A Semi-Markovian Analysis of an Inventory Model with Inventory-Level Dependent Arrival and Service Processes

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Abstract. Two Semi Markov processes are defined to describe the service and inter-arrival times of an $s - S$ inventory model with zero lead time, in which both inter-arrival time and service time depend upon the inventory level. It is assumed that both the service time and the inter-arrival time follow Phase-Type distributions, which are determined by the current inventory level. The marginal distributions of both the service time and the inter-arrival time are obtained. A continuous parameter Markov chain is used to model the queue size. Condition for stability and the steady state characteristics of the system are derived. The impact of interdependence between service and arrival processes, along with inventory level on the system is examined. Furthermore, a numerical analysis is also done to explain the consequences of this dependency on steady-state system characteristics.

Keywords: Interdependent processes · Semi-Markov Process · Matrix analytic method · $s - S$ inventory model

1 Introduction

Inventory models had received significant attention in academic research. While numerous classical models assume negligible or no time for the inventory to be served, real-world scenarios often demand considerations of the time to serve the inventory. Pioneering the exploration of Inventory models with positive service time were Berman et al. [12] and Sigman et al. [14]. Comprehensive insight into studies by various authors in this direction can be found in the survey articles by Krishnamoorthy et al. [8,9].

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Unexpectedly, there have been relatively few prior studies exploring models where the service time and/or inter-arrival time are contingent upon the inventory level. This stands in contrast to the significant emphasis placed on queueing models with state-dependent arrival and service processes in the existing literature. S. K. Gupta [4] analyzed such a model with a finite queue size. In his investigation, both arrival and service rates were treated as arbitrary functions of the number of customers in the system. Other pioneering works in a similar direction were done by Hiller et al. [6] as well as Conway et al. [3]. Later, Bekker et al. [2] provided detailed descriptions of both M/G/1 and G/G/1 models incorporating workload-dependent arrival and service rates.

However, there are classical models (with zero service time) that have analyzed inventory models with stock dependent demands. These models are derived from the observation that maintaining abundant inventory has a favourable impact on demand.

Customers are often enticed to make purchases by prominently displaying a considerable amount of inventory in stores. Larson et al. [10] coined the term "psychic stock" for this displayed inventory. Hadley et al. [5], Wolfe [16], T L Urban [15] and Johnson [7] are among the researchers who have investigated the stimulating impact of inventory level on demand. The influence of inventory level on demand is evident in cases involving distinct inventory items, such as ornaments or clothing materials. A diverse assortment of these items offers customers a wider choice, consequently elevating demand. Another instance is when there's an abundance of inventory or when dealing with perishable items. In these situations, sellers may introduce special offers to entice buyers and clear out excess/old stock, resulting in a notable increase in demand.

Queueing models with interdependent arrival and service processes are introduced by Ranjith et al. [13]. Much before that Guy Latouche [11] derived interdependent phase type processes by a semi Markovian point process. He constructed this by considering a finite state irreducible Markov chain. In this paper we follows a similar approach. But with the difference that our focus is not on the dependence between these processes, but on the interdependence between the state of the embedded Markov chain and the phase type distribution constructed.

Using this method, this paper analyses an $s - S$ inventory model with no lead time and positive service time. In this model both the service and arrival processes depend on the inventory level. The structure of the paper is in the following manner. Section 2 presents the description of the underlying Markov chains for the service and arrival processes. In Sect. 3, the marginal distribution of the service time and inter arrival time are found. In Sect. 4, a continuous-time Markov chain is employed to model a queueing-inventory model with inventory dependent arrival and service processes. It also covers the condition for stability of the system. Section 5 focuses on the steady state analysis and evaluation of the key system performance measures. Furthermore, we conduct a comprehensive numerical investigation of the system in Sect. 6. Finally, in Sect. 7 we conclude the discussion.

2 Semi-Markovian Service and Arrival Processes Depending on Inventory Level

To construct Semi-Markovian point processes suitable for modeling arrival and service processes of a queueing-inventory model in which these two processes depend on the level of inventory, we proceed as follows:-

Consider an irreducible Markov chain $\mathcal{X} = \{X_i | i = 0, 1, 2, 3, ...\}$ with state space $\{1, 2, 3, ..., r\}$. Let $P^X = \left[p_{ij}^X\right]$ where

$$p_{ij}^X = \begin{cases} 1 \text{ if } j = i - 1, 2 \leq i \leq r \\ 1 \text{ if } i = 1, j = r \\ 0 \text{ otherwise.} \end{cases}$$

be the transition probability matrix of the chain $\mathcal{X}$.

