M be a module with an injective hull E, and let H be a continuous overmodule of M.

(I) H is called a *type* I *continuous hull* of M, if $M \subseteq X \subseteq H$ for a continuous module X implies $X = H$.
(II) H is called a *type* II *continuous hull* of M, if for every continuous overmodule X of M, there exists a monomorphism $\mu : H \to X$ over M.
(III) H is called a *type* III *continuous hull* of M (in E), if $M \subseteq H \subseteq E$, and if $H \subseteq X$ for every continuous module $M \subseteq X \subseteq E$.

Observe that a type III continuous hull is uniquely determined as a submodule of a fixed injective hull. They gave an example of a module which has neither a type II nor a type III continuous hull. However they proved the following result.

Theorem 2.8. ([64, Müller and Rizvi]) *Every cyclic module over a commutative ring whose singular submodule is uniform, has a type* III *continuous hull.*

Also, in 1982, Hirano, Hongan, and Ohori [54] defined the Baer hull and the strongly regular hull for a reduced right Utumi ring. Recall that a right nonsingular ring R is called a *right Utumi* ring if every non-essential right ideal of R has a nonzero left annihilator. They defined the *strongly regular hull* of a reduced right Utumi ring to be the intersection of all regular subrings of $Q(R)$. Note that their definition of a strongly regular hull generalizes the epimorphic hull of Storrer [82].

Theorem 2.9. ([54, Hirano, Hongan, and Ohori]) *Let R be a reduced right Utumi ring and let $\mathbf{B}(R)$ be the intersection of all the Baer subrings of $Q(R)$ containing R. Then $\mathbf{B}(R)$ is a Baer ring and coincides with the subring of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$.*

This result generalizes Theorem 2.4 to noncommutative rings.

Corollary 2.10. ([54, Hirano, Hongan, and Ohori]) *Every reduced PI ring has a Baer hull and a strongly regular hull.*

The idempotent closure of a module was introduced by Beidar and Wisbauer [9] in 1993. Recall that $\mathcal{E}_M M_R$ is the quasi-injective hull of M_R by Theorem 2.2. The *idempotent closure* of M_R is the submodule of $\mathcal{E}_M M_R$ generated by $\{e(M) \mid e \in \mathcal{B}(\mathcal{E}_M)\}$. For a ring R, we identify the idempotent closure of R_R with the subring of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$ and denote it by $R\mathcal{B}(Q(R))$. Thus if R is a commutative semiprime ring, then the idempotent closure of R is the Baer hull of R as already shown by Mewborn in 1971 (Theorem 2.4). Beidar and Wisbauer indicated that if $\mathrm{End}(\mathcal{E}_M M_R)$ is Abelian then the idempotent closure of M_R is π-injective (equivalently, quasi-continuous) hull of M_R. In [9] and [10], they showed that information about prime ideals and various types of regularity conditions transfer between R and $R\mathcal{B}(Q(R))$.

Theorem 2.11. ([9, Beidar and Wisbauer]) *Let R be a semiprime ring. Then the following hold.*

(i) *For every prime ideal K of $R\mathcal{B}(Q(R))$, $P = K \cap R$ is a prime ideal of R and $R\mathcal{B}(Q(R))/K = (R+K)/K \cong R/P$.*

(ii) *For any prime ideal P of R, there exists a prime ideal K of $R\mathcal{B}(Q(R))$ with $K \cap R = P$ (i.e., LO (lying over) holds between R and $R\mathcal{B}(Q(R))$).*

Theorem 2.12. ([9, Beidar and Wisbauer]) *Let R be a ring. Then R is biregular if and only if R is semiprime and $R\mathcal{B}(Q(R))$ is biregular.*

Theorem 2.13. ([10, Beidar and Wisbauer]) *Let R be a ring. Then R is regular and biregular if and only if $R\mathcal{B}(Q(R))$ is regular and biregular.*

Burgess and Raphael call a regular ring with bounded index an *almost biregular* ring if and only if for each $x \in R$ there is an $e \in \mathcal{B}(R)$ such that $RxR_R \leq^{\text{ess}} eR_R$ [34]. Recall that if R is a right nonsingular ring, then $Q(R)$ is a Baer ring. From [58, p. 9] each element of a Baer ring R has a central cover (recall that $e \in R$ is a *central cover* for $r \in R$ if e is the smallest central idempotent in the Boolean algebra of the central idempotents of R such that $er = r$). In the following result, Burgess and Raphael show that every regular ring with bounded index is contained in a smallest almost biregular right ring of quotients.

Theorem 2.14. ([34, Burgess and Raphael]) *Let R be a regular ring of bounded index. Define $R^{\#}$ to be the ring generated by R and the central covers from $Q(R)$ of all elements of R. Then $R^{\#}$ is the unique smallest almost biregular ring among the regular rings S such that $R \subseteq S \subseteq Q(R)$. Moreover:*

(i) *If $R^{\#} \subseteq T \subseteq Q(R)$ and T is generated as a subring of $Q(R)$ by R and $\mathcal{B}(T)$, then T is almost biregular; and*

(ii) *Let A be the sub-Boolean algebra of $\mathcal{B}(Q(R))$ generated by the central covers of elements of R. Then $\mathcal{B}(R^{\#}) = A$.*

In 1980, Picavet defined a commutative ring R to be a *weak Baer* ring if and only if R is a PP ring (in our terminology), that is, for each $a \in R$ there exists $e = e^2 \in R$ such that $r_R(a) = eR$ [71]. He defined the *weak Baer envelope* for a commutative reduced ring to be the subring of $Q(R)$ generated by $R \cap \{aq \mid a \in R \text{ and } q \in Q(R) \text{ such that } aqa = a\}$. He showed that the weak Baer envelope of a commutative reduced ring R is the smallest weak Baer subring of $Q(R)$ that contains R. Various applications of the weak Baer envelope appear in [72] and [42].

3. Definitions of a ring hull

In this section, we provide several definitions of the concept of a ring hull to abstract, unify, and encompass the various definitions of particular ring hulls (e.g., Baer extension, Baer hull, epimorphic hull, strongly regular hull, etc.) given in Section 2. These definitions are established in the context of intermediate rings between a base ring R and its injective hull $E(R_R)$ to insure some flow of information between the base ring R and the overrings under consideration. Moreover, our definitions are in terms of abstract classes of rings so as to guarantee their flexibility and versatility.

Henceforth we assume that all right essential overrings of a ring R are contained as right R-modules in a fixed injective hull $E(R_R)$ of R_R and that all right rings of quotients of R are subrings of a fixed maximal right ring of quotients $Q(R)$ of R.

In our next definition we exploit the notion of a right essential overring which is minimal with respect to belonging to a class $\mathfrak{K}$ of rings.

Definition 3.1. ([24, Definition 2.1]) Let $\mathfrak{K}$ denote a class of rings. For a ring R, let S be a right essential overring of R and T an overring of R. Consider the following conditions.

(i) $S \in \mathfrak{K}$.
(ii) If $T \in \mathfrak{K}$ and T is a subring of S, then $T = S$.
(iii) If S and T are subrings of a ring V and $T \in \mathfrak{K}$, then S is a subring of T.
(iv) If $T \in \mathfrak{K}$ and T is a right essential overring of R, then S is a subring of T.

If S satisfies (i) and (ii), then we say that S is a $\mathfrak{K}$ *right ring hull* of R, denoted by $\widetilde{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iii), then we say that S is the $\mathfrak{K}$ *absolute to V right ring hull* of R, denoted by $Q^V_{\mathfrak{K}}(R)$; for the $\mathfrak{K}$ absolute to $Q(R)$ right ring hull, we use the notation $\widehat{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iv), then we say that S is the $\mathfrak{K}$ *absolute right ring hull* of R, denoted by $Q_{\mathfrak{K}}(R)$. Observe that if $Q(R) = E(R_R)$, then $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$. The concept of a $\mathfrak{K}$ absolute right ring hull was already implicit in [64] from their definition of a type III continuous (module) hull (see Definition 2.7).

Moreover, the notions of $\mathfrak{K}$ absolute to $Q(R)$ right ring hull and $\mathfrak{K}$ absolute right ring hull incorporate many of the hull definitions in Section 2 that utilized the intersection of all right rings of quotients from a certain class of rings (e.g., Baer hull, epimorphic hull, etc.) which contain the base ring. This will be illustrated in the next section.

Now we consider generating a right essential overring in a class $\mathfrak{K}$ from a base ring R and some subset of $\mathcal{E}_R$. By using equivalence relations, we can effectively reduce the size of the subsets of $\mathcal{E}_R$ needed to generate a right essential overring of R in $\mathfrak{K}$.

Definition 3.2. ([24, Definition 2.2]) Let $\mathfrak{R}$ denote a class of rings and $\mathfrak{X}$ a class of subsets of rings such that for each $R \in \mathfrak{R}$ all subsets of $\mathcal{E}_R$ are contained in $\mathfrak{X}$. Let $\mathfrak{K}$ be a subclass of $\mathfrak{R}$ such that there exists an assignment $\delta_{\mathfrak{K}} : \mathfrak{R} \to \mathfrak{X}$ such that $\delta_{\mathfrak{K}}(R) \subseteq \mathcal{E}_R$ and $\delta_{\mathfrak{K}}(R)(1) \subseteq R$ implies $R \in \mathfrak{K}$, where $\delta_{\mathfrak{K}}(R)(1) = \{h(1) \in E(R_R) \mid h \in \delta_{\mathfrak{K}}(R)\}$. Let S be a right essential overring of R and ρ an equivalence relation on $\delta_{\mathfrak{K}}(R)$. Note that there may be distinct assignments for the same $\mathfrak{R}$, $\mathfrak{X}$, and $\mathfrak{K}$ say $\delta_{1\mathfrak{K}}$ and $\delta_{2\mathfrak{K}}$ such that for a given R, $\delta_{1\mathfrak{K}}(R) \neq \delta_{2\mathfrak{K}}(R)$; but $\delta_{1\mathfrak{K}}(R)(1) \subseteq R$ implies $R \in \mathfrak{K}$ and $\delta_{2\mathfrak{K}}(R)(1) \subseteq R$ implies $R \in \mathfrak{K}$.

(i) If $\delta_{\mathfrak{K}}(R)(1) \subseteq S$ and $\langle R \cup \delta_{\mathfrak{K}}(R)(1)\rangle_S \in \mathfrak{K}$, then we call $\langle R \cup \delta_{\mathfrak{K}}(R)(1)\rangle_S$ the $\delta_{\mathfrak{K}}$ *pseudo right ring hull of* R *with respect to* S and denote it by $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$. If $S = R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$, then we say that S is a $\delta_{\mathfrak{K}}$ *pseudo right ring hull of* R.

(ii) If $\delta^{\rho}_{\mathfrak{K}}(R)(1) \subseteq S$ and $\langle R \cup \delta^{\rho}_{\mathfrak{K}}(R)(1)\rangle_S \in \mathfrak{K}$, then we call $\langle R \cup \delta^{\rho}_{\mathfrak{K}}(R)(1)\rangle_S$ a $\delta_{\mathfrak{K}}\,\rho$ *pseudo right ring hull of* R *with respect to* S and denote it by $R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$, where $\delta^{\rho}_{\mathfrak{K}}(R)$ is a set of representatives of all equivalence classes of ρ and $\delta^{\rho}_{\mathfrak{K}}(R)(1) = \{h(1) \in E(R_R) \mid h \in \delta^{\rho}_{\mathfrak{K}}(R)\}$. If $S = R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$, then we say that S is a $\delta_{\mathfrak{K}}\,\rho$ *pseudo right ring hull of* R.

If a $\delta_{\mathfrak{K}}$ has been fixed for a class $\mathfrak{K}$, then in the above nomenclature we replace $\delta_{\mathfrak{K}}$ (resp., $\delta_{\mathfrak{K}}\,\rho$) with $\mathfrak{K}$ (resp., $\mathfrak{K}\ \rho$) (e.g., $\delta_{\mathfrak{K}}$ pseudo right ring hull becomes $\mathfrak{K}$ pseudo right ring hull) and delete $\delta_{\mathfrak{K}}$ from the notation (e.g., $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$ becomes $R(\mathfrak{K}, S)$). Observe that if $\delta_{\mathfrak{K}}(R)(1) \subseteq Q(R)$ and S is a right essential overring of R such that $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$ exists, then $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S) = R(\mathfrak{K}, \delta_{\mathfrak{K}}, Q(R))$.

Throughout the remainder of this paper take $\mathfrak{R}$ to be the class of all rings unless indicated otherwise. Some examples illustrating Definition 3.2 are:

(1) $\mathfrak{K} = \mathbf{SI} = \{$right self-injective rings$\}$, $\delta_{\mathfrak{S}\mathfrak{J}}(R) = \mathcal{E}_R$.
(2) $\mathfrak{K} = \mathbf{qCon}$, $\delta_{\mathbf{qCon}}(R) = \mathbf{I}(\mathcal{E}_R)$.
(3) $\mathfrak{K} = \{$right P-injective rings$\}$, $\delta_{\mathfrak{K}}(R) = \{h \in \mathcal{E}_R \mid$ there exist $a \in R$ and an R-homomorphism $f : aR \to R$ such that $h|_{aR} = f\}$.
(4) Let $\mathfrak{R} = \{$right nonsingular rings$\}$, $\mathfrak{K} = \mathbf{B}$, $\delta_{\mathbf{B}}(R) = \{e \in \mathbf{I}(\mathcal{E}_R) \mid$ there exists $\emptyset \neq X \subseteq R$ such that $r_{Q(R)}(X) = eQ(R)\}$.

Also note that Definition 3.2 allows us the flexibility to consider any right essential overring S of a ring R, such that $S \in \mathfrak{K}$ and $S = \langle R \cup \delta(1)\rangle_S$, to be a $R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$ where $\emptyset \neq \delta \subseteq \delta_{\mathfrak{K}}(R)$ and $\delta(1) = \{e(1) \mid e \in \delta\}$. To see this, choose $f \in \delta$. Let $X = \delta_{\mathfrak{K}}(R) \setminus \{e \mid e \in \delta$ and $e \neq f\}$. Then $\{X\} \cup \{\{e\} \mid e \in \delta$ and $e \neq f\}$ is a partition of $\delta_{\mathfrak{K}}(R)$. Let ρ be the equivalence relation induced on $\delta_{\mathfrak{K}}(R)$ by this partition and take $\delta^{\rho}_{\mathfrak{K}}(R)(1) = \delta(1)$. Then $S = R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$.

Observe that the concept of a $\delta_{\mathfrak{K}}$ pseudo right ring hull incorporates that of Goel and Jain [49] for quasi-continuous ring hull (when it exists) by taking $\delta_{\mathbf{qCon}}(R) = \mathbf{I}(\mathcal{E}_R)$, and that of Mewborn [63] for the Baer hull when $\mathfrak{R}$ is the class of commutative semiprime rings by taking $\delta_{\mathbf{B}}(R) = \mathbf{I}(\mathcal{E}_R)$. Also note that several of the hulls considered in Section 2, are types of hulls indicated in both Definitions 3.1 and 3.2.

Definition 3.3. ([24, Definition 1.6]) Let $\mathfrak{R}$ be a class of rings, $\mathfrak{K}$ a subclass of $\mathfrak{R}$, and $\mathfrak{Y}$ a class containing all sets of subsets of every ring. We say that $\mathfrak{K}$ is a class *determined by a property on right ideals* if there exist an assignment $\mathfrak{D}_{\mathfrak{K}} : \mathfrak{R} \to \mathfrak{Y}$ such that $\mathfrak{D}_{\mathfrak{K}}(R) \subseteq \{$right ideals of $R\}$ and a property P such that $\mathfrak{D}_{\mathfrak{K}}(R)$ has P if and only if $R \in \mathfrak{K}$.

If $\mathfrak{K}$ is such a class where P is the property that a right ideal is essential in an idempotent generated right ideal, then we say that $\mathfrak{K}$ is a **D-E** *class* and use $\mathfrak{C}$ to designate a **D-E** class.

Some examples illustrating Definition 3.3 are:

(1) $\mathfrak{K}$ is the class of right Noetherian rings, $\mathbf{D}_{\mathfrak{K}}(R) = \{\text{right ideals of } R\}$, and P is the property that a right ideal is finitely generated;
(2) $\mathfrak{K}$ is the class of regular rings, $\mathbf{D}_{\mathfrak{K}}(R) = \{\text{principal right ideals of } R\}$, and P is the property that a right ideal is generated by an idempotent as a right ideal;
(3) $\mathfrak{K} = \mathbf{B}$, $\mathbf{D}_{\mathbf{B}}(R) = \{r_R(X) \mid \emptyset \neq X \subseteq R\}$, and P is the property that a right ideal is generated by an idempotent as a right ideal;
(4) $\mathfrak{C} = \mathbf{E}$ (resp., $\mathfrak{C} = \mathbf{FI}$, $\mathfrak{C} = \mathbf{eB}$), $\mathbf{D}_{\mathbf{E}}(R) = \{I \mid I_R \leq R_R\}$ (resp., $\mathbf{D}_{\mathbf{FI}}(R) = \{I \mid I \trianglelefteq R\}$, $\mathbf{D}_{\mathbf{eB}}(R) = \{r_R(X) \mid \emptyset \neq X \subseteq R\}$).

Our primary focus in this paper is on classes of rings which are either **D-E** classes or subclasses of **D-E**. Note that any **D-E** class always contains the class of right extending (and hence all right self-injective) rings. Moreover, many known classes of rings are subclasses of a **D-E** class

Theorem 3.4 illustrates the generality achieved by working in the context of a **D-E** class, while Corollary 3.5 demonstrates its application to concrete **D-E** classes.

Theorem 3.4. ([24, Theorem 1.7]) *Assume that $\mathfrak{C}$ is a* **D**-**E** *class of rings.*

(i) *Let T be a right essential overring of R. Suppose that for each $Y \in \mathbf{D}_{\mathfrak{C}}(T)$ there exist $X_R \leq R_R$ and $e \in \mathbf{I}(T)$ such that $X_R \leq^{\text{ess}} eR_R$, $X_R \leq^{\text{ess}} Y_R$, and $eY \subseteq Y$. Then $T \in \mathfrak{C}$.*
(ii) *Let T be a right ring of quotients of R and $R \in \mathfrak{C}$. If $Y \in \mathbf{D}_{\mathfrak{C}}(T)$ implies $Y \cap R \in \mathbf{D}_{\mathfrak{C}}(R)$, then $T \in \mathfrak{C}$.*

Classes of rings which are closed with respect to right rings of quotients (resp., right essential overrings) work especially well with a hull concept in that once one finds a hull from such a class then one has that all right rings of quotients (resp., right essential overrings) of that hull are also in the class. Among our final results of this section, we give several examples of classes of rings that are closed with respect to right rings of quotients or right essential overrings.

Recall the following definitions:

1. A ring R is called right *finitely Σ-extending* if any finitely generated free right R-module is extending [43].
2. A ring R is said to be right *uniform extending* if each uniform right ideal of R is essential as a right R-module in a direct summand of R_R [43].
3. A ring R is said to be right C_{11} if every right ideal of R has a complement which is a direct summand [80].
4. A ring R is called right *G-extending* if for each right ideal Y of R there is a direct summand D of R_R with $(Y \cap D)_R \leq^{\text{ess}} Y_R$ and $(Y \cap D)_R \leq^{\text{ess}} D_R$ [1].
5. A ring R is called *ideal intrinsic over its center*, IIC, if every nonzero ideal of R has nonzero intersection with the center of R [6].

As a consequence of Theorem 3.4, the next corollary exhibits the transfer of the right (FI-) extending property from R to its (right essential overrings) right rings of quotients. Also note that whenever a property is carried from R to its (right essential overrings) right rings of quotients, then a Zorn's lemma argument can be used to show that R has a (right essential overring) right ring of quotients which is maximal with respect to having that property.

Corollary 3.5. ([24, Corollary 1.8])

(i) *Any right essential overring of a right FI-extending ring is right FI-extending.*
(ii) *Any right ring of quotients of a right extending ring is right extending.*
(iii) *Any right ring of quotients of a right finitely Σ-extending ring is right finitely Σ-extending.*
(iv) *Any right ring of quotients of a right uniform extending ring is right uniform extending.*

Theorem 3.6.

(i) ([31, Theorem 3.5]) *If R is a right C_{11}-ring and T is a right essential overring of R, then T is a right C_{11}-ring.*
(ii) ([1]) *If R is a right G-extending ring and T is a right essential overring of R, then T is a right G-extending ring.*
(iii) *If R is an IIC-ring and T is a right essential overring with* $\mathrm{Cen}(R) \subseteq \mathrm{Cen}(T)$ *(e.g., $T = Q(R)$), then T is an IIC-ring.*

We say that a ring R is *right essentially Baer* (resp., *right essentially quasi-Baer*) if the right annihilator of any nonempty subset (resp., ideal) of R is essential in a right ideal generated by an idempotent ([24, Definition 1.1]). We use **eB** (resp., **eqB**) to denote the class of right essentially Baer (resp., right essentially quasi-Baer) rings.

Note that the classes **B** and **qB** are not $\mathfrak{C}$ classes, but they are contained in the $\mathfrak{C}$ classes **eB** and **eqB**, respectively. It can be seen that **eB** (resp., **eqB**) properly contains **E** (resp., **FI**) and **B** (resp., **qB**): If $S = A \oplus B$, where A is a domain which is not right Ore and B is a prime ring with $Z(B_B) \neq 0$ [33, Example 4.4], then S is neither right extending nor Baer. But $S \in \mathbf{eB}$. Next take

$$R = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}.$$

Then the ring R is neither right FI-extending nor quasi-Baer. However $R \in \mathbf{eqB}$.

The following two results provide connections between the classes **FI**, **B**, **qB**, **eB**, and **eqB**.

Proposition 3.7. ([24, Proposition 1.2]) *Assume that R is a right nonsingular ring.*

(i) *If $R \in \mathbf{eB}$ (resp., $R \in \mathbf{eqB}$), then $R \in \mathbf{B}$ (resp., $R \in \mathbf{qB}$).*
(ii) *If $R \in \mathbf{FI}$, then $R \in \mathbf{qB}$.*

Proposition 3.8. ([12, Lemma 2.2] and [18, Theorem 4.7]) *Assume that R is a semiprime ring. Then the following are equivalent.*

(i) $R \in \mathbf{FI}$.
(ii) *For any* $I \trianglelefteq R$, *there is* $e \in \mathcal{B}(R)$ *such that* $I_R \leq^{\text{ess}} eR_R$.
(iii) $R \in \mathbf{qB}$.
(iv) $R \in \mathbf{eqB}$.

Theorem 3.9. ([24, Theorem 1.9])

(i) Let T be a right and left essential overring of R. If $R \in \mathbf{qB}$, then $T \in \mathbf{qB}$.
(ii) Let T be a right essential overring of R which is also a left ring of quotients of R. If $R \in \mathbf{B}$ (resp., $R \in \mathbf{eqB}$), then $T \in \mathbf{B}$ (resp., $T \in \mathbf{eqB}$).
(iii) Let T be a right and left ring of quotients of R. If $R \in \mathbf{eB}$, then $T \in \mathbf{eB}$.

The following corollary generalizes the well-known result that a right ring of quotients of a Prüfer domain is a Prüfer domain [48, pp. 321–323].

Corollary 3.10. ([24, Corollary 1.10]) *Let T be a right and left ring of quotients of R. If R is right semihereditary and every finitely generated free right R-module satisfies the ACC on direct summands, then T is right and left semihereditary.*

4. Existence and uniqueness of ring hulls

In this section, we not only explicitly show how our theory encompasses the particular hulls indicated in Section 2, but how it can be used in a much wider context by applying the theory to many classes of rings not considered in Section 2. Also our results will often show an interplay between the ring hull concept (Definition 3.1) and the pseudo ring hull concept (Definition 3.2). These results also provide answers to Problem I of Section 1.

Our first result illustrates Definitions 3.1 and 3.2 by taking advantage of several well-known facts to provide ring hulls for the classes of semisimple Artinian rings, right self-injective rings, and right duo rings.

Proposition 4.1. ([24, Proposition 2.3])

(i) *Let $\mathbf{A}$ be the class of semisimple Artinian rings and R a right nonsingular ring with finite right uniform dimension. Then* $Q_{\mathbf{A}}(R) = Q(R)$.
(ii) *If* $Q(R) = E(R_R)$, *then* $Q_{\mathbf{SI}}(R) = Q(R) = R(\mathbf{SI}, \delta_{\mathbf{SI}}, Q(R))$, *where* $\mathbf{SI}$ *is the class of right self-injective rings.*
(iii) *If* $Q(R) = E(R_R)$, *then* $Q_{\mathbf{qCon}}(R) = \langle R \cup \mathbf{I}(Q(R)) \rangle_{Q(R)} = R(\mathbf{qCon}, \delta_{\mathbf{qCon}}, Q(R))$.
(iv) *If R is a commutative semiprime ring, then* $Q_{\mathbf{B}}(R) = \langle R \cup \mathbf{I}(Q(R)) \rangle_{Q(R)} = Q_{\mathbf{qCon}}(R)$.
(v) *Assume that R has finite right uniform dimension and S is a right ring of quotients of R. Then* $\mathrm{Mat}_n(S) = \widetilde{Q}_{\mathbf{B}}(\mathrm{Mat}_n(R))$ *for all positive integers n if and only if S is a right and left semihereditary right ring hull of R.*
(vi) *If R is a right Ore domain, then R has a right duo absolute right ring hull.*

For Proposition 4.1(vi), the next example is that of a right Ore domain R which is *not* right duo, but it has a right duo absolute right ring hull properly between R and $Q(R)$.

Example 4.2. ([24, Example 2.4]) Take $A = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$, the integer quaternions. Let $P = 5\mathbb{Z}$ and $\widehat{\mathbb{Z}}_P$ the P-adic completion of $\mathbb{Z}$. Also let

$$R = \widehat{\mathbb{Z}}_P + \widehat{\mathbb{Z}}_P i + \widehat{\mathbb{Z}}_P j + \widehat{\mathbb{Z}}_P k.$$

Then R is a right Ore domain. Note that R is not right duo because $(3 + i)R$ is not a left ideal. Take

$$\lambda = (1/2)(1 + i + j + k) \in Q(A) = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k.$$

Let $S = A + \lambda A$. Then by [74, p. 131, Exercise 2] S is a maximal $\mathbb{Z}$-order in $Q(A)$. Thus the P-adic completion $\widehat{S}_P = \widehat{\mathbb{Z}}_P \otimes_{\mathbb{Z}} S$ of S is a maximal $\widehat{\mathbb{Z}}_P$-order in $Q(R) = Q(\widehat{\mathbb{Z}}_P) \otimes_{\mathbb{Q}} Q(A)$ by [74, p. 134, Corollary 11.6]. Since $\widehat{\mathbb{Z}}_P$ is a complete discrete valuation ring and $Q(R)$ is a division ring, $\widehat{S}_P$ is the unique maximal $\widehat{\mathbb{Z}}_P$-order in $Q(R)$, thus $\widehat{S}_P$ is right duo by [74, p. 139, Theorem 13.2]. So $\widehat{S}_P$ is a proper intermediate right duo ring between R and $Q(R)$. Thus, by Proposition 4.1(vi), there exists a right duo absolute right ring hull properly between R and $Q(R)$.

Let $\mathfrak{U}$ denote the class $\{R \mid R \cap \mathbf{U}(Q(R)) = \mathbf{U}(R)\}$ of rings, where $\mathbf{U}(-)$ is the set of units of a ring. Recall from [84] and [85] that R is called *directly finite* if every one-sided inverse of an element of R is two-sided. Note that if R has finite right uniform dimension, or if R satisfies the condition that $r_R(x) = 0$ implies $\ell_R(x) = 0$, or if R is Abelian, then R is directly finite.

For our next result, let $i < j$ be ordinal numbers. We define $R_1 = \langle R \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in R\}\rangle_{Q(R)}$, $R_j = \langle R_i \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in R_i\}\rangle_{Q(R)}$ for $j = i + 1$, and $R_j = \bigcup_{i<j} R_i$ for j a limit ordinal. The following theorem characterizes $Q^r_{c\ell}(R)$ as a $\mathfrak{U}$ absolute to $Q(R)$ right ring hull.

Theorem 4.3. ([24, Theorem 2.7])

(i) *$\widehat{Q}_{\mathfrak{U}}(R)$ exists and $\widehat{Q}_{\mathfrak{U}}(R) = R_j$ for any j with $|j| > |Q(R)|$.*

(ii) *Assume that T is a directly finite right essential overring of R and T_T satisfies (C_2). Then $\widehat{Q}_{\mathfrak{U}}(R)$ is a subring of T.*

(iii) *If R is a right Ore ring, then $\widehat{Q}_{\mathfrak{U}}(R) = Q^r_{c\ell}(R)$.*

Note that from Theorem 4.3, $\widehat{Q}_{\mathfrak{U}}(R)$ may be thought of as a generalization of $Q^r_{c\ell}(R)$ since $\widehat{Q}_{\mathfrak{U}}(R) = Q^r_{c\ell}(R)$ whenever $Q^r_{c\ell}(R)$ exists. But $\widehat{Q}_{\mathfrak{U}}(R)$ has the advantage in that it always exists which is not the case, in general, for $Q^r_{c\ell}(R)$.

The next results are inspired by the work on continuous module hulls in [64] or [75].

Proposition 4.4. ([24, Proposition 2.9]) *Assume that R is a right Ore ring such that $r_R(x) = 0$ implies $\ell_R(x) = 0$ for $x \in R$. If $Q^r_{c\ell}(R)$ is Abelian and right extending, then $\widehat{Q}_{\mathbf{Con}}(R) = Q^r_{c\ell}(R)$.*

Corollary 4.5. ([24, Corollary 2.10]) *Let R be a right Ore ring. If any one of the following conditions is satisfied, then $\widehat{Q}_{\mathbf{Con}}(R) = Q^r_{c\ell}(R)$.*

(i) *R is Abelian, right extending, and $r_R(x) = 0$ implies $\ell_R(x) = 0$.*
(ii) *R is right uniform and $r_R(x) = 0$ implies $\ell_R(x) = 0$.*
(iii) *R is Abelian, right extending, and $Z(R_R) = 0$.*

The following theorem is an adaptation of [75, Theorem 4.25].

Theorem 4.6. ([24, Theorem 2.11]) *Let R be a right nonsingular ring and S the intersection of all right continuous right rings of quotients of R. Then $Q_{\mathbf{Con}}(R) = S$.*

Theorem 4.7. ([24, Theorem 2.12]) *Let R be a ring such that $Q(R)$ is Abelian.*

(i) *$Q(R)$ is a right extending ring if and only if $\widehat{Q}_{\mathbf{E}}(R) = \widehat{Q}_{\mathbf{qCon}}(R) = R\mathcal{B}(Q(R))$.*
(ii) *Assume that R is a right Ore ring such that $r_R(x) = 0$ implies $\ell_R(x) = 0$ for $x \in R$ and $Z(R_R)$ has finite right uniform dimension. Then $Q(R) \in \mathbf{E}$ if and only if $\widehat{Q}_{\mathbf{Con}}(R)$ exists and $\widehat{Q}_{\mathbf{Con}}(R) = H_1 \oplus H_2$ (ring direct sum), where H_1 is a right continuous strongly regular ring and H_2 is a direct sum of right continuous local rings.*

For commutative rings, the preceding results yield the following corollary which is related to [64, Corollaries 3 and 7], in particular Corollary 4.8 is related to Theorem 2.6.

Corollary 4.8. ([24, Corollary 2.13]) *Let R be a commutative ring.*

(i) *If R or $Q^r_{c\ell}(R)$ is extending, then $\widehat{Q}_{\mathbf{Con}}(R) = Q^r_{c\ell}(R)$.*
(ii) *If R is uniform, then $Q_{\mathbf{Con}}(R) = Q^r_{c\ell}(R)$ and is also a local ring.*
(iii) *If $Z(R_R) = 0$, then $Q_{\mathbf{Con}}(R) = \bigcap \{T \mid \mathcal{B}(Q(R)) \subseteq T$ and T is a regular right ring of quotients of $R\}$.*
(iv) *Assume that $Z(R_R)$ has finite uniform dimension. Then $Q(R)$ is right extending if and only if $\widehat{Q}_{\mathbf{Con}}(R)$ exists and $\widehat{Q}_{\mathbf{Con}}(R) = H_1 \oplus H_2$ (ring direct sum), where H_1 is a continuous regular ring and H_2 is a direct sum of continuous local rings.*

We note that in Corollary 4.8(i), the hypothesis "R or $Q^r_{c\ell}(R)$ is extending" is not superfluous. Let T be a countably infinite direct product of copies of a field F. Take $R = \langle \bigoplus_{i=1}^{\infty} F_i \cup \{1\} \rangle_T$. Then $Q^r_{c\ell}(R)$ is the subring of T whose elements are eventually constant. It can be seen that neither R nor $Q^r_{c\ell}(R)$ is extending. Hence $Q^r_{c\ell}(R)$ is not continuous. Also, in general, R may not satisfy the (C_2) property (e.g., take $F = \mathbb{Q}$); but $Q^r_{c\ell}(R)$ does satisfy the (C_2) property since it is regular.

To develop the theory of pseudo hulls for **D-E** classes $\mathfrak{C}$, we define (and fix)

$$\delta_{\mathfrak{C}}(R) = \{e \in \mathbf{I}(\mathrm{End}\,(E(R_R)) \mid X_R \leq^{\text{ess}} eE(R_R) \text{ for some } X \in \mathbf{D}_{\mathfrak{C}}(R)\}.$$

To find a right essential overring S of R such that $S \in \mathfrak{C}$, one might naturally look for a right essential overring T of R with $\delta_{\mathfrak{C}}(R)(1) \subseteq T$. Then take $S = \langle R \cup \delta_{\mathfrak{C}}(R)(1) \rangle_T$. In order to obtain a right essential overring with some hull-like behavior, we need to determine subsets Ω of $\delta_{\mathfrak{C}}(R)(1)$ for which $\langle R \cup \Omega \rangle_T \in \mathfrak{C}$ in some minimal sense. Moreover, to facilitate the transfer of information between R and $\langle R\cup\Omega\rangle_T$, one would want to include in Ω enough of $\delta_{\mathfrak{C}}(R)(1)$ so that for all (or almost all) $X \in \mathbf{D}_{\mathfrak{C}}(R)$ there is $e \in \delta_{\mathfrak{C}}(R)$ with $X_R \leq^{\text{ess}} e(1)\cdot\langle R\cup\Omega\rangle_T$ and $e(1) \in \Omega$. To accomplish this, we use equivalence relations on $\delta_{\mathfrak{C}}(R)$.

Since we have fixed the $\delta_{\mathfrak{C}}$ assignment for all **D-E** classes $\mathfrak{C}$, we will use the terminology $\mathfrak{C}$ (resp., $\mathfrak{C}$ ρ) pseudo right ring hull for $\delta_{\mathfrak{C}}$ pseudo right ring hull and use $R(\mathfrak{C}, S)$ for $R(\mathfrak{C}, \delta_{\mathfrak{C}}, S)$ and $R(\mathfrak{C}, \rho, S)$ for $R(\mathfrak{C}, \delta_{\mathfrak{C}}, \rho, S)$.

In the next few results, we show that for the concept of idempotent closure [9], we can find a **D-E** class of rings **IC** such that $R\mathcal{B}(Q(R))$ becomes an **IC** absolute to $Q(R)$ right ring hull and a $\delta_{\mathbf{IC}}$ pseudo right ring hull, where $\delta_{\mathbf{IC}}(R) = \mathcal{B}(\mathcal{E}_R)$.

Definition 4.9. ([26, Definition 2.1])

(i) *For a ring R, let $\mathbf{D}_{\mathbf{IC}}(R)=\{I\trianglelefteq R \,|\, I\cap\ell_R(I)=0$ and $\ell_R(I)\cap\ell_R(\ell_R(I))=0\}$.*

(ii) *Let* **IC** *denote the class of rings R such that for each $I \in \mathbf{D}_{\mathbf{IC}}(R)$ there exists some $e \in \mathbf{I}(R)$ such that $I_R \leq^{\text{ess}} eR_R$. We call the class* **IC** *the idempotent closure class.*

The set $\mathbf{D}_{\mathbf{IC}}(R)$ of ideals of R was studied by Johnson and denoted by $\mathfrak{F}'(R)$, who showed that if $Z(R_R) = 0$, then $\mathbf{D}_{\mathbf{IC}}(R) = \{I \trianglelefteq R \mid I\cap\ell_R(I) = 0\}$ [55, p. 538].

Remark 4.10. ([26, Remark 2.2])

(i) R is semiprime if and only if $\mathbf{D}_{\mathbf{IC}}(R)$ is the set of all ideals of R.

(ii) Let $e \in \mathbf{I}(R)$ with $eR \trianglelefteq R$. Then $eR \in \mathbf{D}_{\mathbf{IC}}(R)$ if and only if $e \in \mathbf{B}(R)$.

(iii) For a prime ideal P of R, $P \in \mathbf{D}_{\mathbf{IC}}(R)$ if and only if $P \cap \ell_R(P) = 0$.

(iv) Let P be a prime ideal of R and $P \in \mathbf{D}_{\mathbf{IC}}(R)$. If $I \trianglelefteq R$ such that $P \subseteq I$, then $I \in \mathbf{D}_{\mathbf{IC}}(R)$.

(v) If $I \trianglelefteq R$ such that $\ell_R(I) \cap P(R) = 0$, then $I \in \mathbf{D}_{\mathbf{IC}}(R)$.

(vi) If $Z(R_R) = 0$ and $I \trianglelefteq R$ such that $I \cap P(R) = 0$, then $I \in \mathbf{D}_{\mathbf{IC}}(R)$.

Proposition 4.11. ([26, Proposition 2.4]) *Let R be a ring. Then $\mathbf{D}_{\mathbf{IC}}(R) = \{I \trianglelefteq R \mid$ there exists $J \trianglelefteq R$ with $I \cap J = 0$ and $(I \oplus J)_R \leq^{\text{den}} R_R\}$.*

Theorem 4.12. ([26, Theorem 2.11])

(i) *$\mathbf{D}_{\mathbf{IC}}(R)$ is a sublattice of the lattice of ideals of R.*

(ii) *If $\mathbf{D}_{\mathbf{IC}}(R)$ is a complete sublattice of the lattice of ideals of R, then $\mathbf{B}(Q(R))$ is a complete Boolean algebra.*

(iii) *If R is a ring with unity which is right and left FI-extending, then $\mathbf{D}_{\mathbf{IC}}(R)$ is a complete sublattice of the lattice of ideals of R.*

The following result answers the question: *Which ideals of a ring R are dense in ring direct summands of $Q(R)$?*

Theorem 4.13. ([26, Theorem 2.10]) *Let $I \trianglelefteq R$. Then $I_R \leq^{\text{den}} eQ(R)_R$ for some unique $e \in \mathcal{B}(Q(R))$ if and only if $I \in \mathbf{D}_{\mathbf{IC}}(R)$.*

The next result indicates that $R\mathcal{B}(Q(R))$ is a ring hull according to Definitions 3.1 and 3.2 for the **IC** class of rings. Thus these hulls exist for every ring R. We observe that $\delta_{\mathbf{IC}}(R) = \mathcal{B}(\mathcal{E}_R)$ and $\delta_{\mathbf{IC}}(R)(1) = \mathcal{B}(Q(R))$.

Theorem 4.14. ([26, Theorem 2.7])

(i) *Let T be a right ring of quotients of R. Then $T \in \mathbf{IC}$ if and only if $\mathcal{B}(Q(R)) \subseteq T$.*

(ii) *$R \in \mathbf{IC}$ if and only if $\mathcal{B}(Q(R)) \subseteq R$.*

(iii) *$R\mathcal{B}(Q(R)) = \widehat{Q}_{\mathbf{IC}}(R) = R(\mathbf{IC}, \delta_{\mathbf{IC}}, Q(R))$.*

Our next result is a structure theorem for the idempotent closure $R\mathcal{B}(Q(R))$ when R is a semiprime ring with only finitely many minimal prime ideals. It is used for a characterization of C^*-algebras with only finitely many minimal prime ideals in Section 9. Many well-known finiteness conditions on a ring imply that it has only finitely many minimal prime ideals (see [60, p. 336, Theorem 11.43]).

Theorem 4.15. ([26, Theorem 3.15]) *The following are equivalent for a ring R.*

(i) *R is semiprime and has exactly n minimal prime ideals.*

(ii) *$\widehat{Q}_{\mathbf{IC}}(R) = R\mathcal{B}(Q(R))$ is a direct sum of n prime rings.*

(iii) *$\widehat{Q}_{\mathbf{IC}}(R) = R\mathcal{B}(Q(R)) \cong R/P_1 \oplus \cdots \oplus R/P_n$, where each P_i is a minimal prime ideal of R.*

The following example illustrates Definitions 3.1 and 3.2. In [24] we develop, in detail, the general consequences of Definitions 3.1 and 3.2. The independence of these definitions is beneficial in the sense that they provide distinct tools for analyzing interconnections between a ring and its right essential overrings relative to a class $\mathfrak{K}$. Also the following example shows that there is a quasi-Baer ring R (hence R itself is a quasi-Baer right ring hull of R), but R does not have a unique right FI-extending right ring hull.

Example 4.16. ([27, Example 1.7]) Let F be a field. Consider the following subrings of $\mathrm{Mat}_3(F)$:

$$R = \left\{ \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & b \end{pmatrix} \mid a, b, x, y \in F \right\}, \ H_1 = \begin{pmatrix} F & 0 & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix},$$

$$H_2 = \left\{ \begin{pmatrix} a+b & a & x \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix} \mid a, b, c, x, y \in F \right\},$$

and

$$H_3 = \left\{ \begin{pmatrix} a+b & a & x \\ a & b & y \\ 0 & 0 & c \end{pmatrix} \mid a, b, c, x, y \in F \right\}.$$

Then the following facts are illustrated in [24, Example 3.19].

(i) $Z(R_R) = 0$ and R is quasi-Baer, but R is not right FI-extending.

(ii) H_1, H_2, and H_3 are right FI-extending right ring hulls of R with $H_1 \cong H_2$, but $H_1 \ncong H_3$ for appropriate choices of F.

(iii) H_1 is not a right FI-extending pseudo right ring hull of R.

(iv) $R(\mathbf{FI},\ Q(R)) = \begin{pmatrix} F & F & F \\ F & F & F \\ 0 & 0 & F \end{pmatrix}$.

The following example also illustrates Definition 3.1. In fact, there is a ring R which has mutually isomorphic right FI-extending right ring hulls, but R has no quasi-Baer right essential overring.

Recall from [25, p. 30] that a ring R is right *Osofsky compatible* if $E(R_R)$ has a ring multiplication that extends its R-module scalar multiplication (i.e., $E(R_R)$ has a ring structure that is compatible with its R-module scalar multiplication).

Example 4.17. ([27, Example 1.8]) Assume that $n = p^m$, where p is a prime integer and $m \geq 2$. Let $A = \mathbb{Z}_n$, the ring of integers modulo n and let

$$R = \begin{pmatrix} A & A/J(A) \\ 0 & A/J(A) \end{pmatrix}.$$

Then $Q(R) = R$ by [19]. Further, from [19, Theorem 1]

$$E = \begin{pmatrix} A \oplus A/J(A) & A/J(A) \\ A/J(A) & A/J(A) \end{pmatrix}$$

is an injective hull of R_R, where the addition is componentwise and the R-module scalar multiplication is given by

$$\begin{pmatrix} s + \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} t & \overline{x} \\ 0 & \overline{y} \end{pmatrix} = \begin{pmatrix} st + \overline{at} & \overline{sx} + \overline{ax} + \overline{by} \\ \overline{ct} & \overline{cx} + \overline{dy} \end{pmatrix},$$

where $\overline{a}, \overline{x} \in A/J(A)$, etc. denote canonical images of a, $x \in A$.

It is shown in [19, Theorem 1] that the ring R is right Osofsky compatible. Let $\mathrm{Soc}(A)$ denote the socle of A. By a direct computation using the associativity of multiplication and the distributivity of multiplication over addition, we get that $\{(E, +, \circ_{(\alpha,\beta)}) \mid \alpha, \beta \in \mathrm{Soc}(A)\}$ is the set of all compatible ring structures on $E(R_R)$, where the addition is componentwise and the multiplication $\circ_{(\alpha,\beta)}$ is defined by

$$\begin{pmatrix} s_1 + \overline{a_1} & \overline{b_1} \\ \overline{c_1} & \overline{d_1} \end{pmatrix} \circ_{(\alpha,\beta)} \begin{pmatrix} s_2 + \overline{a_2} & \overline{b_2} \\ \overline{c_2} & \overline{d_2} \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

where

$$x = s_1 s_2 + \alpha a_1 a_2 + \beta c_1 a_2 + (-\beta) s_1 c_2 + \alpha b_1 c_2 + \beta d_1 c_2 + \overline{a_1 a_2} + \overline{a_1 s_2} + \overline{s_1 a_2} + \overline{b_1 c_2},$$

$$y = \overline{a_1 b_2} + \overline{s_1 b_2} + \overline{b_1 d_2},\ z = \overline{c_1 a_2} + \overline{c_1 s_2} + \overline{d_1 c_2},\ \text{and } w = \overline{c_1 b_2} + \overline{d_1 d_2}.$$

Thus E has exactly $|\,\mathrm{Soc}(A)|^2 = p^2$ ring structures extending the R-module scalar multiplication (i.e., compatible ring structures). Define $\theta_{(\alpha,\beta)} : (E, +, \circ_{(\alpha,\beta)}) \to (E, +, \circ_{(0,0)})$ by

$$\theta_{(\alpha,\beta)} \left[\begin{pmatrix} s + \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \right] = \begin{pmatrix} s + \overline{a} + (-\alpha)a + (-\beta)c & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}.$$

Then $\theta_{(\alpha,\beta)}$ is a ring isomorphism. Hence $(E, +, \circ_{(\alpha,\beta)})$ are all isomorphic. Let $e = \begin{pmatrix} 1-\overline{1} & 0 \\ 0 & 0 \end{pmatrix} \in (E, +, \circ_{(0,0)})$ and $f = \begin{pmatrix} \overline{1} & 0 \\ 0 & \overline{1} \end{pmatrix} \in (E, +, \circ_{(0,0)})$. Then e and f are central idempotents in $(E, +, \circ_{(0,0)})$ and $e + f = 1$. Thus $(E, +, \circ_{(0,0)}) \cong e(E, +, \circ_{(0,0)}) \oplus f(E, +, \circ_{(0,0)}) \cong A \oplus \mathrm{Mat}_2(A/J(A))$. Hence $(E, +, \circ_{(0,0)})$ is a QF-ring, and so all $(E, +, \circ_{(\alpha,\beta)})$ are QF-rings for $\alpha, \beta \in \mathrm{Soc}(A)$. Let

$$T = \begin{pmatrix} A \oplus A/J(A) & A/J(A) \\ 0 & A/J(A) \end{pmatrix}.$$

Then T is the only proper R-submodule of E with $R \subseteq T \subseteq E$ (and $R \neq T \neq E$) which can have a ring structure that is compatible with its R-module scalar multiplication. Also, $\{(T, +, \circ_{(\alpha,0)}) \mid \alpha \in \mathrm{Soc}(A)\}$ is the set of all compatible ring structures on T, where the multiplication $\circ_{(\alpha,0)}$ is the restriction of $\circ_{(\alpha,\beta)}$ on E to T for $\beta \in \mathrm{Soc}(A)$. Hence $(T, +, \circ_{(\alpha,0)})$ is a subring of $(E, +, \circ_{(\alpha,\beta)})$ for each $\beta \in \mathrm{Soc}(A)$. Define $\lambda_{(\alpha,0)} : (T, +, \circ_{(\alpha,0)}) \to (T, +, \circ_{(0,0)})$ by

$$\lambda_{(\alpha,0)} \left[\begin{pmatrix} s + \overline{a} & \overline{b} \\ 0 & \overline{d} \end{pmatrix} \right] = \begin{pmatrix} s + (-\alpha)a + \overline{a} & \overline{b} \\ 0 & \overline{d} \end{pmatrix}.$$

Then we see that $\lambda_{(\alpha,0)}$ is a ring isomorphism.

We note that all right essential overrings of R are $\{(E, +, \circ_{(\alpha,\beta)}) \mid \alpha, \beta \in \mathrm{Soc}(A)\}$, $\{(T, +, \circ_{(\alpha,0)}) \mid \alpha \in \mathrm{Soc}(A)\}$, and R itself.

Take $g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Then $g = g^2 \in R$ and $gRg \cong A$. Note that A is not quasi-Baer. Thus R is not quasi-Baer by [39, Lemma 2] or [23, Theorem 3.2]. Next observe that $e = \begin{pmatrix} 1-\overline{1} & 0 \\ 0 & 0 \end{pmatrix} \in T$. Then $e(T, +, \circ_{(0,0)})e \cong A$, which is not quasi-Baer. Thus $(T, +, \circ_{(0,0)})$ is not quasi-Baer by [39, Lemma 2] or [23, Theorem 3.2]. So all $(T, +, \circ_{(\alpha,0)})$ with $\alpha \in \mathrm{Soc}(A)$ cannot be quasi-Baer since $(T, +, \circ_{(\alpha,0)}) \cong (T, +, \circ_{(0,0)})$. Further, $e(E, +, \circ_{(0,0)})e \cong A$ is not quasi-Baer, so $(E, +, \circ_{(0,0)})$ is not quasi-Baer again from [39, Lemma 2] or [23, Theorem 3.2]. Thus $(E, +, \circ_{(\alpha,\beta)})$ cannot be quasi-Baer for $\alpha, \beta \in \mathrm{Soc}(A)$ since $(E, +, \circ_{(\alpha,\beta)}) \cong (E, +, \circ_{(0,0)})$. Hence R has no quasi-Baer right essential overring.

Finally, let $I = \begin{pmatrix} J(A) & 0 \\ 0 & 0 \end{pmatrix} \trianglelefteq R$. Then there is no $h = h^2 \in R$ with $I_R \leq^{\text{ess}} hR_R$. Hence R is not right FI-extending. Note that $f = \begin{pmatrix} \overline{1} & 0 \\ 0 & \overline{1} \end{pmatrix} \in T$. Thus $(T, +, \circ_{(0,0)}) = e(T, +, \circ_{(0,0)}) \oplus f(T, +, \circ_{(0,0)}) \cong A \oplus T_2(A/J(A))$, where $T_2(-)$ is the 2-by-2 upper triangular matrix ring over a ring. From [18, Theorem 1.3 and Corollary 2.5], $(T, +, \circ_{(0,0)})$ is right FI-extending. Thus all $(T, +, \circ_{(\alpha,0)})$ with $\alpha \in \mathrm{Soc}(A)$ are right FI-extending. Therefore the $(T, +, \circ_{(\alpha,0)})$ with $\alpha \in \mathrm{Soc}(A)$ are right FI-extending right ring hulls of R.

In Example 4.16, we have seen that, in general, $\mathfrak{C}$ right ring hulls and $\mathfrak{C}$ pseudo right ring hulls are distinct and may not be unique (when they exist) even if the ring is right nonsingular. Also in Example 4.17, there is a ring where all right FI-extending ring hulls are mutually isomorphic, but it does not have a quasi-Baer right ring hull. However, the semiprime condition on the ring rescues us from this somewhat chaotic situation, for the classes $\mathfrak{C} = \mathbf{FI}$ or $\mathfrak{C} = \mathbf{eqB}$. In the following theorem, we establish the existence and uniqueness of quasi-Baer and right FI-extending right ring hulls of a semiprime ring. This result indicates the ubiquity of the right FI-extending and quasi-Baer ring hulls by showing that every nonzero ring R has a nontrivial homomorphic image, $R/P(R)$, which has each of these hulls. Mewborn [63] (see Theorem 2.4) showed the existence of a Baer (absolute) hull for a commutative semiprime ring. Our next theorem also generalizes Mewborn's result since a commutative quasi-Baer ring is a Baer ring.

Theorem 4.18. ([27, Theorem 3.3]) *Let R be a semiprime ring. Then:*

(i) $\widehat{Q}_{\mathbf{FI}}(R) = R\mathcal{B}(Q(R)) = R(\mathbf{FI}, Q(R))$.
(ii) $\widehat{Q}_{\mathbf{qB}}(R) = \widehat{Q}_{\mathbf{eqB}}(R) = R\mathcal{B}(Q(R)) = R(\mathbf{eqB}, Q(R))$.
(iii) *If R is right Osofsky compatible, then* $R\mathcal{B}(Q(R)) = Q_{\mathbf{FI}}(R) = Q_{\mathbf{qB}}(R) = Q_{\mathbf{eqB}}(R)$.

Corollary 4.19. ([27, Corollary 3.16]) *Let R be a semiprime ring and T a right ring of quotients of R. Then T is quasi-Baer* (*hence right FI-extending*) *if and only if* $\mathcal{B}(Q(R)) \subseteq T$.

Our first corollary to Theorem 4.18 generalizes both the result of Mewborn, Theorem 2.4, and the result of Hirano, Hongan, and Ohori, Theorem 2.8.

Corollary 4.20. (see [27, Theorem 3.8]) *If R is a reduced ring, then* $Q_{\mathbf{B}}(R) = R\mathcal{B}(Q(R))$ (*i.e., R has a Baer hull*).

Corollary 4.21. ([27, Corollary 3.17])

(i) *If R is a semiprime ring, then the central closure of R, the normal closure of R, $Q^m(R)$, $Q^s(R)$, and $Q(R)$ are all quasi-Baer and right FI-extending.*
(ii) *Assume that $Q(R)$ is semiprime. Then $Q(R)$ is quasi-Baer and right FI-extending. Also there exists a right essential overring of R containing $Q(R)$ which is maximal with respect to being quasi-Baer* (*or right FI-extending*).

In [47], Ferrero has shown that $Q^s(R) \in \mathbf{qB}$ for a semiprime ring R. There is a semiprime ring R for which neither $Q^m(R)$ nor $Q^s(R)$ is Baer. In fact, there is a simple ring R given by Zalesski and Neroslavskii [50] which is not a domain and 0, 1 are its only idempotents. Then $Q^m(R) = R$ (and hence $Q^s(R) = R$). In this case, $Q^m(R)$ is not a Baer ring.

In [67] Osofsky poses the question: *If $E(R_R)$ has a ring multiplication which extends its right R-module scalar multiplication, must $E(R_R)$ be a right self-injective ring?* Example 4.23 below shows that this is not true in general. We can, however, show that the ring $E(R_R)$ does satisfy the right FI-extending property – a generalization of right self-injectivity, for the case when the ring R is right FI-extending or when $Q(R)$ is semiprime.

Corollary 4.22. ([27, Corollary 3.18]) *Let R be a right Osofsky compatible ring. If R has a right FI-extending right essential overring which is a subring of $E(R_R)$, then $E(R_R)$ is a right FI-extending ring. In particular, if $Q(R)$ is semiprime, then $E(R_R)$ is a right FI-extending ring.*

The following example, due to Camillo, Herzog, and Nielsen [36] illustrates Corollary 4.22. In fact, in the following example, there exists a right Osofsky compatible ring R which is right extending, but the compatible ring structure on $E(R_R)$ is not right self-injective. However, by Corollary 4.22, the compatible ring structure on $E(R_R)$ is right FI-extending.

Example 4.23. ([27, Example 3.19]) Let $\mathbb{R}\{X_1, X_2, \dots\}$ be the free algebra over the field $\mathbb{R}$ of real numbers with indeterminates $X_1, X_2, \dots$. Put

$$R = \mathbb{R}\{X_1, X_2, \dots\}/\langle X_iX_j - \delta_{ij}\, X_1^2\rangle,$$

where $\langle X_iX_j - \delta_{ij}\, X_1^2\rangle$ is the ideal of $\mathbb{R}\{X_1, X_2, \dots\}$ generated by $X_iX_j - \delta_{ij}\, X_1^2$ with $i, j = 1, 2, \dots$ and δ_{ij} the Kronecker delta. We denote the canonical image of X_i by x_i in R. Set $V = \mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \cdots$, $P = \mathbb{R}x_1^2$ and let the bilinear form on V be given by $B(x_i, x_j) = \delta_{ij}$. Then B is non-degenerate and symmetric. Hence we see that

$$R = \left\{ \begin{pmatrix} k & v & p \\ 0 & k & v \\ 0 & 0 & k \end{pmatrix} \mid k \in \mathbb{R}, v \in V, \text{ and } p \in P \right\},$$

where the addition is componentwise and the multiplication is defined by

$$\begin{pmatrix} k_1 & v_1 & p_1 \\ 0 & k_1 & v_1 \\ 0 & 0 & k_1 \end{pmatrix} \begin{pmatrix} k_2 & v_2 & p_2 \\ 0 & k_2 & v_2 \\ 0 & 0 & k_2 \end{pmatrix}$$

$$= \begin{pmatrix} k_1k_2 & k_1v_2 + k_2v_1 & k_1p_2 + k_2p_1 + B(v_1, v_2)x_1^2 \\ 0 & k_1k_2 & k_1v_2 + k_2v_1 \\ 0 & 0 & k_1k_2 \end{pmatrix}.$$

Let $E_R = [\mathrm{Hom}_{\mathbb{R}}(R_{\mathbb{R}}, \mathbb{R}_{\mathbb{R}})]_R$. Then it is shown in [23] that E_R is an injective hull of R_R. Further, E_R has a compatible ring structure with its R-module scalar multiplication, but it is not right self-injective. Note that R is a commutative local ring. Also

$$\begin{pmatrix} 0 & 0 & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the smallest nonzero ideal of R and it is essential in R. Hence R is uniform, so it is extending. Thus by Corollary 4.22, the compatible ring structure on the injective hull E_R is right FI-extending.

