Parameter Estimation for Kumaraswamy Weibull Distribution Under Ranked Set Sampling

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Abstract In this study, modified maximum likelihood (MML) estimators for the location and scale parameters of the Kumaraswamy Weibull (Kw-Weibull) distribution are derived based on ranked set sampling (RSS) method under the assumption of known shape parameters, see Tiku (1967, 1968) in the context of MML methodology. MML estimators based on RSS are compared with the traditional maximum likelihood (ML) estimators based on simple random sampling (SRS) via Monte-Carlo simulation study in terms of bias, mean squares error (MSE) and relative efficiency (RE) criteria. According to the results of the simulation study, MML estimators based on RSS are found to be more efficient than the ML estimators based on SRS in most of the simulation scenarios.

1 Introduction

The Weibull distribution, which takes its name from Waloddi Weibull, is one of the most popular probability distributions having widespread usage in different areas of science, such as reliability theory, engineering, biology, hydrology, etc., see Weibull (1939). However, it fails to provide a good fit to data sets having bathtub-shaped or upside-down bathtub-shaped failure rates that are commonly observed in the fields of engineering and reliability. In order to model such datasets, many different forms

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of Weibull distribution have been proposed in the literature, for example, inverse Weibull, exponentiated Weibull, modified Weibull, transmuted inverse Weibull, etc. The Kumaraswamy Weibull (Kw-Weibull) distribution proposed by Cordeiro et al. (2010) is another form of the Weibull distribution, for the details of Kumaraswamy distribution, see Kumaraswamy (1980). It has a high degree of flexibility for modeling positive data encountered especially in the area of reliability.

It is worthy to mention that obtaining a sample that is fairly representative of the population is an important step in statistical inference about the real world. Simple random sampling (SRS) is the most widely used sampling method in the literature. Nevertheless, sample selected via SRS may not represent the population well enough. To overcome this problem, the ranked set sampling (RSS) method is used as a powerful alternative to SRS especially for circumstances when the variable of interest is expensive or cannot easily be measured but can easily be ranked at a negligible cost, see McIntyre (1952). It offers the benefit of increasing the efficiency of the estimators for population parameters by containing the information provided not only by measured observations but also by the ranking process.

There is an extensive literature focusing on the performance of RSS and its different forms in parameter estimation. Dell and Clutter (1972) showed that an unbiased estimator for the population mean is obtained based on RSS irrespective of ranking errors and it is at least as efficient as the estimator obtained based on SRS with the same sample size. Stokes (1980a, b) and Stokes and Sager (1988) proved that RSS provides more precise estimators for the variance, Pearson correlation coefficient and cumulative distribution function, respectively. Stokes (1995) examined the maximum likelihood (ML) and the best linear unbiased estimators for the location and scale parameters of the location-scale distribution family based on RSS. Barabesi and El-Sharaawi (2001) showed that parametric inference based on RSS provides more information than SRS. Abu-Dayyeh et al. (2004) proposed different estimators for the location and scale parameters of the logistic distribution using SRS, RSS or some modifications of RSS. Helu et al. (2010) compared ML, method of moments (MoM) and Bayesian estimators for the Weibull distribution parameters based on different sampling schemes such as SRS, RSS and modified ranked set sample (MRSS). Al-Omari and Al-Hadhrami (2011) compared ML estimators for the parameters of the modified Weibull distribution based on the extreme ranked set sampling (ERSS) and SRS. Balci et al. (2013) derived modified maximum likelihood (MML) estimators for the population mean and variance under RSS. Hussian (2014) compared the ML and Bayesian estimators for the Kumaraswamy distribution parameters based on SRS and RSS. Yousef and Al-Subh (2014) compared ML estimators for the Gumbel distribution parameters based on SRS and RSS with MoM and regression estimators based on SRS. Dey et al. (2017) discussed the estimation of the parameter of the Rayleigh distribution using different estimation approaches based on different sampling methods such as SRS, RSS, MRSS and median RSS. Samuh et al. (2020) investigated the ML estimators for the new Weibull-Pareto distribution parameters based on SRS, RSS and some forms of RSS. Taconeli and de Lara (2022) evaluated the performance of nine different estimators for the discrete Weibull distribution parameters based on SRS and RSS.

