On a New Perspective in Longevity Risk Management: The Lifetime Shifting

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Abstract. Longer lives are an achievement and the course of lifespan is increasingly influenced by unobservable risk factors altering the chronological pace of aging. Then, the present work proposes an analytical approach to characterizing the human lifetime based on the concept of non-chronological age. Starting from a chronological Gompertz mortality framework, we define the non-chronological lifespan and characterize it probabilistically by deriving, in closed-form, the expression for the cumulative distribution function, the density of deaths function, and the mortality hazard function. We find that non-chronological death probabilities are a time-dependent affine transformation of chronological death probabilities for a newborn, and we highlight the link between the non-chronological lifetime and the concept of individual frailty in heterogeneous mortality modelling. We believe that our proposal may contribute to shaping a new perspective on longevity risk measurement and management.

Keywords: Lifetime · Non-chronological age · Longevity risk

1 Introduction

In the last decades, the human lifetime has grown continuously and human mortality has shifted to later ages ([8]). Mortality deferment to older ages was empirically observed by investigating changes in the main lifetime indicators expressed in terms of chronological age, such as the mortality hazard, the survival function, and density of deaths function (see, e.g., [6]). While living longer, aka longevity, is a positive achievement at the individual level, it implies significant unexpected financial exposures for governments, annuity providers, and pension schemes, namely 'longevity risk' in actuarial jargon. Such a risk is a compelling matter of interest for both actuaries and policymakers, and both demographic and actuarial literature have seen an enriched focus on mortality modelling and forecasting. Nowadays, thanks to the advances in medical literature, new perspectives on longevity analysis have emerged. In particular, the concept of biological age has been introduced, that is the age indicating how old the human mechanism is at both the cellular and molecular levels (see, e.g., [2]). The biological age may be misaligned with respect to the corresponding chronological age, and it can

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reasonably be acknowledged as a key element in analyzing lifespan randomness. The biological age is usually estimated by collecting data concerning physiological and molecular variables for a large sample of people, and, by means of multivariate regression, the sign of statistically-significant regression coefficients leads an increment or a reduction of the corresponding chronological age. In other words, due to biological (and observable) factors, person's age does not move necessarily in lockstep with calendar time and different individuals may age at different rates. Within the actuarial literature, in [3] the meaning and the use of the biological age is discussed for the first time. On one side, this paper highlights that the biological age is a relevant variable to predict the risk of chronic disease and maximize the health span, but not necessarily lifespan; on the other, the presence of a non-chronological age that differs from the chronological one impacts lifespan and the longevity risk measurement. Therefore, by referring to a Gompertz-Makeham mortality framework, [3] paves the way to construct a non-chronological age, namely longevity-risk-adjusted global age (L-RaG), different from the biological one and in contrast to the chronological age. Another type of non-chronological age is defined in [1], namely survivorship-age (s-age), representing the age at which a proportion of a population is still alive. The underlying idea is to invert the relation between the survival function and the chronological age, so that the latter becomes a function of the survival levels. The authors investigate the behaviour of the mortality hazard associated to the s-age, showing that populations experience a similar risk of dying at specific levels of survivorship. The L-RaG and the s-age are outcomes of distinct approaches, but both state the existence of a non-chronological age determined without the use of observable biological factors. Interestingly, we note that this is what happens when frailty-based models are employed in shaping heterogeneous mortality due to unobservable risk factors (see, e.g., [4,5]). Indeed, some biological factors entailing the gap between the non-chronological age and the chronological one may be not directly observable or not available, and, in addition, they imply a mortality differentiation among individuals. Then, the gap between these ages may be assimilated into an unobserved frailty. To some extent, this is also the intuition behind the work in [7]. The authors assume a Generalized Gompertz distribution (GG) for the lifetime and prove that, under specific assumptions, the frailty can be interpreted as a random correction to the chronological age. However, their proposal allows the presence of negative chronological ages. In the vein of the aforementioned literature, the present work aims to characterize the human lifetime taking into account a random shift of the chronological age. More in detail, we primarly consider a chronological age-based mortality by means of the Gompertz model, and then we assume a Generalized Gompertz distribution for the random shift to probabilistically define a non-chronological lifetime. As a result, we provide closed-form expressions for the cumulative distribution function, the density of death function and the mortality hazard under the non-chronological lifetime. Our proposal contributes to the current literature by posing a new modelling perspective concerning the lifespan randomness due to unobservable risk factors, avoiding the possibility of negative lifetimes.