The following example provides a ring R which is neither semiprime, right (nor left) nonsingular, right (nor left) FI-extending, nor quasi-Baer. However, we have that $Q_{\mathbf{FI}}(R) = R\mathcal{B}(Q(R))$. Thus, even without the semiprime condition, a ring can have a natural unique FI-extending absolute right ring hull. Recall from

[22] that a ring R is right *strongly FI-extending* if for each $I \trianglelefteq R$ there is $e = e^2 \in R$ such that $I_R \leq^{\text{ess}} eR_R$ and $eR \trianglelefteq R$.

Example 4.24. Let A be a QF-ring with $J(A) \neq 0$. Assume that A is right strongly FI-extending, and A has nontrivial central idempotents while the subring of A generated by 1_A contains no nontrivial idempotents (e.g., $A = \mathbb{Q} \oplus \text{Mat}_2(\mathbb{Z}_4)$). Let $1_{\prod_{i=1}^{\infty} A_i}$ denote the unity of $\prod_{i=1}^{\infty} A_i$, where $A_i = A$. Take R to be the subring of $\prod_{i=1}^{\infty} A_i$ generated by $1_{\prod_{i=1}^{\infty} A_i}$ and $\bigoplus_{i=1}^{\infty} A_i$. Observe that $Q(R) = \prod_{i=1}^{\infty} A_i = E(R_R)$ by [85, 2.1]. Now R has the following properties:

(i) R is neither semiprime nor right FI-extending.

(ii) $R\mathcal{B}(Q(R)) = R(\mathbf{FI}, Q(R)) = Q_{\mathbf{FI}}(R)$.

(iii) $R\mathcal{B}(Q(R))$ is neither right extending nor quasi-Baer.

Let c be a nontrivial idempotent of A. Let π_i and κ_i denote the i-th projection and injection, respectively, of the direct product. Let K be the ideal of R generated by $\{\kappa_i(c) \mid 1 \leq i < \infty\}$. Then there exists no $b = b^2 \in R$ such that $K_R \leq^{\text{ess}} bR_R$. Thus R is not right FI-extending.

Now let $I \trianglelefteq R$. Then $\pi_i(I) \trianglelefteq A_i$. By [60, p. 421, Exercise 16], there exists $e_i \in \mathcal{B}(A_i)$ such that $\pi_i(I)_{A_i} \leq^{\text{ess}} e_i A_{i A_i}$, since A_i is right strongly FI-extending by assumption. Let $e \in Q(R)$ such that $\pi_i(e) = e_i$. Then

$$I_R \leq^{\text{ess}} eQ(R)_R \text{ and } e \in \mathcal{B}(Q(R)).$$

Hence $\mathcal{B}(\mathcal{E}_R) = \delta_{\mathbf{FI}}(R)$. Let $S = \langle R \cup \delta_{\mathbf{FI}}(R)(1) \rangle_{Q(R)} = R\mathcal{B}(Q(R))$. Then $\mathbf{D}_{\mathbf{FI}}(S \to R)$ holds (see [24, p. 638]). By [24, Lemma 2.19 and Corollary 2.18], $S = R(\mathbf{FI}, Q(R))$.

Next we show that $S = Q_{\mathbf{FI}}(R)$. Let T be a right FI-extending right ring of quotients of R. Take $e \in \mathcal{B}(Q(R)) = \delta_{\mathbf{FI}}(R)(1)$. Then $eQ(R) \cap T \trianglelefteq T$. Since T is right FI-extending, there is $f = f^2 \in T$ such that $(eQ(R) \cap T)_T \leq^{\text{ess}} fT_T$. So $(eQ(R) \cap T)_R \leq^{\text{ess}} fT_R$ from [24, Lemma 1.4]. Since $fT_R \leq^{\text{ess}} fQ(R)_R$, $(eQ(R) \cap T)_R \leq^{\text{ess}} fQ(R)_R$. Hence $(eQ(R) \cap R)_R \leq^{\text{ess}} fQ(R)_R$. Also $(eQ(R) \cap R)_R \leq^{\text{ess}} eQ(R)_R$. Since $e \in \mathcal{B}(Q(R))$, $fQ(R) \cap eQ(R) = efQ(R)$ and $ef = (ef)^2$. Thus $fQ(R) = eQ(R)$, so $e = f \in T$. Therefore $\mathcal{B}(Q(R)) \subseteq T$. Hence S is a subring of T. Consequently, S is the right FI-extending absolute right ring hull of R.

To see that S, in general, is not right extending, take $A = \mathbb{Q} \oplus \text{Mat}_2(\mathbb{Z}_4)$ and let V be a right ideal of S generated by

$$\left\{ \kappa_i \left[\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \right] \mid 1 \leq i < \infty \right\}.$$

Then V is not right essential in a right direct summand of S_S.

Since $Q(R)$ is a QF-ring, $Q(R) = Q^{\ell}(R) = E(R_R)$. By [60, p. 421, Exercise 16], $\mathbf{S}_{\ell}(Q(R)) = \mathcal{B}(Q(R))$. Note that $Q(R)$ is not semiprime, so $Q(R)$ cannot be right p.q.-Baer from [14, Proposition 1.7]. By Theorem 3.9(i), $R\mathcal{B}(Q(R))$ is not quasi-Baer.

After giving some preliminary results on the class **pqB** of right p.q.-Baer rings, we describe ring hulls for this and related classes over semiprime rings.

Proposition 4.25.

(i) ([15, Proposition 1.8] *and* [14, Proposition 1.12]) *The center of a quasi-Baer (resp., right p.q.-Baer) ring is Baer (resp., PP).*

(ii) ([14, Proposition 3.11]) *Assume that a ring R is semiprime. Then R is quasi-Baer if and only if R is p.q.-Baer and the center of R is Baer.*

(iii) ([81, pp. 78–79] and [15, Theorem 3.5]) *Let a ring R be regular (resp., biregular). Then R is Baer (resp., quasi-Baer) if and only if the lattice of principal right ideals (resp., principal ideals) is complete.*

(iv) *A ring R is biregular if and only if R is right (or left) p.q.-Baer ring and $r_R(\ell_R(RaR)) = RaR$, for all $a \in R$.*

Recall from [30], we say that a ring R is *principally right FI-extending* (resp., *finitely generated right FI-extending*) if every principal ideal (resp., finitely generated ideal) of R is essential as a right R-module in a right ideal of R generated by an idempotent. We use **pFI** (resp., **fgFI**) to denote the class of principally (resp., finitely generated) right FI-extending rings.

Lemma 4.26. ([14, Corollary 1.11]) *Let R be a semiprime ring. Then the following conditions are equivalent.*

(i) *R is right p.q.-Baer.*

(ii) *R is principally right FI-extending.*

(iii) *R is finitely generated right FI-extending.*

Our next result, when applied to a commutative reduced ring yields Picavet's weak Baer envelope [71]; and when it is applied to a regular ring of bounded index, it yields the unique smallest almost biregular ring of Burgess and Raphael [34, Theorem 1.7] (see Section 2).

Theorem 4.27. ([30, Theorem 8]) *Let R be a semiprime ring. Then:*

(i) $\langle R \,\cup\, \delta_{\mathbf{pFI}}(R)(1)\rangle_{Q(R)} = \widehat{Q}_{\mathbf{pFI}}(R) = R(\mathbf{pFI},\, Q(R))$.

(ii) $\langle R \,\cup\, \delta_{\mathbf{pFI}}(R)(1)\rangle_{Q(R)} = \widehat{Q}_{\mathbf{pqB}}(R)$.

(iii) $\langle R \cup \delta_{\mathbf{pFI}}(R)(1)\rangle_{Q(R)} = \widehat{Q}_{\mathbf{fgFI}}(R) = R(\mathbf{fgFI},\, Q(R))$.

Note that $\delta_{\mathbf{pFI}}(R)(1) = \{c \in \mathcal{B}(Q(R)) \mid \text{there is } x \in R \text{ with } RxR_R \leq^{\text{ess}} cR_R\}$.

Corollary 4.28. ([30, Theorem 15]) *Let R be a reduced ring. Then $Q_{\mathbf{pqB}}(R)$ exists and is the PP absolute right ring hull.*

The next two equivalence relations are particularly important to our study.

Definition 4.29. ([24, Definition 2.4])

(i) Let A be a ring and let $\delta \subseteq \mathbf{I}(A)$. We define an equivalence relation α on δ by $e \;\alpha\; c$ if and only if $ce = e$ and $ec = c$.

(ii) We define an equivalence relation β on $\delta_{\mathfrak{C}}(R)$ by $e \;\beta\; c$ if and only if there exists $X_R \leq R_R$ such that $X_R \leq^{\text{ess}} eE(R_R)$ and $X_R \leq^{\text{ess}} cE(R_R)$.

Note that for $e, c \in \delta_{\mathfrak{C}}(R), e\ \alpha\ c$ implies $e\ \beta\ c$. Also note that $\alpha = \beta$ if and only if every element of $\mathbf{D}_{\mathfrak{C}}(R)$ has a unique essential closure in $E(R_R)$. So if $Z(R_R) = 0$, then $\alpha = \beta$.

The following example again indicates the independence of Definition 3.1 and 3.2 for **D-E** classes. Moreover, it shows that a nonsemiprime commutative ring R can have an absolute self-injective right ring hull even when R is not right Osofsky compatible. Recall from [60, Corollary 8.28] that a ring R is right *Kasch* if the left annihilator of every maximal right ideal of R is nonzero.

Example 4.30. ([24, Example 2.15]) For a field F, let $T = F[x]/x^4F[x]$ and $\overline{x}$ be the canonical image of x in T. Then $T = F + F\overline{x} + F\overline{x}^2 + F\overline{x}^3$. Let $R = F + F\overline{x}^2 + F\overline{x}^3$ which is a subring of T. Now R and T have the following properties.

(i) R is right Kasch, so $R = Q(R)$ [60, Corollary 13.24].

(ii) T is a QF right essential overring of R. There is no proper intermediate ring between R and T. Hence $T = Q_{\mathbf{FI}}(R) = Q_{\mathbf{E}}(R) = Q_{\mathbf{SI}}(R)$.

(iii) T is not a $\mathfrak{C}$ ρ pseudo right ring hull of R for any choice of $\mathfrak{C}$ and any equivalence relation ρ on $\delta_{\mathfrak{C}}(R)$. Indeed, there is no $c \in \delta_{\mathfrak{C}}(R)$ such that $c(1) \in T \setminus R$ and $I_R \leq^{\text{ess}} cE(R_R)$ for any nonzero ideal I of R.

(iv) T_R is not FI-extending (hence not extending). In fact, $\overline{x}R_R \trianglelefteq R_R$. But there does not exist $e \in \mathbf{I}(\text{End}\,(T_R))$ such that $\overline{x}R_R \leq^{\text{ess}} eT_R$.

(v) Since T_T is injective, T is maximal among right extending right essential overrings of R.

(vi) By [62, Theorem 4] $E(R_R)$ has no ring multiplication which extends its R-module scalar multiplication.

Our next result shows that when $Q(R) = E(R_R)$ the α pseudo right ring hulls and β pseudo right ring hulls also exist, respectively for the right FI-extending and right essentially quasi-Baer properties.

Corollary 4.31. ([24, Corollary 2.21]) *Assume that* $Q(R) = E(R_R)$.

(i) *For each* $\delta^{\alpha}_{\mathbf{E}}(R)$ (*resp.,* $\delta^{\beta}_{\mathbf{FI}}(R)$), $R(\mathbf{E}, \alpha, Q(R))$ (*resp.,* $R(\mathbf{FI}, \beta, Q(R))$ *exists. Moreover, every right ring of quotients of* R *containing* $R(\mathbf{E}, \alpha, Q(R))$ (*resp.,* $R(\mathbf{FI}, \beta, Q(R))$ *is right extending* (*resp., right FI-extending*).

(ii) *Let* $S = \langle R \cup \delta(1)\rangle_{Q(R)}$. *If* $\delta(1) = \delta^{\alpha}_{\mathbf{eB}}(R)(1)$ (*resp.,* $\delta(1) = \delta^{\beta}_{\mathbf{eqB}}(R)(1)$) *and* S *is a left ring of quotients of* R, *then* $R(\mathbf{eB}, \alpha, Q(R))$ (*resp.,* $R(\mathbf{eqB}, \beta, Q(R))$) *exists. Moreover, any right and left ring of quotients of* R *which also lies between* $R(\mathbf{eB}, \alpha, Q(R))$ (*resp.,* $R(\mathbf{eqB}, \beta, Q(R))$) *and* $Q(R)$ *is right essentially Baer* (*resp., right essentially quasi-Baer*). *If* $Z(R_R) = 0$, *then these intermediate rings are Baer* (*resp., quasi-Baer*).

We remark that the $\mathfrak{K}$ absolute (absolute to $Q(R)$) right ring hull of R is the intersection of all right essential overrings (of all right rings of quotients) of R which are in $\mathfrak{K}$ (see for example, Theorem 4.8). Our next result shows that under suitable conditions, these intersections coincide with the intersections of the α pseudo or the β pseudo right ring hulls for various **D-E** classes (e.g., $\mathbf{E}, \mathbf{FI}$,

eB, and **eqB**). Also under these conditions a $\mathfrak{C}$ right ring hull will be a $\mathfrak{C}$ α or a $\mathfrak{C}$ β pseudo right ring hull. We note that the condition $X \trianglelefteq R$ implies $XT \trianglelefteq T$ holds for example when T is a centralizing extension of R or when R is a right Noetherian ring and T is a right ring of quotients of R contained in $Q^r_{c\ell}(R)$ [60, pp. 314–315]. This condition is useful in the following result.

Corollary 4.32. ([24, Corollary 2.23]) *Let T be a right ring of quotients of R.*

(i) *Suppose that either $\alpha = \beta$ or some $\delta^{\beta}_{\mathbf{E}}(R)(1) \subseteq \mathrm{Cen}(T)$. Then $T \in \mathbf{E}$ if and only if there exists an $R(\mathbf{E}, \alpha, Q(R)$ which is a subring of T.*

(ii) *If $X \trianglelefteq R$ implies $XT \trianglelefteq T$, then $T \in \mathbf{FI}$ if and only if there exists a $R(\mathbf{FI}, \beta, Q(R))$ which is a subring of T.*

(iii) *Suppose that either $\alpha = \beta$ or some $\delta^{\beta}_{\mathbf{eB}}(R)(1) \subseteq \mathrm{Cen}(T)$. If T is also a left ring of quotients of R, then $T \in \mathbf{eB}$ if and only if there exists a $R(\mathbf{eB}, \alpha, Q(R))$ which is a subring of T.*

(iv) *If T is also a left ring of quotients of R and $X \trianglelefteq R$ implies $TX \trianglelefteq T$, then $T \in \mathbf{eqB}$ if and only if there is a $R(\mathbf{eqB}, \beta, Q(R))$ which is a subring of T.*

Proposition 4.33. ([24, Corollary 2.24]) *Assume that $E(R_R) = Q(R)$, $Q(R)$ is a left ring of quotients of R, and T is a right ring of quotients of R. Then:*

(i) *$\delta_{\mathbf{E}}(R) = \delta_{\mathbf{eB}}(R)$.*

(ii) *Assume that $\alpha = \beta$ or some $\delta^{\beta}_{\mathbf{E}}(R)(1) \subseteq \mathrm{Cen}(T)$. Then $T \in \mathbf{E}$ if and only if $T \in \mathbf{eB}$. Also every right extending α pseudo right ring hull of R is a right essentially Baer α pseudo right ring hull of R and conversely.*

(iii) *Assume that $Z(R_R) = 0$. Then $T \in \mathbf{E}$ if and only if $T \in \mathbf{B}$. Moreover every right extending α pseudo right ring hull of R is a right essentially Baer α pseudo right ring hull of R which is Baer and conversely.*

The following result provides an answer to Problem I of Section 1 for the case when $\mathfrak{K} = \mathbf{E}$, the class of right extending rings, and $R = T_2(W)$ by characterizing the right extending right rings of quotients which are intermediate between $T_2(W)$ and $\mathrm{Mat}_2(W)$, where W is from a large class of local right finitely Σ-extending rings (see [43] for finitely Σ-extending modules).

Theorem 4.34. ([24, Theorem 3.11]) *Let W be a local ring, V a subring of W with $J(W) \subseteq V$, $R = \begin{pmatrix} V & W \\ 0 & W \end{pmatrix}$, $S = \begin{pmatrix} V & W \\ J(W) & W \end{pmatrix}$, and $T = \mathrm{Mat}_2(W)$. Then:*

(i) *For each $e \in \mathbf{I}(T)$, there exists $f \in \mathbf{I}(S)$ such that e α f.*

(ii) *$S \in \mathbf{E}$ if and only if $T \in \mathbf{E}$ if and only if $S = R(\mathbf{E}, \rho, T)$ for some ρ.*

(iii) *If W is right self-injective, then $S = R(\mathbf{E}, \alpha, T)$, and $Q_{\mathbf{qCon}}(R) = R(\mathbf{E}, T) = T$.*

(iv) *If $T \in \mathbf{E}$ (resp., W is right self-injective) and at least one of the following conditions is satisfied, then $S = Q^T_{\mathbf{E}}(R)$ (resp., $S = Q_{\mathbf{E}}(R)$):*

(a) $J(W) \subseteq \mathrm{Cen}(W)$;
(b) $\mathbf{U}(W) \subseteq \mathrm{Cen}(W)$;
(c) $J(W)$ *is nil;*
(d) W *is right nonsingular.*

(v) *Assume that* $S = Q^T_{\mathbf{E}}(R)$ *and* M *is an intermediate ring between* R *and* T. *Then* $M \in \mathbf{E}$ *if and only if* $M = \begin{pmatrix} A & W \\ J(W) & W \end{pmatrix}$ *or* $M = T$, *where* A *is an intermediate ring between* V *and* W.

(vi) $R \in \mathbf{FI}$ *if and only if* $W \in \mathbf{FI}$.

5. Transference between R and overrings

In this section, we consider Problem II from the introduction. Since $R\mathcal{B}(Q(R))$ is used in the construction of several hulls, we show how various types of information transfer between R and $R\mathcal{B}(Q(R))$. Indeed, we prove that the properties of lying over, going up, and incomparability of prime ideals hold between R and $R\mathcal{B}(Q(R))$ and so do the π-regularity and classical Krull dimension properties. Moreover, we show that $\varrho(R) = \varrho(R\mathcal{B}(Q(R))) \cap R$, where ϱ is a special radical. We use LO, GU, and INC for "lying over", "going up", and "incomparability" [77, p. 292], respectively.

Lemma 5.1. ([27, Lemma 2.1]) *Assume that* R *is a subring of a ring* T *and* $\mathbb{E}$ *is a subset of* $\mathbf{S}_\ell(T) \cup \mathbf{S}_r(T)$. *Let* S *be the subring of* T *generated by* R *and* $\mathbb{E}$.

(i) *If* K *is a prime ideal of* S, *then* $R/(K \cap R) \cong S/K$.
(ii) *LO, GU, and INC hold between* R *and* S. *In particular, LO, GU, and INC hold between* R *and* $R\mathcal{B}(Q(R))$.

We note that Lemma 5.1 generalizes results of Beidar and Wisbauer [9] for $R\mathcal{B}(Q(R))$ (see Theorem 2.9). Recall that a ring R is *left* π*-regular* if for each $a \in R$ there exist $b \in R$ and a positive integer n such that $a^n = ba^{n+1}$. Observe from [41] that the class of special radicals includes most well-known radicals (e.g., the prime radical, the Jacobson radical, the Brown-McCoy radical, the nil radical, the generalized nil radical, etc.). For a ring R, the *classical Krull dimension* $\mathrm{kdim}(R)$ is the supremum of all lengths of chains of prime ideals of R.

Theorem 5.2. ([27, Theorem 2.2]) *Assume that* R *is a subring of a ring* T *and* $\mathbb{E} \subseteq \mathbf{S}_\ell(T) \cup \mathbf{S}_r(T)$. *Let* S *be the subring of* T *generated by* R *and* $\mathbb{E}$. *Then we have the following.*

(i) $\varrho(R) = \varrho(S) \cap R$, *where* ϱ *is a special radical. In particular,* $\varrho(R) = \varrho(R\mathcal{B}(Q(R))) \cap R$.
(ii) R *is left* π*-regular if and only if* S *is left* π*-regular. Hence,* R *is left* π*-regular if and only if* $R\mathcal{B}(Q(R))$ *is left* π*-regular.*
(iii) $\mathrm{kdim}\,(R) = \mathrm{kdim}\,(S)$. *Thus,* $\mathrm{kdim}\,(R) = \mathrm{kdim}\,(R\mathcal{B}(Q(R)))$.
(iv) *If* S *is regular, then so is* R.

The following corollary complements Theorems 2.12 and 2.13.

Corollary 5.3. ([27, Corollary 3.6]) *For a ring R, the following are equivalent.*

(i) *R is regular.*

(ii) *$R\mathcal{B}(Q(R))$ is regular.*

(iii) *R is semiprime and $\widehat{Q}_{\mathbf{qB}}(R)$ is regular.*

Lemma 5.1 and Corollary 5.3 show a transference of properties between R and $R\mathcal{B}(Q(R))$ or $\widehat{Q}_{\mathbf{qB}}(R)$. Our next example indicates that this transference, in general, fails between R and its right rings of quotients which properly contain $R\mathcal{B}(Q(R))$ or $\widehat{Q}_{\mathbf{qB}}(R)$.

Example 5.4. ([27, Example 3.7]) Let $\mathbb{Z}[G]$ be the group ring of the group $G = \{1, g\}$ over the ring $\mathbb{Z}$. Then $\mathbb{Z}[G]$ is semiprime and $Q(\mathbb{Z}[G]) = \mathbb{Q}[G]$. Note that $\mathbf{B}(\mathbb{Q}[G]) = \{0,\, 1,\, (1/2)(1+g),\, (1/2)(1-g)\}$. Thus, using Theorem 4.18(ii),

$$\mathbb{Z}[G] \neq \widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G])$$

and $\mathbb{Z}[G] \subseteq \widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G]) = \{(a + c/2 + d/2) + (b + c/2 - d/2)g \mid a, b, c, d \in \mathbb{Z}\} \subseteq \mathbb{Z}[1/2][G] \subseteq \mathbb{Q}[G]$, and $\widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G]) \neq \{(a+c/2+d/2)+(b+c/2-d/2)g \mid a, b, c, d \in \mathbb{Z}\} \subseteq \mathbb{Z}[1/2][G] \subseteq \mathbb{Q}[G]$, where $\mathbb{Z}[1/2] = \langle \mathbb{Z} \cup \{1/2\}\rangle_{\mathbb{Q}}$.

In this case, for example, LO does not hold between $\mathbb{Z}[G]$ and $\mathbb{Z}[1/2][G]$. Assume to the contrary that LO holds. From [77, Theorem 4.1], LO holds between $\mathbb{Z}$ and $\mathbb{Z}[G]$. Hence there exists a prime ideal P of $\mathbb{Z}[G]$ such that $P \cap \mathbb{Z} = 2\mathbb{Z}$. By LO, there is a prime ideal K of $\mathbb{Z}[1/2][G]$ such that $K \cap \mathbb{Z}[G] = P$. Now $K \cap \mathbb{Z}[1/2] = K_0$ is a prime ideal of $\mathbb{Z}[1/2]$. So $K_0 \cap \mathbb{Z} = K \cap \mathbb{Z}[1/2] \cap \mathbb{Z} = K \cap \mathbb{Z} = 2\mathbb{Z}$. Thus $2 \in K_0$. But since K_0 is an ideal of $\mathbb{Z}[1/2]$, $1 = 2\cdot(1/2) \in K_0$, a contradiction.

Next, $\mathbb{Q}[G]$ is regular but $\mathbb{Z}[G]$ is not, so Corollary 5.3 does not hold for right rings of quotients properly containing $R\mathcal{B}(Q(R))$ or $\widehat{Q}_{\mathbf{qB}}(R)$.

By [53, Proposition 4] a semiprime ring R with bounded index is right and left nonsingular. Thus in this case $\widehat{Q}_{\mathbf{qB}}(R) = Q_{\mathbf{qB}}(R)$.

Theorem 5.5. ([27, Theorem 3.8]) *Let R be a semiprime ring. Then R has bounded index at most n if and only if $Q_{\mathbf{qB}}(R)$ ($Q_{\mathbf{pqB}}(R)$) has bounded index at most n. In particular, if R is reduced, then $Q_{\mathbf{qB}}(R) = Q_{\mathbf{B}}(R)$ and it is reduced.*

We note that if R is a domain which is not right Ore, then $R = Q_{\mathbf{qB}}(R)$ has bounded index 1, but $Q(R)$ does not have bounded index. So we cannot replace "$Q_{\mathbf{qB}}(R)$" with "$Q(R)$" in Theorem 5.5. An immediate consequence of Corollary 5.3 and Theorem 5.5 is the next result.

Corollary 5.6. ([27, Corollary 3.9]) *A ring R is strongly regular if and only if $R\mathcal{B}(Q(R))$ ($Q_{\mathbf{pqB}}(R)$) is strongly regular.*

In Theorem 4.18, for every semiprime ring R, we show that $\widehat{Q}_{\mathbf{qB}}(R)$ and $\widehat{Q}_{\mathbf{FI}}(R)$ exist. Also as we see in Theorem 5.5, a semiprime ring with bounded index 1 (i.e., a reduced ring) always has a Baer absolute right ring hull. However a Baer absolute right ring hull does not always exist even for prime PI-rings with bounded index 2, as shown in our next example.

Example 5.7. ([27, Example 3.10]) For an infinite field F and a positive integer $k > 1$, let $R = \mathrm{Mat}_k(F[x, y])$, where $F[x, y]$ is the ordinary polynomial ring over F. Then R is a prime PI-ring with bounded index k. (In particular, if $k = 2$, then R has bounded index 2.) Now R has the following properties (observe that $Q(R) = E(R_R)$, hence $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$ for any class $\mathfrak{K}$ of rings).

(i) $Q_{\mathbf{B}}(R)$ does not exist.

(ii) $Q_{\mathbf{E}}(R)$ does not exist.

Since R is prime, $R = Q_{\mathbf{qB}}(R) = Q_{\mathbf{FI}}(R)$. We *claim* that $Q_{\mathbf{B}}(R)$ does not exist (the same argument shows that $Q_{\mathbf{E}}(R)$ does not exist). Assume to the contrary that $Q_{\mathbf{B}}(R)$ exists. Note that $F(x)[y]$ and $F(y)[x]$ are Prüfer domains. So $\mathrm{Mat}_k(F(x)[y])$ and $\mathrm{Mat}_k(F(y)[x])$ are Baer rings [58, p. 17, Exercise 3] (and right extending rings [43, pp. 108–109]). Note that $Q(R) = \mathrm{Mat}_k(F(x, y))$. Hence $Q_{\mathbf{B}}(R) \subseteq \mathrm{Mat}_k(F(x)[y]) \cap \mathrm{Mat}_k(F(y)[x]) = \mathrm{Mat}_k(F(x)[y] \cap F(y)[x])$. To see that $F(x)[y] \cap F(y)[x] = F[x, y]$, let

$$\begin{aligned}\gamma(x, y) &= f_0(x)/g_0(x) + (f_1(x)/g_1(x))y + \cdots \\ &\qquad \cdots + (f_m(x)/g_m(x))y^m \\ &= h_0(y)/k_0(y) + (h_1(y)/k_1(y))x + \cdots \\ &\qquad \cdots + (h_n(y)/k_n(y))x^n \in F(x)[y] \cap F(y)[x]\end{aligned}$$

with $f_i(x), g_i(x) \in F[x]$, $h_j(y), k_j(y) \in F[y]$, and $g_i(x) \neq 0$, $k_j(y) \neq 0$ for $i = 0, 1, \ldots, m, j = 0, 1, \ldots, n$. Let $\overline{F}$ be the algebraic closure of F. If $\deg g_0(x) \geq 1$, then there is $\alpha \in \overline{F}$ with $g_0(\alpha) = 0$. So $\gamma(\alpha, y)$ cannot be defined. But $\gamma(\alpha, y) = h_0(y)/k_0(y)+(h_1(y)/k_1(y))\alpha+\cdots+(h_n(y)/k_n(y))\alpha^n$, a contradiction. Thus $g_0(x) \in F$. Similarly, $g_1(x), \ldots, g_m(x) \in F$. Hence $\gamma(x, y) \in F[x, y]$. Therefore $F(x)[y] \cap F(y)[x] = F[x, y]$. Hence $Q_{\mathbf{B}}(R) = \mathrm{Mat}_k(F(x)[y] \cap F(y)[x]) = \mathrm{Mat}_k(F[x, y])$. Thus $\mathrm{Mat}_k(F[x, y]) \in \mathbf{B}$, a contradiction because $F[x, y]$ is a non-Prüfer domain [58, p. 17, Exercise 3].

A ring is called right *Utumi* [81, p. 252] if it is right nonsingular and right cononsingular (Recall that a ring R is called *right cononsingular* if any right ideal I of R with $\ell_R(I) = 0$ is right essential in R).

Corollary 5.8. ([27, Corollary 3.11]) *A reduced ring R is right Utumi if and only if $R\mathcal{B}(Q(R)) = Q_{\mathbf{E}}(R) = Q_{\mathbf{qCon}}(R)$.*

There is a non-reduced right Utumi ring R for which the equalities $R\mathcal{B}(Q(R)) = Q_{\mathbf{qCon}}(R)$ and $Q_{\mathbf{E}}(R) = Q_{\mathbf{qCon}}(R)$ in Corollary 5.8 do not hold, as the following example shows.

Example 5.9. ([27, Example 3.12]) Let $R = \mathrm{Mat}_k(F[x])$, where $F[x]$ is the polynomial ring over a field F and $k > 1$. Then R is right Utumi by [81, p. 252, Proposition 4.9]. We show that R is *not* right quasi-continuous. For this, let E_{ij} denote the matrix in R with 1 in the (i, j)-position and 0 elsewhere. Take

$$f_1 = xE_{11} + (1 - x)E_{12} + xE_{21} + (1 - x)E_{22} \qquad \text{and} \qquad f_2 = xE_{12} + E_{22}$$

in R. Then $f_1 = f_1^2$, $f_2 = f_2^2$ and $f_1R \cap f_2R = 0$. Also $(f_1R \oplus f_2R)_R \leq^{\text{ess}} fR_R$ since the uniform dimension of fR_R is 2, where $f = E_{11} + E_{22} \in R$. If there is an idempotent $g \in R$ such that $f_1R \oplus f_2R = gR$, then $gR_R \leq^{\text{ess}} fR_R$. So $gR = fR$ by the modular law. But this is impossible because $(x^2+1)E_{11} + E_{12} \in fR \setminus gR$. Therefore R is *not* right quasi-continuous. Now $R\mathcal{B}(Q(R)) = R \neq Q_{\mathbf{qCon}}(R)$. Also by [43, Lemma 12.8 and Corollary 12.10], $R \in \mathbf{E}$, so $R = Q_{\mathbf{E}}(R)$. Thus $Q_{\mathbf{E}}(R) \neq Q_{\mathbf{qCon}}(R)$.

6. How does $Q(R)$ determine R?

In this section, we investigate Problem III listed in the Introduction (i.e., Given classes $\mathfrak{K}$ and $\mathcal{S}$ of rings, determine those $T \in \mathfrak{K}$ such that $Q(T) \in \mathcal{S}$). We take the class $\mathcal{S}$ to be

$$\mathcal{S} := \{\mathrm{Mat}_2(D) \mid D \text{ is a division ring}\}$$

and $\mathfrak{K}$ to be $\mathbf{E}$, $\mathbf{B}$, or related classes.

Our first result of the section characterizes any right extending ring whose maximal right ring of quotients is the 2×2 matrix ring over a division ring.

Theorem 6.1. ([24, Theorem 3.1]) *Let D be a division ring and assume that T is a ring such that $Q(T) = \mathrm{Mat}_2(D)$ (resp., $Q(T) = Q^{\ell}(T) = \mathrm{Mat}_2(D)$). Then T is right extending (resp., T is Baer) if and only if the following conditions are satisfied:*

(i) *there exist $v, w \in D$ such that $\begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix} \in T$ and $\begin{pmatrix} 0 & 0 \\ w & 1 \end{pmatrix} \in T$; and*

(ii) *for each $0 \neq d \in D$ at least one of the following conditions is true:*

(1) $\begin{pmatrix} 0 & d \\ 0 & 1 \end{pmatrix} \in T$,

(2) $\begin{pmatrix} 1 & 0 \\ d^{-1} & 0 \end{pmatrix} \in T$, *or*

(3) *there exists $a \in D$ such that $a - a^2 \neq 0$ and $\begin{pmatrix} a & (1-a)d \\ d^{-1}a & d^{-1}(1-a)d \end{pmatrix} \in T$.*

Corollary 6.2. ([24, Corollary 3.3])

(i) *Let T be a ring such that $Q(T) = \mathrm{Mat}_2(D)$, where D is a division ring and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in T$. If $\begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \subseteq T$ or $\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} \subseteq T$, then T is right extending and Baer.*

(ii) *Let A be a right Ore domain with $D = Q^r_{c\ell}(A)$. Then $\begin{pmatrix} A & D \\ 0 & A \end{pmatrix}$ is a right extending right ring hull of $T_2(A)$ and it is Baer.*

As a consequence of Corollary 6.2, our next example provides a right extending generalized 2-by-2 triangular matrix ring T such that $Q(T) = \mathrm{Mat}_2(D)$, where $D = Q^r_{c\ell}(A)$ and A is a right Ore domain, but T is not necessarily an overring of $T_2(A)$.

Example 6.3. ([24, Example 3.4]) Let A be a right Ore domain with $D = Q^r_{c\ell}(A)$ and B any subring of D. Then $T = \begin{pmatrix} B & D \\ 0 & A \end{pmatrix}$ is right extending and $Q(T) = \text{Mat}_2(D)$. For an explicit example, take $A = \mathbb{Z}[x]$ or $\mathbb{Q}[x]$, and $B = \mathbb{Z}$.

From [58, p. 16, Exercise 2] it is well known that if A is a commutative domain with F as its field of fractions and $A \neq F$, then $T_n(A)$ $(n > 1)$ is not Baer, but by Theorem 3.9 any right ring of quotients of $T_n(A)$ which contains $T_n(F)$ is Baer. This result motivates the question: *If A is a commutative domain, can we find $\mathfrak{C}$ right ring hulls or $\mathfrak{C}\ \rho$ pseudo right ring hulls for $T_n(A)$ and use these to describe all $\mathfrak{C}$ right rings of quotients of $T_n(A)$ when $\mathfrak{C}$ is a class related to the Baer class?* (See Problem I and Problem II in Section 1). Using Theorem 6.1, we answer this question when A is either a PID or a Bezout domain (i.e., every finitely generated ideal of A is principal [48]) and $n = 2$.

Theorem 6.4. ([24, Theorem 3.7]) *Let A be a commutative Bezout domain with F as its field of fractions, $A \neq F$, and T be a right ring of quotients of $T_2(A)$. If any one of the following conditions holds, then T is right extending and Baer.*

(i) $\begin{pmatrix} A & F \\ 0 & A \end{pmatrix}$ *is a subring of T.*

(ii) $\begin{pmatrix} A & a^{-1}A \\ aA & A \end{pmatrix}$ *is a subring of T for some $0 \neq a \in A$.*

(iii) $\begin{pmatrix} A & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1}A \\ aA & A \end{pmatrix}$ *is a subring of T for some $0 \neq a \in A$, where $a = p_1^{k_1} \cdots p_m^{k_m}$, each p_i is a distinct prime, and each k_i is a positive integer.*

The following corollary illustrates how both Definitions 3.1 and 3.2 can be used to characterize all right rings of quotients from a **D-E** class $\mathfrak{C}$ (see Problem I in Section 1).

Corollary 6.5. ([24, Corollary 3.9]) *Let A be a commutative PID with F as its field of fractions, $A \neq F$, and let $R = T_2(A)$.*

(i) *Let T be a right ring of quotients of R. Then T is right extending if and only if either the ring*

$$U = \begin{pmatrix} A & F \\ 0 & A \end{pmatrix}$$

is a subring of T, or the ring

$$V = \begin{pmatrix} A & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1}A \\ aA & A \end{pmatrix}$$

is a subring of T for some nonzero $a = p_1^{k_1} \cdots p_m^{k_m}$, where each p_i is a distinct prime of A.

(ii) $\begin{pmatrix} A & F \\ 0 & A \end{pmatrix}$ *is the unique right extending right ring hull of R.*

(iii) *R has no right extending absolute right ring hull.*

(iv) *In* (i)–(iii) *we can replace "right extending" with "Baer", "right PP", or "right semihereditary".*

We remark that U and V, in Corollary 6.5, are right extending α pseudo right ring hulls of R; whereas $Q(R) = R(\mathbf{E}, Q(R))$. Moreover, if $\{p_1, p_2, \dots\}$ is an infinite set of distinct primes of A and

$$V_i = \begin{pmatrix} A & A \\ p_1 \cdots p_i A & A \end{pmatrix},$$

then $V_1 \supseteq V_2 \cdots$ forms an infinite descending chain of right extending α pseudo right ring hulls none of which contains U. Thus no V_i is a right extending right ring hull.

Corollary 6.6. ([24, Corollary 3.10]) *Let A be a commutative PID with F as its field of fractions, $A \neq F$, and let T be a right ring of quotients of $R = T_2(A)$. Take*

$$S = \begin{pmatrix} A & F \\ 0 & F \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} A & ({p_1}^{k_1-1} \cdots {p_m}^{k_m-1})^{-1} A \\ {p_1}^{k_1} \cdots {p_m}^{k_m} A & A \end{pmatrix},$$

where each p_i is a distinct prime of A.

(i) *If T is right hereditary, then either S or V is a subring of T. The converse holds when T is right Noetherian.*
(ii) *The ring S is the unique right hereditary right ring hull of R; but R has no right hereditary absolute right ring hull.*

7. Hulls of ring extensions

In this section, we seek solutions to Problem IV of Section 1 (i.e., Given a ring R and a class of rings $\mathfrak{K}$, let $\mathbf{X}(R)$ denote some standard type of extension of R (e.g., $\mathbf{X}(R) = R[x]$, or $\mathbf{X}(R) = \mathrm{Mat}_n(R)$, etc.) and let $\mathbf{H}(R)$ denote a right essential overring of R which is "minimal" with respect to belonging to the class $\mathfrak{K}$. Determine when $\mathbf{H}(\mathbf{X}(R))$ is comparable to $\mathbf{X}(\mathbf{H}(R))$), where $\mathfrak{K}$ is $\mathbf{qB}$ or $\mathbf{FI}$ and the types of ring extensions include monoid rings, full and triangular matrix rings, infinite matrix rings, etc.

Theorem 7.1. ([29, Theorem 4]) *Let $R[G]$ be a semiprime monoid ring of a monoid G over a ring R. Then:*

(i) $\widehat{Q}_{\mathbf{qB}}(R)[G] \subseteq \widehat{Q}_{\mathbf{qB}}(R[G])$.
(ii) *If G is a u.p.-monoid, then* $\widehat{Q}_{\mathbf{qB}}(R[G]) = \widehat{Q}_{\mathbf{qB}}(R)[G]$.

In [49] Goel and Jain posed the open question: *If G is an infinite cyclic group and A is a prime right quasi-continuous ring, is it true that $A[G] \in \mathbf{qCon}$?* Since a semiprime right quasi-continuous ring is quasi-Baer (see [24, Proposition 1.3]) and $A[G]$ is semiprime, Theorem 7.1 and [24, Proposition 1.3] show that $A[G] \in \mathbf{FI}$. Thus, from Theorem 7.1, when A is a commutative semiprime quasi-continuous ring and G is torsion-free Abelian, then $A[G] \in \mathbf{E}$, hence $A[G] \in \mathbf{qCon}$. This provides an affirmative answer to this question when A is a commutative semiprime quasi-continuous ring.

Corollary 7.2. ([29, Corollary 5]) *Let R be a semiprime ring. Then:*

(i) $\widehat{Q}_{\mathbf{qB}}(R[x,x^{-1}]) = \widehat{Q}_{\mathbf{qB}}(R)[x,x^{-1}]$.

(ii) $\widehat{Q}_{\mathbf{qB}}(R[X]) = \widehat{Q}_{\mathbf{qB}}(R)[X]$ *and* $\widehat{Q}_{\mathbf{qB}}(R[[X]]) = \widehat{Q}_{\mathbf{qB}}(R)[[X]]$ *for a nonempty set X of not necessarily commuting indeterminates.*

Example 7.3.

(i) ([27, Example 3.7]) Let $\mathbb{Z}[G]$ be the group ring of the group $G = \{1, g\}$ over $\mathbb{Z}$. Then $\mathbb{Z}[G]$ is semiprime, $\widehat{Q}_{\mathbf{qB}}(\mathbb{Z})[G] = \mathbb{Z}[G] \subseteq \widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G]) = \mathbb{Z}[G]\mathbf{B}(\mathbb{Q}[G])$, and $\mathbb{Z}[G] \neq \widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G])$. Thus the "u.p.-monoid" condition is not superfluous in Theorem 7.1(ii).

(ii) Let F be a field. Then $F[x]$ is a semiprime u.p.-monoid ring and $F[x] = Q(F)[x] \neq Q(F[x]) = F(x)$, where $F(x)$ is the field of fractions of $F[x]$. Thus "Q" cannot replace "$\widehat{Q}_{\mathbf{qB}}$" in Theorem 7.1(ii).

Theorem 7.4. ([29, Theorem 7]) *Let $\mathfrak{K}$ be a class of rings such that $\Lambda \in \mathfrak{K}$ if and only if $\mathrm{Mat}_n(\Lambda) \in \mathfrak{K}$ for any positive integer n, and let $H_{\mathfrak{K}}(-)$ denote any of the right ring hulls indicated in Definition* 3.1 *for the class $\mathfrak{K}$. Then for a ring R, $H_{\mathfrak{K}}(R)$ exists if and only if $H_{\mathfrak{K}}(\mathrm{Mat}_n(R))$ exists for any n. In this case, $H_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(H_{\mathfrak{K}}(R))$.*

Corollary 7.5. ([29, Corollary 9]) *Let R be a ring and n a positive integer. Then:*

(i) $\widehat{Q}_{\mathbf{IC}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(\widehat{Q}_{\mathbf{IC}}(R)) = \mathrm{Mat}_n(R\mathcal{B}(Q(R))$.

(ii) $\widehat{Q}_{\mathbf{IC}}(T_n(R)) = T_n(\widehat{Q}_{\mathbf{IC}}(R)) = T_n(R\mathcal{B}(Q(R)))$.

(iii) *If R is semiprime, then $\widehat{Q}_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(\widehat{Q}_{\mathfrak{K}}(R))$, where $\mathfrak{K} = \mathbf{qB}$ or $\mathbf{FI}$.*

Theorem 7.6. ([29, Theorem 11]) *Let R be a semiprime ring. If R and a ring S are Morita equivalent, then $\widehat{Q}_{\mathbf{qB}}(R)$ and $\widehat{Q}_{\mathbf{qB}}(S)$ are Morita equivalent.*

In contrast to Theorem 4.18, the following result provides a large class of nonsemiprime rings T for which $Q_{\mathbf{qB}}(T) = \widehat{Q}_{\mathbf{FI}}(T) = T\mathcal{B}(Q(T))$.

Theorem 7.7. ([29, Theorem 18]) *Let R be a semiprime ring and n a positive integer. Then:*

(i) $\widehat{Q}_{\mathbf{qB}}(T_n(R)) = T_n(\widehat{Q}_{\mathbf{qB}}(R)) = T_n(R)\mathbf{B}(Q(T_n(R)))$.

(ii) $\widehat{Q}_{\mathbf{FI}}(T_n(R)) = T_n(\widehat{Q}_{\mathbf{FI}}(R)) = T_n(R)\mathcal{B}(Q(T_n(R)))$.

For a ring R and a nonempty set Γ, $\mathrm{CFM}_{\Gamma}(R), \mathrm{RFM}_{\Gamma}(R)$, and $\mathrm{CRFM}_{\Gamma}(R)$ denote the column finite, the row finite, and the column and row finite matrix rings over R indexed by Γ, respectively.

In [35, Theorem 1], it was shown that $\mathrm{CRFM}_{\Gamma}(R)$ is a Baer ring for all infinite index sets Γ if and only if R is semisimple Artinian. Our next result shows that the quasi-Baer property is always preserved by infinite matrix rings.

Theorem 7.8. ([29, Theorem 19])

(i) *$R \in \mathbf{qB}$ if and only if* $\mathrm{CFM}_\Gamma(R)$ (*resp.,* $\mathrm{RFM}_\Gamma(R)$ *and* $\mathrm{CRFM}_\Gamma(R)) \in \mathbf{qB}$.

(ii) *If $R \in \mathbf{FI}$, then* $\mathrm{CFM}_\Gamma(R)$ (*resp.,* $\mathrm{CRFM}_\Gamma(R)) \in \mathbf{FI}$.

(iii) *If R is semiprime, then we have that*

$$\widehat{Q}_{\mathbf{qB}}(\mathrm{CFM}_\Gamma(R)) \subseteq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)),$$
$$\widehat{Q}_{\mathbf{qB}}(\mathrm{RFM}_\Gamma(R)) \subseteq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)),$$

and

$$\widehat{Q}_{\mathbf{qB}}(\mathrm{CRFM}_\Gamma(R)) \subseteq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)).$$

Example 7.9. There exist a commutative regular ring R and a set Γ such that

$$\widehat{Q}_{\mathbf{qB}}(\mathrm{CFM}_\Gamma(R)) \subseteq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)), \qquad \widehat{Q}_{\mathbf{qB}}(\mathrm{CFM}_\Gamma(R)) \neq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)),$$
$$\widehat{Q}_{\mathbf{qB}}(\mathrm{RFM}_\Gamma(R)) \subseteq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)), \qquad \widehat{Q}_{\mathbf{qB}}(\mathrm{RFM}_\Gamma(R)) \neq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)),$$

and

$$\widehat{Q}_{\mathbf{qB}}(\mathrm{CRFM}_\Gamma(R)) \subseteq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)), \quad \widehat{Q}_{\mathbf{qB}}(\mathrm{CRFM}_\Gamma(R)) \neq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R))$$

(see [29, Example 20] for details).

8. Modules with FI-extending hulls

In module theory the class of injective modules and, its generalization, the class of extending modules have the property that every submodule of a member is essential in a direct summand of that member. This property, originated by Chatters and Hajarnavis in [37], ensures a rich structure theory for these classes. Although every module has an injective hull, it is usually hard to compute. For many modules a minimal essential extension which belongs to the class of extending modules may not exist (e.g., $\oplus_{n=1}^{\infty}\mathbb{Z}_{\mathbb{Z}}$, see comment above Proposition 8.4). Moreover the class of extending modules lacks some important closure properties (e.g., it is not closed under direct sums).

Recall from [18] that a right R-module M_R is *FI-extending* if every fully invariant submodule of M_R is essential in a direct summand of M_R. A ring R is *right FI-extending* if R_R is FI-extending. Note that the set of fully invariant submodules of a module M_R includes the socle, Jacobson radical, torsion submodule for a torsion theory (e.g., $Z_2(M_R)$ the second singular submodule), and MI for all right ideals I of R, etc. Hence, the FI-extending condition provides an "economical use" of the extending condition by targeting only the fully invariant submodules, and thus some of the most significant submodules of M_R for an essential splitting of M_R. Natural examples of FI-extending modules abound: direct sums of uniform modules, more specifically all finitely generated Abelian groups, and semisimple modules.

We show that over a semiprime ring R, every finitely generated projective module P_R has a smallest FI-extending essential extension $H_{\mathbf{FI}}(P_R)$ (called the absolute FI-extending hull of P_R) in a fixed injective hull of P_R. Moreover, $H_{\mathbf{FI}}(P_R)$

is easily computable (see Theorem 8.2 and Proposition 8.4), it is from a class for which direct sums and direct summands are FI-extending, and since $H_{\mathbf{FI}}(P_R)$ is finitely generated and projective over $\widehat{Q}_{\mathbf{FI}}(R)$, we are assured of a reasonable transfer of information between P_R and $H_{\mathbf{FI}}(P_R)$ (e.g., see Theorem 8.5 and Corollary 8.6).

Since many well-known types of Banach algebras are semiprime (e.g., C^*-algebras), all our results for semiprime rings are applicable. Finitely generated modules over a Banach algebra are considered in [52]. Kaplansky [57] defined AW^*-modules over a C^*-algebra and used them to answer several questions concerning automorphisms and derivations on certain types of C^*-algebras. Furthermore work using these modules appeared in [7]. Moreover, from [32, p. 352], every algebraically finitely generated C^*-module M is projective, hence $H_{\mathbf{FI}}(M)$ exists. Since every C^*-algebra A is both semiprime and nonsingular, $\widehat{Q}_{\mathbf{FI}}(A)$ always exists by Theorem 4.18. Also in [27], we characterized all C^*-algebras with only finitely many minimal prime ideals and showed that for such A, $\widehat{Q}_{\mathbf{FI}}(A)$ is also a C^*-algebra. Thus our results should yield fruitful applications to projective modules over C^*-algebras, as well as many other algebras of Functional Analysis. We shall discuss some of these applications to C^*-algebras in the next section in more detail.

Definition 8.1. ([28, Definition 1]) We fix an injective hull $E(M_R)$ of M_R and a maximal right ring of quotients $Q(R)$ of R. Let $\mathbf{M}$ be a class of right R-modules and M_R a right R-module. We call, when it exists, a module $H_{\mathbf{M}}(M_R)$ the *absolute* $\mathbf{M}$ *hull* of M_R if $H_{\mathbf{M}}(M_R)$ is the smallest essential extension of M_R in $E(M_R)$ that belongs to $\mathbf{M}$.

We first obtain the existence of the absolute FI-extending hull for every finitely generated projective module over a semiprime ring. Also this module hull is explicitly described.

Theorem 8.2. ([28, Theorem 6]) *Every finitely generated projective module P_R over a semiprime ring R has the absolute FI-extending hull $H_{\mathbf{FI}}(P_R)$. Explicitly, $H_{\mathbf{FI}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$ where $P \cong e(\oplus^n R_R)$, for some n and $e = e^2 \in \mathrm{End}(\oplus^n R_R)$.*

Corollary 8.3. ([28, Corollary 7]) *Assume that R is a semiprime right Goldie ring. Then every projective right R-module P_R has the absolute FI-extending hull. Moreover, if $P \cong e(\oplus_\Lambda R_R)$ with $e = e^2 \in \mathrm{End}_R(\oplus_\Lambda R_R)$, then $H_{\mathbf{FI}}(P_R) \cong e(\oplus_\Lambda \widehat{Q}_{\mathbf{FI}}(R)_R)$.*

The FI-extending hull of a module, in general, is distinct from the injective hull of the module or its extending hull (if it exists). From Corollary 8.3, $H_{\mathbf{FI}}(\oplus_\Lambda \mathbb{Z}_\mathbb{Z}) = \oplus_\Lambda \mathbb{Z}_\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. However in $E(\oplus_\Lambda \mathbb{Z}_\mathbb{Z}) = \oplus_\Lambda \mathbb{Q}_\mathbb{Z}$, where Λ is infinite and $\mathbb{Q}$ is the field of rational numbers, there is not even a minimal extending essential extension of $\oplus_\Lambda \mathbb{Z}_\mathbb{Z}$. Our next result gives an alternative description of $H_{\mathbf{FI}}(P_R)$ different from Theorem 8.2.

Proposition 8.4. ([28, Proposition 8]) *Assume that P_R is a finitely generated projective module over a semiprime ring R. Then $H_{\mathbf{FI}}(P_R) \cong P \otimes_R \widehat{Q}_{\mathbf{FI}}(R)$ as $\widehat{Q}_{\mathbf{FI}}(R)$-modules. Hence $H_{\mathbf{FI}}(P_R)$ is also a finitely generated projective $\widehat{Q}_{\mathbf{FI}}(R)$-module.*

From Osofsky [68], there is a prime ring R with $J(R) = 0$ such that $E(R_R)$ is a non-rational extension of R_R. So $Q(R)_R$ is not injective, thus $\mathrm{End}(E(R_R)) \ncong Q(R)$ as rings by [61, p. 95, Proposition 3]. Hence $Q(\mathrm{End}(R_R)) \ncong \mathrm{End}(E(R_R))$ (see also [25, Proposition 2.6]). However, a special case of our next result shows that $\widehat{Q}_{\mathbf{FI}}(R) \cong \mathrm{End}(H_{\mathbf{FI}}(R_R))$ for a semiprime ring R.

Theorem 8.5. ([28, Theorem 12]) *Assume that R is a semiprime ring and P_R is a finitely generated projective module. Then:*

(i) $\widehat{Q}_{\mathbf{FI}}(\mathrm{End}(P_R)) \cong \mathrm{End}(H_{\mathbf{FI}}(P_R))$ *as rings.*
(ii) $\mathrm{Rad}(H_{\mathbf{FI}}(P_R)_{\widehat{Q}_{\mathbf{FI}}(R)}) \cap P = \mathrm{Rad}(P_R)$, *where* $\mathrm{Rad}(-)$ *is the Jacobson radical of a module.*

When P_R is a progenerator, we have the following.

Corollary 8.6. ([28, Corollary 13]) *Let R be a semiprime ring.*

(i) *If P_R is a progenerator of the category* Mod-R, *then $H_{\mathbf{FI}}(P_R)_{\widehat{Q}_{\mathbf{FI}}(R)}$ is a progenerator of the category* Mod-$\widehat{Q}_{\mathbf{FI}}(R)$.
(ii) *If R and S are Morita equivalent, then $\widehat{Q}_{\mathbf{FI}}(R)$ and $\widehat{Q}_{\mathbf{FI}}(S)$ are Morita equivalent.*

Recall from [76] that a module M_R is a *quasi-Baer module* if for any $N_R \trianglelefteq M_R$, there exists $h = h^2 \in \Lambda = \mathrm{End}(M_R)$ such that $\ell_\Lambda(N) = \Lambda h$, where $\ell_\Lambda(N) = \{\lambda \in \Lambda \mid \lambda N = 0\}$. It is clear that R_R is a quasi-Baer module if and only if R is a quasi-Baer ring. Also it is shown in [76] that M_R is quasi-Baer if and only if for any $I \trianglelefteq \Lambda$ there exists $g = g^2 \in \Lambda$ such that $r_M(I) = gM$, where $r_M(I) = \{m \in M \mid \mathrm{Im} = 0\}$. Moreover, if M_R is quasi-Baer, then $\mathrm{End}(M_R)$ is a quasi-Baer ring [76, Theorem 4.1]. Close connections between quasi-Baer modules and FI-extending modules are investigated in [76].

In the next result, we obtain another close connection between FI-extending modules and quasi-Baer modules. It also generalizes some of the equivalences in [18, Theorem 4.7].

Theorem 8.7. ([28, Theorem 14]) *Assume that P_R is a finitely generated projective module over a semiprime ring R. Then the following are equivalent.*

(i) *P_R is FI-extending.*
(ii) *P_R is quasi-Baer.*
(iii) $\mathrm{End}(P_R)$ *is a quasi-Baer ring.*
(iv) $\mathrm{End}(P_R)$ *is a right FI-extending ring.*

9. Applications to rings with involution

In this section, C^*-algebras are assumed to be nonunital unless indicated otherwise. Recall from [11] and [58] that a ring with an involution $*$ is called a *Baer $*$-ring* if the right annihilator of every nonempty subset is generated by a projection (i.e., an idempotent which is invariant under $*$) as a right ideal. (Recall that an ideal I of a ring R with an involution $*$ is called *self-adjoint* if $I^* = I$.) This condition is naturally motivated in the study of Functional Analysis. For example, every von Neumann algebra is a Baer $*$-algebra. With an eye toward returning to the roots of the theory of Baer and Baer $*$-rings (i.e., Functional Analysis), in this section we apply some of our previous results to rings with an involution.

In the first part of this section, we indicate that a ring R with a certain (i.e., semiproper) involution has a quasi-Baer $*$-ring absolute to $Q(R)$ right ring hull. For a reduced ring this hull coincides with a Baer $*$-ring absolute right ring hull. The section culminates with applications to C^*-algebras. We show that a unital C^*-algebra is boundedly centrally closed if and only if it is quasi-Baer. The existence of the boundedly centrally closed hull of a C^*-algebra A (i.e., the smallest boundedly centrally closed intermediate C^*-algebra between A and its local multiplier algebra $M_{\text{loc}}(A)$) is established. Moreover, it is shown that for an intermediate C^*-algebra B between A and $M_{\text{loc}}(A)$, B is boundedly centrally closed if and only if $B\mathcal{B}(Q(A)) = B$. All of the definitions, examples, and results of this section appear in [27].

Definition 9.1. Let R be a ring with an involution $*$.

(i) R is a *quasi-Baer $*$-ring* if the right annihilator of every ideal is generated by a projection as a right ideal ([16 or 20]).

(ii) We say that $*$ is *semiproper* if $xRx^* = 0$ implies $x = 0$.

As in the case for a Baer $*$-ring, the involution can be used to show that the definition of a quasi-Baer $*$-ring is left-right symmetric. If $*$ is a *proper* involution (i.e., $xx^* = 0$ implies $x = 0$ [20, p. 10]), then it is semiproper. Thus all C^*-algebras have a semiproper involution since they have a proper one [11, p. 11]. There is a semiproper involution on a prime ring which is not proper [20, p. 4266]. If R is a (quasi-) Baer $*$-ring, then $*$ is a (semi-) proper involution [11, p. 13] and [16, Proposition 3.4]. Part (ii) of the next lemma is known, but we include it for the readers' convenience.

Lemma 9.2.