In this study, the MML estimators for the location and scale parameters of Kw-Weibull distribution based on RSS are obtained when the shape parameters are assumed to be known. Then, they are compared with the traditional ML estimators based on SRS in terms of bias, mean squares error (MSE) and relative efficiency (RE) criteria via Monte Carlo simulation. The reason why we include MML methodology into the study is its capability to avoid problems due to iterative methods by providing explicit solutions. MML estimators are also asymptotically equivalent to ML estimators as well as having high efficiencies even for small sample sizes. As far as we know, this is the first study investigating the efficiencies of the MML estimators for Kw-Weibull parameters in the context of RSS. Also, note that all the computations are conducted under the assumption of perfect ranking. See Ergenc and Senoglu (2023) for the ML estimators of Kw-Weibull parameters based on SRS.

The remaining sections of this work are organized as follows. In Sect. 2, Kw-Weibull distribution is described and the MML estimators for the location and scale parameters of the Kw-Weibull distribution based on RSS are derived. Section 3 presents the results of the Monte Carlo simulation study. Finally, concluding remarks are given in Sect. 4.

2 Parameter Estimation

In this section, Kw-Weibull distribution is briefly described and the MML estimators based on RSS for its location and scale parameters are obtained with the assumption of known shape parameters.

2.1 Kw-Weibull Distribution

Let X be a random variable with Kw-Weibull density function

$$\begin{aligned} f_X(x) =& \frac{abp}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{p-1}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}\left[1-\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}\right]^{a-1} \\ & \times\left\{1-\left[1-\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}\right]^a\right\}^{b-1} \quad ; x \geq \mu; \sigma > 0; a, b, p \geq 0 \end{aligned} \tag{1}$$

and distribution function

$$F_X(x) = 1-\left\{1-\left[1-exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}\right]^a\right\}^b. \tag{2}$$

Here, a, b and p are the shape parameters, μ and σ are the location and scale parameters, respectively. The random variable X having Kw-Weibull distribution with the mentioned parameters is shortly denoted by $X \sim$ Kw-Weibull(a, b, p, σ, μ).

Kw-Weibull distribution is negatively or positively skewed in addition to being leptokurtic (having kurtosis greater than 3) or platykurtic (having kurtosis less than 3) based on the values of the shape parameters. It is also reduced to some well-known probability distributions, such as Weibull, exponentiated Weibull, Rayleigh, exponentiated Rayleigh, Exponential and exponentiated Exponential for some specific values of its shape parameters. For the details of the Kw-Weibull distribution, one may refer to Cordeiro et al. (2010), Guven and Senoglu (2023).

2.2 MML Estimators Based on RSS

In order to obtain a ranked set sample, firstly m sets of size m are selected by using SRS. Then, the sample units in each set are ranked in ascending order according to some inexpensive criterion, e.g., visual inspection, expert opinion or a concomitant variable without any exact measurement. After ranking, the unit with the smallest rank is selected from the first set. The unit with the second smallest rank is selected from the second set and this procedure is continued until the largest ranked unit is selected from the mth set. The entire process constitutes a cycle and can be repeated for r times yielding a sample of size $n = mr$, if needed. At the end, only the selected units are actually measured.

For this study, let $X_{(i)ic}$, $i = 1, \ldots, m$; $c = 1, \ldots, r$, be the resulted sample with size $n = mr$ from Kw-Weibull(a, b, p, σ, μ) distribution. Realize that $X_{(i)ic}$ denotes the ith order statistics for the ith set from the cth cycle. In the rest of this study, X_{ic} is used instead of $X_{(i)ic}$ for the sake of simplicity.

The probability density function (pdf) of X_{ic} is obtained as

$$f_{X_{ic}}(x_{ic}) = \frac{m!}{(i-1)!(m-i)!}\left[F_X(x_{ic})\right]^{i-1}\left[1 - F_X(x_{ic})\right]^{m-i} f_X(x_{ic}) \tag{3}$$

where f and F are the density and distribution functions of Kw-Weibull distribution given in equations (1) and (2), respectively.