Assume that the transitions of the chain $\mathcal{X}$ occur at random epochs $\gamma_i, i = 1, 2, 3, \ldots$. Let τ_i be the interval of time between the successive transitions.

$$\tau_i = \begin{cases} \gamma_i - \gamma_{i-1}, \text{ if } i = 2, 3, ... \\ \gamma_1 \text{ if } i = 1 \end{cases}$$

For each i, if $X_{i-1} = j$, assume that τ_i follows a Phase type distribution $F_j(.)$ with representation (α_j, D_j), where D_j is an $n_j \times n_j$ matrix. Thus we have a semi-Markov Process $\{Z_X(t) | t \geq 0\}$ defined by

$$Z_X(t) = X_i, \;\; \gamma_i \leq t < \gamma_{i+1}, \;\; i = 0, 1, 2, ...$$

In the present study, states of the chain $\mathcal{X}$ are the inventory levels and each state transition corresponds to a service completion. $\gamma_i, i = 1, 2, 3, ...$ are the epochs of completion of i^{th} service, $Z_X(t)$ represents the inventory level at time t and the distribution of duration of the service happening at time t is $F_{Z_X(t)}(.)$.

For the arrival process, we proceed as follows:- Consider the phase type distributions $G_i(.)$ with representations $(\beta_i, T_i), i = 1, 2, .., r$ where T_i is a square matrix of order m_i. Define a Markov chain $\mathcal{Y} = \{Y_i | i = 0, 1, 2, 3, ...\}$ with state space {0,1,2,...}, $Y_0 = 0$ and having the transition probability matrix $P^Y = \left[p_{ij}^Y\right]$ where

$$p_{ij}^Y = \begin{cases} 1 \text{ if } j = i + 1, i \geq 0 \\ 0 \text{ otherwise.} \end{cases}$$

Starting from time $t = 0$, let ν_i be the epoch at which the chain $\mathcal{Y}$ makes the i^{th} transition, $i = 1, 2, 3, ...$ Let φ_i be the inter occurrence time $\nu_i - \nu_{i-1}$ between the $i-1^{\text{th}}$ and i^{th} transitions of the chain $\mathcal{Y}$. Assume that φ_i follows the distribution $G_j(.)$ where $j = Z_X(\nu_{i-1})$. Hence we have a semi Markov Process

$$Z_Y(t) = Y_i, \nu_i \leq t < \nu_{i+1}, i = 0, 1, 2, ...$$

We may take the states of the chain $\mathcal{Y}$ to be the number of arrivals. The distribution of the inter arrival times are then determined by the inventory level at the epochs of the preceding arrivals.

3 Marginal Distributions of Service Times and Inter-arrival Times

Consider the continuous parameter Markov chain

$$\mathcal{N}_1 = \{(N_1(t), I_X(t), J_\tau(t))|t \geq 0\}$$

where $N_1(t)$ is the number of transitions occurred during the time interval $(0, t]$ of the chain $\mathcal{X}$, $I_X(t)$ is the state of the chain $\mathcal{X}$ and $J_\tau(t)$ is the phase of the distribution of the ongoing service process at time t. The state space of this process is

$$\bigcup_{i=1}^{r}\{(n, i, j)|n = 0, 1, 2, ..., j = 1, 2, 3, ..., n_i\}$$

Since in the steady state all the states of the chain $\mathcal{X}$ are equally likely, the initial probability distribution of the chain $\mathcal{N}_1$ is given by $\frac{1}{r}\tilde{\alpha}$, where $\tilde{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_r)$ The infinitesimal generator of the chain $\mathcal{N}_1$ is

$$Q_s = \begin{bmatrix} U & U^0 & 0 & 0 & ... \\ 0 & U & U^0 & 0 & ... \\ 0 & 0 & U & U^0 & ... \\ . & . & . & . & ... \\ . & . & . & . & ... \end{bmatrix}$$

where

$$U = diag(D_1, D_2, ..., D_r)$$

and

$$U^0 = \begin{bmatrix} 0 & 0 & 0 \,...\, 0 & 0 & D_1^0\alpha_1 \\ D_2^0\alpha_2 & 0 & 0 \,...\, 0 & 0 & 0 \\ 0 & D_3^0\alpha_3 & 0 \,...\, 0 & 0 & 0 \\ . & . & . \,...\, . & . & . \\ . & . & . \,...\, . & . & . \\ 0 & 0 & 0 \,...\, 0 & D_r^0\alpha_r & 0 \end{bmatrix}$$

Here $D_i^0 = -D_i e, i = 1, 2, ..., r$.

The service time τ_i of the i^{th}customer is the time taken by the chain $\mathcal{N}_1$ for the transition from level i to level $i+1$. From the infinitesimal generator of $\mathcal{N}_1$ it follows that τ_i's are identically distributed and that their common marginal distribution $F(t)$ is of phase type with representation $\left(\frac{1}{r}\tilde{\alpha}, U\right)$.

Therefore,

$$\begin{aligned} F(t) &= 1 - \frac{1}{r}\tilde{\alpha}exp(Ut)e \\ &= 1 - \frac{1}{r}\sum_{i=1}^{r}\alpha_i exp(D_i t)e \end{aligned}$$

where e is a column vector of 1's of appropriate order and the density function is

$$f(t) = \frac{1}{r}\sum_{i=1}^{r}\alpha_i exp(D_i t)D_i^0 = \frac{1}{r}\sum_{i=1}^{r}F_i'(t)$$

Thus in the steady state, marginal density of the service time is the mixture of the densities $F_1'(.), F_2'(.), ..., F_r'(.)$.

