The Market Value of Optimal Annuitization and Bequest Motives

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Abstract. Since the seminal contribution of Yaari (1965), who showed that individuals with no bequest motive should convert all their retirement wealth into annuities, a number of papers have analysed the annuitization decision under the so-called *all or nothing* institutional arrangement, where immediate lifetime annuities are purchased just at a one point in time. In this paper, we investigate the effect of linear bequest motives on the annuitization decision for a retired individual who maximizes the market value of future cash flows. Finally, we present numerical examples analyzing optimal annuitization under strong or weak bequest motives.

Keywords: optimal stopping · annuities · bequest motives

1 Introduction

An immediate annuity is an insurance product that pays the annuitant a regular income for as long as he is alive, in exchange for a premium. The annuitization decision has important economic implications because it has a direct effect on the financial resources to support consumption in retirement age. The purchase of an annuity helps individuals to manage the risk of outliving their financial wealth, but it is usually an irreversible transaction, and most annuity contracts impose steep penalties if policyholders want to access their money in the early years of the contract. The natural alternatives to annuitization are the so-called *do-it-yourself* strategies, i.e. the individual asset allocation amongst various financial investment classes. However, it should be taken into account the investment risk as well as the longevity risk to which individuals would be exposed.

Since the seminal paper by Yaari ([10], the study of the annuitization decision has been the subject of a whole research field (see [3–7, 9] among others).

This paper would contribute to this literature by investigating to what extent linear bequest motives (see e.g. [6]) affect the annuitization decision. At this aim, we consider an individual whose retirement wealth is invested in a financial fund which eventually must be converted into an annuity. As in [7], we consider two different mortality forces: a *subjective* one, used by the individual to weight

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future cashflows (denoted μ), and a *objective* one, used by the insurance company to price the annuity (denoted $\hat{\mu}$). The interplay between these two different mortality forces contributes to some key qualitative aspects of the optimal annuitization decision. Before annuitization the individual's wealth is invested in the financial market, and at the time of an annuity purchase, the entire wealth is converted into a lifetime annuity. The central idea is to compare the value deriving from an immediate annuitization with the value of continuing the investment in the financial market. The optimization criterion pursued by the individual is the maximization of the present value of future expected cash-flows, via the optimal timing of the annuity purchase. In particular, the individual takes explicitly into account the presence of linear bequest motives, with a parameter that measures its strength.

The rest of the paper is organized as follows. In Sect. 2 we introduce the financial and actuarial assumptions and then the optimal annuitization problem. In Sect. 3 we perform its analytical study providing the explicit solutions. In Sect. 4 we present some numerical examples to discuss how the presence of bequest motives affects the annuitization decision.

2 Problem Formulation

In this study we are interested in the portion of the individual's wealth dedicated to retirement needs. Such wealth is invested in a financial fund which eventually will be converted into an annuity. The value $(X_t)_{t\geq 0}$ of the financial fund is modelled by a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Letting $(B_t)_{t\geq 0}$ be a Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$, the fund's value evolves according to

$$\begin{cases} dX_t^x = (\theta - \alpha)X_t^x dt + \sigma X_t^x dB_t, & t > 0 \\ X_0^x = x \geq 0, \end{cases} \tag{1}$$

where θ is the average continuous return of the financial investment, α is the constant dividend rate and $\sigma > 0$ is the volatility coefficient.

We consider an individual whose age $\eta \geq 0$ is fixed at time 0. At time $t \geq 0$ the individual uses a constant *subjective* mortality force $\mu \in \mathrm{R}_+$ to compute her self-assessed life expectancy ${}_z p_\eta = e^{-\mu z}$, i.e. the subjective probability to survive z years. Furthermore, the probability that the individual dies during the next z years is ${}_z q_{\eta+t} = 1 -_z p_{\eta+t}$. The insurance company instead relies on a so-called objective survival probability function ${}_z \hat{p}_{\eta+t} = e^{-\hat{\mu} z}$ where $\hat{\mu} \in \mathrm{R}_+$ is the constant *objective* mortality force. The different survival probability functions account for the imperfect information available to the insurer on the individual's risk profile. The value at time $t > 0$ of a unitary life annuity is given by $\hat{a}_{\eta+t} = \int_0^\infty e^{-\hat{\rho} u} {}_u \hat{p}_{\eta+t} du$. Here $\hat{\rho}$ is the interest rate guaranteed by the insurer. The individual evaluates the expected present value of a unitary lifetime annuity by using the coefficient $a_{\eta+t} = \int_0^\infty e^{-\rho u} {}_u p_{\eta+t} du$. In case the annuity is purchased at time t, the constant cash flow paid by the insurer is $P_t = \frac{X_t - K}{\hat{a}_{\eta+t}}$, where the constant K is either a fixed acquisition fee ($K > 0$) or a tax incentive ($K < 0$).