In order to obtain the ML estimators for the location and scale parameters of Kw-Weibull distribution based on RSS, first the likelihood (L) function is written as follows

$$L = \prod_{c=1}^{r}\prod_{i=1}^{m}\frac{m!}{(i-1)!(m-i)!}\frac{1}{\sigma}\left[F(z_{ic})\right]^{i-1}\left[1 - F(z_{ic})\right]^{m-i} f(z_{ic}) \tag{4}$$

where $z_{ic} = \frac{x_{ic}-\mu}{\sigma}$. Then, the log-likelihood ($\ln L$) function is obtained by taking the logarithm of L as given below

$$\ln L = \ln\left(\prod_{c=1}^{r}\prod_{i=1}^{m}\frac{m!}{(i-1)!(m-i)!}\right) - n\ln(\sigma) + \sum_{c=1}^{r}\sum_{i=1}^{m}(i-1)\ln\left[F\left(z_{ic}\right)\right] + \sum_{c=1}^{r}\sum_{i=1}^{m}(m-i)\ln\left[1-F\left(z_{ic}\right)\right] + \sum_{c=1}^{r}\sum_{i=1}^{m}\ln\left[f\left(z_{ic}\right)\right]. \tag{5}$$

By taking the partial derivatives of $\ln L$ with respect to the parameters μ and σ and setting them equal to zero, the likelihood equations are obtained as

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{\sigma}\left(\sum_{c=1}^{r}\sum_{i=1}^{m}(i-1)\frac{f(z_{ic})}{F(z_{ic})} - \sum_{c=1}^{r}\sum_{i=1}^{m}(m-i)\frac{f(z_{ic})}{1-F(z_{ic})} + \sum_{c=1}^{r}\sum_{i=1}^{m}\frac{f'(z_{ic})}{f(z_{ic})}\right) = 0 \tag{6}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\left(n + \sum_{c=1}^{r}\sum_{i=1}^{m}(i-1)z_{ic}\frac{f(z_{ic})}{F(z_{ic})} - \sum_{c=1}^{r}\sum_{i=1}^{m}(m-i)z_{ic}\frac{f(z_{ic})}{1-F(z_{ic})} + \sum_{c=1}^{r}\sum_{i=1}^{m}z_{ic}\frac{f'(z_{ic})}{f(z_{ic})}\right) = 0. \tag{7}$$

Likelihood equations in (6) and (7) cannot be solved in closed form since they include nonlinear functions of the parameters. Therefore, in order to obtain ML estimators, numerical methods should be used. However, they have some drawbacks such as i) non-convergence, ii) convergence to wrong roots and iii) convergence to multiple roots. Thus, in this study Tiku's MML methodology is used, which provides explicit estimators maintaining the same asymptotic properties with ML estimators. In MML methodology, since the complete sums are invariant to ordering, i.e. $\sum_{c=1}^{r}\sum_{i=1}^{m} z_{(i)c} = \sum_{c=1}^{r}\sum_{i=1}^{m} z_{ic}$, first the likelihood equations in (6) and (7) are written in terms of standardized order statistics $z_{(1)c} < z_{(2)c} < \cdots < z_{(m)c}$ $(c = 1, \ldots, r)$ as follows

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{\sigma}\left(\sum_{c=1}^{r}\sum_{i=1}^{m}(i-1)g_1(z_{(i)c}) - \sum_{c=1}^{r}\sum_{i=1}^{m}(m-i)g_2(z_{(i)c}) + \sum_{c=1}^{r}\sum_{i=1}^{m}g_3(z_{(i)c})\right) = 0 \tag{8}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\left(n + \sum_{c=1}^{r}\sum_{i=1}^{m}(i-1)z_{(i)c}g_1(z_{(i)c}) - \sum_{c=1}^{r}\sum_{i=1}^{m}(m-i)z_{(i)c}g_2(z_{(i)c}) + \sum_{c=1}^{r}\sum_{i=1}^{m}z_{(i)c}g_3(z_{(i)c})\right) = 0. \tag{9}$$

Here, $z_{(i)c} = \frac{x_{(i)c}-\mu}{\sigma}$, $g_1(z_{(i)c}) = \frac{f(z_{(i)c})}{F(z_{(i)c})}$, $g_2(z_{(i)c}) = \frac{f(z_{(i)c})}{1-F(z_{(i)c})}$ and $g_3(z_{(i)c}) = \frac{f'(z_{(i)c})}{f(z_{(i)c})}$.