To determine the marginal distribution of the inter arrival time, we consider the Markov chain

$$\mathcal{N}_2 = \{(N_2(t), I_X(\nu), I_X(t), J_\varphi(t))\,|t \geq 0\}$$

where $N_2(t)$ is the number of transitions of the chain $\mathcal{Y}$ occurred in the interval $(0, t]$, $I_X(\nu)$ and $I_X(t)$ are the states of the chain $\mathcal{X}$ at time ν and t respectively where $\nu = max\{\nu_j \in (o, t]\}$ and $J_\varphi(t)$ is the phase of the ongoing arrival process at time t. Note that $N_2(t)$ is the total number of arrivals in $(0, t]$ and $Z_X(\nu)$ is the inventory level at the epoch of previous arrival. This chain has the state space

$$\bigcup_{i=1}^{r}\{(n, i, k, j)|n = 0, 1, 2, ..., k = 1, 2, ..., r, j = 1, 2, 3, ..., m_i\}$$

and the infinitesimal generator

$$Q_V = \begin{bmatrix} V & V^0 & 0 & 0 & ... \\ 0 & V & V^0 & 0 & ... \\ 0 & 0 & V & V^0 & ... \\ . & . & . & . & ... \\ . & . & . & . & ... \end{bmatrix}$$

where

$$V = diag(V_1, V_2, ..., V_r)$$

and

$$V^0 = \begin{bmatrix} V_1^0 \\ V_2^0 \\ \vdots \\ V_r^0 \end{bmatrix}$$

in which

$$V_i = \begin{bmatrix} T_i - \mu_1 I_{m_1} & 0 & 0 & ... & 0 & \mu_1 I_{m_1} \\ \mu_2 I_{m_2} & T_i - \mu_2 I_{m_2} & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & \mu_r I_{m_r} & T_i - \mu_r I_{m_r} \end{bmatrix},$$

$\mu_i = -\alpha_i D_i^{-1}e$ and V_0^i is a block matrix of order $r \times r^2$ with $T_i^0\beta_i$ at positions $(j, (j-1)r + j)$ and zero matrices at every other positions.

Hence in the steady state, the distribution $G(.)$ of the inter arrival time is a Phase type distribution with representation $(\frac{1}{r}\tilde{\beta} \otimes \zeta, V)$ where $\tilde{\beta} = (\beta_1, \beta_2, \ldots, \beta_r)$ and $\zeta = \left(\frac{1}{r}, \frac{1}{r}, \ldots \frac{1}{r}\right)$ is the stationary probability vector of the chain $\mathcal{X}$. Therefore, we have

$$G(t) = 1 - \left(\frac{1}{r}\tilde{\beta} \otimes \zeta\right) exp(Vt)e$$
$$= 1 - \frac{1}{r}\sum_{i=1}^{r}(\beta_i \otimes \zeta) exp(V_i t)e$$

and the density function is

$$g(t) = \frac{1}{r}\sum_{i=1}^{r}(\beta_i \otimes \zeta) exp(V_i t)\left(T_i^0 \otimes e_r\right)$$

where e_r is a column vector of 1's of dimension $r \times 1$. Thus in the steady state, marginal distribution of the service time is the mixture of the phase type distributions with representations $((\beta_i \otimes \zeta), V_i), i = 1, 2, ..., r$.

A simplified version of the model we discussed so far may be obtained by assuming that D_i's and D_j's are linearly dependent and so are T_i's and T_j's. This is made by taking $D_i = \epsilon_i D$, $\alpha_i = \alpha$ and $T_i = \delta_i T$, $\beta_i = \beta$, where (α, D) and (β, D) represents two phase type distributions. This assumption gives us the freedom to switch to the process (β_i, T_i) from (β_{i+1}, T_{i+1}), even before the absorption of the latter, whenever there is a state transition occurs in the chain $\mathcal{X}$. Such a model is introduced in the next section.

4 A Queueing-Inventory Model with Inventory Dependent Arrival and Service Processes

Consider a single server inventory model. The inventory is instantaneously replenished according to $(s - S)$ policy. At any time t, the arrival of customers is according to the inventory level at that time. When the inventory level is $s + i$, the distribution of the inter-arrival time is phase-type with representation $(\beta, \delta_i T)$ where T is a square matrix of order n and δ_i is a real number, $i = 1, 2, ..., r = S - s$. The service time distributions too are determined by the inventory level. While the inventory level is $s + i$, the service time distribution is phase type with representation $(\alpha, \epsilon_i D)$. Here D is of order $m \times m$, ϵ_i are real numbers $i = 1, 2, .., r$ and α is the initial distribution.