The paper is organized as follows. In Sect. 2 we recall the Gompertz mortality framework and we introduce the GG distribution. In Sect. 3 we develop our proposal by defining the non-chronological lifetime and providing analytical results concerning its distribution. Finally, Sect. 4 poses conclusions.

2 Chronological Lifetime in a Gompertz Framework

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let T_0 be the random lifetime for a newborn. We assume that T_0 is Gompertz distributed, i.e., $T_0 \sim \mathrm{G}(h, g)$, with cumulative distribution function (cdf) and probability density function (pdf) given, respectively, by

$$F_{T_0}(x) = 1 - \exp\left\{-\frac{h(e^{gx}-1)}{g}\right\}, \qquad f_{T_0}(x) = h\, e^{gx} \exp\left\{-\frac{h(e^{gx}-1)}{g}\right\}, \quad (1)$$

and the following mortality hazard holds

$$\mu(x) = \frac{f_{T_0}(x)}{1 - F_{T_0}(x)} = h\, e^{gx}. \quad (2)$$

Equation (2) represents the well-known Gompertz mortality law (under the chronological age), where the parameter h is the initial mortality level and the parameter g indicates the rate of aging. For any chronological age $x > 0$, the residual random lifetime is defined as $T_x = T_0 - x \,|\, T_0 > x$, and its cdf, pdf and mortality hazard in the Gompertz mortality framework are, respectively,

$$F_{T_x}(t) = 1 - \exp\left\{-\frac{h}{g} e^{gx}(e^{gt}-1)\right\}, \quad (3)$$

$$f_{T_x}(t) = h\, e^{g(x+t)} \exp\left\{-\frac{h}{g} e^{gx}(e^{gt}-1)\right\}, \quad (4)$$

$$\mu(x+t) = h\, e^{g(x+t)}, \quad (5)$$

where $t > 0$. As argued in [7], the lifetime distribution for a newborn can be described in more general terms by adopting the GG distribution. In detail, we say that T_0 has the Generalized Gompertz distribution, $\mathrm{GG}(a, b, c)$, $a \in \mathbb{R}$, $b, c > 0$, if the cdf and the pdf are, respectively, defined as

$$F_{T_0}(x) = 1 - \frac{\Gamma\left(c, \exp\left(\frac{x-a}{b}\right)\right)}{\Gamma(c)}, \quad f_{T_0}(x) = \frac{1}{b\Gamma(c)} \exp\left\{c\,\frac{x-a}{b} - \exp\left(\frac{x-a}{b}\right)\right\}, \quad (6)$$

where $\Gamma(c, w) = \int_w^{+\infty} u^{c-1} e^{-u} du$ is the upper incomplete Gamma function and $\Gamma(c) = \Gamma(c, 0)$ is the complete Gamma function. We notice that cdf and pdf in (6) are defined for $x \in \mathbb{R}$, that is negative lifetimes may occur with positive probability. Despite this drawback, in [7] it is shown that, when the lifetime under a frailty-based model is considered, a GG-frailty defines a random age correction to the chronological lifetime. In the next section, we propose chronological lifetime shifting by using the GG distribution and avoiding negative lifetimes.

3 Shifting the Chronological Lifetime

Let us introduce the non-chronological lifetime $\widetilde{T}$. We assume that for a newborn $\widetilde{T}_0 = T_0$ almost surely, while the residual lifetime can be defined according to the passage of age in a non-chronological manner:

$$\widetilde{T}_x = T_0 - (x + \Delta) \,|\, T_0 > x, \tag{7}$$

where $(x + \Delta)$ is a non-chronological age, being Δ a random shift in width and sign, and with T_0 and Δ stochastically independent. The cdf of (7) is defined in the following Proposition 1.