(i) *Let $*$ be a semiproper involution on a ring R. Then R is semiprime and every central idempotent is a projection. If R is reduced, then $*$ is a proper involution.*

(ii) *If $*$ is a proper involution on a ring R, then R is right and left nonsingular.*

Since many rings from Functional Analysis have a (semi-) proper involution (e.g., C^*-algebras), Lemma 9.2 and Theorem 4.18 guarantee that such rings have quasi-Baer right ring hulls.

Proposition 9.3. *Let R be a $*$-ring (resp., reduced $*$-ring). Then the following are equivalent.*

(i) *R is a quasi-Baer $*$-ring (resp., Baer $*$-ring).*
(ii) *R is a quasi-Baer ring (resp., Baer ring) in which $*$ is a semiproper (resp., proper) involution.*
(iii) *R is a semiprime quasi-Baer ring and every central idempotent is a projection.*

Thereby the center of a quasi-Baer $*$-ring is a Baer $*$-ring.

Note that Baer $*$-rings are quasi-Baer $*$-rings. But the converse does not hold as follows.

Example 9.4.

(i) ([16, Example 2.2]) Let $R = \mathrm{Mat}_2(\mathbb{C}[x])$. Then R is a Baer ring. We can extend the conjugation on $\mathbb{C}$ to that on $\mathbb{C}[x]$. Let $*$ denote the conjugate transpose involution on R. Then $*$ is a proper involution. The right annihilator $r_R\left[\begin{pmatrix} x & 2 \\ 0 & 0 \end{pmatrix}\right]$ cannot be generated by a projection as a right ideal. So R is not a Baer $*$-ring; but, by Proposition 9.3, R is a quasi-Baer $*$-ring.
(ii) Let $^{-}$ be the conjugation on $\mathbb{C}$. If G is a polycyclic-by-finite group and $*$ is the involution on the group algebra $\mathbb{C}[G]$ defined by $(\sum a_g g)^* = \sum \overline{a}_g g^{-1}$, then the involution $*$ is proper. From [16, Corollary 1.9], $\mathbb{C}[G] \in \mathbf{qB}$. So $\mathbb{C}[G]$ is a quasi-Baer $*$-ring by Proposition 9.3. But in general $\mathbb{C}[G]$ is not a Baer $*$-ring. In fact, let $G = D_\infty \times C_\infty$, where D_∞ is the infinite dihedral group and C_∞ is the infinite cyclic group. Then the group G is polycyclic-by-finite. By [16, Example 1.10] $\mathbb{C}[G]$ is not a Baer $*$-ring.

There is a quasi-Baer ring R with an involution such that R has only finitely many minimal prime ideals, but not all minimal prime ideals are self-adjoint. For example, let F be a field and $R = F \oplus F$, where $*$ is the exchange involution. Then R is a Baer ring with only finitely many minimal prime ideals which are not self-adjoint.

Proposition 9.5. *Let R be a semiprime $*$-ring with only finitely many minimal prime ideals. Then $\widehat{Q}_{\mathbf{qB}}(R)$ is a quasi-Baer $*$-ring if and only if every minimal prime ideal of R is self-adjoint.*

Proposition 9.6. *Let R be a $*$-ring and T a right essential overring of R.*

(i) *If $*$ extends to T and $*$ is semiproper on R, then $*$ is semiproper on T.*
(ii) *If $*$ extends to T, then $*$ is proper on R if and only if $*$ is proper on T.*

Theorem 9.7. *Let R be a ring (resp., reduced ring) with a semiproper involution $*$ and T be a right ring of quotients of R. If $*$ extends to T, then the following are equivalent.*

(i) *T is a quasi-Baer $*$-ring (resp., Baer $*$-ring).*
(ii) *$\widehat{Q}_{\mathbf{qB}}(R)$ is a subring of T.*
(iii) *$\mathcal{B}(Q(R)) \subseteq T$.*

Thus $Q^s(R)$ is a quasi-Baer $*$-ring. Also $\widehat{Q}_{\mathbf{qB}}(R)$ is the quasi-Baer $*$-ring absolute to $Q(R)$ right ring hull of R. If R is reduced, then $\widehat{Q}_{\mathbf{qB}}(R)$ is the Baer $*$-ring absolute right ring hull of R.

In the remainder of this section, we focus on C^*-algebras. Recall that for a C^*-algebra A, the algebra of all double centralizers on A is called its multiplier algebra, $M(A)$, which coincides with the maximal unitization of A in the category of C^*-algebras. It is an important tool in the classification of C^*-algebras and in the study of K-theory and Hilbert C^*-modules.

For a C^*-algebra A, recall that $A^1 = \{a + \lambda 1_{Q(A)} \mid a \in A \text{ and } \lambda \in \mathbb{C}\}$. Then $A^1 = \{a + \lambda 1_{M(A)} \mid a \in A \text{ and } \lambda \in \mathbb{C}\}$ because $1_{Q(A)} = 1_{M(A)}$. Note that $M(A)$ and A^1 are C^*-algebras. For $X \subseteq A$, $\overline{X}$ denotes the norm closure of X in A.

Let A be a C^*-algebra. Then the set $\mathcal{I}_{ce}$ of all norm closed essential ideals of A forms a filter directed downwards by inclusion. The ring $Q_b(A)$ denotes the algebraic direct limit of $\{M(I)\}_{I\in\mathcal{I}_{ce}}$, where $M(I)$ denotes the C^*-algebra multipliers of I; and $Q_b(A)$ is called the *bounded symmetric algebra of quotients* of A in [5, p. 57, Definition 2.23]. The norm closure, $M_{\text{loc}}(A)$, of $Q_b(A)$ (i.e., the C^*-algebra direct limit $M_{\text{loc}}(A)$ of $\{M(I)\}_{I\in\mathcal{I}_{ce}}$) is called the *local multiplier algebra* of A [5, p. 65, Definition 2.3.1]. The local multiplier algebra $M_{\text{loc}}(A)$ was first used by Elliott in [45] and Pedersen in [69] to show the innerness of certain $*$-automorphisms and derivations. Its structure has been extensively studied in [5]. Since A is a norm closed essential ideal of A^1, $M_{\text{loc}}(A) = M_{\text{loc}}(A^1)$ by [5, p. 66, Proposition 2.3.6]. Also note that $Q_b(A) = Q_b(A^1)$. See [5], [45], and [70] for more details on $M_{\text{loc}}(A)$ and $Q_b(A)$.

Lemma 9.8. *Let A be a C^*-algebra. Then we have the following.*

(i) $\mathcal{B}(M_{\text{loc}}(A)) = \mathcal{B}(Q(A)) = \mathcal{B}(Q^s(A)) = \mathcal{B}(Q_b(A))$.

(ii) $\text{Cen}(M_{\text{loc}}(A))$ *is the norm closure of the linear span of* $\mathcal{B}(Q(A))$.

When A is a unital C^*-algebra, Theorem 4.18, Lemma 9.2, and Theorem 9.7 yield that $A\mathcal{B}(Q(A)) = \widehat{Q}_{\mathbf{qB}}(A) = Q_{\mathbf{qB}}(A)$ exists and is the quasi-Baer $*$-ring absolute right ring hull of A. Thus it is of interest to consider unital C^*-algebras which are quasi-Baer $*$-rings.

Recall from [11] that a C^*-algebra is called an *AW^*-algebra* if it is a Baer $*$-ring. In analogy, we say that a unital C^*-algebra A is a *quasi-AW^*-algebra* if it is a quasi-Baer $*$-ring. Thus by Proposition 9.3, a unital C^*-algebra A is a quasi-AW^*-algebra if $A \in \mathbf{qB}$.

The next lemma shows that $Q_b(A)$ is a quasi-Baer $*$-algebra for any C^*-algebra A.

Lemma 9.9. *Let A be a C^*-algebra. Then we have the following.*

(i) $Q_{\mathbf{qB}}(A^1)$ *is a $*$-subalgebra of* $Q_b(A)$.

(ii) $Q_b(A)$ *is a quasi-Baer $*$-algebra.*

By Lemma 9.9, if A is a unital C^*-algebra, then $Q_{\mathbf{qB}}(A)$ is a $*$-subalgebra of $M_{\text{loc}}(A)$.

Definition 9.10. ([5, p. 73, Definition 3.2.1]) For a C^*-algebra A, the C^*-subalgebra $\overline{A\,\mathrm{Cen}(Q_b(A))}$ (the norm closure of $A\,\mathrm{Cen}(Q_b(A))$ in $M_{\mathrm{loc}}(A)$) of $M_{\mathrm{loc}}(A)$ is called the *bounded central closure* of A. If $A = \overline{A\,\mathrm{Cen}(Q_b(A))}$, then A is said to be *boundedly centrally closed*.

A boundedly centrally closed C^*-algebra and the bounded central closure of a C^*-algebra are the C^*-algebra analogues of a centrally closed subring and the central closure of a semiprime ring, respectively. These have been used to obtain a complete description of all centralizing additive mappings on C^*-algebras [3] and for investigating the central Haagerup tensor product of multiplier algebras [4]. Boundedly centrally closed algebras are important for studying local multiplier algebras and have been treated extensively in [5].

It is shown in [5, pp. 75–76, Theorem 3.2.8 and Corollary 3.2.9] that $M_{\mathrm{loc}}(A)$ and $\overline{A\,\mathrm{Cen}(Q_b(A))}$ are boundedly centrally closed. Every AW^*-algebra and every prime C^*-algebra are boundedly centrally closed [5, pp. 76–77, Example 3.3.1]. Moreover, A is boundedly centrally closed if and only if $M(A)$ is so [5, p. 74, Proposition 3.2.3]. However, there exists A which is boundedly centrally closed, but A^1 is not so [5, p. 80, Remarks 3.3.10]. Hence it is of interest to investigate the boundedly centrally closed intermediate C^*-algebras between A and $M_{\mathrm{loc}}(A)$.

Definition 9.11. Let A be a C^*-algebra. The smallest boundedly centrally closed C^*-subalgebra of $M_{\mathrm{loc}}(A)$ containing A is called the *boundedly centrally closed hull* of A.

The following lemma shows that a unital C^*-algebra A is boundedly centrally closed if and only if $A \in \mathbf{qB}$. It is shown that the boundedly centrally closed hull of A is $\overline{Q_{\mathbf{qB}}(A)}$. Moreover, this lemma is a unital C^*-algebra analogue of Theorem 4.18(ii). It generalizes [5, pp. 72–73, Lemma 3.1.3 and Remark 3.1.4].

Lemma 9.12. *Let A be a unital C^*-algebra. Then:*

(i) *A is boundedly centrally closed if and only if $A \in \mathbf{qB}$ (i.e., a quasi-AW^*-algebra).*
(ii) $\overline{Q_{\mathbf{qB}}(A)} = \overline{A\,\mathrm{Cen}(Q_b(A))}$.
(iii) *$\overline{Q_{\mathbf{qB}}(A)}$ is the boundedly centrally closed hull of A.*
(iv) *Let B be an intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$. Then B is boundedly centrally closed if and only if $\mathcal{B}(Q(A)) \subseteq B$.*

From Proposition 9.3 and Lemma 9.12(i), the center of a quasi-AW^*-algebra (i.e., a unital boundedly centrally closed C^*-algebra by Lemma 9.12(i)) is an AW^*-algebra. The next example shows that the class of quasi-AW^*-algebras encompasses more variety than its subclass of AW^*-algebras.

Example 9.13.

(i) ([11, p. 15, Example 1]) There is a quasi-AW^*-algebra which is not an AW^*-algebra. Let A be the set of all compact operators on an infinite-dimensional Hilbert space over $\mathbb{C}$. Then the heart of A^1 is the set of bounded linear operators with finite-dimensional range space. So A^1 is subdirectly irreducible.

Since A^1 is semiprime, A^1 is prime and so $A^1 \in \mathbf{qB}$. Hence A^1 is a quasi-AW^*-algebra. But as shown in [11, p. 15, Example 1], A^1 is not a Baer $*$-ring, thus A^1 is not an AW^*-algebra.

(ii) Every unital prime C^*-algebra is a quasi-AW^*-algebra. There are prime finite Rickart unital C^*-algebras (hence quasi-AW^*-algebras) which are not AW^*-algebras [51].

(iii) From [11, p. 43, Corollary], $\mathbb{C}$ is the only prime projectionless unital AW^*-algebra. Various unital prime projectionless C^*-algebras (hence quasi-AW^*-algebras) are provided in [40, pp. 124–129 and 205–214].

Our next example provides a nonunital C^*-algebra A such that both A and $M_{\mathrm{loc}}(A)$ are boundedly centrally closed, but A^1 is not so.

Example 9.14. Let A be the C^*-direct sum of $\aleph_0$ copies of $\mathbb{C}$. Then $M_{\mathrm{loc}}(A)$ is the C^*-direct product of $\aleph_0$ copies of $\mathbb{C}$. So both A and $M_{\mathrm{loc}}(A)$ are boundedly centrally closed, but A^1 is not so.

Thus Example 9.14 motivates one to seek a characterization of the boundedly centrally closed (not necessarily unital) intermediate C^*-algebras between A and $M_{\mathrm{loc}}(A)$. Our next result provides such a characterization in terms of $\mathcal{B}(Q(A))$ and shows the existence of the boundedly centrally closed hull of a C^*-algebra A.

Theorem 9.15. *Let A be a C^*-algebra and B an intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$. Then:*

(i) $\mathcal{B}(Q(A)) \subseteq \mathrm{Cen}(Q_b(B^1\mathcal{B}(Q(A)))) = \mathrm{Cen}(Q_b(B)) \subseteq \mathrm{Cen}(M_{\mathrm{loc}}(A))$.

(ii) $\overline{B\mathcal{B}(Q(A))} = \overline{B\,\mathrm{Cen}(Q_b(B))}$.

(iii) *B is boundedly centrally closed if and only if* $B = \overline{B\mathcal{B}(Q(A))}$.

(iv) $\overline{A\mathcal{B}(Q(A))}$ *is the boundedly centrally closed hull of A.*

Assume that A is a C^*-algebra and B is an intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$. Then $M(B)$ may not be contained in $M_{\mathrm{loc}}(A)$. However, the next corollary characterizes $M(B)$ to be boundedly centrally closed via $\mathcal{B}(Q(A))$. Moreover, parts (i) and (ii) are of interest in their own rights.

Corollary 9.16. *Let A be a C^*-algebra and B an intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$. Then:*

(i) $\mathcal{B}(Q(B)) = \mathcal{B}(Q(A))$.

(ii) $\mathrm{Cen}(M_{\mathrm{loc}}(B)) = \mathrm{Cen}(M_{\mathrm{loc}}(A))$.

(iii) $\overline{M(B)\,\mathrm{Cen}(Q_b(M(B)))} = \overline{M(B)\mathcal{B}(Q(A))}$.

(iv) *$M(B)$ is boundedly centrally closed if and only if* $\mathcal{B}(Q(A)) \subseteq M(B)$.

Surprisingly, the next result shows that under a mild finiteness condition, $Q_{\mathbf{qB}}(A^1)$ is norm closed.

Corollary 9.17. *Let A be a C^*-algebra and n a positive integer. Then the following are equivalent.*

(i) *A has exactly n minimal prime ideals.*

(ii) *$Q_{\mathbf{qB}}(A^1)$ is a direct sum of n prime C^*-algebras.*

(iii) *The extended centroid of A is $\mathbb{C}^n$.*
(iv) *$M_{\text{loc}}(A)$ is a direct sum of n prime C^*-algebras.*
(v) $\text{Cen}(M_{\text{loc}}(A)) = \mathbb{C}^n$.
(vi) *Some boundedly centrally closed intermediate C^*-algebra between A and $M_{\text{loc}}(A)$ is a direct sum of n prime C^*-algebras.*
(vii) *Every boundedly centrally closed intermediate C^*-algebra between A and $M_{\text{loc}}(A)$ is a direct sum of n prime C^*-algebras.*

Open questions and problems

(i) Determine which classes of rings are closed with respect to right essential overrings. In particular, is the class of right extending rings closed with respect to right essential overrings?
(ii) If a ring R is semiprime, then is $R\mathcal{B}(Q(R)) = Q_{\mathbf{FI}}(R)$?

Note that in [76, Example 4.2], there is an example of a module M_R such that $\text{End}(M_R)$ is a quasi-Baer ring, but M_R is not quasi-Baer. In [28], we have shown that for $\mathfrak{M} = \mathbf{FI}$, if R is a semiprime ring then $H_{\mathbf{FI}}(R_R) = \widehat{Q}_{\mathbf{FI}}(R)$. This motivates:

(iii) For a given class $\mathfrak{M}$ of modules, determine necessary and/or sufficient conditions on R such that $H_{\mathfrak{M}}(R_R) = \widehat{Q}_{\mathfrak{M}}(R)$.

Acknowledgment

The authors express their thanks to the referee for his/her helpful comments for improving the paper. The authors are grateful for the support they received from the Mathematics Research Institute, Columbus, Ohio and for the kind hospitality and support of Busan National University, South Korea, the Ohio State University at Lima, and the University of Louisiana at Lafayette.

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Ring and Module Theory
Trends in Mathematics, 73–81

Rewriting as a Special Case of Noncommutative Gröbner Bases Theory for the Affine Weyl Group $\widetilde{A}_n$

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Abstract. The aim of this work is to find Gröbner-Shirshov bases of the affine Weyl group of type $\widetilde{A}_n$ $(n \geq 2)$ from the point of complete rewriting systems.

Mathematics Subject Classification (2000). Primary 13P10; Secondary 20F55, 68Q42.

Keywords. Rewriting System, Gröbner-Shirshov Bases, affine Weyl Group.

1. Introduction and preliminaries

It is known that the rewriting system for groups, monoids and semigroups is a special case of Gröbner bases theory for noncommutative polynomial algebras. Thus our aim in this paper is to reveal this well-known fact on a specific application, namely, affine Weyl groups of type $\widetilde{A}_n$ $(n \geq 2)$. So this work can also be thought as a transition study between combinatorial group theory (more specify "theoretical computer science") and ring theory over an interesting infinite group.

A good introduction to string-rewriting systems and noncommutative Gröbner bases theory are presented by [2] and [17], respectively. An important subsubject in above both *system* and *base theory* (under a different name), called *critical pair*, can be obtained by two completion methods, namely, Knuth-Bendix and Buchberger's algorithms. The connection between these two algorithms have been pointed out in the commutative case (see, for instance, [4, 14]). In particular it is well known that Buchberger algorithm (in the commutative case) may be applied to presentations of abelian groups to obtain complete rewriting systems. Besides that, in the literaure, since there are no enough work passing through from rewriting systems to noncommutative Gröbner (Gröbner-Shirshov) bases or

The first and third authors are partially supported by Balikesir University BAP-2007/17.
The second author is supported by TÜBA, The Turkish Academy of Sciences.

vice versa except the work in [9], in this paper, we consider the transition from the *system* to *bases* for the affine Weyl group of type $\widetilde{A}_n$ $(n \geq 2)$ and prove the following result.

Theorem 1.1 (Main Theorem). *The affine Weyl group of type* $\widetilde{A}_n$ $(n \geq 2)$ *has a complete rewriting system.*

Affine Weyl groups $\widetilde{A}_n$ $(n \geq 2)$, $\widetilde{B}_m$ $(m \geq 3)$, $\widetilde{C}_l$ $(l \geq 2)$, $\widetilde{D}_k$ $(k \geq 4)$, as family of infinite crystallagraphic Coxeter groups, play an important role in various fields of mathematics, such as Kac-Moody algebras, algebraic groups and their representation theory, combinatorial and geometric group theory, etc. The reduced elements of this special infinite groups are indexing the basis elements of the integral cohomology algebras of infinite-dimensional flag varieties associated Kac-Moody type Lie groups. By Composition Diamond Lemma, the reduced elements of the group are obtained using elements of Gröbner-Shirshov basis of the group. Although it will be just considered the affine Weyl group of type $\widetilde{A}_n$ in this paper, the remaining groups can also be thought in the same direction of this work for a future project. Briefly, the affine Weyl group $\widetilde{A}_n$ is an irreducible Coxeter group whose Coxeter graph is a polygon with n vertices and a presentation for $\widetilde{A}_n$ is

$$\begin{aligned} \langle y_1, y_2, \ldots, y_{n+1} \quad ; \quad & y_i^2 = 1 \ (1 \leq i \leq n+1), \\ & y_i y_{i+1} y_i = y_{i+1} y_i y_{i+1} \ (1 \leq i \leq n), \\ & y_i y_j = y_j y_i \ (1 \leq i < j-1 < n+1 \text{ and } (i,j) \neq (1, n+1)), \\ & y_1 y_{n+1} y_1 = y_{n+1} y_1 y_{n+1} \rangle, \end{aligned} \tag{1.1}$$

where $n \geq 2$ (see [12] for more details).

We should note that, by [1], the affine Weyl group $\widetilde{A}_n$ is a split extension of (n) copies of $\mathbb{Z}$ by the symmetric group S_{n+1} of degree $n+1$. In other words $\widetilde{A}_n$ is a finite extension of a free abelian group, and so the fact that these groups have a finite complete rewriting system follows immediately from the proof that the class of groups with complete rewriting systems is closed under finite extension (see for example [7]). Moreover the fact that the corresponding ideal of the free associative monoid has a finite noncommutative Gröbner basis is also then immediate. Although these results are already seem in the literature, there is new information in this paper, namely that *the complete rewriting system* for $\widetilde{A}_n$ with respect to the Coxeter generators and the degree-lex ordering is finite, and indeed we list the rewriting rules in Theorem 3.2 (below). Since *Coxeter groups in general do not have finite complete rewriting systems for the Coxeter generators and the degree-lex ordering*, this rewriting system (or, equivalently, Gröbner basis) has taken our interest.

2. Proof of Theorem 1.1

In this section, before we proceed the proof of the main result, let us first recall some fundamental material that needed in the proof. We note that the reader is referred to [2, 15] for a detailed survey on (complete) rewriting sytems.

Let X be a set and let X^* be the free monoid consists of all words obtained by the elements of X. A (string) *rewriting system* on X^* is a subset $R \subseteq X^* \times X^*$ and an element $(u, v) \in R$, also written $u \to v$, is called a rule of R. The idea for a rewriting system is an algorithm for substituting the right-hand side of a rule whenever the left-hand side appears in a word. In general, for a given rewriting system R, we write $x \to y$ for $x, y \in X^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$. Also we write $x \to^* y$ if $x = y$ or $x \to x_1 \to x_2 \to \cdots \to y$ for some finite chain of reductions. Furthermore an element $x \in X^*$ is called irreducible with respect to R if there is no possible rewriting (or reduction) $x \to y$; otherwise x is called *reducible*. The rewriting system R is called

- *Noetherian* if there is no infinite chain of rewritings $x \to x_1 \to x_2 \to \cdots$ for any word $x \in X^*$,
- *Confluent* if whenever $x \to^* y_1$ and $x \to^* y_2$, there is a $z \in S^*$ such that $y_1 \to^* z$ and $y_2 \to^* z$,
- *Complete* if R is both Noetherian and confluent.

A rewriting system is *finite* if both X and R are finite sets. Furthermore a *critical pair* of a rewriting system R is a pair of overlapping rules if one of the following forms

(i) (r_1r_2, s), $(r_2r_3, t)\in R$ with $r_2 \neq 1$,
(ii) $(r_1r_2r_3, s)$, $(r_2, t)\in R$,

is satisfied. Also a critical pair is *resolved* in R if there is a word z such that $sr_3 \to^* z$ and $r_1t \to^* z$ in the first case or $s \to^* z$ and $r_1tr_3 \to^* z$ in the second. A Noetherian rewriting system is complete if and only if every critical pair is resolved ([15]). The following lemma is also important to get Noetherian condition.

Lemma 2.1 ([10]). *A rewriting system R on X is Noetherian if and only if there exists a reduction ordering on X^* which is compatible with R.*

Knuth and Bendix have developed an *algorithm* for creating a complete rewriting system R' which is equivalent to R, so that any word over X has an (unique) irreducible form with respect to R'. By considering overlaps of left-hand sides of rules, this algorithm basicly proceeds forming new rules when two reductions of an overlap word result in two distinct reduced forms. The complete rewriting system for Coxeter groups was first constructed in [6] performing the Knuth-Bendix procedure on these groups with a length-lexicographic ordering on words. We note that similar material has also been studied in [8].

Now let us focus on the proof.

Since there is no quite effective algorithm that calculates Gröbner-Shirshov bases for infinite groups, we need to strict ourselves on some countable cases. So, for $n = 4$, let us first consider the affine Weyl group $\widetilde{A}_n$. Clearly, we have the presentation

$$\begin{aligned}\langle y_1, y_2, y_3, y_4, y_5 \quad ; \quad & y_i^2 = 1\ (1 \leq i \leq 5), y_1y_2y_1 = y_2y_1y_2, y_2y_3y_2 = y_3y_2y_3,\\ & y_3y_4y_3 = y_4y_3y_4, y_4y_5y_4 = y_5y_4y_5, y_1y_5y_1 = y_5y_1y_5,\\ & y_1y_3 = y_3y_1, y_1y_4 = y_4y_1, y_2y_4 = y_4y_2,\\ & y_2y_5 = y_5y_2, y_3y_5 = y_5y_3\rangle\end{aligned}$$

for the group $\widetilde{A}_4$, as in (1.1). It is well known that defining relations for a group are actually rewriting rules for this group. In the light of this fact, the set of rewriting rules (i.e., the rewriting system) for $\widetilde{A}_4$ is

$$\begin{aligned}&\{y_i^2 = 1(1 \leq i \leq 5),\ y_1y_2y_1 = y_2y_1y_2,\ y_2y_3y_2 = y_3y_2y_3,\\ &y_3y_4y_3 = y_4y_3y_4,\ y_4y_5y_4 = y_5y_4y_5,\ y_1y_5y_1 = y_5y_1y_5,\ y_1y_3 = y_3y_1\\ &y_1y_4 = y_4y_1,\ y_2y_4 = y_4y_2,\ y_2y_5 = y_5y_2,\ y_3y_5 = y_5y_3\},\end{aligned} \tag{2.1}$$

where the ordering is *DegLex* related to $y_1 > y_2 > y_3 > y_4 > y_5$. (The ordering DegLex can also be called as LengthLex and ShortLex by some studies.) In fact, for any words $w_1, w_2 \in X^*$, this ordering can be summarized as

$w_1 < w_2$ if

- either $\deg(w_1) < \deg(w_2)$,
- or in the case that the degrees (lengths) are equal
  - ◇ if the ith position is the first, working from left to right, in which w_1 and w_2 differ, then the ith letter of w_1 is less than that of w_2 in the ordering given to the alphabet.

The first step that need to check is the existence of Noetherian property for a given rewriting system. On account of the reduction ordering $y_1 > y_2 > y_3 > y_4 > y_5$, by Lemma 2.1, it is easy to see that the rewriting system for $\widetilde{A}_4$ is Noetherian. In addition, to show having confluent property of $\widetilde{A}_4$, we need to consider all overlap words in the set (2.1) and check whether these are resolved. In fact several software packages (for instance, GAP) can be used to compute whether an overlap word has a unique irreducible word or not, and then whether the system for $\widetilde{A}_4$ is confluent. (A manual computation without computer can be obtained from the third author.) Therefore we have the following result.

Corollary 2.2. *The affine Weyl group $\widetilde{A}_4$ has a complete rewriting system.*

The above procedure for the case $n = 4$ can be thought as one of the begining induction steps. Therefore we can adapt this proof to any $n \geq 2$. Hence these all progresses complete the proof of Theorem 1.1, as required.

The existence of a complete rewriting system for a group, monoid or semigroup is very useful. In fact by using this system, we can also investigate the existence of some other algebraic material. The *word problem* for groups can be given an example of this. It was first introduced by M. Dehn in 1911's and basically says that "*for given any words U and V that taken from generators of given group,* *does there exist an algorithm determining whether $U = V$ or not?*". We actually know that if the word problem for a group is solvable, then this group admits a complete rewriting system with finitely many rules. Thus the following result is immediate.

Corollary 2.3. *The affine Weyl group of type $\widetilde{A}_n$ ($n \geq 2$) has solvable word problem.*

An elegant proof for the above corollary was firstly given by Tits [16].

3. Gröbner-Shirshov basis of $\widetilde{A}_n$

In this section we compare Gröbner-Shirshov bases and rewriting systems of the affine Weyl group $\widetilde{A}_n$. We first recall that noncommutative Gröbner bases are also named as Gröbner-Shirshov bases in the literature (see, for example, [17]).

Let X be a linearly ordered set, k be a field and $k\langle X\rangle$ be the free associative algebra over X and k. On the set X^* of words, we impose a well-order $\leq$ that is compatible with the concatenations of words. This order is called *total*. For example, the ordering DegLex is actually total. Now let $f \in k\langle X\rangle$ be a polynomial with leading word (the maximal term by the ordering) $\overline{f}$. We say that f is monic if $\overline{f}$ occurs in f with coefficient 1. Thus for some monic polynomials f and g, the Gröbner-Shirshov basis can be formulated as follows:

I) Let w be a word such that $w = \overline{f}b = a\overline{g}$ with $\deg(\overline{f}) + \deg(\overline{g}) > \deg(w)$. Then the polynomial $(f,g)_w$ is called the *intersection composition* of f and g with respect to w if $(f,g)_w = fb - ag$.

II) Let $w = \overline{f} = a\overline{g}b$. Then the polynomial $(f,g)_w = f - agb$ is called the *inclusion composition* of f and g with respect to w. In this case, the transformation $f \mapsto (f,g)_w = f - agb$ is called the *elimination of the leading word* (ELW) of g in f.

III) Let $R \subset k\langle X\rangle$. A composition $(f,g)_w$ is called *trivial* relative to R (and w) if
$$(f,g)_w = \sum \alpha_i a_i t_i b_i,$$
where $\alpha_i \in k$, $t_i \in R$, $a_i, b_i \in X^*$ and $a_i\overline{t_i}b_i < w$. We usually write it as $(f,g)_w = 0 \bmod (R,w)$. In particular, if $(f,g)_w$ is zero by ELW of polynomials from R, then $(f,g)_w$ is trivial relative to R. We assume that $f_1 \equiv f_2 \bmod (R,w)$ if $f_1 - f_2 \equiv 0 \bmod (R,w)$, for some polynomials f_1 and f_2.

A subset R of $k\langle X\rangle$, as defined in above, is called *Gröbner-Shirshov bases* if any composition of polynomials from R is trivial relative to R. We note that the algebra with generators X and the defining relations R, notationally $\langle X; R\rangle$, will be mean the factor-algebra of $k\langle X\rangle$ by the ideal generated by R. We also note that the following lemma which is important to determine Gröbner-Shirshov basis and was given by Bokut ([5]).

Lemma 3.1 (Composition-Diamond Lemma). *Let $S = \langle X; R\rangle$. The set of defining relations R is a Gröbner-Shirshov bases if and only if the set*

$$\{u \in X^* | u \neq a\overline{f}b, \text{ for any } f \in R\}$$

of R-reduced words consists of a linear basis of S.

If a subset R of $k\langle X\rangle$ is not a Gröbner-Shirshov basis, then one can add to R all nontrivial compositions of poynomials or R, and continue this process many times in order to have a Gröbner-Shirshov bases that contains R. This procedure is called the *Buchberger-Shirshov algorithm* ([3]). We note that, in the paper [9], the author showed that the Knuth-Bendix completion algorithm for given rewriting system R corresponds step-by-step to the Buchberger-Shirshov algorithm for finding a Gröbner bases for the ideal generated by the set $F = \{l - r : (l, r) \in R\}$. Using Buchberger-Shirshov algorithm, Özel and Yilmaz have recently calculated the Gröbner-Shirshov bases of the affine Weyl group $\widetilde{A_n}$ ([13]).

Let $\widetilde{A_n}$ be affine Weyl group generated by $r_0, r_1, r_2, \ldots, r_n$ with defining relations

$$r_i^2 = 1, \quad (i = 0, \ldots, n),$$
$$r_ir_j = r_jr_i, \quad (i = 0, \ldots, n-2,\ j = 2, \ldots, n,\ j - i > 1),$$
$$r_ir_{i+1}r_i = r_{i+1}r_ir_{i+1}, \quad (i = 0, \ldots, n-1)$$

and

$$r_0r_nr_0 = r_nr_0r_n.$$

Let us define the words

$$r_{ij} = \begin{cases} r_ir_{i-1}\cdots r_j & ; \quad i > j \\ r_i & ; \quad i = j \\ r_ir_{i+1}\cdots r_j & ; \quad i < j \end{cases}$$

and $\widehat{r}_{ij} = r_ir_{i-1}r_{i+1}r_i \cdots r_{j+1}r_j$, where $j \geq i - 1$.

Theorem 3.2 (G-S Basis of $\widetilde{A_n}$). *If we identify a relation $u = v$ with polynomial representation $u - v$, then a Gröbner-Shirshov basis for $\widetilde{A_n}$ with respect to Deglex ordering $r_0 > r_1 > \cdots > r_n$ consists of initial relations together with the following polynomials*

(1) $r_{ij}r_i - r_{i+1}r_{ij}$, *where* $j > i, i = 0, \ldots, n-2, j = i+2, \ldots, n$,
(2) $r_0r_nr_j - r_jr_0r_n$, *where* $j = 2, \ldots, n-2$,
(3) $r_0r_nr_{n-1}r_n - r_{n-1}r_0r_nr_{n-1}$,
(4) $r_0r_nr_{n-1}r_{n-2}r_{n-1} - r_{n-2}r_0r_nr_{n-1}r_{n-2}$,
(5) $r_0r_{nj}r_0 - r_nr_0r_{nj}$, *where* $j = 2, \ldots, n-1$,
(6) $r_0r_{nj}\widehat{r}_{1k}r_{k+1} - r_nr_0r_{nj}\widehat{r}_{1k}$, *where* $j = 2, \ldots, n-1$, $k = 0, \ldots, n-1$,
(7) $r_0r_{nj}r_1r_0r_{nk}r_1 - r_nr_0r_{nj}r_1r_0r_{nk}$, *where* $j = 2, \ldots, n-1$, $k = j+1, \ldots, n$,
(8) $r_0r_{nj}r_1r_0r_{2k}\widehat{r}_{2l}r_{l+1} - r_nr_0r_{nj}r_1r_0r_{nk}\widehat{r}_{2p}$,
where $j = 2, \ldots, n-1$, $k = j+1, \ldots, n$, $p = 3, \ldots, n-1$,
(9) $r_0r_n\widehat{r}_{1k}r_{k+1} - r_nr_0r_n\widehat{r}_{1k}$, *where* $k = 0, \ldots, n-1$,

(10) $r_{0j}r_nr_0r_n - r_1r_{0j}r_nr_0$, *where* $j = 2, \dots, n-2$,
(11) $r_{0j}r_nr_{n-1}r_0r_nr_{n-1} - r_1r_{0j}r_nr_0r_{n-1}r_n$, *where* $j = 2, \dots, n-2$,
(12) $r_{0j}r_nr_{n-1}r_{n-2}r_0r_nr_{n-1}r_{n-2} - r_1r_{oj}r_nr_0r_{n-1}r_nr_{n-2}r_{n-1}$,
where $j = 2, \dots, n-2$.

In the previous section, by considering all overlap words in the set (2.1), we mentioned that one can do critical pair analysis for $\widetilde{A}_4$, or basically, the degree-lex rewriting system for $\widetilde{A}_4$ can be checked by several software packages such as GAP. Now by assuming these computations have been done and considering the above theorem, we obtain the following result:

Corollary 3.3. *The elements of* $\widetilde{A_4}$ *in each type given in the above theorem are:*

(1) $r_0r_1r_2r_0 - r_1r_0r_1r_2$,
$r_0r_1r_2r_3r_0 - r_1r_0r_1r_2r_3$,
$r_1r_2r_3r_1 - r_2r_1r_2r_3$,
$r_1r_2r_3r_4r_1 - r_2r_1r_2r_3r_4$,
$r_2r_3r_4r_2 - r_3r_2r_3r_4$,

(2) $r_0r_4r_2 - r_2r_0r_4$,

(3) $r_0r_4r_3r_4 - r_3r_0r_4r_3$,

(4) $r_0r_4r_3r_2r_3 - r_2r_0r_4r_3r_2$,

(5) $r_0r_4r_3r_0 - r_4r_0r_4r_3$,
$r_0r_4r_3r_2r_0 - r_4r_0r_4r_3r_2$,

(6) $r_0r_4r_3r_1r_0r_1 - r_4r_0r_4r_3r_1r_0$,
$r_0r_4r_3r_1r_0r_2r_1r_2 - r_4r_0r_4r_3r_1r_0r_2r_1$,
$r_0r_4r_3r_1r_0r_2r_1r_3r_2r_3 - r_4r_0r_4r_3r_1r_0r_2r_1r_3r_2$,
$r_0r_4r_3r_1r_0r_2r_1r_3r_2r_4r_3r_4 - r_4r_0r_4r_3r_1r_0r_2r_1r_3r_2r_4r_3$,
$r_0r_4r_3r_2r_1r_0r_1 - r_4r_0r_4r_3r_2r_1r_0$,
$r_0r_4r_3r_2r_1r_0r_2r_1r_2 - r_4r_0r_4r_3r_2r_1r_0r_2r_1$,
$r_0r_4r_3r_2r_1r_0r_2r_1r_3r_2r_3 - r_4r_0r_4r_3r_2r_1r_0r_2r_1r_3r_2$,
$r_0r_4r_3r_2r_1r_0r_2r_1r_3r_2r_4r_3r_4 - r_4r_0r_4r_3r_2r_1r_0r_2r_1r_3r_2r_4r_3$,

(7) $r_0r_4r_3r_1r_0r_4r_1 - r_4r_0r_4r_3r_1r_0r_4$,
$r_0r_4r_3r_2r_1r_0r_4r_1 - r_4r_0r_4r_3r_2r_1r_0r_4$,
$r_0r_4r_3r_2r_1r_0r_4r_3r_1 - r_4r_0r_4r_3r_2r_1r_0r_4r_3$,

(8) $r_0r_4r_3r_1r_0r_4r_2r_1r_2 - r_4r_0r_4r_3r_1r_0r_4r_2r_1$,
$r_0r_4r_3r_1r_0r_4r_2r_1r_3r_2r_3 - r_4r_0r_4r_3r_1r_0r_4r_2r_1r_3r_2$,
$r_0r_4r_3r_1r_0r_4r_2r_1r_3r_2r_4r_3r_4 - r_4r_0r_4r_3r_1r_0r_4r_2r_1r_3r_2r_4r_3$,
$r_0r_4r_3r_2r_1r_0r_4r_2r_1r_2 - r_4r_0r_4r_3r_2r_1r_0r_4r_2r_1$,
$r_0r_4r_3r_2r_1r_0r_4r_2r_1r_3r_2r_3 - r_4r_0r_4r_3r_2r_1r_0r_4r_2r_1r_3r_2$,
$r_0r_4r_3r_2r_1r_0r_4r_2r_1r_3r_2r_4r_3r_4 - r_4r_0r_4r_3r_2r_1r_0r_4r_2r_1r_3r_2r_3r_4$,
$r_0r_4r_3r_2r_1r_0r_4r_3r_2r_1r_2 - r_4r_0r_4r_3r_2r_1r_0r_4r_3r_2r_1$,
$r_0r_4r_3r_2r_1r_0r_4r_3r_2r_1r_3r_2r_3 - r_4r_0r_4r_3r_2r_1r_0r_4r_3r_2r_1r_3r_2$,
$r_0r_4r_3r_2r_1r_0r_4r_3r_2r_1r_3r_2r_4r_3r_4 - r_4r_0r_4r_3r_2r_1r_0r_4r_3r_2r_1r_3r_2r_4r_3$,

(9) $r_0r_4r_1r_0r_1 - r_4r_0r_4r_1r_0$,
$r_0r_4r_1r_0r_2r_1r_2 - r_4r_0r_4r_1r_0r_2r_1$,
$r_0r_4r_1r_0r_2r_1r_3r_2r_3 - r_4r_0r_4r_1r_0r_2r_1r_3r_2$,
$r_0r_4r_1r_0r_2r_1r_3r_2r_4r_3r_4 - r_4r_0r_4r_1r_0r_2r_1r_3r_2r_4r_3$,

(10) $r_0r_1r_4r_0r_4 - r_1r_0r_1r_4r_0$,
$r_0r_1r_2r_4r_0r_4 - r_1r_0r_1r_2r_4r_0$,
$r_0r_1r_2r_3r_4r_0r_4 - r_1r_0r_1r_2r_3r_4r_0$,

(11) $r_0r_1r_4r_3r_0r_4r_3 - r_1r_0r_1r_4r_0r_3r_4$,
$r_0r_1r_2r_4r_3r_0r_4r_3 - r_1r_0r_1r_2r_4r_0r_3r_4$,
$r_0r_1r_2r_3r_4r_3r_0r_4r_3 - r_1r_0r_1r_2r_3r_4r_0r_3r_4$,

(12) $r_0r_1r_4r_3r_2r_0r_4r_3r_2 - r_1r_0r_1r_4r_0r_3r_4r_2r_3$,
$r_0r_1r_2r_4r_3r_2r_0r_4r_3r_2 - r_1r_0r_1r_2r_4r_0r_3r_4r - 2r_3$,
$r_0r_1r_2r_3r_4r_3r_2r_0r_4r_3r_2 - r_1r_0r_1r_2r_3r_4r_0r_3r_4r_2r_3$.

By Theorems 1.1, 3.2 and identifications $r_i \leftrightarrow y_{i+1}$ $(0 \leq i \leq 4)$, if we compare the rewriting system and the Gröbner-Shirshov basis for $\widetilde{A_4}$, then we see that elements in the set (2.1) and elements in Corollary 3.3 coincide with each other.

Acknowledgment

The authors would like to thank referee for her/his constructive suggestions which significantly improved the quality of presentation.

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Ring and Module Theory
Trends in Mathematics, 83–90

A Classification of a Certain Class of Completely Primary Finite Rings

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Abstract. We consider the isomorphism problem of a class of completely primary finite rings R such that if $\mathcal{J}$ is the Jacobson radical of R, then $\mathcal{J}^3 = (0)$ and $\mathcal{J}^2 \neq (0)$, in the general case, not necessarily the case in which the maximal Galois subrings lie in the center. We further obtain the number of non-isomorphic classes in some special case of these rings.

Mathematics Subject Classification (2000). Primary 16P10; Secondary 13H99.

Keywords. Completely primary finite rings, unital modules, galois rings.

1. Introduction

A ring R is *completely primary* if the subset $\mathcal{J}$ of all its zero-divisors forms an ideal. These rings have been studied extensively by, among others, Raghavendran [5].

In this paper, we seek an explicit description of the isomorphism classes of a completely primary finite ring R of characteristic p, with Jacobson radical $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$, the annihilator of $\mathcal{J}$ contains the ideal $\mathcal{J}^2$ and $R/\mathcal{J} \cong GF(p^r)$, the finite field of p^r elements, for any prime p and any positive integer r. We leave the cases when the characteristic of R is p^2 or p^3 for future consideration. These rings were studied by the author who gave their constructions for all characteristics, and for details of the general background, the reader is referred to [2] and [3]. In this paper, these rings are given in terms of the basis of their additive groups and the multiplication tables of basis elements. We freely use the definitions and notations introduced in [2], [3] and [5].

The author would like to thank Prof. E.M. Lungu and Prof. D. Theo for inspiring him into this profession.

2. A construction of rings of characteristic p

Let $\mathbb{F}$ be the Galois field $GF(p^r)$. Given three positive integers s, t and λ such that $1 \leq t \leq s^2$, $\lambda \geq 1$, fix $s,\ t,\ \lambda$-dimensional $\mathbb{F}$-spaces $U,\ V,\ W$, respectively. Since $\mathbb{F}$ is commutative we can think of them as both left and right vector spaces. Let $(a_{ij}^k) \in \mathbb{M}_{s\times s}(\mathbb{F})$ $(k = 1, \ldots, t)$ be t linearly independent matrices, $\{\sigma_1, \ldots, \sigma_s\}$, $\{\theta_1, \ldots, \theta_t\}$ and $\{\eta_1, \ldots, \eta_\lambda\}$ be sets of automorphisms of $\mathbb{F}$ (with possible repetitions), and let $\{\sigma_i\}$ and $\{\theta_k\}$ satisfy the additional condition that if $a_{ij}^k \neq 0$, for any k with $1 \leq k \leq t$, then $\theta_k = \sigma_i\sigma_j$.

On the additive group $R = \mathbb{F} \oplus U \oplus V \oplus W$; we select bases $\{u_i\}$, $\{v_k\}$ and $\{w_m\}$ for U, V, and W, respectively, and we define multiplication by the following relations:

$$\begin{aligned} u_iu_j &= \sum_{k=1}^{t} a_{ij}^k v_k, \\ u_iv_k &= v_ku_i = v_kv_l = u_iw_m = w_mu_i = v_kw_m = w_mv_k = w_mw_n = 0, \\ u_i\alpha &= \alpha^{\sigma_i}u_i, \quad v_k\alpha = \alpha^{\theta_k}v_k, \quad w_m\alpha = \alpha^{\eta_m}w_m \\ & \quad (1 \leq i,j \leq s;\ 1 \leq k,l \leq t;\ 1 \leq m,n \leq \lambda)\,; \end{aligned} \tag{2.1}$$

where $\alpha,\ a_{ij}^k \in \mathbb{F}$.

By the above relations, R is a completely primary finite ring of characteristic p with Jacobson radical $\mathcal{J} = U \oplus V \oplus W$, $\text{ann}(\mathcal{J}) = V \oplus W$, $\mathcal{J}^2 = V$ and $\mathcal{J}^3 = (0)$ (see [2] and /or [3]).

Theorem 2.1. *Let R be a ring. Then R is a cube radical zero completely primary finite ring of characteristic p if and only if R is isomorphic to one of the rings given by the above relations.*

Proof. See Theorem 4.1 in [2]. □

3. The isomorphism problem

Let R be a ring of Theorem 2.1 with the linearly independent matrices $A_k = (a_{ij}^k) \in M_s(\mathbb{F})$ $(k = 1, \ldots, t)$ and the automorphisms $\{\sigma_i\}$, $\{\theta_k\}$ and $\{\eta_m\}$. Since $\theta_k = \sigma_i\sigma_j$ if $a_{ij}^k \neq 0$, then up to isomorphism, R is given by t linearly independent matrices $A_k = (a_{ij}^k)$ of size $s \times s$, and the automorphisms $\{\sigma_i\}$ and $\{\eta_m\}$.

Let $\mathcal{A} = \{A_k : k = 1, \ldots, t\}$, and denote the ring R by $R(\mathcal{A},\ \sigma_i,\ \eta_m)$. We call A_k the *structural matrices* of the ring $R(\mathcal{A},\ \sigma_i,\ \eta_m)$. We also recall that if $|R(\mathcal{A},\ \sigma_i,\ \eta_m)| = p^{nr}$, the integers $p,\ n,\ r,\ s,\ t,\ \lambda$ are *invariants* of $R(\mathcal{A},\ \sigma_i,\ \eta_m)$.

We take this opportunity to introduce the symbols M^σ to denote $(\sigma(a_{ij}))$ and M^{σ_j} to denote $(\sigma_1(a_{i1}),\ \sigma_2(a_{i2}),\ \ldots,\ \sigma_t(a_{it}))$, for some automorphisms σ_j, not necessarily distinct, if $M = (a_{ij})$.

Let $R(\mathcal{A},\ \sigma_i,\ \eta_m)$ and $R'(\mathcal{D},\ \sigma_i',\ \eta_m')$ be rings of Theorem 2.1 of the same characteristic p and with the same invariants $p,\ n,\ r,\ s,\ t,\ \lambda$. Also, let us assume that they are constructed from a common coefficient subring $\mathbb{F}$ with associated automorphisms $\sigma_i,\ \eta_m$ and $\sigma_i',\ \eta_m'$, respectively.

3.1. Preliminary results

To determine the isomorphism classes, we first show that the Galois subfield $\mathbb{F} = GF(p^r)$ is invariant under any isomorphism $\phi : R(\mathcal{A},\ \sigma_i,\ \eta_m) \longrightarrow R'(\mathcal{D},\ \sigma_i',\ \eta_m')$. Then we compute the image of the rest of the generators, by a fixed isomorphism. Let U, V and W be the $\mathbb{F}$-vector spaces generated by $\{u_1,\ \dots,\ u_s\}$, $\{v_1,\ \dots,\ v_t\}$ and $\{w_1,\ \dots,\ w_\lambda\}$, respectively. By (2.1), the set $\{u_1,\ \dots,\ u_s\}$ is an $\mathbb{F}$-basis of the vector space $\mathcal{J}/\operatorname{ann}(\mathcal{J}) \cong U$, the set $\{u_iu_j : 1 \le i, j \le s\}$ generates the vector space $V \cong \mathcal{J}^2$ over $\mathbb{F}$ and the set $\{w_1,\ \dots,\ w_\lambda\}$ is a basis for the $\mathbb{F}$-vector space $\operatorname{ann}(\mathcal{J})/\mathcal{J}^2 \cong W$.

Lemma 3.1. *Let $\phi : R(\mathcal{A},\ \sigma_i,\ \eta_m) \longrightarrow R'(\mathcal{D},\ \sigma_i',\ \eta_m')$ be an isomorphism. Then $\phi(\mathbb{F})$ is a maximal subfield of $R'(\mathcal{D},\ \sigma_i',\ \eta_m')$ which is equal to $\mathbb{F}$.*

Proof. It is obvious that $\phi(\mathbb{F})$ is a maximal subfield of $R'(\mathcal{D},\ \sigma_i',\ \eta_m')$ so that there exists an invertible element $x \in R'(\mathcal{D},\ \sigma_i',\ \eta_m')$ such that $x\phi(\mathbb{F})x^{-1} = \mathbb{F}$.

Now, consider the map $\psi : R(\mathcal{A},\ \sigma_i,\ \eta_m) \longrightarrow R'(\mathcal{D},\ \sigma_i',\ \eta_m')$ given by $r \longmapsto x\phi(r)x^{-1}$. Then, clearly ψ is an isomorphism from $R(\mathcal{A},\ \sigma_i,\ \eta_m)$ to $R'(\mathcal{D},\ \sigma_i',\ \eta_m')$ which sends $\mathbb{F}$ to itself. □

Let R be the ring given by the multiplication in (2.1) with respect to the linearly independent matrices $A_k = (a_{ij}^k) \in \mathbb{M}_{s\times s}(\mathbb{F})$ $(k = 1,\ \dots,\ t)$ and associated automorphisms $\{\sigma_i\}$, $\{\theta_k\}$ and $\{\eta_m\}$. Then

$$R = \mathbb{F} \oplus \sum_{i=1}^{s} \mathbb{F}u_i \oplus \sum_{k=1}^{t} \mathbb{F}v_k \oplus \sum_{m=1}^{\lambda} \mathbb{F}w_m;$$

and $u_ir_o = r_o^{\sigma_i}u_i$, $v_kr_o = r_o^{\theta_k}v_k$, $w_mr_o = r_o^{\eta_m}w_m$, for every $r_o \in \mathbb{F}$.

From now on, we simplify our notation by denoting the rings $R(\mathcal{A},\ \sigma_i,\ \eta_m)$ and $R'(\mathcal{D},\ \sigma_i',\ \eta_m')$ by R and R', respectively.

Proposition 3.2. *Let $\phi : R \longrightarrow R'$ be an isomorphism. Then for each $i = 1,\ \dots,\ s$; each $k = 1,\ \dots,\ t$; and each $m = 1,\ \dots,\ \lambda$;*

$$\phi(u_i) = \sum_{\sigma_j=\sigma_i} a_{ji}u_j' + \sum_{\theta_k=\sigma_i} b_{ki}v_k' + \sum_{\eta_m=\sigma_i} c_{mi}w_m';$$

$$\phi(w_m) = \sum_{\theta_l=\eta_m} d_{lm}v_l' + \sum_{\eta_n=\eta_m} e_{nm}w_n';$$

and

$$\phi(v_\mu) = \sum_{\theta_\nu=\theta_\mu} f_{\nu\mu}v_\nu',$$

where $a_{ji},\ b_{ki},\ c_{mi},\ d_{lm},\ e_{nm},\ f_{\nu\mu} \in \mathbb{F}$. In particular, if $b_{ki} \neq 0$ and $c_{mi} \neq 0$, then $\sigma_i = id_{\mathbb{F}}$, and if $d_{lm} \neq 0$, then $\theta_l = \eta_m$.

Proof. Since

$$u_i \in \mathcal{J} = \oplus\sum_{j=1}^{s}\mathbb{F}u_j \oplus \sum_{k=1}^{t}\mathbb{F}v_k \oplus \sum_{m=1}^{\lambda}\mathbb{F}w_m, \quad \text{for all } i = 1, \ldots, s;$$

$$w_m \in \text{ann}(\mathcal{J}) = \oplus\sum_{l=1}^{t}\mathbb{F}v_l \oplus \sum_{n=1}^{\lambda}\mathbb{F}w_n, \quad \text{for all } m = 1, \ldots, \lambda;$$

and

$$v_k \in \mathcal{J}^2 = \oplus\sum_{\mu=1}^{t}\mathbb{F}v_\mu, \quad \text{for all } k = 1, \ldots, t;$$

we can write

$$\phi(u_i) = \sum a_{ji}u'_j + \sum b_{ki}v'_k + \sum c_{mi}w'_m;$$
$$\phi(w_m) = \sum d_{lm}v'_l + \sum e_{nm}w'_n;$$

and

$$\phi(v_\mu) = \sum f_{\nu\mu}v'_\nu,$$

where $a_{ji},\ b_{ki},\ c_{mi},\ d_{lm},\ e_{nm},\ f_{\nu\mu} \in \mathbb{F}$. Now, let $r_o \in \mathbb{F}$ such that $u_i r_o = r_o^{\sigma_i}u_i$, $w_m r_o = r_o^{\eta_m}w_m$, and $v_k r_o = r_o^{\theta_k}v_k$. Then

$$\phi(u_i r_o) = \phi(r_o^{\sigma_i}u_i) = \phi(r_o^{\sigma_i})\phi(u_i) = \phi(r_o^{\sigma_i})[\sum a_{ji}u'_j + \sum b_{ki}v'_k + \sum c_{mi}w'_m].$$

On the other hand,

$$\begin{aligned}\phi(u_i r_o) = \phi(u_i)\phi(r_o) &= [\sum a_{ji}u'_j + \sum b_{ki}v'_k + \sum c_{mi}w'_m]\phi(r_o)\\ &= \sum a_{ji}[\phi(r_o)]^{\sigma_j}u'_j + \sum b_{ki}[\phi(r_o)]^{\theta_k}v'_k\\ &\quad + \sum c_{mi}[\phi(r_o)]^{\eta_m}w'_m.\end{aligned}$$

Similarly

$$\phi(r_o^{\eta_m})[\sum d_{lm}v'_l + \sum e_{nm}w'_n] = \sum d_{lm}[\phi(r_o]^{\theta_l}v'_l + \sum e_{nm}[\phi(r_o]^{\eta_n}w'_n$$

and

$$\phi(r_o^{\theta_\mu})[\sum f_{\nu\mu}v'_\nu] = \sum f_{\nu\mu}[\phi(r_o)]^{\theta_\nu}v'_\nu.$$

From these equalities, we deduce that if $\sigma_j \neq \sigma_i$ then $a_{ji} = 0$; if $\eta_m \neq \eta_n$ then $e_{nm} = 0$; and if $\theta_\nu \neq \theta_\mu$ then $f_{\nu\mu} = 0$. In particular, if $b_{ki} \neq 0$ and $c_{mi} \neq 0$ then $\sigma_i = id_{\mathbb{F}}$, since $\theta_k = \sigma_i\sigma_j$ if $a_{ij}^k \neq 0$ and $\text{ann}(\mathcal{J}) \supseteq \mathcal{J}^2$; and if $d_{lm} \neq 0$, then $\theta_l = \eta_m$. □

4. The main result

We now state and prove the main result of this paper.

Proposition 4.1. *Let R and R' be rings of Theorem* 2.1 *with the same invariants p, n, r, s, t, λ. Also, let us assume that they are constructed from a common coefficient subfield $\mathbb{F}$ with associated automorphisms $\{\sigma_i\}$, $\{\eta_m\}$ and $\{\sigma'_i\}$, $\{\eta'_m\}$, respectively. Then $R \cong R'$, if and only if $\{\sigma_i\} = \{\sigma'_i\}$ (for every $i = 1, \ldots, s$), $\{\eta_m\} = \{\eta'_m\}$ (for every $m = 1, \ldots, \lambda$), and there exist $\sigma \in \mathrm{Aut}(\mathbb{F})$, $B = (\beta_{k\rho}) \in GL(t, \mathbb{F})$ and $C \in GL(s, \mathbb{F})$ such that $C^T D_\rho C^{\sigma_i} = \sum_{k=1}^{t} \beta_{k\rho} A_k^\sigma$.*

Proof. Suppose there is an isomorphism $\psi : R \longrightarrow R'$. Then, as in Lemma 3.1, $\psi(\mathbb{F})$ is a maximal subfield of R' so that there exists an invertible element $x \in R'$ such that $x\psi(\mathbb{F})x^{-1} = \mathbb{F}$.