Let $N(t)$ be the number of customers in the system, $I(t)$ be the inventory level, $J_1(t)$ and $J_2(t)$ be the states of arrival and service processes respectively at time t. Then the system under discussion can be modelled by the continuous time Markov chain

$$\mathcal{N} = \{(N(t), I(t), J_1(t), J_2(t)) / t \geq 0\}$$

with state space

$\{(0,i,j_1)/1 \leq i \leq r, 1 \leq j_1 \leq n\} \cup \{(k,i,j_1,j_2)/k = 1,2,3,...1 \leq i \leq r, 1 \leq j_1 \leq n, 1 \leq j_2 \leq m\}$

The infinitesimal generator of the chain $\mathcal{N}$ is given by

$$Q = \begin{bmatrix} A_{00} & A_{01} & 0 & 0 & 0 & \cdots \\ A_{10} & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & \cdots \end{bmatrix}$$

where

$$\begin{aligned} A_{00} &= \Delta \otimes T \\ A_{01} &= \Delta \otimes T^0\beta \otimes \alpha \\ A_{10} &= E^{\perp} \otimes I_n \otimes D^0 \\ A_0 &= \Delta \otimes T^0\beta \otimes I_m \\ A_1 &= \Delta \otimes T \otimes I_m + E \otimes I_n \otimes D \\ A_2 &= E^{\perp} \otimes I_n \otimes D^0\alpha \end{aligned}$$

Here

$$\begin{aligned} \Delta &= \text{diag}(\delta_1, \delta_2, ..., \delta_r) \\ E &= \text{diag}(\epsilon_1, \epsilon_2, ..., \epsilon_r) \\ E^{\perp} &= \begin{bmatrix} 0 & 0 & 0 \ldots & 0 & \epsilon_1 \\ \epsilon_2 & 0 & 0 \ldots & 0 & 0 \\ 0 & \epsilon_2 & 0 \ldots & 0 & 0 \\ . & . & . \ldots & . & . \\ . & . & . \ldots & . & . \\ 0 & 0 & 0 \ldots & \epsilon_r & 0 \end{bmatrix} \end{aligned}$$

Now let π and θ be the stationary probability vectors of $T + T^0\beta$ and $D + D^0\alpha$ respectively and

$$A = A_0 + A_1 + A_2 = \Delta \otimes (T + T^0\beta) \otimes I_m + E \otimes I_n \otimes D + E^{\perp} \otimes I_n \otimes D^0\alpha.$$

For any row vector φ of length r such that $\varphi E^{\perp} = \varphi E$,

$$\begin{aligned} (\varphi \otimes \pi \otimes \theta) A &= \varphi E \otimes \pi \otimes \theta D + \varphi E^{\perp} \otimes \pi \otimes \theta D^0\alpha \\ &= \varphi E \otimes \pi \otimes \theta \left(D + D^0\alpha\right) \\ &= \mathbf{0}. \end{aligned}$$

Thus we have the following lemma.

Lemma 1. *For any row vector φ of length r such that $\varphi E^{\perp} = \varphi E$, $(\varphi \otimes \pi \otimes \theta)$ is a null vector of A.*

Choose $\phi = \left(\frac{1}{\epsilon_1}, \frac{1}{\epsilon_2}, ..., \frac{1}{\epsilon_r}\right)$. For this ϕ, $\phi E^{\perp} = \phi E$. Hence by lemma 1, $(\phi \otimes \pi \otimes \theta)$ is a null vector of A. Therefore $\Pi = \frac{1}{\phi . e}(\phi \otimes \pi \otimes \theta)$ is the stationary probability vector of A.

Now

$$\Pi A_2 e = \frac{r}{\phi e}\theta D^0 = \frac{r}{\phi e}\mu$$

and

$$\Pi A_0 e = \frac{1}{\phi e}\left(\sum_{i=1}^{r}\frac{\delta_i}{\epsilon_i}\right)\pi T^0 = \frac{1}{\phi e}\left(\sum_{i=1}^{r}\frac{\delta_i}{\epsilon_i}\right)\lambda$$

Hence we have the following theorem.

Theorem 1. *The continuous parameter irreducible Markov chain $\mathcal{N}$ is positive recurrent if and only if*

$$\left(\sum_{i=1}^{S-s}\frac{\delta_i}{\epsilon_i}\right)\lambda < (S-s)\mu$$

Note that when $\delta_i = \epsilon_i \ \forall i$, the condition for stability reduces to $\lambda < \mu$. In particular if $\delta_i = \epsilon_i = 1 \ \forall i$, inventory level and the arrival and service processes are independent.

5 Stationary Distribution of the Markov Chain $\mathcal{N}$

The stationary probability vector $\mathbf{z} = (\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2 \ldots)$ is given by

$$\mathbf{z}_i = \mathbf{z}_1 R^{i-1}, i = 2, 3, 4... \tag{1}$$

$$\mathbf{z}_0\left(\Delta \otimes T\right) + \mathbf{z}_1\left(E^{\perp} \otimes I_n \otimes D^0\right) = 0 \tag{2}$$

$$\mathbf{z}_0\left(\Delta \otimes T^0\beta \otimes \alpha\right) + \mathbf{z}_1\left(\Delta \otimes T \otimes I_m + E \otimes I_n \otimes D\right) + \mathbf{z}_2\left(E^{\perp} \otimes I_n \otimes D^0\alpha\right) = 0 \tag{3}$$

where the matrix R is the minimal solution of the matrix quadratic equation

$$R^2\left(E^{\perp} \otimes I_n \otimes D^0\alpha\right) + R\left(\Delta \otimes T \otimes I_m + E \otimes I_n \otimes D\right) + \left(\Delta \otimes T^0\beta \otimes I_m\right) = 0$$