The optimisation criterion is the maximization of the present value of future expected cash-flows. Let $\tau_d : \Omega \rightarrow \mathrm{R}_+$ be the residual lifetime of the individual of age η (τ_d is assumed to be independent of the Brownian motion for all $t \geq 0$). Letting τ be the time of the annuity purchase, the value function of the optimisation problem is defined by

$$V(x) = \sup_{\tau \geq 0} \mathbb{E}\left[\int_0^{\tau_d \wedge \tau} e^{-\rho t} \alpha X_t^x dt + \mathbb{1}_{\{\tau_d \leq \tau\}} e^{-\rho \tau_d} \nu X_{\tau_d}^x + P_\tau \int_{\tau_d \wedge \tau}^{\tau_d} e^{-\rho t} dt\right], \quad (2)$$

where ρ is the individual's constant discount rate, $\nu \in [0, 1]$ measures the strength of the bequest motives. Before annuitization, i.e. for $t < \tau$, the individual receives the dividends from the fund at rate α. After annuitization, i.e. for $t > \tau$, she gets the annuity payment at a constant rate P_τ. In case the individual dies before the time of the annuity purchase, i.e. on the event $\{\tau_d \leq \tau\}$, she leaves a bequest equal to her wealth.

The optimisation problem (2) may be rewritten as follows

$$V(x) = \sup_{\tau \geq 0} \mathbb{E}\bigg[\int_0^\tau e^{-(\rho+\mu)t}(\alpha + \mu\nu) X_t^x dt + e^{-(\rho+\mu)\tau} \delta (X_\tau^x - K)\bigg], \quad (3)$$

with $\delta := \frac{(\hat{\rho}+\hat{\mu})}{(\rho+\mu)}$. To ensure the finiteness of the value function, throughout this paper, we assume that

Assumption 1. $\theta - \alpha - \mu - \rho < 0$

3 Analysis of the Optimal Stopping Problem

To analyze problem (3), we rely on the geometric approach to the optimal stopping problem (see [2]).

Our first task is to put the optimal stopping problem (3) in the form

$$\sup_\tau \mathbb{E}\Big[e^{-w\tau} G(X_\tau)\Big],$$

where w is a discount rate and $G(\cdot)$ is a reward function.

Noticing that $\mathbb{E}\Big[\int_0^\infty e^{-(\rho+\mu)t}(\alpha+\mu\nu)X_t^x dt\Big] = \beta x$ where $\beta := \frac{\alpha+\mu\nu}{\rho+\alpha+\mu-\theta}$, we may rewrite (3) as follows

$$V(x) = \beta x + \sup_{\tau \geq 0} \mathbb{E}\Big[e^{-(\rho+\mu)\tau}\big((\delta - \beta) X_\tau^x - \delta K\big)\Big] \quad (4)$$

Therefore, we limit ourself to study the problem

$$v(x) = \sup_{\tau \geq 0} \mathbb{E}\big[e^{-(\rho+\mu)\tau} G(X_\tau^x)\big] \quad (5)$$