Now, consider the map $\phi : R \longrightarrow R'$ given by $r \longmapsto x\psi(r)x^{-1}$. Then, clearly ϕ is an isomorphism from R to R' which sends $\mathbb{F}$ to itself. Also,

$$\phi(\sum_i \alpha_i u_i) = \sum_j \sum_i \phi(\alpha_i) a_{ji} u'_j + \sum_k \sum_i \phi(\alpha_i) b_{ki} v'_k + \sum_m \sum_i \phi(\alpha_i) c_{mi} w'_m;$$

$$\phi(\sum_m \beta_m w_m) = \sum_l \sum_m \phi(\beta_m) d_{lm} v'_l + \sum_n \sum_m \phi(\beta_m) e_{nm} w'_n;$$

and

$$\phi(\sum_\mu \gamma_\mu v_\mu) = \sum_\nu \sum_\mu \phi(\gamma_\mu) f_{\nu\mu} v'_\nu.$$

Therefore,

$$\begin{aligned}\phi(\sum_i \alpha_i u_i) \cdot \phi(\sum_i \beta_i u_i) &= (\sum_j \sum_i \phi(\alpha_i) a_{ji} u'_j + \sum_k \sum_i \phi(\alpha_i) b_{ki} v'_k \\ &\quad + \sum_m \sum_i \phi(\alpha_i) c_{mi} w'_m) \cdot (\sum_j \sum_i \phi(\beta_i) a_{ji} u'_j \\ &\quad + \sum_k \sum_i \phi(\beta_i) b_{ki} v'_k + \sum_m \sum_i \phi(\beta_i) c_{mi} w'_m) \\ &= \sum_\rho \sum_{\nu,\mu=1}^{s} \sum_{i,j=1}^{s} \phi(\alpha_i) a_{\nu i} [\phi(\beta_j) a_{\mu j}]^{\sigma'_\nu} (a')^\rho_{\nu\mu} v'_\rho.\end{aligned}$$

On the other hand,

$$\begin{aligned}\phi\left((\sum_i \alpha_i u_i) \cdot (\sum_i \beta_i u_i) \right) &= \phi(\sum_k \sum_{i,j=1}^{s} \alpha_i [\beta_j]^{\sigma_i} a_{ij}^k v_k) \\ &= \sum_\rho \sum_{k=1}^{t} \sum_{i,j=1}^{s} \phi(\alpha_i [\beta_j]^{\sigma_i}) \beta_{\rho k} \phi(a_{ij}^k) v'_\rho.\end{aligned}$$

It follows that

$$\sum_{\nu,\mu=1}^{s}\sum_{i,j=1}^{s}\phi(\alpha_i)a_{\nu i}[\phi(\beta_j)a_{\mu j}]^{\sigma'_\nu}(a')^{\rho}_{\nu\mu}=\sum_{k=1}^{t}\sum_{i,j=1}^{s}\phi(\alpha_i[\beta_j]^{\sigma_i})\beta_{\rho k}\phi(a^k_{ij}).$$

Now, the restriction $\phi|_{\mathbb{F}}$, of ϕ to $\mathbb{F}$, is an automorphism σ of $\mathbb{F}$; and therefore, $\phi(a^k_{ij})=\sigma(a^k_{ij})$ and $\sigma'_i=\sigma_i$, and $\eta'_m=\eta_m$, for every $m=1,\ \ldots,\ \lambda$ (by Proposition 3.2 if the coefficient of v'_i in the transformation of w_m under ϕ is non-zero). Hence, the above equation now implies that $C^TD_\rho C^\theta=\sum_{k=1}^{t}\beta_{k\rho}A_k^\sigma$, with $C=(\alpha_{\mu j})$, and $\sigma_i=\theta$, for every $i=1,\ \ldots,\ s$; $\eta'_m=\eta_m$, for every $m=1,\ \ldots,\ \lambda$; and $\sigma'_i=\sigma_i$, for every $i=1,\ \ldots,\ s$, as required.

Conversely, suppose that the associated automorphisms $\{\sigma_i\}=\{\sigma'_i\}$ and $\{\eta_m\}=\{\eta'_m\}$, respectively, for every $i=1,\ \ldots,\ s$ and $m=1,\ \ldots,\ \lambda$, and there exist $\sigma\in\mathrm{Aut}(\mathbb{F})$, $B=(\beta_{k\rho})\in GL(t,\ \mathbb{F})$ and $C\in GL(s,\ \mathbb{F})$ with $C^TD_\rho C^\theta=\sum_{k=1}^{t}\beta_{k\rho}A_k^\sigma$. Consider the map $\phi:R\to R'$ defined by

$$\begin{aligned}\phi(\alpha_o+&\sum_i\alpha_iu_i+\sum_m\beta_mw_m+\sum_\mu\gamma_\mu v_\mu)\\&=\alpha_o^\sigma+\sum_j\sum_i\alpha_i^\sigma a_{ji}u'_j+\sum_k\sum_i\alpha_i^\sigma b_{ki}v'_k+\sum_m\sum_i\alpha_i^\sigma c_{mi}w'_m\\&\quad+\sum_l\sum_m\beta_m^\sigma d_{lm}v'_l+\sum_n\sum_m\beta_m^\sigma e_{nm}w'_m+\sum_\nu\sum_\mu\gamma_\mu^\sigma f_{\nu\mu}v'_\nu.\end{aligned}$$

Then, it is easy to verify that ϕ is an isomorphism from the ring R to the ring R' of Theorem 2.1. □

Thus, the set $\{\theta,\ \sigma,\ \eta_m\in\mathrm{Aut}(\mathbb{F}),\ \ B=(\beta_{k\rho})\in GL(t,\ \mathbb{F}),\ C\in GL(s,\ \mathbb{F})\}$ determines all the isomorphisms classes of the rings of Theorem 2.1.

4.1. A special case

We obtain the number of non-isomorphic classes of rings of Theorem 2.1 in the special case when the invariants $t=s^2,\ \lambda\geq 1$ and when $\sigma_i=\theta$, for some fixed automorphism θ.

Let R be a ring of Theorem 2.1 with the invariants $p,\ n,\ r,\ s,\ t,\ \lambda$, where $t=s^2$ and $\lambda\geq 1$. Let $\sigma_i=\theta$, η_m $(m=1,\ \ldots,\ \lambda)$ be the associated automorphisms of R with respect to a fixed maximal Galois subfield $\mathbb{F}$. We know that the matrices A_k are linearly independent over $\mathbb{F}$. So, let $\mathcal{A}$ denote the subspace of $\mathrm{M}_s(\mathbb{F})$ generated by the matrices A_k over $\mathbb{F}$. Now, the number of elements in the $\mathbb{F}$-space $\mathrm{M}_s(\mathbb{F})$ is q^{s^2}, where $q=p^r$ is the order of $\mathbb{F}$, and the number of different bases for the s^2-dimensional subspaces of the $\mathbb{F}$-space $\mathrm{M}_s(\mathbb{F})$ is

$$(q^{s^2}-1)(q^{s^2}-q)\ \ldots\ (q^{s^2}-q^{s^2-1}).$$

Of course, if we write the elements of a basis in a different order, we get another basis; and this means that the different bases fall into equivalence classes of $s^2!$

bases each, under the action of permuting the elements. As is well known, it follows that

$$\frac{(q^{s^2}-1)(q^{s^2}-q)\ \ldots\ (q^{s^2}-q^{s^2-1})}{s^2!}$$

is an integer, being the number of equivalence classes, that is, the number of *unordered* bases for the $\mathbb{F}$-space $\mathrm{M}_s(\mathbb{F})$.

We recall from [3, Remark 5.10] that if $\mathcal{A}$ is the set of all t-tuples $(A_1,\ \ldots,\ A_t)$ of $s\times s$ matrices over $\mathbb{F}$, the group $GL_s(\mathbb{F})$ acts on $\mathcal{A}$ by "congruence":

$$(A_1,\ \ldots,\ A_t)\cdot C=(C^TA_1C^\theta,\ C^TA_2C^\theta,\ \ldots,\ C^TA_tC^\theta)$$

and on the left via

$$B\cdot(A_1,\ \ldots,\ A_t)=(\beta_{11}A_1^\sigma+\beta_{12}A_2^\sigma+\cdots+\beta_{1t}A_t^\sigma,\ \cdots,\ \beta_{t1}A_1^\sigma+\beta_{t2}A_2^\sigma+\cdots+\beta_{tt}A_t^\sigma),$$

where $B=(\beta_{k\rho})$. Thus, these two actions are permutable and define a (left) action of $G=GL_s(\mathbb{F})\times GL_t(\mathbb{F})$ on $\mathcal{A}$:

$$(C,\ B)\cdot(A_1,\ \ldots,\ A_t)=B\cdot(A_1^\sigma,\ \ldots,\ A_t^\sigma)\cdot(C^{-1})^\theta,$$

for some fixed automorphisms σ and θ. However, since $t=s^2$, obviously, $\mathcal{A}=\mathrm{M}_s(\mathbb{F})$.

Now, if R' is another ring of the same type with the same invariants as R, and associated automorphisms $\sigma_j'=\theta'$ and η_m' over $\mathbb{F}$, and structural matrices D_l, let $\mathcal{D}$ denote the subspace of $\mathrm{M}_s(\mathbb{F})$ generated by the matrices D_l over $\mathbb{F}$. Then, as before, it follows easily that $\mathcal{D}=\mathrm{M}_s(\mathbb{F})$.

But $\mathcal{A}=\mathcal{D}=\mathrm{M}_s(\mathbb{F})$. Moreover, the action of any automorphism σ on the matrices in $\mathcal{A}$ gives another set of linearly independent matrices.

Thus, up to isomorphism, the rings R and R' are determined by the automorphisms $\sigma_i=\theta$, η_m and $\sigma_j'=\theta'$, η_m', respectively.

Next, the number of ways we can select $\{\eta_1,\ \ldots,\ \eta_\lambda\}$ (η_m not necessarily distinct) from $\mathrm{Aut}(\mathbb{F})$ is the number of solutions of the equation

$$x_1+x_2+\ \cdots\ +x_r=\lambda$$

in non-negative integers $x_1,\ \ldots,\ x_r\in\{0,\ 1,\ \ldots,\ \lambda\}$. This number is well known to be

$$\binom{r+\lambda-1}{\lambda};$$

where $r=|\,\mathrm{Aut}(\mathbb{F})|$.

Proposition 4.2. *The number of mutually non-isomorphic rings of Theorem* 2.1 *with the same invariants p, n, r, s, t, λ such that $t=s^2$, $\lambda\geq 1$ and when $\sigma_i=\theta$, for some fixed automorphism θ, is equal to*

$$r\times\binom{r+\lambda-1}{\lambda};$$

where $r=|\,\mathrm{Aut}(\mathbb{F})|$.

Proof. Follows easily from the above discussion. □

Acknowledgment

Many thanks to the organizing committee of the International Conference on Ring and Module Theory, 2008, for the invitation and encouragement to submit this paper for inclusion in the proceedings. The author would also like to thank the referee for the valuable and careful comments.

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Ring and Module Theory
Trends in Mathematics, 91–114

Higman Ideals and Verlinde-type Formulas for Hopf Algebras

Miriam Cohen and Sara Westreich

Abstract. We offer a comprehensive discussion on Verlinde-type formulas for Hopf algebras H over an algebraically closed field of characteristic 0. Some of the results are new and some are known, but are reproved from the point of view of symmetric algebras and the associated Higman (trace) map. We give an explicit form for the central Casimir element of $C(H)$, which is also known to be χ_{ad}, the character of the adjoint map on H. We then discuss the following variations of the Verlinde formula: (i) Fusion rules for irreducible characters of semisimple Hopf algebras whose character algebras $C(H)$ are commutative. (ii) Structure constants for what we call here conjugacy sums associated to conjugacy classes for these Hopf algebras. (iii) Equality up to rational scalar multiples between the fusion rules of irreducible characters and the structure constants for semisimple factorizable Hopf algebras. (iv) Projective fusion rules for the multiplication of irreducible and indecomposable projective characters for non-semisimple factorizable Hopf algebras.

Mathematics Subject Classification (2000). Primary 16W30, 17W35.

Keywords. Hopf algebras, symmetric algebras, factorizable ribbon Hopf algebras, conjugacy sums and classes, characters, projective center, fusion rules, Verlinde formula.

Introduction

The representation and character theory of semisimple Hopf algebras over an algebraically closed field of characteristic zero has been developed since the 70s, in many cases analogously to the classical theory of finite groups. In this paper we prove new results and also offer a comprehensive discussion of some known results which are reproved using the point of view of symmetric algebras and the associated Higman (trace) map. Symmetric algebras are abundant. Finite group algebras over any field, finite-dimensional semisimple algebras, and more generally Calabi-Yau algebras and Iwahori-Hecke algebras are all examples.

We use the theory of symmetric algebras to prove theorems which we call Verlinde-type formulas. These formulas can be viewed as instances of table algebras (see [1] for a comprehensive survey). The Verlinde formula for semisimple fusion algebras (or categorically for modular tensor categories), can be formulated as '**The matrix S diagonalizes the fusion rules.**' It was proved graphically by various authors. In [4] we used the so-called quantum Fourier transform to prove it in an algebraic setting. In this paper we extend this formula to semisimple Hopf algebras whose character (representation) algebra is commutative in two ways analyzing both the "fusion rules" and the "structure constants" adequately adapted from group theory. We extend these to twistings of Hopf algebras.

Our best results are when H is also factorizable, for then we show that the fusion rules and the structure constants are equal up to rational scalar multiples.

Much less is known about representations and characters of general finite-dimensional Hopf algebras. While the building blocks in the representation theory of finite-dimensional semisimple algebras are the irreducible modules, in the non-semisimple case, a major role is also played by indecomposable projective modules. Here we give a brief summary of a Verlinde-type formula for factorizable Ribbon Hopf algebras proved in [5].

The paper is organized as follows. In Section 1 we study symmetric algebras A and their associated Higman (trace) map τ given by

$$\tau(a) = \sum a_i a b_i$$

where $\{a_i, b_i\}$ are dual bases with respect to the symmetric bilinear form on A. Of special importance are the central Casimir elements $\tau(1) = \sum a_i b_i$. We investigate some interrelations connecting the image under the map τ of primitive idempotents and central primitive idempotents of A. We use these in Section 2 where we focus on the character algebras of semisimple Hopf algebras H over an algebraically closed field of characteristic 0. In this situation both H and its character algebra $C(H)$ are symmetric algebras. We use the primitive idempotents of $C(H)$ and the class equation adapted to our situation we give an explicit form of the central Casimir element of $C(H)$, which is also known to be χ_{ad}, the character of the adjoint map on H.

Let $\{\chi_i\}$ be a complete set of irreducible characters for H. Let $\{E_i\}$ be a complete set of central primitive idempotents for $C(H)$ and let $\{e_i\}$ be the corresponding set of primitive idempotents such that $e_i E_i = e_i$ for all i. We have:

Theorem 2.2. *Let H be a semisimple Hopf algebra over an algebraically closed field k. Then*

$$\chi_{ad} = \sum_{j=1}^{n} \chi_j s(\chi_j) = \sum_{i=1}^{d} n_i E_i$$

where $n_i = \frac{\dim(C(H)e_i)\dim(H^*)}{\dim(H^* e_i)} \in \mathbb{Z}$ for all i.

In Section 3 we assume in addition that $C(H)$ is commutative. We note that in this case H is necessarily quasitriangular, hence we are in the situation described by Witherspoon [28] (see Remark 3.10). Some of our results appear there, but our approach is different.

We first show a Verlinde type formula for the fusion rules $\mathbf{M^i} = (m^i_{jk})$, where $\chi_i\chi_j = \sum_k m^i_{jk}\chi_k$.

Theorem 3.1. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic zero. Assume $C(H)$ is commutative. Let $A = (\alpha_{ij})$ be the change of basis matrix from the full set of primitive idempotents $\{E_j\}_{1\le j\le n}$ to the full set of irreducible characters $\{\chi_i\}_{1\le i\le n}$. Then*

(i) *A diagonalizes the fusion rules. That is, for each i, $\mathbf{D^i} = A^{-1}\mathbf{M^i}A$.*

(ii) *$\chi_i E_j = \alpha_{ij}E_j$ and thus $\{E_j\}$ is a basis of eigenvectors for $l(\chi_i)$ for all i.*

(iii) *All α_{ij} are algebraic integers.*

(iv) *Let $A^{-1} = (\beta_{jk})$. Then $\beta_{jk} = n_j^{-1}\alpha_{k^*j}$.*

(v) *The integers m^i_{js} satisfy*

$$m^i_{js} = \sum_k \frac{\alpha_{ik}\alpha_{jk}\alpha_{s^*k}}{n_k}.$$

For semisimple Hopf algebras, $C(H)$ and the center of H, $Z(H)$, are dual vector spaces. The set of central primitive idempotents of H, $\{F_i\}$, can be obtained up to a scalar multiple as a dual basis to $\{\chi_i\}$ by taking $\Lambda \leftharpoonup s(\chi_i)$ where Λ is an integral for H. In the same spirit class sums and conjugacy classes can be obtained from the central primitive idempotents of $C(H)$. Namely:

Definition 3.3. Let H be a semisimple Hopf algebra with an integral Λ so that $\varepsilon(\Lambda) = 1$. Let E_i be a central primitive idempotent of $C(H)$. Define the i-th **class sum** C_i by:

$$C_i = dE_i \rightharpoonup \Lambda$$

where $d = \dim(H)$. Define the **Conjugacy class** $\mathcal{C}_i$ by:

$$\mathcal{C}_i = H^*E_i \rightharpoonup \Lambda = \text{The right coideal of } H \text{ generated by } C_i.$$

Like in groups we show in Corollary 3.5 that $\mathcal{C}_i$ is stable under the adjoint action by cocommutative elements of H.

Using this approach we prove in Proposition 3.7 a generalization of Littlewood's formula for groups and compute the structure constants of the multiplication of class sums in the spirit of the general Verlinde formula, now taking place in the center of H.

Let $\widehat{\mathbf{D}}^{\mathbf{i}} = \frac{d}{n_i}\mathrm{diag}\{\frac{\alpha_{1i}}{d_1}, \frac{\alpha_{2i}}{d_2}, \dots, \frac{\alpha_{ni}}{d_n}\}$. Let $\widehat{\mathbf{M}^{\mathbf{i}}} \in \mathrm{Mat}_n(k)$ be the matrix of $l(C_i)$ with respect to the basis $\{C_j\}$. Let $\widehat{A} = (\widehat{\alpha}_{ij})$ be the change of basis matrix from $\{C_i\}$ to $\{F_j\}$. We prove:

Theorem 3.8. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic zero. Assume $C(H)$ is a commutative algebra. Let $\{E_i\}$ be a full set of primitive idempotents of $C(H)$ and let $\{F_i\}$ be a full set of primitive idempotents in $Z(H)$. Let $C_i = dE_i \rightharpoonup \Lambda$. Then:*

(i) *$\widehat{A}$ diagonalizes the structure constants. That is, for each i,*

$$\widehat{\mathbf{D}^{\mathbf{i}}} = \widehat{A}^{-1}\widehat{\mathbf{M}^{\mathbf{i}}}\widehat{A}.$$

(ii) *$C_iF_j = \widehat{\alpha}_{ij}F_j$ and thus $\{F_j\}$ is a basis of eigenvectors for $l(C_i)$ for all i.*

(iii) *Let $\widehat{A}^{-1} = (\widehat{\beta}_{kt})$. Then $\widehat{\beta}_{kt} = \frac{d_k}{d}\alpha_{k^*t}$.*

(iv) *$C_iC_j = \sum_t c^i_{jt}C_t$ where c^i_{jt} satisfies*

$$c^i_{jt} = \frac{d}{n_in_j}\sum_k \frac{\alpha_{ki}\alpha_{kj}\alpha_{k^*t}}{d_k}.$$

We end this section by showing

Corollary 3.12. *Let H be as in Theorem 3.1. Then class sums and all the structure constants are invariant under dual Hopf cocycles J.*

In Section 4 we specialize to semisimple factorizable Hopf algebras. Using the Drinfeld map we get the sharpest connection:

Theorem 4.3. *Let (H,R) be a factorizable semisimple Hopf algebra. Let χ_i, C_j, d_i, α_{ij}, c^t_{ij}, m^i_{jt} be as in Theorem 3.1 and Theorem 3.8. Then the structure constants c^i_{jt} are given by:*

$$c^i_{jt} = \frac{d_id_j}{d_t}m^i_{jt}.$$

If we omit the semisimplicity assumption and assume that H is only a symmetric algebra we no longer consider the whole $Z(H)$, but the Higman ideal, sometimes called the projective center since it is isomorphic via a translated Fourier transform to $I(H) \subset C(H)$. The reason for the second name is that $I(H)$ is spanned over k by the set of all characters of finitely generated projective H-modules. We explain the setup and repeat our Verlinde-type formula for this situation:

Let $(H,\mathcal{R},v)$ be a factorizable ribbon Hopf algebra, let $u = \sum(SR^2)R^1$, and $G = u^{-1}v$. Set for all $p \in H^*$, $\mathbf{f}_Q(p) = \sum \langle p, R^2r^1\rangle R^1r^2$ and $\widehat{\mathbf{f}}_Q(p) = \mathbf{f}_Q(p \leftharpoonup G)$. Define also for $a \in H$, $\widehat{\Psi}(a) = \lambda \leftharpoonup S^{-1}(aG)$. Let $\{V_i\}$ be a full set of non-isomorphic irreducible left A-modules of corresponding dimensions d_i, and let $\{f_i\}$ be the corresponding orthogonal primitive idempotents. Let C be the Cartan matrix of H of rank r and let $\mathbf{C}_r$ be the invertible $r \times r$ minor of C. We show:

Theorem 4.4. *Let $(H,\mathcal{R},v)$ be a factorizable ribbon Hopf algebra over an algebraically closed field of characteristic 0. Let $s_{ij} = \left\langle \widehat{\mathbf{f}}_Q(\chi_i), s(\chi_j)\right\rangle$ and let $\mathbf{Q} = (q_{kl}) = \left\langle f_l, \widehat{\Psi}\widehat{\mathbf{f}}_Q(p_{f_k})\right\rangle$. Then $\mathbf{F} = \mathbf{Q}^{-1}\mathbf{C}_r$ diagonalizes the "projective fusion rules" $\mathbf{N}^i$. That is, for all $1 \le i \le n$,*

(i) $\mathbf{F}^{-1}\mathbf{N}^i\mathbf{F} = \mathrm{Diag}\{d_1^{-1}s_{i1}, \ldots, d_r^{-1}s_{ir}\}$.

(ii) *Eigenvectors for* $L_{\widehat{\mathbf{f}}_Q(\chi_i)}$ *are* $\sum \Lambda_1 GS(f_j)S(\Lambda_2)$*, with corresponding eigenvalues* $d_j^{-1}s_{ij}$*.*

(iii) $\{d_j^{-1}s_{ij}\}$ *are algebraic integers.*

Throughout the base field k is assumed to be algebraically closed of characteristic zero.

For each finite-dimensional left A-module V denote its structure map $A \to \mathrm{End}_k(V)$ by ρ_V. Then the character χ_V of V is defined by

$$\chi_V(a) = \mathrm{Trace}(\rho_V(a)).$$

Note that $\chi_V(1) = \dim_k(V)$. We denote by χ_A the character of the left (right) regular representation. That is $V = A$ and the representation is given by $\rho(a) = l(a) =$ left multiplication by a.

1. Frobenius and symmetric algebras

Our basic reference for symmetric algebras will be the paper of Broue [2] who gives a comprehensive presentation of this subject. In what follows we integrate some of the crucial ingredients in the study of symmetric or Frobenius algebras to give a self contained exposition and some new results.

Throughout this paper we regard A^*_A $({}_AA^*)$ as a right (left) A-module via the transpose of multiplication on A as follows:

$$\langle p \leftharpoonup a, b\rangle = \langle p, ab\rangle \qquad \langle a \rightharpoonup p, b\rangle = \langle p, ba\rangle$$

where $a, b \in A$, $p \in A^*$.

Definition 1.1 (e.g., [8, p. 197]). An n-dimensional algebra A is a Frobenius algebra if $A_A \cong A^*_A$ as right A-modules. Equivalently, ${}_AA \cong_A A^*$ as left A-modules. Denote the A-module isomorphism by ϕ. The following are equivalent:

(i) There exists an associative bilinear form $\beta : A \otimes A \to k$ which is non-degenerate. The form β is given by $\beta(a,b) = \langle \phi(a), b\rangle$.

(ii) There exists $t \in A^*$ so that the map ϕ given by $\phi(a) = (t \leftharpoonup a)$ is a right A-modules isomorphism. Equivalently, $a \mapsto (a \rightharpoonup t)$ is a left A-modules isomorphism. t is defined by $t = \phi(1)$ and A^* is a free A-module with a basis t. Conversely, given t, then the A-module map ϕ is determined by $\phi(1) = t$.

(iii) There exist $t \in A^*$, $a_i, b_i \in A$, $i = 1, \ldots, n$, such that for all $x \in A$,

$$x = \sum a_i \langle t, b_i x\rangle \tag{1}$$

We say that $\{a_i, b_i\}$ form dual bases for the form β. In this case $\{t \leftharpoonup b_i, a_i\}$ are standard dual bases of A^* and A. Equivalently,

$$x = \sum \langle t, xa_i\rangle\, b_i$$

and thus $\{a_i \rightharpoonup t, b_i\}$ are dual bases of A^* and A respectively.

The following is well known:

Example 1.2. Any finite-dimensional Hopf algebra H is a Frobenius algebra as follows: Let Λ be a left integral in H, then $t = \lambda$, where λ is a right integral in H^* so that $\langle \lambda, \Lambda \rangle = 1$. The dual bases with respect to t are $\{S(\Lambda_2), \Lambda_1\}$.

For any finite-dimensional vector space V, $V^* \otimes V \cong \mathrm{End}_k(V)$ via $(p \otimes v)(v') = v \langle p, v' \rangle$ for all $v, v' \in V$, $p \in V^*$. In the opposite direction, $f \in \mathrm{End}_k(V)$ corresponds to $\sum_i v_i^* \otimes f(v_i)$ for dual bases of V^* and V. Recall $\mathrm{Tr}(p \otimes v) = \langle p, v \rangle$.

As a result of the above there is a general trace formula for Frobenius algebras. This generalizes Radford trace formula for Hopf algebras.

Lemma 1.3. *Let (A, t) be a Frobenius algebra with dual bases $\{a_i, b_i\}$, and let $f \in \mathrm{End}_k(A)$, then*

$$\mathrm{Tr}(f) = \sum \langle t, f(b_i)a_i \rangle = \sum \langle t, b_i f(a_i) \rangle .$$

Proof. As in Def. 1.1.(iii), let $\{a_i \rightharpoonup t, b_i\}$ be dual bases of A^* and A resp. Express

$$f = \sum b_i^* \otimes f(b_i) = \sum (a_i \rightharpoonup t) \otimes f(b_i).$$

Then $\mathrm{Tr}(f) = \sum \langle (a_i \rightharpoonup t), f(b_i) \rangle = \sum \langle t, f(b_i)a_i \rangle$. The second equality follows if one takes $\{t \leftharpoonup b_i, a_i\}$ as dual bases of A^* and A resp. □

Remark 1.4. For a Frobenius algebra (A, t), $A \otimes A \cong \mathrm{End}_k(A)$ by $a \otimes b \mapsto (a \rightharpoonup t) \otimes b \in A^* \otimes A$. It follows that for any dual bases $\{a_i, b_i\}$ the element $\sum a_i \otimes b_i$ is mapped to Id_A by (1). Thus $\sum a_i \otimes b_i$ is independent of the dual bases. It is called the **Casimir element** of (A, t).

The following lemma appears as a remark in [25], it follows from Remark 1.4.

Lemma 1.5. *Let (A, t) be a Frobenius algebra with dual bases $\{a_i, b_i\}$, then in $A \otimes A$ we have the identity*

$$\sum x a_i \otimes b_i = \sum a_i \otimes b_i x$$

for all $x \in A$.

Let (A, t) be as above, define the Higman (trace) map $\tau : A \to A$ by

$$\tau(a) = \sum a_i a b_i \tag{2}$$

for all $a \in A$.

Lemma 1.6. *Let (A, t) be a Frobenius algebra with dual bases $\{a_i, b_i\}$ and let τ be the map given in (2). Then $\tau(a) \in Z(A)$ for all $a \in A$.*

Proof. For all $a, x \in A$ we have by Lemma 1.5

$$\sum x a_i \otimes a \otimes b_i = \sum a_i \otimes a \otimes b_i x.$$

Multiplying yields $\sum x a_i a b_i = \sum a_i a b_i x$, hence the result follows. □

In particular the element

$$\tau(1) = \sum a_i b_i$$

is called the **central Casimir** element of (A, t).

For each $a \in A$ denote by $r(a), l(a)$ right and left multiplication by a. Define $p_a \in A^*$ by

$$\langle p_a, b\rangle := \mathrm{Tr}(l(b) \circ r(a)) \tag{3}$$

for all $b \in A$.

Remark 1.7.

(i) For a fixed non-idempotent a, the correspondence $b \mapsto l(b) \circ r(a) : A \to A$ is not multiplicative, while for an idempotent e it is. In this case, $\rho(b) = l(b) \circ r(e)$ is a representation of A. In fact, ρ is the representation given by left multiplication on Ae. Thus $p_e = \chi_{Ae}$.

(ii) Since $\langle \chi_e, 1\rangle = \dim(Ae)$ for any idempotenet e, it follows that if $\mathrm{char}(k) = 0$ then $p_e \neq 0$. This is not necessarily true for positive characteristic. It may very well happen that $p_a = 0$ for all $a \in A$ (see, e.g., [5, Ex.1.4].

Definition 1.8. Let (A, t) be a Frobenius algebra. If

$$\langle t, ab\rangle = \langle t, ba\rangle$$

for all $a, b \in A$, then t is called a central form and (A, t) is called a symmetric algebra.

The following are basic examples of symmetric algebras over any field k.

Example 1.9.

1. The group algebra kG, where k is a field and G is a finite group is a motivating example for both Hopf algebras and symmetric algebras. The set $\{g\}_{g\in G}$ is a (standard) k-basis for kG, and as is well known, its k-dual is also a Hopf algebra with a dual basis $\{\pi_g\}_{g\in G}$, where π_g is the projection into the g-component. The symmetric form is defined by π_1, the projection onto $k1$. Dual bases for this form are $\{g, g^{-1}\}_{g\in G}$ and the central Casimir element is $|G| \cdot 1$.
2. The simple algebra $M_n(k)$. The central form is the usual trace of a matrix and dual bases are $\{e_{ij}, e_{ji}\}$ where e_{ij} are the matrix units. The central Casimir element is nI. Similarly, any finite-dimensional semisimple algebra is a symmetric algebra [11].
3. Any finite-dimensional unimodular Hopf algebra so that S^2 is an inner automorphism [22].

When (A, t) is a symmetric algebra then the isomorphism ϕ is a left and right A-module map and the form β is symmetric. We then have $\sum a_i \langle t, b_i x\rangle = \sum a_i \langle t, x b_i\rangle$. Hence if $\{a_i, b_i\}$ are dual bases, so are $\{b_i, a_i\}$. Remark 1.4 implies the following:

Corollary 1.10. *Let* (A, t) *be a symmetric algebra, then*

$$\sum a_i \otimes b_i = \sum b_i \otimes a_i.$$

The following appears in a different context in [2].

Proposition 1.11. *Let* (A, t) *be a symmetric algebra with dual bases* $\{a_i, b_i\}$, *then for all* $a \in A$,

$$t \leftharpoondown \tau(a) = p_a.$$

Proof. For all $b \in A$, we have by Lemma 1.3:

$$\langle p_a, b \rangle = \mathrm{Tr}(l(b) \circ r(a)) = \sum \langle t, bb_i a a_i \rangle .$$

By Corollary 1.10

$$\sum \langle t, bb_i a a_i \rangle = \langle t, ba_i a b_i \rangle = \sum \langle t, b\tau(a) \rangle = \langle t \leftharpoondown \tau(a), b \rangle ,$$

where the last equality follows since t is a central form. This concludes the proof of the proposition. □

As a corollary we have:

Corollary 1.12. *Let* (A, t) *be as in Proposition* 1.11, *then:*

(i) *For all* $a \in A$,

$$\tau(a) = \sum_i \langle p_a, a_i \rangle b_i.$$

(ii) *Let* e *be an idempotent, then*

$$\tau(e) = \sum_i \langle \chi_{Ae}, a_i \rangle b_i.$$

Moreover,

$$\dim(Ae) = \langle t, \tau(e) \rangle .$$

Proof. (i) This follows from (1) and Lemma.1.3, for

$$\tau(a) = \sum \langle t, \tau(a) a_i \rangle b_i = \sum \langle p_a, a_i \rangle b_i.$$

(ii) The first part follows from (i) since $p_e = \chi_{Ae}$. The second part follows since by Proposition 1.11

$$\langle t, \tau(e) \rangle = \langle p_e, 1 \rangle = \langle \chi_{Ae}, 1 \rangle = \dim(Ae).$$ □

In what follows we investigate some interrelations between traces of idempotents.

Proposition 1.13. *Let* (A, t) *be a symmetric algebra with dual bases* $\{a_i, b_i\}$ *over a field of characteristic zero. Let* e *be a primitive idempotent of* A *and let* E *be the unique central primitive idempotent so that* $Ee = e$. *Then*

(i) $$\tau(e) = \alpha E,\ \alpha \neq 0 \text{ and } \tau(E) = \tau(1)E.$$

When A *is semisimple, then*

(ii) $$\tau(E) = \dim(Ae)\tau(e),$$

(iii) $$\langle p_e, \tau(e)\rangle = \frac{\langle p_e, \tau(1)\rangle}{\dim(Ae)},$$

(iv) $$\alpha = \frac{\langle p_e, \tau(e)\rangle}{\dim(Ae)} = \frac{\langle p_e, \tau(1)\rangle}{\dim(Ae)^2}.$$

Proof. (i) Since $\tau(e)$ is central in A and E is central primitive it follows that $\tau(e) = \tau(e)E = \alpha E$. Moreover, $\tau(e) \neq 0$ by Corollary 1.12(ii). Hence $\alpha \neq 0$.

(ii) If $Ae \cong Ae'$ then $p_e = \chi_{Ae} = \chi_{Ae'} = p_{e'}$, hence by Proposition 1.11 $\tau(e) = \tau(e')$. Now, when A is semisimple then $E = \sum_{i=1}^m e_i$ so that $Ae_i \cong Ae_j$ for all $1 \le i, j \le m$, hence $\tau(E) = \sum_{i=1}^m \tau(e_i)$, and $m = \dim(Ae)$).

(iii) By definition of p_e and since $Ee = e$ we have $p_e \leftharpoonup E = p_e$. By (ii), $\tau(e) = \dim(Ae)^{-1}\tau(1)E$. Thus

$$\langle p_e, \tau(e)\rangle = \dim(Ae)^{-1} \langle p_e, \tau(1)E\rangle = \dim(Ae)^{-1} \langle p_e, \tau(1)\rangle .$$

(iv) Applying p_e to both sides of (i) yields

$$\langle p_e, \tau(e)\rangle = \alpha \langle p_e, E\rangle = \alpha \langle p_e, 1\rangle = \alpha \dim(Ae).$$

The result follows now by using (iii). □

Remark 1.14. Kilmoyer's result (see [8, (9.17)]) about split semisimple algebras A over a field of characteristic zero is a special case of 1.11–1.13. Let $V_j \cong Ae_j$ be an irreducible A-module with corresponding character χ_j. The right-hand side of 1.12(ii) is $\sum_i \langle \chi_j, a_i\rangle b_i$ which is denoted there by d_j is $\tau(e_j)$ in our terms.

2. Character algebras of semisimple Hopf algebras

The study of Hopf algebras as Frobenius algebras and sometimes symmetric algebras has appeared for example in works of Oberst-Schneider [22], Schneider [25], Lorenz [20], Kadison-Stolin [17] and Cohen-Westreich [4, 5]. A survey on the representation theory of semisimple Hopf algebras from this point of view appears in Montgomery's survey [21].

Let H be a Hopf algebra over an algebraically closed field k of characteristic zero, with an integral Λ so that $\varepsilon(\Lambda) = 1$. For Hopf algebras H we use Sweedler's notation $\Delta(h) = \sum h_1 \otimes h_2$ for $h \in H$. Let S and s denote the antipodes of H and H^* respectively.

If H is semisimple then $H \cong \prod_{i=1}^n M_{d_i}(k)$. Let F_i be the primitive central idempotent of H such that $HF_i \cong M_{d_i}(k)$. Let V_i be the corresponding irreducible left H-module, ρ_i the irreducible representation $H \mapsto \mathrm{End}(V_i) \cong HF_i$ and let χ_i be the corresponding character. Then $\dim V_i = d_i$ with $d_1 = 1$. Let $\Lambda = F_1$, $\chi_1 = \varepsilon$.

Recall [18] that

$$\lambda = \chi_H = \sum d_i \chi_i.$$

is an integral for H^* satisfying $\langle \lambda, \Lambda \rangle = 1$. Since H is a Hopf algebra, $V \otimes W$ is a left H-module for any left H-modules V and W via

$$h \cdot (v \otimes w) = \sum h_1 \cdot v \otimes h_2 \cdot w$$

for all $v \in V$, $w \in W$, $h \in H$. Since the product in H^* is a convolution product, it follows that the character of $V \otimes W$ satisfies

$$\chi_{V \otimes W} = \chi_V \chi_W.$$

Let $R_{\mathbb{Z}}(H) = \sum_{i=1}^{n} \mathbb{Z}\chi_i$ be its characters ring. Larson [18] extended the orthogonality of characters from groups to semisimple Hopf algebras. That is

$$\langle \Lambda, \chi_i s(\chi_j) \rangle = \delta_{ij}. \tag{4}$$

This implies in particular that $R_{\mathbb{Z}}(H)$ is a finite free $\mathbb{Z}$-module. Let $C(H)$ be the algebra of all cocommutative elements of H^*. It is known that $C(H) = R(H) = k \otimes_{\mathbb{Z}} R_{\mathbb{Z}}(H)$. Moreover, by Kac-Zhu [10, 29], $C(H)$ is a semisimple algebra.

Since $\langle \chi_j, F_i h \rangle = \delta_{ij} \langle \chi_i, h \rangle$ for all $h \in H$, we have in particular, $\langle \chi_j, F_i \rangle = \delta_{ij} d_i$. Hence

$$\left\{ \chi_i, \frac{F_j}{d_j} \right\} \text{ are dual bases for } C(H) \text{ and } Z(H) \text{ respectively.} \tag{5}$$

Now, (4) can be rewritten as $\langle \chi_i, \Lambda \leftharpoonup s(\chi_j) \rangle = \delta_{ij}$, hence (5) implies by the uniqueness of the dual basis, that for all j,

$$\Lambda \leftharpoonup s(\chi_j) = \frac{F_j}{d_j}. \tag{6}$$

Define a form on $C(H)$ by:

$$\beta(p, q) = \langle \Lambda, pq \rangle$$

where $p, q \in C(H)$. That is, the element t corresponding to the symmetric form β is given by $t = \hat{\Lambda} \in (C(H))^*$, where $\left\langle \hat{\Lambda}, p \right\rangle = \langle \Lambda, p \rangle$ for all $p \in C(H)$. By the orthogonality of characters this form is non-degenerate and it is obviously associative. Since Λ is cocommutative it follows that β is symmetric with dual bases $\{\chi_i, s(\chi_i)\}$. That is, its Casimir element is given by:

$$\text{Casimir element } = \sum_{i=1}^{n} \chi_i \otimes s(\chi_i). \tag{7}$$

Denote by τ_C the Higman map corresponding to $(C(H), \Lambda)$. In the following Theorem 2.2 we shall explicitly compute the corresponding central Casimir element.

In [20] Lorenz gave a short proof of the class equation [10, 29]. In what follows we basically use his arguments.

Let E be a central primitive idempotent in $C(H)$ and let e be a primitive idempotent in $C(H)$ so that $eE = e$. Assume $\dim(C(H)e) = m$, m a positive integer. By [18], $\chi_{H^*} = \dim(H^*)\Lambda$. Hence we have:

$$\dim(H^*E) = \langle \chi_{H^*}, E\rangle = \dim(H^*)\langle \Lambda, E\rangle . \tag{8}$$

Ring theory considerations imply also

$$m = \frac{\dim(C(H)E)}{\dim(C(H)e)} = \frac{\dim(H^*E)}{\dim(H^*e)}. \tag{9}$$

The second equality follows since $H^* \cong H^* \otimes_{C(H)} C(H)$, hence $H^*e \cong H^* \otimes_{C(H)} C(H)e$ and $H^*E \cong H^* \otimes_{C(H)} C(H)E \cong H^* \otimes_{C(H)} (C(H)e)^m$.

In what follows we explicitly compute α of Corollary 1.12.

Proposition 2.1. *Let H be a semisimple Hopf algebra and let $C(H)$ be the algebra of cocommutative elements in H^*. Let $e \in C(H)$ be a primitive idempotent and let $E \in C(H)$ be a central primitive idempotent so that $Ee = e$. Then $\tau_C(e) = \alpha E$ and $\tau_C(E) = m\alpha E$ where $\alpha = \frac{\dim(H^*)}{\dim(H^*e)}$ is a positive integer and $m = \dim(C(H)e)$.*

Proof. We first show that $\tau_C(e)$ is integral over $\mathbb{Z}$. Since $R_{\mathbb{Z}}(H)$ is free over $\mathbb{Z}$ it follows that any character χ satisfies a monic polynomial over $\mathbb{Z}$ and hence so does its image under the representation $\rho_{C(H)e}$ and all its eigenvalues as well. It follows that $\langle p_e, \chi\rangle = \mathrm{trace}(\rho_{C(H)e}(\chi)) \in \mathcal{A}$ where $\mathcal{A}$ is the ring of algebraic integers. By Corollary 1.12.1 applied to $e \in C(H)$,

$$\tau_C(e) = \sum \langle p_e, \chi_i\rangle\, s(\chi_i) \in \mathcal{A} \otimes_{\mathbb{Z}} R_{\mathbb{Z}}(H)$$

is integral over $\mathbb{Z}$.

By Corollary 1.12.3, if $p(\tau_C(e)) = 0$ where $p(x) \in \mathbb{Z}[x]$ with leading coefficient 1, then $p(\tau_C(e)) = p(\alpha)E = 0$. It follows that α is an algebraic integer. By Proposition 1.13(i) and Corollary 1.12.(ii) with $t = \hat{\Lambda}$,

$$\langle \Lambda, E\rangle = \alpha^{-1}\langle \Lambda, \tau_C(e)\rangle = m\alpha^{-1}.$$

Substituting it in (8) yields

$$\dim(H^*E) = \dim(H^*)m\alpha^{-1}.$$

Hence

$$\alpha = \frac{m\dim(H^*)}{\dim(H^*E)} = \frac{\dim(H^*)}{\dim(H^*e)} \in Q. \tag{10}$$

The last equality follows from (9). Since α is an algebraic integer we have $\alpha \in \mathbb{Z}$. □

Proposition 2.1 enables us to compute the central Casimir element $\tau_C(1)$ for $C(H)$. Let $\{\chi_j\}_{1\le j\le n}$ be a complete set of irreducible characters for H. Let $\{E_i\}_{1\le i\le d}$ be a complete set of central primitive idempotents for $C(H)$ and let $\{e_i\}$ be a corresponding set of primitive idempotents such that $e_iE_i = e_i$ for all i.

Theorem 2.2. *Let H be a semisimple Hopf algebra over an algebraically closed field k of characteristic* 0. *Then*

$$\chi_{ad} = \sum_{j=1}^{n} \chi_j s(\chi_j) = \sum_{i=1}^{d} n_i E_i$$

where $n_i = \frac{\dim(C(H)e_i)\dim(H^*)}{\dim(H^* e_i)} \in \mathbb{Z}$ *for all* i.

Proof. $C(H)$ is a symmetric algebra with a Casimir element given in (7), hence $\tau_C(1) = \sum \chi_i s(\chi_i)$. By Corollary 1.12.3, each E_i satisfies $\tau_C(1)E_i = \tau_C(E_i) = \beta_i E_i$, where $\beta_i = \dim(C(H)e_i)\alpha_i$ and $\alpha_i = \frac{\dim(H^*)}{\dim(H^* e_i)}$ by (10). Since $\tau_C(1) = \sum_i \tau_C(1)E_i$, the result follows. □

Remark 2.3.

1. When $H = (kG)^*$ then $C(H) = kG$ and the characters are the group elements. In this case all the coefficients n_i equal $\dim H$, which agrees with Example 1.9.1.
2. When $H = kG$ then $C(H)$ is commutative, and thus

$$n_i = \frac{\dim(H^*)}{\dim(H^* E_i)}.$$

Moreover, this fact is true for all H as in Theorem 2.2, whenever $C(H)$ is commutative. This case will be considered in the next section.

3. $C(H)$ is a commutative algebra

Assume $C(H)$ is a commutative algebra. It is well known [14], that if H is a quasitriangular Hopf algebra then $C(H)$ is a commutative algebra. But when H is semisimple and $\mathrm{char}(k) = 0$ then the converse is also true. This is a consequence of the following Tanaka-Krein-type theorem (see, e.g., [15]):

Theorem. *Let* $(A, m, 1)$ *be a finite-dimensional algebra and let* $\mathrm{Rep}(A)$ *be the category of left A-modules. Then there exists a bijection between rigid braided tensor structures on* $\mathrm{Rep}(A)$ *and quasitriangular Hopf algebra structures on* $(A, m, 1)$.

Now, when H is semisimple over k of characteristic zero then there exists a bijection $V \mapsto \chi_V$ between objects of $\mathrm{Rep}(A)$ and the set of characters on H. Assuming $C(H)$ is commutative implies that $\chi_{V\otimes W} = \chi_{W\otimes V}$, hence $V \otimes W \cong W \otimes V$, thus $\mathrm{Rep}(H)$ is braided and by the theorem above, H is quasitriangular. We are thus in the situation described by Witherspoon in [28]. Some of the following results appear also there, but our approach is different.

As a semisimple commutative algebra, $C(H)$ is a direct sum of fields. Thus it has two set of bases: $\{\chi_1, \ldots \chi_n\}$, the set of irreducible characters for H and $\{E_1 \ldots E_n\}$, the complete set of primitive idempotents of $C(H)$. Set

$$n_i = \frac{\dim(H^*)}{\dim(H^* E_i)} = \frac{1}{\langle E_i, \Lambda\rangle} \tag{11}$$

The second equality follows from (8). By Proposition 2.1, we have that n_i are integers for all i.

Recall $l(\chi_i)$ is left multiplication of $C(H)$ by χ_i. Then for all i, j,

$$l(\chi_i)(\chi_j) = \chi_i \chi_j = \sum_s m^i_{js} \chi_s \tag{12}$$

where $m^i_{js} \in \mathbb{N}$. These are the so-called "fusion rules". The change of basis matrix is given by $A = (\alpha_{ij})$, where

$$\chi_i = \sum_j \alpha_{ij} E_j. \tag{13}$$

We denote by i^*, j^* the indexes relating $S(\chi_i), s(E_j)$ respectively. Applying s to both sides of (13) yields

$$\alpha_{ij} = \alpha_{i^* j^*}.$$

Since $\{E_i\}$ are orthogonal idempotents, (13) implies

$$\chi_i E_j = \alpha_{ij} E_j \tag{14}$$

and so E_j is an eigenvector for $l(\chi_i)$. The corresponding eigenvalue is α_{ij}.

Let $\mathbf{M^i} \in \mathrm{Mat}_n(\mathbb{Z})$ be the matrix of $l(\chi_i)$ with respect to the basis $\{\chi_j\}$, that is, $(\mathbf{M^i})_{js} = m^i_{js}$ as in (12). Let $\mathbf{D^i} = \mathrm{diag}\{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}\}$. We are ready to prove a general Verlinde formula for the fusion rules:

Theorem 3.1. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic zero. Assume $C(H)$ is commutative. Let $A = (\alpha_{ij})$ be the change of basis matrix from the full set of primitive idempotents $\{E_j\}_{1 \le j \le n}$ to the full set of irreducible characters $\{\chi_i\}_{1 \le i \le n}$. Then*

(i) *A diagonalizes the fusion rules. That is, for each i, $\mathbf{D^i} = A^{-1}\mathbf{M^i}A$.*
(ii) *$\chi_i E_j = \alpha_{ij} E_j$ and thus $\{E_j\}$ is a basis of eigenvectors for $l(\chi_i)$ for all i.*
(iii) *All α_{ij} are algebraic integers.*
(iv) *Let $A^{-1} = (\beta_{jk})$. Then $\beta_{jk} = n_j^{-1} \alpha_{k^* j}$.*
(v) *The integers m^i_{js} satisfy*

$$m^i_{js} = \sum_k \frac{\alpha_{ik}\alpha_{jk}\alpha_{s^*k}}{n_k}$$

Proof. (i) and (ii) follow from (13).

(iii) Since $l(\chi_i)$ with respect to the basis $\{\chi_j\}$ has integral coefficients, and since α_{ij} are all eigenvalues, it follows that α_{ij} are all algebraic integers.

(iv) Since $\{\chi_i, \frac{F_k}{d_k}\}$ form dual bases, we have

$$E_j = \sum_k \left\langle E_j, \frac{F_k}{d_k} \right\rangle \chi_k.$$

Hence,

$$\begin{aligned}\beta_{jk} &= \left\langle E_j, \frac{F_k}{d_k} \right\rangle \\ &= \langle E_j, \Lambda \leftharpoonup s(\chi_k)\rangle \quad \text{(by (6))} \\ &= \langle s(\chi_k)E_j, \Lambda\rangle \\ &= \alpha_{k^*j}\,\langle E_j, \Lambda\rangle \quad \text{(by (14))} \\ &= \alpha_{k^*j} n_j^{-1} \quad \text{(by (11))}\end{aligned}$$

Thus we obtain

$$\beta_{jk} = \left\langle E_j, \frac{F_k}{d_k} \right\rangle = n_j^{-1}\alpha_{k^*j}. \tag{15}$$

(v) By (i) $\mathbf{M^i} = A\mathbf{D^i}A^{-1}$. Hence the result follows from (iv). □

Remark 3.2. In [7, Theorem 2.3] it was proved that if A is a separable semisimple n-dimensional commutative algebra over a field k, then one can choose a matrix Y so that Y diagonalizes simultaneously all $l(b)$, $b \in A$, and the entries of Y consist of the various eigenvalues. The matrix does not necessarily satisfy (iii)–(v) of Theorem 3.1.

We wish to relate the Verlinde formula for the fusion rules to the structure constants of the product of conjugacy classes in groups. In what follows we define an analogue of the class sums for Hopf algebras, it reduces to the usual definition when applied to finite group. To see this let $H = kG$, G a finite group. Then $\Lambda = \frac{1}{|G|}\sum_{g\in G} g$ and $|G| = \dim(H) = d$. Moreover, $E_i = \sum_{g\in\mathcal{C}_\mathbf{i}} \pi_g$. The conjugacy classes and the class sums for G are defined respectively by:

$$\mathcal{C}_i = \{g^{-1}x_i g,\ g \in G\} \quad \text{and} \quad C_i = \sum_{g\in\mathcal{C}_\mathbf{i}} g.$$

Thus $k\mathcal{C}_\mathbf{i}$ is the left coideal (in fact the subcoalgebra) of kG generated by the central element C_i. We have $C_i = dE_i \rightharpoonup \Lambda$, and $|\mathcal{C}_\mathbf{i}| = \dim(E_i H^*)$. This motivates the following definition:

Definition 3.3. Let H be a semisimple Hopf algebra with an integral Λ so that $\varepsilon(\Lambda) = 1$. Let $\{E_i\}$ be the full set of central primitive idempotents of $C(H)$. Define the i-th **class sum** C_i by:

$$C_i = dE_i \rightharpoonup \Lambda \tag{16}$$

where $d = \dim(H)$. Define the **Conjugacy class** $\mathcal{C}_i$ by:

$$\mathcal{C}_i = H^*E_i \rightharpoonup \Lambda = \text{The left coideal of } H \text{ generated by } C_i.$$

In what follows we prove a general fact which when restricted to groups is natural.

Proposition 3.4. *Let H be a finite-dimensional Hopf algebra so that $S^2 = \mathrm{Id}$. Let z be a central element and let I be the right coideal generated by z. Then $h \cdot_{ad} I \subset I$ for all cocommutative elements h of H.*

Proof. Let $h \in H$ be a cocommutative element and let $p \in H^*$. Then:

$$\begin{aligned} & h \cdot_{ad} (p \rightharpoonup z) \\ & = \sum h_1(p \rightharpoonup z)Sh_2 \\ & = \sum h_1 \left(p_1 \rightharpoonup (z(Sp_2 \rightharpoonup Sh_2))\right) \\ & = \sum p_2 \rightharpoonup \big((s^{-1}(p_1) \rightharpoonup h_1)z(s(p_3) \rightharpoonup Sh_2)\big) \\ & = \sum p_2 \rightharpoonup (\big\langle s^{-1}(p_1), h_2\big\rangle \left\langle s(p_3), Sh_3\right\rangle h_1 z Sh_4 \\ & = \sum \big\langle s^{-1}(p_1), h_1\big\rangle \left\langle p_3, h_2\right\rangle p_2 \rightharpoonup z \text{ (since } z \in Z(H) \text{ and } h \text{ is cocommutative)} \\ & = \sum \left\langle s(p_1)p_3, h\right\rangle p_2 \rightharpoonup z \end{aligned}$$

which is an element of $H^* \rightharpoonup z = I$. □

As a corollary we obtain a generalization of the following obvious fact in groups:

Corollary 3.5. *The conjugacy classes $\mathcal{C}_i$ are stable under the adjoint action by cocommutative elements of H.*

In what follows we discuss various bases for $C(H)$ and $Z(H)$. Note that:

$$\frac{n_i}{d}\left\langle C_i, E_j\right\rangle = \frac{n_j}{d}\left\langle dE_i \rightharpoonup \Lambda, E_j\right\rangle = \delta_{ij} n_j \left\langle \Lambda, E_j\right\rangle \overset{\text{by (11)}}{=} \delta_{ij}.$$

Hence

$$\left\{E_i, \frac{n_i}{d}C_i\right\} \text{ are dual bases for } C(H) \text{ and } Z(H) \text{ respectively.} \tag{17}$$

We wish to compute now change of basis matrix from $\{C_i\}$ to $\{F_j\}$ in $Z(H)$. Observe that the following two bases of $C(H)$, $\mathcal{B}_1 = \{\chi_i\}$ and $\mathcal{B}_2 = \{E_i\}$ have a change of basis matrix $A = (\alpha_{ij})$ given in (13), and that their duals are $\mathcal{B}_1^* = \{\frac{1}{d_i}F_i\}$ and $\mathcal{B}_2^* = \{\frac{n_i}{d}C_i\}$ by (5) and (17) respectively. Hence the change of basis matrix from $\mathcal{B}_2^*$ to $\mathcal{B}_1^*$ is A^t. That is:

$$\frac{n_i}{d}C_i = \sum_j \alpha_{ji}\frac{1}{d_j}F_j.$$

Equivalently,

$$C_i = \frac{d}{n_i}\sum_j \alpha_{ji}\frac{1}{d_j}F_j. \tag{18}$$

Taking the inverse of A^t we obtain by (15)

$$F_k = \frac{d_k}{d}\sum_t \alpha_{k^*t}C_t. \tag{19}$$

Recall, when H is a semisimple Hopf algebra over an algebraically closed field of characteristic 0 then (H, λ) is a symmetric algebra with a Casimir element

$$\sum \Lambda_1 \otimes S\Lambda_2$$

Denote by τ_H the Higman map corresponding to the above Casimir element. Note that the central Casimir element is $\tau_H(1) = \varepsilon(\Lambda) = 1$. We show the following.

Proposition 3.6.

(i) *Let $x \in H$, then $\langle p, x\rangle = \langle p, \tau_H(x)\rangle$ for all $p \in C(H)$, and $\tau_H(x) = 0$ if and only if $\langle p, x\rangle = 0$ for all $p \in C(H)$.*

(ii) *If $x \in \mathcal{C}_i$ then for all k, $\langle E_k, x\rangle = \delta_{ik}\varepsilon(x)$. Hence*

$$\langle \chi_k, x\rangle = \alpha_{ki}\varepsilon(x) \text{ and } \langle \chi_k, C_i\rangle = \frac{d}{n_i}\alpha_{ki}.$$

(iii) *If $x \in \mathcal{C}_i$ then*

$$\tau_H(x) = \frac{n_i}{d}\langle E_i, x\rangle C_i.$$

Thus $\varepsilon(x) = 0$ if and only if $\tau_H(x) = 0$.

Proof. (i) For all $p \in C(H)$, $\langle p, \tau_H(x)\rangle = \langle p, \Lambda_1 x S(\Lambda_2)\rangle = \langle p, x\rangle$. It follows that if $\tau_H(x) = \tau_H(y)$ then $\langle p, x\rangle = \langle p, y\rangle$. Conversely, $\langle p, x\rangle = 0$ implies $\langle p, \tau_H(x)\rangle = 0$ for all $p \in C(H)$. Since $\tau_H(x) \in Z(H)$ which is dual to $C(H)$, it follows that $\tau_H(x) = 0$.

(ii) By definition, each $x \in \mathcal{C}_i$ satisfies $x = pE_i \rightharpoonup \Lambda$ for some $p \in H^*$. We have then $\langle E_j, x\rangle = \langle E_j, pE_i \rightharpoonup \Lambda\rangle$ which since Λ is cocommutative equals $\delta_{ij}\langle pE_i, \Lambda\rangle = \delta_{ij}\varepsilon(x)$. The second part follows now from (11), the definition of C_i and (13).

(iii) By (17), $\tau_H(x) = \sum_j \langle E_j, \tau_H(x)\rangle \frac{n_j}{d} C_j$. Since $E_j \in C(H)$ and $x_i \in \mathcal{C}_i$, this equals by (i) to $\sum_j \langle E_j, x\rangle \frac{n_j}{d} C_j = \frac{n_i}{d}\langle E_i, x\rangle C_i$. The last part is a direct consequence of the results above. □

We use Proposition 3.6 to prove a generalization of Littlewood's formula for groups [8, p. 225]:

$$\chi_i(g_j^{-1}) = |C_G(g_j)| \sum_{x \in C_j} \alpha_x$$

where $g_j \in \mathcal{C}_j$, $f_i = \sum_{x \in G} \alpha_x x$ is a primitive idempotent of kG corresponding to χ_i. Observe that $\sum_{x \in C_j} \alpha_x = \langle E_j, f_i\rangle$ and $|C_G(g_j)| = n_j$.

