Fair Volatility in the Fractional Stochastic Regularity Model

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Abstract. Within the efficient markets framework, discounted stock prices are typically represented through Brownian martingales. The primary measure for evaluating risk is the volatility of log-returns, under the assumption that higher variability indicates greater associated risk. The theoretical foundation of this claim stems from the characterization of the path regularity of price process through the Lévy characterization theorem of Brownian motion. Since this explanation lacks a financial interpretation when considering more realistic models, such as stochastic volatility models, it is necessary to disentangle volatility and regularity. Replacing volatility by the Hölder regularity provides insights into market deviations from the equilibrium of the martingale model, and - within the Fractional Stochastic Regularity Model - contributes to identify the "fair" volatility aimed by the market.

Keywords: Hölder exponent · Volatility · Fractional Stochastic Regularity Model

1 Introduction

This contribution underscores the crucial distinction between *volatility* and *regularity* for the purpose of characterizing risk in financial dynamics. Volatility quantifies the extent to which data deviate from their mean value, while regularity captures the manner in which data are dispersed. In the framework of paradigmatic Efficient Market Hypothesis (EMH) and the consequent (Brownian) martingale model, the determination of "*how*" data are dispersed is uniquely dictated by the quadratic variation of the process. Nevertheless, challenges emerge when questioning or relaxing this model, prompting the need for a distinct consideration of volatility and regularity and advising against a blanket association of volatility with risk. The introduction of memory, triggered by positive or negative autocorrelation, influences the level of regularity in the sequence and introduces a potential error in the assessment of financial risk based solely on volatility, whose value may not be influenced by autocorrelation.

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The need to neatly distinguish between *volatility* and *regularity* also arises from the tendency of literature to support the EMH based on the commonly observed null value of the empirical autocorrelation function of log-returns, a well-known stylized fact. However, a null empirical autocorrelation can also result from a time-varying pointwise regularity of log-returns, a change which is not necessarily detected by volatility. In addition to aiding in the formulation of more realistic models of financial dynamics, the pointwise regularity - owing to its direct connection with the martingale model benchmark - offers insights into market mechanisms that are not captured by volatility alone.

2 Background and Model

Some notions are recalled here which will be combined to show how regularity enhances the informational content of volatility.

Theorem 1. [*Lévy's Characterization Theorem*] *Given the filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$*, let* (X_t) *be a local martingale with* $X_0 = 0$*. Then, the following are equivalent:*

(a) $\{X_t\}$ *is standard Brownian motion on the underlying filtered probability space*
(b) $\{X_t\}$ *is continuous and* $\{X_t^2 - t\}$ *is a local martingale*
(c) $\{X_t\}$ *has quadratic variation* $\langle X \rangle_{2,t} = t$*.*

The equivalence between the condition of local martingale of an $\mathcal{F}_t$-Brownian motion and its quadratic variation is fundamental to justify why, in the context of efficient markets, volatility has traditionally served as a risk indicator: according to the Efficient Market Hypothesis (EMH) [6], if the discounted price process behaves as a martingale, then the only governing factor influencing its randomness is determined by the growth of its quadratic variation, proportional to the time interval t. The absence of alternative possibilities makes the adoption of volatility as a risk indicator a natural choice within this framework.

Pointwise Hölder Exponent. Given the continuous real-valued stochastic process $\{Z_t, t \geq 0\}$, its path roughness at any fixed $\tau > 0$ is usually measured through the *pointwise Hölder exponent* at τ. This is defined as [2,3]:

$$\alpha_Z(\tau) := \sup\{\alpha \in [0,1] : \limsup_{r \to 0^+} r^{-\alpha} \mathrm{Osc}_Z(\tau, r) < +\infty\} \tag{1}$$

where, for all real number $r > 0$ small enough,

$$\mathrm{Osc}_Z(\tau, r) := \sup\{|Z_{t'} - Z_{t''}| : (t', t'') \in [\tau - r, \tau + r]^2\}$$

is the oscillation of $\{Z_t\}$ on the circular neighbourhood of τ with radius r. For specific classes of stochastic processes, such as Gaussian processes, the zero-one law implies the existence of a non-random quantity $a_Z(t)$ for which $\mathbb{P}(a_Z(t) = \alpha_Z(t)) = 1$ [2]. When $\{Z_t\}$ is a semimartingale (e.g. Brownian motion), $\alpha_Z = \frac{1}{2}$. Deviations from $\frac{1}{2}$ characterize non-Markovian processes (whose quintessential example is the well-known fractional Brownian motion, fBm); processes with $\alpha_Z \in \left(\frac{1}{2}, 1\right)$ exhibit excessively high smoothness, while those with $\alpha_Z \in \left(0, \frac{1}{2}\right)$ display insufficient smoothness to satisfy the martingale property. Specifically, the quadratic variation of the process can be proven to be zero if $\alpha_Z > \frac{1}{2}$, and infinite if $\alpha_Z < \frac{1}{2}$.