From Eq. (2),

$$\mathbf{z}_0\left(\Delta \otimes T^0\right) = \mathbf{z}_1\left(E^{\perp} \otimes e_n \otimes D^0\right)$$

So that

$$\begin{aligned}\mathbf{z}_0\left(\Delta \otimes T^0\beta \otimes \alpha\right) &= \mathbf{z}_0\left(\Delta \otimes T^0\right)\left(I_r \otimes \beta \otimes \alpha\right)\\ &= \mathbf{z}_1\left(E^{\perp} \otimes e_n \otimes D^0\right)\left(I_r \otimes \beta \otimes \alpha\right)\\ &= \mathbf{z}_1\left(E^{\perp} \otimes e_n\beta \otimes D^0\alpha\right) \end{aligned} \tag{4}$$

Using Eqs. (3) and (4), we get

$$\mathbf{z}_1\left[E^{\perp}\otimes e_n\beta\otimes D^0\alpha+\Delta\otimes T\otimes I_m+E\otimes I_n\otimes D+R\left(E^{\perp}\otimes I_n\otimes D^0\alpha\right)\right]=0 \tag{5}$$

Therefore the vector z_1 can be uniquely determined up to a multiplicative constant. This constant can be found by normalizing the total probability to one.

We partitioned each steady state vector z_i as $z_i = (z_{i1}, z_{i2}, \ldots, z_{ir})$ where $z_{0j} = (z_{0j1}, z_{0j2}, \ldots, z_{0jn})$, $z_{ij} = (z_{ij11}, z_{ij12}, \ldots, z_{ij1m}, \ldots, z_{ijn1}, z_{ijn2}\ldots, z_{ijnm})$, $j = 1, 2, \ldots, r, i = 1, 2, 3, \ldots$ in which z_{0jk} and $z_{ijkl}, k = 1, 2, \ldots, n, l = 1, 2, \ldots, m$ are scalars.

Some of the important system characteristics in the steady state are as follows.

1. Probability that the server is idle $= z_0 e_{rn}$.
2. For $k > 0$, Probability that there are k customers in the system,

$$P\left(N=k\right)=z_k e_{rnm}.$$

3. Expected number of customers in the system, $E\left(N\right)=\sum\limits_{i=1}^{n} i z_i e_{rnm}$.
4. Probability that the inventory level is j, $P\left(I=j\right)=z_{0j}e_n+\sum\limits_{i=1}^{\infty} z_{ij}e_{nm}$.
5. Expected inventory level, $E\left(I\right)=\sum\limits_{j=1}^{r} jP\left(I=j\right)$.

5.1 Expected Waiting Time

Consider a customer who joins the queue as the k^{th} customer. The waiting time W_k of this customer in the queue is the time until absorption of the Markov chain

$$W(t)=\{(r(t),I(t),J_s(t))/t\geq 0\}$$

where $r(t)$ is the position of the particular customer in the queue, $I(t)$ is the inventory level and $J_s(t)$ is the state of the ongoing service process at time t. The infinitesimal generator of this chain is

$$\tilde{Q}=\begin{bmatrix} Q_w & -Q_w e \\ 0 & 0 \end{bmatrix}$$

where

$$Q_w=\begin{bmatrix} E\otimes D & E^{\perp}\otimes D^0\alpha & 0 & 0 & 0 & \ldots & 0 \\ 0 & E\otimes D & E^{\perp}\otimes D^0\alpha & 0 & 0 & \ldots & 0 \\ 0 & 0 & E\otimes D & E^{\perp}\otimes D^0\alpha & 0 & \ldots & 0 \\ 0 & 0 & 0 & E\otimes D & E^{\perp}\otimes D^0\alpha & \ldots & 0 \\ . & . & . & . & . & & \ldots \\ 0 & 0 & . & . & . & & E\otimes D \end{bmatrix}$$

Hence W_k follows a Phase type distribution with representation (ψ_w, Q_w) where $\psi_w = (\psi_k, \mathbf{0}, \mathbf{0}, ..\mathbf{0})$ in which ψ_k is a vector of length rm. The ij^{th} entry of ψ_k is the conditional probability that the chain is in a state with inventory level i and service phase j given that it is in level k in the steady state.

Hence, when the system is in the steady state, the expected waiting time of this customer is

$$
\begin{aligned}
W_k &= -\psi_w Q_w^{-1} e \\
&= \psi_k \left[\sum_{i=0}^{k} (-1)^i \left[(E \otimes D)^{-1} (E^{\perp} \otimes D^0 \alpha) \right]^i \right] [E \otimes D]^{-1} e \\
&= \psi_k \left[I + \left(\sum_{i=1}^{k} (I^{\perp})^k \right) \otimes e\alpha \right] [E \otimes D]^{-1} e. \\
&\text{where } I^{\perp} = E^{-1} E^{\perp}.
\end{aligned}
$$

Hence the Expected waiting time of an arbitrary customer in the steady state is given by

$$E(W) = \sum_{k=1}^{\infty} P(N = k) W_k$$

6 A Sample Problem on Cost Optimization and Numerical Analysis of the Chain $\mathcal{N}$

In this section we present an example of a cost optimization problem that arises in queueing-inventory situations with interdependent arrival and service processes. In this example we assumed that the demand increases with the inventory level according to the relation $\rho i^{1-\kappa}$, where ρ and $0 < \kappa < 1$ are constants and i is the inventory level. Baker et al. [1] used such a functional to model a situation with inventory level dependent demand which is for a relatively short season. Also we take the service rate to be proportional to the inventory level with σ as the proportionality constant.