Proposition 3.7. *Let $H, E_j, \chi_i, C_j, \mathcal{C}_j, F_i, n_j$ be as above. Let f_i be a primitive idempotent of H so that $f_i F_i = F_i$. Then for all $x \in \mathcal{C}_j$,*

$$\langle s(\chi_i), x\rangle = \varepsilon(x) n_j \langle E_j, f_i\rangle.$$

Proof. By (18), $\langle \chi_i, C_j\rangle = \frac{d}{n_j}\alpha_{ij}$. Hence by Proposition 3.6,

$$\langle s(\chi_i), x\rangle = \langle s(\chi_i), \tau_H(x)\rangle = \frac{n_j}{d}\langle E_j, x\rangle \langle s(\chi_i), C_j\rangle = \varepsilon(x)\alpha_{i^*j}.$$

On the other hand, by Proposition 1.13(ii), using $\tau_H(1) = 1$, we have $\tau_H(f_i) = \frac{1}{d_i}F_i$. By (15) we have $\langle E_j, F_i\rangle = \frac{d_i}{n_j}\alpha_{i^*j}$ for all j. Thus

$$\varepsilon(x)n_j \langle E_j, f_i\rangle = \varepsilon(x)n_j \langle E_j, \tau(f_i)\rangle = \varepsilon(x)\frac{n_j}{d_i} \langle E_j, F_i\rangle = \varepsilon(x)\alpha_{i^*j}. \qquad \square$$

We can compute now the structure constants of the product defined on the C_i's in the spirit of the general Verlinde formula. Since $\{C_i\}$ is a k-basis for $Z(H)$ we have:

$$C_iC_j = \sum_t c^i_{jt}C_t \tag{20}$$

Let $\widehat{\mathbf{D}^{\mathbf{i}}} = \frac{d}{n_i}\mathrm{diag}\{\frac{\alpha_{1i}}{d_1}, \frac{\alpha_{2i}}{d_2}, \ldots, \frac{\alpha_{ni}}{d_n}\}$. Let $\widehat{\mathbf{M}^{\mathbf{i}}} \in \mathrm{Mat}_n(k)$ be the matrix of $l(C_i)$ with respect to the basis $\{C_j\}$, that is, $(\widehat{\mathbf{M}^{\mathbf{i}}})_{jt} = c^i_{jt}$ as in (20). Let $\widehat{A} = (\widehat{\alpha}_{ij})$ be the change of basis matrix from $\{C_i\}$ to $\{F_j\}$. Then by (18), $\widehat{\alpha}_{ij} = \frac{d}{n_id_j}\alpha_{ji}$, where α_{ij} are as in (13).

Theorem 3.8. *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic zero. Assume $C(H)$ is a commutative algebra. Let $\{E_i\}$ be a full set of primitive idempotents of $C(H)$ and let $\{F_i\}$ be a full set of primitive idempotents in $Z(H)$. Let $C_i = dE_i \rightharpoonup \Lambda$. Then:*

(i) *$\widehat{A}$ diagonalizes the structure constants. That is, for each i,*

$$\widehat{\mathbf{D}^{\mathbf{i}}} = \widehat{A}^{-1}\widehat{\mathbf{M}^{\mathbf{i}}}\widehat{A}.$$

(ii) *$C_iF_j = \widehat{\alpha}_{ij}F_j$ and thus $\{F_j\}$ is a basis of eigenvectors for $l(C_i)$ for all i.*

(iii) *Let $\widehat{A}^{-1} = (\widehat{\beta}_{kt})$. Then $\widehat{\beta}_{kt} = \frac{d_k}{d}\alpha_{k^*t}$.*

(iv) *$C_iC_j = \sum_t c^i_{jt}C_t$ where c^i_{jt} satisfies*

$$c^i_{jt} = \frac{d}{n_in_j}\sum_k \frac{\alpha_{ki}\alpha_{kj}\alpha_{k^*t}}{d_k}.$$

Proof. Items (i), (ii) and (iii) follow from (18) and (19). (iv) follows since $c^i_{jt} = \widehat{\mathbf{M}^{\mathbf{i}}}_{jt}$ and $\widehat{\mathbf{M}^{\mathbf{i}}} = \widehat{A}\widehat{\mathbf{D}^{\mathbf{i}}}\widehat{A}^{-1}$ by (i). The result follows now from (iii). $\square$

In the following example we show that when $H = kG$, G a finite group, then the formula in Theorem 3.8(iv) is just the formula for the structure constants recovered from the character table.

Example 3.9. Let $H = kG$, G a finite group. Since $|\mathcal{C}_{\mathbf{i}}| = \dim(H^*E_i)$, we have $n_i = \frac{|G|}{|\mathcal{C}_{\mathbf{i}}|} = |C_G(g_i)|$, where g_i is an element of $\mathcal{C}_{\mathbf{i}}$.

For a group G, its character table (see [8, p. 213])is the matrix (ξ^j_i) whose rows are indexed by the irreducible characters and columns by the conjugacy classes. It takes the form $\xi^j_i = \chi_j(g_i)$. Now, by Proposition 3.6(ii) and since $\varepsilon(g_i) = 1$, we have

$$\xi^j_i = \chi_j(g_i) = \alpha_{ji}.$$

Thus (20) gives the known formula for the structure constants for groups (see, e.g., [23, p. 45]). That is:

$$c^i_{jt} = \frac{|C_i||C_j|}{|G|} \sum_k \frac{\xi^k_i \xi^k_j \xi^k_{t^*}}{d_k}.$$

Remark 3.10. The approach taken in [28], is to use throughout μ_i, irreducible characters on the representation ring of H. Class sums ς_i are defined there as well. By using our approach,

$$\mu_i = \Lambda \leftharpoonup n_i E_i$$

Thus $\mu_i = \frac{n_i}{d} C_i$ and so

$$\varsigma_i = \frac{d}{n_i}\mu_i = C_i$$

Moreover, the coefficients $\mu_i(V_j)$ appearing in the character table in [28], in our setup turn out to be:

$$\mu_i(V_j) = \alpha_{ji} = \left\langle \frac{n_i}{d} C_i, \chi_j \right\rangle = \frac{n_i}{d_j} \langle E_i, F_{j^*} \rangle$$

The concept Conjugacy classes for Hopf algebras is not defined there.

Our next assertion is that the structure constants are invariants under twisting in the sense of Drinfeld [13]. A dual Hopf 2-cocycle (=twist) for H is an invertible element $J \in H \otimes H$ which satisfies:

$$(\Delta \otimes \mathrm{Id})(J)(J \otimes 1) = (\mathrm{Id} \otimes \Delta)(J)(1 \otimes J)$$
$$(\varepsilon \otimes \mathrm{Id})(J) = (\mathrm{Id} \otimes \varepsilon)(J) = 1.$$

Given a twist J, one can define a new Hopf algebra structure H^J on the algebra $(H, m, 1)$. The new comultiplication and the antipode are given by:

$$\Delta^J(a) = J^{-1}\Delta(a)J \qquad S^J(a) = QS(a)Q^{-1}$$

for every $a \in H$, where $Q = m \circ (S \otimes \mathrm{Id})(J)$. When H is finite dimensional then $J \in H \otimes H \cong (H^* \otimes H^*)^*$ can be considered as a linear form $\sigma : H^* \otimes H^* \to k$.

Thus we have the following [12]:

A linear form $\sigma : H \otimes H \to k$ is called a Hopf 2-cocycle for H if it has an inverse σ^{-1} under the convolution product $*$ in $\mathrm{Hom}_k(H \otimes H, k)$, and satisfies the cocycle condition:

$$\sum \sigma(a_1, b_1)\sigma(a_2 b_2, c) = \sum \sigma(b_1, c_1)\sigma(a, b_2 c_2) \tag{21}$$
$$\sigma(a, 1) = \varepsilon(a) = \sigma(1, a)$$

for all $a, b, c \in H$. Given a Hopf 2-cocycle σ for H, one can construct a new Hopf algebra structure H^σ on the coalgebra (H, Δ, ε). The new multiplication is given by

$$a \cdot_\sigma b = \sum \sigma(a_1, b_1) a_2 b_2 \sigma^{-1}(a_3, b_3)$$

for all $a, b \in H$. We have

$$(H^J)^* = (H^*)^\sigma.$$

It is known that when H is finite dimensional then the tensor category of left (right) H-modules is equivalent to that of left (right) H^J-modules. Hence their algebras of characters are the same. We show it in the next lemma for all cocommutative elements of H^*,

Lemma 3.11. *Let H be a finite-dimensional Hopf algebra. Then:*

(i) *$C(H) = C(H^J)$ as algebras.*

(ii) *If H is also semisimple then λ is an integral in $(H^*)^\sigma$, and for all $p \in C(H)$, $\Lambda \leftharpoonup p = \Lambda^J \leftharpoonup p$, where Λ^J denotes the integral Λ with the new comultiplication.*

Proof. (i) If p and q are cocommutative then $p \cdot_\sigma q = \sum \sigma(p_1, q_1)p_2q_2\sigma^{-1}(p_3, q_3) = \sum \sigma(p_2, q_2)p_3q_3\sigma^{-1}(p_1, q_1) = pq$.
(ii) λ is an integral since $\lambda = \chi_H$ which was not changed as the multiplication in H was not changed. The last part follows from the dual basis property (which appears in [24] for the case of Hopf algebras). It states that $\Lambda \leftharpoonup (\lambda \leftharpoonup a) = S^{-1}(a)$ for all $a \in H$. But $\Lambda^J \leftharpoonup (\lambda \leftharpoonup a) = (S^J)^{-1}(a)$ as well. Since $S(a) = S^J(a)$ for all $a \in Z(H)$, and since $\lambda \leftharpoonup Z(H) = C(H)$, the result follows. □

We can show now

Corollary 3.12. *Let H be as in Theorem* 3.1*. Then class sums and all the structure constants are invariant under dual Hopf cocycles J.*

Proof. Lemma 3.11 implies that $\{\chi_i\}$ and $\{E_i\}$ are the same sets for H^* and $(H^*)^\sigma$. It follows that the coefficients α_{ij} in (13) and the fusion coefficients m^s_{jt} in Theorem 3.1(v) are invariants. So are the values $n_i = \frac{1}{\langle E_i, \Lambda\rangle}$ by (11). We have also

$$\dim(H^*E_i) = \dim(H^* \cdot_\sigma E_i) = \frac{d}{\langle E_i, \Lambda\rangle}.$$

Part (iii) of the lemma above implies now that the set $\{C_i\}$ as defined in (16), and thus the structure constants c^i_{jt} as defined in (20), are all invariants. □

4. Factorizable Hopf algebras

Theorem 3.1 and Theorem 3.8 are especially interesting when applied to semisimple factorizable Hopf algebras [14]. In fact, we show that in this case, the fusion rules and the structure constants are equal up to rational scalar multiples.

Let (H, R) be a quasitriangular Hopf algebra. Set $Q = R^\tau R = \sum R^2r^1 \otimes R^1r^2$. The maps $\mathbf{f}_Q$ and its dual $\mathbf{f}^*_Q$ are given by:

$$\mathbf{f}_Q(p) = \sum \langle p, R^2r^1\rangle R^1r^2 \qquad \mathbf{f}^*_Q(p) = \sum \langle p, R^1r^2\rangle R^2r^1$$

for $p \in H^*$. H is factorizable if the map $\mathbf{f}_Q$ is a k-isomorphism between H^* and H. When H is also semisimple, then by [14] $\mathbf{f}_Q, \mathbf{f}^*_Q : C(H) \to Z(H)$ are algebra isomorphisms. This was extended in [6, 26]) showing that $\mathbf{f}_Q^*$ satisfies $\mathbf{f}_Q^*(px) = \mathbf{f}_Q^*(p)\mathbf{f}_Q^*(x)$ for $p \in H^*$, $x \in C(H)$. Set

$$s_{ij} = \langle \mathbf{f}_Q(\chi_i), \chi_j \rangle .$$

It is known that $s_{ij} = s_{ji}$. We need the following:

Lemma 4.1. *Let (H, R) be a factorizable semisimple Hopf algebra. Then $\mathbf{f}_Q = \mathbf{f}_Q^*$ and $S\mathbf{f}_Q = \mathbf{f}_Q s$ when restricted to $C(H)$.*

Proof. For all $x, y \in C(H)$ we have,

$$\langle \mathbf{f}_Q(x), y \rangle = \sum \left\langle x, R^2 r^1 \right\rangle \left\langle R^1 r^2, y \right\rangle = \sum \left\langle x, R^1 r^2 \right\rangle \left\langle R^2 r^1, y \right\rangle = \left\langle \mathbf{f}_Q^*(x), y \right\rangle .$$

Since $\mathbf{f}_Q(x), \mathbf{f}_Q^*(x) \in Z(H)$ and since $Z(H)$ and $C(H)$ are dual vector spaces, $\mathbf{f}_Q = \mathbf{f}_Q^*$ on $C(H)$. The rest follows from [3, Pr. 2.4]. □

Number the central idempotents of $C(H)$ as

$$E_i = \mathbf{f}_Q^{-1}(F_i).$$

Observe that Lemma 4.1 implies that also $E_{i^*} = \mathbf{f}_Q^{-1}(F_{i^*})$. We have (see [4]):

$$\chi_i = \sum_j \frac{s_{ij}}{d_j} E_j .$$

By (13), $\alpha_{ij} = \frac{s_{ij}}{d_j}$. Hence

$$d_j \alpha_{ij} = d_i \alpha_{ji} = s_{ij} \tag{22}$$

Moreover, since $E_i \in C(H)$, we have $\mathbf{f}_Q^*(H^* E_i) = \mathbf{f}_Q^*(H^*)\mathbf{f}_Q^*(E_i) = HF_i$, it follows that

$$\dim(H^* E_i) = \dim(HF_i) = d_i^2 .$$

Hence

$$n_i = \frac{\dim(H^*)}{\dim(H^* E_i)} = \frac{d}{d_i^2}. \tag{23}$$

We have:

Corollary 4.2. *The Verlinde formula for the fusion rules for factorizable semisimple Hopf algebras as given in* [4] *in the following form*

$$m_{jt}^i = \frac{1}{d} \sum_k \frac{s_{tk} s_{ik} s_{jk^*}}{d_k}$$

coincides with the Verlinde formula for the fusion rules as given in Theorem 3.1(v). *This follows by substituting* (22) *and* (23) *in the formula above.*

Most importantly, we have

Theorem 4.3. *Let (H, R) be a factorizable semisimple Hopf algebra. Let χ_i, C_j, d_i, α_{ij}, c_{ij}^t, m_{jt}^i be as in Theorem* 3.1 *and Theorem* 3.8. *Then the structure constants c_{jt}^i are given by:*

$$c_{jt}^i = \frac{d_i d_j}{d_t} m_{jt}^i .$$

Proof. Observe first that since $\mathbf{f}_{Q|C(H)}$ is an algebra map, we have

$$\mathbf{f}_Q(\chi_i) = \sum_j \alpha_{ij} F_j \overset{\text{by (22)}}{=} \sum_j \frac{d_i}{d_j} \alpha_{ji} F_j \overset{\text{by (23)}}{=} \frac{d}{d_i n_i} \sum_j \frac{1}{d_j} \alpha_{ji} F_j \overset{\text{by (18)}}{=} \frac{1}{d_i} C_i.$$

Hence applying the $\mathbf{f}_Q$ to (12) yields the desired result. □

A variation of the Verlinde formula for characters of non-semisimple factorizable ribbon Hopf algebra was given in [5]. We give here a brief summary of the results there. Let H be a finite-dimensional symmetric Hopf algebra over an algebraically closed field k of characteristic 0. Let $\{V_1, \dots, V_n\}$ be a full set of non-isomorphic irreducible left H-modules and $\{f_1, \dots, f_n\}$ the corresponding orthogonal primitive idempotents. Let $P_i = Hf_i$ be the associated projective indecomposable left H-module. The Cartan matrix of H is the $n \times n$ matrix $\mathbf{C}$ with

$$c_{ij} = \dim f_i H f_j.$$

Let p_a be as in (3), set

$$I(H) = \{p_a | a \in H\}.$$

It is known that $I(H)$ contains all characters χ_P of finitely generated projective H-modules P, and $I(H)$ is spanned over k by the set $\{p_e\}$, e a primitive idempotent of H. When $\mathrm{char}(k) = 0$ then $\dim I(H)$ equals the rank of the Cartan matrix associated with H.

Since H is symmetric then by [22], the integral Λ is 2-sided. Let λ a right integral for H^* so that $\langle \lambda, \Lambda \rangle = 1$. In this case we have by [4, Theorem 2.4] that $I(H)$ is an s-stable ideal of $C(H)$.

A quasitriangular Hopf algebra $(H, \mathcal{R}, v)$ is ribbon if there exists a central element v so that $S(v) = v$, $v^2 = uS(u)$, where $u = \sum (SR^2)R^1$, and $\Delta(v) = (v \otimes v)Q^{-1}$. Set $G = u^{-1}v$, then G is a group-like element of H inducing S^{-2}. In this case H is a symmetric algebra with a symmetrizing form $\mathbf{t} = \lambda \leftharpoonup G^{-1}$ and a Casimir element

$$\sum a_i \otimes b_i = \sum \Lambda_1 G \otimes S(\Lambda_2).$$

Hence

$$\tau_H(a) = \sum \Lambda_1 G a S(\Lambda_2).$$

Define $\widehat{\Psi} : H \to H^*$ by:

$$\widehat{\Psi}(a) = \mathbf{t} \leftharpoonup S^{-1}(a).$$

Then we have,

$$\widehat{\Psi}(Z) = C(H) \text{ and } \widehat{\Psi}(\tau_H(H)) = I(H).$$

Recall that $\mathbf{f}_Q : O_{S^2} \to Z$ is an algebra map, where $O_{S^2} = \{p \in H^* \mid \langle p, ab \rangle = \langle p, (S^2 b)a \rangle$ for all $a, b \in H$ [14]. The translated $\mathbf{f}_Q$, $\widehat{\mathbf{f}_Q}$ is given for all $p \in H^*$ by:

$$\widehat{\mathbf{f}}_Q(p) = \mathbf{f}_Q(p \leftharpoonup G).$$

Then $\widehat{\mathbf{f}}_Q : C(H) \to Z(H)$ is an algebra map and

$$\widehat{\mathbf{f}}_Q(I(H)) = \tau_H(H).$$

Assume the rank of the Cartan matrix associated with H is r. Set

$$\mathcal{B} = \{\widehat{f}_Q(p_{f_j})\}_{j=1}^r \qquad \mathcal{C} = \{\widehat{\Psi}^{-1}(p_{f_j})\}_{j=1}^r. \tag{24}$$

Then $\mathcal{B}$ and $\mathcal{C}$ are two bases of $\tau_H(H)$. Let $\mathbf{F} = (f_{ij})_{1\le i,j\le r}$ be the change of bases matrix from $\mathcal{B}$ to $\mathcal{C}$. Thus for all $1 \le j \le r$,

$$\widehat{f}_Q(p_{f_j}) = \sum_{l=1}^r f_{lj}\widehat{\Psi}^{-1}(p_{f_l}). \tag{25}$$

For any projective H-module P, $V_i \otimes P$ is also a projective and $\chi_{V_i\otimes P} = \chi_i\chi_P$. Recall [8, Pr. 16.7] that characters of projective module are linear integer combinations of the characters of the indecomposable projective modules. Hence

$$\chi_i p_{f_j} = \sum_{l=1}^r N_{lj}^i p_{f_l} \tag{26}$$

where $\{N_{lj}^i\}$ are all non-negative integers, which we call the "projective fusion rules" for projective indecomposable modules. Since $\widehat{\mathbf{f}}_Q$ when restricted to $C(H)$ is an algebra isomorphism, we have $\widehat{\mathbf{f}}_Q(\chi_i)\widehat{\mathbf{f}}_Q(p_{f_j}) = \sum_{l=1}^r N_{lj}^i\widehat{\mathbf{f}}_Q(p_{f_l})$. That is, $L_{\widehat{\mathbf{f}}_Q(\chi_i)} : \tau_H(H) \to \tau_H(H)$, the linear operator defined by left multiplication by $\widehat{\mathbf{f}}_Q(\chi_i)$, satisfies

$$L_{\widehat{\mathbf{f}}_Q(\chi_i)}(\widehat{\mathbf{f}}_Q(p_{f_j})) = \sum_{l=1}^r N_{lj}^i\widehat{\mathbf{f}}_Q(p_{f_l}). \tag{27}$$

Thus the matrix of $L_{\mathbf{f}_Q(\widehat{\chi_i})}$ with respect to the basis $\mathcal{B}$ is given by:

$$\mathbf{N^i} = (N_{lj}^i).$$

For $1 \le i, j \le n,\ \ 1 \le k, l \le r$, set

$$s_{ij} := \left\langle \widehat{\mathbf{f}}_Q(\chi_i), s(\chi_j)\right\rangle \qquad q_{kl} = \left\langle f_l, \widehat{\Psi}\widehat{\mathbf{f}}_Q(p_{f_k})\right\rangle. \tag{28}$$

We show in [4, Theorem 3.14]

Theorem 4.4. *Let $(H, \mathcal{R}, v)$ be a factorizable ribbon Hopf algebra over an algebraically closed field of characteristic 0 with a Cartan matrix of rank r and invertible $r \times r$ minor $\mathbf{C}_r$. Let $\mathbf{Q} = (q_{kl})$ and s_{ij} be given in (28). Then $\mathbf{F} = \mathbf{Q}^{-1}\mathbf{C}_r$ diagonalizes the "projective fusion rules" $\mathbf{N}^i$. That is, for all $1 \le i \le n$,*

(i) $\mathbf{F}^{-1}\mathbf{N}^i\mathbf{F} = \mathrm{Diag}\{d_1^{-1}s_{i1}, \dots, d_r^{-1}s_{ir}\}$ *where* $d_j = \dim(V_j)$, V_j *an irreducible representation of H.*
(ii) *Eigenvectors for $L_{\widehat{\mathbf{f}}_Q(\chi_i)}$ are $\sum \Lambda_1 GS(f_j)S(\Lambda_2)$, with corresponding eigenvalues $d_j^{-1}s_{ij}$.*
(iii) *$\{d_j^{-1}s_{ij}\}$ are algebraic integers.*

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Ring and Module Theory
Trends in Mathematics, 115–123

Right Weakly Regular Rings: A Survey

Henry E. Heatherly and Ralph P. Tucci

Abstract. A ring is right weakly regular (r.w.r.) if every right ideal of the ring is idempotent. Such rings are also called fully right idempotent. This paper gives a survey of the theory of r.w.r. rings and some closely allied topics, from its origins in the early 1950's up to the present state-of-the-art. The paper contains sections on: equivalent conditions, examples and constructions, and related conditions.

Mathematics Subject Classification (2000). Primary 16E02; Secondary 16E50.

Keywords. Weakly regular, fully right idempotent, regular.

1. Introduction

A ring is *right weakly regular* (*r.w.r.*) if every right ideal of the ring is idempotent. (Here "ring" means an associative, not necessarily commutative ring which may or may not have unity. Also, R will always denote a nonzero ring.) Similarly one defines *left weakly regular* (*l.w.r.*) rings. The study of such rings, in various equivalent formulations and using several variants in terminology, goes back at least to 1950, when Brown and McCoy introduced the equivalent formulation: $x \in x\langle x\rangle$ for each $x \in R$, [9]. (Here $\langle x\rangle$ is the two-sided ideal generated by $x \in R$.) They called such rings "weakly regular rings". Ramamurthi [31] introduced the terminology "right weakly regular", and it has been used by several others, [6, 10, 22]. The terminology "fully right idempotent" has been frequently used, e.g., [17, 29], as has the variant "right fully idempotent", e.g., [1, 3]. Since 1950 numerous papers on the subject, or closely related to it, have appeared, and the pace of investigation has picked up in the past two decades. Exemplary of this increased interest is that a section of a recent monograph on generalizations of regular rings is devoted to the topic, [34, Section 20]. The point of view of generalizations of regular rings is one motivation for the study of r.w.r. rings, and a source of examples, for all regular rings are both r.w.r. and l.w.r., as are all biregular rings (rings for which every principal ideal is generated by a central idempotent). Simple rings with unity are also always r.w.r. and l.w.r. Further examples and methods of construction of r.w.r. rings will be given in Section 3.

The purpose of this paper is to give a survey of the theory of right weakly regular rings, and closely allied topics, in its present state-of-art. We hope this will serve as an aid to those already working on the subject and a helpful guide to those who wish to enter this field of investigation. Such readers will also find the survey article by Birkenmeier, [5], of complementary interest.

One attribute of the development of the theory of r.w.r. rings is that many of the same results have been discovered and rediscovered independently. This can lead to some confusion, which is aided and abetted by the fact that different terminology is also often used. Here we try to clarify some of this developmental tangle.

Note that for each r.w.r. result there is an obvious l.w.r. dual result, and conversely.

2. Equivalent conditions to r.w.r.

In this section we give numerous equivalent conditions to a ring being r.w.r. This falls naturally into two parts: equivalent conditions in the category of rings and equivalent conditions in the category of rings with unity. Many of the latter type are in terms of modules.

Here we use the notation $\mathbb{R}(R)$ for the multiplicative semigroup of all right ideals of R and $\langle x\rangle_r$ for the right ideal of R generated by $x \in R$.

Most of the equivalences in the first proposition here were given in [31], where the terminology "right weakly regular" was introduced. Proofs can be found in [22] or [31].

Proposition 2.1. *The following are equivalent:*

(i) *R is r.w.r.;*
(ii) *if $b \in R$, then $b \in (bR)^2$;*
(iii) *if $A, B \in \mathbb{R}(R)$ with $A \subseteq B$, then $AB = A$;*
(iv) *every principal right ideal of R is idempotent;*
(v) *if $b \in R$, then $b \in (bR)^n$ for some $n \geq 2$;*
(vi) *if $b \in R$, then $b \in b\langle b\rangle$;*
(vii) *every homomorphic image of R is r.w.r.;*
(viii) *every ideal of R is r.w.r.*

The equivalence of (i) and (ii) above is very useful in developing the theory of r.w.r. rings, as is the property that: R r.w.r. implies $\langle x\rangle_r = xR$. Each of these have been rediscovered several times, e.g., see [22, 31] and

For rings with unity numerous other equivalent conditions to r.w.r. have been noted. These are of two general types: ideal oriented and module oriented. The next result gives a sample of the former type.

Proposition 2.2. *Let R be a ring with unity, Then the following are equivalent:*

(i) *R is r.w.r.;*
(ii) *if B is a right ideal of R and A is an ideal of R, then $B \cap A = BA$;*
(iii) *if B and C are right ideals of R, then $B \cap (RC) = BC$;*
(iv) *every ideal I of R has the property that $x \in xI$ for each $x \in I$.*

The equivalence of (i) through (iv) is stated in [31]. Proofs of these, and further equivalences for rings with unity, are given in [34, p. 171].

Hansen established connections between r.w.r. rings and prime and semiprime ideals, [20]. Recall that a right ideal B of a ring R is said to be *prime* (*semiprime*) if $xRy \subseteq B$, ($xRx \subseteq B$), implies $x \in B$ or $y \in B$ ($x \in B$), [20], [26, p. 365].

Proposition 2.3 ([20, Lemma 21]). *The following are equivalent:*

(i) *R is r.w.r.;*
(ii) *every right ideal of R is semiprime;*
(iii) *every right ideal of R is the intersection of prime right ideals of R.*

Hansen called a ring satisfying these three equivalent conditions a "weakly regular ring". Dauns used condition (ii) in the above as his defining condition for a weakly regular ring, [13, Definition 2.7].

The next equivalence was announced by Armendariz in 2007, [2].

Proposition 2.4. *Let R have unity. Then R is r.w.r. if and only if all three of the conditions hold:*

(i) *every ideal of R is idempotent;*
(ii) *R/P is r.w.r. for every prime ideal P of R;*
(iii) *if T is a right ideal of R and A and B are ideals of R, then $(T+A)\cap(T+B) = T + (A \cap B)$.*

Recently Laszlo Fuchs has made some interesting connections between r.w.r. rings and the relative divisor and the torsion-free ideal conditions, as shown in the next result.

Proposition 2.5 ([18]). *Let R be a ring with unity. Then the following are equivalent:*

(i) *R is r.w.r.;*
(ii) *If L is a left ideal of R and I an ideal of R with $L \subseteq I$, then $xL = (xI) \cap L$ for each $x \in R$;*
(iii) *if I is an ideal of R and $a \in I, x \in R$ with $xa = 0$, then $a \in \mathbf{r}(x) \cdot I$.*

(Note: for any nonempty set X we use $\mathbf{r}(X) = \{a \ \in R \mid Xa = 0\}$.)

In the above, condition (ii) is equivalent to every ideal of R satisfies the (left) relative divisible condition, and condition (iii) is equivalent to every ideal of R is torsion-free. These two conditions are usually given in module theoretic, homological equivalent formulations, e.g., see [14].

Several module related equivalent conditions to a ring being r.w.r. were noted by Fisher, [17]. There he used the terminology right (left) fully idempotent instead of right (left) weakly regular, and he gave the results in the "left" setting.

Proposition 2.6 ([17, Theorem 8]). *Let R be a ring with unity. Then the following are equivalent:*

(i) *R is r.w.r.;*
(ii) *for each ideal I of R, R/I is a flat right R-module;*
(iii) *for each ideal I of R, every injective right R/I-module is injective as a canonical R-module.*

In the above proposition, and hereafter, in the setting of rings with unity, all modules are unital.

In [36] and [24] r.w.r. rings which are algebras (over fields) with unity are considered, and are called "weakly regular algebras". Various module-homological conditions are given to an algebra being weakly regular in this sense. The next proposition is representative of these results.

Proposition 2.7. *Let A be a unital algebra over a field. Then the following are equivalent:*

(i) *A is a r.w.r. ring;*
(ii) *every right A-module M is semiflat, i.e., for each right ideal B of A the sequence $0 \to M \otimes_A B \to M \otimes_A A$ is left exact.*

Equivalent conditions to R being r.w.r. also arise from viewing the property in terms of the semigroup $\mathbb{R}(R)$.

Proposition 2.8 ([21, 22]). *The following are equivalent:*

(i) *R is r.w.r.;*
(ii) *$\mathbb{R}(R)$ is a left normal band (i.e., $A^2 = A$ and $ABC = ACB$ for each $A, B, C \in \mathbb{R}(R)$);*
(iii) *$\mathbb{R}(R)$ is von Neumann regular.*

3. Examples and constructions

Examples of an abstract structure serve to not only motivate the study of the structure but to illustrate both the possible scope and limitations of the theory. The examples and constructions in this section are given with those goals in mind.

As noted in the introduction, all regular rings and all biregular rings are both r.w.r. and l.w.r., as are all simple rings with unity. A generalization of the latter was proved by Andruskiewicz and Puczylowski, [1, Corollary 1], and is given next.

Proposition 3.1. *Let R be a simple ring with $R^2 \neq 0$. Then R is r.w.r. (l.w.r.) if and only if, for each $r \in R$, $r \in rR$ (respectively, $r \in Rr$). Consequently, every simple ring with unity is both r.w.r. and l.w.r.*

Corollary 3.2 ([1, Corollary 2]). *Let R be a simple von Neumannn regular ring, and let L be a proper, nonzero left ideal of R. Then $L/\mathbf{r}(L)$ is a simple ring which is r.w.r.*

Complementing these two results nicely, Andruskiewicz and Puczylowski gave the following classes of examples of rings which are r.w.r. but not l.w.r.

Example 3.3. Let A be a simple ring with unity which is von Neumann regular, but is not left Artinian, and let L be a maximal left ideal of A. Then the ring $R = \begin{bmatrix} L & A \\ L & A \end{bmatrix}$ is r.w.r. and not l.w.r. Such a ring R does not have unity. However, if A is also an F-algebra over a finite prime field F, then R is also an F-algebra and R can be embedded as an ideal in an F-algebra R^*, where R^* has unity and $R^*/R \cong F$. Consequently R^* is r.w.r., but not l.w.r. See [1, p. 156] for details.

Armendariz has given a process that provides a wide class of examples of rings with unity that are r.w.r. and not l.w.r., via the following result.

Proposition 3.4 ([2]). *Let R be a simple regular ring with unity which is not Artinian. If A is a left ideal of R such that $\mathbf{r}(A) = 0$ and $A \neq R$, then A is a simple ring which is r.w.r., but not l.w.r.*

Further examples of r.w.r. rings can be obtained by applying various ring constructions to r.w.r. rings, as the next several propositions illustrate.

Proposition 3.5. *Let R be a r.w.r. ring with unity. Then each of the following is r.w.r.:*

(i) $M_n(R)$, *the full $n \times n$ matrix ring over R, for each n;*
(ii) *the group ring $R[G]$ where G is a locally finite group and the order of each element in G is a unit in R;*
(iii) R_M, *the central localization over a maximal ideal M of the ring $Z(R)$, the center of R;*
(iv) $\mathrm{End}_R(M)$, *where M is a finitely generated right R-module;*
(v) eRe, *for every non-zero idempotent e of R.*

Vanaja stated (i) in an abstract, [35], but never published a proof. Proofs of (i), (iv), and (v) are given in [34, pp. 171–173]. Fisher established (ii) in [17], albeit using l.w.r. Part (iii) was given by Armendariz, Fisher, and Steinberg in [3, Proposition 2].

Ramamurthi stated that every r.w.r. ring can be embedded as an ideal in a r.w.r. ring with unity, [31]. He did not give a proof but remarked that it could be proved by adopting ideas of Fuchs and Halperin in [19]. Feigelstock took a closely related approach and gave a proof of the embedding result in [16, Corollary 2.3]. With this one can extend Proposition 3.5 (i) to r.w.r. rings not necessarily having unity.

Example 3.6. Let $T_1 \subset T_2 \subset \cdots T_n \subset T_{n+1} \subset \cdots$ be a strictly ascending chain of r.w.r. rings, and let $T = \cup_{n=1}^{\infty} T_n$. Lift the operations on the T_n to T to obtain a ring. Observe that this ring T will be r.w.r.

Let $M_\omega(R)$ be the ring of all matrices over R with countably infinite many rows and columns and which are row and column finite, and let T_n be the subring of this ring which consists of a block isomorphic to $M_n(R)$ in the upper left corner and zeroes elsewhere. Then using the construction described above, T is a r.w.r. ring composed of infinite matrices which are row and column finite.

Certain substructures of a r.w.r. ring are r.w.r. rings, as the next result, due to Ramamurthi, [31], shows.

Proposition 3.7. *If R is r.w.r., then every ideal of R is a r.w.r. ring and the center of R, $Z(R)$, is a regular ring.*

The next result gives some ways to build r.w.r. rings from given ones.

Proposition 3.8 ([22]). *Let Λ be a nonempty index set and let R_λ be a r.w.r. ring for each $\lambda \in \Lambda$. Then:*

(i) $\Sigma_\lambda \oplus R_\lambda,\ \lambda \in \Lambda$, *is r.w.r.;*
(ii) $\Pi_\lambda R_\lambda,\ \lambda \in \Lambda$, *is r.w.r.*

Observe that in 3.8(ii) if S_α is the set of all $(\cdots r_\lambda, \cdots)$ such that $r_\lambda \neq 0$ for at most $\aleph_\alpha$ of the subscripts λ, where α is a fixed ordinal, then S_α is an ideal of $\Pi_\lambda R_\lambda$ and hence S_α is a r.w.r. ring.

The class $\mathcal{W}$ of all r.w.r. rings is a hereditary Amitsur-Kurosh radical class. Ramamurthi [31] noted this is implicit in an example given by Brown and McCoy in their pre-Amitsur-Kurosh development of a general theory of radicals, [9, p. 308]. Detailed treatment yielding that $\mathcal{W}$ is a hereditary Amitsur-Kurosh radical class is given in [33, p. 83, p. 197] and in [1]. Thus the sum of all the ideals of R which are all r.w.r. rings, i.e., the sum of all the r.w.r. ideals of R, is a r.w.r. ideal, $\mathcal{W}(R)$, and $\mathcal{W}(R/\mathcal{W}(R)) = 0$. Ramamurthi [31] showed that the Jacobson radical of a r.w.r. ring must be zero. Much earlier Blair had shown that the intersection of any r.w.r. ideal of a ring R and the Jacobson radical, $J(R)$, is zero, [8]. So $J(R) \cap \mathcal{W}(R) = 0$.

4. Related conditions

The relationships between r.w.r. rings and various other well-known classes of rings have received considerable attention. Some of these other classes are: regular rings, V-rings, fully idempotent rings, and weakly π-regular rings.

We give some exemplary results of this nature. Throughout this section all rings have unity.

Proposition 4.1. *Let R be r.w.r. If any one of the following holds, then R is regular:*

(i) *R is left self-injective,* [17, p. 107]*;*
(ii) *all the prime factors of R are regular,* [34, p. 173]*;*
(iii) *every essential maximal right ideal is two-sided,* [23, Lemma 2.4]*;*
(iv) *every maximal right ideal is two-sided,* [37, Theorem 2.7].

Recall that R is a *right V-ring* if every simple right R-module is injective. (See [15, p. 56].) Every right V-ring is r.w.r., [17, Corollary 7].

Proposition 4.2 ([17, Theorem 14]). *The following are equivalent:*

(i) *R is a right V-ring;*
(ii) *R is r.w.r. and each right primitive factor ring of R is a right V-ring.*

Also immediate from [17, p. 107] is that a r.w.r. ring has zero left singular ideal.

Recall that a ring R is *right weakly π-regular* (*r.w.π.r*) if for each $b \in R$ there exists n such that $b^n R = (b^n R)^2$. (For basic results on r.w.π.r. rings, see [34, Section 20].) Observe that r.w.r. implies r.w.π.r., but not conversely. However, for reduced rings much stronger results are known.

Proposition 4.3 ([6, Theorem 8, Corollary 11], [10, Theorem 6]). *Let R be reduced. Then the following are equivalent:*

(i) *R is r.w.π.r;*
(ii) *R is r.w.r.;*
(iii) *R is biregular;*
(iv) *every prime ideal of R is maximal;*
(v) *every prime factor ring is a simple domain;*
(vi) *R is a right p.p. ring* (*every principal right ideal of R is a projective R-module*) *and $RbR = R$ for each nonzero $b \in R$.*

Thus a r.w.r. reduced ring is l.w.r. The term "weakly regular" has become the standard term used for rings which are both r.w.r. and l.w.r. Camillo and Xiao give further properties of reduced weakly regular rings in [10]; also see [34, Section 20]. More on r.w.π.r. rings is given in [6], [7], and [34, Section 20].

Rings (with or without unity) for which every ideal is idempotent were first studied in depth by Courter, [11], who called them "fully idempotent rings". Of course, every r.w.r. ring is fully idempotent, but not conversely. Courter gave a large and varied list of equivalent conditions to fully idempotent, as well as developing an extensive theory for this class of rings, [11, 12]. Fully idempotent rings have been studied from various views and venues, e.g., see [29, 30, 32].

Weak regularity for semigroup rings has been investigated by Fang Li, [27], and by Kuzhokov, [25]. The latter showed that if S is any semigroup and F a field, then there exists a semigroup T, with $T \subseteq S$, such that the semigroup ring $F[T]$ is both r.w.r. and l.w.r. Fang Li gave conditions for a semigroup ring to be l.w.r., in particular for inverse semigroups. He also gave conditions for Munn matrix rings to be l.w.r.

Weak regularity has been generalized to modules by Mabuchi, [28], and to topological weakly regular in the setting of topological rings by Arnautov, [4].

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Ring and Module Theory
Trends in Mathematics, 125–128

Substructures of Hom

Friedrich Kasch

Abstract. Various substructures of $\mathrm{Hom}_R(A, M)$ and their relationships are surveyed.

Mathematics Subject Classification (2000). Primary 16D10; Secondary 16D70.

Keywords. Total, regular, singular, cosingular, radical.

1. Introduction

We present here some typical examples of substructures of Hom. Anybody interested in more results, proofs, and details is referred to the book entitled "Regularity and Substructures of Hom" by myself and Adolf Mader that just appeared (Spring 2009) in the Birkhäuser Verlag series "Frontiers in Mathematics".

The symbol $L \subseteq^* M$ means that the module L is large (or essential) in M, $L \subseteq^\circ M$ means that the L is small (or negligible) in M. All other notations in this note are standard.

2. Regular homomorphisms and the total

First I will explain why I got interested in substructures of Hom. There was a mathematical reason. For a ring R with $1 \in R$ let Mod-R denote the category of all unitary right R-modules. For two of these, A and M, we set $H := \mathrm{Hom}_R(A, M)$, $S := \mathrm{End}(M_R)$, and $T := \mathrm{End}(A_R)$. Then H is an S-T-bimodule. Homomorphisms $f \in \mathrm{Hom}_R(A, M)$ act on the left side on A.

Lemma 2.1. *The following are equivalent for* $f \in \mathrm{Hom}_R(A, M)$.

1. *There exists* $g \in \mathrm{Hom}_R(M, A)$ *such that* $e := gf = e^2 \neq 0$.
2. *There exists* $h \in \mathrm{Hom}_R(M, A)$ *such that* $d := fh = d^2 \neq 0$.
3. *There exists* $k \in \mathrm{Hom}_R(M, A)$ *such that* $kfk = k \neq 0$.
4. *There exist* $0 \neq A_0 \subseteq^\oplus A$ *and* $M_0 \subseteq^\oplus M$, *and such that*

$$A_0 \ni x \mapsto f(x) \in M_0 \quad \text{is an isomorphism.}$$

Definition 2.2. If the conditions of Lemma 2.1 are satisfied, then f is called **partially invertible**. The **total** of A to M is defined to be

$$\mathrm{Tot}(A, M) := \{f \in \mathrm{Hom}_R(A, M) \mid f \text{ is not partially invertible.}\}.$$

When I had defined the total I realized that it has two interesting properties. First, it contains all the classical substructures of Hom, of which we give two examples. Furthermore, it has intersection 0 with all regular substructures.

3. Singular and cosingular submodules

Definition 3.1. The **singular submodule** of ${}_SH_T$ is by definition the S-T-submodule

$$\Delta(A, M) := \{f \in \mathrm{Hom}_R(A, M) \mid \mathrm{Ker}(f) \subseteq^* A\}.$$

The **cosingular submodule** of H is the S-T-submodule

$$\nabla(A, M) := \{f \in \mathrm{Hom}_R(A, M) \mid \mathrm{Im}(f) \subseteq^\circ M\},$$

We prove here only that $\Delta(A, M) \subseteq \mathrm{Tot}(A, M)$.

Proof. To see this, assume to the contrary that $f \in \Delta(A, M)$ is partially invertible. Then there exists $g \in \mathrm{Hom}_R(M, A)$ such that

$$e := gf = e^2 \neq 0.$$

Since $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(e)$, the idempotent e must also have a large kernel. Since the kernel of e is $(1 - e)T$ and is a direct summand of T, it must equal T, eT must be 0 which implies that $e = 0$, a contradiction. □

4. The radicals for $\mathrm{Hom}_R(A, M)$

Several radicals can be defined for $H := \mathrm{Hom}_R(A, M)$: the radical of ${}_SH$ as S-module, and similarly the radical of H_T as a T-module, and finally the most important radical for which there are two equivalent definitions:

$$\begin{aligned} \mathrm{Rad}(\mathrm{Hom}_R(A, M)) &= \mathrm{Rad}(A, M) \\ &= \{f \in H \mid f\,\mathrm{Hom}_R(M, A) \subseteq \mathrm{Rad}(S)\} \\ &= \{f \in H \mid \mathrm{Hom}_R(M, A)f \subseteq \mathrm{Rad}(T)\}. \end{aligned}$$

We verify the equality
$\{f \in H \mid f\,\mathrm{Hom}_R(M, A) \subseteq \mathrm{Rad}(S)\} = \{f \in H \mid \mathrm{Hom}_R(M, A)f \subseteq \mathrm{Rad}(T)\}$

Proof. (1) Let $\emptyset \neq A \subseteq R$ with $AR = A$. It is well known that $A \subseteq \mathrm{Rad}(R)$ if and only if for all $a \in A$, $1 - a$ is an invertible element in R.

(2) Let $f \in \mathrm{Hom}(A, M)$ and $g \in \mathrm{Hom}(M, A)$. If $1_S - fg$ is invertible in S, then it is easy to check that

$$(1_T - gf)^{-1} = 1_T + g(1_S - fg)^{-1}f.$$

Similarly, if $1_T - gf$ is invertible in T, then

$$(1_S - fg)^{-1} = 1_S + f(1_T - gf)^{-1}g.$$

(3) Suppose that $f \in \mathrm{Hom}_R(A, M)$ and for every $g \in \mathrm{Hom}_R(M, A)$, $fg \in \mathrm{Rad}(S)$. Let $g \in \mathrm{Hom}_R(M, A)$, $s \in S$, and $t \in T$. Then also $tgs \in \mathrm{Hom}_R(M, A)$, and we have $ftgs \in \mathrm{Rad}(S)$. Hence the right ideal $ftgS$ is contained in $\mathrm{Rad}(S)$. By (1) it follows that $1_S - ftgs$ is invertible in S. By (2) $1_T - tgsf$ is invertible in T. Then, by (1) again $Tgsf$ is a left ideal contained in $\mathrm{Rad}(T)$. Therefore $f \in \{f \in H \mid \forall\, g \in \mathrm{Hom}_R(M, A), gf \in \mathrm{Rad}(T)\}$. The reverse containment follows in a similar fashion. $\square$

We will show next that $\mathrm{Rad}(A, M) \subseteq \mathrm{Tot}(A, M)$.

Proof. Indirect. Assume that $f \in \mathrm{Rad}(A, M)$, but $f \notin \mathrm{Tot}(A, M)$, so that f is partially invertible. Then there exists $g \in \mathrm{Hom}_R(M, A)$ such that

$$e := fg = e^2 \neq 0.$$

By definition of $\mathrm{Rad}(S)$ we have $e = fg \in \mathrm{Rad}(S)$. It follows from $fgS = eS \subseteq^\circ S_S$ and

$$S = eS \oplus (1 - e)S$$

that $(1 - e)S = S$ and $eS = 0$, and we arrive at the contradiction that $e = 0$. $\square$

5. Regular substructures of Hom

Let $f \in \mathrm{Hom}_R(A, M)$. Then f is **regular** if there exists $g \in \mathrm{Hom}_R(M, A)$ such that $fgf = f$. If so, g is a **quasi-inverse** of f. The following characterization of regularity is well known.

Proposition 5.1. *$f \in \mathrm{Hom}_R(A, M)$ is regular if and only if $\mathrm{Ker}(f) \subseteq^\oplus A$ and $\mathrm{Im}(f) \subseteq^\oplus M$.*

There exists a largest regular S-T-submodule of $H = \mathrm{Hom}_R(A, M)$, denoted by $\mathrm{Reg}(A, M)$. Here "largest" means that any other regular S-T-submodule of H is contained in $\mathrm{Reg}(A, M)$.

Theorem 5.2. *$\mathrm{Reg}(A, M) := \{f \in \mathrm{Hom}_R(A, M) \mid SfT$ is regular$\}$ is the largest regular S-T-submodule of $\mathrm{Hom}_R(A, M)$.*

We now show that $\mathrm{Reg}(A, M)$ and $\mathrm{Tot}(A, M)$ are opposite substructures.

Proposition 5.3.

$$\mathrm{Reg}(A, M) \cap \mathrm{Tot}(A, M) = 0.$$

Proof. This follows because every nonzero regular homomorphism is partially invertible. $\square$

Regular maps produce projective summands.

Theorem 5.4. *Let $0 \neq f \in \mathrm{Hom}_R(A, M)$ be regular. Then the following statements hold.*

1. *Sf is a nonzero S-projective direct summand of* ${}_S\mathrm{Hom}_R(A, M)$ *that is isomorphic to a cyclic left ideal of S that is a direct summand of S.*
2. *fT is a nonzero T-projective direct summand of* $\mathrm{Hom}_R(A, M)_T$ *that is isomorphic to a cyclic right ideal of T that is a direct summand of T.*

There are interesting results on the structure of $\mathrm{Reg}(A, M)$.

Theorem 5.5. *Every finitely or countably generated S-submodule of* $\mathrm{Reg}(A, M)$ *is a direct sum of cyclic S-projective submodules that are isomorphic to left ideals of S that are direct summands of S.*

Every finitely generated S-submodule L of $\mathrm{Reg}(A, M)$ *is S-projective and a direct summand of* ${}_S\mathrm{Hom}_R(A, M)$.

The analogous results hold for $\mathrm{Reg}(A, M)$ as a right T-module.

6. Historical remarks about the total

The first time I defined the total was in 1982 in the lecture notes "Moduln mit LE-Zerlegung und Harada-Moduln" [1]. Subsequently the total was studied in several papers, for which I give here some examples:

– Partiell invertierbare Homomorphismen und das Total, Algebraberichte Nr. 60, Verlag R. Fischer, München (1988)
– The Total in the Category of Modules, General Algebra, Elsevier Science Publishers B.V. (North Holland), (1990)
– The Total of Modules and Rings (mit W. Schneider) Algebraberichte Nr. 69, Verlag R. Fischer, München (1992)

All these papers are mentioned in the bibliograhpy of the two books [2] and [3] which contain more details about the total and related topics.

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Ring and Module Theory
Trends in Mathematics, 129–134

Weak Lifting Modules with Small Radical

Derya Keskin Tütüncü and Saad H. Mohamed

Abstract. Keskin and Tribak (2005) studied the structure of weak lifting modules with small radicals over commutative noetherian (local) rings. They proved that a module M is weak lifting if and only if M is a direct sum of local modules of some special type. Such modules were further studied by Tribak (2007). In this note we study weak lifting modules M with small radical over arbitrary rings. We prove that M is an irredundant sum $M = \sum_{i \in I} M_i$ where each M_i is local and $\sum_{i \in F} M_i$ is a summand of M for every finite subset F of I. Moreover $\sum_{i \in F} M_i = \oplus_{i \in F} K_i$ with K_i local. In particular a finitely generated weak lifting module is a direct sum of local modules. This generalizes the analogous result for finitely generated lifting modules.

Mathematics Subject Classification (2000). Primary 16D10; Secondary 16D99.

Keywords. Weak lifting module.

1. Definitions, notations and preliminaries

R will denote an associative ring with unity and M will denote a unital right R-module. By a summand of M, we mean a direct summand (with the notation $\leq_d$). The notation $X \ll Y$ will mean that X is *small* in Y, that is $X + Z \neq Y$ for any proper submodule Z of Y. Let $A \leq M$. A submodule $C \leq M$ is a *complement* of A if C is maximal such that $A \cap C = 0$. Dually, $S \leq M$ is called a *supplement* of A if S is minimal such that $A + S = M$. We note that S is a supplement of A in M if and only if $M = A + S$ and $A \cap S \ll S$. A submodule $P \leq M$ is a *pseudo supplement* of A if $M = A + P$ and $A \cap P \ll M$. Complements exist by Zorn's Lemma; in fact if $A \cap B = 0$, for $B \leq M$, then B is contained in a complement of A. By contrast, (pseudo) supplements need not exist, in general. M is called (*pseudo*) *supplemented* if every submodule of M has a (pseudo) supplement. M is called *amply supplemented* if B contains a supplement of A whenever $M = A + B$. We call a module M *near supplemented* if M is pseudo supplemented and every maximal submodule has a supplement. (The terminology for supplementation is not followed consistently in the literature, cf. [2], [5] and [9].)

M is called a *CS-module* (*or extending*) if every complement submodule is a summand. M is called an *SS-module* if every supplement submodule is a summand. An SS-module is called *lifting* (*weak lifting*) if M is amply supplemented (supplemented). A module M is called *discrete* if it is lifting and satisfies the following condition:

(D_2) Any submodule N of M with $M/N \cong K \leq_d M$, is a summand of M.

The Jacobson radical of M, denoted by $\operatorname{Rad} M$ is the intersection of all maximal submodules of M. If M has no maximal submodules, then we set $\operatorname{Rad} M = M$. It is known that $\operatorname{Rad} M$ is the sum of all small submodules of M. Thus an arbitrary sum of small submodules of M is small in M if and only if $\operatorname{Rad} M \ll M$. A module H is called *hollow* if every proper submodule of H is small. A hollow module H with $\operatorname{Rad} H \neq H$ is such that $H = xR$ for every $x \notin \operatorname{Rad} H$. Such a module H is called a *local* module.

Lemma 1.1. *Let A be a proper nonsmall submodule of M. If A has a pseudo supplement B such that $B \nleq \operatorname{Rad} M$, then A is contained in a maximal submodule of M.*

Proof. We have $M = A + B$ and $A \cap B \ll M$. Clearly B is a proper submodule of M. There exists a maximal submodule N of M such that $B \nleq N$. Hence $M = N + B$, and so $N \cap B$ is maximal in B. Also $A \cap B \leq \operatorname{Rad} M$, and hence $A \cap B \leq N$. Write $K = A + (N \cap B)$. Since $M/K \cong B/(N \cap B)$, K is a maximal submodule of M. □

Corollary 1.2. *Assume M has small radical. If a proper submodule A of M has a pseudo supplement, then A is contained in a maximal submodule.*

Proof. If A is small in M, then $A \leq \operatorname{Rad} M$, hence A is contained in every maximal submodule of M. So assume A is not small in M. Let B be a pseudo supplement of A. It is clear that B is not small in M and so $B \nleq \operatorname{Rad} M$. The result now follows by Lemma 1.1. □

A module M is called *coatomic* if every proper submodule is contained in a maximal submodule; clearly M has small radical.

Corollary 1.3. *A pseudo supplemented module M with small radical is coatomic.*

The proof of the following theorem is straightforward. We include a short proof for the reader's convenience (cf. [2, 17.2]).

Theorem 1.4. *Let M be a module with small radical. The following are equivalent:*

(1) *M is pseudo supplemented;*
(2) *$M/\operatorname{Rad} M$ is semisimple;*
(3) *M is an irredundant sum of modules M_i such that $M_i \cap \operatorname{Rad} M$ is maximal in M_i;*
(4) *M is a sum of modules M_i such that $M_i \cap \operatorname{Rad} M$ is maximal in M_i.*

Proof. (1)⇒(2): Let $X = A/\operatorname{Rad} M \leq M/\operatorname{Rad} M$. There exists $B \leq M$ such that $M = A + B$ and $A \cap B \ll M$. It follows that $A \cap B \leq \operatorname{Rad} M$. Then clearly $M/\operatorname{Rad} M = X \oplus Y$ where $Y = (B + \operatorname{Rad} M)/\operatorname{Rad} M$. (Here there is no need for $\operatorname{Rad} M$ to be small.)

(2)⇒(1): Let $A \leq M$. There exists $B \leq M$ such that

$$M/\operatorname{Rad} M = (A + \operatorname{Rad} M)/\operatorname{Rad} M \oplus (B + \operatorname{Rad} M)/\operatorname{Rad} M.$$

Then $M = A + B + \operatorname{Rad} M = A + B$ and $A \cap B \leq \operatorname{Rad} M$; hence $A \cap B \ll M$.

(2)⇒(3): $M/\operatorname{Rad} M = \oplus_{i \in I} X_i$, with X_i simple. For each X_i there exists $M_i \leq M$ such that $X_i = (M_i + \operatorname{Rad} M)/\operatorname{Rad} M \cong M_i/(M_i \cap \operatorname{Rad} M)$. It follows that $M = \sum_{i \in I} M_i + \operatorname{Rad} M = \sum_{i \in I} M_i$. The irredundancy of the sum follows from the direct sum of the X_i.

(3)⇒(4): Trivial.

(4)⇒(2): $M = \sum_{i \in I} M_i = \sum_{i \in I} M_i + \operatorname{Rad} M = \sum_{i \in I}(M_i + \operatorname{Rad} M)$. So

$$M/\operatorname{Rad} M = \sum\nolimits_{i \in I}(M_i + \operatorname{Rad} M)/\operatorname{Rad} M = \sum\nolimits_{i \in I}((M_i + \operatorname{Rad} M)/\operatorname{Rad} M)$$

where each $(M_i + \operatorname{Rad} M)/\operatorname{Rad} M$ is isomorphic to $M_i/(M_i \cap \operatorname{Rad} M)$. □

Lemma 1.5. *A summand of an SS-module is SS.*

Proof. Use Lemma 1.7 of [6]. □

Lemma 1.6. *A finite sum of supplemented modules is supplemented.*

Proof. [6] Corollary 1.14. □

2. Main results

A weak lifting module M is a direct sum $M = M_1 \oplus M_2$ where $\operatorname{Rad} M_1 \ll M_1$ and $\operatorname{Rad} M_2 = M_2$ (cf. [5], page 96). So it is natural to study weak lifting modules with small radical. The study of radical weak lifting modules seems to be much more difficult.

Lemma 2.1. *If L is a supplement of a maximal submodule N of M, then L is local.*

Proof. We have $M = N + L$ with $N \cap L \ll L$. Also $N \cap L$ is maximal in L. Hence L is local with $N \cap L$ as its radical. □

Theorem 2.2. *Let M be a module with small radical. Then M is near supplemented if and only if M is an irredundant sum of local modules.*

Proof. 'If': Let $M = \sum_{i \in I} M_i$ be an irredundant sum with M_i local. As $\operatorname{Rad} M_i \leq M_i \cap \operatorname{Rad} M$, we get $\operatorname{Rad} M_i = M_i \cap \operatorname{Rad} M$. Then M is pseudo supplemented by Theorem 1.4. Let N be a maximal submodule of M. Clearly $M_i \not\leq N$ for some $i \in I$. It follows that M_i is a supplement of N.