Proposition 1. *Let the non-chronological lifetime $\widetilde{T}_x$ be defined as in (7), and assume that $\Delta \sim GG(0, b, c)$, with $b, c > 0$ and cdf given by*

$$F_\Delta(\delta) = 1 - \frac{\Gamma\left(c, e^{\frac{\delta}{b}}\right)}{\Gamma(c)}, \quad \delta \in \mathbb{R}. \tag{8}$$

Then, if the chronological lifetime has Gompertz distribution, i.e. $T_0 \sim G(h, g)$, $h, g > 0$, the cdf of the non-chronological lifetime is

$$F_{\widetilde{T}_x}(t) = B(t) + A(t) F_{T_0}(x + t), \quad t > 0, \tag{9}$$

where $F_{T_0}(x + t) = 1 - \exp\left\{-\frac{h}{g}(e^{g(x+t)} - 1)\right\}$, and

$$A(t) = \frac{g^{\frac{c}{gh}-1} b^{\frac{c}{g}-1} h^{\frac{c(h-1)}{hg}} (1 + hb)^{-\frac{c}{g}} \Gamma\left(\frac{c}{b}, e^{-gt}\left(\frac{1+hb}{gb}\right)\right)}{(1 - F_{T_0}(x))\,\Gamma(c)}, \tag{10}$$

$$B(t) = \frac{1}{1 - F_{T_0}(x)} - \frac{\Gamma\left(c, e^{-\frac{t}{b}}\right)}{\Gamma(c)}\left(\frac{2}{1 - F_{T_0}(x)} - 1\right) - A(t). \tag{11}$$

Proof. Since $\widetilde{T}_x = T_0 - (x + \Delta) \,|\, T_0 > x$, then the cdf of the non-chronological lifetime is determined by computing

$$\begin{aligned} F_{\widetilde{T}_x}(t) &= P(T_0 \le x + \Delta + t \,|\, T_0 > x) \\ &= \frac{1}{1 - F_{T_0}(x)} \int_{-t}^{+\infty} P(x < T_0 \le x + \delta + t)\, dF_\Delta(\delta) \\ &= \frac{1}{1 - F_{T_0}(x)} \left\{ \int_{-t}^{+\infty} F_{T_0}(x + \delta + t)\, dF_\Delta(\delta) - F_{T_0}(x)(1 - F_\Delta(-t)) \right\}. \end{aligned} \tag{12}$$

By assuming that $T_0 \sim \mathrm{G}(h, g)$, with $h, g > 0$, the expression of $F_{T_0}(x)$ is the cdf in (1), while the cdf's expression of Δ is given by (8). Then, by substituting in (12), we get

$$\begin{aligned} F_{\widetilde{T}_x}(t) = \exp\left\{\frac{h}{g}(e^{gx} - 1)\right\} - \frac{\Gamma(c, e^{-\frac{t}{b}})}{\Gamma(c)}\left(2\exp\left\{\frac{h}{g}(e^{gx} - 1)\right\} - 1\right) \\ - \frac{\Gamma\left(\frac{c}{b}, e^{-gt}\left(\frac{1+hb}{gb}\right)\right)}{bg\Gamma(c)} \left(\frac{g}{h}\right)^{\frac{c}{gh}} \left(\frac{1 + hb}{hb}\right)^{-\frac{c}{b}} \exp\left\{-\frac{h}{g} e^{gx}(e^{gt} - 1)\right\}. \end{aligned} \tag{13}$$

Due to (1) and (3), it holds that:

$$\exp\left\{\frac{h}{g}(e^{gx}-1)\right\}=\frac{1}{1-F_{T_0}(x)},\quad \exp\left\{-\frac{h}{g}e^{gx}(e^{gt}-1)\right\}=\frac{1-F_{T_0}(x+t)}{1-F_{T_0}(x)}. \tag{14}$$

By substituting (14) in (13), and rearranging the terms, the expressions (9)-(11) follow, completing the proof. □

From Proposition 1, we highlight the following considerations:

- Firstly, (9) provides the probability of death at the non-chronological age $\xi := x + \Delta$, namely ${}_t\tilde{q}_\xi$ by exploiting the actuarial notation, and it differs from the corresponding probability of death at the chronological age x, i.e. ${}_tq_x$. The latter can written as

$${}_tq_x = B + A\,{}_{x+t}q_0, \tag{15}$$

  where ${}_{x+t}q_0$ is the probability of death by the chronological age $(x+t)$ for a newborn, $A=\frac{1}{1-F_{T_0}(x)}$, and $B=-\frac{F_{T_0}(x)}{1-F_{T_0}(x)}$. Looking at (9) and (15), we observe that both the chronological and the non-chronological probabilities of death are affine functions of the probability ${}_{x+t}q_0$. The coefficients of the ${}_tq_x$'s affine transformation are time-invariant, while they become time-dependent (and more complex) when the probability ${}_t\tilde{q}_\xi$ is computed. To some extent, while the chronological probabilities of death are defined in a static way, the non-chronological probabilities stem from a time-dependent adjustment of ${}_{x+t}q_0$;
- The Generalized Gompertz distribution assumption for Δ can be related to the frailty coefficient characterizing the frailty-based mortality models. In particular, a Gamma distribution is usually adopted for the frailty coefficient which is applied, in a multiplicative way, to the population mortality hazard (see, e.g., [4,5]). Then, for all the Gamma realizations in the interval $(0,1)$ the individual mortality hazard is lower than that of the population (lower individual frailty), and the opposite case occurs for realizations in $(1,+\infty)$ (higher individual frailty). For the purposes of our proposal, we highlight that the Gamma and Generalized Gompertz distributions are connected. For instance, $\Delta = b\ln(Y) \sim \mathrm{GG}(0,b,c)$, with $b>0$, if $Y \sim \mathrm{Gamma}(c,1)$, $c>0$. In other words, our proposal supposes a non-chronological lifetime obtained as a frailty-based shift of the chronological lifetime. Then, for every realization $y \in (0,1)$ we attain negative outcomes for Δ, implying a reduction of the chronological age and an increment of the lifetime (i.e. lower frailty); conversely, for every realization $y \in (1,+\infty)$, we have positive values for Δ, a consequent growth of the chronological age and a shortened lifetime (i.e. greater frailty).

Moreover, by differentiating (9), the pdf of the non-chronological lifetime is

$$f_{\widetilde{T}_x(t)} = \frac{d}{dt}F_{\widetilde{T}_x(t)} = C(t) + A(t)f_{T_0}(x+t), \tag{16}$$

where $f_{T_0}(x+t)$ is defined in (1) (with x replaced by $x+t$), and

$$C(t) = B'(t) + A'(t)F_{T_0}(x+t), \tag{17}$$

$$B'(t) = \frac{\exp\left\{-e^{-\frac{t}{b}} - \frac{tc}{b}\right\}\left(1 - \frac{2}{1-F_{T_0}(x)}\right)}{b\Gamma(c)} - A'(t), \tag{18}$$

$$A'(t) = \frac{\exp\left\{-\frac{e^{-gt}(1+bh)+g^2tc}{bg}\right\}\frac{1}{1-F_{T_0}(x)}\left(\frac{1+bh}{b}\right)^{\frac{c(g-b)}{bg}} g^{\frac{c(b-gh)}{ghb}} h^{\frac{c(h-1)}{gh}}}{b\Gamma(c)}. \tag{19}$$

Finally, the non-chronological mortality hazard can be computed as

$$\widetilde{\mu}(x+t) = \frac{f_{\widetilde{T}_x(t)}}{1 - F_{\widetilde{T}_x(t)}}.$$

4 Conclusion

In this work, we have proposed an analytical approach to define a non-chronological lifetime and investigated its main probabilistic features. We have found that non-chronological death probabilities are a time-dependent affine transformation of chronological death probabilities for a newborn. In addition, we have highlighted how the shift between the non-chronological and chronological lifetimes and the concept of individual frailty in heterogeneous mortality models may be related.

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