Secondly, nonlinear functions, $g_1(z_{(i)c})$, $g_2(z_{(i)c})$ and $g_3(z_{(i)c})$ are linearized around the expected values of the ordered statistics, i.e. $t_{(i)c} = E(z_{(i)c})$, by using the first two terms of the Taylor series expansion as follows

$$\begin{aligned} g_1(z_{(i)c}) &\cong \alpha_{1i} - \beta_{1i}z_{(i)c}, \\ g_2(z_{(i)c}) &\cong \alpha_{2i} + \beta_{2i}z_{(i)c}, \\ g_3(z_{(i)c}) &\cong \alpha_{3i} - \beta_{3i}z_{(i)c} \end{aligned} \tag{10}$$

where

$$\begin{aligned} \alpha_{1i} &= \frac{f(t_{(i)c})}{F(t_{(i)c})} + \beta_{1i}t_{(i)c}, \quad \beta_{1i} = -\left(\frac{f'(t_{(i)c})}{F(t_{(i)c})} - \frac{(f(t_{(i)c}))^2}{(F(t_{(i)c}))^2}\right), \\ \alpha_{2i} &= \frac{f(t_{(i)c})}{1-F(t_{(i)c})} - \beta_{2i}t_{(i)c}, \quad \beta_{2i} = \left(\frac{f'(t_{(i)c})}{1-F(t_{(i)c})} + \frac{(f(t_{(i)c}))^2}{(1-F(t_{(i)c}))^2}\right), \\ \alpha_{3i} &= \frac{f'(t_{(i)c})}{f(t_{(i)c})} + \beta_{3i}t_{(i)c} \quad \text{and} \quad \beta_{3i} = -\left(\frac{f''(t_{(i)c})f(t_{(i)c}) - (f'(t_{(i)c}))^2}{(f(t_{(i)c}))^2}\right). \end{aligned}$$

It should be noted that z_{ic} are independent and non-identically (inid) distributed, therefore expected values of standardized order statistics, $t_{(i)c}$, are calculated via Monte Carlo simulation to avoid mathematical difficulties in theory.

By incorporating the linearized functions in (10) into (8) and (9), the modified likelihood equations are obtained as below

$$\frac{\partial \ln L^*}{\partial \mu} = -\frac{1}{\sigma}\sum_{c=1}^{r}\sum_{i=1}^{m}(i-1)(\alpha_{1i} - \beta_{1i}z_{(i)c}) + \frac{1}{\sigma}\sum_{c=1}^{r}\sum_{i=1}^{m}(m-i)\left(\alpha_{2i} + \beta_{2i}z_{(i)c}\right) - \frac{1}{\sigma}\sum_{c=1}^{r}\sum_{i=1}^{m}\left(\alpha_{3i} - \beta_{3i}z_{(i)c}\right) = 0 \tag{11}$$

and

$$\frac{\partial \ln L^*}{\partial \sigma} = -\frac{n}{\sigma} - \frac{1}{\sigma}\sum_{c=1}^{r}\sum_{i=1}^{m}(i-1)z_{(i)c}(\alpha_{1i} - \beta_{1i}z_{(i)c})$$
$$+\frac{1}{\sigma}\sum_{c=1}^{r}\sum_{i=1}^{m}(m-i)z_{(i)c}\left(\alpha_{2i} + \beta_{2i}z_{(i)c}\right)$$
$$-\frac{1}{\sigma}\sum_{c=1}^{r}\sum_{i=1}^{m}z_{(i)c}\left(\alpha_{3i} - \beta_{3i}z_{(i)c}\right) = 0. \tag{12}$$

Simultaneous solutions of the equations (11) and (12) give the following MML estimators

$$\hat{\mu} = K - D\hat{\sigma} \quad \text{and} \quad \hat{\sigma} = \frac{-B + \sqrt{B^2 + 4nC}}{2\sqrt{n(n-1)}} \tag{13}$$

where

$$K = \frac{\sum_{c=1}^{r}\sum_{i=1}^{m}\delta_i x_{(i)c}}{M}, \quad M = r\sum_{i=1}^{m}\delta_i, \quad D = \frac{r\sum_{i=1}^{m}\Delta_i}{M},$$
$$\delta_i = (i-1)\beta_{1i} + (m-i)\beta_{2i} + \beta_{3i}, \quad \Delta_i = (i-1)\alpha_{1i} - (m-i)\alpha_{2i} + \alpha_{3i},$$
$$B = \sum_{c=1}^{r}\sum_{i=1}^{m}\Delta_i\left(x_{(i)c} - K\right) \quad \text{and} \quad C = \sum_{c=1}^{r}\sum_{i=1}^{m}\delta_i\left(x_{(i)c} - K\right)^2.$$