'Only if': Let $K(M)$ denote the sum of all submodules of M that are supplements of maximal submodules. Assume that $K(M) \neq M$. As M is coatomic by Corollary

1.3, $K(M)$ is contained in a maximal submodule N of M. Let L be a supplement of N. Then $L \leq K(M) \leq N$, a contradiction. Hence $M = K(M)$. It follows now by Lemma 2.1 that $M = \sum_{i\in I} M_i$ with M_i local. As M_i is a supplement submodule, $\operatorname{Rad} M_i = M_i \cap \operatorname{Rad} M$. Then the sum is irredundant by Theorem 1.4. □

Corollary 2.3. *A supplemented module with small radical is an irredundant sum of local modules.*

Corollary 2.4. *The following are equivalent for a finitely generated module M:*

(1) *M is near supplemented.*
(2) *M is a finite irredundant sum of local modules.*
(3) *M is a finite sum of local modules.*
(4) *M is supplemented.*

Proof. Follows by Theorem 2.2 and Lemma 1.6. □

We note that a direct sum of hollow modules need not be supplemented; for example the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is a direct sum of hollow modules (see p. 211, [2]). However $\mathbb{Q}/\mathbb{Z}$ is equal to its own radical. We do not know whether a module with small radical which is a (direct) sum of local modules is supplemented!

Lemma 2.5. *Let H be a nonsmall hollow submodule of a module M. Then:*

(1) *H is a supplement submodule in M;*
(2) *If $H \nleq \operatorname{Rad} M$, then H is local.*

Proof. (1) is obvious.

(2) There exists a maximal submodule N of M such that $H \nleq N$. Then clearly H is a supplement of N. Hence H is local by Lemma 2.1. □

Lemma 2.6. *Let $N = A + H$ where A is a summand and H is a hollow submodule of an SS-module M. Then $N = A \oplus K$ with K hollow. Further if K is not small in M, then N is a summand of M.*

Proof. Write $M = A \oplus B$ and $K = N \cap B$. Then $N = A \oplus K$. Now $K \cong N/A = (A + H)/A \cong H/(A \cap H)$. Hence K is hollow. If K is not small in M, then K is a supplement submodule in M by Lemma 2.5. Hence K is a summand of M, and consequently N is a summand of M. □

The following result is, in some sense, a generalization of ([2], 22.18).

Theorem 2.7. *Let M be a near supplemented SS-module with small radical. Then:*

(1) *M is an irredundant sum $M = \sum_{i\in I} M_i$ where each M_i is local;*
(2) *$\sum_{i\in F} M_i$ is a summand of M and weak lifting for every finite subset $F \subseteq I$, and $\sum_{i\in F} M_i = \oplus_{i\in F} K_i$ with K_i local.*

Proof. (1) follows by Theorem 2.2.

(2) Clearly none of the M_i is small in M and so M_i is a supplement submodule in M by Lemma 2.5. Hence M_i is a summand of M. Let F be a finite subset of I. The irredundancy of the sum $M = \sum_{i\in I} M_i$ and Lemma 2.6 imply that $\sum_{i\in F} M_i$ is a summand of M and $\sum_{i\in F} M_i = \oplus_{i\in F} K_i$ with K_i local. By Lemmas 1.5 and 1.6, $\sum_{i\in F} M_i$ is weak lifting. □

Let A and B be two modules. A is said to be B-*co-jective* if any supplement of B in $M = A \oplus B$ is a summand of M (cf. [6]). A is said to be B-*dual-ojective* if, for any epimorphism $\pi : B \longrightarrow X$ and any homomorphism $\varphi : A \longrightarrow X$, there exist decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ together with a homomorphism $\varphi_1 : A_1 \longrightarrow B_1$ and an epimorphism $\varphi_2 : B_2 \longrightarrow A_2$ such that $\pi\varphi_1 = \varphi\mid_{A_1}$ and $\varphi\varphi_2 = \pi\mid_{B_2}$. A module M is said to have the *exchange property* if for any index set I, whenever $M \oplus N = \oplus_{i\in I} A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\oplus_{i\in I} B_i)$ for submodules $B_i \leq A_i$. A decomposition $M = \oplus_{i\in I} M_i$ is *exchangeable* if for any $N \leq_d M$, we have $M = \oplus_{i\in I} M'_i \oplus N$ with $M'_i \leq M_i$.

The following corollary is an extension of a result of Keskin and Lomp ([4]).

Corollary 2.8. *A finitely generated weak lifting module is a finite direct sum of local modules, which are mutually co-jective in pairs.*

Proof. The result follows by Theorem 2.7 and [6, Theorem 2.4]. □

Let $M = \oplus_{i=1}^n M_i$ where M_i is local and M_j- co-jective for $i \neq j$. Does this imply that M is weak lifting? This is indeed true for $n = 2$, by [6, Theorem 2.4]. It is also interesting to investigate the structure of the M_i's without assuming conditions on the ring!

Vasconcelos proved that, over a commutative ring R, any finitely generated R-module M is Hopfian, that is, every surjective endomorphism of M is an isomorphism (cf. [8]). Therefore by [5, Lemma 5.1], over a commutative ring, any local module is discrete. Using this observation, Corollary 2.8 can be improved for modules over commutative rings.

Proposition 2.9. *Let M be a finitely generated weak lifting module over a commutative ring. Then:*

(1) *M is lifting.*
(2) *M has the exchange property.*
(3) *$M = \oplus_{i=1}^n M_i$ where each M_i is local and M_j-dual-ojective for $i \neq j$.*

Proof. (1) M is amply supplemented by Zöschinger's results (see also [3, Lemma 3.2]).

(2) By Corollary 2.8, M has a decomposition $M = \oplus_{i=1}^n M_i$ with each M_i local. By the above remark, each M_i is discrete. It then follows by [5, Corollary 5.5] that M_i has local endomorphism ring. Hence M has the exchange property by [1, Corollary 12.7] and [5, Theorem 2.25].

(3) Consequently, the decomposition $M = \oplus_{i=1}^n M_i$ is exchangeable. Then (3) follows by [6, Proposition 3.5] □

Acknowledgment

The authors thank the referee for valuable suggestions and for carefully reading their manuscript. This work was done during the second author's visit to Department of Mathematics, Hacettepe University. The authors gratefully acknowledge the support from TÜBİTAK (Turkish Scientific Research Council).

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Ring and Module Theory
Trends in Mathematics, 135–142

FP-injective Complexes

Lixin Mao and Nanqing Ding

Abstract. A complex Q is said to be FP-injective if, for any monomorphism $\psi : A \to B$ of complexes with A finitely generated and B finitely generated free and any morphism $\varphi : A \to Q$ of complexes, there exists $\gamma : B \to Q$ such that $\gamma\psi = \varphi$. It is proven that Q is an FP-injective complex if and only if every Q^i is FP-injective and $\mathrm{Hom}^{\cdot}(G, Q)$ is exact for every finitely presented complex G.

Mathematics Subject Classification (2000). Primary 16E05; Secondary 16D50.

Keywords. FP-injective complex, FP-injective module, preenvelope.

1. Introduction

The homological theory of complexes of modules has been studied by many authors such as Avramov, Enochs, Foxby, García Rozas, Goddard, Jenda, Oyonarte and Xu (see [1, 5, 6, 7, 8, 10]). Projective, injective and flat complexes play important roles in the studies of the category of complexes. In this paper, we introduce the concept of FP-injective complexes and study some properties of FP-injective complexes. We prove that Q is an FP-injective complex if and only if every Q^i is FP-injective and $\mathrm{Hom}^{\cdot}(G, Q)$ is exact for every finitely presented complex G. For a left coherent ring R, it is shown that a bounded below left R-module complex Q is FP-injective if and only if Q is an exact complex of left FP-injective modules.

We next recall some known notions and facts needed in the sequel.

Let R be a ring. An R-module complex

$$\cdots \to C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots$$

will be denoted by C. $Z(C)$ and $B(C)$ stand for the subcomplexes of cycles and boundaries of C respectively, and we let the homology group $H(C) = Z(C)/B(C)$.

This research was partially supported by NSFC (No. 10771096), NSF of Jiangsu Province of China (No. BK2008365), Jiangsu 333 Project, and Jiangsu Qinglan Project, Science Research Fund of Nanjing Institute of Technology. The authors would like to thank the referee for the helpful comments and suggestions.

Given a complex C and an integer m, $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ and whose boundary operators are $(-1)^m\delta^{m+n}$.

A complex C is called *bounded above* when $C^i = 0$ for i sufficiently large, *bounded below* when $C^{-i} = 0$ for i sufficiently large and, *bounded* when it is both bounded above and below.

According to [8], C is called a *finitely generated complex* if C is bounded and every C^i is a finitely generated R-module. C is said to be a *finitely presented complex* if C is bounded and every C^i is a finitely presented R-module.

C is called a *projective* (respectively, *injective, flat) complex* if C is an exact complex and every $\ker\delta^i$ is a projective (respectively, injective, flat) module.

If $f : C \to D$ is a morphism of complexes, the *mapping cone* of f, denoted by $M(f)$, is a complex such that $M(f)^n = D^n \oplus C^{n+1}$ and with boundary operators such that $(x, y) \in D^n \oplus C^{n+1}$ is mapped to $(\delta^n(x) + f(x), -\delta^{n+1}(y))$. It is easy to check that there is an exact sequence $0 \to D \to M(f) \to C[1] \to 0$ of complexes.

Let C and D be complexes. Following [8, p. 32–33], $\mathrm{Hom}(C, D)$ means the abelian group of morphisms from C to D, $\mathrm{Ext}^i(C, D)$ for $i \geq 0$ stands for the groups we get from the right derived functor of Hom, and $\mathrm{Hom}^{\cdot}(C, D)$ denotes the complex of abelian groups with $\mathrm{Hom}^{\cdot}(C, D)^n = \Pi_{t\in\mathbb{Z}}\mathrm{Hom}(C^t, D^{t+n})$ and $(\delta^n f)^t = \delta_D^{t+n} f^t - (-1)^n f^{t+1}\delta_C^t$ for $f \in \Pi_{t\in\mathbb{Z}}\mathrm{Hom}(C^t, D^{t+n})$. It is clear that the abelian group $\mathrm{Hom}(C, D)$ is a subgroup of $\mathrm{Hom}^{\cdot}(C, D)^0$.

Given an R-module M, we will denote by $\overline{M}$ the complex

$$\cdots \to 0 \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0 \to 0 \cdots$$

with M in the -1th and 0th positions and 0 in the other positions. Also we mean by $\underline{M}$ the complex

$$\cdots \to 0 \to 0 \to M \to 0 \to 0 \cdots$$

with M in the 0th place and 0 in the other places.

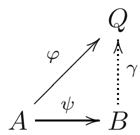
A left R-module M is said to be *FP-injective* (or *absolutely pure*) [11, 12] if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented left R-modules N. R is called a *left coherent ring* if every finitely generated left ideal of R is finitely presented.

Throughout this paper, R is an associative ring with identity and all modules are unitary. We write R-Mod for the category of all left R-modules and $\mathcal{C}$ the category of all complexes of left R-modules. For unexplained concepts and notations, we refer the reader to [4, 8, 9, 13].

2. *FP*-injective complexes

We begin with the following definition.

Definition 2.1. Q is called an *FP-injective complex* if given any monomorphism $A \xrightarrow{\psi} B$ of complexes and any morphism $A \xrightarrow{\varphi} Q$ of complexes, where A is finitely generated and B is finitely generated free, the diagram



can be completed by a morphism $\gamma : B \to Q$ of complexes commutatively.

Remark 2.2.

(1) It can be proven in a standard way that the class of FP-injective complexes is closed under direct products and direct summands. Moreover we claim that the class of FP-injective complexes is closed under direct sums. In fact. Let $\{Q_i : i \in I\}$ be a set of FP-injective complexes, $A \xrightarrow{\psi} B$ any monomorphism of complexes and $A \xrightarrow{\varphi} \oplus_{i\in I} Q_i$ any morphism of complexes, where A is finitely generated and B is finitely generated free. Since $\mathrm{im}(\varphi)$ is finitely generated, $\mathrm{im}(\varphi)$ embeds in $\oplus_{i\in J} Q_i$ for some finite set $J \subseteq I$. So φ factors through $\oplus_{i\in J} Q_i$, i.e., there are $\alpha : A \to \oplus_{i\in J} Q_i$ and $\lambda : \oplus_{i\in J} Q_i \to \oplus_{i\in I} Q_i$ such that $\varphi = \lambda\alpha$. Since $\oplus_{i\in J} Q_i$ is FP-injective, there exists $\beta : B \to \oplus_{i\in J} Q_i$ such that $\alpha = \beta\psi$. Thus $\varphi = (\lambda\beta)\psi$, and so $\oplus_{i\in I} Q_i$ is an FP-injective complex.

(2) If Q is an FP-injective complex, then $Q[i]$ is also an FP-injective complex for any $i \in \mathbb{Z}$.

(3) Note that $\mathrm{Hom}(A, \overline{Q}) \cong \mathrm{Hom}(A^0, Q)$ in the obvious fashion, so if Q is an FP-injective module, then $\overline{Q}$ is an FP-injective complex.

(4) Recall that C is called a *split complex* [13] if there are maps $s^n : C^{n+1} \to C^n$ such that $\delta^n = \delta^n s^n \delta^n$ for all $n \in \mathbb{Z}$. If Q is a split exact complex of FP-injective modules, then Q is a direct sum of complexes of the form $\overline{A^i}[n]$ with each A^i FP-injective modules. So Q is an FP-injective complex.

Theorem 2.3. *The following are equivalent for a complex Q:*

1. *Q is an FP-injective complex.*
2. $\mathrm{Ext}^1(G, Q) = 0$ *for every finitely presented complex G.*
3. *Every Q^i is FP-injective and* $\mathrm{Hom}^{\cdot}(G, Q)$ *is exact for every finitely presented complex G.*
4. *For any short exact sequence of complexes* $0 \to Q \to E \to L \to 0$ *and any finitely presented complex G, the sequence* $\mathrm{Hom}(G, E) \to \mathrm{Hom}(G, L) \to 0$ *is exact.*

Proof. (1) $\Rightarrow$ (2) For a finitely presented complex G, there exists an exact sequence of complexes $0 \to K \to P \to G \to 0$ with P finitely generated free and K finitely generated. So we get the induced exact sequence

$$\mathrm{Hom}(P, Q) \to \mathrm{Hom}(K, Q) \to \mathrm{Ext}^1(G, Q) \to \mathrm{Ext}^1(P, Q) = 0.$$

Since the sequence $\mathrm{Hom}(P, Q) \to \mathrm{Hom}(K, Q) \to 0$ is exact by (1), $\mathrm{Ext}^1(G, Q) = 0$.
(2) $\Rightarrow$ (1) is clear.

(2) $\Rightarrow$ (3) Let $0 \to Q^i \to H \to G \to 0$ be any exact sequence of modules with G finitely presented. Then we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q^{i-1} & \xrightarrow{id} & Q^{i-1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q^{i} & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & Q^{i+1} & \longrightarrow & L & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q^{i+2} & \xrightarrow{id} & Q^{i+2} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Since $\overline{G}[-i-1]$ is a finitely presented complex, the above short exact sequence of complexes is split by (2). Thus the exact sequence $0 \to Q^i \to H \to G \to 0$ is split, and so Q^i is FP-injective.

For a finitely presented complex G, the short exact sequence of complexes

$$0 \to Q[i] \to M(f) \to G[1] \to 0$$

is split for any $i \in \mathbb{Z}$ and any map $f : G \to Q[i]$. So f is homotopic to 0 by [8, Lemma 2.3.2]. It is easy to check that $\mathrm{Hom}^{\cdot}(G, Q)$ is exact.

(3) $\Rightarrow$ (2) Let $0 \to Q \to H \to G \to 0$ be any exact sequence of complexes with G finitely presented. Since every Q^i is FP-injective, the exact sequence is split at the module level. So the exact sequence is isomorphic to

$$0 \to Q \to M(f) \to G \to 0,$$

where $f : G[-1] \to Q$ is a map of complexes. Because $\mathrm{Hom}^{\cdot}(G[-1], Q)$ is exact, f is homotopic to 0. Hence $0 \to Q \to M(f) \to G \to 0$ is a split exact sequence of complexes by [8, Lemma 2.3.2]. Thus $\mathrm{Ext}^1(G, Q) = 0$.

(2) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (2) There exists an exact sequence of complexes $0 \to Q \to E \to L \to 0$ with E injective. For any finitely presented complex G, we have the exact sequence

$$\mathrm{Hom}(G, E) \to \mathrm{Hom}(G, L) \to \mathrm{Ext}^1(G, Q) \to \mathrm{Ext}^1(G, E) = 0.$$

By (4), the sequence

$$\text{Hom}(G,E) \to \text{Hom}(G,L) \to 0$$

is exact, and so $\text{Ext}^1(G,Q) = 0$. □

Corollary 2.4. *Every FP-injective complex is exact.*

Proof. Let Q be an FP-injective complex. Since $\underline{R}[n]$ is finitely presented for any $n \in \mathbb{Z}$, $\text{Ext}^1(\underline{R}[n], Q) = 0$ by Theorem 2.3. So

$$H^{-n+1}(Q) \cong \text{Ext}^1[\underline{R}[n], Q) = 0.$$

Thus Q is an exact complex. □

Corollary 2.5. *The following are equivalent for a complex F:*

1. *F is a flat complex.*
2. *F^+ is an injective complex, where $F^+ \equiv \cdots \to (F^i)^+ \to (F^{i-1})^+ \to \cdots$ and $(-)^+ = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$.*
3. *F^+ is a split FP-injective complex.*

Proof. (1) ⇔ (2) follows from [8, Theorem 4.1.3].
(2) ⇒ (3) is trivial. (3) ⇒ (1) holds by Corollary 2.4. □

Proposition 2.6. *If Q is an exact complex and every* $\ker(\delta^i)$ *is FP-injective, then* $\text{Ext}^1(G,Q) = 0$ *for every finitely presented exact complex G.*

Proof. Let $0 \to Q \to H \to G \to 0$ be any exact sequence in $\mathcal{C}$ with G a finitely presented exact complex. Thus the exact sequence is split at the module level. So the exact sequence is isomorphic to $0 \to Q \to M(f) \to G \to 0$, where $f : G[-1] \to Q$ is a map of complexes. By hypothesis, f is homotopic to 0. Therefore $0 \to Q \to M(f) \to G \to 0$ is split by [8, Lemma 2.3.2], and so $\text{Ext}^1(G,Q) = 0$. □

Theorem 2.7. *If R is a left coherent ring, then a bounded below complex $Q \in \mathcal{C}$ is FP-injective if and only if Q is an exact complex of FP-injective left R-modules.*

Proof. "⇒" follows from Theorem 2.3 and Corollary 2.4.
"⇐" We may assume that

$$Q = \cdots \to 0 \to Q^0 \to Q^1 \to \cdots .$$

Then there is a short exact sequence of complexes

$$0 \to Q \to E \xrightarrow{g} L \to 0$$

such that

$$E = \cdots \to 0 \to E^0 \xrightarrow{\omega^0} E^1 \to \cdots$$

is injective and

$$L = \cdots \to 0 \to L^0 \xrightarrow{\delta^0} L^1 \to \cdots$$

is exact. Since R is a left coherent ring and every Q^i is FP-injective, every L^i is FP-injective and so every $K^i = \ker(L^i \to L^{i+1})$ is FP-injective by [12, Lemma 3.1].

Let

$$G = \cdots \to 0 \to G^0 \xrightarrow{d^0} G^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} G^n \to 0 \to \cdots$$

be any finitely presented complex and $\alpha : G \to L$ any map. We will construct $\beta : G \to E$ such that $g\beta = \alpha$.

For $j > n$, we define $\beta^j = 0$.

For $j = n$, since $\delta^n \alpha^n = 0$, there exists $\varphi^n : G^n \to K^n$ such that $\lambda^n \varphi^n = \alpha^n$, where $\lambda^n : K^n \to L^n$ is the inclusion. Since K^{n-1} is FP-injective, we get the pure exact sequence

$$0 \to K^{n-1} \to L^{n-1} \to K^n \to 0.$$

So there exists $\gamma^{n-1} : G^n \to L^{n-1}$ such that $\pi^{n-1}\gamma^{n-1} = \varphi^n$, where $\pi^{n-1} : L^{n-1} \to K^n$ is the canonical map. But the short exact sequence

$$0 \to Q^{n-1} \to E^{n-1} \to L^{n-1} \to 0$$

is also pure, and hence there exists $\theta^{n-1} : G^n \to E^{n-1}$ such that $g^{n-1}\theta^{n-1} = \gamma^{n-1}$. Define $\beta^n : G^n \to E^n$ as $\beta^n = \omega^{n-1}\theta^{n-1}$. Then $g^n\beta^n = g^n\omega^{n-1}\theta^{n-1} = \delta^{n-1}g^{n-1}\theta^{n-1} = \delta^{n-1}\gamma^{n-1} = \lambda^n\pi^{n-1}\gamma^{n-1} = \lambda^n\varphi^n = \alpha^n$ and $\omega^n\beta^n = 0$.

For $j = n-1$, we have $\delta^{n-1}\alpha^{n-1} = \alpha^n d^{n-1}$ and $\delta^{n-1}\gamma^{n-1} = \alpha^n$. So $\delta^{n-1}(\alpha^{n-1} - \gamma^{n-1}d^{n-1}) = 0$. Thus there exists $\varphi^{n-1} : G^{n-1} \to K^{n-1}$ such that $\lambda^{n-1}\varphi^{n-1} = \alpha^{n-1} - \gamma^{n-1}d^{n-1}$, where $\lambda^{n-1} : K^{n-1} \to L^{n-1}$ is the inclusion. Hence there exists $\gamma^{n-2} : G^{n-1} \to L^{n-2}$ such that $\pi^{n-2}\gamma^{n-2} = \varphi^{n-1}$, where $\pi^{n-2} : L^{n-2} \to K^{n-1}$ is the canonical map. So we get a map $\theta^{n-2} : G^{n-1} \to E^{n-2}$ such that $g^{n-2}\theta^{n-2} = \gamma^{n-2}$. Define $\beta^{n-1} : G^{n-1} \to E^{n-1}$ as $\beta^{n-1} = \omega^{n-2}\theta^{n-2} + \theta^{n-1}d^{n-1}$. Then $g^{n-1}\beta^{n-1} = g^{n-1}\omega^{n-2}\theta^{n-2} + g^{n-1}\theta^{n-1}d^{n-1} = \delta^{n-2}g^{n-2}\theta^{n-2} + \gamma^{n-1}d^{n-1} = \lambda^{n-1}\pi^{n-2}\gamma^{n-2} + \gamma^{n-1}d^{n-1} = \lambda^{n-1}\varphi^{n-1} + \gamma^{n-1}d^{n-1} = \alpha^{n-1}$. We also have $\omega^{n-1}\beta^{n-1} = \omega^{n-1}\theta^{n-1}d^{n-1} = \beta^n d^{n-1}$.

By induction, we can construct β^j for $j < n-1$ and so have $\beta : G \to E$ such that $g\beta = \alpha$, which implies that the sequence

$$\mathrm{Hom}(G, E) \to \mathrm{Hom}(G, L) \to 0$$

is exact. Thus $\mathrm{Ext}^1(G, Q) = 0$, and so Q is FP-injective by Theorem 2.3. □

It is well known that R is a left Noetherian ring if and only if every FP-injective left R-module is injective. By Theorem 2.7, we have

Corollary 2.8. *The following are equivalent for a ring R:*

1. *R is a left Noetherian ring.*
2. *Every bounded below FP-injective complex is injective.*

Corollary 2.9. *The following are equivalent for a ring R:*

1. *R is a left coherent ring.*
2. *A bounded below complex $F \in \mathcal{C}$ is FP-injective if and only if F^+ is flat.*
3. *For any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ with A and B FP-injective, C is FP-injective.*

Proof. (1) ⇔ (2) Note that R being left coherent is equivalent to the condition that a left R-module F is FP-injective if and only if F^+ is flat (see [2, Theorem 1]). So (1) ⇔ (2) follows from Theorem 2.7.

(1) ⇔ (3) is easy by [14, 35.9] and Theorem 2.3. □

Let $\mathcal{F}$ be a class of objects in an abelian category $\mathcal{A}$ and X an object in $\mathcal{A}$. Following [3], we say that a morphism $\phi : X \to F$ is an $\mathcal{F}$*-preenvelope* if $F \in \mathcal{F}$ and the map $\mathrm{Hom}(\phi, F') : \mathrm{Hom}(F, F') \to \mathrm{Hom}(X, F')$ is surjective for each $F' \in \mathcal{F}$. An $\mathcal{F}$-preenvelope $\phi : X \to F$ is said to be an $\mathcal{F}$*-envelope* if every endomorphism $g : F \to F$ such that $g\phi = \phi$ is an isomorphism. $\mathcal{F}$-envelopes may not exist in general, but if they exist, they are unique up to isomorphism.

Proposition 2.10. *The following are true:*

1. *Every complex has an FP-injective preenvelope.*
2. *If $\alpha : Q \to F$ is an FP-injective preenvelope in $\mathcal{C}$, then $\alpha^n : Q^n \to F^n$ is an FP-injective preenvelope in R-Mod for every $n \in \mathbb{Z}$.*
3. *If $\alpha : M \to N$ is an FP-injective preenvelope in R-Mod, then the induced map $\overline{\alpha} : \overline{M}[n] \to \overline{N}[n]$ is an FP-injective preenvelope in $\mathcal{C}$ for any $n \in \mathbb{Z}$.*

Proof. (1) follows from [4, Theorem 7.4.1] and Theorem 2.3.

(2) Let N be an FP-injective left R-module and $\beta : Q^n \to N$ any homomorphism. Then we get an induced map of complexes $\gamma : Q \to \overline{N}[-n]$. Since $\overline{N}[-n]$ is an FP-injective complex, there exists $\varphi : F \to \overline{N}[-n]$ such that $\varphi\alpha = \gamma$. Thus $\varphi^n\alpha^n = \gamma^n = \beta$. So $\alpha^n : Q^n \to F^n$ is an FP-injective preenvelope in R-Mod.

(3) is straightforward. □

Remark 2.11. Though every complex has an FP-injective preenvelope, FP-injective envelopes may not exist in general. In fact, let M be an R-module with no FP-injective envelopes, and let the map $M \to E$ be an FP-injective preenvelope of M (such a module M exists by [9, Theorem 4.1.6 and Corollary 6.3.19]), then we easily see that $\overline{M} \to \overline{E}$ is an FP-injective preenvelope. So an FP-injective envelope of $\overline{M}$, if it exists, will be a direct summand of $\overline{E}$, say $\overline{I}$. But then $M \to I$ will be an FP-injective envelope of M, which is a contradiction.

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Ring and Module Theory
Trends in Mathematics, 143–148

A Generalization of Homological Dimensions

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Abstract. The main aim of this paper is to determine how far a module is from tilting or cotilting. With this in mind, we introduce T_m^n-injective modules and T_m^n-projective modules for any non-negative integers m, n and any Wakamatsu tilting module T, and then give some of their characterizations. In particular, for a tilting module T which satisfies an F.S. condition such that $\operatorname{Add} T$ is closed under submodules, we show that if $\frac{M}{N}$ belongs to $\operatorname{Prod} T$, then submodules of T are T_m^0-projective.

Mathematics Subject Classification (2000). 13D05; 13D07.

Keywords. Resolution, Dimension, Wakamatsu tilting module, pure submodule.

1. Introduction

We know that modules with smaller projective (resp. injective) dimension are nearer to projective (resp. injective) modules than the others are, and so they have similar properties in some cases. In this paper, we will investigate modules that have properties similar to a tilting (resp. cotilting) module T. For this aim, we shall introduce some homological dimensions and generalized functors Ext_R^n and Tor_n^R.

Throughout this paper, R will be an associative ring with non-zero identity, all R-modules are unitary R-modules. We also assume that T is a Wakamatsu tilting module, in the sense of [[1], Proposition 2.1]. We denote by $\mathcal{M}(R)$ the category of R-modules, by $\operatorname{Add} T$ ($\operatorname{Prod} T$) the class of modules isomorphic to direct summands of direct sum (product) of copies of T, by $\operatorname{Pres}^n T$ the set of all $M \in \mathcal{M}(R)$ such that there exists an exact sequence

$$T_n \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_1 \rightarrow M \rightarrow 0,$$

with $T_i \in \operatorname{Add} T$. $\operatorname{Pres}^\infty T$ is defined in the obvious way. In Section 2, we will introduce some homological dimensions and study their properties. In Section 3, we introduce the classes of T_m^n-injective and T_m^n-projective modules and then characterize them, in particular when $n = 0$.

2. T-projective dimension

In this section, we shall give some definitions and propositions which will be used in the proof of the main results. If T is a Wakamatsu tilting module, then for any $M \in \operatorname{Gen} T$ (resp. $M \in \operatorname{Cogen} T$), the T-projective (resp. T-injective) dimension of M is defined to be the least non-negative integer n such that there exists a long exact sequence

$$0 \rightarrow T_n \overset{\delta_n}{\rightarrow} \cdots \overset{\delta_2}{\rightarrow} T_1 \overset{\delta_1}{\rightarrow} T_0 \overset{\delta_0}{\rightarrow} M \rightarrow 0,$$

(resp.

$$0 \rightarrow M \overset{\delta^0}{\rightarrow} T^0 \overset{\delta^1}{\rightarrow} T^1 \overset{\delta^2}{\rightarrow} \cdots \rightarrow T^n \rightarrow 0)$$

with $T_i \in \operatorname{Add} T$ (resp. $T^i \in \operatorname{Prod} T$) for any $0 \leq i \leq n$. If $M \in \operatorname{Gen} T$, then

$$\operatorname{Tor}_n^T(M, B) = \frac{\operatorname{Ker}(\delta_n \otimes 1_B)}{\operatorname{Im}(\delta_{n+1} \otimes 1_B)},$$

where

$$\cdots \rightarrow T_2 \overset{\delta_2}{\rightarrow} T_1 \overset{\delta_1}{\rightarrow} T_0 \overset{\delta_0}{\rightarrow} M \rightarrow 0,$$

is a left $\operatorname{Add} T$-resolution of M, and if $M \in \operatorname{Cogen} T$, then

$$\operatorname{Ext}_T^n(C, M) = \frac{ker\delta_*^n}{\operatorname{Im} \delta_*^{n-1}},$$

where

$$0 \rightarrow M \overset{\delta^0}{\rightarrow} T^0 \overset{\delta^1}{\rightarrow} T^1 \overset{\delta^2}{\rightarrow} T^2 \rightarrow \cdots,$$

is a right $\operatorname{Prod} T$-resolution of M.

We show that T.p.dim$(M) \leq n$ (resp. T.i.dim$(M) \leq n$) if and only if $\operatorname{Ext}_T^i(M, -) = 0$ ($\operatorname{Ext}_T^i(-, M) = 0$) for all $i > n$.

Proposition 2.1. *If M is a generated (resp. cogenerated) module by a Wakamatsu tilting (resp. cotilting) module T, then M has a left $\operatorname{Add} T$-resolution (resp. right $\operatorname{Prod} T$-coresolution).*

Proof. Since T is tilting, it is a 1-star module by [1, Theorem 4.3], and hence $\operatorname{Gen}(T) = \operatorname{Pres}^\infty T$ and $M \in \operatorname{Pres}^\infty T$. This shows that, at least M has a left $\operatorname{Add}(T)$-resolution. Now, it is seen that $M \in \operatorname{Cogen} T$ has a right $\operatorname{Prod} T$-resolution. □

The next definition gives a relative version of the derived functors Ext and Tor.

Definition 2.2. Let T be a Wakamatsu tilting module.

i) For any $M \in \operatorname{Gen} T$, we define

$$\operatorname{Tor}_n^T(M, B) := \frac{\operatorname{Ker}(\delta_n \otimes 1_B)}{\operatorname{Im}(\delta_{n+1} \otimes 1_B)}, \quad \text{where } \cdots \overset{\delta_2}{\rightarrow} T_1 \overset{\delta_1}{\rightarrow} T_0 \overset{\delta_0}{\rightarrow} M \rightarrow 0$$

is a left $\operatorname{Add} T$-resolution of M that we chose once for all.

ii) $\operatorname{Ext}^n_T(C, M) := \frac{\operatorname{Ker}\delta^n_*}{\operatorname{Im}\delta^{n-1}_*}$, where

$$0 \quad \to \quad M \quad \overset{\delta^0}{\to} \quad T^0 \quad \overset{\delta^1}{\to} \quad T^1 \quad \overset{\delta^2}{\to} \quad \cdots$$

is a right $\operatorname{Prod} T$-resolution of M that we chose once for all.

One can easily check that the definition of $\operatorname{Tor}^T_n(M, B)$ and $\operatorname{Ext}^n_T(C, M)$ are independent of the choice of $\operatorname{Add} T$-resolutions and $\operatorname{Prod} T$-coresolutions.

Definition 2.3. Let T be a Wakamatsu tilting module.

i) If M is a module generated by T, then we say that M is of T-projective dimension n (abbreviatedly T.p.dim$(M) = n$) if, n is the least non-negative integer such that there exists a long exact sequence

$$0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to M \to 0$$

with $T_i \in \operatorname{Add} T$ for each $i \geq 0$.

ii) If M is cogenerated by T, then we say that M is of T-injective dimension n (abbreviatedly T.i.dim$(M) = n$) if n is the least non-negative integer such that there exists a long exact sequence $0 \to M \to T^0 \to T^1 \to \cdots \to T^n \to 0$ with $T^i \in \operatorname{Prod} T$ for each $i \geq 0$.

Remark 2.4. Let T be a Wakamatsu tilting module. For any $M \in \operatorname{Gen} T$, the following statements are equivalent:

1) T.p.dim$(M) \leq n$.
2) For any $\operatorname{Add} T$-resolution $0 \to T_{n-1} \to T_{n-2} \to \cdots \to T_1 \to T_0 \to M \to 0$, $\operatorname{Ker}(T_{n-1} \to T_{n-2})$ belongs to $\operatorname{Add} T$.
3) $\operatorname{Ext}^i_T(M, B) = 0$ for any $i > n$ and $B \in \mathcal{M}(R)$.

Proposition 2.5. *Consider the* $\operatorname{Add} T$*-resolution*

$$\cdots \quad \to \quad T_2 \quad \overset{\delta_2}{\to} \quad T_1 \quad \overset{\delta_1}{\to} \quad T_0 \quad \overset{\delta_0}{\to} \quad M \quad \to 0$$

and define $K_i = \operatorname{Ker}(\delta_i)$ *for* $i \geq 0$. *Then,*

i) $\operatorname{Tor}^T_{n+1}(M, B) \cong \operatorname{Tor}^T_n(K_0, B) \cong \cdots \cong \operatorname{Tor}^T_1(K_{n-1}, B)$.
ii) $\operatorname{Ext}^{n+1}_T(M, B) \cong \operatorname{Ext}^n_T(K_0, B) \cong \cdots \cong \operatorname{Ext}^1_T(K_{n-1}, B)$.

Proof. i) It is clear that $\cdots \to T_2 \to T_1 \to K_0 \to 0$ is an $\operatorname{Add} T$-resolution of K_0. Define $S_{n-1} = T_n$ and $\Delta_{n-1} = \delta_n$ for each $n \geq 1$. The $\operatorname{Add} T$-resolution now reads $\cdots S_2 \to S_1 \to S_0 \to K_0 \to 0$ by definition

$$\operatorname{Tor}^T_n(K_0, B) \cong \frac{\operatorname{Ker}(\Delta_n \otimes 1_B)}{\operatorname{Im}(\Delta_{n-1} \otimes 1_B)} = \frac{\operatorname{Ker}(\delta_{n+1} \otimes 1_B)}{\operatorname{Im}(\delta_n \otimes 1_B)} = \operatorname{Tor}^T_{n+1}(M, B).$$

ii) This can be proven in a similar manner. □

3. T_m^n-injective modules and T_m^n-projective modules

As explaind in the preceding section, if M is T-injective(resp. T-projective), then $\mathrm{Ext}_T^1(P,M)=0$ (resp. $\mathrm{Ext}_T^1(M,P)=0$) for any $P\in\mathrm{Gen}\,T$ (resp. $P\in\mathrm{Cogen}\,T$) where T is a Wakamatsu tilting module.

Definition 3.1. Let T be a Wakamatsu tilting module and suppose that m and n are fixed non-negative integers.

i) $M\in\mathrm{Cogen}\,T$ is called T_m^n-injective module if $\mathrm{Ext}_T^{n+1}(P,M)=0$ for any $P\in\mathrm{Pres}^m(T)$.

ii) $M\in\mathrm{Gen}\,T$ is called T_m^n-projective if $\mathrm{Ext}_T^1(M,N)=0$ for any T_m^n-injective module N.

Proposition 3.2.

i) *Every T_s^n-injective module is T_m^n-injective for every $s\le m$.*

ii) *Every T_m^n-projective R-module is T_s^n-projective for every $s\le m$.*

iii) *If $M\in\mathrm{Pres}^m\,T$, then M is T_m^0-projective.*

iv) *If M is an epimorphic image of a module belonging to $\mathrm{Add}\,T$ and $\mathrm{Ext}_T^{k+1}(M,N)=0$, for any T_1^0-injective R-module N and any k, $0\le k\le m-1$, then $M\in\mathrm{Pres}^m\,T$.*

Proof. This is a direct consequence of the definitions. □

Theorem 3.3. *Let $0\to A\to B\to C\to 0$ be a short exact sequence.*

(1) *If A is T_m^n-injective and B is T_{m+1}^n-injective, then C is T_{m+1}^n-injective.*

(2) *If C is T_{m+1}^n-projective and B is T_m^n-projective, then A is T_m^n-projective.*

Proof. (1) Let $M\in\mathrm{Pres}^{m+1}\,T$. Then there exists an exact sequence $0\to K\to T_0\to M\to 0$ in which $T_0\in\mathrm{Add}\,T$. Thus, $K\in\mathrm{Pres}^m\,T$ by definition. Now, we get an induced exact sequence $0=\mathrm{Ext}_T^{n+1}(K,A)\to\mathrm{Ext}_T^{n+2}(M,A)\to\mathrm{Ext}_T^{n+2}(T_0,A)=0$ and hence $\mathrm{Ext}_T^{n+2}(M,A)=0$. On the other hand we have the exact sequence $0=\mathrm{Ext}_T^{n+1}(M,B)\to\mathrm{Ext}_T^{n+1}(M,C)\to\mathrm{Ext}_T^{n+2}(M,A)=0$ and so, $\mathrm{Ext}_T^{n+1}(M,C)=0$. Therefore C is T_{m+1}^n-injective.

(2) Let N be any T_m^n-injective module. We have the induced exact sequence $0=\mathrm{Ext}_T^1(B,N)\to\mathrm{Ext}_T^1(A,N)\to\mathrm{Ext}_T^2(C,N)\to\cdots$. Consider the short exact sequence $0\to N\to T^0\to\frac{T^0}{N}\to 0$ ($N\to T^0$ can be a $\mathrm{Prod}\,T$-preenvelope). The module $\frac{T^0}{N}$ is T_{m+1}^n-injective by (1). Thus $\mathrm{Ext}_T^2(C,N)\cong\mathrm{Ext}_T^1(C,\frac{T^0}{N})=0$, because C is T_{m+1}^n-projective by hypothesis and we get $\mathrm{Ext}_T^1(A,N)=0$. □

Let $\cdots\to T_r\to T_{r-1}\to\cdots\to T_1\to T_0\to N\to 0$ be an $\mathrm{Add}\,T$-resolution for N. We define $K_r=\mathrm{Ker}(T_r\to T_{r-1})$ and call it $r^{th}T$-syzygy of N. Also we denote the class of T_m^n-injective (resp. T_m^n-projective) modules by $\mathcal{I}_{m,n}(T)$ (resp. $\mathcal{P}_{m,n}(T)$).

Lemma 3.4. *Let T be a tilting module. If $\mathrm{Add}\,T$ is closed under submodules, then $\mathrm{Pres}^m\,T$ is closed under syzygies.*

Proof. Let $N \in \operatorname{Pres}^m T$. Then by Proposition 2.1 and this fact that

$$\operatorname{Pres}^m T \subseteq \operatorname{Gen} T,$$

there exists an $\operatorname{Add} T$-resolution

$$\cdots \to T_m \to T_{m-1} \to \cdots \to T_1 \to T_0 \to N \to 0.$$

For n, $n \leq m$, let K_n be $n^{th}T$-syzygy of N. Clearly $K_1 \in \operatorname{Pres}^{m-1} T$, thus K_n as an $(n-1)^{th}T$-syzygy of K_1 belongs to $\operatorname{Pres}^m T$ and hence by induction on m we get $K_n \in \operatorname{Pres}^m T$. □

The following definition gives a generalization of the pure submodules.

Definition 3.5. Let M be an R-module. A submodule N of M is called (n, T)-pure submodule if $\operatorname{Ext}^1_T(B, N) = 0$ for any $B \in \operatorname{Pres}^n T$.

Lemma 3.6. *If T is a tilting module and $M \in \operatorname{Gen} T$, then the following statements are equivalent:*

1. *M is T^0_m-injective.*
2. *There exists an exact sequence $0 \to M \to B \to C \to 0$, where M is an (n, T)-pure submodule of B, and B is T^0_m-injective.*

Definition 3.7. We say that an R-module T satisfies $F.S.$ condition if any R-module can be written as $\bigsqcup_{i=1}^{n} \frac{T}{s_i}$ which S_i' s are submodules of T and n a positive integer.

Theorem 3.8 (Main Theorem). *Let T be a tilting module, $\operatorname{Add} T$ is closed under submodules, and T satisfies F.S. condition, then the following are equivalent:*

1. *If $M \in \operatorname{Prod} T$ and N is an (m, T)-pure submodule of M, then the quotient module $\frac{M}{N}$ belongs to $\operatorname{Prod} T$.*
2. *Every submodule of T^0_m-projective module is T^0_m-projective.*
3. *Every submodule of T is T^0_m-projective.*

Proof. (1) ⇒ (2) Let N be a submodule of T^0_m-projective module M, then for any T^0_m-projective module L, we get the exact sequence $0 \to \operatorname{Ext}^1_T(M, L) \to \operatorname{Ext}^1_T(N, L) \to \operatorname{Ext}^2_T(\frac{M}{N}, L)$. Note that L is an (n, T)-pure submodule of its $\operatorname{Prod} T$-envelope $E(L)$, by definition, it follows that $\frac{E(L)}{L} \in \operatorname{Prod} T$ by (1) and $\operatorname{T.i.dim}(L) \leq 1$. Thus, $\operatorname{Ext}^2_T(\frac{M}{N}, L) = 0$.

(2) ⇒ (3) Since T, itself belongs to $\operatorname{Add} T$, thus it is T^0_m-projective. Therefore (3) is a special case of (2).

(3) ⇒ (1) Let N be an (m, T)-pure submodule of M and $M \in \operatorname{Prod} T$, then N is T^0_m-injective by Lemma 3.6. For any module Q, since T satisfies $F.S.$ condition, we have $Q \cong \coprod_{i=1}^{n} \frac{T}{S_i}$ in which S_i is a submodule of T, for any $1 \leq i \leq n$. We know that $\operatorname{Ext}^1_T(Q, \frac{M}{N}) \cong \prod_{i=1}^{n} \operatorname{Ext}^1_T(\frac{T}{S_i}, \frac{M}{N})$. But the exactness of $0 \to N \to M \to \frac{M}{N} \to$ induces to the exactness of $0 = \operatorname{Ext}^1_T(\frac{T}{S_i}, M) \to \operatorname{Ext}^1_T(\frac{T}{S}, \frac{M}{N}) \to \operatorname{Ext}^1_T(\frac{T}{S}, N)$. On the other hand, the exact sequence $0 \to S_i \to T \to \frac{T}{S_i} \to 0$ gives rise to exact sequence $0 = \operatorname{Ext}^1_T(S_i, N) \to \operatorname{Ext}^1_T(\frac{T}{S_i}, N) \to \operatorname{Ext}^2_T(T, N) = 0$. Thus, $\operatorname{Ext}^1_T(\frac{T}{S_i}, \frac{M}{N}) = 0$ which implies that $\operatorname{Ext}^1_T(Q, \frac{M}{N}) = 0$ that is $\frac{M}{N} \in \operatorname{Prod} T$. □

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Trends in Mathematics, 149–164

Additive Rank Functions and Chain Conditions

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Abstract. Let R be a ring with identity and let mod-R denote the category of finitely generated unitary right R-modules. By an additive rank function on mod-R we mean a mapping ρ from mod-R to the set $\mathbb{N}$ of non-negative integers such that, for every exact sequence $0 \to L \to M \to H \to 0$ in mod-R, we have $\rho(M) = \rho(L) + \rho(H)$. In this note we exhibit a relationship between additive rank functions on mod-R and ascending chain conditions of a certain type. Specifically, we prove that if R is a right Noetherian ring, ρ is an additive rank function on mod-R and a right R-module $M = \Pi_{i \in I} M_i$ is a direct product of finitely generated submodules $M_i\,(i \in I)$ such that $\rho(L) \neq 0$ for every non-zero submodule L of M_i for each $i \in I$ then M satisfies the ascending chain condition on n-generated submodules for every positive integer n. In particular this implies that, for a right Noetherian ring R which is an order in a right Artinian ring, every torsionless right R-module satisfies the ascending chain condition on n-generated submodules for every positive integer n. It is also proved that if R is a right Noetherian ring which satisfies the descending chain condition on right annihilators then every torsionless right R-module M such that every countably generated submodule is contained in a direct sum of finitely generated submodules of M satisfies the ascending chain condition on n-generated submodules for every positive integer n.

Mathematics Subject Classification (2000). 16P40,16P20, 16P70.

Keywords. Noetherian, Artinian, additive rank function, torsionless, fully bounded, prime ideal.

1. Subrings of Artinian rings

All rings are associative with identity and all modules are unital modules. Any unexplained terms can be found in [29]. A recurring theme in the theory of Noetherian rings is the consideration of when a right Noetherian ring can be embedded in a right Artinian ring. The first result that immediately springs to mind is Goldie's Theorem.

Theorem 1.1 (See [17, 18]). *A semiprime right Noetherian ring is a right order in a right Artinian ring.*

Of course, Goldie's Theorem immediately implies that a semiprime right Noetherian ring embeds in a (right and left) Artinian ring. Now let R be any ring. An element c of R is called *regular* provided $rc \neq 0$ and $cr \neq 0$ for every non-zero element r in R. Given any ideal A of R, we denote by $\mathscr{C}(A)$ the set of elements c in R such that $c + A$ is a regular ring of the factor ring R/A. The next theorem we would like to mention is Small's Theorem.

Theorem 1.2 (See [34, Theorems 2.10, 2.11 and 2.12]). *Let R be a right Noetherian ring with maximal nilpotent ideal N. Then R is a right order in a right Artinian ring if and only if $\mathscr{C}(0) = \mathscr{C}(N)$.*

In the same paper Small proves the following result.

Theorem 1.3 (See [34, Theorem 3.13]). *Let R be a ring which is a finitely generated module over a Noetherian central subring S. Then R can be embedded in an Artinian ring.*

In particular, Theorem 1.3 shows that every commutative Noetherian ring S can be embedded in an Artinian ring. This is true for the following reason. The zero ideal $0 = Q_1 \cap \cdots \cap Q_n$ is a finite intersection of primary ideals Q_i $(1 \leq i \leq n)$, for some positive integer n, and Theorem 1.2 can be used to show that the ring S/Q_i is an order in an Artinian ring for each $1 \leq i \leq n$.

Let R be any ring. By a *right annihilator* we mean a right ideal $\mathbf{r}_R(X)$ consisting of all elements r in R such that $xr = 0$ for all x in X, for some non-empty subset X of R. Now suppose that R is a subring of a right Artinian ring S. Let X be any non-empty subset of R. Then it is immediate that $\mathbf{r}_R(X) = R \cap \mathbf{r}_S(X)$ and it follows that R satisfies the ascending and descending chain conditions on right annihilators. This fact is needed in the next example.

Example 1.4 (See [35]). There exists a right Noetherian ring that cannot be embedded in a right Artinian ring.

Proof. Let S be a simple right Noetherian ring which is an algebra over a field K but which is not right Artinian and let E be a proper essential right ideal of S. Let M denote the left K-, right S-bimodule S/E and let R denote the ring

$$\begin{bmatrix} K & M \\ 0 & S \end{bmatrix}$$

consisting of all "matrices" of the form

$$\begin{bmatrix} k & m \\ 0 & s \end{bmatrix}$$

where $k \in K, m \in M$ and $s \in S$. Then R is a right Noetherian ring but R does not satisfy the descending chain on right annihilators because if it did then the singular right S-module M would be unfaithful, a contradiction. Thus, by the above remarks, the ring R cannot be embedded in a right Artinian ring. □

This example of Small led several authors to consider when a right Noetherian ring can be embedded in a right Artinian ring. In 1973 Blair [6] generalized Theorem 1.3 by proving that a right Noetherian ring which is integral over its centre can be embedded in a right Artinian ring. In the following year Gordon [23] showed that every two-sided fully bounded Noetherian ring can be embedded in an Artinian ring. (Recall that a ring R is called *right fully bounded* provided for every prime ideal P of R every essential right ideal of the ring R/P contains a non-zero two-sided ideal of R/P.) The methodology was basically that as outlined for the case of commutative Noetherian rings, namely show that the zero ideal is a finite intersection of ideals with corresponding factor rings embeddable in right Artinian rings. In 1985 A.H. Schofield [33] investigated homomorphisms from a ring R to simple Artinian rings and showed that such homomorphisms were closely related to the existence of what he called *Sylvester module rank functions* which we briefly discuss next.

Let R be any ring. For any positive integer n, $\mathbb{N}(1/n)$ will denote the additive semigroup consisting of all rational numbers of the form m/n for some non-negative integer m. A Sylvester module rank function for R, according to Schofield [33], is a mapping λ from the category of finitely presented right R-modules to $\mathbb{N}(1/n)$, for some positive integer n, such that

(i) $\lambda(R_R) = 1$,
(ii) $\lambda(M_1 \oplus M_2) = \lambda(M_1) + \lambda(M_2)$, and
(iii) $\lambda(H) \leq \lambda(M) \leq \lambda(L) + \lambda(H)$,

for all finitely presented right R-modules M_1, M_2, M, H and L such that the sequence $L \to M \to H \to 0$ is exact. Moreover, λ is called *exact* if $\lambda(M) = \lambda(L) + \lambda(H)$ for all exact sequences

$$0 \to L \to M \to H \to 0$$

of finitely presented right R-modules. For any positive integer k, $M_k(R)$ will denote the ring of $k \times k$ matrices with entries in R. Schofield [33, Theorem 7.12] proved the following theorem.

Theorem 1.5. *Let R be an algebra over a field K. Then any Sylvester rank function λ for R arises from a homomorphism $f : R \to M_n(D)$, for some positive integer n and division ring D, in such a way that the kernel of f consists of all elements a in R such that $\lambda(R/aR) = 1$. Moreover, in this case λ is exact if and only if $M_n(D)$ is a flat left R-module.*

Conversely, a homomorphism $f : R \to M_n(D)$ gives rise to a Sylvester rank function λ_f defined as follows: given any finitely presented right R-module X if the finitely generated right $M_n(D)$-module $X \bigotimes_R M_n(D)$ has composition length s then set $\lambda_f(X) = s/n$.

Moreover, Schofield [33, Theorem 7.13] proved that every right Artinian ring which is an algebra over a field can be embedded in a simple Artinian ring. Using Schofield's work, Blair and Small proved the following theorem.

Theorem 1.6 (See [7, Theorem 1]). *Let R be a right Noetherian algebra over a field K and let N denote the maximal nilpotent ideal of R. Let A denote the set of elements a in R such that for each $r \in R$ there exists $c \in \mathscr{C}(N)$ such that $arc = 0$. Then A is an ideal of R and the ring R/A embeds in a simple Artinian ring. In particular, if $A = 0$ then R embeds in a simple Artinian ring.*

Dean and Stafford [12] proved that if $R = U(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$ then there exists an ideal I of R such that the ring R/I is a Noetherian ring of (Rentschler-Gabriel) Krull dimension 1 but R/I cannot be embedded in a right Artinian ring. Note that, being left Noetherian, R/I satisfies the descending chain condition on right annihilators, unlike Small's example above. Dean [11] showed that the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ can be replaced by any finite-dimensional complex semisimple Lie algebra.

Finally, in this section note that Krause [28, p. 336] points out that a right Noetherian algebra R over a field K embeds in a simple Artinian ring S such that the left S-module R is flat if and only if for each $0 \neq a \in R$ there exists $r \in R$ such that $arc \neq 0$ for all $c \in \mathscr{C}(N)$, where N is again the maximal nilpotent ideal of R, i.e., $A = 0$ in the notation of Theorem 1.6. In [28] Krause gives some sufficient conditions for a right Noetherian ring to be embeddable in a right Artinian ring. Goldie and Krause [21, Corollary 8] generalize Theorem 1.6 to right Noetherian rings (which are not algebras over a field) with the ideal $A = 0$ and with an incomparability condition on prime ideals.

2. Additive rank functions

Let R be a ring with identity and let mod-R denote the category of finitely generated right R-modules. Following [26], by an *additive rank function* on mod-R we mean a rule ρ which assigns to every finitely generated right R-module X a non-negative integer $\rho(X)$ such that $\rho(Y) = \rho(X) + \rho(Z)$ for all exact sequences

$$0 \to X \to Y \to Z \to 0$$

in mod-R. In particular, this gives that $\rho(0) = 0$, $\rho(X) = \rho(X')$ for all finitely generated isomorphic R-modules X and X', and $\rho(X_1 \oplus X_2) = \rho(X_1) + \rho(X_2)$ for all finitely generated R-modules X_1 and X_2. This notion finds its origins in rank functions for von Neumann regular rings (see [22, Chapter 16]) and Sylvester module rank functions in Schofield's work discussed above. Clearly if R is a right Noetherian ring then every exact Sylvester module rank function is an additive rank function on mod-R. Now suppose that R is any ring. There is a trivial additive rank function ρ_z on mod-R defined by $\rho_z(X) = 0$ for all finitely generated right R-modules X. Note that an additive rank function ρ satisfies $\rho = \rho_z$ if and only if $\rho(R_R) = 0$. Let ρ be any additive rank function such that $\rho(R_R) \neq 0$. For each finitely presented right R-module X define $\lambda(X) = \rho(X)/\rho(R_R)$. Then it is clear that λ is an exact Sylvester module rank function for R.

Additive rank functions appear in many places in the literature. The most basic example comes from the theory of right Artinian rings. If R is a right Artinian ring then let $\rho_0(X)$ denote the composition length of any finitely generated right R-module X. It is well known that ρ_0 is an additive rank function on mod-R. This example has been generalized by Krause [26]. Let R be a non-zero right Noetherian ring. Then there exists an ordinal $\alpha \geq 0$ such that R has right Krull dimension α. For the definition and basic properties of Krull dimension see [29, Chapter 6]. For any right R-module M which has Krull dimension, let $k(M)$ denote the Krull dimension of M. Let X be any finitely generated right R-module. Then X has Krull dimension and $k(X) \leq \alpha$. If $k(X) \neq \alpha$ then set $\rho_\alpha(X) = 0$. Now suppose that $k(X) = \alpha$. Construct a chain of submodules $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ such that for all integers $i \geq 1$, X_i is a maximal submodule of X_{i-1} such that $k(X_{i-1}/X_i) = \alpha$. Because $k(X) = \alpha$ there exists a least positive integer n such that $k(X_n) < \alpha$ and in this case we set $\rho_\alpha(X) = n$. Note that n is independent of the choice of the submodules X_i $(1 \leq i \leq n)$ (see [26] for details). The next result is [26, Proposition 4.2].

Proposition 2.1. *Let R be a right Noetherian ring with right Krull dimension α. Then ρ_α is an additive rank function on* mod-*R.*

Let R be any ring and let M be any right R-module. A submodule K of M is called *closed* provided K has no proper essential extension in M. Recall that the *singular submodule* $Z(M)$ of a module M is the submodule consisting of all elements m in M such that $mE = 0$ for some essential right ideal E of R. Then the *second singular submodule* $Z_2(M)$ of M is the submodule containing $Z(M)$ such that $Z_2(M)/Z(M) = Z(M/Z(M))$. Note that $Z_2(M)$ is a closed submodule of M. The module M is called *singular* if $M = Z(M)$ and is called *nonsingular* if $Z(M) = 0$. If R is a *right nonsingular ring*, i.e., R_R is nonsingular, then $Z(M) = Z_2(M)$ for every R-module M by [37, Chapter VI Proposition 6.7]. The module M has *finite uniform dimension* if M does not contain an infinite direct sum of non-zero submodules and in this case the uniform dimension of M will be denoted by $u(M)$. Recall that a non-zero module M has uniform dimension n, for some positive integer n, provided that M contains a direct sum $L_1 \oplus \cdots \oplus L_n$ of non-zero submodules L_i $(1 \leq i \leq n)$ but no direct sum of $n + 1$ non-zero submodules. Note that $u(M)$ is an invariant of M. For the basic properties of uniform dimension see [29, Section 2.2]. Although it is clear that every submodule of a module M with finite uniform dimension also has finite uniform dimension, homomorphic images of M need not have finite uniform dimension. If K is a *closed* submodule of a module M and M has finite uniform dimension then M/K has finite uniform dimension and

$$u(M) = u(K) + u(M/K)$$

(see, for example, [13, 5.10]).

Now let R denote a right Noetherian ring. In this case every finitely generated right R-module is Noetherian and hence has finite uniform dimension. For any finitely generated right R-module X define $\rho_g(X) = u(X) - u(Z(X))$. The next result is due to Goldie (see [19, Lemma 4.1 and remarks on p. 273]).