where

$$G(x) := (\delta - \beta)x - \delta K. \tag{6}$$

As usual in optimal stopping theory, we let $\mathcal{C} = \{x \in \mathrm{R}_+ : v(x) > G(x)\}$ and $\mathcal{S} = \{x \in \mathrm{R}_+ : v(x) = G(x)\}$ be respectively the so-called continuation and stopping regions since, as long as $X_t \in \mathcal{C}$, it is not optimal to stop the diffusion. Assumption (1) and standard optimal stopping results (see [8, Cor. 2.9, Sect. 2]) guarantee that $\mathcal{C}$ is an open connected set and $\tau_* := \inf\{t \geq 0 \ : \ X_t^x \in \mathcal{S}\}$ is optimal for $v(x)$, i.e. the optimal stopping time is the first entry time of X in $\mathcal{S}$. Define the infinitesimal generator $\mathbb{L}$ of X by $(\mathbb{L}u)(x) = \frac{1}{2}\,\sigma^2 x^2 u^{''}(x) + (\theta - \alpha)xu^{'}(x)$, for any $u(\cdot)$ two time continuously differentiable. Then (see e.g. [1], pp. 18–19), there exist two linearly independent, strictly positive solutions of the ordinary differential equation $\mathbb{L}u = (\rho + \mu)u$, i.e. $\psi(x) = x^{\gamma_+}$ and $\phi(x) = x^{\gamma_-}$ where γ_+ and γ_- solve

$$\frac{1}{2}\,\sigma^2\gamma(\gamma - 1) + (\theta - \alpha)\gamma - (\rho + \mu) = 0.$$

Notice that $\gamma_+ > 1$ and $\gamma_- < 0$. As in [2], we define the strictly increasing function $y = F(x) = \frac{\psi(x)}{\phi(x)} = x^{\gamma_+ - \gamma_-}$, together with its inverse function $F^{-1}(y) = y^{\frac{1}{\gamma_+ - \gamma_-}}$, and set

$$\hat{G}(y) := \begin{cases} 0 & \text{if } y = 0, \\ \left(\frac{G}{\phi} \circ F^{-1}\right)(y) & \text{if } y > 0. \end{cases} \tag{7}$$

The following result due to Dayanik and Karatzas (see [2]) relates the convexity of the function $\hat{G}$ to the form of the continuation region and computes the value function.

Theorem 2. *$\hat{G}(y)$ is strictly convex if and only if $(\mathbb{L} - (\rho + \mu))G(x) > 0$. Moreover, let $Q(\cdot)$ be the smallest nonnegative concave function that dominates $\hat{G}(y)$. Then $v(x) = \phi(x)Q(F(x))$.*

In our case,

$$\hat{G}(y) = (\delta - \beta)y^{\frac{1-\gamma_-}{\gamma_+ - \gamma_-}} - \delta K y^{\frac{-\gamma_-}{\gamma_+ - \gamma_-}}, \quad y > 0. \tag{8}$$

Define

$$y_1 = \Big[- \frac{\delta K \gamma_-}{(\delta - \beta)(1 - \gamma_-)} \Big]^{\gamma_+ - \gamma_-},$$

and

$$y_2 = \Big[- \frac{\delta K \gamma_- \gamma_+}{(\delta - \beta)(1 - \gamma_-)(\gamma_+ - 1)} \Big]^{\gamma_+ - \gamma_-}.$$

It is easy to see that $y_1 < y_2$. Depending on the values of the model parameters, we distinguish the following cases

1. **Case** $\delta > \beta$
   (a) If $K \geq 0$ then $\hat{G}'(y) > 0$ for $y > y_1$ and $\hat{G}''(y) < 0$ for $y > y_2$. The smallest nonnegative concave function $Q(y)$ that dominates $\hat{G}(y)$ is

$$Q(y) = \begin{cases} m^* y & \text{if } y < y^*, \\ \hat{G}(y) & \text{if } y \geq y^*, \end{cases}$$

   where m^* and y^* are solutions of the system $\begin{cases} \hat{G}(y) = my \\ \hat{G}'(y) = m. \end{cases}$ In other words, m^* and y^* are such that the line $y = m^* y$ is tangent to $\hat{G}(y)$ at the point y^*. Returning to the variable x, we find that the continuation and stopping regions are respectively $\mathcal{C} = (0, x^*)$ and $\mathcal{S} = [x^*, \infty)$, with $x^* = \frac{\delta K}{(\delta - \beta)} \frac{\gamma_+}{\gamma_+ - 1}$.
   (b) If $K \leq 0$ then $\hat{G}'(y) > 0$ and $\hat{G}''(y) < 0$ for all $y > 0$. Therefore, $\mathcal{S} = [0, \infty)$, i.e. the annuity is immediately purchased whatever is the initial wealth x.
2. **Case** $\delta < \beta$
   (a) If $K \geq 0$ then $\hat{G}'(y) < 0$ and $\hat{G}''(y) > 0$ for all $y > 0$. Therefore, $\mathcal{C} = (0, \infty)$, and it is never optimal to purchase an annuity.
   (b) If $K < 0$ then $\hat{G}'(y) > 0$ for $y < y_1$ and $\hat{G}''(y) > 0$ for $y > y_2$. The smallest nonnegative concave function $Q(y)$ that dominates $\hat{G}(y)$ is

$$Q(y) = \begin{cases} \hat{G}(y) & \text{if } y \leq y_1, \\ \hat{G}(y_1) & \text{if } y \geq y_1. \end{cases} \tag{9}$$