Here, it should be noted if $\hat{\mu}$ is larger than $Min(x_{(1)c})$, it is taken as $Min(x_{(1)c}) - 10^{-4}$. Also, n in the divisor of $\hat{\sigma}$ is replaced by $\sqrt{n(n-1)}$ as a bias-correction. See also Akgül and Şenoğlu (2017) in the context of Weibull distribution.

3 Monte-Carlo Simulation

In this section, ML and MML estimators for the Kw-Weibull location and scale parameters are compared in terms of bias, MSE and RE criteria via an extensive Monte Carlo simulation study. REs are calculated using the following equalities

$$RE_1 = \frac{MSE(\hat{\mu}_{MML,RSS})}{MSE(\hat{\mu}_{ML,SRS})} \times 100 \tag{14}$$

and

$$RE_2 = \frac{MSE(\hat{\sigma}_{MML,RSS})}{MSE(\hat{\sigma}_{ML,SRS})} \times 100. \tag{15}$$

Here, $\hat{\mu}_{MML,RSS}$ and $\hat{\sigma}_{MML,RSS}$ denote the MML estimators for the location and scale parameters of the Kw-Weibull distribution based on RSS, respectively, while $\hat{\mu}_{ML,SRS}$ and $\hat{\sigma}_{ML,SRS}$ denote the ML estimators for the location and scale parameters of the Kw-Weibull distribution based on SRS, respectively. Note that ML estimates based on SRS and MML estimates based on RSS are calculated by using 10,000 replications. ML estimates of the parameters of interest are computed using optim function in R statistical software, while MML estimates are obtained explicitly.

In the simulation scenario, following values of the set size m, cycle size r and the shape parameters a, b and p are used

- (m, r)=(3,4), (4,3) and (6,2) for n=12
  (m, r)=(3,8), (4,6), (6,4) and (8,3) for n=24
- (a, b)= (1,1), (2,2), (5,6) and (6,3.5)

and

- p= 1.5, 3 and 6.

Simulated bias and MSE values for the estimators of the location and scale parameters of the Kw-Weibull distribution and the corresponding RE values are presented in Table 1.

The following conclusions can be drawn from Table 1.

- For small value of n, i.e., $n = 12$, $\hat{\mu}_{ML,SRS}$ and $\hat{\sigma}_{ML,SRS}$ do not perform well in terms of bias criterion while $\hat{\mu}_{MML,RSS}$ and $\hat{\sigma}_{MML,RSS}$ have negligibly small bias in majority of the cases except when $p = 1.5$, $(a, b) = (2, 2)$ and $p = 3$, $(a, b) = (1, 1)$. A point worthy of note for some of these exceptional cases is that biases of $\hat{\mu}_{MML,RSS}$ and $\hat{\sigma}_{MML,RSS}$ become smaller than $\hat{\mu}_{ML,SRS}$ and $\hat{\sigma}_{ML,SRS}$ as the set size m increases, respectively.
- MML estimators based on RSS and ML estimators based on SRS of the location parameter μ and scale parameter σ have negligibly small bias values for $n = 24$ except when $p = 1.5$, $(a, b) = (2, 2)$ and $p = 3$, $(a, b) = (1, 1)$. In the mentioned cases $\hat{\mu}_{MML,RSS}$ and $\hat{\sigma}_{MML,RSS}$ have considerably larger bias than their counterparts $\hat{\mu}_{ML,SRS}$ and $\hat{\sigma}_{ML,SRS}$, respectively.
- Overall, in view of bias, it can be deduced that MML estimators based on RSS of μ and σ have superiority over ML estimators based on SRS of μ and σ.
- $\hat{\mu}_{MML,RSS}$ outperforms $\hat{\mu}_{ML,SRS}$ in terms of efficiency in most of the cases and this superiority becomes more apparent as the set size m increases for a fixed n. Here, it should be noted that when $p = 1.5$, $(a, b) = (2, 2)$ and $p = 3$, $(a, b) = (1, 1)$, $\hat{\mu}_{ML,SRS}$ has higher efficiency than $\hat{\mu}_{MML,RSS}$ only for some small values of m, see RE_1 from Table 1.
- $\hat{\sigma}_{MML,RSS}$ performs better than $\hat{\sigma}_{ML,SRS}$ in terms of efficiency for almost all cases. However, when $n = 24$, $p = 1.5$, $(a, b) = (2, 2)$ and $p = 3$, $(a, b) = (1, 1)$, $\hat{\sigma}_{ML,SRS}$ has slightly better efficiencies than $\hat{\sigma}_{MML,RSS}$ for small values of m, see RE_2 from Table 1.
- RE_1 and RE_2 values also reveal that the efficiencies of MML estimators based on RSS increase as the set size m increases for a fixed sample size n.