Proposition 2.2. *Let R be a right Noetherian ring. Then*

(i) *ρ_g is an additive rank function on* mod-R.

(ii) *For each finitely generated R-module X,*

$$\rho_g(X) = u(X) - u(Z_2(X)) = u(X/Z_2(X)).$$

Given any finitely generated right R-module X over a right Noetherian ring, note that $\rho_g(X) = 0$ if and only if $u(X) = u(Z(X))$ and this occurs if and only if $Z(X)$ is an essential submodule of X or, in other words, if and only if $X = Z_2(X)$. In [19, p. 274], Goldie remarks that the rank of a finitely generated module X given by $\rho_g(X)$ has a number of disadvantages and he introduces a notion which later came to be known as 'reduced rank' and which we shall denote by ρ_r.

Let R be a right Noetherian ring and let X be a finitely generated right R-module. Let N denote the prime radical of R so that $N^k = 0$ for some positive integer k. Consider the chain

$$X = XN^0 \supseteq XN^1 \supseteq \cdots \supseteq XN^k = 0$$

of submodules XN^i $(0 \leq i \leq k)$ of X. In [9, Chapter 2] the *rank* of X, which we shall denote by $\rho_r(X)$, is defined by

$$\rho_r(X) = \sum_{i=1}^{k} \rho_g(XN^{i-1}/XN^i).$$

Note that in [29, Section 4.1], $\rho_r(X)$ is called the *reduced rank* of X. Further note that, for each $1 \leq i \leq k$, $Y_i = XN^{i-1}/XN^i$ is a finitely generated module over the semiprime right Noetherian ring R/N and, because R/N is right nonsingular, we have

$$\rho_r(X) = \sum_{i=1}^{k} u(Y_i/Z(Y_i))$$

by the above remarks. Moreover, as Goldie points out in [19, p. 274], if

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$$

is any finite chain of submodules X_i of X such that $X_{i-1}N \subseteq X_i$ $(1 \leq i \leq m)$ then

$$\rho_r(X) = \sum_{i=1}^{m} \rho_g(X_{i-1}/X_i).$$

Proposition 2.3. *Let R be a right Noetherian ring. Then*

(i) *ρ_r is an additive rank function on* mod-R.

(ii) *$\rho_r(M) = 0$ if and only if for each $m \in M$ there exists $c \in \mathscr{C}(N)$ such that $mc = 0$.*

Proof. See [9, Theorem 2.2]. □

Again let R be a right Noetherian ring and let ρ be any additive rank function on mod-R. Following [20, Proposition 1.1] we let $\mathfrak{F}_\rho$ denote the collection of right ideals A of R such that $\rho(R/A) = 0$. Goldie [20, Proposition 1.1] shows that $\mathfrak{F}_\rho$ is a Gabriel filter of right ideals. For a good discussion of Gabriel filters see [37]. Let M be a right R-module. Given any submodule L of M the set $\bar{L}$ of all elements m in M such that $mF \subseteq L$ for some F in $\mathfrak{F}_\rho$ is a submodule of M containing L and L is called $\mathfrak{F}_\rho$-*closed* provided $L = \bar{L}$. Goldie [20, Proposition 1.1] proves that R satisfies the descending chain condition on $\mathfrak{F}_\rho$-closed right ideals. Moreover, in this case Krause [27, Proposition 1.4] proves that every finitely generated right R-module X satisfies the descending chain condition on $\mathfrak{F}_\rho$-closed submodules. Conversely, let $\mathfrak{G}$ be any Gabriel filter of right ideals of a right Noetherian ring R such that R satisfies the descending chain condition on $\mathfrak{G}$-closed right ideals. For any finitely generated R-module X, define $\rho_{\mathfrak{G}}(X)$ to be the maximal length of a chain of $\mathfrak{G}$-closed submodules of X. Then Krause [27, Theorem 3.1] proves the following result.

Proposition 2.4. *Let $\mathfrak{G}$ be any Gabriel filter of right ideals of a right Noetherian ring R such that R satisfies the descending chain condition on $\mathfrak{G}$-closed right ideals. Then $\rho_{\mathfrak{G}}$ is an additive rank function on* mod-R.

Krause also proves that if ρ is any additive rank function on mod-R, for a right Noetherian ring R, then, for any finitely generated R-module X, $\rho_r(X) = 0$ implies that $\rho(X) = 0$ (see [26, Lemma 1.3]).

Let R be any ring and let $\mathfrak{F}$ be a Gabriel filter of right ideals. It is well known that we can associate with $\mathfrak{F}$ an hereditary torsion theory $\tau_{\mathfrak{F}}$ such that the $\tau_{\mathfrak{F}}$-torsion modules are precisely the right R-modules M such that for each m in M there exist $F \in \mathfrak{F}$ such that $mF = 0$. In particular, if ρ is any additive rank function then the collection of all modules M such that $\rho(mR) = 0$ for all $m \in M$ forms a torsion class for an hereditary torsion theory. In particular, this means that, for any R-module L, the set $t_\rho(L)$ of all elements x in L such that $\rho(xR) = 0$ forms a submodule of L. We shall call an R-module T ρ-*torsion* provided $T = t_\rho(T)$. Moreover, we call an R-module F ρ-*torsion-free* if $t_\rho(F) = 0$. Now note that if R is a right Noetherian ring then a finitely generated right R-module M is ρ_r-torsion if and only if M is ρ-torsion for every additive rank function ρ on mod-R, by [26, Lemma 1.3]. This means that $t_{\rho_r}(L) \subseteq t_\rho(L)$ for every right R-module L. Note further that if the right R-module M is nonsingular then M is ρ_g-torsion-free and hence ρ_r-torsion-free.

There is a relationship between reduced rank and the embedding of a right Noetherian ring in a right Artinian ring. Let R be a right Noetherian ring. Given $a \in R$, $\rho_r(aR) = 0$ if and only if for each $r \in R$ there exists $c \in \mathscr{C}(N)$ such that $arc = 0$, where as usual N denotes the maximal nilpotent ideal of R. Thus in Theorem 1.6, the ideal A is precisely $t_{\rho_r}(R_R)$. Thus Theorem 1.6 can be restated: if R is a right Noetherian algebra over a field K such that the right R-module R is ρ_r-torsion-free then the ring R embeds in a simple Artinian ring. In the next section we shall be interested in rings R such that the module R_R is ρ-torsion-free for some additive rank function ρ on mod-R.

3. Chain conditions

Let R be a ring and let M be an R-module. Given a positive integer n, the module M satisfies *n-acc* provided every ascending chain of n-generated submodules terminates. Moreover, the module M satisfies *pan-acc* in case M satisfies n-acc for every positive integer n. The ring R is said to satisfy *right n-acc*, for a given positive integer n, provided the right R-module R satisfies n-acc. Similarly, the ring R satisfies *right pan-acc* if R_R satisfies pan-acc. These chain conditions are discussed in many places (see, for example, [1]–[5], [8], [10], [14]–[15], [24]–[25], [30]–[32] and [36]).

Consider first Abelian groups. A torsion $\mathbb{Z}$-module A satisfies pan-acc if and only if A satisfies 1-acc and this occurs if and only if A is reduced (i.e., A has no non-zero divisible submodules) and there exists only a finite number of primes p such that $pa = 0$ for some non-zero element a in A (see [5, Theorem 3]). In contrast, for every positive integer n, Fuchs [16, p. 125] gives an example of a torsion-free $\mathbb{Z}$-module B_n which satisfies n-acc but not $(n+1)$-acc. Note that every free $\mathbb{Z}$-module satisfies pan-acc (see [5, Theorem 1 Corollary 1]). However there exists a right Noetherian ring R and a free right R-module F such that F does not satisfy 1-acc, as the following example of Renault [31, Proposition 3.4] shows.

Example 3.1. Let S be a simple right Noetherian ring which is an algebra over a field K but which is not Artinian and let U be a simple right S-module. Let R denote the ring

$$\begin{bmatrix} K & U \\ 0 & S \end{bmatrix}$$

Then a free right R-module F satisfies 1-acc if and only if F is finitely generated.

The reason why Example 3.1 works is that the right R-module $U^{(n)}$ is cyclic for every positive integer n. We do not have an example of a right Noetherian ring R and a free right R-module F such that F satisfies 1-acc but not 2-acc. Note that in Example 3.1 if the ring S is, in addition, left Noetherian then every free left R-module satisfies pan-acc by [4, Corollary 3.13]. In the same paper Renault [31, Théorème 3.2] proved the following theorem.

Theorem 3.2. *Let R be a right Noetherian ring such that for every prime ideal P with $P = \mathbf{r}_R(X)$ for some non-empty subset X of R there exists a finite subset Y of X such that $P = \mathbf{r}_R(Y)$. Then every free right R-module satisfies pan-acc.*

In particular, as Renault [31, Corollaire 3.3] points out, if R is a right Noetherian ring which is either left Noetherian or right fully bounded then every free right R-module satisfies pan-acc. Moreover, if R is a right Noetherian ring which satisfies the descending chain condition on right annihilators then R satisfies the hypotheses of Theorem 3.2. Thus, by the above remarks we have the following result.

Corollary 3.3. *Let R be a right Noetherian ring which can be embedded in a right Artinian ring. Then every free right R-module satisfies pan-acc.*

Let R be any ring. A right R-module M is called *torsionless* provided, for each non-zero element m in M, there exists a homomorphism $\varphi_m : M \to R$ such that $\varphi_m(m) \neq 0$, equivalently, the module M embeds in a direct product $(R_R)^I$ of copies of the R-module R_R, for some index set I. Compare the next result with Theorem 3.2.

Theorem 3.4 (See [36, Theorem 16]). *Let S and R be rings and let M be a left S-, right R-bimodule such that M is Noetherian both as a left S-module and as a right R-module. Then the right R-module M^I satisfies pan-acc, for every index set I.*

Note that, in particular, Theorem 3.4 shows that if R is a (two-sided) Noetherian ring then every torsionless right R-module satisfies pan-acc. This result was proved first in the case of commutative rings by Frohn [15, Theorem 3.3]. There are other situations where torsionless modules satisfy pan-acc and we examine one of these next. First we have an easy lemma.

Lemma 3.5. *Let R be a right Noetherian ring and let ρ be any additive rank function on* mod-R. *Let M be an R-module, let n be a positive integer and let $L_1 \subseteq L_2 \subseteq \cdots$ be any ascending chain of n-generated submodules of M. Then there exists a positive integer k such that $\rho(L_i/L_k) = 0$ for all $i \geq k$.*

Proof. For all $i \geq 1$ there exists an epimorphism from the free R-module $(R_R)^{(n)}$ to L_i and hence $\rho(L_i) \leq n\rho(R_R)$. But, by the properties of ρ, $\rho(L_1) \leq \rho(L_2) \leq \cdots$. Thus there exists a positive integer k such that $\rho(L_k) = \rho(L_{k+1}) = \cdots$ and hence $\rho(L_{i+1}/L_i) = 0$ for all $i \geq k$. The result follows. □

Lemma 3.6. *Let R be a right Noetherian ring and let ρ be any additive rank function on* mod-R. *Let an R-module $M = \Pi_{i \in I} M_i$ be a direct product of ρ-torsion-free R-modules $M_i\,(i \in I)$. Then for each finitely generated submodule N of M there exists a finite subset F of I such that $N \cap (\Pi_{i \in I \setminus F} M_i) = 0$.*

Proof. We shall prove the result by induction on the non-negative integer $\rho(N)$. Note that the module M is ρ-torsion-free by [37, Chapter VI Proposition 2.2]. If $\rho(N) = 0$ then $N = 0$ and the result is clear. Suppose that $\rho(N) \neq 0$ so that $N \neq 0$. There exists $j \in I$ such that $N \nsubseteq \Pi_{i \in I \setminus \{j\}} M_i$. Let $L = \Pi_{i \in I \setminus \{j\}} M_i$. Then

$$N/(N \cap L) \cong (N + L)/L \leq M/L \cong M_j$$

and $N/(N \cap L) \neq 0$ so that $\rho(N/(N \cap L)) \neq 0$. It follows that

$$\rho(N \cap L) = \rho(N) - \rho(N/(N \cap L)) < \rho(N).$$

By induction there exists a finite subset G of $I \setminus \{j\}$ such that

$$N \cap L \cap (\Pi_{i \in I \setminus G} M_i) = 0.$$

Let $F = G \cup \{j\}$. Then $N \cap (\Pi_{i \in I \setminus F} M_i) = 0$, as required. □

Renault [31, Corollaire 2.3] (see also [5, Theorem 8]) proved that if R is a right Noetherian right nonsingular ring then every free right R-module satisfies pan-acc. Sanchez Campos and Smith [32, Corollary 4.3] (see also [2, Theorem 1.5]) proved that if a ring R is right Noetherian, n is a positive integer and M_i $(i \in I)$ is any collection of nonsingular right R-modules each satisfying n-acc then the direct product $\Pi_{i\in I} M_i$ satisfies n-acc. Because every nonsingular R-module is ρ_g-torsion-free, these results are all special cases of the following theorem.

Theorem 3.7. *Let R be a right Noetherian ring and let ρ be any additive rank function on* mod-R. *Let n be any positive integer. Then a direct product $\Pi_{i\in I} M_i$ of ρ-torsion-free right R-modules M_i $(i \in I)$ satisfies n-acc if and only if M_i satisfies n-acc for each $i \in I$.*

Proof. The necessity is clear. Conversely suppose that M_i satisfies n-acc for each $i \in I$. Let $M = \Pi_{i\in I} M_i$ and let $L_1 \subseteq L_2 \subseteq \cdots$ be any ascending chain of n-generated submodules of M. By Lemma 3.5 there exists a positive integer k such that $\rho(L_k) = \rho(L_{k+1}) = \cdots$. Next by Lemma 3.6 there exists a finite subset F of I such that

$$L_k \cap (\Pi_{i\in I\setminus F} M_i) = 0.$$

Let $P = \Pi_{i\in F} M_i$ and let $Q = \Pi_{I\in I\setminus F} M_i$ so that $M = P \oplus Q$. Note that if $L = \cup_{i\geq 1} L_i$ then $L \cap Q = 0$. To see why this is the case, suppose that t is an integer with $t \geq k$. Let $x \in L_t \cap Q$. With the above notation there exists a right ideal $A \in \mathfrak{F}_\rho$ such that $xA \subseteq L_k$. Then $xA \subseteq L_k \cap Q = 0$ and hence $x = 0$, because M is ρ-torsion-free. It follows that $L_t \cap Q = 0$. We have proved that $L \cap Q = 0$. If $\pi : M \to P$ is the canonical projection then $\pi|_L$ is a monomorphism. Moreover, $\pi(L_1) \subseteq \pi(L_2) \subseteq \cdots$ is an ascending chain of n-generated submodules of the module P. By [32, Theorem 4.2], P satisfies n-acc. Thus there exists a positive integer s such that $\pi(L_s) = \pi(L_{s+1}) = \cdots$ and hence $L_s = L_{s+1} = \cdots$. It follows that M satisfies n-acc. □

Corollary 3.8. *Let R be a right Noetherian ring and let ρ be any additive rank function on* mod-R. *Then every direct product of finitely generated ρ-torsion-free right R-modules satisfies pan-acc.*

Proof. By Theorem 3.7. □

Corollary 3.9. *Let R be a right Noetherian ring and let ρ be any additive rank function on* mod-R *such that the right R-module R is ρ-torsion-free. Then every torsionless right R-module satisfies pan-acc.*

Proof. By Corollary 3.8. □

Let R be any non-zero right Noetherian ring. Suppose that R has right Krull dimension α for some ordinal $\alpha \geq 0$. The right R-module R is ρ_α-torsion-free provided $k(A) = \alpha$ for every non-zero right ideal A of R. Thus if R is a right Noetherian ring with right Krull dimension α such that $k(A) = \alpha$ for every non-zero right ideal A then every torsionless right R-module satisfies pan-acc by Corollary

3.9. A non-zero R-module M is called *α-critical* if $k(M) = \alpha$ and $k(M/N) < \alpha$ for every non-zero submodule N of M. It is well known that if M is α-critical then $k(N) = \alpha$ for every non-zero submodule N of M (see [29, Lemma 6.2.11]) and hence M is ρ_α-torsion-free. Another consequence of Theorem 3.7 is the following result.

Theorem 3.10. *Let R be a non-zero right Noetherian ring with right Krull dimension α for some ordinal $\alpha \geq 0$ and let n be any positive integer. Then a direct product $\Pi_{i\in I}M_i$ of α-critical R-modules $M_i\,(i \in I)$ satisfies n-acc if and only if the module M_i satisfies n-acc for every $i \in I$.*

Proof. By Theorem 3.7. □

Let R be a right Noetherian algebra over a field K such that the module R_R is ρ-torsion-free for some additive rank function ρ on mod-R. By the remarks at the end of Section 2, R_R is ρ_r-torsion-free and, by Theorem 1.6, R can be embedded in a simple Artinian ring. This brings us to the following result.

Theorem 3.11. *Let R be a right Noetherian algebra over a field K such that R can be embedded in a simple Artinian ring S in such a way that the left R-module S is flat. Then every torsionless right R-module satisfies pan-acc.*

Proof. By Theorem 1.5 the right R-module R is ρ_r-torsion-free and Corollary 3.9 gives the result. □

Theorem 3.12. *Let R be a right Noetherian ring which is a right order in a right Artinian ring. Then every torsionless R-module satisfies pan-acc.*

Proof. By Theorem 1.2, R_R is ρ_r-torsion-free. Now apply Corollary 3.9. □

4. Modules with the direct sum condition

Let R be a ring. An R-module M will be said to *satisfy the direct sum condition* provided every countably generated submodule is contained in a direct sum of finitely generated submodules of M. Clearly every free module and every semisimple module satisfies the direct sum condition. More generally, every direct sum of finitely generated R-modules satisfies the direct sum condition. Note also that if M_i is an R-module satisfying the direct sum condition, for all i in some index set I, then the R-module $\bigoplus_{i\in I} M_i$ also satisfies the direct sum condition (see [36, p. 74]). Moreover, if R and S are rings and M a left S-, right R-bimodule such that ${}_SM$ is Noetherian and M_R is finitely generated then the right R-module M^I satisfies the direct sum condition for every index set I (see [36, Theorem 5]). Theorem 3.4 is a consequence of this fact. Let $\mathbb{Z}$ denote the ring of integers and let M denote the direct product $\mathbb{Z}^I$ for any infinite index set I. Then the $\mathbb{Z}$-module M satisfies the direct sum condition but M is not a direct sum of finitely generated submodules (see [36, Example 6]). The next result generalizes [36, Theorem 8].

Theorem 4.1. *Let R be a right Noetherian ring and let M be a right R-module that satisfies the direct sum condition. Then the following statements are equivalent for a positive integer n.*

(i) *M satisfies n-acc.*
(ii) *$t_\rho(M)$ satisfies n-acc for any additive rank function ρ on* mod-R.
(iii) *$t_{\rho_r}(M)$ satisfies n-acc.*

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) By [26, Lemma 1.3] $t_{\rho_r}(M) \subseteq t_\rho(M)$ for every additive rank function ρ on mod-R.

(iii) $\Rightarrow$ (i) Suppose that $t_{\rho_r}(M)$ satisfies n-acc. Let $L_1 \subseteq L_2 \subseteq \cdots$ be any ascending chain of n-generated submodules of M. Let $L = \bigcup_{i\geq 1} L_i$. By Lemma 3.5, there exists a positive integer k such that $\rho_r(L_i/L_k) = 0$ for all $i \geq k$. By hypothesis, there exists a submodule H of M such that $L \subseteq H$, $H = H_1 \bigoplus H_2$ for some submodules H_1 and H_2, H_1 is finitely generated and $L_k \subseteq H_1$. Let $\pi : H \to H_2$ denote the canonical projection. Let $x \in L$. There exists $t \geq k$ such that $x \in L_t$. Now $\rho_r(L_t/L_k) = 0$ so that $xA \subseteq L_k$ for some A in $\mathfrak{F}_{\rho_r}$. This implies that $\pi(x)A = 0$ and hence $\pi(x) \in t_{\rho_r}(M)$. It follows that $\pi(L) \subseteq t_{\rho_r}(M)$. Thus $\pi(L_1) \subseteq \pi(L_2) \subseteq \cdots$ is an ascending chain of n-generated submodules of $t_{\rho_r}(M)$. There exists a positive integer s such that $\pi(L_s) = \pi(L_{s+1}) = \cdots$. But H_1 is Noetherian and hence, without loss of generality, $L_s \bigcap H_1 = L_{s+1} \bigcap H_1 = \cdots$. It follows that $L_s = L_{s+1} = \cdots$, as required. □

Let R be any ring and let M be a right R-module. For any non-empty subset X of M we set $\mathrm{ann}_R(X)$ to be the set of elements r in R such that $xr = 0$ for all $x \in X$. We next aim to generalize Theorem 3.2 to modules which satisfy the direct sum condition. Note that the $\mathbb{Z}$-module $\oplus_p(\mathbb{Z}/\mathbb{Z}p)$, where the direct sum is taken over all primes p in $\mathbb{Z}$, satisfies the direct sum condition but does not satisfy 1-acc (see, for example, [32, Lemma 2.3]). First we prove a lemma.

Lemma 4.2. *Let R be a right Noetherian ring and let M be a right R-module which satisfies the direct sum condition but not pan-acc. Let P be an ideal of R maximal in the collection of all ideals A of R such that there exist a positive integer n and a properly ascending chain $L_1 \subset L_2 \subset \cdots$ of n-generated submodules L_i $(i \geq 1)$ of M with $A = \mathrm{ann}_R(\cup_{i\geq 1} L_i)$. Then P is a prime ideal of R.*

Proof. Suppose that s is a positive integer and $N_1 \subset N_2 \subset \cdots$ a properly ascending chain of s-generated submodules N_i $(i \geq 1)$ of M such that $P = \mathrm{ann}_R(\cup_{i\geq 1} N_i)$. Suppose that P is not a prime ideal of R. Then there exist ideals B and C, each properly containing P, such that $BC \subseteq P$. Suppose that the right ideal B can be generated by t elements, for some positive integer t. Then $N_1 B \subseteq N_2 B \subseteq \cdots$ is an ascending chain of (st)-generated submodules of M such that $(\cup_{i\geq 1}(N_i B))C = 0$. By the choice of P, there exists a positive integer k such that $N_k B = N_{k+1} B = \cdots$. Let $N = \cup_{i\geq 1} N_i$. Note that N is a countably generated submodule of M so that, by hypothesis, N is contained in a direct sum of finitely generated submodules of

M. It follows that there exist submodules M_1 and M_2 of M such that $M_1 \cap M_2 = 0$, $N \subseteq M_1 \oplus M_2$, $N_k \subseteq M_1$ and M_1 is finitely generated. Let $\pi : M_1 \oplus M_2 \to M_2$ denote the canonical projection. Observe that $\pi(N_k) \subseteq \pi(N_{k+1}) \subseteq \cdots$ is an ascending chain of s-generated submodules of M such that for all $i \geq k$

$$\pi(N_i)B = \pi(N_iB) = \pi(N_kB) \subseteq \pi(N_k) = 0.$$

By the choice of P, there exists an integer $h \geq k$ such that $\pi(N_h) = \pi(N_{h+1}) = \cdots$. Since the kernel of π is M_1, which is a Noetherian module, it follows that there exists an integer $g \geq h$ such that $N_g = N_{g+1} = \cdots$, a contradiction. Thus P is a prime ideal of R. □

Theorem 4.3. *Let R be a right Noetherian ring and let M be a right R-module which satisfies the direct sum condition such that for every prime ideal P of R with $P = \mathrm{ann}_R(X)$, for some non-empty subset X of M, there exists a finite subset Y of X such that $P = \mathrm{ann}_R(Y)$. Then the right R-module M satisfies pan-acc.*

Proof. Suppose not. Choose P to be the ideal in Lemma 4.2 and adopt the notation of the proof of Lemma 4.2. Again $N = \cup_{i\geq 1} N_i$. Note that N is a right (R/P)-module. Let ρ denote the additive rank functor ρ_g on the category mod-(R/P) of finitely generated right (R/P)-modules. For each positive integer $i \geq 1$, $\rho(N_i) \leq s\rho(R/P)$. Thus there exists a positive integer q such that $\rho(N_q) = \rho(N_{q+1}) = \cdots$. The countably generated submodule N of M is again contained in a direct sum of finitely generated submodules of M. Thus there exist submodules V_1 and V_2 of M such that $V_1 \cap V_2 = 0$, $N \subseteq V_1 \oplus V_2$, $N_q \subseteq V_1$ and V_1 is finitely generated. Let $\pi : V_1 \oplus V_2 \to V_2$ denote the canonical projection. As in the proof of Lemma 4.2, we can suppose without loss of generality that $\pi(N_q) \subset \pi(N_{q+1}) \subset \cdots$ is a properly ascending chain of submodules of M. Note that $\pi(N_i)P = \pi(N_iP) = 0$ for all $i \geq 1$, so that, by the choice of P,

$$P = \mathrm{ann}_R(\cup_{i\geq q}\pi(N_i)) = \mathrm{ann}_R(\pi(N)).$$

By hypothesis, $P = \mathrm{ann}_R(Y)$, for some finite subset Y of $\pi(N)$. For each $y \in Y$ there exists $x \in N$ such that $y = \pi(x)$. Because of the choice of q, $\rho((xR + N_q)/N_q) = 0$, i.e., $(xR + N_q)/N_q$ is a singular right (R/P)-module. In particular this means that $xc_y \in N_q$ for some $c_y \in \mathscr{C}(P)$ and hence $yc_y = \pi(x)c_y = \pi(xc_y) \in \pi(N_q) = 0$. Thus for each $y \in Y$ there exists $c_y \in \mathscr{C}(P)$ such that $yc_y = 0$. Because Y is a finite set, there exists $c \in \mathscr{C}(P)$ such that $yc = 0$ for all y in Y. But this implies that $c \in P$, a contradiction. The result follows. □

Using Theorem 4.3 we can now prove a result which generalizes Theorem 3.2.

Corollary 4.4. *Let R be a right Noetherian ring such that for every prime ideal P with $P = \mathbf{r}_R(X)$ for some non-empty subset X of R there exists a finite subset Y of X such that $P = \mathbf{r}_R(Y)$. Then every torsionless right R-module which satisfies the direct sum condition also satisfies pan-acc.*

Proof. Let M be any torsionless right R-module. Without loss of generality M is a submodule of $\Pi_{i\in I} R_i$, where $R_i = R$ for each $i \in I$, for some index set I. For each $i \in I$, let $\pi_i : M \to R_i$ denote the canonical projection. Let P be a prime ideal of R such that $P = \mathrm{ann}_R(X)$ for some non-empty subset X of M. Let $U = \{\pi_i(x) : x \in X, i \in I\}$. Note that, for any $x \in X$, $r \in R$,

$$xr = 0 \text{ if and only if } \pi_i(x)r = 0 \text{ for all } i \in I.$$

Thus U is a non-empty subset of R such that $P = \mathbf{r}_R(U)$. By hypothesis, there exists a finite subset V of U such that $P = \mathbf{r}_R(V)$. For each $v \in V$ there exists an element $x_v \in X$ such that $v = \pi_i(x_v)$ for some $i \in I$. Let $Y = \{x_v : v \in V\}$. Then Y is a finite subset of X such that

$$P = \mathrm{ann}_R(X) \subseteq \mathrm{ann}_R(Y) \subseteq \mathbf{r}_R(V) = P.$$

Thus $P = \mathrm{ann}_R(Y)$. The result now follows by Theorem 4.3. □

In particular, Corollary 4.4 shows that (as stated in the Abstract) if R is a right Noetherian ring which satisfies the descending chain condition on right annihilators then every torsionless right R-module which satisfies the direct sum condition also satisfies pan-acc. Corollary 4.4 can be applied in other situations. For example, we have the following result. Recall that a ring R is a *right FBN ring* if it is right fully bounded and right Noetherian.

Corollary 4.5. *Let R be a right FBN ring. Then every torsionless right R-module which satisfies the direct sum condition also satisfies pan-acc.*

Proof. By Corollary 4.4. □

Let R be a right Noetherian ring. We do not have an example of a torsionless right R-module which satisfies pan-acc but which does not satisfy the direct sum condition.

Acknowledgement

Some of this work was done while the author attended the *International Conference on Ring and Module Theory* at Hacettepe University, Ankara, on 18–22 August 2008. The author would like to thank TUBITAK for their financial support that enabled him to attend the conference. The author would also like to thank Professor C.R. Hajarnavis for a stimulating remark which prompted this paper.

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Ring and Module Theory
Trends in Mathematics, 165–174

A Note on (α, β)-higher Derivations and their Extensions to Modules of Quotients

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Abstract. We extend some recent results on the differentiability of torsion theories. In particular, we generalize the concept of (α, β)-derivation to (α, β)-higher derivation and demonstrate that a filter of a hereditary torsion theory that is invariant for α and β is (α, β)-higher derivation invariant. As a consequence, any higher derivation can be extended from a module to its module of quotients. Then, we show that any higher derivation extended to a module of quotients extends also to a module of quotients with respect to a larger torsion theory in such a way that these extensions agree. We also demonstrate these results hold for symmetric filters as well. We finish the paper with answers to two questions posed in [L. Vaš, Extending higher derivations to rings and modules of quotients, International Journal of Algebra, 2 (15) (2008), 711–731]. In particular, we present an example of a non-hereditary torsion theory that is not differential.

Mathematics Subject Classification (2000). 16S90, 16W25, 16N80.

Keywords. Derivation, higher derivation, ring of quotients, module of quotients, torsion theory.

1. Preliminaries and summary of results

Recall that a *derivation* on a ring R is an additive mapping $\delta : R \to R$ such that $\delta(rs) = \delta(r)s + r\delta(s)$ for all $r, s \in R$. An additive mapping $d : M \to M$ on a right R-module M is a δ-*derivation* if $d(xr) = d(x)r + x\delta(r)$ for all $x \in M$ and $r \in R$. If α and β are ring automorphisms, the derivation concept generalizes to (α, β)-derivation by requiring that $\delta(rs) = \delta(r)\alpha(s) + \beta(r)\delta(s)$ for all $r, s \in R$.

A *torsion theory* for R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of R-modules such that $\mathcal{T}$ and $\mathcal{F}$ are maximal classes having the property that $\mathrm{Hom}_R(T, F) = 0$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$. The modules in $\mathcal{T}$ are *torsion modules* and the modules in $\mathcal{F}$ are *torsion-free modules*. For a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, $\mathcal{T}(M)$ denotes the largest torsion submodule of a right R-module M and $\mathcal{F}(M)$ denotes the quotient $M/\mathcal{T}(M)$. $\tau = (\mathcal{T}, \mathcal{F})$ is *hereditary* if $\mathcal{T}$ is closed under taking submodules (equivalently $\mathcal{F}$ is closed under formation of injective envelopes). If $\mathcal{T}(R) = 0$, τ is said to be faithful.

If M is a right R-module with submodule N and $m \in M$, denote $\{r \in R \mid mr \in N\}$ by $(m : N)$. Then $(m : 0)$ is the annihilator $\operatorname{ann}(m)$. A *Gabriel filter* $\mathfrak{F}$ on a ring R is a nonempty collection of right ideals such that

1. If $I \in \mathfrak{F}$ and $r \in R$, then $(r : I) \in \mathfrak{F}$.
2. If $I \in \mathfrak{F}$ and J is a right ideal with $(r : J) \in \mathfrak{F}$ for all $r \in I$, then $J \in \mathfrak{F}$.

If $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory, the collection of right ideals $\{I \mid R/I \in \mathcal{T}\}$ is a Gabriel filter. Conversely, if $\mathfrak{F}$ is a Gabriel filter, then the class of modules $\{M \mid \operatorname{ann}(m) \in \mathfrak{F}$ for every $m \in M\}$ is a torsion class of a hereditary torsion theory.

If δ is any additive map on a ring R, a Gabriel filter $\mathfrak{F}$ is said to be δ-*invariant* if for every $I \in \mathfrak{F}$ there is $J \in \mathfrak{F}$ such that $\delta(J) \subseteq I$. If $\mathfrak{F}$ is δ-invariant for all derivations δ, it is said to be a *differential filter*. The hereditary torsion theory determined by $\mathfrak{F}$ is said to be *differential* in this case. By Lemma 1.5 from [1], $\mathfrak{F}$ is δ-invariant iff $d(\mathcal{T}(M)) \subseteq \mathcal{T}(M)$ for every right R-module M and every δ-derivation d on M. In [7], it is shown that Lambek, Goldie and any perfect hereditary torsion theories are differential. Lomp and van den Berg extend these results in [4] by showing that every Gabriel filter that is α and β-invariant is also δ-invariant for any (α, β)-derivation δ (Theorem 2, [4]). As a direct consequence, *every* hereditary torsion theory is differential (Corollary 3, [4]). This answers a question from [7].

If τ is a hereditary torsion theory with Gabriel filter $\mathfrak{F}$ and M is a right R-module, the module of quotients $M_{\mathfrak{F}}$ of M is defined as the largest submodule N of the injective envelope $E(M/\mathcal{T}(M))$ of $M/\mathcal{T}(M)$ such that $N/(M/\mathcal{T}(M))$ is torsion module (i.e., the closure of $M/\mathcal{T}(M)$ in $E(M/\mathcal{T}(M))$). The R-module $R_{\mathfrak{F}}$ has a ring structure and $M_{\mathfrak{F}}$ has a structure of a right $R_{\mathfrak{F}}$-module (see exposition on pages 195–197 in [6]). The ring $R_{\mathfrak{F}}$ is called the right ring of quotients with respect to the torsion theory τ.

Consider the map $q_M : M \to M_{\mathfrak{F}}$ obtained by composing the projection $M \to M/\mathcal{T}(M)$ with the injection $M/\mathcal{T}(M) \to M_{\mathfrak{F}}$. This defines a left exact functor q from the category of right R-modules to the category of right $R_{\mathfrak{F}}$-modules (see [6] pages 197–199).

In Theorem on page 277 and Corollary 1 on page 279 of [3], Golan has shown that if $\mathfrak{F}$ is differential, then any δ-derivation d on any module M extends to a derivation on the module of quotients $M_{\mathfrak{F}}$ such that $dq_M = q_M d$. Bland proved that such extension is unique and that the converse is also true (Propositions 2.1 and 2.3 in [1]). Thus a filter $\mathfrak{F}$ is differential iff every derivation on any module M extends uniquely to a derivation on the module of quotients $M_{\mathfrak{F}}$.

The paper is organized as follows. In Section 2, we generalize the concept of (α, β)-derivation to (α, β)-higher derivation (Definition 2.1). In Section 3, we show that every filter $\mathfrak{F}$ that is α and β-invariant is also Δ-invariant for any (α, β)-higher derivation Δ (Proposition 3.1). As a consequence, we obtain that every Gabriel filter is higher differential (Corollary 3.2) and that every higher derivation on a module extends to its module of quotients (Corollary 3.3). In Section 4, we show that the assumptions for some results from [8] and [9] can be relaxed and that these results hold for *every* two filters $\mathfrak{F}_1$ and $\mathfrak{F}_2$ such that $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ (Corollary 4.1). In Section 5, we show that the results from previous sections hold for symmetric filters as well (Corollary 5.1). Lastly, in Section 6, we present an example of a torsion theory that is not differential (Example 6) thus answering a question from [8]. Using result from Section 3, we also show that there cannot exist a hereditary torsion theory that is differential but not higher differential.

2. (α, β)-higher derivations

Recall that a *higher derivation* (*HD*) on R is an indexed family $\{\delta_n\}_{n\in\omega}$ of additive maps δ_n such that δ_0 is the identity mapping on R and

$$\delta_n(rs) = \sum_{i=0}^{n} \delta_i(r)\delta_{n-i}(s)$$

for all n. For example, if δ is a derivation, the family $\{\frac{\delta^n}{n!}\}$ is a higher derivation.

Let α and β be ring automorphisms. Throughout this, and most of the next section, we assume that R is a ring in which $n1_R$ is invertible for every positive integer n. In case that α and β are both identities, we can drop this additional assumption on R. We generalize the concept of an (α, β)-derivation to higher derivations as follows.

Definition 2.1. An (α, β)*-higher derivation* ((α, β)*-HD*) on R is an indexed family $\Delta = \{\delta_n\}_{n\in\omega}$ of additive maps δ_n such that δ_0 is the identity mapping on R and

$$\delta_n(rs) = \delta_n(r)\alpha^n(s) + \sum_{i=1}^{n} \frac{i!(n-i)!}{n!} \sum_{k_0+\cdots+k_i=n-i} \delta_{k_0} \prod_{j=1}^{i} \beta\delta_{k_j}(r)\ \alpha^{k_0} \prod_{j=1}^{i} \delta_1\alpha^{k_j}(s)$$

where any composition of the form $\delta_1^j\delta_1^k$ in the second product is substituted by δ_{j+k}. Also, in case that β is the identity, $\delta_j\delta_k$ in the first product is substituted by δ_{j+k}.

For $n = 1$ this formula yields the familiar $\delta_1(rs) = \delta_1(r)\alpha(s) + \beta(r)\delta_1(s)$. For $n = 2$ we obtain that $\delta_2(rs) = \delta_2(r)\alpha^2(s) + \frac{1}{2}\beta\delta_1(r)\delta_1\alpha(s) + \frac{1}{2}\delta_1\beta(r)\alpha\delta_1(s) + \beta^2(r)\delta_2(s)$.

Note that the elements of the form $(n1_R)^{-1}$ are in the center of R for any positive integer n since elements of the form $n1_R$ are in the center of R. So, the coefficients $\frac{i!(n-i)!}{n!}$ commute with all ring elements.

If α is an identity, we obtain

$$\delta_n(rs) = \delta_n(r)s + \sum_{i=1}^{n} \frac{i!(n-i)!}{n!} \sum_{k_0+\cdots+k_i=n-i} \delta_{k_0} \prod_{j=1}^{i} \beta\delta_{k_j}(r)\ \delta_i(s)$$

If β is an identity, we obtain

$$\delta_n(rs) = \delta_n(r)\alpha^n(s) + \sum_{i=1}^{n} \frac{i!(n-i)!}{n!} \sum_{k_0+\cdots+k_i=n-i} \delta_{n-i}(r)\ \alpha^{k_0} \prod_{j=1}^{i} \delta_1\alpha^{k_j}(s)$$

In particular, if both α and β are identities, we obtain

$$\begin{aligned}
\delta_n(rs) &= \delta_n(r)s + \sum_{i=1}^{n} \frac{i!(n-i)!}{n!} \sum_{k_0+\cdots+k_i=n-i} \delta_{n-i}(r)\ \delta_i(s) \\
&= \delta_n(r)s + \sum_{i=1}^{n} \frac{i!(n-i)!}{n!}\ \delta_{n-i}(r)\ \delta_i(s) \sum_{k_0+\cdots+k_i=n-i} 1_R \\
&= \delta_n(r)s + \sum_{i=1}^{n} \frac{i!(n-i)!}{n!} \delta_{n-i}(r)\ \delta_i(s) \frac{n!}{i!(n-i)!} \\
&= \delta_n(r)s + \sum_{i=1}^{n} \frac{i!(n-i)!}{n!} \frac{n!}{i!(n-i)!} \delta_{n-i}(r)\ \delta_i(s) = \sum_{i=0}^{n} \delta_{n-i}(r)\delta_i(s).
\end{aligned}$$

The last formula in the chain above is exactly the one that defines a higher derivation.

3. Higher differentiation invariance

If $\Delta = \{\delta_n\}$ is an (α, β)-HD, a Gabriel filter $\mathfrak{F}$ is Δ-*invariant* if for every $I \in \mathfrak{F}$ and every n, there is $J \in \mathfrak{F}$ such that $\delta_i(J) \subseteq I$ for all $i \leq n$ (equivalently, for every $I \in \mathfrak{F}$ and every n, there is $J \in \mathfrak{F}$ such that $\delta_n(J) \subseteq I$). If a filter $\mathfrak{F}$ is Δ-invariant for every (α, β)-HD Δ, $\mathfrak{F}$ is said to be *higher differential* (*HD*). The hereditary torsion theory determined by $\mathfrak{F}$ is said to be *higher differential* in this case.

Proposition 3.1. *Let Δ be a higher (α, β)-derivation. Then any Gabriel filter $\mathfrak{F}$ that is α and β-invariant is Δ-invariant.*

Proof. Let $I \in \mathfrak{F}$. We shall use induction to show that for every n, there is $J \in \mathfrak{F}$ such that $\delta_n(J) \subseteq I$.

For $n = 0$ the claim trivially holds for $J = I$. Assume that the claim holds for all $i < n$. By induction hypothesis for I there are right ideals $J_i \in \mathfrak{F}$ with $\sum_{k_0+\cdots+k_i=n-i} \delta_{k_0} \prod_{j=1}^{i} \beta\delta_{k_j}(J_i) \subseteq I$ for all $0 < i \leq n$. Note that for $i = n$, $\sum_{k_0+\cdots+k_i=n-i} \delta_{k_0} \prod_{j=1}^{i} \beta\delta_{k_j}$ is β^n. Since $\mathfrak{F}$ is β-invariant, there is $J_n \in \mathfrak{F}$ such that $\beta^n(J_n) \subseteq I$.

Take $J_0 = I$, and let J_α be a right ideal in $\mathfrak{F}$ with $\alpha^n(J_\alpha) \subseteq I$. Let $K = \bigcap_{i \leq n} J_i \cap J_\alpha$. Then K is in $\mathfrak{F}$ and $K \subseteq I$. Define $J = \{r \in K | \delta_n(r) \in I\}$. Then J is a right ideal of R, $J \subseteq K \subseteq I$ and $\delta_n(J) \subseteq I$. Also,

$$\sum_{k_0+\cdots+k_i=n-i} \delta_{k_0} \prod_{j=1}^{i} \beta\delta_{k_j}(J) \subseteq \sum_{k_0+\cdots+k_i=n-i} \delta_{k_0} \prod_{j=1}^{i} \beta\delta_{k_j}(J_i) \subseteq I$$

for all $0 < i \leq n$. In order to prove that J is in $\mathfrak{F}$, it is sufficient to show that $(r : J) \in \mathfrak{F}$ for all $r \in K$. To show that, we shall show that $(\alpha^{-n}\delta_n(r) : K) \subseteq (r : J)$. Since $K \in \mathfrak{F}$, $(\alpha^{-n}\delta_n(r) : K) \in \mathfrak{F}$, and so this will be sufficient for $(r : J) \in \mathfrak{F}$.

Let $s \in (\alpha^{-n}\delta_n(r) : K)$. Then $\delta_n(r)\alpha^n(s) \in \alpha^n(K) \subseteq \alpha^n(J_\alpha) \subseteq I$. The terms $\sum_{k_0+\cdots+k_i=n-i} \delta_{k_0} \prod_{j=1}^{i} \beta\delta_{k_j}(r)$ are in I for every $i = 1, \ldots, n$ by construction. Since fractions $\frac{i!(n-i)!}{n!}$ are in the center of R, we obtain that every term on the right side of the formula below is in I as well.

$$\delta_n(rs) = \delta_n(r)\alpha^n(s) + \sum_{i=1}^{n} \frac{i!(n-i)!}{n!} \sum_{k_0+\cdots+k_i=n-i} \delta_{k_0} \prod_{j=1}^{i} \beta\delta_{k_j}(r)\ \alpha^{k_0} \prod_{j=1}^{i} \delta_1\alpha^{k_j}(s).$$

Thus $\delta_n(rs) \in I$. Since $rs \in K$, we have that rs is in J. So $s \in (r : J)$. □

For the remainder of the paper, we drop the condition that the integer multiples of 1_R are invertible and we work with a most general unital ring. Recall that if α and β are identities, the formula in Definition 2.1 becomes $\delta_n(rs) = \sum_{i=0}^{n} \delta_{n-i}(r)\delta_i(s)$. So, the assumption on the invertibility of the integer multiples of 1_R is no longer needed. Note that the proof of Proposition 3.1 still holds in this case as well. Thus, as a direct corollary of Proposition 3.1, we obtain the following.

Corollary 3.2. *Any Gabriel filter is higher derivation invariant (i.e., every torsion theory is higher differential).*

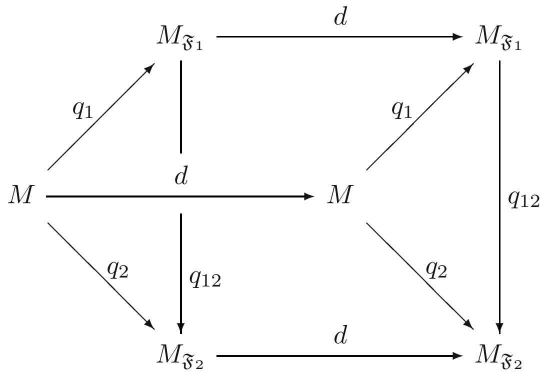
Let $\Delta = \{\delta_n\}_{n\in\omega}$ be a higher derivation on R. If $\{d_n\}_{n\in\omega}$ is an indexed family of additive maps on a right R-module M such that d_0 is the identity mapping on M and $d_n(mr) = \sum_{i=0}^{n} d_i(m)\delta_{n-i}(r)$ for all n, we say that $\{d_n\}$ is *higher Δ-derivation* (Δ-HD for short) on M. If D is such that every d_n extends to the module of quotients $M_{\mathfrak{F}}$ of a Gabriel filter $\mathfrak{F}$ such that $d_n q_M = q_M d_n$ for all n, then we say that D *extends* to a Δ-HD on $M_{\mathfrak{F}}$.

Corollary 3.3. *Let τ be a hereditary torsion theory with filter $\mathfrak{F}$ and Δ be a HD on R. Every Δ-HD D on any module M extends uniquely to the module of quotients $M_{\mathfrak{F}}$.*

Proof. Bland showed that a Gabriel filter is a HD filter iff for every R-module M, every HD $\{d_n\}$ on M, $d_i(\mathcal{T}(M)) \subseteq \mathcal{T}(M)$ for all $i \leq n$ for all n (Lemma 3.5 in [2]). Bland also showed that τ is higher differential iff every Δ-HD D on any module M extends uniquely to a Δ-HD on $M_{\mathfrak{F}}$ (Proposition 4.2, [2]). Since every hereditary torsion theory is higher differential by Corollary 3.2, the result follows. □

4. Extending derivation to different modules of quotients

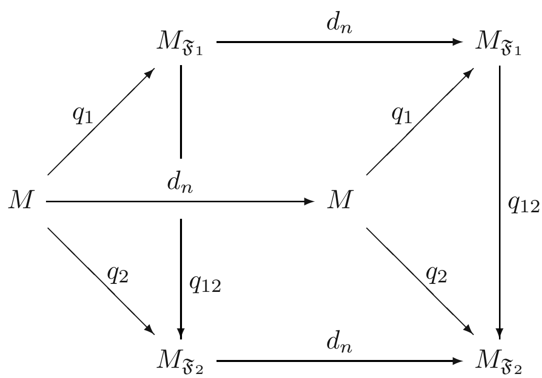
Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two filters such that $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. Let M be a right R-module, q_i the natural maps $M \to M_{\mathfrak{F}_i}$ for $i = 1, 2$, and q_{12} the map $M_{\mathfrak{F}_1} \to M_{\mathfrak{F}_2}$ induced by inclusion $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. In this case, $q_{12}q_1 = q_2$. Let d be a δ-derivation on M. If the diagram below commutes, we say that the extensions of d on $M_{\mathfrak{F}_1}$ and $M_{\mathfrak{F}_2}$ *agree.*



Let d be a derivation on M that extends to $M_{\mathfrak{F}_1}$. Conditions under which d can be extended to $M_{\mathfrak{F}_2}$ so that the extension agree were studied in [9].

By Corollary 3 from [4], any derivation d on M extends to both $M_{\mathfrak{F}_1}$ and $M_{\mathfrak{F}_2}$. By Proposition 2 from [9], the two extensions agree then. This implies that, for any module M, the extensions of a δ-derivation d to $M_{\mathfrak{F}_1}$ and $M_{\mathfrak{F}_2}$ agree. In particular, the extension of d on module of quotients with respect Lambek or Goldie torsion theory agree with the extension of d with respect to any other hereditary and faithful torsion theory.

In [8], the concept of agreeing extensions is generalized to higher derivations. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two filters such that $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$, Δ a HD on R, M be a right R-module, and $\{d_n\}$ a Δ-HD defined on M. If $\{d_n\}$ extends to $M_{\mathfrak{F}_1}$ and $M_{\mathfrak{F}_2}$ in such a way that the following diagram commutes for every n, then we say that the extensions of $\{d_n\}$ on $M_{\mathfrak{F}_1}$ and $M_{\mathfrak{F}_2}$ *agree.*



Corollary 4.1. *If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are two filters such that $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$, and Δ a HD on R, then for any module M the extension of a Δ-HD D to $M_{\mathfrak{F}_1}$ agrees with the extension of D to $M_{\mathfrak{F}_2}$. In particular, the extension of D on module of quotients with respect Lambek or Goldie torsion theory agree with the extension of D with respect to any other hereditary and faithful torsion theory.*

Proof. By Proposition 3 from [8], if a Δ-HD D extends to $M_{\mathfrak{F}_1}$ and $M_{\mathfrak{F}_2}$, then the extensions agree. By Corollary 3.3, D always extends to both $M_{\mathfrak{F}_1}$ and $M_{\mathfrak{F}_2}$ and so the result follows. □

5. Symmetric modules of quotients

In [9], the concept of invariant filters is extended to symmetric filters as well. A symmetric filter ${}_l\mathfrak{F}_r$ induced by a left filter $\mathfrak{F}_l$ and a right filter $\mathfrak{F}_r$ can be defined so that the hereditary torsion theory ${}_l\tau_r$ on R-bimodules that correspond to ${}_l\mathfrak{F}_r$ has the torsion class equal to the intersection of torsion classes of τ_l and τ_r :

$${}_l\mathcal{T}_r = \mathcal{T}_l \cap \mathcal{T}_r.$$

In [5], the *symmetric module of quotients* ${}_{\mathfrak{F}_l}M_{\mathfrak{F}_r}$ of M with respect to ${}_l\mathfrak{F}_r$ is defined to be

$${}_{\mathfrak{F}_l}M_{\mathfrak{F}_r} = \varinjlim_{K \in {}_l\mathfrak{F}_r} \operatorname{Hom}(K, \frac{M}{{}_l\mathcal{T}_r(M)})$$

where the homomorphisms in the formula are $R \otimes R^{op}$ homomorphisms (equivalently R-bimodule homomorphisms). We shorten the notation ${}_{\mathfrak{F}_l}M_{\mathfrak{F}_r}$ to ${}_lM_r$. Just as in the right-sided case, there is a left exact functor q_M mapping M to the symmetric module of quotients ${}_lM_r$ such that $\ker q_M$ is the torsion module ${}_l\mathcal{T}_r(M)$.

Every derivation δ on R determines a derivation on $R \otimes_{\mathbb{Z}} R^{op}$ given by $\overline{\delta}(r \otimes s) = \delta(r) \otimes s + r \otimes \delta(s)$. Similarly, every HD Δ on R determines a HD $\overline{\Delta}$ on $R \otimes_{\mathbb{Z}} R^{op}$ given by

$$\overline{\delta_n}(r \otimes s) = \sum_{i=0}^{n} \delta_i(r) \otimes \delta_{n-i}(s).$$

If M is an R-bimodule, and δ a derivation on R, we say that an additive map $d : M \to M$ is a *δ-derivation* if

$$d(xr) = d(x)r + x\delta(r) \text{ and } d(rx) = \delta(r)x + rd(x)$$

for all $x \in M$ and $r \in R$. Note that d is a $\overline{\delta}$-derivation on M considered as a right $R \otimes_{\mathbb{Z}} R^{op}$-module. Conversely, every $\overline{\delta}$-derivation of a right $R \otimes_{\mathbb{Z}} R^{op}$-module determines a δ-derivation of the corresponding bimodule. Thus, every derivation δ on R is a $\overline{\delta}$-derivation on R considered as a right $R \otimes_{\mathbb{Z}} R^{op}$-module. Conversely, every derivation $\overline{\delta}$ on $R \otimes_{\mathbb{Z}} R^{op}$ is a δ-derivation of $R \otimes_{\mathbb{Z}} R^{op}$ considered as an R-bimodule.

This generalizes to higher derivations as well. If M is an R-bimodule, and Δ a HD on R, we say that an indexed family of additive maps $\{d_n\}$ defined on M is a Δ-*HD* if d_0 is an identity,

$$d_n(xr) = \sum_{i=0}^{n} \delta_i(x)\delta_{n-i}(r) \text{ and } d_n(rx) = \sum_{i=0}^{n} \delta_i(r)\delta_{n-i}(x)$$

for all $x \in M$ and $r \in R$. It is straightforward to check that $\{d_n\}$ is a $\overline{\Delta}$-HD on M considered as a right $R \otimes_{\mathbb{Z}} R^{op}$-module. Conversely, every $\overline{\Delta}$-HD on a right $R \otimes_{\mathbb{Z}} R^{op}$-module M determines a Δ-HD on M considered as an R-bimodule. Specifically, every HD Δ on R is a $\overline{\Delta}$-HD on R considered as a right $R \otimes_{\mathbb{Z}} R^{op}$-module. Conversely, every HD $\overline{\Delta}$ on $R \otimes_{\mathbb{Z}} R^{op}$ is a Δ-HD on $R \otimes_{\mathbb{Z}} R^{op}$ considered as an R-bimodule.

A symmetric filter ${}_l\mathfrak{F}_r$ induced by a left Gabriel filter $\mathfrak{F}_l$ and a right Gabriel filter $\mathfrak{F}_r$ is said to be δ-invariant if for every $I \in {}_l\mathfrak{F}_r$, there is $J \in {}_l\mathfrak{F}_r$ such that $\overline{\delta}(J) \subseteq I$. If we consider the right $R \otimes_{\mathbb{Z}} R^{op}$-ideals I and J as R-bimodules, the condition $\overline{\delta}(J) \subseteq I$ is equivalent with $\delta(J) \subseteq I$ by the above observations. This definition generalizes to higher derivations on a straightforward way just as in one-sided case.

We say that ${}_l\mathfrak{F}_r$ is a *differential filter* if it is δ-invariant for all derivations δ. The hereditary torsion theory determined by ${}_l\mathfrak{F}_r$ is said to be *differential* in this case. Similarly, a HD symmetric filter is defined.

The following proposition proves Corollaries 3.2, 3.3 and 4.1 for symmetric filters.

Corollary 5.1.

1. *Any symmetric Gabriel filter is higher derivation invariant* (*i.e., every symmetric torsion theory is higher differential*).
2. *Let* ${}_l\tau_r$ *be a symmetric hereditary torsion theory with filter* ${}_l\mathfrak{F}_r$ *and* Δ *be a HD on* R. *Every* Δ-*HD* D *on any module* M *extends uniquely to the module of quotients* ${}_lM_r$.
3. *If* ${}_l\mathfrak{F}_r^1$ *and* ${}_l\mathfrak{F}_r^2$ *are two symmetric filters such that* ${}_l\mathfrak{F}_r^1 \subseteq {}_l\mathfrak{F}_r^2$, *and* Δ *is a HD on* R, *then for any bimodule* M, *the extensions of any* Δ-*HD* D *to* ${}_lM_r^1$ *and* ${}_lM_r^2$ *agree.*

Proof. 1. By Proposition 3 of [9], if $\mathfrak{F}_l$ and $\mathfrak{F}_r$ are differential, then ${}_l\mathfrak{F}_r$ is also differential. By Corollary 3 of [4], $\mathfrak{F}_l$ and $\mathfrak{F}_r$ are always differential and so we obtain that every symmetric filter is differential as well. In [8], the results on symmetric filters from [9] are generalized to higher derivations. In particular, it is shown that if $\mathfrak{F}_l$ and $\mathfrak{F}_r$ are HD, that the symmetric filter ${}_l\mathfrak{F}_r$ is HD as well (see Proposition 4 of [8]). This, together with Corollary 3.2, gives us part 1.

2. Part iii) of Proposition 4 in [8] states that part 2 holds provided that the filter ${}_l\mathfrak{F}_r$ is HD. However, any filter is HD by part 1, so the result follows.

3. Proposition 5 in [8] states that part 3 holds provided that ${}_l\mathfrak{F}_r^1$ and ${}_l\mathfrak{F}_r^2$ are HD. Since these conditions are always fulfilled by part 1, the result follows. □

6. Torsion theory that is not differential

Gabriel filter, right ring and modules of quotients and related concepts are defined just for a torsion theory that is hereditary. Thus, if we want to generalize the concept of differential torsion theory to torsion theories that are not necessarily hereditary, we cannot use the second and third of the following three equivalent conditions for differentiability of a hereditary torsion theory.

1. For every derivation δ, and every right R-module M with a δ-derivation d, $d(\mathcal{T}M) \subseteq \mathcal{T}M$;
2. Gabriel filter $\mathfrak{F}$ is δ-invariant (i.e., for every $I \in \mathfrak{F}$ there is $J \in \mathfrak{F}$ with $\delta(J) \subseteq I$) for every derivation δ;
3. For every derivation δ, every δ-derivation on any module uniquely extends to the module of quotients.

Note that just the first condition is meaningful even if a torsion theory is not hereditary. So, let us introduce the following definition.

Definition 6.1. Let τ be a (not necessarily hereditary) torsion theory. Then τ is *differential* if $d(\mathcal{T}M) \subseteq \mathcal{T}M$ for any ring derivation δ and any right R-module M with a δ-derivation d.

The following example shows that not every torsion theory is differential.

Example. Let $R = \mathbb{Z}[x]$ and $I = (x)$. Consider the module $R/I \cong \mathbb{Z}$. Consider the class of right R-modules $\mathcal{T} = \{M | \ker(M \to M \otimes_R \mathbb{Z}) = M\}$. Note that this class is closed under quotients, extensions and direct sums, so it defines a torsion class of a torsion theory τ. In this torsion theory $\mathcal{T}M = \ker(M \to M \otimes_R \mathbb{Z})$ for every module M.