6.1 A Cost Optimization Problem

We assumed that the multipliers ϵ_i and $\delta_i, i = 1, 2, \ldots r$ are related to the inventory level i by the relation $\delta_i = \rho i^{1-\kappa}$ and $\epsilon_i = \sigma i$ where ρ and σ are two positive parameters.

This illustration encompasses four distinct cost categories. The initial type involves the cost associated with providing the service at inventory level i, represented as $c(i)$. The second, denoted as c_0, refers to the cost of maintaining the server in a state of readiness during idle periods. Additionally, there are holding costs, denoted as c_h, incurred to ensure customer comfort within the system, along with the cost c_s associated with preserving the integrity of the inventory. All these costs are calculated per unit time.

Taking all these costs into account, we construct the cost function, $Cost = c_h \times E(N) + c_s \times E(I) + \sum_1^n c(i)P(I=i) + c_0 \times P\,(N=0)$ where $E(N)$ is the expected number of customers in the system, $E(I)$ is the expected inventory level, $P(I=i)$ is the probability that the inventory level is i and $P\,(N=0)$ is the probability that the system is idle.

For illustration, we choose $\rho = 0.7$, $\kappa = 0.4$, $c_h = 2$, $c_s = 0.5$, $c(i) = E(i,i)$, $c_0 = 6$. $T = \begin{bmatrix} -12 & 4 & 6 \\ 3 & -10 & 5 \\ 4 & 3 & -9 \end{bmatrix}$, $D = \begin{bmatrix} -7 & 1 & 2 & 1 \\ 3 & -11 & 2 & 3 \\ 2 & 2 & -10 & 3 \\ 5 & 3 & 4 & -15 \end{bmatrix}$, $\beta = (0.4, 0.35, 0.25)$ and $\alpha = (0.2, 0.3, 0.4, 0.1)$.

Our objective is to determine the rate at which the service rate should increase with the inventory level in order to minimize the incurred cost. That is we would like to find the value of the proportionality constant σ that optimizes the cost function. Figure 1 depicts the cost function plotted against σ. The convex nature of the curve indicates the presence of a minimum cost. Specifically, the cost reaches its minimum value when σ equals 0.54, with the minimal cost being 15.5280. Consequently, by exerting control over the service process, we ensure system stability even in the presence of heightened arrival rates, and we achieve this at the lowest possible cost.

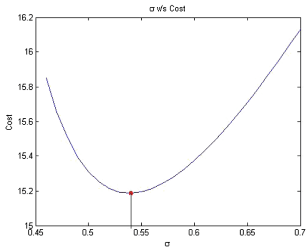

Fig. 1. Variation of cost wrt κ

Our numerical analysis demonstrates that it is possible to adjust the service rate based on the inventory level, thereby enabling control over the system's characteristics.

6.2 Numerical Analysis of the Chain $\mathcal{N}$

For the specified parameter values, we computed the expected number of customers $E(N)$, system idle probability $P(N=0)$, expected inventory level $E(I)$ expected waiting time $E(W)$ for various values of σ and the results are displayed in Figs. 2, 3 4 and 5. The calculated values are given in Table 2 in Appendix.

As σ increases, the service rates corresponding to each inventory level also rise. The escalation in service rate provides support for the growth in the arrival rate, stemming from increased inventory levels. This leads to a decrease in the expected number of customers in the system. With higher σ values, the traffic intensity decreases. Consequently, the system idle probability increases. For a small value of σ, service is delivered at a reduced rate when the inventory level is low. Consequently, it takes more time to complete a service and subsequently replenish the low inventory. As a result, the system experiences prolonged periods with low inventory levels. In contrast, higher values of σ increase the probability that the system is in a state with a higher inventory level.