Table 1 Simulated Bias, MSE and RE values for the MML estimators based on RSS and ML estimators based on SRS

$p = 1.5$								
			$\hat{\mu}$		$\hat{\sigma}$			
			Bias	MSE	Bias	MSE	RE_1	RE_2
		(m, r)	$(a, b) = (2, 2)$					
ML(SRS)	$n = 12$	–	0.072340	0.029763	–0.078016	0.045597		
MML(RSS)		(3, 4)	0.118564	0.034182	–0.070474	0.040843	114	90
		(4, 3)	0.097934	0.028135	–0.066931	0.036501	94	80
		(6, 2)	0.075475	0.021436	–0.058926	0.029582	72	65
ML(SRS)	$n = 24$	–	0.038350	0.013196	–0.041993	0.021870		
MML(RSS)		(3, 8)	0.113396	0.022673	–0.065169	0.022020	172	101
		(4, 6)	0.093946	0.017898	–0.060182	0.019451	136	89
		(6, 4)	0.072783	0.012764	–0.055847	0.015483	97	71
		(8, 3)	0.062998	0.010353	–0.052429	0.013266	79	61
		(m, r)	$(a, b) = (5, 6)$					
ML(SRS)	$n = 12$	–	0.070272	0.047452	–0.068373	0.044535		
MML(RSS)		(3, 4)	0.049735	0.040172	–0.038698	0.038343	85	86
		(4, 3)	0.046873	0.035764	–0.038293	0.033975	75	76
		(6, 2)	0.044277	0.029992	–0.039003	0.028480	63	64
ML(SRS)	$n = 24$	–	0.032074	0.022402	–0.031069	0.020975		
MML(RSS)		(3, 8)	0.041789	0.020389	–0.030900	0.018943	91	90
		(4, 6)	0.045113	0.018845	–0.036734	0.017521	84	84
		(6, 4)	0.043766	0.015529	–0.038443	0.014650	69	70
		(8, 3)	0.043015	0.013745	–0.039000	0.012994	61	62
$(a, b) = (6, 3.5)$								
ML(SRS)	$n = 12$	–	0.089519	0.070353	–0.071578	0.045657		
MML(RSS)		(3, 4)	0.065021	0.061448	–0.036593	0.039812	87	87
		(4, 3)	0.056816	0.055442	–0.034802	0.036263	79	79
		(6, 2)	0.050850	0.044862	–0.034203	0.029368	64	64
ML(SRS)	$n = 24$	–	0.041281	0.032912	–0.032437	0.021271		
MML(RSS)		(3, 8)	0.049610	0.029827	–0.025006	0.019225	91	90
		(4, 6)	0.050599	0.028167	–0.029468	0.017995	86	85
		(6, 4)	0.049945	0.023364	–0.032949	0.014921	71	70
		(8, 3)	0.050030	0.020102	–0.035115	0.013003	61	61
$p = 3$								
			$\hat{\mu}$		$\hat{\sigma}$			
			Bias	MSE	Bias	MSE	RE_1	RE_2
		(m, r)	$(a, b) = (1, 1)$					
ML(SRS)	$n = 12$	–	0.074350	0.036051	–0.080780	0.039964		
MML(RSS)		(3, 4)	0.083876	0.031710	–0.081147	0.035811	88	90
		(4, 3)	0.079154	0.028206	–0.078733	0.032574	78	82
		(6, 2)	0.069088	0.023539	–0.069279	0.027396	65	69