Note that $\ker(R \to R \otimes_R \mathbb{Z}) = I$ so $\mathcal{T}R = I$. $\ker(I \to I \otimes_R \mathbb{Z}) = I^2$. This shows that $I^2 = \mathcal{T}I \neq I \cap \mathcal{T}R = I$, so the torsion theory is not hereditary. Note also that if $\mathbb{Z}$ were flat as a left R-module, then this torsion theory would necessarily have been hereditary.

Now let us consider the map $\delta : R \to R$ given by $\delta = \frac{d}{dx}$, i.e.,

$$\delta(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

It is easy to see that this is a derivation on R. As $\delta(x) = 1$, we have that $x \in \mathcal{T}R$ and $\delta(x) \notin \mathcal{T}R$. $\square$

This answers the first of the three questions from section 6 of [8]: there is a non-hereditary and non-differential torsion theory.

The second question from [8] is asking if an extension of a derivation to module of quotients with respect to larger torsion theory can be restricted to extension of a derivation with respect to a smaller torsion theory. The affirmative answer emerged with Corollary 3 of [4] and Proposition 2 of [9]. Namely, by Corollary 3 of [4], every torsion theory is differential. By Proposition 2 of [9], this implies that all extensions of derivations to module of quotients agree. Moreover, by results of this paper, this result holds for higher derivations as well.

Finally, the third question from [8] is asking if there is a differential hereditary torsion theory that is not higher differential. By Corollary 3.2 any hereditary torsion theory is higher differential so the answer to this question is “no”.

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Ring and Module Theory
Trends in Mathematics, 175–193

Prime Ideals in Noetherian Rings: A Survey

Roger Wiegand and Sylvia Wiegand

Abstract. We consider the structure of the partially ordered set of prime ideals in a Noetherian ring. The main focus is Noetherian two-dimensional integral domains that are rings of polynomials or power series.

Mathematics Subject Classification (2000). 13E05, 13F20, 13F25, 13J15.

Keywords. Prime ideals, Noetherian.

0. Introduction

The authors have been captivated by the partially ordered set of prime ideals for about four decades. Their initial motivation was interest in Kaplansky's question, phrased about 1950: "What partially ordered sets occur as the set of prime ideals of a Noetherian ring, ordered under inclusion?" This has turned out to be an extremely difficult question, perhaps a hopeless one.

Various mathematicians have studied Kaplansky's question and related questions. In 1971, M. Hochster [12] characterized the topological spaces X such that $X \cong \mathrm{Spec}(R)$ for some commutative ring R, where $\mathrm{Spec}(R)$ is considered as a topological space with the Zariski topology. In this topology, the sets of the form $\mathrm{V}(I) := \{P \in \mathrm{Spec}(R) \mid P \supseteq I\}$, where I is an ideal of R, are the closed sets. Of course the topology determines the partial ordering, since $P \subseteq Q$ if and only if $Q \in \overline{\{P\}}$.

In 1973, W.J. Lewis showed that every finite partially ordered set is the prime spectrum of a commutative ring R, and, in 1976, Lewis and J. Ohm found necessary and sufficient conditions for a partially ordered set to be the prime spectrum of a Bézout domain [19, 20]. In [42], S. Wiegand showed that for every *rooted tree* U, there is a Bézout domain R having prime spectrum order-isomorphic to U and such that each localization $R_{\mathbf{m}}$ of R at a maximal ideal $\mathbf{m}$ of R is a maximal valuation domain. (A rooted tree is a finite poset U, with unique minimal element, such that for each $x \in U$ the elements below x form a chain.) The construction in [42] was motivated by another problem of Kaplansky: Characterize the commutative rings

The authors thank the University of Nebraska for its support of our research projects.

for which every finitely generated module is a direct sum of cyclic modules. The solution, which makes heavy use of the prime spectrum, is in [40].

In general, the topology carries more information than the partial ordering. For example, one can build a non-Noetherian domain R with non-zero Jacobson radical $\mathcal{J}(R)$, but whose spectrum is order-isomorphic to $\operatorname{Spec}(\mathbb{Z})$. The partial ordering does not reveal the fact that the radical is non-zero, but the topology does: For this domain R, the set of maximal elements of $\operatorname{Spec}(R)$ is closed, whereas in $\operatorname{Spec}(\mathbb{Z})$ it is not. On the other hand, if a ring is Noetherian, the partial order determines the topology. To see this, we recall that for every ideal I of a Noetherian ring R there are only finitely many prime ideals minimal with respect to containing I; if $P_1, \dots, P_n$ are those primes, then $\mathrm{V}(I) = \bigcup_{i=1}^n \mathrm{V}(P_i) = \bigcup_{i=1}^n \overline{\{P_i\}}$. Therefore the closed subsets of $\operatorname{Spec}(R)$ are exactly the finite unions of the sets $\overline{\{P\}}$, as P ranges over $\operatorname{Spec}(R)$.

We establish some notation and terminology for *posets* (partially ordered sets).

Notation 0.1. The *height* of an element u in a poset U is $\operatorname{ht}(u) := \sup\{n \mid$ there is a chain $x_0 < x_1 < \cdots < x_n = u$ in $U\}$. The *dimension* of U is $\dim(U) = \sup\{\operatorname{ht}(u) \mid u \in U\}$. For a subset S of U, $\min(S)$ denotes the set of minimal elements of S, and $\max(S)$ its set of maximal elements. For $u \in U$, we define

$$u^{\uparrow} := \{v \mid u \leq v\} \quad \text{and} \quad u_{\downarrow} := \{v \mid v \leq u\}.$$

The *exactly less than set* for a subset $S \subseteq U$ is $\mathrm{L}_e(S) := \{v \in U \mid v^{\uparrow} - \{v\} = S\}$. For elements $u, v \in U$, their *minimal upper bound set* is the set $\operatorname{mub}(u,v) := \min(u^{\uparrow} \cap v^{\uparrow})$ and their *maximal lower bound set* is $\operatorname{Mlb}(u,v) := \max(u_{\downarrow} \cap v_{\downarrow})$.

We say v *covers* u (or v is a *cover* of u) and write "$u << v$" provided $u < v$ and there are no elements of U strictly between u and v. A chain $u_0 < u_1 < \cdots < u_n$ is *saturated* provided u_{i+1} covers u_i for each i.

To return to Kaplansky's problem, we begin by listing some well-known properties of a partially ordered set U if U is order-isomorphic to $\operatorname{Spec}(R)$, for R a Noetherian ring:

Proposition 0.2. *Let R be a Noetherian commutative ring and let U be a poset order-isomorphic to* $\operatorname{Spec}(R)$ *for some Noetherian ring R. Then*

(1) *U has only finitely many minimal elements,*
(2) *U satisfies the ascending chain condition.*
(3) *Every element of U has finite height; in particular, U satisfies the descending chain condition.*
(4) $\operatorname{mub}(u,v)$ *is finite, for every pair of elements $u, v \in U$.*
(5) *If $u < v < w$, then there exist infinitely many v_i with $u < v_i < w$.*

Proof. Items (1) and (2) are clear, and (3) comes from the Krull Height Theorem [24, Theorem 13.5], which says that in a Noetherian ring a prime ideal minimal over an n-generated ideal has height at most n. For (4), let P and Q be prime ideals of R, and note that $\operatorname{mub}(P,Q)$ is the set of minimal prime ideals of the ideal $P+Q$. To prove (5), suppose we have a chain $P < V < Q$ of prime ideals in a

Noetherian ring R, but that there are only finitely many prime ideals $V_1, \dots, V_n$ between P and Q. By localizing at Q and passing to R_Q/PR_Q, we may assume that R is a local domain of dimension at least two, with only finitely many non-zero prime ideals V_i properly contained in the maximal ideal Q. By "prime avoidance" [2, Lemma 1.2.2], there is an element $r \in Q - (V_1 \cup \cdots \cup V_n)$. But then Q is a minimal prime of the principal ideal (r), and Krull's Principal Ideal Theorem (the case $n = 1$ of the Krull Height Theorem) says that $\mathrm{ht}(Q) \leq 1$, a contradiction. □

In 1976 [39], the present authors characterized those partially ordered sets that are order-isomorphic to the *j-spectrum* of some countable Noetherian ring. (The *j-spectrum* is the set of primes that are intersections of maximal ideals.) A poset U arises in this way if and only if

(1) U is countable and has only finitely many minimal elements,
(2) U has the ascending chain condition,
(3) every element of U has finite height,
(4) $\mathrm{mub}(u, v)$ is finite for each $u, v \in U$, and
(5) $\min(u^{\uparrow} - \{u\})$ is infinite for each non-maximal element $u \in U$.

An equivalent way of stating the theorem is: A topological space X is homeomorphic to the maximal ideal space of some countable Noetherian ring if and only if

(1) X has only countably many closed sets,
(2) X is T_1 and Noetherian, and
(3) for every $x \in X$ there is a bound on the lengths of chains of closed irreducible sets containing x.

It is still unknown whether or not the theorem is true if all occurrences of "countable" are removed.

1. Bad behavior

Recall that a Noetherian ring R is *catenary* provided, for every pair of primes P and Q, with $P \subset Q$, all saturated chains of primes between P and Q have the same length. Every ring finitely generated as an algebra over a field, or over $\mathbb{Z}$, is catenary. More generally, excellent rings are, by definition, catenary, and the class of excellent rings is closed under the usual operations of passage to homomorphic images, localizations, and finitely generated algebras, cf. [24, p. 260]. Since fields, complete rings (e.g., rings of formal power series over a field), and the ring of integers are all excellent, the rings one encounters in nature are all catenary. Perhaps the first indicator of the rich pathology that can occur in a Noetherian ring was Nagata's example [29] of a Noetherian ring that is not catenary. Every two-dimensional integral domain is catenary, and so Nagata's example is a Noetherian local domain of dimension three; it has saturated chains of length two and length three between (0) and the maximal ideal. Later, in 1979, R. Heitmann [11] showed that every finite poset admits a saturated (i.e., cover-preserving) embedding into $\mathrm{Spec}(R)$ for some Noetherian ring R.

The catenary condition has a connection with the representation theory of local rings. As Hochster observed in 1972 [13], the existence of a maximal Cohen-Macaulay module (a finitely generated module with depth equal to $\dim(R)$) and with support equal to $\operatorname{Spec}(R)$ forces R to be *universally* catenary, that is, every finitely generated R-algebra is catenary. In particular, an integral domain with a maximal Cohen-Macaulay module must be universally catenary. G. Leuschke and R. Wiegand used this connection in [18] to manufacture a two-dimensional domain R with *no* maximal Cohen-Macaulay modules but whose completion $\widehat{R}$ has infinite Cohen-Macaulay type. (This gave a negative answer to a conjecture of Schreyer [34] on ascent of finite Cohen-Macaulay type to the completion.) For other connections between prime ideal structure and representation theory we refer the reader to the survey paper [41] by the present authors.

In Nagata's example, the catenary condition fails because a height-one prime has a cover that has height three. A theorem of McAdam [25] guarantees that such behavior cannot be too widespread:

Theorem 1.1 ([25]). *Let P be a prime of height n in a Noetherian ring. Then all but finitely many covers of P have height $n+1$.*

In response to a question raised by Hochster in 1974 [14], Heitmann [10] and S. McAdam [26] showed independently that there exists a two-dimensional Noetherian domain R with maximal ideals P and Q of height two such that $P \cap Q$ contains no height-one prime ideal. Later, in 1983, S. Wiegand [43] combined Heitmann's procedure with a method for producing non-catenary rings due to A.M. de Souza-Doering and I. Lequain [3], to prove the following:

Theorem 1.2 ([43, Theorem 1]). *Let F be an arbitrary finite poset. There exist a Noetherian ring R and a saturated embedding $\varphi : U \to \operatorname{Spec}(R)$ such that φ preserves minimal upper bounds sets and maximal lower bound sets. In detail, for $u, v \in U$, we have*

(i) *$u < v$ if and only if $\varphi(u) < \varphi(v)$;*
(ii) *v covers u if and only if $\varphi(v)$ covers $\varphi(u)$;*
(iii) *$\varphi(\operatorname{mub}(u,v)) = \operatorname{mub}(\varphi(u), \varphi(v))$; and*
(iv) *$\varphi(\operatorname{Mlb}(u,v)) = \operatorname{Mlb}(\varphi(u), \varphi(v))$.*

Using this theorem, one can characterize the spectra of two-dimensional semi-local Noetherian domains:

Corollary 1.3 ([43, Theorem 2]). *Let U be a countable poset of dimension two. Assume that U has a unique minimal element and $\max(U)$ is finite. Then $U \cong \operatorname{Spec}(R)$ for some Noetherian domain R if and only if $\mathrm{L}_e(u)$ is infinite for each element u with $\operatorname{ht}(u) = 2$.*

Conjecture 1.4. Let U be a two-dimensional poset in which both $\min(U)$ and $\max(U)$ are finite. Then $U \cong \operatorname{Spec}(R)$ for some Noetherian ring R if and only if

(1) $\mathrm{L}_e(u)$ is infinite for each element u with $\operatorname{ht}(u) = 2$, and
(2) $\operatorname{mub}(u, v)$ is finite for all $u, v \in \min(U)$.

2. Affine domains of dimension two

We begin with an example that illustrates the effect of the ground field on delicate properties of the prime spectrum.

Example 2.1. Let k be an algebraically closed field, let $R = k[X, Y]$, let $P = (X^3 - Y^2)$, and let $\mathbf{m}$ be a maximal ideal containing P. There exists a height-one prime ideal Q such that $P^\uparrow \cap Q^\uparrow = \{\mathbf{m}\}$ if and only if either

(i) $\mathbf{m} = (X, Y)$, or
(ii) $\operatorname{char}(k) \neq 0$.

A geometric interpretation is helpful. Let C be the cuspidal curve $y^2 = x^3$, and let $p \in C - \{(0, 0)\}$. Then there is an irreducible plane curve D with $D \cap C = \{p\}$ (set-theoretically) if and only k has non-zero characteristic.

Proof. The curve C is parametrized by

$$x = t^2, \qquad y = t^3 \qquad (t \in k).$$

Since $\mathbf{m} \supset P$, the point corresponding to $\mathbf{m}$ (via the Nullstellensatz) is on C, and we can write $\mathbf{m} = (X - a^2, Y - a^3)$, where $a \in k$.

Suppose (i) and (ii) fail, that is, $\operatorname{char}(k) = 0$ and $a \neq 0$. Suppose there is a height-one prime ideal Q such that $P^\uparrow \cap Q^\uparrow = \{\mathbf{m}\}$. Let g be a monic irreducible polynomial generating Q, and note that $g(t^2, t^3) = 0$ if and only if $t = a$. With $h(T) = g(T^2, T^3)$, we see that a is the only root of h. Since k is algebraically closed, $h(T) = (T - a)^n$ for some positive integer n. But then $h(T)$ has a non-zero linear term, contradicting the fact that $h(T) \in k[T^2, T^3]$.

For the converse, we note that if $\mathbf{m} = (X, Y)$, then $(X)^\uparrow \cap P^\uparrow = \mathrm{V}(X, Y^2) = \{\mathbf{m}\}$. Now assume that $\operatorname{char}(k) = p > 0$ and that $\mathbf{m} \neq (X, Y)$, that is, $a \neq 0$. Write $p = 2r + 3s$ with $r, s \geq 0$, and let $g = X^r Y^s - a^p$. Then g is irreducible (linear, if $p = 2$ or 3). Since $g(T^2, T^3) = (T - a)^p$, (a^2, a^3) is the only point on C where g vanishes. Thus $P^\uparrow \cap (g)^\uparrow = \{\mathbf{m}\}$. □

There is a slightly fancier way to verify the assertions in the example. Notice that there exists a height-one prime $Q = (g)$ of R with $P^\uparrow \cap Q^\uparrow = \{\mathbf{m}\}$ if and only if $\overline{\mathbf{m}} := \mathbf{m}/P$ is the radical of a principal ideal of R/P. The following lemma, from W. Krauter's 1981 Ph.D. dissertation [16] (cf. also [36, Lemma 3] and [31]) explains what's going on:

Lemma 2.2. *Let R be a one-dimensional Noetherian ring such that R_{red} has only finitely many singular maximal ideals. Then* $\operatorname{Pic}(R)$ *is a torsion group if and only if every maximal ideal of R is the radical of a principal ideal.*

Proof. Since nilpotents have no effect on either of the two conditions, we may assume that R is reduced. Suppose $\operatorname{Pic}(R)$ is torsion, and let $\mathbf{m}$ be a maximal ideal of R (possibly of height zero). Choose an element $f \in \mathbf{m}$ and outside every singular maximal ideal (except possibly $\mathbf{m}$) and outside every minimal prime (except possibly $\mathbf{m}$). Write $(f) = I \cap I_1 \cap \cdots \cap I_t$, an intersection of primary ideals with distinct radicals, and with $\sqrt{I} = \mathbf{m}$. Then $(f) = IJ$, where $J = I_1 \ldots I_t$. Each prime containing J is non-singular and of height one, so J is invertible (check locally). Then $J^n = (g)$ for some $n \geq 1$, and $I^n g = (f^n)$. Since g is a non-zerodivisor, it follows that I^n is principal.

Conversely, assume every maximal ideal is the radical of a principal ideal, and let I be an invertible ideal. Then I is isomorphic to an invertible ideal J outside the union of the singular maximal ideals, [28, Lemma 4.3]. Let $\mathbf{m}_1, \ldots, \mathbf{m}_s$ be the maximal ideals containing J. The rings $R_{\mathbf{m}_i}$ are discrete valuation rings, and we let $J_{\mathbf{m}_i} = \mathbf{m}_i^{e_i} R_{\mathbf{m}_i}$. By checking locally, we see that $J = \mathbf{m}_1^{e_1} \ldots \mathbf{m}_s^{e_s}$. Now let $\mathbf{m}_i = \sqrt{(x_i)}$, and write $x_i R_{\mathbf{m}_i} = \mathbf{m}_i^{f_i} R_{\mathbf{m}_i}$. Checking locally again, we have $(x_i) = \mathbf{m}_i^{f_i}$. Now let $g_i = f_1 \ldots \widehat{f_i} \ldots f_s$, and check that $J^{f_1 \cdots f_s} = (x_1^{e_1 g_1} \ldots x_s^{e_s g_s})$. □

We now state the axioms that characterize the posets U that are order-isomorphic to $\operatorname{Spec}(R)$ for an affine domain R over a field k that is algebraic over a finite field:

Axioms 2.3.

(P0) U is countable.
(P1) U has a unique minimal element.
(P2) U has dimension two.
(P3) For each element x of height one, $x^{\uparrow}$ is infinite.
(P4) For each two distinct elements x, y of height one, $x^{\uparrow} \cap y^{\uparrow}$ is finite.
(P5) Given a finite set S of height-one elements and a finite set T of height-two elements, there is a height-one element w such that

(1) $w < t$ for each $t \in T$; and
(2) if $x \in U, s \in S$ and $w < x > s$, then $x \in T$.

Axioms (P0)–(P4) are obviously satisfied for any two-dimensional domain that is finitely generated as an algebra over a countable Noetherian Hilbert ring. (A Hilbert ring is a ring in which each prime ideal is an intersection of maximal ideals, and any finitely generated algebra over a Hilbert ring is again a Hilbert ring.) It is Axiom (P5) that makes a difference. In the special case where $S := \{s\}$ and $T = \{t\}$ with $s < t$, (P5) provides a height-one element w such that $s^{\uparrow} \cap w^{\uparrow} = \{t\}$. Thus Example 2.1 shows that $\operatorname{Spec}(k[X,Y])$ has no such element if $\operatorname{char}(k) = 0$. In fact, much more is true:

Theorem 2.4. *Let k be a field, and let R be a two-dimensional affine domain over k. If* $\operatorname{Spec}(R)$ *satisfies* (P5)*, then k is an algebraic extension of a finite field.*

Proof. Suppose first that $R = k[X,Y]$. Let $P = (X^3 + XY - Y^2)$, the kernel of the map $R \twoheadrightarrow S := k[T(T-1), T^2(T-1)]$ taking X to $T(T-1)$ and Y to $T^2(T-1)$. Let $\mathbf{m}$ be an arbitrary maximal ideal containing P. Since $\operatorname{Spec}(R)$ satisfies (P5), there is a height-one prime Q such that $P^{\uparrow} \cap Q^{\uparrow} = \{\mathbf{m}\}$. Writing $Q = (f)$, we see that $\mathbf{m}/P$ is the radical of the principal ideal $(f + P)$. This shows that every maximal ideal of R/P is the radical of a principal ideal. By Lemma 2.2, $\operatorname{Pic}(R/P)$

is torsion. We can easily compute $\mathrm{Pic}(R/P) = \mathrm{Pic}(S)$ from the Mayer-Vietoris sequence [27] associated to the conductor square for S:

$$\begin{array}{ccc} S & \hookrightarrow & k[T] \\ \downarrow & & \downarrow \\ k & \hookrightarrow & \frac{k[T]}{T(T-1)} \end{array} \tag{2.4.1}$$

By [27], $\mathrm{Pic}(S) \cong G/H$, where $G = (\frac{k[T]}{T(T-1)})^{\times}$, the group of units of $\frac{k[T]}{T(T-1)}$, and H is the join of the images of the horizontal and vertical maps on groups of units. Since $G = k^{\times} \times k^{\times}$ and H is the diagonal embedding of $k^{\times}$ in G, we see that $\mathrm{Pic}(S) \cong k^{\times}$. Thus $k^{\times}$ is a torsion group. Therefore $\mathrm{char}(k) = p > 0$ (else 2 has infinite order in $k^{\times}$), and every non-zero element is algebraic over the prime field. This shows that k is an algebraic extension of a finite field.

In the general case, we use the Noether Normalization Lemma to express R as an integral extension of a subring $T \cong k[X, Y]$ and apply the next lemma, with $A = A' = T$ and $A'' = R$. □

Lemma 2.5 ([37, Lemma 3]). *Let $A' \subseteq A \subseteq A''$ be integral extensions of Noetherian domains of dimension two, and assume that A' is integrally closed. If* $\mathrm{Spec}(A'')$ *satisfies* (P5) *of* (2.3), *so does* $\mathrm{Spec}(A)$.

Proof. Let S be a finite set of height-one prime ideals of A and T a finite set of maximal ideals of A. Let T'' be the finite set of prime ideals, necessarily maximal, lying over primes in T, and let $S'' = \{Q'' \in \mathrm{Spec}(A'') \mid Q'' \cap A' = Q \cap A' \text{ for some } Q \in S\}$. Let P'' be a height-one prime ideal of A'' satisfying (1) and (2) of (P5) for the sets S'' and T'' (cf. Axioms 2.3). We claim that $P := P'' \cap A$ satisfies (1) and (2) for the sets S and T. For (1), let $\mathbf{m} \in T$, and choose any $\mathbf{m}'' \in \mathrm{Spec}(A'')$ lying over $\mathbf{m}$. Then $\mathbf{m}'' \in T''$, so $P'' \subset \mathbf{m}''$; hence $P \subset \mathbf{m}$. As for (2), suppose $P \subset \mathcal{M}$ and $Q \subset \mathcal{M}$, where $\mathcal{M} \in \mathrm{Spec}(A)$ and $Q \in S$. We must show that $\mathcal{M} \in T$. By "going up", there is a prime $\mathcal{M}''$ of A'' such that $P'' \subset \mathcal{M}''$ and $\mathcal{M}'' \cap A = \mathcal{M}$. Now apply "going down" to the extension $A' \subseteq A''$ to get a prime Q'' such that $\mathcal{M}'' \supset Q''$ and $Q'' \cap A' = Q \cap A'$. Since $Q'' \in S''$, (P5)(2) (for the prime P'' and the sets S'' and T'') implies that $\mathcal{M}'' \in T''$, whence $\mathcal{M} = \mathcal{M}'' \cap A \in T$. □

As we shall see, the converse of Theorem 2.4 is true, though the proof is more difficult. At this point, it is not even clear that there exist posets satisfying Axioms 2.3. (Try building one from scratch; it's not easy!) The next theorem shows that there is *at most one* such poset. Before stating the theorem, we define an operation $A \mapsto A^{\#}$ on subsets of a poset X satisfying Axioms 2.3. Given a subset A of X, let $A^{\#}$ be obtained by adjoining to A the unique minimal element of X and the sets $x^{\uparrow} \cap y^{\uparrow}$, where x and y range over distinct height-one elements in A. (*Clarification: Here and in the sequel "height" always refers to height in X, not the relative height in A.*) Clearly $A^{\#\#} = A^{\#}$. Moreover Axiom (P4) guarantees that $A^{\#}$ is finite if A is finite.

Theorem 2.6 ([36, Theorem 1]). *Let U and V be posets satisfying Axioms 2.3. Given finite subsets A and B of U and V, respectively, every height-preserving isomorphism from $A^{\#}$ onto $B^{\#}$ can be extended to an isomorphism from U onto V. (In particular, U and V are isomorphic: take $A = B = \emptyset$.)*

Proof. We may assume that $A = A^{\#}$ and $B = B^{\#}$. It suffices to prove the following: For each height-preserving isomorphism $\theta : A \xrightarrow{\cong} B$ and each $x \in U - A$, θ extends to a height-preserving isomorphism θ' from $A' := (A \cup \{x\})^{\#}$ onto some set $B' = (B')^{\#} \subset Y$. For then, by symmetry, we can extend the domain of $(\theta')^{-1}$ so that it includes an arbitrary $y \in V - B'$. Since U and V are countable, we will get the desired extension of θ by iterating this back-and-forth stepwise procedure. We refer the reader to the proof of [36, Theorem 1] for the details, which are elementary and boring. □

The proof of the converse of Theorem 2.4 has two main ingredients. The first is a variant of the finiteness theorem for the class number of an algebraic number field. We refer the reader to [37] for the technical shenanigans that reduce the following theorem to the classical result on the class number:

Theorem 2.7. *Let R be a finitely generated $\mathbb{Z}$-algebra of dimension one. Then* $\mathrm{Pic}(R)$ *is finite.* □

Corollary 2.8. *Let R be a one-dimensional Noetherian ring that is finitely generated as an algebra over $\mathbb{Z}$ or over a field k that is an algebraic extension of a finite field. Then every maximal ideal of R is the radical of a principal ideal.*

Proof. By Lemma 2.2 it is enough to prove that $\mathrm{Pic}(R)$ is torsion. In view of Theorem 2.7, it will suffice to show that $\mathrm{Pic}(R)$ is torsion when R is a finitely generated k-algebra and k is algebraic over a finite field.

Write $R = k[X_1, \dots, X_m]/(f_1, \dots, f_n)$, and choose a finite field $\mathbb{F}$ such that each f_j is in $\mathbb{F}[X_1, \dots, X_m]$. For each intermediate field F between $\mathbb{F}$ and k, let $R_F = F[X_1, \dots, X_m]/(f_1, \dots, f_m)$. Then $\mathrm{Pic}(R) = \mathrm{Pic}(\varinjlim R_F) = \varinjlim(\mathrm{Pic}(R_F))$. Since each $\mathrm{Pic}(R_F)$ is finite by Theorem 2.7, $\mathrm{Pic}(R)$ is torsion. □

The second main ingredient is the following Bertini-type theorem:

Theorem 2.9 ([36, Lemma 4]). *Let k be an algebraically closed field, let $A = k[x_1, \dots, x_n]$ be a two-dimensional affine domain over k, and let (f, g) be an A-regular sequence. Then there is a non-empty Zariski-open subset U of $\mathbb{A}^{n+1}(k)$ such that $\sqrt{(f + (\alpha + \sum_{i=1}^{n} \beta_i x_i)g)}$ is a prime ideal whenever $(\alpha, \beta_1, \dots, \beta_n) \in U$.* □

Here is the main result of this section.

Theorem 2.10 ([37, Theorem 2]). *Let k be a field, and let R be a two-dimensional affine domain over k. These are equivalent:*

(1) $\mathrm{Spec}(R)$ *satisfies* (P5).
(2) $\mathrm{Spec}(R)$ *is order-isomorphic to* $\mathrm{Spec}(\mathbb{Z}[X])$
(3) *k is an algebraic extension of a finite field.*

Proof. In view of Theorem 2.4 and Theorem 2.6, it will suffice to show that Spec(R) satisfies (P5) whenever $R = \mathbb{Z}[X]$ or R is an affine domain over a field k that is algebraic over a finite field. Let S and $T := \{\mathbf{m}_1, \dots, \mathbf{m}_t\}$ be the finite sets of primes we are given in Axiom (P5). We may harmlessly assume that $T \neq \emptyset$ and, by enlarging S if necessary, that each $\mathbf{m}_j$ contains some prime in S. Put $I = \bigcap S$, and choose, by Corollary 2.8, $f_j \in \mathbf{m}_j$ such that $\mathbf{m}_j = \sqrt{I + (f_j)}$. Put $f = f_1 \cdots f_t$ and $J = \bigcap T$; then $\sqrt{I + (f)} = J$. We seek a height-one prime ideal P such that $\sqrt{I + P} = J$.

Suppose first that k is the algebraic closure of a finite field and that R is a two-dimensional Cohen-Macaulay domain, finitely generated as a k-algebra. Since $I + (f)$ has height two, there is an element $g \in I$ such that (f, g) is A-regular. By Theorem 2.9 there is an element $\lambda \in A$ such that $P := \sqrt{(f + \lambda g)}$ is a prime ideal. Then $\sqrt{I + P} = J$, and so P satisfies (1) and (2) of (P5).

Suppose, now, that k is an algebraic extension of a finite field and that R is a two-dimensional affine domain over k. By the Noether Normalization Lemma [24, §33, Lemma 2] there are elements $\xi, \eta \in A$, algebraically independent over k, such that A is an integral extension of $A' := k[\xi, \eta]$. Let $\bar{k}$ be the algebraic closure of k, and let $B = (A \otimes_k \bar{k})/Q$, where the prime ideal Q is chosen so that $\dim(B) = 2$. Finally, let A'' be the integral closure of B. Then A'' satisfies Serre's condition (S_2) [24, Theorem 23.8] and hence is Cohen-Macaulay. By what we have just shown, Spec(A'') satisfies (P5), and now Lemma 2.5 shows that A satisfies (P5) as well.

Finally, we suppose that $R = \mathbb{Z}[X]$. We seek a height-one prime ideal P of $\mathbb{Z}[X]$ such that $J = \sqrt{I + P}$. Since $I + (f)$ has height two, there is a polynomial $g \in I$ such that f and g are relatively prime. Then, for each $j \geq 1$ the polynomial $f^k + Yg$ is irreducible in $\mathbb{Z}[X, Y]$, and hence irreducible in $\mathbb{Q}[X, Y]$ (cf., e.g., [15, Exercise 2, p. 102]). Since Bertini's Theorem is not available, we use a version of Hilbert's Irreducibility Theorem, as formulated in Chapter VIII of [17]. Combining Corollary 3 of [17, §2, p. 148] with the corollary in [17, §3, p. 152], we find that there are infinitely many prime integers p for which each of the $t + 1$ polynomials $f^j + pg, 1 \leq j \leq t + 1$, is irreducible in $\mathbb{Q}[X]$. Choose such a prime p with the additional property that $p\mathbb{Z} \neq \mathbf{m}_j \cap \mathbb{Z}$ for $1 \leq j \leq t$. For each $j \leq t + 1$, let c_j be the greatest common divisor of the coefficients of $f^j + pg$; then $h_j := \frac{1}{c_j}(f^j + pg)$ is irreducible in $\mathbb{Z}[X]$.

We claim that there exists $j \leq t + 1$ such that $h_j \in J = \mathbf{m}_1 \cap \cdots \cap \mathbf{m}_t$. For suppose not; then there exist i, j, ℓ, with $1 \leq i < j \leq t + 1$ and $1 \leq \ell \leq t$ such that $h_i \notin \mathbf{m}_\ell$ and $h_j \notin \mathbf{m}_\ell$. Let $\mathbf{m}_\ell \cap \mathbb{Z} = q\mathbb{Z}$. Since $c_i h_i$ and $c_j h_j$ are both in $J \subseteq \mathbf{m}_\ell$, we see that the prime q is a common divisor of both c_i and c_j. Therefore $q \mid (c_i h_i - c_j h_j)$. Now $c_i h_i - c_j h_j = f^i - f^j = f^i(1 - f^{j-i})$. Since q and f^{j-i} are in $\mathbf{m}_\ell$, it follows that $q \mid f$. But also $pg = c_i h_i - f^i$ is a multiple of q, and our choice of p now forces $q \mid g$. This contradicts the assumption that f and g are relatively prime, and the claim is proved.

To complete the proof, we choose j as in the claim and put $P = h_j\mathbb{Z}[X]$. Then $P \subset J$, and $f^j = -pg + h_j c_j \in I + P$. It follows that $J = \sqrt{I + P}$ as desired. □

Actually, Theorem 1 of [37] says a bit more:

Theorem 2.11. [37, Theorem1] *Let D be an order in an algebraic number field. Then* Spec($D[X]$) *satisfies* (P5) *and therefore is order-isomorphic to* Spec($\mathbb{Z}[X]$).

We have put in quite a bit of detail in this chapter in order to reawaken interest in the following conjecture from [37]:

Conjecture 2.12. Let R be a two-dimensional domain finitely generated as a $\mathbb{Z}$-algebra. Then Spec(R) satisfies (P5) and hence is order-isomorphic to Spec($\mathbb{Z}[X]$).

It is easy to see that if Spec(R) satisfies (P5) so does Spec($R[\frac{1}{f}]$) for each non-zero $f \in R$. Thus Spec($D[X, \frac{1}{f}]$) is order-isomorphic to Spec($\mathbb{Z}[X]$) whenever D is an order in an algebraic number field. A stronger result is proved in [33]:

Theorem 2.13. [33, Main Theorem 1.2] *Let D be an order in an algebraic number field, let X an indeterminate, let $g_1, \ldots, g_n$ be nonzero elements of the quotient field of $D[X]$, and let $R = D[X, g_1, \ldots, g_n]$. Then* Spec($R$) *is order-isomorphic to* Spec($\mathbb{Z}[X]$).

(The somewhat simpler case where $D = \mathbb{Z}$ is worked out in [22].)

Suppose k is a field that is not algebraic over a finite field. By Theorem 2.10, Spec($k[X, Y]$) is not isomorphic to Spec($\mathbb{Z}[X]$). Still, one can ask whether or not all two-dimensional affine domains over k have order-isomorphic spectra. The answer, in general, is "No":

Example 2.14. [37, Corollary 7] Let k be an algebraically closed field with infinite transcendence degree over $\mathbb{Q}$, and let V be the surface in $\mathbb{A}^3(k)$ defined by the equation $X^4 + Y^4 + Z^4 + 1 = 0$. Then not every point of V is the set-theoretic intersection of two curves on V. Therefore, in the two-dimensional affine domain $R = k[X, Y, Z]/(X^4 + Y^4 + Z^4 + 1)$, there is a maximal ideal $\mathbf{m}$ such that, for each pair P, Q of height-one prime ideals, $\{\mathbf{m}\} \neq P^{\uparrow} \cap Q^{\uparrow}$. On the other hand, in $k[X, Y]$ every maximal ideal is of the form $(X - a, Y - b)$, and $\{(X - a, Y - b)\} = (X - a)^{\uparrow} \cap (Y - b)^{\uparrow}$. Thus Spec($R$) and Spec($k[X, Y]$ are not order-isomorphic.

We know very little about the order-isomorphism classes of two-dimensional affine domains over k if k is not algebraic over a finite field. The following questions indicate the depths of our ignorance:

Questions 2.15. (1) Let k be an algebraic extension of $\mathbb{Q}$, and let R be a two-dimensional affine domain over k. Is Spec(R) order-isomorphic to Spec($k[X, Y]$)? (2) At the other extreme, if R and S are two-dimensional affine domains over k and Spec(R) and Spec(S) are order-isomorphic, are R and S necessarily isomorphic as k-algebras? (3) Let ℓ be another algebraic extension of $\mathbb{Q}$. If Spec($k[X, Y]$) and Spec($\ell[X, Y]$) are order-isomorphic, must k and ℓ be isomorphic fields?

3. Polynomial rings over semilocal one-dimensional domains

Naively one might suppose, since $\text{Spec}(\mathbb{Q}[X])$ is order-isomorphic to $\text{Spec}(\mathbb{Z})$, that also $\text{Spec}(\mathbb{Q}[X,Y])$ is order-isomorphic to $\text{Spec}(\mathbb{Z}[Y])$. The surprising negation of that conclusion, as discussed in the previous section, as well as the mystery surrounding $\text{Spec}(\mathbb{Q}[X,Y])$, led W. Heinzer and S. Wiegand to investigate spectra for "simpler" two-dimensional polynomial rings. What if you started with a one-dimensional ring with spectrum even simpler than $\text{Spec}(\mathbb{Z})$? Would the spectrum of the ring of polynomials be easier to fathom?

In particular Heinzer and S. Wiegand considered the question: What partially ordered sets arise as $\text{Spec}(R[X])$ for R a one-dimensional semilocal Noetherian domain? Just as $\mathbb{Z}[X]$ played a special role in Section 2, the rings $\mathbb{Z}_{(p_1)\cup\cdots\cup(p_n)}[X]$ play a special role in the current section. (Here $p_1,\ldots,p_n$ are distinct prime integers, and $\mathbb{Z}_{(p_1)\cup\cdots\cup(p_n)}$ consists of rational numbers whose denominators are prime to each p_i.) Their investigation led to the following theorem:

Theorem 3.1. [9] *Let R be a countable Noetherian one-dimensional domain with exactly n maximal ideals. Then there exist exactly two possibilities for* $\text{Spec}(R[X])$*:*

1. *If R is not Henselian, then* $\text{Spec}(R[X]) \cong \text{Spec}(\mathbb{Z}_{(p_1)\cup\cdots\cup(p_n)}[X])$*, for distinct prime integers $p_1,\ldots,p_n$.*
2. *If R is Henselian, then $n = 1$ and* $\text{Spec}(R[X]) \cong \text{Spec}(H[X])$*, where H is the Henselization of $\mathbb{Z}_{(2)}$.*

Examples of each are shown in Figures 3.2.1 and 3.3.1 below.

Example 3.2. The spectrum of $\mathbb{Z}_{(2)}[X]$ (where $\mathbb{Z}_{(2)}$ consists of rationals with odd denominators) is crudely drawn in Figure 3.2.1 below.

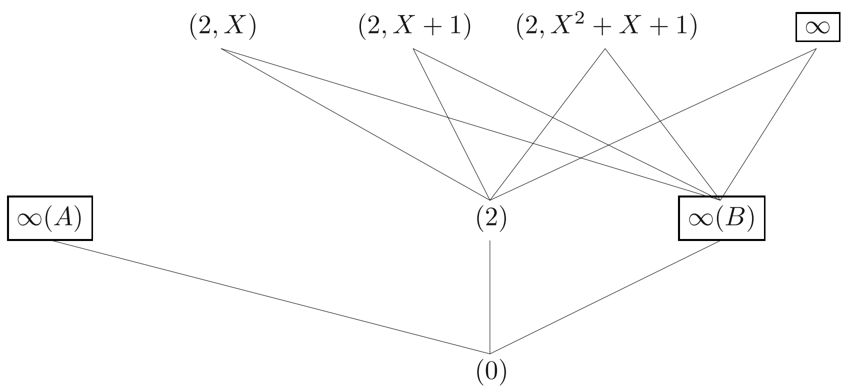


Figure 3.2.1. $\text{Spec}(\mathbb{Z}_{(2)}[X])$

Diagram Notes: The "infinity box" symbol $\boxed{\infty}$ indicates that infinitely many points are in that spot. The relations between the prime ideals in $\boxed{\infty(B)}$ and the top primes are too complicated to draw accurately. Each of the primes in $\boxed{\infty(B)}$ is contained in just finitely many maximal ideals. For example, the two irreducible polynomials X and X^2+X+2 each generate height-one prime ideals in $\boxed{\infty(B)}$; (X) is contained in $(2, X)$ only, but (X^2+X+2) is contained in both $(2, X)$ and $(2, X+1)$. The special height-one prime ideal (2) is in *all* of the height-two maximal primes. The box $\boxed{\infty(A)}$ represents the infinitely many height-one maximal ideals. Each height-two prime contains infinitely many height-one primes.

When $n = 1$, what distinguishes $\operatorname{Spec}(\mathbb{Z}_{(2)}[X])$ from $\operatorname{Spec}(H[X])$ is the following: In $\operatorname{Spec}(\mathbb{Z}_{(2)}[X])$, infinitely many height-one primes are contained in more than one maximal ideal. However, in $\operatorname{Spec}(H[X])$, $\mathbf{m}[X]$ is the *only* height-one prime contained in more than one maximal ideal (and it is contained in infinitely many).

Example 3.3. Although similar to the first picture, the illustration in Figure 3.3.1. of $\operatorname{Spec}(H[X])$, for H a countable Noetherian Henselian discrete rank-one valuation domain with maximal ideal $\mathbf{m}$, is cleaner:

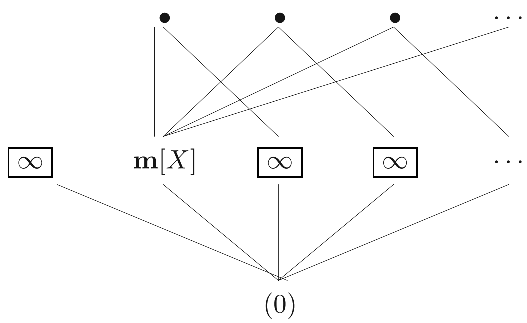


Figure 3.3.1. $\operatorname{Spec}(H[X])$

Remarks 3.4. (1) Loosely speaking, *Henselian* rings are rings for which the conclusion of Hensel's Lemma holds. Complete local rings, e.g., power series rings over a field, are Henselian. (These will come up again in the next section.) Forming the *Henselization* of a local ring is less drastic than going to the completion. For example, the Henselization of a countable local ring is countable, whereas complete local rings of positive dimension are always uncountable. The precise definition of "Henselian" and the construction of the Henselization are given in [30, Section 33].

(2) In [35] C. Shah gave complete sets of invariants for $\operatorname{Spec}(R[X])$ for an arbitrary Noetherian semilocal domain R. If R is Henselian, with maximal ideal P, the two invariants are $r := |R|$ and $k := |(R/P)[X]|$. If R is not Henselian, with maximal ideals $P_1, \dots, P_n$, the invariants are $r := |R|$, n (the number of maximal ideals of R), and $k_1, \dots, k_n$, where $k_i := |(R/P_i)[X]|$. Shah gave examples to show

that some combinations of invariants actually occur. There are some restrictions: Clearly $k_i \leq r$ for each r. Also, $r \leq k_i^{\aleph_0}$ for each i, by Lemma 4.2 below.

As pointed out by Kearnes and Oman [23], Shah assumed, incorrectly, that $a^{\aleph_0} = a$ for each cardinal $a \geq 2^{\aleph_0}$. In the proof of Theorem 3.1 of [8], Heinzer, Rotthaus and S. Wiegand refer to [35]; the arguments in Section 4 of this paper show that the statement of [8, Theorem 3.1] is correct.

(3) There are axioms similar to Axioms 2.3 that characterize the posets $\mathrm{Spec}(H[X])$ and $\mathrm{Spec}(\mathbb{Z}_{(p_1)\cup\cdots\cup(p_n)}[X])$ of Theorem 3.1 up to order-isomorphism. Of course (P3) is missing, and axioms analogous to (P5) distinguish the two cases (see Remark 3.5).

Remark 3.5. To state the distinguishing property between the two possibilities in Theorem 3.1 precisely, we use the "exactly less than" notation introduced in (0.1): In $\mathrm{Spec}(R[X])$, when R as above is *not* Henselian, we have:

(P5′) $L_e(T)$ is infinite for *every* finite set T of height-two maximal ideals.

If H is Henselian, however, we have:

($\mathrm{P5}^H$) Let T be a set of height-two maximal ideals. If $|T| \geq 2$, then $L_e(T) = \emptyset$; if $|T| = 1$, then $L_e(T)$ is infinite.

Remarks 3.6 (Other Related Spectra).

- Let R be a one-dimensional local Noetherian ring and let g, f be elements of $R[X]$ that either generate the unit ideal or form a regular sequence. W. Heinzer, D. Lantz and S. Wiegand characterized j-$\mathrm{Spec}(R[X][\frac{g}{f}])$, and showed its relationship to the polynomials f and g. In some cases (for example, when R is a discrete valuation ring such as $\mathbb{Z}_{(2)}$, or when R is Henselian), they were able to characterize $\mathrm{Spec}(R[X][\frac{g}{f}])$. Knowing j-$\mathrm{Spec}(R[X][\frac{g}{f}])$ is not sufficient, however, to characterize $\mathrm{Spec}(R[X][\frac{g}{f}])$ in general [5], [6], [7].
- In [38], R. Wiegand and W. Krauter found axioms that characterize the projective plane $\mathbb{P}^2(k)$ over the algebraic closure k of a finite field. The axioms are the same, regardless of the characteristic. A surprising consequence of the characterization is that a non-empty proper open subset U of $\mathbb{P}^2(k)$ is homeomorphic either to $\mathbb{P}^2(k) - \{\text{point}\}$ (the complement of a single point) or to $\mathbb{A}^2(k)$.

  The projective line over $\mathbb{Z}$ has been studied too, in [1], [5], [21]. The poset structure is considerably more complex than that of the projective plane over the algebraic closure of a finite field. It is currently being investigated by S. Wiegand and her (current and former) students E. Celikbas and C. Eubanks-Turner.

4. Two-dimensional power series rings

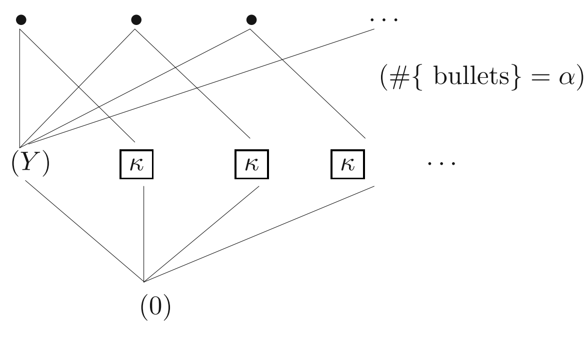
As part of an extensive project using power series rings to construct examples of rings with various properties, W. Heinzer, C. Rotthaus and S. Wiegand described the prime spectra of rings of the form $R[[Y]]$, for R a one-dimensional Noetherian domain and Y an indeterminate [8]. They completely characterized Spec$(R[[Y]])$ in the case where R is a countable domain. They did not, however, work through the cardinality arguments needed for the uncountable case. We review their results here and incidentally fill in the cardinality gap to obtain a characterization for the uncountable case as well.

First observe that, given variables X and Y, one can form "mixed polynomial/power series rings" over a field k in two ways – the second is of infinite transcendence degree over the first:

$$(1)\ k[[Y]][X] \quad \text{and} \quad (2)\ k[X][[Y]].$$

Since $k[[Y]]$ is a Henselian ring, Spec$(k[[Y]][X])$ is characterized in Theorem 3.1(2) of the previous section (actually this is Shah's extension from Remark 3.2(2), cf. [35]).

The interesting fact is that Spec$(k[X][[Y]])$ is pretty similar to Spec$(k[[Y]][X])$ – it just lacks the height-one maximals. Then Spec$(k[X][[Y]])$ in turn is an example of Spec$(R[[Y]])$, for R a general one-dimensional Noetherian domain. As we show in Theorem 4.3, the only variations in the partially ordered sets that occur as Spec$(R[[Y]])$ for different one-dimensional Noetherian domains R of a given cardinality are the numbers of height-two maximal ideals of $R[[Y]]$. This number is the same as the number of maximal ideals of R (of $k[X]$ for that example), because each maximal ideal of $R[[Y]]$ has the form $(\mathbf{m}, Y)R[[Y]]$ where $\mathbf{m}$ is a maximal ideal of R, by [30, Theorem 15.1]. In particular, Spec $R[[Y]]$ has the following picture, by Theorem 4.3 below:



Spec$(R[[Y]])$

In the diagram α is the cardinality of the set of maximal ideals of R. The boxed cardinals κ (one for each maximal ideal of R) indicate that there are κ prime ideals in these positions; that is, $|L_e((\mathbf{m}, Y))| = \kappa$. The upshot of the cardinality addition to the result of [8], which is now included in Theorem 4.3, is that $\kappa = |R[[Y]]|$ and that $|L_e((\mathbf{m}, Y))| = |L_e((\mathbf{m}', Y))|$, for every pair $\mathbf{m}, \mathbf{m}'$ of maximal ideals of R.

We first use a remark from [8].

Remark 4.1 ([8]). Suppose that T is a commutative ring of cardinality δ, that $\mathbf{m}$ is a maximal ideal of T and that γ is the cardinality of $T/\mathbf{m}$. Then

(1) The cardinality of $T[[Y]]$ is $\delta^{\aleph_0}$, because the elements of $T[[Y]]$ are in one-to-one correspondence with $\aleph_0$-tuples having entries in T. If T is Noetherian, then $T[[Y]]$ is Noetherian, and so every prime ideal of $T[[Y]]$ is finitely generated. Since the cardinality of the set of finite subsets of $T[[Y]]$ is $\delta^{\aleph_0}$, it follows that $T[[Y]]$ has in all *at most* $\delta^{\aleph_0}$ prime ideals if T is Noetherian.

(2) If T is Noetherian, there are *at least* $\gamma^{\aleph_0}$ distinct height-one prime ideals of $T[[Y]]$ contained in $(\mathbf{m}, Y)T[[Y]]$. To see this, following the argument of [8], choose a subset $C = \{c_i \mid i \in I\}$ of T so that $\{c_i + \mathbf{m} \mid i \in I\}$ is a complete set of distinct coset representatives for $T/\mathbf{m}$. Then $|C| = \gamma$, and, for $c_i, c_j \in C$ with $c_i \neq c_j$, we have $c_i - c_j \notin \mathbf{m}$. Choose $a \in \mathbf{m}, a \neq 0$. Consider the set

$$G := \{a + \sum_{n\in\mathbb{N}} d_n Y^n \mid d_n \in C \text{ for each } n \in \mathbb{N}\}.$$

Each of the elements of G is in $(\mathbf{m}, Y)T[[Y]] \setminus YT[[Y]]$ and hence each element of G is contained in a height-one prime belonging to $L_e((\mathbf{m}, Y))$. Moreover, $|G| = |C^{\aleph_0}| = \gamma^{\aleph_0}$.

Let $P \in L_e((\mathbf{m}, Y))$. Suppose that two distinct elements of G are both in P, say $f = a + \sum_{n\in\mathbb{N}} d_n Y^n$ and $g = a + \sum_{n\in\mathbb{N}} e_n Y^n$ are in P, where each $d_n, e_n \in C$. Then we have

$$f - g = \sum_{n\in\mathbb{N}} d_n Y^n - \sum_{n\in\mathbb{N}} e_n Y^n = \sum_{n\in\mathbb{N}} (d_n - e_n) Y^n \in P.$$

Let t be the smallest power of Y so that $d_t \neq e_t$. Then $(f-g)/Y^t \in P$, since P is prime and $Y \notin P$. However, the constant term $d_t - e_t$ is not in $\mathbf{m}$, contradicting the fact that $P \subseteq (\mathbf{m}, Y)T[[Y]]$. Thus there must be at least $|C|^{\aleph_0} = \gamma^{\aleph_0}$ distinct height-one primes contained in $L_e((\mathbf{m}, Y)T[[Y]])$, that is,

$$|L_e((\mathbf{m}, Y)T[[Y]])| \geq \gamma^{\aleph_0}.$$

(3) Putting parts (1) and (2) together, we see that, for each maximal ideal $\mathbf{m}$ of T, $\gamma^{\aleph_0} \leq |L_e((\mathbf{m}, Y)T[[Y]])| \leq \delta^{\aleph_0}$, if T is Noetherian.

Lemma 4.2. *Let R be a Noetherian domain, Y an indeterminate and I a proper ideal of R. Let $\delta = |R|$ and $\gamma = |R/I|$. Then $\delta \leq \gamma^{\aleph_0}$, and $|R[[Y]]| = \delta^{\aleph_0} = \gamma^{\aleph_0}$.*

Proof. The first equality holds by Remark 4.1, and of course $\delta^{\aleph_0} \geq \gamma^{\aleph_0}$. For the reverse inequality, we note that the Krull Intersection Theorem [24, Theorem 8.10 (ii)] implies that $\bigcap_{n\geq 1} I^n = 0$. Therefore there is a monomorphism

$$R \hookrightarrow \prod_{n\geq 1} R/I^n. \tag{4.2.1}$$

Now R/I^n has a finite filtration with factors I^{r-1}/I^r for each r with $1 \leq r \leq n$. Since I^{r-1}/I^r is a finitely generated (R/I)-module, $|I^{r-1}/I^r| \leq \gamma^{\aleph_0}$. Therefore

$|R/I^n| \leq (\gamma^{\aleph_0})^n = \gamma^{\aleph_0}$, for each n. Thus (4.2.1) implies $\delta \leq (\gamma^{\aleph_0})^{\aleph_0} = \gamma^{(\aleph_0^2)} = \gamma^{\aleph_0}$. Finally, $\delta^{\aleph_0} \leq (\gamma^{\aleph_0})^{\aleph_0} = \gamma^{\aleph_0}$, and so $\delta^{\aleph_0} = \gamma^{\aleph_0}$. □

Theorem 4.3. *Suppose that R is a one-dimensional Noetherian domain with cardinality $\delta := |R|$, and that the cardinality of the set of maximal ideals of R is α (α can be finite). Let $U = \operatorname{Spec} R[[Y]]$, where Y is an indeterminate over R. Then the poset U is characterized by the following axioms:*

(1) $|U| = \delta^{\aleph_0}$.
(2) *U has a unique minimal element, namely* (0).
(3) $\dim(U) = 2$ *and* $|\{$ *height-two elements of* $U \}| = \alpha$.
(4) *There exists a unique height-one element $u_Y \in U$ (namely $u_Y = (Y)$) such that u is contained in every height-two element of U.*
(5) *Every height-one element of U except for u_Y is in exactly one height-two element.*
(6) *For every height-two element $t \in U$, $|L_e(t)| = |R[[Y]]| = \delta^{\aleph_0}$. If $t_1, t_2 \in U$ are distinct height-two elements, then the element u_Y from (4) is the unique height-one element less than both.*
(7) *There are no height-one maximal elements in U. Every maximal element has height two.* (*This property, implicit in* (5), *is stated for emphasis.*)

Proof. Most of the proof is done in [8]. It remains to check the statement $|L_e(t)| = |R[[Y]]| = \delta^{\aleph_0}$ in item (6). This is immediate from Remark 4.1 and Lemma 4.2. □

Remarks 4.4. C. Eubanks-Turner, M. Luckas, and S. Saydam (former Ph.D. students of S. Wiegand) have characterized $\operatorname{Spec}(R[[X]][g/f])$, for R a one-dimensional Noetherian domain with infinitely many maximal ideals and g, f a generalized $R[[X]]$-sequence, in their recent work [4]. They specify various possibilities for the j-spectrum that depend upon f and g.

For example the diagram in Figure 4.4.1 shows the partially ordered set j-$\operatorname{Spec}(B)$, for $B := \mathbb{Z}[[X]][\frac{g}{f}]$, $f = 11880 + \sum_{i=1}^{\infty} X^i$ and $g = 9900 + \sum_{i=1}^{\infty} X^i$:

Notes 4.5. [8, Remarks 3.6, Corollary 3.7] It is evident from Theorem 4.3 that

$$\operatorname{Spec}(\mathbb{Z}[[Y]]) \cong \operatorname{Spec}(\mathbb{Q}[X][[Y]]) \cong \operatorname{Spec}((\mathbb{Z}/2\mathbb{Z})[X][[Y]]) \not\cong \operatorname{Spec}(\mathbb{R}[X][[Y]]).$$

(The last has uncountably many maximal ideals.) As Theorem 2.10 indicates,

$$\operatorname{Spec}(\mathbb{Z}[Y]) \cong \operatorname{Spec}((\mathbb{Z}/2\mathbb{Z})[X][Y]) \not\cong \operatorname{Spec}(\mathbb{Q}[X][Y]).$$

Thus the situation for power series rings is different from the polynomial case.

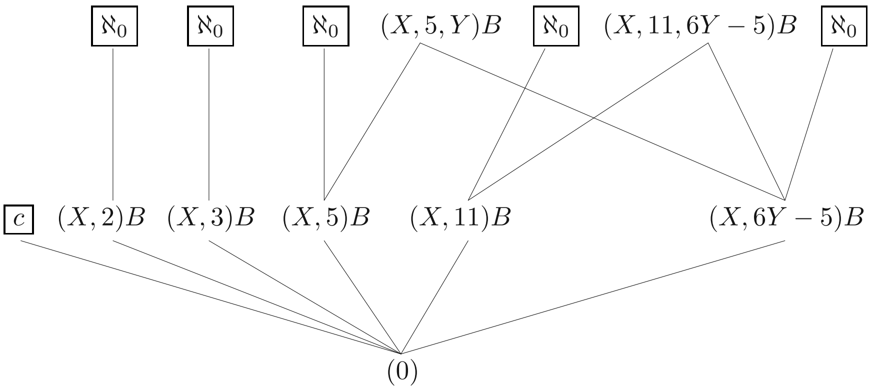


Figure 4.4.1: Spec($\mathbb{Z}[[x]][\frac{9900+\sum_{i=1}^{\infty} X^i}{\sum_{i=1}^{\infty} 11880+X^i}]$)

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Ring and Module Theory
Trends in Mathematics, 195–200

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