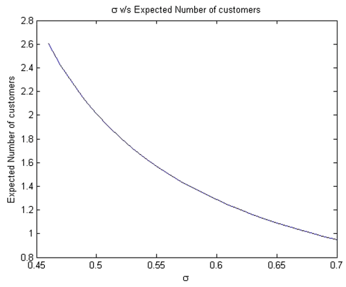

Fig. 2. Expected number of customers

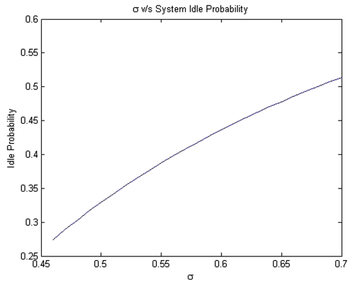

Fig. 3. Idle probability

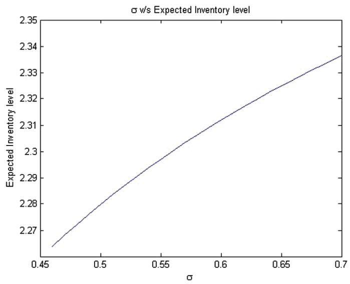

Fig. 4. Inventory level

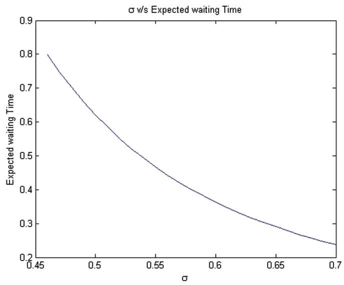

Fig. 5. Expected Waiting time

6.3 A Comparison Between Proposed Model and One with Stock Dependent Arrival Process and Independent Service Process

The model under discussion (Model 1) is compared with a similar one where the service rate remains unaffected by the inventory level (Model 2) giving the following values to the parameters. $\Delta = diag(0.7000, 1.0610, 1.3532, 1.6082, 1.8386)$, and

$$D = \begin{bmatrix} -12 & 1 & 2 & 2 \\ 3 & -16 & 2 & 4 \\ 2 & 3 & -15 & 3 \\ 6 & 3 & 4 & -20 \end{bmatrix}, T = \begin{bmatrix} -10-d & 4 & 2 \\ 3 & -8-d & 1 \\ 1 & 3 & -8-d \end{bmatrix}$$ where d varies from 0 to 2 in increments of 0.1, $r = 5, n = 3, m = 4, \alpha = (.2, .3, .4, .1), \beta = (0.4, 0.35, 0.25), E = diag(0.4567, 0.9133, 1.3700, 1.8267, 2.2833)$

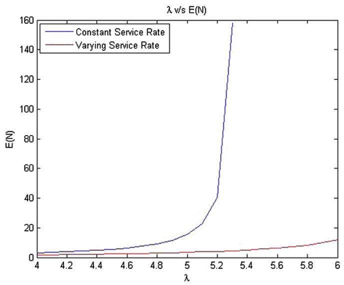

Fig. 6. Copmarison of two models

In Model 2, the service rate remains the same, while the arrival rate escalates with the inventory level. A comparison of the expected number of customer in both systems is presented in Fig. 6. The numerical results are tabulated in Table 1 in the Appendix. Notably, Model 2 experiences a higher inflow of customers compared to Model 1. In both systems, the arrival rate increases with the inventory level. When the inventory level is high, the arrival rate is also elevated. In the case, where the service rate is constant, the inventory level has nearly a uniform change, resulting in approximately equal probabilities for all inventory levels. This, in turn, leads to a high average arrival rate, causing the system to quickly burst out.

Conversely, in Model 1, the service rate diminishes as the inventory level decreases. Consequently, when the inventory level is low, service occurs at a slow pace. Since an increase in inventory level through replenishment occurs only after these long service periods, the system spends a considerable proportion of time in a state of low inventory. Consequently, at these times the expected arrival rate will be low. Therefore, even with higher demand v, the effective arrival rate will be moderate, ensuring the stability of the system.

In scenarios where there is high demand for the inventory, opting for stock-dependent service processes becomes advantageous in regulating the inflow of customers and maintaining system stability. This proves particularly useful in contexts such as ration distribution systems or the distribution of essential commodities to a large population. In such situations, the arrivals increase with the available stock. Consequently, we can exert control over the arrival rate by managing the inventory level. Our numerical study indicates that an effective method to achieve this control is by employing stock-dependent service processes.

7 Conclusion

In our investigation, we delved into a queueing inventory model incorporating interdependent arrival and service processes, along with the inventory level, utilizing two semi-Markov processes. The marginal distributions of both service time and inter-arrival times were identified as mixtures of phase-type distributions. Further analysis focused on a specific instance of this model, with the derivation of conditions for system stability. The stationary distribution was determined numerically. We also explored the distribution of waiting time, its expected value, and other critical system performance measures. This model has the potential for extension to more complex scenarios involving positive lead time. Examining the impact of replenishment time on system performance would be particularly intriguing in such cases.

Appendix

Results of numerical analysis mentioned in Sects. 6.1, 6.2 and 6.3 are tabulated in the following tables.

Table 1. Comparison between the models in terms of expected number of customers

Arrival Rate $-\alpha T^{-1}e$	Expected Number of Customers	
	Model 1	Model 2
4	1.6968	3.0114
4.1	1.8088	3.3473
4.2	1.9302	3.7428
4.3	2.0624	4.2155
4.4	2.2069	4.7899
4.5	2.3656	5.5027
4.6	2.5405	6.4105
4.7	2.7345	7.6053
4.8	2.9508	9.2483
4.9	3.1935	11.6489
5	3.4679	15.4862
5.1	3.7807	22.598
5.2	4.1406	40.2836
5.3	4.5592	157.9844
5.4	5.0521	
5.5	5.6413	
5.6	6.358	
5.7	7.249	
5.8	8.3869	
5.9	9.8911	
6	11.9727	