Table 1 (continued)

ML(SRS)	$n = 24$	–	0.038107	0.016294	–0.040955	0.018091		
MML(RSS)		(3, 8)	0.081924	0.018450	–0.078850	0.020235	113	112
		(4, 6)	0.074480	0.016125	–0.074630	0.018297	99	101
		(6, 4)	0.064266	0.013034	–0.065295	0.015079	80	83
		(8, 3)	0.058952	0.011039	–0.061466	0.012996	68	72
		(m, r)	$(a, b) = (2, 2)$					
ML(SRS)	$n = 12$	–	0.063038	0.040924	–0.066700	0.044006		
MML(RSS)		(3, 4)	0.043504	0.033315	–0.045686	0.037454	81	85
		(4, 3)	0.041683	0.029675	–0.044383	0.033306	73	76
		(6, 2)	0.041134	0.025014	–0.043584	0.028443	61	65
ML(SRS)	$n = 24$	–	0.031525	0.019718	–0.033480	0.021016		
MML(RSS)		(3, 8)	0.036230	0.016268	–0.037837	0.018210	83	87
		(4, 6)	0.036178	0.015217	–0.038019	0.017004	77	81
		(6, 4)	0.037944	0.012768	–0.040078	0.014503	65	69
		(8, 3)	0.038649	0.011388	–0.040665	0.012865	58	61
		(m, r)	$(a, b) = (5, 6)$					
ML(SRS)	$n = 12$	–	0.066067	0.049908	–0.065113	0.046415		
MML(RSS)		(3, 4)	0.023655	0.043534	–0.027627	0.040959	87	88
		(4, 3)	0.028471	0.038423	–0.030532	0.036367	77	78
		(6, 2)	0.028696	0.031618	–0.029825	0.030151	63	65
ML(SRS)	$n = 24$	–	0.031656	0.024048	–0.031556	0.022217		
MML(RSS)		(3, 8)	0.015219	0.020430	–0.019033	0.019377	85	87
		(4, 6)	0.021625	0.018428	–0.024169	0.017512	77	79
		(6, 4)	0.029105	0.015444	–0.030505	0.014800	64	67
		(8, 3)	0.030638	0.013605	–0.031642	0.013070	57	59
$p = 3$								
			$\hat{\mu}$		$\hat{\sigma}$			
			Bias	MSE	Bias	MSE	RE_1	RE_2
		(m, r)	$(a, b) = (6, 3.5)$					
ML(SRS)	$n = 12$	–	0.073271	0.058461	–0.065721	0.045182		
MML(RSS)		(3, 4)	0.025503	0.050083	–0.024275	0.039212	86	87
		(4, 3)	0.029301	0.046109	–0.026956	0.036219	79	80
		(6, 2)	0.034794	0.038262	–0.031707	0.030159	65	67
ML(SRS)	$n = 24$	–	0.035316	0.028058	–0.031097	0.021661		
MML(RSS)		(3, 8)	0.017353	0.023998	–0.016964	0.018845	86	87
		(4, 6)	0.024369	0.022289	–0.022939	0.017479	79	81
		(6, 4)	0.030895	0.019065	–0.028034	0.015003	68	69
		(8, 3)	0.033621	0.016752	–0.030527	0.013242	60	61
$p = 6$								
		(m, r)	$(a, b) = (1, 1)$					
ML(SRS)	$n = 12$	–	0.067451	0.044927	–0.070986	0.043264		
MML(RSS)		(3, 4)	0.034575	0.036711	–0.051248	0.037367	82	86
		(4, 3)	0.034040	0.032620	–0.046275	0.033504	73	77
		(6, 2)	0.033945	0.026890	–0.041317	0.027858	60	64

Table 1 (continued)