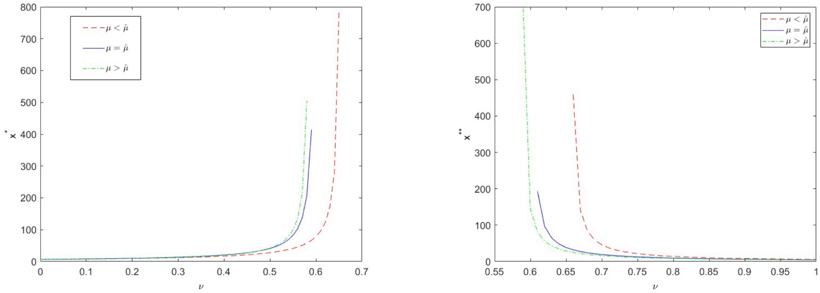
   In other words, for $y \geq y_1$ the function Q is an horizontal line. The continuation and stopping regions are respectively $\mathcal{C} = (x^{**}, \infty)$ and $\mathcal{S} = [0, x^{**}]$, with $x^{**} = F^{-1}(y_1) = \frac{\delta K}{(\delta - \beta)} \frac{\gamma_-}{1 - \gamma_-}$.

4 Numerical Application

Here we present a numerical application of the results obtained in the previous sections. Fix the following set of parameters $\alpha = 2\%$, $\theta = 7\%$, $\sigma = 6\%$, $\rho = \hat{\rho} = 5\%$. Notice that Assumption 1 is satisfied.

In Fig. 1.(a) we look at case 1.(a) and the optimal stopping threshold x^* between the continuation and stopping regions is plotted when the parameter ν increases, in three different mortality scenarios: $\mu = \frac{1}{20}$, $\hat{\mu} = \frac{1}{18}$, $\mu = \hat{\mu} = \frac{1}{20}$, $\mu = \frac{1}{18}$, $\hat{\mu} = \frac{1}{20}$. Notice that x^* increases as ν increases, that is as the strength of bequest motives increases the continuation region enlarges. If $\mu < \hat{\mu}$ ($> \hat{\mu}$) then the individual believes she is healthier (respectively unhealthier) than the average. For a fixed value of ν, moving from the first to the third scenario x^* increases, and then the continuation region becomes progressively larger.

In Fig. 1.(b) we look at case 2.(b) and the optimal stopping threshold x^{**} is plotted when the parameter ν increases, in the three different scenarios. Notice that, in all scenarios, x^{**} decreases as ν increases, i.e. the continuation region progressively enlarges as the strength of the bequest motives increases. For a fixed value of ν, moving from the first to the third scenario x^{**} decreases, and then the continuation region becomes progressively larger.



(a) (b)

Fig. 1. The threshold x^* (a) and x^{**} (b) in three different mortality scenarios

Finally, notice that if $K = 0$, then either $\mathcal{S} = [0, \infty)$ or $\mathcal{C} = (0, \infty)$. In particular, letting for example $\mu = \hat{\mu} = \frac{1}{20}$, we find that if $\nu < 0.6$ then $\mathcal{S} = [0, \infty)$, and if $\nu > 0.6$ then $\mathcal{C} = (0, \infty)$. In other words, in case of actuarially fair annuities and zero acquisition fee/tax incentive, if the strength of bequest motives is low enough then the individual immediately purchases the annuity as Yaari ([10]) found. On the other hand, if the strength of bequest motives is high enough then Yaari's result does not hold anymore and the individual never purchases the annuity.

In sum, these numerical examples show that the continuation region enlarges as the strength of bequest motives increases. This means that consistent bequest motives may explain the scarce propensity to purchase annuities.

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