Table 2. Variation of important performance measures with σ

σ	Inventory Level	$E(N)$	$P(I)$	Cost	Expected Service rate	E(W)
0.46	2.2639	2.6019	0.2744	15.8498	3.0219	0.798
0.47	2.2681	2.4235	0.289	15.6555	3.0876	0.7485
0.48	2.2721	2.2683	0.3031	15.5059	3.1533	0.7026
0.49	2.276	2.1319	0.3165	15.3925	3.219	0.6602
0.5	2.2798	2.0111	0.3295	15.309	3.2847	0.6211
0.51	2.2835	1.9034	0.342	15.2502	3.3504	0.585
0.52	2.287	1.8068	0.354	15.2123	3.4161	0.5518
0.53	2.2905	1.7195	0.3656	15.192	3.4818	0.5211
0.54	2.2938	1.6404	0.3768	15.1868	3.5474	0.4929
0.55	2.2971	1.5683	0.3876	15.1945	3.6131	0.4668
0.56	2.3002	1.5024	0.398	15.2135	3.6788	0.4427
0.57	2.3033	1.4418	0.4081	15.2423	3.7445	0.4204
0.58	2.3062	1.3859	0.4178	15.2796	3.8102	0.3997
0.59	2.3091	1.3342	0.4272	15.3244	3.8759	0.3805
0.6	2.3119	1.2863	0.4363	15.3759	3.9416	0.3627
0.61	2.3147	1.2417	0.4451	15.4333	4.0073	0.3462
0.62	2.3173	1.2001	0.4537	15.496	4.073	0.3307
0.63	2.3199	1.1613	0.462	15.5634	4.1387	0.3163
0.64	2.3225	1.1248	0.47	15.635	4.2044	0.3028
0.65	2.3249	1.0907	0.4778	15.7105	4.2701	0.2902
0.66	2.3273	1.0585	0.4854	15.7893	4.3358	0.2784
0.67	2.3297	1.0282	0.4928	15.8713	4.4015	0.2673
0.68	2.332	0.9996	0.4999	15.9562	4.4672	0.2569
0.69	2.3342	0.9725	0.5069	16.0436	4.5328	0.2471
0.7	2.3364	0.9469	0.5136	16.1333	1.0873	0.2378

References

1. Baker, R.C., Urban, T.L.: Single-period inventory dependent demand models. Omega **16**(6), 605–607 (1988)
2. Bekker, R., Borst, S., Boxma, O.J., Kella, O.: Queues with workload-dependent arrival and service rates: In honor of vladimir kalashnikov (guest editors: Evsey morozov and richard serfozo). Queue. Syst. **46**, 03 (2004)
3. Conway, R.W., Maxwell, W.L.: A queuing model with state dependent service rates. J. Ind. Eng. **12**(2), 132–136 (1962)
4. Gupta, S.K.: Queues with hyper-poisson input and exponential service time distribution with state dependent arrival and service rates. Oper. Res. **15**(5), 847–856 (1967)

5. Hadley, G., Whitin, T.M.: Analysis of inventory systems. International Series in Management. Prentice-Hall (1963)
6. Hillier, F.S., Conway, R.W., Maxwell, W.L.: A multiple server queueing model with state dependent service rate. J. Ind. Eng. **15**(3), 153–157 (1964)
7. Johnson, E.L.: On (s, S) policies. Manage. Sci. **15**(1), 80–101 (1968)
8. Krishnamoorthy, A., Lakshmy, B., Rangaswamy, M.: A survey on inventory models with positive service time. OPSEARCH **48**, 153–169 (2011)
9. Krishnamoorthy, A., Shajin, D., Narayanan, V.: Inventory with positive service time: a survey. Adv. Trends Queue. Theory **2**, 201–238 (2021)
10. Larson, P.D., DeMarais, R.A.: Psychic stock: retail inventory for stimulating demand. In: Dunlap, B.J. (ed.) Proceedings of the 1990 Academy of Marketing Science (AMS) Annual Conference. DMSPAMS, pp. 447–450. Springer, Cham (2015). https://doi.org/10.1007/978-3-319-13254-9_89
11. Latouche, G.: A phase-type semi-Markov point process. SIAM J. Algeb. Discrete Meth. **3**(1), 77–90 (1982)
12. Kaplan, E.H., Berman, O., Shevishak, D.G.: Deterministic approximations for inventory management at service facilities. IIE Trans. **25**(5), 98–104 (1993)
13. Ranjith, K.R., Krishnamoorthy, A., Gopakumar, B., Nair, S.: Analysis of a single server queue with interdependence of arrival and service processes - a semi-Markov approach, pp. 417–429 (2021)
14. Sigman, K., Simchi-Levi, D.: Light traffic heuristic for an $M/G/1$ queue with limited inventory. Ann. Oper. Res. **40**, 371–380 (1993)
15. Urban, T.L.: Inventory models with inventory-level-dependent demand: a comprehensive review and unifying theory. Eur. J. Oper. Res. **162**(3), 792–804 (2005)
16. Wolfe, H.B.: A model for control of style merchandise. Ind. Manag. Rev. **9**, 69–82 (1968)