Multifractional Processes with Random Exponent. When the Hölder exponent is allowed to change through time in a deterministic or stochastic way, a class of stochastic process, named Multifractional Processes with Random Exponent (MPRE), can be defined, subject to some technical constraints [3,8]. A special case of the general MPRE process is

$$K_t^{H,C} = C \int_{-\infty}^{t} \left[(t-s)_+^{H_s - 1/2} - (-s)_+^{H_s - 1/2} \right] dB_s, \tag{2}$$

where C is a scale parameter, $(x)_+ = \max(x, 0)$ and B is the Brownian motion. By introducing a dependence on H_s in the integrand instead of H_t, the integral in Eq. (2) can be formulated in the conventional Itô sense. [8] establish a rescaling limit showing that, for each fixed t, as $h \to 0$,

$$h^{-H_t} \left(K_{t+hr}^{H,C} - K_t^{H,C} \right) \implies C \int_{-\infty}^{r} \left[(r-s)_+^{H_t - 1/2} - (-s)_+^{H_t - 1/2} \right] d\tilde{B}_s \tag{3}$$

where $\tilde{B}_s$ is a Brownian motion independent of H_t. Equation (3), known as Local Asymptotical Self-Similarity property (LASS), states that in the neighborhood of any point t, K_t^H behaves like a fBm with Hurst-Hölder exponent H_t.

(Rough) Fractional Stochastic Volatility Model (RFSV). Introduced by [7] and based on the previous model defined by [5], the Rough Fractional Stochastic Volatility (RFSV) model of the price process S_t reads as:

$$\begin{cases} dS_t = \mu_t S_t dt + S_t \sigma_t dB_t \\ \sigma_t = \exp(X_t) \end{cases} \tag{4}$$

where μ_t is the drift term, B_t is a Brownian motion and X_t is a fractional Ornstein-Uhlenbeck (fOU) process satisfying

$$dX_t = \alpha(m - X_t)dt + \rho dB_t^H, \tag{5}$$

with $m \in \mathbb{R}$, ρ and α positive parameters and with B_t and B_t^H correlated in general. [1] replace the stochastic process in the first line of Eq. (4) by an MPRE driven by a Hölder exponent which follows a proper fOU process related to dX_t by a change of parameters. They show that the stochastic Hölder parameter of the MPRE can replace the log-volatility in the second equation of model (4). This sets up the *Fractional Stochastic Regularity Model* (FSR), defined as

$$\begin{cases} S_t = K_t^{H,C} \\ H_t = m' + \rho' \int_{-\infty}^{t} e^{-\alpha(t-s)} dB_s^H \end{cases} \tag{6}$$

where S_t denotes the log-price of a stock or an index, H_t is the unique pathwise solution of the fOU process of line 2 of (6). Denoting by n the length of the sampled version of MPRE, $m' = -\frac{1}{\log n} \cdot m + \frac{\log C}{\log n}$ and $\rho' = -\frac{1}{\log n} \cdot \rho$. Thus, Eqs. (4) and (6) state that a relation exists between volatility σ_t and regularity H_t when the log-price is modelled by an MPRE and the log-volatility is modeled by a fOU process. In this case, also the Hölder exponent follows a fOU process with parameters which are linear transforms of those used to model the log-volatility. This directly follows from Eq. (3), which entails (see [1]) (Table 1).

$$\log \sigma_{t,n} = \log C - H_t \log n. \tag{7}$$

Figure 1 exhibits the goodness of fit of relation (7) for six global financial indexes: Dow Jones Industrial Average (DJI, USA), Nasdaq Composite (IXIC, USA), Eurostoxx50 (SX5E, Europe), Footsie 100 (UKX, United Kingdom), Hang Seng (HSI, Hong Kong) and Straits Times (STI, Singapore). H_t was estimated as in [1,9].

Table 1. Data set and main statistics of the estimated H_t

	DJI	IXIC	SX5E	UKX	HSI	STI
Start date	1992-01-02	1971-02-05	2000-01-03	1984-01-03	1986-12-31	1987-12-28
End date	2021-12-28	2021-12-28	2021-12-31	2021-12-29	2021-12-29	2022-01-28
# Obs (n)	7,555	12,470	5,730	9,599	5,955	8,409
Mean	0.540	0.512	0.524	0.524	0.512	0.533
St. Dev.	0.0540	0.0549	0.0553	0.0464	0.0498	0.0536
Range	0.308–0.688	0.314–0.670	0.345–0.675	0.329–0.652	0.269–0.622	0.333–0.665

3 Meaning and Financial Interpretation of the Relationship Between Volatility and Regularity

As discussed in the previous section, the Hölder regularity offers insight into the extent to which the process diverges from the martingale property with the baseline value $H_t = 1/2$. Substituting the Hölder exponent for the volatility process is justified by this characterization and brings several advantages:

– Unlike volatility, the Hölder exponent is sensitive to autocorrelation irrespective of the scale parameter. Volatility fails to differentiate between data with high or low correlation when appropriate scale parameters are applied, leading to instances where data with identical volatility levels may exhibit varying degrees of correlation. This incongruity is problematic if volatility is intended to assess financial risk. Conversely, the Hölder parameters of data with differing autocorrelations, are different irrespective of the scale.