ML(SRS)	$n = 24$	–	0.031876	0.021009	–0.033776	0.020162		
MML(RSS)		(3, 8)	0.026835	0.017631	–0.042409	0.018324	83	91
		(4, 6)	0.032119	0.016027	–0.043968	0.016822	76	83
		(6, 4)	0.034754	0.013490	–0.042608	0.014250	64	71
		(8, 3)	0.036360	0.011805	–0.042345	0.012472	56	62
		(m, r)	$(a, b) = (2, 2)$					
ML(SRS)	$n = 12$	–	0.061994	0.046736	–0.065090	0.046791		
MML(RSS)		(3, 4)	0.015073	0.038908	–0.023530	0.039738	83	85
		(4, 3)	0.024646	0.035461	–0.030801	0.036364	76	78
		(6, 2)	0.024914	0.028947	–0.029336	0.030147	62	64
ML(SRS)	$n = 24$	–	0.031791	0.022503	–0.033409	0.022475		
MML(RSS)		(3, 8)	0.014317	0.018875	–0.022367	0.019448	84	87
		(4, 6)	0.021008	0.017164	–0.027384	0.017814	76	79
		(6, 4)	0.017814	0.014135	–0.030223	0.014779	63	66
		(8, 3)	0.030261	0.012393	–0.033767	0.013011	55	58
$p = 6$								
			$\hat{\mu}$		$\hat{\sigma}$			
			Bias	MSE	Bias	MSE	RE_1	RE_2
		(m, r)	$(a, b) = (5, 6)$					
ML(SRS)	$n = 12$	–	0.065423	0.049333	–0.065108	0.047079		
MML(RSS)		(3, 4)	0.015262	0.042233	–0.019471	0.040597	86	86
		(4, 3)	0.024357	0.037928	–0.027087	0.036563	77	78
		(6, 2)	0.025777	0.032181	–0.027499	0.031188	65	66
ML(SRS)	$n = 24$	–	0.029579	0.023694	–0.029604	0.022530		
MML(RSS)		(3, 8)	0.009252	0.021041	–0.013529	0.020209	89	90
		(4, 6)	0.015196	0.019067	–0.018253	0.018450	80	82
		(6, 4)	0.023697	0.015617	–0.025511	0.015168	66	67
		(8, 3)	0.025037	0.013430	–0.026268	0.013090	57	58
$(a, b) = (6, 3.5)$								
ML(SRS)	$n = 12$	–	0.067278	0.053691	–0.063828	0.046907		
MML(RSS)		(3, 4)	0.016559	0.046085	–0.018403	0.040374	86	86
		(4, 3)	0.022479	0.041768	–0.023273	0.036743	78	78
		(6, 2)	0.030241	0.034589	–0.029848	0.030468	64	65
ML(SRS)	$n = 24$	–	0.033218	0.025832	–0.031413	0.022463		
MML(RSS)		(3, 8)	0.011041	0.022555	–0.013359	0.019740	87	88
		(4, 6)	0.017659	0.020266	–0.018731	0.017864	78	80
		(6, 4)	0.027006	0.016983	–0.026798	0.014978	66	67
		(8, 3)	0.030047	0.014780	–0.029262	0.013084	57	58

The reason why we use various different values of m and r in this study is to see the tradeoff between the efficiencies of the MML estimators based on RSS and the practical issues. It is mentioned in the literature that the efficiency of an estimator based on RSS is an increasing function of the set size m, see for example, Patil et al. (1994) and Kowalczyk (2005). Results in Table 1 are consistent with the literature since the efficiencies of the MML estimators based on RSS increase as the set size m increases for a fixed n, see RE_1 and RE_2 values in Table 1. However, large values of set size m may cause an increase in cost and also difficulties in ranking, so in order to obtain enough number of measurements in statistical inference problems, the cycle size r is increased instead of m in RSS procedure.

4 Conclusion

In this study, MML estimators for the location and scale parameters of the Kw-Weibull distribution are obtained under RSS in explicit forms. Then the performances of the proposed estimators are compared with those of the traditional ML estimators based on SRS according to criteria of bias, MSE and RE using Monte Carlo simulation study. MML estimators based on RSS are found to be more efficient compared to the traditional ML estimators based on SRS in most of the cases defined in simulation scenario. Simulation results also indicate that the efficiencies of the MML estimators increase as the set size m increases as expected. Note that MML methodology based on RSS can easily be extended to other distributions belong to the location scale family.

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