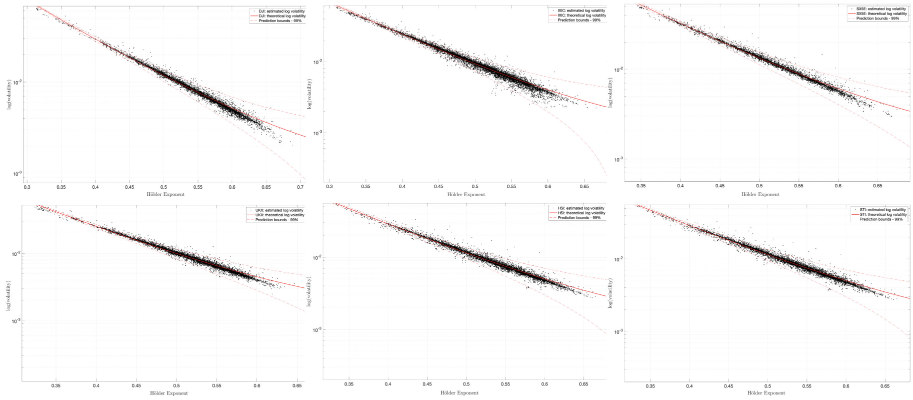

Fig. 1. Realized log volatility versus estimated Hölder exponents (one-day log-changes). X-axis: estimated Hölder exponent; Y-axis: estimated log volatility (black dots), theoretical relation given by Eq. (7) (red line), 99% prediction bounds (dashed red lines). H_t and realized volatility are estimated with a rolling window of 20 trading days.

- Volatility serves as a relative measure, indicating whether a market or asset currently displays more or less variability compared to past periods. However, it does not have the capacity to determine the "optimal" or "fair" level of volatility, one that aligns with an efficient market. In contrast, the Hölder parameter, ranging from 0 to 1, equals $1/2$ only when the process aligns with Brownian motion, a key aspect of the Efficient Market Hypothesis (EMH).
- As markets naturally gravitate towards the equilibrium state associated with $H_t = 1/2$ following deviations, the distance $|H_t - 1/2|$ becomes a significant indicator for determining optimal buying or selling times. The dynamics of H_t are expected to exhibit a fluctuating trend around the value $1/2$, with the rate of return to this equilibrium increasing as the deviation widens. Essentially, this mechanism provides a stochastic formalization and a theoretically grounded explanation for the commonly known trader adage "*What goes up, must come down.*"
- When financial prices exhibit local behavior resembling a fBm (as seen in processes like MPRE), the relationship between volatility and the Hölder exponent can be expressed through Eq. (7). Consequently, using the Hölder parameter instead of volatility does not result in any loss of information.

Table 2 provides a summary of the relationship between the Hölder exponent and the martingale condition, offering a financial interpretation of this connection [4]. Unlike volatility, the Hölder exponent offers a comprehensive assessment of market dynamics, addressing both the magnitude (*how much*) and character (*how*) of price variability. It provides insights into the deviation from equilibrium, represented by the value $H_t = 1/2$, which acts as a benchmark for a semi-martingale. The pointwise Hölder exponent serves as a descriptor of the prevailing dynamics of the discounted price process at a specific moment, distinguishing among a *momentum market* (linked to bullish phases or speculative bubbles), a *sideways market* (indicative of directionless efficiency), and a *reversal market* (resulting from rapid buy-and-sell activities, often following significant price adjustments or periods of uncertainty).

Table 2. Financial interpretation of H_t

H_t	Stochastic patterns	Agents' beliefs	Market patterns
$> \frac{1}{2}$	Persistence - Smooth paths - $\langle X \rangle_{2,t} = 0$	New information confirm outstanding position	"Low" volatility - Momentum Overconfidence - Underreaction
$= \frac{1}{2}$	Independence - Martingale - $\langle X \rangle_{2,t} = t$	Information fully incorporated by price	"Normal" volatility - Sideways market - Efficiency
$< \frac{1}{2}$	Mean-reversion - Rough paths - $\langle X \rangle_{2,t} = \infty$	New information disrupt outstanding position	"High" volatility - Reversals - Overreaction

This interpretation suggests that the apparently conflicting paradigms of Rationality and Behavioral Finance can coalesce within a comprehensive framework of bounded rationality, providing a more nuanced understanding of market dynamics. Within this framework, the pointwise Hölder exponent explicitly identifies *when* rationality transitions to irrationality, a shift that volatility fails to capture because of its insensitiveness to changes in the sign and intensity of autocorrelation. In this sense, assuming the FSR model, the *fair volatility* is the value corresponding to the value 1/2 of the Hölder exponent, via relation (7).

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